Radiality and spokes:

a structural theory of convergence

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Contents

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Abstract

This thesis is a wide-ranging investigation of convergence properties in topological spaces, primarily *Fréchet-Urysohn* and *radial* spaces. The former are spaces such that every point in a closure of a subset is the limit of a sequence from within that set. The latter is a generalisation, defined by replacing 'sequence' with 'transfinite sequence'. Although not all spaces have these properties, they form a large enough class to encompass many important examples of spaces.

These convergence properties can and should be studied *locally* and *structurally*. The first is achieved by removing the quantification over points in the definitions. For the second, we introduce the notion of *spokes* for points in topological spaces, which are subspaces for which the point has a descending neighbourhood base.

In Chapter I, we introduce several convergence properties, and recall how they are connected and characterised by particular quotient maps. We also introduce *p-character* and *quasi-isolation*, to give our results full generality by not assuming the T_1-axiom.

Our main focus is the development of the theory of spokes in Chapter II. Here, we study how spokes can be used to approximate the neighbourhood base of a *radial* point and how (transfinite) sequences converge. We prove several characterisation theorems for radial and Fréchet-Urysohn points and their relationship with *independently-based* points, which are described through *nests*. We also use spokes in productivity problems and variants of the Fréchet-Urysohn property.

In the final chapter, we demonstrate how properties of spokes manifest in different

settings. For example, in compactifications of locally-compact spaces, spoke structures at the point-at-infinity reflect into the compact structure of the original space. Other examples are obtained by dualities, characterising radiality in ring spectra or Stone spaces algebraically. Such results justify using internal structures to investigate convergence properties and the author wishes to continue this line of investigation for the foreseeable future.

Chapter I

A hierarchy of convergence properties

I.1 Introduction

Convergent sequences have played an important role since the beginnings of topology — not only are sequences abundant throughout analysis, some of the first abstract pre-topological structures were *Fréchet's L-spaces*, and later *Urysohn's L^*-spaces*, which axiomatise a relation between sequences and their formal limits. Although sequences are adequate (in a formal sense) to describe many spaces, they are insufficient for general topological spaces. A simple example is the space $\omega_1 + 1$ with the order topology, because $\omega_1 \in \overline{\omega_1}$, but there is no sequence in the *subspace* ω_1 converging to the *point* ω_1. A more important example would be $\beta\omega$, the Stone-Čech compactification of a countably infinite discrete space.

However, the classes of spaces determined by their sequential structure (the *Fréchet-Urysohn spaces*, and in a weaker sense, the *sequential spaces*) are prevalent enough to warrant independent study, in order to further our understanding of these properties and how they arise. In 1965, Stanley Franklin published his celebrated “Spaces in which sequences suffice” paper in which he showed that sequential spaces are the quotients of metrisable spaces and gave a proof of Alexander Arhangel’skiĭ’s claim that Fréchet-Urysohn spaces are the pseudo-open images of metrisable spaces [Fra65, Corollary 1.14, Proposition 2.4]. Even as topologists looked beyond metrisable and sequential spaces, the concrete picture given by these spaces remained appealing: in 1967, Horst Herrlich defined *pseudoradial* and *radial spaces* for the first time [Her67], which include the class of *linearly-ordered* and *generalised-ordered topological spaces*. These are the ‘spaces in which *transfinite* sequences suffice’.

In Section I.2 we will discuss what will be referred to as the *pseudoradial hierarchy* (Figure I.1), as well as preservation and external characterisation theorems for various classes of maps and spaces (Figures I.2, I.3). While new results for semiradial spaces that parallel known theorems for pseudoradial and radial spaces will be proven, but most of the results in this section are known.

In Section I.3 we introduce the notions of *p-character* and *quasi-isolation*, which work in tandem to describe the ‘depth’ of neighbourhood filters. In addition to proving some basic properties of these concepts, we will also use them in investigations of convergence properties which do not assume the T_1-axiom.

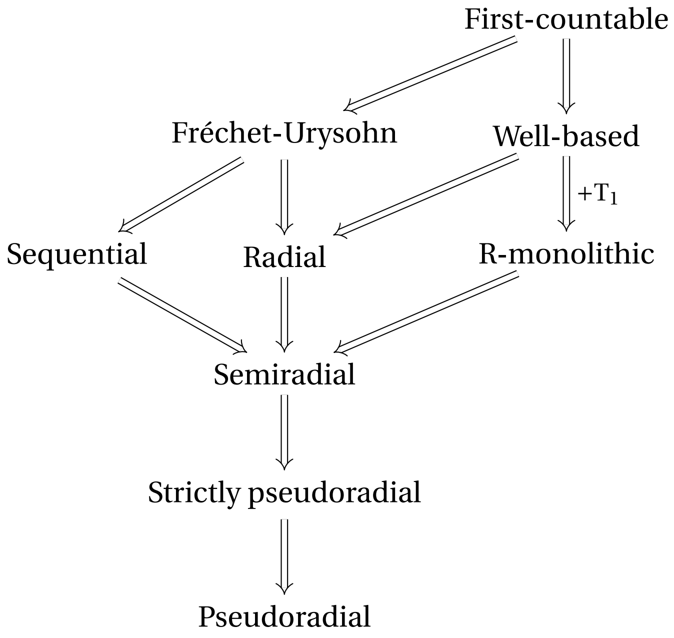

Figure I.1: The pseudoradial hierarchy.

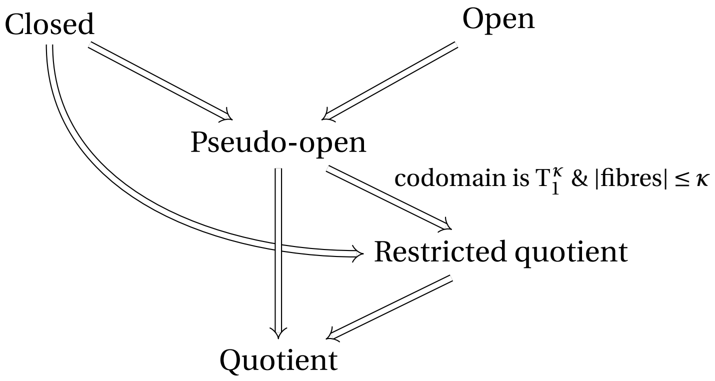

Figure I.2: Properties of continuous surjections.

Property of a space	Type of map that preserves and characterises the property as an image of a well-based space
Pseudoradial	Quotient
Semiradial	Restricted quotient
Radial	Pseudo-open

Figure I.3: Maps corresponding to the pseudoradial hierarchy.

I.1.1 Notation and definitions

We refer to [Eng89; Kun11] for general topology and set-theory background. For all sets X and Y, we denote the set of functions from X to Y by ${}^{X}Y$. For every set X and cardinal κ, define

$$\begin{aligned}[X]^{\leq\kappa} &:= \{Y \subseteq X : |Y| \leq \kappa\},\\ [X]^{<\kappa} &:= \{Y \subseteq X : |Y| < \kappa\},\\ [X]^{\kappa} &:= \{Y \subseteq X : |Y| = \kappa\}.\end{aligned}$$

A net whose domain is a non-zero ordinal λ is called a *transfinite sequence*, or more precisely a *λ-sequence*. We call the domain of a transfinite sequence t its *length*, denoted $\operatorname{len}(t)$. It is said to *lie* in a set A if its range is contained in A. We will often switch between a functional notation $t : \lambda \to X$ and a sequential notation $(x_\alpha)_{\alpha<\lambda}$ for transfinite sequences.

A transfinite sequence t in a topological space X is said to *converge* to a point $x \in X$ if it converges as a net; that is, for all neighbourhoods U of x, there exists an $\alpha < \operatorname{len}(t)$ such that $t[\operatorname{len}(t)\setminus\alpha] \subseteq U$. We call x a *limit* of t and denote this by $t \to x$. We say that a transfinite sequence in a subset A of a topological space X *escapes* A if it converges to a point outside of A. A *subsequence* of a transfinite sequence t is a transfinite sequence of the form $t \circ f$, where $f : \alpha \to \operatorname{len}(t)$ is a strictly increasing, cofinal function from some ordinal α. In particular, $t \circ f$ is a subnet of t, but not every subnet of a transfinite sequence is a subsequence.

Definition I.1.1 (Radially-closed)**.** Let X be a topological space and let $A \subseteq X$ be given. We say that A is *radially-closed* if it contains the limits of all transfinite sequences within A. If we only require this to hold for transfinite sequences of length $\leq\kappa$ / $<\kappa$, where κ is a cardinal, then we say that A is κ-radially-closed / $<\kappa$-radially-closed. The complements of radially-closed subsets are called *radially-open*.

Lemma I.1.2. *Let X be a topological space and let $U \subseteq X$ be given. Then U is radially-open if and only if for every transfinite sequence t in X, if U contains a limit of t, then $t[[\alpha, \operatorname{len}(t))] \subseteq U$ for some $\alpha < \operatorname{len}(t)$.*

Proof. Note that U is radially-open if and only if $X \backslash U$ is radially-closed, which in turn is equivalent to the limits of every transfinite sequence in $X \backslash U$ lieing in $X \backslash U$. So if U is radially-open and t is a transfinite sequence in X with limit $x \in U$, if $t[[\alpha, \operatorname{len}(t))] \not\subseteq U$ for all $\alpha < \operatorname{len}(t)$ then t has a subsequence in $X \backslash U$ that converges to x, which is a contradiction. Hence there exists an $\alpha < \operatorname{len}(t)$ such that $t[[\alpha, \operatorname{len}(t))] \subseteq U$. Now suppose that U is not radially-open, so there exists a transfinite sequence t in $X \backslash U$ that escapes. Then there exists an $x \in U$ such that $t \to x$ and for all $\beta < \operatorname{len}(t)$, $t[[\beta, \operatorname{len}(t))] \not\subseteq U$. □

Definition I.1.3 (LOTS and GO spaces)**.** A topological space X is a LOTS (*Linearly Ordered Topological Space*) if there exists a linear order $<$ on X such that $\{(-\infty, x) : x \in X\} \cup \{(x, \infty) : x \in X\}$ is a subbase for the topology on X. A subspace of a LOTS is called a GO- (*Generalised-Ordered*) space.

Definition I.1.4 ((Pseudo)radial, sequential and Fréchet-Urysohn)**.** Let X be a topological space. We say that X is *sequential* if every non-closed subset has an escaping sequence. If we replace 'sequence' with 'transfinite sequence', then we have the definition of a *pseudoradial* space. Pseudoradial spaces are also known as *chain-net* or *folgenbestimmt* spaces.

Let $x \in X$ be given. We say that X is *Fréchet-Urysohn* at x if for all $A \subseteq X$ with $x \in \overline{A}$, there is a sequence in A that converges to x. By replacing 'sequence' with 'transfinite sequence' we obtain the definition of a *radial* point. If X is radial / Fréchet-Urysohn at every point then we say that X is *radial / Fréchet-Urysohn*. Radial spaces are also known as *Fréchet chain-net* or *stark folgenbestimmt* spaces.

The class of radial spaces includes all metrisable spaces and all GO-spaces (see Corollary II.1.15).

Note that by taking a cofinal subsequence, we can assume that the lengths of the transfinite sequences are regular ordinals and since 1-sequences can be replaced by constant α-sequences for any non-zero ordinal α, we can replace 1 with any other non-zero ordinal. In fact, we can even take a *set* of regular ordinals: if X is a topological space and t is a transfinite sequence in X whose length is a regular ordinal strictly greater than $|X|$, then by regularity it has a constant subsequence which maps to a point x. Thus, if t converges to a point y, then so does any constant transfinite sequence with value x. Therefore we only need consider the transfinite sequences whose lengths are regular ordinals $\alpha \leq |X|$.

Definition I.1.5 (Radial character)**.** Let X be a topological space and let $x \in X$ be given. Then we define the *radial character* of x to be the least cardinal κ such that for every $A \subseteq X$, if there exists a transfinite sequence in A that converges to x then there is one of length at most κ. We denote it by $R\chi_X(x)$.

We define the *radial character* of X to be $R\chi(X) := \sup(R\chi_X(x) : x \in X)$. Equivalently, $R\chi(X)$ is the least cardinal κ such that every κ-radially-closed subset is radially-closed.

Note that by the paragraph preceding the above definitions, $R\chi(X)$ is well-defined and $R\chi(X) \leq |X|$. To close this section, we define the rest of the pseudoradial hierarchy.

Definition I.1.6. Let X be a topological space.

- Let t be a transfinite sequence in X. Then we say that t *converges strictly to* $x \in X$ if $t \to x$ and and for all $\alpha < \text{len}(t), x \notin \overline{t[\alpha]}$. We denote this by $t \dashrightarrow x$. These transfinite sequences are also referred to as *t-*, *thin* or *strict*.
- X is *strictly pseudoradial* if every non-closed subset A has a transfinite sequence that converges strictly to a point outside of A. These were first introduced in [AIT85] (quoted in [DIT87, pg. 752]) as *almost-radial* spaces.[1]
- Let κ be a cardinal and $A \subseteq X$ be given. We say that A is κ*-closed* if for every $B \in [A]^{\leq\kappa}, \overline{B} \subseteq A$. The complements of κ-closed subsets are called κ*-open*.
- X is *semiradial* if for every cardinal κ, every κ-radially-closed subset is κ-closed. This notion was first introduced in [BG92].
- Let $x \in X$ be given and let $\mathcal{B}$ be a neighbourhood base for x that is well-ordered by $\supseteq$. We call $\mathcal{B}$ a *well-ordered neighbourhood base* for x. Moreover, if the order type of $(\mathcal{B}, \supseteq)$ is a regular ordinal, then we say that $\mathcal{B}$ is *regular*.

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[1] We are using this alternative name to avoid confusion, since semiradial spaces are almost-radial / strictly pseudoradial (see Figure 2)

A transfinite sequence of subsets $(B_\alpha)_{\alpha<\lambda}$ of X is also called a *well-ordered neighbourhood base* provided that $\{B_\alpha : \alpha < \lambda\}$ is a well-ordered neighbourhood base and $B_\beta \underset{\neq}{\subsetneq} B_\alpha$ for all $\alpha < \beta < \lambda$.

- If X has a well-ordered neighbourhood base for $x \in X$, then we say that X is *well-based* at x. Moreover, if X is well-based at every point, we say that X itself is *well-based*. We can replace 'well-ordered' with 'linearly-ordered' in the definition of well-ordered neighbourhood base and obtain an equivalent formulation of 'well-based'. These spaces are also known as *funnel-shaped*, *lob-* or *wob*-spaces.
- X is *R-monolithic* if X is pseudoradial and for all $A \subseteq X$, $R\chi(\overline{A}) \leq |A|$, or equivalently for every $C \subseteq X$ closed, $R\chi(C) \leq d(C)$, where $d(C)$ is the density of C. This condition was first introduced in [Bel91].

I.2 The pseudoradial hierarchy

I.2.1 Structure of the hierarchy

We now explore the relationship between various classes of pseudoradial spaces. First, we need to prove that 'pseudoradial' is a closed-hereditary property. We can then easily show the implications in Figure I.1 (see Theorem I.1), which are either referenced results or 'folklore', but we include proofs here for completeness.

Lemma I.2.1. *Let X be a pseudoradial space, $C \subseteq X$ be closed. Then C is pseudoradial.*

Proof. Let $A \subseteq C$ be non-closed in C. Then A is not closed in X, so there exists an escaping transfinite sequence in A with respect to X. Since $\overline{A}^X = \overline{A}^C$ and convergence is absolute between subspaces, it follows that A is not radially-closed in C. Therefore C is pseudoradial. □

Theorem I.2.2. *Let X be a topological space and let $x \in X$ be given.*

1. *If X well-based at x then X is radial at x. [Her67, Theorem 2]*
2. *If X is radial then X is semiradial. [BG92, Lemma 1]*[2]
3. *If X is R-monolithic then X is semiradial. [BG92, Lemma 2]* [2]
4. *If X is semiradial then X is strictly pseudoradial. [BG92, Lemma 1]*
5. *If X is strictly pseudoradial then X is pseudoradial.*
6. *If X is T_1 and well-based then X is R-monolithic.*

Proof.

1. Assume X is well-based at x and let $A \subseteq X$ be given such that $x \in \overline{A}$. Then there exists a well-ordered neighbourhood base $(B_\alpha)_{\alpha<\lambda}$ for x. Now for each $\alpha < \lambda$, pick $y_\alpha \in A \cap B_\alpha$. Then $(y_\alpha)_{\alpha<\lambda}$ converges to x. Therefore X is radial at x.

2. Assume X is radial and let κ be a cardinal and $A \subseteq X$ be non-κ-closed, so there is some $B \in [A]^{\leq\kappa}$ with a point $x \in \overline{B} \setminus A$. Then there exists a transfinite sequence u with

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[2] Stated but not proven.

regular length $\operatorname{len}(u) \leq |B| \leq \kappa$ that converges to x. Hence A is not κ-radially-closed and therefore X is semiradial.

(3) Assume X is R-monolithic and let κ be a cardinal and $A \subseteq X$ be non-κ-closed. Then there exists a $B \in [A]^{\leq\kappa}$ such that $\overline{B} \not\subseteq A$ and so $A \cap \overline{B}$ is not closed in $\overline{B}$. As X is R-monolithic, $R\chi(\overline{B}) \leq |B| \leq \kappa$, and thus there is an escaping transfinite sequence in $A \cap \overline{B}$ with respect to $\overline{B}$, whose length is at most κ. Hence A is not κ-radially-closed and therefore X is semiradial.

(4) Assume X is semiradial and let $A \subseteq X$ be non-closed. Let κ be the least cardinal such that A is not κ-closed. Then by semiradiality, there exists a transfinite sequence t in A with $\operatorname{len}(t) \leq \kappa$ that converges to a point $x \in X \backslash A$. By minimality of κ, A is $<\kappa$-closed, so it must be the case that $\operatorname{len}(t) = \kappa$. Hence for all $\alpha < \kappa, \overline{t[\alpha]} \subseteq A$ as $|t[\alpha]| \leq |\alpha| < \kappa$, so in particular $x \notin \overline{t[\alpha]}$. Therefore X is strictly pseudoradial.

(5) This follows by definition.

(6) Suppose Y is well-based and T_1, and let $A \subseteq Y$ be non-closed, so there exists an $y \in (\overline{A} \backslash A) \cap Y$. Assume A is $d(Y)$-radially-closed and let $(B_\alpha)_{\alpha<\lambda}$ be a regular well-ordered neighbourhood base for y. For all $\alpha < \lambda$, choose a $t(\alpha) \in B_\alpha \cap A$. Then $t \to y$ and so $\lambda > d(Y)$.

Pick a dense $D \in [Y]^{d(Y)}$ and define $D' := D \backslash \{y\}$, so for all $z \in D'$ there exists an $\alpha_z < \lambda$ such that $z \notin B_{\alpha_z}$. Define $\alpha := \sup(\alpha_z : z \in D') < \lambda$, so $B_\alpha \cap D' = \emptyset$. Note that since Y is T_1, λ must be infinite and thus there exists a $z \in B_\alpha \backslash B_{\alpha+1}$ and a neighbourhood U

of z such that $y \notin U$. Then $(B_\alpha \cap U) \cap D = \emptyset$, which is a contradiction. Therefore A is not $d(Y)$-radially-closed and hence $R\chi(Y) \leq d(Y)$.

Now suppose X is T_1 and well-based and let $C \subseteq X$ be closed. Then C is also T_1 and well-based, so by the previous paragraphs $R\chi(C) \leq d(C)$. As radial spaces are pseudoradial, it follows that X is R-monolithic. □

Regarding the other implications:

- There is an compact, Hausdorff, R-monolithic space that isn't radial. [Bel93, pg. 638]
- The class of strictly pseudoradial spaces is properly contained between the classes of radial and pseudoradial spaces. [Bel91, pg. 125]
- There exists a compact Hausdorff strictly pseudoradial space that is not semiradial. [Bel96, pg. 115–6]
- Under CH, there is a compact, Hausdorff, radial space that isn't R-monolithic.[Bel93, pg. 638]
- Under CH, there exists a compact, Hausdorff, semiradial space that is neither radial nor R-monolithic. [Bel96, pg. 115]

To show that the assumption of T_1 in part 2 cannot be removed, consider the space $\mathbb{R} \cup \{\star\}$, where $\star \notin \mathbb{R}$ is isolated and for every $x \in \mathbb{R}$, the neighbourhoods of x are of the form $U \cup \{\star\}$, where U is an $\mathbb{R}$-neighbourhood of x.

I.2.2 Restricted quotient maps

Many of the spaces in the pseudoradial hierarchy are characterised in terms of certain continuous images of well-based spaces or LOTS. We will provide another such characterisation for semiradial spaces in terms of a new class of quotient maps — the *restricted quotient* maps:

Definition I.2.3 (Restricted quotient)**.** Let X, Y be topological spaces and let $f : X \to Y$ be a continuous surjection. If for every cardinal κ and all $A \subseteq X$, A is κ-closed if and only if $f^{-1}[A]$ is κ-closed, then we call f a *restricted quotient* map.

We observe that the forward implication in the definition holds for all continuous functions:

Lemma I.2.4. *Let X, Y be topological spaces, $f : X \to Y$ be continuous and let κ be a cardinal, $A \subseteq Y$ be κ-closed. Then $f^{-1}[A]$ is κ-closed.*

Proof. Let $B \in [f^{-1}[A]]^{\leq\kappa}$ be given. Then $f[B] \in [A]^{\leq\kappa}$, so since A is κ-closed and f is continuous, $f[\overline{B}] \subseteq \overline{f[B]} \subseteq A$. Hence $\overline{B} \subseteq f^{-1}[A]$ and thus $f^{-1}[A]$ is κ-closed. □

Another type of quotient map that characterises radial spaces (see Figure I.3) are the *pseudo-open* maps:

Definition I.2.5 (Pseudo-open)**.** Let X, Y be topological spaces, $f : X \to Y$ be a continuous surjection such that for all $y \in Y$ and all $U \subseteq X$ open, if $f^{-1}[\{y\}] \subseteq U$ then $y \in f[U]^{\circ}$. Then we say that f is *pseudo-open.*

Thus, every open map is pseudo-open. The following lemma provides alternative characterisation of pseudo-open maps.

Lemma I.2.6. *[Eng89, Exercise 2.4.F(a)] Let X, Y be topological spaces, $f : X \to Y$ be continuous and surjective. Then f is pseudo-open if and only if f is hereditarily quotient, i.e. for every $Z \subseteq Y$, $f|_{f^{-1}[Z]} : f^{-1}[Z] \to Z$ is a quotient map.*

Proof. Assume f is pseudo-open and let $Z \subseteq Y, U \subseteq Z$ be given such that $f^{-1}[U] = f|^{-1}_{f^{-1}[Z]}[U]$ is open in $f^{-1}[Z]$, so there exists an open $V \subseteq X$ such that $f^{-1}[U] = V \cap f^{-1}[Z]$. As f is pseudo-open, $U \subseteq f[V]^\circ$ and moreover $f[V] \cap Z \subseteq U$, so it follows that $U = f[V]^\circ \cap Z$ is open in Z. Therefore $f|_{f^{-1}[Z]}$ is a quotient map.

Now assume f is hereditarily quotient and let $U \subseteq X$ be open, $y \in Y$ be given such that $f^{-1}[\{y\}] \subseteq U$. Define $Z := (Y \setminus f[U]) \cup \{y\}$. Then

$$U \cap f^{-1}[Z] = f^{-1}[\{y\}] = f|^{-1}_{f^{-1}[Z]}[\{y\}].$$

Hence $\{y\}$ is open in Z, so there exists an open $V \subseteq Y$ such that $\{y\} = V \cap Z = \{y\} \cup (V \setminus f[U])$. Therefore $y \in V \subseteq f[U]$ and thus $y \in f[U]^\circ$. □

We are now ready to prove the rest of the implications in Figure I.2. The following lemma characterising κ-open subsets will prove useful:

Lemma I.2.7. *Let X be a topological space, κ be a cardinal and let $A \subseteq X$ be given. Then A is κ-open if and only if for every $B \in [X]^{\leq\kappa}$ with $A \subseteq X \setminus B$, $A \subseteq (X \setminus B)^\circ$.*

Proof. By definition, A is κ-open if and only if for all $B \in [X \setminus A]^{\leq\kappa}$, $\overline{B} \subseteq X \setminus A$, or equivalently, for every $B \in [X]^{\leq\kappa}$, if $A \subseteq X \setminus B$ then $A \subseteq X \setminus \overline{B} = (X \setminus B)^\circ$. □

Definition I.2.8 (T_1^κ)**.** Let X be a topological space and κ be a cardinal such that every subset of size strictly less than κ is closed. Then we say that X is T_1^κ. Note that a T_1 space is $\mathrm{T}_1^{\aleph_0}$.

Theorem I.2.9. *Let X, Y be topological spaces and let $f : X \to Y$ be a continuous surjection.*

(1) If f is a restricted quotient map then f is a quotient map.

(2) Let κ be an infinite cardinal such that Y is T_1^κ. If f is pseudo-open and has fibres of cardinality at most κ, then f is a restricted quotient map.

(3) If f is closed then f is a restricted quotient map and pseudo-open.

(4) Let κ be a cardinal such that X is T_1^κ and suppose f is a restricted quotient map. Then Y is T_1^κ.

(5) Let Y be a non-discrete, T_1 topological space. Then there exists a topological space X and an open, continuous surjection $f : X \to Y$ that is not a restricted quotient map. In particular, if $Y = \omega + 1$ then we can take X to be a normal $T_1^{\aleph_1}$-space.

Proof.

(1) Assume f is a restricted quotient map and let $A \subseteq X$ non-closed. Then A is not $|A|$-closed and so $f^{-1}[A]$ is not $|A|$-closed. In particular, $f^{-1}[A]$ is not closed, so f is a quotient map.

(2) Assume f is pseudo-open and let λ be a cardinal, $A \subseteq Y$ be given such that $f^{-1}[A]$ is λ-closed and define $A' := Y \setminus A$. If $\lambda < \kappa$ then since Y is T_1^κ, A is λ-closed. Now

suppose $\lambda \geq \kappa$ and note that $f^{-1}[A'] = X \setminus f^{-1}[A]$ is λ-open. Let $B \in [Y]^{\leq \lambda}$ be given such that $A' \subseteq Y \setminus B$. By assumption,

$$|f^{-1}[B]| = \left| \bigcup_{b \in B} f^{-1}[\{b\}] \right| = \sum_{b \in B} |f^{-1}[\{b\}]| \leq \lambda \cdot \kappa = \lambda.$$

As $f^{-1}[A'] \subseteq f^{-1}[Y \setminus B] = X \setminus f^{-1}[B]$ and $f^{-1}[A']$ is λ-open, it follows that $f^{-1}[A'] \subseteq (X \setminus f^{-1}[B])^{\circ}$. So as f is pseudo-open

$$A' \subseteq f[(X \setminus f^{-1}[B])^{\circ}]^{\circ} \subseteq f[X \setminus f^{-1}[B]]^{\circ} = (Y \setminus B)^{\circ}.$$

Therefore A' is λ-open and hence A is λ-closed. By Lemma I.2.4, it follows that f is a restricted quotient map.

(3) Assume f is closed and let κ be a cardinal, $A \subseteq Y$ be given such that $f^{-1}[A]$ is κ-closed. Let $B \in [A]^{\leq \kappa}$ be given. As f is surjective, for all $b \in B$ there exists a $c_b \in f^{-1}[\{b\}]$. Define $C := \{c_b : b \in B\} \in [f^{-1}[A]]^{\leq \kappa}$. Then since f is closed,

$$\overline{B} = \overline{f[C]} = f[\overline{C}] \subseteq f[f^{-1}[A]] = A.$$

Therefore A is κ-closed and thus by Lemma I.2.4, f is a restricted quotient map. Furthermore, observe that by the Lemma I.2.6 and [Eng89, Exercise 2.4.F(a)], every closed map is also pseudo-open.

(4) Let $A \in [Y]^{<\kappa}$ be given. As X is T_1^{κ}, $f^{-1}[A]$ is $|A|$-closed and hence A is closed. Therefore Y is T_1^{κ}.

(5) Let $A \subseteq Y$ be non-closed and define

$$B := \overline{A} \setminus A, \qquad \kappa := \max(\aleph_0, |A|), \qquad X := (\kappa^{+} \times (Y \setminus B)) \cup (\{\kappa^{+}\} \times B).$$

We topologise X by specifying a base as follows:

- For every $U \subseteq Y$ open, if $U \cap B \neq \emptyset$, then let $(\{\kappa^+\} \times (U \cap B)) \cup ((\kappa^+ \setminus C) \times (U \setminus B))$ be a basic open set for every $C \in [\kappa^+]^{\leq \kappa}$.
- For every $U \subseteq Y$ open, if $U \cap B = \emptyset$, let $\{\alpha\} \times U$ be a basic open set for each $\alpha < \kappa^+$.

Define $f : X \to Y$ to be the projection onto the second coordinate. Then f is continuous, open and surjective. However, A is not κ-closed whereas $f^{-1}[A] = \kappa^+ \times A$ is κ-closed. Therefore f is not a restricted quotient map.

If $Y = \omega + 1$ and $A = \omega$, then $B = \{\omega\}, \kappa = \aleph_0$ and X is a $\mathrm{T}_1^{\aleph_1}$-space with a unique, non-isolated point, so is normal. $\square$

Corollary I.2.10. *Let X, Y be topological spaces such that Y is T_1, and let $f : X \to Y$ be pseudo-open with countable fibres. Then f is a restricted quotient map.*

As quotient maps arise from decompositions of the domain space, we will characterise restricted quotient maps in terms of certain types of decomposition maps.

Definition I.2.11 (Decomposition map)**.** Let X be a topological space and $\mathcal{D}$ be a decomposition (i.e. partition) of X. Define $\pi : X \to \mathcal{D}$ by mapping each $x \in X$ to the unique element of $\mathcal{D}$ containing x. We call π the *decomposition map of $\mathcal{D}$* and consider $\mathcal{D}$ as a topological space by endowing it with the quotient topology.

Proposition I.2.12. *Let X be a topological space, $\mathcal{D}$ be a decomposition of X and $\pi : X \to \mathcal{D}$ be the decomposition map. Then π is a restricted quotient map if and only if for all $\mathcal{A} \subseteq \mathcal{D}$*

and cardinals κ*, if* $\bigcup\mathcal{A}$ *is* κ*-closed then for every* $D \in \mathcal{D}\setminus\mathcal{A}$ *and all* $\mathcal{B} \in [\mathcal{A}]^{\leq\kappa}$*, there exists an open* $\mathcal{U} \subseteq \mathcal{D}$ *such that* $D \in \mathcal{U}$ *and* $\mathcal{U} \cap \mathcal{B} = \emptyset$*.*

Proof. Note that a decomposition map is a quotient map; in particular it is continuous and surjective. Thus π is a restricted quotient map if and only if for all $\mathcal{A} \subseteq \mathcal{D}$ and cardinals κ, if $\pi^{-1}[\mathcal{A}] = \bigcup\mathcal{A}$ is κ-closed then $\mathcal{A}$ is κ-closed. The statement of this proposition is an alternative expression of this equivalence. □

To close this section, we show that the quotient maps in Figure I.3 are closed under composition.

Theorem I.2.13. *Let* X, Y, Z *be topological spaces,* $f : X \to Y, g : Y \to Z$ *be continuous surjections.*

(1) If f, g *are quotients then* $g \circ f$ *is a quotient map.*

(2) If f, g *are restricted quotient maps, then* $g \circ f$ *is a restricted quotient map.*

(3) If f, g *are pseudo-open then* $g \circ f$ *is pseudo-open. [Eng89, Exercise 2.4.F(b)]*

Proof. First, suppose that f and g are both (restricted-)quotient maps and let κ be a cardinal. Assume that $(g \circ f)^{-1}[U] = f^{-1}[g^{-1}[U]]$ is (κ-)closed. Then $g^{-1}[U]$ is (κ-)closed and hence U is (κ-)closed. Therefore $g \circ f$ is a (restricted-)quotient map.

Now assume that f and g are pseudo-open and let $A \subseteq Z$ be given. Then $g|_{g^{-1}[Z]} : g^{-1}[Z] \to Z$ and $f|_{(g\circ f)^{-1}[Z]} : (g \circ f)^{-1}[Z] \to g^{-1}[Z]$ are quotient maps, so it follows that $(g \circ f)|_{(g\circ f)^{-1}[Z]} = g|_{g^{-1}[Z]} \circ f|_{(g\circ f)^{-1}[Z]}$ is a quotient map. Therefore by Lemma I.2.6 $g \circ f$ is pseudo-open. □

I.2.3 Quotient characterisation and preservation theorems

We are now ready to prove the results stated in Figure I.3. To characterise topological spaces that are determined by a type of convergence, the usual approach is to take a certain topological sum of directed sets that are topologised in an appropriate way and define an appropriate quotient (see [Fra65; Her67], [Nyi92, pg. 543–545]). Instead of taking sums, we will consider families of maps and final topologies, which will require a translation of certain quotient-map properties.

But first, we will present the preservation theorems, which utilise the following equivalent characterisation of radiality.

Lemma I.2.14. *Let X be a topological space. Then X is radial if and only if X is hereditarily pseudoradial.*

Proof.

(⇒) Assume X is radial and let $Y \subseteq X, A \subseteq Y$ be non-closed in Y. Then there exists an $x \in \overline{A}^Y \setminus A \subseteq \overline{A}^X \setminus A$, so by radiality there exists a transfinite sequence in A converging to x. Therefore Y is pseudoradial and hence X is hereditarily pseudoradial.

(⇐) Assume X is hereditarily pseudoradial and let $A \subseteq X$ be non-closed, $x \in \overline{A} \setminus A$ be given. Then A is non-closed in $A \cup \{x\}$, which is pseudoradial, so there exists an escaping transfinite sequence in A, with respect to $A \cup \{x\}$. This must converge to x and hence X is radial. □

Corollary I.2.15. *Radiality is a hereditary property.*

Theorem I.2.16. *Let X, Y be topological spaces, $f : X \to Y$ be surjective.*

(1) If X is pseudoradial and f is a quotient map then Y is pseudoradial. [Her67, Theorem 2]

(2) If X is semiradial and f is a restricted quotient map then Y is semiradial.

(3) If X is radial and f is pseudo-open then Y is radial. [Ark80, pg. 50]

Proof.

(1) Assume X is pseudoradial and f is a quotient map. Let $A \subseteq Y$ be non-closed, so $f^{-1}[A]$ is also non-closed. Then there exists a transfinite sequence t in $f^{-1}[A]$ that converges to some point $x \in \overline{f^{-1}[A]} \setminus f^{-1}[A]$ and so $f \circ t \to f(x) \in f[\overline{f^{-1}[A]}] \setminus A \subseteq \overline{A} \setminus A$. Thus A is not radially-closed and therefore Y is pseudoradial.

(2) Assume X is semiradial and f is a restricted quotient map. Let κ be a cardinal and $A \subseteq Y$ be non-κ-closed, so $f^{-1}[A]$ is non-κ-closed. Then there exists a transfinite sequence u in $f^{-1}[A]$ with length at most κ that converges to some point $x \in \overline{f^{-1}[A]} \setminus f^{-1}[A]$, so $f \circ u \to f(x) \in f[\overline{f^{-1}[A]}] \setminus A \subseteq \overline{A} \setminus A$. Therefore Y is semiradial.

(3) Note that by Lemma I.2.6, $f|_{f^{-1}[Z]} : f^{-1}[Z] \to Z$ is a quotient map for all $Z \subseteq Y$. So by Lemma I.2.14 and (1), it follows that Y is radial. □

Now we shall develop the machinery for the characterisation theorems. We will show how to translate between families of maps and final topologies to quotient maps, so that we can state our characterisation theorems in terms of families. First, we shall define the topological sum of spaces and co-diagonal maps.

Definition I.2.17 (Topological sum)**.** Let $(X_i)_{i\in I}$ be a family of topological spaces and define

$$X := \bigcup_{i\in I}(\{i\}\times X_i),$$

$$\forall i \in I, \iota_i : X_i \to X, x \mapsto \langle i, x\rangle.$$

Topologise X with the final topology with respect to $(\iota_i)_{i\in I}$, the finest topology on X making those maps continuous, i.e. a subset U of X is open if and only if $\iota_i^{-1}[U]$ is open for all $i \in I$. We say that X is the *topological sum* of $(X_i)_{i\in I}$ and denote it by $\sum_{i\in I} X_i$. We shall refer to $(\iota_i)_{i\in I}$ as the family of *inclusion maps.*

Definition I.2.18 (Co-diagonal map)**.** Let Y be a set and let $(f_i : X_i \to Y)_{i\in I}$ be a family of functions. Then we define the *co-diagonal map* $\nabla_{i\in I} f_i : \sum_{i\in I} X_i \to Y$ of $(f_i)_{i\in I}$ as follows:

$$\forall i \in I, \forall x \in X_i, \left(\nabla_{i\in I} f_i\right)(i, x) := f_i(x).$$

Lemma I.2.19. *Let X be a topological space, $(Y_i)_{i\in I}$ be a family of topological spaces and let $(f_i)_{i\in I}$ be a family of continuous maps from Y_i to X respectively such that $\bigcup_{i\in I}\operatorname{ran}(f_i) = X$. Then $f := \nabla_{i\in I} f_i$ is a continuous surjection.*

Proof. Note that for every open $U \subseteq X$, $f_i^{-1}[U]$ is open in Y_i and moreover

$$f^{-1}[U] = \bigcup_{i\in I}(\{i\}\times f_i^{-1}[U]).$$

Therefore $f^{-1}[U]$ is open in Y and so f is continuous. Furthermore, f is surjective because $\operatorname{ran}(f) = \bigcup_{i\in I}\operatorname{ran}(f_i) = X$. □

Definition I.2.20 ($\mathcal{P}$ family)**.** Let X be a topological space, $(Y_i)_{i\in I}$ be a family of topological spaces and let $(f_i)_{i\in I}$ be a family of continuous maps from Y_i to X respectively such that $\bigcup_{i\in I}\operatorname{ran}(f_i) = X$. Let $\mathcal{P}$ be a class of continuous surjections. Then we say that $(Y_i, f_i)_{i\in I}$ is a $\mathcal{P}$ *family* if $\nabla_{i\in I} f_i \in \mathcal{P}$ when considered as a map from $\sum_{i\in I} Y_i$ to X.

For 1-element families, the notions of $\mathcal{P}$ map and $\mathcal{P}$ family coincide.

Proposition I.2.21. *Let X, Y be topological spaces, $f : X \to Y$ be a continuous surjection and let $\mathcal{P}$ be a class of continuous surjections closed under pre-composition by homeomorphisms. Then $f \in \mathcal{P}$ if and only if $(f_i)_{i\in\{0\}}$ is a $\mathcal{P}$ family for Y, where $f_0 := f$.*

Proof. Note that $\bigcup\{\operatorname{ran}(f)\} = Y$, so the preliminary condition for the previous lemma is satisfied. Let $\pi : \{0\}\times X \to X$ be the projection map, which is a homeomorphism. Then

$$f \in \mathcal{P} \iff f\circ\pi \in \mathcal{P} \iff (f_i)_{i\in\{0\}} \text{ is a } \mathcal{P} \text{ family.} \qquad \square$$

The particular classes $\mathcal{P}$ that will be used here are the classes of quotient, restricted quotient and pseudo-open maps. We now seek some characterisations of their families.

Lemma I.2.22. *Let $(X_i)_{i\in I}$ be a family of topological spaces and let X be the topological sum with inclusion maps $(\iota_i)_{i\in I}$. Then for all cardinals κ and $A \subseteq X$, A is κ-closed if and only if $\iota_i^{-1}[A]$ is κ-closed for all $i \in I$.*

Proof. As $(\iota_i)_{i\in I}$ is a family of continuous maps, the forward implication follows from Lemma I.2.4. Let $A \subseteq X, \kappa$ be a cardinal such that A is not κ-closed, so there exists a $B \in [A]^{\leq\kappa}$ such that $\overline{B} \not\subseteq A$. Then since $(\operatorname{ran}(\iota_i))_{i\in I}$ is a discrete cover of X,

$$\overline{B} = \overline{\bigcup_{i\in I}(B\cap\operatorname{ran}(\iota_i))} = \bigcup_{i\in I}\overline{B\cap\operatorname{ran}(\iota_i)} = \bigcup_{i\in I}\iota_i[\overline{\iota_i^{-1}[B]}].$$

Thus there exists an $i \in I$ such that $\iota_i[\overline{\iota_i^{-1}[B]}] \not\subseteq A \cap \operatorname{ran}(\iota_i) = \iota_i[\iota_i^{-1}[A]]$. Since ι_i is injective, $\overline{\iota_i^{-1}[B]} \not\subseteq \iota_i^{-1}[A]$. As $\iota_i^{-1}[B] \subseteq \iota_i^{-1}[A]$ and $|\iota_i^{-1}[B]| = |B \cap \operatorname{ran}(\iota_i)| \leq |B| \leq \kappa$, it follows that $\iota_i^{-1}[A]$ is not κ-closed. □

Lemma I.2.23. *Let X be a topological space, $(Y_i)_{i\in I}$ be a family of topological spaces, $(f_i)_{i\in I}$ be a family of continuous maps from Y_i to X respectively such that $\bigcup_{i\in I} \operatorname{ran}(f_i) = X$. Then:*

1. *$(Y_i, f_i)_{i\in I}$ is a quotient family for X if and only if X has the final topology generated by $(f_i)_{i\in I}$; that is, for every open $U \subseteq X$, if $f_i^{-1}[U]$ is open for all $i \in I$, then U is open.*
2. *$(Y_i, f_i)_{i\in I}$ is a restricted quotient family for X if and only if for every $A \subseteq X$ and all cardinals κ, if $f_i^{-1}[A]$ is κ-closed for every $i \in I$, then A is κ-closed.*
3. *$(Y_i, f_i)_{i\in I}$ is a pseudo-open family for X if and only if for all $x \in X, i \in I$ and open $U_i \subseteq X_i$ with $f_i^{-1}[\{x\}] \subseteq U_i$, $x \in \left(\bigcup_{i\in I} f_i[U_i]\right)^{\circ}$.*

Proof. Let Y be the topological sum of $(Y_i)_{i\in I}$ with inclusion maps $(\iota_i)_{i\in I}$ and define $f := \nabla_{i\in I} f_i$. Note that for all $i \in I, f \circ \iota_i = f_i$ and by Lemma I.2.19 f is continuous.

(1) Let $U \subseteq X$ be given. Then

$$f^{-1}[U] \text{ is open} \iff \forall i \in I, \iota_i^{-1}[f^{-1}[U]] = (f \circ \iota_i)^{-1}[U] = f_i^{-1}[U] \text{ is open.}$$

Since f is continuous and surjective, it follows that $(Y_i, f_i)_{i\in I}$ is a quotient family for X if and only if for all $U \subseteq X$, if $f^{-1}[U]$ is open then U is open, or equivalently X has the final topology generated by $(f_i)_{i\in I}$.

(2) Let $A \subseteq X, \kappa$ be a cardinal. Then by the previous lemma

$$f^{-1}[A] \text{ is } \kappa\text{-closed} \iff \forall i \in I, \iota_i^{-1}[f^{-1}[A]] = f_i^{-1}[A] \text{ is } \kappa\text{-closed}.$$

As f is continuous and surjective, it follows that $(Y_i, f_i)_{i\in I}$ is a restricted quotient family for X if and only if for every $A \subseteq X$ and all cardinals κ, if $f^{-1}[A]$ is κ-closed then A is κ-closed, or equivalently, if $f_i^{-1}[A]$ is κ-closed for all $i \in I$, then A is κ-closed.

(3) Suppose $(Y_i, f_i)_{i\in I}$ is a pseudo-open family for X. Let $x \in X$ be given and for all $i \in I$, let $U_i \subseteq Y_i$ be open such that $f_i^{-1}[\{x\}] \subseteq U_i$. Define $U := \bigcup_{i\in I} \iota_i[U_i]$, which is open in Y since for all $i \in I, \iota_i$ is open. Then

$$\begin{aligned} f^{-1}[\{x\}] &= \bigcup_{i\in I} \left(f^{-1}[\{x\}] \cap \operatorname{ran}(\iota_i)\right) \\ &= \bigcup_{i\in I} \iota_i\left[\iota_i^{-1}\left[f^{-1}[\{x\}]\right]\right] \\ &= \bigcup_{i\in I} \iota_i\left[f_i^{-1}[\{x\}]\right] \\ &\subseteq U. \end{aligned}$$

Since f is pseudo-open, $x \in f[U]^\circ$.

Now suppose that for all $x \in X, i \in I$ and for every open $U_i \subseteq Y_i$ with $f_i^{-1}[\{x\}] \subseteq U_i, x \in \left(\bigcup_{i\in I} f_i[U_i]\right)^\circ$. Let $U \subseteq Y$ be open, $x \in X$ be given such that $f^{-1}[\{x\}] \subseteq U$. Then for all $i \in I, f_i^{-1}[\{x\}] = \iota_i^{-1}[f^{-1}[\{x\}]] \subseteq \iota_i^{-1}[U]$ and $\iota_i^{-1}[U]$ is open in Y_i, so it follows that

$$x \in \left(\bigcup_{i\in I} f_i[\iota_i^{-1}[U]]\right)^\circ = f\left[\bigcup_{i\in I} \iota_i[\iota_i^{-1}[U]]\right]^\circ = f[U]^\circ.$$

Therefore f is a pseudo-open map and thus $(Y_i, f_i)_{i\in I}$ is a pseudo-open family for X. □

We now construct the domain space to quotient over and show how convergence properties of the codomain correlate with properties of the quotient map.

Lemma I.2.24. *A topological sum of well-based spaces is well-based.*

Proof. Let $(X_i)_{i\in I}$ be a family of well-based spaces and define X to be the topological sum of X with inclusion maps $(\iota_i)_{i\in I}$. Let $i \in I$ be given. Then the inclusion map $\iota_i : X_i \to X$ is an embedding with open range, since 'well-based' is a local property, it follows that every point in $\operatorname{ran}(\iota_i)$ is well-based in X and therefore X is well-based. □

Definition I.2.25 (S_κ)**.** Let κ be a non-zero, regular ordinal. We define a topology on $\kappa+1$ by setting every point in κ to be isolated and letting $\{[\alpha,\kappa] : \alpha \in \kappa\}$ be a neighbourhood base system for κ. We denote this topological space by S_κ.

Lemma I.2.26. *[Nyi92, Lemma 2.7] Let X be a topological space, κ be a non-zero regular ordinal and let $f : S_\kappa \to X$ be given. Then f is continuous if and only if $f|_\kappa \to f(\kappa)$.*

Theorem I.2.27. *Let X be a topological space and define*

- $\mathcal{S} := \{\kappa : \kappa$ *is a non-zero, regular cardinal,* $\kappa \leq |X|\}$,
- $\Sigma := \bigcup_{\kappa\in\mathcal{S}} {}^{\kappa}X$,
- *For every* $t \in \Sigma, L_t := \{x \in X : t \to x\}$,
- $I := \{\langle t, x\rangle : t \in \Sigma, x \in L_t\}$,

- *For all* $\langle t,x\rangle \in I$, $Y_{\langle t,x\rangle} := S_{\mathrm{len}(t)}$,
- *For all* $\langle t,x\rangle \in I$, $f_{\langle t,x\rangle} := t \cup \{\langle \mathrm{len}(t), x\rangle\} : Y_{\langle t,x\rangle} \to X$.

Then:

(1) *For all* $i \in I$, f_i *is continuous.*

(2) $\bigcup_{i\in I} \mathrm{ran}(f_i) = X$.

(3) X *is pseudoradial if and only if* $(Y_i, f_i)_{i\in I}$ *is a quotient family for* X.

(4) X *is semiradial if and only if* $(Y_i, f_i)_{i\in I}$ *is a restricted quotient family for* X.

(5) X *is radial if and only if* $(Y_i, f_i)_{i\in I}$ *is a pseudo-open family for* X.

Proof.

(1) By definition and the previous lemma, f_i is continuous for every $i \in I$.

(2) Let $x \in X$ be given. Then $1 \in \mathcal{S}$, $t := \{\langle 0, x\rangle\} \in \Sigma$ and $x \in L_t$. Thus $f_{\langle t,x\rangle}(1) = x$ and so $\bigcup_{i\in I} \mathrm{ran}(f_i) = X$.

(3) Suppose that X is pseudoradial and let $U \subseteq X$ be non-open, so $X\backslash U$ is non-closed. Then by the pseudoradiality of X, there exists an $x \in \overline{X\backslash U}\backslash(X\backslash U)$ and there exists a $t \in \Sigma$ such that $x \in L_t$ and $\mathrm{ran}(t) \subseteq X\backslash U$. Then $f^{-1}_{\langle t,x\rangle}[U]$ is not open. Therefore $(Y_i, f_i)_{i\in I}$ is a quotient family for X.

(4) Suppose that X is semiradial and let κ be a cardinal, $A \subseteq X$ be non-κ-closed. Let λ be the least cardinal such that there exists an escaping λ-sequence t in A with a limit

$x \in X \backslash A$. Note that λ is regular and $\lambda \leq \min(\kappa, |X|)$. Then $\langle t, x\rangle \in I$ and $f^{-1}_{\langle t,x\rangle}[A] = \lambda$ is not λ-closed in $Y_{\langle t,x\rangle}$ and in particular not κ-closed. Therefore $(Y_i, f_i)_{i\in I}$ is a restricted quotient family for X.

(5) Suppose that X is radial. Let $x \in X$ be given and for all $i \in I$, let $U_i \subseteq Y_i$ be open such that $f_i^{-1}[\{x\}] \subseteq U_i$. Define $U := \bigcup_{i\in I} f_i[U_i]$ and note that $x \in U$. Suppose $x \notin U^\circ$, so $x \in \overline{X\backslash U}\backslash(X\backslash U)$. As X is radial, there exists a $t \in \Sigma$ with $\mathrm{ran}(t) \subseteq X\backslash U$ such that $t \to x$. Define $\lambda := \mathrm{len}(t)$. Then $\lambda \in f^{-1}_{\langle t,x\rangle}[\{x\}] \subseteq f^{-1}_{\langle t,x\rangle}[U]$, so there exists an $\alpha < \lambda$ such that $(\lambda+1)\backslash\alpha \subseteq f^{-1}_{\langle t,x\rangle}[U]$. Hence $t(\alpha) = f_{\langle t,x\rangle}(\alpha) \in U$, which is a contradiction. Therefore $x \in U^\circ$ and thus $(Y_i, f_i)_{i\in I}$ is a pseudo-open family for X. □

We now arrive at the final statement of characterisation for these spaces. The characterisation for pseudoradial spaces was originally given in [Her67] and used a similar construction as above. This was observed for radial spaces in [Ark80].

Corollary I.2.28 (Quotient characterisation theorem)**.** *Let X be a topological space. Then X is (pseudoradial / semiradial / radial) if and only if X is the (quotient / restricted quotient / pseudo-open) image of a well-based space.*

I.3 Quasi-isolation and p-character

To aid our understanding of pseudoradiality and related properties, we investigate when intersections of neighbourhoods are again neighbourhoods and the effect this has on converging transfinite sequences. To do this, we introduce two related properties: *quasi-isolation,* which is a generalisation of isolated points to a non-T_1-setting, and *p-character,* which captures how badly quasi-isolation fails. We start this section with some theory of these two properties.

I.3.1 Quasi-isolation

A point x in a topological space is *isolated* if and only if $\{x\}$ is open. This implies that it has a 'minimal neighbourhood'. We are going to take this minimal neighbourhood property to define a new property generalising isolated points.

Definition I.3.1 (Quasi-isolated and quasi-discrete)**.** Let X be a topological space and let $x \in X$ be given. If x has a principal neighbourhood filter, i.e. the intersection of all its neighbourhoods is again a neighbourhood, then we say that x is *quasi-isolated.* We denote the (open) neighbourhood filter by ($\mathcal{U}_x^X$) N_x^X and define the *neighbourhood core* of x to be $N_x^X := \bigcap \mathcal{N}_x^X = \bigcap \mathcal{U}_x^X$. The superscripts will be dispensed with if the underlying topological space is unambiguous.

If every point in a topological space is quasi-isolated then we say that the space is *quasi-discrete.* These are also known as *Alexandroff spaces.*

Definition I.3.2 (T_1)**.** Let X be a topological space and let $x \in X$ be given. We say that x is *T_1* if $N_x = \{x\}$. Note that X is T_1 if and only if every point is T_1.

If 'isolated point' translates as 'there is a neighbourhood separating it from all other points', then quasi-isolated point translates as 'there is a neighbourhood separating it from all possible points'. In fact, for T_1-points these notions coincide.

Proposition I.3.3. *Let X be a topological space, $x \in X$ be given.*

(1) x is quasi-isolated if and only if N_x is open.

(2) If x is isolated then x is quasi-isolated.

(3) If x is quasi-isolated and T_1 then x is isolated.

Proof.

(1) If N_x is open then $N_x \in \mathcal{N}_x$, so x is quasi-isolated. Now assume x is quasi-isolated. Then $N_x^\circ \in \mathcal{N}_x$ also, so $N_x \subseteq N_x^\circ \subseteq \bigcap \mathcal{N}_x$. Therefore $N_x = N_x^\circ$ is open.

(2) Assume x is isolated, so $\{x\}$ is open. Then $N_x = \{x\}$ is open and therefore x is quasi-isolated.

(3) If x is quasi-isolated and T_1, then $N_x = \{x\}$ is open, so x is isolated. □

Corollary I.3.4. *Let X be a topological space.*

(1) If X is discrete then X is quasi-discrete.

(2) If X is quasi-discrete and T_1 then X is discrete.

We now generalise the proof that finite T_1 spaces are discrete, using the notion of *κ-additivity.*

Definition I.3.5 (κ-additive)**.** Let X be a topological space and κ be a cardinal such that for every non-empty collection $\mathcal{U}$ of open sets with cardinality strictly less than κ, $\bigcap \mathcal{U}$ is open. Then we say that X is *κ-additive.*

Lemma I.3.6. *Let X be a topological space and κ be a cardinal such that X is κ-additive and let $x \in X$ be given such that $|X \setminus N_x| < \kappa$. Then x is quasi-isolated in X.*

Proof. Let $y \in X \setminus N_x$ be given, so there exists a $U_y \in \mathcal{U}_x$ such that $y \notin U_y$. Then $|\{U_y : y \in X \setminus N_x\}| < \kappa$, so $\bigcap_{y \in X \setminus N_x} U_y = N_x$ is open. Therefore x is quasi-isolated in X. □

Corollary I.3.7. *Let X be a topological space, κ be a cardinal such that $|X| < \kappa$ and X is κ-additive. Then X is quasi-discrete.*

Corollary I.3.8. *Every finite topological space is quasi-discrete.*

Proof. Every topological space is $\aleph_0$-additive by definition. □

Now we show how quasi-isolated points in subspaces and products arise and characterise them for the latter.

Lemma I.3.9. *Let X be a topological space, $Y \subseteq X, x \in Y$ be given. Then $N_x^Y = N_x^X \cap Y$.*

Proof. $N_x^Y = \bigcap \mathcal{N}_x^Y = \bigcap \{N \cap Y : N \in \mathcal{N}_x^X\} = \left(\bigcap \mathcal{N}_x^X\right) \cap Y = N_x^X \cap Y$. □

Theorem I.3.10. *Let X be a topological space, $Y \subseteq X, x \in Y$ be quasi-isolated in X. Then x is quasi-isolated in Y.*

Proof. As x is quasi-isolated in $X, N_x^X \in \mathcal{N}_x^X$ and so $N_x^Y = N_x^X \cap Y \in \mathcal{N}_x^Y$. Therefore x is quasi-isolated in Y. □

Lemma I.3.11. *Let $(X_i)_{i \in I}$ be a non-empty family of topological spaces and define $X := \prod_{i \in I} X_i$. Let $x \in X$ be given. Then $N_x^X = \prod_{i \in I} N_{\pi_i(x)}^{X_i}$.*

Proof. First, note that

$$\begin{aligned}
\mathcal{N}_x^X &= \langle \mathcal{U}_x^X \rangle \\
&= \left\langle \bigcap_{j\in J} \pi_j^{-1}[U_j] : J \subseteq I \text{ finite and non-empty and } \forall\, j \in J, U_j \in \mathcal{U}_{\pi_j(x)}^{X_j} \right\rangle \\
&= \left\langle \pi_i^{-1}[U_i] : i \in I, U_i \in \mathcal{U}_{\pi_i(x)}^{X_i} \right\rangle,
\end{aligned}$$

where $\langle \mathcal{A} \rangle$ denotes the smallest filter containing $\mathcal{A}$. Thus

$$N_x^X = \bigcap_{i\in I} \bigcap_{U \in \mathcal{U}_{\pi_i(x)}^{X_i}} \pi_i^{-1}[U] = \bigcap_{i\in I} \pi_i^{-1}[N_{\pi_i(x)}^{X_i}] = \prod_{i\in I} N_{\pi_i(x)}^{X_i}. \qquad \square$$

Theorem I.3.12. *Let $(X_i)_{i\in I}$ be a non-empty family of topological spaces and define $X := \prod_{i\in I} X_i$. Let $x \in X$ be given. Then x is quasi-isolated in X if and only if for every $i \in I, \pi_i(x)$ is quasi-isolated in X_i and $I' := \{i \in I : \mathcal{N}_{\pi_i(x)}^{X_i} \neq \{X_i\}\}$ is finite.*

Proof.

(⇒) Assume x is quasi-isolated and let $i \in I$ be given. Define

$$\forall\, j \in I, Y_j := \left\{ \begin{array}{ll} X_i & \text{if } j = i \\ \{\pi_j(x)\} & \text{otherwise} \end{array} \right\},$$

$$Y := \prod_{j\in I} Y_j.$$

Then $\pi_i|_Y : Y \to X_i$ is a homeomorphism. Since x is quasi-isolated in X, it is also quasi-isolated in Y and thus $\pi_i(x)$ is quasi-isolated in X_i.

Suppose I' is infinite and let $i \in I'$ be given. Then there exists a $y_i \in X_i$ and $M_i \in \mathcal{N}_{\pi_i(x)}^{X_i}$ such that $y_i \notin M_i$. Since x is quasi-isolated, $\bigcap_{i\in I'} \pi_i^{-1}[M_i] \in \mathcal{N}_x^X$, so there exists a finite and non-empty $L \subseteq I$, and for every $l \in L$ there exists an open $U_l \subseteq X_l$ such that $x \in \bigcap_{l \in L} \pi_l^{-1}[U_l] \subseteq \bigcap_{i \in I'} \pi_i^{-1}[M_i]$. Define:

$$y : I \to \bigcup_{i \in I} X_i, i \mapsto \begin{cases} y_i & \text{if } i \in I' \setminus L \\ \\ \pi_i(x) & \text{otherwise} \end{cases}$$

Then $y \in \bigcap_{l \in L} \pi_l^{-1}[U_l]$. However, as I' is infinite, there exists an $i \in I' \setminus L$ and $\pi_i(y) = y_i \notin M_i$, which is a contradiction. Therefore I' is finite.

(⇐) Assume for all $i \in I, \pi_i(x)$ is quasi-isolated in X_i and I' is finite. Then

$$N_x^X = \prod_{i \in I} N_{\pi_i(x)}^{X_i} = \bigcap_{i \in I'} \pi_i^{-1}[N_{\pi_i(x)}^{X_i}].$$

Now for all $i \in I', N_{\pi_i(x)}^{X_i}$ is open, so it follows that N_x^X is also open. Therefore x is quasi-isolated in X. □

I.3.2 p-character

Here we define a cardinal function that encodes the failure of a point to be quasi-isolated. Thus quasi-isolation and p-character work together to capture information about intersections of neighbourhoods.

Definition I.3.13 (p-character)**.** Let X be a topological space, $x \in X$ be non-quasi-isolated. Then we define the *p-character* of x in X to be the least cardinality of a collection of neighbourhoods of x whose intersection is not a neighbourhood. We designate this cardinal by $p_X(x)$. If the surrounding space is unambiguous then we will dispense with the subscript.

If $x \in X$ is quasi-isolated, then we say that $p_X(x) = \infty$ with the convention that $\infty > \kappa$ for every cardinal κ. In particular, $[A]^{<\infty} = \mathcal{P}(A)$ for any set A.

Theorem I.3.14. *Let X be a topological space, $x \in X$ be non-quasi-isolated. Then $p(x)$ is infinite and regular.*

Proof. As $\mathcal{N}_x$ is a filter, $\kappa := p(x)$ must be infinite. Since x is non-quasi-isolated, there exists $\mathcal{A} = \{A_\alpha : \alpha < \kappa\} \in [\mathcal{N}_x]^\kappa$ such that $\bigcap \mathcal{A} \notin \mathcal{N}_x$. Now let $(\alpha_\beta)_{\beta < \mathrm{CF}(\kappa)}$ be a cofinal, strictly-increasing sequence in $p(x)$. Suppose κ is singular, so $\mathcal{B} := \{\bigcap_{\alpha \le \alpha_\beta} A_\alpha : \beta < \mathrm{CF}(\kappa)\} \subseteq \mathcal{N}_x$ and $\bigcap \mathcal{A} = \bigcap \mathcal{B} \in \mathcal{N}_x$, which is a contradiction. Therefore $p(x)$ is regular. □

The next proposition provides an alternative characterisation of p-character.

Proposition I.3.15. *Let X be a topological space, $x \in X$ be given and let κ be a cardinal. Then the following are equivalent:*

(1) $p(x) \ge \kappa$.

(2) $\mathcal{N}_x$ *is a κ-complete filter, i.e.* $\bigcap \mathcal{U} \in \mathcal{N}_x$ *for all* $\mathcal{U} \in [\mathcal{N}_x]^{<\kappa}$.

Proof.

(1 ⇒ 2) Let $\mathcal{U} \in [\mathcal{N}_x]^{<\kappa}$ be given. If x is quasi-isolated then $\bigcap \mathcal{N}_x \subseteq \bigcap \mathcal{U}$ and $\bigcap \mathcal{N}_x \in \mathcal{N}_x$, so $\bigcap \mathcal{U} \in \mathcal{N}_x$. If not then since $p(x) \ge \kappa, \bigcap \mathcal{U} \in \mathcal{N}_x$.

(2 ⇒ 1) If x is not quasi-isolated then by definition $p(x) \ge \kappa$. Otherwise, $p(x) = \infty \ge \kappa$. □

We now prove some results for subspaces and products.

Theorem I.3.16. *Let X be a topological space, $Y \subseteq X, x \in Y$ be given. Then $p_X(x) \le p_Y(x)$.*

Proof. Let $\mathcal{A} \in [\mathcal{N}_x^Y]^{<p_X(x)}$ be given. Then for every $A \in \mathcal{A}$, there exists a $B_A \in \mathcal{N}_x^X$ such that $A = B_A \cap Y$. As $|\mathcal{A}| < p_X(x)$, it follows that

$$\bigcap_{A\in\mathcal{A}} B_A \in \mathcal{N}_x^X \Rightarrow \bigcap \mathcal{A} = \left(\bigcap_{A\in\mathcal{A}} B_A\right) \cap Y \in \mathcal{N}_x^Y.$$

Therefore by the previous proposition, $p_X(x) \leq p_Y(x)$. □

Theorem I.3.17. *Let $(X_i)_{i\in I}$ be a non-empty family of topological spaces and define $X := \prod_{i\in I} X_i$. Let $x \in X$ be non-quasi-isolated and define*

$$I' := \{i \in I : \mathcal{N}_{\pi_i(x)}^{X_i} \neq \{X_i\}\},$$

$$I^* := \{i \in I : \pi_i(x) \textit{ is non-quasi-isolated in } X_i\}.$$

Then:

$$p_X(x) = \begin{cases} \min(p_{X_i}(\pi_i(x)) : i \in I^*) & \textit{if } I' \textit{ is finite} \\ \aleph_0 & \textit{otherwise} \end{cases}$$

Proof. We divide our argument depending on the cardinality of I'.

Case 1: Assume I' is infinite, so there exists a countably infinite $J \subseteq I'$. Then for all $j \in J$, there exists $y_j \in X_j$ and $M_j \in \mathcal{N}_{\pi_j(x)}^{X_j}$ such that $y_j \notin M_j$. Suppose $\bigcap_{j\in J} \pi_j^{-1}[M_j] \in \mathcal{N}_x^X$, so for some finite and non-empty $F \subseteq I$ and for every $f \in F$, there exists an open $U_f \subseteq X_f$ such that $x \in \bigcap_{f\in F} \pi_f^{-1}[U_f] \subseteq \bigcap_{j\in J} \pi_j^{-1}[M_j]$. Define:

$$y : I \to \bigcup_{i\in I} X_i, i \mapsto \begin{cases} y_i & \text{if } i \in J \backslash F \\ \pi_i(x) & \text{otherwise} \end{cases}$$

Then $y \in \bigcap_{f\in F}\pi_f^{-1}[U_f]$. As J is infinite, there exists a $j \in J\backslash F$ and so $\pi_j(y) = y_j \notin M_j$, which is a contradiction. Therefore $\bigcap_{j\in J}\pi_j^{-1}[M_j] \notin \mathcal{N}_x^X$ and thus $p_X(x) \leq \aleph_0$. As $p_X(x)$ is infinite, it follows that $p_X(x) = \aleph_0$.

Case 2: Assume I' is finite and note that since x is not quasi-isolated, by Theorem I.3.12 $I^* \neq \emptyset$. Then there exists an $i \in I^*$ such that

$$\min(p_{X_j}(\pi_j(x)) : j \in I^*) = p_{X_i}(\pi_i(x)).$$

Define

$$\forall j \in I, Y_j := \left\{ \begin{array}{ll} X_i & \text{if } j = i \\ \\ \{\pi_j(x)\} & \text{otherwise} \end{array} \right\},$$

$$Y := \prod_{j\in I} Y_j$$

Then $\pi_i|_Y : Y \to X_i$ is a homeomorphism and $x \in Y$. Thus

$$p_X(x) \leq p_Y(x) = p_{X_i}(\pi_i(x)).$$

Now let $\mathcal{A} \in [\mathcal{N}_x^X]^{<p_{X_i}(\pi_i(x))}$ be given, so for all $A \in \mathcal{A}$ there exists a finite and non-empty $J_A \subseteq I$ and for every $j \in J_A$, there exists an open $U_{A,j} \subseteq X_j$ such that $x \in \bigcap_{j\in J_A}\pi_j^{-1}[U_{A,j}] \subseteq A$. Define

$$J := \bigcup_{A\in\mathcal{A}} J_A,$$

$$\forall j \in J, \mathcal{B}_j := \{U_{A,j} : A \in \mathcal{A}, j \in J_A\}.$$

Since $I^* \subseteq I'$, by our choice of i it follows that for all $j \in J, \bigcap\mathcal{B}_j \in \mathcal{N}_{\pi_j(x)}^{X_j}$ and so $\bigcap_{B\in\mathcal{B}_j}\pi_j^{-1}[B] = \pi_j^{-1}[\bigcap\mathcal{B}_j] \in \mathcal{N}_x^X$. Observe that $\bigcap\mathcal{B}_j = X_j$ for every $j \in J\backslash I'$. Hence, as p-characters are infinite,

$$\bigcap\mathcal{A} \supseteq \bigcap_{A\in\mathcal{A}}\bigcap_{j\in J_A}\pi_j^{-1}[U_{A,j}] \supseteq \bigcap_{j\in J}\pi_j^{-1}\left[\bigcap\mathcal{B}_j\right] = \bigcap_{j\in J\cap I'} pi_j^{-1}\left[\bigcap\mathcal{B}_j\right] \in \mathcal{N}_x^X.$$

Thus $\bigcap\mathcal{A} \in \mathcal{N}_x^X$ and so $p_X(x) = p_{X_i}(\pi_i(x)) = \min(p_{X_j}(\pi_j(x)) : j \in I^*)$. □

We now conclude this section by introducing the familiar pseudocharacter cardinal function in a general, non-T_1-setting, and prove some relations between pseudocharacter and p-character.

Definition I.3.18 (Pseudocharacter)**.** Let X be a topological space, $x \in X, \mathcal{P} \subseteq \mathcal{N}_x$ be non-empty such that $\bigcap\mathcal{P} = N_x$. Then we say that $\mathcal{P}$ is a *pseudobase* for x. We denote the least cardinality for a pseudobase for x as $\psi_X(x)$. Note that in T_1-spaces, this coincides with the standard definition since $N_x = \{x\}$.

If the surrounding space is unambiguous, then we denote the pseudocharacter of x by $\psi(x)$.

Theorem I.3.19. *Let X be a topological space and let $Y \subseteq X, x \in Y$ be given. Then $\psi_Y(x) \leq \psi_X(x)$.*

Proof. By definition, there exists a pseudobase $\mathcal{P} \in [\mathcal{N}_x^X]^{\psi_X(x)}$ for x in X. Then

$$N_x^Y = N_x^X \cap Y = \left(\bigcap\mathcal{P}\right) \cap Y = \bigcap_{P\in\mathcal{P}}(P \cap Y).$$

Thus $\{P \cap Y : P \in \mathcal{P}\}$ is a pseudobase for x in Y. Therefore $\psi_Y(x) \leq \psi_X(x)$. □

Theorem I.3.20. *Let X be a topological space, $x \in X$ be non-quasi-isolated. Then $p(x) \leq \psi(x)$.*

Proof. Let $\mathcal{P} \in [\mathcal{N}_x]^{\psi(x)}$ be a pseudo-base for x. Since x is not quasi-isolated, $\bigcap \mathcal{P} \notin \mathcal{N}_x$, so $p(x) \leq \psi(x)$. □

I.3.3 Applications to the pseudoradial hierarchy

In this final section, we demonstrate some applications of p-character to the study of the pseudoradial hierarchy. In particular, we show failures of productivity in radial spaces, give conditions to ensure semiradiality, and characterise sequentiality in terms of strict pseudoradiality and countable tightness. First, we will characterise the well-based spaces.

Definition I.3.21 (Cofinality)**.** Let $(X,<)$ be a linearly ordered set, $A \subseteq X$ be given. Then A is *cofinal* if and only if for all $x \in X$, there exists an $a \in A$ such that $x \leq a$. We define the *cofinality* of $(X,<)$ to be the least cardinality of a cofinal subset. This is denoted by $\mathrm{cf}(X,<)$. Of course, $\mathrm{cf}(\alpha,\in) = \mathrm{cf}(\alpha)$ for every ordinal α.

Definition I.3.22 (Character)**.** For a point x in a topological space X, we denote the smallest cardinality of a neighbourhood base of x by $\chi_X(x)$, called the *character* of x in X. If the surrounding space is unambiguous, we will dispense with the subscript.

Lemma I.3.23. *Let X be a topological space, $x \in X, \mathcal{B}$ be a well-ordered neighbourhood base for x. Then there exists a $\mathcal{B}' \subseteq \mathcal{B}$ cofinal in $(\mathcal{B}, \supsetneq)$ that is a regular well-ordered neighbourhood base for x and $|\mathcal{B}'| = \mathrm{cf}(\mathcal{B}, \supsetneq)$.*

Proof. First, there exists an ordinal α and an order isomorphism $f : (\alpha, \in) \to (\mathcal{B}, \supseteq)$. Furthermore $g : \mathrm{cf}(\alpha) \to \alpha$ strictly increasing and cofinal. Define $\mathcal{B}' := \mathrm{ran}(f \circ g)$, so $\mathcal{B}'$ is cofinal in $(\mathcal{B}, \supsetneq)$ and hence $(\mathcal{B}', \supsetneq)$ has order type $\mathrm{cf}(\alpha) = \mathrm{cf}(\mathcal{B}, \supsetneq)$. Now by cofinality, $\langle \mathcal{B}' \rangle =$

$\langle\mathcal{B}\rangle = \mathcal{N}_x$. Therefore $\mathcal{B}'$ is a regular well-ordered neighbourhood base for x and furthermore $|\mathcal{B}'| = \mathrm{cf}(\mathcal{B}, \supsetneqq)$. ☐

Theorem I.3.24. *Let X be a topological space and let $x \in X$ have a regular well-ordered neighbourhood base $\mathcal{B}$. Then $|\mathcal{B}| = \psi(x)$ and if x is not quasi-isolated, $|\mathcal{B}| = p(x)$.*

Proof. Let $\mathcal{B} = \{B_\alpha : \alpha < |\mathcal{B}|\}$, where $B_\beta \subsetneqq B_\alpha$ for all $\alpha < \beta < |\mathcal{B}|$. Assume $\psi(x) < |\mathcal{B}|$, so there exists a pseudobase $\mathcal{P} \in [\mathcal{N}_x]^{<|\mathcal{B}|}$ for x. Then for every $P \in \mathcal{P}$, there exists an $\alpha_P < |\mathcal{B}|$ such that $B_{\alpha_P} \subseteq P$. Define $\alpha := \sup(\alpha_P : P \in \mathcal{P}) < |\mathcal{B}|$, so $B_\alpha \subseteq \bigcap \mathcal{P} = N_x$. As $\psi(x) < |\mathcal{B}|$ and $|\mathcal{B}|$ is regular, it follows that $\alpha + 1 < |\mathcal{B}|$ and $N_x \subseteq B_{\alpha+1} \subsetneqq B_\alpha$, which is a contradiction. As every neighbourhood base is a pseudobase, it follows that $\psi(x) = |\mathcal{B}|$.

Now assume that x is not quasi-isolated and let $\mathcal{A} \in [\mathcal{N}_x]^{<|\mathcal{B}|}$ be given. Then for all $A \in \mathcal{A}$, there exists a $\beta_A < |\mathcal{B}|$ such that $B_{\beta_A} \subseteq A$. Define $\beta := \sup(\beta_A : A \in \mathcal{A}) < |\mathcal{B}|$, so $B_\beta \subseteq \bigcap \mathcal{A}$. Thus $\bigcap \mathcal{A}$ is a neighbourhood of x and so $p(x) \geq |\mathcal{B}|$. Finally, note that $\bigcap \mathcal{B} = N_x \notin \mathcal{N}_x$ as x is not quasi-isolated. Therefore $p(x) = |\mathcal{B}|$. ☐

Theorem I.3.25. *Let X be a topological space and let $x \in X$ be a non-quasi-isolated point such that $\chi(x) = p(x)$. Then x has a well-ordered neighbourhood base of size $\chi(x)$.*

Proof. Let $\mathcal{B} = \{B_\alpha : \alpha < \chi(x)\}$ be a neighbourhood base of x and define for all $\alpha < |\mathcal{B}|$, $A_\alpha := \bigcap_{\beta \leq \alpha} B_\beta$. Then since $\chi(x) = p(x)$, $\mathcal{A} := \{A_\alpha : \alpha < \chi(x)\}$ is a collection of neighbourhoods of x. Moreover, for every $U \in \mathcal{N}_x$ there exists an $\alpha < \chi(x)$ such that $B_\alpha \subseteq A_\alpha \subseteq U$, so $\mathcal{A}$ is a well-ordered neighbourhood base for x. If $|\mathcal{A}| < p(x)$, then $N_x = \bigcap \mathcal{A} \in \mathcal{N}_x$, which is a contradiction as x is not quasi-isolated. Therefore $p(x) \leq |\mathcal{A}| \leq |\mathcal{B}| \leq \chi(x) = p(x)$ and thus $|\mathcal{A}| = \chi(x)$. ☐

Corollary I.3.26. *Let X be a topological space. Then X is well-based if and only if every point is either quasi-isolated or has equal character and p-character.*

Proof. Note that for every quasi-isolated $x \in X, \{N_x\}$ is a well-ordered neighbourhood base for x, so the statement follows from the previous two theorems. □

The following theorem provides an important separation of p-character and pseudocharacter and will be used to demonstrate some failures of radiality.

Theorem I.3.27. *Let X be a topological space with a well-based point $x \in X$ and let t be a transfinite sequence in $X \setminus N_x$ with regular length. Then:*

(1) If t clusters at x then $p(x) \leq \operatorname{len}(t)$.

(2) If t converges to x then $\operatorname{len}(t) \leq \psi(x)$.

Proof.

(1) First, note that x is not quasi-isolated in X, so $p(x)$ is defined. Assume t clusters at x and suppose $\operatorname{len}(t) < p(x)$. Since $\operatorname{ran}(t) \cap N_x = \emptyset$, it follows that for every $\alpha < \operatorname{len}(t)$, there exists a $U_\alpha \in \mathcal{N}_x$ such that $t(\alpha) \notin U_\alpha$. As $\operatorname{len}(t) < p(x)$, it follows that $U := \bigcap_{\alpha < \operatorname{len}(t)} U_\alpha \in \mathcal{N}_x$. However, $U \cap \operatorname{ran}(t) = \emptyset$, which is a contradiction. Therefore $p(x) \leq \operatorname{len}(t)$.

(2) Suppose t converges to x and choose a pseudobase $\mathcal{P} \in [\mathcal{N}_x]^{\psi(x)}$ for x. Assume $\operatorname{len}(t) > \psi(x)$. Since $t \to x$, for every $P \in \mathcal{P}$, there exists an $\alpha_P < \operatorname{len}(t)$ such that $t[\operatorname{len}(t) \setminus \alpha_P] \subseteq P$. Define $\alpha := \sup(\alpha_P : P \in \mathcal{P}) < \operatorname{len}(t)$. Then

$$t(\alpha) \in \bigcap_{P \in \mathcal{P}} t[\operatorname{len}(t) \setminus \alpha_P] \subseteq \bigcap \mathcal{P} = N_x.$$

This is a contradiction. Hence $\psi(x) \geq \operatorname{len}(t)$. $\square$

Corollary I.3.28. *Let x be a well-based point in a topological space and let t be a transfinite sequence in $X \backslash N_x$ that converges to x. Then $t \dashrightarrow x$ and* $\operatorname{len}(t) = \chi(x)$.

Proof. By the previous theorem and Corollary I.3.26,

$$p(x) \leq \operatorname{len}(t) \leq \psi(x) \leq \chi(x) = p(x).$$

Hence $\operatorname{len}(t) = \chi(x)$.

Now for all $\alpha < \operatorname{len}(t), X \backslash \{t(\alpha)\} \in \mathcal{N}_x$ and so for all $\beta < \operatorname{len}(t), X \setminus t[\beta] = \bigcap_{\alpha<\beta}(X \backslash \{t(\alpha)\}) \in \mathcal{N}_x$ and thus $x \notin \overline{t[\beta]}$. Therefore $t \dashrightarrow x$. $\square$

Theorem I.3.29. *Let x be a well-based point in a topological space and let $t : \chi(x) \to X \backslash N_x$ cluster at x. Then t has a subsequence that converges to x.*

Proof. Define $\kappa := \chi(x)$ and let $(A_\alpha)_{\alpha<\kappa}$ be a regular, well-ordered neighbourhood base for x. Define by recursion

$$g : \kappa \to \kappa, \alpha \mapsto \min(\beta \in \kappa \setminus \sup(g(\alpha') + 1 : \alpha' < \alpha) : t(\beta) \in A_\alpha).$$

As κ is regular, g is well-defined and moreover strictly increasing. Let $U \in \mathcal{N}_x$ be given, so there exists an $\alpha < \kappa$ such that $A_\alpha \subseteq U$. Then there exists a $\beta < \kappa$ such that for every $\gamma \in [\beta, \kappa), g(\gamma) \geq \alpha$ and so $(t \circ g)(\gamma) \in A_{g(\gamma)} \subseteq A_\alpha$. Therefore $t \circ g$ converges to x. $\square$

We now demonstrate how a failure of radiality can occur in general for T_1-spaces. Radiality behaves particularly badly under products.

Proposition I.3.30. *Let X, Y be topological spaces with non-quasi-isolated points $x \in X, y \in Y$ respectively such that $\psi_X(x) < p_Y(y)$. Then (x, y) is not radial in $X \times Y$.*

Proof. Define $A := (X \setminus N_x^X) \times (Y \setminus N_y^Y)$, so $(x, y) \in \overline{X \setminus N_x^X} \times \overline{Y \setminus N_y^Y} = \overline{A}$. Suppose there exists a transfinite sequence t in A with regular length that converges to (x, y). Then $\pi_X \circ t \to x, \pi_Y \circ t \to y$ and $\operatorname{ran}(\pi_X \circ t) \cap N_x^X = \emptyset = \operatorname{ran}(\pi_Y \circ t)$. Thus by Theorem I.3.27 $p_Y(y) \leq \operatorname{len}(t) \leq \psi_X(x)$, which is a contradiction. Therefore (x, y) is not radial in $X \times Y$. $\square$

This easily shows that $(\omega + 1) \times (\omega_1 + 1)$ is not radial, where each space has the order topology. Products of radial spaces can fail to be pseudoradial, as was shown in [OT95, Example 2.2] using the space $[0, 1] \times L(\aleph_1)$, where $L(\aleph_1)$ is the one-point Lindelöfisation of a discrete space of size $\aleph_1$. [Sha93] showed that, for the class of compact Hausdorff spaces, pseudoradiality is consistently equivalent to sequential compactness; consequently, compact, Hausdorff pseudoradial spaces are consistently countably productive. Whether ZFC proves this is an open question; see [Tir06] for a general survey.

The following theorem is a generalisation of the result that every sequential space is semiradial (Corollary I.3.33). Note that in a T_1-space, the condition amounts to the space being $T_1^{R\chi(X)}$.

Theorem I.3.31. *Let X be a pseudoradial space such that for every $F \in [X]^{<R\chi(X)}$ and $x \in \overline{F} \setminus F$, x is quasi-isolated in $F \cup \{x\}$. Then X is semiradial.*

Proof. Let κ be a cardinal and let $A \subseteq X$ be κ-radially-closed. If $\kappa \geq R\chi(X)$ then A is radially-closed, so by pseudoradiality is also closed; in particular it is κ-closed. Assume $\kappa < R\chi(X)$ and let $F \in [A]^{\leq\kappa}$ be given. Suppose $\overline{F} \not\subseteq A$ and choose $x \in \overline{F} \setminus A \subseteq \overline{F} \setminus F$. Then

$N_x^{F\cup\{x\}}$ is open in $F\cup\{x\}$, so there exists a $y\in N_x^{F\cup\{x\}}\cap F\subseteq N_x^X\cap F$. Hence $\{\langle 0,y\rangle\}\to x$ and since A is κ-radially-closed (and $\kappa>0$ because $F\neq\emptyset$), it follows that $x\in A$, which is a contradiction. Therefore A is κ-closed and so X is semiradial. □

The assumption in the previous theorem is technical and can be simplified to give the following corollary:

Corollary I.3.32. *Let X be a pseudoradial space such that for every $F\in[X]^{<R\chi(X)}$, F is quasi-discrete. Then X is semiradial.*

Proof. Let $F\in[X]^{<R\chi(X)}, x\in\overline{F}\backslash F$ be given. Then $|F\cup\{x\}|<\max(R\chi(X),\aleph_0)$, so it follows that $F\cup\{x\}$ is quasi-discrete. Hence by the previous theorem, X is semiradial. □

Corollary I.3.33. *Every sequential space is semiradial.*

Proof. Every sequential space has countable radial character and by Corollary I.3.8, every finite subset is quasi-discrete. Therefore by Corollary I.3.32, every sequential space is semiradial. □

To finish this chapter, we present a slight improvement on Arhangel'skiĭ's result that in the class of T_1-spaces, the sequential spaces are precisely the strictly pseudoradial spaces of countable tightness ([AIT85], quoted in [DIT87, pg. 752]), by removing the T_1 assumption.

Definition I.3.34 (Countably-tight)**.** If X is a topological space such that for every $A\subseteq X$ and each $x\in\overline{A}$ there exists a countable $B\subseteq A$ such that $x\in\overline{B}$, then we say that X is *countably-tight.*

Theorem I.3.35. *Let X be a topological space. Then X is sequential if and only if X is strictly pseudoradial and countably-tight.*

Proof. By the corollary above, every sequential space is semiradial and hence strictly pseudoradial. Let $A \subseteq X$ be given and define $C := \bigcup_{B \in [A]^{\leq \aleph_0}} \overline{B}$. Note that $A \subseteq C \subseteq \overline{A}$. Suppose X is sequential and $C \neq \overline{A}$, so C is not closed. Then there exists a sequence s in C and a limit $x \in \overline{C} \backslash C$ of s and thus for every $n \in \omega$, there exists a countable $B_n \subseteq A$ such that $s(n) \in \overline{B_n}$. Define $B := \bigcup_{n \in \omega} B_n \in [A]^{\leq \aleph_0}$ and let $U \in \mathcal{N}_x$ be given. Then there is some $n \in \omega$ such that $s[\omega \backslash n] \subseteq U$, so in particular $s(n) \in U$. Hence $\emptyset \neq N \cap B_n \subseteq N \cap B$ and thus $x \in \overline{B} \subseteq C$, which is a contradiction. Therefore X is countably-tight.

Now assume that X is strictly pseudoradial and countably-tight and let $A \subseteq X$ be non-closed. Then there exists a transfinite sequence t in A with regular length and an $x \in \overline{A} \backslash A$ such that $t \dashrightarrow x$. Then $x \in \overline{\operatorname{ran}(t)}$, so by countable-tightness there exists a countable $A \subseteq \operatorname{len}(t)$ such that $x \in \overline{t[A]}$. If t has uncountable length then A is bounded in $\operatorname{len}(t)$ and thus $x \notin \overline{t[A]}$, which is a contradiction. Hence t has countable length. If t is a 1-sequence, it can be replaced with a constant ω-sequence, as shown in Section I.1.1. Therefore X is sequential. □

Chapter II

Radiality and spoke systems

II.1 Introduction to nests and spokes

In the previous chapter, we presented some external characterisations of spaces in the pseudoradial hierarchy (Corollary I.2.28). However, they are insufficient to (1) truly gain an understanding of these spaces and properties and (2) prove useful theorems. In this chapter we will develop the theory of *spokes* and *spokes systems*, which allows us to investigate radiality *internally*, leading to a deeper understanding of how this property arises.

The most common examples of radial spaces are LOTS, GO-spaces and well-based spaces (e.g. first-countable spaces). These can be viewed as having neighbourhoods generated by *nests* — sets ordered linearly by inclusion. We will generalise from these spaces to create the class of *independently-based* spaces. Independently-based spaces have a clear picture of convergence and, although they do not coincide with radial spaces (as we will show), a slight weakening of the definition gives an exact characterisation of radiality. Informally, radial spaces have 'neighbourhood generators' that describe the different approaches a sequence may take to converge to a point.

In this section, we will introduce and prove some basic properties of nests, spokes and their respective *systems*, which are collections that generate the neighbourhood filter of a point in a formal sense. We will show that all GO-spaces and well-based points are independently-based.

In Section II.2 we will construct spokes and spoke systems for radial points, giving a characterisation radiality in terms of *almost-independent* spoke systems. The construction of spoke systems scales well when considering a restricting set of order-types, so as to characterise Fréchet-Urysohn points as well. We will also order the spokes via 'local containment near a point', thus giving a different characterisation of radiality.

In Section II.3 we will investigate spoke systems in greater detail, and in particular independent systems. By splitting spoke systems into *components* of common character, we construct cofinal collections of spokes and prove that the minimal cardinality of a spoke system for an independently-based point is attained by an independent spoke system. We will also look at *truncations* of spoke systems and provide sufficient conditions such that, when searching for an independent spoke system, we may find one by truncation. This section concludes by comparing independently-based points with *strongly Fréchet* points and show that, although they are in some sense 'orthogonal', they are not complementary.

In the last section, we will characterise radiality in products, as well as some variations of the Fréchet-Urysohn property: strongly Fréchet and Fréchet-Urysohn for n-point sets (where n is a positive natural number).

The results in Sections II.1, II.2.1, II.2.2 and II.3.3 are based on [Lee14].

II.1.1 Nests

Our first attempt at characterising radiality considers spaces with points that are locally-generated by nests:

Definition II.1.1 (Chains and nests)**.** Let X be a set, $\mathcal{C}$ be a non-empty collection of non-empty subsets of X linearly-ordered by $\subseteq$. Then we say that $\mathcal{C}$ is a *chain* on X.

If X is a topological space and $\mathcal{L}$ is a chain of neighbourhoods for a point $x \in X$ then we say that $\mathcal{L}$ is a *nest* of x.

Definition II.1.2 (Selection)**.** Let $\mathcal{A}$ be a non-empty set consisting of non-empty sets. Then a *selection* of $\mathcal{A}$ is a choice function for $\mathcal{A}$; that is, a function f with domain $\mathcal{A}$ such that $f(A) \in A$ for all $A \in \mathcal{A}$. We will use the notation $(a_A)_{A \in \mathcal{A}}$ for a selection of $\mathcal{A}$ with output a_A for input $A \in \mathcal{A}$.

Similarly, a selection for an indexed family $(A_i)_{i \in I}$ is a family $(a_i)_{i \in I}$ such that $a_i \in A_i$ for each $i \in I$.

Definition II.1.3 (Nest system)**.** For a non-empty collection of chains $\mathfrak{C}$, we define its *mesh* to be

$$\mathbf{M}(\mathfrak{C}) := \left\{ \bigcap_{\mathcal{C} \in \mathfrak{C}} A_{\mathcal{C}} : (A_{\mathcal{C}})_{\mathcal{C} \in \mathfrak{C}} \text{ is a selection of } \mathfrak{C} \right\}.$$

If it is understood that each of the chains consists of subsets of some fixed set X, then for each $\mathcal{C} \in \mathfrak{C}$ we define the *$\mathcal{C}$-th spoke of* $\mathfrak{C}$ to be $\mathfrak{C}{\downarrow_{\bullet}}_{\mathcal{C}} := \bigcap_{\mathcal{D} \in \mathfrak{C} \setminus \{\mathcal{C}\}} \left(\bigcap \mathcal{D}\right)$.

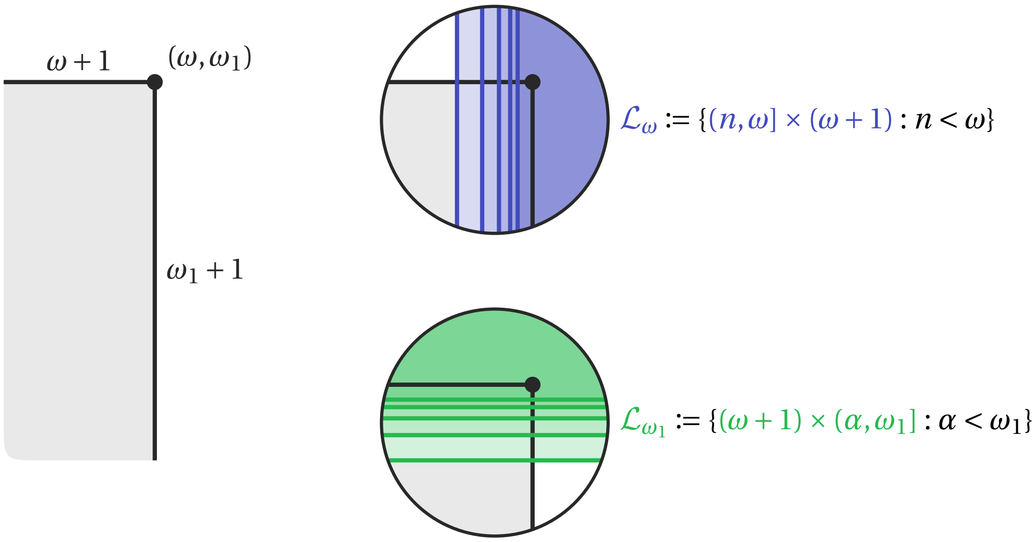

Figure II.1: A nest system $\mathfrak{L} := \{\mathcal{L}_\omega, \mathcal{L}_{\omega_1}\}$ for (ω,ω_1).

Let X be a space, $x \in X$ be given and let $\mathfrak{L}$ be a non-empty family of nests of x. We shall call $\mathfrak{L}$ a *nest system* of x if $\mathbf{M}(\mathfrak{L})$ is a neighbourhood base for x.

As was proven in Chapter 1 (see Theorem I.2.2), well-based points are radial and clearly have a nest system consisting of one nest. In the case of spaces with neighbourhoods generated by two nests, consider the Tychonoff plank $X := (\omega+1) \times (\omega_1+1)$, where each factor is topologised by the the linear ordinal ordering. As observed in Proposition I.3.30 and its following paragraph, X is not radial at (ω,ω_1), but still has a nest system with two nests (Figure II.1).

However, every LOTS is radial and each point in a LOTS has a nest system consisting of at most two nests. The main difference between LOTS and the Tychonoff plank is that each neighbourhood (a,b) of x in a LOTS can be split into two parts, $(a,x]$ and $[x,b)$, each acting independently from the other in the sense that $\overline{A} = \overline{A \cap (a,x]} \cup \overline{A \cap [x,b)}$ and both

$(a,x]$ and $[x,b)$ are well-based at x. LOTS are the inspiration for the definition of this new class of spaces.

Definition II.1.4 (Independence)**.** Let X be a space, $x \in X$ and let $\mathfrak{C}$ be a non-empty family of chains. We say that $\mathfrak{C}$ is *independent* if for every selection $(U_{\mathcal{C}})_{\mathcal{C}\in\mathfrak{C}}$ of $\mathfrak{C}$,

$$\bigcap_{\mathcal{C}\in\mathfrak{C}} U_{\mathcal{C}} = \bigcup_{\mathcal{C}\in\mathfrak{C}} (U_{\mathcal{C}} \cap \mathfrak{C}{\downarrow_{\bullet}}_{\mathcal{C}}).$$

A point is said to be *independently-based* if it has an independent nest system and a space is called *independently-based* if each of its points is independently-based.

To justify this definition and its motivation, we now show that well-based and GO-spaces are independently-based.

Theorem II.1.5. *Every well-based point is independently-based.*

Proof. Let X be a topological space and let $x \in X$ be well-based. Then there exists a well-ordered neighbourhood base $\mathcal{B}$ for x. Thus $\{\mathcal{B}\}$ is obviously an independent nest system of x, so x is independently-based. □

Theorem II.1.6. *Every LOTS is independently-based.*

Proof. Let $(X,<)$ be a linearly ordered set and endow X with the order topology inherited from $<$. Let $x \in X$ be given and define,

$$\mathcal{L} := \{(-\infty, y) : z \in (x,\infty]\}, \qquad \mathcal{R} := \{(y,\infty) : y \in [-\infty, x)\}.$$

Then $\mathfrak{L} = \{\mathcal{L}, \mathcal{R}\}$ is a family of nests of neighbourhoods of x. Note that

$$\mathbf{M}(\mathfrak{L}) = \{(y,z) : y \in [-\infty, x), z \in (x, \infty]\},$$

$$\mathfrak{L}{\downarrow_{\bullet}}_{\mathcal{L}} = (-\infty, x],$$

$$\mathfrak{L}{\downarrow_{\bullet}}_{\mathcal{R}} = [x, \infty).$$

Hence $\mathfrak{L}$ is a nest system of x. Moreover, for every $y \in [-\infty, x)$ and $z \in (x, \infty]$,

$$(y,\infty) \cap (-\infty, z) = (y,z) = (y,x] \cup [x,z) = ((y,\infty) \cap \mathfrak{L}{\downarrow_{\bullet}}_{\mathcal{L}}) \cup ((-\infty, z) \cap \mathfrak{L}{\downarrow_{\bullet}}_{\mathcal{R}}).$$

Therefore $\mathfrak{L}$ is independent and hence X is independently-based. □

The following lemma now implies that all GO-spaces are independently-based:

Lemma II.1.7. *Let X be a space, $Y \subseteq X$ and let $y \in Y$ be independently-based with respect to X. Then y is independently-based with respect to Y.*

Proof. Pick an independent nest system $\mathfrak{L}$ of y with respect to X. For every $\mathcal{L} \in \mathfrak{L}$, define $\mathcal{M}_{\mathcal{L}} := \{U \cap Y : U \in \mathcal{L}\}$. Then $\mathfrak{M} := \{\mathcal{M}_{\mathcal{L}} : \mathcal{L} \in \mathfrak{L}\}$ is a non-empty family of nests of Y-neighbourhoods of y and $\mathfrak{M}{\downarrow_{\bullet}}_{\mathcal{M}_{\mathcal{L}}} = \mathfrak{L}{\downarrow_{\bullet}}_{\mathcal{L}} \cap Y$ for all $\mathcal{L} \in \mathfrak{L}$.

Select $(U_{\mathcal{M}})_{\mathcal{M} \in \mathfrak{M}}$ from $\mathfrak{M}$, so there exists a selection $(V_{\mathcal{L}})_{\mathcal{L} \in \mathfrak{L}}$ of $\mathfrak{L}$ such that $V_{\mathcal{L}} \cap Y = U_{\mathcal{M}_{\mathcal{L}}}$ for all $\mathcal{L} \in \mathfrak{L}$. Then

$$\begin{aligned}\bigcap_{\mathcal{M} \in \mathfrak{M}} U_{\mathcal{M}} &= \left(\bigcap_{\mathcal{L} \in \mathfrak{L}} V_{\mathcal{L}}\right) \cap Y \in \mathcal{N}_y^Y \\ &= \left(\bigcup_{\mathcal{L} \in \mathfrak{L}} (V_{\mathcal{L}} \cap \mathfrak{L}{\downarrow_{\bullet}}_{\mathcal{L}})\right) \cap Y = \bigcup_{\mathcal{M} \in \mathfrak{M}} (U_{\mathcal{M}} \cap \mathfrak{M}{\downarrow_{\bullet}}_{\mathcal{M}}).\end{aligned}$$

Thus $\mathbf{M}(\mathfrak{M}) \subseteq \mathcal{N}_y^Y$ and $\mathfrak{M}$ is independent.

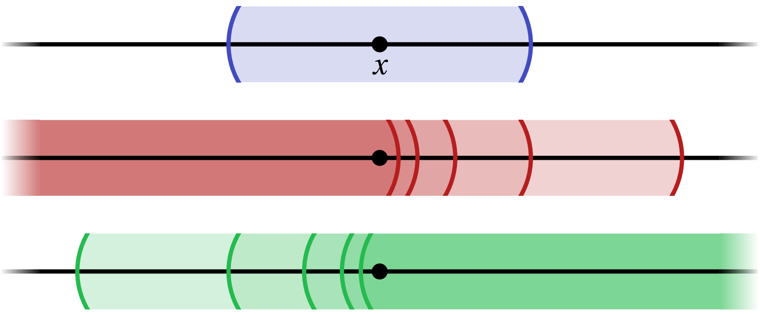

Figure II.2: A generic neighbourhood in a GO-space and two nests $\mathcal{L} = \{(-\infty, y) : z \in (x, \infty]\}$ and $\mathcal{R} = \{(y, \infty) : y \in [-\infty, x)\}$, forming an independent nest system.

Finally, select $(U_{\mathcal{L}})_{\mathcal{L}\in\mathfrak{L}}$ from $\mathfrak{L}$ and for all $\mathcal{M} \in \mathfrak{M}$, let $\mathcal{L}_{\mathcal{M}} \in \mathfrak{L}$ be given such that $\mathcal{M} = \mathcal{M}_{\mathcal{L}_{\mathcal{M}}}$. Then

$$\begin{aligned}\left(\bigcap_{\mathcal{L}\in\mathfrak{L}} U_{\mathcal{L}}\right) \cap Y &= \left(\bigcap_{\mathcal{L}\in\mathfrak{L}} (U_{\mathcal{L}} \cap \mathfrak{L}{\downarrow}_{\mathcal{L}})\right) \cap Y \\ &= \bigcup_{\mathcal{L}\in\mathfrak{L}} (U_{\mathcal{L}} \cap \mathfrak{M}{\downarrow}_{\mathcal{M}_{\mathcal{L}}}) \\ &\supseteq \bigcup_{\mathcal{M}\in\mathfrak{M}} (U_{\mathcal{L}_{\mathcal{M}}} \cap \mathfrak{M}{\downarrow}_{\mathcal{M}}) \\ &= \bigcap_{\mathcal{M}\in\mathfrak{M}} (U_{\mathcal{L}_{\mathcal{M}}} \cap Y).\end{aligned}$$

Therefore $\mathbf{M}(\mathfrak{M})$ is a neighbourhood base for y with respect to Y; hence $\mathfrak{M}$ is an independent nest system of y with respect to Y. Thus y is independently-based in Y. □

II.1.2 Spokes

Rather than approximating a point from outside via nests, we can build up a neighbourhood base using spokes as a primitive notion.

Definition II.1.8 (Spoke systems)**.** Let X be a topological space with a point $x \in X$. If S is a subspace such that $N_x \subseteq S$ and S is well-based at x, then we say that S is a *spoke* of x in X. We denote the set of spokes at x by $\mathbf{Sp}_X(x)$ and we will dispense with the subscript if the space is unambiguous.

Let $\mathfrak{S}$ be a non-empty collection of spokes of x, such that

$$\left\{ \bigcup_{S\in\mathfrak{S}} U_S : (U_S)_{S\in\mathfrak{S}} \text{ is a selection of } (\mathcal{N}_x^S)_{S\in\mathfrak{S}} \right\} \tag{$*$}$$

is a neighbourhood base for x. Then we say that $\mathfrak{S}$ is a *spoke system* of x. Note that $(*)$ will always form a network[3] of x for any collection of spokes.

If $\mathcal{S}$ is a non-empty collection of spokes of x such that $S \cap T = N_x$ for all distinct $S, T \in \mathfrak{S}$, then we say that $\mathfrak{S}$ is *independent.*

The following lemma demonstrates that the spokes of a nest system do in fact form spokes of the point, linking the two definitions. The second part of the lemma will be used to convert independent nest systems to independent spoke systems in the next theorem.

Lemma II.1.9. *Let X be a space, $x \in X$ and let $\mathfrak{L}$ be a nest system of x.*

(1) For all $\mathcal{L} \in \mathfrak{L}, \mathfrak{L}{\downarrow_\bullet}_{\mathcal{L}} \in \mathbf{Sp}(x)$.

(2) If $\mathfrak{L}$ is independent then for all distinct $\mathcal{L}_1, \mathcal{L}_2 \in \mathfrak{L}, \mathfrak{L}{\downarrow_\bullet}_{\mathcal{L}_1} \cap \mathfrak{L}{\downarrow_\bullet}_{\mathcal{L}_2} = N_x$.

Proof.

(1) For every $\mathcal{L} \in \mathfrak{L}, \mathcal{L} \subseteq \mathcal{N}_x$ and so $N_x = \bigcap \mathcal{N}_x \subseteq \bigcap \mathcal{L}$. Hence

$$\mathfrak{L}{\downarrow_\bullet}_{\mathcal{L}} = \bigcap_{\mathcal{M}\in\mathfrak{L}\setminus\{\mathcal{L}\}} \left(\bigcap \mathcal{M}\right) \supseteq N_x.$$

---

[3] A non-empty collection of subsets $\mathcal{A}$, each containing x, is a *network* of x if for each $U \in \mathcal{N}_x$, there exists an $A \in \mathcal{A}$ such that $A \subseteq U$.

Furthermore, $\{U \cap \mathfrak{L}{\downarrow}_{\mathcal{L}} : U \in \mathcal{L}\}$ is a linearly-ordered neighbourhood base for x with respect to $\mathfrak{L}{\downarrow}_{\mathcal{L}}$, so x is well-based in $\mathfrak{L}{\downarrow}_{\mathcal{L}}$. Therefore $\mathfrak{L}{\downarrow}_{\mathcal{L}} \in \mathbf{Sp}(x)$.

(2) Let $\mathcal{L}_1, \mathcal{L}_2 \in \mathfrak{L}$ be distinct. Then since $\mathbf{M}(\mathfrak{L})$ is a neighbourhood base for x,

$$\begin{aligned}\mathfrak{L}{\downarrow}_{\mathcal{L}_1} \cap \mathfrak{L}{\downarrow}_{\mathcal{L}_2} &= \Big(\bigcap_{\mathcal{M} \in \mathfrak{L} \setminus \{\mathcal{L}_1\}} \big(\bigcap \mathcal{M}\big)\Big) \cap \Big(\bigcap_{\mathcal{M} \in \mathfrak{L} \setminus \{\mathcal{L}_2\}} \big(\bigcap \mathcal{M}\big)\Big) \\ &= \bigcap_{\mathcal{M} \in \mathfrak{L}} \big(\bigcap \mathcal{M}\big) \\ &= N_x. \end{aligned}$$

□

Theorem II.1.10. *Let X be a space and let $x \in X$ be given.*

(1) Let $\mathfrak{L}$ be an independent nest system of x. Then $\mathfrak{S} := \{\mathfrak{L}{\downarrow}_{\mathcal{L}} : \mathcal{L} \in \mathfrak{L}\}$ is an independent spoke system of x.

(2) Let $\mathfrak{S}$ be an independent spoke system of x, so for every $S \in \mathfrak{S}$ there exists a well-ordered neighbourhood base $\mathcal{B}_S$ for x with respect to S. For all $S \in \mathfrak{S}$, define

$$\mathcal{L}_S := \Big\{B \cup \bigcup(\mathfrak{S} \setminus \{S\}) : B \in \mathcal{B}_S\Big\}.$$

Then $\mathfrak{L} := \{\mathcal{L}_S : S \in \mathfrak{S}\}$ is an independent nest system of x.

Proof.

(1) Select $(U_S)_{S \in \mathfrak{S}}$ from $(\mathcal{N}_x^S)_{S \in \mathfrak{S}}$, so there exists a selection $(V_{\mathcal{L}})_{\mathcal{L} \in \mathfrak{L}}$ of $\mathfrak{L}$ such that $V_{\mathcal{L}} \cap \mathfrak{L}{\downarrow}_{\mathcal{L}} \subseteq U_{\mathfrak{L}{\downarrow}_{\mathcal{L}}}$ for all $\mathcal{L} \in \mathfrak{L}$. Then

$$\bigcup_{S \in \mathfrak{S}} U_S \supseteq \bigcup_{\mathcal{L} \in \mathfrak{L}} (V_{\mathcal{L}} \cap \mathfrak{L}{\downarrow}_{\mathcal{L}}) = \bigcap_{\mathcal{L} \in \mathfrak{L}} V_{\mathcal{L}} \in \mathcal{N}_x^X$$

Therefore $\mathfrak{S}$ is a spoke system of x. Moreover, it is independent by the previous lemma.

(2) Select $(U_{\mathcal{L}})_{\mathcal{L}\in\mathfrak{L}}$ from $\mathfrak{L}$, so there exists a selection $(B_S)_{S\in\mathfrak{S}}$ from $(\mathcal{B}_S)_{S\in\mathfrak{S}}$ such that $U_{\mathcal{L}_S} = B_S \cup \bigcup(\mathfrak{S}\setminus\{S\})$ for all $S\in\mathfrak{S}$. Then since $\mathfrak{S}$ is independent,

$$\bigcap_{\mathcal{L}\in\mathfrak{L}} U_{\mathcal{L}} = \bigcap_{S\in\mathfrak{S}} \left(B_S \cup \bigcup(\mathfrak{S}\setminus\{S\})\right) = \bigcup_{S\in\mathfrak{S}} B_S \in \mathcal{N}_x^X \tag{$*$}$$

Note that for distinct $S, T \in \mathfrak{S}$,

$$\left(\bigcap\mathcal{L}_S\right)\Delta\left(\bigcap\mathcal{L}_T\right) = \left(N_x \cup \bigcup(\mathfrak{S}\setminus\{S\})\right)\Delta\left(N_x \cup \bigcup(\mathfrak{S}\setminus\{T\})\right) = (S\Delta T)\setminus N_x.$$

Since $N_x \subseteq S\cap T$, it follows that $(S\Delta T)\setminus N_x \neq \emptyset$ and hence $\mathcal{L}_S \neq \mathcal{L}_T$. It follows from $(*)$ that for every selection $(U_S)_{S\in\mathfrak{S}}$ of $(\mathcal{N}_x^S)_{S\in\mathfrak{S}}$, there exists a selection $(B_S)_{S\in\mathfrak{S}}$ of $(\mathcal{B}_S)_{S\in\mathfrak{S}}$ such that $B_S \subseteq U_S$ for each $S\in\mathfrak{S}$, and a selection $(V_{\mathcal{L}})_{\mathcal{L}\in\mathfrak{L}}$ of $\mathfrak{L}$ such that $\bigcap_{\mathcal{L}\in\mathfrak{L}} V_{\mathcal{L}} = \bigcup_{S\in\mathfrak{S}} B_S \subseteq \bigcup_{S\in\mathfrak{S}} U_S$. Therefore $\mathfrak{L}$ is a nest system of x.

Finally, note that for all $S\in\mathfrak{S}$,

$$\mathfrak{L}{\downarrow_{\bullet}}_{\mathcal{L}_S} = \bigcap_{T\in\mathfrak{S}\setminus\{S\}} \left(\bigcap\mathcal{L}_T\right) = \bigcap_{T\in\mathfrak{S}\setminus\{S\}} \left(N_x \cup \bigcup(\mathfrak{S}\setminus\{T\})\right) = S.$$

So since $\mathfrak{S}$ is independent, from $(*)$ it follows that for every selection $(B_S)_{S\in\mathfrak{S}}$ of $(\mathcal{B}_S)_{S\in\mathfrak{S}}$,

$$\bigcap_{S\in\mathfrak{S}} \left(B_S \cap \bigcup(\mathfrak{S}\setminus\{S\})\right) = \bigcup_{S\in\mathfrak{S}} B_S = \bigcup_{S\in\mathfrak{S}} \left(\left(B_S \cup \bigcup(\mathfrak{S}\setminus\{S\})\right) \cap \mathfrak{L}{\downarrow_{\bullet}}_{\mathcal{L}_S}\right)$$

Therefore $\mathfrak{L}$ is independent. □

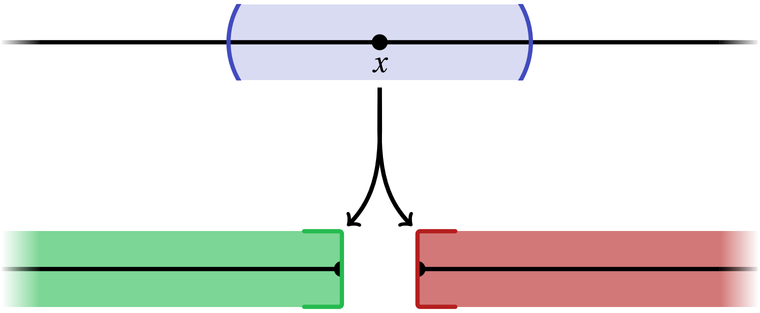

Figure II.3: A generic neighbourhood in a GO-space and two spokes $(-\infty, x], [x, \infty)$ forming an independent spoke system.

Using the previous theorem and applying it to the spoke systems of a GO-space $(X, <)$ obtained from the nest system in Figure II.2, we see that $\{(-\infty, x], [x, \infty)\}$ is an independent spoke system of $x \in X$ (Figure II.3). As another demonstration of Theorem II.1.10, we consider the sequential hedgehog.

Definition II.1.11 (Sequential hedgehog)**.** Define $S(\omega) := \{\star\} \cup (\omega \times \omega)$, where $\star \notin \omega \times \omega$. We topologise $S(\omega)$ by letting $\{\{\star\} \cup \{(m, n) \in \omega \times \omega : n \geq f(m)\} : f \in {}^{\omega}\omega\}$ be a neighbourhood base of $\star$ and let all other points be isolated. We call $S(\omega)$ the *sequential hedgehog*.

It is easy to see that $S(\omega)$ is independently-based, as demonstrated in Figure II.4.

Corollary II.1.12. *A point is independently-based if and only if it has an independent spoke system.*

Thus instead of considering neighbourhood bases created by meshing nests, we can build them by gluing suitable subspaces together. The former viewpoint has it's advantages (for example, we could consider a global collection of nests of open sets, which generates an independent nest system for each point); however, from this point onwards

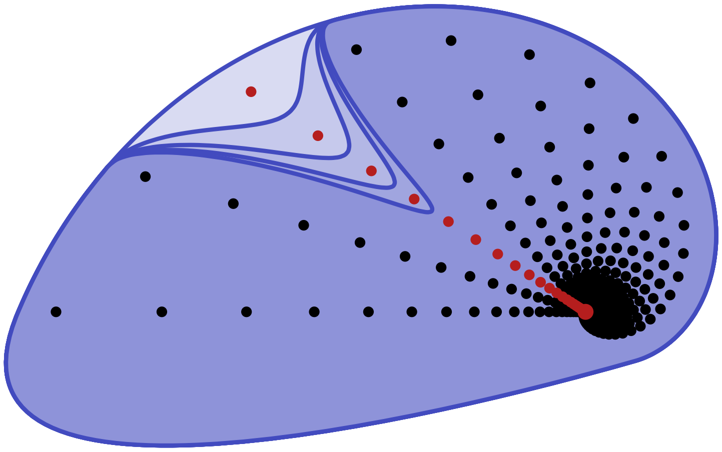

Figure II.4: $\mathfrak{L} := \{\mathcal{L}_n : n < \omega\}$ is an independent nest system of $\star$ in $S(\omega)$, where for each $n < \omega$ we define $\mathcal{L}_n := \{S(\omega) \setminus (\{n\} \times m) : m \in \omega\}$. The spokes of this system are $\mathfrak{L}{\downarrow}_{\mathcal{L}_n} = \{\star\} \cup (\{n\} \times \omega)$.

we will use spoke systems in place of nest systems, as they are more amenable to manipulation and have lower complexity, consisting of subspaces rather than collections thereof, and also characterise radiality exactly (see Corollary II.2.7, paragraph after Theorem II.2.10, Theorem II.3.23, Theorem II.3.24).

The following theorem provides an alternative characterisation of spoke systems, which shows that points with spoke systems are radial.

Theorem II.1.13. *Let X be a topological space, $x \in X$ be given and let $\mathfrak{S}$ be a non-empty collection of spokes at x. Then $\mathfrak{S}$ is a spoke system of x if and only if for every $A \subseteq X$ with $x \in \overline{A}$, there exists an $S \in \mathfrak{S}$ such that $x \in \overline{A \cap S}$.*

Proof. Suppose $\mathfrak{S}$ is a spoke system of x and let $A \subseteq X$ be given such that $x \notin \overline{A \cap S}$ for every $S \in \mathfrak{S}$. Then $U := \bigcup_{S \in \mathfrak{S}} (S \setminus \overline{A \cap S})$ is an X-neighbourhood of x disjoint from A, and so $x \notin \overline{A}$.

Now suppose that $\mathfrak{S}$ is not a spoke system of x, so for all $S \in \mathcal{S}$, there exists a $U_S \in \mathcal{N}_x^S$ such that $U := \bigcup_{S \in \mathfrak{S}} U_S \notin \mathcal{N}_x^X$, hence $x \in \overline{X \backslash U}$. Furthermore, $S \backslash U \subseteq S \backslash U_S$ and $x \notin \overline{S \backslash U_S}$ for all $S \in \mathfrak{S}$, so $x \notin \overline{S \backslash U}$. □

Corollary II.1.14. *Every point with a spoke system is radial.*

Proof. Let X be a topological space, $x \in X$ be given and let $\mathfrak{S}$ be a spoke system of x. Let $A \subseteq X$ be given such that $x \in \overline{A}$, so by the previous theorem there exists an $S \in \mathfrak{S}$ such that $x \in \overline{A \cap S}$. As x is well-based in S, it is also radial in S by Theorem I.2.2, so there exists a transfinite sequence in $A \cap S \subseteq A$ that converges to x. Therefore x is radial. □

Corollary II.1.15. *Every independently-based point is radial. In particular, every GO-space is radial.*

Theorem II.1.13 allows us to push spoke systems down to subspaces:

Corollary II.1.16. *Let $\mathfrak{S}$ be a spoke system of a point x in a topological space X and let $Y \subseteq X$ be given with $x \in Y$. Then $\{S \cap Y : S \in \mathfrak{S}\}$ is a spoke system of x.*

Proof. First, note that by Lemma I.3.9 $N_x^Y = N_x^X \cap Y$, so $S \cap Y \in \mathbf{Sp}_Y(x)$ for all $S \in \mathfrak{S}$. Let $A \subseteq Y$ be given such that $x \in \overline{A}^Y \subseteq \overline{A}^X$, so by Theorem II.1.13 there exists an $S \in \mathfrak{S}$ such that $x \in \overline{A \cap S}^X \cap Y = \overline{A \cap (S \cap Y)}^Y$. Therefore, applying Theorem II.1.13 again, $\{S \cap Y : S \in \mathfrak{S}\}$. □

It is worth observing at this point that any collection of spokes that extends a spoke system is also a spoke system:

Lemma II.1.17. *Let $\mathfrak{S}$ be a spoke system of a point x in a topological space X and let $\mathfrak{T}$ be a collection of spokes of x that extends $\mathfrak{S}$. Then $\mathfrak{T}$ is also a spoke system of x.*

Proof. Select $(U_T)_{T\in\mathfrak{T}}$ from $(\mathcal{N}_x^T)_{T\in\mathfrak{T}}$, so $\bigcup_{T\in\mathfrak{T}} U_T \supseteq \bigcup_{S\in\mathfrak{S}} U_S \in \mathcal{N}_x^X$. Thus $\bigcup_{T\in\mathfrak{T}} U_T \in \mathcal{N}_x^X$ and therefore $\mathfrak{T}$ is a spoke system of x. □

Corollary II.1.18. *If a point x has a spoke system, then* $\mathbf{Sp}(x)$ *is a spoke system of x.*

II.2 An internal characterisation of radiality

In this section, we will construct several characterisations of radial and Fréchet-Urysohn points using spokes and spoke systems. We first need to show how radial points give rise to spokes via strictly convergent sequences.

II.2.1 Basic spokes

Recall that a transfinite sequence t converges strictly to a point x, denoted by $t \dashrightarrow x$, if $t \to x$ and $x \notin \overline{t[\alpha]}$ for all $\alpha < \operatorname{len}(t)$. We can restrict the transfinite sequences that witness radiality to those that strictly converge.

Lemma II.2.1. *Let x be a point in a topological space X and let t be a transfinite sequence that converges strictly to x with regular length. Then* $\operatorname{len}(t) \leq |X|$.

Proof. Suppose $\operatorname{len}(t) > |X|$, so there exists an unbounded $A \subseteq \operatorname{len}(t)$ such that $t|_A$ is constant. Then $t|_A \to x$ and so $y := t(\min(A)) \in N_x$. Since $t \dashrightarrow x$, it follows that $\operatorname{len}(t) = 1$, which is a contradiction. Therefore $\operatorname{len}(t) \leq |X|$. □

Theorem II.2.2. *Let X be a space, $x \in X$ be radial and let $A \subseteq X$ be given such that $x \in \overline{A}$. Then there exists a regular cardinal $\lambda \leq |A|$ and an injective transfinite sequence in A that converges strictly to x.*

Proof. If $A \cap N_x \neq \emptyset$, pick $y \in A \cap N_x$, so $\{\langle 0, y\rangle\} \dashrightarrow x$. Now suppose otherwise, so by radiality there exists a transfinite sequence t in A that converges to x. Assume that $\operatorname{len}(t)$ has minimal length. Note that if $\operatorname{len}(t)$ is not regular then there is an $\gamma < \operatorname{len}(t)$ and a strictly increasing, cofinal map $f : \gamma \to \operatorname{len}(t)$, so $t \circ f \to x$. This is a contradiction by the minimality of $\operatorname{len}(t)$. Hence $\operatorname{len}(t)$ is regular and moreover infinite.

Let $\alpha < \operatorname{len}(t)$ be given and suppose $x \in \overline{t[\alpha]}$. By the above conclusion, there exists a transfinite sequence u with regular length contained in $t[\alpha]$ that converges to x. By minimality $\kappa \geq \operatorname{len}(t) > |\alpha| \geq |\operatorname{ran}(u)|$, so by regularity there exists an unbounded $A \subseteq \operatorname{len}(u)$ such that for all $\alpha, \beta \in A, u(\alpha) = u(\beta)$. Thus $u|_A \to x$ and so $u(\min(A)) \in A \cap N_x$, which is a contradiction. Hence $x \notin \overline{t[\alpha]}$ and therefore $t \dashrightarrow x$. Moreover, for all $\alpha < \operatorname{len}(t), \operatorname{ran}(t) \not\subseteq t[\alpha]$.

Now for every $\alpha < \operatorname{len}(t)$, define by transfinite recursion,

$$g(\alpha) := \min(\beta < \operatorname{len}(t) : \beta > \sup(g[\alpha]) \text{ and } t(\beta) \notin (t \circ g)[\alpha]).$$

Then $g : \operatorname{len}(t) \to \operatorname{len}(t)$ is strictly increasing, and $t \circ g$ is injective and converges strictly to x. Furthermore, $\operatorname{len}(t) \leq |A|$ by injectivity. □

Before constructing spokes with these strictly convergent transfinite sequences, we observe that they interact well with spoke systems.

Lemma II.2.3. *Let $\mathfrak{S}$ be a spoke system of a point x in a topological space X, and let t be a transfinite sequence in X, with regular length, that converges strictly to x. Then there exists an $S \in \mathfrak{S}$ that contains a subsequence of t.*

Proof. By Theorem II.1.13, there exists an $S \in \mathfrak{S}$ such that $x \in \overline{\operatorname{ran}(t) \cap S}$. Since $t \dashrightarrow x$, it follows that $A := t^{-1}[S]$ is unbounded in $\operatorname{len}(t)$, so there exists a strictly increasing bijection $f : \operatorname{len}(t) \to A$ and thus $\operatorname{ran}(t \circ f) \subseteq S$. □

Strictly convergent transfinite sequences allow us to construct spokes:

Theorem II.2.4. *Let X be a topological space, $x \in X$ be given, and let t be a transfinite sequence that converges strictly to x with regular length. Define $B_\alpha := N_x \cup t[\operatorname{len}(t) \backslash \alpha]$ for all $\alpha < \operatorname{len}(t)$. Then $(B_\alpha)_{\alpha < \operatorname{len}(t)}$ is a regular well-ordered neighbourhood base for x with respect to $\mathbb{S}_t := \operatorname{ran}(t) \cup N_x$. In particular, $\mathbb{S}_t$ is a spoke of x.*

Proof. Let $\alpha < \operatorname{len}(t)$ be given. Then $x \notin \overline{t[\alpha]}$, so $B_\alpha \supseteq \mathbb{S}_t \backslash \overline{t[\alpha]} \in \mathcal{N}_x^{\mathbb{S}_t}$. Since $(B_\alpha)_{\alpha < \operatorname{len}(t)}$ is a network at x, it follows that it is a regular well-ordered neighbourhood base for x with respect to $\mathbb{S}_t$, and therefore $\mathbb{S}_t$ is a spoke of x. □

We refer to spokes of the form $\mathbb{S}_t$ as *basic spokes* of x, the collection of which is denoted by $\mathbf{BSp}_X(x)$ or $\mathbf{BSp}(x)$ if the space is unambiguous. If we need to indicate the surrounding space, we will use $\mathbb{S}_t^X$ instead of $\mathbb{S}_t$.

If a spoke system consists of basic spokes, we shall call it a *basic spoke system.* Similarly, if a property $\mathbf{P}$ relates spaces to points, written $X \mathbin{\mathbf{P}} x$, we shall say that a spoke system $\mathfrak{S}$ for a point x in a topological space is a $\mathbf{P}$ *spoke system* provided $S \mathbin{\mathbf{P}} x$ for all $S \in \mathfrak{S}$. For example, we will characterise the Fréchet-Urysohn property in terms of first-countable spokes (Corollary II.2.17).

II.2.2 Neighbourhood characterisation of radiality

Corollary II.1.14 suggests the following question:

> Does every radial point admit a spoke system, independent or otherwise?

If we weaken the condition for independence slightly, we do indeed have an exact characterisation of radiality.

Definition II.2.5 (Almost-independent)**.** Let $\mathcal{S}$ be a non-empty collection of spokes for a point x in a topological space. Then we say that $\mathcal{S}$ is *almost-independent* if $x \notin \overline{(S \cap T) \setminus N_x}$ for all distinct $S, T \in \mathcal{S}$.

Observe that for an independent collection $\mathcal{S}$ of spokes, $(S \cap T) \setminus N_x = \emptyset$ for all distinct $S, T \in \mathcal{S}$.

The following theorem shows that any almost-independent collection of basic spokes can be extended to an almost-independent spoke system. We will use this extension theorem in Section II.4 to characterise stronger versions of the Fréchet-Urysohn property.

Theorem II.2.6. *Let $\mathcal{S}$ be an almost-independent collection of basic spokes for a radial point x in a topological space X. Then there exists an almost-independent, basic spoke system that extends $\mathcal{S}$.*

Proof. If x is quasi-isolated then $\{N_x\}$ is a spoke system for x and $N_x = \mathbb{S}_{\{\langle 0,x\rangle\}}$. Thus by Lemma II.1.17 $\mathcal{S} \cup \{N_x\}$ is an almost-independent, basic spoke system for x.

Now suppose that x is not quasi-isolated. For each $S \in \mathcal{S}$, pick a transfinite sequence t_S with regular length that converges strictly to x such that $S = \mathbb{S}_{t_S}$. Define

$$\mathcal{T} := \{t_S : S \in \mathcal{S}\},$$

$$\mathcal{U} := \{t : t \text{ is a transfinite sequence in } X\backslash N_x, \operatorname{len}(t) \text{ is regular and } t \dashrightarrow x\},$$

$$\mathcal{A} := \{\mathcal{V} \subseteq \mathcal{U} : \mathcal{T} \subseteq \mathcal{V} \text{ and for all distinct } u, v \in \mathcal{V}, x \notin \overline{(\mathbb{S}_u \cap \mathbb{S}_v)\backslash N_x}\}.$$

Note that by Lemma II.2.1 and radiality, $\mathcal{U}$ and $\mathcal{A}$ are non-empty sets. Thus by Tukey's Lemma there exists a maximal $\mathcal{V} \in \mathcal{A}$. Define $\mathfrak{S} := \{\mathbb{S}_v : v \in \mathcal{V}\} \supseteq \mathcal{S}$. Note that given distinct $u, v \in \mathcal{V}$, $x \notin \overline{(\mathbb{S}_u \cap \mathbb{S}_v)\backslash N_x}$, and since $x \in \overline{\mathbb{S}_u\backslash N_x} \cap \overline{\mathbb{S}_v\backslash N_x}$, it follows that $\mathbb{S}_u \neq \mathbb{S}_v$. Therefore the map $\mathcal{V} \to \mathfrak{S}, v \mapsto \mathbb{S}_v$ is bijective.

Now select $(U_S)_{S\in\mathfrak{S}}$ from $(\mathcal{N}_x^S)_{S\in\mathfrak{S}}$, so there exists a selection $(\alpha_v)_{v\in\mathcal{V}}$ of $(\operatorname{len}(v))_{v\in\mathcal{V}}$ such that $N_x^{\mathbb{S}_v} \cup v[[\alpha_v, \operatorname{len}(v))] \cup N_x \subseteq U_{\mathbb{S}_v}$ for all $v \in \mathcal{V}$. Define

$$U := \bigcup_{S\in\mathfrak{S}} U_S,$$

$$V := \bigcup_{v\in\mathcal{V}} (v[\operatorname{len}(v)\backslash\alpha_v] \cup N_x).$$

Note that $V \subseteq U$ and suppose $V \notin \mathcal{N}_x^X$. Then by Theorem II.2.2 there exists a transfinite sequence u in $X\backslash V$ that converges strictly to x. Let $v \in \mathcal{V}$ be given. Then since $\operatorname{ran}(u) \subseteq X\backslash V$ and $v[[\alpha_v, \operatorname{len}(v))] \subseteq U$, it follows that $(\mathbb{S}_u \cap \mathbb{S}_v)\backslash N_x \subseteq v[\alpha_v]$ and so $x \notin \overline{(\mathbb{S}_u \cap \mathbb{S}_v)\backslash N_x}$. Hence by maximality, $u \in \mathcal{V}$, which is a contradiction because $u(\alpha_u) \in V$. Therefore $V \in \mathcal{N}_x^X$ and thus $\mathfrak{S}$ is an almost-independent, basic spoke system for x. □

Applying this theorem to the empty collection, we arrive at a characterisation of radiality:

Theorem II.2.7. *Let x be a point in a topological space. Then the following are equivalent:*

(1) x is radial.

(2) $\mathbf{Sp}(x)$ *is a spoke system of x.*

(3) x has a spoke system.

(4) x has an almost-independent, basic spoke system.

Proof. The previous theorem, applied to the empty collection of basic spokes, shows that (1) implies (4), which in turn vacuously implies (3). Then by Corollary II.1.18, (3) implies (2), which, by Corollary II.1.14, implies (1). □

II.2.3 Radial spectrum

We learn more about radial points by considering characters of spokes and partitioning spoke systems into sub-collections with common character. For instance, we scale the result of the previous theorem to characterise Fréchet-Urysohn points without needing to create a whole new construction. To achieve this, we introduce the notion of *radial spectrum*, which is a measure of which lengths of transfinite sequences are necessary as witnesses for the radial property.

Definition II.2.8 (Radial spectrum)**.** Let x be a radial point in a topological space X and let Λ be a class of regular, non-zero ordinals such that for all $A \subseteq X$ with $x \in \overline{A}$, there exists a transfinite sequence in A with length in Λ that converges to x. Then we say that x is Λ-*radial.*

We define the *radial spectrum of* x, denoted by $\Sigma_X^{\text{rad}}(x)$, to be the intersection of all sets of regular ordinals Λ such that x is Λ-radial. We will dispense with the subscript if the surrounding space is unambiguous.

Note that every radial point is

$$\{\lambda \in |X|^+ : \lambda \text{ is a regular, non-zero ordinal}\}\text{-radial,}$$

by Theorem II.2.2, so the radial spectrum is well-defined.

Theorem II.2.9. *Let x be a radial point in a topological space X and define*

$$\mathcal{T} := \{t : t \textit{ is a transfinite sequence in } X \setminus N_x, \operatorname{len}(t) \textit{ is regular and } t \dashrightarrow x\},$$

$$\Lambda := \{\operatorname{len}(t) : t \in \mathcal{T}\}.$$

(1) If x is quasi-isolated then $\Sigma^{rad}(x) = \emptyset$.

(2) If x is non-quasi-isolated then x is Λ-radial.

(3) $\Sigma^{rad}(x) = \Lambda$.

Proof. Define

$$\mathcal{M} := \{M \subseteq |X|^+ : M \text{ is a set of regular ordinals and } x \text{ is } M\text{-radial}\}$$

and let $t \in \mathcal{T}$ be given, so by Lemma II.2.1 $\operatorname{len}(t) \leq |X|$. Let $M \in \mathcal{M}$ be given, so since $x \in \overline{\operatorname{ran}(t)}$, there exists a transfinite sequence u in $\operatorname{ran}(t) \subseteq \mathbb{S}_t \in \mathbf{Sp}(x)$ that converges to x with $\operatorname{len}(u) \in M$. Then by Corollary I.3.28, $\lambda = \chi_{\mathbb{S}_t}(x) = \operatorname{len}(u) \in M$ and therefore $\lambda \in \bigcap \mathcal{M} = \Sigma^{\text{rad}}(x)$. Hence $\Lambda \subseteq \Sigma^{\text{rad}}(x)$.

Note that if $y \in N_x$ then $\lambda \times \{y\} \to x$ for all regular ordinals λ. Suppose x is quasi-isolated and let $A \subseteq X$ be given such that $x \in \overline{A}$. Then there exists a $y \in A \cap N_x$, so $1 \times \{y\}$ and $\omega \times \{y\}$ both converge to x. Hence x is $\{1\}$-radial and $\{\omega\}$-radial, so $\Sigma^{\mathrm{rad}}(x) \subseteq \{1\} \cap \{\omega\} = \emptyset$. Therefore $\Lambda = \Sigma^{\mathrm{rad}}(x)$.

Now assume that x is non-quasi-isolated, so $x \in \overline{X \backslash N_x}$ and thus by Theorem II.2.2 $\Lambda \neq \emptyset$. Let $A \subseteq X$ be given such that $x \in \overline{A}$. If $x \in \overline{A \backslash N_x}$ then by Theorem II.2.2 there exists a transfinite sequence in $A \backslash N_x$ with length in Λ that converges to x. If $x \in \overline{A \cap N_x}$ then there exists a $y \in A \cap N_x$ and a $\lambda \in \Lambda$, so $\lambda \times \{y\} \to x$. Therefore x is Λ-radial and so $\Lambda \in \mathcal{M}$ by Lemma II.2.1. Hence $\Sigma^{\mathrm{rad}}(x) = \Lambda$ and thus x is $\Sigma^{\mathrm{rad}}(x)$-radial. $\square$

Recall that the radial character of a radial point x in a space X is the least ordinal $R\chi(x)$ such that for every $A \subseteq X$, if $x \in \overline{A}$ then there exists a transfinite sequence t in A that converges to x with $\mathrm{len}(t) \leq R\chi(x)$. The radial spectrum allows us to calculate the radial character and p-character of a radial point, as well as a lower bound for the pseudocharacter.

Theorem II.2.10. *Let x be a non-quasi-isolated radial point in a topological space X. Then:*

(1) $p(x) = \min(\Sigma^{rad}(x))$.

(2) $R\chi(x) = \sup(\Sigma^{rad}(x))$.

(3) $\psi(x) \geq \sup(\Sigma^{rad}(x))$, *with equality if x is independently-based.*

Proof. First, note that by Theorem II.2.9 $\Sigma^{\mathrm{rad}}(x) \neq \emptyset$. Let $\mathcal{U} \in [\mathcal{N}_x]^{<\min(\Sigma^{\mathrm{rad}}(x))}$ be given. By Theorem II.2.7 and Lemma II.2.12, there exists a non-trivial spoke system $\mathfrak{S}$ of x, and by Corollary I.3.26 $\{\chi_S(x) : S \in \mathfrak{S}\} = \{p_S(x) : S \in \mathfrak{S}\} = \Sigma^{\text{rad}}(x)$. Hence for all $S \in \mathfrak{S}, U_S := S \cap \bigcap \mathcal{U} = \bigcap_{U \in \mathcal{U}}(U \cap S) \in \mathcal{N}_x^S$ and so $\bigcup_{S \in \mathfrak{S}} U_S \subseteq \bigcap \mathcal{U}$. Therefore $\bigcap \mathcal{U} \in \mathcal{N}_x^X$, so by Proposition I.3.15 $p_X(x) \geq \min(\Sigma^{\text{rad}}(x))$. However by Theorem II.2.9, there exists a transfinite sequence t in $X \setminus N_x$ that converges strictly to x with length $\min(\Sigma^{\text{rad}}(x))$. Then $X \setminus \operatorname{ran}(t) \notin \mathcal{N}_x^X$ but $X \setminus \{t(\alpha)\}$ for all $\alpha < \min(\Sigma^{\text{rad}}(x))$. Therefore $p_X(x) = \min(\Sigma^{\text{rad}}(x))$.

By Theorem II.2.9, x is $\Sigma^{\text{rad}}(x)$-radial and so by definition $R\chi(x) \leq \sup(\Sigma^{\text{rad}}(x))$. Let t be a transfinite sequence in $X \setminus N_x$ that converges strictly to x, with regular length. Then $x \in \overline{\operatorname{ran}(t)}$, so there exists a transfinite sequence u in $\operatorname{ran}(t) \subseteq \mathbb{S}_t$ that converges to u with $\operatorname{len}(u) < R\chi(x)$. By taking a cofinal subsequence, we assume without loss of generality that $\operatorname{len}(u)$ is regular. Hence by Corollary I.3.28, $\operatorname{len}(u) = \chi_{\mathbb{S}_t}(x) = \operatorname{len}(t)$ and so by Theorem II.2.9, $\Sigma^{\text{rad}}(x) \subseteq R\chi(x)$. Therefore $R\chi(x) = \sup(\Sigma^{\text{rad}}(x))$.

Let $\mathcal{U} \in [\mathcal{N}_x]^{<\sup(\Sigma^{\text{rad}}(x))}$ be given, so there exists a $\lambda \in \Sigma^{\text{rad}}(x)$ such that $|\mathcal{U}| < \lambda$. Then by Theorem II.2.13 there exists an $S \in \mathfrak{S}$ such that $\psi_S(x) = \chi_S(x) = \lambda$. Hence $N_x \neq \bigcap_{U \in \mathcal{U}}(U \cap S) \subseteq \bigcap \mathcal{U}$ and therefore $\psi(x) \geq \sup(\Sigma^{\text{rad}}(x))$.

Finally, suppose that x has an independent spoke sysetm $\mathfrak{S}$. For each $\lambda \in \Sigma^{\text{rad}}(x)$ and $S \in \mathbf{C}_\lambda(\mathfrak{S})$, pick a well-ordered neighbourhood base $(B_\alpha^S)_{\alpha<\lambda}$ for x with respect to S. Then by independence:

$$\begin{aligned}
&\bigcap_{\lambda \in \Sigma^{\text{rad}}(x)} \bigcap_{\alpha<\lambda} \left(N_x \cup \left(\bigcup (\mathfrak{S} \setminus (\mathbf{C}_1(\mathfrak{S}) \cup \mathbf{C}_\lambda(\mathfrak{S}))) \right) \cup \left(\bigcup_{S \in \mathbf{C}_\lambda(\mathfrak{S})} B_\alpha^S \right) \right) \\
=& \bigcap_{\lambda \in \Sigma^{\text{rad}}(x)} \left(N_x \cup \bigcup (\mathfrak{S} \setminus (\mathbf{C}_1(\mathfrak{S}) \cup \mathbf{C}_\lambda(\mathfrak{S}))) \right) \\
=& N_x
\end{aligned}$$

Therefore $\psi(x) \leq |\Sigma^{\text{rad}}(x)| \cdot \sup(\Sigma^{\text{rad}}(x)) = \sup(\Sigma^{\text{rad}}(x))$, since $\Sigma^{\text{rad}}(x)$ is non-empty and infinite by the previous theorem. Hence $\psi(x) = \sup(\Sigma^{\mathrm{rad}}(x))$. $\square$

It now follows that there are radial points that are not independently-based. For instance, let $X = D \cup \{\star\}$ be the one-point compactification of an uncountable discrete space (sometimes referred to as a Fort space). Then X is Fréchet-Urysohn so $R\chi(\star) = \aleph_0$, but $\psi(\star) = |D| > \aleph_0$. In the next section we will construct counterexamples with stronger properties.

If a spoke is quasi-isolated at its well-based point then it is in effect redundant. We call these spokes *trivial.*

Definition II.2.11 (Trivial spokes)**.** A spoke S for a point x in a topological space X is *trivial* if x is quasi-isolated in S. A spoke system for a point x is said to be *non-trivial* if each of its spokes is non-trivial.

Thus a spoke system $\mathfrak{S}$ is almost-independent precisely when the intersection of any two distinct spokes is trivial.

Every radial point that is not quasi-isolated has a non-trivial spoke system by the following lemma:

Lemma II.2.12. *Let $\mathfrak{S}$ be a spoke system for a non-quasi-isolated point x in a topological space X and define $\mathfrak{T} := \{S \in \mathfrak{S} : S$ is non-trivial$\}$. Then $\mathfrak{T}$ is a spoke system for x.*

Proof. If $\mathfrak{T} = \emptyset$ then $N_x \in \mathcal{N}_x^S$ for each $S \in \mathfrak{S}$, so $N_x = \bigcup_{S \in \mathfrak{S}} N_x \in \mathcal{N}_x^X$, which is a contradiction. Thus $\mathfrak{T}$ is non-empty.

Select $(U_T)_{T \in \mathfrak{T}}$ from $(\mathcal{N}_x^T)_{T \in \mathfrak{T}}$ and note that $N_x \subseteq U_T$ for each $T \in \mathfrak{T}$. Define $U_S := N_x \in$

$\mathcal{N}_x^S$ for each $S \in \mathfrak{S} \setminus \mathfrak{T}$. Then $\bigcup_{T \in \mathfrak{T}} U_T = \bigcup_{S \in \mathfrak{S}} U_S \in \mathcal{N}_x^X$. Therefore $\mathfrak{T}$ is a spoke system for x. □

We can identify the radial spectrum of a point by using the characters of spokes, rather than the lengths of strictly convergent transfinite sequences as in Theorem II.2.9.

Theorem II.2.13. *Let x be a point in a topological space X and let $\mathfrak{S}$ be a spoke system for x. Then $\Sigma^{rad}(x) = \{\chi_S(x) : S \in \mathfrak{S}$ is non-trivial$\}$.*

Proof. First, note that if x is quasi-isolated then every spoke is trivial and by Theorem II.2.9 $\Sigma^{\mathrm{rad}}(x) = \emptyset$. Suppose x is not quasi-isolated and define

$$\Lambda := \{\chi_S(x) : S \in \mathfrak{S} \text{ is non-trivial}\}.$$

Let $A \subseteq X \setminus N_x$ be given such that $x \in \overline{A}$. Then by Theorem II.1.13 there exists an $S \in \mathfrak{S}$ such that $x \in \overline{A \cap S}$, so by Theorem II.2.2 and Corollary I.3.28 there exists a transfinite sequence in $A \cap S \subseteq S \setminus N_x$ with length $\chi_S(x)$ that converges strictly to x. Hence by Theorem II.2.9, $\chi_S(x) \in \Sigma^{\mathrm{rad}}(x)$. Therefore $\Lambda \subseteq \Sigma^{\mathrm{rad}}(x)$.

Now let t be a transfinite sequence in $X \setminus N_x$, with regular length, that converges strictly to x. Then by Lemma II.2.3 there exists a subsequence u of t contained in an $S \in \mathfrak{S}$ that converges to x. Therefore by Corollary I.3.28 and regularity, $\mathrm{len}(t) = \mathrm{len}(u) = \chi_S(x)$. Moreover, $x \in \overline{S \setminus N_x}$, so S in non-trivial and hence $\mathrm{len}(t) \in \Lambda$. Therefore by Theorem II.2.9 $\Sigma^{\mathrm{rad}}(x) = \Lambda$. □

Corollary II.2.14. *Let x be a point in a topological space and let $\mathfrak{S}, \mathfrak{T}$ be spoke systems for x. Then $\{\chi_S(x) : S \in \mathfrak{S}$ is non-trivial$\} = \{\chi_T(x) : T \in \mathfrak{T}$ is non-trivial$\}$.*

The previous corollary can be improved slightly when two spokes overlap arbitrarily close to the radial point.

Lemma II.2.15. *Let x be a point and $S, T \in \mathbf{Sp}(x)$ be non-trivial such that $x \in \overline{(S \cap T)\setminus N_x}$. Then $\chi_S(x) = \chi_T(x)$.*

Proof. By Theorem I.2.2, x is radial in S, so since $x \in \overline{(S \cap T)\setminus N_x}^S$, there exists a transfinite sequence t in $(S \cap T)\setminus N_x$ that converges to x. By Corollary I.3.28, $\chi_S(x) = \mathrm{len}(t) = \chi_T(x)$. $\square$

Using the radial spectrum, we can scale Theorem II.2.7 to characterise Λ-radial spaces for any class Λ of non-zero regular ordinals. This is summarised in the following theorem:

Theorem II.2.16. *Let Λ be a non-empty class of regular, non-zero ordinals and let x be a point in a topological space X. Then the following are equivalent:*

(1) x is Λ-radial.

(2) x has a spoke system $\mathfrak{S}$ such that $\chi_S(x) \in \Lambda$ for each non-trivial $S \in \mathfrak{S}$.

(3) x has an almost-independent, basic spoke system $\mathfrak{S}$ such that $\chi_S(x) \in \Lambda$ for every non-trivial $S \in \mathfrak{S}$.

Proof. Note that by Theorems II.2.7 and II.2.13, (1) implies (3), which in turn vacuously implies (2). Now suppose that (2) holds and let $\mathfrak{S}$ be such a spoke system. Let $A \subseteq X$ be given such that $x \in \overline{A}$, so by Theorem II.1.13 there exists an $S \in \mathfrak{S}$ such that $x \in \overline{A \cap S}$. If S is non-trivial then there exists a transfinite sequence in $A \cap S$ with length $\chi_S(x) \in \Lambda$ that converges to x. Otherwise, there exists a $y \in A \cap N_X^x$ and a $\lambda \in \Lambda$, so $\lambda \times \{y\} \to x$. Therefore x is Λ-radial. $\square$

Since a point is Fréchet-Urysohn if and only if it is $\{\omega\}$-radial, we have the following characterisation for Fréchet-Urysohn points:

Corollary II.2.17. *Let x be a point in a topological space X. Then the following are equivalent:*

(1) x is Fréchet-Urysohn.

(2) x has a first-countable spoke system.

(3) x has an almost-independent, first-countable, basic spoke system.

II.2.4 Ordering spokes

We conclude this section by proving a characterisation of radiality in terms of cofinal collections of spokes with respect to local-containment. In fact, this will provide conditions for when a *subspace* is radial at a point.

Definition II.2.18 (Locally contained)**.** Let X be a topological space, $x \in X, A, B \subseteq X$ be given. Then we say that *near x, A is locally-contained in B*, denoted by $A \subseteq_x^X B$, if there exists $U \in \mathcal{N}_x^X$ such that $A \cap U \subseteq B$, or equivalently $x \notin \overline{A \setminus B}$. If $A \subseteq_x^X B \subseteq_x^X A$ then we say that A and B are *locally-equal near x* and denote this by $=_x^X$. Otherwise, we say that A and B are *locally-distinct near x*, denoted by $\neq_x^X$. Note that $A =_x^X B$ if and only if there exists a $U \in \mathcal{N}_x^X$ such that $A \cap U = B \cap U$. If $x \in A \cap B$ then $A =_x^X B$ if and only if $\mathcal{N}_x^A \cap \mathcal{N}_x^B \neq \emptyset$. We will dispense with the superscripts if the surrounding space is unambiguous.

Given a collection $\mathcal{S} \subseteq \mathbf{Sp}(x)$, we say that *$\mathcal{S}$ is (basically) cofinal* if for every $T \in \mathbf{Sp}(x)$ ($T \in \mathbf{BSp}(x)$), there exists an $S \in \mathcal{S}$ such that $T \subseteq_x S$.

Given a spoke system, we can select one representative from each $=_x$-equivalence-class and still obtain a spoke system. Thus we can convert any spoke system to a spoke system that is pairwise locally-distinct near x.

Lemma II.2.19. *Let $\mathfrak{S}$ be a spoke system for a point x in a topological space X and select $(T_E)_{E\in\mathfrak{S}/=_x}$ from $\mathfrak{S}/=_x$. Then $\mathfrak{T} := \{T_E : E \in \mathfrak{S}/=_x\}$ is a spoke system of x.*

Proof. First note that for distinct $E_1, E_2 \in \mathfrak{S}/=_x$, $T_{E_1} \neq_x T_{E_2}$ and so $T_{E_1} \neq T_{E_2}$.

Select $(U_E)_{E\in\mathfrak{S}/=_x}$ from $\mathcal{N}_x^X$ and let $S \in \mathfrak{S}$ be given, so $T_{[S]_{=x}} =_x S$. Then there exists a $V_S \in \mathcal{N}_x^X$ such that $V_S \cap T_{[S]_{=x}} = V_S \cap S$. Define $W_S := U_{[S]_{=x}} \cap V_S \in \mathcal{N}_x^X$. Then

$$\begin{aligned}\bigcup_{S\in\mathfrak{S}} (W_S \cap S) &= \bigcup_{E\in\mathfrak{S}/=_x} \bigcup_{S\in E} (U_E \cap V_S \cap S) \\ &= \bigcup_{E\in\mathfrak{S}/=_x} \left(U_E \cap T_E \cap \left(\bigcup_{S\in E} V_S \right) \right) \\ &\subseteq \bigcup_{E\in\mathfrak{S}/=_x} (U_E \cap T_E)\end{aligned}$$

Therefore $\bigcup_{E\in\mathfrak{S}/=_x}(U_E \cap T_E) \in \mathcal{N}_x^X$ and thus $\mathfrak{T}$ is a spoke system of x. □

When considering cofinal collections of spokes, for trivial spokes the core spoke N_x suffices by the following lemma:

Lemma II.2.20. *Let S be a spoke for a point x in a topological space X. Then the following are equivalent:*

(1) S is trivial.

(2) $S =_x N_x$.

(3) $S \subseteq_x N_x$.

Proof. Suppose S is trivial, so $N_x^X = N_x^S \in \mathcal{N}_x^S$. Thus there exists a $U \in \mathcal{N}_x^X$ such that $U \cap S = N_x^X = U \cap N_x^X$. Hence $S =_x N_x^X$, which in turn implies that $S \subseteq_x N_x^X$.

Now suppose that $S \subseteq_x N_x^X$, so there exists a $U \in \mathcal{N}_x^X$ such that $U \cap S \subseteq N_x^X$. Hence $N_x^S = N_x^X \in \mathcal{N}_x^S$, so x is quasi-isolated in S. Therefore S is trivial. □

We can characterise radiality of points using (basically) cofinal collection of spokes. Moreover, we will provide conditions for when a *subspace* is radial at a point in terms of the spokes of the larger space. First, we need the following lemma to lift spokes from a subspace to a larger space.

Lemma II.2.21. *Let X be a topological space, $Y \subseteq X, x \in Y, S \in \mathbf{Sp}_Y(x)$ be given. Then $S \cup N_x^X \in \mathbf{Sp}_X(x)$. Furthermore, if $S \in \mathbf{BSp}_Y(x)$ then $S \cup N_x^X \in \mathbf{BSp}_X(x)$.*

Proof. Let $(B_\alpha)_{\alpha<\lambda}$ be a well-ordered neighbourhood base for x with respect to Y and let $U \in \mathcal{N}_x^X$ be given, so there exists $\alpha < \lambda$ such that $B_\alpha \subseteq U \cap S$. Then $B_\alpha \cup N_x^X \subseteq U \cap (S \cup N_x^X)$. Moreover, for all $\alpha < \lambda$, there exists a $V \in \mathcal{N}_x^X$ such that $B_\alpha = V \cap S$ and so $B_\alpha \cup N_x^X = V \cap (S \cup N_x^X) \in \mathcal{N}_x^{S \cup N_x^X}$ since $N_x^X \subseteq V$. Therefore $(B_\alpha \cup N_x^X)_{\alpha<\lambda}$ is a well-ordered neighbourhood base for x with respect to $S \cup N_x^X$. Hence $S \cup N_x^X \in \mathbf{Sp}_X(x)$.

Finally, assume $S = \mathbb{S}_t^Y$ for some transfinite sequence t in Y that converges strictly to x. Then by Lemma I.3.9,

$$S \cup N_x^X = (\operatorname{ran}(t) \cup N_x^Y) \cup N_x^X = \operatorname{ran}(t) \cup N_x^X = \mathbb{S}_t^X \in \mathbf{BSp}_X(x)$$ □

Theorem II.2.22. *Let X be a topological space, $Y \subseteq X, x \in Y$ be given. Then Y is radial at x if and only if $Y \subseteq_x^X \bigcup \mathcal{S}$ for all (basically) cofinal $\mathcal{S} \subseteq \mathbf{Sp}_X(x)$.*

Proof. Suppose Y is radial at x, so by Theorem II.2.7 there exists a basic spoke system $\mathfrak{S}$ for x with respect to Y. Let $\mathcal{S} \subseteq \mathbf{Sp}_X(x)$ be basically cofinal, so by the previous lemma for all $S \in \mathfrak{S}$, there exists a $T_S \in \mathcal{S}$ such that $S \cup N_x^X \subseteq_x^X T_S$, and thus there exists a $U_S \in \mathcal{N}_x^X$ such that $(S \cap U_S) \cup N_x^X = (S \cup N_x^X) \cap U_S \subseteq T_S \cap U_S$. Define $U := \bigcup_{S \in \mathcal{S}} (S \cap U_S) \in \mathcal{N}_x^Y$ so there exists a $V \in \mathcal{N}_x^X$ such that $U = V \cap Y$. As $U \subseteq \bigcup \mathcal{S}$, it follows that $Y \subseteq_x^X \bigcup \mathcal{S}$.

Note that every cofinal collection of spokes is trivially basically cofinal, so now suppose that for every cofinal $\mathcal{S} \subseteq \mathbf{Sp}_X(x), Y \subseteq_x^X \bigcup \mathcal{S}$. We will show that $\mathbf{Sp}_Y(x)$ is a spoke system of x with respect to Y. Select $(U_S)_{S \in \mathbf{Sp}_Y(x)}$ from $(\mathcal{N}_x^S)_{S \in \mathbf{Sp}_Y(x)}$, so there exists a selection $(V_S)_{S \in \mathbf{Sp}_Y(x)}$ from $\mathcal{N}_x^X$ such that $U_S = V_S \cap S$ for every $S \in \mathbf{Sp}_Y(x)$. Since $\{S \cap V_{S \cap Y} : S \in \mathbf{Sp}_X(x)\}$ is cofinal in $\mathbf{Sp}_X(x)$, it follows that there exists a $W \in \mathcal{N}_x^X$ such that

$$W \cap Y \subseteq \bigcup_{S \in \mathbf{Sp}_X(x)} (S \cap V_{S \cap Y})$$

$$\Rightarrow W \cap Y \subseteq \bigcup_{S \in \mathbf{Sp}_X(x)} (S \cap Y \cap V_{S \cap Y}) = \bigcup_{S \in \mathbf{Sp}_Y(x)} (S \cap V_S).$$

Thus $W \cap Y \in \mathcal{N}_x^Y$ and so $\mathbf{Sp}_Y(x)$ is a spoke system of x with respect to Y. Therefore x is radial in Y by Theorem II.2.7. □

Corollary II.2.23. *Let X be a topological space, $x \in X$ be given. Then X is radial at x if and only if for all (basically) cofinal collections of spokes $\mathcal{S}, \bigcup \mathcal{S}$ is a neighbourhood of x.*

Proof. Note that for all $A \subseteq X, X \subseteq_x A$ if and only if there exists a $U \in \mathcal{N}_x^X$ such that $U = U \cap X \subseteq A$, that is, precisely when A is a neighbourhood of x. Therefore the statement of the corollary follows immediately from the previous theorem. $\square$

We now demonstrate how to use the Theorem II.2.22 and the previous corollary by investigating the Tychonoff plank $(\omega+1)\times(\omega_1+1)$. Intuitively, all transfinite sequences that converge to (ω,ω_1) must have an eventually constant projection, and any subspace which is radial at (ω,ω_1) must avoid $\omega\times\omega_1$. This is formalised below:

Example II.2.24. *Define*

$$P := (\omega+1)\times(\omega_1+1),$$

$$S_\omega := (\omega+1)\times\{\omega_1\},$$

$$S_{\omega_1} := \{\omega\}\times(\omega_1+1).$$

Then $\{S_\omega, S_{\omega_1}\}$ *is a cofinal collection of spokes of* (ω,ω_1). *Furthermore, for all* $X\subseteq P$ *containing* (ω,ω_1), X *is radial at* (ω,ω_1) *if and only if* $X\subseteq_{(\omega,\omega_1)} Y := S_\omega\cup S_{\omega_1}$.

Proof. Let $S\in\mathbf{Sp}((\omega,\omega_1))$ be given and assume that $(\omega,\omega_1)\in\overline{S\backslash Y}$. Then by Theorem I.2.2 there exists a transfinite sequence in $S\backslash Y\subseteq\omega\times\omega_1$ that converges to (ω,ω_1), which is a contradiction as observed in Section II.1.1. Thus $S\subseteq_{(\omega,\omega_1)} Y$, so there exists an $n<\omega$ and an $\alpha<\omega_1$ such that $U\cap S\subseteq Y$, where $U := (n,\omega]\times(\alpha,\omega_1]$. Now assume that $U\cap S\not\subseteq_{(\omega,\omega_1)} S_\beta$ and so $(\omega,\omega_1)\in\overline{(U\cap S)\backslash S_\beta}$, for both $\beta=\omega,\omega_1$. Since $(U\cap S)\backslash S_\omega\subseteq S_{\omega_1}$ and $(U\cap S)\backslash S_{\omega_1}\subseteq S_\omega$, it follows that $x\in\overline{(S\cap S_\omega)\backslash N_{(\omega,\omega_1)}}\cap\overline{(S\cap S_{\omega_1})\backslash N_{(\omega,\omega_1)}}$, and thus by Lemma II.2.15 $\aleph_0 = \chi_{S_\omega}((\omega,\omega_1)) = \chi_S((\omega,\omega_1)) = \chi_{S_{\omega_1}}((\omega,\omega_1)) = \aleph_1$, which is a contradiction. Therefore $U\cap S\subseteq_{(\omega,\omega_1)} S_\beta$ for some $\beta=\omega,\omega_1$, so there exists a $V\in\mathcal{N}_{(\omega,\omega_1)}$ such that $(U\cap V)\cap S\subseteq S_\beta$ and thus $\{S_\omega,S_{\omega_1}\}$ is cofinal.

Now by Theorem II.2.22, if $X \subseteq P$ is radial at $(\omega, \omega_1) \in X$ then $X \subseteq_{(\omega,\omega_1)} Y$. Note that if $A \subseteq Y$ is such that $(\omega, \omega_1) \in \overline{A}^Y$ then there is a $\beta = \omega, \omega_1$ such that $(\omega, \omega_1) \in \overline{A \cap S_\beta}^Y$, so it follows that $\{S_\omega, S_{\omega_1}\}$ is a spoke system of (ω, ω_1) with respect to Y by Theorem II.1.13. Thus by Theorem II.2.7, Y is radial at (ω, ω_1).

Let $X \subseteq P$ be given such that $(\omega, \omega_1) \in X \subseteq_{(\omega,\omega_1)} Y$, so there exists a $U \in \mathcal{N}_{(\omega,\omega_1)}$ such that $U \cap X \subseteq Y$. Let $A \subseteq X$ be given such that $(\omega, \omega_1) \in \overline{A}^X$. Then $(\omega, \omega_1) \in \overline{U \cap A}^Y$ and thus there exists a transfinite sequence in $U \cap A \subseteq A$ that converges to (ω, ω_1). Therefore X is radial at (ω, ω_1). $\square$

Of course, not every spoke system is basically cofinal; for instance, $\{E \cup \{\omega\}, O \cup \{\omega\}\}$ is a spoke system of ω in $\omega + 1$, where E and O are the set of even and odd natural numbers respectively. However, $t := (3n)_{n<\omega} \dashrightarrow \omega$ and $\mathbb{S}_t \not\subseteq_\omega A \cup \{\omega\}$ for both $A = E, O$. However, for independent spoke systems we can coalesce 'small' collections from the *components* of a spoke system to form a cofinal collection of spokes — this will be explored in the next section.

II.3 Independence

In this section, we will investigate independent spoke systems and independently-based points. We first show that for independently-based points, we can find (independent) spoke systems of minimal cardinality by using the *components* of a spoke system corresponding to spokes of common character. Then we will investigate the *truncation* of spoke systems, that is, the process of (potentially) converting an almost-independent spoke system into an independent one. Finally, we construct several Fréchet-Urysohn spaces with non-independently-based points, showing that the assumption of almost-independence in Theorem II.2.7 cannot be weakened.

II.3.1 Components

Having shown the usefulness of radial spectra, we continue by splitting spoke systems into *components* with common character.

Definition II.3.1 (Components)**.** Let $\mathfrak{S}$ be a spoke system for a point x and for every regular, non-zero ordinal λ, define

$$\mathbf{C}_\lambda(\mathfrak{S}) := \{S \in \mathfrak{S} : \chi_S(x) = \lambda\}$$

We call $\mathbf{C}_\lambda(\mathfrak{S})$ the λ*-component* of $\mathfrak{S}$. Note that the 1-component of a spoke system consists of the trivial spokes.

By focusing on the components of a spoke system, we show that certain independent spoke systems have the smallest cardinality amongst *all* spoke systems. The following lemma is the first step to paring down the number of spokes.

Lemma II.3.2. *Let x be a point in a topological space X and let λ be a regular, non-zero ordinal and $\mathcal{S}$ be a non-empty collection of spokes of x, each of character λ at x. Suppose $p(x) \geq \lambda$ and $|\mathcal{S}| < \lambda$. Then $T := \bigcup \mathcal{S} \in \mathbf{Sp}(x)$ and $\chi_T(x) = \lambda$.*

Proof. For each $S \in \mathcal{S}$, choose a well-ordered neighbourhood base $(B^S_\alpha)_{\alpha<\lambda}$. Then for each

$S \in \mathcal{S}$ and $\alpha < \lambda$, there exists a $U^S_\alpha \in \mathcal{N}^X_x$ such that $B^S_\alpha = U^S_\alpha \cap S$. Define for every $\alpha < \lambda$,

$$B_\alpha := \bigcup_{S \in \mathcal{S}} B^S_\alpha.$$

Then for every $\alpha < \lambda$,

$$\left(\bigcap_{S \in \mathcal{S}} U^S_\alpha \right) \cap T \subseteq \bigcup_{S \in \mathcal{S}} U^S_\alpha \cap S = B_\alpha.$$

Since $p(x) \geq \lambda$, it follows that $B_\alpha \in \mathcal{N}^T_x$.

Now let $U \in \mathcal{N}^T_x$ be given, so for every $S \in \mathcal{S}, U \cap S \in \mathcal{N}^S_x$ and thus there exists an $\alpha < \lambda$ such that $B^S_\alpha \subseteq U \cap S$. By regularity of $\lambda, \alpha := \sup(\alpha_S : S \in \mathcal{S}) < \lambda$, so it follows that

$$B_\alpha \subseteq \bigcup_{S \in \mathcal{S}} (U \cap S) = U.$$

Therefore $(B_\alpha)_{\alpha < \lambda}$ is a well-ordered neighbourhood base for x with respect to T. Thus $T \in \mathbf{Sp}(x)$. Moreover, since $B_{\sup(A)} \subseteq \bigcap_{\alpha \in A} B_\alpha$ for every $A \in [\lambda]^{<\lambda}$, by Corollary I.3.26 we have $\lambda \leq p_T(x) = \chi_T(x) \leq \lambda$. □

Corollary II.3.3. *Let λ be a non-zero, regular ordinal and let x be a $\{\lambda\}$-radial point in a topological space with a spoke system $\mathfrak{S}$ such that $|\mathfrak{S}| < \lambda$. Then x is well-based and $\chi(x) \in \{1, \lambda\}$.*

Proof. Define $\mathcal{S} := \{S \in \mathfrak{S} : S \text{ is non-trivial}\}$, so by Theorem II.2.13 $\chi_S(x) = \lambda$ for all $S \in \mathcal{S}$. Define $S := N_x \cup \bigcup \mathcal{S}$, so by the preceding lemma S is a spoke of x and if $\mathcal{S} \neq \emptyset$ then $\chi_S(x) = \lambda$; otherwise, $\chi_S(x) = 1$. As $\mathfrak{S}$ is a spoke system, $S \in \mathcal{N}^X_x$, so x is well-based in X and $\chi_X(x) = \chi_S(x)$. □

Of course, we can also split a spoke into a 'small' number of spokes.

Proposition II.3.4. *Let S be a non-trivial spoke at x and let $\kappa \leq \chi_S(x)$ be a non-zero cardinal. Then there exists an independent collection of spokes $\mathcal{S} \in [\mathbf{Sp}(x)]^\kappa$ such that $S = \bigcup \mathcal{S}$.*

Proof. Define $\lambda := \chi_S(x)$ and let $(B_\alpha)_{\alpha<\lambda}$ be a well-ordered neighbourhood base for x with respect to S, where $B_\beta \subsetneqq B_\alpha$ for all $\alpha < \beta < \lambda$. Define for all $\alpha < \lambda$,

$$C_\alpha := \bigcap_{\beta<\alpha} B_\beta.$$

Note that by Corollary I.3.26, $(C_\alpha)_{\alpha<\lambda}$ is again a well-ordered neighbourhood base for x and moreover, $C_\beta \subsetneqq C_\alpha$ for all $\alpha < \beta < \lambda$.

As S is non-trivial, λ is infinite and so $\lambda = \lambda \cdot \kappa$. Hence there exists a partition $\mathcal{P} \subseteq [\lambda]^\lambda$ of cardinality κ. Define for all $P \in \mathcal{P}, S_P := N_x \cup \bigcup_{\alpha \in P}(C_\alpha \setminus C_{\alpha+1})$. Then S_P is a spoke at x and $\bigcup_{P \in \mathcal{P}} S_P = S$. Moreover, since $\{C_\alpha \setminus C_{\alpha+1} : \alpha < \lambda\}$ is a partition of $S \setminus N_x$, it follows that $\mathfrak{S} := \{S_P : P \in \mathcal{P}\}$ is independent and $S_{P_1} \neq S_{P_2}$ for all distinct $P_1, P_2 \in \mathcal{P}$. Therefore $|\mathfrak{S}| = |\mathcal{P}| = \kappa$. □

Using Lemma II.3.2, we can collapse the small components of an independent spoke system to obtain a new spoke system.

Theorem II.3.5. *Let $\mathfrak{S}$ be a non-trivial, independent spoke system for a point x in a topological space X. Define*

$$\Lambda := \{\lambda \in \Sigma^{rad}(x) : |\mathbf{C}_\lambda(\mathfrak{S})| < \lambda\},$$

$$\mathfrak{T} := \left\{\bigcup \mathbf{C}_\lambda(\mathfrak{S}) : \lambda \in \Lambda\right\} \cup \bigcup_{\lambda \in \Sigma^{rad}(x) \setminus \Lambda} \mathbf{C}_\lambda(\mathfrak{S}).$$

Then $\mathfrak{T}$ *is a non-trivial, independent spoke system for* x*. Furthermore, for all* $\lambda \in \Sigma^{rad}(x)$*:*

$$\mathbf{C}_\lambda(\mathfrak{T}) = \begin{cases} \{\bigcup \mathbf{C}_\lambda(\mathfrak{S})\} & \textit{if } \lambda \in \Lambda \\ \mathbf{C}_\lambda(\mathfrak{S}) & \textit{otherwise} \end{cases}$$

Proof. Let $\lambda \in \Lambda$ be given and define $S_\lambda := \bigcup \mathbf{C}_\lambda(\mathfrak{S})$. Note that $\mathbf{C}_\lambda(\mathfrak{S}) \neq \emptyset$ by Corollary II.2.13. Let $\mathcal{U} \in [\mathcal{N}_x^{S_\lambda}]^{<\lambda}$ be given, so for every $U \in \mathcal{U}$, there exists a selection $(V_{U,S})_{S\in\mathfrak{S}}$ of $(\mathcal{N}_x^S)_{S\in\mathfrak{S}}$ such that

$$\bigcup_{S\in\mathbf{C}_\lambda(\mathfrak{S})} V_{U,S} = \left(\bigcup_{S\in\mathfrak{S}} V_{U,S}\right) \cap S_\lambda \subseteq U.$$

Then by Corollary I.3.26, for each $S \in \mathbf{C}_\lambda(\mathfrak{S})$, $W_S := \bigcap_{U\in\mathcal{U}} V_{U,S} \in \mathcal{N}_x^S$. Thus

$$\left(\left(\bigcup_{S\in\mathbf{C}_\lambda(\mathfrak{S})} W_S\right) \cup \bigcup(\mathfrak{S}\setminus\mathbf{C}_\lambda(\mathfrak{S}))\right) \cap S_\lambda = \bigcup_{S\in\mathbf{C}_\lambda(\mathfrak{S})} W_S \subseteq \bigcap \mathcal{U}.$$

Therefore $\bigcap \mathcal{U} \in \mathcal{N}_x^{S_\lambda}$ and hence by Proposition I.3.15, $p_{S_\lambda}(x) \geq \lambda$. Since $\mathbf{C}_\lambda(\mathfrak{S}) \subseteq \mathbf{Sp}_{S_\lambda}(x)$, it follows from Lemma II.3.2 that S_λ is a spoke of x and $\chi_{S_\lambda}(x) = \lambda$.

Now let $A \subseteq X$ be given such that $x \in \overline{A}$. Then by Corollary II.1.13, there exists an $S \in \mathfrak{S}$ such that $x \in \overline{A \cap S}$. If $\chi_S(x) \notin \Lambda$ then $S \in \mathfrak{T}$. Otherwise, $x \in \overline{A \cap S} \subseteq \overline{A \cap S_{\chi_S(x)}}$. Therefore, by Corollary II.1.13 again, $\mathfrak{T}$ is a non-trivial spoke system for x.

Let $T_1, T_2 \in \mathfrak{T}$ be distinct and define $\lambda_i := \chi_{T_i}(x)$ for $i = 1,2$. If $\lambda_1, \lambda_2 \notin \Lambda$ then $T_1, T_2 \in \mathfrak{S}$ and so $T_1 \cap T_2 = N_x$. If $\lambda_j \notin \Lambda \ni \lambda_i$ for distinct $i, j \in \{1,2\}$ then for each $S \in \mathbf{C}_{\lambda_i}(\mathfrak{S})$, $S \cap T_j = N_x$, and so $T_i \cap T_j = N_x$. Finally, note that for distinct $\mu_1, \mu_2 \in \Lambda$, $\chi_{S_{\mu_1}}(x) = \mu_1 \neq \mu_2 = \chi_{S_{\mu_2}}(x)$ and thus $S_{\mu_1} \neq S_{\mu_2}$. Thus, if $\lambda_1, \lambda_2 \in \Lambda$ then $T_i = S_{\lambda_i}$ for $i = 1,2$ and $\mathbf{C}_{\lambda_1}(\mathfrak{S}) \cap \mathbf{C}_{\lambda_2}(\mathfrak{S}) = \emptyset$. Hence by the independence of $\mathfrak{S}$, $T_1 \cap T_2 = \emptyset$ and therefore $\mathfrak{T}$ is independent.

Finally, note that each $\chi_{S_\lambda}(x) = \lambda$ for all $\lambda \in \Lambda$, from which the final statement of the theorem follows immediately. □

We will now show that all independently-based points have a spoke system of the form of $\mathfrak{T}$ from above. In fact, these will have the smallest cardinality of *all* spoke systems for that point.

As spokes prescribe the different approaches convergence may take, it is possible that a spoke may overlap with one or many spokes from a system, even arbitrarily close to the radial point. This is formalised in the following definition:

Definition II.3.6 (Thread)**.** Let x be a point in a topological space and let $\mathcal{S} \cup \{T\}$ be a collection of spokes of x. Then we define the *thread of T through $\mathcal{S}$* to be

$$\mathbf{Thr}_{\mathcal{S}}(T) := \{S \in \mathcal{S} : x \in \overline{(S \cap T) \backslash N_x}\}.$$

Observe that for all collections of spokes $\mathcal{S}, \mathcal{T}$ and all $S \in \mathcal{S}, T \in \mathcal{T}, S \in \mathbf{Thr}_{\mathcal{S}}(T)$ if and only if $T \in \mathbf{Thr}_{\mathcal{T}}(S)$.

Lemma II.3.7. *Let $\mathfrak{S}$ be a spoke system of a point x in a topological space. Then for all non-trivial spokes T of x, $\emptyset \neq \mathbf{Thr}_{\mathfrak{S}}(T) \subseteq \mathbf{C}_{\chi_T(x)}(\mathfrak{S})$.*

Proof. As $x \in \overline{T \backslash N_x}$, by Theorem II.1.13 there exists an $S \in \mathfrak{S}$ such that $x \in \overline{(S \cap T) \backslash N_x}$ and hence $S \in \mathbf{Thr}_{\mathfrak{S}}(T)$. Finally, note from Lemma II.2.15 that $\mathbf{Thr}_{\mathfrak{S}}(T) \subseteq \mathbf{C}_{\chi_T(x)}(\mathfrak{S})$. □

Theorem II.3.8. *Let x be a point in a topological space X and let S be a spoke and $\mathfrak{T}$ be a spoke system for x.*

(1) If $\mathfrak{T}$ is almost-independent then $|\mathbf{Thr}_{\mathfrak{T}}(S)| \neq \chi_S(x)$.

(2) If $\mathfrak{T}$ is independent then $|\mathbf{Thr}_{\mathfrak{T}}(S)| < \chi_S(x)$.

Proof. First, suppose that $\mathfrak{T}$ is almost-independent and $|\mathbf{Thr}_{\mathfrak{T}}(S)| = \lambda := \chi_S(x)$ and enumerate $\mathbf{Thr}_{\mathfrak{T}}(S) = \{T_\alpha : \alpha < \lambda\}$. As $\mathbf{Thr}_{\mathfrak{T}}(S) \neq \emptyset$, it follows that S and all spokes in $\mathbf{Thr}_{\mathfrak{T}}(S)$ are non-trivial, so by Lemma II.2.15 $\chi_{T_\alpha}(x) = \lambda$ for all $\alpha < \lambda$.

Let $(B_\alpha)_{\alpha<\lambda}$ be a well-ordered neighbourhood base for x with respect to S. For $\alpha < \beta < \lambda$, $x \notin \overline{(T_\alpha \cap T_\beta)\backslash N_x}$ and thus $T_\beta\backslash(T_\alpha\backslash N_x) \in \mathcal{N}_x^{T_\beta}$. By Corollary I.3.26, $p_{T_\beta}(x) = \lambda$ and so for $\beta < \lambda$, $\bigcap_{\alpha<\beta}(T_\beta\backslash(T_\alpha\backslash N_x)) = N_x \cup (T_\beta\backslash(\bigcup_{\alpha<\beta} T_\alpha)) \in \mathcal{N}_x^{T_\beta}$. Hence there exists an $x_\beta \in (T_\beta \cap B_\beta)\backslash(N_x \cup \bigcup_{\alpha<\beta} T_\alpha)$. Then $(x_\beta)_{\beta<\lambda} \dashrightarrow x$ by Corollary I.3.28, so by Theorem II.1.13 there exists a $T \in \mathfrak{T}$ such that $x \in \overline{\{x_\alpha : \alpha < \lambda\} \cap T}$. Thus $\{\alpha < \lambda : x_\alpha \in T\}$ is unbounded in λ and $T \in \mathbf{Thr}_{\mathfrak{T}}(S)$, so there exists an $\alpha < \lambda$ such that $T = T_\alpha$. But then there exists a $\beta \in (\alpha, \lambda)$ such that $x_\beta \in T = T_\alpha$, which is a contradiction. Therefore $|\mathbf{Thr}_{\mathfrak{T}}(S)| \neq \lambda$.

Now suppose that $\mathfrak{T}$ is independent and $|\mathbf{Thr}_{\mathfrak{T}}(S)| \geq \lambda$, so for all $\alpha < \lambda$, there exists a $T_\alpha \in \mathfrak{T}$ and an $x_\alpha \in (B_\alpha \cap T_\alpha)\backslash N_x$ such that $T_\alpha \neq T_\beta$ for all distinct $\alpha, \beta < \lambda$. Note that $x \in \overline{(S \cap T_0)\backslash N_x}$, so x is not quasi-isolated in S and thus λ is infinite. Then by Corollary I.3.28, $(x_\alpha)_{\alpha<\lambda} \dashrightarrow x$ and thus by Lemma II.2.3, there exists a strictly increasing $f : \lambda \to \lambda$ and a $T \in \mathfrak{T}$ such that $\{x_{f(\alpha)} : \alpha < \lambda\} \subseteq T$. By independence, it follows that $T = T_{f(\alpha)}$ for all $\alpha < \lambda$, so f must be constant, which is a contradiction. Therefore $|\mathbf{Thr}_{\mathfrak{T}}(S)| < \chi_S(x)$. □

For an almost-independent spoke system $\mathfrak{T}$ and spoke S for which $|\mathbf{Thr}_{\mathfrak{T}}(S)| > \chi_S(x)$, consider an infinite maximal almost-disjoint family $\mathcal{A}$ of infinite subsets of ω (see pg. 134), which must be uncountable [Kun11, Lemma III.1.19, pg. 159]. Then $\mathfrak{T} := \{\{\omega\} \cup A : A \in \mathcal{A}\}$ is an uncountable, almost-independent spoke system of ω in $\omega + 1$ and $|\mathbf{Thr}_{\mathfrak{T}}(\omega + 1)| = |\mathcal{A}| > \aleph_0 = \chi_{\omega+1}(\omega)$.

Theorem II.3.9. *Let x be a point in a topological space with a non-trivial, independent spoke system $\mathfrak{S}$ such that for all $\lambda \in \Sigma^{rad}(x)$, either $|\mathbf{C}_\lambda(\mathfrak{S})| = 1$ or $|\mathbf{C}_\lambda(\mathfrak{S})| \geq \lambda$. Then every spoke system of x has cardinality at least $|\mathfrak{S}|$.*

Moreover, every non-quasi-isolated, independently-based point has such a spoke system.

Proof. Let $\mathfrak{T}$ be a spoke system for x. Note that there is an $S \in \mathfrak{S}$ and $x \in \overline{S \setminus N_x}$, so x is not quasi-isolated. By Lemma II.2.12, we can without loss of generality assume that $\mathfrak{T}$ is non-trivial, and by Corollary II.2.13,

$$\Lambda := \{\chi_S(x) : S \in \mathfrak{S}\} = \{\chi_T(x) : T \in \mathfrak{T}\}.$$

We claim that $|\mathbf{C}_\lambda(\mathfrak{S})| \leq |\mathbf{C}_\lambda(\mathfrak{T})|$ for every $\lambda \in \Lambda$. From this, it immediately follows that $|\mathfrak{S}| = |\bigcup_{\lambda \in \Lambda} \mathbf{C}_\lambda(\mathfrak{S})| \leq |\bigcup_{\lambda \in \Lambda} \mathbf{C}_\lambda(\mathfrak{T})| = |\mathfrak{T}|$.

Let $\lambda \in \Lambda$ be given. If $|\mathbf{C}_\lambda(\mathfrak{S})| = 1$ then vacuously $|\mathbf{C}_\lambda(\mathfrak{S})| \leq |\mathbf{C}_\lambda(\mathfrak{T})|$. Suppose otherwise, so by assumption $|\mathbf{C}_\lambda(\mathfrak{S})| \geq \lambda$. Note that by the previous theorem, $|\mathbf{Thr}_{\mathfrak{S}}(T)| < \lambda$ for each $T \in \mathbf{C}_\lambda(\mathfrak{T})$. Let $S \in \mathbf{C}_\lambda(\mathfrak{S})$ be given. Then since S is non-trivial, by Lemma II.3.7 there exists a $T \in \mathbf{Thr}_{\mathfrak{T}}(S) \subseteq \mathbf{C}_\lambda(\mathfrak{T})$ and hence $S \in \mathbf{Thr}_{\mathfrak{S}}(T)$. Therefore $\mathbf{C}_\lambda(\mathfrak{S}) \subseteq \bigcup_{T \in \mathbf{C}_\lambda(\mathfrak{T})} \mathbf{Thr}_{\mathfrak{S}}(T)$ and so $|\mathbf{C}_\lambda(\mathfrak{S})| \leq |\mathbf{C}_\lambda(\mathfrak{T})| \cdot |\mathbf{Thr}_{\mathfrak{S}}(T)| \leq |\mathbf{C}_\lambda(\mathfrak{T})| \cdot \lambda = |\mathbf{C}_\lambda(\mathfrak{T})|$. Thus $|\mathfrak{S}| \leq |\mathfrak{T}|$.

To finish the proof, let x be a non-quasi-isolated point and let $\mathfrak{S}$ be an independently-based spoke system for x. Without loss of generality (by Lemma II.2.12) and by Theorem II.3.5, we can create a new independent spoke system $\mathfrak{T}$ such that either $|\mathbf{C}_\lambda(\mathfrak{T})| = 1$ or $|\mathbf{C}_\lambda(\mathfrak{T})| \geq \lambda$ for all $\lambda \in \Sigma^{\mathrm{rad}}(x)$. □

We conclude this section by constructing cofinal collections of spokes from the threads of an independent spoke system.

Lemma II.3.10. *Let $\mathfrak{S}$ be a spoke system of a point x in a topological space X and let T be a spoke of x. Then $T \subseteq_x \bigcup \mathbf{Thr}_{\mathfrak{S}}(T)$.*

Proof. Suppose $T \not\subseteq_x \bigcup \mathbf{Thr}_{\mathfrak{S}}(T)$, so $x \in \overline{T \setminus \bigcup \mathbf{Thr}_{\mathfrak{S}}(T)}$ and thus by Theorem II.1.13 there exists an $S \in \mathfrak{S}$ such that $x \in \overline{(S \cap T) \setminus \bigcup \mathbf{Thr}_{\mathfrak{S}}(T)} \subseteq \overline{(S \cap T) \setminus N_x}$. But then $S \in \mathbf{Thr}_{\mathfrak{S}}(T)$, which is a contradiction. Therefore $T \subseteq_x \bigcup \mathbf{Thr}_{\mathfrak{S}}(T)$. □

Theorem II.3.11. *Let $\mathfrak{S}$ be an independent spoke system for a point x in a topological space X and define*

$$\mathcal{S} := \{N_x\} \cup \left\{ \bigcup \mathcal{T} : \lambda \in \Sigma^{rad}(x), \mathcal{T} \in [\mathbf{C}_\lambda(\mathfrak{S})]^{<\lambda} \right\}.$$

Then $\mathcal{S}$ is a cofinal collection of spokes of x.

Proof. Let T be a non-trivial spoke of x and define $\lambda := \chi_T(x)$. Then by Theorem II.3.8 $\mathbf{Thr}_{\mathfrak{S}}(T) \in [\mathbf{C}_\lambda(\mathfrak{S})]^{<\lambda}$, so $\bigcup \mathbf{Thr}_{\mathfrak{S}}(T) \in \mathbf{Sp}(x)$ by Lemma II.3.2. As T is non-trivial, by Theorem II.2.13 $\lambda \in \Lambda$ and hence $\bigcup \mathbf{Thr}_{\mathfrak{S}}(T) \in \mathcal{S}$. Since every trivial spoke is locally-contained at x in N_x, it follows from the previous lemma that $\mathcal{S}$ is a cofinal collection of spokes of x. □

II.3.2 Truncation

We now investigate how to convert a spoke system to an independent one, if possible. The simplest process of (potential) conversion involves 'truncating' the spokes.

Definition II.3.12 (Truncation)**.** Let $\mathcal{S}$ be a collection of spokes for a point x and select $(U_S)_{S \in \mathcal{S}}$ from $(\mathcal{N}_x^S)_{S \in \mathcal{S}}$. Then we call $\{U_S : S \in \mathcal{S}\}$ a *truncation* of $\mathcal{S}$.

We see that every truncation of a spoke system is a spoke system once again.

Lemma II.3.13. *Let $\mathfrak{S}$ be a spoke system for a point x in a topological space X and select $(U_S)_{S\in\mathfrak{S}}$ from $(\mathcal{N}_x^S)_{S\in\mathfrak{S}}$. Then $\{U_S : S \in \mathfrak{S}\}$ is a spoke system of x.*

Proof. Define $\mathfrak{T} := \{U_S : S \in \mathfrak{S}\}$ and select $(V_T)_{T\in\mathfrak{T}}$ from $(\mathcal{N}_x^T)_{T\in\mathfrak{T}}$. For all $S \in \mathfrak{S}$, define $W_S := V_{U_S} \in \mathcal{N}_x^{U_S} \subseteq \mathcal{N}_x^S$. Then $\bigcup_{T\in\mathfrak{T}} V_T = \bigcup_{S\in\mathfrak{S}} W_S \in \mathcal{N}_x^X$. Therefore $\mathfrak{T}$ is a spoke system of x. □

Theorem II.3.14. *Let x be a point in a topological space X and let $\mathfrak{S}$ be a spoke system of x that is pairwise locally-distinct near x. Then $\mathfrak{S}$ has no independent truncation if and only if for every selection $(U_S)_{S\in\mathfrak{S}}$ of $(\mathcal{N}_x^S)_{S\in\mathfrak{S}}$,*

$$x \in \overline{\left(\bigcup_{\{S,T\}\in[\mathfrak{S}]^2} (U_S \cap U_T)\right)\setminus N_x}.$$

Proof. First, suppose that $\mathfrak{S}$ has no independent truncation. Let $V \in \mathcal{N}_x^X$ be given and select $(U_S)_{S\in\mathfrak{S}}$ from $(\mathcal{N}_x^S)_{S\in\mathfrak{S}}$. Then $\{U_S \cap V : S \in \mathfrak{S}\}$ is not independent, so there exist distinct $S, T \in \mathfrak{S}$ such that $(U_S \cap U_T) \cap V \neq N_x$, and thus $V \cap ((\bigcup_{\{P,Q\}\in[\mathfrak{S}]^2} (U_P \cap U_Q))\setminus N_x) \neq \emptyset$. Therefore $x \in \overline{(\bigcup_{\{S,T\}\in[\mathfrak{S}]^2} (U_S \cap U_T))\setminus N_X}$.

Now suppose that there exists a selection $(U_S)_{S\in\mathfrak{S}}$ of $(\mathcal{N}_x^S)_{S\in\mathfrak{S}}$ such that $\{U_S : S \in \mathfrak{S}\}$ is independent and define $U := \bigcup_{S\in\mathfrak{S}} U_S \in \mathcal{N}_x^X$. Since $\mathfrak{S}$ is pairwise locally-distinct near x, it follows that $U_S \cap U_T = N_x$ for all distinct $S, T \in \mathfrak{S}$ and thus

$$x \notin \overline{\emptyset} = \overline{\left(\bigcup_{\{S,T\}\in[\mathfrak{S}]^2} (U_S \cap U_T)\right)\setminus N_x}.$$ □

Lemma II.3.15. *Let $\mathfrak{S}$ be a spoke system for a point x in a topological space X and suppose $\mathfrak{S}$ has an independent truncation. Then for all distinct $S, T \in \mathfrak{S}$, either $S =_x T$ or*

$x \notin \overline{(S \cap T)\backslash N_x}$. *In particular, if* $\mathfrak{S}$ *is pairwise locally-distinct near* x *then* $\mathfrak{S}$ *is almost-independent.*

Proof. Select $(U_S)_{S\in\mathfrak{S}}$ from $(\mathcal{N}_x^S)_{S\in\mathfrak{S}}$ such that $\mathfrak{T} := \{U_S : S \in \mathfrak{S}\}$ is independent. Then there exists a selection $(V_S)_{S\in\mathfrak{S}}$ of $\mathcal{N}_x^X$ such that $U_S = V_S \cap S$ for all $S \in \mathfrak{S}$. Let $S, T \in \mathfrak{S}$ be distinct, so either $U_S = U_T$ or $U_S \cap U_T = N_x$. If $U_S = U_T$ then $(V_S \cap V_T) \cap S = U_S \cap T = U_T \cap S = (V_S \cap V_T) \cap T$, so $S =_x T$. If $U_S \cap U_T = N_x$ then $(V_S \cap V_T) \cap (S \cap T) = U_S \cap U_T = N_x$ and so $x \notin \overline{(S \cap T)\backslash N_x}$. □

This suggests, together with Lemma II.2.19, that our study of truncating spoke systems can be restricted to pairwise locally-distinct, almost-independent spoke systems.

The next lemma shows that, for 'small' collections of spokes of equal character, we can perform a diagonalisation procedure and find an independent truncation.

Lemma II.3.16. *Let* x *be a point in a topological space* X *and let* $\lambda \in \Sigma^{rad}(x), \mathcal{S} \in [\mathbf{Sp}(x)]^{\leq\lambda}$ *be almost-independent, where each* $S \in \mathcal{S}$ *has character* λ *at* x. *Then there is a selection* $(U_S)_{S\in\mathcal{S}}$ *of* $(\mathcal{N}_x^S)_{S\in\mathcal{S}}$ *such that* $U_S \cap U_{S'} = N_x$ *for all distinct* $S, S' \in \mathcal{S}$. *In particular,* $\mathcal{S}$ *has an independent truncation.*

Proof. Enumerate $\mathcal{S} = \{S_\alpha : \alpha < \kappa\}$, where $\kappa \leq \lambda$, and let $\alpha < \beta < \kappa$ be given. Then $x \notin \overline{(S_\alpha \cap S_\beta)\backslash N_x}$, so $S_\beta \backslash (S_\alpha \backslash N_x) \in \mathcal{N}_x^{S_\beta}$. Define $U_{S_\beta} := S_\beta \backslash ((\bigcup_{\alpha<\beta} S_\alpha)\backslash N_x)$ and note that by Theorem I.3.26 $U_{S_\beta} \in \mathcal{N}_x^{S_\beta}$. Moreover, for distinct $\beta, \gamma < \kappa, U_{S_\beta} \cap U_{S_\gamma} = N_x$. Therefore $\{U_{S_\beta} : \beta < \kappa\}$ is an independent truncation of $\mathcal{S}$. □

The following technical theorem allows us to limit the search for an independent spoke system to finding an independent truncation. We will be using it in the next section to construct a strong example of a Fréchet-Urysohn space with a non-independently-based point.

Theorem II.3.17. *Let* $\mathfrak{S}$ *be an almost-independent, non-trivial spoke system for a point* x *in a topological space* X *such that* $S \cap S' = N_x$ *for all* $S, S' \in \mathfrak{S}$ *with* $\chi_S(x) \neq \chi_{S'}(x)$*. Suppose there exists an independent, non-trivial spoke system* $\mathfrak{T}$ *such that for all* $\lambda \in \Sigma^{rad}(x)$ *and all* $T \in \mathbf{C}_\lambda(\mathfrak{T}), |\mathbf{Thr}_{\mathfrak{S}}(T)| \leq \lambda$*. Then* $\mathfrak{S}$ *has an independent truncation.*

Proof. Let $\lambda \in \Sigma^{\text{rad}}(x), S \in \mathbf{C}_\lambda(\mathfrak{S})$ be given and note that $|\mathbf{Thr}_{\mathfrak{S}}(T)| < \lambda$ by Theorem II.3.8. By Lemma II.3.10, $S \subseteq_x \bigcup \mathbf{Thr}_{\mathfrak{T}}(S)$, so $U_S := S \cap (\bigcup \mathbf{Thr}_{\mathfrak{T}}(S)) \in \mathcal{N}_x^S$. Let $S, S' \in \mathbf{C}_\lambda(\mathfrak{S})$ be distinct such that $U_S \cap U_{S'} \neq N_x$, so there exist $T \in \mathbf{Thr}_{\mathfrak{T}}(S)$ and $T' \in \mathbf{Thr}_{\mathfrak{T}}(S')$ such that $T \cap T' \neq N_x$. As $\mathfrak{T}$ is independent, it follows that $T = T' \in \mathbf{Thr}_{\mathfrak{T}}(S) \cap \mathbf{Thr}_{\mathfrak{T}}(S')$ and so $S' \in \mathbf{Thr}_{\mathfrak{S}}(T) \subseteq \bigcup_{R \in \mathbf{Thr}_{\mathfrak{T}}(S)} \mathbf{Thr}_{\mathfrak{S}}(R)$. Define for all $S \in \mathbf{C}_\lambda(\mathfrak{S})$,

$$\mathcal{U}_S := \{U_{S'} \setminus N_x : S' \in \mathbf{C}_\lambda(\mathfrak{S}) \setminus \{S\}, U_S \cap U_{S'} \neq N_x\}.$$

Then by the regularity of λ, for all $S \in \mathbf{C}_\lambda(\mathfrak{S})$,

$$|\mathcal{U}_S| \leq \left| \bigcup_{R \in \mathbf{Thr}_{\mathfrak{T}}(S)} \mathbf{Thr}_{\mathfrak{S}}(R) \right| \leq \sum_{R \in \mathbf{Thr}_{\mathfrak{T}}(S)} |\mathbf{Thr}_{\mathfrak{S}}(R)| < \lambda.$$

By almost-independence, for all distinct $S, S' \in \mathbf{C}_\lambda(\mathfrak{S})$ with $U_S \cap U_{S'} \neq N_x$,

$$U_S \setminus (U_{S'} \setminus N_x) \in \mathcal{N}_x^S \Rightarrow V_S := U_S \setminus (\bigcup \mathcal{U}_S) \in \mathcal{N}_x^S.$$

Now let $S_1, S_2 \in \mathfrak{S}$ be distinct. If $\chi_{S_1}(x) \neq \chi_{S_2}(x)$ then by assumption, $S_1 \cap S_2 = N_x$ and so $V_{S_1} \cap V_{S_2} = N_x$. If not define $\lambda := \chi_{S_1}(x) = \chi_{S_2}(x)$. If $U_{S_1} \cap U_{S_2} = N_x$ then $V_{S_1} \cap V_{S_2} = N_x$. Otherwise, $U_{S_2} \setminus N_x \in \mathcal{U}_{S_1}$ and so $V_{S_1} \cap V_{S_2} \subseteq (U_{S_1} \setminus (U_{S_2} \setminus N_x)) \cap U_{S_2} = N_x$, hence $V_{S_1} \cap V_{S_2} = N_x$. Therefore $\{V_S : S \in \mathfrak{S}\}$ is an independent truncation of $\mathfrak{S}$. □

Corollary II.3.18. *Let $\mathfrak{S}$ be an almost-independent, non-trivial spoke systems for a point x in a topological space X such that $S \cap S' = N_x$ for all $S, S' \in \mathfrak{S}$ with $\chi_S(x) \neq \chi_{S'}(x)$. Suppose for every spoke $T \in \mathbf{Sp}(x), |\mathbf{Thr}_{\mathfrak{S}}(T)| \leq \chi_T(x)$. Then x is independently-based if and only if $\mathfrak{S}$ has an independent truncation.*

Proof. By Lemma II.3.13, if $\mathfrak{S}$ has an independent truncation then x is independently-based, so suppose that x has an independent spoke system $\mathfrak{T}$. As there exists an $S \in \mathfrak{S}$ and $x \in \overline{S \setminus N_x}$, it follows that x is non-quasi-isolated in X and thus by Lemma II.2.12 we can without loss of generality assume that $\mathfrak{T}$ is non-trivial. Therefore by the previous theorem, $\mathfrak{S}$ has an independent truncation. □

II.3.3 Strongly Fréchet spaces

The characterisation of Fréchet-Urysohn points in Corollary II.2.17 is optimal for independence, since for any topological space being independently-based and strongly Fréchet is equivalent to first-countability.

Definition II.3.19 (Strongly Fréchet)**.** Let x be a point in topological space. Then x is *strongly Fréchet* if for every decreasing sequence of sets $(A_n)_{n<\omega}$ with $x \in \bigcap_{n\in\omega} A_n$, there exists a selection $(x_n)_{n<\omega}$ of $(A_n)_{n<\omega}$ that converges to x.

The property that every point in a topological space is strongly Fréchet is also known as *countably-bisequential*, and is equivalent to the space being Fréchet-Urysohn and having Arhangel′skiĭ's α_4 (see Definition II.4.6).

Lemma II.3.20. *[Arh81, Proposition 5.11, Theorem 5.23] Every topological space is strongly Fréchet at each of its first-countable points.*

Theorem II.3.21. *Let X be a topological space, $x \in X$ be given such that x is independently-based and strongly Fréchet. Then x is first-countable.*

Proof. Suppose otherwise. Then there exists an independent spoke system $\mathfrak{S}$ for x and without loss of generality (by Lemma II.2.12), assume that $\mathfrak{S}$ is non-trivial. As x is strongly Fréchet, it follows that $\Sigma^{\mathrm{rad}}(x) = \{\omega\}$ so $\mathfrak{S}$ must be infinite by Corollary II.3.3. Choose $\mathcal{S} = \{S_n : n < \omega\} \subseteq \mathfrak{S}$, where each S_n is distinct from the others, and define for every $n < \omega, A_n := (\bigcup_{m \geq n} S_m) \backslash N_x$. As $x \in \overline{S_n \backslash N_x^X}$ for each $n < \omega$, it follows that $(A_n)_{n<\omega}$ is descending sequence of subsets of X with $x \in \bigcap_{n<\omega} \overline{A_n}$. Since x is strongly Fréchet, there exists a selection $(x_n)_{n<\omega}$ of $(A_n)_{n<\omega}$ that converges to x and thus there exists a unique $j_n \in [n, \omega)$ such that $x_n \in S_{j_n}$. Hence for all $m < \omega, \{x_n : n \in \omega\} \cap S_m$ is finite, so there exists a $U_m \in \mathcal{N}_x^{S_m}$ disjoint from $\{x_n : n < \omega\}$. Then $U := (\bigcup_{m<\omega} U_m) \cup \bigcup(\mathfrak{S} \backslash \mathcal{S})$ is an X-neighbourhood of x and $U \cap \{x_n : n < \omega\} = \emptyset$, which is a contradiction. Therefore x is first-countable. □

Recall from Figure II.4 that $S(\omega)$ was independently-based. However, it is not first-countable at $\star$: define for all $f : \omega \to \omega, U_f := \{\star\} \cup \{(m,n) : n \geq f(m)\}$. Let $(V_n)_{n<\omega} \subseteq \mathcal{N}_\star^{S(\omega)}$ be given, so there exists a sequence $(f_n)_{n<\omega} \subseteq {}^\omega\omega$ such that $U_{f_n} \subseteq V_n$. Define for all $n < \omega, g(n) := \max(f_0(n), \dots, f_n(n)) + 1$. Then $V_n \not\subseteq U_g$ for all $n < \omega$, so $\{V_n : n < \omega\}$ is not a neighbourhood base of $\star$ with respect to $S(\omega)$. Thus $\star$ is not first-countable in $S(\omega)$ and so by the previous theorem, $S(\omega)$ is not strongly Fréchet at $\star$. Since the property of a point being strongly Fréchet is hereditary, it implies that if X is a topological space and $x \in Y \subseteq X$, if $Y \cong S(\omega)$ and x is non-isolated in A, then x is not strongly Fréchet in X.

Using this previous theorem, we now show how the classes of strongly Fréchet and independently-based spaces intersect. In Section II.4, we will characterise the strongly Fréchet points using spoke systems, from which we will also obtain the following corollary:

Corollary II.3.22. *Let X be a topological space and let $x \in X$ be given. Then X is first-countable at x if and only if X is independently-based and strongly Fréchet at x.*

An example of a strongly Fréchet space that is not first-countable is the one-point compactification of an uncountable discrete space (see [Arh81, Example 5.12]) and, as we saw earlier, the one-point is not independently-based.

The previous corollary could be understood as saying that Fréchet-Urysohn, independently-based spaces and strongly Fréchet spaces are 'orthogonal'. However, they are not 'complementary'.

Theorem II.3.23. *There exists a Fréchet-Urysohn space with a point that is neither strongly Fréchet nor independently-based.*

Proof. Define X to be the space obtained by quotienting the non-isolated points of $\alpha(D(\aleph_1)) = D(\aleph_1) \cup \{\star_1\}$ and $S(\omega) = (\omega \times \omega) \cup \{\star_2\}$ together and let $\star \in X$ be the unique non-isolated point, where $\alpha(D(\aleph_1))$ denotes the one-point compactification of a discrete space $D(\aleph_1)$ of size $\aleph_1$. Then $\star$ is not first-countable in $D(\aleph_1) \cup \{\star\}$, nor is it strongly-Fréchet in $(\omega \times \omega) \cup \{\star\} \cong S(\omega)$. However, $\star$ is strongly-Fréchet in $D(\aleph_1) \cup \{\star\}$ (given a descending sequence $(B_n)_{n<\omega}$ in $D(\aleph_1)$ with $\star \in \bigcap_{n<\omega} \overline{B_n}$, any selection $(x_n)_{n<\omega}$ from $(B_n)_{n<\omega}$ converges to $\star$), so by the previous corollary $\star$ is not independently-based in $D(\aleph_1) \cup \{\star\}$. As both of these properties are hereditary, it follows that $\star$ is not strongly Fréchet nor independently based in X. However, $Y := \aleph(D(\aleph_1)) \cup S(\omega)$ is Fréchet-Urysohn and $\{\star_1, \star_2\}$ is closed in X, so the quotient map from Y to X is closed. Therefore by Theorem I.2.16, X is Fréchet-Urysohn. □

We conclude this section by creating another example. This space cannot be decomposed around the specified point into finitely-many subspaces, in each of which the point is either strongly Fréchet and not independently-based, or vice versa. Furthermore, this has many of the properties of a Fréchet-Urysohn, independently-based space.

Theorem II.3.24. *There exists a Fréchet-Urysohn space X with a point x that is neither strongly Fréchet nor independently-based in X. Moreover:*

(1) $\psi(x) = \aleph_0$.

(2) If Y is a subspace containing x that is strongly Fréchet at x then $\chi_Y(x) \leq \aleph_0$.

(3) If $\mathcal{Y}$ is a finite collection of subspaces, each containing x, such that $\bigcup\mathcal{Y}$ is a neighbourhood of x then there exists a $Y \in \mathcal{Y}$ such that x is neither strongly Fréchet nor independently-based in Y.

Proof. Let $\|\cdot\|$ be the Euclidean norm on $\mathbb{R}^2$ and for each $x \in \mathbb{R}^2$ and $\varepsilon > 0$, let $B_\varepsilon(x)$ denote the open ε-ball around x given by the norm. Denote the origin by $\mathbf{0}$ and for every $x \in$

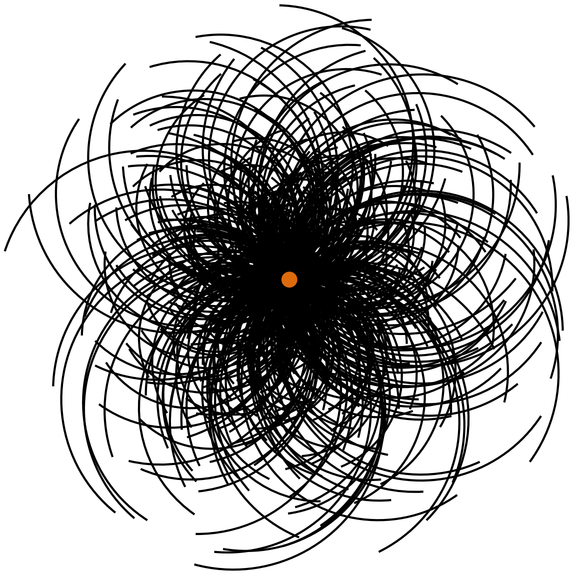

Figure II.5: A(n approximation to a) basic neighbourhood of $\mathbf{0}$.

$\mathbb{R}^2\setminus\{\mathbf{0}\}$, define

$$S_x := \{y \in \mathbb{R}^2 : \|y - x\| = \|x\|\},$$

$$\forall x \in \mathbb{R}^2\setminus\{\mathbf{0}\}, \forall \varepsilon > 0, B_\varepsilon^x := B_\varepsilon(\mathbf{0}) \cap S_x,$$

$$\mathcal{B} := \left\{ \bigcup_{x\in\mathbb{R}^2\setminus\{\mathbf{0}\}} B_{\varepsilon_x}^x : (\varepsilon_x)_{x\in\mathbb{R}^2\setminus\{\mathbf{0}\}} \text{ is a selection of } (0,\infty) \right\}.$$

Endow $\mathbb{R}^2$ with the unique topology with neighbourhood base $\mathcal{B}$ at the origin, and all other points isolated. Denote this space by X. Let $x, y \in \mathbb{R}^2\setminus\{\mathbf{0}\}$ be distinct. Then since distinct circles intersect in at most two points, there exists an $\varepsilon_y > 0$ such that $B_{\varepsilon_y}^y \cap S_x = \{\mathbf{0}\}$. Thus for all $\delta > 0$,

$$\left(B_\delta^x \cup \bigcup_{y\in\mathbb{R}^2\setminus\{\mathbf{0},x\}} B_{\varepsilon_y}^y \right) \cap S_x = B_\delta^x.$$

Hence $\{B_\delta^x : \delta > 0\}$ is a neighbourhood base for $\mathbf{0}$ with respect to S_x, so S_x is well-based at $\mathbf{0}$. Therefore $\mathfrak{S} := \{S_x : x \in \mathbb{R}^2\setminus\{\mathbf{0}\}\}$ is a spoke system of $\mathbf{0}$, so by Theorem II.2.7 X is radial.

Furthermore, $\chi_{S_x}(\mathbf{0}) = \aleph_0$ for all $x \in \mathbb{R}^2 \backslash \{\mathbf{0}\}$, so by Corollary II.2.17 X is Fréchet-Urysohn. Note that $N_{\mathbf{0}} = \{\mathbf{0}\}$ and define for all $F \subseteq \mathbb{R}^2 \backslash \{\mathbf{0}\}$,

$$Y_F := \{S \in \mathfrak{S} : |S \cap F| \geq 3\}.$$

Claim. $\mathbf{Thr}_{\mathfrak{S}}(T)$ *is finite for every* $T \in \mathbf{Sp}(\mathbf{0})$.

Proof of claim. Let $T \in \mathbf{Sp}(\mathbf{0})$ and note that by Theorem II.2.7 and II.2.13, T is first-countable at $\mathbf{0}$. Let $(C_n)_{n<\omega}$ be a descending T-neighbourhood base of $\mathbf{0}$. Suppose $\mathbf{Thr}_{\mathfrak{S}}(T)$ is infinite and for every $n < \omega$, pick an $x_n \in \mathbf{Thr}_{\mathfrak{S}}(T)$ and a

$$y_n \in ((C_n \cap S_{x_n}) \backslash \{\mathbf{0}\}) \backslash \left(\bigcup \{S_x : x \in Y_{\{y_j : j<n\}}\}\right)$$

such that for all distinct $m, k < \omega, x_m \neq x_k$. Then $(y_n)_{n<\omega} \dashrightarrow \mathbf{0}$ so by Lemma II.2.3 there exists a strictly increasing $f : \omega \to \omega$ and a $y \in \mathbb{R}^2 \backslash \{\mathbf{0}\}$ such that $\{y_{f(n)} : n < \omega\} \subseteq S_y$. Then $y \in Y_{\{y_j : j<g(3)\}}$, so $y_{g(3)} \notin S_y$, which is a contradiction. Therefore $\mathbf{Thr}_{\mathfrak{S}}(T)$ is finite. □

So by Corollary II.3.18, if $\mathbf{0}$ is independently-based then $\mathfrak{S}$ has an independent truncation. We now show that the latter is not the case. Select $(\varepsilon_x)_{x \in \mathbb{R}^2 \backslash \{\mathbf{0}\}}$ from $(0, \infty)$ and assume $\{B^x_{\varepsilon_x} : x \in \mathbb{R}^2 \backslash \{\mathbf{0}\}\}$ is independent. Since

$$\{x \in \mathbb{R}^2 : \|x\| > 1\} = \bigcup_{n \in \mathbb{N}^+} \left\{x \in \mathbb{R}^2 : \|x\| > 1 \text{ and } \varepsilon_x \geq \frac{1}{n}\right\},$$

the Baire category theorem implies that there exists an $n \in \mathbb{N}^+$ such that

$$Z := \left\{x \in \mathbb{R}^2 : \|x\| > 1 \text{ and } \varepsilon_x \geq \frac{1}{n}\right\}$$

is not nowhere-dense in the Euclidean topology. Thus there exists an open, non-empty subset U of $\mathbb{R}^2$ such that $U \cap Z$ is dense in U with respect to the Euclidean topology. Pick an

$x \in U$, so there exists an $\varepsilon \in (0,1)$ and a $\theta \in \mathbb{R}$ such that $B_\varepsilon(x) \subseteq U$ and $x = \|x\|(\cos(\theta), \sin(\theta))$. By continuity of the sine function, there exists a $\delta \in (-\pi/2, \pi/2)$ such that for all $a \in (-\delta, \delta)$,

$$|\sin(a)| < \frac{\varepsilon}{4n\|x\|(\|x\| + \varepsilon)}.$$

Define $L := \operatorname{Span}(x)$, which is closed in $\mathbb{R}^2$. Note that

$$V := \{c(\cos(\varphi), \sin(\varphi)) : c \in (\|x\| + \varepsilon/2, \|x\| + \varepsilon), \varphi \in (\theta - \delta, \theta + \delta)\}$$

is open in $\mathbb{R}^2$ and $(U \cap V) \backslash L$ is a non-empty open subset of U with respect to the Euclidean topology. Thus there exists a $c \in (\|x\| + \varepsilon/2, \|x\| + \varepsilon)$ and $\varphi \in (\theta - \delta, \theta + \delta)$ such that $y := c(\cos(\varphi), \sin(\varphi)) \in (V \cap Z) \backslash L$. Define $\rho : \mathbb{R}^2 \to \mathbb{R}^2, (x, y) \mapsto (-y, x)$, which is the anticlockwise rotation about the origin by $\pi/2$ radians. Also define

$$z := \frac{2(y \cdot \rho^{-1}(x))}{\|x - y\|^2} \rho(y - x).$$

By the choice of y, it follows that $y \cdot \rho^{-1}(x) = \|x\| \|y\| \cos(\pi/2 \pm (\theta - \varphi))$ and $z \neq \mathbf{0}$. Hence

$$\|z\| \leq \frac{2\|x\|(\|x\| + \varepsilon)|\sin(\theta - \varphi)|}{\varepsilon/2} < \frac{1}{n}.$$

Now let $x = (x_1, x_2), y = (y_1, y_2), z = (z_1, z_2)$. Then:

$$\begin{aligned}
\frac{2(y\cdot\rho^{-1}(x))}{\|x-y\|^2} &= \frac{2(y_1x_2 - y_2x_1)}{(x_1-y_1)^2 + (x_2-y_2)^2} \\
z_1 &= \frac{2(y_1x_2 - y_2x_1)(x_2-y_2)}{(x_1-y_1)^2 + (x_2-y_2)^2} \\
z_2 &= \frac{2(y_1x_2 - y_2x_1)(y_1-x_1)}{(x_1-y_1)^2 + (x_2-y_2)^2} \\
\Rightarrow z_1^2 + z_2^2 &= \frac{4(y_1x_2 - y_2x_1)^2}{(x_1-y_1)^2 + (x_2-y_2)^2} \\
&= 2 \cdot \frac{2(y_1x_2 - y_2x_1)}{(x_1-y_1)^2 + (x_2-y_2)^2}((x_2-y_2)x_1 + (y_1-x_1)x_2) \\
&= 2(z_1x_1 + z_2x_2)
\end{aligned}$$

$$\begin{aligned}
\Rightarrow (z_1-x_1)^2 + (z_2-x_2)^2 &= z_1^2 + z_2^2 + x_1^2 + x_2^2 - 2(z_1x_1 + z_2x_2) \\
&= x_1^2 + x_2^2.
\end{aligned}$$

Therefore $\|z - x\| = \|x\|$, so $z \in B^x_{\varepsilon_x}$. Since ρ is an isometry,

$$\frac{2(x\cdot\rho^{-1}(y))}{\|y-x\|^2}\rho(x-y) = \frac{2(\rho^{-1}(x)\cdot\rho^{-2}(y))}{\|x-y\|^2}(-\rho(y-x)) = z.$$

So by symmetry $z \in B^y_{\varepsilon_y}$, which is a contradiction. Hence **0** has no independent spoke system and is therefore not independently-based.

* * *

We now move on to proving parts (1–3) of the theorem. First, observe that $\{B_{1/n}(\mathbf{0}) : n \in \mathbb{N}^+\}$ is a pseudobase of **0** which is not quasi-isolated, so $\psi(\mathbf{0}) = \aleph_0$. Now let $Y \subseteq X$ be given such that $\mathbf{0} \in Y$ and **0** is not first-countable in Y. Then $\{S \cap Y : S \in \mathfrak{S}\}$ is a spoke system of **0** with respect to Y by Corollary II.1.16, so by Lemma II.2.12, $\mathfrak{T} := \{S \cap Y : S \in \mathfrak{S}, \mathbf{0} \in \overline{(S \cap Y) \setminus \{\mathbf{0}\}}\}$ is also a spoke system of **0** with respect to Y. By Corollary II.3.3 there exists an injective sequence $(T_n)_{n<\omega}$ in $\mathfrak{T}$ and by Theorem I.2.2 for each $n < \omega$, there exists a sequence $(x_{n,m})_{m<\omega}$ in $(T_n \cap Y)\setminus\{\mathbf{0}\}$ that converges to $\mathbf{0}$. Define $C := \{x_{n,m} : n, m < \omega\}$ and note that $Y_C = \bigcup_{F\in[C]^3} Y_F$ is countable. Enumerate

$$\{x \in Y_C : S_x \neq T_n \text{ for all } n < \omega\} = \{y_n : n < \alpha\}$$

where $\alpha \leq \omega$ (possibly $\alpha = 0$). Define for all $l < \omega$,

$$D_l := C\setminus\left(\bigcup_{k<\min(\alpha,l)} S_{y_k}\right).$$

Observe that $(D_l)_{l<\omega}$ is a descending chain of subsets. Furthermore, since distincts circles intersect in at most two points, it follows that D_l contains a tail of $(x_{l,m})_{m<\omega}$ for each $l < \omega$ and so $\mathbf{0} \in \overline{D_l}$.

Suppose there exists a selection $(z_l)_{l<\omega}$ of $(D_l)_{l<\omega}$ that converges (strictly) to $\mathbf{0}$. Without loss of generality, by Corollary II.2.3 there exists a $w \in \mathbb{R}^2\setminus\{\mathbf{0}\}$ such that S_w that contains $Z := \{z_l : l < \omega\}$. Then $w \in Y_Z \subseteq Y_C$ and for all $l < \omega, Z \cap D_l$ is finite, so $S_w \neq T_l$. Thus there exists an $m < \alpha$ such that $w = y_m$. But $z_{m+1} \in D_{m+1}$, so $z_{m+1} \notin S_{y_m} = S_w$, which is a contradiction. Therefore $\mathbf{0}$ is not strongly Fréchet in Y. Moreover, as $\mathbf{0}$ is not independently-based in X, it is also not first-countable and thus not strongly Fréchet.

Finally, let $\mathcal{Y}$ be a finite collection of subspaces, each containing $\mathbf{0}$, such that $\bigcup\mathcal{Y} \in \mathcal{N}_{\mathbf{0}}^X$ and assume that for each $Y \in \mathcal{Y}, \mathbf{0}$ is either strongly Fréchet or independently-based with respect to Y. Since first-countable points are independently-based, it follows that $\mathbf{0}$ is independently-based with respect to each $Y \in \mathcal{Y}$. For every non-empty $\mathcal{Z} \subseteq \mathcal{Y}$, define

$$P_{\mathcal{Z}} := \left(\left(\bigcap\mathcal{Z}\right)\setminus\left(\bigcup(\mathcal{Y}\setminus\mathcal{Z})\right)\right) \cup \{\mathbf{0}\}.$$

Note that $\{P_{\mathcal{Z}}\setminus\{\mathbf{0}\} : \mathcal{Z} \in \mathcal{P}(\mathcal{Y})\setminus\{\emptyset\}\}$ is a partition of $(\bigcup\mathcal{Y})\setminus\{\mathbf{0}\}$ and for each non-empty $\mathcal{Z} \subseteq \mathcal{Y}$, there exists a $Z \in \mathcal{Z}$ and so $P_{\mathcal{Z}} \subseteq Z$. Hence $\mathbf{0}$ is also independently-based in every $P_{\mathcal{Z}}$ by Lemma II.1.7.

For each non-empty $\mathcal{Z} \subseteq \mathcal{Y}$, choose an independent spoke system $\mathfrak{T}_{\mathcal{Z}}$ of $\mathbf{0}$ with respect to $P_{\mathcal{Z}}$ and define $\mathfrak{T} := \bigcup_{\emptyset \neq \mathcal{Z} \subseteq \mathcal{Y}} \mathfrak{T}_{\mathcal{Z}}$. Let $\mathcal{Z}_0, \mathcal{Z}_1 \in \mathcal{P}(\mathcal{Y}) \setminus \{\emptyset\}$ be distinct, so there exists a $Z \in \mathcal{Z}_i \Delta \mathcal{Z}_{1-i}$. Then

$$P_{\mathcal{Z}_1} \cap P_{\mathcal{Z}_2} \subseteq (Z \cup \{\mathbf{0}\}) \cap ((X \setminus Z) \cup \{\mathbf{0}\}) = \{\mathbf{0}\}.$$

Hence $\mathfrak{T}$ is an independent collection of spokes of $\mathbf{0}$.

Now let $A \subseteq X$ be given such that $\mathbf{0} \in \overline{A}$. Then since $\bigcup\{P_{\mathcal{Z}} : \mathcal{Z} \in \mathcal{P}(\mathcal{Y}) \setminus \{\emptyset\}\} = \bigcup \mathcal{Y} \in \mathcal{N}_{\mathbf{0}}^{X}$ and $\mathcal{Y}$ is finite, there exists a non-empty $\mathcal{Z} \subseteq \mathcal{Y}$ such that $\mathbf{0} \in \overline{A \cap P_{\mathcal{Z}}}$ and thus by Theorem II.1.13 there exists a $T \in \mathfrak{T}_{\mathcal{Z}} \subseteq \mathfrak{T}$ such that $\mathbf{0} \in \overline{A \cap T}$. By the same theorem, $\mathfrak{T}$ is an independent spoke system of $\mathbf{0}$, which is a contradiction. Therefore there exists a $Y \in \mathcal{Y}$ such that $\mathbf{0}$ is neither strongly Fréchet nor independently-based with respect to $\mathbf{0}$. $\square$

We conclude with a few open questions concerning independently-based spaces and spoke systems.

Question II.3.25. *Is there a 'natural' characterisation of radial spaces in terms of nests?*

Question II.3.26. *Do independently-based spaces coincide with a subclass of radial spaces with stronger convergence properties?*

II.4 Products and strong convergence properties

In this last section, we will show how certain strong convergence properties are reflected into spokes. As some of these properties are equivalently characterised by preservation theorems under products, we first investigate how to take appropriate products of spoke systems.

II.4.1 Productivity

The obvious ‘product’ of a collection $\mathfrak{C}$ of spoke systems would be

$$\left\{ \prod_{\mathfrak{S}\in\mathfrak{C}} S_{\mathfrak{S}} : (S_{\mathfrak{S}})_{\mathfrak{S}\in\mathfrak{C}} \text{ is a selection from } \mathfrak{C} \right\}.$$

However, for this to form a spoke system, we must ensure that these products do form *spokes.*

Definition II.4.1 (Coherence)**.** Let $(X_i)_{i\in I}$ be a finite non-empty family of topological spaces and select $(x_i)_{i\in I}$ from $(X_i)_{i\in I}$ and $(S_i)_{i\in I}$ from $(\mathbf{Sp}_{X_i}(x_i))_{i\in I}$. Then we say that $(S_i)_{i\in I}$ is *coherent* if $\prod_{i\in I} S_i$ is a spoke of $(x_i)_{i\in I} \in \prod_{i\in I} X_i$.

Lemma II.4.2. *Let $(X_i)_{i\in I}$ be a finite non-empty family of topological spaces and select $(x_i)_{i\in I}$ from $(X_i)_{i\in I}$ and $(S_i)_{i\in I}$ from $(\mathbf{Sp}_{X_i}(x_i))_{i\in I}$. Then $(S_i)_{i\in I}$ is coherent if and only if for all $i, j \in I$ where S_i and S_j are non-trivial, $\chi_{S_i}(x_i) = \chi_{S_j}(x_j)$.*

Proof. First, define

$$X := \prod_{i\in I} X_i,$$

$$x := (x_i)_{i\in I},$$

$$J := \{i \in I : S_i \text{ is non-trivial}\}.$$

Note that by Lemma I.3.11 $N_x^X = \prod_{i \in I} N_{x_i}^{X_i} \subseteq S := \prod_{i \in I} S_i$. Suppose there exists $i, j \in J$ such that $\chi_{S_i}(x_i) \neq \chi_{S_j}(x_j)$. Then by Theorem I.3.20, Corollary I.3.26 and Proposition I.3.30, $S_i \times S_j$ is not radial at (x_i, x_j). Since there is an embedding of $S_i \times S_j$ into S that carries (x_i, x_j) to x, it follows that S is not radial at x; in particular (by Theorem I.2.2) S is not a spoke of x.

Now suppose there exists a non-zero cardinal λ such that $\chi_{S_j}(x_j) = \lambda$ for all $j \in J$ and let $(B_\alpha^j)_{\alpha<\lambda}$ be well-ordered neighbourhood base of x with respect to S_j. By Lemma I.3.23, we can assume without loss of generality that λ is regular and infinite. Define

$$\forall \alpha < \lambda, \forall i \in I \backslash J, B_\alpha^i := N_{x_i}^{X_i},$$

$$\mathcal{B} := \left\{ \prod_{i \in I} B_\alpha^i : \alpha < \lambda \right\}$$

Then $\mathcal{B} \subseteq \mathcal{N}_x^S$ is well-ordered with respect to $\supsetneqq$. Let $U \in \mathcal{N}_x^S$ be given, so for all $i \in I$, there exists a $V_i \in \mathcal{N}_{x_i}^{S_i}$ such that $\prod_{i \in I} V_i \subseteq U$. Then for all $i \in I$, there exists an $\alpha_i < \lambda$ such that $B_{\alpha_i}^i \subseteq V_i$. Define $\alpha := \max(\alpha_i : i \in I) < \lambda$, so then $\prod_{i \in I} B_\alpha^i \subseteq U$. Therefore $\mathcal{B}$ is a well-ordered neighbourhood base for x with respect to S, so S is a spoke at x. □

Definition II.4.3 (Product of spoke systems)**.** Let $(X_i)_{i \in I}$ be a finite non-empty family of topological spaces and select $(x_i)_{i \in I}$ from $(X_i)_{i \in I}$ and $(\mathcal{S}_i)_{i \in I}$ from $(\mathcal{P}(\mathbf{Sp}_{X_i}(x_i)))_{i \in I}$. Then we define the *(coherent) product* of $(\mathcal{S}_i)_{i \in I}$ to be

$$\bigotimes_{i \in I} \mathcal{S}_i := \left\{ \prod_{i \in I} S_i : (S_i)_{i \in I} \text{ is a coherent selection from } (\mathcal{S}_i)_{i \in I} \right\}.$$

If $X_i = X, x_i = x$ and $\mathcal{S}_i = \mathcal{S}$ for all $i \in I$, then we denote this product by $\mathcal{S}^{\otimes n}$.

If $(X_i)_{i<n}$ is a finite non-empty sequence of topological spaces, with $n \geq 2$, and $(x_i)_{i<n}$

is a selection of $(X_i)_{i<n}$ and $(\mathcal{S}_i)_{i<n}$ is a selection of $(\mathcal{P}(\mathbf{Sp}_{X_i}(x_i)))_{i<n}$, then we define

$$\mathcal{S}_0 \otimes \cdots \otimes \mathcal{S}_{n-1} := \bigotimes_{i<n} \mathcal{S}_i.$$

If there are two factors where the projections are non-quasi-isolated then the productivity of radiality could fail if they have different radial spectra. The following shows that the radial spectrum of the product and the non-quasi-isolated factors indeed coincide if the product point is radial.

Lemma II.4.4. *Let $(X_i)_{i\in I}$ be a finite non-empty family of topological spaces and select $x := (x_i)_{i\in I}$ from $(X_i)_{i\in I}$. Suppose x is radial and non-quasi-isolated in $X := \prod_{i\in I} X_i$, and that there are at least two $i \in I$ where x_i is non-quasi-isolated in X_i. Then $\Sigma^{rad}_X(x)$ is a singleton and for all $i \in I$, if x_i is non-quasi-isolated then $\Sigma^{rad}_{X_i}(x_i) \subseteq \Sigma^{rad}_X(x)$.*

Proof. First note that by Lemma I.3.11, $N^X_x = \prod_{i\in I} N^{X_i}_{x_i}$. Let $i, j \in I$ be distinct such that x_i, x_j are both non-quasi-isolated in X_i, X_j respectively and let $\lambda \in \Sigma^{\mathrm{rad}}_{X_i}(x_i), \mu \in \Sigma^{\mathrm{rad}}_{X_j}(x_j)$ be given. Then by Theorem II.2.9 there exist transfinite sequences t, u in $X_i \setminus N^{X_i}_{x_i}$ and $X_j \setminus N^{X_j}_{x_j}$ that converge strictly to x_i, x_j respectively, with $\mathrm{len}(t) = \lambda$ and $\mathrm{len}(u) = \mu$. Then x_i, x_j are non-quasi-isolated in $\mathbb{S}_t, \mathbb{S}_u$ respectively. As there exists an embedding from $X_i \times X_j$ into X that maps (x_i, x_j) to x, it follows that (x_i, x_j) is radial in $\mathbb{S}_t \times \mathbb{S}_u$. Hence by Proposition I.3.30, Theorem I.3.24 and Corollary I.3.26, $\lambda = p_{\mathbb{S}_t}(x_i) \le \psi_{\mathbb{S}_u}(x_j) = \chi_{\mathbb{S}_u}(x_j) = p_{\mathbb{S}_u}(x_j) = \mu$ and vice versa. Therefore $\Sigma^{\mathrm{rad}}_{X_i}(x_i) = \Sigma^{\mathrm{rad}}_{X_j}(x_j)$ is a singleton, so there exists a unique non-zero regular ordinal ν such that $\Sigma^{\mathrm{rad}}_{X_i}(x_i) = \{\nu\}$ for all $i \in I$ where x_i is non-quasi-isolated in X.

Now let t be a transfinite sequence in $X \backslash N_x^X$ with regular length that converges strictly to x and let $i \in I$ be given. As projections are open and continuous, $\pi_i[\mathbb{S}_t] \in \mathbf{Sp}_{X_i}(x_i)$ and $\chi_{\pi_i[\mathbb{S}_t]}(x_i) \leq \chi_{\mathbb{S}_t}(x) = \operatorname{len}(t)$. However, $\mathbb{S}_t \subseteq \prod_{i\in I} \pi_i[\mathbb{S}_t]$ and so $\operatorname{len}(t) \leq \prod_{i\in I} \chi_{\pi_i[\mathbb{S}_t]}(x_i)$. As t must have infinite length, it follows that there exists an $i \in I$ such that $\operatorname{len}(t) = \chi_{\pi_i[\mathbb{S}_t]}(x_i)$.

Define $A := (\pi_i \circ t)^{-1}[X_i \backslash N_{x_i}^{X_i}]$ and note that $\pi_i[\mathbb{S}_t] = N_{x_i}^{X_i} \cup \operatorname{ran}(\pi_i \circ t)$, so it follows that A is unbounded in $\operatorname{len}(t)$. By Corollary I.3.26, $p_{\pi_i[\mathbb{S}_t]}(x_i) = \operatorname{len}(t)$ and thus by Corollary I.3.28 $(\pi_i \circ t)|_A \dashrightarrow x_i$. Therefore $\operatorname{len}(t) \in \Sigma_{X_i}^{\text{rad}}(x_i)$ and hence $\Sigma_X^{\text{rad}}(x) \subseteq \Sigma_{X_i}^{\text{rad}}(x_i) = \{\nu\}$. Furthermore, by Theorem II.2.9 $\Sigma_X^{\text{rad}}(x) \neq \emptyset$, completing the proof. □

Theorem II.4.5. *Let $(X_i)_{i\in I}$ be a finite non-empty family of topological spaces and let $x := (x_i)_{i\in I}$ be a selection of radial points from $(X_i)_{i\in I}$. Then the following are equivalent:*

(1) x is radial in $X := \prod_{i\in I} X_i$.

(2) There exists a selection $(\mathfrak{S}_i)_{i\in I}$, where $\mathfrak{S}_i$ is a spoke system of x_i in X_i for all $i \in I$, such that $\otimes_{i\in I}\mathfrak{S}_i$ is a spoke system of x.

(3) For every selection $(\mathfrak{S}_i)_{i\in I}$, where $\mathfrak{S}_i$ is a spoke system of x_i in X_i for all $i \in I$, $\otimes_{i\in I}\mathfrak{S}_i$ is a spoke system of x.

Proof. By Theorem II.2.7, each x_i has a spoke systems, so it follows that (3) ⇒ (2) ⇒ (1).

Assume x is radial in X and for each $i \in I$, let $\mathfrak{S}_i$ be a spoke system of x_i. Define

$$J := \{j \in I : x_j \text{ is non-quasi-isolated in } X_j\},$$

$$\mathfrak{S} := \bigotimes_{i \in I} \mathfrak{S}_i,$$

$$\forall i \in I, U_i := \begin{cases} X_i & \text{if } i = j, \\ N_{x_i}^{X_i} & \text{otherwise.} \end{cases}$$

Let $A \subseteq X$ be given such that $x \in \overline{A}$. As $U := \prod_{i \in I} U_i \in \mathcal{N}_x^X$, it follows that $x \in \overline{A \cap U}$.

Case 1: J is empty. Select $(S_i)_{i \in I}$ from $(\mathfrak{S}_i)_{i \in I}$. By Theorem I.3.12, x is quasi-isolated in X and thus quasi-isolated in $S := \prod_{i \in I} S_i$ by Theorem I.3.10. Hence $S \in \mathfrak{S}$, and by Theorem I.3.11, $N_x^X = \prod_{i \in I} N_{x_i}^{X_i} \subseteq S$, so $x \in \overline{A \cap S}$.

Case 2: J has a unique element j. As $x_j \in \overline{\pi_j[A \cap U]}$, there exists an $S_j \in \mathfrak{S}_j$ such that $x_j \in \overline{\pi_j[A \cap U] \cap S_j}$. Select $(S_i)_{i \in I \setminus \{j\}}$ from $(\mathfrak{S}_i)_{i \in I \setminus \{j\}}$, so by Lemma II.4.2 $S := \prod_{i \in I} S_i \in \mathfrak{S}$. Let $V \in \mathcal{N}_x^X$ be given, so there exists a selection $(V_i)_{i \in I}$ from $(\mathcal{N}_{x_i}^{X_i})_{i \in I}$ such that $\prod_{i \in I} V_i \subseteq V$. Then there exists an $a \in A \cap U$ such that $\pi_j(a) \in S_j \cap V_j$ and, moreover, $\pi_i(a) \in N_{x_i}^{X_i} \subseteq S_i \cap V_i$ for each $i \in I \setminus \{j\}$. Hence $a \in A \cap S \cap V$ and thus $x \in \overline{A \cap S}$.

Case 3: $|J| \geq 2$. Then by Theorem I.3.12, x is non-quasi-isolated and so there exists an infinite regular cardinal λ such that $\Sigma_X^{\mathrm{rad}}(x) = \{\lambda\}$ and $\Sigma_{X_j}^{\mathrm{rad}}(x_j) = \{\lambda\}$ for each $j \in J$.

By Theorem II.2.9, x is $\Sigma_X^{\mathrm{rad}}(x)$-radial, so there exists a λ-sequence t in $A \cap U$ that converges to x. Enumerate $J = \{j_k : k < m\}$ and let $k < m$ be given such that there exists a selection $(S_{j_l})_{l<k}$ of $(\mathfrak{S}_{j_l})_{l<k}$ and a subsequence u from t such that $\operatorname{ran}(\pi_{j_l} \circ u) \subseteq S_{j_l}$ for all $l < k$. As $x \in \overline{\operatorname{ran}(\pi_{j_k} \circ u)}$, there exists an $S_{j_k} \in \mathfrak{S}_{j_k}$ such that $x \in$

$\overline{\operatorname{ran}(\pi_{j_k} \circ u) \cap S_{j_k}}$. By Corollary II.2.10 $p_{X_{j_k}}(x_{j_k}) = \lambda$, so it follows $Z := (\pi_{j_k} \circ u)^{-1}[S_{j_k}]$ is unbounded in λ. Therefore by induction, there exists a subsequence u of t and a selection $(S_j)_{j \in J}$ of $(\mathfrak{S}_j)_{j \in J}$ such that $\operatorname{ran}(\pi_j \circ u) \subseteq S_j$. For all $i \in I \backslash J$, there exists an $S_i \in \mathfrak{S}_i$ and so $\operatorname{ran}(u) \subseteq S := \prod_{i \in I} S_i$. By Lemma II.4.2, $S \in \mathfrak{S}$ and $x \in \overline{A \cap S}$.

Therefore by Theorem II.1.13, $\mathfrak{S}$ is a spoke system of x. □

II.4.2 Strongly Fréchet

Recall that a point x in a topological space X is *strongly Fréchet* if whenever $(A_n)_{n<\omega}$ is a descending sequence of subsets of X with $x \in \bigcap_{n<\omega} \overline{A_n}$, there exists a selection $(x_n)_{n<\omega}$ of $(A_n)_{n<\omega}$ that converges to x. The following definition was first investigated in [Arh81].

Definition II.4.6 (α_4)**.** A point x in a topological space X is α_4 if whenever $\mathcal{S}$ is a countably-infinite collection of disjoint sequences in $X \backslash N_x$, each converging to x, there exists a sequence t in $X \backslash N_x$ that converges to x such that $\{s \in \mathcal{S} : \operatorname{ran}(s) \cap \operatorname{ran}(t) \neq \emptyset\}$ is infinite.

Theorem II.4.7. *[Mic72, Proposition 4.D.5, pg. 113] [Arh81, Theorem 5.23, pg. 190] Let x be a point in a topological space. Then the following are equivalent:*

(1) x is strongly Fréchet.

(2) x is Fréchet-Urysohn and α_4.

(3) (x, ω) *is Fréchet-Urysohn in* $X \times (\omega + 1)$.[4]

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[4] The original proofs use the unit interval in place of $\omega + 1$ and characterise when the whole space is strongly Fréchet, but they are easily adapted to prove this theorem.

Using Theorem II.4.5 and adapting the α_4 definition to spokes, we get the following characterisation for strongly Fréchet points:

Theorem II.4.8. *Let x be a Fréchet-Urysohn point in a topological space X. Then the following are equivalent:*

(1) x is strongly Fréchet.

(2) For every almost-independent spoke system $\mathfrak{S}$ of x and all collections of non-trivial spokes $\mathcal{S} \in [\mathfrak{S}]^{\aleph_0}$, there exists a $T \in \mathfrak{S}$ such that $\{S \in \mathcal{S} : S \cap T \neq N_x\}$ is infinite.

(3) There exists a spoke system $\mathfrak{S}$ of x, $\mathfrak{S} \otimes \{\omega+1\}$ is a spoke system of (x, ω) in $X \times (\omega+1)$.

(4) For every spoke system $\mathfrak{S}$ of x, $\mathfrak{S} \otimes \{\omega+1\}$ is a spoke system of (x, ω) in $X \times (\omega+1)$.

Proof. The equivalence of (1), (3) and (4) follows from Theorems II.4.5 and II.4.7. Suppose x is strongly Fréchet and let $\mathfrak{S}$ be an almost-independent spoke system, $\mathcal{S} \in [\mathfrak{S}]^{\aleph_0}$ be a collection of non-trivial spokes. Then by Lemma II.3.16, there exists a selection $(U_S)_{S \in \mathcal{S}}$ of $(\mathcal{N}_x^S)_{S \in \mathcal{S}}$ such that $U_S \cap U_{S'} = N_x$ for all distinct $S, S' \in \mathcal{S}$.

For each $S \in \mathcal{S}, x \in \overline{U_S \setminus N_x}$, so there exists a sequence t_S in $U_S \setminus N_x$ that converges to x. By Theorem II.4.7 x is α_4, so there exists a sequence u in $X \setminus N_x$ such that $\mathcal{R} := \{S \in \mathcal{S} : \operatorname{ran}(t_S) \cap \operatorname{ran}(u) \neq \emptyset\}$ is infinite. Thus $x \in \overline{\operatorname{ran}(u) \cap (\bigcup \mathcal{S})}$, so by Theorem II.1.13 there exists a $T \in \mathfrak{S}$ such that $x \in \overline{\operatorname{ran}(u) \cap T \cap (\bigcup \mathcal{S})}$. Hence $A := u^{-1}[T \cap (\bigcup \mathcal{S})]$ is infinite and therefore $\{S \in \mathcal{S} : S \cap T \neq N_x\}$ is infinite.

Now assume (2) and let $\mathcal{S}$ be a countably infinite collection of disjoint sequences in $X \setminus N_x$ that converge (strictly) to x. Then by Lemma II.2.6, there exists an almost-independent, basic spoke system $\mathfrak{S}$ of x that contains $\mathbb{S}_s$ for all $s \in \mathcal{S}$. So by assumption, there exists a $T \in \mathfrak{S}$ such that $\{s \in \mathcal{S} : \mathbb{S}_s \cap T \neq N_x\}$ is infinite. Then there exists a transfinite sequence t that converges strictly to x such that $T = \mathbb{S}_t$ and T is non-trivial. It follows that t has infinite length and we can assume that $\operatorname{ran}(t) \cap N_x = \emptyset$ without loss of generality. Then by Theorem II.2.13, $\operatorname{len}(t) \in \Sigma^{\mathrm{rad}}(x) \subseteq \{\omega\}$ and hence $\{s \in \mathcal{S} : \operatorname{ran}(s) \cap \operatorname{ran}(t) \neq \emptyset\}$ is infinite. Therefore x is α_4. □

II.4.3 Fréchet-Urysohn for n-point sets

We can strengthen the Fréchet-Urysohn property by considering *networks* and sequences of finite *subsets*. This idea was first introduced in [GS05].

Definition II.4.9. Let x be a point in a topological space X and let $(A_n)_{n<\omega}$ be a sequence of subsets of X. Then we say that $(A_n)_{n<\omega}$ *converges to x*, written $(A_n)_{n<\omega} \to x$, if for every $U \in \mathcal{N}_x$, there exists an $N < \omega$ such that $A_n \subseteq U$ for all $n \in [N, \omega)$. Thus, a sequence of points $(x_n)_{n<\omega}$ in X converges in the usual sense to x precisely when $(\{x_n\})_{n<\omega}$ converges to x.

Let n be a positive natural number. Then we say that x is *Fréchet-Urysohn for n-point sets* if for every network $\mathcal{A} \subseteq [X]^n$ of x, there exists a sequence $(A_m)_{m<\omega}$ in $\mathcal{A}$ that converges to x. Thus a point is Fréchet-Urysohn if and only if it is Fréchet-Urysohn for 1-point sets.

Ol'ga Sipacheva gave the following characterisation of Fréchet-Urysohn for n-point sets in terms of the n-fold product.

Theorem II.4.10. *[Sip02, Proposition 1, pg. 308] Let x be a point in a topological space X*

and let n be a positive natural number. Then x is Fréchet-Urysohn for n-point sets if and only if $(x,\dots,x)$ is Fréchet-Urysohn in X^n.

The following characterisation follows immediately by Theorem II.4.5:

Theorem II.4.11. *Let x be a Fréchet-Urysohn point in a topological space X and let n be a positive natural number. Then the following are equivalent:*

(1) x is Fréchet-Urysohn for n-point sets.

(2) There exists a spoke system $\mathfrak{S}$ of x such that $\mathfrak{S}^{\otimes n}$ is a spoke system of $(x,\dots,x)$ in X^n.

(3) For all spoke systems $\mathfrak{S}$ of x, $\mathfrak{S}^{\otimes n}$ is a spoke system of $(x,\dots,x)$ in X^n.

II.5 Future directions

II.5.1 Convergence spaces

Throughout this chapter, we have developed the theory of spokes for points in topological spaces but we did not use the full strength of a topology. We can express our ideas in the weaker setting of *pretopologies*. First, we require a few definitions. We refer to [Dol09] for the background on convergence spaces and pretopologies.

Definition II.5.1 (Filters)**.** Let X be a set and $\mathcal{F}$ be a collection of subsets of X with the following properties:

- $\emptyset \notin \mathcal{F} \neq \emptyset$.

- For all $A, B \in \mathcal{F}$, $A \cap B \in \mathcal{F}$.

- For all $A \in \mathcal{F}$ and $B \subseteq X$, if $A \subseteq B$ then $B \in \mathcal{F}$.

Then we call $\mathcal{F}$ a *filter on* X. We define $\mathfrak{F}(X)$ to be the set of all filters on X. For each $x \in X$, we define the *principal ultrafilter of* x to be $\mathcal{P}_x := \{A \subseteq X : x \in A\}$.

Given a collection $\mathcal{B}$ of subsets of X, we say that $\mathcal{B}$ has the *finite-intersection property* if for every finite $\mathcal{F} \subseteq \mathcal{B}, \bigcap \mathcal{F} \neq \emptyset$. If $\mathcal{B}$ has the finite-intersection property then we define the *filter generated by* $\mathcal{B}$ to be

$$\langle \mathcal{B} \rangle := \left\{ A \subseteq X : \bigcap \mathcal{G} \subseteq A \text{ for some finite } \mathcal{G} \subseteq \mathcal{B} \right\}.$$

A filter $\mathcal{F}$ is *generated by* $\mathcal{B}$, if $\mathcal{F} = \langle \mathcal{B} \rangle$. Moreover, if $\mathcal{B}$ is well-ordered by $\supseteq$, then we say that $\mathcal{F}$ *is well-based.*

A filter $\mathcal{F}$ is *principal* if there exists a non-empty subset $A \subseteq X$ such that $\mathcal{F} = \langle \{A\} \rangle$. Given an arbitrary non-principal filter $\mathcal{F}$, define $\mathcal{F}^\circ := \langle \mathcal{F} \cup \{X \setminus (\bigcap \mathcal{F})\} \rangle$. We call $\mathcal{F}^\circ$ the *free part of* $\mathcal{F}$. Note that since $\mathcal{F}$ is non-principal, its free part is again a filter on X.

We define an ordering on $\mathfrak{F}(X)$ given by $\subseteq$. This turns $\mathfrak{F}(X)$ into a complete meet-semilattice (i.e. every non-empty collection of filters has an infimum), where the meet is given by intersection.

Definition II.5.2 (Convergence space)**.** Let X be a set and let $\mathfrak{F}(X)$ denote the set of all proper filters on X. Then a *convergence on* X is a relation ξ between $\mathfrak{F}(X)$ and X that satisfies the following properties:

(1) For all $x \in X, \mathcal{P}_x \, \xi \, x$.

(2) For all $\mathcal{F},\mathcal{G} \in \mathfrak{F}(X)$ and all $x \in X$, if $\mathcal{F}\ \xi\ x$ and $\mathcal{F} \subseteq \mathcal{G}$ then $\mathcal{G}\ \xi\ x$.

We will typically write $\mathcal{F} \to_\xi x$ instead of $\mathcal{F}\ \xi\ x$ and will dispense with the subscript ξ if the convergence is clear. We refer to (X,ξ) as a *convergence space*.

Definition II.5.3 (Pretopology)**.** Let (X,ξ) be a convergence space such that for all $\mathfrak{G} \subseteq \mathfrak{F}(X)$ and all $x \in X$, if $\mathcal{G} \to x$ for each $\mathcal{G} \in \mathfrak{G}$ then $\bigcap \mathfrak{G} \to x$. Then we say that ξ is a *pretopology* and (X,ξ) is a *pretopological space*.

If (X,ξ) is a pretopological space and $x \in X$, then since $\mathcal{P}_x \to x$ it follows that $\mathcal{V}_x^\xi \to x$, where $\mathcal{V}_x^\xi$ is the intersection of all filters on X that converge to x. We call $\mathcal{V}_x^\xi$ the *vicinity filter of x* and will dispense with the subscript if the convergence is unambiguous.

Thus a convergence ξ on X is a pretopology if and only if for all $\mathcal{F} \in \mathfrak{F}(X)$ and all $x \in X, \mathcal{F} \to_\xi x$ precisely when $\mathcal{V}_x^\xi \subseteq \mathcal{F}$.

Definition II.5.4 (Mesh and adherence)**.** Let (X,ξ) be a convergence space and let $A \subseteq X, \mathcal{F} \in \mathfrak{F}(X)$ be given. Then we say that $\mathcal{F}$ *meshes with* A, denoted $\mathcal{F}\#A$, if for all $F \in \mathcal{F}, F \cap A \neq \emptyset$.

Given $A \subseteq X$, we define the *adherence of* A to be

$$\operatorname{adh}_\xi(A) := \bigcup \{\mathcal{F} \in \mathfrak{F}(X) : \mathcal{F}\#A\}.$$

It is an easy exercise to prove that for all $x \in X, x \in \operatorname{adh}_\xi(A)$ if and only if $\mathcal{V}_x^\xi\#A$.[Dol09, Remark 21, pg. 128]

Definition II.5.5 (Trans-sequential filter)**.** Let X be a set and let t be a transfinite sequence in X. Then we define $\mathcal{T}_t := \{T \subseteq X : \exists \alpha < \lambda \text{ such that } t[[\alpha,\lambda)] \subseteq T\}$ and call it the *trans-*

sequential filter based on t. Given a filter $\mathcal{F} \in \mathfrak{F}(X)$, we say that $\mathcal{F}$ is *trans-sequential* if there exists a transfinite-sequence t on X such that $\mathcal{F} = \mathcal{T}_t$.

Given a transfinite sequence t and a convergence ξ on X, we say that *t ξ-converges to x* (written $t \to_\xi x$) if $\mathcal{T}_t \to_\xi x$. If ξ is a pretopology, then we say that *t strictly ξ-converges to x* (written $t \dashrightarrow_\xi x$) if t ξ-converges to x and for all $\alpha < \mathrm{len}(t), x \notin \mathrm{adh}_\xi(t[\alpha])$.

Definition II.5.6 (Radial)**.** Let (X,ξ) be a convergence space and let $x \in X$ be given. Then we say that x is *radial* if for all $A \subseteq X$ with $x \in \mathrm{adh}(A)$, there exists a transfinite sequence t on A that converges to x.

If $\mathcal{F}$ is a filter on X, then we say that $\mathcal{F}$ *is radial* if whenever there is a subset $A \subseteq X$ which meshes with $\mathcal{F}$, there exists a transfinite sequence on A such that $\mathcal{F} \subseteq \mathcal{T}_t$. Thus, if ξ is a pretopology, x is radial precisely when $\mathcal{V}_x^\xi$ is radial.

Given a pretopological space (X,ξ) and a radial point $x \in X$, we can adapt the proof of Theorem II.2.2 to show that if $x \in \mathrm{adh}_\xi(A)$ then there exists a transfinite sequence on A that converges *strictly* to x.

Theorem II.5.7. *Let (X,ξ) be a pretopological space and let $x \in X$ be given. Then x is radial if and only if there exists a non-empty collection $\mathfrak{S}$ of subspaces of X such that:*

- *For all $S \in \mathfrak{S}$, $V_x^\xi \subseteq S$ and $\langle \mathcal{V}_x^\xi \cup \{S\}\rangle$ is well-based.*
- $\mathcal{V}_x^\xi = \bigwedge_{S\in\mathfrak{S}} \langle \mathcal{V}_x^\xi \cup \{S\}\rangle$.

Moreover, $\mathfrak{S}$ can be taken to also satisfies the following conditions:

- *For all distinct $S, T \in \mathfrak{S}$, $\langle \mathcal{V}_x^\xi \cup \{S, T\}\rangle$ is principal.*

- *For every $S \in \mathfrak{S}$, if $\langle \mathcal{V}_x^\xi \cup \{S\} \rangle$ is non-principal then its free part is strictly transsequential for each $S \in \mathfrak{S}$.*

Proof. Define a topology τ on X by declaring every point apart from x isolated and let $\mathcal{V}_x^\xi$ be the neighbourhood filter of x with respect to X. Then for all $A \subseteq X, x \in \mathrm{adh}_\xi(A)$ if and only if $x \in \overline{A}^\tau$. Therefore by Theorem II.2.7, x is radial with respect to ξ if and only if there exists an almost-independent spoke system $\mathfrak{S}$ for x with respect to (X, τ).

Observe that $\mathcal{V}_x^\xi = \mathcal{N}_x^{(X,\tau)}$ and thus $V_x^\xi = N_x^{(X,\tau)}$. Moreover, given a subspace S containing $N_x^{(X,\tau)}$, x is well-based in S (with respect to (X, τ)), if and only if $\langle \mathcal{N}_x^{(X,\tau)} \cup \{S\} \rangle$ is well-based.

Let $\mathfrak{S}$ be a non-empty collection of subspaces of x. Then $\mathcal{V}_x^\xi \subseteq \langle \mathcal{V}_x^\xi \cup \{S\} \rangle$ for each $S \in \mathfrak{S}$. Moreover, $\mathcal{V}_x^\xi = \bigwedge_{S \in \mathfrak{S}} \langle \mathcal{V}_x^\xi \cup \{S\} \rangle$ if and only if for each $V \in \mathcal{V}_x^\xi$ there exists a selection $(W_S)_{S \in \mathfrak{S}}$ from $\mathcal{V}_x^\xi$ such that $\bigcup_{S \in \mathfrak{S}} (W_S \cap S) \subseteq V$. Therefore $\mathfrak{S}$ satisfies the first two conditions precisely when $\mathfrak{S}$ is a spoke system of x with respect to (X, τ). Hence by Theorem II.2.7, x is radial with respect to ξ if and only if an $\mathfrak{S}$ with the first two properties exists.

Now suppose that x is ξ-radial, so by Theorem II.2.7 there exists an almost-independent, basic spoke system $\mathfrak{S}$ of x with respect to (X, τ). Then for all distinct $S, T \in \mathfrak{S}$, there exists a $V \in \mathcal{N}_x^\tau = \mathcal{V}_x^\xi$ such that $V \cap S \cap T = N_x^\tau = \mathcal{V}_x^\xi$, i.e. $\langle V_x^\xi \cup \{S, T\} \rangle$ is principal. Finally, let $S \in \mathfrak{S}$ be given, so there exists a transfinite sequence t in S that converges strictly to x such that $S = \mathbb{S}_t^\tau$. By strict convergence, it follows that if $\langle \mathcal{V}_x^\xi \cup \{S\} \rangle$ is non-principal then its free part is $\mathcal{T}_t$. □

We conclude by demonstrating a characterisation of independently-based points in terms of the convergence space of a topological space, thus allowing us to extend this notion to pretopological spaces.

Theorem II.5.8. *Let X be a topological space, $x \in X$ be given. Then x is independently-based if and only if there exists a non-empty collection $\mathfrak{W}$ of well-based filters contained in $\mathcal{N}_x$ such that*

$$\mathcal{N}_x = \left\langle \bigcap_{\mathcal{W}\in\mathfrak{W}} U_{\mathcal{W}} : (U_{\mathcal{W}})_{\mathcal{W}\in\mathfrak{W}} \text{ is a selection from } \mathfrak{W} \right\rangle = \bigwedge_{\mathcal{W}\in\mathfrak{W}} \langle \mathcal{W} \cup \{\mathfrak{W}{\downarrow}_{\mathcal{W}}\}\rangle,$$

where $\mathfrak{W}{\downarrow}_{\mathcal{W}} = \bigcap_{\mathcal{W}'\in\mathfrak{W}\setminus\{\mathcal{W}\}} \bigcap \mathcal{W}'$.

Proof. Suppose there exists such a collection $\mathfrak{W}$. Then

$$\left\{ \bigcup_{\mathcal{W}\in\mathfrak{W}} (U_{\mathcal{W}} \cap \mathfrak{W}{\downarrow}_{\mathcal{W}}) : (U_{\mathcal{W}})_{\mathcal{W}\in\mathfrak{W}} \text{ is a selection from } \mathfrak{W} \right\}$$

is a neighbourhood base for x with respect to X. Moreover, for each selection $(U_{\mathcal{W}})_{\mathcal{W}\in\mathfrak{W}}$ from $\mathfrak{W}$,

$$\begin{aligned}\left(\bigcup_{\mathcal{W}'\in\mathfrak{W}} (U_{\mathcal{W}'} \cap \mathfrak{W}{\downarrow}_{\mathcal{W}'}) \right) \cap \mathfrak{W}{\downarrow}_{\mathcal{W}} &= (U_{\mathcal{W}} \cap \mathfrak{W}{\downarrow}_{\mathcal{W}}) \cup \left(\bigcap_{\mathcal{W}'\in\mathfrak{W}} \bigcap \mathcal{W}' \right) \\ &= (U_{\mathcal{W}} \cap \mathfrak{W}{\downarrow}_{\mathcal{W}}) \cup N_x \\ &= U_{\mathcal{W}} \cap \mathfrak{W}{\downarrow}_{\mathcal{W}}.\end{aligned}$$

Therefore $\{W \cap \mathfrak{W}{\downarrow}_{\mathcal{W}} : W \in \mathcal{W}\}$ is a neighbourhood base for x with respect to $\mathfrak{W}{\downarrow}_{\mathcal{W}}$. As $\mathcal{W}$ is well-based, it follows that $\mathfrak{W}{\downarrow}_{\mathcal{W}}$ is a spoke of x. Furthermore, for distinct $\mathcal{W}, \mathcal{W}' \in \mathfrak{W}, \mathfrak{W}{\downarrow}_{\mathcal{W}} \cap \mathfrak{W}{\downarrow}_{\mathcal{W}'} = N_x$.

Define $\mathfrak{S} := \{\mathfrak{W}{\downarrow}_{\mathcal{W}} : \mathcal{W} \in \mathfrak{W}\}$ and select $(U_S)_{S\in\mathfrak{S}}$ from $(\mathcal{N}_x^S)_{S\in\mathfrak{S}}$. Then for each $\mathcal{W} \in \mathfrak{W}$, there exists a $V_{\mathcal{W}} \in \mathcal{W}$ such that $V_{\mathcal{W}} \cap \mathfrak{W}{\downarrow}_{\mathcal{W}} \subseteq U_{\mathfrak{W}{\downarrow}_{\mathcal{W}}}$. Thus

$$\bigcup_{S \in \mathfrak{S}} U_S \supseteq \bigcup_{\mathcal{W} \in \mathfrak{W}} (V_{\mathcal{W}} \cap \mathfrak{W}{\downarrow}_{\mathcal{W}}) \in \bigwedge_{\mathcal{W} \in \mathfrak{W}} \langle \mathcal{W} \cup \{\mathfrak{W}{\downarrow}_{\mathcal{W}}\} \rangle = \mathcal{N}_x^X.$$

Therefore $\mathfrak{S}$ is an independent spoke system for x, so by Corollary II.1.12 x is independently-based.

Now suppose that x is independently-based. If x is quasi-isolated then define $\mathfrak{W} := \{\langle\{N_x\}\rangle\}$. Otherwise, by Theorem II.1.10 and Lemma II.2.12 there exists a non-trivial, independent spoke system $\mathfrak{S}$. Define for all $S \in \mathfrak{S}, \mathcal{W}_S := \langle\{U \cup \bigcup(\mathfrak{S} \setminus \{S\}) : U \in \mathcal{N}_x^S\}\rangle$. As x is well-based in each $S \in \mathfrak{S}$, it follows that $\mathcal{W}_S$ is well-based and is contained in $\mathcal{N}_x^X$.

Define $\mathfrak{W} := \{\mathcal{W}_S : S \in \mathfrak{S}\}$ and note that $\mathfrak{W}{\downarrow}_{\mathcal{W}_S} = S$ for each $S \in \mathfrak{S}$. Since $\mathfrak{S}$ non-trivial, it follows that $\mathcal{W}_S \neq \mathcal{W}_T$ for distinct $S, T \in \mathfrak{S}$. Finally

$$\begin{aligned}\bigwedge_{\mathcal{W} \in \mathfrak{W}} \langle \mathcal{W} \cup \{\mathfrak{W}{\downarrow}_{\mathcal{W}}\} \rangle &= \bigwedge_{S \in \mathfrak{S}} \langle \mathcal{W}_S \cup \{S\} \rangle \\ &= \left\langle \bigcup_{S \in \mathfrak{S}} (U_S \cap S) : (U_S)_{S \in \mathfrak{S}} \text{ is a selection from } (\mathcal{N}_x^S)_{S \in \mathfrak{S}} \right\rangle \\ &= \mathcal{N}_x^X. \end{aligned}$$

□

II.5.2 $\mathbb{D}$-radiality

Instead of taking well-ordered nets in the definition of radiality, we can generalise to an arbitrary class of nets, as shown in [Nyi92, pg. 543–5].

Definition II.5.9 ($\mathbb{D}$-radiality)**.** Let $\mathbb{D}$ be a non-empty class of directed nets and let X be a topological space and $x \in X$ be given. Then we say that x is $\mathbb{D}$-radial if for every subset $A \subseteq X$ with $x \in \overline{A}$, there exists a $D \in \mathbb{D}$ a net $f : D \to A$ that converges to x.

Given a direct set D, a net $f : D \to X$ *converges strictly to* x, written $f \dashrightarrow x$, if $f \to x$ and for all $d \in D, x \notin \overline{\{e \in D : d \not\leq e\}}$. If we request that the net in the definition of $\mathbb{D}$-radiality strictly converges then we say that x is *strictly* $\mathbb{D}$*-radial.*

Examples of potentially interesting classes include the class of nets based on directed sets of cardinality less than a specific infinite cardinal, or the κ-nets as studied in [Hod10].

Theorem II.5.10. *Endow with* $\omega \times \omega_1$ *with the product order. Then there exists a topological space with a* $\{\omega \times \omega_1\}$*-radial point that is not strictly* $\{\omega \times \omega_1\}$*-radial.*

Proof. Define $X := (\omega \times \omega_1) \cup \{\star\}$, where $\star \notin \omega \times \omega_1$. We endow X with a topology by making each point in $\omega \times \omega_1$ isolated, and let $\{X \setminus (n \times \omega_1) : n < \omega\}$ be a neighbourhood base for $\star$.

Let $A \subseteq \omega \times \omega_1$ be given such that $\star \in \overline{A}$, so for each $n < \omega$, there exists an $(\alpha_n, \beta_n) \in A$ such that $\alpha_n \geq n$. Define for all $m < \omega$ and $\gamma < \omega_1, f((m, \gamma)) := x_m$. Then f is a $\{\omega \times \omega_1\}$-net in A that converges to $\star$ and hence $\star$ is $\{\omega \times \omega_1\}$-radial.

Now let $f : \omega \times \omega_1 \to \omega \times \omega_1$ converge to $\star$. Then for all $n < \omega$, there exists an $m_n < \omega$ and $\alpha_n < \omega_1$ such that $\pi_\omega(f(k, \beta)) \geq n$ for all $(k, \beta) \in \omega \times \omega_1$ such that $(k, \beta) \geq (m_n, \alpha_n)$. Define $\alpha := \sup(\alpha_n : n < \omega) + 1 < \omega_1$. Then $\star \in \overline{\{f((m_n, \alpha_n)) : n < \omega\}} \subseteq \overline{f[\omega \times (\omega_1 \setminus \alpha)]}$, so $f \not\dashrightarrow \star$. Therefore $\star$ is not strictly $\{\omega \times \omega_1\}$-radial. □

There are three potential fixes for this problem:

(1) Focus exclusively on strictly $\mathbb{D}$-radial points / spaces.

(2) Find conditions that ensure that $\mathbb{D}$-radiality and its strict version coincide.

(3) Enlarge the class $\mathbb{D}$ such that these versions agree.

The author feels that the last solution might be most useful and could involve analysing the Tukey reduction structure on directed sets (see [Tuk40]). A particularly amenable case would be $\mathbb{D} = \{1, \omega, \omega_1, \omega \times \omega_1, ([\omega_1]^{<\aleph_1}, \subseteq)\}$ – Stevo Todorčević has shown that under PFA, all directed sets of cardinality at most $\aleph_1$ are Tukey equivalent to one in $\mathbb{D}$ [Tod85, Theorem 9, pg. 718]. More specifically,

Question II.5.11. *Let* $\mathbb{D}$ *be a class of directed sets that contains well-ordered sets of every non-zero, regular order type. Does* $\mathbb{D}$*-radiality and strict* $\mathbb{D}$*-radiality coincide for all points in all topological spaces?*

Chapter III

Compactifications and dualities

III.1 Introduction

The previous chapter exhibited some local characterisations of radiality using spokes and spoke systems. In this chapter, we will use these characterisations to investigate radiality of compactifications, ring spectra and Stone spaces.

Closed spokes and spoke systems are better behaved than arbitrary spokes or systems; for instance, they reflect into properties of the ideal of compact subsets for compactifications, of ideals for ring spectra, and of filters and quotient algebras for Stone spaces. In this first section, we will show that, for semi-regular spaces, closure of spokes are also spokes. Consequently, closed spokes suffice to characterise radiality for these spaces.

In Section III.2, we investigate compactifications that are radial or Fréchet-Urysohn in their remainder, focusing exclusively on locally compact spaces. In this setting, having a radial compactification implies that the one-point compactification is also radial. By using radiality at the point-at-infinity, we can improve these compactification results, first considering finite, countable and ordinal compactifications, then using small cardinals (cardinals that describe some form of infinite combinatorics on countable sets) to achieve consistently stronger theorems. We also prove some results for pseudoradial and sequential compactifications.

In the last section of this chapter, we discuss radiality of points in ring spectra and Stone spaces. We first state our spoke-characterisation theorems in the general context of spectra of commutative rings with identity. This requires a translation of the notions of spokes, spoke systems and cofinal collections of spokes into ideal-theoretic language, using generalisations of the radical of an ideal. Consequently we obtain an algebraic characterisation of the radiality of the Stone-Čech compactification of a Tychonoff space X, using the ring of bounded, continuous, real-valued functions $C^*(X)$. Next we translate our results for Stone spaces of Boolean algebras, which is a specific case of the previous work on ring spectra. We state these results with filters instead of ideals, as is usual when dealing with Stone spaces of Boolean algebras. Finally, we use ω^*, the Stone-Čech remainder of a countably-infinite discrete space, as a test case and show some strong negative results on constructing radial, non-well-based points in ω^*.

The material in Sections III.2 and III.3 (excluding III.2.2) is based on [Lee15].

III.1.1 Enlarging spokes

We conclude this introduction by showing how to enlarge spokes by taking their closure, which is fundamental to the rest of this chapter. Besides paring down the possibilities a spoke may take (thus aiding our search for a spoke system / cofinal collections of spokes), it also allows some algebraic characterisations of radiality, which we will prove later.

The simplest method of enlarging spokes is to take a neighbourhood base and try to lift it to a neighbourhood base of a larger space. If the neighbourhood base consists of open sets, we can attempt to take the largest such extension.

Definition III.1.1 (Lift)**.** Let X be a topological space and let $U \subseteq Y \subseteq X$ be given, where U is open in Y. Then we define the *lift of U to X*, denoted $U{\upharpoonleft}^X_Y$, to be the largest open subset of X such that $U{\upharpoonleft}^X_Y \cap Y = X$, that is,

$$U{\upharpoonleft}^X_Y := \bigcup \{V \subseteq X \text{ open} : V \cap Y = U\}.$$

Thus we consider conditions ensuring that if $\mathcal{B}$ is a neighbourhood base for $x \in Y \subseteq X$ with respect to Y, then $\{B{\upharpoonleft}^X_Y : B \in \mathcal{B}\}$ is a neighbourhood base of x with respect to X. First, let us describe some properties of the lift operation.

Proposition III.1.2. *Let X be a topological space and let $U \subseteq V \subseteq Z \subseteq Y \subseteq X$ be given, where U, V is open in Z. Then:*

(1) $U{\upharpoonleft}^Y_Z = Y \setminus \overline{Z \setminus U}$.

(2) $U{\upharpoonleft}^Y_Z \subseteq V{\upharpoonleft}^Y_Z$.

(3) $U{\upharpoonleft}^Y_Z = U{\upharpoonleft}^X_Z \cap Y$.

Proof. First, note that $Y \setminus \overline{Z \setminus U}$ is open in Y and $Z \cap (Y \setminus \overline{Z \setminus U}) = Z \setminus \overline{Z \setminus U}^Z = U$ since U is open in Z. Thus $Y \setminus \overline{Z \setminus U} \subseteq U{\upharpoonleft}^Y_Z$ by definition. However, $U{\upharpoonleft}^Y_Z \subseteq Y$ and $U{\upharpoonleft}^Y_Z \cap (Z \setminus U) =$

$(U{\upharpoonleft}^Y_Z \cap Z)\setminus U = \emptyset$. Since $U{\upharpoonleft}^Y_Z$ is open in Y, it follows that $U{\upharpoonleft}^Y_Z \subseteq Y\setminus\overline{Z\setminus U}^{Y} = Y\setminus\overline{Z\setminus U}$. Therefore $U{\upharpoonleft}^Y_Z = Y\setminus\overline{Z\setminus U}$.

From this, we immediately get

$$U{\upharpoonleft}^Y_Z = Y\setminus\overline{Z\setminus U} \subseteq Y\setminus\overline{Z\setminus V} = V{\upharpoonleft}^Y_Z,$$

$$U{\upharpoonleft}^Y_Z = Y\setminus\overline{Z\setminus U} = (X\setminus\overline{Z\setminus U})\cap Y = U{\upharpoonleft}^X_Z \cap Y. \qquad \square$$

Definition III.1.3 (Regular open and semi-regular)**.** Let X be a topological space and let $U \subseteq X$ be given such that $\overline{U}^{\circ} = U$. Then we say that U is a *regular open* set. If a point $x \in X$ has a neighbourhood base consisting of regular open subsets, then we say that x is *semi-regular*. If every point of x is semi-regular, then we say that X *is semi-regular*.

Lemma III.1.4. *Let X be a topological space and let $Y \subseteq X$ be dense, $U \subseteq X$ be regular open. Then $U = X\setminus\overline{Y\setminus U} = (U\cap Y){\upharpoonleft}^X_Y$.*

Proof. Since Y is dense, $\overline{V} = \overline{V\cap Y}$ for all open $V \subseteq X$. Thus

$$X\setminus U = X\setminus\overline{U}^{\circ} = \overline{X\setminus\overline{U}} = \overline{Y\setminus\overline{U}} \subseteq \overline{Y\setminus U} \subseteq X\setminus U.$$

Therefore by the previous proposition, $U = X\setminus\overline{Y\setminus U} = (U\cap Y){\upharpoonleft}^X_Y$. $\square$

Now we define a 'weak-closure' of a space, which is an upper bound for superspaces such that neighbourhood bases lift, for a fixed point.

Definition III.1.5 (Weak closure)**.** Let X be a topological space and let $Y \subseteq X, x \in Y$ be given. Then we define the *weak closure of Y with basepoint x (with respect to X)* to be

$$\mathbf{L}^X_x(Y) := N^X_x \cup \bigcup_{U\in\mathcal{N}^Y_x} \overline{Y\setminus U}.$$

If the ambient space X is unambiguous, then we will dispense with the superscript.

Theorem III.1.6. *Let X be a topological space and let $x \in Y \subseteq Z \subseteq X$ be given. Let $\mathcal{B}$ be a neighbourhood base of x with respect to Y and define $\mathcal{B}\!\uparrow := \{B\!\uparrow_Y^Z : B \in \mathcal{B}\}$.*

(1) If $\mathcal{B}\!\uparrow$ is a neighbourhood base of x with respect to Z then $Z \subseteq \mathbf{L}_x(Y)$.

(2) If x is semi-regular in X and $Z \subseteq \overline{Y} \cup N_x^X$ then $\mathcal{B}\!\uparrow$ is a neighbourhood base of x with respect to Z.

(3) If x is semi-regular in X then $\mathbf{L}_x(Y) = \overline{Y} \cup N_x^X$.

Proof.

(1) Suppose $\mathcal{B}\!\uparrow$ is a neighbourhood base of x with respect to Z and let $z \in Z \backslash N_x^X$ be given. Then there exists a $U \in \mathcal{N}_x^X$ such that $z \notin U$ and so there exists a $B \in \mathcal{B}$ such that $B\!\uparrow_Y^Z \subseteq U$. Thus $z \in Z \backslash B\!\uparrow_Y^Z = \overline{Y \backslash B} \subseteq \mathbf{L}_x(Y)$ and therefore $Z \subseteq \mathbf{L}_x(Y)$.

(2) Assume x is semi-regular with respect to X and $Z \subseteq \overline{Y} \cup N_x^X$. Define $W := Y \cup N_x^Z$, which is dense in Z, and let $U \in \mathcal{N}_x^X$ be regular-open, so there exists a $B \in \mathcal{B}$ such that $B \subseteq U \cap Y$. Then $B \cup N_x^Z \subseteq (U \cap Y) \cup N_x^Z = U \cap W$, so by the previous lemma

$$B\!\uparrow_Y^Z = Z \backslash \overline{Y \backslash B} \subseteq Z \backslash \overline{W \backslash (B \cup N_x^Z)} \subseteq Z \backslash \overline{W \backslash (U \cap W)} = (U \cap W)\!\uparrow_W^Z = U \cap Z.$$

Therefore $\mathcal{B}\!\uparrow$ is a neighbourhood base of x with respect to Z.

(3) By applying (2) to $Z = \overline{Y} \cup N_x^X$, it follows from (1) that $\overline{Y} \cup N_x^X = \mathbf{L}_x(Y)$. □

Applying this to spokes and spoke systems, we see that in semi-regular spaces we can take the spokes in a system to be closed. We denote the collection of closed spokes of a point x by $\overline{\mathbf{Sp}}_X(x)$, removing the subscript if the surrounding space is unambiguous. The following theorem summarises the characterisation theorems for semi-regular points:

Theorem III.1.7. *Let x be a semi-regular point in a topological space X.*

(1) *If S is a spoke of x then $\overline{S}$ is a spoke of x and $\chi_{\overline{S}}(x) = \chi_S(x)$.*

(2) *If $\mathfrak{S}$ is a spoke system of x then $\{\overline{S} : S \in \mathfrak{S}\}$ is a spoke system of x.*

(3) *Let $\mathcal{S} \subseteq \overline{\mathbf{Sp}}(x)$ be given such that for all $T \in \overline{\mathbf{Sp}}(x)$, there exists an $S \in \mathcal{S}$ such that $T \subseteq_x S$. Then $\mathcal{S}$ is cofinal.*

(4) *If $Y \subseteq X$ and $x \in Y$ then x is radial in Y if and only $Y \subseteq_x \bigcup \mathcal{S}$ for all cofinal collections $\mathcal{S}$ of closed spokes of x in X.*

(5) *The following are equivalent:*

    (a) *x is radial.*

    (b) *$\overline{\mathbf{Sp}}(x)$ is a spoke system of x.*

    (c) *x has a closed spoke system.*

    (d) *$\bigcup \mathcal{S} \in \mathcal{N}_x^X$ for all cofinal collections $\mathcal{S}$ of closed spokes of x.*

Proof.

(1) Let S be a spoke of x, so by Corollary I.3.26 and Theorem I.3.25, there exists a well-ordered neighbourhood base $\mathcal{B}$ of x with respect to S, whose cardinality is $\chi_S(x)$. Without loss of generality, assume that all elements of $\mathcal{B}$ are open. Then by the previous theorem and Proposition III.1.2, $\{B\!\uparrow^{\overline{S}}_S : B \in \mathcal{B}\}$ is a well-ordered neighbourhood

base of x with respect to $\overline{S}$. As $N_x^X \subseteq S \subseteq \overline{S}$, it follows that $\overline{S}$ is also a spoke of x. Furthermore, $\chi_{\overline{S}}(x) \leq |\mathcal{B}| = \chi_S(x)$ and since $S \subseteq \overline{S}$, it follows that $\chi_{\overline{S}}(x) = \chi_S(x)$.

(2) Let $\mathfrak{S}$ be a spoke system of x and define $\mathfrak{T} := \{\overline{S} : S \in \mathfrak{S}\}$. Select $(U_T)_{T\in\mathfrak{T}}$ from $(\mathcal{N}_x^T)_{T\in\mathfrak{T}}$. Then $U := \bigcup_{S\in\mathfrak{S}}(U_{\overline{S}} \cap S) \subseteq \bigcup_{T\in\mathfrak{T}} U_T$ and $U \in \mathcal{N}_x^X$. Therefore $\mathfrak{T}$ is a spoke system of x.

(3) Let $S \in \mathbf{Sp}(x)$ be given, so by (1) $\overline{S} \in \overline{\mathbf{Sp}}(x)$ and hence there exists a $T \in \mathcal{S}$ such that $x \notin \overline{\overline{S}\setminus T} \supseteq \overline{S\setminus T}$. Therefore $\mathcal{S}$ is cofinal.

(4) Let $Y \subseteq X$ be given such that $x \in Y$, and suppose that $Y \subseteq_x \bigcup\mathcal{S}$ for all cofinal collections $\mathcal{S}$ of closed spokes of x with respect to X. We aim to show that $\overline{\mathbf{Sp}}_Y(x)$ is a spoke system of x with respect to Y. Select $(U_S)_{S\in\overline{\mathbf{Sp}}_Y(x)}$ from $(\mathcal{N}_x^S)_{S\in\overline{\mathbf{Sp}}_Y(x)}$, so there exists a selection $(V_S)_{S\in\overline{\mathbf{Sp}}_Y(x)}$ from $\mathcal{N}_x^X$ such that $U_S = V_S \cap S$. Then $\{V_{S\cap Y} \cap S : S \in \overline{\mathbf{Sp}}_X(x)\}$ is a cofinal collection of spokes of x in X, so there exists a $W \in \mathcal{N}_x^X$ such that

$$W \cap Y \subseteq \bigcup_{S\in\overline{\mathbf{Sp}}_X(x)} (V_{S\cap Y} \cap S) \Rightarrow W \cap Y \subseteq \bigcup_{S\in\overline{\mathbf{Sp}}_X(x)} (V_{S\cap Y} \cap S \cap Y) = \bigcup_{S\in\overline{\mathbf{Sp}}_Y(x)} U_S.$$

Therefore as $W \cap Y \in \mathcal{N}_x^Y$, it follows that $\overline{\mathbf{Sp}}_Y(x)$ is a spoke system of x with respect to Y and hence x is radial in Y. The converse follows from Theorem II.2.22.

(5) By applying (2) and (3) to Theorem II.2.7 and noting that $X \subseteq_x A$ if and only if $A \in \mathcal{N}_x^X$ for any $A \subseteq X$, (4) is proven. □

In the next section we will be considering locally-compact, non-compact, Hausdorff spaces exclusively. However, we do not necessarily have a spoke system that is both closed and almost-independent, even for compact Hausdorff spaces.

We need to introduce some more notation: we denote the one-point compactification of a space X by αX, with its point-at-infinity denoted by $\star$. Also, let $\mathcal{K}(X)$ denote the set of compact subsets of a topological space X.

In Theorem II.4.5 we found equivalent conditions for when the finite product of radial points is again radial in the product space. The next theorem shows that the property of the one-point compactification being radial at infinity is preserved by products of locally-compact Hausdorff spaces.

Theorem III.1.8. *Let $(X_i)_{i\in I}$ be a non-empty family of locally-compact Hausdorff spaces such that $\alpha X_i = X_i \cup \{\star_i\}$ is radial at $\star_i$ for all non-compact X_i. If $X := \prod_{i\in I} X_i$ is non-compact and locally-compact*[5] *then $\alpha X = X \cup \{\star\}$ is radial at $\star$.*

Proof. Assume X is non-compact and locally-compact and define

$$F := \{i \in I : X_i \text{ is non-compact}\},$$

which is finite and non-empty Let $A \subseteq X$ be given such that $\star \in \overline{A}^{\alpha X}$ and suppose for all $i \in F, \star_i \notin \overline{\pi_i[A]}^{\alpha X_i}$, so $\overline{\pi_i[A]}^{X_i}$ is compact. Then $A \subseteq \bigcap_{i\in F} \pi_i^{-1}\left[\overline{\pi_i[A]}^{X_i}\right]$ which is compact and thus we have a contradiction. Therefore there exists an $i \in F$ such that $\star_i \in \overline{\pi_i[A]}^{\alpha X_i}$. As αX_i is radial at $\star_i$, there exists a transfinite sequence t in A such that $\pi_i \circ t \to \star_i$.

Let $K \in \mathcal{K}(X)$ be given. Then $\pi_i[K]$ is compact, so there exists $\alpha \in \mathrm{len}(t)$ such that for all $\beta \in [\alpha, \mathrm{len}(t)), \pi_i(t(\beta)) \notin \pi_i[K]$ and hence $t(\beta) \notin K$. Therefore $t \to \star$ and $\star$ is radial in αX. □

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[5] Which only occurs precisely only a non-zero, finite number of factors is non-compacts - see [Eng89, Theorem 3.3.13].

Theorem III.1.9. *There exists a compact Hausdorff space X and a radial point $x \in X$ with no closed, almost-independent spoke system.*

Proof. Define $X := \alpha(\omega_1 \times \omega_2)$ and note that for all $K \in \mathcal{K}(\omega_1 \times \omega_2), \pi_{\omega_1}[K], \pi_{\omega_2}[K]$ are bounded in ω_1, ω_2 respectively, and hence $K \subseteq \alpha \times \beta$ for some $\alpha < \omega_1$ and $\beta < \omega_2$. In particular, every σ-compact subset of $\omega_1 \times \omega_2$ has compact closure, i.e. $\star$ is a p-point.[6] Observe that X is radial at $\star$ by the previous theorem since $\alpha\omega_1 \cong \omega_1 + 1$ and $\alpha\omega_2 \cong \omega_2 + 1$ are radial.

Assume there exists a closed, almost-independent spoke system $\mathfrak{S}$ for $\star$ and define $\Lambda := \{\lambda < \omega_2 : \mathrm{cf}(\lambda) = \omega_1\}$. We claim that for all $\lambda \in \Lambda$, there exists an $\alpha_\lambda < \omega_1$, a $\beta_\lambda < \lambda$ and an $S_\lambda \in \mathfrak{S}$ such that $[\alpha_\lambda, \omega_1) \times [\beta_\lambda, \lambda) \subseteq S_\lambda$. Before proving this claim, we will show how it allows us to derive a contradiction.

Suppose that $\{S_\lambda : \lambda \in \Lambda\}$ is uncountable and pick $f : \omega_1 \to \Lambda$ strictly increasing such that for all distinct $\alpha, \beta < \omega_1, S_{f(\alpha)} \neq S_{f(\beta)}$. Define $\lambda := \sup(\mathrm{ran}(f)) \in \Lambda$. Then f is cofinal in λ, so there exists a $\gamma < \omega_1$ such that $f(\gamma) > \beta_\lambda$. Thus for all $\delta \in [\gamma, \omega_1)$,

$$[\max(\alpha_{f(\delta)}, \alpha_\lambda), \omega_1) \times \{\max(\beta_{f(\delta)}, \beta_\lambda)\} \subseteq S_{f(\delta)} \cap S_\lambda.$$

Hence $\star \in \overline{(S_{f(\delta)} \cap S_\lambda)} \setminus \{\star\}$. Since $\mathfrak{S}$ is almost-independent, it follows that $S_{f(\delta)} = S_\lambda$ and in particular $S_{f(\gamma)} = S_{f(\gamma+1)}$, which is a contradiction. Therefore $\{S_\lambda : \lambda \in \Lambda\}$ is countable and so there exists an $L \subseteq \Lambda$ of cardinality $\aleph_2$ such that $S_\lambda = S_\mu$ for all $\lambda, \mu \in L$. As each S_λ contains an ω_1-sequence converging to $\star$, it follows from Corollary I.3.28 that $\chi_{S_\lambda}(\star) = \aleph_1$. However, since L is unbounded in $\Lambda, |S_{\min(L)}| = \aleph_2$ and any injective ω_2-sequence in $S_{\min(L)}$

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[6] A point x in a topological space is a *p-point* if countable intersections of neighbourhoods of x are again a neighbourhood. Equivalently, $p(x) \geq \aleph_1$.

converges to $\star$, which is a contradiction again by Corollary I.3.28. Therefore $\star$ does not have a closed, almost-independent spoke system.

We now prove our claim. Fix a $\lambda \in \Lambda$ and let $t : \omega_1 \to \omega_1 \times \lambda$ be given such that $t \to \star$. Then $\star \in \overline{\operatorname{ran}(t)}$ and thus by Theorem II.1.13 there exists an $S \in \mathfrak{S}$ such that $\star \in \overline{\operatorname{ran}(t) \cap S}$. Since $\star$ is a p-point, it follows that $\operatorname{ran}(t) \cap S$ is uncountable.

Now let $h : \omega_1 \to \lambda$ be cofinal, strictly increasing and continuous. Define for all $\alpha < \omega_1, t(\alpha) := (\alpha, h(\alpha))$. Then $t \to \star$, so by the work above there exists an $S_\lambda \in \mathfrak{S}$ such that $\operatorname{ran}(t) \cap S_\lambda$ is uncountable. Define $A := \pi_{\omega_1}[\operatorname{ran}(t) \cap S_\lambda]$ and suppose that for all $\alpha < \omega_1$, there exists an $x_\alpha \in ([\alpha, \omega_1) \times [h(\alpha), \lambda)) \setminus S_\lambda$. Then $(x_\alpha)_{\alpha < \omega_1} \to \star$, so again there exists a $T \in \mathfrak{S}$ such that $B := \pi_{\omega_1}[\{x_\alpha : \alpha < \omega_1\} \cap T]$ is uncountable. Since $\{x_\alpha : \alpha < \omega_1\} \cap S_\lambda = \emptyset$, it follows that T is distinct from S_λ, so $(S_\lambda \cap T) \setminus \{\star\}$ has compact closure in $\omega_1 \times \omega_2$. In particular its projection onto ω_1 is bounded.

Let $\beta < \omega_1$ be given. As λ has uncountable cofinality, $A' \cap B'$ is a club in ω_1, where C' is defined to be the set of limit points of $C \subseteq \omega_1$. Let $\gamma \in [\beta, \omega_1) \cap A' \cap B'$ be given, so there exist strictly increasing sequences $(\delta_n)_{n<\omega} \subseteq A, (\varepsilon_n)_{n<\omega} \subseteq B$ with supremum γ. Then by the continuity of h,

$$(\gamma, h(\gamma)) \in \overline{\{(\delta_n, h(\delta_n)) : n < \omega\}} \subseteq \overline{\operatorname{ran}(t) \cap S_\lambda} \subseteq S_\lambda.$$

Moreover, for each $n < \omega$ there exists an $\alpha_n < \omega_1$ such that $x_{\alpha_n} \in T$ and $\pi_{\omega_1}(x_{\alpha_n}) = \varepsilon_n$. Therefore, since λ is sequentially compact (by virtue of being an ordinal with uncountable cofinality), there exists a subsequence of $(\pi_{\omega_2}(x_{\alpha_n}))_{n<\omega}$ that converges to some ordinal $\theta < \lambda$, and thus $(\gamma, \theta) \in \overline{\{x_\alpha : \alpha \in B\} \cap T} \subseteq T$. Hence $\gamma \in \pi_{\omega_1}[(S_\lambda \cap T) \setminus \{\star\}]$. But this then shows that $\pi_{\omega_1}[(S_\lambda \cap T)\setminus\{\star\}]$ is unbounded, which is a contradiction. Therefore there exists an $\alpha_\lambda < \omega_1$ such that $[\alpha_\lambda, \omega_1) \times [h(\alpha_\lambda), \lambda) \subseteq S_\lambda$. By defining $\beta_\lambda := h(\alpha_\lambda)$, we conclude the proof of our claim and the theorem. □

III.2 Compactifications

For this section, we assume (unless otherwise stated) that our spaces are locally-compact, non-compact and Hausdorff. We will use our characterisations from the previous section to characterise radiality in compactifications, starting with one-point compactifications.

III.2.1 One-point compactifications

Using closed spokes, we can reflect radiality of compactifications down to the structure of the compact subsets of X. We first show considering the points in the remainder is sufficient.

Lemma III.2.1. *Let X be a topological space, $U \subseteq X$ be open. If U is radial and X is radial at every point outside U then X is radial.*

Proof. Let $u \in U, A \subseteq X$ be given such that $u \in \overline{A}$. Then for each open $V \subseteq X$, if $u \in V$ then $V \cap A \neq \emptyset$. In particular, for each open $V \subseteq U, V \cap A \neq \emptyset$ and so $u \in \overline{A \cap U}^U$. Thus there exists a transfinite sequence contained in $A \cap U$ that converges to u and therefore X is radial at u. □

Thus to find radial compactifications, it suffices to find remainders which are radial in the whole compactification. If we only require that there is *some* radial compactification, the one-point compactification suffices by Theorem I.2.16, since the one-point compactification is the continuous (and hence closed) image of every compactification, and closed maps are pseudo-open — see Figure I.2.

Thus we will be investigating when $\star$ is radial in αX. For this, we need to translate spokes the notions of spokes of $\star$ in αX into the internal language of compact subsets of X. Note that since $\star$ is not (quasi-)isolated in αX, when considering spokes and spokes systems of $\star$ we may restrict ourselves to non-trivial ones by Lemma II.2.12. This motivates the following definition.

Definition III.2.2 (Spoke at infinity)**.** Let $S \subseteq X$ be given such that $S \cup \{\star\}$ is a non-trivial spoke at $\star$ in αX. Then we say that S is a *spoke at infinity* of X. We denote the set of (closed) spokes at infinity by $\mathbf{Sp}^{\infty}(X)$ $(\overline{\mathbf{Sp}}^{\infty}(X))$.

A collection $\mathfrak{S} \subseteq \mathbf{Sp}^{\infty}(X)$ is called a *spoke system at infinity* if $\{S \cup \{\star\} : S \in \mathfrak{S}\}$ is a spoke system of $\star$ in αX. We say that $\mathfrak{S}$ is *closed* if it consists of closed spokes at infinity.

Lemma III.2.3. *Let $S \subseteq X$ be closed. Then S is a spoke at infinity if and only if S is non-compact and there exists a cofinal chain in* $(\mathcal{K}(S), \subseteq)$.

Proof. Assume S is a spoke at infinity, so there exists a well-ordered neighbourhood base $\mathcal{B}$ of $\star$ in $S \cup \{\star\}$. By taking interiors, we can assume that $\mathcal{B}$ consists of open sets. Then $\{S \backslash B : B \in \mathcal{B}\}$ is a chain in $\mathcal{K}(S)$. Moreover, for all $K \in \mathcal{K}(S)$, there exists a $B \in \mathcal{B}$ such that $B \subseteq \alpha X \backslash K$ and so $K \subseteq S \backslash B$. Therefore $\{S \backslash B : B \in \mathcal{B}\}$ is cofinal in $(\mathcal{K}(S), \subseteq)$. Furthermore, since $\star$ is not (quasi-)isolated in $S \cup \{\star\}$, it follows that $\star \in \overline{S}^{\alpha X}$ and thus S is non-compact.

Now assume that S is non-compact and there exists a cofinal chain in $(\mathcal{K}(S), \subseteq)$, so by considering its cofinality, there exists an increasing, cofinal, transfinite sequence $(K_\alpha)_{\alpha<\lambda} \subseteq \mathcal{K}(S)$. Then $(S \cup \{\star\}) \backslash K_\alpha$ is a neighbourhood of $\star$ in S for each $\alpha < \lambda$. Let $V \subseteq \alpha X$ be open with $\star \in V$, so $X \backslash V$ is compact and hence $S \backslash V$ has compact closure in S. Thus there exists an $\alpha < \lambda$ such that $S \backslash V \subseteq K_\alpha$ and so $(S \cup \{\star\}) \backslash K_\alpha \subseteq S \cap V$. Therefore $((S \cup \{\star\}) \backslash K_\alpha)_{\alpha<\lambda}$ is a well-ordered neighbourhood base for $\star$ in $S \cup \{\star\}$. Finally, as S is non-compact, $\star \in \overline{S}^{\alpha X}$ and hence $S \cup \{\star\}$ is a non-trivial spoke of $\star$ in αX. □

Theorem III.2.4. *Let* $\mathfrak{S} \subseteq \overline{\mathbf{Sp}}^{\infty}(X)$ *be given. Then* $\mathfrak{S}$ *is a spoke system at infinity if and only if* $\bigcup_{S\in\mathfrak{S}}(S \backslash K_S)$ *has co-compact*[7] *interior for every selection* $(K_S)_{S\in\mathfrak{S}}$ *from* $(\mathcal{K}(S))_{S\in\mathfrak{S}}$.

Proof. By definition, $\mathfrak{S}$ is a spoke system at infinity if and only if $\bigcup_{S\in\mathfrak{S}} U_S \in \mathcal{N}_\star^{\alpha X}$ for every selection $(U_S)_{S\in\mathfrak{S}}$ from $(\mathcal{U}_\star^{S\cup\{\star\}})_{S\in\mathfrak{S}}$. Since αX is compact, this is equivalent to $\bigcup_{S\in\mathfrak{S}}(S \backslash K_S)$ having co-compact interior in X for all selections $(K_S)_{S\in\mathfrak{S}}$ from $(\mathcal{K}(S))_{S\in\mathfrak{S}}$. □

The following theorem uses just the radiality property to characterise radiality of $\star$ in αX.

Theorem III.2.5. *αX is radial at $\star$ if and only if for all $Y \subseteq X$ with non-compact closure in X, there exists an $S \in \overline{\mathbf{Sp}}^{\infty}(X)$ such that $S \subseteq Y$ and $K_\alpha = \overline{K_\alpha \cap S}$ for all $\alpha < \lambda$, where $(K_\alpha)_{\alpha<\lambda}$ is a cofinal chain in $(\mathcal{K}(S), \subseteq)$.*

Proof. Suppose that αX is radial at $\star$ and let $Y \subseteq X$ have non-compact closure in X, so $\star \in \overline{Y}$. Then by Theorem II.2.2, there exists an injective transfinite sequence $t : \lambda \to Y$ that

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[7] A subset is *co-compact* if its complement is compact.

converges strictly to $\star$, and by Theorem III.1.7, $\overline{\mathbb{S}_t^{\alpha X}} \in \overline{\mathbf{Sp}}_{\alpha X}(\star)$ and $(\overline{t[\beta]})_{\beta<\lambda}$ is a cofinal chain in $(\mathcal{K}(\overline{\operatorname{ran}(t)}^X), \subseteq)$. Let $\beta < \lambda$ be given. Then

$$t[\beta] \subseteq \overline{t[\beta]} \cap Y \subseteq \overline{t[\beta]} \Rightarrow \overline{t[\beta]} = \overline{\overline{t[\beta]} \cap Y}.$$

Moreover, $\overline{\operatorname{ran}(t)}^X$ is non-compact, since $\star \in \overline{\operatorname{ran}(t)}^{\alpha X}$.

Now suppose the latter holds and let $A \subseteq X$ be given such that $\star \in \overline{A}$, so A has non-compact closure in X. Then there exists an $S \in \overline{\mathbf{Sp}}^{\infty}(X)$ and a strictly increasing, cofinal chain $(K_\beta)_{\beta<\lambda}$ in $\mathcal{K}(S)$ such that $K_\beta = \overline{K_\beta \cap A}$ for all $\beta < \lambda$. Since S is non-compact, λ must be a limit ordinal.

Let $\beta < \lambda$ be given, so $\overline{K_\beta \cap A} = K_\beta \subsetneq K_{\beta+1} = \overline{K_{\beta+1} \cap A}$ and hence there exists an $x_\beta \in (K_{\beta+1} \setminus K_\beta) \cap A$. Since $(\{\star\} \cup (S \setminus K_\beta))_{\beta<\lambda}$ is a neighbourhood base for $\star$ with respect to $S \cup \{\star\}$ (by the proof of Lemma III.2.3), it follows that $(x_\beta)_{\beta<\lambda}$ converges to $\star$ and is contained in A. Therefore αX is radial at $\star$. □

Corollary III.2.6. *Suppose αX is radial at $\star$. Then for all $A \subseteq X$ closed and non-compact, there exists an $S \in \overline{\mathbf{Sp}}^{\infty}(X)$ contained in A.*

Proof. By picking S and $(K_\alpha)_{\alpha<\lambda}$ from the previous theorem, it follows that $K_\alpha = \overline{K_\alpha \cap A} \subseteq A$ for all $\alpha < \lambda$ and so $S = \bigcup_{\alpha<\lambda} K_\alpha \subseteq A$. □

The preceding corollary is not surprising when our spokes at infinity are σ-compact, as we can then take an ω-sequence converging to $\star$. However, this is an artefact of the T_1 condition implying that finite subsets are closed. If $\star$ is a p-point, then the spokes will contain closures of countably-infinite subsets, which could be potentially large. Unfortunately, even though this corollary is a more natural condition, it is not equivalent to radiality at $\star$.

Theorem III.2.7. *There exists a locally-compact, non-compact Hausdorff space such that for all $A \subseteq X$ closed and non-compact, there exists an $S \in \overline{\mathbf{Sp}}^{\infty}(X)$ with $S \subseteq A$, yet αX is not radial at $\star$.*

Proof. Define the *deleted Tychonoff plank* to be $X := ((\omega+1)\times(\omega_1+1))\setminus\{(\omega,\omega_1)\}$ and observe that $\alpha X \cong (\omega+1)\times(\omega_1+1)$, so αX is not radial at $\star = (\omega,\omega_1)$ (as noted in the comments after Proposition I.3.30). Let $A \subseteq X$ be closed and non-compact and suppose $\pi_\omega[A\cap(\omega\times\{\omega_1\})]$ and $\pi_{\omega_1}[A\cap(\{\omega\}\times\omega_1)]$ are bounded in ω and ω_1 respectively. Then there exists an $n<\omega$ and $\alpha<\omega_1$ such that

$$A \subseteq ((\omega+1)\times(\omega_1+1))\setminus([n,\omega]\times[\alpha,\omega_1]) = (n\times(\omega_1+1))\cup((\omega+1)\times(\alpha+1)).$$

Thus A has compact closure in X, which is a contradiction. Therefore either $\pi_\omega[A\cap(\omega\times\{\omega_1\})]$ is unbounded in ω or $\pi_{\omega_1}[A\cap(\{\omega\}\times\omega_1)]$ is unbounded in ω_1. As $\{\omega\}\times\omega_1$ and $\omega\times\{\omega_1\}$ are easily seen to be spokes at infinity, it follows that one of $A\cap(\omega\times\{\omega_1\})$, $A\cap(\{\omega\}\times\omega_1)$ is a closed spoke at infinity, completing the proof. □

Now we translate the ordering on spokes of $\star$ to spokes at infinity.

Definition III.2.8. Let X be a non-compact, locally-compact Hausdorff space. For all $S,T\in\mathbf{Sp}^\infty(X)$, we define $S\leq_K T$ if $S\setminus T$ has compact closure. Observe that for $S,T\in\mathbf{Sp}^\infty(X)$, $S\cup\{\star\}\subseteq_\star^{\alpha X} T\cup\{\star\}$ if and only if $S\leq_K T$. We endow $\mathbf{Sp}^\infty(X)$ with this quasi-order and say that a collection $\mathcal{S}\subseteq\mathbf{Sp}^\infty(X)$ is *cofinal* if it is cofinal in this order, that is, for all $T\in\mathbf{Sp}^\infty(X)$, there exists an $S\in\mathbf{Sp}^\infty(X)$ such that $T\leq_K S$.

By translating the characterisation of spoke systems from Theorem III.1.7, we have the following characterisations of radiality at $\star$:

Theorem III.2.9. *The following are equivalent:*

(1) αX *is radial at* $\star$*.*

(2) There exists a collection $\mathfrak{S} \subseteq \overline{\mathbf{Sp}}^{\infty}(X)$ *such that* $\bigcup_{S\in\mathfrak{S}}(S\backslash K_S)$ *has co-compact interior for every selection* $(K_S)_{S\in\mathfrak{S}}$ *from* $(\mathcal{K}(S))_{S\in\mathfrak{S}}$*.*

(3) For every cofinal collection $\mathcal{S} \subseteq \overline{\mathbf{Sp}}^{\infty}(X), \bigcup\mathcal{S}$ *has co-compact interior.*

Proof. The equivalence of (1) and (2) follows Theorem III.2.5, noting that if $\mathfrak{T}$ is a closed, non-trivial spoke system of $\star$ in αX, then $\{T\backslash\{\star\} : T \in \mathfrak{T}\}$ is a closed spoke system at infinity, by Lemma II.2.12.

Now observe that for every collection $\mathcal{S} \subseteq \overline{\mathbf{Sp}}^{\infty}(X), \mathcal{S}$ is cofinal in $(\mathbf{Sp}^{\infty}(X), \leq_K)$ if and only if $\{S \cup \{\star\} : S \in \mathcal{S}\} \subseteq \overline{\mathbf{Sp}}_{\alpha X}(\star)$ is cofinal in $(\mathbf{Sp}_{\alpha X}(\star), \subseteq_{\star}^{\alpha X})$. Moreover, for every $\mathcal{S} \subseteq \overline{\mathbf{Sp}}^{\infty}(X), \bigcup\mathcal{S}$ has co-compact interior if and only if $\bigcup_{S\in\mathcal{S}}(S \cup \{\star\}) \in \mathcal{N}_{\star}^{\alpha X}$. Therefore by Theorem III.1.7, (1) implies (3). Now suppose (3) and let $\mathcal{S}$ be a cofinal collection of closed spokes of $\star$ in αX and let $T \in \overline{\mathbf{Sp}}^{\infty}(X)$ be given. Then $T \cup \{\star\}$ is a closed spoke at infinity, so there exists an $S \in \mathcal{S}$ such that $T \cup \{\star\} \subseteq_{\star}^{\alpha X} S$ and thus $T \leq_K S\backslash\{\star\}$ and $S\backslash\{\star\}$ is non-compact. Therefore $\mathcal{T} := \{S\backslash\{\star\} : S \in \mathcal{S} \text{ is non-trivial}\} \subseteq \overline{\mathbf{Sp}}^{\infty}(X)$ is cofinal, and thus $\bigcup\mathcal{T}$ has co-compact interior and $\bigcup\mathcal{T} \subseteq \bigcup_{S\in\mathcal{S}}(S\backslash\{\star\})$. Therefore $\bigcup\mathcal{S} \in \mathcal{N}_{\star}^{\alpha X}$ and so by Theorem III.1.7, $\star$ is radial in αX. □

We conclude this section with an investigation of one-point compactifications of Mrówka spaces and their spokes at infinity.

Mrówka spaces

Let $\mathcal{A}$ be an infinite *almost-disjoint* family of subsets of ω; that is, a collection of infinite subsets of ω such that any two distinct elements intersect finitely. We define a topology on $\omega \cup \mathcal{A}$ as follows: let each $n \in \omega$ be isolated and for all $A \in \mathcal{A}$, let $\{\{A\} \cup (A \setminus F) : F \in [\omega]^{<\aleph_0}\}$ be a neighbourhood base for A. We denote this space by $\Psi_{\mathcal{A}}$ and call it a *Mrówka space*. Observe that it is locally-compact, Hausdorff and non-compact, since $\mathcal{A}$ is an infinite, closed, discrete subset of $\Psi_{\mathcal{A}}$. For every countably-infinite set X and every collection $\mathcal{C} \subseteq \mathcal{P}(X)$, we define the *perpendicular ideal of* $\mathcal{C}$ to be

$$\mathcal{C}^{\perp X} := \{D \subseteq X : C \cap D \text{ is finite, for all } C \in \mathcal{C}\}.$$

Also given a collection $\mathcal{C}$ and a set D, we define the *trace of* $\mathcal{C}$ *on* D to be

$$\mathcal{C}{\downarrow_D} := \{C \cap D : C \in \mathcal{C}, C \cap D \text{ is infinite}\}.$$

Proposition III.2.10. $\mathcal{S} := \{\mathcal{B} \cup C : \mathcal{B} \in [\mathcal{A}]^{\aleph_0}, C \in (\mathcal{A} \setminus \mathcal{B})^{\perp\omega}\}$ *is a cofinal collection of closed spokes at infinity for* $\alpha\Psi_{\mathcal{A}}$.

Proof. Let $\mathcal{B} \in [\mathcal{A}]^{\aleph_0}, C \in (\mathcal{A} \setminus \mathcal{B})^{\perp\omega}$ be given. Then for all $A \in \mathcal{A} \setminus \mathcal{B}, \{A\} \cup (A \setminus C)$ is an open neighbourhood of A disjoint from $\mathcal{B} \cup C$, so $\mathcal{B} \cup C$ is closed and countable. We claim that $\mathcal{B} \cup C$ is *anti-compact*, that is, $\mathcal{K}(\mathcal{B} \cup C) = [\mathcal{B} \cup C]^{<\aleph_0}$. Then we can trivially construct a countable, cofinal chain in $(\mathcal{K}(\mathcal{B} \cup C), \subseteq)$. As $\mathcal{B} \cup C$ is non-compact (because it contains the infinite, closed, discrete subset $\mathcal{B}$), this shows that $\mathcal{B} \cup C$ is a closed spoke at infinity.

Let $K \subseteq \mathcal{B} \cup C$ be compact, so $K \cap \mathcal{A}$ is finite since $\mathcal{A}$ is closed and discrete in $\Psi_{\mathcal{A}}$. If $K \cap \omega$ is infinite then there exists an $B \in \overline{K \cap \omega} \setminus (K \cap \omega) \subseteq \mathcal{A}$. But then $B \in \mathcal{B}$ and $C \cap B$ is infinite, which is a contradiction. Therefore $K \cap \omega$ is also finite, so K is finite, proving that $\mathcal{B} \cup C$ is anti-compact.

We now show that $\mathcal{S}$ is cofinal. Let $S \in \overline{\mathbf{Sp}}^{\infty}(\Psi_{\mathcal{A}})$ be given. Define $\lambda := \mathrm{cf}(\mathcal{K}(S), \subseteq)$ and let $(K_\alpha)_{\alpha<\lambda}$ be a strictly increasing, cofinal chain in $(\mathcal{K}(S), \subseteq)$. Observe that $K \cap \mathcal{A}$ is finite for all $K \in \mathcal{K}(\Psi_{\mathcal{A}})$ and thus all such K are countable. As $(K_\alpha)_{\alpha<\lambda}$ is a strictly increasing chain, it follows that it also must be countable and hence $\lambda = \omega$. Therefore $S \cap \mathcal{A} = \bigcup_{n<\omega}(K_n \cap \mathcal{A})$ is countable, so we may pick a $\mathcal{B} \in [\mathcal{A}]^{\aleph_0}$ such that $S \cap \mathcal{A} \subseteq \mathcal{B}$. Define $C := S \cap \omega$ and note that since S is closed, $C \in (\mathcal{A} \setminus S)^{\perp \omega} \subseteq (\mathcal{A} \setminus \mathcal{B})^{\perp \omega}$. Hence $S \subseteq \mathcal{B} \cup C \in \mathcal{S}$ and thus $S \leq_K \mathcal{B} \cup C$. □

Theorem III.2.11. *Let $X \subseteq \alpha\Psi_{\mathcal{A}}$ be given such that $\star \in X$ is non-isolated. Then the following are equivalent:*

(1) X is radial at $\star$.

(2) X is Fréchet-Urysohn at $\star$.

(3) $\mathfrak{S} := [\mathcal{A} \cap X]^{\aleph_0} \cup ((\mathcal{A} \cap X)^{\perp (X \cap \omega)} \cap [X]^{\aleph_0})$ is a closed spoke system at infinity.

(4) $(\mathcal{A} \cap X)\downarrow_Y$ is not m.a.d. for any $Y \in [X \cap \omega]^{\aleph_0}$.

Proof. Assume X is radial at $\star$ and note that by the previous proposition, $\mathfrak{S}$ is a collection of closed spokes at infinity because they are all non-compact. We claim that it is a spoke system at infinity. Let $D \subseteq X \setminus \{\star\}$ be given such that $\star \in \overline{D}^X = \overline{D \cap \mathcal{A}}^X \cup \overline{D \cap \omega}^X$. If $\star \in \overline{D \cap \mathcal{A}}^X$ then there exists a $\mathcal{B} \in [D \cap \mathcal{A}]^{\aleph_0}$ and so $\star \in \overline{\mathcal{B}}^X$. Otherwise, $\star \in \overline{D \cap \omega}^X$, so by Theorem II.2.2 there exists an injective sequence $(x_n)_{n<\omega}$ in $D \cap \omega$ that converges to $\star$. Define $C := \{x_n : n < \omega\}$ and let $A \in \mathcal{A}$ be given. Then $(\{A\} \cup A) \cap C = A \cap C$ must be finite, so it follows that $C \in (\mathcal{A} \cap X)^{\perp(X\cap\omega)}$. Thus in any case, there exists an $S \in \mathfrak{S}$ such that $\star \in \overline{D \cap S}^X$ and therefore by Theorem II.1.13, $\mathfrak{S}$ is a closed spoke system at infinity. As each spoke in $\mathfrak{S}$ is σ-compact, it follows from Theorem II.2.13 that $\Sigma_X^{\mathrm{rad}}(\star) = \{\omega\}$ and so X is Fréchet-Urysohn at $\star$. Therefore (1–3) are equivalent.

Let $Y \in [X \cap \omega]^{\aleph_0}$ be given and suppose $(\mathcal{A} \cap X)\downarrow_Y$ is m.a.d., and in particular infinite. Thus for all finite $\mathcal{F} \subseteq \mathcal{A} \cap X$, $Y \setminus (\bigcup \mathcal{F})$ is still infinite, so $\star \in \overline{Y}^X$. However, for all $Z \in [Y]^{\aleph_0}$, there exists an $A \in (\mathcal{A} \cap X)\downarrow_Y$ such that $A \cap Z$ is infinite and thus Z is not the range of a sequence in Y that converges to $\star$. Therefore $\star$ is not Fréchet-Urysohn in X.

Finally, assume $(\mathcal{A} \cap X)\downarrow_Y$ is not m.a.d. for all $Y \in [X \cap \omega]^{\aleph_0}$ and let $D \subseteq X \setminus \{\star\}$ be given such that $\star \in \overline{D}^X = \overline{D \cap \mathcal{A}}^X \cup \overline{D \cap \omega}^X$. If $\star \in \overline{D \cap \mathcal{A}}^X$ then $D \cap \mathcal{A}$ is infinite, so $D \cap \mathcal{A} \in \mathfrak{S}$. Otherwise, $D \cap \omega$ is infinite. Define $\mathcal{B} := (\mathcal{A} \cap X)\downarrow_{D\cap\omega}$. Note that $\overline{D}^{X\setminus\{\star\}} = \mathcal{B} \cup D$ is non-compact. If $\mathcal{B}$ is finite, it follows that $E := (D \cap \omega) \setminus (\bigcup \mathcal{B})$ is infinite. Otherwise, by assumption there exists an $E \in [D \cap \omega]^{\aleph_0}$ such that $E \cap B$ is finite for all $B \in \mathcal{B}$. In any case, $E \in (\mathcal{A} \cap X)^{\perp(X\cap\omega)}$, showing that $\mathfrak{S}$ is a spoke system at infinity. □

III.2.2 Countable core

In [Arh07], Arhangel′skiĭ developed the concept of the *core* of a space and showed that for non-compact, locally-compact Hausdorff spaces X with countable core, $\star$ is Fréchet-Urysohn in αX precisely when it has a countable neighbourhood base (see III.2.17). We will show how the core concept connects with spokes at infinity.

Definition III.2.12. Let X be a topological space and $A \subseteq X$ be given.

- A is *saturated* if A is closed and for all $C \subseteq X$ closed and non-compact, $A \cap C \neq \emptyset$.

- A is *compact in X from inside* if A is open and for all closed $C \subseteq X$, if $C \subseteq A$ then C is compact.

Lemma III.2.13 ([Arh07, Lemma 1.7, pg. 626])**.** *Let X be a topological space, $A \subseteq X$ be given. Then A is saturated if and only if $X \backslash A$ is compact in X from inside.*

Definition III.2.14 (Core)**.** Let X be a non-compact, locally-compact Hausdorff space. Then we define the *core of X* to be the least cardinality $\operatorname{core}(X)$ of a family of saturated subsets with empty intersection. Equivalently, $\operatorname{core}(X)$ is the least cardinality of an open cover of subsets that are compact in X from inside.

As every compact subset of X is compact in X from inside, the core of X is well-defined.

Proposition III.2.15 ([Arh07, Proposition 1.5, pg. 626])**.** *Let $Y \subseteq X$ be closed and non-compact. Then* $\operatorname{core}(Y) \leq \operatorname{core}(X)$.

We will extend Arhangel′skiĭ's theorem to include the condition 'αX is radial at $\star$' to the list of equivalent statements. To achieve this, we show how the core of a space interacts with its closed spokes at infinity.

Lemma III.2.16. *Let $S \in \overline{\mathbf{Sp}}^{\infty}(X)$ be given. Then $\chi_{S \cup \{\star\}}(\star) = \operatorname{core}(S)$.*

Proof. Let $(B_\beta)_{\beta<\lambda}$ be a well-ordered neighbourhood base of $\star$ with respect to $S \cup \{\star\}$. By Lemma I.3.23, Theorem I.3.24 and Corollary I.3.26, we can assume that λ is regular and each B_β is open in $S \cup \{\star\}$. Then $\{S \backslash B_\beta : \beta < \lambda\}$ is cofinal in $(\mathcal{K}(S), \subseteq)$, so $\operatorname{core}(X) \leq \lambda$.

Let $\mathcal{S}$ be a collection of saturated subsets of S with cardinality $\mathrm{core}(S)$ such that $\bigcap\mathcal{S} = \emptyset$. Assume without loss of generality that $\mathcal{S}$ is closed under finite intersection.

Let $\beta < \lambda$ be given. As $\bigcap_{S\in\mathcal{S}}(S \cup \{\star\}) = \{\star\}$, it follows by compactness that there exists a finite non-empty subset $\mathcal{F}$ of $\mathcal{S}$ such that $\bigcap_{S\in\mathcal{F}}(S \cup \{\star\}) \subseteq B_\beta$. Define $S_\beta := \bigcap\mathcal{F} \in \mathcal{S}$. If $\lambda > \mathrm{core}(S)$ then by regularity there exists $S \in \mathcal{S}$ such that $\Lambda := \{\beta < \lambda : S_\beta = S\}$ has cardinality λ and thus $S \subseteq \bigcap_{\beta\in\Lambda}(B_\beta\backslash\{\star\}) = \bigcap_{\beta<\lambda}(B_\beta\backslash\{\star\}) = \emptyset$, which is a contradiction. Therefore $\mathrm{core}(S) = \lambda$. □

The equivalence of the last three statements in following theorem is stated in [Arh07, Corollary 3.17, pg. 633]. We present here a full proof.

Theorem III.2.17. *Let X be a non-compact, locally-compact, Hausdorff space with countable core. Then the following are equivalent:*

(1) αX is radial at $\star$.

(2) αX is Fréchet-Urysohn at $\star$.

(3) αX is first-countable at $\star$.

(4) X is σ-compact.

Proof. It is obvious that the reverse implications hold, so assume that αX is radial at $\star$. Then by Proposition III.2.15 and Lemma III.2.16, every closed, non-compact spoke at infinity S is σ-compact and thus $\chi_{S\cup\{\star\}}(\star) = \aleph_0$. Hence by Theorems II.2.16, III.2.4 and III.2.9, $\star$ is $\{\omega\}$-radial in αX, i.e. $\star$ is Fréchet-Urysohn in αX.

Now assume that αX is Fréchet-Urysohn at $\star$ and let $U \subseteq X$ be compact in X from inside. Suppose $\overline{U}^X$ is not compact, so $\star \in \overline{U}^{\alpha X}$. Then there exists a sequence $(x_n)_{n<\omega} \subseteq U$ that converges to $\star$. But then $\{x_n : n < \omega\}$ is closed in X and contained in U, and so is compact, which is a contradiction since $\star \in \overline{\{x_n : n < \omega\}}$. Hence U has compact closure in X and therefore since X has countable core it is σ-compact.

The equivalence of σ-compactness of X and first-countability of $\star$ in αX is trivial by [Eng89, Problem 3.1.F(a), pg. 135]. $\square$

We can obtain a similar implication as (2) implies (4) in the above theorem for arbitrary cores — however, this requires $\star$ to be independently-based in αX. Under this hypothesis, we prove that $\chi_{\alpha X}(\star) = \mathrm{core}(X)$, i.e. if we take an open cover of X of cardinality $\mathrm{core}(X)$ consisting of subsets that are compact in X from inside, we cannot improve upon taking compact sets.

Lemma III.2.18. $\mathrm{core}(X) \leq \chi_{\alpha X}(\star)$.

Proof. Let $\mathcal{B}$ be a closed neighbourhood base of $\star$ with respect to αX and define $\mathcal{S} := \{B \backslash \{\star\} : B \in \mathcal{B}\}$. Then for every closed and non-compact $C \subseteq X$, $\star \in \overline{C}^{\alpha X}$ and thus $(B \backslash \{\star\}) \cap C \neq \emptyset$ for every $B \in \mathcal{B}$. Therefore $\mathcal{S}$ is a collection of saturated subsets of X with empty intersection and thus $\mathrm{core}(X) \leq \chi_{\alpha X}(\star)$. $\square$

Theorem III.2.19. *Suppose αX is independently-based at $\star$. Then* $\mathrm{core}(X) = \chi_{\alpha X}(\star)$.

Proof. Let $\mathfrak{S}$ be an independent spoke system of $\star$ with respect to αX. By Lemma II.2.12 we can assume that $\mathfrak{S}$ is non-trivial. Then by Lemma III.2.16, Proposition III.2.15 and

Theorem III.1.7,

$$\chi_{S\cup\{\star\}}(\star) = \chi_{\overline{S}^X\cup\{\star\}}(\star) = \operatorname{core}(\overline{S}^X) \leq \operatorname{core}(X).$$

Hence by Theorems II.2.10 and II.2.13, $\psi_{\alpha X}(\star) = \sup(\Sigma^{\mathrm{rad}}_{\alpha X}(\star)) \leq \operatorname{core}(X)$. However, by [Eng89, Problem 3.1.F(a), pg. 135] $\psi_{\alpha X}(\star) = \chi_{\alpha X}(\star)$, so by the previous lemma $\chi_{\alpha X}(\star) = \operatorname{core}(X)$. □

III.2.3 Finite, countable and ordinal compactifications

We will now investigate larger compactifications, assuming that αX is radial at $\star$. We start with finite and countable compactifications; in fact, we will demonstrate results for *ordinal compactifications* — those whose remainder is homeomorphic to some ordinal. Recall that an ordinal space α is compact if and only if α is a non-limit ordinal [Eng89, Problem 3.12.3(a), pg. 221]. We also obtain conditions for the existence of sequential and pseudo-radial compactifications.

We can construct the one-point compactification of X by identifying the remainder of any compactification to a single point (see [Eng89, Theorems 3.5.12 & 3.5.13, pg. 170]). We will be implicitly using this identification from now on.

The following lemma is fundamental to our study of spaces X where αX is radial at $\star$:

Lemma III.2.20. *Let γX be an ordinal compactification of X and suppose αX is radial at $\star$. Let $t : \lambda \to X$ be a transfinite sequence that converges to $\star$ in αX. Then there exists a subsequence of t that converges to some point in $\gamma X \backslash X$.*

Proof. Pick an ordinal α such that $\gamma X\backslash X \cong \alpha + 1$ and identify $\gamma X\backslash X$ with $\alpha + 1$. Assume

t has no such subsequence, so α must be non-zero. Define $\lambda := \operatorname{len}(t)$ and suppose there exists an $m < \omega$, a strictly decreasing sequence of ordinals $(\beta_n)_{n \le m}$ in $\alpha+1$, and a sequence of open subsets $(U_n)_{n<m}$ of γX such that:

- $\beta_0 = \alpha$,
- $\beta_m > 0$,
- $U_n \setminus X = (\beta_{n+1}, \beta_n]$ for all $n < m$,
- $D := \lambda \setminus t^{-1}[\bigcup_{n<m} U_n]$ is unbounded.

Then since no subsequence of t converges to β_m, there exists an open subset $V \subseteq \gamma X$ such that $\beta_m \in V$ and $D \setminus t|_D^{-1}[V] = \lambda \setminus t^{-1}[V \cup \bigcup_{n<m} U_n]$ is unbounded. Assume $[0, \beta_m) \subseteq V$. Then $U := V \cup \bigcup_{n<m} U_n$ is a neighbourhood of $\gamma X \setminus X$ and $\lambda \setminus t^{-1}[U]$ is unbounded, which is a contradiction since $t \to \star$ in αX. Thus there exists a $\beta_{m+1} \in (0, \beta_m)$ such that $[\beta_{m+1}, \beta_m] \subseteq V$, and furthermore there exists an open subset $W \subseteq \gamma X$ such that $(\beta_{m+1}, \beta_m] = W \setminus X$. Define $U_m := V \cap W$ and note that $\lambda \setminus t^{-1}[\bigcup_{n \le m} U_n]$ is unbounded, and $(\beta_{m+1}, \beta_m] = U_m \setminus X$.

Therefore by recursion, we produce a strictly decreasing sequence in $\alpha + 1$, which is a contradiction. Hence there is a subsequence of t that converges to some point in $\gamma X \setminus X$.

□

Using the previous lemma, we now prove several compactification theorems. Recall that a compactification is *finite* / *countable* if it has *finite* / *countable* remainder.

Theorem III.2.21. *Suppose αX is radial at $\star$ and let φX be a finite compactification of X. Then φX is radial on $\varphi X \setminus X$.*

Proof. Let $A \subseteq \varphi X, z \in \varphi X \backslash X$ be given such that $z \in \overline{A}^{\varphi X}$ and $z \notin A$. Since φX is a finite compactification, there exists a closed neighbourhood $C \subseteq \varphi X$ such that $\{z\} = C \backslash X$, and thus $z \in \overline{A \cap C \cap X}^{\varphi X}$. Then $\star \in \overline{A \cap C \cap X}^{\alpha X}$, so by radiality there exists a transfinite sequence t contained in $A \cap C \cap X$ that converges to $\star$. By the previous lemma, t has a subsequence that converges to some point in $\varphi X \backslash X$. This point must be z, and so φX is radial on $\varphi X \backslash X$. □

Corollary III.2.22. *If αX is radial then every finite compactification of X is radial.*

Proof. This follows from the previous theorem and Lemma III.2.1. □

We now show how we can obtain sequential and pseudoradial compactifications. As both proofs are similar, we define some new notation to simplify matters.

Definition III.2.23 (Λ-pseudoradial)**.** Let X be a topological space and let Λ be a non-empty set of regular, non-zero ordinals. Then we say that X is Λ-*pseudoradial* if for every non-closed $A \subseteq X$, there exists an $x \in \overline{A} \backslash A$ and a transfinite sequence t in A such that $t \to x$ and $\text{len}(t) \in \Lambda$.

Thus a space is pseudoradial if it is {regular, non-zero ordinals}-pseudoradial and is sequential if it is $\{\omega\}$-pseudoradial. Recall that a countable, compact Hausdorff space is homeomorphic to an ordinal [Sou04, pg. 351].

Lemma III.2.24. *Let Λ be a non-empty set of regular, non-zero ordinals and suppose X is Λ-pseudoradial and every closed, non-compact subset of X contains a closed spoke at infinity S*

such that $\chi_{S\cup\{\star\}}(\star)\in\Lambda$[8]*. Let* γX *be a compactification of* X *with* Λ*-pseudoradial remainder. Suppose for every transfinite sequence in* X*, with length in* Λ*, that converges to* $\star$ *in* αX *has a subsequence that converges to some point in* $\gamma X\backslash X$*. Then* γX *is* Λ*-pseudoradial.*

Proof. Let $A\subseteq\gamma X$ be non-closed and suppose $A\cap X$ is not closed in X. Then since X is Λ-pseudoradial, there is a transfinite sequence in $A\cap X$, with length in Λ, that converges to a point in $X\backslash A$. Now assume that $A\cap X$ is closed in X. If $A\backslash X$ is not closed, then since $\gamma X\backslash X$ is Λ-pseudoradial, there exists a $z\in\overline{A\backslash X}^{\gamma X}\backslash A$ and a transfinite sequence contained in $A\backslash X$ that converges to z, with length in Λ.

Finally, assume that $A\backslash X$ is closed. Then because A is not closed in γX, there exists a $z\in\overline{A}^{\gamma X}\backslash A$ and thus $z\in\gamma X\backslash(A\cup X)$. As $A\backslash X$ is closed, it follows that there is some closed neighbourhood $C\subseteq\gamma X$ of z such that $C\cap(A\backslash X)=\emptyset$ and so $z\in\overline{A\cap C}^{\gamma X}=\overline{A\cap C\cap X}^{\gamma X}$. Hence $\star\in\overline{A\cap C\cap X}^{\alpha X}$, so $\overline{A\cap C\cap X}^{X}$ is closed and non-compact. By assumption, there exists an $S\in\overline{\mathbf{Sp}}^{\infty}(X)$ such that $S\subseteq A\cap C\cap X$ and $\chi_{S\cup\{\star\}}(\star)$. Then $S\cup\{\star\}$ is a non-trivial spoke of $\star$ in αX, so by Corollary I.3.28 there exists a transfinite sequence t in $S\subseteq A\cap C\cap X$ that converges to $\star$ in αX with length in Λ. By assumption, there then exists a $w\in\gamma X\backslash X$ and a subsequence u of t that converges to w. As C is closed in γX, it follows that $w\in C$ and so $w\notin A$. Therefore γX is Λ-pseudoradial. □

Theorem III.2.25. *Suppose* X *is pseudoradial and* αX *is radial at* $\star$*. Then every ordinal compactification of* X *is pseudoradial.*

Proof. Note that every ordinal is (pseudo)radial by Corollary II.1.15, so by Lemmas III.2.20

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[8] Observe that the second condition is strictly weaker than αX being Λ-radial at $\star$ by Corollary III.2.6 and Theorem III.2.7.

and III.2.24, we conclude our proof. □

Theorem III.2.26. *Let* Λ *be a set of non-zero regular ordinals with* $\omega \in \Lambda$ *and suppose* X *is* Λ*-pseudoradial and every closed, non-compact subset of* X *contains a closed spoke at infinity* S *with* $\chi_{S\cup\{\star\}}(\star) \in \Lambda$. *Then every countable compactification of* X *is* Λ*-pseudoradial.*

Proof. As $\gamma X \setminus X$ is homeomorphic to an ordinal, it is sequential and thus Λ-pseudoradial. By Lemmas III.2.20 and III.2.24, we conclude our result. □

Corollary III.2.27. *If* X *is sequential and every closed, non-compact subset of* X *contains a* σ*-compact, closed spoke at infinity. Then every countable compactification of* X *is sequential.*

III.2.4 Using small cardinals

In this section, we will use *small cardinals* to improve the results from the last section. These are uncountable cardinals bounded above by $\mathfrak{c} := 2^{\aleph_0}$. The small cardinals we will be using are defined below.

Definition III.2.28 (Small cardinals)**.**

- For $f, g : \omega \to \omega$, we say that f is *eventually bounded by* g if $\{n \in \omega : f(n) > g(n)\}$ is finite. We denote this relation by $f \leq^* g$. Observe that $\leq^*$ is a quasi-order on ${}^{\omega}\omega$. The *bounding number*, denoted by $\mathfrak{b}$, is the smallest cardinality of an unbounded subset of $({}^{\omega}\omega, \leq^*)$.

- For $A, B \subseteq \omega$, we say that A is *almost-contained in* B, written $A \subseteq^* B$, if $A \setminus B$ is finite. If $\mathcal{F}$ is a family of subsets of ω, then a subset $P \subseteq \omega$ is a *pseudo-intersection of* $\mathcal{P}$ if $P \subseteq^* F$ for all $F \in \mathcal{F}$.

- A family $\mathcal{P}$ of infinite subsets of ω has the *strong finite intersection property* if $\bigcap \mathcal{F}$ is infinite for all finite and non-empty $\mathcal{F} \subseteq \mathcal{P}$. The *pseudo-intersection number*, denoted by $\mathfrak{p}$, is the smallest cardinality of a family of infinite subsets of ω with the strong finite intersection property, yet has no infinite pseudo-intersection.

- A *tower* is a transfinite sequence $(T_\alpha)_{\alpha<\lambda}$ of infinite subsets of ω such that $T_\beta \subseteq^* T_\alpha \not\subseteq^* T_\beta$ for all $\alpha < \beta < \lambda$. The *tower number*, denoted by $\mathfrak{t}$, is the smallest cardinality of a tower with no infinite pseudo-intersection.

All three cardinals $\mathfrak{b}, \mathfrak{p}, \mathfrak{t}$ are well-defined small cardinals - see [Dou84]. Furthermore, it was proven in [MS15, Theorem 14.1, pg. 58] that $\mathfrak{p} = \mathfrak{t}$. However, in our theorems we will use the small cardinal that corresponds to the relavent property for our proof.

We also need a ‘not so small’ cardinal:

Definition III.2.29 (Novak number)**.** We define the *Novak number*, denoted by $\mathfrak{n}$, to be the smallest cardinality of a nowhere-dense cover of ω^*. Recall that a subset A of a topological space X is *nowhere-dense* if $\overline{A}^\circ = \emptyset$.

We recall the following from [BN10]: $\mathfrak{t} < \mathfrak{n} \leq 2^{\mathfrak{c}}$ and it is independent of ZFC whether $2^{\mathfrak{t}}$ is less than or equal to $\mathfrak{n}$, or vice versa. Moreover, if we define μ to be the least cardinality of a compact Hausdorff space that is not sequentially compact, then μ is well-defined and

$\max(2^{\mathfrak{t}}, \mathfrak{n}) \leq \mu \leq 2^{\mathfrak{c}}$. Furthemore, under Martin's axiom $\mathfrak{p} = \mathfrak{t} = \mathfrak{b} = \mathfrak{c}$ and $\mu = 2^{\mathfrak{c}}$ [Kun11, Lemma III.3.22, pg. 176].

We now use these cardinals to obtain Fréchet-Urysohn and sequential compactifications, provided we know that our remainder is already Fréchet-Urysohn and sequential respectively. The following theorems are similar in spirit and can be summarised by the following meta-theorem:

> We can use conditions that imply certain convergence properties (e.g. sequentiality implying Fréchet-Urysohn, subsequentiality, sequential compactness) to obtain compactification results.

First, we investigate sequential compactifications:

Theorem III.2.30. *Assume X is sequential and every closed, non-compact subset of X contains a σ-compact, closed spoke at infinity. Then every compactification of X with sequential remainder of cardinality strictly less than μ is sequential.*

Proof. Let γX be a compactification of X such that $\gamma X \backslash X$ is sequential and $|\gamma X \backslash X| < \mu$. Note that if $A \subseteq X$ is a sequence that converges to $\star$ in αX, then $\overline{A}^{\gamma X} = A \cup (\overline{A}^{\gamma X} \backslash X)$, so $|\overline{A}^{\gamma X}| < \mu$ and hence is sequentially compact. Therefore by Lemma III.2.24 γX is sequential. $\square$

Theorem III.2.31. *Suppose X is sequential, $\chi_X(x) < \mathfrak{b}$ for all $x \in X$, and every closed, non-compact subset of X contains a σ-compact, closed spoke at infinity. Let γX be a compactification of X such that:*

- *$\gamma X \setminus X$ is sequential.*

- *$|\gamma X \setminus X| < \mu$.*

- *$\chi_{\gamma X}(y) < \mathfrak{b}$ for all $y \in \gamma X \setminus X$.*

Then γX is Fréchet-Urysohn.

Proof. By Theorem III.2.30, γX is sequential, so by [BBM13, Proposition 3.4, pg. 534] it follows that γX is Fréchet-Urysohn, because $\chi_X(x) = \chi_{\gamma X}(x)$ for all $x \in X$ since X is locally-compact and thus open in γX. ☐

Now we show that certain compactifications of countable spaces are Fréchet-Urysohn. We first observe some simple facts about sequential-density and sequential-separability:

Definition III.2.32 (Sequentially-dense and sequentially-separable)**.** Let X be a topological space, $D \subseteq X$ be given. We say that D is *sequentially-dense* if every point in X is a limit of a sequence in D. We say that X is *sequentially-separable* if it has a countable, sequentially-dense subset.

Note that if X is countable and γX is a Fréchet-Urysohn compactification, then X is sequentially-dense in γX and therefore γX is sequentially-separable.

The following theorem is an adaptation of [Dou84, Theorem 6.2, pg. 129], from which we recall the definition of a subsequential point:

Definition III.2.33 (Subsequential)**.** Let X be a topological space and let $x \in X$ be given such that for all countable $A \subseteq X$, if $x \in \overline{A}$ then there exists a sequence in A that converges to x. Then we say that x is *subsequential*[9]. Equivalently, x is subsequential if every sequence that clusters at x has a subsequence that converges to x.

Theorem III.2.34. *Let X be a topological space and let $x \in X$ be given such that $\chi_X(x) < \mathfrak{p}$. Then x is subsequential.*

Proof. Let $A \in [X]^{\aleph_0}$ be given such that $x \in \overline{A}$. If there exists a $y \in A \cap N_x$ then $(y)_{n<\omega} \to x$, so suppose $N_x \cap A = \emptyset$. Let $\mathcal{B}$ be a neighbourhood base for x with $|\mathcal{B}| < \mathfrak{p}$. Then $\{A \cap B : B \in \mathcal{B}\}$ has the strong finite intersection property, so there exists an infinite subset $C \subseteq A$ such that $C \subseteq^* B$ for all $B \in \mathcal{B}$. Hence $(c_n)_{n<\omega} \to x$ for any enumeration $\{c_n : n \in \omega\} = C$ and so x is subsequential. $\square$

Lemma III.2.35. *Assume X is countable and let γX be a compactification of X such that $\chi_{\gamma X}(y) < \mathfrak{p}$ for all $y \in \gamma X \setminus X$. Then for all $A \subseteq X$ and every $x \in \overline{A}^{\gamma X}$, there exists a sequence in A that converges to x.*

Proof. As αX is homeomorphic to an ordinal, X is Fréchet-Urysohn. Let $A \subseteq X, x \in \overline{A}^{\gamma X}$ be given. If $x \in X$ then $x \in \overline{A}^{X}$ and so there exists a sequence in A that converges to x. If $x \notin X$ then by the previous theorem there exists a sequence in A that converges to x. $\square$

Theorem III.2.36. *Suppose X is countable and let γX be a compactification of X such that $\gamma X \setminus X$ is Fréchet-Urysohn and $\chi_{\gamma X}(x) < \mathfrak{p}$ for all $x \in \gamma X \setminus X$. Then γX is Fréchet-Urysohn.*

Proof. As αX is homeomorphic to an ordinal, X is Fréchet-Urysohn. Let $A \subseteq \gamma X, x \in \overline{A} \setminus A$ be given. If $x \in X$ then there exists a sequence contained in $A \cap X$ that converges to x.

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[9] *Subsequential* has also been used for spaces which can be embedded into a sequential space. We will only use the definition taken from [Dou84] given above.

Suppose $x \notin X$. Then $x \in \overline{A} = \overline{A \cap X} \cup \overline{A \backslash X}$. If $x \in \overline{A \backslash X}$ then there exists a sequence in A that converges to x, since $\gamma X \backslash X$ is Fréchet-Urysohn. Otherwise, by the previous lemma, there also exists a sequence in $A \cap X$ that converges to x. Therefore γX is Fréchet-Urysohn. $\square$

III.2.5 Open questions

The focus so far in this section has been on building up compactifications from below, starting with the one-point compactification and extending results beyond there. The following question is crucial to understanding these compactifications:

Question III.2.37. *Does every Tychonoff space that has a radial / Fréchet-Urysohn compactification have a maximal or greatest such compactification? If not, what conditions ensure its existence?*

Question III.2.38. *Are any of the bounds in Section III.2.4 strict?*

To give a partial answer to Question III.2.37, we require the following simple lemma:

Lemma III.2.39. *The infinite T_1 quotients of $\omega + 1$ are homeomorphic to $\omega + 1$.*

Proof. Let $f : \omega + 1 \to X$ be a quotient map, so X is compact. Let $x, y \subseteq X$ be distinct, so $f^{-1}[\{x\}], f^{-1}[\{y\}]$ are closed and disjoint. Then either $\omega \notin f^{-1}[\{x\}]$ or $\omega \notin f^{-1}[\{y\}]$, so one of $f^{-1}[\{x\}], f^{-1}[\{y\}]$ must be clopen. Therefore if X is infinite it has a unique non-isolated point $x \in X$. Moreover, we must have $x = f(\omega)$ and hence the neighbourhoods of x are co-finite. Thus $X \cong \omega + 1$. $\square$

Theorem III.2.40. *$X := (\omega+1)\times\omega_1$ is a locally-compact, non-compact, first-countable Hausdorff space with no infinite radial compactification, yet αX is radial. Consequently, X has no maximal radial compactification.*

Proof. As $\omega+1$ is compact, $\alpha\omega_1 \cong \omega_1+1$ and both are radial, it follows from Theorem III.1.8 that αX is radial at $\star$. Furthermore, by Corollary III.2.22, every finite compactification of X is radial.

Let γX be a compactification of X with infinite remainder, so there exists a continuous surjection $f : P \to \gamma X$ that extends id_X. Then $f|_{P\setminus X} : P\setminus X \to \gamma X\setminus X$ is a closed surjection and thus a quotient map. As $P\setminus X \cong \omega+1$, it follows from the previous lemma that $\gamma X\setminus X \cong \omega+1$. Let $\delta \in \gamma X\setminus X$ be the unique non-isolated point and define $A := \pi_{\omega+1}[f^{-1}[\{\delta\}]], B := X\setminus(A\times\omega_1)$. We claim that $\delta \in \overline{B}^{\gamma X}$ but no transfinite sequence in B converges to δ, thus showing that there is no infinite radial compactification of X.

Let $U \subseteq \gamma X$ be an open neighbourhood of δ. If $f(\omega,\omega_1) \neq \delta$ then $f^{-1}[\{f(\omega,\omega_1)\}]$ is an open neighbourhood of (ω,ω_1) and so contains $[n,\omega]\times\{\omega_1\}$ for some $n<\omega$. But then $f[P\setminus X] = \gamma X\setminus X$ is finite, which is a contradiction. Therefore $f(\omega,\omega_1) = \delta$ and so there exists an $n<\omega$ and an $\varepsilon<\omega_1$ such that $[n,\omega]\times[\varepsilon,\omega_1] \subseteq f^{-1}[U]$. Moreover, $\gamma X\setminus X$ is infinite, so there exists an $m \in [n,\omega]$ such that $f((m,\omega_1)) \neq \delta$ and hence $(m,\varepsilon) \in U$. Thus $\delta \in \overline{B}^{\gamma X}$.

Now suppose that there exists a transfinite sequence t in B that converges to δ. Then t converges to $\star$ in αX, so by Lemma III.2.20 there exists an $m \leq \omega+1$ and a subsequence u of t that converges to (m,ω_1) in P. However, by Proposition I.3.30 and following paragraph, no transfinite sequence in $\omega\times\omega_1$ converges to (ω,ω_1) in P. Since $f(\omega,\omega_1) = \delta$, it follows that $m < \omega$. Moreover, $f \circ u \to f((m,\omega_1))$, so $f((m,\omega_1)) = \delta$ and hence $m \in A$. But $\{m\} \times (\omega_1 + 1)$ is an open neighbourhood of (m,ω_1) disjoint from B, which is a contradiction. Therefore no transfinite sequence in B converges to δ, so γX is not radial, proving our claim.

Finally, note that by [Eng89, Problem 3.12.20(c), pg. 237] the Tychonoff plank $P := (\omega + 1) \times (\omega_1 + 1)$ is the Stone-Čech compactification of X with remainder homeomorphic to $\omega + 1$. Thus X has no maximal finite compactifications — every compactification of X is obtained by forming a closed partition of $\omega + 1$ and any finite, closed partition can be refined to a larger, finite closed partition. Thus X also has no maximal radial compactification. $\square$

Any future investigation of the structure of radial / Fréchet-Urysohn compactifications must seriously consider the following question:

Question III.2.41. *Do the radial / Fréchet-Urysohn compactifications form an ideal in the join-semilattice of compactifications? If not, what conditions ensure this?*

III.3 Spectra and dualities

In this section, we apply a few categorial dualities to obtain algebraic characterisations of radiality. As shown at the start of this chapter, in semi-regular spaces it suffices to consider closed spokes. As closed subsets pass through the dualities we consider nicely, we will restrict our attention to semi-regular spaces.

III.3.1 Ring spectra

Throughout this section, let R be a commutative ring with identity. We first recall some basic facts about rings, ideals and their spectra — [AM69, Chapter 1] will be our reference for this material.

Definition III.3.1 (Rings and spectra)**.** A *unit* is an element $r \in R$ such that $rs = 1$ for some $s \in R$. We denote the set of units of R by $R^{\times}$.

A subset $I \subseteq R$ is an *ideal*, written $I \lhd R$, if it satisfies the following properties:

- $0 \in I$.
- $x + y \in I$ for all $x, y \in I$.
- $xy \in I$ for all $x \in I$ and $y \in R$.

We say that I is *proper* if $I \neq R$.

A proper ideal $\mathbf{p} \lhd R$ is called *prime* if for all $r, s \in R$, if $rs \in \mathbf{p}$ then either $r \in \mathbf{p}$ or $s \in \mathbf{p}$. A proper ideal is said to be *maximal* if it is a maximal with respect to inclusion amongst all proper ideals of R. The *prime* and *maximal spectra* are the collections of all prime / maximal ideals of R and are denoted by $\mathsf{Spec}(R)$ and $\mathsf{Max}(R)$ respectively. We note that $\mathsf{Max}(R) \subseteq \mathsf{Spec}(R)$.

Define for every subset $A \subseteq R$, $V^R(A) = \{\mathbf{p} \in \mathsf{Spec}(R) : A \subseteq \mathbf{p}\}$. We will use $V^R(\{r\})$ instead of $V^R(r)$ for each $r \in R$, and will dispense with the superscripts if the ring is unambiguous. Note that by primeness, $V(r) \cup V(s) = V(rs)$ for all $r, s \in R$ and also $V(A) = \bigcap_{a \in A} V(a)$ for each $A \subseteq R$. Therefore $\{V(A) : A \subseteq R\}$ is the collection of closed subsets of a unique topology on $\mathsf{Spec}(R)$ with base $\{V(r) : r \in R\}$, which we call the *Zariski topology*. We implicitly endow $\mathsf{Spec}(R)$ with this topology and consider $\mathsf{Max}(R)$ as a subspace of $\mathsf{Spec}(R)$.

To study radiality in ring spectra, we ideally want to characterise it using closed spokes (since these correspond to ideals of the ring) and thus we require the semi-regular ring spectra. However, we still want non-Hausdorff ring spectra, as 'most' ring spectra are not Hausdorff. To this end, we will construct a non-Hausdorff, semi-regular maximal ring spectrum.

Wolfgang Rump has characterised maximal spectra of Bézout domains — integral domains such that every finitely-generated ideal is principal. We first need to define the serial property of a topological space.

Definition III.3.2 (Serial)**.** A collection $\mathcal{A}$ of subsets of a topological space X is *serial* if for all $A, B \in \mathcal{A}$ and for all $x \in X$, either $A \subseteq_x B$ or $B \subseteq_x A$. We then say that a topological space is *serial* if it has a serial base.

Theorem III.3.3 ([Rum14a, Section 4, Theorem 4, pg. 2215])**.** *A topological space X is homeomorphic to the maximal spectrum of a Bézout domain R if and only if it is compact, T_1 and serial.*

We note here some facts that will prove useful:

- [Rum14b, pg. 322] A base $\mathcal{B}$ is *serial* if and only if $\overline{A \backslash B} \cap \overline{B \backslash A} = \emptyset$ for all $A, B \in \mathcal{B}$.
- [Rum14a, Section 3, Example 1, pg. 2212] The base $\{(q, r) : q \in \mathbb{Q}, r \in \mathbb{R}\backslash\mathbb{Q}\}$ for $\mathbb{R}$ is serial.

- [Rum14a, Section 3, Proposition 6, pg. 2211] If $\mathcal{B}$ is a serial base for a space X and $Y \subseteq X$, then $\{B \cap Y : B \in \mathcal{B}\}$ is a serial base for Y.

Theorem III.3.4. *There exists a second countable, semi-regular, non-Hausdorff space that is homeomorphic to the maximal spectrum of a Bézout domain.*

Proof. Define

$$\alpha := \frac{1}{4} \in \left(0, \frac{1}{2}\right) \cap \mathbb{Q}, \beta := \frac{1}{\sqrt{2}} \in \left(\frac{1}{2}, 1\right) \setminus \mathbb{Q},$$

$$X := \{(0,0), (1,0)\} \cup \left([0,1] \times \left\{\frac{1}{n} : n \in \mathbb{N}^+\right\}\right),$$

$$\forall n \in \mathbb{N}^+, B_{0,n} := \{(0,0)\} \cup \left([0,\beta) \times \left\{\frac{1}{m} : m \in \mathbb{N}^+, m \geq n\right\}\right),$$

$$\forall n \in \mathbb{N}^+, B_{1,n} := \{(1,0)\} \cup \left((\alpha,1] \times \left\{\frac{1}{m} : m \in \mathbb{N}^+, m \geq n\right\}\right),$$

$$\mathcal{B}_0 := \{B_{0,n} : n \in \mathbb{N}^+\}, \mathcal{B}_1 := \{B_{1,n} : n \in \mathbb{N}^+\},$$

$$\mathcal{B}_2 := \left\{((q,r) \cap [0,1]) \times \left\{\frac{1}{n}\right\} : q \in \mathbb{Q}\setminus\{0\}, r \in \mathbb{R}\setminus\mathbb{Q}, q < r, n \in \mathbb{N}^+\right\}.$$

Then $\mathcal{B} := \mathcal{B}_0 \cup \mathcal{B}_1 \cup \mathcal{B}_2$ is a base for a topology on X, which we endow X with. Note that every element of $\mathcal{B}_2$ is regular-open and for all $n \in \mathbb{N}^+$,

$$\overline{B_{0,n}} = \{(0,0), (1,0)\} \cup \left([0,\beta] \times \left\{\frac{1}{m} : m \in \mathbb{N}^+, n \geq m\right\}\right)$$

Hence every element of $\mathcal{B}_0$ is also regular-open and similarly so is every element of $\mathcal{B}_1$. Therefore X is semi-regular. Moreover, $N_x = \{x\}$ for all $x \in X$, so X is T_1. However, for all $m, n \in \mathbb{N}^+$,

$$\left(\frac{1}{2}, \frac{1}{\max(m,n)}\right) \in B_{0,m} \cap B_{1,n},$$

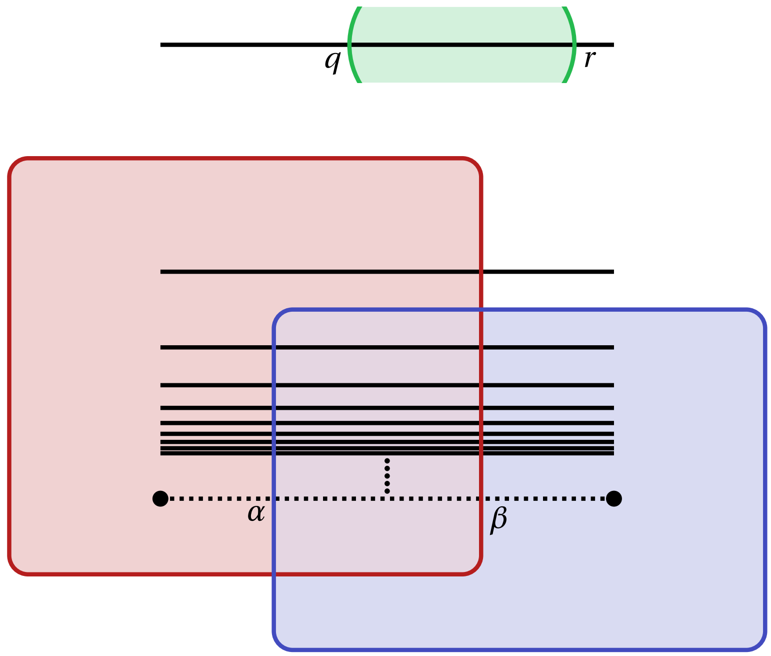

Figure III.1: Basic open subsets of X.

so X is not Hausdorff. To show that X is compact, observe that for every basic open cover $\mathcal{U}$, there exists $n \in \mathbb{N}^+$ and $U_0, U_1 \in \mathcal{U}$ such that $X \setminus ([0,1] \times (1/n, 1]) \subseteq U_0 \cup U_1$, and since the subspace topology on $X \cap ([0,1] \times (1/n, 1]) = [0,1] \times \{1/k : k \in \mathbb{N}^+, k \leq n\}$ is also induced by the Euclidean metric on $\mathbb{R}^2$, it must be compact.

Now note that $\mathcal{B}_2$ is serial. Also, for all $m, n \in \mathbb{N}^+$ with $m \leq n$,

$$\begin{aligned}\overline{B_{0,m} \setminus B_{1,n}} \cap \overline{B_{1,n} \setminus B_{0,m}} = &\left[\{(0,0)\} \cup \left([0,\alpha] \times \left\{\frac{1}{k} : k \in \mathbb{N}^+, k \geq n\right\}\right)\right. \\ &\left.\cup \left([0,\beta] \times \left\{\frac{1}{k} : k \in \mathbb{N}^+, m \leq k < n\right\}\right)\right] \\ &\cap \left[\{(1,0)\} \cup \left([\beta,1] \times \left\{\frac{1}{k} : k \in \mathbb{N}^+, k \geq n\right\}\right)\right] \\ = &\varnothing\end{aligned}$$

Thus by fixing $U \in \mathcal{B}_0$ and $V \in \mathcal{B}_2$, we only need to check that $\overline{U \setminus V} \cap \overline{V \setminus U} = \varnothing$. By symmetry, this will imply that this condition holds also when $U \in \mathcal{B}_1$ and hence that $\mathcal{B}$ is serial.

Let $q \in \mathbb{Q}\backslash\{0\}, r \in \mathbb{R}\backslash\mathbb{Q}, m, n \in \mathbb{N}^+$ be given and define

$$V := ((q,r) \cap [0,1]) \times \left\{\frac{1}{n}\right\}.$$

Without loss of generality, assume $V \not\subseteq B_{0,m}$ and note that

$$\overline{B_{0,m}} \subseteq \{(0,0),(1,0)\} \cup \left([0,\beta] \times \left\{\frac{1}{k} : k \in \mathbb{N}^+, k \geq m\right\}\right),$$
$$\overline{V} \subseteq ([q,r] \cap [0,1]) \times \left\{\frac{1}{n}\right\}.$$

Therefore if either $m > n$ or $q > \beta$ then $\overline{B_{0,m}} \cap \overline{V} = \emptyset$, so assume $m \leq n$ and $q \leq \beta$ and thus $q < \beta$. As $V \not\subseteq B_{0,m}$, it follows that $r > \beta$ and hence

$$\begin{aligned}\overline{B_{0,m}\backslash V} \cap \overline{V\backslash B_{0,m}} &\subseteq \left(\overline{[0,\beta)\backslash(q,r)} \cap \overline{((q,r)\cap[0,1])\backslash[0,\beta)}\right) \times \left\{\frac{1}{n}\right\} \\ &\subseteq ([0,q] \cap [\beta,1]) \times \left\{\frac{1}{n}\right\} \\ &= \emptyset.\end{aligned}$$

Therefore $\mathcal{B}$ is serial. □

We now characterise the radiality of points in subspaces of $X \subseteq \mathsf{Spec}(R)$. It will prove useful to limit the choices of $A \subseteq R$ witnessing closed subsets $C = V(A) \cap X$ of X. Recall that for every ideal $I \lhd R$, there is a natural bijection between

$$\{J \lhd R : I \subseteq J\} \longleftrightarrow \{J \lhd R/I\} \tag{$*$}$$

given by sending $J \lhd R$ with $I \subseteq J$ to J/I. Furthermore, this correspondence restricts to a bijection between $V^R(I)$ and $\mathsf{Spec}(R/I)$ and also to a bijection between $V^R(I) \cap \mathsf{Max}(R)$ and $\mathsf{Max}(R/I)$.

Definition III.3.5 (X-radical)**.** Let $X \subseteq \mathsf{Spec}(R)$ and $A \subseteq R$ be given. Then we define the *X-radical* of A to be $r_X(A) := \bigcap(V(A) \cap X)$. As an intersection of ideals is again an ideal, $r_X(A) \lhd R$. Of course, if this intersection is empty then we take $r_X(A)$ to be R. We denote $r_{\mathsf{Spec}(R)}(A)$ by $r(A)$ and $r_{\mathsf{Max}(R)}(A)$ by $r_{\mathfrak{M}}(A)$.

We say that an ideal $I \lhd R$ is *X-radical* if $I = r_X(I)$.

Proposition III.3.6. *Let $A \subseteq R, X \subseteq \mathsf{Spec}(R)$ be given. Then:*

(1) $V(A) \cap X = V(r_X(A)) \cap X$.

(2) $r_X(A)$ *is X-radical.*

Proof. First, note that $A \subseteq r_X(A)$ for all $A \subseteq R$ and by definition of $r_X(A)$, an ideal $\mathbf{p} \in X$ extends A if and only if it extends $r_X(A)$. Therefore $V(A) \cap X = V(r_X(A)) \cap X$ and $r_X(r_X(A)) = r_X(A)$. □

We can characterise the radicals and max-radicals of ideals using $(*)$ and the *nilradical* and *Jacobson radical* of rings, defined below.

Definition III.3.7 (Nilpotent and nilradical)**.** Let R be a ring and $n \in R$ be given. Then we say that n is *nilpotent* if there exists an $m \in \mathbb{N}$ such that $n^m = 0_R$. We define the *nilradical* of R to be the set of nilpotent elements of R and denote it by $\mathfrak{N}(R)$.

Definition III.3.8 (Jacobson radical)**.** Let R be a ring and define $\mathfrak{R}(R)$ to be the intersection of all maximal ideals of R. We call $\mathfrak{R}(R)$ the *Jacobson radical* of R. Observe that $\mathfrak{R}(R) = r_{\mathfrak{M}}(\{0\})$.

Proposition III.3.9 ([AM69, Proposition 1.8, pg. 5])**.** *Let R be a ring. Then* $\mathfrak{N}(R) = r(\{0\})$.

Proposition III.3.10 ([AM69, Proposition 1.9, pg. 6])**.** *Let R be a ring. Then*

$$\mathfrak{R}(R) = \{r \in R : \forall s \in R, 1 - rs \in R^\times\}.$$

Corollary III.3.11. *Let R be a ring and let $I \lhd R$ be given. Then:*

(1) $r(I) = \{r \in R : r^n \in I$ *for some* $n \in \mathbb{N}\}$*. [AM69, Chapter 1, Proposition 1.14]*

(2) $r_{\mathfrak{M}}(I) = \{r \in R : \forall s \in R, \exists t \in R$ *such that* $(1 - rs)t - 1 \in I\}$.

Proof. By $(*)$ and the two previous propositions,

$$\begin{aligned}
r(I) &= \{r \in R : r + I \in \mathfrak{N}(R/I)\} \\
&= \{r \in R : (r + I)^n = r^n + I = I \text{ for some } n \in \mathbb{N}\} \\
&= \{r \in R : r^n \in I \text{ for some } n \in \mathbb{N}\}, \\
r_{\mathfrak{M}}(I) &= \{r \in R : r + I \in \mathfrak{R}(R/I)\} \\
&= \{r \in R : \forall s \in R, \exists t \in R \text{ such that } ((1 + I) - (r + I)(s + I))(t + I) = 1 + I\} \\
&= \{r \in R : \forall s \in R, \exists t \in R \text{ such that } (1 - rs)t - 1 \in I\}.
\end{aligned}$$

□

We now translate the characterisation of radiality given in Theorem III.1.7 into ring-theoretic language. First, we need to describe the neighbourhood core and spokes of points in a subspace of $\mathsf{Spec}(R)$.

Lemma III.3.12. *Let* $X \subseteq \mathsf{Spec}(X), \mathbf{p} \in X$ *be given. Then*

$$N_{\mathbf{p}}^X = \{\mathbf{q} \in X : \mathbf{q} \subseteq \mathbf{p}\}.$$

Proof. By definition,

$$\begin{aligned}
N_{\mathbf{p}}^{\mathsf{Spec}(R)} &= \bigcap \mathcal{N}_{\mathbf{p}}^{\mathsf{Spec}(R)} \\
&= \{\mathbf{q} \in \mathsf{Spec}(R) : \forall U \in \mathcal{N}_{\mathbf{p}}^{\mathsf{Spec}(R)}, \mathbf{q} \in U\} \\
&= \{\mathbf{q} \in \mathsf{Spec}(R) : \mathbf{p} \in \overline{\mathbf{q}}^{\mathsf{Spec}(R)}\}.
\end{aligned}$$

By [AM69, Chapter 1, Exercise 18], $\overline{\mathbf{q}}^{\mathsf{Spec}(R)} = V(\mathbf{q})$ for all $\mathbf{q} \in \mathsf{Spec}(R)$, so it follows that $N_{\mathbf{p}}^{\mathsf{Spec}(R)} = \{\mathbf{q} \in \mathsf{Spec}(R) : \mathbf{q} \subseteq \mathbf{p}\}$. We conclude this proof by applying Lemma I.3.9. □

Thus in our notation, the T_1-points of prime spectra are the minimal prime ideals.

Lemma III.3.13. *Let X be a topological space, $x \in X$ be well-based and let $\mathcal{B}$ be a neighbourhood base for x. Then there exists a well-ordered neighbourhood base $(B_\alpha)_{\alpha<\lambda}$ of x contained in $\mathcal{B}$.*

Proof. As x is well-based, by Lemma I.3.23 and Theorem I.3.24, there exists a well-ordered neighbourhood base $(A_\alpha)_{\alpha<\lambda}$ of x such that λ is regular and $p(x) \geq \lambda$.

Let $\gamma < \lambda$ be given and let $(\alpha_\beta)_{\beta<\gamma}$ be a strictly increasing sequence in λ and let $(B_\beta)_{\beta<\gamma}$ be a transfinite sequence in $\mathcal{B}$ such that:

- For all $\beta < \gamma, B_\beta \subseteq A_{\alpha_\beta}$.
- For all $\beta < \gamma$ with $\beta + 1 < \gamma, A_{\alpha_{\beta+1}} \subseteq B_\beta$.

Then since $p(x) \geq \lambda$, there exists a $B_\gamma \in \mathcal{B}$ such that $B_\gamma \subseteq \bigcap_{\beta<\gamma} A_{\alpha_\beta}$ and hence by regularity there exists an $\alpha_\gamma < \lambda$ such that $A_{\alpha_\gamma} \subseteq B_\beta$ and $\alpha_\gamma > \sup(\alpha_\beta : \beta < \gamma)$. Therefore by recursion, there exists a transfinite sequence $(B_\alpha)_{\alpha<\lambda}$ in $\mathcal{B}$ such that $B_\beta \subseteq B_\alpha$ for all $\alpha < \beta < \lambda$

and for all $\alpha < \lambda$, there exists a $\beta < \lambda$ such that $B_\beta \subseteq A_\alpha$. Thus $(B_\alpha)_{\alpha<\lambda}$ is a well-ordered neighbourhood base of x. $\square$

Lemma III.3.14. *Let $X \subseteq \mathsf{Spec}(R), \mathbf{p} \in X$ be given and let I be an ideal of R contained in $\mathbf{p}$. Then $V(I) \cap X$ is a well-based at $\mathbf{p}$ if and only if there exists a transfinite sequence $(x_\alpha)_{\alpha<\lambda}$ in $R \backslash \mathbf{p}$ such that:*

(1) For all $\alpha < \beta < \lambda, x_\beta \in r_X(I \cup \{x_\alpha\})$.

(2) For every $y \in R \backslash \mathbf{p}$, there exists a $\gamma < \lambda$ such that $x_\gamma \in r_X(I \cup \{y\})$.

Proof. Firstly, we note that for $x, y \in R$, if $x \in r_X(I \cup \{y\})$ then $r_X(I \cup \{x\}) \subseteq r_X(I \cup \{y\})$. Suppose we have a transfinite sequence $(x_\alpha)_{\alpha<\lambda}$ with these properties and define for all $\alpha < \lambda, U_\alpha := (V(I) \cap X) \backslash V(x_\alpha)$, which is a neighbourhood of $\mathbf{p}$ with respect to $V(I) \cap X$. Let $\alpha < \beta < \lambda$ be given, so in particular $r_X(I \cup \{x_\beta\}) \subseteq r_X(I \cup \{x_\alpha\})$, and hence by Proposition III.3.6

$$\begin{aligned}
(V(I) \cap X) \cap V(x_\alpha) &= V(I \cup \{x_\alpha\}) \cap X \\
&= V(r_X(I \cup \{x_\alpha\})) \cap X \\
&\subseteq V(r_X(I \cup \{x_\beta\})) \cap X \\
&= V(I \cup \{x_\beta\}) \cap X \\
&= (V(I) \cap X) \cap V(x_\beta).
\end{aligned}$$

Therefore $U_\beta \subseteq U_\alpha$.

Now let $W \subseteq X$ be an open neighbourhood of $\mathbf{p}$, so there exists $y \in R\backslash\mathbf{p}$ such that $X\backslash V(y) \subseteq W$. Then there exists $\gamma < \lambda$ such that $r_X(I \cup \{x_\gamma\}) \subseteq r_X(I \cup \{y\})$ and hence

$$\begin{aligned} V(I) \cap V(y) \cap X &= V(I \cup \{y\}) \cap X \\ &= V(r_X(I \cup \{y\})) \cap X \\ &\subseteq V(r_X(I \cup \{x_\gamma\})) \cap X \\ &= V(I \cup \{x_\gamma\}) \cap X \\ &= V(I) \cap V(x_\gamma) \cap X \\ \Rightarrow U_\gamma &= (V(I) \cap X)\backslash V(x_\gamma) \\ &\subseteq (V(I) \cap X)\backslash V(y) \\ &\subseteq V(I) \cap X \cap W. \end{aligned}$$

Therefore $\{U_\gamma : \gamma < \lambda\}$ is a well-ordered neighbourhood base for $\mathbf{p}$ with respect to $V(I) \cap X$.

Now suppose that $\mathbf{p}$ is well-based in $V(I) \cap X$. Then by the previous lemma, since $\{(V(I) \cap X)\backslash V(x) : x \in R\backslash\mathbf{p}\}$ is a neighbourhood base of $\mathbf{p}$ with respect to $V(I) \cap X$, there exists a transfinite sequence $(x_\alpha)_{\alpha<\lambda}$ in $R\backslash\mathbf{p}$ such that $((V(I) \cap X)\backslash V(x_\alpha))_{\alpha<\lambda}$ is a well-ordered neighbourhood base of x with respect to $V(I) \cap X$. Moreover, for all $\alpha < \beta < \lambda$,

$$V(I \cup \{x_\alpha\}) \cap X = V(I) \cap V(x_\alpha) \cap X \subseteq V(I) \cap V(x_\beta) \cap X = V(I \cup \{x_\beta\}) \cap X$$

$$\Rightarrow x_\alpha \in r_X(I \cup \{x_\alpha\}) \subseteq r_X(I \cup \{x_\beta\}).$$

Finally, let $y \in R\backslash\mathbf{p}$ be given, so $(V(I) \cap X)\backslash V(y)$ is a neighbourhood of $\mathbf{p}$ with respect to

$V(I) \cap X$. Then there exists a $\alpha < \lambda$ such that

$$(V(I) \cap X) \backslash V(x_\alpha) \subseteq (V(I) \cap X) \backslash V(y)$$

$$\Rightarrow V(I \cup \{y\}) \cap X = V(I) \cap V(\{y\}) \cap X \subseteq V(I) \cap V(x_\alpha) \cap X = V(I \cup \{x_\alpha\}) \cap X$$

$$\Rightarrow x_\alpha \in r_X(I \cup \{x_\alpha\}) \subseteq r_X(I \cup \{y\}). \qquad \square$$

Thus, if $\mathbf{p} \in X \subseteq \mathsf{Spec}(R)$ and $L \lhd R$ is such that $V(L) \cap X \in \overline{\mathbf{Sp}}_X(\mathbf{p})$, then we can view L as 'lineariser' of $\mathbf{p}$ in the sense that $\mathbf{p}/L$ has a linear structure (or rather $(R/L)\backslash(\mathbf{p}/L)$ has such a structure, given by the previous lemma) in R/L.

Definition III.3.15 (Lineariser)**.** Let R be a ring and let $X \subseteq \mathsf{Spec}(R), \mathbf{p} \in X$ be given. An X-radical ideal L contained in $\mathbf{p}$ is called an *X-lineariser of* $\mathbf{p}$ if $V(L) \cap X$ is a spoke of $\mathbf{p}$. Note that by Lemma III.3.12, we must have $L \subseteq \mathbf{q}$ for every $\mathbf{q} \in X$ with $\mathbf{q} \subseteq \mathbf{p}$. We denote the collection of X-linearisers of $\mathbf{p}$ by $\mathbf{Lin}_X(\mathbf{p})$.

If $X = \mathsf{Spec}(R)$ we refer to X-linearisers as *prime linearisers* and if $X = \mathsf{Max}(R)$ then we call them *maximal linearisers.* We write $\mathbf{Lin}(\mathbf{p}), \mathbf{Lin}_{\mathfrak{M}}(\mathbf{p})$ for $\mathbf{Lin}_{\mathsf{Spec}(R)}(\mathbf{p}), \mathbf{Lin}_{\mathsf{Max}(R)}(\mathbf{p})$ respectively.

Observe that by Proposition III.3.6, if S is a closed spoke of $\mathbf{p} \in X \subseteq \mathsf{Spec}(R)$ with respect to X, there is a unique lineariser L of $\mathbf{p}$ with respect to X such that $S = V(L) \cap X$.

To utilise the 'cofinal' characterisation of radiality, we also need to translate the locally-contained ordering:

Lemma III.3.16. *Let $X \subseteq \mathsf{Spec}(R), \mathbf{p} \in X, I, J \lhd R$ be given. Then $V(I) \cap X \subseteq^X_{\mathbf{p}} V(J) \cap X$ if and only if there exists an $x \in R \backslash \mathbf{p}$ such that $r_X(xJ) \subseteq r_X(I)$.*[10]

---

[10] $xJ := \{xj : j \in J\} \lhd R$.

Proof. Suppose $V(I) \cap X \subseteq_{\mathbf{p}}^{X} V(J) \cap X$, so there exists an $x \in R \backslash \mathbf{p}$ such that

$$(V(I) \cap X) \backslash V(x) \subseteq V(J) \cap X$$

$$\Rightarrow V(I) \cap X \subseteq (V(J) \cup V(x)) \cap X = V(xJ) \cap X$$

$$\Rightarrow r_X(xJ) \subseteq r_X(I).$$

Now suppose there exists an $x \in R \backslash \mathbf{p}$ such that $r_X(xJ) \subseteq r_X(I)$. Then by Proposition III.3.6,

$$(V(J) \cup V(x)) \cap X = V(xJ) \cap X = V(r_X(xJ)) \cap X \supseteq V(r_X(I)) \cap X = V(I) \cap X$$

$$\Rightarrow (V(I) \cap X) \backslash V(x) \subseteq (V(J) \cap X) \backslash V(x).$$

Since $X \backslash V(x) \in \mathcal{N}_{\mathbf{p}}^{X}$, it follows that $V(I) \cap X \subseteq_{\mathbf{p}}^{X} V(J) \cap X$. $\square$

Definition III.3.17. Let R be a ring and let $X \subseteq \mathsf{Spec}(R), \mathbf{p} \in X$ be given. Let $I, J \lhd R$ be X-radical. Then we write $I \sqsubseteq_{\mathbf{p}}^{X} J$ if and only if there exists an $x \in R \backslash \mathbf{p}$ such that $xJ \subseteq r_X(I)$. Since I is X-radical, by the previous lemma this is equivalent to $V(I) \cap X \subseteq_{\mathbf{p}}^{X} V(J) \cap X$.

If $\mathcal{L}$ is a collection of X-linearisers of $\mathbf{p}$ ideals, then we say that $\mathcal{L}$ is *X-cofinal at* $\mathbf{p}$ if for every X-lineariser K of $\mathbf{p}$, there exists an $L \in \mathcal{L}$ such that $K \sqsubseteq_{\mathbf{p}}^{X} L$; that is, $\mathcal{L}$ is cofinal in $(\mathbf{Lin}_X(\mathbf{p}), \sqsubseteq_{\mathbf{p}}^{X})$.

Thus by the previous lemma and the observation preceding it, the following is an order-isomorphism:

$$(\mathbf{Lin}_X(\mathbf{p}), \sqsubseteq_{\mathbf{p}}^{X}) \to (\overline{\mathbf{Sp}}_X(\mathbf{p}), \subseteq_{\mathbf{p}}^{X}), \tag{†}$$

$$L \mapsto V(L) \cap X.$$

Now we are in a position to translate Theorem III.1.7.

Theorem III.3.18. *Let R be a ring, $X \subseteq \mathsf{Spec}(R), \mathbf{p} \in X$ be given such that $\mathbf{p}$ is semi-regular in X. Then the following are equivalent:*

(1) $\mathbf{p}$ is a radial in X.

(2) There exists a non-empty collection $\mathcal{L}$ of X-linearisers of $\mathbf{p}$ such that for every selection $(x_L)_{L\in\mathcal{L}}$ from $R\backslash\mathbf{p}$, there is some $y \in R\backslash\mathbf{p}$ such that for every $\mathbf{q} \in X$, either $y \in \mathbf{q}$ or $x_L \notin \mathbf{q} \supseteq L$ for some $L \in \mathcal{L}$.

(3) For every selection $(x_L)_{L\in\mathbf{Lin}_X(\mathbf{p})}$ from $R\backslash\mathbf{p}$, there is some $y \in R\backslash\mathbf{p}$ such that for every ideal $\mathbf{q} \in X$, either $y \in \mathbf{q}$ or $x_L \notin \mathbf{q} \supseteq L$ for some $L \in \mathbf{Lin}_X(\mathbf{p})$.

(4) For every collection $\mathcal{L} \subseteq \mathbf{Lin}_X(\mathbf{p})$ that is X-cofinal at $\mathbf{p}$, there is some $x \in R\backslash\mathbf{p}$ such that for every $\mathbf{q} \in X$, either $x \in \mathbf{q}$ or $L \subseteq \mathbf{q}$ for some $L \in \mathcal{L}$.

Proof. Let $\mathcal{L}$ be a non-empty set of X-linearisers of $\mathbf{p}$ with the property stated in (2). We claim that $\mathfrak{S} := \{V(L) \cap X : L \in \mathcal{L}\}$ is a spoke system of $\mathbf{p}$ with respect to X. First, note that from Proposition III.3.6, if $L_1, L_2 \in \mathcal{L}$ are distinct then $V(L_1) \cap X \neq V(L_2) \cap X$. Select $(U_L)_{L\in\mathcal{L}}$ from $\mathcal{N}_{\mathbf{p}}^X$, so there exists a selection $(x_L)_{L\in\mathcal{L}}$ from $R\backslash\mathbf{p}$ such that $X \backslash V(x_L) \subseteq U_L$ for each $L \in \mathcal{L}$. Then there exists an $y \in R\backslash\mathbf{p}$ such that $X \subseteq V(y) \cup \bigcup_{L\in\mathcal{L}}((V(L) \cap X) \backslash V(x_L))$ and hence

$$\bigcup_{L\in\mathcal{L}} (U_L \cap (V(L) \cap X)) \supseteq \bigcup_{L\in\mathcal{L}} ((V(L) \cap X) \backslash V(x_L)) \supseteq X \backslash V(y) \in \mathcal{N}_{\mathbf{p}}^X.$$

Therefore $\mathfrak{S}$ is a closed spoke system of $\mathbf{p}$ with respect to X.

Now let $\mathfrak{T}$ be a closed spoke system of $\mathbf{p}$ with respect to X, so by Proposition III.3.6 there exists a selection $(L_T)_{T\in\mathfrak{T}}$ from $\mathbf{Lin}_X(\mathbf{p})$ such that $T = V(L_T) \cap X$ for every $T \in \mathfrak{T}$.

$$\begin{tikzcd}
X \arrow[r, hook, "\beta"] \arrow[dr, "f"'] & \beta X \arrow[d, dashed, "!"', "\beta f"] \\
& Y
\end{tikzcd}$$

Figure III.2: The universal property of the Stone-Čech compactification.

Observe that for distinct $T_1, T_2 \in \mathfrak{T}, L_{T_1} \neq L_{T_2}$. Define $\mathcal{L} := \{L_T : T \in \mathfrak{T}\}$ and select $(x_T)_{T\in\mathfrak{T}}$ from $R\backslash\mathbf{p}$. Then $(T\backslash V(x_T))_{T\in\mathfrak{T}}$ is a selection from $(\mathcal{N}_{\mathbf{p}}^{T})_{T\in\mathfrak{T}}$, so $U := \bigcup_{T\in\mathfrak{T}}(T\backslash V(x_T)) \in \mathcal{N}_{\mathbf{p}}^{X}$. Thus there exists an $y \in R\backslash\mathbf{p}$ such that $X\backslash V(y) \subseteq U$ and hence $X \subseteq V(y) \cup U$. Therefore $\mathcal{L}$ has the property given in (2).

Applying Theorem III.1.7 to the translations above and by the definition of lineariser, we see that (1–3) are equivalent. Finally, by Theorem III.1.7 and the isomorphism (†), (1) and (4) are equivalent. □

Rings of continuous functions

Recall that the *Stone-Čech compactification* of a Tychonoff space X is a compact Hausdorff space βX and a dense embedding $\beta : X \to \beta X$ with the property that if Y is a compact Hausdorff space and $f : X \to Y$ is continuous, then there exists a unique continuous map $\beta f : \beta X \to Y$ such that $\beta f \circ \beta = f$. Usually we think of X as a subspace of βX and β as the inclusion map.

There are several approaches to proving the existence of the Stone-Čech compactification. One is to take the supremum over all compactifications [Eng89, Theorem 3.5.9, pg. 169]. However, we can also construct it explicitly — this is the approach taken in [GJ60, 6.5, pg. 86–8] and [GJ60, 7.10–11, pg. 105–6]. The former constructs βX using *z-ultrafilters*, whilst the latter essentially shows that $\beta X \cong \mathsf{Max}(C^*(X))$, where $C^*(X)$ is the ring of bounded, continuous real-valued functions on X with pointwise algebra operations. The embedding is defined as

$$\begin{aligned} \beta : X &\to \mathsf{Max}(C^*(X)), \\ x &\mapsto \{f \in C^*(X) : f(x) = 0\}. \end{aligned}$$

Moreover, $\mathsf{Max}(C^*(X)) \cong \mathsf{Max}(C^*(\beta X))$; the isomorphism is given as

$$\begin{aligned} \mathsf{Max}(C^*(X)) &\to \mathsf{Max}(C^*(\beta X)), \\ \mathbf{m} &\mapsto \{\beta f : f \in \mathbf{m}\}. \end{aligned}$$

In fact, the maximal ideals of $C^*(X)$ are of the form $\mathbf{m}_p := \{f \in C^*(X) : (\beta f)(p) = 0\}$, where $p \in \beta X$ [GJ60, Theorem 7.2, pg. 101].

By applying the previous theorem to the ring $C^*(X)$, we obtain an algebraic characterisation of radial points in βX. Of course, $\beta X \cong \mathsf{Max}(C(X))$ also [GJ60, 7.11, pg. 105] but here we only focus on $C^*(X)$ to simplify matters. There are several descriptions of the max-radical of ideals; we quote a few below.

Theorem III.3.19. *[GJ60] Let X be a Tychonoff space, $I \lhd C^*(X)$ be given. Then the following are equivalent:*

(1) I is an intersection of maximal ideals, i.e. I is max-radical.

(2) *I is closed in $C_u^*(X)$, the space $C^*(X)$ endowed with the uniform norm topology.*

(3) *I is an* e-ideal*; that is, given $f \in C^*(X)$, if for every $\varepsilon > 0$ there exists a $g \in I$ and $\delta > 0$ such that $f^{-1}[[-\varepsilon, \varepsilon]] = g^{-1}[[-\delta, \delta]]$, then $f \in I$.*

Proof.

(1) ⇒ (3): By [GJ60, Problem 2L.4 & 2L.9, pg. 33], every maximal ideal is an e-ideal and every intersection of e-ideals is again an e-ideal.

(2) ⇒ (1): [GJ60, Problem 6A.2, pg. 93].

(3) ⇒ (2): [GJ60, Problem 2M.5, pg. 34]. □

Since the max-radical of an ideal is the smallest max-radical ideal containing it, then, for example, the max-radical is the closure in $C_u^*(X)$.

In future research, we hope to use similar ideas for other rings of continuous functions to investigate, for example, the ideal of compactifications of X (see e.g. Questions III.2.37 & III.2.41).

III.3.2 Stone duality

We now focus our attention on Stone spaces and Boolean algebras. A *Stone space* is a compact Hausdorff space with a base of clopen sets. Before discussing Stone spaces and Stone duality, we first need to recall some material on Boolean algebras. We refer to [CN74, Chapter 2] for the basic background on Boolean algebras. There are several equivalent definitions — we take the following as our formal definition:

Definition III.3.20 (Boolean algebra)**.** Let $\mathbb{B}$ be a non-empty set with a partial order $\leq$. Then we say that $(\mathbb{B}, \leq)$ is a *Boolean algebra* if the following properties hold:

- Every pair of elements $a, b \in \mathbb{B}$ has a supremum $a \vee b$ and infimum $a \wedge b$.
- There is a greatest element $\mathbb{1}$ and a least element $\mathbb{0}$.
- Every element $b \in \mathbb{B}$ has a (unique) *complement* $\neg b \in \mathbb{B}$, which has the defining property $b \vee \neg b = \mathbb{1}$ and $b \wedge \neg b = \mathbb{0}$.
- $\wedge$ and $\vee$ *distribute over each other*; that is, for every $a, b, c \in \mathbb{B}$,

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c),$$

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c).$$

An example of such a Boolean algebra is the powerset algebra $\mathcal{P}(X)$ of any set X, with the ordering given by $\subseteq$.

We can extend the binary suprema and infima to finitary suprema and infima, designated by $\bigwedge B$ and $\bigvee B$ for a finite set $B \subseteq \mathbb{B}$, and define the *symmetric difference* of $a, b \in \mathbb{B}$ to be

$$a \Delta b := (a \wedge \neg b) \vee (\neg a \wedge b).$$

We will usually refer to $\mathbb{B}$ as a Boolean algebra, leaving the order implicit.

Every Boolean algebra $\mathbb{B}$ is a commutative ring, with addition Δ, multiplication $\wedge$, zero element $\mathbb{0}$ and identity element $\mathbb{1}$ [AM69, Chapter 1, Problem 24, pg. 14]; with this structure, we refer to $\mathbb{B}$ as a *Boolean ring*. However, rather than discussing ideals on Boolean rings, it is more usual to discuss *filters* on Boolean algebras.

Definition III.3.21 (Filters and filter bases)**.** Let $\mathbb{B}$ be a Boolean algebra, $\mathcal{B} \subseteq \mathbb{B}$ be given. Then we say that $\mathcal{B}$ is a *filter-base on* $\mathbb{B}$ if it is non-empty and for every $x, y \in \mathcal{B}$, there exists a $z \in \mathcal{B}$ such that $z \leq x \wedge y$. Moreover, if for all $a \in \mathcal{B}$ and $b \in \mathbb{B}$, $a \leq b$ implies $b \in \mathcal{B}$, then we say that $\mathcal{B}$ is a *filter* on $\mathbb{B}$.

If $\mathcal{F}$ is a filter and $\mathbb{0} \notin \mathcal{F}$, then we say that $\mathcal{F}$ is a *proper filter* on $\mathbb{B}$; otherwise, $\mathcal{F} = \mathbb{B}$ and we call $\mathcal{F}$ the *improper filter*. We say that a proper filter $\mathcal{U}$ is an *ultrafilter* if it is not strictly contained in another proper filter.

Given a collection $\mathcal{B} \subseteq \mathbb{B}$, we define the *filter generated by* $\mathcal{B}$ to be the intersection of all filters that contain $\mathcal{B}$, which we denote by $\langle \mathcal{B} \rangle$. Note that $\langle \mathcal{B} \rangle$ is a proper filter precisely when $\mathcal{B}$ has the *finite-intersection property*: for every finite $B \subseteq \mathcal{B}, \bigwedge B \neq \mathbb{0}$. If $\mathcal{F}$ is a filter on $\mathbb{B}$ and $\mathcal{B} \subseteq \mathcal{F}$ is a filter-base such that $\mathcal{F} = \langle \mathcal{B} \rangle$, then we say that $\mathcal{B}$ is a *filter-base for* $\mathcal{F}$.

There is a natural correspondence between ideals in the Boolean algebra ring structure and filters. Given a subset $A \subseteq \mathbb{B}$, define $\neg[A] := \{\neg a : a \in A\}$. Then for any filter $\mathcal{F}$ and ideal I on $\mathbb{B}, \neg[\mathcal{F}]$ is an ideal and $\neg[I]$ is a filter. Moreover, if either $A \subseteq \mathbb{B}$ is a maximal ideal / ultrafilter, then $\neg[A]$ is an ultrafilter / maximal ideal. A simple consequence of this is that every prime ideal of $\mathbb{B}$ is maximal (see [CN74, Theorem 2.19, pg. 39]) and every filter is the intersection of all ultrafilters extending it. Thus every ideal is radical.

Definition III.3.22 (Quotient algebra)**.** Let $\mathbb{B}$ be a Boolean algebra, $\mathcal{F}$ be a filter on $\mathbb{B}$. Then we define the *quotient algebra* of $\mathbb{B}$ over $\mathcal{F}$, denoted by $\mathbb{B}/\mathcal{F}$, as follows: for all $a, b \in \mathcal{B}$, $a \sim_{\mathcal{F}} b$ if and only if there exists $c \in \mathcal{F}$ such that $a \wedge c = b \wedge c$. Then $\sim_{\mathcal{F}}$ is an equivalence relation on $\mathbb{B}$. Denote by $[a]_{\mathcal{F}}$ the $\sim_{\mathcal{F}}$-equivalence class of $a \in \mathbb{B}$ and let $\mathbb{B}/\mathcal{F}$ be the set of all $\sim_{\mathcal{F}}$-

equivalence classes. Define for all $a, b \in \mathbb{B}$, $[a]_{\mathcal{F}} \leq_{\mathcal{F}} [b]_{\mathcal{F}}$ if and only if there exists $c \in \mathcal{F}$ such that $a \wedge c \leq b$. Then $\mathbb{B}/\mathcal{F}$ is a Boolean algebra and the operations commute with taking equivalence classes:

- $\mathbb{1}_{\mathbb{B}/\mathcal{F}} = [\mathbb{1}_{\mathbb{B}}]_{\mathcal{F}} = \mathcal{F}$.
- $\mathbb{0}_{\mathbb{B}/\mathcal{F}} = [\mathbb{0}_{\mathbb{B}}]_{\mathcal{F}} = \neg\,[\mathcal{F}]$.
- For all $a, b \in \mathbb{B}$,

$$[a]_{\mathcal{F}} \wedge_{\mathbb{B}/\mathcal{F}} [b]_{\mathcal{F}} = [a \wedge_{\mathbb{B}} b]_{\mathcal{F}},$$

$$[a]_{\mathcal{F}} \vee_{\mathbb{B}/\mathcal{F}} [b]_{\mathcal{F}} = [a \vee_{\mathbb{B}} b]_{\mathcal{F}},$$

$$\neg_{\mathbb{B}/\mathcal{F}}[a]_{\mathcal{F}} = [\neg_{\mathbb{B}} a]_{\mathcal{F}}.$$

For each ultrafilter $\mathcal{U}$ on $\mathbb{B}$ that contains $\mathcal{F}$, $\mathcal{U}/\mathcal{F} := \pi_{\mathcal{F}}[\mathcal{U}]$ is an ultrafilter on $\mathbb{B}/\mathcal{F}$.

We are now ready to discuss Stone duality and radiality in Stone spaces. We refer to [AHS] for the basic background on categories.

Definition III.3.23 (Stone duality)**.** We define **BoolAlg** to be the category whose objects are Boolean algebras and whose arrows are Boolean algebra homomorphisms. We also define **Stone** to be the full subcategory of **Top**, the category of topological spaces, whose objects are Stone spaces and whose arrows are continuous maps.

For each Boolean algebra $\mathbb{B}$, we define $\mathsf{St}(\mathbb{B})$ to be the topological space whose domain consists of all ultrafilters on $\mathbb{B}$ and whose basic (cl)open subsets are of the form

$$\{b\}\uparrow := \{\mathcal{U} \in \mathsf{St}(\mathbb{B}) : b \in \mathcal{U}\}$$

for each $b \in \mathbb{B}$. By [CN74, Theorem 2.21, pg. 40] and [CN74, Lemma 2.24, pg. 43], $\mathsf{St}(\mathbb{B})$ is a Stone space.

For each topological space X, we define $\mathsf{Cl}(X)$ to be the Boolean algebra of clopen subsets of X, whose ordering is $\subseteq$ and suprema, infima and complement are given by $\cup, \cap$ and $X \backslash C$ for each $C \in \mathsf{Cl}(X)$ respectively. Moreover, $\mathbb{0}_{\mathsf{Cl}(X)} = \emptyset$ and $\mathbb{1}_{\mathsf{Cl}(X)} = X$.

By [CN74, Theorem 2.29, pg. 46], it is possible to extend the maps St and Cl to arrows of **BoolAlg** and **Stone** in such a way that $\mathsf{St} : \mathbf{BoolAlg} \to \mathbf{Stone}$ and $\mathsf{Cl} : \mathbf{Stone} \to \mathbf{BoolAlg}$ are contravariant functors that form a dual equivalence between **BoolAlg** and **Stone**. In particular, $\mathsf{St}(\mathsf{Cl}(X)) \cong X$ and $\mathsf{Cl}(\mathsf{St}(\mathbb{B})) \cong \mathbb{B}$ for all Stone spaces X and Boolean algebras $\mathbb{B}$.

We are finally in a position to investigate radiality in Stone spaces. By the definition of spectrum and Stone space of a Boolean algebra $\mathbb{B}$,

$$\neg\,[\cdot] : \mathsf{St}(\mathbb{B}) \to \mathsf{Spec}(\mathbb{B}), \mathcal{U} \mapsto \neg\,[\mathcal{U}]$$

is a homeomorphism. Using this, we can transfer over the concepts from Section III.3.1. For example, for every collection $\mathcal{B} \subseteq \mathbb{B}$, define

$$\mathcal{B}\!\uparrow\, := \{\mathcal{U} \in \mathsf{St}(\mathbb{B}) : \mathcal{B} \subseteq \mathcal{U}\}.$$

Then $\{\neg\,[\mathcal{U}] : \mathcal{U} \in \mathcal{B}\!\uparrow\} = V(\neg\,[\mathcal{B}])$. Furthermore,

$$\begin{aligned}
\neg\,[r(\mathcal{B})] &= \neg\left[\left(\bigcap\{\mathbf{p} : \mathbf{p} \in \mathsf{Spec}(\mathbb{B}), \mathcal{B} \subseteq \mathbf{p}\}\right)\right] \\
&= \bigcap\{\neg\,[\mathbf{p}] : \mathbf{p} \in \mathsf{Spec}(\mathbb{B}), \mathcal{B} \subseteq \mathbf{p}\} \\
&= \bigcap\{\mathcal{U} : \mathcal{U} \in \mathsf{St}(\mathbb{B}) : \neg\,[\mathcal{B}] \subseteq \mathcal{U}\} \\
&= \langle \neg\,[\mathcal{B}] \rangle.
\end{aligned}$$

Definition III.3.24 (Well-based)**.** Let $\mathcal{F}$ be a filter on a Boolean algebra $\mathbb{B}$. A *well-ordered filter base of* $\mathcal{F}$ is a transfinite sequence $(b_\alpha)_{\alpha<\lambda}$ in $\mathcal{F}$ such that $\{b_\alpha : \alpha < \lambda\}$ is a filter-base of $\mathcal{F}$ and $b_\beta < b_\alpha$ for all $\alpha < \beta < \lambda$. We say that the filter $\mathcal{F}$ is well-based if it has a well-ordered filter base. This generalises the previous definition, since a point x in a topological space X is well-based precisely when $\mathcal{N}_x$ is well-based in $\mathcal{P}(X)$.

Lemma III.3.25. *Let* $\mathbb{B}$ *be a Boolean algebra and let* $\mathcal{F}$ *be a filter on* $\mathbb{B}$, $\mathcal{U} \in \mathcal{F}\!\uparrow$ *be given. Then* $\mathcal{U}$ *is well-based in* $\mathcal{F}\!\uparrow$ *if and only if* $\mathcal{U}/\mathcal{F}$ *is well-based.*

Proof. First, note that $\mathcal{U}$ is well-based in $\mathcal{F}\!\uparrow$ if and only if $\neg[\mathcal{U}]$ is well-based in $V(\neg[\mathcal{F}])$. By Lemma III.3.14, this is in turn equivalent to the existence of a transfinite sequence $(x_\alpha)_{\alpha<\lambda}$ in $\mathbb{B}\backslash\neg[\mathcal{U}] = \mathcal{U}$ such that:

- For each $\alpha < \beta < \lambda$,

$$x_\beta \in r(\neg[\mathcal{F}] \cup \{x_\alpha\}) = r(\neg[\mathcal{F} \cup \{\neg x_\alpha\}]) = \neg[\langle \mathcal{F} \cup \{\neg x_\alpha\}\rangle].$$

- For all $y \in \mathbb{B}\backslash\neg[\mathcal{U}] = \mathcal{U}$, there exists an $\gamma < \lambda$ such that

$$x_\gamma \in r(\neg[\mathcal{F}] \cup \{y\}) = \neg\left[\langle \mathcal{F} \cup \{\neg y\}\rangle\right].$$

Given $y, z \in \mathbb{B}$,

$$\begin{aligned} y \in \neg[\langle \mathcal{F} \cup \{\neg z\}\rangle] &\iff \neg y \in \langle \mathcal{F} \cup \{\neg z\}\rangle \\ &\iff \exists w \in \mathcal{F} \text{ such that } \neg y \geq w \wedge \neg z \\ &\iff [\neg y]_{\mathcal{F}} \geq_{\mathcal{F}} [\neg z]_{\mathcal{F}} \\ &\iff [y]_{\mathcal{F}} \leq_{\mathcal{F}} [z]_{\mathcal{F}}. \end{aligned}$$

Therefore $\mathcal{U}$ is well-based in $\mathcal{F}\upharpoonleft$ if and only if there exists a transfinite sequence $(x_\alpha)_{\alpha<\lambda}$ in $\mathcal{U}$ such that:

- For all $\alpha < \beta < \lambda$, $[x_\beta]_\mathcal{F} \le_\mathcal{F} [x_\alpha]_\mathcal{F}$.
- For all $y \in \mathcal{U}$, there exists a $\gamma < \lambda$ such that $[x_\gamma]_\mathcal{F} \le_\mathcal{F} [y]_\mathcal{F}$.

The latter is equivalent to $\mathcal{U}/\mathcal{F}$ being well-based in $\mathbb{B}/\mathcal{F}$. □

Definition III.3.26 (Lineariser)**.** Let $\mathbb{B}$ be a Boolean algebra, $\mathcal{F}$ be a filter on $\mathbb{B}$ and let $\mathcal{U} \in \mathcal{F}\upharpoonleft$ be given. Then we say that $\mathcal{F}$ is a *lineariser of* $\mathcal{U}$ if $\mathcal{U}/\mathcal{F}$ is well-based. We denote the set of linearisers of $\mathcal{U}$ be $\mathbf{Lin}(\mathcal{U})$.

Lemma III.3.27. *Let $\mathbb{B}$ be a Boolean algebra, $\mathcal{F}$ and $\mathcal{G}$ be filters on $\mathbb{B}$ and let $\mathcal{U} \in \mathsf{St}(\mathbb{B})$ be given. Then $\mathcal{F}\upharpoonleft \subseteq_\mathcal{U} \mathcal{G}\upharpoonleft$ if and only if there exists an $x \in \mathcal{U}$ such that $\mathcal{G} \subseteq \langle \mathcal{F} \cup \{x\}\rangle$.*

Proof. By Lemma III.3.16,

$$\mathcal{F}\upharpoonleft \subseteq_{\mathcal{U}}^{\mathsf{St}(\mathbb{B})} \mathcal{G}\upharpoonleft \iff V(\neg[\mathcal{F}]) \subseteq_{\neg[\mathcal{U}]}^{\mathsf{Spec}(\mathbb{B})} V(\neg[\mathcal{G}])$$

$$\iff \exists x \in \mathbb{B}\setminus\neg[\mathcal{U}] = \mathcal{U} \text{ such that } r(x\cdot\neg[\mathcal{G}]) \subseteq r(\neg[\mathcal{F}])$$

$$\iff \exists x \in \mathcal{U} \text{ such that } \{x \wedge \neg y : y \in \mathcal{G}\} \subseteq \neg[\mathcal{F}]$$

$$\iff \exists x \in \mathcal{U} \text{ such that } \{\neg x \vee y : y \in \mathcal{G}\} \subseteq \mathcal{F}$$

$$\iff \mathcal{G} \subseteq \langle \mathcal{F} \cup \{x\}\rangle.$$

For the last equivalence, let $y \in \mathbb{B}$ be given and suppose that $\neg x \vee y \in \mathcal{F}$. Then $x \wedge y = x \wedge (\neg x \vee y) \le y$, so $y \in \langle \mathcal{F} \cup \{x\}\rangle$. Now assume that there exists $z \in \mathcal{F}$ such that $z \wedge x \le y$.

Then $z = (z \wedge \neg x) \vee (z \wedge x) \leq \neg x \vee (z \wedge x) \leq \neg x \vee y$, so $\neg x \vee y \in \mathcal{F}$. Therefore the last two statements are equivalent. $\square$

Definition III.3.28. Let $\mathbb{B}$ be a Boolean algebra and let $\mathcal{U} \in \mathsf{St}(\mathbb{B}), \mathcal{L}, \mathcal{M} \in \mathbf{Lin}(\mathcal{U})$ be given. We define $\mathcal{L} \sqsubseteq_{\mathcal{U}} \mathcal{M}$ if and only if there exists an $x \in \mathcal{U}$ such that $\mathcal{M} \subseteq \langle \mathcal{L} \cup \{x\} \rangle$. This gives a quasi-order on $\mathbf{Lin}(\mathcal{U})$. We say that a collection $\mathfrak{L} \subseteq \mathbf{Lin}(\mathcal{U})$ is *cofinal* provided it is cofinal with respect to this ordering; that is, for every $\mathcal{M} \in \mathbf{Lin}(\mathcal{U})$, there exists an $\mathcal{L} \in \mathfrak{L}$ such that $\mathcal{M} \sqsubseteq_{\mathcal{U}} \mathcal{L}$.

Notice that for a filter $\mathcal{F}$ on a Boolean algebra $\mathbb{B}$ and an ultrafilter $\mathcal{U} \in \mathsf{St}(\mathbb{B}), \mathcal{F} \in \mathbf{Lin}(\mathcal{U})$ if and only if $\neg[\mathcal{F}] \in \mathbf{Lin}(\neg[\mathcal{U}])$. Thus we arrive at the characterisation theorem:

Theorem III.3.29. *Let $\mathbb{B}$ be a Boolean algebra, $\mathcal{U} \in \mathsf{St}(\mathbb{B})$ be given. Then the following are equivalent:*

(1) $\mathcal{U}$ is radial in $\mathsf{St}(\mathbb{B})$.

(2) There exists a non-empty collection of linearisers $\mathfrak{L}$ of $\mathcal{U}$ such that for every selection $(x_{\mathcal{L}})_{\mathcal{L} \in \mathfrak{L}}$ from $\mathcal{U}$, there is some $y \in \mathcal{U}$ such that every $\mathcal{V} \in \{y\}\uparrow$ contains $\mathcal{L} \cup \{x_{\mathcal{L}}\}$ for some $\mathcal{L} \in \mathfrak{L}$.

(3) For every selection $(x_{\mathcal{L}})_{\mathcal{L} \in \mathbf{Lin}(\mathcal{U})}$ from $\mathcal{U}$, there is some $y \in \mathcal{U}$ such that every $\mathcal{V} \in \{y\}\uparrow$ contains $\mathcal{L} \cup \{x_{\mathcal{L}}\} \subseteq \mathcal{V}$ for some $\mathcal{L} \in \mathbf{Lin}(\mathcal{U})$.

(4) For every cofinal $\mathfrak{L} \subseteq \mathbf{Lin}(\mathcal{U})$, there exists an $x \in \mathcal{U}$ such that every $\mathcal{V} \in \{x\}\uparrow$ contains some $\mathcal{L} \in \mathfrak{L}$.

Proof. This follows from Theorem III.3.18, noting that for every ultrafilter $\mathcal{U} \in \mathsf{St}(\mathbb{B})$, $\mathbb{B} \backslash \neg[\mathcal{U}] = \mathcal{U}$, so for all $x \in \mathbb{B}, x \in \mathcal{U}$ if and only if $x \notin \neg[\mathcal{U}]$. □

III.3.3 Radial points in ω^*

As a test case, we investigate radial points in $\omega^* := \beta\omega \backslash \omega$. It is known that $\beta\omega$ is not pseudoradial [Dow+96, pg. 191]; furthermore, the only radial points in $\beta\omega$ are the isolated points, since ω is dense in $\beta\omega$ yet there are no (transfinite) sequences in ω that converge to any point in ω^* because every convergent sequence in $\beta\omega$ is eventually constant [Eng89, Corollary 3.6.15, pg. 175].

Define $\iota : \omega \to \mathsf{St}(\mathsf{Cl}(\omega)) = \mathsf{St}(\mathcal{P}(\omega)), n \mapsto \{A \subseteq \omega : n \in A\}$. Then by [CN74, Lemma 2.23, pg. 42], $\beta\iota : \beta\omega \to \mathsf{St}(\mathcal{P}(\omega))$ is a homeomorphism. We identify $\beta\omega$ with $\mathsf{St}(\mathcal{P}(\omega))$ via this homeomorphism. Since $\operatorname{ran}(\iota)$ consists of all the fixed ultrafilters on ω, ω^* will then consist of all free ultrafilters on ω.[11] By [CN74, Lemma 7.2, pg. 144] and the discussion preceding it, an ultrafilter on ω is free if and only if it contains $\mathfrak{Fr} := \langle \omega \backslash F : F \in [\omega]^{<\aleph_0} \rangle$, which we call the *Fréchet filter on* ω. Hence $\omega^* = \mathfrak{Fr}\uparrow$.

Proposition III.3.30. *Let $X \subseteq \omega^*, x \in X$ be radial in X. Then x is a p-point in X.*

Proof. If x is isolated in X then $p(x) = \infty$, so x is a p-point. Otherwise, by Theorem II.2.9, x is $\Sigma_X^{\mathrm{rad}}(x)$-radial and since there are no non-trivial convergent sequences in ω^*, it follows that $\omega \notin \Sigma^{\mathrm{rad}}(x)$. Hence by Theorem II.2.10, $p_X(x) = \min(\Sigma_X^{\mathrm{rad}}(x))$ is uncountable, i.e. x is a p-point in X. □

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[11] We say that $\mathcal{U}$ is a *filter on a set* X if it is a filter on $\mathcal{P}(X)$. Similarly, an ultrafilter on X is an ultrafilter of $\mathcal{P}(X)$.

p-points exist under the Continuum Hypothesis [Rud56, Theorem 4.2, pg. 415] and it is possible (assuming $\mathfrak{t} = \mathfrak{c}$, which follows from Martin's Axiom) to construct well-based points in ω^* with character $\mathfrak{c}$, by adapting Walter Rudin's construction. Thus it is consistent that radial points exist in ω^*. However, Saharon Shelah has proven that it is consistent with ZFC that there are no p-points and thus no radial points in ω^* [Wim82, Theorem 6.5, pg. 47]. So the existence of radial points in ω^* is independent of ZFC. However, in these models, the radial points constructed are all well-based, which raises the following question:

Question III.3.31. *Is it consistent with ZFC that there exists a radial, non-well-based point in ω^*?*

The simplest method of constructing such a point is to have a spoke system consisting of two spokes with different characters. Andreas Blass, Heike Mildenberger and Saharon Shelah claim to have proven that it is consistent there are well-based points in ω^* of different characters — in particular, with characters $\aleph_1$ and $\aleph_2$.[12] There may be a way to combine these two ultrafilters to produce a radial, non-well-based point in ω^*.

Question III.3.32. *If there exists well-based points in ω^* with different characters, does there exist a radial, non-well-based point in ω^*?*

We describe one such attempt, using the *product* of ultrafilters.

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[12] A gap was identified in Andreas Blass and Saharon Shelah's proof [BS87, Theorem 6.1, pg. 239] by Alan Dow (credited in [BMS15, pg. 2]). The authors, together with Heike Mildenberger, have released a preprint [BMS15] claiming to have fixed the flaw.

Definition III.3.33 (Filter product)**.** Let $\mathcal{F}, \mathcal{G}$ be two filters on ω. Then we define the *(Fubini) product of $\mathcal{F}$ and $\mathcal{G}$* to be

$$\mathcal{F} \otimes \mathcal{G} := \{A \subseteq \omega \times \omega : \{m \in \omega : \{n \in \omega : (m,n) \in A\} \in \mathcal{G}\} \in \mathcal{F}\}.$$

A base of $\mathcal{F} \otimes \mathcal{G}$ is given by sets of the form

$$\bigcup_{m \in F} (\{m\} \times G_m)$$

where $F \in \mathcal{F}$ and $(G_m)_{m \in F}$ is a selection from $\mathcal{G}$.

By [CN74, Lemmas 7.20 & 7.21, pg. 156–157], the filter product of filters is always a filter on $\omega \times \omega$ and if $\mathcal{U}, \mathcal{V}$ are ultrafilters on ω then $\mathcal{U} \otimes \mathcal{V}$ is an ultrafilter on $\omega \times \omega$. Since $\omega \times \omega \cong \omega$, it follows that $(\omega \times \omega)^* \cong \omega^*$, so we work with the former instead. Of course, if we are to construct a radial point, it needs to be a p-point.

Definition III.3.34 (p-filter)**.** Let $\mathcal{F}$ be a filter on a set X. Then we say that $\mathcal{F}$ is a *p-filter* if for every countable $\mathcal{C} \subseteq \mathcal{F}$, there exists an $F \in \mathcal{F}$ such that $F \subseteq^* C$ for each $C \in \mathcal{C}$.

Thus if $\mathcal{F}$ is a free filter on ω then it is a p-filter if and only if $\mathcal{F}/\mathfrak{Fr}$ is *countably-closed*, that is for every countable $\mathcal{C} \subseteq \mathcal{F}/\mathfrak{Fr}$, there exists an $F \in \mathcal{F}/\mathfrak{Fr}$ such that $F \leq_{\mathfrak{Fr}} C$ for each $C \in \mathcal{C}$. Consequently, a free ultrafilter on ω is a p-filter precisely when it is a p-point in ω^*.

Lemma III.3.35. *Let $\mathcal{F}, \mathcal{G}$ be filters on ω.*

(1) If $\mathcal{F}$ is free then $\mathcal{F} \otimes \mathcal{G}$ is not a p-filter.

(2) If $\mathcal{F}, \mathcal{G}$ are free p-filters, then $(\mathcal{F} \otimes \mathcal{G})/\mathfrak{Fr}^{\otimes 2}$ is countably closed.

Proof.

(1) Assume $\mathcal{F}$ is free, so $(\omega \backslash n) \times \omega \in \mathcal{F} \otimes \mathcal{G}$ for each $n < \omega$. Let $F \in \mathcal{F}$ be given and select $(G_m)_{m \in F}$ from $\mathcal{G}$. If $H := \bigcup_{m \in F}(\{m\} \times G_m) \subseteq^* (\omega \backslash n) \times \omega$ then $F \cap n = \emptyset$. Thus there exists an $n < \omega$ such that $H \not\subseteq^* (\omega \backslash n) \times \omega$, so $\mathcal{F} \otimes \mathcal{G}$ is not a p-filter.

(2) Assume $\mathcal{F}, \mathcal{G}$ are free p-filters, so $\mathfrak{Fr}^{\otimes 2} \subseteq \mathcal{F} \otimes \mathcal{G}$. Let $\mathcal{H} \subseteq \mathcal{F} \otimes \mathcal{G}$ be countable and let $H \in \mathcal{H}$ be given, so there exists an $F_H \in \mathcal{F}$ and a selection $(G_{H,m})_{m \in F_H}$ from $\mathcal{G}$ such that

$$\bigcup_{m \in F_k} (\{m\} \times G_{H,m}) \subseteq H.$$

As $\mathcal{F}$ and $\mathcal{G}$ are p-filters, there exists an $F \in \mathcal{F}$ and a $G \in \mathcal{G}$ such that $F \subseteq^* F_H$ for each $H \in \mathcal{H}$ and $G \subseteq^* G_{H,m}$ for each $H \in \mathcal{H}$ and $m \in F_H$.

Let $H \in \mathcal{H}$ be given and define

$$C_H := ((\omega \backslash F) \times \omega) \cup \bigcup_{m \in F \cap F_H} (\{m\} \times (\omega \backslash (G \backslash G_{H,m}))) \in \mathfrak{Fr}^{\otimes 2}.$$

Then $(F \times G) \cap C_H = \bigcup_{m \in F \cap F_H}(\{m\} \times (G \cap G_{H,m})) \subseteq H$ and thus $[F \times G]_{\mathfrak{Fr}^{\otimes 2}} \leq [H]_{\mathfrak{Fr}^{\otimes 2}}$. Therefore $(\mathcal{F} \otimes \mathcal{G}) / \mathfrak{Fr}^{\otimes 2}$ is countably closed. □

So although we cannot create radial points in ω^* by this method, it seems like it might be possible for this to work in $\mathfrak{Fr}^{\otimes 2}\uparrow$. The next theorem shows this is impossible:

Theorem III.3.36. *Let $\mathcal{U}, \mathcal{V}$ be well-based points in ω^* with distinct characters. Then $(\mathcal{U} \otimes \mathcal{V}) / \mathfrak{Fr}^{\otimes 2}$ is a non-radial p-point in $\mathfrak{Fr}^{\otimes 2}\uparrow$.*

Proof. Define $\mathcal{D} := \mathfrak{Fr}^{\otimes 2}, \mathcal{W} := (\mathcal{U} \otimes \mathcal{V}) / \mathcal{D}$ and note that by the previous lemma, $\mathcal{W}$ is a p-point in $\mathcal{D}\uparrow$. Also define $\kappa := \chi(\mathcal{U}), \lambda := \chi(\mathcal{V})$ and let $([U_\alpha]_{\mathfrak{Fr}})_{\alpha < \kappa}, ([V_\beta]_{\mathfrak{Fr}})_{\beta < \lambda}$ be descending filter-bases for $\mathcal{U}/\mathfrak{Fr}$ and $\mathcal{V}/\mathfrak{Fr}$ respectively, which exist by Lemma III.3.13. Define $\mathcal{W} := (\mathcal{U}\otimes\mathcal{V})/\mathcal{D}$ and let $W \in \mathcal{U}\otimes\mathcal{V}$ be given, so there exists an $A \in \mathcal{U}$ and a selection $(B_m)_{m\in A}$ from $\mathcal{V}$ such that

$$\bigcup_{m\in A}(\{m\}\times B_m) \subseteq W.$$

Then there is some $\alpha < \kappa$ such that $U_\alpha \subseteq^* A$ and for each $m \in A\cap U_\alpha$, there is some $\beta_m < \lambda$ such that $V_{\beta_m} \subseteq^* B_m$.

Define $\beta := \sup(\beta_m : m \in U_\alpha)$. Since there are no non-trivial convergent sequences in $\omega^*, \mathcal{V}/\mathfrak{Fr}$ is not first-countable and thus λ is uncountable. Hence $\beta < \lambda$, so for all $m \in U_\alpha, V_\beta \subseteq^* B_m$. Define

$$D := ((\omega\setminus U_\alpha)\times\omega) \cup \bigcup_{m\in A\cap U_\alpha}(\{m\}\times(\omega\setminus(V_\beta\setminus B_m))) \in \mathcal{D}$$

$$\Rightarrow (U_\alpha\times V_\beta)\cap D = \bigcup_{m\in A\cap U_\alpha}(\{m\}\times(V_\beta\cap B_m)) \subseteq W.$$

Thus $[U_\alpha\times V_\beta]_{\mathcal{D}} \le W$, so $\{[U_\alpha\times V_\beta]_{\mathcal{D}} : \alpha<\kappa, \beta<\lambda\}$ is a filter base for $\mathcal{W}$.

Define $\mathcal{K} := \langle [\omega\times V_\beta]_{\mathcal{D}} : \beta<\lambda\rangle$ and $\mathcal{L} := \langle [U_\alpha\times\omega]_{\mathcal{D}} : \alpha<\kappa\rangle$. Then $\{[[U_\alpha\times\omega]_{\mathcal{D}}]_{\mathcal{K}} : \alpha<\kappa\}$ and $\{[[\omega\times V_\beta]_{\mathcal{D}}]_{\mathcal{L}} : \beta<\lambda\}$ are filter-bases for $\mathcal{W}/\mathcal{K}$ and $\mathcal{W}/\mathcal{L}$ respectively, so $\mathcal{K},\mathcal{L} \in \mathbf{Lin}(\mathcal{W})$.

We claim that $\{\mathcal{K},\mathcal{L}\}$ is cofinal. Suppose this is the case and let $\alpha<\kappa, \beta<\lambda$ be given. Then since $U_{\alpha+1} \subseteq^* U_\alpha \not\subseteq^* U_{\alpha+1}$ and $V_{\beta+1} \subseteq^* V_\beta \not\subseteq^* V_{\beta+1}$, it follows that $W := (U_\alpha\setminus U_{\alpha+1})\times(V_\beta\setminus V_{\beta+1})$ is infinite and meshes with $\mathcal{D}$, so there exists a $\mathcal{Z} \in \mathcal{D}\uparrow$ that contains W. Then

$$(\omega\times V_{\beta+1})\cap W = \emptyset = (U_{\alpha+1}\times\omega)\cap W \Rightarrow [\omega\times V_{\beta+1}]_{\mathcal{D}}, [U_{\alpha+1}\times\omega]_{\mathcal{D}} \notin \mathcal{Z}/\mathcal{D}.$$

Hence $\mathcal{K} \not\subseteq \mathcal{Z}/\mathcal{D} \not\supseteq \mathcal{L}$, so by Theorem III.3.29 $\mathcal{Z}/\mathcal{D}$ is not radial in $\mathcal{D}\uparrow$.

Now we prove our claim. Let $\mathcal{M} \in \mathbf{Lin}(\mathcal{W})$ be given and define $\mu := \chi(\mathcal{W}/\mathcal{M})$, so $\mu \le$

$\max(\kappa, \lambda)$. Let $(W_\gamma)_{\gamma<\mu} \subseteq \mathcal{W}$ be given such that $([W_\gamma]_{\mathcal{M}})_{\gamma<\mu}$ is a descending filter-base for $\mathcal{W}/\mathcal{M}$.

Case 1: $\mu = \kappa$. Then for each $\beta < \lambda$, there exists a $\gamma_\beta < \mu$ such that $W_{\gamma_\beta} \leq_{\mathcal{M}} [\omega \times V_\beta]_{\mathcal{D}}$.

> If $\lambda < \kappa$ then by regularity $\gamma := \sup(\gamma_\beta : \beta < \lambda) < \kappa$ and thus for every $\beta < \lambda, W_\gamma \leq_{\mathcal{M}} [\omega \times V_\beta]_{\mathcal{D}}$. If $\lambda > \kappa$ then by regularity there exists a $\Lambda \in [\lambda]^\lambda$ such that $\gamma_\beta = \gamma_{\beta'}$ for all $\beta, \beta' \in \Lambda$. Thus for all $\beta < \lambda$, there exists a $\beta' \in \Lambda$ such that $\beta \leq \beta'$ and hence $W_{\min(\Lambda)} = W_{\beta'} \leq_{\mathcal{M}} W_\beta \leq_{\mathcal{M}} [\omega \times V_\beta]_{\mathcal{D}}$.
>
> In any case, there exists a $\delta < \lambda$ such that $W_\delta \leq_{\mathcal{M}} [\omega \times V_\beta]_{\mathcal{D}}$ for each $\beta < \lambda$. So given a $\beta < \lambda$, there exists an $M \in \mathcal{M}$ such that $W_\delta \wedge M \leq [\omega \times V_\beta]_{\mathcal{D}}$ and therefore $\mathcal{K} \subseteq \langle \mathcal{M} \cup \{W_\delta\} \rangle$; i.e. $\mathcal{M} \sqsubseteq_{\mathcal{W}} \mathcal{K}$.

Case 2: $\mu = \lambda$. This is similar to Case 1 – in this case, $\mathcal{M} \sqsubseteq_{\mathcal{W}} \mathcal{L}$.

Case 3: $\mu \neq \kappa, \lambda$. Then for each $\gamma < \mu$, there exists $\alpha_\gamma < \kappa$ and $\beta_\gamma < \lambda$ such that $[U_{\alpha_\gamma} \times V_{\beta_\gamma}]_{\mathcal{D}} \leq W_\gamma$. By similar arguments as those above, there exists an $\alpha < \kappa$ and $\beta < \lambda$ such that $[U_\alpha \times V_\beta]_{\mathcal{D}} \leq W_\gamma$ for each $\gamma < \lambda$. Thus $\{[[U_\alpha \times V_\beta]_{\mathcal{D}}]_{\mathcal{M}}\}$ is a filter-base for $\mathcal{W}/\mathcal{M}$, so $\mu = 1$. Thus $\langle \mathcal{M} \cup \{[U_\alpha \times V_\beta]_{\mathcal{D}}\} \rangle = \mathcal{W} \supseteq \mathcal{K}$, so $\mathcal{K} \sqsubseteq_{\mathcal{W}} \mathcal{M}$.

Therefore $\{\mathcal{K}, \mathcal{L}\}$ is cofinal, completing our proof. $\square$

The method used above cannot be extended to a product of three or more p-points with different characters. By a theorem of Peter Nyikos, any well-based point in ω^* has character $\mathfrak{b}$ or $\mathfrak{d}$ [BS87, pg. 214], where $\mathfrak{d}$ is the *dominating number*, defined to be the least cardinal of a family $\mathcal{D} \subseteq {}^{\omega}\omega$ such that for every $f : \omega \to \omega$, there exists a $d \in \mathcal{D}$ such that $f \leq^* d$.

Notation

Index

References

[AHS] Jiří Adámek, Horst Herrlich and George E. Strecker. *Abstract and Concrete Categories: The Joy of Cats.* Online, pg. 524. URL: `http://katmat.math.uni-bremen.de/acc` (visited on 30th March 2012).

[Arh07] A. V. Arhangel’skii. ‘Locally compact spaces of countable core and Alexandroff compactification’. In: *Topology and its Applications* 154.3 (2007), pg. 625–634.

[Arh81] A. V. Arhangel′skiĭ. ‘The frequency spectrum of a topological space and the product operation’. In: *Transactions of the Moscow Mathematical Society* 1981.2 (1981), pg. 163–200.

[AIT85] A. V. Arhangel′skiĭ, R. Isler and G. Tironi. ‘Pseudoradial spaces and another generalization of sequential spaces’. In: *Convergence structures 1984.* Ed. by Josef Novák et al. Vol. 24. Mathematical Research. Berlin: Akademie-Verlag, 1985, pg. 33–37.

[Ark80] A. V. Arkhangel′skii. ‘Some properties of radial spaces’. In: *Mathematical Notes* 27.1-2 (1980), pg. 50–54.

[AM69] M. F. Atiyah and I. G. Macdonald. *Introduction to Commutative Algebra.* Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1969, pg. ix+128.

[BG92] A. Bella and J. Gerlits. 'On a condition for the pseudo radiality of a product'. In: *Commentationes Mathematicae Universitatis Carolinae* 33.2 (1992), pg. 311–313.

[Bel91] Angelo Bella. 'More on the product of pseudo radial spaces'. In: *Commentationes Mathematicae Universitatis Carolinae* 32.1 (1991), pg. 125–128.

[Bel93] Angelo Bella. 'Countable productivity of a class of pseudoradial spaces'. In: *Proceedings of the American Mathematical Society* 119.2 (1993), pg. 637–640.

[Bel96] Angelo Bella. 'Few remarks and questions on pseudoradial and related spaces'. In: *Topology and its Applications* 70.2-3 (1996), pg. 113–123.

[BBM13] Angelo Bella, Maddalena Bonanzinga and Mikhail Matveev. 'Sequential + separable vs sequentially separable and another variation on selective separability'. English. In: *Central European Journal of Mathematics* 11.3 (2013), pg. 530–538.

[BN10] Angelo Bella and Peter Nyikos. 'Sequential compactness vs. countable compactness'. In: *Colloquium Mathematicum* 120.2 (2010), pg. 165–189.

[BMS15] Andreas Blass, Heike Mildenberger and Saharon Shelah. 'A Simple $P_{\aleph_1}$-Point and a Simple $P_{\aleph_2}$-Point'. May 2015. URL: `http://home.mathematik.uni-freiburg.de/mildenberger/postings/paperspdf/bs242_2015_05_22.pdf` (visited on 1st July 2015).

[BS87] Andreas Blass and Saharon Shelah. 'There may be simple $P_{\aleph_1}$- and $P_{\aleph_2}$-points and the Rudin-Keisler ordering may be downward directed'. In: *Annals of Pure and Applied Logic* 33.3 (1987), pg. 213–243.

[CN74] W. W. Comfort and S. Negrepontis. *The Theory of Ultrafilters*. Die Grundlehren der mathematischen Wissenschaften 211. Springer-Verlag, New York-Heidelberg, 1974, pg. x+482.

[DIT87] G. D. Dimov, R. Isler and G. Tironi. ‘On functions preserving almost radiality and their relations to radial and pseudo-radial spaces’. In: *Commentationes Mathematicae Universitatis Carolinae* 28.4 (1987), pg. 751–781.

[Dol09] Szymon Dolecki. ‘An Initiation into Convergence Theory’. In: *Beyond topology*. Ed. by Frédéric Mynard and Elliot Pearl. Contemporary Mathematics 486. Providence, RI: American Mathematical Society, 2009, pg. 115–161.

[Dou84] Eric K. van Douwen. ‘The Integers and Topology’. In: *Handbook of Set-Theoretic Topology*. Ed. by Kenneth Kunen and Jerry E. Vaughan. Amsterdam: North-Holland Publishing Co., 1984, pg. 111–167.

[Dow+96] A. Dow et al. ‘More on sequentially compact implying pseudoradial’. In: *Topology and its Applications* 73.2 (1996), pg. 191–195.

[Eng89] Ryszard Engelking. *General Topology*. Revised and completed. Sigma Series in Pure Mathematics 6. Berlin: Heldermann Verlag, 1989, pg. viii+529.

[Fra65] S. P. Franklin. ‘Spaces in which sequences suffice’. English. In: *Fundamenta Mathematicae* 57 (1965), pg. 107–115.

[GJ60] Leonard Gillman and Meyer Jerison. *Rings of Continuous Functions*. The University Series in Higher Mathematics. D. Van Nostrand Co., Inc., Princeton, N.J.-Toronto-London-New York, 1960, pg. ix+300.

[GS05] Gary Gruenhage and Paul J. Szeptycki. ‘Fréchet-Urysohn for finite sets’. In: *Topology and its Applications* 151.1-3 (2005), pg. 238–259.

[Her67] Horst Herrlich. ‘Quotienten geordneter Räume und Folgenkonvergenz’. In: *Fundamenta Mathematicae* 61 (1967), pg. 79–81.

[Hod10] R. E. Hodel. ‘A theory of convergence and cluster points based on κ-nets’. In: *Topology Proc.* 35 (2010), pg. 291–330.

[Kun11] Kenneth Kunen. *Set theory*. Studies in Logic, Mathematical Logic and Foundations 34. College Publications, London, 2011, pg. vii+401.

[Lee14] Robert Leek. 'An internal characterisation of radiality'. In: *Topology and its Applications* 177 (2014), pg. 10–22.

[Lee15] Robert Leek. 'Convergence properties and compactifications'. 18th May 2015. URL: `http://arxiv.org/abs/1412.8701`.

[MS15] M. Malliaris and S. Shelah. 'Cofinality spectrum theorems in model theory, set theory, and general topology'. In: *Journal of the American Mathematical Society* (2015), pg. 1–61.

[Mic72] E.A. Michael. 'A quintuple quotient quest'. In: *General Topology and its Applications* 2.2 (1972), pg. 91–138.

[Nyi92] Peter J. Nyikos. 'Convergence in Topology'. In: *Recent Progress in General Topology*. Ed. by Miroslav Hušek and Jan van Mill. Amsterdam: North-Holland Publishing Co., 1992, pg. 537–570.

[OT95] Franco Obersnel and Gino Tironi. 'Products of pseudoradial spaces'. In: *Mathematica Pannonica* 6.1 (1995), pg. 29–38.

[Rud56] Walter Rudin. 'Homogeneity problems in the theory of Čech compactifications'. In: *Duke Mathematical Journal* 23 (1956), pg. 409–419.

[Rum14a] Wolfgang Rump. 'Abelian lattice-ordered groups and a characterization of the maximal spectrum of a Prüfer domain'. In: *Journal of Pure and Applied Algebra* 218.12 (2014), pg. 2204–2217.

[Rum14b] Wolfgang Rump. 'Serial compact spaces and the maximal spectrum of a Bézout domain'. In: *Topology and its Applications* 178 (2014), pg. 320–330.

[Sha93] Boris Shapirovskiĭ. ‘The Equivalence of Sequential Compactness and Pseudoradialness in the Class of Compact T_2-Spaces, Assuming CH’. In: *Papers on general topology and applications*. Ed. by Susan Andima et al. Annals of the New York Academy of Sciences 704. New York: New York Academy of Sciences, 1993, pg. 322–327.

[Sip02] Ol’ga V. Sipacheva. ‘Spaces Fréchet-Urysohn with respect to families of subsets’. In: 121.1-2 (2002): *Proceedings of the First Joint Japan-Mexico Meeting in Topology*, pg. 305–317.

[Sou04] Lajos Soukup. ‘Scattered spaces’. In: *Encyclopedia of General Topology*. Ed. by Klaas Pieter Hart, Jun-iti Nagata and Jerry E. Vaughan. Elsevier Science Publishers, B.V., Amsterdam, 2004, pg. 350–353.

[Tir06] Gino Tironi. ‘Pseudoradial Spaces: Results and Problems’. In: *Mediterranean Journal of Mathematics* 3.2 (2006), pg. 313–325.

[Tod85] Stevo Todorčević. ‘Directed sets and cofinal types’. In: *Trans. Amer. Math. Soc.* 290.2 (1985), pg. 711–723.

[Tuk40] John W. Tukey. *Convergence and Uniformity in Topology*. Annals of Mathematics Studies, no. 2. Princeton University Press, Princeton, N. J., 1940, pg. ix+90.

[Wim82] Edward L. Wimmers. ‘The Shelah P-point independence theorem’. In: *Israel Journal of Mathematics* 43.1 (1982), pg. 28–48.