



Rational points of varieties with ample cotangent bundle over function fields

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Abstract Let K be the function field of a smooth curve over an algebraically closed field k . Let X be a scheme, which is smooth and projective over K . Suppose that the cotangent bundle $\Omega_{X/K}$ is ample. Let $R := \text{Zar}(X(K) \cap X)$ be the Zariski closure of the set of all K -rational points of X , endowed with its reduced induced structure. We prove that for each irreducible component \mathfrak{R} of R , there is a projective variety \mathfrak{R}'_0 over k and a finite and surjective K^{sep} -morphism $\mathfrak{R}'_{0,K^{\text{sep}}} \rightarrow \mathfrak{R}_{K^{\text{sep}}}$, which is birational when $\text{char}(K) = 0$. This improves on results of Noguchi and Martin-Deschamps in characteristic 0. In positive characteristic, our result can be used to give the first examples of varieties, which are not embeddable in abelian varieties and satisfy an analog of the Bombieri–Lang conjecture.

1 Introduction

Recall that the Bombieri–Lang conjecture (see [22, middle of p. 108]) asserts that the set of rational points of a variety of general type over a number field is not dense. This conjecture can be proven in the situation where the variety is embeddable in an abelian variety (this is a deep result of Faltings, see [7]) but it is not known to be true in any other situation, as far as the authors know.

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Over function fields, it seems reasonable to make the following conjecture, which must have been part of the folklore for some time.

Conjecture 1.1 (‘Bombieri–Lang’ conjecture over function fields) *Let K_0 be the function field of a variety over an algebraically closed field k_0 . Let Z be a variety of general type over K_0 . Suppose that $\text{Zar}(Z(K_0)) = Z$. Then there exists a variety Z_0 over k_0 and a rational, dominant, generically finite K_0^{sep} -map $g : Z_{0, K_0^{\text{sep}}} \rightarrow Z_{K_0^{\text{sep}}}$.*

One might speculate that the rational map g appearing in Conjecture 1.1 can also be taken to be generically purely inseparable.

In his article [14, p. 781], Lang gave a loose formulation of Conjecture 1.1 for $\text{char}(k_0) = 0$. If Z is embeddable in an abelian variety and $\text{char}(k_0) = 0$, Conjecture 1.1 can be proven (see [4] and [8]). It can also be proven in the situation where $\text{char}(k_0) = 0$, the variety is smooth and its cotangent bundle is ample. This is a result of Noguchi, which was also proved independently by Martin-Deschamps (see [22] and [19]). When Z is embeddable in an abelian variety and $\text{char}(k_0) > 0$, Conjecture 1.1 is a theorem of Hrushovski (see [12]). See also [23, 27] and [2] for different proofs of Hrushovski’s theorem. When Z is of dimension 1 and $\text{char}(k_0) = 0$, Conjecture 1.1 was first proven by Manin and Grauert and several other proofs were given in the course of the 1970s (eg. by Parshin and Arakelov). See the articles [9, 17, 18] and [5, chap. I]. When Z is of dimension 1 and $\text{char}(k_0) > 0$, Conjecture 1.1 was first proven by Samuel and other proofs were given later by Szpiro and Voloch. See the articles [25, 26] and [1].

In the following paper, we shall prove Conjecture 1.1 in the situation where K_0 has transcendence degree 1 over its prime field and Z is a subvariety of a larger variety Z' , where Z' is smooth and has ample cotangent bundle over K_0 . See Theorem 1.2 below. Theorem 1.2 can be used to give the first examples (to the authors knowledge) of varieties of general type in positive characteristic that satisfy Conjecture 1.1 and are not embeddable in abelian varieties (see [6] about this). Theorem 1.2 also provides a strengthening of Noguchi’s result in the situation where K_0 has transcendence degree 1 over \mathbb{Q} (because unlike Noguchi we do not assume that $Z = Z'$).

Here is a precise formulation of our result.

Let K be the function field of a smooth curve U over an algebraically closed field k . Let X be a scheme, which is smooth and projective over K .

We prove:

Theorem 1.2 *Suppose that the cotangent bundle $\Omega_X := \Omega_{X/K}$ is ample. Let $R := \text{Zar}(X(K) \cap X)$ be the Zariski closure of the set of all K -rational points of X , endowed with its induced reduced structure. For each irreducible component \mathfrak{R} of R , there is a projective variety \mathfrak{R}'_0 over k and a finite and surjective K^{sep} -morphism $h : \mathfrak{R}'_{0, K^{\text{sep}}} \rightarrow \mathfrak{R}_{K^{\text{sep}}}$.*

If $\text{char}(k) = 0$ then there exists a morphism h as above, which is birational.

If $R = \mathfrak{R} = X$ and

$$\text{char}(k) > \dim(X)^2 \int_X c_1(\Omega_X)^{\dim(X)}$$

then there exists a morphism h as above, which is birational (and thus an isomorphism).

Here $c_1(\Omega_X)$ denotes the first Chern class of the vector bundle Ω_X in the Chow intersection ring $\mathrm{CH}^\bullet(X)$ of X and

$$\int_X : \mathrm{CH}^{\dim(X)}(X) \rightarrow \mathrm{CH}^0(\mathrm{Spec} K) \simeq \mathbb{Z}$$

refers to the push-forward morphism.

Note that the number $\dim(X)^2 \int_X c_1(\Omega_X)^{\dim(X)}$ is always positive, since Ω_X is ample.

Remark In Theorem 1.2, we assume that K is the function field of a curve (see also the discussion above). This restriction, which can probably be removed at the price of added technicality, comes from the fact that we need to consider a smooth compactification of the curve U in the proof and also from the fact that our proof depends on the existence of Néron desingularisations. It seems difficult to reduce the case of a high dimensional base to the case of a base of dimension one by a slicing argument, because it is not clear (to us) how the assumption that the rational points are Zariski dense in a certain closed set behaves when one passes to a fibre in a family.

We now describe the strategy of the proof of Theorem 1.2, which can be viewed as a refinement of the method of Grauert (see [16, chap. VI] for a nice overview of this method), in which all the higher jet schemes are brought into the picture (unlike Grauert, who considers only the first jet scheme).

Here is how Grauert and his followers proceed in characteristic 0, in the situation where the rational points are dense in the whole variety. One first shows that the rational points of X lift to rational points of the first jet scheme. Next, one shows that the “non-constant” rational points concentrate on a closed subscheme Σ (say), which is finite and generically inseparable over X . To establish this last fact, one needs to consider a non-singular compactification of X/K over a compactification of U and use the height machine over function fields. The proof can now be completed quickly because generically inseparable morphisms are birational in characteristic 0 and thus the projection of Σ onto X is an isomorphism. In view of the definition of the first jet scheme, this means that the Kodaira–Spencer class of X vanishes. Using the exponential map, one can conclude from this that X descends to k (up to a separable extension of K).

If one tries to carry through the above proof in positive characteristic, the first problem that one faces is that one cannot easily construct a non-singular compactification of X , unless one assumes the existence of resolutions of singularities in positive characteristic.

Next, even if one supposes that this first problem can be solved, one is faced with the basic problem that the projection $\Sigma \rightarrow X$ might not be an isomorphism.

Finally, even if the projection $\Sigma \rightarrow X$ can be shown to be an isomorphism, i.e. even if the Kodaira–Spencer class of X vanishes, one cannot conclude that X descends to k . For example, any smooth proper curve over K , which descends to K^p , has a vanishing Kodaira–Spencer class.

Here is how we deal with these issues. For the first problem, we replace the non-singular compactification by a Néron desingularisation, which is not compact, but

suffices for our purposes. The second and third issues are dealt with simultaneously. We show that after a finite purely inseparable base-change the entire tower of jet schemes becomes trivial, in the following sense: the base-change of the first jet scheme has a section, the base-change of the second jet scheme has a section over the image of the first section, the base-change of the third jet scheme has a section over the image of the second section and so on. We show that the trivialisation of the tower of jet schemes is a consequence of a generalisation of a cohomological result of Szpiro and Lewin-Ménégaux (see before Proposition 2.3 below). Our proof of this generalisation is not based on the same principle as the result of Szpiro and Lewin-Ménégaux. Specializing all this to a closed point u_0 of U , we obtain a morphism of formal schemes between a constant formal scheme and the completion at u_0 of (a suitable model of) X . Applying Grothendieck's formal GAGA theorem and using the fact that the completion of U at u_0 is an excellent discrete valuation ring, we can construct the required morphism $\mathfrak{R}'_{0,K^{\text{sep}}} \rightarrow \mathfrak{R}_{K^{\text{sep}}}$. The condition given in the theorem for this morphism to be birational in positive characteristic is a consequence of a result of Langer (see below for detailed references) and of the fact that the minimal slope of Ω_X must be positive, since Ω_X is ample.

The method that we just outlined also allows us to treat the situation where the rational points are not Zariski dense (i.e. when $R \neq X$). This was (apparently) not accessible before even in characteristic 0. Note that when R is smooth then R will also have an ample cotangent bundle but we do not need to make this assumption here (the fact that the ambient variety X is smooth with ample cotangent bundle suffices).

Here is the structure of the text.

In Sect. 2.1, we recall various facts about the geometry of torsors under vector bundles (in particular, ample vector bundles). In Sect. 2.2, we prove an injectivity criterion for purely inseparable pull-back maps between first cohomology groups of vector bundles (Corollary 2.2) and we prove a basic vanishing result (Proposition 2.3) for the group of global sections of a coherent sheaf, which is twisted by a sufficiently high power of Frobenius pull-backs of an ample bundle. In Sect. 3, we prove Theorem 1.2. As explained above, our proof does not use the exponential map but uses formal schemes directly and thus differs in nature from the proofs of Noguchi and Martin-Deschamps (see op. cit.) even when $\text{char}(k) = 0$.

We recommend to the reader to first read the proof with the supplementary assumption that U is proper over k and that X extends to a smooth and projective scheme over U . Many technicalities of the proof disappear when that (unrealistic...) supplementary assumption is made. A more technical outline of the proof is given at the beginning of Sect. 3.

Finally, we would like to extend our heartfelt thanks to the anonymous referee for his thorough checking of the text. Without him, this article would be a lot less clearly written.

Notations A scheme of positive characteristic is a scheme S , such that for all points $s \in S$, the local ring \mathcal{O}_s is a ring of positive characteristic. A commutative ring R is of positive characteristic if there exists a prime number p_0 such that $p_0 \cdot 1_R = 0$, where 1_R is the unit element of R . If S is a scheme of positive characteristic, we write F_S for the absolute Frobenius endomorphism of S . The acronym wrog stands for “without

restriction of generality". If Y is an integral scheme, we write $\kappa(Y)$ for the function field of Y . "Almost all" means "for all but a finite number".

We are deeply grateful to the anonymous referee for his thorough checking of the text. Without him, this article would be a lot less clearly written.

2 Preliminaries

2.1 The geometry of the compactifications of torsors under vector bundles

In this subsection, we recall various results proven in [19, par. 1].

Let S be a scheme, which is of finite type over a field k_0 .

If V a locally free sheaf over S , we shall write $\mathbb{P}(V)$ for the S -scheme representing the functor on S -schemes

$$T \mapsto \{ \text{iso. classes of surjective morphisms of } \mathcal{O}_T\text{-modules } V_T \rightarrow Q, \\ \text{where } Q \text{ is locally free of rank 1} \}.$$

By construction, $\mathbb{P}(V)$ comes with a universal line bundle $\mathcal{O}_P(1)$. Let now

$$\mathcal{E} : 0 \rightarrow \mathcal{O}_S \rightarrow E \rightarrow F \rightarrow 0 \quad (1)$$

be an exact sequence of locally free sheaves over S . Consider the S -group scheme $\underline{F} := \text{Spec}(\text{Sym}(F))$ representing the group functor on S -schemes sending T to $F_T^\vee(T)$. Let $R_{\mathcal{E}}$ be the functor from S -schemes to sets given by

$$T \mapsto \{ \text{morphisms of } \mathcal{O}_T\text{-modules } E_T \mapsto \mathcal{O}_T \text{ splitting } \mathcal{E}_T \}. \quad (2)$$

There is an obvious (group functor-)action of \underline{F} on $R_{\mathcal{E}}$.

- (i) (see [19, Prop. 1 and proof]) The natural morphism $\mathbb{P}(F) \rightarrow \mathbb{P}(E)$ is a closed immersion and there is an isomorphism of line bundles $\mathcal{O}(\mathbb{P}(F)) \simeq \mathcal{O}_P(1)$.
- (ii) (see [19, Prop. 1]) The complement $\mathbb{P}(E) \setminus \mathbb{P}(F)$ represents the functor $R_{\mathcal{E}}$. The isomorphism of functors on S -schemes $R_{\mathcal{E}} \rightarrow \mathbb{P}(E) \setminus \mathbb{P}(F)$ can be described as follows. There is a natural transformation of functors $R_{\mathcal{E}} \rightarrow \mathbb{P}(E)$ sending a morphism of \mathcal{O}_T -modules $E_T \mapsto \mathcal{O}_T$ splitting \mathcal{E}_T to the same morphism $E_T \rightarrow \mathcal{O}_T$, viewed as a morphism from E_T onto a locally free sheaf of rank 1 (the latter being the trivial sheaf). This gives a morphism of schemes $R_{\mathcal{E}} \rightarrow \mathbb{P}(E)$, which is an open immersion onto $\mathbb{P}(E) \setminus \mathbb{P}(F)$.

Thus

- (iii) the scheme $R_{\mathcal{E}}$ with its \underline{F} -action is an S -torsor under \underline{F} (see also [19, top of p. 42]).

Further, by (i):

- (iv) if E is ample and S is proper over k_0 then the scheme $\mathbb{P}(E) \setminus \mathbb{P}(F)$ is affine (point (iv) will actually not be used in the text).

Let us now suppose until the end of this section that F is ample and that S is proper over k_0 .

- (v) (see [19, Prop. 2]) if $Z \hookrightarrow R_{\mathcal{E}}$ is a subscheme, which is closed in $\mathbb{P}(E)$, then the induced map $Z \rightarrow S$ is finite and has only a finite number of fibres that contain more than one point.
- (vi) (see [19, Prop. 3]) for all sufficiently large $n \in \mathbb{N}$, the line bundle $\mathcal{O}_P(n+1)$ is generated by its global sections and in this case the induced k_0 -morphism

$$\phi_n : \mathbb{P}(E) \rightarrow \mathbb{P}(\Gamma(\mathcal{O}_P(n+1))) \simeq \mathbb{P}_{k_0}^n$$

is generically finite;

- (vii) (see [19, Prop. 3]) if the line bundle $\mathcal{O}_P(n)$ is generated by its global sections, the positive-dimensional fibres of the morphism ϕ_n are disjoint from $\mathbb{P}(F)$ (where ϕ_n is as in (vi)).

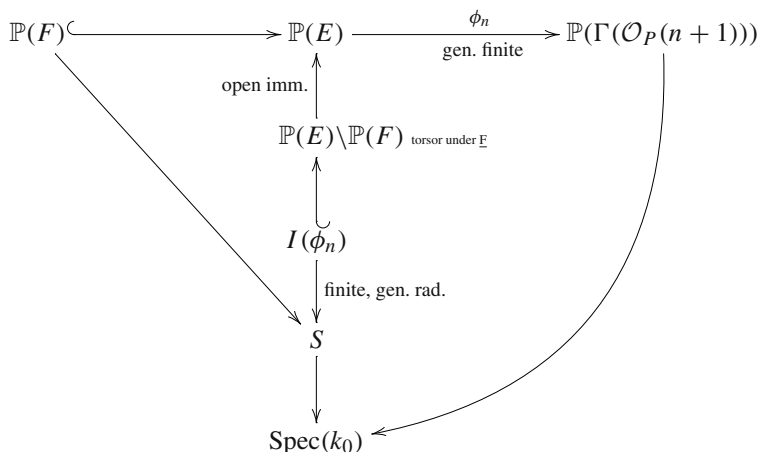
From the fact that fibre dimension is upper semi-continuous (see [10, IV, 13.1.5]) and (vii) we deduce that

- (viii) the union $I(\phi_n)$ of the positive dimensional fibres of ϕ_n is closed in $\mathbb{P}(E)$ and is contained in $R_{\mathcal{E}}$.

We endow $I(\phi_n)$ with its reduced-induced structure. From (v) we deduce that

- (ix) the morphism $\lambda_n : I(\phi_n) \rightarrow S$ is finite and has only a finite number of fibres that contain more than one point; in particular, if $T \subseteq S$ is irreducible and closed, then $I(\phi_n) \cap \lambda_n^{-1}(T)$ has at most one irreducible component dominating T , and if it exists, this component is generically radicial over T .

The following diagram summarises the relationships between and the various properties of the various morphisms introduced in this subsection.



We shall also record the following geometric consequence of (v):

- (x) If $Z \hookrightarrow R_{\mathcal{E}}$ and $Z' \hookrightarrow R_{\mathcal{E}}$ are integral subschemes, such that the induced morphisms $Z \rightarrow S$ and $Z' \rightarrow S$ are finite and surjective then $Z = Z'$.

In particular, if S is integral, then any two sections of $R_{\mathcal{E}}$ must coincide.

Finally, we shall also need the

- (xi) Every torsor under \underline{F} is isomorphic to a torsor $R_{\mathcal{E}}$ for some exact sequence \mathcal{E} as in (1). The class in $H^1(S, F^{\vee}) \simeq \text{Ext}^1(\mathcal{O}_S, F^{\vee})$ corresponding to $R_{\mathcal{E}}$ is the image of $1 \in H^0(S, \mathcal{O}_S)$ in $H^1(S, F^{\vee})$ under the connecting map in the long exact sequence

$$0 \rightarrow H^0(S, F^{\vee}) \rightarrow H^0(S, E^{\vee}) \rightarrow H^0(S, \mathcal{O}_S) \rightarrow H^1(S, F^{\vee}) \rightarrow \dots$$

associated with the dual exact sequence \mathcal{E}^{\vee} .

We leave the details of the proof of (xi) (which is but unravelling the definitions) to the reader. The assumptions that S is proper over k_0 and that F is ample are not necessary in (xi).

2.2 Torsors under vector bundles and purely inseparable base-change

If W is a quasi-coherent \mathcal{O}_Y -module on a integral scheme Y , we shall write

$$\Gamma(Y, W)_g := \{e \in W_{\kappa(Y)} \mid \exists \sigma \in \Gamma(Y, W) : \sigma_{\kappa(Y)} = e\}.$$

Lemma 2.1 *Let Y be a normal, noetherian and integral scheme. Let W be a vector bundle over Y . Let $T \rightarrow Y$ be a torsor under W and let $Z \hookrightarrow T$ be a closed immersion, where Z is an integral scheme. Suppose that the induced morphism $f : Z \rightarrow Y$ is quasi-finite, radicial and dominant. Suppose that $\Gamma(Z, \Omega_f \otimes f^*W)_g = 0$. Then f is an open immersion.*

Proof Let $\pi : T \times_Y T \rightarrow Y$. We consider the scheme $T \times_Y (T \times_Y T)$. Via the projection on the second factor $T \times_Y T$, this scheme is naturally a torsor under the vector bundle π^*W . This torsor has two sections:

- the section σ_1 defined by the formula $t_1 \times t_2 \mapsto t_1 \times (t_1 \times t_2)$;
- the section σ_2 defined by the formula $t_1 \times t_2 \mapsto t_2 \times (t_1 \times t_2)$.

Since $T \times_Y (T \times_Y T)$ is a torsor under π^*W , there is a section $s \in \Gamma(T \times_Y T, \pi^*W)$ such that $\sigma_1 + s = \sigma_2$ and by construction $s(t_1 \times t_2) = 0$ iff $t_1 = t_2$. In other words, s vanishes precisely on the diagonal of $T \times_Y T$.

Now let $\Delta_Z : Z \hookrightarrow Z \times_Y Z$ be the diagonal immersion. Let $\Delta_Z^{(1)} : Z^{(1)} \hookrightarrow Z \times_Y Z$ be the first infinitesimal neighborhood of Δ_Z . By the definition of the differentials, we have an exact sequence of sheaves

$$0 \rightarrow \Omega_f \rightarrow \mathcal{O}_{Z^{(1)}} \rightarrow \mathcal{O}_Z \rightarrow 0$$

which gives rise to an exact sequence

$$0 \rightarrow \Gamma(Z, \pi^*W|_Z \otimes \Omega_f) \rightarrow \Gamma(Z^{(1)}, \pi^*W|_{Z^{(1)}}) \rightarrow \Gamma(Z, \pi^*W|_Z)$$

Now by construction, the image of the section $s \in \Gamma(T \times_Y T, \pi^*W)$ in $\Gamma(Z, \pi^*W|_Z)$ vanishes. Hence the image of the section $s \in \Gamma(T \times_Y T, \pi^*W)$ in $\Gamma(Z^{(1)}, \pi^*W|_{Z^{(1)}})$ is the image of a section $s_0 \in \Gamma(Z, \pi^*W|_Z \otimes \Omega_f)$. By assumption, we have $s_{0,\kappa}(Z) = 0$ and thus by the construction of s the immersion $Z \hookrightarrow Z^{(1)}$ is generically an isomorphism. Hence $\Omega_{f,\kappa}(Z) = 0$ and thus the extension of function fields $\kappa(Z)|\kappa(Y)$ must be of degree one because it is a purely inseparable extension. Thus f is birational. Zariski's main theorem now implies that f is an open immersion. \square

Corollary 2.2 *Let Y be a normal, noetherian and integral scheme of positive characteristic. Suppose that F_Y is a finite morphism. Let W be a vector bundle over Y . If $\Gamma(Y, \Omega_{F_Y} \otimes F_Y^*(W))_g = 0$ then the natural map of abelian groups $H^1(Y, W) \rightarrow H^1(Y, F_Y^*W)$ is injective.*

Recall that $F_Y : Y \rightarrow Y$ is the absolute Frobenius morphism (see the conventions on notations at the end of the introduction).

Proof Consider an element in the kernel of the map $H^1(Y, W) \rightarrow H^1(Y, F_Y^*W)$. Let $T \rightarrow Y$ be a torsor under W corresponding to this element. By assumption, we have a diagram

$$\begin{array}{ccc} F_Y^*T & \xrightarrow{J} & T \\ \downarrow \sigma & & \downarrow \\ Y & \xrightarrow{F_Y} & Y \end{array}$$

where the square is cartesian and σ is a Y -morphism. Now define Z as the scheme-theoretic image $J_*(\sigma(Y))$ by J . By construction, we have a factorisation $Y \xrightarrow{\phi} Z \xrightarrow{f} Y$, where the arrow f is the natural projection and $f \circ \phi = F_Y$. Now using [20, Th. 26.5, p. 202], we see that we have

$$\begin{aligned} p^{\mathrm{rk}_{\kappa}(Z)(\Omega_{f,\kappa}(Z))} &= [\kappa(Z) : \kappa(Y)], \\ p^{\mathrm{rk}_{\kappa}(Y)(\Omega_{F_Y,\kappa}(Y))} &= [\kappa(Y)^{p^{-1}} : \kappa(Y)] \end{aligned}$$

and

$$p^{\mathrm{rk}_{\kappa}(Y)(\Omega_{\phi,\kappa}(Y))} = [\kappa(Y) : \kappa(Z)].$$

Thus in the standard exact sequence

$$\phi^*\Omega_{f,\kappa}(Y) \rightarrow \Omega_{F_Y,\kappa}(Y) \rightarrow \Omega_{\phi,\kappa}(Y) \rightarrow 0.$$

the leftmost arrow is injective. Now from this and from the fact that $\Gamma(Y, \Omega_{F_Y} \otimes F_Y^*(W))_g = 0$, we conclude that $\Gamma(Z, \Omega_f \otimes f^*W)_g = 0$. Thus, by Lemma 2.1, f is an isomorphism and thus the torsor $T \rightarrow Y$ is trivial. \square

Note that if Y is projective over an algebraically closed field, then a weaker form of Corollary 2.2 (which is not sufficient for our purposes) is contained in [26, exp. 2, Prop. 1]. The proof given there depends on the existence of the Cartier isomorphism.

Let now S be an integral scheme, which is projective and geometrically reduced over a field k_0 of characteristic p . As before, let $F_S : S \rightarrow S$ be the absolute Frobenius morphism on S .

Proposition 2.3 *Suppose that $\dim(S) > 0$. Let V be an ample vector bundle of rank r over S . Let W be a coherent sheaf over S . Then we have $H^0(S, F_S^{n,*}(V^\vee) \otimes W)_g = 0$ for all sufficiently large $n \geq 0$.*

Proof We may assume wlog that k_0 is algebraically closed.

The proof is by induction on the dimension $d \geq 1$ of S .

Suppose that $d > 1$. Consider a pencil of hypersurfaces in S and let $b : \tilde{S} \rightarrow S$ the total space of the pencil. By construction, we are given a morphism $m : \tilde{S} \rightarrow \mathbb{P}_{k_0}^1$. Let $\eta \in \mathbb{P}_{k_0}^1$ be the generic point. Let n_0 be sufficiently large, so that

$$H^0(\tilde{S}_\eta, F_{\tilde{S}_\eta}^{n,*}((b^*V^\vee)_\eta) \otimes (b^*W)_\eta)_g = 0$$

for all $n \geq n_0$. This is possible by the induction hypothesis and because $(b^*V)_\eta$ is ample. The fact that $(b^*V)_\eta$ is ample is a consequence of the fact that the restriction of V to any closed fibre of m is ample and of the fact that ampleness on the fibre of m is a constructible property (see [10, IV, 9, Cor. 9.6.4]). Now if we had $H^0(S, F_S^{n,*}(V^\vee) \otimes W)_g \neq 0$ for some $n \geq n_0$ then the pull-back $b^*(F_S^{n,*}(V^\vee) \otimes W)$ would have a section that does not vanish at the generic point of \tilde{S}_η , which is a contradiction.

Thus we are reduced to prove the statement for $d = 1$. We may replace wlog S by its normalisation. Since S is now a non-singular curve, we know that V is cohomologically p -ample (see [21, Rem. 6], p. 91]). Also, W is now the direct sum of a torsion sheaf and of a locally free sheaf so we may assume wlog that W is locally free. Now using Serre duality, we may compute

$$H^0(S, F_S^{n,*}(V^\vee) \otimes W) = H^1(S, F_S^{n,*}(V) \otimes W^\vee \otimes \Omega_{S/k_0})^\vee$$

and the vector space $H^1(S, F_S^{n,*}(V) \otimes W^\vee \otimes \Omega_{S/k_0})$ vanishes for $n \gg 0$ because V is cohomologically p -ample. \square

Corollary 2.4 *Suppose that V is an ample bundle on S and that S is normal. Let $n_0 \in \mathbb{N}$ be such that $H^0(S, F_S^{n,*}(V^\vee) \otimes \Omega_{F_S})_g = 0$ for all $n > n_0$. Let S' be an irreducible scheme and let $\phi : S' \rightarrow S$ be a finite surjective morphism, which is generically inseparable. Then the map*

$$H^1(S, F_S^{n_0,*}(V^\vee)) \rightarrow H^1(S', \phi^*(F_S^{n_0,*}(V^\vee)))$$

is injective.

Proof We may suppose that S' is a normal scheme, since we may replace S' by its normalization without restriction of generality. Let H be the function field of S and let H' be the function field of S' . We are provided with a finite purely inseparable field extension $H \hookrightarrow H'$ and thus for some sufficiently large $\ell_0 \geq 1$, there exists a field extension $H' \hookrightarrow H$ and a commutative diagram

$$\begin{array}{ccc} H & \hookrightarrow & H' \\ \uparrow = & & \downarrow \\ H & \xrightarrow{x^{p^{\ell_0}}} & H \end{array}$$

Now notice that by Zariski's main theorem, S' is canonically isomorphic to the normalisation of S in H' . Similarly, the morphism $F_S^{\ell_0} : S \rightarrow S$ gives a presentation of S as its own normalization in H via the p^{ℓ_0} -th power map $H \xrightarrow{x^{p^{\ell_0}}} H$. There is thus a natural factorization $S \rightarrow S' \xrightarrow{\phi} S$, where the morphism $S \rightarrow S$ is given by $F_S^{\ell_0}$. Using Corollary 2.2, we see that there is a natural injection

$$H^1(S, F_S^{n_0,*}(V^\vee)) \hookrightarrow H^1(S, F_S^{\ell_0,*}(F_S^{n_0,*}(V^\vee)))$$

and thus an injection

$$H^1(S, F_S^{n_0,*}(V^\vee)) \rightarrow H^1(S', \phi^*(F_S^{n_0,*}(V^\vee))).$$

□

2.3 Jet schemes

The jet scheme construction described in [23, sec. 2] provides a covariant functor

$$\mathcal{Y} \mapsto J^i(\mathcal{Y}/U)$$

from the category of quasi-projective schemes \mathcal{Y} over U to the category of quasi-projective schemes over U .

The construction also provides an infinite tower of U -morphisms

$$\cdots \rightarrow J^2(\mathcal{Y}/U) \xrightarrow{\Lambda_{2,\mathcal{Y}}} J^1(\mathcal{Y}/U) \xrightarrow{\Lambda_{1,\mathcal{Y}}} \mathcal{Y}.$$

If Y is smooth over U , then the scheme $J^i(\mathcal{Y}/U)$, viewed as a $J^{i-1}(\mathcal{Y}/U)$ -scheme via $\Lambda_{i,\mathcal{Y}}$, is a torsor under the vector bundle $\mathrm{T}\mathcal{Y} \otimes \mathrm{Sym}^i(\Omega_{U/k})$, where $\mathrm{T}\mathcal{Y} := (\Omega_{\mathcal{Y}/U}^1)^\vee$. Here the bundle $\mathrm{T}\mathcal{Y}$ (resp. $\mathrm{Sym}^i(\Omega_{U/k})$) is implicitly pulled back from \mathcal{Y} (resp. U) to the scheme $J^{i-1}(\mathcal{Y}/U)$.

The functor $J^i(\cdot/U)$ preserves closed immersions and smooth morphisms. In particular, for any $i \in \mathbb{N}$, there is a natural map

$$\psi_{i,\mathcal{Y}} : \mathcal{Y}(U) = \text{Mor}_U(U, \mathcal{Y}) \rightarrow \text{Mor}_U(J^i(U/U), J^i(\mathcal{Y}/U)) = \text{Mor}_U(U, J^i(\mathcal{Y}/U)), \quad (3)$$

and these maps are compatible with the morphisms $\Lambda_{i,\mathcal{Y}}$.

Lastly, if a quasi-projective scheme \mathcal{Y} over U has the property that $\mathcal{Y} \simeq \mathcal{Y}_0 \times_k U$, where \mathcal{Y}_0 is a quasi-projective scheme over k , then we are provided with a commutative diagram

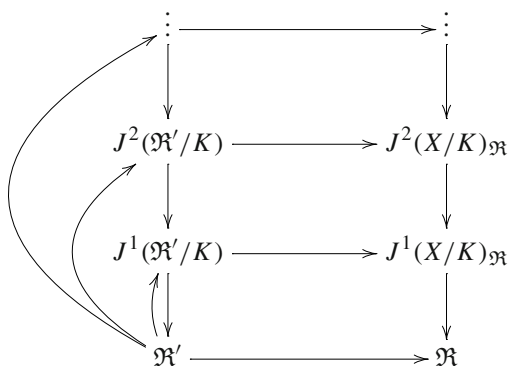
In particular, in this situation, the scheme $J^i(\mathcal{Y}/U)$, viewed as a $J^{i-1}(\mathcal{Y}/U)$ -torsor under the vector bundle $T\mathcal{Y} \otimes \text{Sym}^i(\Omega_{U/k})$, is a trivial torsor. The curved arrows are functorial in an obvious manner with respect to the scheme \mathcal{Y}_0/k and the isomorphism $\mathcal{Y} \simeq \mathcal{Y}_0 \times_k U$.

We refer to [23, sec. 2] for the construction of jet schemes and for the proofs of the various assertions about them made above.

3 Proof of Theorem 1.2

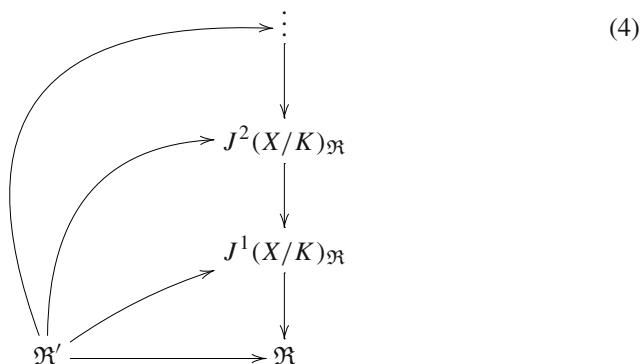
Outline of the proof

Suppose for the purposes of this outline that X/K extends to a (not necessarily proper) smooth and quasi-projective U -scheme $\pi : \mathcal{X} \rightarrow U$. First notice the following fact. If the conclusion of Theorem 1.2 holds, then after possibly replacing K by one of its finite separable extensions, there is an infinite commutative diagram



where $\mathfrak{R}' \simeq \mathfrak{R}'_0 \times_k K$ and $J^i(X/K)_{\mathfrak{R}} = J^i(X/K) \times_X \mathfrak{R}$. Here the horizontal arrows arise from the arrow $\mathfrak{R}' \rightarrow \mathfrak{R}$ by functoriality. The curved arrows arise from the isomorphism $\mathfrak{R}' \simeq \mathfrak{R}'_0 \times_k K$ (see Sect. 2.3).

In particular, we have an infinite commutative diagram



Step IV of the proof below will show that the existence of a diagram (4) is actually *equivalent* to the conclusion of Theorem 1.2. The method used in Step IV is analytic. Using the curved arrows in diagram 4 and the definition of jet schemes we first construct a formal morphism $\mathfrak{R}' \rightarrow \mathcal{X}$, i.e. a morphism from \mathfrak{R}' to \mathcal{X} viewed as formal schemes over the completion of U along a suitable closed point. We then show using Grothendieck's formal GAGA theorem that this morphism has an algebraic model.

In view of Step IV, it is thus sufficient to provide a diagram (4). To do this, we use the fact that there are natural lifting maps $\psi_i : \mathcal{X}(U) \rightarrow J^i(\mathcal{X}/U)$ (see Sect. 2.3). Using a Néron desingularisation to construct \mathcal{X} , we may assume that the natural map $\mathcal{X}(U) \rightarrow X(K)$ is surjective so that $\mathcal{X}(U)$ can be assumed to be large. The idea is now to consider the Zariski closure of the image of the map ψ_i and to try to show that the base change to $\text{Spec}(K)$ of one of these irreducible components qualifies as the image of a curved arrow $\mathfrak{R}' \rightarrow J^i(X/K)_{\mathfrak{R}}$. We will see in Step II that there is a natural geometric candidate for the image of this arrow: the irreducible component Z_i of positive dimension of the scheme $I(\phi_{n_i})$ —see below. The crux of the proof is to

show that the image of the lifting map $\mathcal{X}(U) \rightarrow J^i(\mathcal{X}/U)(U)$ meets Z_i in a Zariski dense subset. To show this, we first compactify over k all the objects in sight (i.e. the scheme U , the jet schemes $J^i(\mathcal{X}/U)$ etc.), in order to avail ourselves of the Hilbert schemes of morphisms from the compactification of U to the compactifications of the jet schemes. This is what we do in Step I, where these compactifications are constructed as canonically as possible by using the natural compactifications of torsors described in Sect. 2.1.

In Step II, a height argument based on the existence of Hilbert schemes is then used to show that the schemes $I(\phi_{n_i})$ must meet ‘most’ of the image of ψ_i (see in particular Lemma 3.3 and Corollary 3.4). The proof is made somewhat messy by the fact that some rational maps have to be made into morphisms in an arbitrary way (see especially after Lemma 3.1).

Once it is shown that the schemes Z_i have the required properties, it remains to show that the morphisms between them eventually become birational (otherwise the diagram (4) cannot exist). This is what is shown in Step III. It is not difficult to show that the morphisms between the schemes Z_i are generically purely inseparable but to show that they eventually become birational, we use the cohomological results of Sect. 2.2, especially Corollary 2.2. Here the torsor structure of the jet schemes is used and plugged into a cohomological machinery.

The proof.

We now suppose that the assumptions of Theorem 1.2 are in force. We may suppose wlog that \mathfrak{X} has positive dimension. Next, we may suppose wlog that U is proper and smooth over k , since U has dimension one.

Step I Compactifications.

By a compactification \bar{T} of an S -scheme T , we shall mean a proper scheme $\bar{T} \rightarrow S$, which comes with an open immersion $T \hookrightarrow \bar{T}$ with dense image.

Now choose any projective model $\bar{\mathcal{X}}_0 \rightarrow U$ of X (this may for instance be obtained by embedding X into some projective space over K). By applying Néron desingularization to $\bar{\mathcal{X}}_0$ (see [3, chap. 3, th. 2]), we obtain another projective model $\bar{\mathcal{X}}_{00}$ of X over U , with the property that the injection $\bar{\mathcal{X}}_{00}^{\text{sm}}(U) \hookrightarrow \bar{\mathcal{X}}_{00}(U) = X(K)$ is a bijection. Here $\bar{\mathcal{X}}_{00}^{\text{sm}} \subset \bar{\mathcal{X}}_{00}$ is the largest open subscheme $\bar{\mathcal{X}}_{00}^{\text{sm}}$ of $\bar{\mathcal{X}}_{00}$, such that $\bar{\mathcal{X}}_{00}^{\text{sm}} \rightarrow U$ is smooth. We now define $\mathcal{X} := \bar{\mathcal{X}}_{00}^{\text{sm}}$ and $\bar{\mathcal{X}} := \bar{\mathcal{X}}_{00}$.

To summarize: \mathcal{X} is a smooth model of X over U such that the natural map $\mathcal{X}(U) \rightarrow X(K)$ is a bijection and we have an open immersion of U -schemes $\mathcal{X} \hookrightarrow \bar{\mathcal{X}}$, where $\bar{\mathcal{X}}$ is projective over U . In particular the map $\mathcal{X} \rightarrow U$ is surjective, since $\mathfrak{X} \neq \emptyset$.

We now choose specific compactifications $\bar{J}^i(\mathcal{X}/U)$ over U for the jet schemes $J^i(\mathcal{X}/U)$.

For $i = 0$, we let $\bar{J}^0(\mathcal{X}/U) = \bar{\mathcal{X}}$. We shall define the schemes $\bar{J}^i(\mathcal{X}/U)$ inductively for $i \geq 0$.

So suppose that the compactification $\bar{J}^i(\mathcal{X}/U)$ has already been constructed. As explained in Sect. 2.3, the $J^i(\mathcal{X}/U)$ -scheme $J^{i+1}(\mathcal{X}/U)$ is a torsor under F_i^\vee , where $F_i := (T\mathcal{X} \otimes \text{Sym}^{i+1}(\Omega_{U/k}))^\vee$ (viewed as a vector bundle over $J^i(\mathcal{X}/U)$). Let

$$0 \rightarrow \mathcal{O}_{J^i(\mathcal{X}/U)} \rightarrow E_i \rightarrow F_i \rightarrow 0 \quad (5)$$

be an extension (unique up to non-unique isomorphism) associated with the class of $J^{i+1}(\mathcal{X}/U)$ in $H^1(J^i(\mathcal{X}/U), F_i^\vee)$ (see (xi) in Sect. 2.1). It was explained in (ii) Sect. 2.1 that the $J^i(\mathcal{X}/U)$ -scheme $J^{i+1}(\mathcal{X}/U)$ can be realized as the complement $\mathbb{P}(E_i) \setminus \mathbb{P}(F_i)$. We now define the compactification $\bar{J}^{i+1}(\mathcal{X}/U)$ to be some $\bar{J}^i(\mathcal{X}/U)$ -compactification of $\mathbb{P}(E_i)$, such that the diagram

$$\begin{array}{ccc} \mathbb{P}(E_i) & \hookrightarrow & \bar{J}^{i+1}(\mathcal{X}/U) \\ \downarrow & & \downarrow \\ J^i(\mathcal{X}/U) & \hookrightarrow & \bar{J}^i(\mathcal{X}/U) \end{array}$$

is cartesian. This is possible because we may extend E_i to a coherent sheaf \bar{E}_i on $\bar{J}^i(\mathcal{X}/U)$ and define $\bar{J}^{i+1}(\mathcal{X}/U) := \mathbb{P}(\bar{E}_i)$. We call $\bar{\Lambda}_{i+1} : \bar{J}^{i+1}(\mathcal{X}/U) \rightarrow \bar{J}^i(\mathcal{X}/U)$ the corresponding morphism.

The following diagram summarizes the resulting geometric configuration:

$$\begin{array}{ccccc} J^{i+1}(\mathcal{X}/U) & \hookrightarrow & \mathbb{P}(E_i) & \hookrightarrow & \bar{J}^{i+1}(\mathcal{X}/U) \\ \Lambda_{i+1} \downarrow & & \downarrow & & \downarrow \bar{\Lambda}_{i+1} \\ J^i(\mathcal{X}/U) & = & J^i(\mathcal{X}/U) & \hookrightarrow & \bar{J}^i(\mathcal{X}/U) \end{array}$$

Here the hooked horizontal arrows are open immersions and the square on the right is cartesian.

Recall the following key properties. The scheme U is proper over k and the schemes $\bar{J}^i(\mathcal{X}/U)$ are proper over U . The morphisms $\bar{J}^{i+1}(\mathcal{X}/U) \rightarrow \bar{J}^i(\mathcal{X}/U)$ are proper. The schemes $J^i(\mathcal{X}/U)$ and $\mathbb{P}(E_i)$ are smooth and surjective onto U .

Now remember that for each $i \geq 0$, there is a natural map $\psi_i : \mathcal{X}(U) \rightarrow J^i(\mathcal{X}/U)(U)$, which is a section of the projection map $J^i(\mathcal{X}/U)(U) \rightarrow \mathcal{X}(U)$ (see Sect. 2.3). Abusing notation, we shall also call ψ_i the map ψ_i composed with the open immersion $J^i(\mathcal{X}/U) \hookrightarrow \bar{J}^i(\mathcal{X}/U)$.

Finally, define $J^i(X/K) := J^i(\mathcal{X}/U)_K$ and $\bar{J}^i(X/K) := \bar{J}^i(\mathcal{X}/U)_K$.

Step II The schemes $Z_i \hookrightarrow J^i(X/K)$.

We shall inductively construct closed integral subschemes $Z_i \hookrightarrow J^i(X/K)$ with the following properties. They are sent onto each other by the morphisms $\Lambda_{i,K}$. The morphisms $Z_{i+1} \rightarrow Z_i$ are finite, surjective and generically radicial and Z_i is proper over K . In particular, Z_i is closed in $\bar{J}^i(X/K)$. Finally, the image of $\mathcal{X}(U) = X(K)$ by $\psi_{i,K}$ in $J^i(X/K)$ meets $Z_i(K)$ in a dense subset of Z_i and $Z_0 = \mathfrak{R}$.

The schemes Z_i are defined via the following inductive procedure.

We define $Z_0 := \mathfrak{R}$.

To define Z_{i+1} from Z_i notice that by Step I, we have an identification $\bar{J}^{i+1}(\mathcal{X}/U)_{Z_i} = \mathbb{P}(E_{i,Z_i})$.

Notice also that F_{i,Z_i} is ample (over K) since $Z_i \rightarrow X$ is finite. Thus by (vi) in Sect. 2.1, there exists a K -morphism

$$\phi_{n_i} : \mathbb{P}(E_{i,Z_i}) \rightarrow \mathbb{P}_K^{n_i},$$

for some $n_i \in \mathbb{N}$ that we fix. Using the notation of (viii) in Sect. 2.1, call $I(\phi_{n_i})$ the union of the positive dimensional fibres of ϕ_{n_i} .

Substep II.1 The closed set $I(\phi_{n_i})$ dominates Z_i .

Write $H_i \subseteq \mathbb{P}_K^{n_i}$ for a hyperplane such that $\phi_{n_i}^{-1}(H_i) = \mathbb{P}(F_{i,Z_i})$. To find such a hyperplane, consider that by (i) and (vi) in Sect. 2.1, the section of $\mathcal{O}(F_{i,Z_i})^{\otimes(n_i+1)}$ corresponding to the Cartier divisor $(n_i + 1)F_{i,Z_i}$ gives a section of $\mathcal{O}_{\mathbb{P}_K^{n_i}}(n_i + 1)$. By construction, the hyperplane H_i can be taken to be the reduced scheme underlying the zero scheme of this section.

Let $\bar{\mathbb{P}}(E_{i,Z_i})$ be the Zariski closure of $\mathbb{P}(E_{i,Z_i})$ in $\bar{J}^{i+1}(\mathcal{X}/U)$ and let $\bar{\mathbb{P}}(F_{i,Z_i})$ be the Zariski closure of $\mathbb{P}(F_{i,Z_i})$ in $\bar{J}^{i+1}(\mathcal{X}/U)$.

Now call $\Sigma_i \subseteq \mathcal{X}(U) = X(K)$ the set of sections $\sigma \in \mathcal{X}(U)$ such that $\psi_{i,K}(\sigma) \in Z_i(K)$.

Lemma 3.1 *Let $\sigma \in \Sigma_i$. We have*

$$\psi_{i+1}(\sigma) \in \bar{\mathbb{P}}(E_{i,Z_i})(U)$$

and

$$\psi_{i+1}(\sigma)(U) \cap \bar{\mathbb{P}}(F_{i,Z_i}) = \emptyset. \quad (6)$$

Proof The first equation follows from the definitions. The second equation follows from the fact that $\psi_{i+1}(\sigma)(U) \subseteq J^{i+1}(\mathcal{X}/U) = \mathbb{P}(E_i) \setminus \mathbb{P}(F_i)$ and from the fact that we have a set-theoretic identity $\bar{\mathbb{P}}(F_{i,Z_i}) \cap \mathbb{P}(E_i) \subseteq \mathbb{P}(F_i)$, since $\mathbb{P}(F_i)$ is a closed subset of $\mathbb{P}(E_i)$. \square

Now choose a proper birational U -morphism $b_i : \bar{\mathbb{P}}'(E_{i,Z_i}) \rightarrow \bar{\mathbb{P}}(E_{i,Z_i})$, which is an isomorphism over K and such that there exists a proper U -morphism

$$\tilde{\phi}_i : \bar{\mathbb{P}}'(E_{i,Z_i}) \rightarrow \mathbb{P}_U^{n_i},$$

with the property that $\phi_{n_i} \circ b_{i,K} = \tilde{\phi}_{i,K}$.

Let $\bar{\mathbb{P}}'(F_{i,Z_i})$ be reduced Zariski closure of $b_{i,K}^{-1}(\mathbb{P}(F_{i,Z_i}))$ in $\bar{\mathbb{P}}'(E_{i,Z_i})$. Let $\bar{H}_i \subseteq \mathbb{P}_U^{n_i}$ be the Zariski closure of H_i .

Now consider the Stein factorisation (see [11, III, Cor. 11.5] for this)

$$\bar{\mathbb{P}}'(E_{i,Z_i}) \xrightarrow{\rho_i} \bar{\mathbb{P}}''(E_{i,Z_i}) \xrightarrow{\bar{v}_i} \mathbb{P}_U^{n_i}$$

of $\tilde{\phi}_i$. Notice that by construction the morphism ρ_i is birational and the morphism \bar{v}_i is finite.

The following commutative diagram sums up the geometric situation:

$$\begin{array}{ccccccc}
 & & \tilde{\mathbb{P}}(F_i, Z_i) & & \bar{J}^{i+1}(\mathcal{X}/U) & \longleftarrow & \bar{J}^{i+1}(X/K) & \longleftarrow & \mathbb{P}(E_i)_K \\
 & \nearrow & & \nwarrow & \uparrow & & & & \nearrow \\
 \tilde{\mathbb{P}}'(F_i, Z_i) & \hookrightarrow & \tilde{\mathbb{P}}'(E_i, Z_i) & \xrightarrow{b_i} & \tilde{\mathbb{P}}(E_i, Z_i) & \longleftarrow & \mathbb{P}(E_i, Z_i) & \longleftarrow & \mathbb{P}(F_i, Z_i) = \phi_{n_i}^{-1}(H_i) \\
 & & \downarrow \rho_i & & \searrow \tilde{\phi}_i & & \downarrow \phi_{n_i} & & \\
 & & \tilde{\mathbb{P}}''(E_i, Z_i) & \xrightarrow{\bar{v}_i} & \mathbb{P}_U^{n_i} & \longleftarrow & \mathbb{P}_K^{n_i} & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & \bar{H}_i & \longleftarrow & H_i & &
 \end{array}$$

By the valuative criterion of properness, there is a natural map

$$\psi'_{i+1} : \mathcal{X}(U) \rightarrow \tilde{\mathbb{P}}'(E_i, Z_i)(U),$$

such that $b_i \circ \psi'_{i+1} = \psi_{i+1}$.

Lemma 3.2 *There exists a constant $\beta_{i+1} \geq 0$, which is independent of σ , such that for all $\sigma \in \Sigma_i$, we have*

$$\text{length}(\rho_i(\psi'_{i+1}(\sigma))(U) \cap \bar{v}_i^*(\bar{H}_i)) \leq \beta_{i+1}. \quad (7)$$

Here \cap refers to the scheme-theoretic intersection.

Proof Notice that $\bar{v}_i^*(\bar{H}_i)$ has a finite number of irreducible components. Among those, the only horizontal (ie dominating U) irreducible component is $\rho_i(\tilde{\mathbb{P}}'(F_i, Z_i))$. Furthermore, we have $\psi'_{i+1}(\sigma) \cap \tilde{\mathbb{P}}'(F_i, Z_i) = \emptyset$ by equation (3.1). Thus $\rho_i(\psi'_{i+1}(\sigma))$ meets only the vertical irreducible components of $\bar{v}_i^*(\bar{H}_i)$. To conclude, notice that the intersection multiplicity of $\rho_i(\psi'_{i+1}(\sigma))$ with a fixed vertical component of $\bar{v}_i^*(\bar{H}_i)$ can be bounded independently of σ . \square

N.B. The fact that we use Néron desingularisations is what makes it possible to prove the existence of the bound β_{i+1} given in Lemma 7. Indeed, the fact that we use a Néron desingularisation implies that the Zariski closure in $\bar{J}^{i+1}(\mathcal{X}/U)$ of the lifting of an element of $X(K)$ to $J^{i+1}(X/K)$ will lie over $J^i(\mathcal{X}/U)$ and the restriction $\bar{J}^{i+1}(\mathcal{X}/U)|_{J^i(\mathcal{X}/U)}$ is the natural compactification $\mathbb{P}(E_i)$, so that (3.1) can be made to hold.

Now we have

Lemma 3.3 *There exist a scheme M_i , which is quasi-projective over k and a U -morphism $\mu_i : M_i \times_k U \rightarrow \tilde{\mathbb{P}}''(E_i, Z_i)$, with the following properties:*

- for all $P \in M_i(k)$, we have

$$\deg((\mu_i \circ (P \times \text{Id}_U))^*(\bar{v}_i^*(\mathcal{O}(\bar{H}_i)))) \leq \beta_{i+1};$$

- for any U -morphism $\kappa : U \rightarrow \tilde{\mathbb{P}}''(E_{i,Z_i})$ such that

$$\deg((\mu_i \circ (P \times \text{Id}_U))^*(\bar{v}_i^*(\mathcal{O}(\bar{H}_i)))) \leq \beta_{i+1}$$

there is a $P \in M_i(k)$ such that $\kappa = \mu_i \circ (P \times \text{Id}_U)$.

Here β_{i+1} is the constant appearing in Lemma 7.

Proof Notice that the morphism \bar{v}_i is finite and thus the divisor $\bar{v}_i^*(\bar{H}_i)$ is ample. The existence of M_i is now a consequence of the theory of Hilbert schemes. \square

Corollary 3.4 For almost all the sections $\sigma \in \Sigma_i$, we have

$$\psi_{i+1,K}(\sigma) \in (I(\phi_{n_i}))(K).$$

Proof Let M_i be as in Lemma 3.3. We may suppose wlog that Σ_i is infinite. In view of (7), there is a natural map $\text{gr} : \Sigma_i \rightarrow M_i(k)$, such that

$$(\mu_i \circ (\text{gr}(\sigma) \times \text{Id}_U)) = \sigma$$

for all $\sigma \in \Sigma_i$.

On the other hand, consider N , a reduced subscheme of M_i . We have by construction a K -morphism $N_K \rightarrow \tilde{\mathbb{P}}''(E_{i,Z_i})_K$. Let $U_i''' \subseteq \mathbb{P}_K^{n_i}$ be the locus of points where the fiber of $\tilde{\phi}_{i,K}$ is finite and let $U_i'' := \bar{v}_i^{-1}(U_i''')$. Note that by the construction of the Stein factorisation, the morphism $\rho_i|_{U_i''} : \rho_i^{-1}(U_i'') \rightarrow U_i''$ is an isomorphism. If the conclusion of the corollary fails, we may thus find N as above such that N is integral and $\dim(N) > 0$, such that N_K intersects U_i'' and such that $N \cap \text{gr}(\Sigma_i)$ is dense in N .

We suppose that such an N exists to obtain a contradiction. Restricting the size of N , we may then suppose that the image of $N_K \rightarrow \tilde{\mathbb{P}}''(E_{i,Z_i})_K$ lies inside U_i'' and thus we obtain a morphism $l_N : N_K \rightarrow \mathbb{P}(E_{i,Z_i}) \subseteq \bar{J}^{i+1}(X/K)$.

Consider that we also have by functoriality a morphism

$$J^{i+1}(N_K/K) \rightarrow J^{i+1}(X/K)$$

arising from the morphism $N_K \rightarrow X$, which is the composition of l_N with the projection $\mathbb{P}(E_{i,Z_i}) \rightarrow X$. The morphism $J^{i+1}(N_K/K) \rightarrow J^{i+1}(X/K)$ can be composed with the natural section $N_K \rightarrow J^{i+1}(N_K/K)$ (see Sect. 2.3) to obtain a second K -morphism

$$l'_N : N_K \rightarrow J^{i+1}(X/K) \subseteq \bar{J}^{i+1}(X/K).$$

By construction, the morphisms l_N and l'_N coincide on a dense set of K -points and thus $l_N = l'_N$.

Now let $C \subseteq N$ be a smooth curve. Repeating the construction of l'_N for C , we obtain a morphism $l'_C : C_K \rightarrow J^{i+1}(X/K)$ and furthermore the composition of $l'_N = l_N$ with $C \rightarrow N$ is l'_C . Now consider a smooth compactification \bar{C} of C over

k . By the valuative criterion of properness, there is a unique K -morphism $\bar{C}_K \rightarrow X$ extending the morphism $C_K \rightarrow X$. Following the steps of the construction of l'_C , we again obtain a K -morphism

$$l'_{\bar{C}} : \bar{C}_K \rightarrow J^{i+1}(X/K),$$

which extends l'_C . On the other hand, $l'_{\bar{C}}(\bar{C}_K) \subseteq \mathbb{P}(E_{i,Z_i}) \setminus \mathbb{P}(F_{i,Z_i})$ by construction and the image $\phi_{n_i}(l'_{\bar{C}}(\bar{C}_K))$ is closed in $\mathbb{P}_K^{n_i}$. Furthermore, by the definition of H_i ,

$$\phi_{n_i}(l'_{\bar{C}}(\bar{C}_K)) \cap H_i = \emptyset.$$

Thus $\phi_{n_i}(l'_{\bar{C}}(\bar{C}_K))$ is the underlying set of an integral scheme, which is affine and proper over K and thus consists of a closed point. Since by construction $l'_C(C_K) \not\subseteq I(\phi_{n_i})$, we see that $l'_C(C_K)$ is a closed point. Now any two closed points in the smooth locus N^{sm} of N can be connected by a smooth curve lying in N^{sm} (see for instance [13]). Hence the image of $N^{\text{sm}}(k)$ by l'_N is a closed point P . Since the image of $N^{\text{sm}}(k)$ in $N^{\text{sm}}(K)$ is dense N_K the fibre $(l'_N)^{-1}(P)$ is dense in N_K and thus $l'_N(N_K) = P$. This contradicts the fact that $N \cap \text{gr}(\Sigma_i)$ is dense in N (because the elements of Σ_i correspond to pairwise distinct points in $X(K)$). \square

We can now conclude that $I(\phi_{n_i})$ dominates Z_i , since by Corollary 3.4 the image of $I(\phi_{n_i})$ in Z_i contains the image by $\psi_{i,K}$ of a cofinite set of Σ_i and this set is dense in Z_i by assumption.

Substep II.2 Definition of Z_{i+1} .

Since $\psi_{i,K}(\Sigma_i)$ meets Z_i in a dense subset by assumption, we can now deduce from (ix) in Sect. 2.1 that $I(\phi_{n_i})$ has a single irreducible component of positive dimension, which is finite, surjective and generically radicial over Z_i .

We now define Z_{i+1} to be this irreducible component.

We have shown that Z_{i+1} has all the required properties.

Step III Purely inseparable trivialization.

Let $\tilde{Z}_0 \rightarrow Z_0$ be the normalisation of Z_0 . We note the important fact that \tilde{Z}_0 is geometrically reduced over K because $\tilde{Z}_0(K)$ is dense in \tilde{Z}_0 (see eg. [1, Lemma 1] for this). For every $i \geq 0$, write $\tilde{Z}_i \rightarrow \tilde{Z}_0$ for the \tilde{Z}_0 -scheme obtained from Z_i by base-change. Similarly, write $\tilde{J}^i(X/K)$ for the pull-back of $J^i(X/K)$ to \tilde{Z}_0 . We denote by \tilde{F}_0 the pull-back to \tilde{Z}_0 of the vector bundle $F_0 \simeq (TX \otimes \Omega_{K/k})^\vee$.

Suppose that $\text{char}(k) = 0$.

The scheme $\tilde{J}^1(X/K)$ is by construction a torsor under \tilde{F}_0^\vee . Furthermore, the morphism $\tilde{Z}_1^{\text{red}} \rightarrow \tilde{Z}_0$ is an isomorphism by Zariski's main theorem and it trivializes the torsor $\tilde{J}^1(X/K)$ by construction.

Repeating this reasoning for \tilde{Z}_2^{red} over $\tilde{Z}_1^{\text{red}} \simeq \tilde{Z}_0$, \tilde{Z}_3^{red} over \tilde{Z}_2^{red} and so forth, we see that the natural morphisms $\tilde{Z}_{i+1}^{\text{red}} \rightarrow \tilde{Z}_i^{\text{red}}$ are isomorphisms for all $i \geq 0$.

Now suppose until the end of step III that $\text{char}(k) > 0$.

Write $F_{\tilde{Z}_0}^{m_0,*} \tilde{Z}_i$ for the base-change of $\tilde{Z}_i \rightarrow \tilde{Z}_0$ by $F_{\tilde{Z}_0}^{m_0}$. Let

$$\pi_{i,m_0} : (F_{\tilde{Z}_0}^{m_0,*} \tilde{Z}_i)^{\text{red}} \rightarrow \tilde{Z}_0$$

be the natural morphism.

Similarly, write $F_{\tilde{Z}_0}^{m_0,*} \tilde{J}^i(X/K)$ for the base-change of $\tilde{J}^i(X/K)$ by $F_{\tilde{Z}_0}^{m_0}$.

We now define

$$m_0 := \sup\{m \in \mathbb{N} \mid H^0(\tilde{Z}_0, F_{\tilde{Z}_0}^{m,*}(\tilde{F}_0^\vee) \otimes \Omega_{F_{\tilde{Z}_0}})_g \neq 0\}.$$

By Proposition 2.3, the quantity m_0 is finite.

The scheme $F_{\tilde{Z}_0}^{m_0,*} \tilde{J}^1(X/K)$ is by construction a torsor under $F_{\tilde{Z}_0}^{m_0,*}(\tilde{F}_0)$. Furthermore, the morphism π_{1,m_0} trivializes this torsor by construction, since there is a \tilde{Z}_0 -morphism $(F_{\tilde{Z}_0}^{m_0,*} \tilde{Z}_1)^{\text{red}} \rightarrow F_{\tilde{Z}_0}^{m_0,*} \tilde{J}^1(X/K)$. By the assumption on m_0 and because π_{1,m_0} is finite and generically radicial, the $F_{\tilde{Z}_0}^{m_0,*}(\tilde{F}_0)$ -torsor $F_{\tilde{Z}_0}^{m_0,*} \tilde{J}^1(X/K)$ is actually trivial (use Corollary 2.4). Let

$$t : \tilde{Z}_0 \rightarrow F_{\tilde{Z}_0}^{m_0,*} \tilde{J}^1(X/K)$$

be a section. By (x) in Sect. 2.1 we have that $t_*(\tilde{Z}_0) = (F_{\tilde{Z}_0}^{m_0,*} \tilde{Z}_1)^{\text{red}}$ and thus the morphism $(F_{\tilde{Z}_0}^{m_0,*} \tilde{Z}_1)^{\text{red}} \rightarrow \tilde{Z}_0$ is an isomorphism.

Repeating this reasoning for $(F_{\tilde{Z}_0}^{m_0,*} \tilde{Z}_2)^{\text{red}}$ over $(F_{\tilde{Z}_0}^{m_0,*} \tilde{Z}_1)^{\text{red}} \simeq \tilde{Z}_0$, $(F_{\tilde{Z}_0}^{m_0,*} \tilde{Z}_3)^{\text{red}}$ over $(F_{\tilde{Z}_0}^{m_0,*} \tilde{Z}_2)^{\text{red}}$ and so forth, we see that the natural morphisms

$$(F_{\tilde{Z}_0}^{m_0,*} \tilde{Z}_{i+1})^{\text{red}} \rightarrow (F_{\tilde{Z}_0}^{m_0,*} \tilde{Z}_i)^{\text{red}}$$

are isomorphisms from all $i \geq 0$.

Step IV Formalization and utilization of Grothendieck's GAGA.

In this fourth step, the argument will be written up under the assumption that $\text{char}(k) > 0$. The argument goes through verbatim when $\text{char}(k) = 0$, if one sets $m_0 = 0$ in the text below.

We now choose a non empty affine open subset $U_0 \subseteq U$ such that $\mathcal{X}_{U_0} \rightarrow U_0$ is smooth and projective and we let Z_0 be the reduced Zariski closure of $Z_0 = \mathfrak{R}$ in \mathcal{X}_{U_0} . We let \tilde{Z}_0 be the normalisation of Z_0 . By construction, the natural morphism $\tilde{Z}_0 \rightarrow Z_0$ is finite and birational and the morphism $\tilde{Z}_{0,K} \rightarrow Z_{0,K} = Z_0$ can be identified with the morphism $\tilde{Z}_0 \rightarrow Z_0$.

We may assume wlog that the natural map

$$H^1(\tilde{Z}_0, F_{\tilde{Z}_0}^{m_0,*}(\text{T}\mathcal{X}_{U_0} \otimes \Omega_{U_0/k})) \rightarrow H^1(\tilde{Z}_0, F_{\tilde{Z}_0}^{m_0,*}(\text{TX} \otimes \Omega_{K/k})) \quad (8)$$

is injective (here as before, $\text{T}\mathcal{X}_{U_0}$ and $\Omega_{U_0/k}$ are identified with their pull-backs to \tilde{Z}_0 and TX and $\Omega_{K/k}$ are identified with their pull-backs to \tilde{Z}_0).

Indeed, let $\pi_{\tilde{Z}_0 \rightarrow U_0} : \tilde{Z}_0 \rightarrow U_0$ and $\pi_{\tilde{Z}_0 \rightarrow \text{Spec}(k)} : \tilde{Z}_0 \rightarrow \text{Spec}(k)$ be the structural morphisms. Since U_0 is affine, we have

$$H^1(\tilde{Z}_0, F_{\tilde{Z}_0}^{m_0,*}(\mathcal{T}\mathcal{X}_{U_0} \otimes \Omega_{U_0/k})) = H^0(U_0, R^1\pi_{\tilde{Z}_0 \rightarrow U_0,*}(F_{\tilde{Z}_0}^{m_0,*}(\mathcal{T}\mathcal{X}_{U_0} \otimes \Omega_{U_0/k})))$$

and the map $H^1(\tilde{Z}_0, F_{\tilde{Z}_0}^{m_0,*}(\mathcal{T}\mathcal{X}_{U_0} \otimes \Omega_{U_0/k})) \rightarrow H^1(\tilde{Z}_0, F_{\tilde{Z}_0}^{m_0,*}(\mathcal{T}X \otimes \Omega_{K/k}))$ is induced by the morphism of sheaves

$$R^1\pi_{\tilde{Z}_0 \rightarrow U_0,*}(F_{\tilde{Z}_0}^{m_0,*}(\mathcal{T}\mathcal{X}_{U_0} \otimes \Omega_{U_0/k})) \rightarrow R^1\pi_{\tilde{Z}_0 \rightarrow \text{Spec}(k),*}(F_{\tilde{Z}_0}^{m_0,*}(\mathcal{T}X \otimes \Omega_{K/k}))$$

which is injective if the sheaf $R^1\pi_{\tilde{Z}_0 \rightarrow U_0,*}(F_{\tilde{Z}_0}^{m_0,*}(\mathcal{T}\mathcal{X}_{U_0} \otimes \Omega_{U_0/k}))$ is torsion free. This will be the case if U_0 is sufficiently small, since the sheaf $R^1\pi_{\tilde{Z}_0 \rightarrow U_0,*}(F_{\tilde{Z}_0}^{m_0,*}(\mathcal{T}\mathcal{X}_{U_0} \otimes \Omega_{U_0/k}))$ is a coherent sheaf.

We write $\tilde{J}^i(\mathcal{X}_{U_0}/U_0)$ for the base change of $J^i(\mathcal{X}_{U_0}/U_0)$ to \tilde{Z}_0 and $F_{\tilde{Z}_0}^{m_0,*}\tilde{J}^i(\mathcal{X}_{U_0}/U_0)$ for the base change of $\tilde{J}^i(\mathcal{X}_{U_0}/U_0)$ by $F_{\tilde{Z}_0}^{m_0}$.

Because of the existence of the injection (8) and the end of Step III, the $F_{\tilde{Z}_0}^{m_0,*}(\mathcal{T}\mathcal{X}_{U_0} \otimes \Omega_{U_0/k})$ torsor $F_{\tilde{Z}_0}^{m_0,*}\tilde{J}^1(\mathcal{X}_{U_0}/U_0)$ is trivial. Furthermore, if we choose a section

$$\tilde{Z}_0 \hookrightarrow F_{\tilde{Z}_0}^{m_0,*}\tilde{J}^1(\mathcal{X}_{U_0}/U_0)$$

then by (x) in Sect. 2.1 and again by (8), the pull-back by this section of the $F_{\tilde{Z}_0}^{m_0,*}(\mathcal{T}\mathcal{X}_{U_0} \otimes \Omega_{U_0/k})$ torsor $F_{\tilde{Z}_0}^{m_0,*}\tilde{J}^2(\mathcal{X}_{U_0}/U_0)$ is also trivial. Continuing in this way, we get a sequence of sections $\tilde{Z}_0 \hookrightarrow F_{\tilde{Z}_0}^{m_0,*}\tilde{J}^i(\mathcal{X}_{U_0}/U_0)$. The following commutative diagram summarises the situation:

$$\begin{array}{ccccc}
 & \vdots & \longleftarrow & \vdots & \longrightarrow \vdots \\
 & \downarrow & & \downarrow \sim & \downarrow \\
 & F_{\tilde{Z}_0}^{m_0,*}\tilde{J}^2(\mathcal{X}_{U_0}/U_0) & \longleftarrow & (F_{\tilde{Z}_0}^{m_0,*}\tilde{Z}_2)^{\text{red}} & \longrightarrow \tilde{Z}_2 \\
 & \downarrow & & \downarrow \sim & \downarrow \\
 & F_{\tilde{Z}_0}^{m_0,*}\tilde{J}^1(\mathcal{X}_{U_0}/U_0) & \longleftarrow & (F_{\tilde{Z}_0}^{m_0,*}\tilde{Z}_1)^{\text{red}} & \longrightarrow \tilde{Z}_1 \\
 & \updownarrow & & \downarrow \sim & \downarrow \\
 & \tilde{Z}_0 & \longleftarrow & \tilde{Z}_0 & \xrightarrow{F_{\tilde{Z}_0}^{m_0}} \tilde{Z}_0
 \end{array}$$

There are also curved arrows from the top \tilde{Z}_0 to the middle \tilde{Z}_0 and from the middle \tilde{Z}_0 to the bottom \tilde{Z}_0 .

Now write $\tilde{J}^i(Z_0/U_0)$ for the base-change of $J^i(Z_0/U_0)$ to \tilde{Z}_0 and $F_{\tilde{Z}_0}^{m_0,*}\tilde{J}^i(Z_0/U_0)$ for the base-change of $\tilde{J}^i(Z_0/U_0)$ by $F_{\tilde{Z}_0}^{m_0}$. Similarly, write $\tilde{J}^i(Z_0/K)$ for the base-change of $J^i(Z_0/K)$ to \tilde{Z}_0 and $F_{\tilde{Z}_0}^{m_0,*}\tilde{J}^i(Z_0/K)$ for the base-change of $\tilde{J}^i(Z_0/K)$ by $F_{\tilde{Z}_0}^{m_0}$.

Notice that we have a natural identification $\tilde{J}^i(Z_0/K) \simeq \tilde{J}^i(Z_0/U_0)_K$ and

$$(F_{\tilde{Z}_0}^{m_0,*} \tilde{J}^i(Z_0/U_0))_K \simeq F_{\tilde{Z}_0}^{m_0,*} \tilde{J}^i(Z_0/K).$$

We claim that each of the sections $\tilde{Z}_0 \hookrightarrow F_{\tilde{Z}_0}^{m_0,*} \tilde{J}^i(\mathcal{X}_{U_0}/U_0)$ factors through $F_{\tilde{Z}_0}^{m_0,*} \tilde{J}^i(Z_0/U_0)$ viewed as a closed subscheme of $F_{\tilde{Z}_0}^{m_0,*} \tilde{J}^i(\mathcal{X}_{U_0}/U_0)$. Since \tilde{Z}_0 is integral and dominates U_0 , to establish this, it is sufficient to show that the image of the corresponding section $\tilde{Z}_0 \rightarrow F_{\tilde{Z}_0}^{m_0,*} \tilde{J}^i(X/K)$ is contained in the closed subscheme

$$F_{\tilde{Z}_0}^{m_0,*} \tilde{J}^i(Z_0/K) \hookrightarrow F_{\tilde{Z}_0}^{m_0,*} \tilde{J}^i(X/K).$$

To see this, first notice that we have a commutative diagram

$$\begin{array}{ccccc} \vdots & \hookrightarrow & \vdots & \hookrightarrow & \vdots \\ \downarrow & & \downarrow & & \downarrow \\ Z_2 & \hookrightarrow & J^2(Z_0/K) & \hookrightarrow & J^2(X/K) \\ \downarrow & & \downarrow & & \downarrow \\ Z_1 & \hookrightarrow & J^1(Z_0/K) & \hookrightarrow & J^1(X/K) \\ & & \downarrow & & \downarrow \\ & & Z_0 & \hookrightarrow & X \end{array} \quad (9)$$

This does *not* follow from the general properties of jet schemes but follows from the fact that for all $i \geq 0$, the set $\psi_i(Z_0(K)) \cap Z_i(K)$ is (by construction) dense in Z_i and from the fact that $\psi_i(Z_0(K)) \subseteq J^i(Z_0/K)(K)$ by the functoriality of jet schemes. Applying base-change to the commutative diagram (9), we deduce that there is a commutative diagram

$$\begin{array}{ccccc} \vdots & \hookrightarrow & \vdots & \hookrightarrow & \vdots \\ \downarrow \sim & & \downarrow & & \downarrow \\ (F_{\tilde{Z}_0}^{m_0,*} \tilde{Z}_2)^{\text{red}} & \hookrightarrow & F_{\tilde{Z}_0}^{m_0,*} \tilde{J}^2(Z_0/K) & \hookrightarrow & F_{\tilde{Z}_0}^{m_0,*} \tilde{J}^2(X/K) \\ \downarrow \sim & & \downarrow & & \downarrow \\ (F_{\tilde{Z}_0}^{m_0,*} \tilde{Z}_1)^{\text{red}} & \hookrightarrow & F_{\tilde{Z}_0}^{m_0,*} \tilde{J}^1(Z_0/K) & \hookrightarrow & F_{\tilde{Z}_0}^{m_0,*} \tilde{J}^1(X/K) \\ & \searrow \sim & \downarrow & & \downarrow \\ & & \tilde{Z}_0 & \xrightarrow{=} & \tilde{Z}_0 \end{array}$$

and this establishes our claim.

Now choose a closed point $u_0 \in U_0$. View u_0 as a closed subscheme of U_0 . For any $i \geq 0$, let u_i be the i -th infinitesimal neighborhood of $u_0 \simeq \operatorname{Spec} k$ in U_0 (so that there is no ambiguity of notation for u_0). Notice that u_i has a natural structure of k -scheme. We may assume wlog that the morphism $\tilde{Z}_{0,u_0} \rightarrow Z_{u_0}$ is birational. Recall that by the definition of jet schemes (see [23, sec. 2]), the scheme $J^i(Z_0/U_0)_{u_0}$ represents the functor on k -schemes

$$T \mapsto \operatorname{Mor}_{u_i}(T \times_k u_i, Z_{0,u_i}).$$

Thus the infinite chain of compatible morphisms $\tilde{Z}_0 \hookrightarrow F_{Z_0}^{m_0,*} \tilde{J}^i(Z_0/U_0)$ gives rise to u_i morphisms

$$\tilde{Z}_{0,u_0}^{(p^{-m_0})} \times_{k_0} u_i \rightarrow Z_{0,u_i} \quad (10)$$

compatible with each other under base-change. Here $\tilde{Z}_{0,u_0}^{(p^{-m_0})}$ denotes the base-change of the k -scheme \tilde{Z}_{0,u_0} by the m_0 -th power of the inverse of the Frobenius automorphism of k .

Let \hat{U}_{u_0} be the completion of the local ring of U at u_0 . View the \hat{U}_{u_0} -schemes $\tilde{Z}_{0,u_0}^{(p^{-m_0})} \times_k \hat{U}_{u_0}$ and $Z_{0,\hat{U}_{u_0}}$ as formal schemes over \hat{U}_{u_0} in the next sentence. The family of morphisms (10) provides us with a morphism of formal schemes

$$\tilde{Z}_{0,u_0}^{(p^{-m_0})} \times_k \hat{U}_{u_0} \rightarrow Z_{0,\hat{U}_{u_0}}$$

and since both schemes are projective over \hat{U}_{u_0} , Grothendieck's formal GAGA theorem shows that this morphism of formal schemes comes from a unique morphism of \hat{U}_{u_0} -schemes

$$\iota : \tilde{Z}_{0,u_0}^{(p^{-m_0})} \times_k \hat{U}_{u_0} \rightarrow Z_{0,\hat{U}_{u_0}}.$$

By construction, at the closed point u_0 of \hat{U}_{u_0} , the morphism ι specializes to the morphism $F_{\tilde{Z}_{0,u_0}}^{m_0}$ composed with the finite and birational morphism $\tilde{Z}_{0,u_0} \rightarrow Z_{u_0}$. Now the set of points of $\mathcal{X}_{\hat{U}_{u_0}}$, where the fibres of ι are non-empty and of dimension > 0 is closed (see [10, IV,13.1.5]). On the other hand, since $\mathcal{X}_{\hat{U}_{u_0}}$ is proper over $\operatorname{Spec}(\hat{U}_{u_0})$ and since this set does not meet the special fibre \mathcal{X}_{u_0} , it must be empty. Thus the morphism ι is finite over the generic point of \hat{U}_{u_0} .

For use in Step V, we also record the fact that if ι can be shown to be flat then its degree $\deg(\iota)$ is equal to the degree $\deg(\iota_{u_0})$ of its base change to the special fibre \mathcal{X}_{u_0} , which is $p^{\dim(\mathfrak{R})m_0}$. (**)

Let \hat{K} be the function field of \hat{U}_{u_0} . Since k is an excellent field, we know that the field extension $\hat{K}|K$ is separable (see eg. [15, 8.2, Ex. 2.34]). On the other hand the just constructed finite morphism $\tilde{Z}_{0,u_0}^{(p^{-m_0})} \times_k \hat{K} \rightarrow Z_{0,\hat{K}}$ is defined over a finitely generated subfield K' (as a field over K) of \hat{K} . The field extension $K'|K$ is then still separable, so that by the theorem on separating transcendence bases, there exists

a variety U'/K , which is smooth over K and whose function field is K' . Furthermore, possibly replacing U' by one of its open subschemes, we may assume that the morphism $\widetilde{Z}_{0,u_0}^{(p^{-m_0})} \times_k K' \rightarrow Z_{0,K'}$ extends to a finite morphism

$$\alpha : \widetilde{Z}_{0,u_0}^{(p^{-m_0})} \times_k U' \rightarrow Z_{0,U'}.$$

Furthermore, we may assume wlog that α is flat if ι is flat.

Let $P \in U'(K^{\text{sep}})$ be a K^{sep} -point over K (the set $U'(K^{\text{sep}})$ is not empty because U' is smooth over K). The base change of the morphism α by P gives a morphism

$$\widetilde{Z}_{0,u_0}^{(p^{-m_0})} \times_k K^{\text{sep}} \rightarrow Z_{0,K^{\text{sep}}}.$$

This is the morphism h advertised in Theorem 1.2. If ι is flat, then we may assume that $\deg(\iota) = \deg(h)$.

Step V Degree estimates.

In this last step, we shall prove that if $\mathfrak{R} = X$ then

- the morphism ι is flat;
- the inequality

$$\text{char}(k) > \dim(X)^2 \int_X c_1(\Omega_X)^{\dim(X)} \quad (11)$$

implies that $m_0 = 0$.

According to the end of Step V (see in particular (**)), if ι is flat and $m_0 = 0$ then the morphism h can be assumed to be an isomorphism. This will thus conclude the proof of Theorem 1.2.

So suppose until the end of Step IV that $\mathfrak{R} = X$. Since both the source and the target of the morphism ι are regular schemes, the morphism ι is automatically flat by ‘miracle flatness’ (see [20, Th. 23.1]). Next recall that by definition

$$m_0 := \sup\{m \in \mathbb{N} \mid H^0(\widetilde{Z}_0, F_{\widetilde{Z}_0}^{m,*}(\widetilde{F}_0^\vee) \otimes \Omega_{F_{\widetilde{Z}_0}})_g \neq 0\}.$$

Since $X = \mathfrak{R}$ is smooth over K , we have $\widetilde{F}_0 = \Omega_{X/K}$ and $F_{\widetilde{Z}_0} = F_X$. Furthermore, let $f : X \rightarrow \text{Spec}(K)$ be the structural morphism. Notice that we have an exact sequence

$$F_X^*(\Omega_{X/k}) \xrightarrow{F_X^*} \Omega_{X^{(p)}/k} \rightarrow \Omega_{F_X} \rightarrow 0$$

where $X^{(p)}$ is the k -scheme X with the structural morphism $(F_{\text{Spec}(k)})^{-1} \circ f$. For a function x defined on an open set of X we have $F_X^*(x) = x^p$, where $p := \text{char}(K)$. Thus we may compute

$$F_X^*(d(x)) = d(F_X^*(x)) = d(x^p) = p \cdot x^{p-1} \cdot d(x) = 0$$

and thus the morphism $F_X^* : F_X^*(\Omega_{X/k}) \xrightarrow{F_X^*} \Omega_{X^{(p)}/k}$ vanishes. So we have a natural isomorphism $\Omega_{X^{(p)}/k} \simeq \Omega_{F_X}$. Note that we also have an isomorphism $\Omega_{X^{(p)}/k} \simeq \Omega_{X/k}$, since $(F_{\text{Spec}(k)})^{-1}$ is an isomorphism.

We also have an exact sequence

$$f^*(\Omega_{K/k}) \rightarrow \Omega_{X/k} \rightarrow \Omega_{X/K} \rightarrow 0$$

and since the morphism f has a section (since $X(K) \neq \emptyset$ because $X(K)$ is dense in X !), this sequence is exact on the left and splits. Thus we have a non-canonical isomorphism

$$f^*(\Omega_{K/k}) \oplus \Omega_{X/K} \simeq \Omega_{F_X}$$

In particular, Ω_{F_X} is locally free and thus we have

$$\begin{aligned} m_0 &= \sup\{m \in \mathbb{N} \mid H^0(X, F_X^{m,*}(\Omega_{X/K}^\vee) \otimes \Omega_{F_X}) \neq 0\} = \\ &= \max\left(\sup\{m \in \mathbb{N} \mid H^0(X, F_X^{m,*}(\Omega_{X/K}^\vee) \otimes f^*(\Omega_{K/k})) \neq 0\}, \right. \\ &\quad \left. \sup\{m \in \mathbb{N} \mid H^0(X, F_X^{m,*}(\Omega_{X/K}^\vee) \otimes \Omega_{X/K}) \neq 0\}\right) \end{aligned}$$

Now $f^*(\Omega_{K/k}) \simeq \mathcal{O}_X$ (non canonically) and for all $m \geq 0$, the sheaf $F_X^{m,*}(\Omega_{X/K})$ is ample. Thus the sheaf $F_X^{m,*}(\Omega_{X/K}^\vee)$ has no global sections. In particular, for all $m \geq 0$, we have $H^0(X, F_X^{m,*}(\Omega_{X/K}^\vee) \otimes f^*(\Omega_{K/k})) = 0$ and thus we have

$$m_0 = \sup\{m \in \mathbb{N} \mid H^0(X, F_X^{m,*}(\Omega_{X/K}^\vee) \otimes \Omega_{X/K}) \neq 0\}.$$

Now it is a special case of [24, Lemma-Def. 3.3] that we have

$$(\text{char}(\bar{K}))^{\sup\{m \in \mathbb{N} \mid H^0(X_{\bar{K}}, F_{X_{\bar{K}}}^{m,*}(\Omega_{X_{\bar{K}}/\bar{K}}^\vee) \otimes \Omega_{X_{\bar{K}}/\bar{K}}) \neq 0\}} \leq \frac{\bar{\mu}_{\max}(\Omega_{X_{\bar{K}}/\bar{K}})}{\bar{\mu}_{\min}(\Omega_{X_{\bar{K}}/\bar{K}})}$$

where $\bar{\mu}_{\max}(\Omega_{X_{\bar{K}}/\bar{K}})$ (resp. $\bar{\mu}_{\min}(\Omega_{X_{\bar{K}}/\bar{K}})$) is the Frobenius stabilised maximal (resp. minimal) slope of $\Omega_{X_{\bar{K}}/\bar{K}}$ with respect to some arbitrary ample line bundle L on $X_{\bar{K}}$. See [24, sec. 3] for the definition of Frobenius stabilised slopes and further references. Furthermore, the computation in [24, sec. 5] shows that if

$$\text{char}(\bar{K}) > \text{rk}(\Omega_{X_{\bar{K}}/\bar{K}})^2 \deg_L(\Omega_{X_{\bar{K}}/\bar{K}})$$

then we have $\bar{\mu}_{\max}(\Omega_{X_{\bar{K}}/\bar{K}}) < \text{char}(\bar{K}) \cdot \bar{\mu}_{\min}(\Omega_{X_{\bar{K}}/\bar{K}})$. Here

$$\deg_L(\Omega_{X_{\bar{K}}/\bar{K}}) := \int_{X_{\bar{K}}} c_1(\Omega_{X_{\bar{K}}/\bar{K}}) c_1(L)^{\dim(X_{\bar{K}})-1}.$$

In our situation, we may choose $L = \det(\Omega_{X_{\bar{K}}/\bar{K}})$, since $\Omega_{X_{\bar{K}}/\bar{K}}$ is ample and thus we see that the inequality

$$\begin{aligned} \text{char}(k) = \text{char}(\bar{K}) &> \text{rk}(\Omega_{X_{\bar{K}}/\bar{K}})^2 \int_{X_{\bar{K}}} c_1(\Omega_{X_{\bar{K}}/\bar{K}})^{\dim(X_{\bar{K}})} \\ &= \text{rk}(\Omega_{X/K})^2 \int_X c_1(\Omega_{X/K})^{\dim(X)} \end{aligned}$$

ensures that

$$\begin{aligned} m_0 &= \sup\{m \in \mathbb{N} \mid H^0(X, F_X^{m,*}(\Omega_{X/K}^\vee) \otimes \Omega_{X/K}) \neq 0\} \\ &= \sup\{m \in \mathbb{N} \mid H^0(X_{\bar{K}}, F_{X_{\bar{K}}}^{m,*}(\Omega_{X_{\bar{K}}/\bar{K}}^\vee) \otimes \Omega_{X_{\bar{K}}/\bar{K}}) \neq 0\} = 0 \end{aligned}$$

This completes Step V.

N.B. At the time of submission of the present article, the article [24] by the second author (quoted in Step V) has not been accepted for publication yet. This article is only used in Step V and thus only the degree estimate in Theorem 1.2 depends on it.

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