A practical model for aerodynamic probe-system response estimation (with review of existing models)

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Abstract. The accurate estimation of the unsteady response (bandwidth) of pneumatic pressure probe-systems (probe, line and transducer volume) is a common practical problem encountered in the design of aerodynamic experiments. Understanding the bandwidth of the probe-system is necessary to ensure that unsteady flow features are accurately captured. Where traversing probes are used, the desired traverse speed and spatial gradients in the flow dictate the minimum probe-system bandwidth required to fully resolve the flow. Existing approaches for bandwidth estimation are either complex or inaccurate in implementation, so probes are often designed based on experience. Where probe-system bandwidth *is* characterized, it is often done experimentally, requiring careful experimental set-up and analysis. There is a need for a relatively simple but accurate model for estimation of probe-system bandwidth.

A new model is presented for the accurate estimation of pressure probe bandwidth for simple probes commonly used in wind tunnel environments. Experimental validation is provided.

1. Keywords

Response, bandwidth, frequency, model, pressure, probe, transient, experimental, acoustic, pneumatic.

2. Nomenclature

‾	Mean speed of sound (m/s)		
C	Electrical capacitance (Farads)	*R*	Electrical resistance (Ohms)
d	Diameter (inner) of the tube (m)	*t*	Time (s)
dB	Decibels e.g. 20 $\log_{10}(p_1/p_2)$	*u*	Velocity (m/s)
DC	Direct current	*U*	Voltage (V)
		v	Kinematic fluid viscosity (xx)
		V	Volume (m^3)
			Damped natural frequency
G	Electrical conductance (Siemens)		Undamped natural frequency
GND	Ground (0 V) potential	‾	Mean fluid density (kg/m^3)
			Mean fluid dynamic viscosity (Pa s)
			Damping ratio
j	$\sqrt{\quad}$	**Subscripts**	

l	Length of the cylindrical tube (m)	S	Source (e.g. voltage source U_S)
L	Electrical inductance (Henries)	t	Quantity in or parameter of the probe tube
		$t0$	Quantity at tube open end
		$t1$	Quantity at tube-transducer interface
LTI	Linear time-invariant	Tot	Combined volume of tube and transducer (i.e. V_{tot})
n	Polytropic exponent ($pV^n = C$)	T	Quantity in the transducer
N	Number of tube (length)lumps		
p	Pressure (Pa)		
$\bar{p}$	Mean pressure (Pa)		

3. Introduction

Pressure probe-systems (probe, line and transducer volume) are used in many research fields for experimental aerodynamic characterization of flows, as well as for routine monitoring in many process industries, in power generation machines and on airplanes, for example. Probe-systems may be wall mounted (static pressure tappings) or fully immersed (probes for measurement of total pressure, flow direction etc.).

In general, it is desirable to place pressure transducers as close to the measurement point as possible to maximize measurement bandwidth. In some applications it is possible to install very small transducers (which tend to be expensive and are limited on operating temperature) directly at the point of interest. It is more conventional to have a pressure sensing hole, or multiple holes in the case of probes, located at the point of interest, connected via circular tubing (the line) of given length (typically anywhere from a few cm to a few m) to a remote pressure transducer. In the simplest system (where the probe and line have the same internal diameter) the probe-system reduces to a circular tube of constant diameter open at one end and connected to a pressure transducer at the other, as in Figure 1. More complex systems are discussed.

The problem of probe-system response arises because the pressure transducer indicates the pressure at the transducer diaphragm (p_T), not that at the point of interest (p_{t0}), which is separated from the transducer diaphragm by probe-system (probe, line and transducer volume). Only if the frequency of variations in p_{t0} lie within the bandwidth of the probe-system will p_{t0} and p_T be equal. The frequency content of p_{t0} above the probe-system bandwidth will be distorted, and p_T will differ from p_{t0} in both the frequency and time domain. In practice, the bandwidth is defined as the frequency band from 0 Hz (DC) to the -3 dB cutoff frequency or to +3 dB at the first resonance peak, depending on whether the probe-transducer system is under-damped or over-damped.

At the probe-system design stage, a relatively simple method to predict probe-system response would inform decisions regarding transducer choice (in terms of transducer volume V_T), tube length (l) and diameter (d). It is widely understood that minimizing the values of V_T and l/d^2 maximizes bandwidth [1], however accurate quantitative prediction remains non-trivial, and is important in preliminary design and for setting, for example, traverse speeds etc.

3.1. Summary of existing models

A number of models aimed at the characterization of pressure probe-systems have been proposed. The review of van Ommen et. al. [2] provided a comparison of some of the models in literature up to the time of publication (1999), focusing on the characterization simple system of Figure 1. This review demonstrated by way of comparison to experiments that the formulation of Bergh and Tijdeman [3] (the earliest model reviewed) based on the work of Iberall [4] and Nichols [5] was the only model of those reviewed capable of accurately describing the probe-systems under investigation across the entire frequency spectrum. A number of low-order models—based on linear ordinary differential equation (ODE) descriptions of the system—were examined. The review considered the second order ‘isothermal’ model of Clarke and Atkinson [6] from which an ‘adiabatic’ variant was also derived by the authors. The model of Edelman [7] was also reviewed although the derivation and certain assumptions were not available in the original publication. It was shown that these ODE-based models were able to predict the first resonance frequencies with limited accuracy (for reasons discussed below), while agreement decayed rapidly for higher frequencies. Finally, the first order model of Xie and Geldart [1] was reviewed. This is a first order model which is inherently unable to predict resonance.

A much earlier model by Taback [8], also produced results in good agreement with experiments. A shortcoming of the model is the use of constants which must be determined experimentally on a case-by-case basis, however. Richards [9] presented an analogous model to that of Bergh and Tijdeman [3], but considers non-circular tubes (although easily adapted to circular tubes) and presents the results in a transfer matrix format which was simpler to implement for complex systems. The Richards model was based on an analogy between acoustic wave propagation in the probe tube and electrical wave propagation in a transmission line, which was comprehensively covered by Kirshner and Katz [10].

The relevant literature, either that of specific probe models (as above), or that related to modelling individual components (e.g. the tube), is spread across numerous engineering fields, including general instrumentation [3][4][11], instrumentation for fluidised bed combustion research [1][2][6], more general studies in acoustics [12][13][14], control, and fluid transmission in tubes (e.g. for fluid power transmission) [15][16][17]. The large body of existing research on fluid transients in cylindrical tubes has been overlooked in some recent studies [1][2][6], including existing textbook characterizations of the specific problem of the simple tube-transducer system, like that of [11]. The proposed model attempts to bring the most relevant work (from a diverse range of fields) to bear on the specific problem of probe-system response.

3.2. The need for a simplified model

The general tube-plena response problem was rigorously described in 1965 by the Bergh and Tijdeman [3] model and again by Richards [9] in an analogous method, both of which are suited to systems of greater complexity than the one of typical interest. Nachtigal and Martin [11] presented a simple method for systems that were approximately lossless (i.e. low l/d^2). Furthermore, the various approaches to the problem of fluidic components such as tubes and volumes were generalised in the 1975 reference text of Kirshner and Katz [10]. Despite these resources, attempts to simplify the probe-system response problem for simple systems are (the commonly-cited models for probe system response) are often either unwieldy to implement or inaccurate. A simple but accurate method to estimate probe-system response is seen to be helpful.

Fluid transmission in tubes is a wave propagation problem dealt with analytically using mathematical models (e.g. [4]) that include transcendental terms that are not amenable to hand calculations. It is possible to derive general rules of thumb from the full equations [2] for the case of simple pressure probes. Alternatively, employing methods to remove (by approximation) the transcendental terms (e.g. [15][18]), yields a generalized analytical tool and is discussed below.

The aim of this paper is to present a simple but accurate low-order model for probe-system bandwidth estimation using hand calculations (no scripting), that is able to additionally provide a meaningful Bode plot with limited computational effort.

4. Description

4.1. Comparison of existing models

To date, a number of models (outlined above) have been presented for either specific pressure probe configurations [1][2][6][7] or generalized formulations [3][9] that can be built in to models of more complex fluidic systems. We now compare models in the literature using analogy to electrical circuits to understand the assumptions inherent in each model.

4.1.1. Basis of comparison.

The transmission line analogy draws on the earlier work of electrical transmission lines in communication ([10] provides a good summary, with more detail in e.g. [15][16][17]). Each continuous circular tube section is treated as a lossy transmission line, while connected volumes are treated as capacitive components. Voltage is analogous to pressure difference and current to volume flow. At the most general level, this is described in Figure 2 for the simplest probe-transducer system shown Figure 1.

To gain further understanding of this analogy, a differential element of the transmission line is illustrated in Figure 3. A transmission line element consists of distributed series resistance (*R'* d*x*) and distributed series inductance (*L'* d*x*) per unit length of the conductor – collectively the series impedance per unit length (*Z*' d*x*). The fluidic analogies are viscous pressure loss and fluid inertance respectively. Additionally, the transmission line consists of distributed shunt admittance (*G'* d*x*) and distributed shunt capacitance (*C'* d*x*) per unit length between the conductor pair – collectively the shunt admittance per unit length (*Y* d*x'*). The shunt admittance (*G'*) is related to the heat transfer at the wall and would be zero for an adiabatic tube. The shunt capacitance is analogous to the storage of compressible fluid within the tube. Finally, a transmission line is treated as one-dimensional, which is typically acceptable for axisymmetric fluid problems as is the case for a cylindrical tube.

The 'distributed-parameter' model [12] of a transmission line is produced by considering the system as d*x* approaches zero - this yields wave equations. These equations are typically manipulated to yield the transfer matrix representation of a transmission line in terms of hyperbolic expressions:

$$\begin{matrix} \cosh\Gamma & \sinh\Gamma \\ / \quad \sinh\Gamma & \cosh\Gamma \end{matrix} \qquad (1)$$

A number of simplified methods exist for approximating the transmission line. A number of the existing models [1][2][6] effectively use a 'lumped-parameter' approach, in which the tube is divided in to a finite number of elements (as in Figure 3) of finite length (i.e. d*x* >> 0). These are discussed in section 4.1.3. It is also possible to consider only a finite number of harmonic modes [18].

The propagation constant (Γ) and characteristic impedance (Z_C) of equation 1 are defined in the most general case:

$$\Gamma \quad \sqrt{} \qquad (2)$$

$$(3)$$

It is evident from equations 1 to 3 that *Z* and *Y* completely describe the tube system. In the case of ideal electrical components (e.g. linearly resistive elements), the above system of equations can be solved directly in terms of the variables in Figure 3. Clearly for the fluidic case, relationships for viscous loss, fluid impedance and fluid capacitance in terms of pressure and volume flow (or velocity) must be derived. The accurate formulation of these terms for the fluidic case is non-trivial, however. The assumptions made in deriving these terms delineate the existing models and are considered in further detail in the following section.

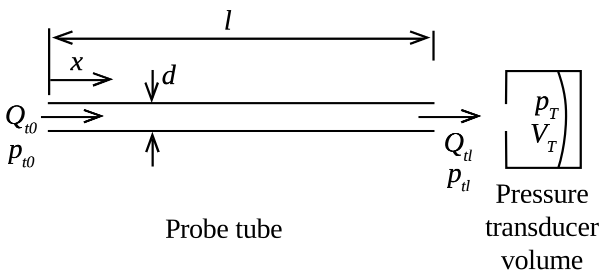

Figure 1: Simple probe-transducer system as subcomponents.

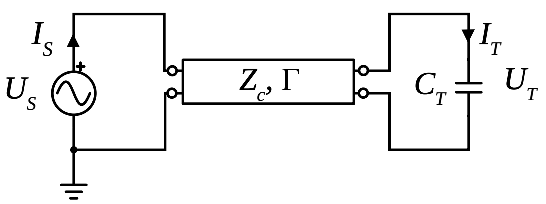

Figure 2: Electrical analogy of non-linear model for a simple tube-transducer system.

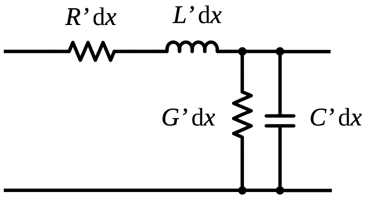

Figure 3: Differential element representation of a transmission line. Primed values are per unit length.

4.1.2. Distributed-parameter, dissipative friction models.

The model of Bergh and Tidjeman [3] and by extension the equivalent model of Richards [9] have been shown experimentally to describe the physical probe-transducer system with good accuracy [2][3][9]. Both models utilize the distributed-parameter treatment of the tube (or tubes), and it is this that allows them to describe the harmonic response, which is dominant in many configurations. Additionally, both models use the so-called ‘dissipative friction’ model [12] based on the work of Iberall [4] which considers the frequency dependence of the viscous effects and heat transfer in the tube.

The response for a simple system (Figure 1) has been modeled using a distributed-parameter approach with the dissipative friction model and the magnitude response is shown in Figure 4 along with the first 3 resonant frequencies for an open tube of the same dimensions. In this case the transducer volume (that of a FirstSensor CTE-8000) is significant in comparison to the tube volume. The figure illustrates that the system consists of a harmonic response similar to that of an open organ pipe, and an additional resonant peak at lower frequency (which would define the useful bandwidth), which results from the low-order impedance-capacitance system of the tube and volume. As the transducer volume tends to zero, the harmonic response dominates, as shown in Figure 5.

The distributed-parameter treatment is capable of describing both the low-order and harmonic response. While this enables the models to produce the correct characteristic curves, it is also important that the tube series impedance (Z) and shunt admittance (Y) are well described for the fluidic problem. If Y and Z are poorly described, the characteristic curve may be similar, however the damping and frequency of each resonance mode will be incorrect. The dotted lines of Figure 4 and Figure 5 illustrate the effect of replacing the dissipative friction model of the Bergh and Tijdeman model [3] with a simplified friction model – the ‘average friction’ described in [12]. The characteristic shape of the curve remains the same, however the magnitude and frequency of the peaks is affected.

and Z_C for the dissipative friction model are presented in Appendix A (equations A1 and A2) along with and Y (equations A3-5) on which they depend. The derivation of these terms can be found in [3][9]. The accuracy of the model is a result of the frequency dependent treatment of heat transfer and viscous effects (which can be inferred from the equations). This treatment relies on the Bessel function of the first kind which makes reducing the resulting model to simple terms describing bandwidth impossible.

A number of approximations can be made for equations 1 to 8 in order to yield a simplified solution. For example, one could replace the Bessel function with an approximation which utilized algebraic terms [refxx]. Removal of the Bessel function altogether could be achieved with an alternative friction model. The hyperbolic trigonometric terms of equation 1 could be approximated by an infinite product as described in [12]. The coarsening and simplification of this kind of model represents one possible basis for a new model. This is discussed in section 4.2.

4.1.3. Low-order models with lumped-parameter

A number of low-order models exist in the literature. These consider a simple momentum balance across the tube and transducer volume to yield first and second order ODE-based models. The second order model of Clark and Atkinson [6], the proposed variation of van Ommen et al [2] and the first order model of Xie and Geldart [1] are all such models. The model of Edelman [7], while producing reasonable results, is not discussed at length since its derivation was not published.

The magnitude response predicted by the low-order models (excluding the Xie and Geldart model) is shown in Figure 6. The figure illustrates that compared to the prediction of a distributed-parameter, dissipative friction model, the models of Clark and Atkinson [6] and van Ommen et al [2] do not predict the first resonance peak accurately and do not predict harmonic resonance (as expected for a second order model). The model of Edelman predicts the first resonance and first harmonic frequency with some accuracy for this configuration. The disagreement in each case can be understood through further analysis of the formulation of the respective models.

The low-order models are relatively easily compared amongst each other on the basis of their common momentum balance formulation (as was done in [2]). It is not immediately clear how these formulations compare to the distributed-parameter models in terms of simplifying assumptions. Simply comparing results from each model for particular cases does not necessarily provide insight in to the shortcomings of the individual models or by extension any means by which to improve them. To understand these models, the momentum balance (see [2]), presented here in terms of volume flow rate Q, is first considered:

$$\frac{}{}\frac{}{} \quad - \frac{\mathrm{d}}{\mathrm{d}}\frac{}{} \qquad (4)$$

The first right hand side term is the driving pressure at the open end of the tube, the second is the viscous pressure loss (based on the Hagan-Poiseuille equation – referred to as the linear friction model [12]), the third term is the inertia of the fluid in the tube (which is ignored in the model of Xie and Geldart [1]). The fluid volume flow rate is derived from the ideal gas equation [2]:

$$\frac{}{-}\frac{\mathrm{d}}{\mathrm{d}} \qquad (5)$$

Equation 10 and its time-derivative can be substituted in to equation 9 to yield a second order ODE relating the pressure of interest (p_{t0}) to the transducer-measured pressure (p_T).

It is apparent from equation 9 that the pressure is explicitly considered at only two locations—each end of the tube—and that the flow rate is assumed to be constant along the tube. The tube is therefore quasi-zero-dimensional. This precludes the prediction of any harmonic modes by these models. Furthermore, equation 10 implies that the expansion of the fluid in the tube and the transducer is treated as a lumped system, which is likely to be an additional source of error.

These models can be more easily understood by considering an electrical circuit analogy. This also allows for convenient comparison to the transmission line models of the previous section (Figure 2). This is illustrated in Figure 7. This circuit (in which dx = l) implies that I_S is constant across the series resistive and inductive components, which collectively represent a zero-dimensional (in space) system, confirming the previous conclusion regarding the treatment of the tube. As per the previous conclusions, the system describes a single capacitive component (noting that parallel connection of C_t and C_T).

It is apparent that the formulation of these models includes some significant simplifications of the physical system. Considering only the momentum balance (equations 9 and 10), it is not necessarily clear whether the models describes the tube in a meaningful way – the quasi-zero-dimensionality, and lumping of tube and transducer volumes being particularly counterintuitive. By considering the circuit analogy (Figure 7), however, it becomes clear that the components representing the tube (R_t, L_t and C_t – defined in Appendix A) relate to a coarse approximation of a transmission line (c.f. Figure 3) –

specifically an approximation where d*x* is finite (in this case d*x* = *l*) rather than infinitesimal as in the distributed-parameter method. These models are therefore a lumped (by length) parameters approximation with the number of lumps (*N*) equal to 1. By reducing d*x* the solution would approach that of a distributed-parameter solution (as in the previous section) as the number of pi sections approached infinity. The use of a lumped-parameter approach in the context of a new model is discussed further in section 4.2.

It is likely that the model of Edelman was based on a lumped-parameter formulation with *N* = 2, possibly with a correction for the damped natural frequency.

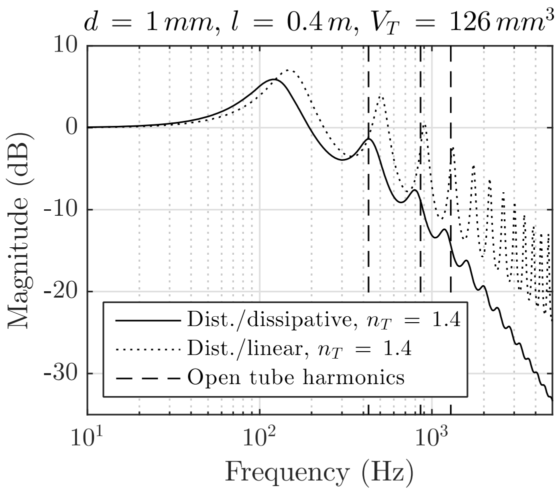

Figure 4: Magnitude of a simple probe-transducer system using distributed-parameter model of tube and first 3 harmonic frequencies for an open-ended organ pipe (*d*, *l*)

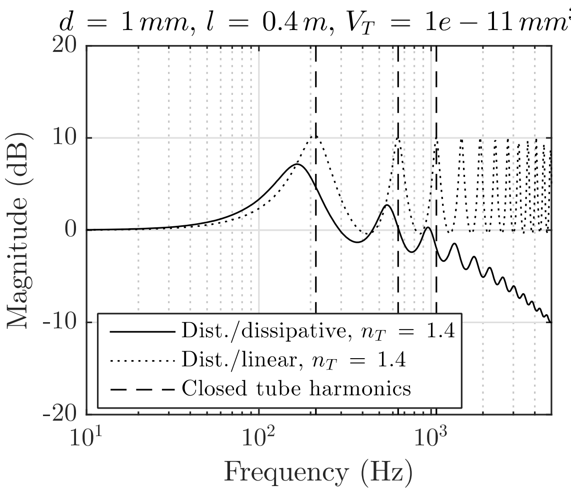

Figure 5: Magnitude of a simple probe-transducer system using distributed-parameter model of tube and first 3 harmonic frequencies for a closed-ended organ pipe (*d*, *l*)

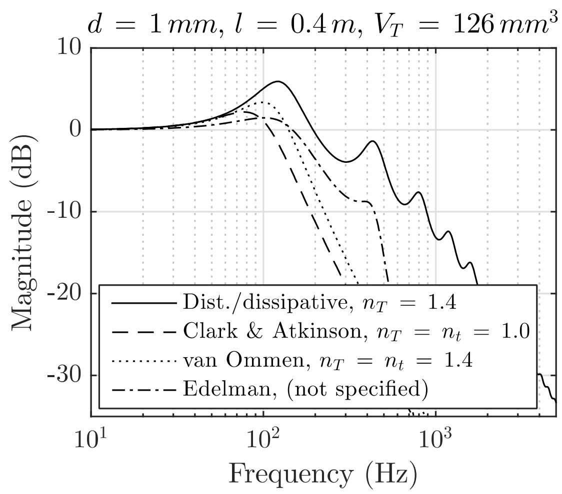

Figure 6: Comparison of the low-order lumped-parameter models for a simple probe-transducer system. The distributed-parameter, dissipative friction model is included for reference.

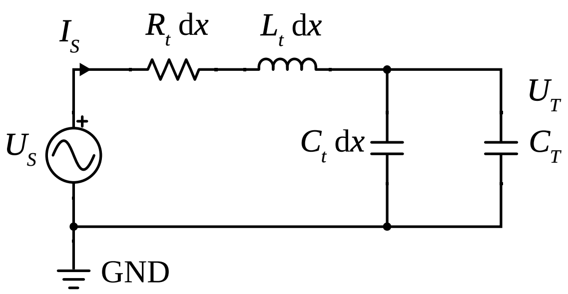

Figure 7: Electrical analogy of Xie and Geldart [1] (L_{tube} = 0), Clark and Atkinson [6] and van Ommen et al [2] models. *dx* = *l* in all cases.

4.2. Development of a new model

None of the existing probe-system response models are both accurate and easy to use. Distributed-parameter models (with dissipative friction), while flexible and accurate, are relatively time consuming to implement (requiring a mathematical coding environment), while the simpler low-order models lack accuracy and fail to predict harmonics or account for frequency dependent loss. A simple model for the tube-transducer configuration (Figure 1) which strikes a balance between accuracy and ease of implementation is thought to be useful. The Edelman model is in this category, but the model derivation was not published, which may be one of the reasons it was not more widely adopted. It also predicts the magnitude of the resonance peaks poorly.

We now develop a new model which has the simplicity and ease of use of a low-order model but with improved accuracy. A low-order transfer-function model was used because expressions for resonant frequency and cut-off frequency can be derived from the system poles (or from system coefficients for a second order system). This means that an estimate of the probe bandwidth can be made from simple equations.

Two fundamental aspects of model formulation are now discussed: the treatment of the tube (either *continuous*, as in the distributed-parameter method, or *discrete* as in the lumped-parameter method); and the propagation/friction model (treatment of friction, fluid inertia and heat transfer).

4.2.1. Tube treatment

The numerical treatment of the tube provides the framework for a model of a tube-transducer system, the treatment of the volume being comparatively simple. In the previous sections, two approaches have been identified in existing probe models – the distributed-parameter approach and the lumped-parameter approach. The latter approach, when coupled with simple fluidic expressions for R, L and C (e.g. the linear friction model), yields relatively simple equations which can be reduced to an s-domain polynomial transfer-function, although as N increases the system order increases and the polynomial coefficients become unmanageably large in their exact form. On the other hand, the distributed-parameter model, as represented by equation 1, does not reduce to an easily used form due to its transcendental terms.

It is possible to simplify equation 1 by employing Taylor series approximations of each (tube) matrix element [10] or by approximation of the hyperbolic functions with infinite product expressions, which are referred to as modal approximations [12]. The approximation of the hyperbolic trigonometric terms by infinite products has the advantage of preserving the roots of the equations, which is not the case for the Taylor series expansion method [10] nor if the tube is approximated using the lumped-parameter method. The infinite products method, consisting of N products will resolve the N^{th} harmonic. Hence for bandwidth estimation alone, N = 1 might be sufficient.

The modal approximation of Goodson [18] can be easily used to demonstrate the usefulness of this approach, but does not yield a transfer-function which is easily separable in terms of s, making further analysis difficult. Addressing a similar set of problems (transcendental, irreducible expressions) for the purpose of creating control models, Yang and Tobler [15] developed a modal formulation based on second order ¯-domain (defined below) polynomial expressions, which lends itself well to the problem at hand.

4.2.2. Propagation models

A number of fluid propagation models applicable to the circular tube of a probe-transducer system are summarised in [12]. The simplest model is the lossless model which does not account for the effect of viscosity or heat transfer, instead considering only fluid inertia and tube capacitance. The second common model [2] is the linear friction model, in which viscous effects proportional to mean (cross sectional) velocity are considered, although heat transfer is not. A more accurate dissipative model was developed by Iberall [4] and by Nichols [5] which accounts for the frequency dependence of viscous effects and heat transfer.

Each of these 3 models can be applied to certain types of tube configuration with accuracy. Goodson and Leonard [12] used the dissipation number (D_n) to delineate the types of system which were suited to each model:

(6)

Where dissipation along the tube is small ([12] suggest $D_n < 0.0001$, while [11] proposed a method for bandwidth estimation valid up to $D_n < 0.01$) the lossless model gives reasonable results, while the dissipative model provided the best accuracy where D_n is relatively large. A dependence on the line termination—a function of the transducer volume (V) also exists. For pressure probes measuring air at atmospheric conditions (1 atm, 273 K), and considering a criteria of $D_n < 0.01$, the maximum length of a 1 mm tube for lossless treatment is approximately 50 mm. The lossless approach is therefore of limited use for most wind tunnel probes.

In general, the dissipative model is accurate across all cases with no mean flow (subsonic, acoustic propagation) although it also holds for small mean flow [10], while the lossless case is a useful simplification for tubes of small l/d^2 and where the transducer volume approaches zero. The average friction model should offer some improvement as viscous effects in the tube increase from near-zero, however for air (and other gasses with 1.0) the frequency dependence of the polytropic exponent (n_t) and therefore viscous effects and heat transfer make its region of applicability less easily definable. Goodson and Leonard [12] suggested that in the case of gasses with 1.0, the dispersive model is the only suitable choice (for accurate results).

Recognising the need for an algebraically simplified way of describing a tube with the accuracy of the dissipative friction model and distributed-parameter tube treatment, Yang and Tobler [15] developed a modal approximation of the distributed-parameter treatment. The approximation used second order polynomial terms to represent each harmonic mode, and was based on the linear friction model, but included also a correction for frequency dependent effects. This method provides an ideal means to develop a simple and accurate model for a simple probe-transducer system.

4.2.3. Proposed model

In the development of a new, simplified model, the first consideration was the simplification of the tube treatment – either lumped (by length) parameters or by modal approximation. As illustrated in [10], the infinite product method of Goodson [18] preserves the roots of the hyperbolic terms, while lumped-parameter approximations and Taylor series expansions of the hyperbolic terms do not.

The transducer volume affects the tube harmonic frequencies such that the preservation of the hyperbolic roots does not necessarily mean a preservation of harmonic frequencies. This is demonstrated in Figure 8, which compares a modal approximation (N = 1) to the distributed-parameter model (both using dissipative friction) – the first harmonics do not agree. Agreement for the N^{th} mode improves as the N+1 mode is included. The use of linear friction in the modal approximation introduces further error, as expected. Figure 8 also includes a lumped-parameter approximation (with N = 2), which shows surprisingly good agreement given its linear friction assumptions. The latter model deteriorates in extreme cases, such as when the tube-transducer geometry approaches that of a closed-end organ pipe.

Both the standard infinite products method [12] where N = 1 and the lumped-parameter approach where N = 2 (which both resolve the first tube harmonic only) result in relatively high order transfer-functions with cumbersome coefficients. This makes calculations for estimating bandwidth algebraically awkward. The corrected modal approximation of Yang and Tobler [15] provides a useful method which deals with the error associated with the linear friction model and results in a low (second) order transfer-function.

The corrected modal approximation of Yang and Tobler [15] is based around the following equation (a modal approximation of the hybrid arrangement of equation 1):

$$\sum^{N} \qquad (7)$$

Where p_{t0} and Q_{t0} are the pressure and volume flow rate at the open end of the tube (probe tip) and p_{t1} and Q_{t1} are values at the transducer interface. The natural frequency and damping ratio correction terms and can be determined from Figure 14 in Appendix A. The remaining terms as defined by [15] are also included in Appendix A. The normalised Laplace frequency $\bar{}$ is defined:

(8)

Where a finite number of modes are used in the summation, a DC offset error occurs. This is discussed in [15]. Where N = 1, this offset is given by the infinite sum:

$$\sum \qquad (9)$$

The relationship for the pressure in the transducer ($p_{T,}$ which is equal to p_{t1} at the transducer interface) and the volume flow rate in to the transducer (Q_{t1}) in terms of $\bar{}$ can be approximated (assuming no diaphragm deflection):

(10)

Using the first row of equation 7 for N = 1, equation 10 and considering the DC offset (which must be applied to the system gain and damping ratio), the following ratio of input pressure (p_{t0}) to transducer pressure (p_T) can be defined in terms of the standard *s*-domain variable (by ultimately substituting equation 8 for $\bar{}$):

(11)

Although the equation reduces to that of a second order system, it will resolve either the low-order impedance-capacitance response or where the transducer volume approaches zero, the first tube harmonic. Having substituted numerical values describing a tube-transducer system into equation 11, the system parameters of interest are yielded by standard second order system relations where the denominator is of the standard form :

(12), $\sqrt{}$ (3),

The -3 dB frequency ($_{-3dB}$) can then be calculated:

$$1 \quad 2 \qquad 4 \qquad (15)$$

For underdamped systems (< 0) where the resonance magnitude is greater than +3 dB, it would be better to consider the upper limit of the bandwidth to be +3dB which occurs at a frequency less than *n*. That is when:

— √2 (16), 0.55 (17).

The model is applied across a range of examples in the following section and compared to experimental results.

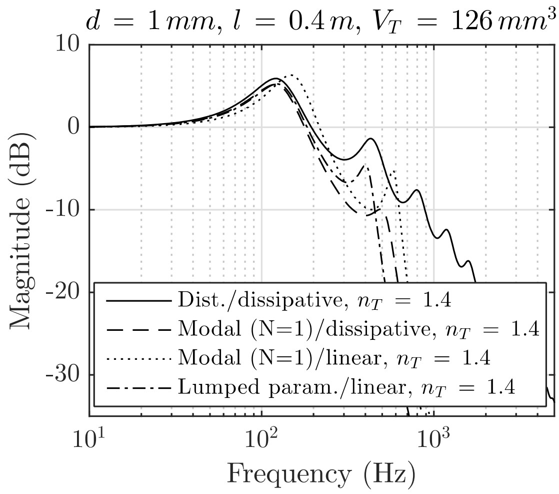

Figure 8: Comparison of the modal approximation for N=1 and for dissipative and linear friction, and the lumped-parameter model with linear friction. The distributed model with dissipative friction is included for reference.

4.2.4. Experimental method

Experimental frequency responses were determined for a number of tube lengths and transducers volumes. The experimental set up consisted of an aluminium cup with stainless steel tubular connections (Figure 9). By stretching a latex balloon over the cup and inflating until failure, an input pressure step of approximation -0.1 bar was generated. The (output) pressure at the end of each tube was measured with a FirstSensor KMA-8000 2 bar absolute pressure transducer (uncalibrated combined uncertainty of 0.2 mbar) connected via a precision machined fitting. The transducer volume was measured to be 200 ± 5 mm^3 using a 10 mm^3 pipette. The input pressure was measured with a FirstSensor CTE-8000 2 bar absolute transducer the same specifications. The volume of the output transducers was reduced to approximately zero by filling each with a mixture of water and detergent (to avoid air bubbles). The pressure transducers were sampled at 20 kHz with a National Instruments 16-bit analogue to digital converter. The data was digitally lowpass filtered at 1 kHz, which was well above the frequencies of interest (e.g. the first and second resonant frequencies).

Each tube assembly effectively consisted of a length (l_p) of flexible plastic tube with nominal internal diameter of 1.37 mm connected to hardware at each end which consisted of a length (l_{ss} = 25.4 mm) of stainless steel tube with a nominal internal diameter of 1.06 mm. This is shown in Figure 10. This style of assembly is common in laboratory aerodynamics experiments. Each assembly of total length l was replicated 4 times and tested simultaneous with ….

From the measured input and output signals in the time domain, low-order transfer-function models and corresponding Bode plots could be estimated using MATLAB. This method focused on estimating the low frequency modes of the system, which was sufficient for validation of the proposed low-order modal model. 4 pole transfer-functions were used where the relative order of the transfer-function was 1. This captured the first 1 to 2 modes of each system. The results are presented in the following section.

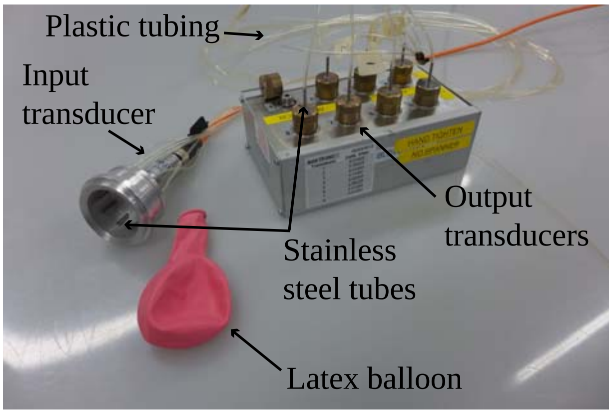

Figure 9: Photograph of complete experimental set up.

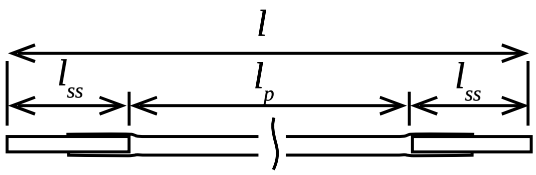

Figure 10: Schematic of effective experimental tube set up.

5. Experimental results compared to proposed model

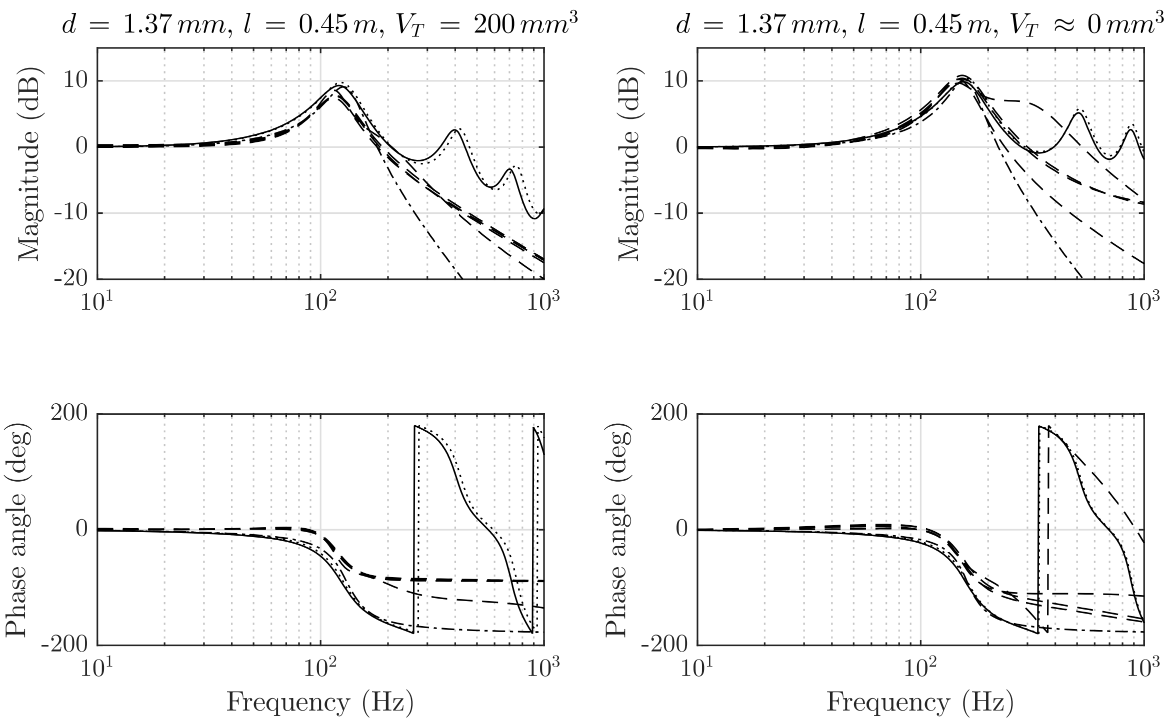

Figure 11: Experimental results for l = 0.45 m, V_T = 200 mm^3 and ~0 mm^3. See Figure 12 for legend.

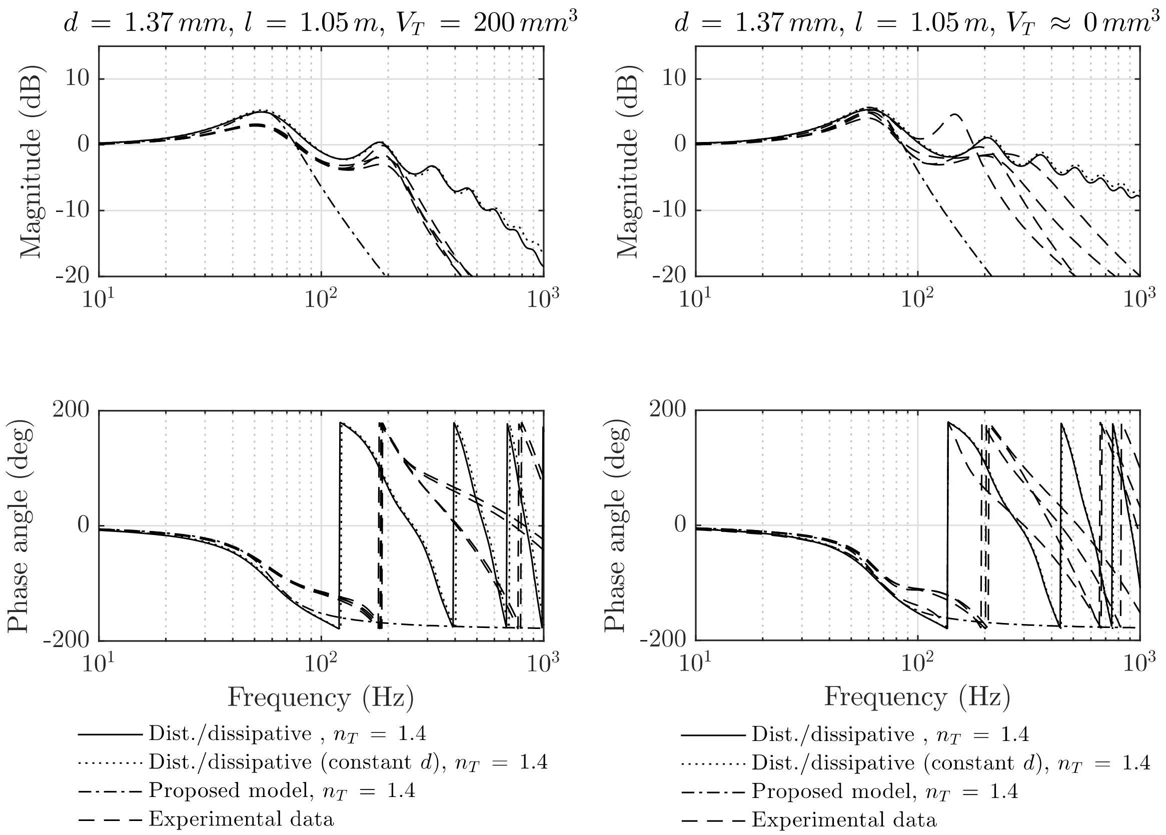

Figure 12: Experimental results for l = 1.05 m, V_T = 200 mm³ and ~0 mm³.

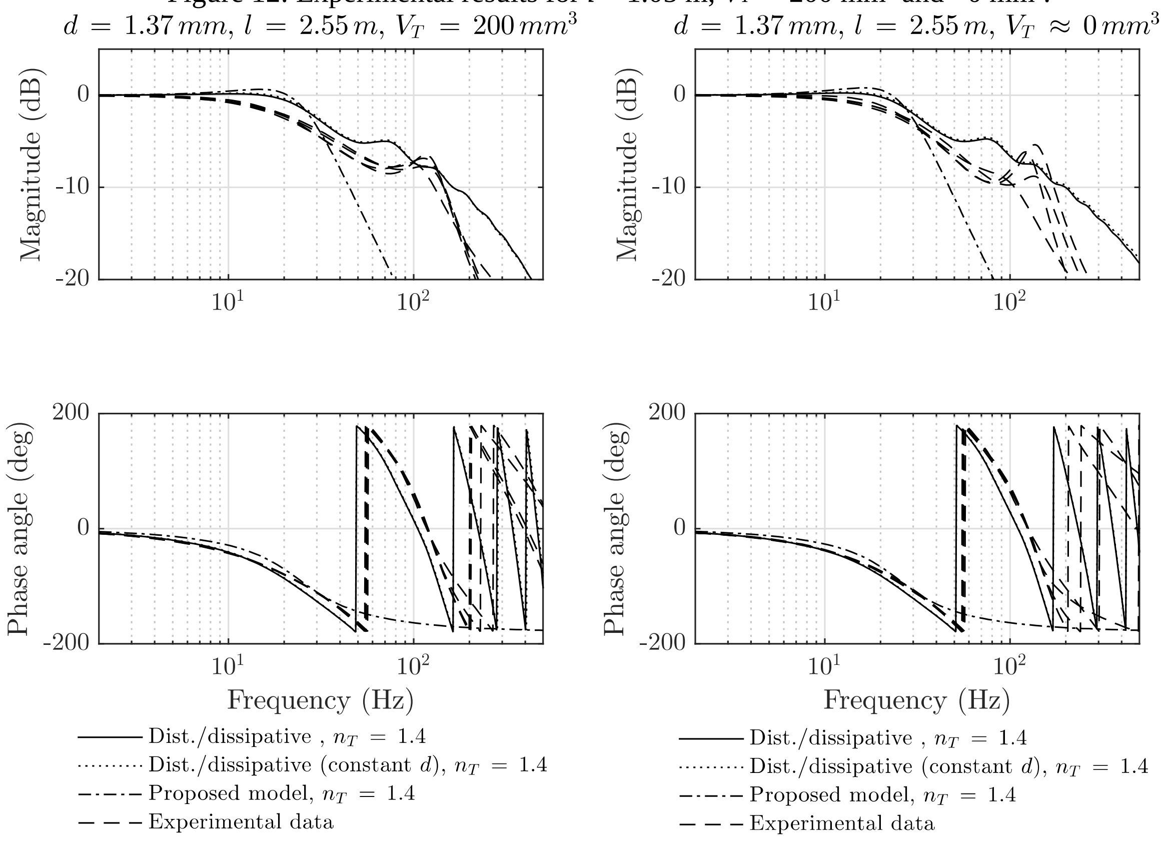

Figure 13: Experimental results for *l* = 2.55 m, V_T = 200 mm^3 and ~0 mm^3.

The results for total tube length (*l*) equal to 0.45 m, 1.05 m and 2.55 m are presented in Figures 11 to 13. For each tube length, four nominally identical tube assemblies were constructed and tested. The agreement between nominally identical cases was generally very good (in the cases of $V_T \approx 0$ and *l* = 1.05 m and 2.55 m, an outlier exists: in these cases water contamination was observed post-test, and this is the likely origin of the disagreement).

For each tube length, the response predicted by the dissipative friction, distributed-parameter model is shown for two assumptions: lengths and internal diameters of each of the tube sections modelled separately; the total length (*l*) and the internal diameter of the longest (plastic) tube. In the cases examined, approximating the three-tube system as a single tube produced only a small difference in the prediction. For the underdamped cases (figure 11, *l* = 0.45 m, and figure 12, *l* = 1.55 m) the magnitude and frequency of the first mode was well predicted by both models. The phase angle agreement was also good in all cases up to and including the first harmonic. For the overdamped cases of Figure 13, particularly where V_T = 200 mm^3, the agreement in terms of the magnitude was less good.

6. Discussion

We have shown good agreement between the proposed model and the distributed-parameter, dissipative friction model for the underdamped cases (small l/d^2), and reasonable agreement for an overdamped case (high l/d^2). We conclude that where the goal is to estimate bandwidth (cut-off value for low-pass filtering), the proposed model should perform as well as the distributed-parameter, dissipative friction model, and offer improvement over simpler models. Neither the DPDF model or the proposed model predicted the PSD well for underdamped cases, highlighting that care should be taken in the case of relatively long tubes.

Bergh and Tidjeman [3] noted that the deviation of the physical system from the nominal dimensions (particularly for *d*) is a likely source of error. This is a problem common to all modelling approaches. In order to minimise overall error in estimating bandwidth, it is important that the model used is at least capable of accurately describing the low-order response of the idealised geometry. Where accuracy is critical (or where a signal reconstruction is necessary), experimental characterisation of the frequency response should be carried out.

A useful observation was that for a system of mixed tube diameters (specifically where *d* is smaller at each end - Figure 10), the response did not deviate significantly from a simplified case assuming a single diameter (*d* for the longest tube) and a summed length. For the tube configuration and diameters considered in 4.2.4. this appears to be a reasonable assumption where $l_{ss} < 0.1l$.

Tube length (m)	Step response 5% settling time (s)	Step response 1% settling time (s)	Step response 0.1% settling time (s)	Frequency of first resonant peak (Hz)	Error at 1st resonance frequency for sine wave magnitude (% peak to trough)
0.01				1000	
0.1				100	
0.5				10	
1.0					
2.0					
5.0	5				

10.0					

Table 1: Step response settling times, and first resonant peak frequency and error as a function of tube length for XX mm diameter tube with XX mm3 transducer volume.

7. Conclusion

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8. Appendix A – Equations

Table 2: Summary of electrical and fluidic analogies for the linear friction model - primed variables are per unit length.

Circuit component		Fluid analogy	
Resistance (/m)	⟨ ⟩	Viscosity	⟨32 ⟩ d
Inductance (H/m)	d ⟨ ⟩	Inertance	d d ⟨ ⟩
Capacitance (F/m)	d dt ⟨ ⟩	Fluid capacitance (tube)	d dt ⟨ ⟩ d
Capacitance (F)	d dt ⟨ ⟩	Fluid capacitance (transducer)	d dt

The following equations are used to define the dissipative tube model of section 4.1.2.

$$\Gamma \quad (A1)$$

$$(A2)$$

$$(A3)$$

$$(A4)$$

$$(A5)$$

The following equations are used to define the corrected model of section 4.2.3.

$$(A6)$$

$$(A7)$$

$$(A9)$$

$$(A10)$$

$$(A11)$$

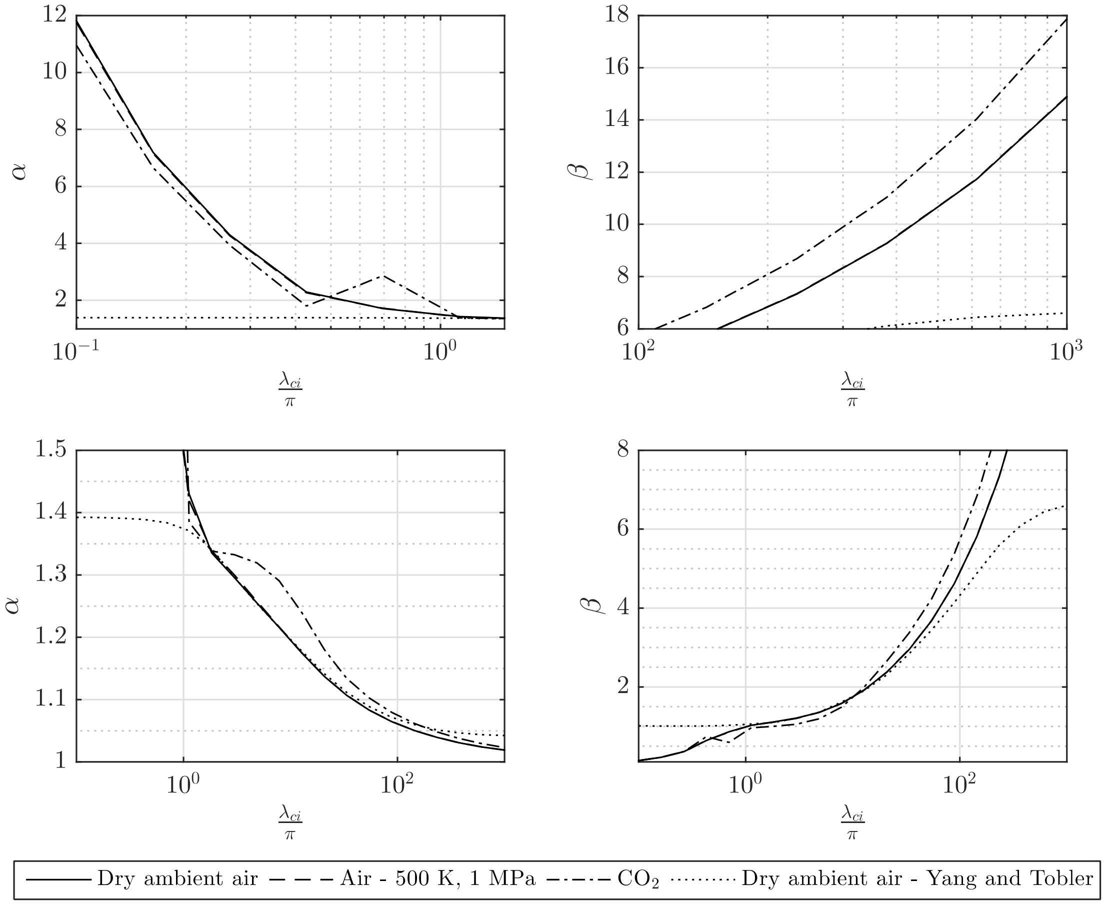

Figure 14: The natural frequency and damping ratio modification parameters (and   respectively) calculated using a methodology based on [19]