Exact Categories, Koszul Duality, and Derived Analytic Algebra

Jack Kelly
Christ Church
University of Oxford

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Abstract

Recent work of Bambozzi, Ben-Bassat, and Kremnitzer suggests that derived analytic geometry over a valued field k can be modelled as geometry relative to the quasi-abelian category of Banach spaces, or rather its completion $Ind(Ban_k)$. In this thesis we develop a robust theory of homotopical algebra in $Ch(\mathcal{E})$ for $\mathcal{E}$ any sufficiently 'nice' quasi-abelian, or even exact, category.

Firstly we provide sufficient conditions on weakly idempotent complete exact categories $\mathcal{E}$ such that various categories of chain complexes in $\mathcal{E}$ are equipped with projective model structures. In particular we show that as soon as $\mathcal{E}$ has enough projectives, the category $Ch_{+}(\mathcal{E})$ of bounded below complexes is equipped with a projective model structure. In the case that $\mathcal{E}$ also admits all kernels we show that it is also true of $Ch_{\geqslant 0}(\mathcal{E})$, and that a generalisation of the Dold-Kan correspondence holds. Supplementing the existence of kernels with a condition on the existence and exactness of certain direct limit functors guarantees that the category of unbounded chain complexes $Ch(\mathcal{E})$ also admits a projective model structure. When $\mathcal{E}$ is monoidal we also examine when these model structures are monoidal.

We then develop the homotopy theory of algebras in $Ch(\mathcal{E})$. In particular we show, under very general conditions, that categories of operadic algebras in $Ch(\mathcal{E})$ can be equipped with transferred model structures. Specialising to quasi-abelian categories we prove our main theorem, which is a vast generalisation of Koszul duality. We conclude by defining analytic extensions of the Koszul dual of a Lie algebra in $Ind(Ban_k)$.

Statement of Originality

I declare that the work contained in this thesis is, to the best of my knowledge, original and my own work, unless indicated otherwise.

I also declare that the work contained in this thesis has not been submitted towards any other degree at this institution or at any other institution.

Jack Kelly

Contents

Chapter 1

Introduction

1.1 Background and Motivation

Derived Geoemetry

Derived geometry has proved crucial for understanding intersection theory, deformation theory and moduli theory in algebraic, smooth, and, recently, complex analytic geometry.

There are two dominating abstract models for derived geometry. Lurie's approach [49] uses a higher-categorical generalization of ringed spaces, namely structured $(\infty, 1)$-topoi. This is an $(\infty, 1)$-topos $\mathcal{X}$ together with a limit-preserving functor

$$\mathcal{O} : \mathcal{G} \to \mathcal{X}$$

where $\mathcal{G}$ is a geometry - an $(\infty, 1)$-category satisfying certain properties. For example taking $(\mathcal{G})^{op}$ to be the $(\infty, 1)$-category of simplicial rings gives a reasonable notion of derived algebraic stacks. David Spivak [76] considers derived smooth manifolds by taking as $(\mathcal{G})^{op}$ the category of simplicial C^{∞}-rings. Mauro Porta and Tony Yue Yu [60],[57] [58], [59],[63], [61], [62] are developing derived analytic geometry by taking $\mathcal{G}^{op}$ to be the category of simplicial rings equipped with a holomorphic functional calculus. In particular they have proven GAGA, base-change, and Riemann Hilbert type theorems. They have also announced a Hochschild-Kostant-Rosenberg theorem.

Toën and Vezzosi's model for derived geometry is inspired by the theory of (non-derived) geometry relative to a symmetric monoidal category (developed for instance in [21] and [6]). This is a category-theoretic framework which views geometry as the unification of algebra and topology. The algebra describes local pieces and a Grothendieck topology allows one to glue these local pieces and obtain global objects. In [79] they introduce the notion of a homotopical algebraic geometry context. Up to some technical details, a homotopical algebraic geometry context consists of a monoidal model category $\mathcal{M}$ such that the category $\mathcal{Alg}_{\mathfrak{Comm}}(\mathcal{M})$ of unital commutative monoids in $\mathcal{M}$ is a model category with the transferred model structure, and $(\mathcal{Alg}_{\mathfrak{Comm}}(\mathcal{M}))^{op}$ is equipped with a homotopy Grothendieck topology τ. We regard $\mathcal{Aff}_{\mathcal{M}} := (\mathcal{Alg}_{\mathfrak{Comm}}(\mathcal{M}))^{op}$ as a category of affine spaces. The category of derived stacks on $\mathcal{M}$ is then the category of functors $\mathcal{X} : \mathcal{Aff}_{\mathcal{M}} \to s\mathcal{Set}$ satisfying descent for τ-hypercovers. For derived algebraic geometry one considers either the category $\mathcal{M} = Ch(R)$ of chain complexes of modules over a ring R (in characteristic zero), or the category $\mathcal{M} = s_R\mathcal{Mod}$ of simplicial R-modules.

We expect that a good model of derived analytic geometry along the lines of [79] would vastly simplify and conceptually clarify many results of Porta, Yu, and collaborators. Generalisations of notions such as shifted symplectic structures would also become obvious. The main results of this work on Koszul duality also suggests that it provides a very convenient formal setup for analytic deformation theory.

Monoidal Categories and Analytic Geometry

A systematic formulation of derived analytic geometry using homotopical algebraic geometry contexts is the subject of a forthcoming work [8] and will not appear in this thesis. However for the purposes of motivation we will give a brief overview of one approach to it. Let $(X, \mathcal{O}_X)$ be a complex manifold. For each open set U the set $\mathcal{O}_X(U)$ has a canonical structure of a Fréchet space. Moreover, the restriction maps $\mathcal{O}_X(V) \to \mathcal{O}_X(U)$ are continuous. Let $\mathcal{F}$ be a coherent sheaf on X. Cartan's Theorem B implies that on a coordinate neighbourhood (or more generally a Stein neighbourhood), there is an exact sequence

$$\mathcal{O}_X^m(V) \to \mathcal{O}_X^n(V) \to \mathcal{F}(V) \to 0$$

The quotient topology on $\mathcal{F}(V)$ makes it a Fréchet space. Thus sheaves on complex spaces have natural topological structures.

It is therefore tempting to view (non-derived) complex analytic geometry as geometry relative to the symmetric monoidal category of Fréchet spaces. Unfortunately this does not seem possible. However in [5] the authors construct a Grothendieck topology τ^{fhZ} on a subcategory $\mathcal{St}$ of $(\mathcal{Alg}_{\mathfrak{Comm}}(\mathcal{Fr}))^{op}$. $\mathcal{St}$ is equivalent to the category of (dagger) Stein spaces and when $k = \mathbb{C}$ the coverings in their topology correspond to coverings of Stein spaces by Stein spaces. In particular the category of complex analytic spaces embeds in the category of schemes on this site.

This construction is somewhat ad hoc but as usual passing to the derived world proves enlightening. The Grothendieck topology τ^{fhZ} of [5] makes use of the homological structure on $\mathcal{Fr}$ which is a quasi-abelian, and therefore exact, category. It is an additive category with classes of admissible monomorphisms and admissible epimorphisms which provide a well-defined notion of homology. There are also notions of projective objects, exact functors, derived categories, and derived functors. If $\mathpzc{E}$ is a monoidal exact category with a left-derivable tensor product $\otimes$, then we say a map $A \to B$ of commutative monoids in $\mathpzc{E}$ is a **homotopy epimorphism** if the map $B \otimes^{\mathbb{L}}_{A} B \to B$ is a quasi-isomorphism. The opposites of these maps make up the covers in τ^{fhZ}. The obstacle to such covers defining a topology on the entire category $(\mathpzc{Alg}_{\mathfrak{Comm}}(\mathpzc{Fr}))^{op}$ is that they are not stable under base-change (because of the derived tensor product).

If $Ch(\mathpzc{Fr})$ were a good enough monoidal model category then we could easily extend the definition of a homotopy epimorphism. Moreover as a homotopy cover in such a model category the issue of base change would disappear and would give a genuine model topology on $(\mathpzc{Alg}_{\mathfrak{Comm}}(Ch(\mathpzc{Fr})))^{op}$. Tragically $\mathpzc{Fr}$ is not good enough. It is neither complete nor cocomplete and does not have enough projectives. Fortunately it does nicely embed in a complete and cocomplete exact category with enough projectives, namely the category $CBorn_{\mathbb{C}}$ of complete bornological spaces over $\mathbb{C}$. It is sometimes convenient to pass to the even bigger category $Ind(Ban_{\mathbb{C}})$, the formal completion of the category of Banach spaces by filtered colimits.

1.2 Goals and Layout

Our goal in this thesis is to put the local theory of derived analytic geometry, i.e. homotopical algebra in $Ch(Ind(Ban_{\mathbb{C}}))$, on a firm footing. Much more generally we develop a robust theory of homotopical algebra in $Ch(\mathpzc{E})$ for $\mathpzc{E}$ any sufficiently 'nice' quasi-abelian, or even exact, category. We also connect with future work on derived analytic geometry by proving a vast generalisation of Koszul duality.

Exact Category Generalities

Building on work of [14] in Chapter 2 we establish some technical results about exact categories in general which we will need in subsequent chapters. After recalling some basic facts we introduce various useful notions of acyclicity. We then discuss bounded and unbounded resolutions in exact categories. In particular we generalize the famous result of Spaltenstein [75] to exact categories satisfying very general conditions.

Theorem 1.2.1 (Corollary 2.1.55)**.** *Let $\mathpzc{E}$ be an exact category with kernels in which the direct limit functor $lim_{\to_{\mathbb{N}}}$ exists and is exact. Let $\mathcal{P}$ be a class of objects such that for each object X in $\mathpzc{E}$ there is an object P in $\mathcal{P}$ together with an admissible epimorphism $P \twoheadrightarrow X$. Suppose further that $\mathcal{P}$ is closed under $\mathbb{N}$-indexed extensions. Then for any complex $X_{\bullet}$ in $Ch(\mathpzc{E})$ there is a complex $P_{\bullet}$ in $Ch(\mathcal{P})$ and an admissible epimorphism $P_{\bullet} \to X_{\bullet}$ which is a quasi-isomorphism. Moreover, $X_{\bullet}$ is the limit of a $Ch_{+}(\mathcal{P})$-special direct system.*

We then introduce a suitable idea of generators, before defining so-called elementary and weakly elementary exact categories. These technical notions will be crucial for controlling the homotopy theory of an exact category and avoiding set-theoretic smallness concerns. Next we define monoidal exact categories and establish some basic properties of them. In particular we prove the existence of an induced exact structure on modules for commutative monoids internal to such categories. More generally we study monads on exact categories and their categories of algebras.

Model Structures on Exact Categories

In Chapter 3 we discuss model structures on exact categories. In particular we give very general conditions on an exact category $\mathpzc{E}$ such that its category of complexes $Ch(\mathpzc{E})$ is equipped with the projective model structure. There is a general theory of model structures on weakly idempotent complete exact categories due to [40], [31] and [78] using **cotorsion pairs**. A pair of classes of objects $(\mathfrak{L}, \mathfrak{R})$ in an exact category $\mathpzc{E}$ is said to be a cotorsion pair if $L \in \mathfrak{L}$ if and only if $Ext^1(L, R) = 0$ for all $R \in \mathfrak{R}$, and $R \in \mathfrak{R}$ if and only if $Ext^1(L, R) = 0$ for all $L \in \mathfrak{L}$. The example we have in mind is $(Proj(\mathpzc{E}), Ob(\mathpzc{E}))$ where $Proj(\mathpzc{E})$ is the class of projective objects in $\mathpzc{E}$. In [28] Gillespie suggests a strategy for producing a model structure on $Ch(\mathpzc{E})$, given a cotorsion pair on an abelian category $\mathpzc{E}$, which can easily be adapted to exact categories more generally. There are no general results regarding when this strategy works. However we prove the following result.

Theorem 1.2.2 (Theorem 3.2.3)**.** *Let $\mathpzc{E}$ be an exact category satisfying the following conditions*

1. *$\mathpzc{E}$ has enough projectives.*

2. *$\mathpzc{E}$ has kernels.*

3. *Let $Fun_{adm}(\mathbb{N}, \mathpzc{E})$ be the full subcategory of the functor category $Fun(\mathbb{N}, \mathpzc{E})$ consisting of functors F such that for each $i \leqslant j$ the map $F(i \leqslant j)$ is an admissible monic. We suppose that $colim : Fun_{adm}(\mathbb{N}, \mathpzc{E}) \rightarrow \mathpzc{E}$ exists and is exact.*

Then, applied to the cotorsion pair $(Proj(\mathpzc{E}), Ob(\mathpzc{E}))$, Gillespie's strategy produces a model structure on $Ch(\mathpzc{E})$.

We call this the **projective model structure** on $Ch(\mathpzc{E})$. If $E = {}_{R}\mathpzc{Mod}$ is the category of R-modules over a ring R then this is the usual projective model structure. Under some stronger assumptions, namely that the category $\mathpzc{E}$ has generators which are compact relative to the class of admissible monics, the result of the above theorem can be deduced from results of [78]. An advantage of our result is that it avoids many set-theoretic concerns and we suggest examples where this is useful. In particular at the end of the chapter we give natural examples of exact categories with very different set-theoretic properties which satisfy the conditions of Theorem 3.2.3.

In order to study the homotopy theory of algebras in monoidal exact categories we need to know how the monoidal structure interacts with the model structure. We call a monoidal exact category **monoidal elementary** if it is elementary and its projectives are flat and closed under the tensor product.

Theorem 1.2.3 (Theorem 3.2.13)**.** *Let $\mathcal{E}$ be a monoidal elementary exact category. Then the projective model structure on $Ch(\mathcal{E})$ is monoidal and satisfies the monoid axiom.*

We then prove a generalisation of the Dold-Kan correspondence.

Theorem 1.2.4 (Theorem 3.2.23)**.** *Let $\mathcal{E}$ be a small elementary exact category. Endow $Ch_{\geqslant 0}(\mathcal{E})$ and $\boldsymbol{s}\mathcal{E}$ with their projective model structures. Then the functors*

$$\Gamma : Ch_{\geqslant 0}(\mathcal{E}) \to \boldsymbol{s}\mathcal{E}$$

and

$$N : \boldsymbol{s}\mathcal{E} \to Ch_{\geqslant 0}(\mathcal{E})$$

form a Quillen equivalence.

We then discuss model structures for graded and filtered objects in quasi-abelian categories before concluding with some comments about homotopy theory in additive model categories. Note that a preprint [45] based on a large proportion of the first two chapters is available.

Homotopy Theory of Operads and Koszul Duality

Chapter 4 is the main event. The existence of a monoidal model structure on the category $Ch(\mathcal{E})$, for $\mathcal{E}$ monoidal elementary, allows us to investigate the homotopy theory of operadic algebras. Precisely, let $\mathfrak{P}$ be either a symmetric or non-symmetric operad in $Ch(\mathcal{E})$, and $Alg_{\mathfrak{P}}(Ch(\mathcal{E}))$ the category of $\mathfrak{P}$-algebras in $Ch(\mathcal{E})$. Under certain conditions on $\mathfrak{P}$ the free-forgetful adjunction may be used to endow $Alg_{\mathfrak{P}}(Ch(\mathcal{E}))$ with the transferred model structure. Recall that a symmetric operad is **split** if for each n the action of the symmetric group $\Sigma_n \otimes \mathfrak{P}(n) \to \mathfrak{P}(n)$ has a splitting which is compatible with operadic composition. We show, generalising results of [36], that the transferred model structure exists on algebras over split operads in such categories. More generally we have the following.

Theorem 1.2.5 (Theorem 4.1.19)**.** *Let $\mathcal{E}$ be a monoidal elementary exact category and let $R \in \mathcal{Alg}_{\mathfrak{Comm}}(Ch(\mathcal{E}))$.*

1. *If $\mathfrak{P}$ is a non-symmetric operad in ${}_R\mathcal{Mod}$ then the transferred model structure exists on $Alg_{\mathfrak{P}}({}_R\mathcal{Mod})$.*
2. *If $\mathfrak{P}$ is a split symmetric operad in ${}_R\mathcal{Mod}$ then the transferred model structure exists on $Alg_{\mathfrak{P}}({}_R\mathcal{Mod})$.*

We then set about proving a vast generalisation of Vallete's co-operadic Koszul duality [82] for monoidal elementary quasi-abelian categories. Before outlining our results we first review some of the vast history of Koszul duality and its various manifestiations. Vallette's work generalises previous work of Hinich [37] which interprets duality between Lie algebras and cocommutative coalgebras in terms of formal stacks. This in turn generalises results from the seminal work of Quillen [69] on rational homotopy theory. Getzler and Jones [27] have also done crucial work on Koszul duality. Motivated by studying differential forms on interated loops spaces they in particular study duality for E_n-algebras. In the process of establishing chiral Koszul duality, Francis and Gaitsgory [24] prove a general Koszul duality result in the context of pro-nilpotent ∞-categories. An operadic version of Koszul duality has been established by Ginzburg and Kapranov [32]. There is also a curved operadic version due to Hirsh and Milles [39]. Ching and Harper have recently proved a spectral version of Koszul duality [17]. The relationship between Koszul duality and deformation theory has also been extensively studied by Kontsevich and Soibelman in [47] and [46], as well as by Lurie [50] and Hennion [35].

Let us now recall Vallette's Theorem 2.1 in [82]. In any complete and cocomplete monoidal additive category $\mathcal{E}$ a twisting morphism $\alpha : \mathfrak{C} \to \mathfrak{P}$ from a co-operad to an operad in $(Ch(\mathcal{E}))$ induces the **bar-cobar** adjunction

$$\Omega_\alpha \colon co\mathcal{Alg}^{nil}_{\mathfrak{C}} \rightleftarrows \mathcal{Alg}_{\mathfrak{P}} : B_\alpha$$

where $co\mathcal{Alg}^{nil}_{\mathfrak{C}}$ is the category of co-nilpotent co-algebras over $\mathfrak{C}$ and $\mathcal{Alg}_{\mathfrak{P}}$ is the category of algebras over $\mathfrak{P}$. In the case that $\mathcal{E} = {}_k\mathcal{Mod}$ for some field k and α is a so-called **Koszul morphism** Vallette shows that there is a model category structure on $co\mathcal{Alg}^{nil}_{\mathfrak{C}}$ and that the adjunction is a Quillen equivalence.

With significant modification we show that Vallette's version generalises to monoidal elementary quasi-abelian categories. Let $\mathcal{Alg}^{c}_{\mathfrak{P}}$ denote the full subcategory of cofibrant algebras. We say that a conilpotent $\mathfrak{C}$-co-algebra is **cof-nilpotent** if, equipped with its coradical filtration, it is cofibrant as a filtered object. Denoting the full subcategory of cof-nilpotent coaglebras by $\mathcal{Cof}_{\mathfrak{C}}$ we prove the following:

Theorem 1.2.6 (Corollary 4.2.28). *Let α be a Koszul morphism.The bar-cobar adjunction induces a equivalence of relative categories.*

$$\Omega_\alpha\colon Cof_{\mathfrak{C}} \rightleftarrows \mathcal{Alg}^c_{\mathfrak{P}} : B_\alpha$$

Next we discuss operadic Koszul duality. Given a Koszul morphism $\alpha : \mathfrak{C} \to \mathfrak{P}$ we consider the dual operad $\mathfrak{C}^\vee$ and the composite functor $\hat{C}_\alpha := (-)^\vee \circ B_\alpha : \mathcal{Alg}_{\mathfrak{P}} \to (\mathcal{Alg}_{\mathfrak{C}^\vee})^{op}$ (strictly speaking we consider a shift of this functor). Assuming that $\mathcal{Alg}_{\mathfrak{C}^\vee}$ is equipped with a transferred model structure we show the following

Theorem 1.2.7 (Theorem 4.2.33). *The functor $\hat{C}_\alpha$ induces a functor of $(\infty,1)$-categories $\hat{\boldsymbol{C}}_\alpha : \boldsymbol{Alg}_{\mathfrak{P}} \to (\boldsymbol{Alg}_{\mathfrak{C}^\vee})^{op}$ which admits a right adjoint $\boldsymbol{D}_\alpha$.*

We then specialise to operadic Koszul duality between Lie algebras and augmented commutative algebras and generalise results of [50] and [35]. We show in Proposition 4.3.3 that the underlying complex of $\mathbf{D}_\alpha(A)$ is naturally equivalent to the shifted tangent complex of A at its canonical point. We conclude by showing that under certain boundedness conditions on a Lie algebra $\mathfrak{g}$, the unit $\mathfrak{g} \to \mathbf{D}_\alpha\hat{\mathbf{C}}_\alpha(\mathfrak{g})$ is an equivalence. To conclude the thesis we introduce the notion of the analytic Koszul dual augmented commutative algebra of a Banach Lie algebra. Precisely, we define a contravariant functor $C^{an,\infty}_\alpha$ from the category of Banach Lie algebras $\mathfrak{g}$ concentrated in negative degrees to the category of augmented commutative algebras concentrated in positive degrees. If $\mathfrak{g}$ is degree-wise finite dimensional then the degree 0 part of $C^{an,\infty}_\alpha(\mathfrak{g})$ is naturally isomorphic to the algebra of entire analytic functions on $\mathfrak{g}^\vee_{-1}$. Moreover the shifted tangent complex of $C^{an,\infty}_\alpha(\mathfrak{g})$ is equivalent to $\mathfrak{g}$.

Appendices

The appendices include both recollections of some standard results and also some original and non-trivial results which do not fit neatly in to the main body of work. Appendix A includes some rudimentary facts about model categories, transferred model structures, and $(\infty,1)$-categories which we use throughout this thesis. In Appendix B we establish some basic facts about operads and algebras over them in monoidal additive categories. Many of these follow mutatis-mutandis from the corresponding results in categories of vector spaces proved in [48] and so we do not reproduce those proofs. Finally in Appendix C we discuss spaces of holomorphic functions between Banach spaces. In particular we recall notions of holomorphy types. We also introduce contracting and multiplicative holomorphy types before giving a novel description of algebras of holomorphic functions of bounded type as objects in the category $Pro(Ban_{\mathbb{C}})$ of Pro-Banach spaces.

1.3 Future Work

A major application of this work will be to develop derived analytic and smooth geometry. In a forthcoming work [8] we establish the foundations of derived analytic geometry. In particular we prove descent properties for the topology which mixes homotopy monomorphisms and formal étale morphisms. With this infrastructure in hand standard results such as an analytic Hochschild-Kostant-Rosenberg theorem along the lines of [80] and base change should be easily accessible. We also expect that the techniques of Chapter 2 can be modified to prove the existence of a flat model structure for categories of sheaves, and indeed that we can prove a Koszul duality theorem as in Chapter 3 using the flat model model structure. This could potentially lead to generalisations of chiral Koszul duality including, by considering categories of bornological C^{∞} rings and modules, a smooth version. Finally we expect that our operadic Koszul duality results for Lie algebras and commutative algebras should also work for E_n-duality and duality for E_{∞}- and L_{∞}- algebras.

1.4 Notation and Conventions

Throughout this work we will use the following notation.

- 1-categories will be denoted using the mathpzc font $\mathpzc{C}, \mathpzc{D}, \mathpzc{E}$, etc. In particular we denote by $\mathpzc{Ab}$ the category of abelian groups and ${}_{\mathbb{Q}}\mathpzc{Vect}$ the category of $\mathbb{Q}$-vector spaces. If $\mathpzc{M}$ is a model category, or a category with weak equivalences, its associated $(\infty, 1)$-category will be denoted $\mathbf{M}$.

- Operads and co-operads will be denoted using capital fractal letters $\mathfrak{C}, \mathfrak{P}$, etc. Algebras over an operad will generally be denoted using small fractal letters $\mathfrak{g}, \mathfrak{h}$, etc. The category of (co)algebras over a (co)operad will be denoted $(\mathpzc{co})\mathpzc{Alg}_{\mathfrak{P}}$

- We denote the operads for unital associative algebras, unital commutative algebras, non-unital commutative algebras, and Lie algebras by $\mathfrak{Ass}, \mathfrak{Comm}, \mathfrak{Comm}^{nu}$, and $\mathfrak{Lie}$ respectively. We also denote by $\mathfrak{coComm}$ and $\mathfrak{coComm}^{nu}$ the co-operads of cocommutative and non-unital cocommutative coalgebras.

- If $\mathfrak{P}$ is a (co)operad in category $\mathpzc{E}$ and V is an objet of $\mathpzc{E}$, then typically we will denote the (co)free (co)algebra on V by $\mathfrak{P}(V)$. For the operad $\mathpzc{Ass}, \mathpzc{Comm}, \mathpzc{Lie}$ we will denote the corresponding free algebras by $T(V), S(V)$, and $L(V)$ respectively. We also denote by $\hat{S}(V)$ the commutative algebra of formal power series on an object V and by $U(L)$ the universal enveloping algebra of a Lie algebra L.

- Unless stated otherwise, the unit in a monoidal category will be denoted by k, the tensor functor by $\otimes$, and for a closed monoidal category the internal hom functor will be denoted by $\underline{\mathrm{Hom}}$. Monoidal categories will always be assumed to be symmetric, with symmetric braiding σ.
- Filtered colimits will be denoted by $\lim_{\rightarrow}$. Projective limits will be denoted $\lim_{\leftarrow}$.

Let us now introduce some conventions for chain complexes.

Definition 1.4.1. *A* ***chain complex*** *in a pre-additive category $\mathcal{E}$ is a sequence*

$$K_{\bullet} = \ldots \longrightarrow K_n \xrightarrow{d_n} K_{n-1} \xrightarrow{d_{n-1}} K_{n-2} \longrightarrow \ldots$$

where the K_i are objects and the d_i are morphisms such that $d_{n-1} \circ d_n = 0$. The morphisms are called ***differentials***. *A* ***morphism of chain complexes*** *$f_{\bullet} : K_{\bullet} \to L_{\bullet}$ is a collection of morphisms $f_n : K_n \to L_n$ such that the following diagram commutes for each n:*

$$\begin{array}{ccccccccc}
\ldots & \longrightarrow & K_{n+1} & \xrightarrow{d^K_{n+1}} & K_n & \xrightarrow{d^K_n} & K_{n-1} & \longrightarrow & \ldots \\
 & & \Big\downarrow{\scriptstyle f_{n+1}} & & \Big\downarrow{\scriptstyle f^n} & & \Big\downarrow{\scriptstyle f^{n-1}} & & \\
\ldots & \longrightarrow & L_{n+1} & \xrightarrow{d^L_{n+1}} & L_n & \xrightarrow{d^L_n} & L_{n-1} & \longrightarrow & \ldots
\end{array}$$

The category whose objects are chain complexes and whose morphisms are as described above is called the **category of chain complexes in** $\mathcal{E}$, denoted $Ch(\mathcal{E})$. We also define $Ch_{\geqslant 0}(\mathcal{E})$ to be the full subcategory of $Ch(\mathcal{E})$ on complexes $A_{\bullet}$ such that $A_n = 0$ for $n < 0$, $Ch_{\leqslant 0}(\mathcal{E})$ to be the full subcategory of $Ch(\mathcal{E})$ on complexes $A_{\bullet}$ such that $A_n = 0$ for $n > 0$, $Ch_{+}(\mathcal{E})$, the full subcategory of chain complexes $A_{\bullet}$ such that $A_n = 0$ for $n << 0$, $Ch_{-}(\mathcal{E})$, the full subcategory of chain complexes $A_{\bullet}$ such that $A_n = 0$ for $n >> 0$ and $Ch_b(\mathcal{E})$ to be the full subcategory of $Ch(\mathcal{E})$ on complexes $A_{\bullet}$ such that $A_n \neq 0$ for only finitely many n. A lot of the statements in the rest of this document apply to several of these categories at once. In such cases we will write $Ch_{*}(\mathcal{E})$, and specify that $*$ can be any element of some subset of $\{\geqslant 0, \leqslant 0, +, -, b, \varnothing\}$, where by definition $Ch_{\varnothing}(\mathcal{E}) = Ch(\mathcal{E})$.

We will frequently use the following special chain complexes.

Definition 1.4.2. *If E is an object of a pointed category $\mathcal{E}$ we let $S^n(E) \in Ch(\mathcal{E})$ be the complex whose nth entry is E, with all other entries being 0. We also denote by $D^n(E) \in Ch(\mathcal{E})$ the complex whose nth and $(n-1)$st entries are E, with all other entries being 0, and the differential d_n being the identity.*

Let us also introduce some notation for truncation functors.

Definition 1.4.3. *Let $\mathcal{E}$ be an additive category which has kernels. For a complex $X_\bullet$ we denote by $\tau_{\geqslant n}X$ the complex such that $(\tau_{\geqslant n}X)_m = 0$ if $m < n$, $(\tau_{\geqslant n}X)_m = X_m$ if $m > n$ and $(\tau_{\geqslant n}X)_n = Ker(d_n)$. The differentials are the obvious ones. The construction is clearly functorial.*

All of the above categories are naturally enriched over $Ch(\mathcal{Ab})$. We denote the enriched hom by $\mathbf{Hom}(-,-)$. For notational clarity we recall its definition here.

Definition 1.4.4. *Let $X_\bullet, Y_\bullet \in Ch(\mathcal{E})$. We define $\boldsymbol{Hom}(X_\bullet, Y_\bullet) \in Ch(\mathcal{Ab})$ to be the complex with*

$$\boldsymbol{Hom}(X_\bullet, Y_\bullet)_n = \prod_{i \in \mathbb{Z}} Hom_{\mathcal{E}}(X_i, Y_{i+n})$$

and differential d_n defined on $Hom_{\mathcal{E}}(X_i, Y_{i+n})$ by

$$df = d^Y_{i+n} \circ f - (-1)^n f \circ d^X_i$$

Let $(\mathcal{E}, \otimes, k)$ be a monoidal additive category, i.e. $\otimes$ is an additive bifunctor. There is an induced monoidal structure on $Ch_*(\mathcal{E})$ for $* \in \{\geqslant 0, \leqslant 0, +, -, b, \varnothing\}$. The unit is $S^0(k)$. If $X_\bullet$ and $Y_\bullet$ are chain complexes then we set

$$(X_\bullet \otimes Y_\bullet)_n = \bigoplus_{i+j=n} X_i \otimes Y_j$$

If $i + j = n$, then we define the differential on the summand $X_i \otimes Y_j$ of $(X_\bullet \otimes Y_\bullet)_n$ by

$$d_n^{X_\bullet \otimes Y_\bullet}|_{X_i \otimes Y_j} = d_i^{X_\bullet} \otimes id_{Y_\bullet} + (-1)^i id_{X_\bullet} \otimes d_j^{Y_\bullet}$$

If $* \in \{\geqslant 0, \leqslant 0, +, -, b, \varnothing\}$ then $(Ch_*(\mathcal{E}), \otimes, S^0(k))$ is a monoidal additive category.

If $(\mathcal{E}, \otimes, k, \underline{\text{Hom}})$ is a closed monoidal additive category then we define a functor

$$\underline{\text{Hom}}(-,-) : Ch(\mathcal{E})^{op} \times Ch(\mathcal{E}) \to Ch(\mathcal{E})$$

$$\underline{\text{Hom}}(X_\bullet, Y_\bullet)_n = \prod_{i \in \mathbb{Z}} \underline{\text{Hom}}_{\mathcal{E}}(X_i, Y_{i+n})$$

and differential d_n defined on $\text{Hom}_{\mathcal{E}}(X_i, Y_{i+n})$ by

$$d = \underline{\text{Hom}}(d_i^{X_\bullet}, id) + (-1)^i \underline{\text{Hom}}(id, d_{i+n}^{Y_\bullet})$$

This does define an internal hom on the monoidal category

$$(Ch(\mathcal{E}), \otimes, S^0(k))$$

The internal hom on chain complexes also restricts to a bifunctor

$$\underline{\text{Hom}}(-,-) : Ch_b(\mathcal{E})^{op} \times Ch_b(\mathcal{E}) \to Ch_b(\mathcal{E})$$

Then

$$(Ch_b(\mathcal{E}), \otimes, S^0(k), \underline{\text{Hom}})$$

is a closed monoidal additive category. In fact, in both of these categories there are natural isomorphisms of chain complexes of abelian groups.

$$\mathbf{Hom}(X_\bullet, \underline{\text{Hom}}(Y_\bullet, Z_\bullet)) \cong \mathbf{Hom}(X_\bullet \otimes Y_\bullet, Z_\bullet)$$

The categories $Ch_*(\mathcal{E})$ for $* \in \{+,-,b,\varnothing\}$ also come equipped with a shift functor. It is given on objects by $(A_\bullet[1])_i = A_{i+1}$ with differential $d_i^{A[1]} = -d_{i+1}^A$. The shift of a morphism $f^\bullet$ is given by $(f_\bullet[1])_i = f_{i+1}$. $[1]$ is an auto-equivalence with inverse $[-1]$. We set $[0] = \text{Id}$ and $[n] = [1]^n$ for any integer n.

Finally, we define the mapping cone as follows.

Definition 1.4.5. *Let $X_\bullet$ and $Y_\bullet$ be chain complexes in an additive category $\mathcal{E}$ and $f_\bullet : X_\bullet \to Y_\bullet$. The **mapping cone of** $f_\bullet$, denoted $cone(f_\bullet)$ is the complex whose components are*

$$cone(f_\bullet)_n = X_{n-1} \oplus Y_n$$

and whose differential is

$$d_n^{cone(f)} = \begin{pmatrix} -d_{n-1}^X & 0 \\ -f_{n-1} & d_n^Y \end{pmatrix}$$

There are natural morphisms $\tau : Y_\bullet \to \text{cone}(f)$ induced by the injections $Y_i \to X_{i-1} \oplus Y_i$, and $\pi : \text{cone}(f) \to X_\bullet[-1]$ induced by the projections $X_{i-1} \oplus Y_i \to X_{i-1}$. The sequence

$$Y_\bullet \to \text{cone}(f) \to X_\bullet[-1]$$

is split exact in each degree.

Chapter 2

Constructions in Exact Categories

In this chapter we will establish some technicalities about exact categories which will be used throughout the thesis. In particular we will discuss acyclicity of complexes and the existence of unbounded resolutions. We will also discuss various notions of generation and compactness in such categories, and introduce the notion of a monoidal exact category. Finally we will see when exact structures can be lifted to categories of algebras for some monad acting on an exact category. The results in this chapter will prove crucial for studying the homotopy theory of exact categories in Chapter 3.

2.1 Exact Category Generalities

In this section we review the rudiments of exact categories, following [14]. In the following $\mathcal{E}$ will be an additive category. A **kernel-cokernel pair** in $\mathcal{E}$ is a pair of composable maps (i, p), $i : A \to B, p : B \to C$ such that $i = \text{Ker}(p)$ and $p = \text{Coker}(i)$. If $\mathcal{Q}$ is a class of kernel-cokernel pairs and $(i, p) \in \mathcal{Q}$, then we say that i is an admissible monic and p is an admissible epic with respect to $\mathcal{Q}$.

Definition 2.1.1. *A* ***Quillen exact structure*** *on an additive category $\mathcal{E}$ is a collection $\mathcal{Q}$ of kernel-cokernel pairs such that*

1. *Isomorphisms are both admissible monics and admissible epics.*
2. *Both the collection of admissible monics and the collection of admissible epics are closed under composition.*
3. *If*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & & \downarrow \\ X & \xrightarrow{f'} & Y \end{array}$$

is a push out diagram, and f is an admissible monic, then f' is as well.

4. *If*

$$\begin{array}{ccc} A & \xrightarrow{f'} & B \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

is a pullback diagram, and f is an admissible epic, then f' is as well.

Let $(\mathcal{E}, \mathcal{Q})$ be an exact category. We call a null sequence

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{p} C \longrightarrow 0$$

short exact if (i, p) is a kernel-cokernel pair in $\mathcal{Q}$. We will use interchangeably the notion of kernel-cokernel pair and short exact sequence. In the context of diagrams in exact categories $\rightarrowtail$ will be used to denote an admissible monic, and $\twoheadrightarrow$ an admissible epic. When it is not likely to cause confusion, we will suppress the notation $(\mathcal{E}, \mathcal{Q})$ to $\mathcal{E}$.

When studying exact categories it is natural to consider so-called exact functors:

Definition 2.1.2. *Let $(\mathcal{E}, \mathcal{P})$, $(\mathcal{F}, \mathcal{Q})$ be exact categories. A functor $F : \mathcal{E} \to \mathcal{F}$ is said to be **exact** (with respect to $\mathcal{P}$ and $\mathcal{Q}$) if for any short exact sequence*

$$0 \to X \to Y \to Z \to 0$$

in $\mathcal{P}$,

$$0 \to F(X) \to F(Y) \to F(Z) \to 0$$

is a short exact sequence in $\mathcal{Q}$.

Definition 2.1.3. *Let $(\mathcal{E}, \mathcal{P})$ be an exact category. An **exact subcategory** of $(\mathcal{E}, \mathcal{P})$ is an exact category $(\mathcal{F}, \mathcal{Q})$ where $\mathcal{F}$ is a subcategory of $\mathcal{E}$ and the inclusion functor is exact.*

On any additive category one can define the split exact structure for which the kernel-cokernel pairs are the split exact sequences. Any exact category contains this is an exact subcategory. At the other extreme we have quasi-abelian exact structures.

Definition 2.1.4. *An additive category $\mathcal{E}$ with all kernels and cokernels is said to be **quasi-abelian** if the class $\mathcal{qac}$ of all kernel-cokernel pairs forms an exact structure on $\mathcal{E}$.*

The following is then tautological.

Proposition 2.1.5. *Let $\mathcal{E}$ be a quasi-abelian category, and let $\mathcal{Q}$ be a class of kernel-cokernel pairs on $\mathcal{E}$ such that $(\mathcal{E}, \mathcal{Q})$ is an exact category. Then the identity functor $id_{\mathcal{E}}$ is an exact functor $(\mathcal{E}, \mathcal{Q}) \to (\mathcal{E}, \mathcal{qac})$.*

We will study quasi-abelian structures in more detail later. For now let us note that abelian categories are quasi-abelian. In an abelian category all monics are kernels of their cokernels, and all epics are cokernels of their kernels. It therefore trivially follows that both classes are closed under composition. It is also clear that both classes contain all isomorphisms. It is a standard exercise that in an abelian category, monomorphisms are pushout-stable and epimorphisms are pullback-stable. See for example [26] Theorem 2.54. Let us now record some basic results about exact categories which will prove useful.

Proposition 2.1.6. *Let*

$$\begin{array}{ccc} A & \xrightarrow{i} & B \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} \\ A' & \xrightarrow{i'} & B' \end{array}$$

be a commutative diagram in which the horizontal morphisms are admissible monics. Then the following are equivalent

1. *The square above is a push-out.*
2. *The sequence*

$$0 \longrightarrow A \xrightarrow{\begin{pmatrix} i \\ -f \end{pmatrix}} B \oplus A' \xrightarrow{\begin{pmatrix} f' & i' \end{pmatrix}} B' \longrightarrow 0$$

   is short exact.
3. *The square above is bicartesian.*
4. *The square is part of a commutative diagram*

$$\begin{array}{ccccc} A & \xrightarrow{i} & B & \xrightarrow{p} & C \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} & & \| \\ A' & \xrightarrow{i'} & B' & \xrightarrow{p'} & C \end{array}$$

   with short exact rows.

Proof. See [14] Proposition 2.12. □

Proposition 2.1.7. *Let $\mathcal{E}$ be an exact category and $\mathcal{A} \subset \mathcal{E}$ a full additive subcategory. Suppose that for every morphism $f : A \to B$ which is admissible in $\mathcal{E}$, a kernel and cokernel of f in $\mathcal{E}$ exist in $\mathcal{A}$. Then the collection of all kernel-cokernel pairs $(i : A \to B, p : B \to C)$ which are exact in $\mathcal{E}$ where $A, B, C \in \mathcal{A}$ defines an exact structure on $\mathcal{A}$ which makes it an exact subcategory of $\mathcal{E}$.*

Proof. It is clearly sufficient to show that this collection of kernel-cokernel pairs endows $\mathcal{A}$ with an exact structure. The first and second conditions are clearly satisfied. Let

$$\begin{array}{ccc} A & \xrightarrow{i} & B \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} \\ A' & \xrightarrow{i'} & B' \end{array}$$

be a pushout diagram in $\mathcal{E}$ with f an admissible monic, and i and i' in $\mathcal{A}$. We need to show that Y is (isomorphic to) and object of $\mathcal{A}$. But there is an exact sequence

$$0 \longrightarrow A \xrightarrow{\binom{i}{-f}} B \oplus A' \xrightarrow{(f'\ i')} B' \longrightarrow 0$$

in $\mathcal{E}$. Now a cokernel of the map $A \to B \oplus A'$ in $\mathcal{E}$ exists in $\mathcal{A}$, so B' is isomorphic to an object of $\mathcal{A}$. The last condition is dual to this one. □

For technical reasons, unless stated otherwise we will assume from now on that all exact categories are **weakly idempotent complete**. This means that every retraction has a kernel, or equivalently, that every coretraction has a cokernel. Note that the condition is self-dual. Quasi-abelian categories are in particular weakly idempotent complete. In weakly idempotent complete exact categories, we then have the following useful result, often called the **Obscure Axiom**.

Proposition 2.1.8 (The Obscure Axiom)**.**

1. *Suppose that $i : A \to B$ is a morphism. If there exists a morphism $j : B \to C$ such that the composite $ji : A \to C$ is an admissible monic, then i is an admissible monic.*

2. *Suppose that $i : A \to B$ is a morphism. If there exists a morphism $j : C \to A$ such that $i \circ j$ is an admissible epic, then i is an an admissible epic.*

Proof. See [14] Proposition 2.16. □

2.1.1 Abelianizations

Let $(\mathcal{E}, \mathcal{Q})$ be an exact category. Let $\mathcal{F}$ be a full subcategory of $\mathcal{E}$. Suppose that $\mathcal{F}$ is closed under extensions, that is if

$$0 \to A \to B \to C \to 0$$

is a short exact sequence in $(\mathcal{E}, Q)$ with A and C objects of $\mathcal{F}$, then B is an object of $\mathcal{F}$ as well. Let $\mathcal{Q}_{\mathcal{F}}$ consist of those kernel-cokernel pairs $(i : A \to B, q : B \to C)$ in $\mathcal{F}$ which when regarded as pairs of morphisms in $\mathcal{E}$ are kernel-cokernel pairs in $\mathcal{Q}$. It is then straightforward to show ([7]) that $(\mathcal{F}, \mathcal{Q}_{\mathcal{F}})$ is an exact subcategory of $(\mathcal{E}, \mathcal{Q})$. It turns out that any small exact category can be obtained as a full subcategory of an abelian category which is closed under extensions. This is the main content of the Quillen Embedding Theorem which provides an invaluable tool for studying exact categories.

Theorem 2.1.9 (The Quillen Embedding Theorem)**.** *Let $\mathcal{E}$ be a small exact category. Then there is an abelian category $\mathcal{A}(\mathcal{E})$ and a fully faithful additive functor $I : \mathcal{E} \to \mathcal{A}(\mathcal{E})$ which is exact, reflects exactness, and preserves all kernels. Moreover the essential image of I is closed under extensions. $\mathcal{A}(\mathcal{E})$ may be chosen to be the category of left-exact functors $\mathcal{E} \to \mathcal{Ab}$. If in addition $\mathcal{E}$ is weakly idempotent complete then a morphism $f : E \to F$ in $\mathcal{E}$ is an admissible epic if and only if $I(f)$ is an epic in $\mathcal{A}(\mathcal{E})$.*

Proof. See Appendix A in [14]. □

Definition 2.1.10. *We call an embedding $I : \mathcal{E} \to \mathcal{A}$ of an exact category into an abelian category a* ***left abelianization*** *of $\mathcal{E}$ if*

1. *I is fully faithful.*
2. *I is exact.*
3. *I reflects exactness.*
4. *The essential image of I is closed under extensions.*
5. *I preserves all kernels which exist.*
6. *If f is a morphism in $\mathcal{E}$, then f is an admissible epic if and only if $I(f)$ is an epic.*

In particular, Theorem 2.1.9 says that any compact exact category admits a left abelianization. There is an obvious dual notion of a **right abelianization**. It is clear that right abelianizations of small exact categories exist. Indeed, if $\mathcal{E}^{op} \to \mathcal{A}$ is a left-abelianization of $\mathcal{E}^{op}$, then $\mathcal{E} \to \mathcal{A}^{op}$ is a right-abelianization of $\mathcal{E}$.

2.1.2 Generation of Exact Subcategories

Let $\mathcal{E}$ be a locally small additive category and $\mathcal{A}$ a small full subcategory. By an argument similar to [42] we can find a small full exact subcategory of $\mathcal{E}$ containing $\mathcal{A}$. In the rest of the section we assume that given a small subcategory $\mathcal{E}$ of $\mathcal{A}$ one can choose direct sums for finite collections of objects in $\mathcal{E}$, and kernels and cokernels of morphisms in $\mathcal{E}$. For example one might assume that such limits and colimits can be made functorial in the ambient category (e.g. if $\mathcal{E}$ is locally presentable).

Proposition 2.1.11. *There is a small full additive subcategory $\Sigma(\mathcal{A}; \mathcal{E})$ of $\mathcal{E}$ containing $\mathcal{A}$.*

Proof. We let $\Sigma(\mathcal{A};\mathcal{E})$ be the full subcategory whose objects are the zero object and a choice of a direct sum in $\mathcal{E}$ for each finite collection of objects of $\mathcal{A}$. This is clearly additive, contains $\mathcal{A}$, and is small. □

Now let $\mathcal{E}$ be an exact category and $\mathcal{A}$ a full subcategory.

Proposition 2.1.12. *There is a small full exact subcategory $Ex(\mathcal{A},\mathcal{E})$ of $\mathcal{E}$ containing $\mathcal{A}$.*

Proof. By Proposition 2.1.11 we may assume that $\mathcal{A}$ is additive. Let $Ex^1(\mathcal{A},\mathcal{E})$ denote the full subcategory of $\mathcal{E}$ consisting of a choice of kernels and cokernels of morphisms $f : A \to A'$ which are admissible in $\mathcal{E}$. We set $Ex^{n+1}(\mathcal{A};\mathcal{E}) := Ex^1(Ex^n(\mathcal{A};\mathcal{E});\mathcal{E})$ We claim that

$$Ex(\mathcal{A};\mathcal{E}) := \bigcup_{n=1}^{\infty} Ex^n(\mathcal{A};\mathcal{E})$$

works. Since $Ex^1(\mathcal{A};\mathcal{E})$ is small for $\mathcal{A}$ small this would prove the claim. $\bigcup_{n=1}^{\infty} Ex^n(\mathcal{A};\mathcal{E})$ is clearly closed under taking kernels and cokernels of those morphisms which are admissible in $\mathcal{E}$. By Proposition 2.1.7 it is an exact subcategory. □

The point of this is that even if a category $\mathcal{E}$ is not small, when working with small diagrams in $\mathcal{E}$ we can pass to an abelianization.

2.1.3 Notions of Acyclicity

In a general exact category, arbitrary kernels and cokernels may not exist. Therefore it is not in general possible even to write down candidates for the homology objects of a chain complex. Even if all kernels and cokernels do exist, then there are multiple candidates for the homology which are not isomorphic in general. For example, given a null sequene

$$\Gamma = E \xrightarrow{f} F \xrightarrow{g} G$$

i.e. $g \circ f = 0$, one could consider both $\text{Coker}(\text{Im}(f) \to \text{Ker}(g))$ and $\text{Im}(\text{Ker}(g) \to \text{Coker}(f))$. In an abelian category these are isomorphic, but for general additive categories this is not the case. Despite these ambiguities, there are still various useful notions of acyclicity in exact categories, which we discuss below. First let us define several classes of morphisms.

Definition 2.1.13. *A morphism $f : E \to F$ in an exact category is said to be*

1. ***weakly left admissible*** *if it has a kernel and the map*

$$Ker(f) \to E$$

is admissible.

2. ***weakly right admissible*** *if it has a cokernel, and the map*

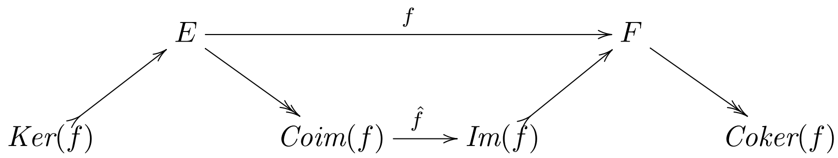
$$F \to Coker(f)$$

is admissible.

3. ***weakly admissible*** *if it is both weakly left admissible and weakly right admissible.*

The following characterisation of weakly admissible morphisms is immediate.

Proposition 2.1.14. *A morphism $f : E \to F$ in an exact category $\mathcal{E}$ is weakly admissible if and only if it admits a decomposition*



where the sequences

$$Ker(f) \rightarrowtail E \twoheadrightarrow Coim(f)$$

and

$$Im(f) \rightarrowtail F \twoheadrightarrow Coker(f)$$

are short exact.

Definition 2.1.15. *Let f be a morphism in exact category. Then f is said to be* ***admissible*** *if it is weakly admissible and the map $Coim(f) \to Im(f)$ is an isomorphism.*

Remark 2.1.16. *Admissible epimorphisms and admissible monomorphisms are admisssible morphisms in the sense above.*

This is not how admissible morphisms are usually defined (see e.g. [14]). However the notions are equivalent:

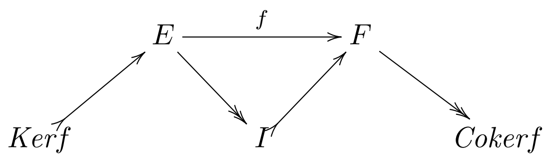
Proposition 2.1.17. *Let $f : E \to F$ be a morphism in an exact category $\mathcal{E}$. Then the following are equivalent.*

1. *f is admissible.*

2. *f admits a decomposition*

$$E \twoheadrightarrow I \rightarrowtail F$$

3. There is a commutative diagram



where the sequences

$$Kerf \rightarrowtail E \twoheadrightarrow I$$

and

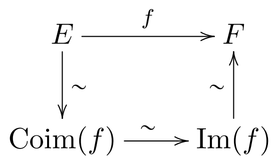
$$I \rightarrowtail F \twoheadrightarrow Coker(f)$$

are short exact.

Proof. 1 and 3 are clearly equivalent thanks to Proposition 2.1.14. Also 3 $\Rightarrow$ 2 trivially. Let us show that 2 $\Rightarrow$ 1. Since $I \rightarrowtail F$ is an admissible monic, the kernel of f exists, and coincides with the kernel of $E \twoheadrightarrow I$. Hence $\text{Ker}(f) \to E$ is an admissible monic and in particular $E \to I$ is a coimage of f. Dually, the cokernel of f exists, it coincides with the cokernel of $G \rightarrowtail F$, and $I \rightarrowtail F$ is an image of f. □

Corollary 2.1.18. *A morphism $f : E \to F$ in an exact category is an isomorphism if and only if it is both an admissible epic and an admissible monic.*

Proof. Axiomatically an isomorphism is both an admissible monic and an admissible epic. Conversely, suppose f is both an admissible monic and an admissible epic. Since it is an admissible monic the map $E \to \text{Coim}(f)$ is an isomorphism. Since it is an admissible epic the map $\text{Im}(f) \to E$ is an isomorphism. Since f is admissible the map $\text{Coim}(f) \to \text{Im}(f)$ is an isomorphism. The claim now follows from the commutative diagram



□

We are now ready to introduce our various notions of acyclic sequences.

Definition 2.1.19. *A null-sequence*

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

is said to be

1. ***weakly acyclic*** *if f is weakly right admissible, g has a kernel, and the natural map $Im(f) \to Ker(g)$ is an isomorphism.*

2. ***weakly coacyclic*** *if g is weakly left admissible, f has a cokernel, and the natural map $Coker(f) \to Coim(g)$ is an isomorphism.*

3. ***admissibly acyclic*** *if it is weakly acyclic and f is admissible,*

4. ***admissibly coacyclic*** *if it is weakly coacyclic and g is admissible*

5. ***admissible*** *if both f and g are admissible.*

6. ***acyclic*** *if it is both admissibly acyclic and admissibly coacyclic.*

Remark 2.1.20. *If a null sequence*

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

is weakly acyclic then g is automatically weakly left admissible.

Definition 2.1.21. *A complex*

$$X_n \xrightarrow{f_n} X_{n-1} \xrightarrow{f_{n-1}} \ldots \longrightarrow X_0$$

is said to be weakly acyclic/ weakly coacyclic/ admissibly acyclic/ admissibly coacyclic/ admissible/ acyclic if for each $1 \leqslant i \leqslant n-1$ each sequence

$$X_{i+1} \xrightarrow{f_{i+1}} X_i \xrightarrow{f_i} X_{i-1}$$

is weakly acyclic/ weakly coacyclic/ admissibly acyclic/ admissibly coacyclic/ admissible/ acyclic.

Let us now set up some tools for determining whether a complex is acyclic. We can partially test acyclicity by passing to a left abelianisation:

Proposition 2.1.22. *Let $I : \mathcal{E} \to \mathcal{A}$ be a left abelianization of $\mathcal{E}$.*

1. *If*

$$X_n \xrightarrow{f_n} X_{n-1} \xrightarrow{f_{n-1}} \ldots \longrightarrow X_0$$

is admissibly acyclic in $\mathcal{E}$ then

$$I(X_n) \xrightarrow{I(f_n)} I(X_{n-1}) \xrightarrow{I(f_{n-1})} \ldots \longrightarrow I(X_0)$$

is exact in $\mathcal{A}$.

2. *If f_i is weakly admissible for $2 \leqslant i \leqslant n$ and*

$$I(X_n) \xrightarrow{I(f_n)} I(X_{n-1}) \xrightarrow{I(f_{n-1})} \ldots \longrightarrow I(X_0)$$

is exact, then

$$X_n \xrightarrow{f_n} X_{n-1} \xrightarrow{f_{n-1}} \ldots \longrightarrow X_0$$

is admissibly acyclic.

3. *If f_i is weakly left admissible for $1 \leqslant i \leqslant n-1$ and*

$$I(X_n) \xrightarrow{I(f_n)} I(X_{n-1}) \xrightarrow{I(f_{n-1})} \ldots \longrightarrow I(X_0)$$

is exact in $\mathcal{A}$, then

$$X_n \xrightarrow{f_n} X_{n-1} \xrightarrow{f_{n-1}} \ldots \longrightarrow X_0$$

is admissibly acyclic.

Proof. Clearly it is sufficient to prove the claims for sequences

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

1. Suppose the above sequence is admissibly acyclic. Since f is admissible I preserves $\text{Im}(f)$. By assumption I preserves all kernels. Hence

$$I(X) \xrightarrow{I(f)} I(Y) \xrightarrow{I(g)} I(Z)$$

is exact.

2. Suppose now that

$$I(X) \xrightarrow{I(f)} I(Y) \xrightarrow{I(g)} I(Z)$$

is exact and that f is weakly admissible. Since I preserves all kernels, and cokernels of admissible morphisms, we have $I(\text{Coim}(f)) \cong \text{Coim}I(f)$. Now

$$\text{Coim}I(f) \cong \text{Im}I(f) \cong \text{Ker}I(g)$$

Since I is fully faithful, $\text{Coim}(f)$ is a kernel of g. Finally, note that we have a factorisation of $\text{Coim}(f) \to \text{Ker}(g)$

$$\text{Coim}(f) \to \text{Im}(f) \rightarrowtail \text{Ker}(g)$$

By Proposition 2.1.8 $\text{Im}(f) \rightarrowtail \text{Ker}(g)$ is also an (admissible) epic. By Corollary 2.1.18 it is an isomorphism. Therefore $\text{Coim}(f) \to \text{Im}(f)$ is as well. By Proposition 2.1.17 we are done.

3. We can factor f as

$$X \xrightarrow{f'} \mathrm{Ker}(g) \longrightarrow Y$$

with $\mathrm{Ker}(g) \to Y$ an admissible monic. We need to show f' is an admissible epic. Since I preserves kernels, it sends the diagram above to

$$I(X) \xrightarrow{I(f')} \mathrm{Ker} I(g) \longrightarrow I(Y)$$

Since

$$I(X) \xrightarrow{I(f)} I(Y) \xrightarrow{I(g)} I(Z)$$

is exact, $I(f')$ is an epic. thus f' is an admissible epic, and we are done.

□

Part 1) of the above proposition says that the functor I is admissibly exact. This is a stronger notion than exactness. It will be useful in later contexts, so we make a definition.

Definition 2.1.23. *A functor $F : \mathcal{E} \to \mathcal{F}$ between exact categories is said to be* ***admissibly (co)exact*** *if for any admissibly (co)acyclic sequence*

$$X \to Y \to Z$$

in $\mathcal{E}$, the sequence

$$F(X) \to F(Y) \to F(Z)$$

is admissibly (co)acyclic. A functor which is both admissibly exact and admissibly coexact is said to be ***strongly exact****.*

Moreover, the proof of Part 1) also gives the following result.

Proposition 2.1.24. *Let $F : \mathcal{E} \to \mathcal{D}$ be an exact functor which preserves kernels. Then F is admissibly exact.*

Example 2.1.25. *It is easy to show that taking finite direct sums is a strongly exact functor. Indeed being both a limit and a colimit, this functor commutes with all limits and colimits.*

Although the functor I reflects short exact sequences, it need not in general reflect acyclicity of unbounded complexes. However it does for a certain nice class of complexes.

Definition 2.1.26. *A complex $X_\bullet$ in an exact category is said to be* ***good*** *if for each n there is $m < n$ such that d_m has a kernel. $X_\bullet$ is said to be* ***cogood*** *if for each n there is $m > n$ such that d_m has a cokernel.*

Example 2.1.27. *Bounded below complexes are good.*

We will frequently use the following trick for good complexes.

Proposition 2.1.28. *Let $X_\bullet$ be a good complex in an exact category. Suppose that for any n such that d_n^X has a kernel, the induced map*

$$d'_{n+1} : X_{n+1} \to Z_n X$$

is an admissible epic. Then $X_\bullet$ is acyclic.

Proof. Suppose d_m has a kernel. By assumption d_{m+1} factors as

$$X_{m+1} \twoheadrightarrow Z_m X \to X_m$$

A priori $Z_m X \to X_m$ is not admissible. However it is a monomorphism. Therefore, since $X_{m+1} \twoheadrightarrow Z_m X$ is admissible its kernel exists and it coincides with the kernel $Z_{m+1}X$ of d_{m+1}. Since $X_{m+1} \twoheadrightarrow Z_m X$ is admissible it is in particular weakly left admissible. Therefore d_{m+1} is also weakly left admissible. Now consider d_{m+2}. By assumption it factors as

$$d_{m+2} : X_{m+2} \twoheadrightarrow Z_{m+1}X \rightarrowtail X_{m+1}$$

Thus d_{m+2} is an admissible morphism whose image is $Z_{m+1}X$. An easy induction then shows that $X_\bullet$ is acyclic. $\square$

Since I preserves kernels and reflects admissible epimorphisms, Proposition 2.1.28 gives the following.

Corollary 2.1.29. *Let $(X_\bullet, d_\bullet)$ be a complex in $\mathcal{E}$. Let $I : \mathcal{E} \to \mathcal{A}$ be a left abelianisation of $\mathcal{E}$. Suppose $X_\bullet$ is good. Then $X_\bullet$ is acyclic if and only if $I(X_\bullet)$ is.*

Proof. Suppose $I(X_\bullet)$ is a acyclic, and d_n^X has a kernel $Z_n X$. By assumption $I(d'_{n+1}) : I(X_{n+1}) \to Z_n I(X) = I(Z_n X)$ is an epimorphism. Thus $d'_{n+1} : X_{n+1} \to Z_n X$ is an admissible epimorphism. $\square$

2.1.4 Homotopies and Quasi-Isomorphisms

Let us now discuss homological properties of maps between complexes.

Definition 2.1.30. *A **homotopy** between morphisms of chain complexes $f_\bullet, g_\bullet : K_\bullet \to L_\bullet$ is a collection of morphisms $D_i : A_i \to B_{i+1}$ such that*

$$f_i - g_i = D_{i-1} \circ d_i^K + d_{i+1}^L \circ D_i$$

We then say $f_\bullet \sim g_\bullet$.

Definition 2.1.31. *Two complexes $K_\bullet$ and $L_\bullet$ are said to be **homotopy equivalent** if there are maps $g : K_\bullet \to L_\bullet$ and $f : L_\bullet \to K_\bullet$ such that $f \circ g \sim id_{K_\bullet}$ and $g \circ f \sim id_{L_\bullet}$.*

If

$$\begin{array}{ccccc} A & \xrightarrow{p} & B & \xrightarrow{q} & C \\ \downarrow & & \downarrow{\scriptstyle \alpha} & & \downarrow \\ X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \end{array}$$

is a diagram with the top and bottom row being null-sequences, we will also say that it is homotopic to zero if there are two maps $D : B \to X$ and $D' : C \to Y$ such $\alpha = f \circ D - D' \circ q$.

We can use homotopies in an exact category to test for acyclicity.

Proposition 2.1.32. *Let $\mathcal{E}$ be an exact category, and let*

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

be a null sequence. Suppose that g has a kernel. Then the induced map $f' : X \to Ker(g)$ is an admissible epimorphism if and only there is a diagram

$$\begin{array}{ccccc} A & \xrightarrow{p} & B & \xrightarrow{q} & C \\ \downarrow & & \downarrow{\scriptstyle \alpha} & & \downarrow \\ X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \end{array}$$

which is homotopic to zero, and such that the induced map $\tilde{\alpha} : Ker(q) \to Ker(g)$ is an admissible epic.

Proof. Suppose that g has a kernel and that the induced map $f' : X \to \text{Ker}(g)$ is an admissible epimorphism. Consider the diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & X & \longrightarrow & 0 \\ \downarrow & & \downarrow{\scriptstyle f} & & \downarrow \\ X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \end{array}$$

By assumption the induced map $\tilde{f} : X \to \text{Ker}(g)$ is an admissible epic. Moreover the diagram is clearly homotopic to 0 via the maps $D = id : X \to X$ and $D' = 0 : 0 \to Y$. Conversely suppose we have a diagram

$$\begin{array}{ccccc} A & \xrightarrow{p} & B & \xrightarrow{q} & C \\ \downarrow & {\scriptstyle D}\swarrow & \downarrow{\scriptstyle \alpha} & {\scriptstyle D'}\swarrow & \downarrow \\ X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \end{array}$$

such that g has a kernel, $\alpha = f \circ D - D' \circ q$, and $\tilde{\alpha}$ is an admissible epic. We have the factorisation of f

$$X \xrightarrow{\tilde{f}} \text{Ker}(g) \longrightarrow Y$$

Moreover, $\tilde{\alpha} = \tilde{f} \circ D|_{\text{Ker}(q)}$. By Proposition 2.1.8 $\tilde{f}$ is an admissible epic. $\square$

Corollary 2.1.33. *Let $\mathcal{E}$ be an exact category, and let*

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

be a null sequence. The sequence is admissibly acyclic if and only if g is weakly left admissible and there is a diagram

$$\begin{array}{ccccc} A & \xrightarrow{p} & B & \xrightarrow{q} & C \\ \downarrow & & \downarrow \alpha & & \downarrow \\ X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \end{array}$$

which is homotopic to zero, and such that the induced map $\tilde{\alpha} : Ker(q) \to Ker(g)$ is an admissible epic.

Proof. Suppose the sequence is admissibly acyclic. By Remark 2.1.20 g is weakly left admissible.

For the converse, note that by Proposition 2.1.32 and the fact that $\mathrm{Ker}(g) \to Y$ is admissible, we have a decomposition of f

$$X \twoheadrightarrow \mathrm{Ker}(g) \rightarrowtail Y$$

By Proposition 2.1.17 f is an admissible morphism whose image is $\mathrm{Ker}(g)$. □

We can also test split exactness by looking at homotopy.

Proposition 2.1.34. *Let $\mathcal{E}$ be an exact category, and let*

$$\Gamma := X \xrightarrow{f} Y \xrightarrow{g} Z$$

be a null-sequence. The sequence is admissibly acyclic in the split exact structure if and only if g is weakly left admissible and the diagram

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \\ \downarrow id_X & & \downarrow id_Y & & \downarrow id_Z \\ X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \end{array}$$

is homotopic to zero.

Proof. Suppose the diagram is homotopic to the zero. If we can show that g is also weakly left admissible in the split exact structure, then the claim follows from Corollary 2.1.33. By Corollary 2.1.33 we already know that the sequence is admissibly acyclic, so $\mathrm{Im}(f) \cong \mathrm{Ker}(g)$. Let $D : Y \to X$ and $D' : Z \to Y$ be maps such that $id_Y = f \circ D - D' \circ g$. The map $f \circ D : Y \to Y$ factors as

$$Y \xrightarrow{\tilde{(f \circ D)}} \mathrm{Im}(f) \xrightarrow{i} Y$$

where i is the inclusion. But

$$f \circ D \circ i = f \circ D \circ i - D \circ g \circ i = i$$

since $g \circ i = 0$. It follows that $(f \tilde{\circ} D) \circ i = \mathrm{Id}_{\mathrm{Im}(f)}$. This implies that the map $\mathrm{Ker}(g) \cong \mathrm{Im}(f) \to Y$ is split, and so is an admissible monic in the split exact structure. □

Corollary 2.1.35. *Let $X_\bullet$ be a good complex.*

1. *$X_\bullet$ is acyclic whenever there is a complex $Y_\bullet$, a morphism of complexes $f_\bullet : Y_\bullet \to X_\bullet$ which is homotopic to 0, and such that the induced maps $\tilde{f}_n : Ker(d_n^Y) \to Ker(d_n^X)$ are admissible epimorphisms.*

2. *$X_\bullet$ is split exact whenever $id_{X_\bullet}$ is homotopic to 0.*

Proof. The first assertion follows from Proposition 2.1.28 and Proposition 2.1.32. For the second assertion note that $X_\bullet$ is acyclic by the first. In particular each

$$X_{n+1} \to X_n \to X_{n-1}$$

is acyclic, and $X_n \to X_{n-1}$ is (weakly left) admissible. Thus we may use Proposition 2.1.34. □

Quasi-isomorphisms

Recall that in an abelian category a map of complexes induces a map on homology. The map is said to be a quasi-isomorphism if the induced map on homology is an isomorphism. Quasi-isomorphisms can also be characterised in terms of their mapping cone. A map of chain complexes in an abelian category is a quasi-isomorphism if and only if its mapping cone is acyclic. As remarked previously, in an exact category we cannot in general define the homology of a complex. However the construction of the mapping cone makes sense in any additive category. By the previous remarks, the following definition is sensible.

Definition 2.1.36. *Let $\mathcal{E}$ be an exact category. A map $f_\bullet : X_\bullet \to Y_\bullet$ of complexes of $\mathcal{E}$ is said to be a **quasi-isomorphism** if $cone(f_\bullet)$ is acyclic.*

Proposition 2.1.37. *Homotopy equivalences are quasi-isomorphisms.*

Proof. See [14] Proposition 10.9. □

The next proposition is an immediate consequence of Corollary 2.1.29.

Proposition 2.1.38. *Let $I : \mathcal{E} \to \mathcal{A}$ be a left abelianisation of an exact category $\mathcal{E}$. Let $f_\bullet : X_\bullet \to Y_\bullet$ be a morphism of complexes. Suppose $cone(f)$ is good. Then f is a quasi-isomorphim if and only if $I(f)$ is.*

Remark 2.1.39. *As for abelian categories, one can define the derived category $D_*(\mathcal{E})$ of an exact category $\mathcal{E}$ by localizing $Ch_*(\mathcal{E})$ at the quasi-isomorphisms. For details see for example [14].*

2.1.5 Ext Groups

In order to study cotorsion pairs in exact categories in Section 3, we will need the notion of Ext groups in exact categories. Recall for an abelian category $\mathcal{A}$ one can define the groups $\text{Ext}^n(A, B)$ for any pair of objects $A, B \in \mathcal{A}$ regardless of whether $\mathcal{A}$ has enough projectives by the Yoneda construction. This construction goes through mutatis-mutandis for exact categories. The elements are Yoneda equivalences classes of n-extensions and the binary operation is the Baer sum. All the proofs for the above facts work as the abelian case. The interested reader can adapt the relevant proofs in [13] for example. The first ext group $\text{Ext}^1(A, B)$ can also be computed by passing to a left abelianization. More generally we have the following straight-forward result.

Proposition 2.1.40. *Let $\mathcal{E}$ and $\mathcal{F}$ be exact categories. Let $F : \mathcal{E} \to \mathcal{F}$ be a fully faithful exact functor which reflects exactness. Suppose that the essential image of $\mathcal{E}$ is closed under extensions. Then F induces a natural isomorphism of abelian groups*

$$Ext^1_{\mathcal{E}}(-,-) \cong Ext^1_{\mathcal{F}}(F(-), F(-))$$

Remark 2.1.41. *In the above we make the implicit assumption that each $Ext^n(A, B)$ is a set. This always holds for exact categories with enough projectives, which can be seen from the discussion in the following section.*

2.1.6 Projective Objects and Resolutions in Exact Categories

At this point we recall the notion of a projective object in an exact category, and mention how they relate to the Ext functor.

Definition 2.1.42. *An object P in an exact category $\mathcal{E}$ is said to be* ***projective*** *if the functor $Hom(P, -) : \mathcal{E} \to \mathcal{Ab}$ is exact.*

Remark 2.1.43. *By Proposition 2.1.24, for any projective object P the functor $Hom(P, -)$ is admissibly exact.*

Example 2.1.44. *In the split exact structure every object is projective.*

As in the abelian case one has the following result.

Proposition 2.1.45. *The following are equivalent.*

1. *P is projective.*

2. *Given a map $f : P \to C$ and an admissible epic $e : B \to C$, there is a morphism $g : P \to B$ such that the following diagram commutes*

$$\begin{array}{ccc} & & B \\ & \overset{g}{\nearrow} & \downarrow e \\ P & \xrightarrow{f} & C \end{array}$$

3. *Any admissible epic with codomain P splits.*

4. *$Ext^1(P, A)$ vanishes for any object A.*

5. *$Ext^n(P, A)$ vanishes for any object A and any $n \geqslant 1$.*

We will need some results about projective resolutions in exact categories later.

Bounded Resolutions

Definition 2.1.46. *An exact category $\mathcal{E}$ is said to* ***have enough projectives*** *if for any object X of $\mathcal{E}$,there is a projective object P and an admissible epimorphism $P \twoheadrightarrow X$.*

Lemma 2.1.47. *Let $\mathcal{P}$ be a subclass of $\boldsymbol{Ob}(\mathcal{E})$, the object of $\mathcal{E}$. Assume that for any object E of $\mathcal{E}$ there is an object $P \in \mathcal{P}$ and an admissible epimorphism $P \twoheadrightarrow E$. Then, for any bounded below complex E of $Ch_+(\mathcal{E})$, there is a bounded below complex P whose entries are objects of $\mathcal{P}$, and a quasi-isomorphism*

$$u : P \to E$$

where each $u_k : P_k \to E_k$ is an admissible epimorphism. Moreover, this construction can be made functorial if the choice of admissible epimorphism $P \twoheadrightarrow E$ can be made functorial.

Proof. This is proved in [14] for the case that $\mathcal{P}$ is the class of projectives in an exact category with enough projectives. However the proof goes through the same. □

Lemma 2.1.48. *Let A, B be objects in an exact category $\mathcal{E}$. Let $f : A \to B$ be a morphism. Let $P_\bullet$ be a complex with $P_{-1} = A, P_n = 0$ for $n < -1$ and P_n projective for $n > 0$. Also let $Q_\bullet$ be an acyclic complex with $Q_{-1} = B$ and $Q_n = 0$ for $n < -1$. Then there is a chain map $f_\bullet : P_\bullet \to Q_\bullet$ with $f_{-1} = f$. Moreover, $f_\bullet$ is unique up to homotopy.*

Proof. See [14] Theorem 12.4. □

As in the abelian case one can define derived functors between derived categories of exact categories. There are also notions of adapted classes for functors. Proposition 2.1.45 and Lemma 2.1.47 essentially say that as in the abelian case, if a category $\mathcal{E}$ has enough projectives, then the class of projective objects is adapted to the functor $\text{Hom}(-, A) : \mathcal{E}^{op} \to \mathcal{Ab}$. It can be shown that $R^n\text{Hom}(-, A) := H_n(R\text{Hom}(-, A)) \cong \text{Ext}^n(-, A)$.

Unbounded Resolutions

When dealing with the model structures on unbounded chain complexes., we will also need to have unbounded resolutions. For this we will modify the famous Theorem 3.4 in [75] and it proof to work for more general exact categories. In the following we shall let $\mathcal{B}$ be a class of complexes in $\mathcal{E}$ which is stable under shifts, and we shall assume that for any bounded below complex $X_\bullet$ there is a bounded below complex $B_\bullet$ in $\mathcal{B}$ and a quasi-isomorphism $B_\bullet \to X_\bullet$ which is an admissible epimorphism in each degree. We will call such a class a **bounded resolving class**.

Before continuing we introduce some terminology. Let $\mathcal{I}$ be a category, $\mathcal{E}$ an exact category and $\mathcal{S}$ a class of morphisms in $\mathcal{E}$.

Definition 2.1.49. *We say that $\mathcal{E}$ has $(\mathcal{I};\mathcal{S})$-**(co)limits** if for any functor $D : \mathcal{I} \to \mathcal{E}$ such that $D(i \to j)$ is in $\mathcal{S}$ for any morphism $i \to j$ in $\mathcal{I}$, a (co)limit of D exists.*

Let $Ch(\mathcal{S})$ denote the class of morphisms in $Ch(\mathcal{E})$ consisting of those morphisms $f_\bullet : A_\bullet \to B_\bullet$ such that $f_n \in \mathcal{S}$ for each n. Clearly if $\mathcal{E}$ has $(\mathcal{I};\mathcal{S})$-(co)limits then $Ch(\mathcal{E})$ has $(\mathcal{I}; Ch(\mathcal{S}))$-(co)limits.

Definition 2.1.50. *Suppose that $\mathcal{E}$ has $(\mathcal{I};\mathcal{S})$-(co)limits. We say that $(\mathcal{I};\mathcal{S})$-(co)limits are exact in $\mathcal{E}$ if for any functor*

$$F : \mathcal{I} \to Ch(\mathcal{E})$$

with $F(i \leqslant j) \in Ch(\mathcal{S})$ for any $i \leqslant j$ in $\mathcal{I}$ and $F(i)$ acyclic for any object i in $\mathcal{I}$, the (co)limit

$$lim_{\to_{\mathcal{I}}} F(i)$$

is acyclic.

Let us now recall some notions from Splatenstein's paper.

Definition 2.1.51. *Let $\mathcal{B}$ be a class of complexes. A direct system $(P^n_\bullet)_{n\in E}$ in $Ch(\mathcal{E})$ is a $\mathcal{B}$-**special direct system** if it satisfies the following conditions.*

1. *E is well-ordered.*

2. *If $n \in E$ has no predecessor then $P^n_\bullet = lim_{\to_{m<n}} P^m_\bullet$.*

3. *If $n \in E$ has a predecessor $n-1$ then the natural chain map $P_\bullet^{n-1} \to P_\bullet^n$ is injective, its cokernel $C_\bullet^n$ belongs to $\mathcal{B}$, and the short exact sequence*

$$0 \to P_\bullet^{n-1} \to P_\bullet^n \to C_\bullet^n \to 0$$

is split exact in each degree.

We denote by $\lim_{\to}\mathcal{B}$ the class of complexes which are limits of $\mathcal{B}$-special direct systems.

Proposition 2.1.52. *Let $\mathcal{E}$ be an exact category which has kernels. Suppose that $\mathcal{B}$ is a bounded resolving class. Then for any complex $X_\bullet$ there exists a $\mathcal{B}$-special direct system $(P_\bullet^n)_{n \geqslant -1}$ and a direct system of chain maps $f^n : P_\bullet^n \to \tau_{\geqslant n} X_\bullet$ such that*

1. *f^n is a quasi-isomorphism for every $n \geqslant 0$.*
2. *f^n is an admissible epimorphism in each degree.*

Proof. We construct the data $(P_\bullet^n)_{n \geqslant -1}$ and $(f^n)_{n \geqslant -1}$ by induction. For $n = -1$ we take $P_\bullet^{-1} = 0$ and so $f^{-1} = 0$. Let now $n \geqslant 1$, and suppose that $P_\bullet^{-1}, \ldots, P_\bullet^{n-1}$ and $f^{-1}, \ldots, f^{n-1}$ have been constructed. Let $P_\bullet = P_\bullet^{n-1}$ and $Y_\bullet = \tau_{\geqslant n} X_\bullet$. Denote by f the composite $P_\bullet^{n-1} \to \tau_{\geqslant n-1} X_\bullet \to Y_\bullet$. By assumption we can find a quasi-isomorphism $g : Q_\bullet \to \mathrm{cone}(f)[1]$ which is an admissible epimorphism in each degree, and $Q_\bullet[-1] \in \mathcal{B}$. Now we have a degree wise splitting, $\mathrm{cone}(f)[1] = P_\bullet \oplus Y_\bullet[1]$. We therefore get two maps $g' : Q_\bullet \to P_\bullet$ and $g'' : Q_\bullet \to Y_\bullet[1]$ which are admissible epimorphisms in each degree, and such that g' is a chain map. Define $P^n := \mathrm{cone}(-g')$ and let $f^n : \mathrm{cone}(-g') = Q[1] \oplus P \to Y$ be defined by $f^n = g''[1] + f$. As in [75], by direct calculation f^n is a chain map and $\mathrm{cone}(f^n) = \mathrm{cone}(g)[1]$. Since g is a quasi-isomorphism f^n is as well. Moreover the sequence

$$0 \to P_\bullet^{n-1} \to P_\bullet^n \to Q_\bullet[1] \to 0$$

is split exact in each degree. □

Corollary 2.1.53. *Let $\mathcal{E}$ be an exact category with kernels and such that $(\mathbb{N}_0, \textbf{\textit{AdMon}})$-colimits exist and are exact. Let $\mathcal{B}$ be a bounded resolving class. Then any chain complex $X_\bullet$ in $\mathcal{E}$ admits a $\lim_{\to}\mathcal{B}$ resolution which is an admissible epimorphism in each degree.*

Proof. Fix a $\mathcal{B}$-special direct system $(P_\bullet^n)_{n \geqslant -1}$ and a direct system of chain maps $f^n : P_\bullet^n \to \tau_{\geqslant n} X_\bullet$ such that

1. f^n is a quasi-isomorphism for every $n \geqslant 0$.
2. f^n is an admissible epimorphism in each degree.

Let $P_\bullet$ be the direct limit of the special direct system. For each n the composition $P^n_\bullet \to P_\bullet \to X_\bullet$ is an admissible epimorphism in degrees $> n$. Thus $P_\bullet \to X_\bullet$ is an admissible epimorphism in all degrees. □

Now let $\mathcal{P}$ be any class of objects in $\mathcal{E}$. Suppose that for each object X in $\mathcal{E}$ there is an object P in $\mathcal{P}$ together with an admissible epimorphism $P \twoheadrightarrow X$. By Lemma 2.1.47 the class $Ch_+(\mathcal{P})$ of chain complexes with entries in $\mathcal{P}$ is a bounded resolving class. Let us introduce the following notion.

Definition 2.1.54. *Let $\mathcal{E}$ be an exact category such that for some ordinal λ, λ-indexed transfinite compositions of admissible monomorphisms exist. We say that a class of objects $\mathcal{P}$ in $\mathcal{E}$ is **closed under λ-indexed extensions** if for any continuous functor $X : \lambda \to \mathcal{E}$ functor such that for each $i < j$ in λ the map $X_i \to X_j$ is an admissible monic whose cokernel is in $\mathcal{P}$, then the limit X_λ is in $\mathcal{P}$.*

From the proof of Corollary 2.1.53 we then immediately have the following.

Corollary 2.1.55. *Let $\mathcal{E}$ be an exact category with kernels in which $(\mathbb{N}_0, \boldsymbol{AdMon})$-colimits exist and are exact. Let $\mathcal{P}$ be a class of objects such that for each object X in $\mathcal{E}$ there is an object P in $\mathcal{P}$ together with an admissible epimorphism $P \twoheadrightarrow X$. Suppose further that $\mathcal{P}$ is closed under $\mathbb{N}$-indexed extensions. Then for any complex $X_\bullet$ in $Ch(\mathcal{E})$ there is a complex $P_\bullet$ in $Ch(\mathcal{P})$ and an admissible epimorphism $P_\bullet \to X_\bullet$ which is a quasi-isomorphism. Moreover, $X_\bullet$ is the limit of a $Ch_+(\mathcal{P})$-special direct system.*

We will also need the following acyclicity result, also proved in [75] for abelian categories.

Proposition 2.1.56. *Let $\mathcal{T}$ be a class of complexes in $Ch(\mathcal{E})$. The class of all complexes $A_\bullet \in Ch(\mathcal{E})$ such that $\boldsymbol{Hom}(A_\bullet, T_\bullet)$ is acyclic for every $T_\bullet$ in $\mathcal{T}$ is closed under special direct limits.*

Proof. It is clear from the definition of the contravariant functor $\mathbf{Hom}(-, T_\bullet)$ that it transforms colimits into limits. If $(P^n_\bullet)_{n \in E}$ is a $\mathcal{B}$-special direct system then $(\mathbf{Hom}(P^n_\bullet, T_\bullet))_{n \in E}$ is a $\mathcal{B}$-special inverse system of acyclic complexes of abelian groups, where we use the terminology of [75]. Lemma 2.3 in [75] says that the inverse limit of such a system is again acyclic. □

2.1.7 Exact Structures on Chain Complexes

Let $\mathcal{E}$ be an exact category and consider the category $Ch_*(\mathcal{E})$ for $* \in \{\varnothing, b, \geqslant, \leqslant, +, -\}$. Say that a sequence $0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0$ is exact precisely if for each $i \in \mathbb{Z}$ the sequence $0 \to A_i \to B_i \to C_i \to 0$ is exact. Since limits and colimits in $Ch_*(\mathcal{E})$ are computed degree-wise this is an exact structure on $Ch(\mathcal{E})$.

Proposition 2.1.57. *Let $F : \mathcal{A} \to \mathcal{B}$ be a fully faithful exact functor which reflects exactness and whose essential image is closed under extensions. Then for $* \in \{\geqslant 0, \leqslant 0, +, -, b, \emptyset\}$ the induced functor*

$$Ch_*(F) : Ch_*(\mathcal{A}) \to Ch_*(\mathcal{B})$$

is a fully faithful exact functor which reflects exactness and whose essential image is closed under extensions.

Proof. Since exactness of chain complexes is defined level wise, $Ch_*(F)$ is clearly exact and reflects exactness. It is clearly faithful. Let us check that it is full. Let $(X_\bullet, d_\bullet)$ and $(Y_\bullet, \delta_\bullet)$ be chain complexes in $\mathcal{A}$. Let $f_\bullet : F(X_\bullet) \to F(Y_\bullet)$ be a chain map. For each n there is some $g_n : X_n \to Y_n$ with $f_n = F(g_n)$. Moreover

$$F(g_n \circ d_{n+1}) = F(g_n) \circ F(d_{n+1}) = f_n \circ F(d_{n+1}) = F(\delta_{n+1}) \circ f_{n+1} = F(\delta_{n+1} \circ g_{n+1})$$

Since F is faithful, $g_n \circ d_{n+1} = \delta_{n+1} \circ g_{n+1}$. It remains to show that the essential image of $Ch_*(F)$ is closed under extensions. So suppose we have an exact sequence of chain complexes.

$$0 \longrightarrow F(X_\bullet, d_\bullet) \xrightarrow{f_\bullet} (Q_\bullet, \gamma_\bullet) \xrightarrow{g_\bullet} F(Y_\bullet, \delta_\bullet) \longrightarrow 0$$

For each n pick an object $P_n \in \mathcal{A}$ and an isomorphism $p_n : Q_n \tilde{\to} F(P_n)$. Let $\gamma'_n = p_{n-1} \circ \gamma_n \circ p_n^{-1} : F(P_n) \to F(P_{n-1})$. Then $(P_\bullet, \gamma'_\bullet)$ is a chain complex. Moreover by construction we have an isomorphism $p_\bullet : Q_\bullet \to F(P_\bullet)$ whose nth component is p_n. □

Corollary 2.1.58. *Let $I : \mathcal{E} \to \mathcal{A}(\mathcal{E})$ is a left abelianization of $\mathcal{E}$. Then $Ch_*(I) : Ch_*(\mathcal{E}) \to Ch_*(\mathcal{A}(\mathcal{E}))$ is a left abelianization of $Ch_*(\mathcal{E})$.*

Proof. By the previous proposition, it remains to check that $Ch(I)$ preserves kernels, and $Ch(I)(f_\bullet)$ is an admissible epimorphism if and only if $f_\bullet$ is. However this is clear since everything is computed degree-wise. □

A Useful Example: The Degree-Wise Exact Structure

Let $\mathcal{E}$ be an additive category, and endow it with the split exact structure. The induced exact structure on $\mathcal{C}h(\mathcal{E})$ is called the **degree-wise split** exact structure, and we denote the ext functors in this structure by Ext^n_{dw}. We conclude this section with a brief discussion of the relation between extensions in the degree-wise split exact structure and the $\mathcal{C}h(\mathbf{Ab})$-enriched structure on $\mathcal{C}h(\mathcal{E})$. This is also done in a model theoretic context for modules over a ring in [31].

Proposition 2.1.59. *A sequence of chain complexes* $0 \longrightarrow X_\bullet \xrightarrow{p_\bullet} Z_\bullet \xrightarrow{q_\bullet} Y_\bullet \longrightarrow 0$ *is split exact in each degree if and only if it is isomorphic to a complex of the form*

$$0 \to X_\bullet \to \mathit{cone}(f_\bullet) \to Y_\bullet \to 0$$

for some morphism of complexes $f_\bullet : Y_\bullet[1] \to X_\bullet$.

Proof. The sequence

$$0 \to X_\bullet \to \mathrm{cone}(f_\bullet) \to Y_\bullet \to 0$$

is clearly split exact in each degree, so any complex isomorphic to it is split exact in each degree as well. Suppose

$$0 \longrightarrow X_\bullet \xrightarrow{p_\bullet} Z_\bullet \xrightarrow{q_\bullet} Y_\bullet \longrightarrow 0$$

is split exact in each degree. Let $\alpha_n : Z_n \to X_n$ be such that $\alpha_n \circ p_n = \mathrm{id}_{X_n}$ and $\beta_n : Y_n \to Z_n$ be a map such that $q_n \circ \beta_n = \mathrm{id}_{Y_n}$. We may assume also that $\alpha_n \circ \beta_n = 0$. Define $f_\bullet : Y_\bullet[1] \to X_\bullet$ by $f_n = \alpha_n \circ d^Z_{n+1} \circ \beta_{n+1}$. This is easily seen to be a map of chain complexes. Let $\alpha_n : Z_n \to X_n \oplus Y_n$ denote the isomorphism induced by the degree-wise splitting. A straight-forward computation shows that this gives a map of chain complexes $\alpha_\bullet : Z_\bullet \to \mathrm{cone}(f_\bullet)$. Thus we get an isomorphism of exact sequences.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X_\bullet & \xrightarrow{p_\bullet} & Z_\bullet & \xrightarrow{q_\bullet} & Y_\bullet & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle \alpha_\bullet} & & \| & & \\
0 & \longrightarrow & X_\bullet & \longrightarrow & \mathrm{cone}(f_\bullet) & \longrightarrow & Y_\bullet & \longrightarrow & 0
\end{array}$$

□

Proposition 2.1.60. *A map of chain complexes* $f_\bullet : X_\bullet \to Y_\bullet$ *is homotopic to* 0 *if and only if the sequence*

$$0 \to Y_\bullet \to \mathit{cone}(f_\bullet) \to X_\bullet[-1] \to 0$$

is split exact.

Proof. Suppose that $f_\bullet$ is homotopic to 0. Let $\{D_n : X_n \to Y_{n+1}\}$ be a homotopy. We then get a map $\alpha_n = (\mathrm{id}_{X_{n-1}}, D_{n-1}) : X_{n-1} \to \mathrm{cone}(f)_n$. It is straight-forward to check that this gives a chain map $\alpha_\bullet : X_\bullet[-1] \to \mathrm{cone}(f_\bullet)$. Moreover it obviously gives a splitting of $\mathrm{cone}(f_\bullet) \to X_\bullet[-1]$. Conversely suppose the sequence is split exact. Let $\alpha_\bullet : X_\bullet[-1] \to \mathrm{cone}(f_\bullet)$ be a splitting of the map $\mathrm{cone}(f_\bullet) \to X_\bullet[-1]$. It is an easy computation to check that the collection of compositions $\{D_{n-1} : X_{n-1} \to \mathrm{cone}(f_\bullet)_n \to Y_n\}$ is a homotopy between f and 0. □

We recover the following standard result

Corollary 2.1.61. *For chain complexes $X_\bullet, Y_\bullet$ in an additive category $\mathcal{E}$. we have*

$$Ext^1_{dw}(X, Y[n-1]) \cong H_n \boldsymbol{Hom}(X_\bullet, Y_\bullet) = Hom_{Ch(\mathcal{E})}(X, Y[n])/\sim$$

where $\sim$ is chain homotopy.

Proof. By direct computation, one finds that $f \in \prod_i \mathrm{Hom}(X_i, Y_{i+n})$ defines a chain map $f_\bullet : X_\bullet \to Y_\bullet[n]$ if and only if $f \in \mathrm{Ker}(d_n)$. Similarly, $f_\bullet$ is then null-homotopic if and only if it is in $\mathrm{Im}(d_{n+1})$. This gives the isomorphism

$$H_n \mathbf{Hom}(X_\bullet, Y_\bullet) = \mathrm{Hom}_{Ch(\mathcal{E})}(X, Y[n])/\sim$$

The isomorphism $\mathrm{Ext}^1_{dw}(X, Y[n-1]) \cong \mathrm{Hom}_{Ch(\mathcal{E})}(X, Y[n])/\sim$ follows from Proposition 2.1.59 and Proposition 2.1.60. □

2.1.8 Monoidal Exact Categories and Monads in Exact Categories

We conclude this section with a brief note on monoidal exact categories, and exact structures on categories of modules over monoids. More generally we put an exact structure on the category of algebras for an additive monad which is compatible in a precise sense with the exact structure on the underlying category. First we need a general definition.

Definition 2.1.62. *A covariant functor $F : \mathcal{E} \to \mathcal{F}$ between exact categories is said to be **right exact** if for any short exact sequence*

$$0 \to X \to Y \to Z \to 0$$

in $\mathcal{E}$, the sequence

$$F(X) \to F(Y) \to F(Z) \to 0$$

is admissibly coacyclic in $\mathcal{F}$.

*A contravariant functor $F : \mathcal{E} \to \mathcal{F}$ between exact categories is said to be **right exact** if for any short exact sequence*

$$0 \to X \to Y \to Z \to 0$$

in $\mathcal{E}$, the sequence

$$F(Z) \to F(Y) \to F(X) \to 0$$

is admissibly coacyclic in $\mathcal{F}$.

Dually one defines left exactness.

Definition 2.1.63. *Let $\mathcal{E}$ be an exact category. A symmetric monoidal structure with additive tensor functor $\otimes$ is said to be* ***compatible*** *if for any object X of $\mathcal{E}$ the functor $X \otimes (-)$ preserves all colimits which exist and is right exact. A* ***monoidal exact category*** *is a symmetric monoidal category $(\mathcal{E}, \otimes, k)$ where $\mathcal{E}$ is an exact category and the monoidal structure is compatible.*

Definition 2.1.64. *Let $\mathcal{E}$ be an exact category. A closed symmetric monoidal structure with additive tensor functor $\otimes$ and additive internal hom $\underline{Hom}$ is said to be* ***compatible*** *with $\mathcal{E}$ if for each object X of $\mathcal{E}$, the functor $X \otimes (-)$, is right exact and the functors $\underline{Hom}(A, -)$ and $\underline{Hom}(-, A)$ are left exact. A* ***closed monoidal exact category*** *is a closed symmetric monoidal category $(\mathcal{E}, \otimes, \underline{Hom}, k)$ where $\mathcal{E}$ is an exact category and the closed monoidal structure is compatible.*

Note that if $(\mathcal{E}, \otimes, \underline{\text{Hom}}, k)$ is a closed monoidal exact category, then $(\mathcal{E}, \otimes, k)$ is automatically a monoidal exact category. Indeed for each object X, $X \otimes (-)$ is a left adjoint so it preserves colimits.

Definition 2.1.65. *Let $(\mathcal{E}, \otimes, k)$ be an exact category equipped with a (not necessarily compatible) symmetric monoidal structure where the tensor functor is additive. An object F of $\mathcal{E}$ is said to be* ***(strongly) flat*** *if the functor $F \otimes (-)$ is (strongly) exact.*

In the familiar category of R-modules over some ring R with the usual monoidal structure, projectives are always flat. Moreover the tensor product of two projective R-modules is again projective. This is not always guaranteed for an arbitrary monoidal exact category. However it is a useful property to have, in particular when dealing with the projective model structure later. We therefore make a definition.

Definition 2.1.66. *A monoidal exact category in which projective objects are flat and $P \otimes P'$ is projective whenever both P and P' are is said to be* ***projectively monoidal*** *. A projectively monoidal exact category is said to be* ***strongly projectively monoidal*** *if projectives are strongly flat.*

In closed exact categories we have the following observation.

Observation 2.1.67. *Let $(\mathcal{E}, \otimes, k, \underline{Hom})$ be a closed monoidal exact category with enough projectives such that the underlying monoidal category is projectively monoidal. Then for any projective P, the functor $\underline{Hom}(P, -) : \mathcal{E} \to \mathcal{E}$ is exact. The proof follows immediately from the adjunction between $\otimes$ and $\underline{Hom}$. It is shown in the quasi-abelian case in [72], for example, and the proof works identically in the exact case.*

Now let R be a unital associative monoid internal to a monoidal exact category $(\mathcal{E}, \otimes, k)$. It turns out that there is an exact structure on the additive category ${}_R\mathcal{Mod}$ where a null sequence

$$0 \to X \to Y \to Z \to 0$$

in ${}_RMod$ is exact if and only if it is a short exact sequence when regarded as a null-sequence in $\mathcal{E}$. This follows from a more general result about compatible monads in exact categories.

Definition 2.1.68. *An additive monad T on an exact category $\mathcal{E}$ is said to be* ***compatible*** *if it preserves all colimits and is a right exact functor.*

Proposition 2.1.69. *Let $\mathcal{E}$ be an exact category and let $T : \mathcal{E} \to \mathcal{E}$ be a compatible monad. There is an exact structure on $\mathcal{E}^T$ where a null sequence*

$$0 \to X \to Y \to Z \to 0$$

in $\mathcal{E}^T$ is exact if and only if it is a short exact sequence when regarded as a null-sequence in $\mathcal{E}$. We call this exact structure the ***induced exact structure****.*

Proof. This follows from the general fact that if T is a cocontinuous monad on any category $\mathcal{E}$ then the forgetful functor $\mathcal{E}^T \to \mathcal{E}$ creates limits and colimits and reflects isomorphisms. For a proof see [12] Proposition 4.3.1 and Proposition 4.3.2. □

This exact structure inherits a lot from the exact structure on $\mathcal{E}$. In fact we have the following lemma.

Lemma 2.1.70. *Let $| - | : \mathcal{D} \to \mathcal{E}$ be a functor between exact categories which reflects exactness and creates both kernels and cokernels. Then $| - |$ reflects admissible monomorphisms, admissible epimorphisms, weakly admissible morphisms, admissible morphisms, and admissibly acyclic sequences.*

Proof. Let $f : X \to Y$ be a morphism in $\mathcal{D}$. Supposethat $|f|$ is an admissible monomorphism. Then there is an exact sequence

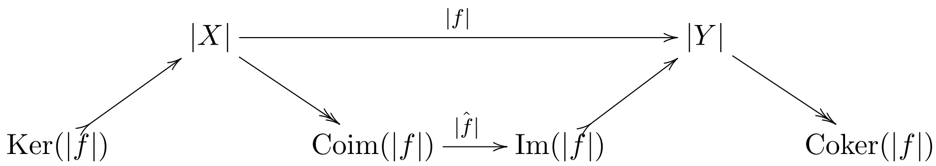
$$0 \to |X| \to |Y| \to \text{Coker}(|f|) \to 0$$

Since $| - |$ creates cokernels and reflects exactness

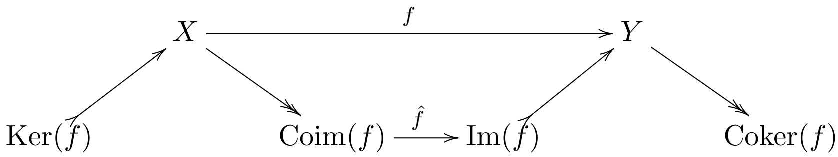
$$0 \to X \to Y \to \text{Coker}(f) \to 0$$

is an exact sequence in $\mathcal{D}$. Thus f is an admissible monomorphism. That $| - |$ reflects admissible epimorphisms is proved similarly. Note in particular that this means $| - |$ reflects isomorphisms.

Suppose now that $|f| : |X| \to |Y|$ is weakly admissible. Then there is a decomposition



Since $|-|$ reflects exactness and creates both kernels and cokernels there is a decomposition in $\mathcal{D}$



Thus f is weakly admissible. If in addition $|f|$ is admissible then $|\hat{f}|$ is an isomorphism. Since $|-|$ reflects isomorphisms $\hat{f}$ is an isomorphism, so f is admissible.

Finally suppose

$$|X| \xrightarrow{|f|} |Y| \xrightarrow{|g|} |Z|$$

is admissibly acyclic. Then $|f|$ is admissible, $|g|$ has a kernel and the map $\mathrm{Im}(|f|) \to \mathrm{Ker}(|g|)$ is an isomorphism. By the above f is admissible. Since $|-|$ creates kernels and cokernels and also reflects isomorphisms $\mathrm{Im}(f) \to \mathrm{Ker}(g)$ is also an isomorphism. $\square$

As a consequence of this and Remark 2.1.77 later, if $\mathcal{E}$ is (quasi)-abelian, then so is $\mathcal{E}^T$. In particular categories of modules for monoid objects in monoidal (quasi)-abelian categories are themselves (quasi)-abelian.

Before concluding this discussion of monoidal exact categories, let us briefly mention induced monoidal structures on chain complexes. So, let $(\mathcal{E}, \otimes, k)$ be a monoidal exact category. Recall from Section 1.4 there is an induced additive monoidal structure $(Ch_*(\mathcal{E}), \otimes, S^0(k))$ on $Ch_*(\mathcal{E})$ for $* \in \{\geqslant 0, \leqslant 0, +, -, b, \varnothing\}$. Since colimits of chain complexes are computed degreewise, finite direct sums are strongly exact, and a null-sequence of chain complexes is admissibly coacyclic if and only if it is so in each degree, it is clear that this monoidal structure is compatible, so that $(Ch_*(\mathcal{E}), \otimes, S^0(k))$ is a monoidal exact category for $* \in \{\geqslant 0, \leqslant 0, +, -, b\}$.

Now suppose $(\mathcal{E}, \otimes, S^0(k), \underline{\mathrm{Hom}})$ is a closed monoidal exact category. Then

$$(Ch_b(\mathcal{E}), \otimes, S^0(k), \underline{\mathrm{Hom}})$$

is a closed monoidal exact category. Note that the closed symmetric monoidal category

$$(Ch(\mathcal{E}), \otimes, S^0(k), \underline{\mathrm{Hom}})$$

need not be a closed monoidal exact category since infinite direct sums/ products need not be admissibly coexact/ admissibly exact. When we deal with unbounded complexes later we shall assume this to be the case. We shall see shortly that this is guaranteed for a closed monoidal structure on a quasi-abelian category.

2.1.9 Quasi-Abelian Categories

Let us apply what we have seen so far to the particular case of quasi-abelian categories. The theory of quasi-abelian categories is developed significantly in [72] which is our main reference here. Applications to categories of topological vector spaces can be found in [67].

Strict Morphisms

First we show that Definition 2.1.4 is equivalent to the one given in [72]. Recall that in a finitely bicomplete additive category, any morphism $f : E \to F$ gives rise to a commutative diagram

$$\begin{array}{ccc} E & \xrightarrow{f} & F \\ \downarrow & & \uparrow \\ \mathrm{Coim}(f) & \longrightarrow & \mathrm{Im} f \end{array}$$

In any abelian category the map $\mathrm{Coim}(f) \to \mathrm{Im}(f)$ is an isomorphism. However this is not true in general. For example, consider the standard example of the category $\mathcal{F}r$ of Fréchet spaces. Then $\mathrm{Coim}(f) = E/f^{-1}(0)$, $\mathrm{Im}(f) = \overline{f(E)}$ and the natural map $E/f^{-1}(0) \to \overline{f(E)}$ is the obvious one. By the Open Mapping Theorem $\mathrm{Coim}(f) \to \mathrm{Im}(f)$ is an isomorphism if and only if f has closed range, which is not always the case.

Definition 2.1.71. *Let $\mathcal{E}$ be an additive category with all kernels and cokernels. A morphism $f : E \to F$ in $\mathcal{E}$ is said to be **strict** if $Coim(f) \to Im(f)$ is an isomorphism.*

Proposition 2.1.72. *Let $\mathcal{E}$ be a finitely bicomplete additive category.*

1. *A monic is strict if and only if it is the kernel of some morphism. In this case it is the kernel of its cokernel.*

2. *An epic is strict if and only if it is the cokernel some morphism. In this case it is the cokernel of its kernel.*

Proof. 1. Let $f : E \to F$ and write $i_f : \mathrm{Ker}(f) \to E$ for the canonical map. Let us show that i_f is strict. First note that for any monic $A \to B$, the coimage is $\mathrm{id} : A \to A$. Let us compute the image of i_f. It is given by

$$\mathrm{Ker}(\mathrm{Coker}(\mathrm{Ker}(f) \to E) \to E$$

By some abstract nonsense this is just $\mathrm{Ker}(f) \to E$. Conversely suppose $m : X \to E$ is a strict monic. Then the maps $E \to \mathrm{Coim}(m) \to \mathrm{Im}(m)$ are all isomorphisms, i.e. we get a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{m} & E \\ \downarrow \wr & & \uparrow \\ \mathrm{Coim}(m) & \xrightarrow{\sim} & \mathrm{Im}(m) \end{array}$$

Since $\mathrm{Im}(m) \to E$ is a kernel of $\mathrm{Coker}(m)$, so is $m : X \to E$.

2. This is dual to the first part.

□

Proposition 2.1.73. *The class of strict epics (resp. monics) in a quasi-abelian category $\mathcal{E}$ is stable by composition.*

Proof. See [72] Proposition 1.1.7. □

Corollary 2.1.74. *A finitely bicomplete additive category $\mathcal{E}$ is quasi-abelian if and only if the following two conditions hold:*

1. *If*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & & \downarrow \\ X & \xrightarrow{f'} & Y \end{array}$$

is a push out diagram, and f is a strict monic, then f' is as well.

2. *If*

$$\begin{array}{ccc} A & \xrightarrow{f'} & B \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

is a pullback diagram, and f is a strict epic, then f' is as well.

Let us now describe the admissible morphisms in the quasi-abelian exact structure.

Proposition 2.1.75. *Let $\mathcal{E}$ be a finitely bicomplete additive category. A morphism $f : E \to F$ in $\mathcal{E}$ is strict if and only if it can be written as $f = i \circ p$ where $p : E \to I$ is a strict epic and $i : I \to F$ is a strict monic.*

Proof. Suppose f admits a decomposition $f = i \circ p$ as in the statement. Then $\text{Ker}(f) = \text{Ker}(p)$. So $\text{Coim}(f) = \text{Coim}(p)$. Since p is strict $\text{Coim}(p) \cong \text{Im}(p)$. Since p is an epic, $\text{Im}(p) = I$. Similarly $\text{Im}(f) = \text{Im}(i) = I$. Conversely suppose f is a strict morphism. Now $E \to \text{Coim}(f)$ is a strict epic, and $\text{Im}(f) \to F$ is a strict monic. But since f is strict, $\text{Coim}(f) \cong \text{Im}(f)$, so this gives the decomposition of f. □

Corollary 2.1.76. *A morphism in a quasi-abelian category is admissible in the quasi-abelian exact structure if and only if it is strict.*

Remark 2.1.77. *An exact structure on a finitely bicomplete additive category coincides with the quasi-abelian structure if and only if every morphism is weakly admissible. Then as a consequence of Proposition 2.1.72, a finitely bicomplete additive category is abelian if and only if every morphism is admissible.*

The Left Heart

Homology in quasi-abelian is significantly easier than in more general exact categories. For example, there is an even stronger abelian embedding.

Theorem 2.1.78. *Let $\mathcal{E}$ be a quasi-abelian category. There exists a left abelianization $I : \mathcal{E} \to LH(\mathcal{E})$ of $\mathcal{E}$ such that I has a left adjoint $C : LH(\mathcal{E}) \to \mathcal{E}$ with $C \circ I \cong id_{\mathcal{E}}$, i.e. $\mathcal{E}$ is a reflective subcategory of $LH(\mathcal{E})$. Moreover the induced functor on derived categories*

$$D(I) : D(\mathcal{E}) \to D(LH(\mathcal{E}))$$

is an equivalence.

Proof. See [72] Proposition 1.1.26, Corollary 1.2.27, Proposition 1.2.28, and Proposition 1.2.31. □

$LH(\mathcal{E})$ is called the **left heart** of $\mathcal{E}$. The embedding of $\mathcal{E}$ into its left heart also behaves extremely well with respect to projectives, namely:

Proposition 2.1.79.

1. *An object P of $\mathcal{E}$ is projective if and only if $I(P)$ is projective in $LH(\mathcal{E})$.*

2. *$\mathcal{E}$ has enough projectives if and only if $LH(\mathcal{E})$ has enough projectives. In this case an object of $LH(\mathcal{E})$ is projective if and only if it is isomorphic to $I(P)$ where P is projective in $\mathcal{E}$.*

Proof. See [72] Proposition 1.3.24. □

Moreover left abelianizations of quasi-abelian categories allow us to test acyclicity of any unbounded complex. Indeed as a consequence of Remark 2.1.77 and Corollary 2.1.29 we get.

Corollary 2.1.80. *Let $I : \mathcal{E} \to \mathcal{A}$ be a left abelianisation of $\mathcal{E}$ where $\mathcal{E}$ is a quasi-abelian category. Then a complex $X_{\bullet}$ in $\mathcal{E}$ is acyclic if and only if $I(X_{\bullet})$ is acyclic. In particular a map of complexes $f : X \to Y$ is a quasi-isomorphism if and only if $I(f)$ is.*

Monoidal Quasi-Abelian Categories

Let us briefly discuss (strongly) projectively monoidal quasi-abelian categories, i.e. a (strongly) projectively monoidal exact category in which the underlying exact category is quasi-abelian. We first make the following observation.

Observation 2.1.81. *An additive functor $F : \mathcal{E} \to \mathcal{F}$ between quasi-abelian categories is right exact if and only if it preserves cokernels of strict morphisms.*

This implies that if $(\mathcal{E}, \otimes, k)$ is a monoidal category with $\mathcal{E}$ quasi-abelian and $\otimes$ additive, then it is a monoidal quasi-abelian category if and only if $X \otimes (-)$ preserves colimits for each object X of $\mathcal{E}$. In particular if $(\mathcal{E}, \otimes, \underline{\mathrm{Hom}}, k)$ is a closed monoidal category with $\mathcal{E}$ quasi-abelian and $\otimes$, $\underline{\mathrm{Hom}}$ additive functors, then $(\mathcal{E}, \otimes, \underline{\mathrm{Hom}}, k)$ is in fact a closed monoidal quasi-abelian category.

Proposition 2.1.82. *Let $(\mathcal{E}, \otimes, \underline{Hom}, k)$ be a bicomplete closed monoidal quasi-abelian category which is also projectively monoidal. Then there is a monoidal structure $\tilde{\otimes}, \underline{\tilde{Hom}}$ on $LH(\mathcal{E})$ such that $(LH(\mathcal{E}), \tilde{\otimes}, \underline{\tilde{Hom}}, I(k))$ is a closed monoidal abelian category. Moreover $I : \mathcal{E} \to LH(\mathcal{E})$ is a lax monoidal functor. If $(\mathcal{E}, \otimes, \underline{Hom}, k)$ is strongly projectively monoidal then $(LH(\mathcal{E}), \tilde{\otimes}, \underline{\tilde{Hom}}, I(k))$ is projectively monoidal.*

Proof. See [72] Proposition 1.5.3 and Corollary 1.5.4. □

2.2 Generators in Exact Categories

In this section we will introduce suitable notions of a generating set in an exact category. This will come into play later when we discuss cofibrant generation of model structures, where some compactness assumptions are required. For our definition of generating set we will generalise an equivalent characterisation ([72] Proposition 2.1.7) of Schneiders' notion of a **strictly generating set** in a quasi-abelian caegory, [72] Definition 2.1.5. If $\mathcal{G}$ is a collection of objects in an exact category we denote by $\bigoplus \mathcal{G}$ the collection of all small coproducts of objects in $\mathcal{G}$. We will use the word 'collection' because we will also be interested in proper classes of generators.

Definition 2.2.1. *A collection of objects $\mathcal{G}$ in an exact category $\mathcal{E}$ is said to be an **admissible generating collection** if for each object E of $\mathcal{E}$ there is an object Q of $\bigoplus \mathcal{G}$ and an admissible epimorphism $Q \twoheadrightarrow E$. An admissible generating collection $\mathcal{G}$ is said to be a **projective generating collection** if all objects in $\mathcal{G}$ are projective.*

The next two results are adaptations of the proof of [72] Proposition 1.3.23 to the exact case.

Proposition 2.2.2. *Let $\mathcal{G}$ be an admissible generating collection. Suppose $f : E \to F$ is a morphism such that for each G in $\mathcal{G}$ then map $Hom(G, E) \to Hom(G, F)$ is an epimorphism. Then f is an admissible epimorphism.*

Proof. Pick an admissible epimorphism $\epsilon : P \to F$ where $P \in \bigoplus \mathcal{G}$. By assumption there is a morphism $\epsilon' : P \to E$ such that $\epsilon = f \circ \epsilon'$. By Proposition 2.1.8 f is then an admissible epimorphism. □

Proposition 2.2.3. *Let $\mathcal{G}$ be a generating collection in an exact category $\mathcal{E}$. A complex*

$$0 \longrightarrow E \xrightarrow{e'} E \xrightarrow{e''} E''$$

with e'' weakly left admissible is admissibly acyclic if and only if for each $G \in \mathcal{G}$ the sequence

$$0 \longrightarrow Hom(G, E') \longrightarrow Hom(G, E) \longrightarrow Hom(G, E'')$$

is acyclic in $\mathcal{A}b$. If in addition the objects of $\mathcal{G}$ are projective, then a sequence

$$E \xrightarrow{e'} E \xrightarrow{e''} E''$$

with e'' weakly left admissible is admissibly acyclic if and only if for each $P \in \mathcal{G}$ the sequence

$$Hom(P, E') \longrightarrow Hom(P, E) \longrightarrow Hom(P, E'')$$

is acyclic in $\mathcal{A}b$.

Proof. Suppose that for each $G \in \mathcal{G}$ the sequence

$$0 \longrightarrow \mathrm{Hom}(G, E') \longrightarrow \mathrm{Hom}(G, E) \longrightarrow \mathrm{Hom}(G, E'')$$

is acyclic in $\mathcal{A}b$. Since e'' is weakly left admissible it is sufficient to show that e' is a kernel of e''. Then e' is automatically an admissible monic. To show this one can follow the proof in [72]. At one point in that proof the existence of a resolution of X by objects of $\oplus\mathcal{G}$ is used. Here instead we may use Lemma 2.1.47

Finally let us consider the assertion about projective generators. Proposition 2.1.24 implies that

$$\mathrm{Hom}(P,E') \longrightarrow \mathrm{Hom}(P,E) \longrightarrow \mathrm{Hom}(P,E'')$$

is acyclic. For the converse first consider the sequence

$$0 \longrightarrow \mathrm{Ker}(e'') \longrightarrow E \xrightarrow{e''} E''$$

Since $\mathrm{Hom}(P,-)$ preserves kernels, Proposition 2.2.2 implies that

$$E \xrightarrow{e'} E \xrightarrow{e''} E''$$

is admissibly acyclic. □

In particular if $\mathcal{E}$ is quasi-abelian, then every morphism is weakly admissible, so in this case one has that a sequence

$$E \xrightarrow{e'} E \xrightarrow{e''} E''$$

is admissibly acyclic if and only if for each $P \in \mathcal{P}$ the sequence

$$\mathrm{Hom}(P,E') \longrightarrow \mathrm{Hom}(P,E) \longrightarrow \mathrm{Hom}(P,E'')$$

is acyclic in $\mathcal{A}b$. For general exact categories we still have the following result.

Corollary 2.2.4. *Let $\mathcal{G}$ be a projective generating collection in an exact category $\mathcal{E}$. Let $X_\bullet$ be a complex. Suppose that $X_\bullet$ is good. Then $X_\bullet$ is acyclic if and only if Hom$(G,X_\bullet)$ is acyclic for each $G \in \mathcal{G}$.*

Proof. Since each $G \in \mathcal{G}$ is projective the functors $\mathrm{Hom}(G,-)$ preserve acyclic complexes. Conversely suppose $\mathrm{Hom}(G,X_\bullet)$ is acyclic for each $G \in \mathcal{G}$, and d_n^X has a kernel Z_nX. By assumption $\mathrm{Hom}(G,d'_{n+1}) : \mathrm{Hom}(G,X_{n+1}) \to Z_n\mathrm{Hom}(G,X) = \mathrm{Hom}(G,Z_nX)$ is an epimorphism for each n. Thus $d'_{n+1} : X_{n+1} \to Z_nX$ is an admissible epimorphism. Now apply Proposition 2.1.28. □

2.2.1 Elementary Exact Categories

It is convenient to have generators satisfying some compactness conditions. Recall that a poset $\mathcal{J}$ is said to be λ-**filtered** for a cardinal λ if any subset S of $\mathcal{J}$ with $|S| < \lambda$ has an upper bound.

Definition 2.2.5. *Let $\mathcal{E}$ be an additive category, $\mathcal{S}$ a class of morphisms in $\mathcal{E}$, and κ a cardinal. An object E of $\mathcal{E}$ is said to be*

1. *$(\mathcal{S},\kappa)$-**small** if the canonical map $lim_{\to_\lambda} Hom(E,F_i) \to Hom(E, lim_{\to_{i\in\mathcal{I}}} F_i)$ is an isomorphism for any regular cardinal $\lambda \geqslant \kappa$ and any λ-indexed transfinite sequence.*

2. *$\mathcal{S}$-**small** if it is $(\mathcal{S},\kappa)$-**small** for some cardinal κ*

3. *$(\mathcal{S},\kappa)$-**compact** if the natural map*

$$lim_{\to_{i\in\mathcal{I}}} Hom(E,F_i) \to Hom(E, lim_{\to_{i\in\mathcal{I}}} F_i)$$

is an isomorphism for any λ-filtered inductive system $E : \mathcal{I} \to \mathcal{E}$ whose direct limits exists where $\lambda \geqslant \kappa$ is regular, and such that $E(\alpha) \in \mathcal{S}$ for any morphism α in $\mathcal{I}$.

4. *$\mathcal{S}$-compact if it is $(\mathcal{S},\kappa)$-compact for some cardinal κ.*

5. *$\mathcal{S}$-**tiny** if it is $(\mathcal{S},\kappa)$-compact for alll cardinals κ (i.e. if $Hom(E,-)$ commutes with all direct limits whose morphisms are in $\mathcal{S}$).*

6. ***tiny** if it is $\mathcal{S}$-**tiny** for $\mathcal{S} = \mathcal{M}or(\mathcal{E})$.*

The terminology is inspired by [72].

Definition 2.2.6. *Let $\mathcal{E}$ be an exact category $\mathcal{E}$ is said to be*

1. ***projectively generated** if it has a projective generating set.*

2. *$\mathcal{S}$-**elementary** if it is complete, cocomplete and has a projective generating set consisting of $\mathcal{S}$-tiny objects.*

3. ***quasi-elementary** if it is complete, cocomplete and has a projective generating set consisting of $\mathcal{S}$-tiny objects, where $\mathcal{S}$ is the class of split monomorphisms.*

4. ***AdMon-elementary** if it is elementary for the class of admissible monomorphisms.*

5. ***elementary** if it is $\mathcal{S}$-elementary for $\mathcal{S} = \mathcal{M}or(\mathcal{E})$.*

Proposition 2.2.7. *A cocomplete quasi-abelian category is (quasi)-elementary if and only if its left heart is (quasi)-elementary.*

Proof. See [72] Proposition 2.1.12. □

The following proposition is immediate from Proposition 2.2.3 and Corollary 2.2.4 but it has a useful consequence.

Proposition 2.2.8. *Let $\mathcal{E}$ be a complete and cocomplete elementary(resp. quasi-elementary) exact category. Then filtering inductive limits (resp. direct sums) in $\mathcal{E}$ are exact. If in addition $\mathcal{E}$ is quasi-abelian elementary (resp. quasi-elementary), then filtering inductive limits (resp. direct sums) are admissibly exact.*

This also motivates a more general definition 2.2.9 below.

Definition 2.2.9. *Let $\mathcal{E}$ be an exact category and $\mathcal{S}$ a collection of morphisms in $\mathcal{E}$. $\mathcal{E}$ is said to be*

1. ***weakly $\mathcal{S}$-elementary** if for any ordinal λ $\mathcal{E}$ has $(\lambda;\mathcal{S})$-colimits and $(\lambda;\mathcal{S})$-colimits are exact.*

2. ***weakly AdMon-elementary** if it is weakly $\mathcal{S}$-elementary for $\mathcal{S}$ = **AdMon**.*

3. ***weakly elementary** if it is weakly $\mathcal{S}$-elementary for $\mathcal{S} = \mathcal{M}or(\mathcal{E})$.*

In particular $\mathcal{S}$-elementary exact categories are weakly $\mathcal{S}$-elementary.

Proposition 2.2.10. *Let $\mathcal{E}$ be a weakly **AdMon**-elementary exact category. Then transfinite compositions of admissible monics are admissible monics.*

Proof. The proof is by transfinite induction. Since finite compositions of admissible monics are admissible, the successor case is clear. For the limit case let Λ be a limit ordinal, and consider the commutative diagram

$$\begin{array}{ccccccc}
E_0 & \longrightarrow & E_0 & \longrightarrow & E_0 & \longrightarrow & \dots \\
\downarrow & & \downarrow & & \downarrow & & \\
E_0 & \xrightarrow{c_\lambda} & E_\lambda & \longrightarrow & E_{\lambda'} & \longrightarrow & \dots \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Coker}(c_\lambda) & \longrightarrow & \mathrm{Coker}(c_{\lambda'}) & \longrightarrow & \dots
\end{array}$$

with short exact columns. Taking the direct limit over Λ, we get a short exact sequence

$$0 \to E_0 \to E \to C \to 0$$

In particular $E_0 \to E$ is admissible. □

2.2.2 Generators in Categories of Chain Complexes

Our goal now is to show that if $\mathcal{E}$ is an elementary exact category then so is $Ch_*(\mathcal{E})$, for $* \in \{+, \leqslant 0, -, b, \geqslant 0, \varnothing\}$. Much of this is based on the following technical result.

Lemma 2.2.11. *Let $\mathcal{E}$ be a weakly idempotent complete exact category. For any object $C \in \mathcal{E}$ and $X, Y \in Ch(\mathcal{E})$ we have natural isomorphisms:*

1. *$Hom_{\mathcal{E}}(C, Y_n) \cong Hom_{Ch(\mathcal{E})}(D^n(C), Y)$*

2. *$Hom_{\mathcal{E}}(X_{n-1}, C) \cong Hom_{Ch(\mathcal{E})}(X, D^n(C))$*

3. *$Ker(Hom_{\mathcal{E}}(C, d_n^Y)) \cong Hom_{Ch(\mathcal{E})}(S^n(C), Y)$. In particular if $Ker(d_n^Y)$ exists then $Hom_{\mathcal{E}}(C, Ker(d_n^Y)) \cong Hom_{Ch(\mathcal{E})}(S^n(C), Y)$*

4. *$Ker(Hom_{\mathcal{E}}(d_{n+1}^X, C)) \cong Hom_{Ch(\mathcal{E})}(X, S^n(C))$ In particular if $Coker(d_{n+1}^X)$ exists then $Hom_{\mathcal{E}}(Coker(d_{n+1}^X), C) \cong Hom_{Ch(\mathcal{E})}(X, S^n(C))$*

5. *$Ext^1_{\mathcal{E}}(C, Y_n) \cong Ext^1_{Ch(\mathcal{E})}(D^nC, Y)$*

6. *$Ext^1_{\mathcal{E}}(X_n, C) \cong Ext^1_{Ch(\mathcal{E})}(X, D^{n+1}C)$*

7. *Let X be a complex such that $Kerd_n^X$ exists. Then there is a monomorphism*

$$Ext^1(C, Ker(d_n^X)) \hookrightarrow Ext^1(S^nC, X)$$

If X is acyclic then this is an isomorphism.

8. *Let X be a complex such that $Coker(d_{n+1}^X)$ exists. Then there is a monic*

$$Ext^1(Coker(d_{n+1}^X), C) \hookrightarrow Ext^1(X, S^nC)$$

If X is acyclic then this is an isomorphism.

Proof. By Proposition 2.1.40 and Corollary 2.1.58 it is sufficient to prove statements $1-3, 5, 6, 7$ under the assumption that $\mathcal{E}$ is abelian. In this context the result is Lemma 3.1 in [28] and Lemma 4.2 in [30]. Statement 4 is dual to to 3, and statement 8 is dual to 7. □

Remark 2.2.12. *It is possible to prove most of this lemma internally in an exact category without passing to an abelianisation.*

At this point we can prove the following lemma. It provides one of our main applications of generating sets, namely a convenient method for testing acyclicity. It is a modification of Lemma 3.7 in [29].

Lemma 2.2.13. *Let $\mathcal{E}$ be an exact category with a collection of generators $\mathcal{G}$. Let X be a chain complex. Suppose that $X_\bullet$ is good. If for every $G \in \mathcal{G}$ each map $f : S^n(G) \to X$ extends to $D^{n+1}(G)$, then X is acyclic.*

Proof. By Proposition 2.1.28 it is enough to show that whenever d_m has a kernel, the induced map

$$d' : X_{m+1} \to Z_mX$$

is an admissible epic. For this it is enough to show that for each $G \in \mathcal{G}$,

$$\text{Hom}(G, d') : \text{Hom}(G, X_{m+1}) \to \text{Hom}(G, Z_mX)$$

is surjective, i.e. that any map $f : G \to Z_m X$ lifts to a diagram

$$\begin{array}{ccc} & & X_{m+1} \\ & \nearrow & \downarrow{\scriptstyle d'} \\ G & \xrightarrow{f} & Z_n X \end{array}$$

But this is equivalent to showing that the chain map $S^n(G) \to X$ induced by f extends to a morphism $D^{n+1}(G) \to X$.

□

Next we characterise projective objects in categories of chain complexes. It is well known that projective objects in the category of chain complexes in an abelian category are precisely the split exact complexes with projective entries. See for example [41] Proposition 2.3.10. We generalise the result to exact categories.

Proposition 2.2.14. *Let $\mathcal{E}$ be a weakly idempotent complete exact category, and let $* \in \{\geqslant 0, \leqslant 0, +, -, b, \varnothing\}$. Then split exact complexes of projectives are projective objects in $Ch_*(\mathcal{E})$. In addition, if P is projective in $\mathcal{E}$ then $S^0(P)$ is projective in $Ch_{\geqslant 0}(\mathcal{E})$. Conversely, if a complex $X_\bullet$ is a projective in $Ch_*(\mathcal{E})$ for $* \in \{+, -, b, \geqslant 0, \leqslant 0, \varnothing\}$ then every X_n is projective. Moreover, if $* \in \{+, -, b, \varnothing\}$ and $X_\bullet$ is good then $X_\bullet$ is a split exact complex of projective objects of $\mathcal{E}$. In particular if $\mathcal{E}$ has all kernels then the projective objects in $Ch_*(\mathcal{E})$, for $* \in \{+, -, b, \varnothing\}$ are precisely the split exact complexes of projectives contained in $Ch_*(\mathcal{E})$.*

Proof. By Lemma 2.2.11, split exact complexes of projectives are projective objects in $Ch_*(\mathcal{E})$ for $* \in \{+, -, b, \geqslant 0, \leqslant 0, \varnothing\}$. Let us show that $S^0(P)$ is a projective object in $Ch_{\geqslant 0}(\mathcal{E})$ whenever P is projective in $\mathcal{E}$. Indeed in this case, Lemma 2.2.11 implies that $\mathrm{Hom}_{Ch(\mathcal{E})}(S^0(P), Y_\bullet) \cong \mathrm{Hom}_{\mathcal{E}}(P, Y_0)$. Since P is projective, $S^0(P)$ is as well. Conversely if a complex $X_\bullet$ is a projective complex, then it follows immediately from Lemma 2.2.11 that each X_n is projective in $\mathcal{E}$. Suppose that $* \in \{+, -, b, \varnothing\}$ and that $X_\bullet$ is good. Let P be the cone of the identity $id : X_\bullet \to X_\bullet$. Consider the surjection $P \to X_\bullet[-1]$. Since $X_\bullet[-1]$ is projective this map splits by Proposition 2.1.45. The second factor of this splitting gives a homotopy between $id_{X_\bullet[-1]}$ and the 0 map. By Corollary 2.1.35, $X_\bullet[-1]$ is split acyclic so $X_\bullet$ is as well. □

We can now show that $Ch_*(\mathcal{E})$ has enough projectives. (This is well known for $Ch(\mathcal{A})$ with $\mathcal{A}$ abelian. See for example [83] Exercise 2.2.2).

Corollary 2.2.15. *Let $\mathcal{E}$ be an exact category with enough projectives. Then $Ch_*(\mathcal{E})$ has enough projectives for $* \in \{+, -, b, \leqslant 0, \geqslant 0, \varnothing\}$*

Proof. By Proposition 2.2.14 $D^n(P)$ is projective in $Ch_*(\mathcal{E})$ for $* \in \{+, -, b, \varnothing\}$ whenever P is projective. Also $D^n(P)$ for $n \leqslant 0$ is projective in $Ch_{\leqslant 0}(\mathcal{E})$. Let $X_\bullet \in Ch_*(\mathcal{E})$ for $* \in \{+, -, b, \leqslant 0, \varnothing\}$. For each n pick a projective P_n and an admissible epimorphism $P_n \twoheadrightarrow X_n$. This induces a map $D^n(P_n) \to X_\bullet$ which is an admissible epimorphism in degree n. Let $P_\bullet = \bigoplus_n D^n(P_n)$. By the above discussion we have an admissible epimorphism $P_\bullet \twoheadrightarrow X_\bullet$.

Now let $X_\bullet \in Ch_{\geqslant 0}(\mathcal{E})$. For $n > 0$ the object $D^n(P)$ is projective in $Ch_{\geqslant 0}(\mathcal{E})$. $S^0(P)$ is also projective in $Ch_{\geqslant 0}(\mathcal{E})$. For $n > 0$, as before there is a projective object P_n and a morphism $D^n(P_n) \to X_\bullet$ which is an admissible epimorphism in degree n. For $n = 0$ pick a projective object P_0 and an admissible epimorphism $P_0 \to X_0$. Since $X_{-1} = 0$, this induces a map $S^0(P_0) \to X_\bullet$ which is an admissible epimorphism in degree 0. Let $P_\bullet = \Big(\bigoplus_{n>0} D^n(P_n)\Big) \oplus S^0(P_0)$. Then we have an admissible epimorphism $P_\bullet \twoheadrightarrow X_\bullet$.

□

In particular we have shown that $Ch_*(\mathcal{E})$ has a set of projective generators whenever $\mathcal{E}$ does.

Corollary 2.2.16. *Suppose $\mathcal{P}$ is a collection of admissible generators for an exact category $\mathcal{E}$. Then $D^*(\mathcal{P}) = \{D^n(P) : P \in \mathcal{P}, n \in \mathbb{Z}\} \cap Ch_*(\mathcal{E})$ is a collection of generators for $Ch_*(\mathcal{E})$ and $* \in \{+, -, b, \leqslant 0, \varnothing\}$. For $* \in \{\geqslant 0\}$, $\tilde{D}^*(\mathcal{P}) := D^*(\mathcal{P}) \cup \{S^0(P) : P \in \mathcal{P}\}$ is a collection of generators for $Ch_*(\mathcal{E})$. They are projective generating collections if $\mathcal{P}$ is.*

Proof. The proof of Corollary 2.2.15 shows that the collection in the statement of the proposition are admissible generating collection. Proposition 2.2.14 establishes the second assertion. □

We are nearly ready to show that $Ch_*(\mathcal{E})$ is elementary for $* \in \{+, \geqslant 0, \leqslant 0, -, b, \varnothing\}$. It remains to identify some suitably compact objects in complexes. However by Lemma 2.2.11 we have the following.

Proposition 2.2.17. *Let E be an object satisfying one of the smallness conditions of Definition 2.2.5. Then $D^n(E)$ and $S^n(E)$ satisfy the same smallness condition in $Ch(\mathcal{E})$.*

As a consequence we have

Corollary 2.2.18. *Let $\mathcal{E}$ be an elementary exact category. Then $Ch_*(\mathcal{E})$ is elementary for $* \in \{+, \leqslant 0, \geqslant 0, -, b, \varnothing\}$.*

Proof. Let $\mathcal{P}$ be a projective generating set consisting of compact objects. The sets $D^*(\mathcal{P})$ (resp. $\tilde{D}^*(\mathcal{P})$) are projective generating sets in $Ch_*(\mathcal{E})$ for $* \in \{\leqslant 0, +, -, b, \varnothing\}$ (resp. $* \in \{\geqslant 0\}$). For each $n \in \mathbb{Z}$ $D^n(P)$ is tiny, as is $S^n(P)$, by Proposition 2.2.17. □

2.2.3 Generators in Monoidal Exact Categories

Let us briefly mention a useful compatibility condition between generators and monoidal structure.

Definition 2.2.19. *A monoidal exact category which has a collection of flat admissible generators is said to be* ***flatly generated****.*

Definition 2.2.20. *A projectively monoidal exact category which is also (quasi)-elementary is said to be* ***monoidal (quasi)-elementary***

Proposition 2.2.21. *Suppose that $(\mathcal{E}, \otimes, k)$ is a flatly generated monoidal exact category in which direct sums are exact. Then every projective object is flat.*

Proof. In this case every projective will be a summand of a flat object, and therefore flat. □

In particular to check that a category is projectively monoidal, it suffices to find a collection of flat generators.

2.2.4 Generators and Adjunctions

We conclude this section with a note about passing generating collections through adjunctions. The specific application we have in mind is to categories of algebras over compatible monads. We have the following general setup $F : \mathcal{E} \rightarrow \mathcal{D}$ and $|-| : \mathcal{D} \rightarrow \mathcal{E}$ are additive functors between exact categories. Moreover these functors form an adjoint pair $F \dashv |-|$. We have the following result which is standard for abelian categories.

Proposition 2.2.22. *Let $F \dashv |-|$ be an adjunction as above. Suppose that $|-|$ is an exact functor. If P is a projective object of $\mathcal{E}$ then $F(P)$ is a projective object of $\mathcal{D}$.*

Proof. Let

$$0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$$

be a short exact sequence in $\mathcal{D}$, and let P be projective in $\mathcal{E}$. Then we have a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \text{Hom}(F(P), X) & \longrightarrow & \text{Hom}(F(P), Y) & \longrightarrow & \text{Hom}(F(P), Z) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \text{Hom}(P, |X|) & \longrightarrow & \text{Hom}(P, |Y|) & \longrightarrow & \text{Hom}(P, |Z|) & \longrightarrow & 0 \end{array}$$

The vertical arrows are isomorphisms and the bottom row is exact since $|-|$ is exact and P is projective. Hence the top row is short exact as well. □

We know how adjunctions act on projectives. Let us now see what happens on generating collections.

Proposition 2.2.23. *Let $F \dashv |-|$ be an adjunction as above. Suppose that $|-|$ reflects admissible epimorphisms, and that $\mathcal{E}$ has an admissible generating collection $\mathcal{G}$. Let $F(\mathcal{G})$ denote the collection $\{F(G) : G \in \mathcal{G}\}$ of objects of $\mathcal{D}$. Then $F(\mathcal{G})$ is an admissible generating collection in $\mathcal{D}$.*

Proof. Let X be an object of $\mathcal{D}$. Suppose there is some object Q of $\mathcal{E}$ and an admissible epimorphism $p : Q \to |X|$. There is an induced morphism $\tilde{p} : F(Q) \to X$. Then p coincides with the composition $Q \to |F(Q)| \to |X|$. By Proposition 2.1.8, the map $|\tilde{p}|$ is an admissible epimorphism. Since $|-|$ reflects admissible epimorphisms, $\tilde{p}$ is an admissible epimorphism in $\mathcal{D}$.

Now let $\mathcal{G}$ be an admissible generating collection in $\mathcal{E}$, and let X be an object of $\mathcal{D}$. Since $\mathcal{G}$ is an admissible generating collection, there is an object G of $\bigoplus \mathcal{G}$ and an admissible epimorphism $G \twoheadrightarrow |X|$. The induced morphism $F(G) \to X$ is an admissible epimorphism by the above remarks. Since F is a left adjoint it preserves colimits, so $F(G)$ is an element of $\bigoplus F(\mathcal{G})$. □

Proposition 2.2.24. *Let $F \dashv |-|$ be an adjunction as above.*

1. *Suppose that $|-|$ is exact and reflects admissible epimorphisms. If $\mathcal{G}$ is a projective generating collection in $\mathcal{E}$ then $F(\mathcal{G})$ is a projective generating collection in $\mathcal{D}$.*

2. *Suppose that $|-|$ is exact, reflects epimorphisms and preserves direct sums (resp. filtered colimits). If $\mathcal{E}$ satisfies any of the smallness properties of Definition 2.2.6 then so does $\mathcal{D}$.*

Proof.
1. The first assertion follows from Proposition 2.2.22 and Proposition 2.2.23.

2. This follows since $|-|$ preserves direct sums (resp. filtered colimits).

□

Example 2.2.25. *Let T be a compatible monad on an exact category $\mathcal{E}$. Then the forgetful functor $|-| : \mathcal{E}^T \to \mathcal{E}$ has a right adjoint $F : \mathcal{E} \to \mathcal{E}^T$ assigning to an object the free T-algebra on it. By construction of the exact structure on $\mathcal{E}^T$ in Proposition 2.1.69, the functor $|-|$ is admissibly exact and reflects exactness. Moreover it creates limits and colimits. By Lemma 2.1.70, Proposition 2.2.24 is applicable in such categories.*

2.3 Examples

In the final section of this chapter we give examples of interesting exact categories which satisfy have very different set-theoretic properties but which are all weakly **AdMon**-elementary. In the next section we shall see that $\mathcal{E}$ being weakly **AdMon**-elementary and having kernels is enough for the category $Ch(\mathcal{E})$ to be equipped with the projective model structure. The moral of the story is that often difficult to check set-theoretic assumptions can be ignored to some extent when discussing such model structures.

2.3.1 Categories of Topological Vector Spaces

In this section we let k be a Banach ring, that is, a unital commutative ring k together with a map $|-|: k \to \mathbb{R}_{>0}$ such that for all $x, y \in k$ we have

1. $|x| = 0 \Leftrightarrow x = 0$
2. $|x + y| \leqslant |x| + |y|$
3. $|xy| \leqslant |x||y|$
4. k is complete with respect to the topology defined by $|-|$.

k is said to be **non-Archimedean** if $|x + y| \leqslant \max\{|x|, |y|\}$ and **Archimedean** otherwise. Over such rings we can consider categories of topological k-modules. For details of claims made in this section consult [72], [4], [10], and [5].

Categories of Normed and Banach Modules

Definition 2.3.1. *A **normed** k**-module** is a k-module V together with a map $||-|| : V \to \mathbb{R}_{>0}$ such that for all $\lambda \in k$ and for all $x, y \in V$ we have*

1. $||x|| = 0 \Leftrightarrow x = 0$
2. $||x + y|| \leqslant ||x|| + ||y||$
3. $||\lambda x|| \leqslant |\lambda|||x||$

*If V is complete with respect to the metric defined by $||-||$ then V is said to be a **Banach** k**-module**.*

If k is non-Archimedean then V is said to be non-Archimedean if $||x+y|| \leqslant max\{||x||, ||y||\}$. We denote by $Norm_k$ the category whose objects are normed k-modules and whose morphisms are bounded k-linear maps. Ban_k is the full subcategory of $Norm_k$ on Banach k-modules. For k non-Archimedean we also consider the full subcategories $Norm_k^{nA}$ and

Ban_k^{nA} of non-Archimedean normed and Banach spaces respectively. All of these categories are additive, finitely complete, and finitely cocomplete. The inclusions

$$Ban_k \rightarrow Norm_k, \;\; Ban_k^{nA} \rightarrow Norm_k^{nA}$$

have left-adjoint functors given by completion.

They are also symmetric monoidal. If E and F are objects in $Norm_k$ then we define $E \otimes_\pi F$ to be their usual module tensor product endowed with the cross-norm

$$||u|| = \inf\Big\{\sum_{i=1}^{n} ||e_i|| ||f_i|| : u = \sum_{i=1}^{n} e_i \otimes f_i\Big\}$$

If E and F are objects in $Norm_k^{nA}$ we define $E \otimes_\pi^{nA} F$ to be their usual module tensor product endowed with the norm.

$$||x||_\pi = \inf\Big\{\max\{||a_i|| ||b_i||\}_{i=1}^{n} : \; x = \sum_{i=1}^{n} a_i \otimes b_i\Big\}$$

We refer to both of these constructions as the **projective tensor product**. If E and F are Banach spaces then $E\hat{\otimes}_\pi F$ is the completion of their projective tensor product as normed spaces. These constructions are functorial in each of the categories defined above and form part of symmetric monoidal structures on them with unit the ground ring k. These monoidal structures are in fact closed. The module $\text{Hom}_k(E, F)$ of bounded maps between E and F can be given the structure of a normed space. The norm of $T : E \rightarrow F$ is

$$||T|| = \sup_{e \in E \setminus \{0\}} \frac{||T(e)||_F}{||e||_E}$$

This gives an internal Hom functor, which we denote by $\underline{\text{Hom}}$. Thus $(Ban_k, \hat{\otimes}_\pi, \underline{\text{Hom}})$ is a monoidal quasi-abelian category. Finally, the projective objects $l^1(I)$ are flat by [4]. By Proposition 2.2.21 this category is projectively monoidal. There are unfortunately some problems with this category. Although it is finitely complete and cocomplete it does not even have countable colimits in general. The larger category $\hat{\mathcal{T}}_c$ of complete locally convex topological spaces is complete and cocomplete, but tragically it is not quasi-abelian ([72]). Instead we pass to the formal completion $Ind(Ban_k)$ of Ban_k by filtered colimits.

Ind and Pro Categories

Recall that if $\mathcal{C}$ is a $\mathbb{U}$-compact category for some universe $\mathbb{U}$, and $\mathbb{V}$ is a universe, then the $\mathbb{V}$-ind-completion of $\mathcal{C}$ is a category constructed as follows. Objects are diagrams $E : \mathcal{I} \rightarrow \mathcal{C}$ where $\mathcal{I}$ is a $\mathbb{V}$-compact filtrant category. If $E : \mathcal{I} \rightarrow \mathcal{C}$ and $F : \mathcal{J} \rightarrow \mathcal{C}$ are objects in $Ind(\mathcal{C})$ (where we suppress universes in the notation) then we write

$$\text{Hom}_{Ind(\mathcal{C})}(E, F) = \lim_{\leftarrow \mathcal{I}} \lim_{\rightarrow \mathcal{J}} \text{Hom}_{\mathcal{C}}(E_i, F_j)$$

Proposition 2.3.2. *Let $\mathcal{E}$ be a quasi-abelian category with enough projectives. Then $Ind(\mathcal{E})$ is a cocomplete elementary quasi-abelian category. Moreover, if $\mathcal{E}$ is a closed monoidal exact category, then its ind-completion has a canonical exact closed monoidal structure extending the one on $\mathcal{E}$. Finally if $\mathcal{E}$ is projectively monoidal then so is $Ind(\mathcal{E})$.*

Proof. See [72] Proposition 2.1.16 and Proposition 2.1.19. □

Corollary 2.3.3. *The category $Ind(Ban_k)$ is a locally presentable, closed monoidal elementary quasi-abelian category.*

The category $Ind(Ban_k)$ is not concrete. However it does have a natural concrete full subcategory $Ind^m(Ban_k)$. An object of $Ind^m(Ban_k)$ is a formal colimit "$lim_{\rightarrow}$"E_i such that any map $E_i \rightarrow E_j$ is a monomorphism (not necessarily admissible!). It is shown in [4] that this category is equivalent to the concrete category $CBorn_k$ of complete **bornological** k-modules. These are spaces equipped with an appropriate notion of 'bounded subsets'. To a (complete) locally convex space E one can functorially assign both the von Neumann bornology $vN(E)$ and the compact bornology $Cpt(E)$. The von Neumann bornology is composed of the subsets of E absorbed by all zero neighbourhoods. The compact bornology is composed of subsets with compact closure. There is a natural transformation of functors $Cpt \rightarrow vN$. For details see [54].

There is also the dual notion of the $\mathbb{V}$-pro-completion of $\mathcal{C}$, which is defined to be

$$Pro(\mathcal{C}) = Ind(\mathcal{C}^{op})^{op}$$

It is the formal completion of $\mathcal{C}$ by projective limits.

For k a Banach ring $Pro(Ban_k)$ contains $\hat{\mathcal{T}}_{c,k}$ as a full subcategory. Indeed if E is an object of $\hat{\mathcal{T}}_{c,k}$ defined by a family of seminorms $\mathcal{P}$ then define $PB(E) =$ "$lim_{\leftarrow_{p\in\mathcal{P}}}$"$\hat{E}_p$ where $\hat{E}_p$ is the completion of E with respect to the metric defined by the semi-norm p. This construction is functorial, lax monoidal, and $PB : \hat{\mathcal{T}}_{c,k} \rightarrow Pro(Ban_k)$ is fully faithful.

If $\mathcal{E}$ is a quasi-abelian category enough projectives and injectives then by Proposition 2.1.15 in [72] both $Ind(\mathcal{E})$ and $Pro(\mathcal{E})$ are both complete and cocomplete. In particular by an obvious Kan extension there is a canonical functor

$$PI : Pro(\mathcal{E}) \rightarrow Ind(\mathcal{E})$$

Again this is lax monoidal. There is a natural isomorphism of functors $PI \circ PB \cong vN$ (see [5]), and therefore a natural transformation $Cpt \rightarrow PI \circ PB$. Let $k = \mathbb{C}$ In [68] for $k = \mathbb{C}$ the composite functor

$$\hat{\mathcal{T}}_c \xrightarrow{Cpt} CBorn_{\mathbb{C}} \longrightarrow Ind(Ban_{\mathbb{C}})$$

is denoted IB. It is lax monoidal. When restricted to the category of nuclear Fréchet spaces it is strong monoidal and fully faithful (again see [68] and [54]). Moreover in this case the functor $Cpt \to PI \circ PB$ is an isomorphism. In particular the category of nuclear Fréchet algebras over $\mathbb{C}$ embeds fully faithfully in the category of commutative complete bornological algebras. Since the category $CBorn_{\mathbb{C}}$ has good categorical properties, in particular it is closed monoidal elementary, this is evidence that it provides a convenient setting in which to study analytic algebra.

2.3.2 The Non-Expanding Normed and Banach Categories

Each of the normed and Banach categories considered in the previous section has a corresponding 'non-expanding' subcategory. If E and F are normed spaces and $s \in \mathbb{R}_{\geqslant 0}$ then we denote by $Hom_k^{\leqslant s}(E,F) \subset Hom_k(E,F)$ the set of maps of k-modules of norm at most s. Composition gives a map

$$Hom_k^{\leqslant r}(E,F) \otimes Hom_k^{\leqslant s}(F,G) \to Hom_k^{\leqslant rs}(F,G)$$

In particular there wide subcategories of $Norm_k, Ban_k, Norm_k^{nA}, Ban_k^{nA}$ consisting of maps of norm at most 1 which we denote by $Norm_k^{\leqslant 1}, Ban_k^{\leqslant 1}, Norm_k^{nA,\leqslant 1}, Ban_k^{nA,\leqslant 1}$. They are equipped with closed symmetric monoidal structures by restricting the ones on the larger categories. If k is non-Archimedean then these categories are also additive and in fact quasi-abelian.

In both the Archimedean and non-Archimedean case these categories are complete and co-complete. Details of this can be found in [10]. For convenience we recall how to construct arbitrary coproducts in $Ban_k^{\leqslant 1}$ and, for k non-Archimedean, $Ban_k^{nA,\leqslant 1}$. For k Archimedean the coproduct $\coprod_{i\in I}^{\leqslant 1} A_i$ of a collection $\{A_i\}_{i\in I}$ of Banach spaces in $Ban_k^{\leqslant 1}$ is

$$\{(a_i)_{i\in I} \subset \prod_{i\in I}^{k\mathcal{M}od} A_i : \sum_{i\in I} ||a_i|| < \infty\}$$

with the norm $||(a_i)|| = \sum_{i\in I} ||a_i||$. Here $\prod^{k\mathcal{M}od}$ denotes the set-theoretic product in the category of k-modules. For k non-Archimedean the coproduct in both $Ban_k^{\leqslant 1,nA}$ and $Ban_k^{\leqslant 1}$ the coproduct $\coprod_{i\in I}^{\leqslant 1} A_i$ of a collection $\{A_i\}_{i\in I}$ of Banach spaces is the subspace

$$\{(a_i)_{i\in I} \subset \prod_{i\in I}^{k\mathcal{M}od} A_i : \lim_{i\in I} ||v_i|| = 0\}$$

endowed with the norm $||(a_i)_{i\in I}|| = \sup_{i\in I} ||a_i||$.

Rescaling Functors

For $r \in \mathbb{R}_{>0}$ we denote by $(-)_r : Norm_k \to Norm_k$ the endofunctor which sends a normed space E to E_r which has the same underlying k-vector space as E but with norm rescaled by r. On morphisms it does nothing. It is evidently an autoequivalence, and in fact an automorphism, with inverse given by $(-)_{\frac{1}{r}}$. Moreover it restricts to an auto-equivalence on all the normed and Banach categories defined above. These functors satisfy the following sueful property.

Proposition 2.3.4. *Let E and F be Banach k-modules. $Hom_k^{\leqslant r}(E_s, F_t) = Hom_k^{\leqslant \frac{sr}{t}}(E, F)$*

Proof. Let $f : E_s \to F_t$ have norm at most r, so that for any $e \in E$,

$$||f(e)||_F = \frac{1}{t}||f(e)||_{F_t} \leqslant \frac{r}{t}||e||_{E_s} = \frac{sr}{t}||e||_E$$

Conversely suppose $f : E \to F$ has norm at most $\frac{sr}{t}$. Then we get the same inequality as above. □

The Quasi-Abelian Exact Structure

Proposition 2.3.5. *In both $Norm_k^{\leqslant 1,nA}$ and $Ban_k^{\leqslant 1,nA}$ we have the following.*

1. *A monomorphism $f : A \to B$ is admissible in the quasi-abelian exact structure if and only if it is an isometry with closed image.*

2. *An epimorphism $g : B \to C$ is admissible in the quasi-abelian exact structure if and only if it is a set-theoretic epimorphism and $||g(b)|| = inf_{a \in Ker(g)}||b - a||$.*

Proof.
1. Suppose that $f : A \to B$ is admissible in the quasi-abelian exact structure. Then it is the kernel of its cokernel $g : B \to C$. Therefore f induces an isometric isomorphism with the normed subspace $K = \{b \in B : g(b) = 0\}$. In particular f is an isometry. Conversely suppose that f is an isometry with closed image. Then $A \cong f(A)$ in $Norm_k^{\leqslant 1,nA}$. The cokernel of f is isometrically isomorphic to the quotient space $B/f(A)$, and the kernel of $B \to B/f(A)$ is $\{b \in B : g(b) = 0\} = f(A) \cong A$.

2. Suppose that $g : B \to C$ is an admissible epimorphism in the quasi-abelian exact structure. Then it is the cokernel of its kernel, which is the subspace $A = \{b \in B : g(b) = 0\}$. In particular g induces an isometric isomorphsim $\overline{g} : B/A \cong C$. So

$$||g(b)|| = ||[b]|| = \inf_{a \in A}||b - a||$$

Moreover $B \to B/A$ is a set-theoretic epimorphism, so g is as well. Conversely suppose that g is a set-theoretic epimorphism, and that $||g(b)|| = \inf_{a \in \mathrm{Ker}(g)} ||b - a||$. Then g clearly induces an isometric isomorphism.

$\square$

Remark 2.3.6. *In the case of $Ban_k^{\leqslant 1, nA}$ we may remove the assumption in Proposition 2.3.5 1) that f has closed image, since an isometry of Banach spaces always has closed image.*

The Strong Exact Structure

We introduce a different exact structure on $Norm_k^{\leqslant 1, nA}$ (resp. $Ban_k^{\leqslant 1, nA}$).

Definition 2.3.7.

1. *We say that a morphism $f : A \to B$ in $Norm_k^{\leqslant 1, nA}$ is a **strong monomorphism** if it is an isometry, and any $b \in B$ has a closest point $a_b \in f(A)$.*
2. *We say that a morphism $g : B \to C$ in $Norm_k^{\leqslant 1, nA}$ is a **strong epmorphism** if for any $c \in C$ there is a $b_c \in B$ with $g(b_c) = c$ and $||b_c|| = ||c||$.*
3. *We say that a morphism $f : A \to B$ in $Ban_k^{\leqslant 1, nA}$ is a **strong monomorphism** (resp. **strong monomorphism**) if it is a strong monomorphism (resp. strong epimorphism) in $Norm_k^{\leqslant 1, nA}$.*

Corollary 2.3.8. *A strong monomorphism is an admissible monomorphism in the quasi-abelian exact structure. A strong epimorphism is an admissible epimorphism in the quasi-abelian exact structure.*

Proposition 2.3.9. *A map $f : A \to B$ is a strong monomorphism if and only if it is the kernel of a strong epimorphism. A map $g : B \to C$ is a strong epimorphism if and only if it is the cokernel of a strong monomoprhism.*

Proof. Suppose that $f : A \to B$ is a strong monomorphism. Then in particular it is an admissible monomorphism in the quasi-abelian exact structure so it is the kernel of its cokernel $g : B \to C$. Let us show that g is a strong epimorphism. Let $c = [b] \in C = B/f(A)$. Now $||[b]|| = \inf_{a \in A} ||b - f(a)||$. By assumption there is some a_b such that $||[b]|| = ||b - f(a_b)||$. Moreover $g(b - f(a_b)) = [b]$. So g is a strong epimorphism. Conversely suppose that $f : A \to B$ is the kernel of a strong epimorphism $g : B \to C$. Let $b \in B$. There is $b' \in B$ such that $g(b) = g(b')$ and $||g(b)|| = ||b'||$. Now $b - b' \in A$. We claim that $b - b'$ is a closest point to b in A. Indeed for any $a \in A$

$$||b - (b - b')|| = ||b'|| = ||g(b)|| = ||g(b - a)|| \leqslant ||b - a||$$

$\square$

So we get a class of kernel-cokernel pairs

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

where f is a strong monomorphism and g is a strong epimorphism. We denote this class by *strong*. We are going to prove the following.

Theorem 2.3.10. *The collection strong of strong kernel-cokernel pairs is an exact structure on both $Norm_k^{\leqslant 1,nA}$ and $Ban_k^{\leqslant 1,nA}$.*

We do this in several steps. It is clear that isomorphisms are strong epimorphisms and strong monomorphisms. It is also clear that the projection $A \oplus B \to B$ is a strong epimorphism and the inclusion $A \to A \oplus B$ is a strong monomorphism.

Proposition 2.3.11. *Let*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g'} \\ X & \xrightarrow{f'} & Y \end{array}$$

be a pushout diagram in $Norm_k^{\leqslant 1,nA}$ or $Ban_k^{\leqslant 1,nA}$. If f is a strong monomorphism then so is f'.

Proof. We shall prove it for $Norm_k^{\leqslant 1,nA}$. The case of $Ban_k^{\leqslant 1,nA}$ is similar. The space Y is isometrically isomorphic to the quotient normed space

$$X \oplus B/\{-g(a), f(a)\}$$

Let us show that the map

$$X \to X \oplus B/\{-g(a), f(a)\}, x \mapsto [(x, 0)]$$

is an isometry with closed image. First we show that it is an isometry

$$||[(x,0)]|| = \text{inf}_{a\in A}||(x-g(a), f(a))|| = \text{inf}_{a\in A}\max\{||x-g(a)||, ||f(a)||\} = \text{inf}_{a\in A}\max\{||x-g(a)||, ||a||\}$$

Now if $||a|| < ||x||$ then $||g(a)|| < ||x||$, so $||x - g(a)|| = ||x||$, and $\max\{||x - g(a)||, ||a||\} = ||x||$. If $||a|| \geqslant ||x||$ then $||x - g(a)|| \leqslant ||a||$, and so $\max\{||x - g(a)||, ||a||\} \leqslant ||a||$. Hence $\text{inf}_{a\in A}\max\{||x - g(a)||, ||a||\} = ||x||$, so the map is an isometry.

Now let us show that it has closed image. Let $[(x_n, 0)]$ converge to some $[(x, y)]$. Then for every $\epsilon > 0$ there is an N_ϵ such that for every $n > N$ there is an $a_n \in A$ such that

$$||x - x_n - g(a_n)|| < \epsilon,\ ||y - f(a_n)|| < \epsilon$$

In particular there is a sequence (a_n) in A with $(f(a_n))$ converging to y. Since f has closed image this means $y = f(a)$. Therefore

$$[(x,y)] = [(x, f(a))] = [(x + g(a), 0)]$$

which is in the image of $X \to X \oplus B/\{-g(a), f(a)\}$. □

Proposition 2.3.12. *Let*

$$\begin{array}{ccc} A & \xrightarrow{f'} & B \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Y \end{array}$$

be a pullback diagram in $Norm_k^{\leqslant 1,nA}$ *or* $Ban_k^{\leqslant 1,nA}$. *If f is a strong epimorphism then so is f'.*

Proof. A is (isometrically isomorphic to) the subspace $\{(x,b) : f(a) = g(b)\}$ of $X \oplus B$, with f' being $(x,b) \mapsto b$. Let $b \in B$ and let $x \in X$ be such that $f(x) = g(b)$ and $||g(b)|| = ||x||$. Then $(x,b) \in A$ and $f'(x,b) = b$. Moreover $||(x,b)|| = \max\{||x||, ||b||\}$. If $||x|| \leqslant ||b||$ then we are done. Suppose $||x|| \geqslant ||b||$. Then $||b|| \leqslant ||x|| = ||g(b)|| \leqslant ||b||$ so $||x|| = ||b||$. In either case $||(x,b)|| = ||b||$. □

It is clear that the composition of strong epimorphisms is a strong epimorphism. To conclude the proof of Theorem 2.3.10 let us next show that compositions of strong monomorphisms are strong. More generally we have the following.

Proposition 2.3.13. *Let (C,d) be an ultrametric space and $A \subset B \subset C$ be subspaces. Let $c \in C$. Suppose that c has a nearest point b_c in B and that b_c has a nearest point $a_c \in A$. Then c has a nearest point in A.*

Proof. Let $a \in A$. If $d(a,c) = d(b_c,c)$ then a is a nearest point to C in B and hence therefore in A. Hence we may assume that $d(a,c) > d(b_c,c)$ for all $a \in A$. In particular $d(b_c,a) = d(a,c)$. So $d(a_c,c) = d(b_c,a_c) < d(b_c,a) = d(a,c)$ for all $a \in c$ and a_c is a closest point to c in A. □

Corollary 2.3.14. *The composition of strong monomorphisms $f : A \to B$ and $g : B \to C$ in* $Norm_k^{\leqslant 1,nA}$, *and hence in* $Ban_k^{\leqslant 1,nA}$ *is a strong monomorphism.*

Now let us establish some properties of this exact structure. The following is clear.

Proposition 2.3.15. *Let $f : A \to B$ be a strong monomorphism. Then for $[b] \in B/f(A)$ we have $||[b]|| = ||b - f(a)||$ where $f(a)$ is a closest point to b in $f(A)$.*

Proposition 2.3.16. *In both $Norm_k^{\leqslant 1,nA}$ and $Ban_k^{\leqslant 1,nA}$ products and coproducts preserve strong monomorphisms, strong epimorphisms and kernels. Coproducts preserve cokernels and products preserve cokernels of admissible monomorphisms. In particular they are exact for the strong exact structures.*

Proof. Let us first prove the claims about products. It suffices to show this for $Norm_k^{\leqslant 1,nA}$. First note that products always commute with kernels. Now let

$$0 \longrightarrow A_i \xrightarrow{f_i} B_i \xrightarrow{g_i} C_i \longrightarrow 0$$

be a strong exact sequence. We write the product sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

We need to show that this sequence is exact.

Let us show that the map g is a strong epimorphism. Indeed by Proposition 2.3.15 $||([b_i])|| = \sup_{i\in I}||b_i - f_i(a_i)||$ where a_i is such that $f_i(a_i)$ is a closest point to b_i in $f_i(A_i)$. Now

$$||a_i|| = ||f_i(a_i)|| = ||(f_i(a_i) - b_i) + b_i|| \leqslant \max\{||f_i(a_i) - b_i||, ||b_i||\} \leqslant ||b_i||$$

So $(a_i) \in A$. Moreover $||(b_i - f_i(a_i))|| = ||([b_i])||$ and $\pi((b_i - f_i(a_i)) = ([b_i])$. Now let us show that f is a strong monomorphism. It is clearly an isometry. Let $c = (c_i) \in \prod_{i\in I} C_i$. For each i pick $b_i \in B_i$ with $g_i(b_i) = c_i$ and $||c_i|| = ||b_i||$. Then clearly $\sup_{i\in I}||b_i|| = \sup_{i\in I}||c_i||$. Set $b = (b_i) \in \prod_{i\in I} B_i$. Then $g(b) = c$ and $||c|| = ||b||$. A sequence (b_i^n) converges to (b_i) in $\prod_{i\in I} B_i$ if and only if each b_i^n converges to b_i in B_i. It follows that the image of f is closed in B. Finally let $(b_i) \in B$ and for each i pick a closest point $f_i(a_i)$ to b_i in $f_i(A_i)$. Now

$$||([b_i])|| = \sup_{i\in I}\inf_{a_i\in A_i}||b_i - f_i(a_i)||$$

Pick a_i such that $f_i(a_i)$ is a closest point to b_i in $f_i(A_i)$. Then $||([b_i])|| = \sup_{i\in I}||b_i - f_i(a_i)$ By a computation similar to the previous part of the proof $\sup_i||a_i|| \leqslant \sup_i||b_i|| < \infty$ and $(a_i) \in A$. Moreover for any $(\tilde{a}_i) \in A$ we have

$$||(b_i) - f((a_i))|| = \sup_{i\in I}||b_i - f_i(a_i)|| \leqslant \sup_{i\in I}||b_i - f_i(\tilde{a}_i)|| = ||(b_i) - (a_i)||$$

So $f((a_i))$ is a closest point to (b_i) in $f(A)$.

Finally it is clear that f is a kernel of g and therefore the sequence is exact.

Coproducts always preserve cokernels. It is obvious that coproducts preserve strong epimorphisms in $Norm_k^{\leqslant 1,nA}$ and for $Ban_k^{\leqslant 1,nA}$ the proof is similar to the proof that products preserve strong epimorphisms. It is clear that coproducts preserve kernels. □

Corollary 2.3.17. *In* $Norm_k^{nA,\leqslant 1}$ *and* $Ban_k^{nA,\leqslant 1}$ *products are admissibly coexact and coproducts are strongly exact for the strong exact structure.*

Completion Functors

There is a completion functor $Cpl : Norm_k^{nA} \to Ban_k^{nA}$ which sends a normed space A to its separated completion $\hat{A}$. It is left adjoint to the inclusion functor $\iota : Ban_k^{nA} \to Norm_k^{nA}$. It restricts to a functor $Cpl^{\leqslant 1} : Norm_k^{\leqslant 1,nA} \to Ban_k^{\leqslant 1,nA}$. Again it is left adjoint to the inclusion functor $\iota^{\leqslant 1} : Ban_k^{\leqslant 1,nA} \to Norm_k^{\leqslant 1,nA}$. From the

Proposition 2.3.18. *The functor Cpl is exact for the quasi-abelian exact structure.*

Proof. This is in [67] 3.1.13 for $k = \mathbb{C}$, but the proof works for any Banach ring. □

We are going to show the following.

Proposition 2.3.19. *The functor* $Cpl^{\leqslant 1}$ *is exact for the strong exact structure.*

First we need two basic facts about Cauchy sequences in non-Archimedean fields.

Proposition 2.3.20. *Let* (a_n) *be a sequence in* k *such that* $||a_{n+1} - a_n|| \to 0$. *Then* (a_n) *is a Cauchy sequence.*

Proof. For any pair $m > n$ we have $||a_m - a_n|| \leqslant \sup_{n \leqslant i \leqslant m-1}\{||a_{i+1} - a_i||\}$. Let $\delta > 0$ and let N be such that $||a_{j+1} - a_j|| < \delta$ for $j > N$. Then for $m > n > N$ we have $||a_m - a_n|| < \delta$. □

By Lemma 2.19 in [3] we have

Proposition 2.3.21. *Let* (a_n) *be a Cauchy sequence in* k. *If* (a_n) *does not converge to zero then the sequence* $(|a_n|)$ *is eventually constant.*

Combining these two propositions we get the following.

Proposition 2.3.22. *Let*

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

be an strong exact sequence in $Norm_k^{\leqslant 1,nA}$. *Then*

$$0 \longrightarrow \hat{A} \xrightarrow{\hat{f}} \hat{B} \xrightarrow{\hat{g}} \hat{C} \longrightarrow 0$$

is a strong exact sequence in $Ban_k^{\leqslant 1,nA}$.

Proof. By Proposition 2.3.18 the complex is a kernel-cokernel pair. Thus it remains to show that hat $\hat{g}$ is a strong epimorphism. Let $[(c_n)]$ be a non-zero equivalence class of Cauchy sequences in C. By Proposition 2.3.21 we may assume that $||c_n||$ is a constant r. Pick $\tilde{b}_0$ such that $g(\tilde{b}_0) = c_0$ and $||b_0|| = ||c_0||$. For each $n+1$ pick $\tilde{b}_{n+1}$ such that $g(\tilde{b}_{n+1}) = c_{n+1} - c_n$ and $||\tilde{b}_{n+1}|| = ||c_{n+1} - c_n||$. Write $b_n = \sum_{k=0}^{n} \tilde{b}_n$. Then $g(b_n) = c_n$. Moreover

$$r = ||c_n|| \leqslant ||b_n|| \leqslant \max_{k \leqslant n} ||\tilde{b}_n|| = \max\{||c_0||, \max_{1 \leqslant k \leqslant n} ||c_k - c_{k-1}||\} \leqslant r$$

Hence $||b_n|| = r$. Moreover $||b_{n+1} - b_n|| = ||\tilde{b}_{n+1}|| = ||c_{n+1} - c_n|| \to 0$, so by Proposition 2.3.20, (b_n) is a Cauchy sequence. □

Proposition 2.3.23. *For each $r \in \mathbb{R}_{>0}$ the object k_r is projective in both $Norm_k^{\leqslant 1,nA}$ and $Ban_k^{\leqslant 1,nA}$. In particular the strong exact structures on both $Norm_k^{\leqslant 1,nA}$ and $Ban_k^{\leqslant 1,nA}$ have enough functorial projectives.*

Proof. Let us first prove the proposition for $Norm_k^{\leqslant 1,nA}$. It suffices to show that the functor $\text{Hom}(k_r, -) : Norm_k^{\leqslant 1,nA} \to \mathcal{Ab}$ preserves cokernels. Let $f : A \to B$ be a strong monomorphism with cokernel $g : B \to C$. We need to show that the map

$$B_B\left(0, \frac{1}{r}\right) \to B_C\left(0, \frac{1}{r}\right)$$

on open balls is an epimorphism. This follows immediately from the definition of strong epimorphism.

For the second assertion, let $E \in Norm_k^{\leqslant 1,nA}$. Write $\mathcal{P}(E) := \bigoplus_{e \in E} k_{||e||}$. There is a map $\mathcal{P}(E) \to E$ induced by the isometry

$$k_{||e||} \to E, \ \lambda \mapsto \lambda e$$

This is clearly a strong epimorphism.

For $Ban_k^{\leqslant 1,nA}$ we use the fact that $Cpl^{\leqslant 1}$ is exact and preserves projectives since it is left adjoint to an exact functor. □

Compactness and Smallness

Let $D : \mathcal{I} \to Norm_k^{\leqslant 1,nA}$ be a diagram with $\mathcal{I}$ a directed category and f_{ji} isometries. Write $A_i = D(i)$ and $f_{ji} = D(i \leqslant j)$ The direct limit $A := \lim_{\rightarrow} A_i$ is constructed as follows. The underlying vector space $\lim_{\rightarrow} A_i$ is the direct limit of the underlying vector spaces of the A_i. Namely it is the disjoint union of the A_i quotient by the relation $a_i \sim f_{ji}(a_i)$ for any $j > i$. If $a_i \in A_i$ and $a_j \in A_j$ then $[a_i] + [a_j] := [f_{Ki}(a_i) + f_{Kj}(a_j)]$ where K is any upper bound of i and j. If $\lambda \in k$ then $\lambda[a_i] := [\lambda a_i]$. We define a norm on this vector space by

$||[a_i]|| := ||a_i||$. This is well-defined because if $a_i \sim a_j$ then $a_j = f_{ji}(a_i)$, so $||a_j|| = ||a_i||$. Clearly $||[a_i]|| = 0$ if and only if $a_i = 0$ and if $\lambda \in k$ then $||\lambda[a_i]|| = |\lambda|||a_i||$. Finally

$$\begin{aligned}
||[a_i] + [a_j]|| &= ||[f_{Ki}(a_i) + f_{Kj}(a_j)]|| \\
&= ||f_{Ki}(a_i) + f_{Kj}(a_j)|| \\
&\leqslant \max\{||f_{Ki}(a_i)||, ||f_{Kj}(a_j)||\} \\
&= \max\{||a_i||, ||a_j||\} \\
&= \max\{||[a_i]||, ||[a_j]||\}
\end{aligned}$$

So this is a non-Archimedean norm. The map $f_i : A_i \to A$ sends a_i to $[a_i]$.

Proposition 2.3.24. *The normed space described above is the direct limit in $Norm_k^{\leqslant 1,nA}$.*

Proof. Let $g_i : A_i \to C$ be a cocone from D. There is a unique map of vector spaces $g : A \to C$ such that $g \circ f_i = g_i$. It remains to show that g is bounded with $||g|| \leqslant 1$. Let $[a_i] \in A$ with $[a_i] = f_i(a_i)$. Then $||g([a_i])|| = ||g_i(a_i)|| \leqslant ||a_i|| = ||[a_i]||$. □

Corollary 2.3.25. 1. *Suppose that for each $j < k$, $f_{kj} : A_j \to A_k$ is an admissible monomorphism in the quasi-abelian exact structure on $Norm_k^{\leqslant 1,nA}$. Then for each i the map $A_i \to A$ is an admissible monomorphism in the quasi-abelian exact structure.*

2. *Suppose that for each $j < k$, $f_{kj} : A_j \to A_k$ is an admissible monomorphism in the strong exact structure on $Norm_k^{\leqslant 1,nA}$. Then for each i the map $A_i \to A$ is an admissible monomorphism in the strong exact structure.*

Proof. 1. It is clear from the definition of the norm on A that f_i is an isometry. Suppose that $(f_i(a_i^n))$ converges to $[a_j]$ with $a_j \in A_j$. Let K be an upper bound of i and j. Then $([f_{Ki}(a_i^n)])$ converges to $[f_{Kj}(a_j)]$. But by the definition of the norm on A this clearly means that $f_{Ki}(a_i^n)$ converges to $f_{Kj}(a_j)$ in A_K. Since f_{Ki} has closed image, $f_{Kj}(a_j) = f_{Ki}(a_i)$. Since f_{Ki} is an isometry (a_i^n) converges to a_i, so $(f_i(a_i^n))$ converges to $f_i(a_i)$, and f_i has closed image.

2. Let $[a_j] \in A$ with $a_j \in A_j$. Let K be an upper bound of i and j. The map $f_{Ki} : A_i \to A_K$ is a strong monomorphism. Therefore $f_{Kj}(a_j)$ has a closest point $f_{Ki}(a_i)$ in $f_{Ki}(A_i)$. We claim that $[a_i]$ is a closest point to $[a_j]$ in $f_i(A_i)$. Indeed let $[a_i'] \in f_i(A_i)$ with $a_i' \in A_i$. Then

$$||[a_j] - [a_i']|| = ||[f_{Kj}(a_j) - f_{Ki}(a_i')]|| = ||f_{Kj}(a_j) - f_{Ki}(a_i')||$$

$$\geqslant ||f_{Kj}(a_j) - f_{Ki}(a_i)|| = ||[a_j] - [a_i]||$$

□

Proposition 2.3.26. *For $r \in \mathbb{R}_{>0}$ the objects k_r are compact with respect to the class of admissible monomorphisms in the strong exact structure on $Norm_k^{\leqslant 1,nA}$.*

Proof. We need to show that for any $r \in \mathbb{R}_{>0}$ the map

$$\lim_{\rightarrow} B_{A_i}(0,r) \to B_{\lim_{\rightarrow} A_i}(0,r)$$

is an isomorphism. It suffices to prove that it is an epimorphism. Let $[a_i] \in \lim_{\rightarrow} A_i$ be such that $||[a_i]|| = ||a_i|| \leqslant r$. Then $a_i \in B_{A_i}(0,r) \hookrightarrow \lim_{\rightarrow} B_{A_i}(0,r)$ maps to $[a_i]$. □

Proposition 2.3.27. *For any $r \in \mathbb{R}_{>0}$ the k-Banach space k_r is not compact with respect to the class of admissible monomorphisms in the strong exact structure (or even the split exact structure). However every object is $\aleph_1$-small.*

Proof. Consider the sequence with $X_i = k_r^i$ and the map $X_i \rightarrowtail X_{i+1}$ being the inclusion of the first i copies of k. The group $\lim_{\rightarrow}\text{Hom}(k_r, X_i)$ is the ascending union of the closed balls in X_i of radius r, while $\text{Hom}(k_r, \lim_{\rightarrow} X_i)$ is the closed ball of radius r in $\lim_{\rightarrow} X_i$. The map

$$\lim_{\rightarrow}\text{Hom}(k_r, X_i) \to \text{Hom}(k_r, \lim_{\rightarrow} X_i)$$

is the obvious inclusion. Consider the example with $X_i = k^{\oplus i}$ with $X_i \to X_{i+1}$ being the split injection $k^i \to k^{i+1}$ which is the inclusion of the first i copies of k. Then $\lim_{\rightarrow} X_i$ is the space of sequences in k converging to 0 with the supremum norm. The group $\lim_{\rightarrow}\text{Hom}(k_r, X_i)$ is the group of finite sequences of norm at most $\frac{1}{r}$, while $\text{Hom}(k_r, \lim_{\rightarrow} X_i)$ is the group of sequences converging to 0 with norm at most $\frac{1}{r}$. It is clear that for a non-discrete field the map

$$\lim_{\rightarrow}\text{Hom}(k_r, X_i) \to \text{Hom}(k_r, \lim_{\rightarrow} X_i)$$

is not an epimorphism. The last claim is [1] 1.48. □

Recall that a Banach space E is said to have the **Hahn-Banach extension property** if for every subspace D of E, every bounded functional $f : D \to k$ there is an extension $g : E \to k$ of f with $||g|| = ||f||$.

Theorem 2.3.28 ([65] Theorem 4.12)**.** *If k is spherically complete then every Banach space over k has the Hahn-Banach extension property.*

Proposition 2.3.29. *Let E be a non-zero Banach space with the Hahn-Banach extension property and let $e \in E$. Then there is a Banach space E' and an isometric isomorphism $E \cong E' \oplus k_{||e||}$. In particular if k is spherically complete then there are no non-nonzero compact objects in $Ban_k^{\leqslant 1,nA}$.*

Proof. Let $\left\langle e \right\rangle$ be the span of e in E. The map $f : \left\langle e \right\rangle \to k_{||e||}$ sending e to 1 is an isometric isomorphism with inverse g sending 1 to e. Therefore f extends to a map $\overline{f} : E \to k_{||e||}$ with $||\overline{f}|| = 1$. Moreover $\overline{f} \circ g = \mathrm{Id}_{k_{||e||}}$. Since $Ban^{\leqslant 1}_{k_{||e||}}$ is quasi-abelian and in particular weakly-idempotent complete this gives a splitting. □

The Monoidal Structure

The following is straightforward.

Proposition 2.3.30. *Consider the functors*

$$Cpl \circ \otimes_{\pi} : Norm^{nA}_k \otimes Norm^{nA}_k \to Ban^{nA}_k$$

and

$$\hat{\otimes}_{\pi} \circ Cpl \times Cpl : Norm^{nA}_k \otimes Norm^{nA}_k \to Ban^{nA}_k$$

There is a natural isometric isometric isomorphism

$$\phi : Cpl \circ \otimes_{\pi} \to \hat{\otimes}_{\pi} \circ Cpl \times Cpl$$

In particular we get a natural isomorphism

$$\phi^{\leqslant 1} : Cpl^{\leqslant 1} \circ \otimes_{\pi} \to \hat{\otimes}_{\pi} \circ Cpl^{\leqslant 1} \times Cpl^{\leqslant 1}$$

Proposition 2.3.31. *Let $r \in \mathbb{R}_{>0}$. Consider the functors*

$$(-)_r \circ Cpl : Norm^{nA}_k \to Ban^{nA}_k$$

and

$$Cpl \circ (-)_r : Norm^{nA}_k \to Ban^{nA}_k$$

Then there is a natural isometric isomorphism

$$\zeta : (-)_r \circ Cpl \to Cpl \circ (-)_r$$

In particular this induces a natural isomorphism of functors

$$\zeta^{\leqslant 1} : (-)_r \circ Cpl^{\leqslant 1} \cong Cpl^{\leqslant 1} \circ (-)_r$$

Proposition 2.3.32. *Let $s, r \in \mathbb{R}_{>0}$ and consider the functors*

$$(-)_{rs} \circ \otimes_{\pi} : Norm^{nA}_k \times Norm^{nA}_k \to Norm^{nA}_k$$

and

$$\otimes_{\pi} \circ (-)_r \times (-)_s : Norm^{nA}_k \times Norm^{nA}_k \to Norm^{nA}_k$$

Then there is an natural isometric isomorphism

$$\eta : \otimes_{\pi} \circ (-)_r \times (-)_s \to (-)_{rs} \circ \otimes_{\pi}$$

At this point let us make the following remark

Remark 2.3.33. *The rescaling functors are exact for both the quasi-abelian and strong exact structures.*

By Proposition 2.3.31 and Proposition 2.3.30 we get

Corollary 2.3.34. *Let $s, r \in \mathbb{R}_{>0}$ and consider the functors*

$$(-)_{rs} \circ \otimes_{\pi} : Ban_k^{nA} \times Ban_k^{nA} \to Ban_k^{nA}$$

and

$$\otimes_{\pi} \circ (-)_r \times (-)_s : Ban_k^{nA} \times Ban_k^{nA} \to Ban_k^{nA}$$

Then there is an natural isometric isomorphism

$$\eta : \otimes_{\pi} \circ (-)_r \times (-)_s \to (-)_{rs} \circ \otimes_{\pi}$$

Corollary 2.3.35. *Projective objects in $Norm_k^{\leqslant 1,nA}$ an $Ban_k^{\leqslant 1,nA}$ are flat in both the quasi-abelian and strong exact structures.*

Proof. By Proposition 2.3.32 and Corollary 2.3.34, we only need to note that tensoring with k is the identity functor and hence is exact. $\square$

Corollary 2.3.36. *The tensor product of projective objects in $Norm_k^{\leqslant 1,nA}$ an $Ban_k^{\leqslant 1,nA}$ is a projective object.*

Proof. It suffices to prove this in $Norm_k^{\leqslant 1,nA}$ for objects of the form k_r with $r \in \mathbb{R}_{>0}$. But $k_r \otimes k_s \cong k_{rs}$ which is projective. $\square$

We summarise this section with the following result.

Theorem 2.3.37. *$Ban_k^{\leqslant 1}$ is a projectively monoidal weakly **AdMon**-elementary exact category which is $\aleph_1$-presentable but not $\aleph_0$-presentable. $Norm_k^{\leqslant 1}$ is a monoidal elementary exact category.*

2.3.3 The Split Exact Structure

We conclude with an example of a category which has no small generating set whatsoever but is still weakly **AdMon**-elementary. Let $\mathcal{E}$ be an additive category and endow it with the split exact structure.

Proposition 2.3.38. *If $\mathcal{E}$ is an additive category with kernels and countable coproducts then for the split exact structure $(\aleph_0, \textbf{AdMon})$ colimits exist and are exact.*

Proof. It has kernels by assumption. It trivially has enough projectives since every object is projective. The fact $(\aleph_0, \mathbf{AdMon})$ colimits exist and are exact is similar to the proof of Lemma 3.1.1 in the next section. □

For $\mathcal{E} = \mathcal{Ab}$ this category has no small generating set by [18]. This can be generalised to other exact structures defined by projective classes as discussed in the same paper.

Chapter 3

Model Structures on Exact Categories

In this chapter we discuss model structures on categories of chain complexes in exact categories. We give very general conditions under which unbounded complexes are equipped with the projective model structure. We also investigate when such a model structure is monoidal and satisfies the monoid axiom, which will be crucial for studying homotopical algebra in exact categories in the next section. Finally we generalise the Dold-Kan correspondence.

3.1 Cotorsion Pairs

In [40], Hovey introduced the notion of a **compatible model structure** on an abelian category. He showed that there is a 1-1 correspondence between such model structures and purely homological data now known as **Hovey triples**. Gillespie noticed that this correspondence generalises to weakly idempotent complete exact categories, and explains in [31] how to adapt Hovey's proofs. In the next two subsections we will recall some of Hovey's/ Gillespie's results both for the reader's convenience and because we will need many of the individual propositions later anyway. We shall modify the exposition somewhat, by first extracting from Hovey's proof a bijection between cotorsion pairs and compatible weak factorisation systems (this has been noticed in [78]). For basic facts about weak factorisation systems and model structures in general see Appendix A.

Let $\mathcal{S}$ be a class of objects in an exact category $\mathcal{E}$. We shall denote by ${}^{\perp}\mathcal{S}$ the class of all objects X such that $\mathrm{Ext}^1(X,S)=0$ for all $S\in\mathcal{S}$, and by $\mathcal{S}^{\perp}$ the class of all objects X such that $\mathrm{Ext}^1(S,X)=0$ for all $S\in\mathcal{S}$. The class $\mathcal{S}^{\perp}$ is called the class of $\mathcal{S}$-**injectives**, and the class ${}^{\perp}\mathcal{S}$ is called the class of $\mathcal{S}$-**projectives**. The following technical result will be useful. The proof is a straightforward generalisation of Lemma 6.2 in [40].

Lemma 3.1.1. *Let $\mathcal{E}$ be an exact category. Let $\mathcal{S}$ be a class of objects in $\mathcal{E}$, and let $\mathfrak{L} = {}^{\perp}\mathcal{S}$. Then $\mathfrak{L}$ is closed under retracts and finite extensions. If $\mathcal{E}$ is cocomplete it is closed under transfinite extensions.*

Proof. First we show that $\mathfrak{L}$ is closed under retracts. Note that it is sufficient to show that for a given $Y \in \mathcal{E}$, the collection of objects X such that $\mathrm{Ext}^1(X, Y) = 0$ is closed under retracts. Let X be such that $\mathrm{Ext}^1(X, Y) = 0$ and let X' be a retract of X. Then X' is a summand of X, and so $\mathrm{Ext}^1(X', Y) = 0$.

Let us show that $\mathfrak{L}$ is closed under transfinite extensions. Again it is sufficient to show that for any object $Y \in \mathcal{E}$ the collection of all X with $\mathrm{Ext}^1(X, Y) = 0$ is closed under transfinite extensions and retracts. More generally, suppose ϕ is an ordinal such that for any $\phi' \leqslant \phi$, $\mathcal{E}$ has ϕ'-colimits. Suppose λ is an ordinal with $\lambda \leqslant \phi$ and $X : \lambda \to \mathcal{E}$ is a colimit-preserving functor such that $\mathrm{Ext}^1(X_0, Y) = 0$, $X_\alpha \to X_{\alpha+1}$ is an admissible monic for all $\alpha < \lambda$ and $\mathrm{Ext}^1(\mathrm{Coker}(X_\alpha \to X_{\alpha+1}), Y) = 0$ for all $\alpha < \lambda$. We shall prove by transfinite induction that $\mathrm{Ext}^1(X_\beta, Y) = 0$ for all $\beta \leqslant \lambda$, where $X_\lambda = \mathrm{colim}_{\alpha<\lambda} X_\alpha$. If $\lambda = 0$ then this is clear. Suppose that λ is a successor ordinal, so that $\lambda = \alpha + 1$ for some ordinal α. We have $\mathrm{Ext}^1(X_\alpha, Y) = 0$ and $\mathrm{Ext}^1(\mathrm{Coker}(X_\alpha \to X_\lambda), Y) = 0$. The long exact Ext sequence then shows that $\mathrm{Ext}^1(X_\lambda, Y) = 0$.

Now suppose that $\beta \leqslant \lambda$ is a limit ordinal, and that $\mathrm{Ext}^1(X_\alpha, Y) = 0$ for all $\alpha < \beta$. Let

$$0 \longrightarrow Y \xrightarrow{f} N \xrightarrow{p} X_\beta \longrightarrow 0$$

represent an element of $\mathrm{Ext}^1(X_\beta, Y)$. For each $\alpha \leqslant \beta$, pull this short exact sequence back through the map $j_\alpha : X_\alpha \to X_\beta$ for $\alpha \leqslant \beta$. For $\alpha \leqslant \gamma < \beta$ we get a commutative diagram.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Y & \xrightarrow{f} & N & \xrightarrow{p} & X_\beta & \longrightarrow & 0 \\
 & & \| & & \uparrow{\scriptstyle k_\gamma} & & \uparrow{\scriptstyle j_\gamma} & & \\
0 & \longrightarrow & Y & \xrightarrow{f_\gamma} & N_\gamma & \xrightarrow{p_\gamma} & X_\gamma & \longrightarrow & 0 \\
 & & \| & & \uparrow{\scriptstyle k_{\alpha,\gamma}} & & \uparrow{\scriptstyle j_{\alpha,\gamma}} & & \\
0 & \longrightarrow & Y & \xrightarrow{f_\alpha} & N_\alpha & \xrightarrow{p_\alpha} & X_\alpha & \longrightarrow & 0
\end{array}$$

Since f is an admissible monic, f_α is as well by Proposition 2.1.8. Since $\mathrm{Ext}^1(X_\alpha, Y) = 0$ there is some splitting $t_\alpha : X_\alpha \to N_\alpha$ of p_α. We are going to modify the t_α to s_α so that they are compatible, i.e. $k_{\alpha,\gamma} s_\alpha = s_\gamma j_{\alpha,\gamma}$ for all $\alpha \leqslant \gamma$. We will do this by transfinite induction.

Set $s_0 = t_0$. If γ is a limit ordinal let $s_\gamma : \mathrm{colim}_{\alpha<\gamma} X_\alpha \to N_\gamma$ be the map whose restriction to X_α is $k_{\alpha,\gamma} s_\alpha$, where $k_{\alpha,\gamma} : N_\alpha \to N_\gamma$ is the transfinite composition of the continuous functor $\gamma \to \mathcal{E}$, $\beta \mapsto N_\beta$. Then by construction $k_{\alpha,\gamma} s_\alpha = s_\gamma j_{\alpha,\gamma}$.

Now for the successor case. Suppose we have constructed s_α. Let us construct $s_{\alpha+1}$. We have

$$\begin{aligned}p_{\alpha+1}(k_{\alpha,\alpha+1}s_\alpha - t_{\alpha+1}j_{\alpha,\alpha+1}) &= j_{\alpha,\alpha+1}\circ p_\alpha \circ s_\alpha - p_{\alpha+1}\circ t_{\alpha+1}\circ j_{\alpha,\alpha+1}\\ &= j_{\alpha,\alpha+1} - j_{\alpha,\alpha+1}\\ &= 0\end{aligned}$$

Therefore there is a map $h : X_\alpha \to Y$ such that $f_{\alpha+1}h = k_{\alpha,\alpha+1}s_\alpha - t_{\alpha+1}j_{\alpha,\alpha+1}$. Since $j_{\alpha,\alpha+1} : X_\alpha \to X_{\alpha+1}$ is an admissible monic and $\text{Ext}^1(\text{Coker}(j_{\alpha,\alpha+1}), Y) = 0$, the long exact Ext sequence implies that there is a map $g : X_{\alpha+1} \to Y$ such that $gj_{\alpha,\alpha+1} = h$. Let $s_{\alpha+1} = t_{\alpha+1} + f_{\alpha+1}g$. Then clearly $s_{\alpha+1}$ is a section of $p_{\alpha+1}$. Moreover

$$\begin{aligned}s_{\alpha+1}j_{\alpha,\alpha+1} &= t_{\alpha+1}\circ j_{\alpha,\alpha+1} + f_{\alpha+1}g\circ j_{\alpha,\alpha+1}\\ &= k_{\alpha,\alpha+1}s_\alpha - f_{\alpha+1}\circ h + f_{\alpha+1}\circ h\\ &= k_{\alpha,\alpha+1}s_\alpha\end{aligned}$$

as required. □

Let us now define cotorsion pairs, and discuss their relation with weak factorisation systems. We shall largely follow the notation of [78].

Definition 3.1.2. *Let $\mathpzc{E}$ be an exact category. A* ***cotorsion pair*** *on $\mathpzc{E}$ is a pair of families of objects $(\mathfrak{L}, \mathfrak{R})$ of $\mathpzc{E}$ such that $\mathfrak{L} = {}^\perp\mathfrak{R}$ and $\mathfrak{R} = \mathfrak{L}^\perp$.*

Definition 3.1.3. *A cotorsion pair $(\mathfrak{L}, \mathfrak{R})$ is said to have* ***enough (functorial) projectives*** *if for every $X \in \mathpzc{E}$ there is an admissible epic $p : Y \to X$, (functorial in X), such that $Y \in \mathfrak{L}$ and Ker$(p) \in \mathfrak{R}$. It is said to have* ***enough (functorial) injectives*** *if, for every X, there is an admissible monic $i : X \to Z$, (functorial in X), such that $Z \in \mathfrak{R}$ and Coker$(i) \in \mathfrak{L}$. A cotorsion pair is said to be* ***(functorially) complete*** *if it has enough (functorial) projectives and enough (functorial) injectives.*

Example 3.1.4. *Our main example is the projective cotorsion pair. Let $\mathpzc{E}$ be an exact category. Let* ***Proj***$(\mathpzc{E})$ *denote the collection of projective objects of $\mathpzc{E}$. Then* $($***Proj***$(\mathpzc{E}),$ ***Ob***$(\mathpzc{E}))$ *is clearly a cotorsion pair. Suppose that $\mathpzc{E}$ has enough (functorial) projectives. Then the cotorsion pair* $($***Proj***$(\mathpzc{E}),$ ***Ob***$(\mathpzc{E}))$ *is trivially (functorially) complete.*

Notation 3.1.5. *Let $\mathpzc{E}$ be an exact category and $(\mathcal{L}, \mathcal{R})$ a weak factorisation system on $\mathpzc{E}$. Denote by Coker$\mathcal{L}$ the collection of objects L such L is a cokernel of some map in, $\mathcal{L}$ and by Ker$\mathcal{R}$ the collection of objects R such that R is the kernel of some map in $\mathcal{R}$.*

Given classes of objects $\mathfrak{A}, \mathfrak{B}$ in $\mathpzc{E}$, we denoteby Infl$(\mathfrak{A})$ the class of admissible monics with cokernel in $\mathfrak{A}$ and by Defl$(\mathfrak{B})$ the class of admissible epics with kernel in $\mathfrak{B}$.

Definition 3.1.6. *Let $\mathcal{E}$ be an exact category. A weak factorisation system $(\mathcal{L}, \mathcal{R})$ on $\mathcal{E}$ is said to be* ***compatible*** *if*

1. *$f \in \mathcal{L}$ if and only if f is an admissible monic and $0 \to Coker(f)$ belongs to $\mathcal{L}$.*

2. *$f \in \mathcal{R}$ if and only if f is an admissible epic and $Ker(f) \to 0$ belongs to $\mathcal{R}$.*

The following result is Theorem 5.13 in [78].

Theorem 3.1.7. *Let $\mathcal{E}$ be an exact category. Then*

$$(\mathcal{L}, \mathcal{R}) \mapsto (Coker\mathcal{L}, Ker\mathcal{R}) \text{ and } (\mathfrak{A}, \mathfrak{B}) \mapsto (Infl(\mathfrak{A}), Defl(\mathfrak{B}))$$

define mutually inverse bijective mappings between compatible weak factorisation systems and complete cotorsion pairs. The bijections restrict to mutually inverse mappings between compatible functorial weak factorisation systems and functorially complete cotorsion pairs.

3.1.1 Compatible Model Structures

Having described the bijection between cotorsion pairs and compatible weak factorisation systems, we now introduce compatible model structures, and explain how they too correspond to purely homological data. Remember that we do not assume our model categories are complete or cocomplete.

Let $(\mathcal{C}, \mathcal{F}, \mathcal{W})$ be a model structure on an additive category $\mathcal{E}$.

Definition 3.1.8. *Let $\mathcal{E}$ be an exact category. Let $(\mathcal{C}, \mathcal{F}, \mathcal{W})$ be a model structure on $\mathcal{E}$. The model structure is said to be* ***compatible*** *if both $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ and $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ are compatible weak factorisation systems.*

Let us now define the corresponding homological data. As for abelian categories, we will call a subcategory $\mathcal{D}$ of an exact category $\mathcal{E}$ **thick** if whenever

$$0 \to A \to B \to C \to 0$$

is a short exact sequence and two of the objects are in $\mathcal{D}$, then so is the third.

Definition 3.1.9. *A* ***Hovey triple*** *on an exact category $\mathcal{E}$ is a triple $(\mathfrak{C}, \mathfrak{W}, \mathfrak{F})$ of collections of objects of $\mathcal{E}$ such that the full subcategory on $\mathfrak{W}$ is closed under retracts and thick, and that both $(\mathfrak{C}, \mathfrak{F} \cap \mathfrak{W})$ and $(\mathfrak{C} \cap \mathfrak{W}, \mathfrak{F})$ are complete cotorsion pairs.*

We then have the following theorem (Theorem 6.9 in [78]). It is originally due to [40] in the abelian case and [31] in the more general exact case.

Theorem 3.1.10. *Let $\mathcal{E}$ be a weakly idempotent complete exact category. Then there is a bijection between Hovey triples and compatible model structures. The correspondence assigns to a Hovey triple $(\mathfrak{C}, \mathfrak{W}, \mathfrak{F})$ the model structure $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ such that*

1. $\mathcal{C} = Infl(\mathfrak{C})$

2. $\mathcal{F} = Defl(\mathfrak{F})$

3. *$\mathcal{W}$ consists of morphisms of the form $p \circ i$ where $i \in Infl(\mathfrak{C} \cap \mathfrak{W})$ and $p \in Defl(\mathfrak{F} \cap \mathfrak{W})$.*

Before we move on let us mention a more general notion than compatible model structures. We will need it when we consider the projective model structure on $Ch_{\geqslant 0}(\mathcal{E})$.

Definition 3.1.11. *Let $\mathcal{E}$ be an exact category. A model structure $(\mathcal{C}, \mathcal{F}, \mathcal{W})$ on $\mathcal{E}$ is said to be* ***left pseudo-compatible*** *if there are classes of objects $\mathfrak{C}$ and $\mathfrak{W}$ such that*

1. *The full subcategory on $\mathfrak{W}$ is thick.*

2. *A map f is in $\mathcal{C}$ (resp. $\mathcal{C} \cap \mathcal{W}$) if and only if it is an admissible monic with cokernel in $\mathfrak{C}$ (resp. $\mathfrak{C} \cap \mathfrak{W}$).*

3. *An admissible monic is in $\mathcal{W}$ if and only if its cokernel is in $\mathfrak{W}$.*

As before $\mathfrak{C}$/ $\mathfrak{W}$ /$\mathfrak{C} \cap \mathfrak{W}$ are called the ***cofibrant /trivial/ trivially cofibrant*** *objects. The pair $(\mathfrak{C}, \mathfrak{W})$ will be called the* ***left homological Waldhausen pair*** *of the model structure. Dually one defines* ***right pesudo-compatible*** *model structures and* ***right homological Waldhausen pairs***

The terminology comes from the notion of a Waldhausen category, in which classes of weak equivalences and cofibrations are specified. Clearly any compatible model structure is left pseudo-compatible.

Definition 3.1.12. *Let $\mathcal{E}$ be an exact category. A left-pseudo compatible model structure on $\mathcal{E}$ defined by a left homological Waldhausen pair $(\mathfrak{C}, \mathfrak{W})$ is said to be* ***strong*** *if a map $f : B \to C$ is an acyclic fibration if and only if it is an admissible epimorphism whose kernel is in $\mathfrak{W}$. The corresponding Waldhausen pair is then also called* ***strong****.*

3.1.2 Small Cotorsion Pairs and Cofibrant Generation

When working with model categories, it is computationally convenient that they be generated by suitably compact objects (see Appendix A for exactly what we mean here). In this section, we study what conditions on the cotorsion pairs defining a compatible model structure guarantee that the model structure is cofibrantly small. The material here is adapted from [40] §6 to exact categories.

Definition 3.1.13. *Let $\mathcal{E}$ be an exact category. A cotorsion pair $(\mathfrak{L}, \mathfrak{R})$ on $\mathcal{E}$ is said to be **cogenerated by a set** if there is a set of objects $\mathcal{G}$ in $\mathfrak{L}$ such that $X \in \mathfrak{R}$ if and only if $Ext^1(G, X) = 0$ for all $G \in \mathcal{G}$.*

Definition 3.1.14. *Suppose $\mathcal{E}$ is an exact category. A cotorsion pair $(\mathfrak{L}, \mathfrak{R})$ is said to be **small** if the following conditions hold*

1. *$\mathfrak{L}$ contains a set of admissible generators.*
2. *$(\mathfrak{L}, \mathfrak{R})$ is cogenerated by a set $\mathcal{G}$.*
3. *For each $G \in \mathcal{G}$ there is an admissible monic i_G with cokernel G such that, if $Hom_{\mathcal{E}}(i_G, X)$ is surjective for all $G \in \mathcal{G}$, then $X \in \mathfrak{R}$.*

*The set of i_G together with the maps $0 \rightarrow U_i$ for some generating set $\{U_i\}$ contained in $\mathfrak{L}$ is called a set of **generating morphisms** of $(\mathfrak{L}, \mathfrak{R})$.*

There is an easy example.

Example 3.1.15. *Recall the projective cotorsion pair $(\textbf{Proj}(\mathcal{E}), \textbf{Ob}(\mathcal{E}))$. Suppose that the category $\mathcal{E}$ is projectively generated, with $\mathcal{P}$ a generating set of projectives. We claim that in this case the projective cotorsion pair is small. Indeed by assumption $\textbf{Proj}(\mathcal{E})$ contains a set of generators $\mathcal{P}$. This set trivially cogenerates the cotorsion pair as well. The third condition is also trivial.*

We now come to the connection between cofibrantly small model structures and cotorsion pairs. The proof of the following is a straightforward modification of [40] Lemma 6.7.

Lemma 3.1.16. *Let $\mathcal{E}$ be an exact category together with a compatible weak factorisation system $(\mathcal{L}, \mathcal{R})$ with corresponding cotorsion pair $(\mathfrak{L}, \mathfrak{R})$. If the cotorsion pair is small, then this weak factorisation system is cofibrantly small. If in addition the generating morphisms have compact domain, the weak factorisation system is cellular.*

3.1.3 Cotorsion Pairs on Monoidal Exact Categories

In this section $(\mathcal{E}, \otimes, k)$ is a monoidal exact category.

We will now study sufficient conditions on cotorsion pairs defining a model category structure so that the resulting structure is monoidal. We generalise the work of [40] §7 to exact categories.

Definition 3.1.17. *A short exact sequence in a monoidal exact category $\mathcal{E}$ is said to be **pure** if it remains exact after tensoring with any object of $\mathcal{E}$. An admissible monic is said to be **pure** if it remains an admissible monic after tensoring with any object of $\mathcal{E}$.*

Theorem 3.1.18. *Let $\mathcal{E}$ be a closed symmetric monoidal exact category. Suppose that $\mathcal{E}$ has a left pseudo-compatible model structure with Waldhausen pair $(\mathfrak{C}, \mathfrak{W})$. Suppose the following conditions are satisfied.*

1. *Every cofibration is pure.*
2. *If $X, Y \in \mathfrak{C}$ then $X \otimes Y \in \mathfrak{C}$.*
3. *If $X, Y \in \mathfrak{C}$ and one of them is in $\mathfrak{W}$, then $X \otimes Y \in \mathfrak{C} \cap \mathfrak{W}$.*
4. *The unit I of the monoidal structure is in $\mathfrak{C}$.*

Then $\mathcal{E}$ is a monoidal model category.

In order to prove this we need the following two results

Proposition 3.1.19. *Let $\mathcal{E}$ be a weakly idempotent complete exact category, Suppose we have a diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{h} & Y & \xrightarrow{i} & Z & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \epsilon} & & \downarrow{\scriptstyle \phi} & & \\ 0 & \longrightarrow & P & \xrightarrow{j} & Q & \xrightarrow{k} & R & \longrightarrow & 0 \end{array}$$

with the top and bottom rows being short exact and the vertical arrows being admissible morphisms. Then there is an exact sequence

$$0 \to Ker(\delta) \rightarrowtail Ker(\epsilon) \to Ker(\phi) \to Coker(\delta) \to Coker(\epsilon) \twoheadrightarrow Coker(\phi) \to 0$$

Proof. This is [14] Corollary 8.13. □

Proposition 3.1.20. *Let*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{h} & Y & \xrightarrow{i} & Z & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \epsilon} & & \downarrow{\scriptstyle \phi} & & \\ 0 & \longrightarrow & P & \xrightarrow{j} & Q & \xrightarrow{k} & R & \longrightarrow & 0 \end{array}$$

be a commutative diagram with short-exact rows. Suppose that the map $\phi : Z \to R$ is an admissible monomorphism with cokernel $l : R \to S$ and that δ is an isomorphism. Then $\epsilon : Y \to Q$ is an admissible monomorphism with cokernel $l \circ k : Q \to S$.

Proof. This can be proven by passing to an abelianisation. □

Proof of Theorem 3.1.18. Let $i : A \to B$ and $j : A' \to B'$ be cofibrations with respective cokernels $f : B \to C$ and $g; B' \to C$. Consider the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A\otimes A' & \xrightarrow{1\otimes j} & A\otimes B' & \xrightarrow{1\otimes g} & A\otimes C' & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle i\otimes 1} & & \downarrow & & \| & & \\
0 & \longrightarrow & B\otimes A' & \longrightarrow & P & \longrightarrow & A\otimes C' & \longrightarrow & 0\\
 & & \| & & \downarrow{\scriptstyle i\boxtimes j} & & \downarrow{\scriptstyle i\otimes 1} & & \\
0 & \longrightarrow & B\otimes A' & \xrightarrow{1\otimes j} & B\otimes B' & \xrightarrow{1\otimes g} & B\otimes C' & \longrightarrow & 0
\end{array}$$

where the top left square is a push-out. Since cofibrations are pure by assumption, the rows of the diagram are exact. Moreover, both $i \otimes id_{A'}$ and $i \otimes id_{C'}$ are admissible monomorphisms, and the cokernel of $i \otimes id_{C'}$ is $C \otimes C'$. By Proposition 3.1.20, $i \boxtimes j$ is an admissible monomorphism with cokernel $C \otimes C'$. By assumption $C \otimes C' \in \mathfrak{C}$, so that $i \boxtimes j$ is a cofibration. Again by assumption, if either of C or C' is in $\mathfrak{W}$ then so is $C \otimes C'$, and hence in this case $i \boxtimes j$ is a trivial cofibration. □

Remark 3.1.21. *The statement of Theorem 3.1.18 also holds without the assumption that the monoidal structure is compatible with the exact structure, since it was not used at all in proof. This is also shown in [78]. However the remaining results do require this assumption.*

The next lemma says that if cofibrant objects are flat then condition 1 in Theorem 3.1.18 is automatically satisfied.

Lemma 3.1.22. *Suppose $\mathcal{E}$ is a symmetric monoidal exact category with enough flat objects. If $C \in \mathcal{E}$ is flat then every short exact sequence*

$$0 \to A \to B \to C \to 0$$

is pure.

Proof. Suppose Z is arbitrary and let

$$0 \to X \to Y \to Z \to 0$$

be a short exact sequence with Y flat. We have a diagram

$$\begin{array}{ccccccc}
A\otimes X & \longrightarrow & A\otimes Y & \longrightarrow & A\otimes Z & \longrightarrow & 0\\
\downarrow & & \downarrow & & \downarrow & & \\
B\otimes X & \longrightarrow & B\otimes Y & \longrightarrow & B\otimes Z & \longrightarrow & 0\\
\downarrow & & \downarrow & & \downarrow & & \\
C\otimes X & \longrightarrow & C\otimes Y & \longrightarrow & C\otimes Z & \longrightarrow & 0\\
\downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & &
\end{array}$$

with admissibly coacyclic rows and columns. The bottom row is short exact since C is flat. Since Y is flat the middle column is short exact. We need to prove that the right-hand column is short exact. In order to do this we may pass to a right abelianization of $\mathcal{E}$, and so without loss of generality assume that $\mathcal{E}$ is abelian. Then the argument becomes a simple diagram chase. $\square$

Proposition 3.1.23. *Pure monics are stable under push out.*

Proof. Let $i : A \to B$ be a pure monic. Consider a pushout diagram

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

Since tensoring with Z preserves colimits,

$$\begin{array}{ccc} A \otimes Z & \longrightarrow & B \otimes Z \\ \downarrow & & \downarrow \\ X \otimes Z & \longrightarrow & Y \otimes Z \end{array}$$

is a push out. But by assumption $A \otimes Z \to B \otimes Z$ is an admissible monic. Hence $X \otimes Z \to Y \otimes Z$ is also an admissible monic. $\square$

Theorem 3.1.24. *Let $\mathcal{E}$ be a bicomplete, monoidal exact category Suppose that $\mathcal{E}$ has a left pseudo-compatible model structure satisfying the hypotheses of Theorem 3.1.18. In addition, suppose that the following conditions hold*

1. *If $X \in \mathfrak{C} \cap \mathfrak{W}$ and Y is arbitrary, then $X \otimes Y$ is in $\mathfrak{W}$.*

2. *Transfinite compositions of weak equivalences which are also pure monics are still weak equivalences.*

Then the model structure satisfies the monoid axiom.

Proof. The first condition implies that if i is an acyclic cofibration, then $i \otimes Y$ is a weak equivalence. By Propositions 3.1.23 and the fact that pushouts commute with cokernels any push out of $i \otimes Y$ is a weak equivalence as well as a pure monic. By the second condition, any transfinite composition of such maps is a weak equivalence. $\square$

If in $\mathcal{E}$ transfinite compositions of admissible monics are admissible monics (e.g. if $\mathcal{E}$ is weakly **AdMon**-elementary) then one can replace the second condition by requiring that the class $\mathfrak{W}$ is closed under transfinite compositions of pure monomorphisms. By this we mean that if λ is some ordinal, and $X : \lambda \to \mathcal{E}$ a continuous functor such that $0 \to X_0$ is a weak equivalence, and for each $i < j$ in λ the map $X_i \to X_j$ is a pure monic which is also a weak equivalence, then X_λ is in $\mathfrak{W}$. (This is the condition used in [40] Theorem 7.4). Since $\mathfrak{W}$ forms a thick subcategory and $X_0 \to X_\lambda$ is an admissible monic, this is equivalent to the cokernel of the map $X_0 \to X_\lambda$ being in $\mathfrak{W}$ which in turn is equivalent to $X_0 \to X_\lambda$ being a weak equivalence.

3.1.4 Model Structures on Chain Complexes

Generalising results of [28], in this section we describe a method for constructing compatible model structures on categories of chain complexes $Ch_*(\mathcal{E})$ from cotorsion pairs on $\mathcal{E}$. Note that what we describe below will not always produce a model structure. However we will show in the next chapter that it does in the case that $\mathcal{E}$ has enough projectives, and the cotorsion pair is the projective one (Example 3.1.4). First we define the collections of objects which will be candidates for the (trivially) fibrant and (trivially) cofibrant objects.

Definition 3.1.25. *Let $(\mathfrak{L}, \mathfrak{R})$ be a cotorsion pair on an exact category $\mathcal{E}$. Let $X \in Ch(\mathcal{E})$ be a chain complex.*

1. *X is called an $\mathfrak{L}$ complex if it is acyclic and $Z_n X \in \mathfrak{L}$ for all n. The collection of all $\mathfrak{L}$ complexes is denoted $\tilde{\mathfrak{L}}$.*
2. *X is called an $\mathfrak{R}$ complex if it is acyclic and $Z_n X \in \mathfrak{R}$ for all n. The collection of all $\mathfrak{R}$ complexes is denoted $\tilde{\mathfrak{R}}$.*
3. *X is called a dg$\mathfrak{L}$ complex if $X_n \in \mathfrak{L}$ for each n, and $\textbf{Hom}(X, B)$ is exact whenever B is an $\mathfrak{R}$ complex. The collection of all dg$\mathfrak{L}$ complexes is denoted $\widetilde{dg\mathfrak{L}}$.*
4. *X is called a dg$\mathfrak{R}$ complex if $X_n \in \mathfrak{R}$ for each n, and $\textbf{Hom}(A, X)$ is exact whenever A is an $\mathfrak{L}$ complex. The collection of all dg$\mathfrak{R}$ complexes is denoted $\widetilde{dg\mathfrak{R}}$.*

Notation 3.1.26. *We define the collections $\tilde{\mathfrak{L}}, \tilde{\mathfrak{R}}, \widetilde{dg\mathfrak{L}}, \widetilde{dg\mathfrak{R}}$ similarly in the categories $Ch_*(\mathcal{E})$ for $* \in \{\geqslant 0, \leqslant 0, +, -, b\}$. We will use the same notation for these collections irrespective of which category of chain complexes we are working in.*

Remark 3.1.27. *In $Ch_*(\mathcal{E})$ for $* \in \{+, -, \geqslant 0, b, \varnothing\}$ all of the above classes are closed under shifts $[n]$ for $n \leqslant 0$. For $* \in \{+, -, \leqslant 0, b, \varnothing\}$ they are closed under shifts $[n]$ for $n \geqslant 0$.*

Let us start to populate these collections. We first make the following easy observation.

Proposition 3.1.28. *Let X be an $\mathfrak{R}$-complex. Then $X_n \in \mathfrak{R}$ for each n.*

Proof. For each n we have a short exact sequence

$$0 \to Z_n X \to X_n \to Z_{n-1} X \to 0$$

and $Z_n X, Z_{n-1} X \in \mathfrak{R}$. By Proposition 3.1.1 $\mathfrak{R}$ is closed under extensions. $\square$

With this in hand the result belows generalises immediately from [28] Lemma 3.4.

Lemma 3.1.29. *1. Bounded below complexes with entries in $\mathfrak{L}$ are $dg\mathfrak{L}$ complexes.*

2. Bounded above complex with entries in $\mathfrak{R}$ are $dg\mathfrak{R}$ complexes.

Gillespie's crucial Proposition 3.6 in [28] does not hold in arbitrary exact categories. However some of it can be salvaged to give the following two results.

Proposition 3.1.30. *Let $(\mathfrak{L}, \mathfrak{R})$ be a cotorsion pair in an exact category $\mathcal{E}$. Then in $Ch_*(\mathcal{E})$ for $* \in \{+, -, b, \varnothing\}$ we have*

1. $dg\tilde{\mathfrak{L}} = {}^{\perp}\tilde{\mathfrak{R}}$.
2. $dg\tilde{\mathfrak{R}} = \tilde{\mathfrak{L}}^{\perp}$
3. $\tilde{\mathfrak{R}} \subseteq (dg\tilde{\mathfrak{L}})^{\perp}$
4. $\tilde{\mathfrak{L}} \subseteq {}^{\perp}(dg\tilde{\mathfrak{R}})$
5. *Suppose $\mathcal{E}$ has enough $\mathfrak{L}$-objects. Let $X \in (dg\tilde{\mathfrak{L}})^{\perp}$ be good. Then X is an $\mathfrak{R}$-complex.*
6. *Suppose $\mathcal{E}$ has enough $\mathfrak{R}$-objects. Let $X \in {}^{\perp}dg(\tilde{\mathfrak{R}})$ be cogood. Then X is an $\mathfrak{L}$-complex.*

Proof. Parts 1) and 3) are easily seen to generalise to the exact case from the Gillespie's proof.

1. Let $X \in {}^{\perp}\tilde{\mathfrak{R}}$. Then $\text{Ext}^1(X, B) = 0$ whenever B is an $\mathfrak{R}$ complex. In particular $\text{Ext}^1_{dw}(X, B) = 0$. Hence $\textbf{Hom}(X, B)$ is exact whenever B is an $\mathfrak{R}$ complex by Corollary 2.1.61. It remains to show $X_n \in \mathfrak{L}$. Let $B \in \mathfrak{R}$. By Lemma 2.2.11 we have

$$\text{Ext}^1(X_n, B) = \text{Ext}^1(X, D^{n+1}B) = 0$$

since $D^{n+1}B \in \tilde{\mathfrak{R}}$. So $X_n \in \mathfrak{L}$, and ${}^{\perp}\tilde{\mathfrak{R}} \subset dg\tilde{\mathfrak{L}}$. Now let $X \in dg\tilde{\mathfrak{L}}$. Since the entries of X are in $\mathfrak{L}$, for any $Y \in \tilde{\mathfrak{R}}$, any short exact sequence

$$0 \longrightarrow Y \longrightarrow Z \longrightarrow X \longrightarrow 0$$

is split exact in each degree. But also $\text{Ext}^1_{dw}(X, Y) = 0$. Hence, any sequence as above must be split exact, i.e. $\text{Ext}^1(X, Y) = 0$.

2. This is dual to the previous part.

3. Let $X \in \tilde{\mathfrak{R}}$ and $A \in dg\tilde{\mathfrak{L}}$. Note that since $X_n \in \mathfrak{R}$, $\mathrm{Ext}^1(X, A) = \mathrm{Ext}^1_{dw}(X, A)$. Now since $\mathbf{Hom}(A, X)$ is exact, $\mathrm{Ext}^1_{dw}(X, A) = 0$.

4. This is dual to the previous part.

5. Let us show that X is acyclic. We will again use Proposition 2.1.28. Let n be such that d_n has a kernel. Since we have enough $\mathfrak{L}$-objects, we may choose an admissible epic $f'_n : A' \to Z_n X$ for some $A' \in \mathfrak{L}$. By Lemma 2.2.11 this induces a map $f : S^n(A') \to X$. Now $\mathrm{Ext}^1_{dw}(S^n(A')[-1], X) \subset \mathrm{Ext}^1(S^n(A')[-1], X) = 0$ by assumption. Hence f is homotopic to 0. Applying Proposition 2.1.32 the map $d'_{n+1} : X_{n+1} \to Z_n X$ is an admissible epic. By Proposition 2.1.28 X is acyclic. To see that $Z_n X \in \mathfrak{R}$, we note that since X is acyclic, we have for any $A \in \mathfrak{L}$,

$$\mathrm{Ext}^1_{\mathcal{E}}(A, Z_n X) \cong \mathrm{Ext}^1(S^n A, X) = 0$$

Since $(\mathfrak{L}, \mathfrak{R})$ is a cotorsion pair, $Z_n X \in \mathfrak{R}$. Hence $X \in \tilde{\mathfrak{R}}$ and so $(dg\tilde{\mathfrak{L}})^{\perp} \subseteq \tilde{\mathfrak{R}}$.

6. The proof for the second part is dual.

□

We also have the following

Proposition 3.1.31. *Let $* \in \{\geqslant 0\}$, and let $(\mathfrak{L}, \mathfrak{R})$ be a cotorsion pair in $\mathcal{E}$ with enough $\mathfrak{L}$-objects. Then $dg\tilde{\mathfrak{L}} = {}^{\perp}\tilde{\mathfrak{R}}$ and $\tilde{\mathfrak{R}} = (dg\tilde{\mathfrak{L}})^{\perp}$. Dually, if the cotorsion pair has enough $\mathfrak{R}$-objects, then for $* \in \{\leqslant 0\}$ $dg\tilde{\mathfrak{R}} = \tilde{\mathfrak{L}}^{\perp}$ and $\tilde{\mathfrak{L}} = {}^{\perp}dg(\tilde{\mathfrak{R}})$.*

Proof. The proofs of parts (3) and (5) in the previous proposition go through here, as does the proof that $dg\tilde{\mathfrak{L}} \subset {}^{\perp}\tilde{\mathfrak{R}}$. Now let $X \in {}^{\perp}\tilde{\mathfrak{R}}$. The same proof as in part (1) of the previous proposition shows that each X_n must be an object in $\mathfrak{L}$. Thus X is a bounded below complex of objects in $\mathfrak{L}$ and hence a $dg\tilde{\mathfrak{L}}$ complex. □

We get as an immediate corollary:

Corollary 3.1.32. *Let $(\mathfrak{L}, \mathfrak{R})$ be a cotorsion pair on an exact category $\mathcal{E}$ with enough $\mathfrak{L}$-objects and enough $\mathfrak{R}$-objects.*

1. *$(dg\tilde{\mathfrak{L}}, \tilde{\mathfrak{R}})$ is a cotorsion pair on $Ch_{\geqslant 0}(\mathcal{E})$ and $Ch_{+}(\mathcal{E})$. If $\mathcal{E}$ has all kernels then it is a cotorsion pair on $Ch(\mathcal{E})$.*

2. *$(\tilde{\mathfrak{L}}, dg\tilde{\mathfrak{R}})$ is a cotorsion pair on $Ch_{\leqslant 0}(\mathcal{E})$ and $Ch_{-}(\mathcal{E})$. If $\mathcal{E}$ has all cokernels then it is a cotorsion pair in $Ch(\mathcal{E})$.*

3. $(\tilde{\mathfrak{L}}, dg\tilde{\mathfrak{R}})$ *and* $(dg\tilde{\mathfrak{L}}, \tilde{\mathfrak{R}})$ *are cotorsion pairs in* $Ch_b(\mathcal{E})$.

4. *If* $\mathcal{E}$ *has all kernels and cokernels, in particular if* $\mathcal{E}$ *is quasi-abelian, then* $(\tilde{\mathfrak{L}}, dg\tilde{\mathfrak{R}})$ *and* $(dg\tilde{\mathfrak{L}}, \tilde{\mathfrak{R}})$ *are cotorsion pairs in* $Ch(\mathcal{E})$.

Existence of dg-Model Structures

The hope now is that the class $\mathfrak{W}$ of acyclic complexes satisfies

$$\tilde{\mathfrak{L}} = dg\tilde{\mathfrak{L}} \cap \mathfrak{W}, \quad \tilde{\mathfrak{R}} = dg\tilde{\mathfrak{R}} \cap \mathfrak{W}$$

and that the cotorsion pairs $(dg\tilde{\mathfrak{L}}, \tilde{\mathfrak{R}})$ and $(\tilde{\mathfrak{L}}, dg\tilde{\mathfrak{R}})$ are functorially complete. It is not at all clear that this will be the case. In [84] it is shown that for a bicomplete abelian category in which infinite products are exact (i.e. an $AB4^*$ abelian category) it is always the case. We suspect this result can be easily adapted for bicomplete exact categories satisfying a similar condition. In general we do not know how to give useable conditions on a cotorsion pair $(\mathfrak{L}, \mathfrak{R})$ which guarantee that $(dg\tilde{\mathfrak{L}}, \tilde{\mathfrak{R}})$ and $(\tilde{\mathfrak{L}}, dg\tilde{\mathfrak{R}})$ induce a model structure. However we will obtain some partial results in this direction. First we need acyclic complexes to form a thick subcategory.

Proposition 3.1.33. *Let* $\mathcal{E}$ *be an exact category. Then for* $* \in \{\geqslant 0, \leqslant 0, +, -, b\}$ *the full subcategory on* $\mathfrak{W}$ *is a thick subcategory of* $Ch_*(\mathcal{E})$. *If* $\mathcal{E}$ *has all kernels then this is also true for* $* = \{\varnothing\}$.

Proof. One may assume that $\mathcal{E}$ is abelian by passing to a left abelianization for $* \in \{\geqslant 0, +, b\}$, (or a right abelianization for $* \in \{\leqslant 0, -\}$). The result in this case follows from the long exact sequence on homology. □

It turns out that we always have the inclusions $\tilde{\mathfrak{L}} \subset dg\tilde{\mathfrak{L}} \cap \mathfrak{W}$, and $\tilde{\mathfrak{R}} \subset dg\tilde{\mathfrak{R}} \cap \mathfrak{W}$. This follows from the next result, which is an easy modification of the proof of [28] Lemma 3.9.

Lemma 3.1.34. *Every chain map from an* $\mathfrak{L}$ *complex to an* $\mathfrak{R}$ *complex is homotopic to* 0.

Corollary 3.1.35. *Let* $(\mathfrak{L}, \mathfrak{R})$ *be a cotorsion pair in an exact category. Then* $\tilde{\mathfrak{L}} \subset dg\tilde{\mathfrak{L}} \cap \mathfrak{W}$, *and* $\tilde{\mathfrak{R}} \subset dg\tilde{\mathfrak{R}} \cap \mathfrak{W}$.

In order to have any chance of getting the reverse inclusion, we'll need the cotorsion pair on $\mathcal{E}$ to be hereditary. The following definition and the subsequent proposition are immediate generalisations of [71] §1.2.3 from abelian categories to exact categories.

Definition 3.1.36. *A cotorsion pair* $(\mathfrak{L}, \mathfrak{R})$ *is said to be* ***hereditary*** *if*

$$Ext^i(A, B) = 0$$

for any $A \in \mathfrak{L}, B \in \mathfrak{R}$ *and* $i \geqslant 1$.

Example 3.1.37. *Clearly the projective cotorsion pair is hereditary.*

Proposition 3.1.38. *Let $(\mathfrak{L}, \mathfrak{R})$ be a hereditary cotorsion pair on an exact category $\mathcal{E}$. Then*

1. *$\mathfrak{L}$ is resolving. That is $\mathfrak{L}$ is closed under taking kernels of admissible epis.*

2. *$\mathfrak{R}$ is coresolving. That is $\mathfrak{R}$ is closed under taking cokernels of admissible monics.*

If $\mathcal{E}$ has enough $\mathfrak{R}$-projectives then $(\mathfrak{L}, \mathfrak{R})$ is hereditary if and only if $\mathfrak{L}$ is resolving. Dually if $\mathcal{E}$ has enough $\mathfrak{L}$-injectives then $(\mathfrak{L}, \mathfrak{R})$ is hereditary if and only if $\mathfrak{R}$ is coresolving.

With this result in hand [28] Theorem 3.12 generalises immediately to the exact setting.

Theorem 3.1.39. *Let $(\mathfrak{L}, \mathfrak{R})$ be a hereditary cotorsion pair in an exact category $\mathcal{E}$. If $\mathcal{E}$ has enough projectives then in $Ch_*(\mathcal{E})$ for $* \in \{\geqslant 0, +, \varnothing\}$, $dg\tilde{\mathfrak{R}} \cap \mathfrak{W} = \tilde{\mathfrak{R}}$. If $\mathcal{E}$ has enough injectives then in $Ch_*(\mathcal{E})$ for $* \in \{\leqslant 0, -, \varnothing\}$ $dg\tilde{\mathfrak{L}} \cap \mathfrak{W} = \tilde{\mathfrak{L}}$. In particular, if $\mathcal{E}$ has enough projectives and injectives, then the induced cotorsion pairs on $\mathcal{E}$ are compatible.*

Lemma 3.14 in [28], which partially handles the case in which we may not have enough injectives or projectives also passes essentially unaffected to exact categories.

Lemma 3.1.40. *Let $\mathcal{E}$ be an exact category and $(\mathfrak{L}, \mathfrak{R})$ a cotorsion pair on $\mathcal{E}$. Consider the categories $Ch_*(\mathcal{E})$ for any $* \in \{\geqslant 0, \leqslant 0, +, -, b, \varnothing\}$.*

1. *If $(\tilde{\mathfrak{L}}, dg\tilde{\mathfrak{R}})$ is a cotorsion pair with enough projectives and $dg\tilde{\mathfrak{R}} \cap \mathfrak{W} = \tilde{\mathfrak{R}}$ then $dg\tilde{\mathfrak{L}} \cap \mathfrak{W} = \tilde{\mathfrak{L}}$.*

2. *If $(dg\tilde{\mathfrak{L}}, \tilde{\mathfrak{R}})$ is a cotorsion pair with enough injectives and $dg\tilde{\mathfrak{L}} \cap \mathfrak{W} = \tilde{\mathfrak{L}}$ then $dg\tilde{\mathfrak{R}} \cap \mathfrak{W} = \tilde{\mathfrak{R}}$.*

These next two results partially deal with the issue of completeness.

Lemma 3.1.41. *Let $\mathcal{E}$ be an exact category. Suppose*

$$0 \longrightarrow B \longrightarrow A \xrightarrow{f} X \longrightarrow 0$$

is a short exact sequence of complexes in the degree wise exact structure with both B and cone(f) either good or cogood. Then B is acyclic if and only if f is a quasi-isomorphism.

Proof. Let $I : \mathcal{E} \to \mathcal{A}$ a suitable abelianization . Then by [83] Exercise 1.59 there is a long exact sequence

$$\ldots \longrightarrow H_{n+1}(\mathrm{Ker}(I(f_\bullet))) \longrightarrow H_n(\mathrm{cone}(I(f_\bullet))) \longrightarrow H_n(\mathrm{Coker}(I(f_\bullet))) \longrightarrow \ldots$$

$$\longrightarrow H_{n-1}(\mathrm{Ker}(I(f))) \longrightarrow \ldots$$

If $f_\bullet$ is a quasi-isomorphism, then $\text{cone}(I(f_\bullet))$ is acyclic. It is also an admissible epimorphism, so $\text{Coker}(I(f_\bullet)) = 0$. Hence $\text{Ker}(I(f_\bullet)) = I(B)$ is acyclic.

If B is acyclic then again since $\text{Coker}(I(f_\bullet)) = 0$, $H_n(\text{cone}(I(f_\bullet))) = 0$ as well. Thus $I(f)$ is a quasi-isomorphism, so f is as well.

□

Proposition 3.1.42. *Let $(\mathfrak{L}, \mathfrak{R})$ be a functorially complete cotorsion pair on an exact category $\mathcal{E}$. Then the cotorsion pair $(dg\tilde{\mathfrak{L}}, \tilde{\mathfrak{R}})$ on both $Ch_{\geqslant 0}(\mathcal{E})$ and $Ch_{+}(\mathcal{E})$ has enough functorial projectives.*

Proof. Let $X_\bullet$ be an object of $Ch_*(\mathcal{E})$ where $* \in \{\geqslant 0, +\}$. By an easy adaptation of the proof of Lemma 2.1.47, one can find a (functorial) quasi-isomorphism $f_\bullet : L_\bullet \to X_\bullet$ with each L_n an object of $\mathfrak{L}$, which is an admissible epimorphism, and whose kernel is a complex $R_\bullet$ with $R_n \in \mathfrak{R}$. Now $L_\bullet$ is a $dg\mathfrak{L}$ complex by Lemma 3.1.29. By Lemma 3.1.41 $R_\bullet$ is acyclic, and therefore an $\mathfrak{R}$-complex. So the cotorsion pair has enough (functorial) projectives. □

This is essentially all that can be said at this level of generality.

Properties of dg-Model Structures

Definition 3.1.43. *Let $\mathcal{E}$ be an exact category and $(\mathfrak{L}, \mathfrak{R})$ a cotorsion pair on $\mathcal{E}$. If $(\tilde{\mathfrak{L}}, dg\tilde{\mathfrak{R}})$ and $(dg\tilde{\mathfrak{L}}, \tilde{\mathfrak{R}})$ are (functorially) complete cotorsion pairs on $Ch_*(\mathcal{E})$ for $* \in \{\geqslant 0, \leqslant 0, b, +, -\varnothing\}$ satisfying $dg\mathfrak{L} \cap \mathfrak{W} = \tilde{\mathfrak{L}}$ and $dg\mathfrak{R} \cap \mathfrak{W} = \tilde{\mathfrak{R}}$, then we say $(\mathfrak{L}, \mathfrak{R})$ is $dg_* -$* ***compatible***.

In particular, if $(\mathfrak{L}, \mathfrak{R})$ is dg_*-compatible, then there is an induced compatible model structure on $Ch_*(\mathcal{E})$. The resulting model structure will have quasi-isomorphisms as its weak equivalences.

Proposition 3.1.44. *Suppose that $* \in \{\geqslant 0, \leqslant 0, +, -, b\}$ cotorsion pair on an exact category $\mathcal{E}$. The weak equivalences in the induced model structure are precisely the quasi-isomorphisms. If $\mathcal{E}$ has all kernels then this is also true for $* \in \{\varnothing\}$.*

Proof. First we show that admissible monics and admissible epics which are weak equivalences are quasi-ismorphisms. We will show it for monics, the case of epics being dual. Let $f : A \to B$ be an an admissible monic which is a weak equivalence. It is sufficient to show that $I(f)$ is quasi-isomorphism, where $I : \mathcal{E} \to \mathcal{A}(\mathcal{E})$ is a suitable abelianization. Now we have an exact sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

with $C \in \mathfrak{W}$. In particular, C is acyclic. By (the dual of) Lemma 3.1.41, f is a quasi-isomorphism.

Let f be a morphism of $Ch_*(\mathcal{E})$. Factor it as $p \circ i$ where i is a fibration, p is a cofibration and either p or i is trivial, and therefore a quasi-isomorphism. By the exact triangle (after passing to an abelianisation)

$$\mathrm{cone}(i) \to \mathrm{cone}(f) \to \mathrm{cone}(p) \to^{+1}$$

and the fact that acyclic complexes form a thick subcategory, we find that f is a quasi-isomorphism if and only the other factor is trivial. □

Remark 3.1.45. *The previous result says that the homotopy category of a model structure arising from a dg-compatible cotorsion pair is the derived category (for $* \in \{+, -, b, \varnothing\}$).*

Such model structures are also both left and right proper. More generally, we have the following.

Proposition 3.1.46. *Let $\mathcal{E}$ be an exact category. Let $* \in \{\geqslant 0, \leqslant 0, +, -, b\}$. Suppose there is a model structure on $Ch_*(\mathcal{E})$ whose weak equivalences are the quasi-isomorphisms and such that any cofibration is an admissible monomorphism in each degree. Then the model structure is left proper. If $\mathcal{E}$ has all kernels then this is also true for $Ch(\mathcal{E})$. Dually, if any fibration is an admissible epimorphism in each degree then the model structure is right proper.*

Proof. The dual case is slightly easier to write down, so we will prove that. We need to check that, given a pull-back diagram

$$\begin{array}{ccc} A_\bullet & \xrightarrow{p'} & B_\bullet \\ \downarrow{\scriptstyle q'} & & \downarrow{\scriptstyle q} \\ X_\bullet & \xrightarrow{p} & Y_\bullet \end{array}$$

where p is an admissible epic, and q is a quasi-isomorphism, then q' is a quasi-isomorphism. By Lemma 2.1.6 without loss of generality, we may assume that the category $\mathcal{E}$ is actually abelian. We argue by elements. $A_\bullet$ is isomorphic to

$$\{(x, b) \in X_\bullet \times B_\bullet : p(x) = q(b)\}$$

with q' and p' being the restrictions of the projections. Suppose $(x, b) \in \mathrm{Ker} d_n^A$ is such that $q'(x, b) = x = 0$. But then $q(b) = p(x) = 0$. So $b = d_{n+1}^B(\tilde{b})$ for some b, and $(x, b) = d_{n+1}^A((0, \tilde{b}))$. Now suppose $x \in \mathrm{Ker} d_n^X$. Then $p(x) \in \mathrm{Ker} d_n^Y$. Thus there is a $b \in \mathrm{Ker} d_n^B$ and a $\tilde{y} \in Y_{n+1}$ such that $q(b) = p(x) + d_{n+1}^Y(\tilde{y})$. Now, p is an epic, so there is $\tilde{x} \in X_{n+1}$ such that $\tilde{y} = p(\tilde{x})$. Write $a = (x + d_{n+1}^X(\tilde{x}), b)$. Then $a \in A_\bullet$ and $q'(a) = x + d_{n+1}^X(\tilde{x})$. This shows that q' is a quasi-isomorphism. □

Small dg-Cotorsion Pairs

Let us now examine when the cotorsion pair $(\tilde{\mathfrak{L}}, dg\tilde{\mathfrak{R}})$ is small.

Proposition 3.1.47. *Let $(\mathfrak{L}, \mathfrak{R})$ be a cotorsion pair in an exact category $\mathcal{E}$ which has a set of admissible generators $\mathcal{G}$. Suppose that $(\mathfrak{L}, \mathfrak{R})$ is cogenerated by a set $\{A_i\}_{i \in I}$. Then $(dg\tilde{\mathfrak{L}}, \tilde{\mathfrak{R}})$ is cogenerated by the set*

$$S = \{S^n(G) : G \in \mathcal{G}, n \in \mathbb{Z}\} \cup \{S^n(A_i) : n \in \mathbb{Z}, i \in I\}$$

for $* \in \{+\}$ (and. $* \in \{\emptyset\}$ if $\mathcal{E}$ has kernels) and

$$S = \{S^n(G) : G \in \mathcal{G}, n \geqslant 0\} \cup \{S^n(A_i) : n \geqslant 0, i \in I\}$$

for $* \in \{\geqslant 0\}$.

Furthermore, suppose $(\mathfrak{L}, \mathfrak{R})$ is small with generating morphisms the map $\{0 \to G : G \in \mathcal{G}\}$ together with monics k_i as below (one for each $i \in I$):

$$0 \longrightarrow Y_i \xrightarrow{k_i} Z_i \longrightarrow A_i \longrightarrow 0$$

Then $(dg\tilde{\mathfrak{L}}, \tilde{\mathfrak{R}})$ is small with generating morphisms the set

$$\tilde{I} = \{0 \to D^n(G)\} \cup \{S^{n-1}(G) \to D^n(G)\} \cup \{S^n(k_i) : S^n(Y_i) \to S^n(Z_i)\}$$

for $* \in \{+\}$ (and. $* \in \{\emptyset\}$ if $\mathcal{E}$ has kernels) and

$$\tilde{I} = \{0 \to S^0(G)\} \cup \{0 \to D^n(G) : n > 0\} \cup \{S^{n-1}(G) \to D^n(G) : n > 0\}$$
$$\cup \{S^n(k_i) : S^n(Y_i) \to S^n(Z_i) : n \geqslant 0\}$$

for $* \in \{\geqslant 0\}$.

Proof. For $* \in \{+, \emptyset\}$ the proof of [29] Proposition 3.8 generalises immediately to exact categories. Now consider the case $* \in \{\geqslant 0\}$. The only difference in the proof is that now the generating set for $Ch_{\geqslant 0}(\mathcal{E})$ is $\{D^n(G) : G \in \mathcal{G} : n > 0\} \cup \{S^0(G) : G \in \mathcal{G}\}$. This is also a subset of $dg\tilde{\mathfrak{L}}$. ☐

Remark 3.1.48. *In the situation of the previous proposition, if the domains of the generating morphisms for the cotorsion pair $(\mathfrak{L}, \mathfrak{R})$ are compact, then the domains of the maps in I are also compact by Proposition 2.2.17.*

3.2 The Projective Model Structure and the Dold-Kan Correspondence

3.2.1 The Projective Model Structure

In this section $\mathcal{E}$ is an exact category with enough functorial projectives. We denote the collection of all projective objects in $\mathcal{E}$ by $\mathbf{Proj}(\mathcal{E})$

Definition 3.2.1. *Let $\mathcal{E}$ be an exact category. If it exists, the **projective model structure** on $Ch_*(\mathcal{E})$, for $* \in \{+, \varnothing\}$ is the model structure in which*

- *Weak equivalences are quasi-isomorphisms.*
- *Fibrations are degree-wise admissible epics.*
- *Cofibrations are maps which have the left-lifting property with respect to acyclic fibrations.*

Proposition 3.2.2. *Let $\mathcal{E}$ be an exact category. Suppose that the cotorsion pair $(dg\widetilde{\mathbf{Proj}}(\mathcal{E}), \widetilde{\mathbf{Ob}}(\mathcal{E}))$ on $Ch_*(\mathcal{E})$ for $* \in \{+, \geqslant 0, \varnothing\}$ has enough functorial projectives. Then it has enough functorial injectives.*

Proof. Let $X_\bullet$ be an object of $Ch_*(\mathcal{E})$, and let $f_\bullet : L_\bullet \to X_\bullet$ be a quasi-isomorphism and admissible epimorphism with acyclic kernel, and $L_\bullet \in dg\widetilde{\mathbf{Proj}}(\mathcal{E})$.

We have a short exact sequence

$$0 \to X_\bullet \to \text{cone}(f_\bullet) \to L_\bullet[-1] \to 0$$

$\text{cone}(f_\bullet)$ is an acyclic complex, so it is in $\widetilde{\mathbf{Ob}}(\mathcal{E})$. Clearly $L_\bullet[-1] \in dg\widetilde{\mathbf{Proj}}(\mathcal{E})$. □

We are now ready to prove the following theorem.

Theorem 3.2.3. *Let $\mathcal{E}$ be an exact category with enough projectives. Then the projective model structure exists on $Ch_+(\mathcal{E})$ and is compatible. It is functorial if $\mathcal{E}$ has enough functorial projectives. It is cellular if $\mathcal{E}$ is elementary, and combinatorial if $\mathcal{E}$ is locally presentable. If $\mathcal{E}$ has all kernels and $(\mathbb{N}_0, \mathbf{AdMon})$-colimits exist and are exact, then this is all true for $Ch(\mathcal{E})$ as well.*

Proof. Consider the projective cotorsion pair $(\mathbf{Proj}(\mathcal{E}), \mathbf{Ob}(\mathcal{E}))$ on $\mathcal{E}$. By Corollary 3.1.32, $(dg\widetilde{\mathbf{Proj}}(\mathcal{E}), \widetilde{\mathbf{Ob}}(\mathcal{E}))$ is a cotorsion pair on $Ch_+(\mathcal{E})$. It is functorially complete by Proposition 3.1.42 and Proposition 3.2.2

We claim that $(\widetilde{\mathbf{Proj}}(\mathcal{E}), dg\widetilde{\mathbf{Ob}}(\mathcal{E}))$ is also a cotorsion pair on $Ch_+(\mathbf{Ob}(\mathcal{E}))$. First note that $\widetilde{\mathbf{Proj}}(\mathcal{E})$ consists of split exact complexes of projectives. By Proposition 2.2.14 this is precisely the class of projective objects in $Ch_+(\mathcal{E})$. Then by Proposition 3.2.11 $dg\widetilde{\mathbf{Ob}}(\mathcal{E}) = Ch_+(\mathbf{Ob}(\mathcal{E}))$. Hence $(\widetilde{\mathbf{Proj}}(\mathcal{E}), dg\widetilde{\mathbf{Ob}}(\mathcal{E}))$ is just the projective cotorsion pair. Now $\widetilde{\mathbf{Ob}}(\mathcal{E})$ is the class of all acyclic complexes, $\mathfrak{W}$. Thus $dg\widetilde{\mathbf{Ob}}(\mathcal{E}) \cap \mathfrak{W} = Ch_+(\mathcal{E}) \cap \mathfrak{W} = \mathfrak{W} = \widetilde{\mathbf{Ob}}(\mathcal{E})$. Moreover $Ch_+(\mathcal{E})$ has enough projectives by Corollary 2.2.15. By Lemma 3.1.40 it remains to prove that $(\widetilde{\mathbf{Proj}}(\mathcal{E}), dg\widetilde{\mathbf{Ob}}(\mathcal{E}))$ is (functorially) complete. But in a category with enough (functorial) projectives the projective cotorsion pair is always (functorially) complete by Example 3.1.4.

Assume further that $\mathcal{E}$ is elementary. Then by Example 3.1.15, the cotorsion pair $(\widetilde{\mathbf{Proj}}(\mathcal{E}), dg\widetilde{\mathbf{Ob}}(\mathcal{E}))$ is small and by Proposition 3.1.47 the cotorsion pair $(dg\widetilde{\mathbf{Proj}}(\mathcal{E}), \widetilde{\mathbf{Ob}}(\mathcal{E}))$ is small. By Lemma 3.1.16, the model structure is cellular. The fact about combinatoriality is clear.

The proof for unbounded complexes works in almost exactly the same way. All that needs to be verified in this case is that $(dg\widetilde{\mathbf{Proj}}(\mathcal{E}), \widetilde{\mathbf{Ob}}(\mathcal{E}))$ is complete. Now the class of projectives is closed under $\mathbb{N}$-indexed extensions by Lemma 3.1.1. Completeness therefore follows from from Corollary 2.1.55, Proposition 2.1.56 and Proposition 3.2.2.

□

Remark 3.2.4. *The existence of the projective model structure on bounded below chain complexes on a quasi-abelian category with enough projectives was already known to Bühler [15] (see Appendix C). The proof there is more direct. In fact the proof works for any idempotent complete exact category in which the class of all kernel-cokernel pairs forms the exact structure (all kernels and cokernels need not exist).*

Recall that if $\mathcal{E}$ is (quasi)-elementary quasi-abelian, then Proposition 2.2.7 says that $LH(\mathcal{E})$ is as well. Thus the projective model structure exists on $Ch(LH(\mathcal{E}))$. Moreover the induced functor $I : Ch(\mathcal{E}) \to Ch(LH(\mathcal{E}))$ is then right Quillen. Indeed it is left adjoint to the induced functor $C : Ch(LH)(\mathcal{E}) \to Ch(\mathcal{E})$. It preserves fibrations since $I : \mathcal{E} \to LH(\mathcal{E})$ is a left abelianization, and it preserves quasi-isomorphisms by Corollary 2.1.80. Moreover by Theorem 2.1.78, Proposition 2.1.79 and Proposition 3.1.44 it induces an equivalence between the homotopy categories. We therefore have

Proposition 3.2.5. *Let $\mathcal{E}$ be an elementary quasi-abelian category. Then the adjunction*

$$Ch(LH(\mathcal{E})) \underset{I}{\overset{C}{\rightleftarrows}} Ch(\mathcal{E})$$

is a Quillen equivalence between the projective model structures.

We claim that the projective model structure exists also on $Ch_{\geqslant 0}(\mathcal{E})$ for $\mathcal{E}$ an exact category with kernels. It will be strongly left pseudo-compatible, but not compatible.

Definition 3.2.6. *Let $\mathcal{E}$ be an exact category. If it exists, the* ***projective model structure*** *on $Ch_{\geqslant 0}(\mathcal{E})$, is the model structure in which*

- *Weak equivalences are quasi-isomorphisms.*
- *Fibrations are degree-wise admissible epics in each strictly positive degree.*
- *Cofibrations are maps which have the left-lifting property with respect to acyclic fibrations.*

Theorem 3.2.7. *Let $\mathcal{E}$ be an exact category with enough projectives and which has all kernels. Then the projective model structure exists on $Ch_{\geqslant 0}(\mathcal{E})$. Moreover it is a strong left pseudo-compatible model structure with Waldhausen pair $(dg\tilde{\boldsymbol{Proj}}(\mathcal{E}), \mathfrak{W})$. In particular the acyclic cofibrations are the degree-wise admissible monics whose cokernels are split exact complexes of projectives. If $\mathcal{E}$ is elementary then it is cellular. In particular if $\mathcal{E}$ is locally presentable and elementary then the projective model structure is combinatorial.*

Proof. The class of weak equivalences satisfies the 2-out-of-6 property since it does so in $Ch_{+}(\mathcal{E})$. Denote the class of fibrations by $\mathcal{F}$ and of weak equivalences by $\mathcal{W}$. Also denote the class of admissible monomorphisms with degree-wise projective cokernel by $\mathcal{C}$. Let us show that $\mathcal{F} \cap \mathcal{W}$ consists of quasi-isomorphisms which are admissible epimorphisms in each degree. In order to do this, one may first pass to a left abelianization and assume that $\mathcal{E}$ is abelian. Here the argument is a simple diagram chase. By Proposition 3.1.42 and Proposition 3.2.2, it follows $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ is a (compatible) weak factorisation system with corresponding cotorsion pair $(dg\tilde{\mathbf{Proj}}(\mathcal{E}), \mathfrak{W})$. In particular the cofibrations in the sense of Definition 3.2.6 coincide with the class $\mathcal{C}$. It therefore remains to check that $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ is a weak factorisation system.

Let us first check the lifting conditions. First suppose a map $A_\bullet \rightarrow B_\bullet$ in $Ch_{\geqslant 0}(\mathcal{E})$ has the left lifting property with respect to maps $X_\bullet \rightarrow Y_\bullet$ in $Ch_{\geqslant 0}(\mathcal{E})$ which are admissible epimorphisms in each strictly positive degree. Let $E_\bullet \rightarrow F_\bullet$ be a map between any complexes in $Ch(\mathcal{E})$ which is an admissible epimorphism in all degrees. Consider a diagram

$$\begin{array}{ccc} A_\bullet & \longrightarrow & E_\bullet \\ \downarrow & & \downarrow \\ B_\bullet & \longrightarrow & F_\bullet \end{array}$$

Since $A_\bullet$ and $B_\bullet$ are in $Ch_{\geqslant 0}$ we can factor the above diagram as

$$\begin{array}{ccccc} A_\bullet & \longrightarrow & \tau_{\geqslant 0}E_\bullet & \longrightarrow & E_\bullet \\ \downarrow & & \downarrow & & \downarrow \\ B_\bullet & \longrightarrow & \tau_{\geqslant 0}F_\bullet & \longrightarrow & F_\bullet \end{array}$$

Now the map $\tau_{\geqslant 0}E_\bullet \to \tau_{\geqslant 0}F_\bullet$ is an epimorphism in each strictly positive degree. By assumption we can find a lift as follows.

$$\begin{array}{ccccc} A_\bullet & \longrightarrow & \tau_{\geqslant 0}E_\bullet & \longrightarrow & E_\bullet \\ \downarrow & \nearrow & \downarrow & & \downarrow \\ B_\bullet & \longrightarrow & \tau_{\geqslant 0}F_\bullet & \longrightarrow & F_\bullet \end{array}$$

Thus the map $A_\bullet \to B_\bullet$ has the left-lifting property with respect to all degree-wise epimorphisms in $Ch_+(\mathcal{E})$. By Theorem 3.2.3 $A_\bullet \to B_\bullet$ is an admissible monic whose cokernel is a split exact complex of projectives. Now, any acyclic cofibration is of the form $A_\bullet \to A_\bullet \oplus \Big(\oplus_{n>0} D^n(P_n)\Big)$ where each P_n is a projective object in $\mathcal{E}$, and the map is the inclusion into the first factor of the direct sum. Clearly then it is enough to show that the collection of maps $\{0 \to D^n(P) : n > 0, P \text{ is projective }\}$ has the left lifting property with respect to $\mathcal{F}$, and that a map is in $\mathcal{F}$ if and only if it has the right-lifting property with respect to these maps. However this follows from Lemma 2.2.11 and Proposition 2.2.2.

It remains to find a (functorial) factorisation. Let $f_\bullet : X_\bullet \to Y_\bullet$ be a map in $Ch_{\geqslant 0}(\mathcal{E})$. We can factor it in $Ch_+(\mathcal{E})$ as

$$X_\bullet \to X_\bullet \oplus \Big(\oplus_{n\geqslant 0} D^n(P_n)\Big) \to Y_\bullet$$

where $X_\bullet \to X_\bullet \oplus \Big(\oplus_{n\geqslant 0} D^n(P_n)\Big)$ is the inclusion into the first factor, and $X_\bullet \oplus \Big(\oplus_{n\geqslant 0} D^n(P_n)\Big) \to Y_\bullet$ is an admissible epimorphism in each degree. Then

$$X_\bullet \to X_\bullet \oplus \Big(\oplus_{n>0} D^n(P_n)\Big) \to Y_\bullet$$

is also a factorisation of $f_\bullet$, $X_\bullet \to X_\bullet \oplus \Big(\oplus_{n>0} D^n(P_n)\Big)$ is an acyclic cofibration in $Ch_{\geqslant 0}(\mathcal{E})$, and $X_\bullet \oplus \Big(\oplus_{n>0} D^n(P_n)\Big) \to Y_\bullet$ is an admissible epimorphism in each strictly positive degree.

We prove the statement about cellularity. Suppose that $\mathcal{P}$ is a projective generating set consisting of compact objects. It follows from Proposition 3.1.47 that the weak factorsiation system $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ is cellular. From our proof above that $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ is a weak factorisation system, it follows that $\{0 \to D^n(P) : n > 0, P \in \mathcal{P}\}$ is a set of generating morphisms for $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$, so it is also a cellular weak factorisation system. The claim about combinatoriality is clear. $\square$

Remark 3.2.8. *The existence of the projective model structure on $Ch_{\geqslant 0}(\mathcal{E})$ in the case that $\mathcal{E}$ is quasi-abelian was also known. This is mentioned in a math.stackexchange.com exchange, [81], as an adaptation of the proof for $Ch_{+}(\mathcal{E})$ in [15].*

3.2.2 The Projective Model Structure on Monoidal Exact Categories

We now turn our attention to monoidal model structures on categories of chain complexes.

Proposition 3.2.9. *Let $(\mathcal{E}, \otimes, k)$ be an additive symmetric monoidal category with $\mathcal{E}$ an exact category. For $* \in \{\geqslant 0, \leqslant 0, b, +, -\}$ the flat objects in $(Ch_*(\mathcal{E}), \otimes, S^0(k))$ are precisely the complexes $F_\bullet$ in $Ch_*(\mathcal{E})$ such that for each $n \in \mathbb{Z}$, F_n is flat. If in addition countable direct sums exist and are exact, then the flat objects in $(Ch(\mathcal{E}), \otimes, S^0(k))$ are also the complexes $F_\bullet$ that for each $n \in \mathbb{Z}$, F_n is flat*

Proof. Let

$$0 \longrightarrow X_\bullet \longrightarrow Y_\bullet \longrightarrow Z_\bullet \longrightarrow 0$$

be a short exact sequence in $Ch_*(\mathcal{E})$. Let $F_\bullet$ be a complex. Then the nth row of

$$0 \longrightarrow X_\bullet \otimes F_\bullet \longrightarrow Y_\bullet \otimes F_\bullet \longrightarrow Z_\bullet \otimes F_\bullet \longrightarrow 0$$

is

$$0 \longrightarrow \bigoplus_{i+j=n} X_i \otimes F_j \longrightarrow \bigoplus_{i+j=n} Y_i \otimes F_j \longrightarrow \bigoplus_{i+j=n} Z_i \otimes F_j \longrightarrow 0$$

Since the direct sums involved are exact, this sequence is short exact if for each i, j,

$$0 \longrightarrow X_i \otimes F_j \longrightarrow Y_i \otimes F_j \longrightarrow Z_i \otimes F_j \longrightarrow 0$$

is short exact. It follows immediately that a complex whose entries are flat in $\mathcal{E}$ is itself a flat object in $Ch_*(\mathcal{E})$. To see that a flat complex must have flat entries, simply take a short exact sequence in $\mathcal{E}$, and regard it as a short exact sequence in $Ch_*(\mathcal{E})$ concentrated in degree 0. ☐

We are going to use Theorem 3.1.18. In order to deal with the third condition of that theorem we are going to need the following notion.

Definition 3.2.10. *An acyclic complex $F_\bullet \in Ch(\mathcal{E})$ is said to be $\otimes$-**stably acyclic** if for any complex $X_\bullet$, $F_\bullet \otimes X_\bullet$ is acyclic. An acyclic complex $F_\bullet \in Ch(\mathcal{E})$ is said to be **Hom-stably acyclic** if for any complex $X_\bullet$, $\mathbf{Hom}(F_\bullet, X_\bullet)$ is acyclic.*

Proposition 3.2.11. *Let F be a flat object in $\mathcal{E}$. Then for all n, the complex $D^n(F)$ is $\otimes$-stably acyclic. In particular, split exact complexes of flat objects are $\otimes$-stably acyclic. If F is projective then $D^n(F)$ is **Hom**-stably acyclic. Hence split exact complexes of projectives are **Hom**-stably acyclic.*

Proof. Clearly it is sufficient to prove the proposition for $n = 1$. In this case, $D^1(F) \otimes X_\bullet \cong F \otimes \text{cone}(id_{X_\bullet})$. Since F is flat this complex is acyclic. The proof for projectives is similar. □

We would also like our model structure to satisfy the monoid axiom. Towards this we note the following.

Proposition 3.2.12. *Let $\mathcal{E}$ be an exact category and $\mathcal{S}$ a class of morphisms in $\mathcal{E}$ closed under direct sums. Suppose that $\mathcal{E}$ is weakly $\mathcal{S}$-elementary. Then transfinite compositions of quasi-isomorphisms in $Ch(\mathcal{E})$ which are also maps in $\mathcal{S}$ are quasi-isomorphisms.*

Proof. The proof is by transfinite induction. Since a finite composition of quasi-isomorphisms is a quasi-isomorphism, the successor part of the induction is finished. Now let λ be a limit ordinal and $F : \lambda \to Ch(\mathcal{E})$ a continuous functor with $F(\alpha \leqslant \beta)_n \in \mathcal{S}$ for any morphism $\alpha \leqslant \beta$ in λ and $n \in \mathbb{Z}$. For $\alpha \leqslant \beta \leqslant \lambda$ denote by $f_{\alpha,\beta}$ the map $F_\alpha \to F_\beta$. For $\beta \leqslant \lambda$ write $f_\beta = f_{0,\beta}$. It is clear that

$$\text{cone}(f_\lambda) \cong \lim_{\to_{\beta<\lambda}} \text{cone}(f_\beta)$$

Since each f_β is a quasi-isomorphism, $\text{cone}(f_\beta)$ is acyclic. Since $\mathcal{E}$ is weakly $\mathcal{S}$-elementary, this implies $\lim_{\to_{\beta<\lambda}} \text{cone}(f_\beta)$ is acyclic, means that $\text{cone}(f_\lambda)$ is acyclic and hence that f_λ is a quasi-isomorphism. □

We are now ready to prove the following

Theorem 3.2.13. *Let $\mathcal{E}$ be a projectively monoidal exact category with enough projectives. Then the projective model structure on $Ch_+(\mathcal{E})$ is monoidal. If $\mathcal{E}$ also has kernels, then the projective model structure on $Ch_{\geqslant 0}(\mathcal{E})$ is monoidal. If in addition $\mathcal{E}$ is weakly **AdMon**-elementary then $Ch_{\geqslant 0}(\mathcal{E})$ satisfies the monoid axiom.*

Proof. By Propososition 3.2.9 the cofibrant objects are flat. By Lemma 3.1.22, all cofibrations are pure. Now since $\mathbf{Proj}(\mathcal{E})$ is closed under $\otimes$, $dg\tilde{\mathbf{Proj}}(\mathcal{E})$ is closed under $\otimes$ by Lemma 3.1.29. Now let L, L' be $dg\tilde{\mathbf{Proj}}(\mathcal{E})$-complexes with L' acyclic. We have to show that $L \otimes L'$ is acyclic. However any complex in $\tilde{\mathbf{Proj}}(\mathcal{E})$ is a split exact complex of objects in $\mathbf{Proj}(\mathcal{E})$. By Proposition 3.2.11 such an object is $\otimes$-stably acyclic.

For the assertions about the monoid axiom we must check the conditions of Theorem 3.1.24. The first condition is guaranteed, again because the trivially cofibrant objects are $\otimes$-stably acyclic. The second condition follows from Proposition 3.2.12 and by Proposition 3.1.44. □

We would like a version for unbounded complexes. We are going to prove the following.

Theorem 3.2.14. *Let $(\mathcal{E},\otimes,k)$ be a projectively monoidal, weakly* ***AdMon****-elementary exact category which has a projective generating set, such that countable product functors are admissibly exact and and countable coproduct functors are admissibly coexact. Then $(Ch(\mathcal{E}),\otimes,\underline{Hom},S^0(k))$ is a monoidal model category which satisfies the monoid axiom.*

Proof. The condition on products and coproducts guarantee that $(Ch(\mathcal{E}),\otimes,\underline{\text{Hom}},S^0(k))$ is actually a monoidal exact category. Now the proof goes through in almost exactly the same way as Proposition 3.2.13. All that remains to prove is that the tensor product of two complexes in $dg\tilde{\mathbf{Proj}}(\mathcal{E})$ is again in $dg\tilde{\mathbf{Proj}}(\mathcal{E})$. This is a consequence of the proposition below, and its corollary. □

Proposition 3.2.15. *Let $\mathcal{E}$ be a projectively monoidal exact category with a projective generating set $\mathcal{P}$. If $X_\bullet$ is in $dg\tilde{\mathbf{Proj}}(\mathcal{E})$ then for any acyclic complex $Y_\bullet$, $\underline{Hom}(X_\bullet,Y_\bullet)$ is acyclic.*

Proof. Let $\mathcal{P}$ be a generating set of projectives. Then by Observation 2.1.67, for each $P\in\mathcal{P}$, $\underline{\text{Hom}}(S^0(P),Y_\bullet)$ is acyclic. Moreover $S^0(P)\in dg\tilde{\mathbf{Proj}}(\mathcal{E})$, so $\underline{\text{Hom}}(S^0(P),Y_\bullet)$is acyclic by assumption. Now

$$\begin{aligned}\mathbf{Hom}(S^0(P),\underline{\text{Hom}}(X_\bullet,Y_\bullet)) &\cong \mathbf{Hom}(S^0(P)\otimes X_\bullet,Y_\bullet)\\ &\cong \mathbf{Hom}(X_\bullet,\underline{\text{Hom}}(S^0(P),Y_\bullet))\end{aligned}$$

Since $X_\bullet$ is in $dg\tilde{\mathbf{Proj}}(\mathcal{E})$, $\mathbf{Hom}(X_\bullet,\underline{\text{Hom}}(S^0(P),Y_\bullet))$ is acyclic. Since $\mathcal{P}$ is a projective generating set in a quasi-abelian category $\underline{\text{Hom}}(X_\bullet,Y_\bullet)$ is acyclic. □

Corollary 3.2.16. *If $X_\bullet$ and $Z_\bullet$ in $dg\tilde{\mathbf{Proj}}(\mathcal{E})$, so is $X_\bullet\otimes Z_\bullet$*

Proof. Since $\mathbf{Proj}(\mathcal{E})$ is closed under countable direct sums the entries of the tensor product are objects in $\mathbf{Proj}(\mathcal{E})$. Let $Y_\bullet$ be an acyclic complex.

$$\mathbf{Hom}(X_\bullet\otimes Z_\bullet,Y_\bullet)\cong\mathbf{Hom}(X_\bullet,\underline{\text{Hom}}(Z_\bullet,Y_\bullet))$$

By the Proposition, $\underline{\text{Hom}}(Z_\bullet,Y_\bullet)$ is acyclic. Since $X_\bullet$ is in $dg\tilde{\mathbf{Proj}}(\mathcal{E})$, $\mathbf{Hom}(X_\bullet,\underline{\text{Hom}}(Z_\bullet,Y_\bullet))$ is acyclic. □

Duality

In any closed monoidal category $(\mathcal{E},\otimes,k,\underline{Hom})$ one can consider the functor

$$(-)^\vee:\mathcal{E}\to\mathcal{E}^{op},\ E\mapsto\underline{Hom}(E,k)$$

This functor is contravariantly self-adjoint.

Proposition 3.2.17. *Let $\mathcal{E}$ be a monoidal elementary exact category. The functor $(-)^{\vee}$: $Ch_*(\mathcal{E}) \to Ch_*(\mathcal{E})^{op}$ is left Quillen for the projective model structure on the left and its opposite model structure on the right.*

Proof. Since any object of $Ch_*(\mathcal{E})^{op}$ is cofibrant and $\underline{Hom}(-,k)$ clearly preserves degree-wise split exact sequences all that remains to prove is that it sends trivially cofibrant objects to acyclic objects. Indeed if $P_{\bullet}$ is trivially cofibrant then $P_{\bullet} \to 0$ is a homotopy equivalence. Hence $0 \to (P_{\bullet})^{\vee}$ is a homotopy equivalence and we're done. □

3.2.3 The Dold-Kan Correspondence

In this section we generalise the Dold-Kan correspondence for abelian groups to elementary exact categories. If $\mathcal{C}$ is a category, we denote by $\mathbf{s}\mathcal{C}$ the functor category $[\Delta^{op}, \mathcal{C}]$, where Δ is the usual simplicial category. We use this to show that when $\mathcal{E}$ is elementary the projective model structure on $Ch(\mathcal{E})$ and $Ch_{\geqslant 0}(\mathcal{E})$ are Kan complex-enriched.

Let us recall the Dold-Kan correspondence for abelian categories. The exposition here follows [83] 8.4. For an abelian category $\mathcal{A}$, there are functors

$$\Gamma : Ch_{\geqslant 0}(\mathcal{A}) \to \mathbf{s}\mathcal{A},\ N : \mathbf{s}\mathcal{A} \to Ch_{\geqslant 0}(\mathcal{A})$$

constructed as follows:

Given an object $A \in \mathbf{s}\mathcal{A}$ set

$$NA_n = \bigcap_{i=0}^{n-1} \mathrm{Ker}(d_i)$$

Define a differential $\delta_n = (-1)^n d_n : NA_n \to NA_{n-1}$. It follows from the simplicial relations that $NA_{\bullet}$ is a chain complex. Moreover, since by definition a map of simplicial objects commutes with the face maps, this construction is functorial.

The construction of Γ is more involved. For a chain complex $C \in Ch_{\geqslant 0}(\mathcal{A})$, one sets

$$\Gamma(C)_n = \bigoplus_{\eta:[n]\twoheadrightarrow[p], p\leqslant n} C_{\eta}$$

where for $\eta : [n] \twoheadrightarrow [p]$, $C_{\eta} = C_p$. Given a morphism $\alpha : [n] \to [m]$ in Δ, define a morphism $\Gamma(C)(\alpha) : \Gamma_m(C) \to \Gamma_n(C)$ by its restriction $\Gamma(\alpha,\eta) : C_{\eta} \to \Gamma(C)$ to each summand C_{η} as follows. For each surjection $\eta : [n] \to [p]$ we consider its epi-mono factorisation $\epsilon\eta'$ of $\eta\alpha$.

$$\begin{array}{ccc} [m] & \xrightarrow{\alpha} & [n] \\ \downarrow{\scriptstyle \eta'} & & \downarrow{\scriptstyle \eta} \\ [q] & \xrightarrow{\epsilon} & [p] \end{array}$$

If $p = q$ so that $\eta\alpha = \eta'$ then we take $\Gamma(\alpha, \eta)$ to be the natural identification of C_η with the summand $C_{\eta'}$ of Γ_m. If $p = q + 1$ and $\epsilon = \epsilon_p$, so that the image of $\eta\alpha$ is $\{0, \ldots, p-1\}$, then we take $\Gamma(\alpha, \eta)$ to be the composition

$$C_\eta = C_p \xrightarrow{d} C_{p-1} = C_{\eta'} \longrightarrow \Gamma_m(C)$$

Otherwise we take $\Gamma(\alpha, \eta)$ to be 0.

The Dold-Kan Correspondence says the following

Theorem 3.2.18 (Dold-Kan for Abelian Categories)**.** *Let $\mathcal{A}$ be an abelian category. Then the functors*

$$\Gamma : Ch_{\geqslant 0}(\mathcal{A}) \to \boldsymbol{s}\mathcal{A}, \; N : \boldsymbol{s}\mathcal{A} \to Ch_{\geqslant 0}(\mathcal{A})$$

form an equivalence of categories.

Proof. See [83] §8.4. □

The constructions of Γ and N make sense in any exact category which has kernels. Thus for an exact category $\mathcal{E}$ with kernels we get functors

$$\Gamma : Ch_{\geqslant 0}(\mathcal{E}) \to \mathbf{s}\mathcal{E}, \;\; N : \mathbf{s}\mathcal{E} \to Ch_{\geqslant 0}(\mathcal{E})$$

constructed mutatis mutandis as above.

Corollary 3.2.19 (Dold-Kan for Exact Categories)**.** *Let $\mathcal{E}$ be an elementary exact category. The functors*

$$\Gamma : Ch_{\geqslant 0}(\mathcal{E}) \to \boldsymbol{s}\mathcal{E}, \;\; N : \boldsymbol{s}\mathcal{E} \to Ch_{\geqslant 0}(\mathcal{E})$$

defined above are weakly inverse to each other. In particular they give equivalences of categories.

Proof. Pick a left abelianization $I : \mathcal{E} \to \mathcal{A}$. Then I extends to functors $\mathbf{s}\mathcal{E} \to \mathbf{s}\mathcal{A}$ and $Ch_{\geqslant 0}(\mathcal{E}) \to Ch_{\geqslant 0}(\mathcal{A})$, which we will also denote by I. Since I preserves kernels we get a commutative diagram.

$$\begin{array}{ccc} \mathbf{s}\mathcal{A} & \xrightarrow{N} & Ch_{\geqslant 0}(\mathcal{A}) \\ {\scriptstyle I}\uparrow & & \uparrow{\scriptstyle I} \\ \mathbf{s}\mathcal{E} & \xrightarrow{N} & Ch_{\geqslant 0}(\mathcal{E}) \end{array}$$

It is also clear from the construction of Γ that the following diagram commutes

$$\begin{array}{ccc} \mathbf{s}\mathcal{A} & \xleftarrow[\Gamma]{} & Ch_{\geqslant 0}(\mathcal{A}) \\ {\scriptstyle I}\uparrow & & \uparrow{\scriptstyle I} \\ \mathbf{s}\mathcal{E} & \xleftarrow[\Gamma]{} & Ch_{\geqslant 0}(\mathcal{E}) \end{array}$$

Since the functor I is fully faithful, Theorem 3.2.18 implies the result. □

Remark 3.2.20. *This result is actually overkill. It has been pointed out to us by Theo Buehler that the Dold-Kan equivalence is valid for any weakly idempotent complete additive category. A proof (which in fact works on the level of quasi-categories) can be found in [44] Section 35.*

If $\mathcal{A} = \mathcal{Ab}$ is just the category of abelian groups, then there is a well-known model structure on the category $\mathbf{s}\mathcal{Ab}$. The weak equivalences (resp. fibrations) are those maps of simplicial abelian groups which are weak equivalences (resp. fibrations) on the underlying simplicial set. As usual, the cofibrations are maps of simplicial abelian groups which have the left-lifting property with respect to the trivial fibrations. Moreover, the category $\mathcal{Ab}$ is an elementary abelian category. As a set of compact projective generators we can take $\mathcal{P} = \{\mathbb{Z}\}$. Thus there is a projective model structure on $Ch_{\geqslant 0}(\mathcal{Ab})$. In this case the functors N and Γ also form a Quillen equivalence between these model categories. For a proof see [33] Chapter 3 Section 2. The model structure on $\mathbf{s}\mathcal{Ab}$ is a special case of a much more general model structure.

Notation 3.2.21.

1. *Let Z be an object in a category $\mathcal{C}$. We denote by $\boldsymbol{s}Z$ the constant simplicial object in $s\mathcal{C}$ which is Z in each degree, and such that the face and degeneracy maps are all id_Z.*
2. *If $\mathcal{C}$ is additive, then the category $\boldsymbol{s}\mathcal{C}$ is enriched over $\boldsymbol{s}\mathcal{Ab}$ in an obvious way. We denote the enriched hom functor by $\boldsymbol{Hom}_{\boldsymbol{s}\mathcal{C}}$*

Theorem 3.2.22. *Suppose that $\mathcal{C}$ is a small bicomplete category, and let $\mathcal{Z} = \{Z_i : i \in I\}$ be a set of compact objects of $\mathcal{C}$. Then $\boldsymbol{s}\mathcal{C}$ is a simplicial model category with $A \to B$ a weak equivalence (respectively fibration) if and only if the induced map*

$$\boldsymbol{Hom}_{\boldsymbol{s}\mathcal{C}}(\boldsymbol{s}Z_i, A) \to \boldsymbol{Hom}_{\boldsymbol{s}\mathcal{C}}(\boldsymbol{s}Z_i, B)$$

is a weak equivalence (respectively fibration) for all $I \in I$.

Proof. See [33] Theorem 6.9. □

In particular if $\mathcal{E}$ is a small bicomplete elementary exact category, then there is a model category structure on $\mathbf{s}\mathcal{E}$ where for the set $\mathcal{Z}$ in Theorem 3.2.22 we take a generating set $\mathcal{P}$ of compact projective objects. We shall call this the **projective model structure on** $\mathbf{s}\mathcal{E}$. We are now going to show the following

Theorem 3.2.23 (Model Dold-Kan for Elementary Exact Categories)**.** *Let $\mathcal{E}$ be a small bicomplete elementary exact category. Endow $Ch_{\geqslant 0}(\mathcal{E})$ and $\boldsymbol{s}\mathcal{E}$ with their projective model structures. Then the functors*

$$\Gamma : Ch_{\geqslant 0}(\mathcal{E}) \to \boldsymbol{s}\mathcal{E},\ N : \boldsymbol{s}\mathcal{E} \to Ch_{\geqslant 0}(\mathcal{E})$$

form a Quillen equivalence.

We use the following notion:

Definition 3.2.24. *Let $\mathcal{M}, \mathcal{N}$ be model categories. $\mathcal{M}$ is said to be* ***generated*** *by a collection of functors $\{F_i : \mathcal{M} \to \mathcal{N}\}_{i \in I}$ if a map $f : X \to Y$ in $\mathcal{M}$ is a fibration (resp. weak equivalence) if and only if $F_i(f)$ is a fibration (resp. weak equivalence) for each $i \in I$.*

By construction the model structure on $\mathbf{s}\mathcal{E}$ is generated by the functors

$$\{\mathbf{Hom}_{\mathbf{s}\mathcal{E}}(\mathbf{s}P, -) : \mathbf{s}\mathcal{E} \to \mathbf{s}\mathcal{Ab}\}_{P \in \mathcal{P}}$$

where we endow $\mathbf{s}\mathcal{Ab}$ with its projective model structure.

The model structure on $Ch_{\geqslant 0}(\mathcal{E})$ is generated by a similar set of functors:

Proposition 3.2.25. *Let $\mathcal{E}$ be an elementary exact category with a projective generating set $\mathcal{P}$. The projective model structure on $Ch_{\geqslant 0}(\mathcal{E})$ is generated by the functors*

$$\{\boldsymbol{Hom}(S^0(P), -) : Ch_{\geqslant 0}(\mathcal{E}) \to Ch_{\geqslant 0}(\mathcal{Ab}) : P \in \mathcal{P}\}$$

where we endow $Ch_{\geqslant 0}(\mathcal{Ab})$ with its projective model structure.

Proof. The fibrations in $Ch_{\geqslant 0}(\mathcal{E})$ are the degree-wise admissible epics in positive degree, and the fibrations in $Ch_{\geqslant 0}(\mathcal{Ab})$ are the degree-wise epics in positive degree. Let $f_\bullet : X_\bullet \to Y_\bullet$ be a morphism in $Ch_{\geqslant 0}(\mathcal{E})$. Then the components of $\mathbf{Hom}(S^0(P), f_\bullet)$ are $\mathrm{Hom}_{\mathcal{E}}(P, f_n)$. Now $f_\bullet$ is a fibration if and only if each f_n is an admissible epimorphism for $n > 0$. This is true if and only if $\mathrm{Hom}_{\mathcal{E}}(P, f_n)$ is an epic for each $n > 0$ and each $P \in \mathcal{P}$, i.e. if and only if $\mathbf{Hom}(S^0(P), f_\bullet)$ is a fibration for each $P \in \mathcal{P}$.

It is clear that $\mathbf{Hom}(S^0(P), \mathrm{cone}(f_\bullet)) \cong \mathrm{cone}(\mathbf{Hom}(S^0(P), f_\bullet))$. Now by Corollary 2.2.4, $\mathrm{cone}(f_\bullet)$ is acyclic if and only if $\mathbf{Hom}(S^0(P), \mathrm{cone}(f_\bullet))$ is acyclic for all $P \in \mathcal{P}$. Equivalently, $f_\bullet$ is a weak equivalence if and only if $\mathbf{Hom}(S^0(P), f_\bullet)$ is a weak equivalence for each $P \in \mathcal{P}$. □

With these structures in hand, we will use the following result in order to prove the theorem.

Proposition 3.2.26. *Let $\mathcal{M}, \mathcal{N}, \mathcal{M}', \mathcal{N}'$ be model categories. Suppose $\mathcal{M}$ is generated by functors $\{F_i : \mathcal{M} \to \mathcal{N}\}_{i \in I}$, and $\mathcal{M}'$ is generated by functors $\{F'_i : \mathcal{M}' \to \mathcal{N}'\}_{i \in I}$. Let $G : \mathcal{M} \to \mathcal{M}'$ and $H : \mathcal{M}' \to \mathcal{M}$ be adjoint functors*

$$G \dashv H$$

Suppose also that there is a Quillen adjunction $P \dashv Q$, with $P : \mathcal{N} \to \mathcal{N}'$ and $Q : \mathcal{N}' \to \mathcal{N}$ such that for each $i \in I$ the diagram

$$\begin{array}{ccc} \mathcal{M} & \xleftarrow[H]{} & \mathcal{M}' \\ \big\downarrow{\scriptstyle F_i} & & \big\downarrow{\scriptstyle F_i'} \\ \mathcal{N} & \xleftarrow[Q]{} & \mathcal{N}' \end{array}$$

commutes. Then $G \dashv H$ is a Quillen adjunction.

Proof. We need to show that H preserves (acyclic) fibrations. Let f be an (acyclic) fibration in $\mathcal{M}'$. By assumption, for each i, $F_i'(f)$ is an (acyclic) fibration in $\mathcal{N}'$. Since Q is right Quillen, $Q \circ F_i'(f)$ is an (acyclic) fibration. By commutativity of the diagram

$$\begin{array}{ccc} \mathcal{M} & \xleftarrow[H]{} & \mathcal{M}' \\ \big\downarrow{\scriptstyle F_i} & & \big\downarrow{\scriptstyle F_i'} \\ \mathcal{N} & \xleftarrow[Q]{} & \mathcal{N}' \end{array}$$

$F_i \circ H(f)$ is an (acyclic) fibration for each $i \in I$. Again by assumption, $H(f)$ is an (acyclic) fibration. □

Before proving the theorem, we shall make the following easy observation.

Proposition 3.2.27. *Let $\mathcal{M}$ and $\mathcal{M}'$ be model categories, and $G : \mathcal{M} \to \mathcal{M}'$ and $H : \mathcal{M}' \to \mathcal{M}$ be Quillen adjoint functors*

$$G \dashv H$$

Suppose further that

1. *The unit and counit maps of the adjunction are weak equivalences.*
2. *G preserves weak equivalences of the form $X \to HY$ where X is cofibrant and Y is fibrant.*
3. *H preserves weak equivalences of the form $GX \to Y$ where X is cofibrant and Y is fibrant.*

Then $G \dashv H$ is a Quillen equivalence.

Proof. Let X be a cofibrant object of $\mathcal{M}$ and Y a fibrant object of $\mathcal{M}'$. Suppose that $f : GX \to Y$ is a weak equivalence. Then by assumption $HGX \to HY$ is a weak equivalence. Also by assumption $X \to HGX$ is a weak equivalence. Hence $X \to HY$ is a weak equivalence.

Conversely suppose that $X \to HY$ is a weak equivalence. Then $GX \to GHY$ is a weak equivalence by assumption. Also by assumption $GHY \to Y$ is a weak equivalence. Thus $GX \to Y$ is a weak equivalence.

□

Proof of Theorem 3.2.23. We first note that the following diagrams commute (up to natural isomorphism).

$$\begin{array}{ccc} \mathbf{s}\mathcal{E} & \xleftarrow[\Gamma]{} & Ch_{\geqslant 0}(\mathcal{E}) \\ \Big\downarrow{\scriptstyle \mathbf{Hom}_{\mathbf{s}\mathcal{E}}(\mathbf{s}P,-)} & & \Big\downarrow{\scriptstyle \mathbf{Hom}(S^0(P),-)} \\ \mathbf{s}\mathcal{Ab} & \xleftarrow[\Gamma]{} & Ch_{\geqslant 0}(\mathcal{Ab}) \end{array}$$

$$\begin{array}{ccc} \mathbf{s}\mathcal{E} & \xrightarrow{N} & Ch_{\geqslant 0}(\mathcal{E}) \\ \Big\downarrow{\scriptstyle \mathbf{Hom}_{\mathbf{s}\mathcal{E}}(\mathbf{s}P,-)} & & \Big\downarrow{\scriptstyle \mathbf{Hom}(S^0(P),-)} \\ \mathbf{s}\mathcal{Ab} & \xrightarrow{N} & Ch_{\geqslant 0}(\mathcal{Ab}) \end{array}$$

The second diagram follows from the fact that $\mathrm{Hom}(P,-) : \mathcal{E} \to \mathcal{Ab}$ preserves kernels (and therefore intersections). The first diagram follows from the fact that $\mathrm{Hom}(P,-) : \mathcal{E} \to \mathcal{Ab}$ preserves finite direct sums. By Proposition 3.2.26 the adjunction is a Quillen adjunction. Let us now check the hypotheses of Proposition 3.2.27. The unit and counit maps are isomorphisms. In particular they are weak equivalences. In the Dold-Kan correspondence for abelian groups, it can be shown that the functors $N : \mathbf{s}\mathcal{Ab} \to Ch_{\geqslant 0}(\mathcal{Ab})$ and $\Gamma : Ch_{\geqslant 0}(\mathcal{Ab}) \to \mathbf{s}\mathcal{Ab}$ both preserve all weak equivalences. By the commutativity of the above diagrams, this also implies that the functors $N : \mathbf{s}\mathcal{E} \to Ch_{\geqslant 0}(\mathcal{E})$ and $\Gamma : Ch_{\geqslant 0}(\mathcal{E}) \to \mathbf{s}\mathcal{E}$ also preserve all weak equivalences. □

3.2.4 The Simplicial Model Structure

In this section we show that for $\mathcal{E}$ elementary the projective model structure on $Ch(\mathcal{E})$ is simplicial.

Definition 3.2.28. *Let $\mathcal{M}$ be a monoidal model category. A functor $F : s\mathcal{Set} \to \mathcal{M}$ which preserves colimits and sends (acyclic) cofibrations to (acyclic) cofibrations will be called a* ***simplicial enrichment functor****.*

Let a simplicial enrichment functor F be given. Define bifunctors as follows

$$\otimes : \mathcal{M} \times s\mathcal{Set} \to \mathcal{M}, \quad E \otimes X := E \otimes F(X)$$

Now also define

$$(-)^{(-)} : \mathcal{M} \times s\mathcal{Set}^{op} \to \mathcal{M}, \quad E^X := \underline{\mathrm{Hom}}(F(X), E)$$

and

$$\mathrm{Map} : \mathcal{M}^{op} \times \mathcal{M} \to s\mathcal{S}et, \quad \mathrm{Map}(M, N)_n := \mathrm{Hom}(M \otimes \Delta^n, N)$$

Proposition 3.2.29. *The functors defined above endow $\mathcal{M}$ with the structure of a simplicial model category.*

Proof. First let us check that they give a two-variable adjunction. The isomorphism

$$\mathrm{Hom}_{\mathcal{M}}(E \otimes X, F) \cong \mathrm{Hom}_{\mathcal{M}}(E, F^X)$$

is tautological. For the other, note that we have

$$\begin{aligned} \mathrm{Hom}_{\mathcal{M}}(E \otimes \Delta^n, F) &= \mathrm{Map}(M, N)_n \\ &= \mathrm{Hom}_{s\mathcal{S}et}(\Delta^n, \mathrm{Map}(M, N)) \end{aligned}$$

Since $E \otimes - : s\mathcal{S}et \to \mathcal{M}$ preserves colimits, and every simplicial set is a colimit of the standard simplicial sets Δ^n, we get isomorphisms

$$\mathrm{Hom}_{\mathcal{M}}(E \otimes X, F) \cong \mathrm{Hom}_{s\mathcal{S}et}(X, \mathrm{Map}(E, F))$$

The pushout-product axiom follows from the one for the monoidal structure on $\mathcal{M}$ and the fact that F preserves (acyclic) cofibrations. □

If $\mathcal{C}$ is a closed monoidal category with all small coproducts then there is a strong monoidal functor $k[-] : \mathcal{S}et \to \mathcal{C}$. It sends a set X to the object $\coprod_{x \in X} k$ of $\mathcal{C}$. This induces a strong monoidal functor

$$k[-] : s\mathcal{S}et \to s\mathcal{C}$$

Let $\mathcal{Z} = \{Z_i : i \in I\}$ be a set of compact objects in $\mathcal{C}$, and consider the model structure on $s\mathcal{C}$ induced by $\mathcal{Z}$. Theorem 6.9 in [33] in fact says that $k[-]$ is an enrichment functor. The following is clear.

Proposition 3.2.30. *Let $F : s\mathcal{S}et \to \mathcal{M}$ be an enrichment functor. Let $\mathcal{N}$ be a model category, and suppose that there are functors $G : \mathcal{M} \to \mathcal{N}$ and $H : \mathcal{N} \to \mathcal{M}$ such that*

$$G \colon \mathcal{M} \rightleftarrows \mathcal{N} : H$$

is a Quillen adjunction. Then $G \circ F$ is an enrichment functor.

Corollary 3.2.31. *Let $\mathcal{E}$ be a monoidal elementary exact category. The projective model structures on $Ch_{\geqslant 0}(\mathcal{E})$ and $Ch(\mathcal{E})$ are Kan complex-enriched.*

Proof. By Proposition 3.2.23 and Proposition 3.2.30 the model structures are simplicial. In fact they are enriched in simplicial abelian groups, and all simplicial abelian groups are Kan complexes. □

3.2.5 Examples and the Injective Model Structure

Examples

Example 3.2.32. *1. All the examples of Section 2.3 satisfy the assumptions of Theorem 3.2.3 such that their categories of unbounded complexes have projective model structures. The model structures for $Ch(Ind(Ban_k))$, $Ch(CBorn_k)$, and for unbounded complexes in the contracting normed and Banach categories are monoidal and satisfy the monoid axiom.*

2. In fact if any quasi-abelian category has enough projectives then the projective model structure exists on $Ch(Ind(\mathcal{E}))$.

The Injective Model Structure

Definition 3.2.33. *Let $\mathcal{E}$ be an exact category. If it exists, the* ***injective model structure*** *on $Ch_*(\mathcal{E})$, for $* \in \{+, b, \emptyset\}$ is the model structure in which*

- *Weak equivalences are quasi-isomorphisms.*
- *Cofibrations are degree-wise admissible monics.*
- *Fibrations are maps which have the right lifting property with respect to acyclic cofibrations.*

By duality we have the following.

Proposition 3.2.34. *If a quasi-abelian category $\mathcal{E}$ has enough injectives then $Ch(Pro(\mathcal{E}))$ is equipped with the injective model structure.*

This was proven by Pridham for $\mathcal{E} = Ban_{\mathbb{C}}$ in [64]. In [78], Št'ovíček introduces the notions of efficient exact categories and exact categories of Grothendieck type and shows that they are equipped with injective model structures. Essentially such categories generalise Grothendieck abelian categories, and as in that case one shows that such categories have enough injectives. It is not clear to us whether the categories we are interested in, namely $CBorn_k$ and $Ind(Ban_k)$ are of Grothendieck type.

3.3 Filtered and Graded Model Structures

In [72] Schneiders shows that the category of filtered abelian groups is an elementary quasi-abelian category. In particular the category of unbounded chain complexes of filtered abelian groups is equipped with the projective model structure. In this section we generalise this result to any quasi-abelian category. This will be useful for establishing a version of Koszul duality in the final chapter

3.3.1 Model Structures on Graded Objects

Let $\mathcal{E}$ be an additive category with kernels and cokernels.

Definition 3.3.1. *The category of* $\mathbb{N}_0$-***graded objects in*** $\mathcal{Gr}(\mathcal{E})$ *is the full subcategory of* $Ch_{\geqslant 0}(\mathcal{E})$ *on the chain complexes whose differentials are all zero.*

This is an exact subcategory of $Ch(\mathcal{E})$ (but it is not extension closed). There is also an exact functor $|-| : Ch(\mathcal{E}) \to \mathcal{Gr}(\mathcal{E})$ which just sends differentials to 0. Both functors preserve admissible monomorphisms and epimorphisms. In particular we get the following result.

Proposition 3.3.2. *If* $\mathcal{E}$ *is an elementary exact category then so is* $\mathcal{Gr}(\mathcal{E})$.

Proof. If $\mathcal{P}$ is a generating set of compact projective objects in $\mathcal{E}$, then $\{|S^n(P)| : n \in \mathbb{Z}, P \in \mathcal{P}\}$ is a generating set of compact projective objects in $\mathcal{Gr}(\mathcal{E})$. □

Corollary 3.3.3. *If* $\mathcal{E}$ *is an elementary exact category then the projective model structure exists on* $Ch(\mathcal{Gr}(\mathcal{E}))$.

If $\mathcal{E}$ is monoidal, then $\mathcal{Gr}(\mathcal{E})$ inherits the monoidal structure from $Ch(\mathcal{E})$. Moreover $|-|$ preserves flatness, so we get

Proposition 3.3.4. *If* $\mathcal{E}$ *is a (closed) monoidal, elementary exact category then so is* $\mathcal{Gr}(\mathcal{E})$.

3.3.2 Model Structures on Filtered Objects

Filtered Objects

Let $\mathcal{E}$ be an exact category.

Definition 3.3.5. *Let A be an object of $\mathcal{E}$. A **admissible subobject** of $\mathcal{E}$ is an admissible monomorphism $s : A' \to A$. A **filtration** of A consists of a collection of admissible subobjects of A, $\{\alpha_i : A_i \to A\}_{i \in \mathbb{N}_0}$ together with morphisms $a_i : A_i \to A_{i+1}$ such that $\alpha_{i+1} \circ a_i = \alpha_i$. A **filtered object** of $\mathcal{E}$ is tuple of data $(A_\infty, \alpha_i, a_i)$ where A_∞ is an object of $\mathcal{E}$ and (α_i, a_i) is a filtration of A_∞. A **morphism of filtered objects** $g : (A_\infty, \alpha_i, a_i) \to (B_\infty, \beta_i, b_i)$ consists of a collection of morphisms $\{g_i : A_i \to B_i\}_{i \in \mathbb{N}_{\geqslant 0}}$, and $g_\infty : A_\infty \to B_\infty$ such that $g_{i+1} \circ \alpha_i = b_i \circ g_i$ and $g_\infty \circ a_i = \beta_i \circ g_i$ for all $i \in \mathbb{N}_0$. Filtered objects and morphisms of filtered objects can then be organised into a category $\mathcal{Filt}(\mathcal{E})$.*

We often extend an $\mathbb{N}_0$-indexed filtration as in the definition to a $\mathbb{Z}$-indexed filtration by declaring $A_i = 0$ for $i < 0$.

Definition 3.3.6. *A filtration* (α_i, a_i) *of an object* A_∞ *is said to be* ***exhaustive*** *if* A_∞ *together with the maps* $\alpha_i : A_i \to A_\infty$ *is a direct limit of the diagram*

$$A_0 \xrightarrow{a_0} A_1 \xrightarrow{a_1} A_2 \longrightarrow \dots$$

The full subcategory of $\mathcal{Filt}(\mathcal{E})$ *on objects equipped with an exhaustive filtration will be denoted by* $\overline{\mathcal{Filt}}(\mathcal{E})$*. Note that a morphism* $g : (A_\infty, \alpha_i, a_i) \to (B_\infty, \beta_i, b_i)$ *of exhaustively filtered objects is completely determined by the maps* $g_i : A_i \to B_i$ *for* $0 \leqslant i < \infty$*.*

Proposition 3.3.7. *The inclusion functor* $i : \overline{\mathcal{Filt}}(\mathcal{E}) \hookrightarrow \mathcal{Filt}(\mathcal{E})$ *has a right adjoint.*

Proof. Let $A = (A_\infty, \alpha_i, a_i)$ be an object of $\mathcal{Filt}(\mathcal{E})$. Define $\overline{A}$ as a follows. $\overline{A}_i = A_i$ and $\overline{a}_i = a_i$, while $\overline{A}_\infty = \lim_{\to} A_i$ and $\overline{\alpha}_i : \overline{A}_i \to \overline{A}_\infty$ are the canonical colimit maps. By the universal property of the colimit there is a unique map $\epsilon_\infty(A) : \overline{A}_\infty \to A_\infty$ which satisfies $\epsilon_\infty(A) \circ \overline{\alpha}_i = \alpha_i$. In particular $\overline{\alpha}_i$ is an admissible monomorphism. Hence $\overline{A} = (\overline{A}_\infty, \overline{\alpha}_i, \overline{a}_i)$ is a well-defined object of $\overline{\mathcal{Filt}}(\mathcal{E})$. This construction is clearly functorial. We denote the functor by $\overline{(-)}$. It remains to check that it is a right adjoint. Let us construct a unit and a counit. The unit $\eta : Id \to \overline{(-)} \circ i$ is defined on an exhaustively filtered object $A = (A_\infty, \alpha_i, a_i)$ as the inverse of the canonical isomorphism $\lim_{\to} A_i \to A_\infty$. The counit $\epsilon : i \circ \overline{(-)} \to \mathrm{Id}$ is constructed as follows. For a filtered object $A = (A_\infty, \alpha_i, a_i)$, $\epsilon_j(A) : (i \circ \overline{A})_j \to A_j$ is the identity. $\epsilon_\infty(A) : \overline{A}_\infty \to A_\infty$ is the map constructed above. The unit and counit identities are easy to check. ☐

Filtered and Graded Objects

There is a functor $\mathcal{Filt} : \mathcal{Gr}(\mathcal{E}) \to \overline{\mathcal{Filt}}(\mathcal{E})$. It sends a graded object $A_\bullet$ to the filtered object $\mathcal{Filt}(A_\bullet)$ which has $\mathcal{Filt}(A_\bullet)_\infty = \bigoplus A_i$, and $\mathcal{Filt}(A_\bullet)_i = \bigoplus_{j \leqslant i} A_j$. The inclusions which give the filtered structure are the obvious ones. The functor acts on maps in the obvious way.

There is also a functor $\mathrm{gr} : \mathcal{Filt}(\mathcal{E}) \to \mathcal{Gr}(\mathcal{E})$, called the **associated graded functor** defined as follows. To a filtered object $A = (A_\infty, \alpha_i, a_i)$ it assigns the graded object $\mathrm{gr}(A)_\bullet$ with $\mathrm{gr}(A)_i = A_i / A_{i-1}$. Again it acts on morphisms in the obvious way.

Limits and Colimits of Filtered Objects

We want to put an exact structure on the categories $\mathcal{Filt}(\mathcal{E})$ and $\overline{\mathcal{Filt}}(\mathcal{E})$. In order to do this we first examine limits and colimits in these categories. Let $D : \mathcal{J} \to \mathcal{Filt}(\mathcal{E})$ be a diagram. For each object $j \in \mathcal{J}$ denote the object $D(j)$ by $(D_\infty(j), \delta_i(j), d_i(j))$, and for each morphism $\alpha : j \to j'$ in $\mathcal{J}$, denote the induced morphism $D(\alpha) : (D_\infty(j), \delta_i(j), d_i(j)) \to (D_\infty(j'), \delta_i(j'), d_i(j'))$ by $(\alpha_\infty, \alpha_i)$. For each $0 \leqslant i \leqslant \infty$, there is an induced diagram $D_i : \mathcal{J} \to \mathcal{E}$ sending an object j to $D_i(j)$ and a morphism $\alpha : j \to j'$ to α_i.

Proposition 3.3.8. *Let $D : \mathcal{J} \to \mathcal{Filt}(\mathcal{E})$ be a diagram. Suppose that for each $0 \leqslant i < \infty$ the induced maps (co)lim$(\delta_i(j))$: (co)lim$_j D_i(j) \to$ (co)lim$_j D_{i+1}(j)$ and (co)lim$(d_i(j))$: (co)lim$_j D_i(j) \to$ (co)lim$_j D_\infty(j)$ are admissible monos. Then*

$$((co)lim_j D_\infty(j), (co)lim(\delta_i(j)), (co)lim(d_i(j)))$$

is a (co)limit of D in $\mathcal{Filt}(\mathcal{E})$. If each $D(j)$ is exhaustively filtered, and (co)limits of diagrams of shape $\mathcal{J}$ commute with transfinite compositions of admissible monos, then the formula above is also a (co)limit in $\overline{\mathcal{Filt}}(\mathcal{E})$.

Proof. By assumption $((\text{co})\lim_j D_\infty(j), (\text{co})\lim(\delta_i(j)), (\text{co})\lim(d_i(j)))$ is an object of $\mathcal{Filt}(\mathcal{E})$. The universal property is checked directly. □

Proposition 3.3.9. *Let $\mathcal{E}$ be an elementary quasi-abelian category. Then$\overline{\mathcal{Filt}}(\mathcal{E})$ is complete and cocomplete*

Proof. By Proposition 3.3.8 $\mathcal{Filt}(\mathcal{E})$ has filtered colimits and kernels. It remains to show that cokernels and products exist. As for filtered abelian groups in [72], the cokernel of a map $g : (A_\infty, \alpha_i, a_i) \to (B_\infty, \beta_i, b_i)$ is given by $(coker(g_\infty), \gamma_i, g_i)$ where $\gamma_i : Im(A_i \to coker(g_\infty)) \to coker(g_\infty)$ is the obvious inclusion. The product $\prod_k A^k$ of a family $(A^k)_{k \in K}$ of filtered objects is constructed as follows. Write $A^k = (A^k_\infty, \alpha^k_i, a^k_i)$ Define $(\prod_{k \in K} A^k)_i := \prod_{k \in K} A^k_i$ and set $(\prod_{k \in K} A^k)_\infty := lim_{\to}(\prod_{k \in K} A^k)_i$. There are obvious maps $p_i : (\prod_{k \in K} A^k)_i \to (\prod_{k \in K} A^k)_\infty$. Since infinite products in an elementary quasi-abelian category are exact they are admissible monomorphisms, and this gives the filtered structure. □

The Exact Structure on Filtered Objects

Let $\mathcal{E}$ be an exact category. We denote by **AdMon** the class of admissible monomorphisms. We say that $\mathcal{E}$ has **admissible intersections** if the class **AdMon** is **AdMon**-pullback stable. We say that $\mathcal{E}$ has **admissible preimages** if the class **AdMon** is pullback stable along admissible morphisms. These notions are in fact equivalent due to the following result, which is Proposition 2.15 in [14].

Proposition 3.3.10. *Let $\mathcal{E}$ be a weakly idempotent complete exact category. Then the pullback of an admissible monomorphism along an admissible epimorphism yields an admissible monic.*

By pasting for pullback squares we immediately get the following.

Corollary 3.3.11. *A weakly idempotent complete exact category $\mathcal{E}$ has admissible intersections if and only if it has admissible preimages.*

Proposition 3.3.12. *Let $\mathcal{E}$ be a quasi-abelian category. Then $\mathcal{E}$ has admissible preimages.*

Proof. By Corollary 3.3.11 it suffices to show that $\mathcal{E}$ has admissible intersections. Let

$$\begin{array}{ccc} A & \xrightarrow{f'} & B \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Y \end{array}$$

be a pullback diagram with f and g being admissible monics. Then $A \to Y$ is the kernel of the composition

$$Y \xrightarrow{\Delta} Y \oplus Y \longrightarrow \mathrm{coker}(g) \oplus \mathrm{coker}(f)$$

In particular it is an admissible monic.

□

Definition 3.3.13. *A null sequence*

$$0 \longrightarrow (A_\infty, \alpha_i, a_i) \xrightarrow{(f_\infty, f_i)} (B_\infty, \beta_i, b_i) \xrightarrow{(g_\infty, g_i)} (C_\infty, \gamma_i, c_i) \longrightarrow 0$$

in $\mathcal{Filt}(\mathcal{E})$ is said to be ***exact*** *if each null sequence*

$$0 \longrightarrow A_i \longrightarrow B_i \longrightarrow C_i \longrightarrow 0$$

is exact for $0 \leqslant i \leqslant \infty$

Proposition 3.3.14. *Suppose that $\mathcal{E}$ be an elementary quasi-abelian category. Then with the class of short exact sequences defined above, $\mathcal{Filt}(\mathcal{E})$ is an exact category.*

Proof. Clearly the classes of admissible monics and admissible epics both contain isomorphisms and are closed under composition. By Proposition 3.3.8 the required pushouts and pullbacks exist. Now let

$$\begin{array}{ccc} (A_\infty, \alpha_i, a_i) & \xrightarrow{(f_\infty, f_i)} & (B_\infty, \beta_i, b_i) \\ \downarrow & & \downarrow \\ (X_\infty, \chi_i, x_i) & \xrightarrow{(f'_\infty, f_i)} & (Y_\infty, \xi_i, y_i) \end{array}$$

be a push-out diagram with (f_∞, f_i) an admissible monic. By Proposition 3.3.8, the diagram

$$\begin{array}{ccc} A_\infty & \xrightarrow{f_\infty} & B_\infty \\ \downarrow & & \downarrow \\ X_\infty & \xrightarrow{f'_\infty} & Y_\infty \end{array}$$

is a pushout diagram. The filtration on Y is given by $Y_i = Im(B_i \oplus X_i) \to Y$. It remains to see that $X_i \to Im(B_i \oplus X_i)$ is an admissible monomorphism. But $X_i \to Im(B_i \oplus X_i) \to Y$

coincides with the composition $X_i \to X \to Y$. $X \to Y$ is an admissible monomorphism as the pushout of an admissible monomorphism. The axiom for pushouts follow from Proposition 3.3.8. $\square$

Proposition 3.3.15. *Let*

$$0 \longrightarrow (A_\infty, \alpha_i, a_i) \xrightarrow{(f_\infty, f_i)} (B_\infty, \beta_i, b_i) \xrightarrow{(g_\infty, g_i)} (C_\infty, \gamma_i, c_i) \longrightarrow 0$$

be a short exact sequence of filtered objects. Suppose that $(\mathbb{N}, \boldsymbol{AdMon})$*-colimits exist and are exact. If* $(A_\infty, \alpha_i, a_i)$ *and* $(C_\infty, \gamma_i, c_i)$ *are exhaustive then so is* (B_∞, β_i, b_i).

Proof. Consider the diagram of short exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \lim_{\to} A_i & \longrightarrow & \lim_{\to} B_i & \longrightarrow & \lim_{\to} C_i & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_\infty & \longrightarrow & B_\infty & \longrightarrow & C_\infty & \longrightarrow & 0
\end{array}$$

The two outer vertical maps are isomorphisms so the middle one is as well. $\square$

In particular the category $\overline{\mathcal{F}ilt}(\mathcal{E})$ is an extension-closed subcategory of $\mathcal{F}ilt(\mathcal{E})$, and so is an exact category. Let us now classify the projective objects in the category $\overline{\mathcal{F}ilt}(\mathcal{E})$. First consider the following functors. For each $l \in \mathbb{N}$ we denote by $(-)_l$ the exact functor

$$\mathcal{F}ilt(\mathcal{E}) \to \mathcal{E}$$

which sends a filtered object

$$(A_\infty, \alpha_i, a_i)$$

to A_l. It sends a morphism $(f_\infty, f_i) : (A_\infty, \alpha_i, a_i) \to (B_\infty, \beta_i, b_i)$ to f_l. This functor is clearly exact. We denote by $Q_i : \mathcal{F}ilt(\mathcal{E}) \to \mathcal{E}$ defined on objects by

$$Q_i(A_\infty, \alpha_i, a_i) = A_\infty / A_i$$

It is defined on morphisms in the obvious way. Finally we denote by $F_i : \mathcal{E} \to \mathcal{F}ilt(\mathcal{E})$ the functor which sends an object A of $\mathcal{E}$ to the following filtered object. $(F_i(A))_j$ is 0 for $j < i$ and $(F_i(A))_j = A$ for $i \leqslant j \leqslant \infty$. with the structure maps being the obvious ones. Again it is defined on morphisms in the obvious way, and this functor is clearly exact.

Proposition 3.3.16.

$$Q_i \dashv F_{i+1} \dashv (-)_{i+1}$$

Proof. Let us first prove the second adjunction. Fix an object A of $\mathcal{E}$, and a filtered object (B_∞, β_i, b_i). Let $f : A \to (B)_{i+1}$ be a map in $\mathcal{E}$. There is an induced map $\tilde{f} : F_{i+1}A \to B$ defined as follows. $\tilde{f}_j = 0$ for $j < i+1$ and $\tilde{f}_j$ is the composition $A \to B_{i+1} \to B_j$ for $i+1 \leqslant j < \infty$. $\tilde{f}_\infty$ is given by the composition $\beta_{i+1} \circ f$. This gives a map

$$\mathrm{Hom}_{\mathcal{E}}(A, (B)_{i+1}) \to \mathrm{Hom}_{\mathcal{F}ilt(\mathcal{E})}(F_{i+1}A, B)$$

It is straightforward to verify that it is natural in both A and B. It is clearly an isomorphism of abelian groups.

Let us now show the first adjunction. Let (B_∞, β_i, b_i) be a filtered object, and let $f : B_\infty / B_i \to A$ be a morphism in $\mathcal{E}$. There is an induced map $\tilde{f} : (B_\infty, \beta_i, b_i) \to F_{i+1}A$ defined as follows. $\tilde{f}_j$ is 0 for $j < i+1$, and for $i+1 \leqslant j \leqslant \infty$ $\tilde{f}_j$ is given by the composition

$$B_j \to B_\infty \to B_\infty / B_{i+1} \to A$$

This gives a homomorphism of abelian groups

$$\mathrm{Hom}_{\mathcal{E}}(Q_i(B_\infty, \beta_i, b_i), A) \to \mathrm{Hom}_{\mathcal{F}ilt(\mathcal{E})}((B_\infty, \beta_i, b_i), F_{i+1}A)$$

which is clearly natural in (B_∞, β_i, b_i) and A. It is also clearly an isomorphism. □

Proposition 3.3.17. *Assume that $\mathcal{E}$ is a cocomplete elementary quasi-abelian category. If a filtered object $(A_\infty, \alpha_i, a_i)$ is projective in $\mathcal{F}ilt(\mathcal{E})$ then A_∞ / A_i is projective in $\mathcal{E}$ for all $i \in \mathbb{Z}$. In $\overline{\mathcal{F}ilt}(\mathcal{E})$ A is projective if and only if A_∞ / A_i is projective for each $i \geqslant 0$.*

Proof. The first assertion is a consequence of the fact that the functor $Q_i : \mathcal{F}ilt(\mathcal{E}) \to \mathcal{E}$ is left adjoint to the exact functor $F_{i+1} : \mathcal{E} \to \mathcal{F}ilt(\mathcal{E})$. Thus $Q_i(A_\infty, \alpha_i, a_i) = A_\infty / A_i$ is projective. The second assertion is a consequence of Proposition 3.1.1, and the fact that the functor $\overline{(-)}$ is right adjoint to an exact functor, and so preserves projectives. □

Next we classify the compact objects.

Proposition 3.3.18. *Let $A = (A_\infty, \alpha_i, a_i)$ be an object of $\mathcal{F}ilt(\mathcal{E})$. Suppose that each A_i and $A_\infty$is satisfy one of the smallness conditions of Definition 2.2.5, and for sufficiently large i, a_i is an isomorphism. Then A satisfies the same smallness condition in $\mathcal{F}ilt(\mathcal{E})$.*

Proof. Let $D : \mathcal{I} \to \overline{\mathcal{F}ilt}(\mathcal{E})$ be a relevant filtered diagram. By Proposition 3.3.8 the colimit is computed by taking the colimit in each degree of the filtration. For each $k \in \mathbb{N}_0$, there is an $i_k \in \mathcal{I}$ such that $A_k \to colim(-)_k \circ D$ factors through $(-)_k(i_k)$. Let n be such that $A_n \to A_{n+i}$ is an isomorphism for any $i \in \mathbb{N}$. Let $i = max_{0 \leqslant k \leqslant n} i_{i_k}$. Then the map $A \to colimD$ factors through $D(i)$. □

Proposition 3.3.19. *Let $\mathcal{G}$ be an admissible generating set. Suppose that transfinite compositions of admissible monomorphisms indexed by $\mathbb{N}$ are exact. Then $\bigcup_i F_i(\mathcal{G})$ is an admissible generating set for $\overline{\mathcal{F}ilt}(\mathcal{E})$.*

Proof. Let $(A_\infty, \alpha_i, a_i)$ be a filtered object. For each i pick some $G_i \in \mathcal{G}$ and an admissible epimorphism $\mathcal{G}_i \twoheadrightarrow A_i$. Then $\bigoplus_i F_i G_i \rightarrow A$ is an admissible epimorphism. □

Corollary 3.3.20. *If $\mathcal{E}$ is an elementary cocomplete quasi-abelian category then $\overline{\mathcal{F}ilt}(\mathcal{E})$ is an elementary exact category.*

The Monoidal Structure on Filtered Objects

Recall in an additive category $\mathcal{E}$ a morphism is said to be a **regular monomorphism** if it is the kernel of a morphism. A subobject $u : X \rightarrow A$ in an additive category is said to be regular if u is a regular monomorphism.

Definition 3.3.21. *Let $\{u_i : X_i \hookrightarrow A\}_{i=1}^n$ be regular subobjects of an object A in an additive category $\mathcal{E}$ which has kernels and cokernels. The **regular union** of X_i, denoted $u : \bigcup_{i=1}^n \hookrightarrow A$ is the image of the induced map $\tilde{u} : \bigoplus_{i=1}^n X_i \rightarrow A$.*

Unions satisfy the following universal property

Proposition 3.3.22. *Let $\{m_i : X_i \hookrightarrow A\}_{i=1}^n$ be regular subobjects of an object A in an additive category which has kernels and cokernels. There is a regular monomorphism $u : \bigcup_{i=1}^n X_i \hookrightarrow A$ together with monomorphisms $u_i : X_i \rightarrow \bigcup_{i=1}^n X_i$ which satisfy $u \circ u_i = m_i$. If $v : V \rightarrow A$ is a regular monomorphism together with monomorphisms $v_i : X_i \rightarrow V$ which satisfy $v \circ v_i = u_i$ then there is a map $w : \bigcup_{i=1}^n X_i \rightarrow V$ such that $v \circ w = u$.*

Proof. The image of a map is a regular monomorphism by its definition. Let $v_i : X_i \rightarrow V$ and $v : V \rightarrow A$ be as in the statement of the proposition. There is an induced map $\tilde{w} : \bigoplus X_i \rightarrow V$ which fits into a commutative diagram

$$\begin{array}{ccc} \bigoplus X_i & \xrightarrow{\tilde{w}} & V \\ & \searrow^{\tilde{u}} & \downarrow{\scriptstyle v} \\ & & A \end{array}$$

Taking images, and noting that v is regular, we get a commutative diagram

$$\begin{array}{ccc} \bigcup X_i & \xrightarrow{w} & V \\ & \searrow^{u} & \downarrow{\scriptstyle v} \\ & & A \end{array}$$

□

Definition 3.3.23. *Let $\mathcal{S}$ be a class of regular monomorphisms in a additive category with kernels and cokernels. We say that $\mathcal{S}$* ***admits regular unions*** *if whenever $\{m_i : X_i \hookrightarrow A\}_{i=1}^{n}$ is a collection of morphisms in $\mathcal{S}$, then the induced map $\bigcup_{i=1}^{n} X_i \to A$ is in $\mathcal{S}$.*

Definition 3.3.24. *An exact category $\mathcal{E}$ is said to* ***have admissible unions*** *if the class* ***AdMon*** *admits regular unions.*

Remark 3.3.25. *A quasi-abelian category has admissible unions.*

Let $(\mathcal{E}, \otimes, k)$ be a monoidal exact category with kernels and cokernels. Suppose that for any pair of admissible monos $s : E \to F$, $t : X \to Y$, the map $\mathrm{Im}(s \otimes t) \to X \otimes Y$ is is an admissible monomorphism, and for any finite collection of monomorphisms $\{s_i : X_i \to X\}_{i=1}^{n}$ the induced map $\bigcup_{i=1}^{n} X_i \to X$ is an admissible monomorphism. Note that this is automatically true for the class of admissible monomorphisms in a quasi-abelian category.

For filtered objects $A = (A_\infty, \alpha_i, a_i)$ and $B = (B_\infty, \beta_i, b_i)$, define a filtered object $A \otimes B$ as follows. $(A \otimes B)_\infty := A_\infty \otimes B_\infty$.

$$(A \otimes B)_n = \bigcup_{i+j=n} \mathrm{Im}(A_i \otimes B_j \to A_\infty \otimes B_\infty)$$

The maps $(\alpha \otimes \beta)_n : (A \otimes B)_n \to A \otimes B$ and $(a \otimes b)_n : (A \otimes B)_n \to (A \otimes B)_{n+1}$ are constructed as follows. Write $T = A \otimes B$ and $T_{i,j} = \mathrm{Im}(A_i \otimes B_j \to A \otimes B)$. Denote by $\phi_{ij} : T_{i,j} \to T_{i+1,j}$ the map induced from $\alpha_i \otimes Id_{B_j} : A_i \otimes B_j \to A_i \otimes B_{j+1}$ and by $\psi_{ij} : T_{i,j} \to T_{i,j+1}$ the map induced from $Id_{A_i} \otimes \beta_j : A_i \otimes B_j \to A_{i+1} \otimes B_j$. Also denote by $\omega_{ij} : T_{i,j} \to T$ the map induced from $a_i \otimes b_j : A_i \otimes B_j \to A \otimes B$. Finally write $T_n = \bigcup_{i+j=n} T_{i,j}$. There is a unique map $\omega_n : T_n \to T$ such that the canonical inclusions $u_{i,j} : T_{i,j} \to T_n$ for $i + j = n$ satisfy $\omega_n \circ u_{i,j} = \omega_{i,j}$. Now the compositions $u_{i+1,j} \circ \phi_{i,j} : T_{i,j} \hookrightarrow T_{n+1}$ and $u_{i,j+1} \circ \psi_{i,j} : T_{i,j} \hookrightarrow T_{n+1}$ induce maps $\phi_{n,n+1} : T_n \to T_{n+1}$ and $\psi_{n,n+1} : T_n \to T_{n+1}$. We claim that these maps coincide. Indeed it is sufficient to show that the maps $u_{i+1,j} \circ \phi_{i,j}$ and $u_{i,j+1} \circ \psi_{i,j}$ coincide. But $\omega_{n+1} \circ u_{i,j+1} \circ \psi_{i,j} = \omega_{i,j+1} \circ \psi_{i,j} = \omega_{i,j}$ and $\omega_{n+1} \circ u_{i+1,j} \circ \phi_{i,j} = \omega_{i+1,j} \circ \phi_{i,j} = \omega_{i,j}$. Since ω_{n+1} is a monomorphism we get the required result. We set $(s \otimes t)_n = \phi_{n,n+1} = \psi_{n,n+1}$. This is an admissible mono. Moreover the following diagrams commute

$$\begin{array}{ccc} T_{i,j} & \xrightarrow{u_{i,j}} & T_n \\ \downarrow{\scriptstyle \phi_{i,j}} & & \downarrow{\scriptstyle (\alpha \otimes \beta)_n} \\ T_{i+1,j} & \xrightarrow{u_{i+1,j}} & T_{n+1} \end{array}$$

$$\begin{array}{ccc} T_{i,j} & \xrightarrow{u_{i,j}} & T_n \\ \downarrow{\scriptstyle \psi_{i,j}} & & \downarrow{\scriptstyle (\alpha \otimes \beta)_n} \\ T_{i,j+1} & \xrightarrow{u_{i,j+1}} & T_{n+1} \end{array}$$

Proposition 3.3.26. *$(A_\infty \otimes B_\infty, (\alpha \otimes \beta)_n, (a \otimes b)_n)$ is a filtered object. Suppose that*

1. *Filtered colimits commute with kernels.*

2. *$\otimes$ preserves colimits in each variable.*

If A and B are exhaustive, then so is $A \otimes B$.

Proof. By the preceding remarks there is a factorisation.

$$\lim_{\to_i}\lim_{\to_j}\mathrm{Im}(A_i \otimes B_j \to A \otimes B) \to \lim_{\to_n}(A \otimes B)_n = \lim_{\to_n} \bigcup_{i+j=n} \mathrm{Im}(A_i \otimes B_j \to A_\infty \otimes B_\infty) \to A \otimes B$$

Since each map in the factorisation is an admissible monomorphism it suffices to show that the composite is an isomorphism.

$$\begin{aligned}
\lim_{\to_i}\lim_{\to_j}\mathrm{Im}(A_i \otimes B_j \to A_\infty \otimes B_\infty) &\cong \mathrm{Im}(\lim_{\to_i}\lim_{\to_j} A_i \otimes B_j \to A_\infty \otimes B_\infty) \\
&\cong \mathrm{Im}(A_\infty \otimes B_\infty \to A_\infty \otimes B_\infty) \\
&= A_\infty \otimes B_\infty
\end{aligned}$$

□

Proposition 3.3.27. *A filtered object (H_∞, t_i, h_i) in an elementary quasi-abelian category is flat if H_∞, and H_∞ / H_i are flat.*

Proof. Suppose that (H_∞, t_i, h_i) is flat. Let

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

Then

$$0 \longrightarrow F_0 A \longrightarrow F_0 B \longrightarrow F_0 C \longrightarrow 0$$

is exact in $\mathcal{Filt}(\mathcal{E})$. Therefore

$$0 \longrightarrow H \otimes F_0 A \longrightarrow H \otimes F_0 B \longrightarrow H \otimes F_0 C \longrightarrow 0$$

is exact. In particular

$$0 \longrightarrow H_\infty \otimes A \longrightarrow H_\infty \otimes B \longrightarrow H_\infty \otimes C \longrightarrow 0$$

is exact. Hence H_∞ is flat. Moreover, by assumption we have the following diagram with exact columns and exact top two rows.

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Im}(H_i \otimes A \to H_\infty \otimes A) & \longrightarrow & \mathrm{Im}(H_i \otimes B \to H_\infty \otimes B) & \longrightarrow & \mathrm{Im}(H_i \otimes C \to H_\infty \otimes C) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H_\infty \otimes A & \longrightarrow & H_\infty \otimes B & \longrightarrow & H_\infty \otimes C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \left(H_\infty/H_i\right) \otimes A & \longrightarrow & \left(H_\infty/H_i\right) \otimes B & \longrightarrow & \left(H_\infty/Hi\right) \otimes C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

Therefore the third row is exact, so H_∞/H_i is flat. □

We're not sure how to precisely classify flat objects. However we have the following.

Proposition 3.3.28. *Let X be a flat object of $\mathcal{E}$. Then for any $i \geqslant 0$, F_iX is a flat object of $\mathcal{F}ilt(\mathcal{E})$.*

Proof. If X is flat and $A = (A_\infty, \alpha_i, a_i)$ is an object of $\mathcal{F}ilt(\mathcal{E})$, then $(F_iX \otimes A)_j$ is 0 for $j < i$ and $X \otimes A_j$ for $j \geqslant i$. From the definition of exact sequenecs in $\mathcal{F}ilt\mathcal{E}$ it is clear that $F_iX \otimes A$ is exact. □

Corollary 3.3.29. *Let $\mathcal{E}$ be a monoidal elementary quasi-abelian category. Then $\overline{\mathcal{F}ilt}(\mathcal{E})$ is a monoidal elementary quasi-abelian category.*

Model Structure on Filtered Chain Complexes

The results in the previous sections imply the following.

Proposition 3.3.30. *Let $\mathcal{E}$ be an elementary quasi-abelian category. Then the projective model structure exists on $Ch(\overline{\mathcal{F}ilt}(\mathcal{E}))$ and is cellular and combinatorial.*

Proposition 3.3.31. *Let $f : A \to B$ be a map of exhaustively filtered complexes. Then*

1. *$gr(f)$ is an admissible monic if and only if f is.*

2. *$gr(f)$ is an admissible epic if and only if f is.*

3. *$gr(f)$ is a cofibration then so is f.*

4. *gr(f) is a quasi-isomorphism if and only if f is.*

Proof. 1. Suppose that gr(f) is an admissible monic. Let C be the cokernel of f. Let us show by induction that for each i

$$0 \to A_i \to B_i \to C_i \to 0$$

is exact. For $i = 0$ this is true by assumption. Suppose it has been shown for $i \leqslant n$. Consider the following commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_n & \longrightarrow & B_n & \longrightarrow & C_n & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_{n+1} & \longrightarrow & B_{n+1} & \longrightarrow & C_{n+1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_{n+1}/A_n & \longrightarrow & B_{n+1}/B_n & \longrightarrow & C_{n+1}/C_n & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & & 0
\end{array}$$

The columns are exact. By assumption the bottom row is exact. By the inductive step the top row is exact. Hence by the 3×3 lemma the middle row is exact. The converse is proved similarly, by applying the 3×3 lemma this time with the top two rows exact.

2. This is proved in exactly the same way as part 1.

3. By Part 1 it remains to show that each C_i is cofibrant. Again we do this by induction. For $i = 0$ again this is true by assumption. Suppose it has been shown that C_i is cofibrant for $i \leqslant n$. There is an exact sequence

$$0 \longrightarrow C_n \longrightarrow C_{n+1} \longrightarrow C_{n+1}/C_n \longrightarrow 0$$

The first and last terms are cofibrant. Hence the middle term is as well.

4. By Parts 1 and 2 the functor gr reflects admissible morphisms. Thus it is sufficient to show that it reflects short exact sequences. This is proved in exactly the same way as Part 1.

□

3.4 Homotopy in Additive Model Categories

In this section the goal is to investigate when homotopy equivalences in additive model categories are in fact weak equivalences. Let us first make the following straightforward observations.

Proposition 3.4.1. *Let $\mathscr{E}$ be an additive category. Let $\mathcal{M}$ be a class of morphisms in $\mathscr{E}$. Then the classes of morphisms $\mathcal{M}^{\boxslash}$ and ${}^{\boxslash}\mathcal{M}$ are closed under taking direct sums.*

Immediately we get the following result.

Corollary 3.4.2. *Let $f : X \to Y$ and $g : A \to B$ be weak equivalences. Then $f \oplus g : X \oplus A \to Y \oplus B$ is a weak equivalence.*

Proposition 3.4.3. *Let $f : X \to Y$ be a weak equivalence, and let $g : X \to Y$ be a map which factors through a trivial object W. Then $f + g$ is a weak equivalence.*

Proof. Write $g = q \circ p$ where $p : X \to W$ and $q : W \to Y$, with W being a trivial object. We can factor $f + g$ as

$$X \longrightarrow X \oplus W \longrightarrow Y \oplus W \longrightarrow Y$$

The middle map is a weak equivalence by assumption. It therefore suffices to show that the maps $X \to X \oplus W$ and $Y \oplus W \to Y$ are weak equivalences. By duality, it is sufficient to show that $X \to X \oplus W$ is a weak equivalence. Now the projection $X \oplus W \to X$ is a weak equivalence by Corollary 3.4.2. The composition $X \to X \oplus W \to X$ is the identity, and is therefore a weak equivalence. By the 2-out-of-3 property, $X \to X \oplus W$ is a weak equivalence. □

Chapter 4

Homotopy Theory of Algebras in Exact Categories

In this chapter we develop the homotopy theory of algebras in monoidal elementary exact categories. We begin showing that transferred model structures exist on categories of algebras over operads in very general additive model categories. We then study co-operadic and operadic versions of Koszul duality in monoidal elementary quasi-abelian categories. We show that many results known for categories of vector spaces work in this very broad setting. In particular we give an interpretation of Koszul duality for Lie and commutative algebras in terms of the shifted cotangent complex. We conclude by studying analytic (non-formal) fattenings of formal neighbourhoods of differentially graded Lie algebras. For basic definitions and results about operads and their algebras in monoidal additive categories consult Appendix B.

4.1 Higher Algebra Settings

In this section we will let $\mathpzc{E}$ be a complete and cocomplete, locally presentable additive category. We further assume that $\mathpzc{E}$ is endowed with an additive Kan-complex enriched monoidal model structure which satisfies the monoid axiom, is combinatorial, is proper, and has an additive homotopy category. We do not assume that either the model structure or the monoidal structure are compatible with the exact structure. An additive category satisfying all of the above assumptions will be called a **higher algebra setting (HAS)** (c.f. the notion of HAG in [79]). An HAS is said to be a **strong HAS** if there is a set of generating acyclic cofibrations which are split monomorphisms with trivially cofibrant cokernel. A strong HAS is said to be a **rigid HAS** if tensoring with cofibrant objects preserves weak equivalences. Note that by [51] Theorem 5.5.1.1 a HAS presents a locally presentable $(\infty,1)$-category $\mathbf{E}$. We have the following obvious but extremely useful technical property of strong higher algebra settings.

Proposition 4.1.1. *Let $\mathcal{E}$ be a strong HAS. Then any acyclic cofibration $f : A \to B$ is a retract of a map of the form $X \to X \oplus Y$ where Y is trivially cofibrant.*

Example 4.1.2. *Let $(\mathcal{E}, \otimes, \underline{Hom}, k)$ be a locally presentable closed monoidal elementary exact category. Then $(Ch_{\geqslant 0}(\mathcal{E}), \otimes, S^0(k))$ is a rigid HAS. If countable coproducts are admissibly coexact and countable products are admissibly exact then $(Ch(\mathcal{E}), \otimes, \underline{Hom}, S^0(k))$ is also a rigid HAS.*

Proof. By Theorem 3.2.13 and Theorem 3.2.14 these are monoidal model categories which satisfy the monoid axiom. By Theorem 3.2.3 and Theorem 3.2.7 they are combinatorial, and have generating acyclic cofibrations which are split monomorphisms with trivially cofibrant cokernel. The standard proof that derived categories of abelian categories are additive goes through for exact categories. By Corollary 3.2.31 the projective model structure on $Ch(\mathcal{E})$ is Kan complex-enriched. To see that this HAS is rigid, let C be cofibrant. Then $0 \to C$ is a cofibration. In particular C is the cokernel of a cofibration. Thus it suffices to show that tensoring with the cokernel of a cofibration preserves weak equivalences. It in fact suffices to show that tensoring with cokernels of generating cofibrations preserves weak equivalences. But the cokernel of a generating cofibration is either $S^n(P)$ or $D^m(P')$ for P, P' projective. Since projectives are flat, tensoring with such objects clearly preserves weak equivalences. □

4.1.1 Familiar Homotopical Algebra in Higher Algebra Settings

Modules

Proposition 4.1.3. *Let $(\mathcal{E}, \otimes, k)$ be a HAS and let R be a commutative monoid in $\mathcal{E}$. Then with its transferred model structure and induced closed symmetric monoidal structure, $({}_R\mathcal{M}od, \otimes_R, \underline{Hom}_R)$ is a HAS. If $(\mathcal{E}, \otimes, k)$ is a rigid or strong HAS then so is $({}_R\mathcal{M}od, \otimes_R, \underline{Hom}_R)$.*

Proof. The transferred model structure exists by Theorem A.1.25. and is cofibrantly generated by Corollary A.1.16. Also by Theorem A.1.25 it is monoidal and satisfies the monoid axiom. ${}_R\mathcal{M}od$ is locally presentable by [53]. To see that its homotopy category is additive, let M and N be R-modules. We may assume that M is cofibrant, and in fact that it is free on a cofibrant object P in $\mathcal{E}$. Then $Hom_{Ho({}_R\mathcal{M}od)}(R \otimes P, N) \cong Hom_{Ho(\mathcal{E})}(P, N)$ which is an abelian group by assumption. Finite biproducts are constructed in the obvious way.

Suppose that I is a set of generating acyclic cofibrations for $\mathcal{E}$ which are split monomorphisms. Then $\{\mathrm{id}_R \otimes i : i \in I\}$ is a set of generating acyclic cofibrations in ${}_R\mathcal{M}od$. Tensoring with R clearly preserves split exactness of a sequence, so ${}_R\mathcal{M}od$ also has a set of generating acyclic cofibrations which are split monomorphisms with trivially cofibrant cokernel.

Finally suppose that $\mathcal{E}$ is rigid. Let X be a cofibrant object of ${}_R\mathcal{Mod}$. Then X is a retract of an object of the form $R \otimes A$ where A is a cofibrant object of $\mathcal{E}$. Thus it suffices to show that tensoring with $R \otimes A$ over R preserves weak equivalences. This is clear. $\square$

Let $(\mathcal{E}, \otimes, \underline{\text{Hom}}, k)$ be a locally presentable closed, projectively monoidal, elementary exact category. Suppose that countable coproducts are admissibly coexact and countable products are admissibly exact and let R be a commutative monoid in $\mathcal{E}$. By Proposition 2.2.24 ${}_R\mathcal{Mod}$ is again an elementary exact category, and it is locally presentable. Thus $Ch({}_R\mathcal{Mod})$ is equipped with a combinatorial projective model structure, and has a set of generating acyclic cofibrations which are split monomorphisms with trivially cofibrant cokernel. However the induced monoidal structure on ${}_R\mathcal{Mod}$, namely $- \otimes_R -$ need not be compatible with the exact structure. This one of our motivations for considering higher algebra settings rather than just pseudo-compatible model structures on monoidal exact categories. We also want to consider model structures on modules over commutative differential graded algebras. It is however useful to know that in these cases the model structures are left pseudo-compatible (resp. compatible).

Categories of Algebras

Recall that if we have an adjunction $F\colon \mathcal{C} \rightleftarrows \mathcal{D} : G$ and $\mathcal{C}$ is a model category, we can investigate when the **transferred model structure** exists on $\mathcal{D}$. For details see Section A.1.4. We are interested in the case of the free-forgetful adjunction for algebras over an operad.

Definition 4.1.4. *An operad $\mathfrak{P}$ in $\mathcal{E}$ is said to be **admissible** if the transferred model structure exists on $\mathcal{Alg}_{\mathfrak{P}}(\mathcal{E})$.*

The next result follows immediately from Theorem A.1.25

Proposition 4.1.5. *Let $\mathcal{E}$ be a HAS. Then the associative operad $\mathfrak{Ass}$ is admissible.*

Now we turn to commutative monoids. If a (strong/ rigid) HAS $\mathcal{E}$ is enriched over $\mathcal{Vect}_{\mathbb{Q}}$ rather than just $\mathcal{Ab}$, we shall call it a (strong/ rigid) $\mathbb{Q}$-HAS. We denote the category of unital commutative monoids by $\mathcal{Alg}_{\mathfrak{Comm}}(\mathcal{E})$ and of non-unital commutative monoids by $\mathcal{Alg}_{\mathfrak{Comm}^{nu}}(\mathcal{E})$.

Proposition 4.1.6. *Let $\mathcal{E}$ be a strong $\mathbb{Q}$-HAS. Then the operads $\mathfrak{Comm}$ and $\mathfrak{Comm}^{nu}$ are admissible.*

Proof. The forgetful functor $\mathcal{Alg}_{\mathfrak{Comm}}(\mathcal{E}) \to \mathcal{E}$ preserves filtered colimits. Thus we may apply Corollary A.1.16. The transferred model structure exists on $\mathcal{Ass}(\mathcal{E})$ by Proposition 4.1.5. In particular the functor $T : \mathcal{E} \to \mathcal{Ass}(\mathcal{E})$ preserves acyclic cofibrations. By Proposition B.1.1, for any map $X \to Y$ in $\mathcal{E}$, the map $S(X) \to S(Y)$ is a retract of $T(X) \to T(Y)$. In particular if $X \to Y$ is an acyclic cofibration in $\mathcal{E}$, then $S(X) \to S(Y)$ is a weak equivalence in $\mathcal{E}$. Now suppose g is a generating acyclic cofibration in $\mathcal{E}$. We may assume g is an inclusion as a direct summand, i.e. of the form $X \to X \oplus Z$ where Z is trivially cofibrant. Since S is a left adjoint it preserves colimits, so $S(X \oplus Z) \cong S(X) \otimes S(Z)$, and $S(g)$ is the map $\mathrm{id}_{S(X)} \otimes 1_{S(Z)}$ where $1_{S(Z)}$ is the unit of the commutative monoid $S(Z)$. Consider a push-out diagram

$$\begin{array}{ccc} S(X) & \longrightarrow & A \\ \downarrow{\scriptstyle S(g)} & & \downarrow{\scriptstyle S(g)'} \\ S(X) \otimes S(Z) & \longrightarrow & B \end{array}$$

Then B is isomorphic to $A \otimes_{S(X)} (S(X) \otimes S(Z)) \cong A \otimes S(Z)$ and under this isomorphism $S(g)'$ is $\mathrm{id}_A \otimes 1_{S(Z)}$. Any transfinite composition of such maps will again be of the form $t : A \to A \otimes S(Y)$ with Y trivially cofibrant, since both $\otimes$ and S preserve colimits and coproducts of trivially cofibrant objects are trivially cofibrant. Now $k = S(0) \to S(Y)$ is a split monomorphism with cokernel $\bigoplus_{n \geqslant 1} S^n(Y)$. $\bigoplus_{n \geqslant 1} S^n(Y)$ is trivially cofibrant. Therefore $S(0) \to S(Y)$ is an acyclic cofibration. By assumption $(-) \otimes A$ sends acyclic cofibrations to weak equivalences. In particular t is a weak equivalence.

For the category $\mathcal{Alg}_{\mathfrak{Comm}^{nu}}(\mathcal{E})$ the proof is similar. In this category the coproduct of two non-unital commutative monoids A and B is $A \oplus B \oplus A \otimes B$. If $g : X \to X \oplus Z$ is a generating acyclic cofibration as before Then $S^{nu}(X) \to S^{nu}(X \oplus Z) \cong S^{nu}(X) \oplus S^{nu}(Z) \oplus S^{nu}(X) \otimes S^{nu}(Z)$. The map $S^{nu}(g)$ is the natural inclusion. If

$$\begin{array}{ccc} S^{nu}(X) & \longrightarrow & A \\ \downarrow{\scriptstyle S(g)} & & \downarrow{\scriptstyle S(g)'} \\ S^{nu}(X) \oplus S^{nu}(Z) \oplus (S^{nu}(X) \otimes S^{nu}(Z)) & \longrightarrow & B \end{array}$$

Then B is isomorphic to $A \oplus S^{nu}(Z) \oplus A \otimes S^{nu}(Z)$ and $S(g)'$ is the natural inclusion. The cokernel of this map is $S^{nu}(Z) \oplus (A \otimes S^{nu}(Z))$ which is trivially cofibrant by the first part of the proof. $\square$

Finally we turn to Lie monoids.

Proposition 4.1.7. *Let $(\mathcal{E}, \otimes, k)$ be a $\mathbb{Q}$-HAS. Then $\mathfrak{Lie}$ is admissible.*

Proof. Let $f : X \to Y$ be a generating trivial cofibration in $\mathcal{E}$ and suppose

$$\begin{array}{ccc} L(X) & \xrightarrow{f} & A \\ \downarrow{\scriptstyle L(g)} & & \downarrow{\scriptstyle g'} \\ L(Y) & \xrightarrow{f'} & B \end{array}$$

is a pushout diagram in $\mathcal{L}ie(\mathcal{E})$. Since U is a left-adjoint the following diagram

$$\begin{array}{ccc} T(X) & \xrightarrow{U(f)} & U(A) \\ \downarrow{\scriptstyle T(g)} & & \downarrow{\scriptstyle U(g')} \\ T(Y) & \xrightarrow{U(f')} & U(B) \end{array}$$

is a pushout in $\mathcal{A}ss(\mathcal{E})$. Now as a left adjoint, the functor U preserves colimits. Thus if m is a transfinite composition of pushouts of images $L(g)$ of generating acyclic cofibrations g, then $U(m)$ is a transfinite composition of pushouts of images $T(g)$ of generating acyclic cofibrations g. By Theorem 4.1.5 and Theorem A.1.14 $U(m)$ is acyclic. But m is a retract of $U(m)$ by Theorem B.1.2. Hence m is also a weak equivalence. □

In strong pseudo-left compatible model structures we also have the following technical, but useful, fact.

Proposition 4.1.8. *Let $\mathcal{E}$ be a $\mathbb{Q}$-HAS whose underlying model structure is a strong left pseudo-compatible model structure. Then the model category $\mathcal{A}lg_{\mathfrak{Comm}}(\mathcal{E})$ is left proper.*

Proof. Let $f : X \to Y$ be a weak equivalence and $g : X \to Z$ a cofibration. Consider the pushout diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g'} \\ Z & \xrightarrow{f'} & Z \otimes_X Y \end{array}$$

Since acyclic cofibrations are stable under pushout along any map we may assume that f is an acyclic fibration. So there is an exact sequence $0 \to W \to X \to Y \to 0$ with $W \in \mathfrak{W}$. Then $Z \otimes_X Y \cong coker(W \to Z)$. The map $W \to Z$ is an admissible mono, so there is an exact sequence $0 \to W \to Z \to Z \otimes_X Y \to 0$. In particular f' is an admissible epimorphism with kernel $W \in \mathfrak{W}$, and is in fact an acyclic fibration. □

4.1.2 Interval Higher Algebra Settings

To prove that categories of algebras over more general operads are equipped with transferred model structures we need our model category to have an interval object (see Section A.1.5).

Definition 4.1.9. *An* ***interval higher algebra setting (IHAS)*** *is a rigid HAS together with a good coassociative coalgebra interval object* $([0,1], \Delta, \epsilon)$*, and a generating set of acyclic cofibrations of the form* $\{0 \to C_\gamma\}_{\gamma \in \Gamma}$ *such that the objects* C_γ *are* $[0,1]$*-contractible.*

Our main example is of course chain complexes in a monoidal elementary quasi-abelian category.

Example 4.1.10. *Let* $\mathcal{E}$ *be a monoidal exact category. Consider in* $Ch_{\geqslant 0}(\mathcal{E}) \hookrightarrow Ch(\mathcal{E})$ *the complex* $[0,1]$

$$0 \longrightarrow k_q \xrightarrow{d} k_{p_1} \oplus k_{p_1} \longrightarrow 0$$

Here for each index $\alpha \in \{q, p_1, p_2\}$ k_α *is just a copy of* k*. The differential* $k_q \to k_{p_1} \oplus k_{p_2}$ *is given by*

$$d = \begin{pmatrix} Id \\ -Id \end{pmatrix}$$

As in the case of R-modules (see for example [74]) this can be endowed with a coassocative coalgebra structure. If $\mathcal{E}$ *is a monoidal elementary exact category then it is a good coassociative coalgebra interval object for the projective model structure.*

Proposition 4.1.11. *Let* $\mathcal{E}$ *be an IHAS Let* $f : A \to B$ *and* $g : A \to B$ *be maps such that* $f \sim^{l}_{[0,1]} g$*. If* g *is a weak equivalence then so is* f*.*

Proof. The map $f - g$ is homotopic to 0, so it factors through $\operatorname{coker}(k \otimes A \to [0,1] \otimes A) \cong \operatorname{coker}(k \to [0,1]) \otimes A$. By assumption this is a trivial object, so we conclude by Proposition 3.4.3. □

In particular we have that homotopy equivalences relative to $[0,1]$ are in fact weak equivalences.

Proposition 4.1.12. *Let* $\mathcal{E}$ *be an IHAS and* $f : A \to B$ *a homotopy equivalence relative to* $[0,1]$*. Then* f *is a weak equivalence.*

Proof. Pick a homotopy inverse g for f. Then $g \circ f \sim^{l}_{[0,1]} Id_A$ and $f \circ g \sim^{l}_{[0,1]} Id_B$. By Proposition 4.1.11 $f \circ g$ and $g \circ f$ are weak equivalences. By the 2-out-of-6 property, f and g are weak equivalences. □

Let us briefly discuss properties of $[0,1]$-contractible objects in interval higher algebra settings. Let M be $[0,1]$-contractible. That is the 0 map $M \to 0$ is a homotopy equivalence relative to $[0,1]$. This means that there is a map $H : [0,1] \otimes M \to M$ with $H_0 = 0$ and $H_1 = Id_M$.

Proposition 4.1.13. *Let* $Y = X \oplus C$*. If* Y *is* $[0,1]$*-contractible then so is* X*.*

Proof. Let $H : [0,1] \otimes Y \to Y$ be a map such that $H_0 = 0$ and $H_1 = Id_Y$. Consider the map $H_X : [0,1] \otimes X \to X$ defined by the composition.

$$[0,1] \otimes X \longrightarrow ([0,1] \otimes X) \oplus ([0,1] \otimes Y) \cong [0,1] \otimes (X \oplus C) \xrightarrow{H} X \oplus C \longrightarrow X$$

This gives a homotopy between id_X and 0. □

Proposition 4.1.14. *Let $\mathcal{E}$ be an IHAS and $f : A \to B$ an trivial cofibration in $\mathcal{E}$ with cokernel C. Then C is $[0,1]$-contractible. In particular trivially cofibrant objects in $\mathcal{E}$ are $[0,1]$-contractible.*

Proof. By Proposition 4.1.1 f is a retract of a trivial cofibration map of the form $X \to X \oplus Y$ where the map includes X as a direct summand and Y is a coproduct of the objects $\{C_\gamma\}_{\gamma\in\Gamma}$. In particular C is a summand of Y. □

Proposition 4.1.15. *Let $\mathcal{E}$ be an IHAS and R a commutative monoid object in $\mathcal{E}$. Then ${}_R\mathcal{Mod}$ is an IHAS.*

Proof. Let $([0,1], \nabla, \epsilon)$ be a coalgebra interval in $\mathcal{E}$. The functor $R \otimes (-) : \mathcal{E} \to {}_R\mathcal{Mod}$ is strong monoidal, so $(R \otimes [0,1], R \otimes \nabla, R \otimes \epsilon)$ is a co-associative coalgebra interval in ${}_R\mathcal{Mod}$. Moreover the functor $R\otimes$ preserves cofibrations and weak equivalences between cofibrant objects, so the coalgebra interval is good. □

Homotopy of Σ-modules

Let $\mathcal{E}$ be an IHAS with coassociative coalgebra interval $([0,1], \Delta, \epsilon)$, and a generating set of acyclic cofibrations of the form $\{0 \to C_\gamma\}_{\gamma\in\Gamma}$ with C_γ being $[0,1]$-contractible. Consider the category $\mathcal{Gr}(\mathcal{E})$. This can be endowed with a monoidal model structure in which weak equivalences, fibrations and cofibrations are defined degree-wise. It is clearly a higher algebra setting. Regarding $([0,1], \Delta, \epsilon)$ as a graded object concentrated in degree 0 gives a good coassociative coalgebra interval in $\mathcal{Gr}(\mathcal{E})$. A generating set of acyclic cofibrations is given by $\{0 \to C_\gamma\}_{\gamma\in\Gamma, n\geqslant 0}$ and each of these is $[0,1]$-contractible. In particular $\mathcal{Gr}(\mathcal{E})$ is an IHAS.

Moreover as modules over an associative monoid in $\mathcal{Gr}(\mathcal{E})$ the category of Σ-modules (see Appendix B) has a transferred model structure.

Proposition 4.1.16.

1. *Let $f : M \to N$ be a morphism of Σ-modules and X an object of $\mathcal{E}$. If f is an acyclic cofibration then $f \circ Id_X$ is a weak equivalence.*

2. *Let $f : M \to N$ be a morphism in $\mathcal{E}$ and X a Σ-module. If X is cofibrant and f is an acyclic cofibration then $id_X \circ f$ is a weak equivalence.*

Proof. 1. We may assume f is a generating acyclic cofibration, so it is of the form $0 \to P \otimes \Sigma_n$ where P is trivailly cofibrant in $\mathcal{E}$. Then $f \circ Id_X$ is 0 except in arity n, where it is $0 \to P \otimes X^{\otimes n}$ which is a weak equivalence by assumption.

2. This is similar to part 1, again noting that X is a retract of an object of the form $Y \otimes \Sigma$ with Y cofibrant in $\mathcal{Gr}(\mathcal{E})$.

□

Homotopy of Operadic Algebras

Let (A, γ_A, η_A) be an algebra over an operad $\mathfrak{P}$. Since the unit η_A satisfies $\gamma_A \circ \eta_A = Id_A$ we immediately get the following.

Proposition 4.1.17. *Let A be a $\mathfrak{P}$-algebra. The map $\gamma_A : \mathfrak{P}(A) \to A$ is a split epimorphism. Moreover, $Ker(\gamma_A)$ is the regular ideal generated by the image of the endomorphism $Id_{\mathfrak{P}(A)} - \eta_A \circ \gamma_A \in End_{\mathcal{E}}(\mathfrak{P}(A))$.*

Split Operads

Recall the adjunction between the categories of symmetric and non-symmetric operads.

$$(-)^{\Sigma} \colon \mathcal{Op}_{ns} \rightleftarrows \mathcal{Op} :| - |$$

As we shall see it is reasonably straightforward to put a model structure on categories of algebras over a non-symmetric operad. If a symmetric operad $\mathfrak{P}$ were 'split', i.e. the counit of the adjunction $|(\mathfrak{P})^{\Sigma}| \to \mathfrak{P}$ had a section then it would also be easy to put a model category structure on $\mathcal{Alg}_{\mathfrak{P}}$. Following [36] this is possible even with a weaker notion of splitting. Denote by $< n >$ the ordered set $\{1, \ldots, n\}$ and by Σ_n the nth symmetric group. A monotone injective map $f :< s > \to < n >$ induces an injective group homomorphism $\iota_f : \Sigma_s \to \Sigma_n$ as follows

$$\iota_f(\rho) = \begin{cases} i & i \notin Im(f) \\ f(\rho(j)) & i = f(j) \end{cases}$$

It also induces a map of sets $\rho_f : \Sigma_n \to \Sigma_s$ by $\rho = \rho_f(\sigma)$ precisely if

$$\rho(i) < \rho(j) \Leftrightarrow \sigma(f(i)) < \sigma(f(j))$$

Also define the set

$$T_f = \{\sigma \in \Sigma_n : \sigma \circ f :< s > \to < n > \text{ is monotone } \}$$

By a Lemma in [36] any $\sigma \in \Sigma_n$ can be written uniquely as

$$\sigma = \tau \iota_f(\rho)$$

where $\tau \in T_f$ and $\rho \in \Sigma_s$. Moreover $\rho = \rho_f(\sigma)$. If M is a Σ_n-module then we also denote by ρ_f the map

$$\rho_f : M \otimes \Sigma_n \to M \otimes \Sigma_s$$

given by

$$M \otimes \Sigma_n \xrightarrow{\tau \otimes \rho_f} M \otimes \Sigma_s$$

where τ is the automorphism of M determined by $\tau \in \Sigma_n$.

Definition 4.1.18. *Let $\mathfrak{P}$ be an operad in $\mathcal{E}$. A Σ-**splitting** of $\mathfrak{P}$ is a collection $t(n) : \mathfrak{P}(n) \to |\mathfrak{P}|^{\Sigma}(n)$ of maps in $\mathcal{E}$ such that*

1. *$t(n)$ is Σ_n-equivariant.*

2. *$\pi \circ t(n) = id : \mathfrak{P}(n) \to \mathfrak{P}(n)$*

3. *For any $m, n > 0$ and $1 \leqslant k \leqslant n$ the diagram below commutes.*

$$\begin{array}{ccc}
\mathfrak{P}(n) \otimes \mathfrak{P}(m) & \xrightarrow{\circ_k} & \mathfrak{P}(n+m-1) \\
\downarrow{\scriptstyle t \otimes id} & & \\
\mathfrak{P}(n) \otimes k[\Sigma_n] \otimes \mathfrak{P}(m) & & \downarrow{\scriptstyle t} \\
 & & \mathfrak{P}(n+m-1) \otimes k[\Sigma_{n+m-1}] \\
\downarrow{\scriptstyle \rho_f \otimes id} & & \\
\mathfrak{P}(n) \otimes k[\Sigma_{n-1}] \otimes \mathfrak{P}(m) & & \downarrow{\scriptstyle \rho_g} \\
\downarrow{\scriptstyle \sigma_{23}} & & \\
\mathfrak{P}(n) \otimes \mathfrak{P}(m) \otimes k[\Sigma_{n-1}] & \xrightarrow{\circ_k \otimes id} & \mathfrak{P}(n+m-1) \otimes k[\Sigma_{n-1}]
\end{array}$$

*An operad which admits a Σ-splitting is said to be Σ-**split**.*

If $\mathcal{E}$ is a $\mathbb{Q}$-HAS then any operad is split via the map

$$t(n) = \frac{1}{n!} \sum_{\sigma \in \Sigma_n} \sigma^{-1} \otimes \sigma$$

We are going to prove the following generalization of [36] Theorem 4.1.1.

Theorem 4.1.19. *Let $\mathcal{E}$ be an IHAS.*

1. *If $\mathfrak{P}$ is a non-symmetric operad then the transferred model structure exists on $\mathcal{Alg}_{\mathfrak{P}}$.*

2. *Suppose that the underlying additive category of $\mathcal{E}$ is quasi-abelian. If $\mathfrak{P}$ is a split symmetric operad then the transferred model structure exists on $\mathcal{Alg}_{\mathfrak{P}}$.*

Proof. 1. We need to check that for a $\mathfrak{P}$-algebra A, and for X trivially cofibrant, the map $i : A \to A \coprod \mathfrak{P}(X)$ is a weak equivalence. We may assume that $0 \to X$ is a generating trivial cofibration which is $[0,1]$-contractible. We are going to prove the more general statement that for an $[0,1]$-contractible object X, the map $A \to A \coprod \mathfrak{P}(X)$ is a homotopy equivalence.

Write $V = |A| \oplus X$. Let $\tilde{h} : [0,1] \otimes X \to X$ be a homotopy between the identity of X and the map $0 : X \to X$. Then the map h given by the composition

$$[0,1] \otimes (|A| \oplus X) \xrightarrow{\cong} ([0,1] \otimes |A|) \oplus ([0,1] \otimes X) \xrightarrow{\epsilon \otimes id_{|A|} \oplus \tilde{h}} |A| \oplus X$$

gives a homotopy between id_V and the endomorphism α which is given by the following composition

$$V \to X \to V$$

with the first map being the projection and the second the inclusion. Now there is a natural map $p : A \coprod \mathfrak{P}(X) \to A$ defined as follows. Its restriction to A is the identity while it restriction to $\mathfrak{P}(X)$ is the unique map of algebras induced by the 0 map $X \to A$. By construction $p \circ i = id_A$. We claim that $i \circ p$ is homotopy equivalent to $id_{A \coprod \mathfrak{P}(X)}$. The map p can also be described as follows. Consider the map $\mathfrak{P}(V) \to \mathfrak{P}(A)$ induced by the projection $V \to |A|$. Let K be the kernel of the map $\mathfrak{P}(|A|) \to A$. Its preimage J in $\mathfrak{P}(V)$ is the ideal in $\mathfrak{P}(V)$ generated by K. Moreover $\mathfrak{P}/J$ is $A \coprod \mathfrak{P}(X)$, and the induced map

$$A \coprod \mathfrak{P}(X) \to A$$

is p. Now by Proposition A.1.24 h extends to a homotopy H between $id_{\mathfrak{P}(V)}$ and the composition $\tilde{p}$

$$\mathfrak{P}(V) \to \mathfrak{P}(|A|) \to \mathfrak{P}(V)$$

It is sufficient to show that $H|_{[0,1] \otimes J}$ factors through K. Now since J is generated by $\mathrm{Im}\mathfrak{P}(i_A) \circ (Id_{\mathfrak{P}(A)} - \eta_A \circ \gamma_A)$, by Propositions B.3.14 and B.3.15 it is sufficient to check that $H \circ Id_{[0,1]} \otimes (\mathfrak{P}(i_A) \circ (Id_{\mathfrak{P}(A)} - \eta_A \circ \gamma_A))$ factors through K, i.e. that $\gamma_A \circ H \circ Id_{[0,1]} \otimes (Id_{\mathfrak{P}(A)} - \eta_A \circ \gamma_A) = 0$. But by construction of H, $H \circ Id_I \otimes (Id_{\mathfrak{P}(A)} - \eta_A \circ \gamma_A) : [0,1] \otimes \mathfrak{P}(A) \to \mathfrak{P}(A)$ is just $p \otimes (Id_{\mathfrak{P}(A)} - \eta_A \circ \gamma_A)$.

The composition is then clearly 0.

2. Consider the composition

$$\tilde{H} = [0,1] \otimes (\mathfrak{P} \circ V) \xrightarrow{Id \otimes (t \circ id_V)} [0,1] \otimes ((\mathfrak{P} \otimes \Sigma) \circ V) \xrightarrow{\sim} [0,1] \otimes (\mathfrak{P}^{ns} \circ V)$$

$$\xrightarrow{H} \mathfrak{P}^{ns} \circ V \longrightarrow \mathfrak{P} \circ V$$

Here $\mathfrak{P}^{ns}$ is the non-symmetric operad associated to $\mathfrak{P}$. H is the homotopy between $id_{\mathfrak{P}^{ns}(V)}$ and the composition

$$\mathfrak{P}^{ns}(V) \to \mathfrak{P}^{ns}(|A|) \to \mathfrak{P}^{ns}(V)$$

and the map $\mathfrak{P}^{ns} \otimes V \to \mathfrak{P} \otimes V$ is given in each arity n by the projection $\mathfrak{P}(n) \otimes V^{\otimes n} \to \mathfrak{P}(n) \otimes_{\Sigma_n} V^{\otimes n}$. Since t is not a morphism of operads we cannot mimic the proof for non-symmetric operads. For each $r \in \mathbb{N}$ let γ_r^{ns} denote the restriction of the composition $\mathfrak{P}^{ns} \circ A \to \mathfrak{P} \circ A \to A$ to $\mathfrak{P}(r) \otimes A^{\otimes r}$, and let K_r^{ns} denote the image of the composition

$$\mathfrak{P}(r) \otimes A^{\otimes r} \xrightarrow{(Id_{\mathfrak{P}(r) \otimes A^{\otimes r}}, -\eta_A^{ns} \circ \gamma_r^{ns})} (\mathfrak{P}(r) \otimes A^{\otimes r}) \oplus \mathfrak{B}(1) \otimes A \longrightarrow (\mathfrak{P}^{ns} \circ A) \otimes (\mathfrak{P}^{ns} \circ A) \xrightarrow{\Delta} \mathfrak{P}^{ns} \circ A$$

where the last map is the diagonal, and the second last map is the natural inclusion. For $r, s \in \mathbb{N}$ let $J_{r,s}^{ns}$ denote the image of the composition

$$\mathfrak{P}(s) \otimes K_r^{ns} \otimes V^{s-1} \longrightarrow \bigoplus_m \mathfrak{P}(s) \otimes \mathfrak{P}(m) \otimes V^{m+s-1} \xrightarrow{\bigoplus_m \circ_1 \otimes Id_{V^{m+s-1}}}$$

$$\longrightarrow \bigoplus_m \mathfrak{P}(m+s-1) \otimes V^{m+s-1} \longrightarrow \mathfrak{P}^{ns} \circ V$$

Finally let $J_{r,s}$ denote the image of $J_{r,s}^{ns}$ under the projection $\mathfrak{P}^{ns} \circ V \to \mathfrak{P} \circ V$. Then $J = \sum_{r,s} J_{r,s}$. A tedious diagram chase using condition (3) of Definition 4.1.18 shows that $\tilde{H}$ maps $[0,1] \otimes J_{r,s}$ to $J_{r,s}$. Therefore $\tilde{H}$ maps $[0,1] \otimes J$ to J and we are done.

□

Corollary 4.1.20. *Let $\mathcal{E}$ be a $\mathbb{Q}$-IHAS and $\mathfrak{P}$ any operad. Then $\mathfrak{P}$ is admissible*

4.1.3 Homotopical Algebra Contexts

Before moving on let us make a connection with geometry. Recall that in [79] Toën and Vezzosi introduce an abstract categorical framework in which one can 'do' homotopical algebra, namely a homotopical algebra context. Let us recall the (slightly modified) definition.

Definition 4.1.21. *Let $\mathcal{M}$ be a combinatorial symmetric monoidal model category. We say that $\mathcal{M}$ is an* ***homotopical algebra context*** *(or HA context) if for any $A \in \mathcal{Alg}_{\mathfrak{Comm}}(\mathcal{M})$.*

1. *The model category $\mathcal{M}$ is proper, pointed and for any two objects X and Y in $\mathcal{M}$ the natural morphisms*

$$QX \coprod QY \to X \coprod Y \to RX \times RY$$

are equivalences.

2. *$Ho(\mathcal{M})$ is an additive category.*

3. *With the transferred model structure and monoidal structure $-\otimes_A$, the category ${}_A\mathcal{Mod}$ is a combinatorial, proper, symmetric monoidal model category.*

4. *For any cofibrant object $M \in {}_A\mathcal{Mod}$ the functor*

$$- \otimes_A M : {}_A\mathcal{Mod} \to {}_A\mathcal{Mod}$$

preserves equivalences.

5. *With the transferred model structures $\mathcal{Alg}_{\mathfrak{Comm}}({}_A\mathcal{Mod})$ and $\mathcal{Alg}_{\mathfrak{Comm}_{nu}}({}_A\mathcal{Mod})$ are combinatorial proper model categories.*

6. *If B is cofibrant in $\mathcal{Alg}_{\mathfrak{Comm}}({}_A\mathcal{Mod})$ then the functor*

$$B \otimes_A - : {}_A\mathcal{Mod} \to {}_A\mathcal{Mod}$$

preserves equivalences.

Proposition 4.1.22. *Let $\mathcal{E}$ be a rigid $\mathbb{Q}$-HAS. Then it is a homotopical algebra context.*

Proof. $\mathcal{E}$ is assumed to be proper, and its homotopy category is assumed to be additive. It is clearly pointed. Since it is additive the natural maps

$$QX \coprod QY \to X \coprod Y \to RX \times RY$$

are clearly equivalences. All that remains to prove is the final claim. Now if B is a cofibrant A-algebra then it is a retract of the free A-algebra on a cofibrant A-module. But the free A-algebra on a cofibrant A-module is cofibrant as an A-module. Hence $B \otimes_A (-)$ preserves equivalences by 4). □

The Cotangent Complex

By [79] Section 1.2 a homotopical algebra context has sufficient structure to define the relative cotangent complex of a map $f : A \to B$ in $\mathcal{A}lg_{\mathfrak{Comm}}(\mathcal{E})$. Let us briefly recall the discussion here. For a commutative monoid B write $\mathcal{A}lg^{aug}_{\mathfrak{Comm}}({}_B\mathcal{M}od) := \mathcal{A}lg_{\mathfrak{Comm}}({}_B\mathcal{M}od)\big/B$ for the category of augmented commutative B-algebras. There is an adjunction

$$K\colon \mathcal{A}lg_{\mathfrak{Comm}^{nu}}({}_B\mathcal{M}od) \rightleftarrows \mathcal{A}lg^{aug}_{\mathfrak{Comm}}({}_B\mathcal{M}od) :I$$

where K is the trivial extension functor and I sends an algebra C to the kernel of the map $C \to B$. This is both an equivalence of categories and a Quillen equivalence of model categories. There is also a Quillen adjunction

$$Q\colon \mathcal{A}lg_{\mathfrak{Comm}^{nu}}({}_B\mathcal{M}od) \rightleftarrows {}_B\mathcal{M}od :Z$$

where $Q(C)$ is defined by the pushout

$$\begin{array}{ccc} C\otimes_B C & \longrightarrow & C \\ \downarrow & & \downarrow \\ \bullet & \longrightarrow & Q(C) \end{array}$$

and Z just equips a module M with the trivial non-unital commutative monoid structure. Now given a map $f : A \to B$ in $\mathcal{A}lg_{\mathfrak{Comm}}(\mathcal{E})$ we define the **relative cotangent complex** by

$$\mathbb{L}_{B/A} := \mathbb{L}Q\mathbb{R}I(B \otimes^{\mathbb{L}}_A B)$$

It is shown in [79] that $\mathbb{L}_{B/A}$ corepresents the functor of $(\infty,1)$-categories

$${}_B\mathbf{Mod} \to \mathbf{sSet},\ M \mapsto Map_{\mathbf{Alg}_{\mathfrak{Comm}}({}_A\mathcal{M}od)/B}(B, B \ltimes M)$$

where $B \ltimes M$ is the square-zero extension of B by M (see Section B.3.2). Here Map is the simplicial mapping space. We also write $\mathbb{L}_B := \mathbb{L}_{B/k}$. Now let C be any A-algebra and consider the category $\mathbf{Alg}_{\mathfrak{Comm}}({}_A\mathcal{M}od)\big/C$. There is a functor

$$\mathbf{Alg}_{\mathfrak{Comm}}({}_A\mathcal{M}od)\big/C \to {}_C\mathbf{Mod},\ B \mapsto \mathbb{L}_{B/A} \otimes^{\mathbb{L}}_B C$$

It is left adjoint to the functor sending a C-module M to the square zero extension $C \ltimes M$. When $C = A = k$ so that $\mathbf{Alg}_{\mathfrak{Comm}}({}_A\mathcal{M}od)\big/k = \mathbf{Alg}^{aug}_{\mathfrak{Comm}}$ we denote this functor by $\mathbb{L}_0$. For cofibrant algebras we have the following result. The proof for vector spaces over a field is standard (for more general operads it can be found in Section 12.3.19, [48]), and goes through with minor modifications.

Proposition 4.1.23. *Let $A \to k$ be a cofibrant augmented algebra. Then $\mathbb{L}_0(A) \cong coker(I \otimes I \to I)$ where $I = Ker(A \to k)$ is the augmentation ideal. Suppose further that $\mathbb{L}_0(A)$ is cofibrant. Then $\mathbb{L}_A \cong A \otimes \mathbb{L}_0(A)$.*

Proof. Since A is a cofibrant we may assume everything is underived. We need to show that the functor sending A to the k-module $coker(I \otimes I \to I)$ is left adjoint to the square-zero extension functor. Let $f : A \to k \ltimes M$ be map of augmented algebras. This induces a map $I \to M$ of augmentation ideals which clearly descends to a map $\tilde{f} : coker(I \otimes I \to I) \to M$. Conversely suppose we are given a map $g : coker(I \otimes I \to I) \to M$. Consider the map of modules $A \cong k \coprod I \to k \coprod I/I^2 \to k \coprod M$. This is in fact a map of algebras $A \to k \ltimes M$. These maps on hom sets are inverse, realising the adjunction. For the second assertion note that A is cofibrant in the slice category over A. Thus we need to show that

$$Hom_{\mathcal{Alg}_{\mathcal{E}}/A}(A, A \ltimes M) \cong Hom_{{}_A\mathcal{Mod}}(A \otimes \mathbb{L}_0(A), M) \cong Hom_{\mathcal{E}}(\mathbb{L}_0(A), M)$$

$$\cong Hom_E(I/I^2, M)$$

This computation is entirely similar to the first part. $\square$

We will repeatedly make use of the following facts which constitute Proposition 1.2.1.6 in [79].

Proposition 4.1.24. *1. Let $f : A \to B$ and $g : B \to C$ be morphisms of algebras. Then there is a homotopy cofiber sequence in ${}_C\mathcal{Mod}$.*

$$\mathbb{L}_{B/A} \otimes^{\mathbb{L}}_B C \to \mathbb{L}_{C/A} \to \mathbb{L}_{C/B}$$

2. If

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ A' & \longrightarrow & B' \end{array}$$

is a homotopy pushout in $\mathcal{Alg}_{\mathfrak{Comm}}(\mathcal{E})$ then the natural map $\mathbb{L}_{B/A} \otimes^{\mathbb{L}}_B B' \to \mathbb{L}_{B'/A'}$ is an equivalence.

Homotopy Epimorphisms

Let us now briefly discuss the homotopy epimorphisms which appear in the covers of the Stein topology studied in [5]. Recall that in a category $\mathcal{C}$ with pushouts a map $A \to B$ is an epimorphism if and only if the induced map $B \coprod_A B \to B$ is an isomorphism. This motivates the following derived notion of an epimorphism.

Definition 4.1.25. *Let $\mathcal{M}$ be a model category. A morphism $f : A \to B$ in $\mathcal{M}$ is said to be a* ***homotopy epimorphism*** *if the induced map $B \coprod_A^{\mathbb{L}} B \to B$ is an equivalence.*

Let us make the following straightforward observation.

Proposition 4.1.26. *Let $f : A \to B$ be a homotopy epimorphism. Let $g : X \to Y$ and $h : X \to A$ be any maps. Then the induced map*

$$A \coprod_X^{\mathbb{L}} Y \to B \coprod_X^{\mathbb{L}} Y$$

is a homotopy epimorphism. In particular if $\mathcal{M}$ is left proper and g is a cofibration then the map

$$A \coprod_X Y \to B \coprod_X Y$$

is a homotopy epimorphism.

Proof. We have

$$(B \coprod_X^{\mathbb{L}} Y) \coprod_{A \coprod_X^{\mathbb{L}} Y} (B \coprod_X^{\mathbb{L}} Y) \cong (B \coprod_A^{\mathbb{L}} B) \coprod_X^{\mathbb{L}} (Y \coprod_Y^{\mathbb{L}} Y) \cong B \coprod_X^{\mathbb{L}} Y$$

The second assertion follows because in this case the normal pushout is the homotopy pushout. □

The following is an obvious generalisation of a calculation in [10] Lemma 4.40

Proposition 4.1.27. *Let $f : A \to B$ be a map. Then f is a homotopy epimorphism if and only if for any map $g : B \to C$ the map*

$$B \coprod_A^{\mathbb{L}} C \to C$$

is an equivalence.

Proof. We have

$$B \coprod_A^{\mathbb{L}} C \cong B \coprod_A^{\mathbb{L}} B \coprod_B^{\mathbb{L}} C \cong B \coprod_B^{\mathbb{L}} C \cong C$$

□

Proposition 4.1.28. *Let h be the composition*

$$A \xrightarrow{f} B \xrightarrow{g} C$$

If h and f are homotopy epimorphisms then so is g. If f and g are homotopy epimorphisms then so is h.

Proof. Suppose that h and f are a homotopy epimorphisms. Then we have

$$C \cong C \coprod_A^{\mathbb{L}} C \cong C \coprod_B^{\mathbb{L}} B \coprod_A^{\mathbb{L}} C \cong C \coprod_B^{\mathbb{L}} C$$

Now suppose that f and g are homotopy epimorphisms. Then we have

$$C \cong C \coprod_B^{\mathbb{L}} C \cong C \coprod_{B \coprod_A^{\mathbb{L}} B}^{\mathbb{L}} C \cong C \coprod_A^{\mathbb{L}} C$$

□

The following is clear and applies to all the higher algebra settings we consider.

Proposition 4.1.29. *Suppose that direct limits preserve weak equivalences in $\mathcal{M}$. Let $A : \mathcal{I} \to \mathcal{M}$, $B : \mathcal{I} \to \mathcal{M}$ be direct systems and $f : A \to B$ a natural transformation. Suppose that for each i the map $f_i : A_i \to B_i$ is a homotopy epimorphism. then*

$$lim_{\to} f_i : lim_{\to} A_i \to lim_{\to} B_i$$

is a homotopy epimorphism.

We will exploit the following relationship between homotopy epimorphisms and cotangent complexes. It is a direct consequence of [79] Corollary 1.2.6.6. Note that chain complexes are stable model categories.

Proposition 4.1.30. *Let $\mathcal{E}$ be a homotopical algebra context which is stable as a model category and let $f : A \to B$ be a homotopy epimorphism. Then $\mathbb{L}_{B/A} \cong 0$ and $\mathbb{L}_A \otimes_A^{\mathbb{L}} B \to \mathbb{L}_B$ is an equivalence.*

4.2 Koszul Duality

In this section $\mathcal{E}$ we will be a monoidal elementary quasi-abelian category. We shall consider operads and their algebras in the category $\mathrm{Ch}(\mathcal{E})$. We fix an operad $\mathfrak{P}$, a co-operad $\mathfrak{C}$ and a twisting morphism $\alpha : \mathfrak{C} \to \mathfrak{P}$. Recall (see Appendix B for details) that this gives rise to the bar-cobar ajdunction

$$\Omega_\alpha \colon co\mathcal{Alg}_{\mathfrak{C}}^{nil} \rightleftarrows \mathcal{Alg}_{\mathfrak{P}} : B_\alpha$$

Our goal is to generalize Koszul duality results of [82] which for $\mathcal{E} = {}_k\mathcal{Vect}$ say that this is a Quillen equivalence.

4.2.1 Cofibrant Co-operads and Cof-nilpotent Coalgebras

First we need to refine the class of coalgebras under consideration.

Definition 4.2.1. *A filtered co-operad* $(\mathfrak{C}, \Delta)$ *in* $Ch(\overline{\mathcal{F}ilt}(\mathcal{E}))$ *is said to be* ***cofibrant*** *if its underlying filtered* Σ*-module is cofibrant.*

Definition 4.2.2. *A filtered* $\mathfrak{C}$*-coalgebra* $(\mathfrak{C}, \Delta)$ *is said to be cofibrant if it is cofibrant as a filtered object.*

From now on we shall assume that our co-operad is **weight graded**, $\mathfrak{C} = \bigoplus_0^\infty \mathfrak{C}^{(\omega)}$ where $\mathfrak{C}^0 = I$. There is of course an associated **weight filtration**. A coalgebra C is equipped with a canonical **coradical filtration**, where C_n is given by the following pullback

$$\begin{array}{ccc} C_n & \longrightarrow & C \\ \downarrow & & \downarrow{\scriptstyle \Delta} \\ \bigoplus_{\omega=0}^{n} \mathfrak{C}^{(\omega)}(C) & \longrightarrow & \mathfrak{C}(C) \end{array}$$

We also write $\overline{\Delta}_C := \Delta_C - i$ where $i : C \to I \circ C \to \mathfrak{C} \circ C$ is the inclusion.

Definition 4.2.3. *A coalgebra* C *is said to be* ***conilpotent*** *if its coradical filtration is exhaustive.*

We need a more homotopical notion.

Definition 4.2.4. *Let* $\mathfrak{C}$ *be a co-operad. A* $\mathfrak{C}$*-colgebra* C *is said to be* ***cof-nilpotent*** *if it is a conilpotent* $\mathfrak{C}$*-coalgebra, and with its coradical filtration is cofibrant as an object of* $Ch(\overline{\mathcal{F}ilt}(\mathcal{E}))$*.*

Proposition 4.2.5. *Let* $(\mathfrak{C}, \Delta)$ *be a filtered cofibrant co-operad. Then for any cofibrant object* V *of* $\mathcal{E}$*,* $\mathfrak{C} \circ V$ *is cof-nilpotent.*

Proof. Since $\mathfrak{C}$ is cofibrant as a filtered Σ-module, it is a retract of a Σ-module of the form $P \otimes k[\Sigma]$ for some filtered cofibrant object P. Thus $\mathfrak{C} \circ V$ is a retract of $P \circ_{ns} V$ which is filtered cofibrant. Let K denote the cokernel of the map $\Delta : \mathfrak{C} \to \mathfrak{C} \circ \mathfrak{C}$. Then K is also filtered cofibrant. Moreover, $\circ$ preserves cokernels, so $\text{coker}(\Delta \circ Id_V) \cong K \circ V$. By the same proof as before, this is cofibrant. □

4.2.2 Koszul Morphisms

In this and the following section we generalize Theorem 2.1 parts (1) and (3) of [82] to monoidal elementary quasi-abelian categories. Essentially we shall set up the necessary technical machinery so that we can generalise the proof in [82] (and also some of the proofs in [39]) to quasi-abelian categories. As will become clear, some of the proof goes through with only minor modifications, while some requires significant effort to generalise. We regard $\mathrm{Ch}(\mathcal{E})$ as an IHAS using the interval object $[0,1]$ defined in Example 4.1.10.

Definition 4.2.6. *Let $\alpha : \mathfrak{C} \to \mathfrak{P}$ be a twisting morphism.*

1. *A morphism $f : C \to D$ of $\mathfrak{C}$-coalgebras is said to be a α-**weak equivalence** if $\Omega_\alpha f$ is a quasi-ismorphism of $\mathfrak{P}$-algebras.*
2. *A morphism $f : C \to D$ between filtered coalgebras is said to be a **cofibration** if $|f|$ is a cofibration of filtered objects.*
3. *A morphism $f : C \to D$ of filtered coalgebras is said to be a α-**fibration** if it has the right lifting property with respect to those maps which are both cofibrations and α-weak equivalences.*

We are specifically interested in the case that α is a Koszul morphism. We shall assume from now on that $\mathfrak{P}$ is a filtered operad with $F_0\mathfrak{P} \cong I$. However when we consider the category of algebras over $\mathfrak{P}$ we shall forget this filtration.

Definition 4.2.7. *A twisting morphism is said to be **Koszul** if*

1. *$\mathfrak{P}$ is split, has cofibrant entries, and the differential lowers the filtration.*
2. *$\mathfrak{C}$ is filtered cofibrant when equipped with its weight filtration, and the differential lowers the weight filtration.*
3. *α preserves the filtration, and $\alpha|_{F_0\mathfrak{C}} = 0$.*
4. *$\mathfrak{P} \circ_\alpha \mathfrak{C} \circ_\alpha \mathfrak{P} \to \mathfrak{P}$ is a filtered homotopy equivalence relative to $[0,1]$.*
5. *$I \to \mathfrak{P} \circ_\alpha \mathfrak{C}$ is a filtered acyclic cofibration.*

Note that this indeed gives the usual definition of a Koszul morphism from [82] when $\mathcal{E} = {}_k\mathcal{V}ect$, for k a field.

Theorem 4.2.8. *Let $\alpha : \mathfrak{C} \to \mathfrak{P}$ be Koszul. Then in $\mathcal{C}of_{\mathfrak{C}}$ every morphism can be factored into admissible monomorphisms followed by α-acyclic fibrations, or admissible monomorphisms which are α-acyclic followed by fibrations. If all objects in $Ch(\mathcal{E})$ are cofibrant then this forms a model structure.*

We shall prove this in several steps.

Definition 4.2.9. *A **filtered quasi-isomorphism** of filtered coalgebras is a quasi-isomorphism of the underlying filtered chain complexes.*

With the machinery of filtered objects in quasi-abelian categories developed earlier, the next two propositions generalise easily from [82].

Proposition 4.2.10. *Suppose that $\mathfrak{P}$ is an admissible operad. Then filtered quasi-isomorphisms are α-weak equivalences.*

Proof. Let $f : C \to D$ be a filtered quasi-isomorphism of filtered $\mathfrak{C}$ co-algebras. Consider the following filtration on $\Omega_\alpha(C) = (\mathfrak{P}(C), d_1 + d_2)$:

$$F_n\Omega_\alpha C = \sum_{k\geqslant 1,\ n_1+\ldots+n_k\leqslant n} \mathfrak{P}(k) \otimes_{\Sigma_k} (F_{n_1}C \otimes \ldots \otimes F_{n_k}C)$$

Recall that $d_1 = \mathfrak{P} \circ' d_C$. Moreover by inspecting the formula defining d_2 it lowers the filtration. Thus $\text{gr}(\Omega_\alpha(C)) \cong (\mathfrak{P}(\text{gr}(C)))$. By assumption $\text{gr}(f) : \text{gr}(C) \to \text{gr}(D)$ is a weak-equivalence of graded objects. Hence $\text{gr}(\Omega_\alpha f) = \mathfrak{P}(\text{gr}(f))$ is a weak-equivalence. By Proposition 3.3.31 $\Omega_\alpha f$ is a quasi-isomorphism. □

Proposition 4.2.11. *Suppose $f : A \to A'$ is a quasi-isomorphism of dg $\mathfrak{P}$-algebras whose underlying chain complexes are cofibrant. Then $B_\alpha f$ is a filtered quasi-isomorphism of dg $\mathfrak{C}$-coalgebras.*

Proof. The weight filtration on the underlying graded object of $B_\alpha A$ is given by

$$(B_\alpha A)_n = \bigoplus_{\omega=0}^{n} \mathfrak{C}^{(\omega)}(A)$$

Its differential is given by the sum $d_{\mathfrak{C}} \circ Id_A + Id_{\mathfrak{C}} \circ' d_A + d_2$ where d_2 is the unique coderivation extending the map

$$\mathfrak{C} \circ A \xrightarrow{\alpha\circ Id_A} \mathfrak{P} \circ A \xrightarrow{\gamma_A} A$$

The formula defining this coderivation implies that d_2 lowers the filtration. By assumption $d_{\mathfrak{C}} \circ Id_A$ also lowers the filtration. So

$$\text{gr}(B_\alpha f) = \mathfrak{C}(f) : (\mathfrak{C}(A), Id_{\mathfrak{C}} \circ' d_A) \to (\mathfrak{C}(A), Id_{\mathfrak{C}} \circ' d_A)$$

which is a quasi-isomorphism. □

Proposition 4.2.12. *Let α be a Koszul morphism and C a cof-nilpotent $\mathfrak{C}$-coaglebra. Then the unit $\nu_\alpha(C) : C \to B_\alpha\Omega_\alpha C$ is an α-weak equivalence and an admissible monomorphism.*

Proof. On underlying graded objects, the unit is the composition

$$C \xrightarrow{\Delta_C} \mathfrak{C}(C) \cong \mathfrak{C} \circ I \circ C \longrightarrow \mathfrak{C} \circ \mathfrak{P} \circ C$$

Both of these maps are clearly admissible monomorphisms.

Let us now show that $\nu_\alpha(C)$ is an α-weak equivalence. Consider the filtration on $\Omega_\alpha C$ given by

$$F_n\Omega_\alpha C = \sum_{k\geqslant 1, m+n_1+\ldots+n_k\leqslant n} F_m\mathfrak{P}(k) \otimes_{\Sigma_k} (F_{n_1}C \otimes \ldots \otimes F_{n_k}C)$$

and the one on $\Omega_\alpha B_\alpha \Omega_\alpha C$ given by

$$F_n\Omega_\alpha B_\alpha \Omega_\alpha C = \sum_{k\geqslant 1, p+q+m+n_1+\ldots+n_k\leqslant n} (F_p\mathfrak{P} \circ F_q\mathfrak{C} \circ F_m\mathfrak{P})(k) \otimes_{\Sigma_k} (F_{n_1}C \otimes \ldots \otimes F_{n_k}C)$$

$\Omega_\alpha(\nu_\alpha(C))$ preserves these filtrations. Passing to associated graded objects gives

$$gr(\Omega_\alpha(\nu_\alpha(C))) : gr(\Omega_\alpha C) \to gr(\Omega_\alpha B_\alpha \Omega_\alpha C)$$

The underlying object on the left-hand side is $gr(\mathfrak{P}) \circ gr(C)$ and the underlying object on the right-hand side is $gr(\mathfrak{P} \circ \mathfrak{C} \circ \mathfrak{P}) \circ gr(C)$. Now consider the filtration on $gr(\Omega_\alpha C)$ given by

$$F_n gr(\Omega_\alpha C) = \sum_{k\geqslant 1, n_1+\ldots+n_k\leqslant n} gr(\mathfrak{P}(k)) \otimes gr_{n_1}(C) \otimes \ldots \otimes gr_{n_k}(C)$$

and the filtration on $gr(\Omega_\alpha B_\alpha \Omega_\alpha C)$ given by

$$F_n gr(\Omega_\alpha B_\alpha \Omega_\alpha C) = \sum_{k\geqslant 1, n_1+\ldots+n_k\leqslant n} (\mathfrak{P} \circ \mathfrak{C} \circ \mathfrak{P})(k) \otimes_{\Sigma_k} (gr_{n_1}(C) \otimes \ldots \otimes gr_{n_k}(C))$$

Then $gr(\Omega_\alpha(\nu_\alpha(C)))$ preserves these filtrations. The associated graded of the filtration on $gr(\Omega_\alpha C)$ is $gr(\mathfrak{P}) \circ gr(C)$, and the associated graded of the filtration on $gr(\Omega_\alpha B_\alpha \Omega_\alpha C)$ is $gr(\mathfrak{P} \circ_\alpha \mathfrak{C} \circ_\alpha \mathfrak{P}) \circ gr(C)$. Denote by $\tilde{gr}(\Omega_\alpha(\nu_\alpha(C)))$ the associated graded of $gr(\Omega_\alpha(\nu_\alpha(C)))$. The composite

$$gr(\mathfrak{P}) \circ gr(C) \xrightarrow{\tilde{gr}(\Omega_\alpha(\nu_\alpha(C)))} gr(\mathfrak{P} \circ_\alpha \mathfrak{C} \circ_\alpha \mathfrak{P}) \circ gr(C) \longrightarrow gr(\mathfrak{P}) \circ gr(C)$$

is the identity. The map $gr(\mathfrak{P} \circ_\alpha \mathfrak{C} \circ_\alpha \mathfrak{P}) \circ gr(C) \to gr(\mathfrak{P}) \circ gr(C)$ is a homotopy equivalence relative to $[0, 1]$ and hence is a weak equivalence. By the 2-out-of-3 property $\tilde{gr}(\Omega_\alpha(\nu_\alpha(C)))$ is an equivalence. Therefore $\Omega_\alpha(\nu_\alpha(C))$ is an equivalence.

□

Standard Cofibrations

Because complexes in a general quasi-abelian category don't split the discussion of standard cofibrations as defined in [82] Section 2.4 is significantly more involved Let $(V_\bullet, d_V)$ be a cofibrant object in $\mathrm{Ch}(\mathcal{E})$, $(A_\bullet, d_A)$ a $\mathfrak{P}$-algebra, and $\alpha : V \to A$ be a degree -1 map of graded objects. There is then an induced map of graded objects

$$V \to A \to A \coprod \mathfrak{P}(V)$$

By Proposition B.3.11 there is then a unique derivation of degree -1

$$d_\alpha : A \coprod \mathfrak{P}(V) \to A \coprod \mathfrak{P}(V)$$

whose restriction to V is α. We denote this algebra equipped with the derivation given by $d_A + d_\alpha$ by $A \coprod_\alpha \mathfrak{P}(V)$.

Proposition 4.2.13. *Suppose that* $\alpha : V[1] \to A$ *is a morphism of chain complexes. Then* $A \coprod_\alpha \mathfrak{P}(V)$ *is a chain complex.*

Proof. The derivation $A \coprod \mathfrak{P}(V) \to A \coprod \mathfrak{P}(V)[1]$ is induced from the derivation

$$d_{\mathfrak{P}} \circ Id_{A \oplus B} + Id_{\mathfrak{P}} \circ_{(1)} (d_A + \alpha + d_B) : \mathfrak{P}(A \oplus B) \to \mathfrak{P}(A \oplus B)[1]$$

Therefore it suffices to show that this derivation squares to 0. This is a straightforward computation. □

Lemma 4.2.14. *The embedding* $A \to A \coprod_\alpha \mathfrak{P}(V)$ *is a cofibration of* $\mathfrak{P}$*-algebras.*

First let us prove some auxiliary results. Let $(V_\bullet, d_V)$ and $(W_\bullet, d_W)$ be chain complexes, and let $f : V_\bullet \to W_\bullet$ be a morphism of chain complexes. Suppose there are degree -1 maps $\nu : V_\bullet \to A_\bullet$ and $\omega : W_\bullet \to A_\bullet$ such that $\omega \circ f = \nu$, and both

$$A \coprod_\nu \mathfrak{P}(V) \text{ and } A \coprod_\omega \mathfrak{P}(W)$$

are complexes. Then clearly the morphism $Id_A \coprod \mathfrak{P}(f)$ induces a morphism of chain complexes

$$A \coprod_\nu \mathfrak{P}(V) \to A \coprod_\omega \mathfrak{P}(W)$$

Now, let $V^i_\bullet$ be a diagram of chain complexes. Suppose there is a degree -1 map $\alpha : \mathrm{colim} V^i_\bullet \to A$. Composing with the maps $f_i : V_i \to \mathrm{colim} V^i_\bullet$ gives degree -1 maps $\alpha_i = \alpha \circ f_i : V_i \to A$. Suppose that for each i $A \coprod_{\alpha_i} \mathfrak{P}(V^i)$ is a chain complex.

Proposition 4.2.15. *There is an isomorphism of algebras*

$$colim(A \coprod_{\alpha_i} \mathfrak{P}(V^i)) \cong A \coprod_{\alpha} \mathfrak{P}(V)$$

Proof. By the above remarks there is a map of algebras $\mathrm{colim}(A \coprod_{\alpha_i} \mathfrak{P}(V^i)) \cong A \coprod_{\alpha} \mathfrak{P}(V)$. To see that is is an isomorphism we may forget the differentials, it which case it reduces to the fact that coproducts and colimits commute. □

Let $f : V \to W$ and $\nu : V \to A$ be degree 0 maps and let $\omega : W \to A$ be a degree -1 map. There is an induced degree -1 map $\nu + \omega : \mathrm{cone}(f) \to A$. There is also a degree -1 map

$$V[1] \xrightarrow{(\nu,-f)} A \oplus W \longrightarrow A \textstyle\coprod_{-\omega} \mathfrak{P}(W)$$

which we denote by $\nu \cup (-f)$.

Proposition 4.2.16. *Suppose that $\omega : W[1] \to A$ is a map of complexes and that ν satisfies*

$$\omega_n \circ f_n = d_n^A \circ \nu_n - \nu_{n-1} d_n^V$$

Then

1. *$\nu \cup (-f)$ is a map of chain complexes.*

2. *There is an isomorphism*

$$A \coprod_{\nu+\omega} \mathfrak{P}(cone(f)) \cong (A \coprod_{-\omega} \mathfrak{P}(W)) \coprod_{\nu \cup f} \mathfrak{P}(V[1])$$

Proof. The first part is a direct computation. For the second part, let us forget the differentials for the moment. Then we have

$$(A \coprod_{\omega} \mathfrak{P}(W)) \coprod_{\nu \cup f} \mathfrak{P}(V[1]) \cong A \coprod \mathfrak{P}(W) \coprod \mathfrak{P}(V[1]) \cong A \coprod \mathfrak{P}(W \oplus V[1])$$
$$= A \coprod \mathfrak{P}(\mathrm{cone}(f)) = A \coprod_{\nu+\omega} \mathfrak{P}(\mathrm{cone}(f))$$

We need to check that this isomorphism preserves the differentials. Again this is a direct computation. □

Proof of Lemma 4.2.14. In fact we are going to show the following. Suppose that $f : V \to W$ is a cofibration of chain complexes. Let $\alpha : W \to A$ be a degree -1 map. Then the induced map

$$A \coprod_{\alpha} \mathfrak{P}(V) \to A \coprod_{\alpha \circ f} \mathfrak{P}(W)$$

is a cofibration of $\mathfrak{P}$-algebras. By Proposition 4.2.15 it is sufficient to show that this is the case for the generating cofibrations $S^n(P) \to D^{n+1}(P)$. First note that $D^{n+1}(P) = \mathrm{cone}(id_{S^n(P)})$. Therefore by Proposition 4.2.16 we reduce to showing that, given a degree -1 map $\alpha : S^n(P) \to A$, the map $A \to A \coprod_\alpha \mathfrak{P}(S^n(P))$ is a cofibration. But in this case we have pushout diagram

$$\begin{array}{ccc} \mathfrak{P}(S^{n-1}(P)) & \xrightarrow{\gamma_A \mathfrak{P}(\alpha[-1])} & A \\ \downarrow & & \downarrow \\ \mathfrak{P}(D^n(P)) & \longrightarrow & A \coprod_\alpha \mathfrak{P}(S^n(P)) \end{array}$$

The left-hand vertical map is a cofibration, so as a pushout of a cofibration the right-hand map is also a cofibration.

□

This result has numerous applications. Following [48] B.6.13 we define triangulated quasi-free algebras.

Definition 4.2.17. *A $\mathfrak{P}$-algebra A is said to be **quasi-free** if there is a cofibrant object V of $Ch(\mathcal{E})$ such that the underlying graded algebra of A is isomorphic to the underlying graded algebra of $\mathfrak{P}(V)$. A quasi-free algebra A is said to be **triangulated** if V is equipped with a cofibrant filtration V_i such that $d|_{V_i}$ factors through $\mathfrak{P}(V_{i-1})$.*

In particular quasi-free algebras on bounded below cofibrant objects are triangulated.

Corollary 4.2.18. *A triangulated quasi-free algebra is cofibrant.*

Proof. It is clear that the map $0 \to A$ is a standard cofibration, which is a cofibration by Lemma 4.2.14 □

Immediately we get the following result.

Corollary 4.2.19. *If A is quasi-free on a bounded below complex of projectives then A is a cofibrant $\mathfrak{P}$-algebra.*

This also allows us to generalise Proposition 2.8 of [82].

Proposition 4.2.20. *Let (C, Δ_C) and $(C', \Delta_{C'})$ be $\mathfrak{C}$-coalgebras, and let $i : C' \to C$ be a morphism of coalgebras whose cokernel is a cofibrant chain complex. Suppose that $Im(\overline{\Delta}_C(C))$ factors through $\mathfrak{C}(C')$. Then $\Omega_\alpha(i)$ is a cofibration of $\mathfrak{P}$-algebras.*

Proof. As graded objects there is an isomorphism $C \cong C' \oplus E$ for some graded-projective E, so (as graded objects)

$$\Omega_\alpha C \cong \mathfrak{P}(C') \coprod \mathfrak{P}(E)$$

Under the decomposition $C \cong C' \oplus E$, d_C is the sum of three degree -1 maps

$$d_{C'} : C' \to C',\ d_E : E \to E, \alpha : E \to C'$$

By assumption the composition

$$\beta : E \rightarrowtail C \xrightarrow{\Delta_C} \mathfrak{C}(C) \xrightarrow{\alpha(C)} \mathfrak{P}(C)$$

inducing the twisted differential on $\Omega_\alpha C$ factors through $\mathfrak{P}(C')$. Thus $\Omega_\alpha C' \to \Omega_\alpha C$ is given by the standard cofibration $\Omega_\alpha C' \rightarrowtail \Omega_\alpha C' \coprod_{\alpha+\beta} \mathfrak{P}(E)$ which is a cofibration by Lemma 4.2.14 □

Theorem 4.2.21. *1. The cobar construction Ω_α preserves cofibrations and weak equivalences between cofibrant objects.*

2. The bar construction B_α preserves fibrations and weak equivalences between objects whose underlying chain complex is cofibrant.

Proof. 1. Ω_α preserves weak equivalences by definition. Now let $f : C \to D$ be a cofibration of conilpotent $\mathfrak{C}$-coalgebras with cokernel E. For any $n \in \mathbb{N}$ consider the sub coalgebra of D defined by

$$D^{[n]} = f(C) + F_{n-1}D$$

for $n \geqslant 1$ and

$$D^{[0]} = C$$

The definition of the weight filtration implies that $\overline{\Delta}_{D^{[n+1]}}(D^{[n+1]}) \subset \mathfrak{C}(D^{[n]})$. It remains to check that $D^{[n]} \to D^{[n+1]}$ has a cofibrant cokernel. Let $g : D \to E$ be the cokernel of f. We claim that $\mathrm{coker}(D^{[n]} \to D^{[n+1]}) = \mathrm{gr}_n(E)$. Since E is a cofibrant object of $\overline{\mathcal{F}ilt}(\mathcal{E})$ this proves the claim. Let us first show that the map $C \oplus F_n D \to D$ is admissible. It is sufficient to show that it is admissible after forgetting the differentials. In this case $D \cong C \oplus E$, as filtered objects. and we have the following commutative diagram

$$\begin{array}{ccc} C \oplus F_n D & \longrightarrow & D \\ \downarrow \sim & & \downarrow \sim \\ C \oplus F_n C \oplus F_n E & \longrightarrow & C \oplus E \end{array}$$

The bottom map is clearly admissible, so the top one is as well. Consider the map

$$(0, g_n) : C \oplus F_n D \twoheadrightarrow \mathrm{gr}_n(E)$$

$C \oplus F_n D = F_n C$ is contained in its kernel, so we get a well defined map

$$C + F_n D \to \mathrm{gr}_n(E)$$

which is an admissible epimorphism. We claim that its kernel is the map

$$D^{[n]} \to D^{[n+1]}$$

Again we may ignore differentials. Then in the direct sum composition this map corresponds to the inclusion

$$C \oplus F_{n-1} E \hookrightarrow C \oplus F_n E$$

which clearly has cokernel equal to $\mathrm{gr}_n(E)$.

2. The fact that B_α preserves fibrations follows from the Part 1, definition of fibration and the fact that B_α is a right adjoint. The second assertion follows from Proposition 4.2.10 and Proposition 4.2.11.

$\square$

Proposition 4.2.22. *Let $\mathfrak{C}$ be a cooperad in a monoidal elemetnary exact category so that each $\mathfrak{C}(n)$ is a retract of an object of the form $\tilde{\mathfrak{C}}(n) \otimes \Sigma_n$ with $\tilde{\mathfrak{C}}(n)$ being flat in $\mathcal{E}$. Let $f : A \to B$ and $g : X \to Y$ be morphisms of cof-nilpotent coalgebras which are admissible monomorphisms in $\mathcal{E}$. Then $g \times f : A \times X \to B \times Y$ is an admissible monomorphism in $\mathcal{E}$.*

Proof. The construction of products of coalgebras is dual to the construction of coproducts of algebras. We therefore have a commutative diagram

$$\begin{array}{ccc} \mathfrak{C}(A \oplus X) & \xrightarrow{\mathfrak{C}(f \oplus g)} & \mathfrak{C}(B \oplus Y) \\ \uparrow & & \uparrow \\ A \times X & \xrightarrow{f \times g} & B \times Y \end{array}$$

The top horizontal map is an admissible monomorphism. An explicit construction of the product $A \times X$ realises the map $A \times X \to \mathfrak{C}(A \oplus X)$ as a kernel so it is anadmissible monomorphism. By the obscure lemma the bottom horizontal map is an admissible monomorphism as well. $\square$

Using this proposition, we have the following version of Lemma B.1 from [82] which can be proved in the same way as in [82] .

Lemma 4.2.23. *Let D be a cofibrant $\mathfrak{C}$-coalgebra and $p : A \twoheadrightarrow \Omega_\alpha D$ be a fibration of $\mathfrak{P}$-algebras. Then in the following pullback diagram*

$$\begin{array}{ccc} B_\alpha A \times_{B_\alpha \Omega_\alpha D} D & \longrightarrow & D \\ \big\downarrow{\scriptstyle j} & & \big\downarrow{\scriptstyle \nu_\alpha D} \\ B_\alpha A & \xrightarrow{\;B_\alpha p\;} & B_\alpha \Omega_\alpha D \end{array}$$

the map j is an α-weak equivalence and an admissible monomorphism.

Remark 4.2.24. *We expect that j should also be a cofibration. We originally had a proof but found a gap shortly before submission. It is definitely true if all complexes involved are bounded below. We expect the optimum set-up is to also consider the filtered model structure on the $\mathfrak{P}$-algebra side.*

With the technology developed above the proof of theorem 2.1 1) works in our setup, as we show below.

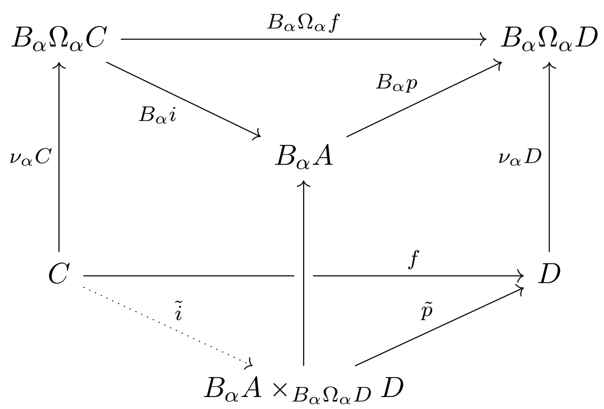
Proof of Theorem 4.2.8. Suppose that $f : C \to D$ is a morphism of cof-nilpotent $\mathfrak{C}$-coalgebras. In $\mathcal{A}lg_{\mathfrak{P}}$ we get the the following commutative diagram

$$\begin{array}{ccccc} \Omega_\alpha C & & \xrightarrow{\;\Omega_\alpha f\;} & & \Omega_\alpha D \\ & {\scriptstyle i}\searrow & & \nearrow\!\!\!\nearrow{\scriptstyle p} & \\ & & A & & \end{array}$$

where i is a cofibration, p a fibration, and one of them is a quasi-isomorphism. Applying B_α we get the following commutative diagram.

$$\begin{array}{ccccc} B_\alpha\Omega_\alpha C & & \xrightarrow{\;B_\alpha\Omega_\alpha f\;} & & B_\alpha\Omega_\alpha D \\ & {\scriptstyle B_\alpha i}\searrow & & \nearrow\!\!\!\nearrow{\scriptstyle B_\alpha p} & \\ & & B_\alpha A & & \end{array}$$

and using the universal property we get the following commutative diagram



We first show that $\tilde{i}$ is an admissible monomorphism and $\tilde{p}$ is a fibration. $\tilde{p}$ is a fibration since it is the pullback of $B_\alpha p$, which is a fibration by Proposition 4.2.21. The map $B_\alpha i \circ \nu_\alpha C$ is given by the composite $\mathfrak{P}(i_C) \circ \Delta_C$ where i_C is the restriction of i to C. It is clearly an admissible monomorphism. By Lemma 4.2.23 f is an admissible monomorphism. Hence $\tilde{i}$ is as well. Suppose that i is a weak equivalence. Then $B_\alpha i$ is a weak equivalence by Proposition 4.2.11. By the two out of three-property $\tilde{i}$ is a weak-equivalence. A similar proof shows that $\tilde{p}$ is a weak equivalence assuming p is.

Now suppose that all complexes are cofibrant. Then cofibrations are just admissible monomorphisms so we get the factorisation axioms for a model category. It remains to prove the lifting property. Consider the following commutative diagram in $\mathrm{Cof}_{\mathfrak{C}}^{nil}$

$$\begin{array}{ccc} E & \longrightarrow & C \\ {\scriptstyle c}\downarrow & \overset{\alpha}{\nearrow} & \downarrow{\scriptstyle f} \\ F & \longrightarrow & D \end{array}$$

where c is a cofibration and f is a fibration. The right lifting property for fibrations against acyclic cofibrations is built into the definition of fibration. Thus it remains to check the right lifting property for acyclic fibrations against cofibrations. Therefore we suppose now that f is a weak equivalence. By the previous part of the proof we may factor f as $\tilde{p} \circ \tilde{i}$ where $\tilde{i}$ is a cofibration, $\tilde{p}$ is a fibration, and by the 2-out-of-3 property, both are weak equivalences. Again by the definition of fibration, there is a lift in the following diagram

$$\begin{array}{ccc} C & \xrightarrow{id_C} & C \\ {\scriptstyle \tilde{i}}\downarrow & \overset{r}{\nearrow} & \downarrow{\scriptstyle f} \\ B_\alpha A \times_{B_\alpha \Omega_\alpha D} D & \overset{\tilde{p}}{\twoheadrightarrow} & D \end{array}$$

It therefore remains to find a lift in the diagram

$$\begin{array}{ccc} E & \longrightarrow & B_\alpha A \times_{B_\alpha \Omega_\alpha D} D \\ {\scriptstyle c}\downarrow & \nearrow & \downarrow{\scriptstyle \tilde{p}} \\ F & \longrightarrow & D \end{array}$$

By the universal property of the pullback, it is sufficient to find a lift in the diagram

$$\begin{array}{ccc} E & \longrightarrow & B_\alpha A \\ {\scriptstyle c}\downarrow & \nearrow & \downarrow{\scriptstyle B_\alpha p} \\ F & \longrightarrow & B_\alpha \Omega_\alpha D \end{array}$$

By adjunction we can instead consider the diagram

$$\begin{array}{ccc} \Omega_\alpha E & \longrightarrow & A \\ {\scriptstyle \Omega_\alpha c}\downarrow & \nearrow & \downarrow{\scriptstyle p} \\ F & \longrightarrow & B_\alpha \Omega_\alpha D \end{array}$$

By Proposition 4.2.21 $\Omega_\alpha c$ is a fibration, and by assumption p is an acyclic fibration. Using the model category structure on $\mathcal{Alg}_{\mathfrak{P}}$ gives the required lifting. □

4.2.3 The 'Quillen' Adjunction

We denote by $\mathcal{Alg}_{\mathfrak{P}}^{|c|}$ the full subcategory of $\mathcal{Alg}_{\mathfrak{P}}$ consisting of $\mathfrak{P}$-algebras whose underlying chain complex is cofibrant. It inherits classes of fibrations, cofibrations and weak equivalecnes from the full category $\mathcal{Alg}_{\mathfrak{P}}$, but this does not necessarily form a model structure. We also denote by $\mathcal{Alg}_{\mathfrak{P}}^{c}$ the full subcategory of $\mathcal{Alg}_{\mathfrak{P}}$ consisting of cofibrant objects. This is a model category with the model structure inherited from $\mathcal{Alg}_{\mathfrak{P}}$.

Proposition 4.2.25. *Suppose that an operad $\mathfrak{P}$ is split and each $\mathfrak{P}(n)$ is cofibrant. Then the forgetful functor $|-| : \mathcal{Alg}_{\mathfrak{P}} \to Ch(\mathcal{E})$ preserves cofibrant objects.*

Proof. Suppose that A is a cofibrant $\mathfrak{P}$-algebra. Then $|A|$ is a retract of $|\mathfrak{P}(V)|$ for some cofibrant V in $\mathrm{Ch}(\mathcal{E})$. Therefore it is sufficient to check that $|\mathfrak{P}(V)|$ is cofibrant. Since $\mathfrak{P}$ is split, it is sufficient to check that $|(\mathfrak{P}\otimes\Sigma)(V)|$ is cofibrant. But this is just $\bigoplus_{i=0}^{\infty}\mathfrak{P}(n)\otimes V^{\otimes n}$. By assumption each $\mathfrak{P}(n)$ is cofibrant, as is V. □

Proposition 4.2.26. *Let $\alpha : \mathfrak{C} \to \mathfrak{P}$ be a Koszul morphism, and let A be a $\mathfrak{P}$-algebra which is cofibrant as a chain complex. Then $B_\alpha A$ is cof-nilpotent. If C is a cof-nilpotent $\mathfrak{C}$-coalgebra then $\Omega_\alpha C$ is a cofibrant algebra.*

Proof. By Proposition 3.3.31 it is sufficient to show that $\mathrm{gr}(B_\alpha A)$ is cof-nilpotent. But $\mathrm{gr}(B_\alpha A) \cong \mathrm{gr}(\mathfrak{C}(A))$. Therefore it is sufficient to prove that $\mathfrak{C}(A)$ is cof-nilpotent for any cofibrant chain complex A. This is obvious. The second assertion follows from the fact that Ω_α is a left adjoint functor which preserves cofibrations. □

With our slightly more complicated definition of a Kosuzl morphism, we can generalise the proof of [48] Theorem Corollary 11.3.8 to our setting.

Proposition 4.2.27. *Let $A \in \mathcal{Alg}_{\mathfrak{P}}^{|c|}$. The counit $\epsilon_\alpha(A) : \Omega_\alpha B_\alpha A \to A$ is a weak equivalence.*

Proof. We filter the complex $\Omega_\alpha B_\alpha A$ as follows. The underlying graded object of $\Omega_\alpha B_\alpha A$ is $\mathfrak{P} \circ \mathfrak{C} \circ A$. We filter it by

$$F_n \Omega_\alpha B_\alpha A = \sum_{k \geqslant 1, n_1+\ldots+n_k \leqslant n} (\mathfrak{P})(k) \otimes_{\Sigma_k} (F_{n_1}\mathfrak{C}(A) \otimes \ldots \otimes F_{n_k}\mathfrak{C}(A))$$

This induces a filtration of the chain complex $\Omega_\alpha B_\alpha A$. We regard A as a filtered algebra via the functor F_0. The counit then gives a morphism of filtered complexes. We consider the spectral sequence associated to these filtered complexes. For a filterd complex $C_\bullet$ we denote by $E^i(C_\bullet)$ the direct sum of the complexes on the ith page of the spectral sequence associated to $C_\bullet$. Then $E^0(\Omega_\alpha B_\alpha A) \cong (\mathfrak{P} \circ_\alpha \mathfrak{C}) \circ A$, while $E^0(F_0(A)) \cong A \cong I \circ A$. Since α is a Koszul morphism, the map $j : I \to \mathfrak{P} \circ_\alpha \mathfrak{C}$ is an acyclic cofibration. Therefore the map $j_A : A \to (\mathfrak{P} \circ_\alpha \mathfrak{C}) \circ A$ is a weak equivalence by Proposition 4.2.6. Moreover $E^0(\epsilon_\alpha(A)) \circ j_A = Id_A$. Therefore $E^0(\epsilon_\alpha(A))$ is also a quasisomorphism. We now apply the convergence theorem for spectral sequences of filtered complexes which can be bootstrapped from the abelian version by passing to an abelianization. □

Corollary 4.2.28. *The bar and cobar constructions restrict to adjunctions*

$$\Omega_\alpha \colon \mathcal{Cof}^{nil}_{\mathfrak{C}} \rightleftarrows \mathcal{Alg}^{|c|}_{\mathfrak{P}} : B_\alpha$$

$$\Omega_\alpha \colon \mathcal{Cof}^{nil}_{\mathfrak{C}} \rightleftarrows \mathcal{Alg}^{c}_{\mathfrak{P}} : B_\alpha$$

In both cases Ω_α preserves cofibrations and acyclic cofibrations, while B_α preserves fibrations and acyclic fibrations. Moreover the counit $\epsilon_\alpha(A)$ is a weak equivalence for any algebra A, and the unit $\nu_\alpha(C)$ is a weak equivalence for any cof-nilpotent coalgebra $\mathfrak{C}$.

In particular if $\mathcal{E} = \mathcal{V}ect_k$ for a field k then the first adjunction recovers the Koszul duality theorem of [82]. We regard $\mathcal{Cof}_{\mathfrak{C}}$ and $\mathcal{Alg}^{c}_{\mathfrak{P}}$ as relative categories By Corollary 2.9 in [34] we have the following result.

Theorem 4.2.29. *There is an adjoint equivalence of $(\infty, 1)$-categories.*

$$\Omega_\alpha \colon \boldsymbol{Cof}_{\mathfrak{C}} \rightleftarrows \boldsymbol{Alg}_{\mathfrak{P}} : B_\alpha$$

4.2.4 The Koszul Dual Operad

Let $\mathfrak{C}$ be a co-operad. The dualizing functor $(-)^\vee : \underline{\mathrm{Hom}}(-, k) : \mathcal{E} \to \mathcal{E}^{op}$ is lax monoidal, so it induces a functor

$$(-)^\vee : co\mathcal{Alg}_{\mathfrak{C}} \to (\mathcal{Alg}_{\mathfrak{C}^\vee})^{op}$$

Now let $\alpha : \mathfrak{C} \to \mathfrak{P}$ be a Koszul morphism.

Proposition 4.2.30. *The functor $(-)^\vee : \mathcal{Cof}_{\mathfrak{C}} \to (\mathcal{Alg}_{\mathfrak{C}^\vee})^{op}$ sends α-weak equivalences to quasi-isomorphisms.*

Proof. Dualizing preserves quasi-isomorphisms between cofibrant complexes and α-weak equivalences are in particular quasi-ismorphisms. □

Definition 4.2.31. *A Koszul morphism $\alpha : \mathfrak{C} \to \mathfrak{P}$ is said to be **operadic** if $\mathfrak{C}^{\vee}$ is an admissible operad.*

From now on $\alpha : \mathfrak{C} \to \mathfrak{P}$ will be an operadic Koszul morphism. We let

$$\hat{C}_{\alpha} : \mathcal{Alg}_{\mathfrak{P}} \to \mathcal{Alg}_{\mathfrak{S}^c \otimes_H \mathfrak{C}^{\vee}}$$

denote the composition $[-1] \circ (-)^{\vee} \circ B_{\alpha}$. Here $\otimes_H$ is the Hadamard tensor product defined in Appendix 2.

Remark 4.2.32. *The hat notation is supposed to invoke comparisons with formal power series. This will be clear in the case of duality for the Lie operad as we shall see later. The underlying graded algebra of $\hat{C}_{\alpha}(\mathfrak{g})$ is $\prod_n ((\mathfrak{S}^c(n) \otimes \mathfrak{C}(n)) \otimes_{\Sigma_n} \mathfrak{g}^{\otimes n}[1])^{\vee}$. This has a graded 'polynomial' subalgebra $\bigoplus_n ((\mathfrak{S}^c(n) \otimes \mathfrak{C}(n)) \otimes_{\Sigma_n} \mathfrak{g}^{\otimes n}[1])^{\vee}$. One can check that it is closed under the differential on $\hat{C}_{\alpha}(\mathfrak{g})$ whenever $\mathfrak{P}$ and A are nuclear objects. We denote this subalgebra by $C_{\alpha}(\mathfrak{g})$.*

By Corollary 4.2.28 and Proposition 4.2.30 there is an induced functor of ∞-categories

$$\hat{\mathbf{C}}_{\alpha} : \mathbf{Alg}_{\mathfrak{P}} \to \mathbf{Alg}_{\mathfrak{S}^c \otimes_H \mathfrak{C}^{\vee}}$$

We are going to prove the following. The approach is a generalisation of [50] Proposition 2.2.12.

Theorem 4.2.33. *The functor $\hat{\mathbf{C}}_{\alpha} : \mathbf{Alg}_{\mathfrak{P}} \to (\mathbf{Alg}_{\mathfrak{C}^{\vee}})^{op}$ admits a right adjoint $\mathbf{D}_{\alpha}$*

Before doing so we note the following technical result

Proposition 4.2.34. *Let $\mathbf{M}$ be an $(\infty, 1)$-categor, $\mathcal{E}$ a monoidal elementary quasi-abelian category, and $F : \mathbf{M} \to \mathbf{Ch}(\overline{Filt}(\mathcal{E}))$ a functor. If $\mathbf{gr} \circ F$ and $(-)_0 \circ F$ preserve sifted colimits then so does F.*

Proof. The proof is an easy induction. We show that each $(-)_n \circ F$ preserves sifted colimits. Then we conclude by Proposition 3.3.8. By assumption it is true for $n = 0$. We suppose it has been shown for $n = k$. Now there is a homotopy fiber sequence of functors

$$(-)_k \circ F \to (-)_{k+1} \circ F \to gr_{k+1} \circ F$$

Since the left and right-hand functors preserves sifted colimits so does the middle functor □

Proof of Theorem 4.2.33. Using Lurie's $(\infty, 1)$-adjoint functor theorem and noting that $\mathbf{Alg}_{\mathfrak{P}}$ is locally presentable, we need to show that $\hat{\mathbf{C}}_{\alpha}$ preserves colimits. Since the functor $|-| : \mathbf{Alg}_{\mathfrak{C}^{\vee}} \to \mathbf{Ch}(\mathcal{E})$ is conservative, and preserves and reflects limits, and preserves sifted colimits by Section A.2, it suffices to show that $|-|\circ\hat{\mathbf{C}}_\alpha : \mathbf{Alg}_{\mathfrak{P}} \to (\mathbf{Ch}(\mathcal{E}))^{op}$ preserves colimits. Let us first show that it preserves sifted colimits. Now we have a commutative diagram

$$\begin{array}{ccccc} \mathbf{Alg}_{\mathfrak{P}} & \xrightarrow{\mathbf{B}_\alpha} & \mathbf{Cof}_{\mathfrak{C}} & \xrightarrow{(-)^\vee} & \mathbf{Alg}_{\mathfrak{C}^\vee} \\ \downarrow{\scriptstyle \mathbf{B}_\alpha} & & & & \downarrow{\scriptstyle |-|} \\ \mathbf{Cof}_{\mathfrak{C}} & \xrightarrow{|-|} & \mathbf{Ch}(\mathcal{E}) & \xrightarrow{(-)^\vee} & \mathbf{Ch}(\mathcal{E})^{op} \end{array}$$

and both compositions are equal to $\hat{\mathbf{C}_\alpha}$. The functor $(-)^\vee : \mathbf{Ch}(\mathcal{E}) \to \mathbf{Ch}(\mathcal{E})^{op}$ is a left adjoint so it preserves all colimits. Therefore we reduce to showing that $|-|\circ\mathbf{B}_\alpha$ preserves sifted colimits. There is a factorisation of $|-|\circ\mathbf{B}_\alpha$

$$\mathbf{Alg}_{\mathfrak{P}} \xrightarrow{\mathbf{B}_\alpha} \mathbf{Cof}_{\mathfrak{C}} \xrightarrow{|-|_{filt}} \mathbf{Filt}(\mathbf{Ch}(\mathcal{E})) \xrightarrow{(-)_\infty} \mathbf{Ch}(\mathcal{E})$$

where $|-|_{filt}$ is the forgetful functor. The functor $(-)_\infty$ is colimit preserving. Therefore it remains to show that $|-|_{filt}\circ\mathbf{B}_\alpha$ preserves colimits. Now $(-)_0\circ|-|_{filt}\circ\mathbf{B}_\alpha = |-|$ which preserve sifted colimits. By Proposition 4.2.34 we finally reduce to showing that the composition $\mathbf{gr}\circ|-|_{filt}\circ\mathbf{B}_\alpha$ is colimit preserving. But this functor is equivalent to the composition

$$\mathbf{Alg}_{\mathfrak{P}} \xrightarrow{|-|} \mathbf{Ch}(\mathcal{E}) \xrightarrow{|-|\circ\mathfrak{C}(-)} \mathbf{Ch}(\mathcal{E})$$

All the functors in this composition preserve sifted colimits by Section A.2 so we are done. It remains to show that $\hat{\mathbf{C}}_\alpha$ preserves products. Now by Section A.2 the category $\mathbf{Alg}_{\mathfrak{P}}$ is generated under sifted colimits by free objects $\mathfrak{P}(V)$ on cofibrant objects V. Thus it is enough to show that $\hat{\mathbf{C}}_\alpha$ preserves coproducts of the form $\mathfrak{P}(V)\coprod\mathfrak{P}(W)\cong\mathfrak{P}(V\oplus W)$. But

$$\hat{\mathbf{C}}_\alpha(\mathfrak{P}(V)) \cong (\mathfrak{C}\circ_\alpha\mathfrak{P}(V))^\vee \cong V^\vee$$

So if

$$\begin{array}{ccc} \mathfrak{P}(0) & \longrightarrow & \mathfrak{P}(V) \\ \downarrow & & \downarrow \\ \mathfrak{P}(W) & \longrightarrow & \mathfrak{P}(V\oplus W) \end{array}$$

is a coproduct diagram in $\mathbf{Alg}_{\mathfrak{P}}$ then applying $\hat{\mathbf{C}}_\alpha$ gives the diagram

$$\begin{array}{ccc} V^\vee\oplus W^\vee & \longrightarrow & V^\vee \\ \downarrow & & \downarrow \\ W^\vee & \longrightarrow & 0 \end{array}$$

which is a product diagram in $\mathbf{Alg}_{\mathfrak{C}^\vee}$. □

Since we use the adjoint functor theorem the proof of the existence of $\mathbf{D}_\alpha$ is not constructive. However for Koszul duality between Lie algebras and commutative algebras we will give an interpretation of $\mathbf{D}_\alpha$ in terms of the shifted tangent complex.

4.3 Main Example: The Lie Operad

Recall that in the category ${}_{\mathbb{Q}}\mathcal{V}ect$ of vector spaces over $\mathbb{Q}$ there is a Koszul morphism

$$\kappa : \mathfrak{S}^c \otimes_H \mathfrak{coComm}^{nu} \to \mathfrak{Lie}$$

This follows from the fact, which we will not explore in detail here, that $\mathfrak{S}^c \otimes_H \mathfrak{coComm}^{nu}$ is the quadratic dual co-operad of $\mathfrak{Lie}$. See [48] and [20] for details. This duality also extends to monoidal elementary quasi-abelian categories.

4.3.1 Lie Algebras and Commutative Algebras

If $\mathcal{E}$ is a monoidal elementary quasi-abelian category then we can bootstrap the Koszul morphism in ${}_{\mathbb{Q}}\mathcal{V}ect$ to one in $\mathcal{E}$.

Proposition 4.3.1. *Let $\alpha : \mathfrak{C} \to \mathfrak{P}$ be a Koszul morphism in $Ch({}_{\mathbb{Q}}\mathcal{V}ect)$. Let $\mathcal{E}$ be a monoidal elementary quasi-abelian category with unit k. Then*

$$k \otimes \alpha : k \otimes \mathfrak{C} \to k \otimes \mathfrak{P}$$

is a Koszul morphism in $Ch(\mathcal{E})$.

Proof. Together with Proposition B.4.4, this follows immediately from the fact that $k\otimes(-)$ is a strong monoidal, kernel and cokernel preserving functor whose image consists of cofibrant objects. □

In particular if $\kappa : \mathfrak{S}^c \otimes_H \mathfrak{coComm} \to \mathfrak{Lie}$ is the canonical Koszul morphism in ${}_{\mathbb{Q}}\mathcal{M}od$ then we get a Koszul morphism

$$\mathfrak{S}^c \otimes_H (k \otimes \mathfrak{coComm}) \xrightarrow{\cong} k \otimes (\mathfrak{S}^c \otimes_H \mathfrak{coComm}) \xrightarrow{k\otimes\alpha} k \otimes \mathfrak{Lie}$$

in $Ch(\mathcal{E})$. This generalises classical Koszul duality between Lie algebras and cocommutative coalgebras, and between Lie algebras and (non-unital) commutative algebras, to arbitrary elementary quasi-abelian categories. Let us now study this example in greater detail.

The Shifted Tangent Complex

Recall the functor $\mathbb{L}_0 : \mathbf{Alg}_{\mathfrak{Comm}}(Ch(\mathcal{E})) \to \mathbf{Ch}(E)$ defined in section 4.1.3. The **shifted tangent complex functor** is the composition $\mathbb{T}_0 := (-)^{\vee} \circ \mathbb{L}_0$. We shall abuse notation and write $\mathbf{D}_{\kappa} : \mathbf{Alg}_{\mathfrak{Comm}}(Ch(\mathcal{E})) \to \mathbf{Alg}_{\mathfrak{Lie}}(Ch(\mathcal{E}))^{op}$ for the composite $\mathbf{D}_{\kappa} \circ I : \mathbf{Alg}_{\mathfrak{Comm}}(Ch(\mathcal{E})) \to \mathbf{Alg}_{\mathfrak{Comm}^{nu}}(Ch(\mathcal{E})) \to \mathbf{Alg}_{\mathfrak{Lie}}(Ch(\mathcal{E}))^{op}$. In classical Koszul duality for Lie algebras and commutative algebras the functor

$$|-| \circ \mathbf{D}_{\kappa} : \mathbf{Alg}_{\mathfrak{Comm}}(Ch(\mathcal{V}ec_k)) \to \mathbf{Alg}_{\mathfrak{Lie}}(\mathbf{Ch}(\mathcal{V}ec_k))^{op} \to \mathbf{Ch}(\mathcal{V}ec_k)$$

is equivalent to the shifted tangent complex. We will now see that this generalises to monoidal elementary quasi-abelian categories enriched over $\mathbb{Q}$.

Proposition 4.3.2. *Let $\mathcal{E}$ be a monoidal elementary quasi-abelian category. Let $A \in \mathcal{A}lg^{aug}_{\mathfrak{Comm}}(Ch(\mathcal{E}))$ be a quasi-free object on a bounded below cofibrant complex V. Then $\mathbb{L}_0(A) \cong V$ and $\mathbb{L}_A \cong A \otimes V$*

Proof. Since A by is cofibrant by Corollary 4.2.19 we may use Proposition 4.1.23. This gives $\mathbb{L}_0 \cong V$. Now V is cofibrant so again by Proposition 4.1.23 $\mathbb{L}_A \cong A \otimes V$. □

Proposition 4.3.3. *The functor $|-| \circ \boldsymbol{D}_{\kappa}$ is naturally equivalent to the shifted tangent complex functor $\mathbb{T}_0[1]$.*

Proof. We show that $|-| \circ \mathbf{D}_{\kappa}$ and $\mathbb{T}_0$ are both right adjoint to the same functor. Now $|-| \circ \mathbf{D}_{\kappa}$ is right adjoint to the functor $\hat{\mathbf{C}}_{\kappa} \circ \mathfrak{Lie}(-)$ which is equivalent to the functor $k \oplus (-)^{\vee}[-1]$. But this functor is left adjoint to $\mathbb{T}_0[1]$. □

Definition 4.3.4. *A Lie algebra $\mathfrak{g}$ is said to be **very good** if the underlying complex $|\mathfrak{g}|$ is cofibrant and is of the form $\bigoplus_{i=1}^{\infty} k^{p_i}[i]$. $\mathfrak{g}$ is said to be **good** if it is equivalent to a very good algebra.*

Let us populate this class of algebras.

Example 4.3.5. *1. Say that a Lie algebra $\mathfrak{g}$ is cellular finite if $\mathfrak{g} = lim_{\to} L_n$ where $L_0 = 0$, for each $n \geqslant 0$ there is a pushout diagram*

$$\begin{array}{ccc} L(S^{m_n - 1}(k^{p_{m_n}})) & \longrightarrow & L_n \\ \downarrow & & \downarrow \\ L(D^{m_n}(k^{p_{m_n}})) & \longrightarrow & L_{n+1} \end{array}$$

where $m_n \leqslant -1$, and for each integer $l \leqslant -1$ there are only finitely many n such that $m_n - 1 = l$. Then $\mathfrak{g}$ is very good. The last condition is automatically satisfied if the filtration L_n terminates.

2. *Consider the categories ${}_k\mathcal{V}ect$ for k a field and $Ind(Ban_k)$ for k a Banach field. Then if $\mathfrak{g}$ is concentrated in negative degrees and each $\mathfrak{g}_n$ is free of finite rank, $\mathfrak{g}$ is very good. Indeed in these cases any complex of free objects of finite rank is split. In particular it is a coproduct of objects of the form $S^n(k^m)$ and $D^r(k^s)$, so it is cofibrant. In particular this recovers Lurie's conditions in Lemma 2.3.5 of [50].*

Definition 4.3.6. *Let $\mathcal{E}$ be a rigid $\mathbb{Q}$-HAS. We say that $\mathcal{E}$ is* ***decent*** *if for any object V which is free of finite rank there is an equivalence*

$$\mathbb{L}_{\hat{S}(V)/S(V)} \otimes^{\mathbb{L}}_{\hat{S}(V)} k \cong 0$$

in $\mathbf{Ch}(\mathcal{E})$.

This assumption holds in the case that $\mathcal{E} = {}_R\mathcal{M}od$ for R a Noetherian ring. We shall see later that it also holds for $\mathcal{E} = Ind(Ban_k)$ for k any Banach ring. In fact in this case by Proposition 4.5.10 the stronger condition $\mathbb{L}_{\hat{S}(V)/S(V)} \cong 0$ holds. Recall the subalgebra $C_\kappa(\mathfrak{g}) \subset \hat{C}_\kappa(\mathfrak{g})$ considered in Remark 4.2.32.

Proposition 4.3.7. *Let $\mathfrak{g}$ be very good. The map $\hat{S}(\mathfrak{g}^\vee_{-1}) \otimes^{\mathbb{L}}_{S(\mathfrak{g}^\vee_{-1})} C_\kappa(\mathfrak{g}) \to \hat{C}_\kappa(\mathfrak{g})$ is an equivalence.*

Proof. The map $S(\mathfrak{g}^\vee_{-1}) \to C_\kappa(\mathfrak{g})$ is a (standard) cofibration. Therefore the tensor product doesn't need to be derived. Thus it remains to check that the map $\hat{S}(\mathfrak{g}^\vee_{-1}) \otimes_{S(\mathfrak{g}^\vee_{-1})} C_\kappa(\mathfrak{g}) \to \hat{C}_\kappa(\mathfrak{g})$ is a degreewise isomorphism. This is clear. □

The abstract machinery we have set up allows us to generalise Lurie's [50] Lemma 2.3.5 and its proof.

Theorem 4.3.8. *Let $\mathcal{E}$ be a monoidal elementary quasi-abelian category such that $Ch(\mathcal{E})$ is decent, and let $\mathfrak{g}$ be a good Lie algebra in $Ch(\mathcal{E})$. Then the unit $\eta_{\mathfrak{g}} : \mathfrak{g} \to \boldsymbol{D}_\kappa \circ \hat{\boldsymbol{C}}_\kappa(\mathfrak{g})$ is an equivalence.*

Proof. Without loss of generality we may assume that $\mathfrak{g}$ is very good. It suffices to show that the map of complexes $|\mathfrak{g}| \to \mathbb{T}_0(\hat{C}(\mathfrak{g}))[1]$ is an equivalence. This is dual to the map

$$\mathbb{L}_0(\hat{C}_\kappa(\mathfrak{g})) \to |\mathfrak{g}^\vee|[-1]$$

so it suffices to show that this is an equivalence. By Proposition 4.3.7 we have a an equivalence $\mathbb{L}_{\hat{S}(\mathfrak{g}^\vee_{-1})/S(\mathfrak{g}^\vee_{-1})} \otimes^{\mathbb{L}}_{\hat{S}(\mathfrak{g}^\vee_{-1})} \hat{C}_\kappa(\mathfrak{g}^\vee_{-1}) \cong \mathbb{L}_{\hat{C}_\kappa(\mathfrak{g})/C_\kappa(\mathfrak{g})}$. Tensoring with k over $\hat{C}_\kappa(\mathfrak{g})$ and using that $Ch(\mathcal{E})$ is decent then gives $\mathbb{L}_{\hat{C}_\kappa(\mathfrak{g})/C_\kappa(\mathfrak{g})} \otimes^{\mathbb{L}}_{\hat{C}_\kappa(\mathfrak{g})} k \cong 0$. Finally, considering the homotopy cofiber sequence

$$\mathbb{L}_{\hat{C}_\kappa(\mathfrak{g})/C_\kappa(\mathfrak{g})} \otimes^{\mathbb{L}}_{\hat{C}_\kappa(\mathfrak{g})} k \to \mathbb{L}_0(C_\kappa(\mathfrak{g})) \to \mathbb{L}_0(\hat{C}_\kappa(\mathfrak{g}))$$

gives that $\mathbb{L}_0(C_\kappa(\mathfrak{g})) \to \mathbb{L}_0(\hat{C}_\kappa(\mathfrak{g}))$ is an equivalence. In particular it suffices to show that

$$\mathbb{L}_0(C_\kappa(\mathfrak{g})) \to |\mathfrak{g}^\vee|[-1]$$

is an equivalence.This follows from Proposition 4.3.2. Since $\mathfrak{g}$ is degree-wise free of finite rank, $C(\mathfrak{g})$ is quasi-free on a cofibrant complex. □

We conclude this section with some remarks.

Remark 4.3.9. 1. *Essentially all of this chapter generalises to interval higher algebra settings of the form ${}_R\mathcal{Mod}$ for R a cdga in $Ch(\mathcal{E})$ for $\mathcal{E}$ elementary quasi-abelian. If $\mathcal{E} = {}_{\mathbb{Q}}\mathcal{Mod}$ then ${}_R\mathcal{Mod}$ is decent if R is concentrated in non-negative degrees and is Noetherian. This recovers Lemma 1.4.12 of [35]. If $\mathcal{E} = Ind(Ban_k)$ for some Banach ring k containing $\mathbb{Q}$, then ${}_R\mathcal{Mod}$ is decent if R is concentrated in non-negative degrees, R_0 is a Banach ring (or more generally $\aleph_1$-filtered) and R is flat over R_0.*

2. *We expect that results above extends to the situation that $\alpha : \mathfrak{C} \to \mathfrak{P}$ is a Koszul morphism, $\mathfrak{P}(n)$ is concentrated in degree at most n, is bounded, each $\mathfrak{P}(n)$ is free of finite type, and $\mathfrak{C}$ is the augmentation coideal of a counital cooperad. This could be useful for L_∞ and E_∞ duality over rings which do not contain $\mathbb{Q}$.*

3. *We expect that operadic Koszul duality works in exact categories which are not quasi-abelian, since we did not need to use the filtered exact structure.*

4.4 Analytic and Derived Analytic Algebra

In this section we specialise to the monoidal elementary quasi-abelian category $Ind(Ban_k)$ where k is a fixed Banach ring. We are going to study the homotopy theory of algebras in this category and establish some foundational results which in the future will be applied to study derived analytic geometry. We will begin by studying analytic notions of algebras over Banach operads in general (with a view to dealing with E_∞-operads in the future) before focusing on the commutative operad.

4.4.1 Algebras of Power Series

Let us begin by introducing some classes of operads and algebras we wish to consider.

Definition 4.4.1. *Let $\mathcal{E}$ be a complete and cocomplete monoidal category. For $V \in \mathcal{E}$ we denote by*

$$\hat{\mathfrak{P}}(V) := \prod_{n=0}^{\infty} \mathfrak{P}(n) \otimes_{\Sigma_n} V^{\otimes n}$$

the free power series $\mathfrak{P}$-algebra on V.

If V is a Banach module regarded as an object of $Ind(Ban_k)$ and $\mathfrak{P}$ is an operad in Ban_k then $\hat{\mathfrak{P}}(V)$ is a bornological k-module. An element of $\hat{\mathfrak{P}}(V)$ is a formal sum $\sum_{i=0}^{\infty} v_i$ where $v_i \in \mathfrak{P}(n) \otimes_{\Sigma_n} V^{\otimes n}$. For $\underline{\epsilon} \in \mathbb{R}^{\mathbb{N}_0}$ we let $B_{\underline{\epsilon}} = \{\sum_{i=0}^{\infty} v_i : ||v_i|| \leqslant \epsilon_i\}$. These sets precisely constitute the bounded sets forming the bornology on $\hat{\mathfrak{P}}(V)$. We are particularly interested in subalgebras of this bornological algebra. From now on we suppose that $\mathfrak{P}$ is non-expanding, namely that the partial composition maps $\mathfrak{P}(m) \otimes \mathfrak{P}(n) \to \mathfrak{P}(m+n-1)$ are of norm at most 1. In particular it may also be regarded as an operad in $Ban_k^{\leqslant 1}$.

Definition 4.4.2. *Let V be a Banach k-module. The* ***free contracting*** *$\mathfrak{P}$-algebra on V, denoted $\mathfrak{P}^{\leqslant 1}(V)$ is the free algebra on V taken in the category $Ban_k^{\leqslant 1}$, and then included into Ban_k. For $r \in (0, \infty)$ we write $\mathfrak{P}^{\leqslant r}(V) := \mathfrak{P}^{\leqslant 1}(V_r)$ where V_r is the rescaling of V.*

There is clearly a monomorphism $\mathfrak{P}^{\leqslant r}(V) \to \hat{\mathfrak{P}}(V)$ in $CBorn_k \subset Ind(Ban_k)$.

Definition 4.4.3. *Let $\mathfrak{P}$ be a contracting operad in Banach k-modules. A $\mathfrak{P}$-algebra A is said to be* ***almost contracting*** *if there is a $C \geqslant 0$ such that for each $n \in \mathbb{N}$, the map*

$$\mu_A^n : \mathfrak{P}(n) \otimes A^{\otimes n} \to A$$

is bounded by C^{n-1}. If A is almost contracting the ***magnitude*** *of A is $\gamma_A = min\{C : ||\mu_E^n|| \leqslant C^{n-1} \ \forall n \geqslant 1\}$. A $\mathfrak{P}$-algebra A is said to be* ***contracting*** *if it is almost contracting of magnitude $\gamma_A \leqslant 1$. The full subcategory of $\mathcal{Alg}_{\mathfrak{Comm}}(Ban)$ consisting of contracting algebras is denoted $Alg_{\mathfrak{P}}^{\leqslant 1}$.*

Note that there is a faithful functor $Alg_{\mathfrak{P}}(Ban^{\leqslant 1}) \to Alg_{\mathfrak{P}}^{\leqslant 1}(Ban)$. It is not full.

Proposition 4.4.4. *There is an equivalence of categories $Alg_{\mathfrak{P}}^{\leqslant 1}(Ban) \cong Alg_{\mathfrak{P}}^{ac}(Ban)$.*

Proof. The inclusion of $Alg_{\mathfrak{P}}^{\leqslant 1}(Ban)$ into $Alg_{\mathfrak{P}}^{ac}$ is clearly fully faithful. It remains to show that it is essentially surjective. Let A be almost contracting of magnitude γ_A. Then A_{γ_A} is a contracting algebra which is isomorphic to A in $Alg_{\mathfrak{P}}^{ac}$. Note that the functor sending A to A_{γ_A} is a quasi-inverse for the inclusion. $\square$

Let us show that free almost contracting algebras are in some sense left adjoints.

Proposition 4.4.5. *For V a Banach space the algebra $\mathfrak{P}^{\leqslant r}(V)$ is almost contracting of magnitude r. Moreover if F is an almost contracting algebra of magnitude γ_F then there is a natural isomorphism*

$$Hom^{\leqslant \frac{1}{\gamma_F}}(\mathfrak{P}^{\leqslant r\gamma_F}(E), F) \cong Hom^{\leqslant r}(E, F)$$

Proof. Let $\phi : \mathfrak{P}^{\leqslant r\gamma_F}(E) \to F$ be a map of norm at most $\frac{1}{\gamma_F}$. Now $i : E \to \mathfrak{P}^{\leqslant r\gamma_F}(E)$ has norm $r\gamma_F$. Thus $\phi \circ i$ is of norm at most r. Conversely suppose we are given a map $f : E \to F$ of norm r. Consider the induced map of algebras in the category of vector spaces $\mathfrak{P}(n)(f) : \mathfrak{P}(n)(E) \to F$

$$\begin{aligned}
||\mathfrak{P}(n)(f)(p \otimes e_1 \otimes \ldots \otimes e_n)||_F &= ||\mu_n^F(p \otimes f(e_1) \otimes \ldots \otimes f(e_n))||_F \\
&\leqslant ||p||_{\mathfrak{P}(n)} \gamma_F^{n-1} r ||e_1||_E \ldots \gamma_F r ||e_n|| \\
&= \frac{1}{\gamma_F} ||p||_{\mathfrak{P}(n)} ||e_1||_{r\gamma_F} \ldots ||e_n||_{r\gamma_F} \\
&= \frac{1}{\gamma_F} ||p \otimes e_1 \otimes \ldots \otimes e_n||_{\mathfrak{P}^{\leqslant \gamma_F r}(n;E)}
\end{aligned}$$

Thus $\mathfrak{P}(n)(f)$ is bounded of norm at most $\frac{1}{\gamma_F}$ as a map from $E_{\gamma_F r}$ to F. It therefore extends to a continuous map $\mathfrak{P}^{\leqslant \gamma_F r}(E) \to F$ of norm at most $\frac{1}{\gamma_F}$. □

Next we consider algebras which are intended as analogues of algebras of holomorphic functions (at least over $\mathbb{C}$). Let E be a Banach k-module and consider the projective system.

$$\ldots E_{n+1} \to E_n \to \ldots \to E_2 \to E_1$$

in $Ban_k^{\leqslant 1}$. We get an object

$$\text{“}\mathfrak{P}^{an,\infty}\text{”}(E) := \text{“}\lim_{\leftarrow_{\mathbb{N}^{op}}}\text{”}\mathfrak{P}^{\leqslant 1}(E_n)$$

in $Pro(Alg^{ac}_{\mathfrak{P}}(Ban_k)) \subset ProInd(Alg^{ac}_{\mathfrak{P}}(Ban_k))$. We shall see shortly that these are the free algebras in the category of pro-multiplicatively convex $\mathfrak{P}$-algebras.

Definition 4.4.6. *The category of* ***pro-multiplicatively convex*** *$\mathfrak{P}$-algebras is the category*

$$pmc\mathcal{A}lg_{\mathfrak{P}} := ProInd(Alg^{ac}_{\mathfrak{P}}(Ban)_k)$$

In particular “$\mathfrak{P}^{an,\infty}$”(E) is a pro-multiplicatively convex algebra. The assignment $E \mapsto$ “$\mathfrak{P}^{an,\infty}$”(E) for E a Banach module is in fact functorial. Since $Ind(Alg^{ac}_{\mathfrak{P}}(Ban))$ has filtering inductive limits, $ProInd(Alg^{ac}_{\mathfrak{P}}(Ban))$ does as well. Thus there is an induced functor

$$\text{“}\mathfrak{P}^{an,\infty}\text{”} : Ind(Ban) \to ProInd(Alg^{ac}_{\mathfrak{P}}(Ban))$$

It sends an object “$\lim_{\to_{\mathcal{I}}} E_i$” to $\lim_{\to_{\mathcal{I}}}$“$\lim_{\leftarrow_{\mathbb{N}^{op}}}$”$\mathfrak{P}^{\leqslant n}(E_i)$. There is a natural ‘forgetful’ functor $|-| : ProInd(Alg^{ac}_{\mathfrak{P}}(Ban_k)) \to Ind(Ban_k)$. We claim that this has a left adjoint. In fact we are going to prove the following.

Theorem 4.4.7. *There is an adjunction*

$$\text{“}\mathfrak{P}^{an,\infty}\text{”} \vdash |-|$$

Proof. We prove this in several steps. First suppose that E is a Banach space, and F is an almost contracting $\mathfrak{P}$-algebra of magnitude γ_F. First note that $\lim_{\to_{\mathbb{N}}}\text{Hom}^{\leqslant n}_{Ban}(E,F) \to \text{Hom}_{Ban}(E,F)$ is an isomorphism. By the previous proposition it is sufficient to show that the obvious map

$$\lim_{\to_{\mathbb{N}}}\text{Hom}^{\leqslant \frac{1}{\gamma_F}}(\mathfrak{P}^{\leqslant n\gamma_F}(E),F) \to \lim_{\to_{\mathbb{N}}}\text{Hom}(\mathfrak{P}^{\leqslant n\gamma_F}(E),F)$$

is an isomorphism. Since monomorphisms are stable under direct limits in $\mathcal{S}et$ it remains to show that the map is surjective. Let $[f_n : \mathfrak{P}^{\leqslant \gamma_F n}(E) \to F]$ be in $\lim_{\to_{\mathbb{N}}}\text{Hom}_{Alg^{ac}_{\mathfrak{P}}}(\mathfrak{P}^{\leqslant \gamma_F n}(E),F)$, with $||f_n|| \leqslant \frac{m}{\gamma_F}$ for some integer m. We claim that the map $\mathfrak{P}^{\leqslant mn\gamma_F}(E) \to F$ is of norm at most $\frac{1}{\gamma_F}$. But by Proposition 4.4.5 this follows from the fact that the map $E \to \mathfrak{P}^{\leqslant n\gamma_f}(E)$ is of norm at most $n\gamma_F$. This shows that the map is surjective.

Now suppose E is a Banach space and F is an inductive limit of Banach $\mathfrak{P}$-algebras. So, write $F = \text{“}\lim_{\to_{\mathcal{J}}}\text{”}F_j$. Then

$$\begin{aligned}\text{Hom}(\text{“}\mathfrak{P}^{an,\infty}\text{”}(E),F) &= \text{Hom}(\text{“}\lim_{\leftarrow_{n\in\mathbb{N}^{op}}}\text{”}\mathfrak{P}^{\leqslant r}(E), \text{“}\lim_{\to_{\mathcal{J}}}\text{”}F_j)\\ &= \lim_{\to_{\mathcal{J}}}\lim_{\to_{n\in\mathbb{N}^{op}}}\text{Hom}(\mathfrak{P}^{\leqslant n}(E),F_j)\\ &= \lim_{\to_{\mathcal{J}}}\text{Hom}_{Ban}(E,|F_j|)\\ &= \text{Hom}_{Ind(Ban)}(E,|F|)\end{aligned}$$

Finally we prove the general case.Let $E = \text{“}\lim_{\to_{\mathcal{I}}}\text{”}E_i$ be an inductive system, and let $F = \text{“}\lim_{\leftarrow_K}\text{”}F_k$ with $F_k \in Ind(Alg^{ac}_{\mathfrak{P}})$ be a pro-multiplicatively convex $\mathfrak{P}$-algebra. Then

$$\begin{aligned}\text{Hom}(\text{“}\mathfrak{P}^{an,\infty}\text{”}(E),F) &= \lim_{\leftarrow_{\mathcal{I}}}\lim_{\leftarrow_K}\text{Hom}(\text{“}\mathfrak{P}^{an,\infty}\text{”}(E_i),F_k)\\ &= \lim_{\leftarrow_{\mathcal{I}}}\lim_{\leftarrow_K}\text{Hom}_{Ind(Ban)}(E_i,|F_k|)\\ &= \text{Hom}_{Ind(Ban)}(\text{“}\lim_{\to_{\mathcal{I}}}\text{”}E_i, \lim_{\leftarrow_K}|F_k|)\\ &= \text{Hom}_{Ind(Ban)}(\text{“}\lim_{\to_{\mathcal{I}}}\text{”}E_i, |\lim_{\leftarrow_K}F_k|)\end{aligned}$$

□

Note that there is a natural isomorphism $\text{“}\mathfrak{B}^{an,\infty}\text{”}(E) \cong \text{“}lim_{\leftarrow_{r\in\mathbb{R}_{>0}}}\text{”}\mathfrak{P}^{\leqslant r}(E)$. For each $\epsilon \in (0,\infty]$ and each Banach space E one can also define

$$\text{“}\mathfrak{P}^{an,\epsilon}\text{”}(E) := \text{“}lim_{\leftarrow_{r\in(0,\epsilon)}}\text{”}\mathfrak{P}^{\leqslant r}(E)$$

In general for $\epsilon < \infty$ this only defines a functor on the category $Ban^{\leqslant 1}_k$.

Finally for $\epsilon \in (0,\infty]$ we define $\mathfrak{B}^{an,\epsilon} := PI \circ \text{“}\mathfrak{P}^{an,\epsilon}\text{”} : Ban^{\leqslant 1}_k \to Ind(Ban_k)$. Since the projective limit of monomorphisms is a monomorphism the map $\mathfrak{P}^{an,\epsilon}(V) \to \hat{\mathfrak{P}}(V)$ is a monomorphism for any $\epsilon \in (0,\infty]$, so these are power series algebras.

Example: The Commutative Operad

We now give our main example which is a consequence of Proposition C.2.14. Let $k = \mathbb{C}$ and $\mathfrak{P} = \mathfrak{Comm}$. Let E be a Banach space. First we introduce a holomorphy type from $E^{\vee}$ to $\mathbb{C}$ (for details on holomorphy types see Section C.2).

Definition 4.4.8. *Let E and F be Banach spaces. For $m \in \mathbb{N}$ write $E^{-m} := (E^{\vee})^{-m}$. We define the **tame holomorphy type**, τ from $E^{\vee}$ to F to be subspace of $\mathcal{P}^m(E^{\vee}, F)$ consisting of maps of the form $A_m x^m$, where $A_m : E^{-m} \to F$ is in the image of $E^m \otimes F \to Hom(E^{-m}, F)$.*

Recall that a Banach space E is said to have the λ-**approximation property** for some $1 \leqslant \lambda < \infty$ if for every compact set $K \subset X$ and every $\epsilon > 0$ there is a finite rank operator $T : E \to E$ so that $||T|| \leqslant \lambda$ and $||Te - e|| \leqslant \epsilon$ for all $e \in E$. E is said to have the **approximation property** if it it has the λ-approximation property for some $\leqslant \lambda < \infty$. The Proposition below can be found in [16] Section 3.

Proposition 4.4.9. *Let E be a Banach space. If $E^{\vee\vee}$ has the approximation property then the map*

$$X^{\vee\vee} \otimes \ldots X^{\vee\vee} \to (X \otimes \ldots \otimes X)^{\vee\vee}$$

is an isometric embedding. In particular it is an admissible monomorphism.

In particular the image of the map $E^m \to Hom(E^{-m}, \mathbb{C})$ is isomorphic to E^m. Using Proposition C.2.14 we immediately get the following result.

Corollary 4.4.10. *There is a natural transformation*

$$\text{“}S^{an,\infty}(-)\text{”} \to PB(\mathcal{H}^b_{\tau}((-)^{\vee}))$$

If E has the approximation property it is an isomorphism. In particular if E is finite-dimensional then there is a (non-canonical) isomorphism.

$$\text{“}S^{an,\infty}(E)\text{”} \cong PB(\mathcal{H}(E))$$

and therefore an isomorphism

$$S^{an,\infty}(E) \cong IB(\mathcal{H}(E))$$

This gives an interpretation of the algebra of holomorphic functions on a finite dimensional Banach space as a ‘free’ commutative algebra in the category of pro-multiplicatively convex algebras.

Dagger Analytic Algebras

Let E be a Banach space and $\epsilon \in [0, \infty)$. There corresponds an object "$lim_{\rightarrow_{r>\epsilon}}$"$E_r$ of $Ind(Ban_k^{\leqslant 1})$ and hence an object $\mathfrak{P}^{\dagger,\epsilon}(E) := lim_{\rightarrow_{r>\epsilon}}$"$\mathfrak{P}(E_r)$ of $Ind(Alg_{\mathfrak{P}}^{ac}(Ban_k))$. This construction gives a functor

$$\mathfrak{P}^{\dagger,\epsilon}(-) : Ban_k^{\leqslant 1} \rightarrow Ind(\mathcal{Alg}_{\mathfrak{P}}^{ac})$$

Since filtered colimits commute with finite limits in $Ind(Ban)$ the map $\mathfrak{P}^{\dagger,\epsilon}(E) \rightarrow \hat{\mathfrak{P}}(V)$ is a monomorphism. Hence these are power series algebras.

Note that for $\infty \geqslant \zeta > \epsilon \geqslant \delta \geqslant \gamma > \beta > \alpha \geqslant 0$ we get canonically defined maps

$$\mathfrak{B}(V) \rightarrow \mathfrak{B}^{an,\zeta}(V) \rightarrow \mathfrak{B}^{\leqslant \epsilon}(V) \rightarrow \mathfrak{B}^{\dagger,\delta}(V) \rightarrow \mathfrak{B}(V) \rightarrow \mathfrak{B}^{\leqslant \beta}(V) \rightarrow \mathfrak{B}^{\dagger,\alpha}(V) \rightarrow \mathfrak{B}(V)$$

4.5 Analytic Commutative Algebras

We now specialise this discussion to the case $\mathfrak{P} = \mathfrak{Comm}$. We shall write $S^{\leqslant r}(E) := \mathfrak{Comm}^{\leqslant r}(E)$, $S^{an,\epsilon}(E) := \mathfrak{Comm}^{an,\epsilon}(E)$, and $S^{\dagger,\epsilon}(E) := \mathfrak{Comm}^{\dagger,\epsilon}(E)$. Let us establish some identities between these algebras.

4.5.1 $\aleph_1$-Filtered Objects

Following [54], an object $V \in Ind(Ban_k)$ is said to be λ-filtered if it can be written as "$lim_{j \in \mathcal{J}}$"V_j where $\mathcal{J}$ is λ-filtered. In a forthcoming work [9] Ben-Bassat and Kremnitzer study $\aleph_1$-filtered objects in detail. By using explicit descriptions of products in the category $Ind(Ban_k)$ they prove the following crucial results.

Proposition 4.5.1. *Let $\{V_k\}_{k \in K}$ be a countable collection of Banach k-modules. Then $\prod_{k \in K} V_k$ is $\aleph_1$-filtered.*

Proposition 4.5.2. *If V is $\aleph_1$-filtered and for each $i \in \mathcal{I}$, W_i is an object of $Ind(Ban_k)$, then the natural map*

$$V \otimes_k \prod_{i \in \mathcal{I}} W_i \rightarrow \prod_{i \in \mathcal{I}} (V \otimes_k W_i)$$

is an isomorphism.

As a consequence they can prove a result which allows one to commute tensor products and projective limits.

Proposition 4.5.3. *Let $A : \mathcal{I} \to \mathcal{Alg}_{\mathfrak{Comm}}(\mathcal{E})$ be a projective diagram where $\mathcal{I}$ has countably many objects and morphisms. Suppose that for any countable collection of J of objects of $\mathcal{I}$, $\prod_{j \in J} A_j$ is $\aleph_1$-filtered. Let $E : \mathcal{I} \to \mathcal{E}$ and $F : \mathcal{I} \to \mathcal{E}$ be projective systems of A-modules such that for any countable collection of J of objects of $\mathcal{I}$ $\prod_{j \in J} F_j$ is $\aleph_1$-filtered. Then the natural map*

$$\mathbb{R}lim_{\leftarrow} E_i \otimes^{\mathbb{L}}_{\mathbb{R}lim_{\leftarrow} A_i} \mathbb{R}lim_{\leftarrow} F_i \to \mathbb{R}lim_{\leftarrow}(E_i \otimes^{\mathbb{L}}_{A_i} F_i)$$

is an equivalence.

The idea of their proof is to use clever deformation functors for the tensor product and projective limit functors. For the derived tensor product one uses the Bar complex (defined for example in [10]) which give free resolutions over A. For the derived projective limit one uses the Roos complex (defined in [66]). The left hand-side of the isomorphism in the statement of Proposition 4.5.3 is computed by totalising a double complex after taking a bar resolution and then a Roos resolution. The right-hand side is computed by first taking a Roos resolution, then a bar resolution, and totalising. Proposition 4.5.1 and Proposition 4.5.2 ensure that the double complexes are naturally isomorphic, so their totalisations are equivalent.

Remark 4.5.4. *The result of Ben-Bassat and Kremnitzer actually proves that the map*

$$F \otimes_A lim_{\leftarrow_{i \in I}} E_i \to lim_{\leftarrow_{i \in I}}(F \otimes_A E_i)$$

is an isomorphism whenever F and A are $\aleph_1$-filtered, F is transverse to E_i over A for each i, and the system E_i is lim-acyclic. Their proof essentially uses the argument presented above. Then, under their assumptions, everything becomes underived.

Corollary 4.5.5. *Let $\mathcal{A} : \mathcal{I} \to \mathcal{Alg}_{\mathfrak{Comm}}(\mathcal{E})$ and $\mathcal{B} : \mathcal{I} \to \mathcal{Alg}_{\mathfrak{Comm}}(\mathcal{E})$ be projective diagrams where $\mathcal{I}$ has countably many objects and morphisms. Suppose that for any countable collection of J of objects of $\mathcal{I}$, $\prod_{j \in J} A_j$ and $\prod_{j \in J} B_j$ are $\aleph_1$-filtered. Let $f : \mathcal{A} \to \mathcal{B}$ be a map such that $f_i : A_i \to B_i$ is a homotopy epimorphism for each i. Then the induced map*

$$\mathbb{R}f : \mathbb{R}lim_{\leftarrow} \mathcal{A} \to \mathbb{R}lim_{\leftarrow} \mathcal{B}$$

is a homotopy epimorphism.

4.5.2 Relations Between Algebras

We introduce some notation. For $\underline{V} = (V_1, \ldots, V_n)$ a tuple of Banach modules and $\underline{\epsilon} \in (0, \infty)^n$ we set $S^{\leqslant \underline{\epsilon}}(\underline{V}) := S^{\leqslant 1}(\bigoplus_{i=1}^n (V_i)_{\epsilon_i})$ where $\bigoplus_{i=1}^n (V_i)_{\epsilon_i}$ is equipped with the norm $||(v_1, \ldots, v_n)|| = max_{1 \leqslant i \leqslant n} \epsilon_i ||v_i||$. For $\underline{\delta} \in (0, \infty]^n$ we write $S^{an, \underline{\delta}}(\underline{V}) :=$ "$lim_{\leftarrow_{\underline{\epsilon} < \underline{\delta}}}$"$S^{\leqslant \underline{\epsilon}}(\underline{V})$. For $\underline{r} \in [0, \infty)^n$ we write $S^{\dagger, \underline{r}}(\underline{V}) :=$ "$lim_{\rightarrow_{\underline{\epsilon} > \underline{r}}}$"$S^{\leqslant \underline{\epsilon}}(\underline{V})$.

Proposition 4.5.6. *Let $\{V_i\}_{i\in I}$ be a collection of flat Banach spaces. Then $\bigoplus_{i\in I}^{\leqslant 1} V_i$ is flat.*

Proof. Let $0 \to E \to F \to G \to 0$ be an exact sequence of Banach spaces. By rescaling we may assume that all maps are non-expanding. Now in the category $Ban_k^{\leqslant 1}$ the tensor product is a left adjoint, so it commutes with contracting direct sums. So $(\bigoplus_i^{\leqslant 1} V_i)\hat{\otimes}(-)$ is a composition of the functors $(V_i\hat{\otimes}(-)) : Ban_k^{\leqslant 1} \to (Ban_k^{\leqslant 1})^{\mathcal{I}}$ and $\bigoplus_{i\in I}^{\leqslant 1} : (Ban_k^{\leqslant 1})^{\mathcal{I}} \to Ban_k^{\leqslant 1}$. Since each V_i is flat the first functor is kernel and cokernel preserving. The second functor is clearly kernel and cokernel preserving. This implies that the composite is exact. $\square$

Corollary 4.5.7. *Let V be a flat k-module. Then $S^{\leqslant r}(V)$ and $S^{\dagger,\delta}(V)$ are flat.*

Proposition 4.5.8. *Let k be a Banach ring and $V_1, \ldots, V_n$ flat k-modules. Then the natural maps*

$$S^{\leqslant r_1}(V_1) \otimes^{\mathbb{L}} \ldots \otimes^{\mathbb{L}} S^{\leqslant r_n}(V_n) \to S^{\leqslant (r_1,\ldots,r_n)}(V_1, \ldots, V_n)$$

$$\hat{S}(V_1) \otimes^{\mathbb{L}} \ldots \otimes^{\mathbb{L}} \hat{S}(V_n) \to \hat{S}(V_1 \oplus \ldots \oplus V_n)$$

$$S^{\dagger,\delta_1}(V_1) \otimes^{\mathbb{L}} \ldots \otimes^{\mathbb{L}} S^{\dagger,\delta_n}(V_n) \to S^{\dagger,(\delta_1,\ldots,\delta_n)}(V_1, \ldots, V_n)$$

$$\mathbb{R}S^{an,\epsilon_1}(V_1) \otimes^{\mathbb{L}} \cdots \otimes^{\mathbb{L}} \mathbb{R}S^{an,\epsilon_n}(V_n) \to \mathbb{R}S^{an,(\epsilon_1,\ldots,\epsilon_n)}(V_1, \ldots, V_n)$$

are equivalences. In particular for $k = \mathbb{C}$ and $V_1, \ldots, V_n$ finite dimensional then

$$S^{an,\epsilon_1}(V_1) \otimes^{\mathbb{L}} \cdots \otimes^{\mathbb{L}} S^{an,\epsilon_n}(V_n) \to S^{an,(\epsilon_1,\ldots,\epsilon_n)}(V_1, \ldots, V_n)$$

is an equivalence.

Proof. We prove the result by induction. For $n = 1$ all results are trivial. Suppose they have been proven for some k and let $n = k + 1$. The first assertion follows from Proposition 4.5.7 and the universal property of coproducts and the symmetric algebra in $Ban_k^{\leqslant 1}$. The second isomorphism is not formal. For example it is not true in the algebraic category. However it follows from Proposition 4.5.3, Proposition 4.5.1, and the assumption that V is flat. The third assertion follows from the first and the fact that direct limits are exact and commute with the tensor product. The fourth assertion agains follows from Proposition 4.5.3. For the final assertion we use the fourth assertion, Lemma 3.60, and Corollary 3.80 in [5], and Montel's theorem. $\square$

4.5.3 Homotopical Power Series Algebras

Let us write $k[x_1,\ldots,x_n] := S(k^{\oplus n})$ and $k[[x_1,\ldots,x_n]] := \hat{S}(k^{\oplus n})$. Denote by $s : k[[y,z]] \to k[[y,z]]$ the map which sends a power series f to $\frac{f(y,z)-f(z,z)}{y-z}$. This is well-defined (i.e. bounded). Indeed it can be defined in any complete and cocomplete additive category. In [9] Ben-Bassat and Kremnitzer prove the following.

Proposition 4.5.9. *Let k be a Banach ring. Then the map $S(k^{\oplus n}) \to S^{\leqslant 1}(k^{\oplus n})$ is a homotopy epimorphism.*

Their proof can be distilled into the following two points.

Proposition 4.5.10. *Let $k[x] \subset B \subset k[[x]]$ be a power series algebra. Suppose that*

1. *The map $k[x] \to B$ is an epimorphism (i.e. polynomials are dense in B).*
2. *$B \otimes^{\mathbb{L}} B \to B \otimes B$ is an equivalence.*
3. *$B \otimes B \to k[[y,z]]$ is a monomorphism.*
4. *The composition*

$$B \otimes B \longrightarrow k[[y,z]] \xrightarrow{s} k[[y,z]]$$

factors through $B \otimes B$.

Then for each n the map

$$k[x_1,\ldots,x_n] \to B^{\otimes n}$$

is a homotopy epimorphism.

Let us sketch their proof. The second condition means that we only have to check this for $n = 1$. Let $A = k[x]$. It suffices to prove that $B \otimes^{\mathbb{L}}_A B \to B \otimes_A B$ is an equivalence. To prove this they consider the Koszul resolution $K_A \to A$ of the algebra $A = k[x]$. Then using this resolution of A as an $A \otimes A$-module one finds that $B \otimes^{\mathbb{L}}_A B$ is the complex in degrees $[1,0]$ $B \otimes_k B \to B \otimes_k B$ where the differential is given by multiplication by $(y-z)$. Then s gives a splitting which completes the proof of Proposition 4.5.10. To prove Proposition 4.5.9 it remains to show that $S^{\leqslant 1}(k)$ satisfies the conditions of Proposition 4.5.10. The only remaining point to check is the last one. Ben-Bassat and Kremnitzer show this as follows. For convenience let us write $k\{r_1x_1,\ldots,r_nx_n\} := S^{\leqslant(r_1,\ldots,r_n)}(k^{\oplus n})$. Note that for $f(y,z) \in k\{y,z\}$, $f(y,z) - f(z,z) \in k\{y,z\}$. Consider the isomteric isomorphism $\phi : k\{2\xi,\eta\} \to k\{y,z\}$ sending ξ to $y-z$ and η to z. Let ψ be its inverse. Then

$$\frac{|g|}{|(y-z)|g} = \frac{|g \circ \psi|}{|\xi g \circ \psi|} = \frac{1}{2}$$

In particular if g is a formal power series such that $(y-z)g$ is in $k\{y,z\}$ then $g \in \{y,z\}$. Thus for an element $f \in k\{y,z\}, s(f) \in k\{y,z\}$.

Corollary 4.5.11. *Let $\infty \geqslant \zeta > \epsilon \geqslant \delta \geqslant \gamma > \beta > \alpha \geqslant 0$ There is a sequence of homotopy epimorphisms*

$$S(k^{\oplus n}) \to S^{\leqslant \epsilon}(k^{\oplus n}) \to S^{\dagger,\delta}(k^{\oplus n}) \to S^{\leqslant \beta}(k^{\oplus n}) \to S^{\dagger,\alpha}(k^{\oplus n}) \to \hat{S}(k^{\oplus n})$$

which for $k = \mathbb{C}$ can be extended to sequence of homotopy epimorphisms

$$S(k^{\oplus n}) \to S^{an,\zeta}(k^{\oplus n}) \to S^{\leqslant \epsilon}(k^{\oplus n}) \to S^{\dagger,\delta}(k^{\oplus n}) \to S^{an,\gamma}(k^{\oplus n}) \to S^{\leqslant \beta}(k^{\oplus n}) \to S^{\dagger,\alpha}(k^{\oplus n}) \to \hat{S}(k^{\oplus n})$$

Proof. Using the two-out-of-three property for homotopy epimorphisms Proposition 4.1.28, Proposition 4.5.9, Proposition 4.5.5, and Proposition 4.1.29, all that remains to prove is that $S(k^{\oplus n}) \to \hat{S}(k^{\oplus n})$ is a homotopy epimorphism. However using Proposition 4.5.8, and the fact that $\hat{S}(k)$ is clearly closed under the map s, this follows from Proposition 4.5.10. □

4.5.4 Analytic Koszul Duality

Let $\mathfrak{g}$ be a very good Lie algebra and $S(\mathfrak{g}^{\vee}_{-1}) \subset A \subset \hat{S}(\mathfrak{g}^{\vee}_{-1})$ a power series algebra. Since $S(\mathfrak{g}^{\vee}_{-1}) \to C_{\kappa}(\mathfrak{g})$ is a cofibration we have $C_{\kappa,A} := A \otimes^{\mathbb{L}}_{S(\mathfrak{g}^{\vee}_{-1})} C_{\kappa}(\mathfrak{g}) \cong A \otimes_{S(\mathfrak{g}^{\vee}_{-1})} C_{\kappa}(\mathfrak{g})$. As established in Remark 4.3.7 we have $C_{\kappa,\hat{S}}(\mathfrak{g}) \cong \hat{C}_{\kappa}(\mathfrak{g})$. The map $C_{\kappa}(\mathfrak{g}) \to \hat{C}_{\kappa}(\mathfrak{g})$ then factors as

$$C_{\kappa}(\mathfrak{g}) \to C_{\kappa,A}(\mathfrak{g}) \to \hat{C}_{\kappa}(\mathfrak{g})$$

We also write

$$C^{\leqslant r}_{\kappa}(\mathfrak{g}) := C_{\kappa,S^{\leqslant r}(\mathfrak{g}^{\vee}_{-1})}(\mathfrak{g})\ , C^{an,\epsilon}_{\kappa}(\mathfrak{g}) := C_{\kappa,S^{an,\epsilon}(\mathfrak{g}^{\vee}_{-1})}(\mathfrak{g}), C^{\dagger,\delta}_{\kappa}(\mathfrak{g}) := C_{\kappa,S^{\dagger,\delta}(\mathfrak{g}^{\vee}_{-1})}(\mathfrak{g})$$

where $r \in (0,\infty), \epsilon \in (0,\infty], \delta \in [0,\infty)$. Using Corollary 4.5.11 and Proposition 4.1.26 we immediately get the following.

Theorem 4.5.12. *Let $\infty \geqslant \zeta > \epsilon \geqslant \delta \geqslant \gamma > \beta > \alpha \geqslant 0$ and let $\mathfrak{g}$ be a very good Lie algebra. Then in the following diagram all maps are homotopy epimorphisms.*

$$C_{\kappa}(\mathfrak{g}) \to C^{\leqslant \epsilon}_{\kappa}(\mathfrak{g}) \to C^{\dagger,\delta}_{\kappa}(\mathfrak{g}) \to C^{\leqslant \beta}_{\kappa}(\mathfrak{g}) \to C^{\dagger,\alpha}_{\kappa}(\mathfrak{g}) \to \hat{C}_{\kappa}(\mathfrak{g})$$

If $k = \mathbb{C}$ then this sequence can be extended to

$$C_{\kappa}(\mathfrak{g}) \to C^{an,\zeta}_{\kappa}(\mathfrak{g}) \to C^{\leqslant \epsilon}_{\kappa}(\mathfrak{g}) \to C^{\dagger,\delta}_{\kappa}(\mathfrak{g}) \to C^{an,\gamma}_{\kappa}(\mathfrak{g}) \to C^{\leqslant \beta}_{\kappa}(\mathfrak{g}) \to C^{\dagger,\alpha}_{\kappa}(\mathfrak{g}) \to \hat{C}_{\kappa}(\mathfrak{g})$$

In particular for each of the algebras A above we have $\mathbb{L}_0(C_A(\mathfrak{g})) \cong g^{\vee}[1], \mathbb{T}_0(C_A(\mathfrak{g}))[1] \cong \mathfrak{g}$, and $\mathbb{L}_{C_A(\mathfrak{g})} \cong C_A \otimes g^{\vee}[1]$. The algebras C_A may be interpreted as analytic 'fattenings' of the formal Koszul dual.

Appendix A

Model Categories and $(\infty, 1)$-Categories

A.1 Model Categories

A.1.1 Weak Factorization Systems and Model Structures

Here we briefly recall the definition of a model structure by means of weak factorisation systems. Details can be found in [70].

Definition A.1.1. *Let $\mathcal{C}$ be a class of morphisms in a category $\mathcal{M}$. A morphism f in $\mathcal{M}$ is said to have the* ***left lifting property*** *with respect to $\mathcal{C}$ if in any diagram of the form*

$$\begin{array}{ccc} A & \longrightarrow & C \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle c} \\ B & \longrightarrow & D \end{array}$$

with $c \in C$, there exists a morphism $h : B \to C$ such that the following diagram commutes

$$\begin{array}{ccc} A & \longrightarrow & C \\ {\scriptstyle f}\downarrow & \overset{h}{\nearrow} & \downarrow{\scriptstyle c} \\ B & \longrightarrow & D \end{array}$$

We denote the class of all morphisms which have the left-lifting property with respect to $\mathcal{C}$ by ${}^{\boxslash}\mathcal{C}$. Dually one defines the morphisms having the right lifting property with respect to $\mathcal{C}$. The class of all such morphisms is denoted $\mathcal{C}^{\boxslash}$.

The following is straightforward

Proposition A.1.2. *Let $\mathcal{C}$ be a class of morphisms in a category $\mathcal{M}$. Then ${}^{\boxslash}\mathcal{C}$ is closed under retracts, push-outs and transfinite composition (whenever they exist).*

Proof. See [70] Lemma 11.1.4. □

Definition A.1.3. *A **weak factorisation system** on a category $\mathcal{C}$ is a pair $(\mathcal{L}, \mathcal{R})$ such that*

1. *Any map in $\mathcal{C}$ can be factored as a map in $\mathcal{L}$ followed by a map in $\mathcal{R}$.*
2. *$\mathcal{L} = {}^{\boxslash}\mathcal{R}$ and $\mathcal{R} = \mathcal{L}^{\boxslash}$.*

*A weak factorisation system is said to be **functorial** if the factorisation in* (1) *can be made functorial.*

We can now give a definition of the notion of a model structure in terms of weak factorisation systems.

Definition A.1.4. *A **model structure** on a category $\mathcal{M}$ is a collection of three wide subcategories $(\mathcal{C}, \mathcal{F}, \mathcal{W})$ such that*

1. *The class $\mathcal{W}$ satisfies the* 2*-out-of-*6 *property (see [70]).*
2. *Both $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ and $(\mathcal{C}, \mathcal{W} \cap \mathcal{F})$ are weak factorization systems.*

We do not assume completeness or cocompleteness of $\mathcal{M}$.

Definition A.1.5. *A model structure on a category $\mathcal{M}$ is said to be **functorial** if the factorisation systems are functorial.*

Definition A.1.6. *A **(functorial) model category** a category together with a (functorial) model structure.*

A.1.2 Cofibrant Generation

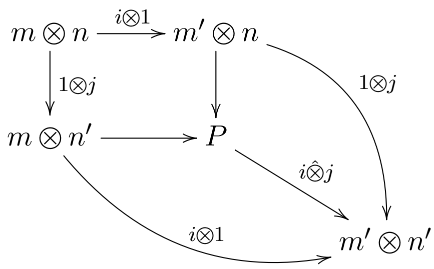
We state here our conventions regarding cofibration generation.

Definition A.1.7. *Let $\mathcal{C}$ be a category. A weak factorisation system $(\mathcal{L}, \mathcal{R})$ on $\mathcal{C}$ is said to be **cofibrantly small** if there is a set I of maps in $\mathcal{L}$ such that $\mathcal{R} = I^{\boxslash}$. I is called a set of **generating morphisms**. If in addition I admits the small object argument then the weak factorisation system is said to be **cofibrantly generated**. If I can be chosen such that the domains are compact with respect to $\mathcal{L}$, then the weak factorisation system is said to be **cellular**. If $\mathcal{C}$ is locally presentable and cofibrantly generated, then the weak factorisation system is said to be **combinatorial**. A model category $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ is said to be cofibrantly bsmall/ cofibrantly generated/ cellular/ combinatorial if both the weak factorisation systems $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ and $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ are cofibrantly small/ cellular/ combinatorial.*

Remark A.1.8. *A cofibrantly generated weak factorisation system (resp. model structure) on a locally presentable category is automatically cellular.*

A.1.3 Monoidal Model Categories

Definition A.1.9. *Let $\mathcal{M}, \mathcal{N}, \mathcal{P}$ be model categories. A bifunctor $- \otimes - : \mathcal{M} \times \mathcal{N} \to \mathcal{P}$ is said to be* ***left Quillen*** *if whenever $i : m \to m'$ and $j : n \to n'$ are cofibrations then so is $i \hat{\otimes} j$, and it is an acyclic cofibration if either i or j is. Here $i \hat{\otimes} j$ is the following map*



where the square is a push out.

Definition A.1.10. *A* ***(closed) monoidal model category*** *is a (closed) symmetric monoidal category $(\mathcal{V}, \otimes, k)$ $((\mathcal{V}, \otimes, k, \underline{Hom}))$ with a model structure so that the monoidal product is a left Quillen bifunctor, and the maps*

$$Q(k) \otimes v \to k \otimes v \cong v$$

and

$$v \otimes Q(k) \to v \otimes k \cong v$$

are weak equivalences whenever v is cofibrant. Here Q is the cofibrant replacement functor.

Another condition that is often asked of a monoidal model category is that it satisfies the so-called monoid axiom. Under certain additional technical assumptions on the model category, this guarantees the existence of a model structure on the category of algebras over any cofibrant operad.

Definition A.1.11. *A monoidal model category $(\mathcal{V}, \otimes, k, \underline{Hom})$ is said to satisfy the* ***monoid axiom*** *if every morphisms which is obtained a a transfinite composition of pushouts of tensor products of acyclic cofibrations with any object is a weak equivalence.*

A.1.4 Transferred Model Structures

Definition A.1.12. *Let $\mathcal{D}$ and $\mathcal{E}$ be categories with $\mathcal{D}$ a model category. Suppose $F : \mathcal{D} \to \mathcal{E}$ and $G : \mathcal{E} \to \mathcal{D}$ are functors with $F \dashv G$. If it exists, the* ***transferred model structure*** *on $\mathcal{E}$ is the one defined as follows.*

1. *A map f in $\mathcal{E}$ is a weak equivalence precisely if $G(f)$ is a weak equivalence in $\mathcal{D}$.*

2. *A map f in $\mathcal{E}$ is a fibration precisely if $G(f)$ is a fibration in $\mathcal{D}$.*

3. *A map f in $\mathcal{E}$ is a cofibration precisely if it has the left lifting property with respect to acyclic cofibrations.*

Remark A.1.13. *If the transferred model structure exists on $\mathcal{E}$ then $F \dashv G$ is a Quillen adjunction.*

We need the following important result, which is Theorem 3.3 in [19].

Theorem A.1.14. *Let $\mathcal{D}$ and $\mathcal{E}$ be categories, with $\mathcal{D}$ a cocomplete cellular model category and $\mathcal{E}$ having finite limits and all colimits. Suppose $F : \mathcal{D} \to \mathcal{E}$ and $G : \mathcal{E} \to \mathcal{D}$ are functors with $F \dashv G$. If F preserves compact objects, then the transferred model structure on $\mathcal{E}$ exists if and only if the weak equivalences in $\mathcal{E}$ contain any sequential colimit of pushouts of images $F(g)$, where g is allowed to vary over the generating trivial cofibrations in $\mathcal{D}$. Moreover the transferred model structure is cellular.*

Remark A.1.15. *Note that in [19] it is actually proved that if an adjunction satisfying the above condition then the transferred model structure exists. The converse is clear however since as a left Quillen functor F preserves acyclic cofibrations and colimits.*

We will actually use the following immediate corollary.

Corollary A.1.16. *Let $\mathcal{D}$ and $\mathcal{E}$ be categories, with $\mathcal{D}$ a cococomplete cellular model category and $\mathcal{E}$ having finite limits and all colimits. Suppose $F : \mathcal{D} \to \mathcal{E}$ and $G : \mathcal{E} \to \mathcal{D}$ are functors with $F \dashv G$. If G preserves filtered colimits, then the transferred model structure on $\mathcal{E}$ exists if and only if the weak equivalences in E contain any transfinite composition of pushouts of images $F(g)$, where g is a generating trivial cofibration in $\mathcal{D}$. Moreover the transferred model structure is cellular.*

A.1.5 Cylinder Objects and Interval Objects

In this section we follow [38] and [11].

Definition A.1.17. *Let $\mathcal{M}$ be a model category and M an object of $\mathcal{M}$. A **cylinder object** for M is a factorization*

$$M \coprod M \to Cyl(M) \tilde{\to} M$$

*of the fold map $M \coprod M \to M$ such that $Cyl(M) \tilde{\to} M$ is a weak equivalence. The cylinder object is said to be **good** if $M \coprod M \to Cyl(M)$ is a cofibration, and **very good** if in addition $Cyl(M) \tilde{\to} M$ is a fibration.*

Cylinder objects allow us to define (left) homotopies of maps. Let M be an object and

$$M \coprod M \xrightarrow{j_M} \mathrm{Cyl}(M) \xrightarrow{p_M} M$$

a cylinder object for M. Denote by $i_0 : M \to M \coprod M$ the inclusion of M as the first factor of the coproduct and $i_1 : M \to M$ the inclusion as the second factor. Let N be any other object and $H : \mathrm{Cyl}(M) \to N$ a map. We write H_0 for the composition $H \circ j_M \circ i_0$ and H_1 for the composition $H \circ j_M \circ i_1$.

Definition A.1.18. *Let $f, g : M \to N$ be maps and $Cyl(M)$ a cylinder object for M. f is said to be **left homotopic to** g relative to $Cyl(M)$ if there is a map $H : Cyl(M) \to N$ such that $H_0 = f$ and $H_1 = g$.*

The following result about the interaction of coproducts and homotopic maps is obvious, but usefl.

Proposition A.1.19. *Let $\{M_\gamma\}_{\gamma\in\Gamma}$ be a collection of objects. Suppose that coproducts of acyclic fibrations are acyclic fibrations. For each M_γ let*

$$M_\gamma \coprod M_\gamma \xrightarrow{j_{M_\gamma}} Cyl(M_\gamma) \xrightarrow{p_{M_\gamma}} M_\gamma$$

be a very good cylinder object for M_γ.

1.

$$(\textstyle\coprod_\gamma M_\gamma) \coprod (\coprod_\gamma M_\gamma) \cong \coprod_\gamma (M_\gamma \coprod M_\gamma) \xrightarrow{\coprod_\gamma j_{M_\gamma}} \coprod_\gamma Cyl(M_\gamma) \xrightarrow{\coprod_\gamma p_{M_\gamma}} \coprod_\gamma M_\gamma$$

is a very good cylinder object for $\prod_\gamma M_\gamma$.

2. *Suppose that $f_\gamma, g_\gamma : M_\gamma \to N_\gamma$ are maps such that f_γ is left homotopic to g_γ relative to $Cyl(M_\gamma)$. Then $\coprod_\gamma f_\gamma$ is left homotopic to $\coprod_\gamma g_\gamma$ relative to $\coprod_\gamma Cyl(M_\gamma)$.*

Typically we would like our cylinder objects to be functorial.

Definition A.1.20. *A **(good/very good) cylinder functor** for a model category $\mathcal{M}$ is an endofunctor $Cyl : \mathcal{M} \to \mathcal{M}$ together with natural transformations $p : Cyl \to Id$ and $j : (- \coprod -) \circ \Delta \to Cyl$ such that for each object M of $\mathcal{M}$*

$$M \coprod M \xrightarrow{j_M} Cyl(M) \xrightarrow{p_M} M$$

is a (good/ very good) cylinder object for M.

In a monoidal model category one can define cylinder objects using so-called interval-objects.

Definition A.1.21. *Let $(\mathcal{E}, \otimes, k)$ be a monoidal model category. A* ***(good/ very good) coassociative coalgebra interval*** *is a coassociative comonoid $([0,1], \Delta, \eta)$ together with morphisms of coalgebras $i_k : k \oplus k \to [0,1]$ and $p_k : [0,1] \to k$ such that*

$$k \coprod k \xrightarrow{i_k} [0,1] \xrightarrow{p_k} k$$

is a (good/ very good) cylinder object for k.

The next results follows directly from the definitions.

Proposition A.1.22. *Let $(\mathcal{E}, \otimes, k)$ be a monoidal model category which satisfies the monoid axiom, with a very good interval object $([0,1], \Delta, \eta)$. Suppose in addition that whenever $A \to B$ is an acyclic fibration between cofibrant objects and C is any object, the map $C \otimes A \to C \otimes B$ is a weak equivalence. Then the functor $Cyl = [0,1] \otimes (-)$ together with the natural transformations $i_k \otimes (-) : \Big(- \coprod - \Big) \circ \Delta \to Cyl$ and $p_k \otimes (-) : Cyl \to Id$ is a cylinder object. Moreover the restriction of Cyl to cofibrant objects provides a functorial good cylinder object.*

Definition A.1.23. *If two maps $f, g : A \to B$ are left homotopic using the cylinder object induced by an interval object $[0,1]$, we say that f and g are* ***homotopic relative to*** *$[0,1]$, and we write $f \sim^l_{[0,1]} g$.*

A map $f : A \to B$ is said to be a **homotopy equivalence relative to** $[0,1]$ if there is a map $g : B \to A$ such that $f \circ g \sim^l_{[0,1]} Id_B$ and $g \circ f \sim^l_{[0,1]} Id_A$. If $*$ is the terminal object then an object A is said to be $[0,1]$-**contractible** if the map $A \to *$ is a homotopy equivalence relative to $[0,1]$.

Proposition A.1.24. *Let $(\mathcal{C}, \otimes, k)$ be a symmetric monoidal model category which satisfies the monoid axiom, and let $([0,1], \nabla, \epsilon)$ be a very good coassociative coalgebra interval. Suppose $f, g : A \to B$ and $s, t : X \to Y$ are maps, that $f \sim^l_{[0,1]} g$ and $s \sim^l_{[0,1]} t$, and that both B and Y are fibrant. Then $f \otimes s \sim^l_{[0,1]} g \otimes t$.*

Proof. There exists a left homotopy using $[0,1] \otimes X$ and $[0,1] \otimes A$. Let $\eta_X : [0,1] \otimes X \to Y$ realise the left homotopy between f and g, and $\eta_A : [0,1] \otimes A \to B$ realise the homotopy between s and t. Consider the composition

$$[0,1] \otimes (X \otimes A) \xrightarrow{\nabla \otimes \mathrm{Id}} ([0,1] \otimes [0,1]) \otimes (X \otimes A) \longrightarrow ([0,1] \otimes X) \otimes ([0,1] \otimes A) \xrightarrow{\eta_X \otimes \eta_A}$$

$$\longrightarrow Y \otimes B$$

This gives a left homotopy between $f \otimes s$ and $g \otimes t$ relative to I. $\square$

A.1.6 Algebra in Monoidal Model Categories

Let $(\mathcal{C}, \otimes)$ be a monoidal model category. We recall here a major result regarding the existence of transferred model structures on categories of monoids and modules internal to $\mathcal{C}$.

Theorem A.1.25 ([73]). *Let $(\mathcal{C}, \otimes)$ be a bicomplete monoidal model category and R a monoid object in $\mathcal{C}$. Suppose that*

1. *$(\mathcal{C}, \otimes)$ satisfies the monoid axiom.*
2. *$\mathcal{C}$ is a combinatorial model category.*

Then

1. *The transferred model structure on ${}_R\mathcal{M}od$ exists and is cofibrantly generated.*
2. *If R is commutative, then the transferred model structure on ${}_R\mathcal{M}od$ is monoidal and satisfies the monoid axiom.*
3. *If R is commutative then the transferred model structure exists on the category of monoids in ${}_R\mathcal{M}od$. Moreover it is cofibrantly generated. Every cofibration of R-algebras whose source is cofibrant is also a cofibration of R-modules.*

Proof. This is Theorem 4.1 in [73]. □

A.2 $(\infty, 1)$-Categories

Quasi-Categories and Localization of Model Cateogries

For concreteness we fill fix as quasi-categories our model for $(\infty, 1)$-categories. If $\mathcal{M}$ is a model category we can associated to it a quasi-category $\mathbf{M}$. If $\mathcal{M}$ is a combinatorial simplicial model category then $\mathbf{M}$ is a locally presentable $(\infty, 1)$-category by Proposition A.3.7.6 in [51].

Generation under Sifted Colimits

Let $\mathbf{C}$ be a cocomplete $(\infty, 1)$-category and $\mathbf{C}_0$ a full subcategory. We denote by $\mathcal{P}_\Sigma(\mathbf{C}_0)$ the free cocompletion of $\mathbf{C}_0$ by sifted colimits, i.e. by filtered colimits and geometric realisations. There is a natural functor

$$\mathcal{P}_\Sigma(\mathbf{C}_0) \to \mathbf{C}$$

Let $\mathbf{T}$ be a monad on $\mathbf{C}$ which preserves sifted colimits and let $\mathbf{C}^{\mathbf{T}}$ its category of Eilenberg-Moore algebras. Consider the corresponding adjunction.

$$Free_{\mathbf{T}} \colon \mathbf{C} \rightleftarrows \mathbf{C}^{\mathbf{T}} : |-|_{\mathbf{T}}$$

Let $Free_{\mathbf{T}}(\mathbf{C}_0)$ denote the full subcategory of $\mathbf{C}^{\mathbf{T}}$ spanned by free $\mathbf{T}$-algebras on $\mathbf{C}_0$.

$$\begin{array}{ccc} \mathcal{P}_{\Sigma}(\mathbf{C}_0) & \longrightarrow & \mathcal{P}_{\Sigma}(Free_{\mathbf{T}}(\mathbf{C}_0)) \\ \downarrow & & \downarrow \\ \mathbf{C} & \longrightarrow & \mathbf{C}^{T} \end{array}$$

Suppose that the left-hand vertical map is essentially surjective. By Proposition 4.7.3.14 in [52] every object in $\mathbf{C}^{T}$ can be obtained as a colimit of a simplicial diagram of objects in the image of $Free_{\mathbf{T}}$. Therefore by the previous proposition every object in $\mathbf{C}^{T}$ can be obtained as a sifted colimit of objects in $Free_{\mathbf{T}}(\mathbf{C}_0)$. In particular the functor $\mathcal{P}_{\Sigma}(Free_{\mathbf{T}}(\mathbf{C}_0)) \to \mathbf{C}^{\mathbf{T}}$ is essentially surjective.

We are particularly interested in the case that $\mathbf{C}$ is presented by a Kan complex enriched monoidal model category $\mathcal{C}$ and $\mathfrak{P}$ is an admissible operad on $\mathcal{C}$. This gives rise to a monadic Quillen adjunction

$$\mathfrak{P}(-) \colon \mathcal{C} \rightleftarrows \mathcal{Alg}_{\mathfrak{P}}(\mathcal{C}) : |-|$$

which by localization induces an adjunction of $(\infty, 1)$-categories

$$\mathfrak{P}(-) \colon \mathbf{C} \rightleftarrows \mathbf{Alg}_{\mathfrak{P}}(\mathcal{C}) : |-|$$

According to [43] this is also a monadic adjunction. Since $\mathcal{C}$ is cofibrantly generated $\mathbf{C}$ is generated under sifted colimits by some small subcategory $\mathbf{C}_0$ of cofibrant objects in $\mathcal{C}$. Therefore $\mathbf{Alg}_{\mathfrak{P}}(\mathcal{C})$ is generated under sifted colimits by the full category of free $\mathfrak{P}$-algebras on objects in $\mathbf{C}_0$.

Remark A.2.1. *The argument given above is a significant component of the one in [35] Proposition 1.2.2, which shows that the category of chain complexes of vector spaces over a field, and the category of Lie algebras over it, are in fact the formal completion of subcategories of certain compact objects by sifted colimits.*

Appendix B

Algebra in Additive Categories

Throughout this section $(\mathcal{E}, \otimes, k)$ is a symmetric monoidal category, with monoidal functor $\otimes$. The symmetric braiding will be denoted by σ. We further assume that $\mathfrak{E}$ is finitely bicomplete. What follows is largely standard. Much of it can be found in [10] for example.

B.1 Familiar Algebras

Associative Monoids

We denote the category of (unital) associative monoids internal to $\mathcal{E}$ by $\mathcal{Ass}(\mathcal{E})$. There is a faithful forgetful functor $|-|_{\mathcal{Ass}} : \mathcal{Ass}(\mathcal{E}) \to \mathcal{E}$. If $\mathcal{E}$ has countable products then $|-|$ has a left adjoint T constructed in the usual way as the tensor algebra. Namely, $T(V) = \bigoplus_{n=0}^{\infty} T_n(V)$ with $T_n(V) = V^{\otimes n}$.

Commutative Monoids

We denote the category of (unital) commutative monoids by $\mathcal{Comm}(\mathcal{E})$. If $\mathcal{E}$ has finite coequalizers and countable coproducts then the forgetful functor $|-|_{\mathcal{Comm}} : \mathcal{Comm}(\mathcal{E}) \to \mathcal{E}$ has a left-adjoint, which can be constructed explicitly as follows. The symmetric group on n letters Σ_n acts on $T_n(V) = V^{\otimes n}$. Let $S_n(V) = T_n(V)_{\Sigma_n}$ be the coinvariants for this action. We then set $S(V) = \bigoplus_{n=0}^{\infty} S_n(V)$. The associative monoid structure on $T(V)$ descends to an associative monoid structure on $S(V)$. One checks easily that it is commutative and that it is a left adjoint.

Modules

Given objects A and B of $\mathcal{Ass}(\mathcal{E})$ we denote by ${}_A\mathcal{Mod}$ the category of left modules for A, by $\mathcal{Mod}_A$ the category of right modules for A, and by ${}_A\mathcal{Mod}_B$ the category of $A-B$ bimodules. There is a forgetful functor $|-|_{{}_A\mathcal{Mod}} : {}_A\mathcal{Mod} \to \mathcal{E}$. This functor has a left adjoint. It sends an object E to the object $A \otimes E$ with the obvious left action of A.

Let E be a right A-module with action morphism $a_E : E \otimes A \to E$ and F a left A-module with action morphism $a_F : A \otimes F \to F$. If the category $\mathcal{E}$ has finite equalisers, then we define $E \otimes_A F$ to be the coequaliser of the maps

$$E \otimes A \otimes F \underset{a_F}{\overset{a_E}{\rightrightarrows}} E \otimes F$$

This defines a bifunctor

$$\otimes_A : \mathcal{M}od_A \times {}_A\mathcal{M}od \to \mathcal{E}$$

If E is a $B - A$ bimodule and F is an $A - C$ bimodule, then $E \otimes_A F$ is naturally a $B - C$-bimodule, i.e. $\otimes_A$ gives a bifunctor

$${}_B\mathcal{M}od_A \times {}_A\mathcal{M}od_C \to {}_B\mathcal{M}od_C$$

If A is a commutative monoid then this gives a bifunctor

$${}_A\mathcal{M}od \times {}_A\mathcal{M}od \to {}_A\mathcal{M}od$$

which endows ${}_A\mathcal{M}od$ with a monoidal structure.

Suppose further that the monoidal structure is closed, and let $\underline{\mathrm{Hom}}(-,-)$ denote the internal hom functor. Then one can also construct an internal hom, $\underline{\mathrm{Hom}}_A(-,-)$ functor on ${}_A\mathcal{M}od$ by a similar method as used to construct $\otimes_A$. This makes $(\mathcal{E}, \otimes_A, \underline{\mathrm{Hom}}_A(-,-), A)$ a closed monoidal category. See for example [10] for details.

Lie Monoids

Now we suppose $(\mathcal{E}, \otimes, k)$ is a monoidal additive category. Then one can define the category of Lie monoids internal to $\mathcal{E}$. Denote the symmetric braiding by σ. A **Lie monoid** in $\mathcal{E}$ is a pair $(L, [-,-])$ consisting an object L of $\mathcal{E}$ together with a morphism $[-,-] : L \otimes L \to L$ satisfying the Jacobi identity

$$[-,[-,-]] + [-,[-,-]] \circ (\mathrm{id}_L \otimes \sigma_{L,L}) + [-,[-,-]] \circ (\sigma_{L,L} \otimes \mathrm{id}_L) \circ (\mathrm{id}_L \otimes \sigma_{L,L}) = 0$$

and the antisymmetry condition

$$[-,-] + [-,-] \otimes \sigma_{L,L} = 0$$

Morphisms of Lie monoids are defined in the obvious way. This gives a category $Lie(\mathcal{E})$ of Lie monoids internal to $\mathfrak{C}$.

There is of course a forgetful functor $|-|_{Lie} : Lie(\mathcal{E}) \to \mathcal{E}$. If $\mathcal{E}$ is enriched over ${}_{\mathbb{Q}}\mathcal{V}ect$ rather than $\mathcal{A}b$ we will also see that this functor has a left adjoint L which can be constructed explicitly.

Now let A be an associative monoid in $\mathcal{E}$ with multiplication m. Define $[-,-] : A \otimes A \to A$ by $[-,-] = m - m \circ \sigma_{A,A}$. It is easy to see that $(A, [-,-])$ is a Lie monoid. Moreover this structure is clearly functorial, and we get a faithful functor $\mathcal{A}ss(\mathcal{E}) \to Lie(\mathcal{E})$. As we shall see later, if $\mathcal{E}$ is enriched over ${}_{\mathbb{Q}}\mathcal{V}ect$ then this functor has a left adjoint U.

Algebra in ${}_{\mathbb{Q}}\mathcal{V}ect$-Enriched Symmetric Monoidal Categories

We now assume that our monoidal additive category $\mathcal{E}$ is enriched over ${}_{\mathbb{Q}}\mathcal{V}ect$ rather than just $\mathcal{A}b$. We also assume that $\mathcal{E}$ is finitely bicomplete and has countable coproducts. Let us relate the functors U, L, T, S.

The easiest identity is $U \circ L \cong T$. This follows from the fact that both $U \circ L$ and T are left adjoints to the forgetful functor $|-|_{\mathcal{A}ss} : \mathcal{A}ss(\mathcal{E}) \to \mathcal{E}$.

Now consider T and S. The following is an easy generalisation of the same fact for $\mathbb{Q}$-vector spaces. It is done for dg-vector spaces in [69] for example.

Proposition B.1.1. *The natural transformation $|-|_{\mathcal{A}ss} \circ T \to |-|_{\mathcal{C}omm} \circ S$ admits a section.*

Proof. Let V be an object of $\mathcal{E}$. Define a map $\rho_V : T(V) \to T(V)$ of graded objects in $\mathcal{E}$ by

$$\rho_{V,\,n} = \frac{1}{n!} \sum_{\sigma \in \Sigma_n} \sigma : T_n(V) \to T_n(V)$$

This clearly induces a map

$$T_n(V)_{\Sigma_n} = S_n(V) \to T_n(V)$$

which is a section of the projection $T_n(V) \to S_n(V)$. It is also clear that ρ_V is natural in V, i.e. we get a natural transformation $\rho : |-|_{\mathcal{C}omm} \circ S \to |-|_{\mathcal{A}ss} \circ T$ which is a section of $|-|_{\mathcal{A}ss} \circ T \to |-|_{\mathcal{C}omm} \circ S$ □

Let us now explain how U and S are related. In [22] it is shown that if $\mathcal{E}$ is $\mathbb{Q}$-linear then a left adjoint U to the forgetful functor $\mathcal{A}ss(\mathcal{E}) \to Lie(\mathcal{E})$ exists, and there is a natural isomorphism

$$|-|_{\mathcal{A}ss} \circ U \cong |-|_{\mathcal{C}omm} \circ S \circ |-|_{Lie}$$

$U(L)$ is called the **universal enveloping algebra** of L. The proof in fact works in the following setup

Theorem B.1.2 (Poincaré-Birkhoff-Witt)**.** *Let $(\mathcal{E}, \otimes, k)$ a monoidal additive category enriched over ${}_{\mathbb{Q}}\mathcal{V}ect$ with countable coproducts and finite coequalizers. Then a left adjoint U to the forgetful functor $\mathcal{A}ss(\mathfrak{C}) \to \mathcal{L}ie(\mathfrak{C})$ exists, and there is a natural isomorphism*

$$|-|_{\mathcal{A}ss} \circ U \cong |-|_{\mathcal{C}omm} \circ S \circ |-|_{\mathcal{L}ie}$$

Corollary B.1.3. *Let $\mathfrak{g}$ be a Lie monoid and let $i : \mathfrak{g} \to U(\mathfrak{g})$ denote the natural map in $\mathcal{E}$. Then the map $\mathfrak{g} \to Im(i)$ is an isomorphism.*

Finally we relate T and L. First we give an explicit construction of L. Consider the tensor algebra $T(V)$ as a Lie algebra with Lie bracket $[-,-]$ the one induced from the associative algebra structure. Let $L_0(V) = V \hookrightarrow T(V)$. Inductively define a subobject $L_{r+1}(V)$ of $T(V)$ as the image of the restriction of $[-,-]$ to $V \otimes L_r(V)$. Define

$$L(V) = \bigoplus_{r=0}^{\infty} L_r(V)$$

The Lie bracket on $T(V)$ pulls back to one on $L(V)$. The construction is clearly functorial. To see that it is a left adjoint we follow the method of [77]. Suppose $\mathfrak{g}$ is a Lie monoid and $V \to \mathfrak{g}$ is a morphism in $\mathcal{E}$. This induces a morphism $V \to \mathfrak{g} \to U(\mathfrak{g})$ and therefore a morphism of associative algebras $T(V) \to U(\mathfrak{g})$. The image of $L(V)$ under this map is clearly contained in the image of $\mathfrak{g}$ in $U(\mathfrak{g})$. But by Corollary B.1.3 this is isomorphic to $\mathfrak{g}$. Thus we get a lift of $V \to \mathfrak{g}$ to a map of Lie algebras $L(V) \to \mathfrak{g}$. Such a map is clearly unique.

We are going to show that the natural inclusion $L(V) \hookrightarrow T(V)$ is split. First we introduce some notation. Let $\mathfrak{g}$ be a Lie monoid with bracket $[-,-]$. Define $[-,-]_n : \mathfrak{g}^{\otimes n} \to \mathfrak{g}$ inductively as follows. We set $[-,-]_1 = [-,-]$ and define $[-,-]_{n+1}$ to be the composite.

$$\mathfrak{g}^{\otimes n+1} \xrightarrow{id_{\mathfrak{g}} \otimes [-,-]_n} \mathfrak{g} \otimes \mathfrak{g} \xrightarrow{[-,-]} \mathfrak{g}$$

We then get the following result, which is a generalisation of Lemma 2.2 in [69].

Lemma B.1.4. *The graded natural transformation $\rho : |-|_{\mathcal{A}ss} \circ T \to |-|_{\mathcal{L}ie} \circ L$ of graded objects in $\mathcal{E}$ given by*

$$\rho_n = \begin{cases} 0 & n = 0 \\ [-,-]_n & n > 0 \end{cases}$$

is a left inverse for the map $L(V) \to T(V)$.

Proof. Fix an object V of $\mathcal{E}$. Denote the Lie bracket on $L(V)$ by $[-,-]$. Define a Lie monoid endomorphism D of $L(V)$ whose action on $L_n(V)$ is multiplication by n. Consider the Lie monoid $L'(V) = L(V) \oplus k$ with Lie bracket

$$[-.-]' : (L(V) \oplus k) \otimes (L(V) \oplus k) \cong L(V) \otimes L(V) \oplus L(V) \oplus L(V) \oplus k \to L(V) \oplus k$$

given by the matrix

$$\begin{pmatrix} [-,-] & D & -D & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The inclusion $L(V) \to L(V) \oplus k$ is a morphism of Lie monoids whose image is an ideal of $L(V) \oplus k$. Thus $L(V) \oplus k$ is an $L(V)$-module, and hence a $U(L(V)) = T(V)$ module. Consider the composition

$$T(V) \cong T(V) \otimes k \to T(V) \otimes L'(V) \to L'(V)$$

In degree n this map is $n\rho_n$. But when restricted to $L_n(V)$ it is given by $D_n = n\mathrm{id}_{L_n(V)}$. Thus $\rho_n|_{L_n(V)} = \mathrm{id}_{L_n(V)}$. $\square$

B.2 Operadic Algebra in Additive Categories

In this section we recall some basic results on operads and co-operads. Our reference is[48] where, unless stated otherwise, anything left unproved in this section can be found. While the book works in the context of vector spaces, most of the proofs work mutatis mutandis for monoidal additive categories with some mild assumptions. Note that all the algebras studied in the previous section are algebras over certain operads.

Discrete Groups in Monoidal Categories

Let $(\mathcal{E}, \otimes, k)$ be a symmetric monoidal category with all small coproducts. We denote by $k[-] : \mathcal{S}et \to \mathcal{E}$ the functor which sends a set S to the object $k[S] = \bigoplus_S k$. If $f : S \to T$ is a map of sets, then $k[f] : k[S] \to k[T]$ is the morphism which sends the copy of k indexed by $s \in S$ to the copy indexed by $f(s) \in T$. Objects and morphisms in the essential image of the functor $k[-] : \mathcal{S}et \to \mathcal{E}$ will be called **discrete**.

Proposition B.2.1. *Let $(\mathcal{E}, \otimes, k)$ be a symmetric monoidal category. Suppose that $\otimes$ preserves all coproducts. Endow $\mathcal{S}et$ with its Cartesian monoidal structure. Then the functor $k[-] : \mathcal{S}et \to \mathcal{E}$ is strong monoidal.*

Proof. Let S and T be sets. Then

$$\Big(\coprod_S k\Big)\otimes\Big(\coprod_T k\Big) \cong \coprod_S\coprod_T k\otimes k \cong \coprod_S\coprod_T k \cong \coprod_{S\times T} k$$

□

In particular $\mathcal{Set} \to \mathcal{E}$ sends groups to Hopf monoids. If G is a group we call $k[G]$ the group monoid of G in $\mathcal{E}$.

The Category of Σ-Modules

We denote by $k[\Sigma]$ the monoid in $\mathcal{Gr}(\mathcal{E})$ defined as follows. In degree n it is given by the monoid $k[\Sigma_n]$, the free monoid on the symmetric group in n letters.

Definition B.2.2. *The* ***category of*** Σ-***modules*** *in* $\mathcal{E}$ *is the category of* $k[\Sigma]$*-modules.*

Definition B.2.3. *Let M be a Σ-module. Its associated* ***Schur functor*** *is the endofunctor $\tilde{M} : \mathcal{E} \to \mathcal{E}$ defined by*

$$\tilde{M}(V) = \bigoplus_{n\geqslant 0} M(n)\otimes_{\Sigma_n} V^{\otimes n}$$

The assignment $M \mapsto \tilde{M}$ is functorial in a natural way. We denote the functor $\mathcal{Mod}_\Sigma \to End(\mathcal{E},\mathcal{E})$ by Sch.

The Tensor Product of Σ-Modules

We are going to define a monoidal structure on the category of Σ-Modules.

Definition B.2.4. *Let M and N be two Σ-modules. The* ***tensor product*** *of M and N, denote $M\otimes N$, is the Σ-module define by*

$$(M\otimes N)(n) = \bigoplus_{i+j=n} Ind^{\Sigma_n}_{\Sigma_i\times\Sigma_j}(M(i)\otimes N(j))$$

Definition B.2.5. *The Σ-module I is defined by $M(i) = 0$ for $i \neq 1$ and $M(1) = k$.*

The following can be proven as in [48]

Proposition B.2.6. *$(\mathcal{Mod}_\Sigma, \otimes, I)$ is a symmetric monoidal category. Moreover if we endow $[\mathcal{E},\mathcal{E}]$ with its object wise monoidal structure, Sch is a strong monoidal functor.*

The Composite of Σ-Modules

Definition B.2.7. *Let M and N be two Σ-modules. The* ***composite product*** *of M and N, denote $M \circ N$ is defined by*

$$M \circ N(n) = \bigoplus_{k \geqslant 0} (M(k) \otimes_{\Sigma_k} N^{\otimes k}(n)$$

Proposition B.2.8. *$(\mathcal{Mod}_{\Sigma}, \circ, I)$ is a monoidal category.*

Another useful notion is the infinitesimal composite product of Σ-modules

Definition B.2.9. *The functor $- \circ (-;-) : \mathcal{Mod}_{\Sigma}^{3} \to \mathcal{Mod}_{\Sigma}$ is the subfunctor of $- \circ (- \oplus -) : \mathcal{Mod}_{\Sigma}^{3} \to \mathcal{Mod}_{\Sigma}$ which is linear in the last variable.*

Definition B.2.10. *The* ***infinitesimal composite product*** *functor $- \circ_{(1)} - : \mathcal{Mod}_{\Sigma}^{2} \to \mathcal{Mod}_{\Sigma}$ is the functor*

$$- \circ_{(1)} - := - \circ (I; -)$$

Definition B.2.11. *Let $f : M_1 \to M_2$ an $g : N_1 \to N_2$ be morphisms of Σ-modules. The* ***infinitesimal composite*** *of f and g, denoted $f \circ' g$ is the map*

$$f \circ' g : M_1 \circ N_1 \to M_2 \circ (N_1; N_2)$$

given by the formula

$$f \circ' g = \sum_i f \otimes (Id_{N_1} \otimes \ldots \otimes Id_{N_1} \otimes g \otimes Id_{N_1} \otimes \ldots \otimes Id_{N_1})$$

Operads and Co-Operads

Definition B.2.12. *The* ***category of operads*** *denoted Op is the category of associative monoids $(\mathfrak{P}, \gamma, \eta)$ in the monoidal category $(\mathcal{Mod}_{\Sigma}, I, \circ)$. The* ***category of cooperads*** *denoted coOp is the category of coassociative comonoids in the monoidal category $(\mathcal{Mod}_{\Sigma}, I, \circ)$.*

Definition B.2.13. *Let $(\mathfrak{P}, \gamma, \eta)$ be an operad. The* ***infinitesimal composition map*** *$\gamma_{(1)} : \mathfrak{P} \circ_{(1)} \mathfrak{P} \to \mathfrak{P}$ given by the composition.*

$$\mathfrak{P}_{\circ_{(1)}} \mathfrak{P} = \mathfrak{P} \circ (I; \mathfrak{P}) \rightarrowtail \mathfrak{P} \circ (I \oplus \mathfrak{P}) \xrightarrow{Id_{\mathfrak{P}} \circ (\eta + Id_{\mathfrak{P}})} \mathfrak{P} \circ \mathfrak{P} \xrightarrow{\gamma} \mathfrak{P}$$

Definition B.2.14. *Let $(\mathfrak{C}, \Delta, \epsilon)$ be a co-operad. The* ***infinitesimal decomposition map*** *$\Delta_{(1)} \mathfrak{C} \to \mathfrak{C} \circ_{(1)} \mathfrak{C}$ given by the composition.*

$$\mathfrak{C} \xrightarrow{\Delta} \mathfrak{C} \circ \mathfrak{C} \xrightarrow{Id_{\mathfrak{C}} \circ' Id_{\mathfrak{C}}} \mathfrak{C} \circ (\mathfrak{C}; \mathfrak{C}) \xrightarrow{Id_{\mathfrak{C}} \circ (\epsilon; Id_{\mathfrak{C}})} \mathfrak{C} \circ (I; \mathfrak{C}) = \mathfrak{C}_{\circ_{(1)}} \mathfrak{C}$$

Partial Compositions

An operadic structure on a Σ-module $(\mathfrak{P}, \gamma, \eta)$ can also be described by the so-called partial composition maps. Let $\mathfrak{P}$ be an operad. For $m, n \in \mathbb{N}$ and $1 \leqslant i \leqslant m$ we define the partial composition map

$$\circ_i : \mathfrak{P}(m) \otimes \mathfrak{P}(n) \to \mathfrak{P}(m+n-1)$$

as the composition

$$\mathfrak{P}(m) \otimes \mathfrak{P}(n) \cong \mathfrak{P}(m) \otimes k \otimes \ldots \otimes k \otimes \mathfrak{P}(n) \otimes k \otimes \ldots \otimes k \to$$

$$\to \mathfrak{P}(m) \otimes \mathfrak{P}(1) \otimes \ldots \otimes \mathfrak{P}(1) \otimes \mathfrak{P}(n) \otimes \mathfrak{P}(1) \otimes \ldots \otimes \mathfrak{P}(1) \hookrightarrow \mathfrak{P} \circ \mathfrak{P} \to \mathfrak{P} \to \mathfrak{P}(m+n-1)$$

It turns out that the operadic structure can be completely recovered from these maps.

The Hadamard Tensor Product

If $\mathfrak{O}$ and $\mathfrak{P}$ are Σ-modules then we define their **Hadamard tensor product** by $(\mathfrak{O} \otimes_H \mathfrak{P})(n) := \mathfrak{O}(n) \otimes \mathfrak{P}(n)$ equipped with the diagonal action of Σ_n. If $\mathfrak{O}$ and $\mathfrak{P}$ are operads then so is their Hadamard tensor product. By duality the Hadamard tensor product of two co-operads is a co-operad.

Operadic Modules and Ideals

Definition B.2.15. *Let $\mathfrak{P}$ be an operad. A **(non-unital) left $\mathfrak{P}$-module** is a Σ-module M together with a map*

$$\rho_M : \mathfrak{P} \circ (\mathfrak{P}; M) \to M$$

such that the following diagrams commutes

$$\begin{array}{ccccc} \mathfrak{P} \circ (\mathfrak{P}; \mathfrak{P} \circ (\mathfrak{P}; M))) & & \xrightarrow{Id_{\mathfrak{P}} \circ (Id_{\mathfrak{P}}; \gamma_M)} & & \mathfrak{P} \circ (\mathfrak{P}; M) \\ \downarrow \sim & & & & \downarrow \gamma_M \\ (\mathfrak{P} \circ \mathfrak{P}) \circ (\mathfrak{P}; M) & \xrightarrow{\mu_{\mathfrak{P}} \circ (Id; Id)} & \mathfrak{P} \circ (\mathfrak{P}; M) & \xrightarrow{\gamma_M} & M \end{array}$$

a

There is an obvious notion of a morphism of $\mathfrak{P}$-modules, and this gives a category $\mathfrak{P} - \mathcal{M}od$.

Let M be a $\mathfrak{P}$-module and $j_M : N \hookrightarrow M$ a sub-Σ-module. Consider the composition

$$\gamma_N = \mathfrak{P} \circ (\mathfrak{P}; N) \to \mathfrak{P} \circ (\mathfrak{P}; M) \to M$$

Write $\left\langle N \right\rangle_1 = \mathrm{Im}(\gamma_N)$. Inductively define $\left\langle N \right\rangle_{k+1} = \left\langle \left\langle N \right\rangle_k \right\rangle_1$. Finally define $\left\langle N \right\rangle = \lim_{\rightarrow} \left\langle N \right\rangle_k$. Let us endow $\left\langle N \right\rangle$ with a left $\mathfrak{P}$-module structure. First note that the natural map

$$\lim_{\rightarrow} \mathfrak{P} \circ \left(\mathfrak{P}; \left\langle N \right\rangle_k\right) \rightarrow \mathfrak{P} \circ \left(\mathfrak{P}; \lim_{\rightarrow} \left(\left\langle N \right\rangle_k\right)\right)$$

is an isomorphism. We now construct a map $\lim_{\rightarrow} \mathfrak{P} \circ (\mathfrak{P}; \left\langle N \right\rangle_k) \rightarrow \lim_{\rightarrow} \left\langle N \right\rangle_k$. We have the following commutative diagram

$$\begin{array}{ccc} \mathfrak{P} \circ (\mathfrak{P}; \left\langle N \right\rangle_k) & \longrightarrow & \mathfrak{P} \circ (\mathfrak{P}; \left\langle N \right\rangle_{k+1}) \\ \downarrow & & \downarrow \\ \left\langle N \right\rangle_{k+1} & \longrightarrow & \left\langle N \right\rangle_{k+2} \end{array}$$

which induces the desired map.

Proposition B.2.16. *$\left\langle N \right\rangle$ is a left $\mathfrak{P}$-module. There is a monomorphism $j_{\left\langle N \right\rangle} : N \hookrightarrow \left\langle N \right\rangle$ which satisfies the following universal property. $\phi_M \circ j_{\left\langle N \right\rangle} = j_M$.*

Suppose further that transfinite compositions of regular monomorphisms are regular. Then $j_{\left\langle N \right\rangle}$ is a regular monomorphism. If there is a commutative diagram

$$\begin{array}{ccc} N & \xrightarrow{j_{N'}} & N' \\ & {\scriptstyle j_M} \searrow \quad \swarrow {\scriptstyle i_{N'}} & \\ & M & \end{array}$$

with all maps being regular monomorphisms and N' being a sub-$\mathfrak{P}$-module of M, then there is a unique map of $\mathfrak{P}$-modules $\phi_{N'} : \left\langle N \right\rangle \rightarrow N'$ such that the following diagram commutes

$$\begin{array}{ccc} \left\langle N \right\rangle & \xrightarrow{\phi_{N'}} & N' \\ & {\scriptstyle \phi_M} \searrow \quad \swarrow {\scriptstyle i_{N'}} & \\ & M & \end{array}$$

In particular, if ϕ_M is a regular monomorphism then $\left\langle N \right\rangle$ is the regular sub-$\mathfrak{P}$module of M generated by N. Further suppose that $\mathtt{E}$ is quasi-abelian. Then $\left\langle N \right\rangle = \left\langle N \right\rangle_1$.

Proof. It is clear that $\tilde{\gamma_N} \circ \eta_M|_N = i_N$. In particular the inclusion i_N factors as

$$N \rightarrow \mathrm{Im}(\gamma_N) \rightarrow M$$

We let $j_{\left\langle N \right\rangle}$ be the composite

$$N \to \operatorname{Im}(\gamma_N) = \left\langle N \right\rangle_1 \to \lim_{\to} \left\langle N \right\rangle_k$$

It follows from the construction of the map ϕ_M that $\phi_M \circ j_{\left\langle N \right\rangle} = j_M$. Suppose there is a commutative diagram

$$\begin{array}{ccc} N & \xrightarrow{j_{N'}} & N' \\ & {\scriptstyle j_M} \searrow \quad \swarrow {\scriptstyle i_{N'}} & \\ & M & \end{array}$$

with all maps being monomorphisms and N' being a sub-$\mathfrak{P}$-module of M. Then the following diagram commutes

$$\begin{array}{ccccc} \mathfrak{P} \circ (\mathfrak{P}; N) & \xrightarrow{Id_{\mathfrak{P}} \circ (Id_{\mathfrak{P}}; j_{N'})} & \mathfrak{P} \circ (\mathfrak{P}; N') & \xrightarrow{\gamma'_N} & N' \\ & \searrow & & & \downarrow {\scriptstyle i_{N'}} \\ & & & & M \end{array}$$

This implies that $\left\langle N \right\rangle_1 = \operatorname{Im}(\gamma_N) \hookrightarrow M$ factors (uniquely) through $i_{N'}$. To prove the second claim we check that $\left\langle N \right\rangle_1 = \operatorname{Im}(\gamma_N)$ is a sub-module. To do this we must show that the map $\gamma_{\left\langle N \right\rangle_1}$ factors through $\left\langle N \right\rangle_1$. The map $\mathfrak{P} \circ (\mathfrak{P}; N)) \to \left\langle N \right\rangle_1$ is an epimorphism since $\mathcal{E}$ is quasi-abelian. Therefore the map $\mathfrak{P} \circ (\mathfrak{P}; \mathfrak{P} \circ (\mathfrak{P}; N)) \to \mathfrak{P} \circ (\mathfrak{P}; \left\langle N \right\rangle_1))$ is an epimorphism. Moreover, the following diagram commutes

$$\begin{array}{ccc} \mathfrak{P} \circ (\mathfrak{P}; \mathfrak{P} \circ (\mathfrak{P}; N)) & \xrightarrow{Id_{\mathfrak{P}} \circ (Id_{\mathfrak{P}}; \gamma_N)} & \mathfrak{P} \circ (\mathfrak{P}; M) \\ \downarrow & & \downarrow {\scriptstyle \gamma_M} \\ \mathfrak{P} \circ (\mathfrak{P}; \left\langle N \right\rangle_1)) & \xrightarrow{\gamma_{\left\langle N \right\rangle_1}} & M \end{array}$$

Thus it is sufficient to prove that the image of the composition

$$\mathfrak{P} \circ (\mathfrak{P}; \mathfrak{P} \circ (\mathfrak{P}; N)) \xrightarrow{Id_{\mathfrak{P}} \circ (Id_{\mathfrak{P}}; \gamma_N)} \mathfrak{P} \circ (\mathfrak{P}; M) \xrightarrow{\gamma_M} M$$

factors through $\operatorname{Im}(\tilde{\gamma_N})$. But we also have the following commutative diagram.

$$\begin{array}{ccccc} \mathfrak{P} \circ (\mathfrak{P}; \mathfrak{P} \circ (\mathfrak{P}; N))) & & \xrightarrow{Id_{\mathfrak{P}} \circ (Id_{\mathfrak{P}}; \gamma_N)} & & \mathfrak{P} \circ (\mathfrak{P}; M) \\ \downarrow {\scriptstyle \sim} & & & & \downarrow {\scriptstyle \gamma_M} \\ (\mathfrak{P} \circ \mathfrak{P}) \circ (\mathfrak{P}; N) & \xrightarrow{\mu_{\mathfrak{P}} \circ (Id; Id)} & \mathfrak{P} \circ (\mathfrak{P}; N) & \xrightarrow{\gamma_N} & M \end{array}$$

This proves the claim. □

B.3 Algebras Over Operads

Definition B.3.1. *Let* $(\mathfrak{P}, \gamma, \eta)$ *be an operad. A* $\mathfrak{P}$-***algebra*** *is a pair* (A, γ_A) *where* A *of* $\mathcal{E}$ *and* $\gamma_A : \mathfrak{P}(A) \to A$ *is a morphism in* $\mathcal{E}$ *such that the following diagrams commute*

$$\begin{array}{ccccc} (\mathfrak{P}\circ\mathfrak{P})(A) & = & \mathfrak{P}(\mathfrak{P}(A)) & \xrightarrow{\mathfrak{P}(\gamma_A)} & \mathfrak{P}(A) \\ \downarrow{\scriptstyle \gamma(A)} & & & & \downarrow{\scriptstyle \gamma_A} \\ \mathfrak{P}(A) & & \xrightarrow{\quad\gamma_A\quad} & & A \end{array}$$

$$\begin{array}{ccc} I(A) & \xrightarrow{\eta(A)} & \mathfrak{P}(A) \\ & \searrow\!\!\!\!= & \downarrow{\scriptstyle \gamma_A} \\ & & A \end{array}$$

Definition B.3.2. *Let* $\mathfrak{C}$ *be a cooperad. A* $\mathfrak{C}$-***coalgebra*** *is an object* C *of* $\mathcal{E}$ *together with a map* $\Delta_C : C \to \mathfrak{C} \circ C$ *such that* (C^{op}, Δ_C^{op}) *is a* $\mathfrak{C}^{op}$ *algebra in the category* $\mathcal{E}^{op}$.

There are obvious notions of morphisms of algebras and coalgebras, giving categories $\mathcal{Alg}_{\mathfrak{P}}$ and $co\mathcal{Alg}_{\mathfrak{P}}$.

B.3.1 Modules Over Algebras and Ideals

Definition B.3.3. *Let* $(\mathfrak{P}, \gamma, \eta)$ *be an operad and* (A, γ_A) *an algebra over* $\mathfrak{P}$. *A* ***module over*** A *together with maps* $\gamma_M : \mathfrak{P} \circ (A; M) \to M$ *and* $\eta_M : M \to \mathfrak{P} \circ (A; M)$ *such that the following diagrams commute*

$$\begin{array}{ccccc} (\mathfrak{P}\circ\mathfrak{P})\circ(A;M) & \xrightarrow{\sim} & \mathfrak{P}\circ(\mathfrak{P}(A);\mathfrak{P}\circ(A;M)) & \xrightarrow{Id\circ(\gamma_A;\gamma_M)} & \mathfrak{P}\circ(A;M) \\ \downarrow{\scriptstyle \gamma\circ(Id;Id)} & & & & \downarrow{\scriptstyle \gamma_M} \\ \mathfrak{P}\circ(A;M) & & \xrightarrow{\quad\gamma_M\quad} & & M \end{array}$$

$$\begin{array}{ccc} M & \xrightarrow{\eta_M} & \mathfrak{P}\circ(A;M) \\ & \searrow\!\!\!\!= & \downarrow{\scriptstyle \gamma_M} \\ & & M \end{array}$$

Note that A is canonically a module over itself. An **ideal** of A is a module I over A which is a submodule of A regarded as a module over itself.

Definition B.3.4. *Let* M *be an* A*-module and* $N \hookrightarrow M$ *a (regular) subobject in the ambient quasi-abelian category* $\mathcal{E}$. *A (regular) sub-*A*-module* $K \hookrightarrow M$ *is said to be* ***generated by*** N *if the inclusion* $N \hookrightarrow M$ *factors through* $K \hookrightarrow M$, *and* K *is universal with this property.*

Proposition B.3.5. *Let M be an A-module and $N \hookrightarrow M$ a subobject in the ambient quasi-abelian category. Consider the composition*

$$\gamma_N = \mathfrak{P}(A;N) \to \mathfrak{P}(A;M) \to M$$

Then $Im(\gamma_N)$ is athe regular submodule of M generated by N.

Proof. Similar (in fact easier than) Proposition B.2.16. □

B.3.2 Derivations

For most of this section we follow [2]

Definition B.3.6. *Let $\mathfrak{P}$ be an operad, A a $\mathfrak{P}$-algebra and M an A-module. Then the **square zero extension** of A by M is the $\mathfrak{P}$-algebra $A \ltimes M$ whose underlying object is $A \oplus M$ and whose algebra structure is given by the composite*

$$\mathfrak{P}(A \oplus M) \to \mathfrak{P}(A) \oplus \mathfrak{P}(A;M) \to A \oplus M$$

The algebra $A \ltimes M$ is equipped with a map of algebras $A \ltimes M \to A$ which on underlying objects is just the projection $A \oplus M \to A$. This gives an object $A \ltimes M \to A$ in $\mathcal{A}lg_{\mathfrak{P}}/A$.

Definition B.3.7. *Let B be an object in $\mathcal{A}lg_{\mathfrak{P}}/A$ and let M be an A module. The abelian group of **derivations** from B to M is*

$$Der_{\mathfrak{P},A}(B,M) = Hom_{\mathcal{A}lg_{\mathfrak{P}}/A}(B, A \ltimes M)$$

Proposition B.3.8. *An element of $Der_{\mathfrak{P},A}(B,M)$ is determined by a morphism $D : B \to M$ in $\mathcal{E}$ so that the following diagram commutes*

$$\begin{array}{ccccc}
\mathfrak{P}(B) & \xrightarrow{\Delta} & \mathfrak{P} \circ (B;B) & \xrightarrow{Id_{\mathfrak{P}} \circ (Id_B, D)} & \mathfrak{P} \circ (B,M) \\
\downarrow{\scriptstyle \gamma_B} & & & & \downarrow{\scriptstyle \gamma_M} \\
B & & \xrightarrow{\quad D \quad} & & M
\end{array}$$

In the case that $A = \mathfrak{P}(V)$ is a free algebra one can show the following (see [48])

Proposition B.3.9. *Let $V \in \mathcal{E}$. Suppose we are given a map $\phi : V \to \mathfrak{P}(V)$. There is a unique derivation d_ϕ on the free $\mathfrak{P}$-algebra $\mathfrak{P}(V)$ which extends ϕ. It is given by the formula*

$$d_\phi = d_{\mathfrak{P}} \circ Id_V + (\gamma_{(1)} \circ Id_V)(Id_{\mathfrak{P}} \circ' V)$$

where we have used the isomorphism

$$\mathfrak{P} \circ (V; \mathfrak{P}(V)) \cong (\mathfrak{P} \circ_{(1)} \mathfrak{P})(V)$$

In fact, by Observation.4.4.2, [25] one has

Proposition B.3.10. *Let $V \in \mathcal{E}$. Then there is a natural isomorphism*

$$Hom(V, M) \cong Der_{\mathfrak{P},\mathfrak{P}(0)}(\mathfrak{P}(V), M)$$

Proposition B.3.11. *Let A and B be $\mathfrak{P}$-algebras and M an $A \coprod B$-module. In particular it is both an A-module and a B-module. Suppose $d : A \to M$ and $\delta : B \to M$ are derivations. There is a unique derivation $d + \delta : A \coprod B \to M$ whose restriction to A is d and whose restriction to B is δ.*

Proof. $A \coprod B$ can be constructed as a quotient of $\mathfrak{P}(A \oplus B)$. There is a map of Σ-modules $\overline{d + \delta} : A \oplus B \to M$ which uniquely extends to a derivation $\mathfrak{P}(A \oplus B) \to M$. One checks that this descends to a map $A \coprod B \to M$. □

B.3.3 Twists by Coalgebras

Throughout this section $\mathfrak{P}$ will be a non-symmetric operad and (C, Δ, ϵ) a coassociative coalgebra. This section provides a useful formalism for dealing with homotopies of $\mathfrak{P}$-algebras.

Let us define maps

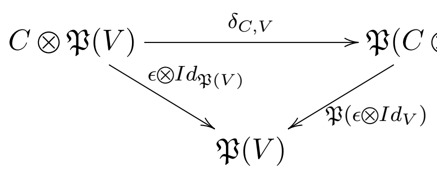
$$\delta_{C;V}(k) : C \otimes \mathfrak{P}(k) \otimes V^{\otimes k} \to \mathfrak{P}(k) \otimes (C \otimes V)^{\otimes k}$$

The symmetric monoidal structure gives a canonical isomorphism

$$\sigma : C^{\otimes k} \otimes \mathfrak{P}(k) \otimes V^{\otimes k} \cong \mathfrak{P}(k) \otimes (C \otimes V)^{\otimes k}$$

The isomorphism does not swap any of the Cs past each other. We now define $\delta_{C;V}(k)$ to be $\sigma \circ \Delta^{k-1}$. So we get a map $\delta_{C,V} : C \otimes \mathfrak{P}(V) \to \mathfrak{P}(C \otimes V)$. It is natural in C and V.

Proposition B.3.12. *Let $\mathfrak{P}$ be a non-symmetric operad, V an object of $\mathcal{E}$, and (C, Δ, ϵ) a coassociative coalgebra. The following diagram commutes*



Definition B.3.13. *Let A and B be $\mathfrak{P}$-algebras and C a coalgebra. A C-**twisted morphism** of $\mathfrak{P}$-algebras from A to B is a map*

$$\phi : C \otimes A \to B$$

such that the following diagram commutes

$$\begin{array}{ccccc}
C \otimes \mathfrak{P}(A) & & \xrightarrow{Id_C \otimes \gamma_A} & & C \otimes A \\
\big\downarrow{\scriptstyle \delta_{C,A}} & & & & \big\downarrow{\scriptstyle \phi} \\
\mathfrak{P}(C \otimes A) & \xrightarrow{\mathfrak{P}(\phi)} & \mathfrak{P}(B) & \xrightarrow{\gamma_B} & B
\end{array}$$

Proposition B.3.14. *Let C be a coassociative coalgebra, V, W objects of $\mathcal{E}$ and $h : C \otimes V \to W$ a map. Then the map $\mathfrak{P}(h) \circ \delta_{C,V} : C \otimes \mathfrak{P}(V) \to \mathfrak{P}(W)$ is a C-twisted map of $\mathfrak{P}$-algebras from $\mathfrak{P}(V)$ to $\mathfrak{P}(W)$.*

Proof. Let $\psi : \mathfrak{P}(C \otimes \mathfrak{P}(V)) \to \mathfrak{P}(W)$ denote the unique map of algebras extending $\mathfrak{P}(h) \circ \delta_{C,V}$. The coassociativity of C implies that the following diagram commutes

$$\begin{array}{ccccc}
C \otimes (\mathfrak{P}(\mathfrak{P}(V))) & & \xrightarrow{\delta_{C,\mathfrak{P}(V)}} & & \mathfrak{P}(C \otimes \mathfrak{P}(V)) \\
\big\downarrow{\scriptstyle Id \otimes \gamma_{\mathfrak{P}(V)}} & & & & \big\downarrow{\scriptstyle \psi} \\
C \otimes \mathfrak{P}(V) & \xrightarrow{\delta_{C,V}} & \mathfrak{P}(C \otimes V) & \xrightarrow{\mathfrak{P}(h)} & \mathfrak{P}(W)
\end{array}$$

which says precisely that $\mathfrak{P}(h) \circ \delta_V$ is a C-twisted morphism. □

Proposition B.3.15. *Let $\mathcal{E}$ be a closed monoidal quasi-abelian category, A and B be $\mathfrak{P}$-algebras, C a coassociative coalgebra, and suppose that $H : C \otimes A \to B$ is a C-twisted morphism of $\mathfrak{P}$-algebras. Let $K \hookrightarrow A$ be a subobject of A and $L \to B$ an admissible sub-algebra of B. If $H|_{C \otimes K}$ factors through L, then so does $H|_{C \otimes \left\langle K \right\rangle}$.*

Proof. First note that we have the following commutative diagram

$$\begin{array}{ccccccc}
C \otimes \mathfrak{P}(K) & \longrightarrow & C \otimes \left\langle K \right\rangle & \longrightarrow & C \otimes A & \longrightarrow & B \\
\big\downarrow & & & & & & \big\| \\
\mathfrak{P}(C \otimes K) & \longrightarrow & \mathfrak{P}(L) & \longrightarrow & L & \longrightarrow & B
\end{array}$$

Since in quasi-abelian category any map is an epimorphism onto its image, and the tensor product preserves cokernels, the first horizontal map is an epimorphism. Therefore the image of the top horizontal map coincides with the image of the composition

$$C \otimes \left\langle K \right\rangle \to C \otimes A \to B$$

But by the commutativity of the diagram the image of the top map factors through the image of $L \to B$ which is L, since L is a regular subobject. □

B.4 (Co)Operads in Chain Complexes

In this section we let $(\mathcal{E}, \otimes, k)$ be a monoidal additive category with kernels and cokernels, and consider operads and co-operads in the monoidal additive category $Ch(\mathcal{E})$.

B.4.1 The Convolution Operad

Let $(\mathfrak{C}, \Delta, \epsilon)$ be a cooperad and $(\mathfrak{P}, \gamma, \eta)$ be an operad in $\mathcal{E}$, and consider the collection in $Ch(\mathcal{Ab})$ given by

$$\underline{\mathrm{Hom}}(\mathfrak{C}, \mathfrak{P})(n) = \mathrm{Hom}_{\mathcal{E}}(\mathfrak{C}(n), \mathfrak{P}(n))$$

It is a (right) Σ module in $Ch(\mathcal{Ab})$. For $f \in \mathrm{Hom}_{\mathcal{E}}(\mathfrak{C}(n), \mathfrak{P}(n))$ $\sigma \in \Sigma_n$ acts on the right by

$$f \mapsto \sigma \circ f \circ \sigma^{-1}$$

Using the methods of [48] Section 6.4.1 this Σ-module in $Ch(\mathcal{Ab})$ can be made into an operad, called the **convolution operad** of $\mathfrak{C}$ and $\mathfrak{P}$. Recall that to any *dg*-operad $\mathfrak{P}$ there is an associated dg pre-Lie algebra whose underlying differentially graded abelian group is $\prod_n \mathfrak{P}(n)$. Denote by

$$\mathrm{Hom}_{\Sigma}(\mathfrak{C}, \mathfrak{P}) := \prod_{n \geqslant 0} \mathrm{Hom}_{\Sigma_n}(\mathfrak{C}(n), \mathfrak{P}(n))$$

the subobject of $\prod_{n \geqslant 0} \mathrm{Hom}(\mathfrak{C}(n), \mathfrak{P}(n))$ consisting of Σ-equivariant maps.

Proposition B.4.1. *$Hom_{\Sigma}(\mathfrak{C}, \mathfrak{P})$ is a sub dg pre-Lie algebra of the dg pre-Lie algebra associated to the convolution operad.*

Definition B.4.2. *$Hom_{\Sigma}(\mathfrak{C}, \mathfrak{P})$ is called the **convolution dg pre-Lie algebra**.*

B.4.2 Twisting Morphisms and Twisted Composite Products

Details for the next two sections can be found in Chapter 11 of [48]. Let $\mathfrak{C}$ be a cooperad and $\mathfrak{P}$ an operad. We consider the convolution operad $\underline{\mathrm{Hom}}(\mathfrak{C}, \mathfrak{P})$ in $Ch(\mathcal{Ab})$.

Definition B.4.3. *A **twisting morphism** is a Maurer-Cartan element in the convolution dg pre-Lie algebra $Hom_{\Sigma}(\mathfrak{C}, \mathfrak{P})$.*

Typically the twisting morphisms we consider on additive categories are induced from ones in $Ch(\mathcal{Ab})$ (or $Ch({}_{\mathbb{Q}}\mathcal{Mod})$). To this end we note the following result.

Proposition B.4.4. *Let $\mathcal{D}$ and $\mathcal{E}$ be monoidal additive categories and $F : \mathcal{D} \to \mathcal{E}$ a strict monoidal functor. Let $\alpha : \mathfrak{C} \to \mathfrak{P}$ be a twisting morphism in $Ch(\mathcal{E})$. Then $F(\alpha) : F(\mathfrak{C}) \to F(\mathfrak{P})$ is a twisting morphism.*

Proof. This follows from the fact that F induces a homomorphism of convolution Lie algebras, and homomorphisms of Lie algebras send Maurer-Cartan elements to Maurer-Cartan elements. $\square$

Let $\alpha : \mathfrak{C} \to \mathfrak{P}$ be a degree -1 morphism. Denote by d^r_α the unique derivation on $\mathfrak{C} \circ \mathfrak{P}$ which extends the composition

$$\mathfrak{C} \xrightarrow{\Delta_{(1)}} \mathfrak{C} \circ_{(1)} \mathfrak{C} \xrightarrow{Id_{\mathfrak{C}} \circ_{(1)} \alpha} \mathfrak{C} \circ_{(1)} \mathfrak{P} \longrightarrow \mathfrak{C} \circ \mathfrak{P}$$

Denote by d^l_α the unique derivation on $\mathfrak{P} \circ \mathfrak{C}$ the unique derivation which extends

$$\mathfrak{C} \xrightarrow{\Delta} \mathfrak{C} \circ \mathfrak{C} \xrightarrow{\alpha \circ Id_{\mathfrak{C}}} \mathfrak{P} \circ \mathfrak{C}$$

Lemma B.4.5. *On $\mathfrak{P} \circ \mathfrak{C}$ the derivation d_α satisfies*

$$d^2_\alpha = d^l_{\partial(\alpha)+\alpha\star\alpha}$$

On $\mathfrak{C} \circ \mathfrak{P}$ the derivation d_α satisfies

$$d^2_\alpha = d^r_{\partial(\alpha)+\alpha\star\alpha}$$

Corollary B.4.6. *If α is a twisting morphism then*

$$\mathfrak{P} \circ_\alpha \mathfrak{C} := (\mathfrak{P} \circ \mathfrak{C}, d_\alpha)$$

and

$$\mathfrak{C} \circ_\alpha \mathfrak{P} := (\mathfrak{C} \circ \mathfrak{P}, d_\alpha)$$

are chain complexes.

Finally we denote by $\mathfrak{P} \circ_\alpha \mathfrak{C} \circ_\alpha \mathfrak{P}$ the complex whose underlying graded Σ-module is given by $\mathfrak{P} \circ \mathfrak{C} \circ \mathfrak{P}$, with differential given by $d_{\mathfrak{P}\circ\mathfrak{C}\circ\mathfrak{P}} + Id_{\mathfrak{P}} \circ' d^r_\alpha - d^l_\alpha \circ Id_{\mathfrak{P}}$. This is called the **two-sided twisted composite product**. It is a complex by [48].

B.4.3 Bar and Cobar Constructions for Algebras

Let $\alpha : \mathfrak{C} \to \mathfrak{P}$ be a twisting morphism. There is an construct an adjoint pair of functors.

$$\Omega_\alpha \colon co\mathcal{Alg}^{nil}_{\mathfrak{C}} \rightleftarrows \mathcal{Alg}_{\mathfrak{P}} : B_\alpha$$

Let C be a $\mathfrak{C}$-coalgebra. The underlying graded algebra of $\Omega_\alpha C$ is the free algebra $\mathfrak{P}(C)$ on C. The differential however is augmented using the twisting morphism. Namely, it is the sum of the derivations $d_1 = -d_{\mathfrak{P}} \circ C$ and d_2, where $-d_2$ is the unique derivation extending the map

$$C \xrightarrow{\Delta} C \circ C \xrightarrow{\alpha \circ Id_C} \mathfrak{P} \circ C$$

Proposition B.4.7. *There is a natural isomorphism of algebras with derivations*

$$(\mathfrak{P}(C), -d_1 - d_2) \cong ((\mathfrak{P} \circ_\alpha \mathfrak{C}) \circ^{\mathfrak{C}} C, d_\alpha)$$

In particular $d_1 + d_2$ is a square-zero derivation.

Definition B.4.8. $(\mathfrak{P}(C), d_1 + d_2)$ *is called the* ***cobar construction of*** A ***with respect to*** α.

Now let A be a $\mathfrak{P}$-algebra. The underlying graded algebra of $B_\alpha A$ is the co-free co-algebra $\mathfrak{C}(A)$. Denote by d_1 the square zero coderivation $d_{\mathfrak{C}} \circ Id_A$. There is a unique coderivation d_2 extending the degree -1 map

$$\mathfrak{C} \circ A \xrightarrow{\alpha \circ Id_A} \mathfrak{P} \circ A \xrightarrow{\gamma_A} A$$

Proposition B.4.9. *There is a natural isomorphism*

$$(\mathfrak{C}(A), d_1 + d_2) \cong ((\mathfrak{C} \circ_\alpha \mathfrak{P}) \circ_{\mathfrak{P}} A, d_\alpha)$$

In particular $d_1 + d_2$ is a square zero coderivation.

Definition B.4.10. $B_\alpha A := (\mathfrak{C}(A), d_1 + d_2)$ *is called the* ***bar construction of*** A ***with respect to*** α.

Proposition B.4.11. *There is an adjunction*

$$\Omega_\alpha \colon co\mathcal{A}lg^{nil}_{\mathfrak{C}} \rightleftarrows \mathcal{A}lg_{\mathfrak{P}} : B_\alpha$$

In the context of the bar-cobar adjunction it is convenient to introduce the following operad.

Definition B.4.12. *The* ***shifting cooperad****, denoted $\mathfrak{S}^c$, is the cooperad with $\mathfrak{S}(n) = S^{n-1}(k)$. The* ***shifting operad*** *$\mathfrak{S}$ is the dual operad of $\mathfrak{S}$, with $\mathfrak{S}(n) = S^{-n+1}(k)$.*

By [48] Page 187 we have the following.

Proposition B.4.13. *For any complex $V \in Ch(\mathcal{E})$ and any Σ-module $\mathfrak{P}$ there is an isomorphism, natural in V,*

$$\mathfrak{P}(V)[1] \cong (\mathfrak{S} \otimes_H \mathfrak{P})(V[1])$$

In particular in $\mathfrak{P}$ is an operad the shift functor induces an equivalence of categories

$$[1] : \mathfrak{Alg}_{\mathfrak{P}} \to \mathfrak{Alg}_{\mathfrak{S} \otimes_H \mathfrak{P}}$$

Similarly If $\mathfrak{C}$ is a co-operad the shift functor induces an equivalence of categories.

$$[-1] : \mathfrak{coAlg}_{\mathfrak{C}} \to \mathfrak{coAlg}_{\mathfrak{S}^c \otimes_H \mathfrak{C}}$$

B.4.4 Non-Symmetric Operads

There is a non-symmetric version of the theory detailed above, where we just consider $\mathbb{N}$-graded objects of $\mathcal{E}$ without any Σ-action. Recall from Section 3.3 this is a monoidal category.

Definition B.4.14. *Let M and N be graded objects. The* ***non-symmetric composite product*** *of M and N, denoted $M \circ^{ns} N$ is the graded object given by*

$$(M \circ^{ns} N)(n) = \bigoplus_k M(k) \otimes N^{\otimes k}(n)$$

Definition B.4.15. *The graded object I is defined by $M(i) = 0$ for $i \neq 1$ and $M(1) = k$.*

With these the preceding constructions on results for (symmetric) operads go through mutatis mutandis. We denote the category of non-symmetric operads by $\mathcal{O}p_{ns}$.

As with any module category there is an adjunction

$$k[\Sigma] \otimes (-) \colon \mathcal{G}rad(\mathcal{E}) \rightleftarrows \Sigma - \mathcal{M}od : | - |$$

which induces an adjunction

$$(-)^{\Sigma} \colon \mathcal{O}p_{ns} \rightleftarrows \mathcal{O}p : | - |$$

Appendix C

Holomorphic Functions on Banach Spaces

In this appendix we recall some notions from the theory of holomorphic functions on Banach spaces of possibly infinite dimension. We then introduce contracting and multiplicative holomorphy types before giving a description of algebras of holomorphic functions after embedding them in the category $Pro(Ban)$ introduced in Section 2.3.1. Our main references are [56] and [55]. Proofs of all unproven claims can be found there. We will consider the category $Ban_{\mathbb{C}}$ of Banach spaces over $\mathbb{C}$ with its projective closed monoidal structure.

C.1 Definitions and Basic Properties

If E is a Banach space space there is for each $n \in \mathbb{N}$ a continuous map

$$\Delta_E^n : E \to E^{\otimes n}, \; e \mapsto e^{\otimes n}$$

which is natural in the following sense. Whenever $f : E \to F$ is a linear map the following diagram commutes

$$\begin{array}{ccc} E & \xrightarrow{\Delta_E^n} & E^{\otimes n} \\ \downarrow f & & \downarrow f^{\otimes n} \\ F & \xrightarrow{\Delta_F^n} & F^{\otimes n} \end{array}$$

Let $A \in Hom_{Ban}(E^{\otimes m}, F)$. Denote by $Ax^m : E \to F$ the map of topological spaces given by the composition $A \circ \Delta_E^m$.

Definition C.1.1. *A* ***continuous homogeneous polynomial of degree*** *M from E to F is a map $P : E \to F$ of the form Ax^m for some $A \in Hom_{Ban}(E^{\otimes m}, F)$. We then write $P = \hat{A}$. The space of such maps is denoted $\mathcal{P}^m(E, F)$.*

We endow this space with the topology of uniform convergence on bounded subsets of E, and denote the resulting topological vector space by $\mathcal{P}_\omega^m(E, F)$. It is a Banach space [23].

Let us now define holomorphic functions.

Definition C.1.2. *A **power series** from E to F about $\chi \in E$ is a sum of the form*

$$\sum_{m=0}^{\infty} P_m(x-\chi)$$

where $P_m \in \mathcal{P}^m(E,F)$.

Definition C.1.3. *Let E and F be Banach spaces and $U \subset E$ open. A mapping $f : U \to F$ is said to be **holomorphic** on U if for every $\xi \in U$ there is a power series*

$$\sum_{m=0}^{\infty} P_m(x-\chi)$$

from E to F about χ and some $\rho > 0$ such that $B_\rho(\xi) \subset U$, and

$$f(x) = \sum_{m=0}^{\infty} P_m(x-\chi)$$

converges uniformly on $B_\rho(\chi)$.

It can be shown that the sequence P_m is unique at every point χ. We then write

$$\hat{d}^m f(\chi) = m! P_m$$

As in the finite dimensional case we have the Cauchy integral theorem

Proposition C.1.4. *Let $f \in \mathcal{H}(U;F)$, $\chi \in U$, $x \in E$ and $\rho > 0$ be such that $\chi + \lambda x \in U$ for every $\lambda \in \mathbb{C}$, $|\lambda| \leqslant \rho$. Then*

$$\frac{1}{m!}\hat{d}^m f(\chi)(x) = \frac{1}{2\pi i}\int_{|\lambda|=\rho} \frac{f(\chi+\lambda x)}{\lambda^{m+1}} d\lambda$$

for all $m \in \mathbb{Z}_{\geqslant 0}$.

which gives the Cauchy estimates

Corollary C.1.5.

$$\left|\left|\frac{1}{m!}\hat{d}^m f(\chi)\right|\right| \leqslant \frac{1}{\rho^m} sup_{||x-\chi||=\rho}||f(x)||$$

for all $m \in \mathbb{Z}_{\geqslant 0}$.

In infinite dimensions it is not necessarily true that the radius of convergence of the power series of a holomorphic function f at a point χ is the maximal radius of an open ball around χ on which f is holomorphic.

Definition C.1.6. *Let $f \in \mathcal{H}(U;F)$ and $\chi \in U$. The* ***radius of boundedness*** *of f at χ is the largest r, $0 < r \leqslant \infty$, such that $B_r(\chi) \subset U$ and f is bounded on every $\overline{B}_\rho(\chi)$, $0 \leqslant \rho < r$.*

We then have the following.

Proposition C.1.7. *Let $f \in \mathcal{H}(U;F)$ and $\chi \in U$. The radius of boundedness r_b of f at χ is the infimum of the radius of convergence r_c of the Taylor series of f at χ and the distance d of χ to the boundary of U,*

Definition C.1.8. *Let $U \subset E$ be an open set. $A \subset U$ is said to be U-**bounded** if A is bounded and $d(A, E\backslash U) > 0$.*

Definition C.1.9. *A mapping $f \in \mathcal{H}(U;F)$ is said to be of* ***bounded type*** *if f is bounded on all U-bounded sets. The linear subspace of all such holomorphic mappings is denoted by $\mathcal{H}^b(U;F)$.*

Remark C.1.10. *The radius of convergence of the power series of a holomorphic function f of bounded type at a point χ is the maximal radius of an open ball around χ on which f is holomorphic.*

We endow $\mathcal{H}^b(E,F)$ with the **topology of uniform convergence on bounded sets**. It is defined by the following system of semi-norms. For each integer n, let B_n denote the open ball around 0 of radius n. Then the seminorm ρ_{B_n} is

$$\rho_{B_n}(f) = \sup_{x \in B_n} ||f(x)||_E$$

We denote this topology by ω.

C.2 Holomorphy Types

It is useful to consider more manageable subspaces of the space of all holomorphic functions between Banach spaces.

Definition C.2.1. *Let E and F be Banach spaces. A* ***holomorphy type*** *Θ from E to F is for each n a linear subspace $\mathcal{P}^n_\Theta(E,F)$ of $\mathcal{P}^n(E,F)$ which is a Banach space when equipped with a norm $|| - ||_\Theta$, such that*

1. *$\mathcal{P}^0_\Theta(E;F) = F$ as a normed linear space.*
2. *There is a $\sigma \geqslant 1$ such that for any $n \in \mathbb{N}$, $k \leqslant n$, $a \in E$ and $P \in \mathcal{P}^n_\Theta(E;F)$, the kth differential $\hat{d}^k P(a)$ is in $\mathcal{P}^k_\Theta(E;F)$ and it satisfies*

$$\left|\left|\frac{1}{k!}\hat{d}^k P(a)\right|\right|_\Theta \leqslant \sigma^n ||P||_\Theta ||a||^{n-k}$$

Example C.2.2. *The **current holomorphy type** is given by* $(\mathcal{P}_\Theta(E^n, F), || - ||_\Theta) = \mathcal{P}_\omega(E^n, F)$.

Definition C.2.3. *Let E and F be Banach spaces, $U \subset E$ open and Θ a holomorphy type from E to F. $f \in \mathcal{H}(U; F)$ is said to be of Θ-**holomorphy type** at $\xi \in U$ if*

1. $\hat{d}^m f(\xi) \in \mathcal{P}^m_\Theta(E, F)$ *for* $m \in \mathbb{N}$
2. *There are real numbers $C \geqslant 0$ and $c \geqslant 0$ such that*

$$\Big|\Big|\frac{1}{m!}\hat{d}^m f(\xi)\Big|\Big|_\Theta \leqslant C.c^m$$

   for $m \in \mathbb{N}$.

*f is said to be of Θ-**holomorphy type** on U if it is of Θ-holomorphy type at every point of U. The linear subspace consisting of all such functions is denoted $\mathcal{H}_\Theta(E, F)$.*

We're going to define a topology on $\mathcal{H}_\Theta(U, F)$.

Proposition C.2.4. *Let p be a seminorm on $\mathcal{H}_\Theta(U; F)$ and K be a compact subset of U. Then the following conditions are equivalent*

1. *Given any real number $\epsilon > 0$ we can find a real number $c(\epsilon) > 0$ such that*

$$p(f) \leqslant c(\epsilon) \sum_{m=0}^{\infty} \epsilon^m sup_{x \in K} \Big|\Big|\frac{1}{m!}\hat{d}^m f(x)\Big|\Big|_\Theta$$

   for every $f \in \mathcal{H}_\Theta(U; F)$.

2. *Given any real number $\epsilon > 0$ and any open subset V of U containing K, we can find a real number $c(\epsilon, V) > 0$ such that*

$$p(f) \leqslant c(\epsilon, V) \sum_{m=0}^{\infty} \epsilon^m sup_{x \in V} \Big|\Big|\frac{1}{m!}\hat{d}^m f(x)\Big|\Big|_\Theta$$

   for every $f \in \mathcal{H}_\Theta(U; F)$.

Definition C.2.5. *A seminorm on $\mathcal{H}_\Theta(U; F)$ is said to be **ported** by a compact subset K of U if the equivalent conditions of the preceding proposition are satisfied. The topology generated by all such seminorms is denoted $\mathcal{I}_{\omega,\Theta}$.*

We are now going to define a notion of holomorphic functions of Θ-bounded type.

Definition C.2.6. *A function $f \in \mathcal{H}_\Theta(E,F)$ is said to be of Θ-**bounded type** if for each $n \in \mathbb{N}$ there is a $C_n \geqslant 0$ such that for each $m \in \mathbb{N}$*

$$||\frac{1}{m!}\hat{d}^m f(0)||_\Theta \leqslant C_n \frac{1}{n^m}$$

We write

$$\rho_n(f) = inf\{C_n \in \mathbb{R} : ||\frac{1}{m!}\hat{d}^m f(0)||_\Theta \leqslant C_n \frac{1}{n^m}\ \forall m \geqslant 0\}$$

We define $\mathcal{H}^b_\Theta(E,F)$ to be the subspace of $\mathcal{H}_\Theta(E,F)$ consisting of functions of Θ-bounded type. The following is clear.

Proposition C.2.7. *ρ_n is a semi-norm on $\mathcal{H}^b_\Theta(E,F)$.*

We endow $\mathcal{H}^b_\Theta(E,F)$ with the topology generated by the seminorms ρ_n.

Proposition C.2.8.

1. *Endow $\mathcal{H}_\Theta(E,F)$ with the topology $\mathcal{I}_{\omega,\Theta}$. Then the inclusion $\mathcal{H}^b_\Theta(E,F) \to \mathcal{H}_\Theta(E,F)$ is continuous.*
2. *Let $f \in \mathcal{H}^b_\Theta(E,F)$. Then under the inclusion $\mathcal{H}^b_\Theta(E,F) \to \mathcal{H}_\Theta(E,F) \to \mathcal{H}(E,F)$, f is in $\mathcal{H}^b(E,F)$. Moreover the inclusion $\mathcal{H}^b_\Theta(E,F) \hookrightarrow \mathcal{H}^b(E,F)$ is continuous.*
3. *Let Θ be the current holomorphy type. Then $\mathcal{H}^b_\Theta(E,F) = \mathcal{H}^b(E,F)$ as topological vector spaces.*

Proof. 1. Let p be a seminorm ported by a compact subset K and let $0 < \epsilon < 1$. Let $n \in \mathbb{Z}$ be such that $K \subset B_n(0)$. Then

$$||\frac{1}{m!}\hat{d}^m f(x)|| \leqslant \frac{\rho_n(f)}{n^m}$$

for all $m \in \mathbb{Z}$. Then for any $x \in B_n(0)$ and any $m \in \mathbb{Z}$,

$$||\frac{1}{m!}\hat{d}^m f(x)||_\Theta \leqslant \frac{\rho_n(f)}{1-\sigma}\left(\frac{\sigma}{n}\right)^m$$

Choose $r > \sigma$ so that the sequence $\left\{\left(\frac{\sigma}{n}\right)^m\right\}_m$ is summable.

$$\begin{aligned} p(f) &\leqslant c(\epsilon)\frac{\rho_n(f)}{1-\sigma}\sum_{m=0}^{\infty}\epsilon^m\left(\frac{\sigma}{n}\right)^m \\ &= \left(c(\epsilon)\frac{\sigma n}{(1-\sigma)(n-\sigma)}\right)\rho_n(f) \end{aligned}$$

2. Fix $r \in \mathbb{R}$ and let n be such that $r\sigma < n$. We have $||\frac{1}{m!}\hat{d}^m f(0)||_\Theta \leqslant \rho_n(f)\frac{1}{n^m}$. Therefore $||\frac{1}{m!}\hat{d}^m f(0)|| \leqslant \rho_n(f)\left(\frac{\sigma}{n}\right)^m$, and so for x with $||x||_E \leqslant r$, $||\frac{1}{m!}\hat{d}^m f(0)x^m|| \leqslant \rho_n(f)\left(\frac{r\sigma}{n}\right)^m$. So

$$\begin{aligned} \rho_{B_r}(f) &\leqslant \sum \rho_{B_r}(\frac{1}{m!}\hat{d}^m f(0)) \\ &\leqslant \frac{n}{n-r\sigma}\rho_n(f) \end{aligned}$$

3. By Part (2) the inclusion $\mathcal{H}^b_\Theta(E,F) \to \mathcal{H}^b(E,F)$ is continuous. Conversely let $f \in \mathcal{H}^b(E,F)$. Then clearly $f \in \mathcal{H}^b_\Theta(E,F)$ and $\rho_n(f) \leqslant \rho_{B_n}(f)$.

□

Corollary C.2.9. *There is a commutative diagram of topological spaces*

$$\begin{CD} \mathcal{H}^b_\Theta(E,F) @>>> \mathcal{H}_\Theta(E,F) \\ @VVV @VVV \\ \mathcal{H}^b(E,F) @>>> \mathcal{H}(E,F) \end{CD}$$

Next we introduce the notion of contracting holomorphy types.

Definition C.2.10. *Let E and F be Banach spaces. A* ***contracting holomorphy type*** *from E to F is for each $r \in (0,\infty)$ a holomorphy type Θ_r from E_r to F such that*

1. *For any $r,s > 0$ and $m \in \mathbb{N}$ there is an isomorphism*

$$\phi_{r,s} : \mathcal{P}^m_{\Theta_r}(E_r,F) \cong \mathcal{P}^m_{\Theta_s}(E_s,F)$$

such that the following diagram commutes

$$\begin{CD} \mathcal{P}^m_{\Theta_r}(E_r,F) @>>> \mathcal{P}^m(E_r,F) \\ @V{\phi_{r,s}}VV @VV{\psi_{r,s}}V \\ \mathcal{P}^m_{\Theta_s}(E_s,F) @>>> \mathcal{P}^m(E_s,F) \end{CD}$$

where $\psi_{r,s}$ is the isomorphism induced from the isomorphism of Banach spaces $Id_E : E_s \to E_r$

2. *For any $r,s > 0$ and $P \in \mathcal{P}_{\Theta_s}(E_s,F)$ one has*

$$||P||_{\Theta_s} = r^m ||\psi_{s,rs}(P)||_{\Theta_{rs}}$$

We clearly have the following result.

Proposition C.2.11. *Let Θ be a holomorphy type from E to F. For $r \in (0,\infty)$ define a holomorphy type Θ_r from E_r to F as follows. The space $\mathcal{P}^m_{\Theta_r}(E_r,F)$ is the (isomorphic) image of $\mathcal{P}^m_\Theta(E,F)$ under the map $\psi_{1,r} : \mathcal{P}^m(E,F) \to \mathcal{P}^m(E_r,F)$. The norm is*

$$||P||_{\Theta_r} = \frac{1}{r^m}||P||_\Theta$$

Then this defines a contracting holomorphy type from E to F.

Let E be a Banach space and F a Banach algebra.

Definition C.2.12. *Let E be a Banach space and F a Banach algebra. A holomorphy type Θ from E to F is said to be **mutliplicative** if for each $m, n \in \mathbb{N}$ the image of the restriction of the product map to $\mathcal{P}^m_\Theta(E,F) \times \mathcal{P}^n_\Theta(E,F)$ is in $\mathcal{P}^{m+n}_\Theta(E,F)$.*

Proposition C.2.13. 1. *The tame (defined in Definition 4.4.8) and current holomorphy types are contracting.*

2. *Suppose that F is a Banach algbera. The tame and current holomorphy types are multiplicative.*

Proposition C.2.14. *Let Θ be a contracting holomorphy type from E to F. There is a natural isomorphism*

$$\text{“}lim_{\leftarrow_{r\in\mathbb{N}}}\text{”}\hat{\bigoplus}_m \mathcal{P}^m_\Theta(E_{\frac{1}{\sigma r}}, F) \cong PB(\mathcal{H}^b_\Theta(E,F))$$

If F is a Banach algebra and Θ is multiplicative then this is an isomorphism of algebras.

Proof. Let $P_m \in \mathcal{P}^m(E,F)$. Then by assumption

$$\begin{aligned}\left|\left|\frac{1}{m!}\hat{d}^l P_m(0)\right|\right|_\Theta &\leqslant \sigma^m ||P_m||_\Theta \\ &= \frac{\sigma^m}{r^m}||P_m||_{\Theta_{\frac{1}{r}}} \\ &= \frac{||P_m||_{\Theta_{\frac{1}{\sigma r}}}}{r^m}\end{aligned}$$

Hence $\rho_r(P_m) \leqslant ||P||_{\Theta_{\frac{1}{\sigma r}}}$ This induces a map

$$\hat{\bigoplus_m} \mathcal{P}^m_\Theta(E_{\frac{1}{\sigma r}}, F) \to \mathcal{H}_\Theta(E,F)_{B_r}$$

and in turn a map

$$\text{“}\lim_{\leftarrow_{r\in\mathbb{N}}}\text{”}\hat{\bigoplus}_m \mathcal{P}^m_\Theta(E_{\frac{1}{\sigma r}}, F) \to PB(\mathcal{H}^b_\Theta(E,F))$$

If F is a Banach algebra and Θ is multiplicative then this map is clearly multiplicative. To construct an inverse map, consider a globally convergent power series

$$f(x) = \sum A_m x^m$$

For any $s, r \in \mathbb{R}$ one has

$$\begin{aligned}||A_m x^m||_{\Theta_{\frac{1}{\sigma s}}} &= \sigma^m s^m ||A_m x^m||_\Theta \\ &\leqslant \left(\frac{\sigma s}{r}\right)^m \rho_r(f)\end{aligned}$$

Thus for $\frac{\sigma s}{r} < 1$,

$$\sum ||A_m x^m|| \leqslant \frac{r}{r - \sigma s}\rho_r(f)$$

so we get a bounded map $\mathcal{H}^b_\Theta(E,F)_{B_r} \to \hat{\bigoplus}_m \mathcal{P}^m_\Theta(E_{\frac{1}{\sigma s}}, F)$. In the projective limit this gives an inverse. □

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