Minimal coupling, the strong equivalence principle, and the adaptation of matter to spacetime geometry

A thesis submitted for examination in the BPhil in Philosophy

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To Molly and Aleisha.

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But in summer, the garden was filled with California poppies.

Abstract

I provide a systematic exploration of one set of precise technical conditions under which matter fields might be said to be 'adapted' to a relativistic spacetime geometry—namely, that the equations governing those matter fields be minimally coupled, quasilinear, and symmetric hyperbolic. I discuss the implications of this for the role and status of the 'strong equivalence principle' and the geometry-dynamics debate.

Contents

1 Introduction

It has been 20 years since the publication of Harvey Brown's *Physical Relativity*, and during that time, numerous articles have been devoted both to explicating and interrogating the position defended in that book. In chapter 9 of *Physical Relativity*, Brown put forward a view—the so-called 'dynamical approach' to general relativity—according to which the metric field of general relativity has its 'chronogeometric significance' in virtue of the fact that the dynamics of general-relativistic matter fields are 'adapted' to spacetime geometry. This was contrasted by Brown with the opposing 'geometrical approach' to general relativity, according to which the metric field has its chronogeometric significance 'of necessity'.

This thesis finds its home at something of a turning point in the development of this literature. In particular, in recent work, Read (2020a,b) and Weatherall (2019, 2021) have suggested that the locus of dispute between advocates of the dynamical and geometrical approaches should not be understood—at least in the context of general relativity—as about whether or not the dynamics of general-relativistic matter fields are, or must be, 'adapted' to spacetime geometry. That the dynamics of general-relativistic matter fields are, or must be, 'adapted' to spacetime geometry is something on which both sides of the debate can agree. But even so, as stressed by Weatherall (2021), it would be desirable to have technical control over the conditions under which a system of partial differential equations (i.e. a candidate dynamics for matter fields) on a relativistic spacetime will be 'adapted' to spacetime geometry. That is, one might want to identify:

(a) Features that 'adapted' dynamics will (necessarily) have.

(b) Necessary and/or sufficient conditions for a system of partial differential equations to exhibit the features of adapted dynamics.

On (a), some widely-accepted candidates include:

Motion: Solutions 'track' (only) timelike or null geodesics.

Causality: Solutions do not propagate superluminally.

Time: Suitable periodic solutions measure elapsed proper time (the 'clock hypothesis').

Of course, as it stands, these are not precise conditions. However, in the literature, there exist a variety of approaches to making these conditions precise. For instance, one can take small-body motion (the geodesic principle) as an explication of **motion**—and, drawing on well-known results by e.g. Ehlers and Geroch (2004), Geroch and Jang (1975), and Geroch and Weatherall (2018), show that this condition will be satisfied by any matter fields which can be associated with a rank-two symmetric tensor field which is covariantly conserved and satisfies a certain energy condition.[1] Earman (2014) and Weatherall (2014) discuss a precise criterion for **causality** proposed by Geroch (1996, 2011), including the ways in which it is (and is not) related to other criteria such as the dominant energy condition (DEC). And there exist various constructions for explicating **time**, including discussions of pendulum clocks (Misner et al. 1973, pp. 394–395), harmonic oscillators (Møller 1955, 1956), muons (Eisele 1987), and 'light clocks' (Perlick 1987, 2008), see especially Fletcher (2013), who shows that for any timelike curve, there exists a light clock which measures its arclength to arbitrary degrees of accuracy, as well as Fletcher (2025, §2.2) for further discussion.

On (b), too, various candidate sufficient conditions have been proposed in the literature by advocates of the dynamical approach. Chief among them are the following:

MC: The system is 'minimally coupled' to the metric.

SEP: The system satisfies the 'strong equivalence principle'.

Symmetry: The system is invariant under 'local (approximate) Poincaré symmetries'.

LSR: The system is 'locally special relativistic'.

[1] In the case of Ehlers and Geroch (2004) and Geroch and Weatherall (2018) this is the dominant energy condition; in the case of Geroch and Jang (1975) this is the strengthened dominant energy condition.

I state these as separate conditions, here, because it is not *a priori* clear what (if any) logical relationships these conditions bear to one another. (As I will discuss, this is because these conditions are generally not stated clearly or precisely, and it is not clear how one might go about establishing any logical relationships between them in the absence of such a statement.) But it is worth noting that various relationships between these conditions are often taken to hold. For instance, Brown (2005) takes **MC** to be sufficient for both **symmetry** and **LSR** (followed, in the former case, by Read et al. (2018), and in the latter case, by Knox (2019)). **Symmetry** and **LSR**, on the other hand, are often taken as explications of what is meant by **SEP**—in the case of **symmetry**, by Brown (2005), Brown and Pooley (2001), Read (2020a), and Read et al. (2018), and in the case of **LSR**, by Brown (2005) and Knox (2013, 2019). And **symmetry** itself is sometimes taken as an explication of what is meant by **LSR**—for instance, by Read (2020a) and Read et al. (2018).

One can then ask: do any of these various candidate sufficient conditions (**MC**, **SEP**, **symmetry**, **LSR**) imply the necessary ones (**motion**, **causality**, **time**)? And here we run up against the problem I alluded to above—namely, that these various candidate sufficient conditions (**MC**, **SEP**, **symmetry**, **LSR**) are generally not stated clearly or precisely, and in any case not in a way which makes contact with the various approaches to explicating conditions such as **motion**, **causality**, and **time** mentioned earlier. For instance, **MC** is often identified in the literature (e.g. in Brown (2005) and Read et al. (2018), following standard physics practice, see e.g. Misner et al. (1973, §16.3)) with the so-called 'comma goes to semicolon' rule—which, recall, is the 'prescription' for constructing general-relativistic matter dynamics from special-relativistic ones whereby one replaces all instances in the expression of an equation of the Minkowski metric with an arbitrary Lorentzian metric, and all instances of the flat Minkowskian connection with the Levi-Civita connection for that metric. **LSR**, on the other hand, is described by Brown (2005) as the condition that given any point in a relativistic spacetime, systems of partial differential equations governing matter fields, expressed in Lorentz normal coordinates at that point, take their 'special-relativistic form'. **Symmetry** is identified by Read et al. (2018) as the condition that the 'form' of those equations, thus expressed, is invariant under Poincaré transformations at that point. And here is Knox (2013, p. 352) on **SEP**:

> To any required degree of approximation, given a sufficiently small region of spacetime, it is possible to find a reference frame with respect to whose associated coordinates the metric field takes Minkowskian form, and the connection and its derivatives do not appear in any of the fundamental field equations of matter.

As Fletcher (2020), Fletcher and Weatherall (2023b), and Weatherall (2021) argue, the problem with these explications of **MC**, **SEP**, **symmetry**, and **LSR** is that they are not clear, in the sense that given a system of partial differential equations on a relativistic spacetime, it is ambiguous whether that system of partial differential equations satisfies them or not. This is because whether or not an equation takes a certain 'form', expressed in certain coordinates, is generally not preserved under arbitrary syntactic manipulations of that equation which preserve its space of solutions. At worst, this means that they are "hyperintensional" (Fletcher 2020, p. 266) (a fact acknowledged by Brown (2005) and Read et al. (2018), and in the case of **MC**, since at least Trautman (1965, §6.2), see also Misner et al. (1973, §16.3)), in the sense that they distinguish between cases which should be regarded as equivalent. Put somewhat differently, one might expect that a satisfactory definition of **MC**, **SEP**, **symmetry**, **LSR** would capture something 'intrinsic' about that system of equations, rather than the syntactic expression of those equations, in certain coordinates, using certain conventions, and certain background assumptions (Fletcher and Weatherall 2023b; Weatherall 2021). But in any case, in the absence of a sufficiently clear and precise statement of conditions such as **MC**, **SEP**, **symmetry**, and **LSR**, it is not clear how one might go about establishing that these conditions do, in fact, ensure that a system of partial differential equations will be adapted to spacetime geometry in the senses specified by **motion**, **causality**, and **time**.

Recently, however, there have been various attempts at progress in this direction. For instance, Fletcher (2020) discusses a precise definition of 'local approximate symmetry' which could enter into the explication of **symmetry** (on which, see also previous work in the physics literature, e.g. Harte (2008)). Fletcher and Weatherall (2023b) discuss a variety of precise criteria for when a system of partial differential equations might be taken to satisfy **LSR**, and show that on at least some of these criteria, standard examples such as Maxwell's equations and the Klein–Gordon equation count as 'locally special relativistic'. And even more recently, Weatherall and March (2025) have proposed a precise explication of **MC** which avoids the ambiguity worries raised by Fletcher (2020), Fletcher and Weatherall (2023b), and Weatherall (2021).

Drawing on these ideas, the aims of this thesis are twofold. The first is to provide a systematic exploration of one set of precise technical conditions under which matter fields might be said to be 'adapted' to a relativistic spacetime geometry—namely, that the equations governing those matter fields are minimally coupled, quasilinear, and symmetric hyperbolic. In particular, I will show that minimally coupled, quasilinear, symmetric hyperbolic systems of partial differential equations necessarily satisfy **motion** and **causality**, on plausible ways of making those latter two conditions precise. In doing so, I substantiate the link between **MC** and adapted dynamics proposed by Brown (2005), Knox (2019), Read (2020a), and Read et al. (2018).

The second aim of this thesis is to connect this up with discussions of **SEP**, **symmetry**, and **LSR**. To this end, I propose a precise statement of **SEP**, and show that it is both entailed by **MC**, and entails **MC** under fairly minimal additional conditions. This substantiates the link between **MC** and **SEP** proposed by Brown (2005), and the link between **SEP** and adapted dynamics proposed by various defenders of the dynamical approach (Brown 2005; Knox 2013, 2019; Read 2020a; Read et al. 2018). I then provide a precise statement of **symmetry** and show that it is entailed by **MC**, and discuss the relationship between this condition and **motion** and **causality**, thus substantiating the link between **MC** and **symmetry** proposed by Read (2020a) and Read et al. (2018). And finally, building on Fletcher and Weatherall (2023b), I provide general proofs to the effect that minimally coupled theories satisfy **LSR**, on at least two plausible ways of making that condition precise.

Along the way, I prove several intermediate results which are of foundational interest in their own right. These include (i) uniqueness results for minimal coupling, thus resolving the aforementioned longstanding issue concerning the extent to which minimal coupling is ambiguous as a prescription for constructing general-relativistic matter dynamics from special-relativistic ones, (ii) a rigourous schema for understanding, in geometric terms, how higher-order partial differential equations can be transformed into lower-order ones, and (iii) a derivation of the DEC, for minimally coupled, quasilinear, symmetric hyperbolic theories, thus addressing a question raised by Curiel (2017) about whether such a derivation is possible, and a question raised by Weatherall (2020) about the relationship between the DEC and Geroch–Earman causality.

In a little more detail, then, the structure of this thesis is as follows. First, in §2, I present some technical background on minimally coupled, quasilinear, symmetric hyperbolic theories, including the explication of **MC** proposed by Weatherall and March (2025), and state and prove a uniqueness theorem for minimal coupling understood in this sense. In §3, I present and discuss in some detail the various approaches to explicating **motion** and **causality** mentioned above, focussing on the results of Geroch and Weatherall (2018) and the Geroch–Earman causality condition discussed by Earman (2014), Geroch (2011), and Weatherall (2014). In §4, I then state and prove a series of results to the effect that minimally coupled, quasilinear, symmetric hyperbolic theories satisfy (i) the Geroch–Earman causality condition, (ii) the DEC, and (iii) in the absence of sources, the covariant conservation condition, in senses I make precise there. This suffices for my claim that minimally coupled, quasilinear, symmetric hyperbolic theories necessarily satisfy **motion** and **causality**. In §5, I propose precise explications of **SEP** (in §5.1) and **symmetry** (in §5.2), and discuss their relationship to **MC**. I also provide a precise explication of **LSR** and discuss some of the explications of **LSR** proposed by Fletcher and Weatherall (2023b), and show that these are satisfied by minimally coupled theories (in the latter case, under certain additional regularity assumptions). I close in §6 with future directions.

As a final health warning before I begin: I am aware that the methods, notation, and math-physics ideology of this thesis are not always taken to occupy neutral ground in the geometry-dynamics debate and related discussions. I do not want to engage in that controversy here. They reflect my own preference for a certain way of doing philosophy of physics—one which I find the most conceptually clear, but which need not be the most conceptually clear to anyone else, nor do I want to suggest is somehow the most conceptually clear *simpliciter*. Different problems call for different tools, and I take the tools used in this thesis to be valuable insofar as they shed light on the problems mentioned above. More generally: philosophy of physics is pluralistic, and if there are any general morals to be drawn from this thesis, I hope that one of them is that that pluralism is something to be embraced.

2 Minimally coupled, quasilinear, symmetric hyperbolic theories

2.1 Preliminaries: fibre bundles

I will begin by recalling some basic details of fibre bundles. (Further relevant background on fibre bundles, including principal bundles, associated bundles, pullback bundles, fibred products, connections etc. can be found in appendix A. An index of notation can be found in appendix C.) A smooth fibre bundle $B \xrightarrow{\pi} M$ with standard fibre S consists of smooth manifolds B (the total space) and M (the base space), and a smooth surjective map $\pi : B \to M$ with the property that for any $p \in M$, there exists an open neighbourhood O of p and a local trivialisation of B over O, i.e. a diffeomorphism $\varphi : O \times S \to \pi^{-1}(O)$. A smooth bundle morphism (Φ, φ) between smooth fibre bundles $B \xrightarrow{\pi} M$, $B' \xrightarrow{\pi'} M'$ is a pair of maps $\Phi : B \to B'$, $\varphi : M \to M'$ such that $\pi' \circ \Phi = \varphi \circ \pi$. In this case, we will say that Φ is a smooth bundle morphism covering φ. In what follows, I will sometimes omit reference to the map φ, and denote the bundle morphism simply as Φ.

The above definition of a fibre bundle is often specialised by requiring that the standard fibre S is endowed with some further structure, e.g. as a vector space, affine space, etc. In this case, we amend the definitions of a local trivialisation and smooth bundle morphism to require that they preserve that structure. For example, if S is a vector space, one can obtain the definition of a smooth vector bundle by requiring that each local trivialisation $\varphi : O \times S \to \pi^{-1}(O)$, restricted to $\pi^{-1}(p)$ for any $p \in O$, is a vector space isomorphism. A smooth vector bundle morphism would then be a smooth bundle morphism $(\Phi, \varphi) : B \to B'$ such that the map Φ, restricted to $\pi^{-1}(p)$ for any $p \in M$, is a linear map.

Meanwhile, a smooth local section of a fibre bundle $B \xrightarrow{\pi} M$ is a smooth map $\psi : O \subset M \to B$ with the property that for any $p \in O$, $\psi(p) \in \pi^{-1}(p)$. If O happens to be M itself, then we say that the section is global. Note that smooth local sections of a fibre bundle always exist, but generically, bundles need not admit smooth global sections. (Global) sections of a bundle $B \xrightarrow{\pi} M$ can be thought of as fields on M. For instance, sections of the bundle $\mathbb{R} \times M \xrightarrow{\pi} M$, where π is the projection onto M, would be smooth scalar fields; sections of the tangent bundle TM would be smooth vector fields; sections of the cotangent bundle T^*M would be smooth covector fields, and so on.

If $B \xrightarrow{\pi} M$ is a smooth fibre bundle, I will use lowercase Latin indices to denote vectors, tensors, etc. on M.[2] We can also consider vectors, tensors etc. on B (since B is, in particular, a smooth manifold), which will be denoted with lowercase Greek indices. Given a vector ξ^α on B, we will say that it is vertical iff its pushforward along π vanishes, and horizontal otherwise. Vertical vectors on B will be denoted with primed indices—i.e. if ξ^α is vertical, I will write $\xi^{\alpha'}$. Likewise, covectors on B restricted to act on vertical vectors will be denoted with primed indices. We can also consider mixed index tensor fields on B—for instance, the pushforward along the projection map can be thought of as a mixed index tensor field $(\mathrm{d}\pi)^a{}_\alpha$ on B, which acts on vectors on B to produce vectors on M.

Finally, note that, since the vertical vectors at each point $x \in B$ arise as the kernel of a linear map, they form a vector space at that point, which will be denoted V_xB. (Likewise for the dual space of vertical covectors, V_x^*B.) These vertical vector spaces form a vector bundle over B, which will be denoted VB. In particular, if B is itself a vector bundle, then VB is canonically isomorphic to $B \times_M B$; likewise, if B is an affine bundle, then VB is canonically isomorphic to $U \times_M B$, where U is the modelling vector bundle for B. By contrast, the horizontal vectors at a point in B do not form a vector space—instead, one has many horizontal subspaces H_xB, each complementary to V_xB, in the sense that $T_xB = V_xB \oplus H_xB$.

2.2 Natural bundles, jet bundles, and natural operators

Natural bundles are a way of capturing the idea of a geometric object developed originally by Schouten, Veblen, and others (Schouten and Haantjes 1936; Schouten and van Dantzig 1935; Veblen and Thomas 1926; Veblen and White-

[2] Here, and throughout, all indices are abstract unless stated otherwise.

head 1932), and which culminated in the work of Nijenhuis (1952).[3] The idea, very roughly, is that natural bundles are bundles that depend only on (or: are 'constructed out of') the structure of their base space, in the sense that (1) given any appropriate smooth manifold, the bundle can always be constructed, and (2) any appropriate smooth map on the base space 'lifts', uniquely, to a smooth bundle morphism.

To make this idea precise, we need some basic category theory. (For relevant background on category theory, see appendix B, or any standard textbook treatment, e.g. Mac Lane (1998).) Let $\mathcal{M}_n$ denote the category whose objects are smooth n-manifolds, and whose arrows are smooth embeddings.[4] And let $\mathcal{FB}$ denote the category whose objects are smooth fibre bundles, and whose arrows are smooth bundle morphisms. (I will use calligraphic script for categories.) Following the presentations in March and Weatherall (2024) and Weatherall and March (2025), we define:

Definition 2.2.1 (Natural bundle)**.** *Let $\mathcal{M}_n$ and $\mathcal{FB}$ be as above. A natural bundle over n-manifolds is a functor $X : \mathcal{M}_n \to \mathcal{FB}$ such that for each $M \in \mathrm{ob}(\mathcal{M}_n)$,[5] XM is a bundle with base space M, and for each $\varphi \in \mathrm{mor}(\mathcal{M}_n)$, $X\varphi$ is a smooth bundle morphism covering φ.*[6]

In this definition, the two informal conditions mentioned above get realized in the requirement of functoriality. Natural bundles associate smooth bundles to every smooth n-manifold, and smooth bundle morphisms to every smooth embedding between n-manifolds (and do so in a way that preserves composition and identity). Meanwhile, the restriction to 'appropriate' smooth manifolds and

[3] A systematic treatment of natural bundles (and natural operators) is given by (Kolář et al. 1993), see also Fatibene and Francaviglia (2003). The modern presentation of natural bundles was first introduced by Nijenhuis (1972).

[4] This definition of $\mathcal{M}_n$ follows Palais and Terng (1977). An alternative approach, following Kolář et al. (1993), would be to define the category of n-manifolds to have local diffeomorphisms as arrows.

[5] I am using set-theoretic notation, here, because it is compact—in doing so, I do not intend to make any commitment on the issue of set-theoretic foundations for category theory, nor on whether the categories under consideration here are small (or locally small). For anyone who is worried about size considerations, we can assume that all the collections of objects and morphisms considered here have cardinality less than some inaccessible cardinal, κ.

[6] Note that a third condition which Kolář et al. (1993, §14.1) call 'regularity', is sometimes included, but this in fact follows from the other assumptions, though the proof is non-trivial.

smooth maps in these two informal conditions gets realized in the definition of the category $\mathcal{M}_n$. To get an intuitive handle on how this works, consider e.g. tangent bundles. Any smooth n-manifold M determines a bundle $TM \overset{\pi_{TM}}{\to} M$, and any smooth embedding $\varphi : M \to M'$ between n-manifolds determines, uniquely, a smooth map from TM to TM' via the pushforward construction. Thus tangent bundles are natural. Other examples include cotangent bundles, frame bundles, tensor bundles, bundles of m-forms for fixed m, bundles of affine connections etc.

At this point, two health warnings are in order. First, note that a natural bundle is not a bundle, but a functor from the category of smooth n-manifolds to the category of smooth fibre bundles. That said, I will on occasion abuse language and use the term 'natural bundle' to refer to both the functor and the image of objects $M \in \mathcal{M}_n$ under that functor. Second, note that whilst I have defined natural bundles above as functors from the category $\mathcal{M}_n$, for some purposes, it is helpful to consider natural bundles defined over some full subcategory of $\mathcal{M}_n$. For example, one would like to say that the bundle of Lorentzian metrics is natural, but as we have set things up, it is not, since not every smooth n-manifold admits any Lorentzian metrics at all (Geroch and Horowitz 1979; O'Neill 1983). To remedy this, I will often consider the category $\overline{\mathcal{M}}_n$ of smooth n-manifolds admitting a Lorentzian metric.

Finally, note that the collection of natural bundles over n-manifolds (or some full subcategory thereof) itself has the structure of a category, which I will denote $\mathcal{NB}_n$ (or, e.g., $\overline{\mathcal{NB}}_n$, in the case of natural bundles over $\overline{\mathcal{M}}_n$), whose objects are natural bundles over n-manifolds and whose arrows are natural transformations. A systematic treatment of the category $\mathcal{NB}_n$ would take us too far afield here (and as far as I am aware, no such treatment exists in the literature), but throughout what follows, we will meet several examples of constructions on natural bundles which are usefully (and perhaps most cleanly) thought of as instances of standard categorical constructions (pullbacks, products, subobjects, quotient objects, etc.) in $\mathcal{NB}_n$. In fact, we already have the resources to introduce one such construction. Recall that an object 1 of a category $\mathcal{C}$ is called a *terminal object* iff, for any object Y of $\mathcal{C}$, there exists a unique arrow $y : Y \to 1$. (The terminal object, if it exists, is unique up to unique isomorphism.) Now consider the natural bundle 1_n such that $1_n M = M \overset{\mathrm{id}_M}{\to} M$ for all $M \in \mathrm{ob}(\mathcal{M}_n)$ and $1_n\varphi = (\varphi, \varphi)$ for all $\varphi \in \mathrm{mor}(\mathcal{M}_n)$. It is straightforward to see that 1_n is the terminal object in $\mathcal{NB}_n$—which follows trivially from the fact that the component of any natural transformation between natural bundles over n-manifolds covers the identity at each $M \in \mathrm{ob}(\mathcal{M}_n)$ (Kolář et al. 1993, §14.11), and for any object X in $\mathcal{NB}_n$, there always exists a natural transformation $\pi_X : X \to 1_n$ whose component π_{XM} at each $M \in \mathrm{ob}(\mathcal{M}_n)$ is the projection map of XM. Thus, the projection map of a natural bundle $X : \mathcal{M}_n \to \mathcal{FB}$ can be thought of as the unique arrow $\pi_X : X \to 1_n$ in $\mathcal{NB}_n$.[7]

I will now turn to jet bundles. First, recall that if M and N are smooth manifolds, O, O' are open neighbourhoods of some $p \in M$, and $f : O \to N$, $g : O' \to N$ are smooth maps, then f and g are said to be k-equivalent at p iff they agree on all their partial derivatives up to order k at p (in any, and hence every, pair of local coordinate charts containing p, $f(p)$, $g(p)$).[8] A k-jet of smooth maps from M to N at p is an equivalence class of smooth maps $f : O \subset M \to N$ that are k-equivalent at p, and the k-jet at p containing f is denoted $j^k_p f$.[9] Note that k-jets have a natural composition structure: for any smooth maps $f : O \subset M \to N$ and $g : O' \subset N \to P$ with $f(p) = q \in O'$, one defines $j^k_q g \circ j^k_p f := j^k_p(g \circ f)$. A k-jet $j^k_p f$, $f : O \subset M \to N$ is invertible iff there exists a k-jet $j^k_q g$, for some $g : O' \subset N \to M$ with $q = f(p)$, such that $j^k_q g \circ j^k_p f = j^k_p \mathrm{id}_M$ and $j^k_p f \circ j^k_q g = j^k_q \mathrm{id}_N$.

Now let $B \overset{\pi}{\to} M$ be a smooth fibre bundle. We can construct a smooth

[7]Conversely, a natural section of a natural bundle X would be an arrow $s : 1_n \to X$ in $\mathcal{NB}_n$. One way of capturing the idea that, generically, sections of a bundle are not natural (and hence are in some sense 'more structure' than the bundle itself) is to note that, in general, a natural bundle admits no natural sections. (There are exceptions. For example, the natural bundle of real scalar fields over n-manifolds admits continuum many natural sections, one for each element of $\mathbb{R}$. Likewise, natural vector bundles generically admit exactly one natural section, namely the zero section.)

[8]Note that we include partial derivatives at order 0 in this definition.

[9]Algebraic characterisations of k-jets are also available, but would require a discussion of higher-order tangent bundles, which would take us too far afield here. See Kolář et al. (1993, §12.9).

bundle J^kB—the kth jet bundle of B—whose fibre, at each $p \in M$, is the space of all k-jets of local sections of B at p. This bundle is equipped with a variety of projection maps: for any J^kB, $J^\ell B$, $k \geq \ell$, one has a projection map $\pi_B^{(k,\ell)} : J^kB \to J^\ell B$ defined via $\pi_B^{(k,\ell)}(j_p^k\psi) = j_p^\ell\psi$ for all $p \in M$ and all (local) sections ψ, which make it into a smooth bundle over M and the $J^\ell B$. In particular, note that for any bundle B, and any k, $\pi_B^{(k,k-1)} : J^kB \to J^{k-1}B$ is a smooth affine bundle, whose modelling vector bundle is the pullback of $VB \otimes S^kT^*M$ over $J^{k-1}B$ with respect to the projection map $\pi_B^{(k-1,0)}$ (where S^k denotes the k-times symmetric tensor product bundle and T^*M is naturally considered as a vector bundle over B via the pullback with respect to π). In what follows, we will often omit reference to these pullbacks for notational hygiene when discussing the affine structure on $\pi_B^{(k,k-1)} : J^kB \to J^{k-1}B$.

Given a smooth fibre bundle $B \xrightarrow{\pi} M$ with kth jet bundle J^kB, one can define two important constructions: prolongation of morphisms and prolongation of sections. These allow us to 'lift' smooth bundle morphisms and smooth sections of B to, respectively, smooth bundle morphisms and smooth sections of J^kB. I will begin with the latter. Given any section $\psi : M \to B$, we can define a section $j^k\psi : M \to J^kB$ (its kth jet prolongation) via $j^k\psi(p) = j_p^k\psi$ for all $p \in M$. Similarly, any smooth bundle morphism $(\Phi, \varphi) : B \to B'$, where φ is a smooth embedding (i.e. a diffeomorphism onto its image), will determine, uniquely, a smooth bundle morphism $j^k\Phi : J^kB \to J^kB'$ (its kth jet prolongation) covering φ via the condition $j^k\Phi(j_p^k\psi) = j_{\varphi(p)}^k(\Phi \circ \psi \circ \varphi^{-1})$ (where I have suppressed restriction of φ^{-1} to the image of φ for notational hygiene).

In this way, one can define a functor $J^k : \mathcal{FB} \to \mathcal{FB}$ (really, from the subcategory of $\mathcal{FB}$ whose arrows, restricted to their base spaces, are smooth embeddings) taking every smooth fibre bundle to its kth jet bundle, and every smooth bundle morphism to its kth jet prolongation. (The case $k = 0$ is, of course, just the identity functor on $\mathcal{FB}$.) For any pair of such functors J^k, J^ℓ, $k \geq \ell$, one then has an epic natural transformation $\pi^{(k,\ell)} : J^k \to J^\ell$ whose component, at each object $B \xrightarrow{\pi_B} M$ of $\mathcal{FB}$, is the smooth bundle morphism $(\pi_B^{(k,\ell)}, \mathrm{id}_M)$. The same point applies to jets of natural bundles. That is, for any

k, we have a functor $J^k : \mathcal{NB}_n \to \mathcal{NB}_n$,[10] which takes every natural bundle over n-manifolds X to the bundle whose image, at each $M \in \text{ob}(\mathcal{M}_n)$, is the kth jet bundle of XM and each natural transformation f between natural bundles to the natural transformation whose component, at each $M \in \text{ob}(\mathcal{M}_n)$, is $j^k f_M$, and given any two such functors J^k, J^ℓ, $k \geq \ell$, an epic natural transformation $\pi^{(k,\ell)} : J^k \to J^\ell$.

Note that, if ℓ_i, $i = 1, 2, ..., r$ are (natural) numbers with $\sum_i \ell_i = k$, then the functors $J^{\ell_1} J^{\ell_2} ... J^{\ell_r}$ and J^k are not naturally isomorphic to one another—or in other words, one does not have, for any smooth fibre bundle $B \xrightarrow{\pi} M$, $J^{\ell_1} J^{\ell_2} ... J^{\ell_r} B = J^k B$. Instead, for any such ordered r-tuple of numbers, one has a monic natural transformation $\iota^{(\ell_1, \ell_2, ..., \ell_r)} : J^k \to J^{\ell_1} J^{\ell_2} ... J^{\ell_r}$, whose component, at each $B \in \text{ob}(\mathcal{FB})$, is the map $\iota_B^{(\ell_1, \ell_2, ..., \ell_r)} : J^k B \to J^{\ell_1} J^{\ell_2} ... J^{\ell_r} B$, $\iota_B^{(\ell_1, \ell_2, ..., \ell_r)}(j^k_p \phi) = j^{\ell_1}_p (j^{\ell_2} ... j^{\ell_r} \phi)$, and likewise for objects of $\mathcal{NB}_n$.

At this point, it is helpful to mention something about the order of natural bundles. To do so, we begin by introducing an important class of natural bundles: the frame bundles of order r. Let M be a smooth manifold of dimension n. An r-frame at $p \in M$ is an invertible r-jet $j^r_{\mathbf{0}} f$, for some $f : \mathbb{R}^n \to M$ such that $f(\mathbf{0}) = p$, where $\mathbf{0}$ denotes the zero element of $\mathbb{R}^n$. The set of all r-frames on M, $L^r M$ is a principal fibre bundle (the r-frame bundle) over M with structure group $\text{GL}^r(n, \mathbb{R})$, where $\text{GL}^r(n, \mathbb{R})$ is the group of invertible r-jets from $\mathbb{R}^n$ to itself at $\mathbf{0}$, with group multiplication as composition of r-jets.[11] It is straightforward to check that $\text{GL}^1(n, \mathbb{R}) = \text{GL}(n, \mathbb{R})$ and $L^1 M$ is canonically isomorphic to the classical frame bundle LM.

The following (classic, fundamental) result in the theory of natural bundles, essentially due to Palais and Terng (1977), makes clear the significance of r-frame bundles:

Theorem 2.2.2 (Finite-order theorem)**.** *Let M be a smooth manifold of di-*

[10] That the kth jet bundle of a natural bundle is itself natural follows trivially from the discussion of prolongation of bundle morphisms above.

[11] This terminology follows e.g. Kolář et al. (1993), though note that $L^r M$ is also sometimes called the bundle of holonomic r-frames (to distinguish it from, e.g. the bundle of (non-holonomic) r-frames obtained by $r - 1$ times recursively taking the bundle of first jets of sections of LM).

mension n and let S be a smooth manifold which carries a (left) $\mathrm{GL}^r(n, \mathbb{R})$ *action. Then the associated bundle $L^r M \times_{\mathrm{GL}^r} S \to M$ is a natural bundle, and conversely, any natural bundle over M can be constructed in this way, for some finite $r \geq 0$.*

So every natural bundle can be understood as an associated bundle to some frame bundle, say, of order r. In this case, we will say that the bundle is of order r. Standard examples include: 1_n and the bundle of real scalar fields (0th-order); bundles of tensor fields, m-forms for fixed m, and (Lorentzian) metrics (1st-order); and the bundle of affine connections (2nd-order). Note that if X is a natural bundle of order r, then $J^k X$ is a natural bundle of order $r + k$.

Finally, I will introduce natural operators. Suppose we are given bundles $B \xrightarrow{\pi} M$, $B' \xrightarrow{\pi'} M$ over some n-manifold M. In general, a kth-order operator from B to B' can be thought of as a rule transforming sections of B into sections of B', in a way that depends only on the derivatives of those sections of B up to order k. In other words:

Definition 2.2.3 (Operator)**.** *Let $B \xrightarrow{\pi} M$, $B' \xrightarrow{\pi'} M$ be smooth fibre bundles over M. A kth-order operator $F : B \to B'$ is a smooth bundle morphism $f : J^k B \to B'$ covering the identity on M.*

This definition has an obvious extension to natural operators on natural bundles. That is, a kth-order natural operator between natural bundles $X : \mathcal{M}_n \to \mathcal{FB}$, $Y : \mathcal{M}_n \to \mathcal{FB}$ should associate, to each $M \in \mathrm{ob}(\mathcal{M})$, a kth-order operator from XM to YM, in a way that is itself natural. We have the following definition:

Definition 2.2.4 (Natural operator)**.** *Let $X : \mathcal{M}_n \to \mathcal{FB}$, $Y : \mathcal{M}_n \to \mathcal{FB}$ be natural bundles. A kth-order natural operator $F : X \to Y$ is a natural transformation $f : J^k X \to Y$.*[12]

[12]A somewhat different, and more general characterisation of natural operators, which is equivalent to mine in the cases considered here (Kolář et al. 1993, §14.17), is given by Kolář et al. (1993, §18).

Given a kth-order natural operator $F : X \to Y$, some $M \in \mathrm{ob}(\mathcal{M}_n)$, and a section $\psi : M \to XM$, I will write $F(M, \psi) := f_M \circ j^k\psi$ for its action on that section.

2.3 Equations and natural equations

We now turn to equations and natural equations. First, we recall that a system of kth-order partial differential equations on sections of a bundle B can be understood as a closed embedded submanifold of J^kB:

Definition 2.3.1 (Equation)**.** *Let $B \to M$ be a smooth fibre bundle. A kth-order partial differential equation on B is a closed embedded submanifold E of J^kB.*

To understand this definition, note that a kth-order system of partial differential equations on M specifies, at each $p \in M$, a distinguished collection of field values (points in B) at p and their derivatives up to order k at p (points in J^kB), such that these points would solve the equation at p, so any (local) system of kth order partial differential equations determines a closed embedded submanifold E of J^kB. Conversely, given any closed embedded submanifold E of J^kB, the implicit function theorem entails that given any choice of local (fibred) coordinates for J^kB, (x_i, b^α_I),[13] E can locally be expressed as an equation as follows: $f(x_i, b^\alpha_I) = 0$ for some f. A *solution* of a system of kth-order partial differential equations on B is then a section $\psi : M \to B$ such that $j^k\psi \subset E$.

A similar idea applies to natural equations on natural bundles. That is, a natural kth-order equation on a natural bundle $X : \mathcal{M}_n \to \mathcal{FB}$ should associate, to each $M \in \mathrm{ob}(\mathcal{M}_n)$, an embedded submanifold of the kth jet bundle J^kXM, in a way that is functorial over smooth manifolds. We have the following definition:

Definition 2.3.2 (Natural equation)**.** *Let $X : \mathcal{M}_n \to \mathcal{FB}$ be a natural bundle. A kth-order natural equation on X is a subobject $\mathfrak{E}$ of J^kX in $\mathcal{NB}_n$.*

[13] Here, i indexes M-coordinates, α indexes coordinates on the typical fibre of B, and I is a multi-index taking values in k-tuples of i-indices (some of which may be zero), which indexes partial derivatives (up to order k) of the b^α with respect to the x_i. For details, see Saunders (1989, §6.1).

Given a kth-order natural equation $\mathfrak{E}$ on a natural bundle X, I will denote by $\mathfrak{E}M$ the submanifold of J^kXM defined via $\mathfrak{E}M = e_M(EM)$, where $e : E \to J^kX$ is any representative of $\mathfrak{E}$.

This definition requires some unpacking. First, a subobject $\mathfrak{E}$ of J^kX in $\mathcal{NB}_n$ is an equivalence class of monic natural transformations with target J^kX under the equivalence relation $e \sim e'$, $e : E \to J^kX$, $e' : E' \to J^kX$, iff there exists a natural isomorphism $i : E \to E'$ such that $e = e' \circ i$, where $E, E' : \mathcal{M}_n \to \mathcal{FB}$ are natural bundles. The components of each element e of $\mathfrak{E}$, with source E, tell us how to think of the image EM of E at each $M \in \mathrm{ob}(\mathcal{M}_n)$ as an embedded submanifold of J^kXM, in a way that is functorial over $\mathcal{M}_n$. One could just stop at this point, and define kth-order natural equations as tuples (E, e, X), where E and X are natural bundles, and $e : E \to J^kX$ is a monic natural transformation. But in fact, we do not really care about the object E—we care about the image $e_M(EM)$ in J^kXM at each $M \in \mathrm{ob}(\mathcal{M}_n)$. (This is precisely analogous to the fact that, when talking about embedded submanifolds of a smooth manifold, we do not really care about the particular choice of domain used to define the embedding—we care about its image in the codomain.) This is why the subobject construction is helpful: it provides a way of 'forgetting' about the particular choice of natural bundle we have used as the source of e, since any other isomorphic natural bundle would do equally well. We will denote a natural equation on a natural bundle as a pair $(\mathfrak{E}, X)$.

Second, it is worth emphasising that our definition of natural equations as subobjects of a natural jet bundle in $\mathcal{NB}_n$ implicitly imposes several further regularity conditions on the (closed, embedded) submanifolds we are considering as equations. Namely, and as mentioned above, it imposes that for each $M \in \mathrm{ob}(\mathcal{M}_n)$, $\mathfrak{E}M$ is a subbundle of $J^kXM \to M$. This is harmless, in the sense that were we to define natural equations in a more general way, these additional regularity conditions would hold anyway as a consequence of functoriality (see Weatherall and March (2025, Appendix A) for a precise statement of this and proof).

Finally, suppose we are given kth-order equations E, E' on bundles $B \overset{\pi_B}{\to} M$,

$B' \overset{\pi_{B'}}{\to} M'$. Of course, we say that they are isomorphic iff there exists a smooth bundle isomorphism $(\Phi, \varphi) : B \to B'$ such that $j^k\Phi^*(E') = E$. Similarly, we can define a notion of isomorphism for natural equations:

Definition 2.3.3 (Natural equation isomorphism)**.** *Let $\mathfrak{N} = (\mathfrak{E}, X)$, $\mathfrak{N}' = (\mathfrak{E}', X')$ be kth-order natural equations on n-manifolds. A natural equation isomorphism $i : \mathfrak{N} \to \mathfrak{N}'$ is a natural isomorphism $i : X \to X'$ such that the following diagram is a pullback (in $\mathcal{NB}_n$):*[14]

$$\begin{tikzcd} \mathfrak{E} \arrow[r] \arrow[d] & J^kX \arrow[d, "j^k i"] \\ \mathfrak{E}' \arrow[r] & J^kX' \end{tikzcd}$$

Note that these definitions apply only to pairs of (natural) equations of some fixed order, k. But one might be interested in a broader notion of equivalence between (natural) equations which relaxes this condition, to capture, e.g. the idea that lower-order systems of equations imply systems at higher order, or that higher-order systems of equations can be reduced to lower-order systems of equations by treating lower-order derivatives as independent auxiliary variables. This will be the task of the next section.[15]

2.4 Order reduction

I will begin with some motivation. What should one mean by an order reduction of a system of equations? Here, it is helpful to consider an example. The vacuum Maxwell's equations, formulated as a second-order system on the bundle of one-forms, $2\nabla_n\nabla^{[n}A^{a]} = \mathbf{0}$, are in some sense equivalent to the first-order system $\nabla_n F^{na} = \mathbf{0}$, $\nabla_{[a}F_{bc]} = \mathbf{0}$ on the bundle of two-forms. Namely, if one makes

[14]It is important, here, to keep in mind that this notation is slightly misleading, because $\mathfrak{E}$ and $\mathfrak{E}'$ are not really objects in $\mathcal{NB}_n$, and the pullback of $\mathfrak{E}'$ along $j^k i$ when it exists is, formally, the equivalence class consisting of morphisms $j^k i^* e'$ for each $e' : E' \to J^kX'$ in $\mathfrak{E}'$. (That the result of this construction is an equivalence class, and in fact a subobject of J^kX, follows from the fact that pullbacks preserve monomorphisms and isomorphisms.)

[15]As far as I am aware, no such general and geometric account has been given in the mathematics literature to date.

the identification $F_{ab} = 2\mathrm{d}_a A_b$, then one can show that any solution of the first system induces a solution of the second system, and conversely, any solution of the second system gives rise to (a family of) solutions of the first system. In other words, an order reduction of a system of equations E (of some order, say, k) on a bundle B consists of a lower-order system of equations E' (say, of order ℓ) on a bundle B', plus a rule for transforming kth derivatives of B fields into ℓth derivatives of B' fields, such that E' is obtained from E upon application of this rule, and conversely the original system E is recovered from E' in the same way.

These considerations suggest the following definition:

Definition 2.4.1 (Order reduction)**.** *Let E be a kth-order equation on a bundle $B \to M$. A reduction of E to order ℓ consists of a bundle B', a smooth bundle morphism $(\Phi, \varphi) : J^k B \to J^\ell B'$, and an ℓth-order equation E' on B' such that $\Phi^*(E') = E$.*

In this definition, the smooth bundle morphism $(\Phi, \varphi) : J^k B \to J^\ell B'$ tells us how to think about k-jets of B fields as ℓ-jets of B' fields, whilst the condition $\Phi^*(E') = E$ captures the requirement that, under this identification, E' is obtained from E and conversely E is recovered from E'. Note that the definition of an isomorphims of equations given in §2.3 is clearly a special case of order reduction, where $\ell = k$ and (Φ, φ) is the kth jet prolongation of a smooth bundle isomorphism between B and B'.

There is an obvious analogue of definition 2.4.1 for natural equations on natural bundles:

Definition 2.4.2 (Natural order reduction)**.** *Let $(\mathfrak{E}, X)$ be a natural, kth-order equation on X. A natural reduction of $(\mathfrak{E}, X)$ to order $\ell \leq k$ is an ℓth-order natural equation $(\mathfrak{E}', X')$ and a natural transformation $f : J^k X \to J^\ell X'$ such that the following diagram is a pullback (in $\mathcal{NB}_n$):*

$$\begin{tikzcd}
\mathfrak{E} \arrow[r] \arrow[d] & J^k X \arrow[d, "f"] \\
\mathfrak{E}' \arrow[r] & J^\ell X'
\end{tikzcd}$$

In other words, a natural order reduction of a natural kth-order equation $(\mathfrak{E}, X)$ to order ℓ consists of the specification of a natural, ℓth-order equation $(\mathfrak{E}', X')$ and a natural transformation $f : J^k X \to J^\ell X'$ such that our original equation $(\mathfrak{E}, X)$ is (uniquely, up to unique isomorphism) recovered from the data $(\mathfrak{E}', X', f)$. Again, note that our definition of a natural equation isomorphism from §2.3 is clearly a special case of natural order reduction, where $\ell = k$ and f is the kth jet prolongation of a natural isomorphism between X and X'.

These definitions are abstract, and very general. How they work is perhaps clearest when one considers examples. First, consider the idea that lower-order systems of equations imply equations at higher order. In our current framework, we can understand this as follows. Let $E' \subset J^\ell B \to ... \to B \to M$ be an ℓth-order equation on B, let $k > \ell$, and consider the kth-order equation E on B defined via $E = (\pi_B^{(k,\ell)})^*(E')$. Then, clearly, $(E', B, (\pi_B^{(k,\ell)}, \mathrm{id}_M))$ is a reduction of E to order ℓ. By contrast, note that generically, no order reduction exists in the reverse direction. That is, if E is a kth-order equation on B, and $\ell < k$, then in general $(\pi_B^{(k,\ell)}(E), B, (\pi_B^{(k,\ell)}, \mathrm{id}_M))$ is not an order reduction of E, because $(\pi_B^{(k,\ell)})^*(\pi_B^{(k,\ell)}(E))$ will not coincide with E—or in other words, one generally loses information about an equation by simply discarding its higher derivatives.

Second, consider the idea that equations at higher order can be transformed into lower-order ones by treating lower-order derivatives as independent auxiliary variables. Let $E \subset J^k B \to ... \to B \to M$ be a kth-order equation on B, and let $\ell < k$. As discussed in §2.2, there is a monic natural transformation $\iota^{(\ell,k-\ell)} : J^k \to J^\ell J^{k-\ell}$, which induces a smooth bundle morphism $\iota_B^{(\ell,k-\ell)} : J^k B \to J^\ell J^{k-\ell} B$. Thus, given a bundle $B' \to M$ and a smooth bundle morphism $(\Phi, \mathrm{id}_M) : J^{k-\ell} B \to B'$, where Φ is a monomorphism, then defining $E' = j^\ell \Phi \circ \iota_B^{(\ell,k-\ell)}$, we have that $(E', B', j^\ell \Phi \circ \iota_B^{(\ell,k-\ell)})$ is a reduction of E to order ℓ. Thus, we have obtained an order reduction of E by treating $(k-\ell)$th derivatives of B fields as independent auxiliary variables, i.e. as B' fields. Note that this framework already captures the idea that, in treating such derivatives as independent auxiliary variables, one must generally impose 'additional' equations to guarantee consistency (e.g. that auxiliary variables which correspond to exterior derivatives must be closed differential forms). That comes about at the level of the map $\iota_B^{(\ell,k-\ell)}$—in particular, as noted in §2.2, the fact that it is a monomorphism but not an isomorphism. One can think of the points in $J^\ell J^{k-\ell}B$ which lie outside of the image of $\iota_B^{(\ell,k-\ell)}$ as precisely those points which are ruled out by these additional 'consistency' equations.

To make this concrete, let us return to the Maxwell example with which we began. Let $A : \mathcal{M}_n \to \mathcal{FB}$ be the natural bundle of one-forms, and $F : \mathcal{M}_n \to \mathcal{FB}$ the natural bundle of two-forms. There is a family of natural transformations $\alpha \mathrm{d} : J^1 A \to F$ given by constant multiples of the exterior derivative on (local) sections of A—we take $\alpha = 2$.[16] Now let (M, g_{ab}) be a relativistic spacetime, let E be the second-order equation on AM which consists of all points satisfying $2\nabla_n \nabla^{[n} A^{a]} = \mathbf{0}$, relative to the metric and its associated Levi-Civita connection at that point, and likewise let E' be the first-order equation on FM which consists of all points satisfying $\nabla_n F^{na} = \mathbf{0}$, $\nabla_{[a} F_{bc]} = \mathbf{0}$, relative to the metric and its associated Levi-Civita connection at that point. Then $(E', FM, j^1(2\mathrm{d}) \circ \iota_{AM}^{(1,1)})$ is a reduction of E to order 1. If one wishes, one can then define the bundle $AM \times_M FM$, and a 1st-order equation on $AM \times_M FM$ consisting of all points $(j^1_p\psi, j^1_p\chi)$ such that $2\mathrm{d}(j^1_p\psi) = \chi(p)$ and $j^1_p\chi \in E'$, whose solutions, restricted to AM, coincide with the solutions of E.

2.5 Minimal coupling

I will now move on to discuss minimal coupling. As we saw in §2.3, the subobject construction in $\mathcal{NB}_n$ is important for thinking about natural equations on natural bundles. As such, one might wonder about the significance of the

[16] In fact, these are the only such natural transformations (Kolář et al. 1993, §25.4).

dual concept of a quotient object (or co-subobject) in $\mathcal{NB}_n$. It turns out that quotient objects in $\mathcal{NB}_n$ are closely related to the notion of what it is for an equation to 'depend' on some 'background structure', which in turn is closely related to the criterion of minimal coupling. Explicating that criterion will be the business of this section.

We begin, following Geroch (1996), with a definition:

Definition 2.5.1 (Quotient bundle)**.** *Let $B \xrightarrow{\pi} M$ be a smooth fibre bundle. A quotient bundle of B is a tuple $(B, \check{\pi}, \hat{B})$, where $\hat{B} \xrightarrow{\hat{\pi}} M$ is a smooth fibre bundle and $\check{\pi} : B \to \hat{B}$ is a smooth map, such that $\check{\pi} : B \to \hat{B}$ is a smooth fibre bundle and $\pi = \hat{\pi} \circ \check{\pi}$.*

In other words, a quotient bundle $B \xrightarrow{\check{\pi}} \hat{B} \xrightarrow{\hat{\pi}} M$ 'inserts a fibre bundle' $\hat{B}$ in between B and M, in such a way that there is created a fibre bundle on either side of $\hat{B}$. There is an obvious extension of the definition of a quotient bundle to the case of natural bundles:

Definition 2.5.2 (Natural quotient bundle)**.** *Let $X : \mathcal{M}_n \to \mathcal{FB}$ be a natural bundle. A natural quotient bundle of X is a tuple $(X, \mathfrak{Y})$, where $\mathfrak{Y}$ is a quotient object of X in $\mathcal{NB}_n$ such that, for any $y \in \mathfrak{Y}$ and any $M \in \mathrm{ob}(\mathcal{M}_n)$, $XM \xrightarrow{y_M} YM$ is a smooth fibre bundle, where Y denotes the target of y. If $y : X \to Y$ is an element of $\mathfrak{Y}$, we will say that we have a natural quotient of X over Y.*

Again, it is worth unpacking this definition carefully. A quotient object $\mathfrak{Y}$ of X in $\mathcal{NB}_n$ is an equivalence class of epic natural transformations with source X, under the equivalence relation $y \sim y'$, $y : X \to Y$, $y' : X \to Y'$, iff there exists a natural isomorphism $i : Y \to Y'$ such that $y' = i \circ y$, where $Y, Y' : \mathcal{M}_n \to \mathcal{FB}$ are natural bundles. For any such $y : X \to Y$ in $\mathfrak{Y}$, and any $M \in \mathrm{ob}(\mathcal{M}_n)$, our definition says that (XM, y_M, YM) is a quotient bundle. (Why? Because we know, from the definition of a natural quotient bundle, that $XM \xrightarrow{y_M} YM$ is a smooth fibre bundle, and since 1_n is terminal, we automatically have $\pi_X = \pi_Y \circ y$ and hence $\pi_{XM} = \pi_{YM} \circ y_M$.) As before, we could just stop here, and define a natural quotient bundle as a tuple (X, y, Y), where X and Y

are natural bundles, and $y : X \to Y$ is an epic natural transformation satisfying the above conditions. But (as will become clear in what follows) we do not really care about the particular choice of object Y. The quotient object construction allows us to 'forget' about the particular choice of bundle we have used as the target of y, since any other isomorphic natural bundle would do equally well.

The above definitions of a (natural) quotient bundle have the following simple consequence. I will begin with the not-necessarily-natural case, since this will bring out most clearly the advantages of thinking about natural quotient bundles as quotient objects in $\mathcal{NB}_n$, and will also bring out the reason why it is less straightforward to do so in the case of not-necessarily-natural bundles. So let $(B, \breve{\pi}, \hat{B})$ be a quotient bundle, and let $\psi : M \to \hat{B}$ be a smooth (global) section. (This structure may arise on its own, or from a natural quotient $(X, \mathfrak{Y})$ for a particular element of $\mathfrak{Y}$ and choice of $M \in \mathrm{ob}(\mathcal{M}_n)$.) Then we can define a new bundle $B_{|\psi}$ over M, with total space $B_{|\psi} = \breve{\pi}^{-1}(\psi(M))$, and projection map $\pi_{|\psi}$ which is the restriction of π to $B_{|\psi}$. In fact, letting $i : B_{|\psi} \to B$ denote the subspace embedding, the following diagram is a pullback (in the category $\mathcal{FB}$):

$$\begin{tikzcd}
B_{|\psi} \arrow[r, "{(i, \mathrm{id}_M)}"] \arrow[d, "{(\pi_{|\psi}, \mathrm{id}_M)}"] & B \arrow[d, "{(\breve{\pi}, \mathrm{id}_M)}"] \\
1_n M \arrow[r, "{(\psi, \mathrm{id}_M)}"] & \hat{B}
\end{tikzcd}$$

In what follows, we will primarily be interested in quotient bundles from the perspective of this construction. But this also reveals a certain sense in which the definition of a quotient bundle, as presented above, gives us more than we want. In particular, given any other quotient bundle $(B, \breve{\pi}', \hat{B}')$ of B and a smooth bundle isomorphism $(\Phi, \mathrm{id}_M) : \hat{B} \to \hat{B}'$ with $\breve{\pi}' = \Phi \circ \breve{\pi}$, one would have, applying the same construction to the section $\Phi \circ \psi$ of $\hat{B}'$, $B_{|\psi} = B_{|\Phi \circ \psi}$. From this perspective, and as noted above, we do not really care about the particular choice of bundle $\hat{B}$—we care about the bundle structure of B over $\hat{B}$, or which points in B 'live in the same fibre' over $\hat{B}$, as expressed by the map $\breve{\pi}$. This is precisely what the quotient object construction in the definition of a natural quotient bundle allows us to express. That is, given a natural quotient bundle $(X, \mathfrak{Y})$ and any $M \in \mathrm{ob}(\mathcal{M}_n)$, any section $\psi : M \to YM$ of some YM, where $y : X \to Y$ is in $\mathfrak{Y}$, will induce a section $\psi' : M \to Y'M$ for every other $y' : X \to Y'$ in $\mathfrak{Y}$, all of which give rise to the same bundle $XM_{|\psi}$. In this case, I will either work with a canonical representative of $\mathfrak{Y}$ (as when, e.g., X is a product, see below), or abuse terminology and speak of a section of $\mathfrak{Y}M$, by which I mean an equivalence class of sections $\psi : M \to YM$, one for each $y : X \to Y$ in $\mathfrak{Y}$, such that for any other $y' : X \to Y'$ in $\mathfrak{Y}$, there exists a natural isomorphism $i : Y \to Y'$ such that the following diagram commutes:

$$\begin{tikzcd}
1_n M \arrow[r, "{(\psi, \mathrm{id}_M)}"] \arrow[dr, "{(\psi', \mathrm{id}_M)}"'] & YM \arrow[d, "i_M"] & XM \arrow[l, "y_M"'] \arrow[dl, "y'_M"] \\
& Y'M &
\end{tikzcd}$$

In what follows, two sections $\psi : M \to \mathfrak{Y}M$, $\chi : M \to \mathfrak{Y}M$ will be said to be k-equivalent at some $p \in M$ iff their representatives are k-equivalent at p for any, and hence every, $y : X \to Y$ in $\mathfrak{Y}$.

Could we do something similar for not-necessarily-natural quotient bundles? Yes—but the situation is less straightforward. In particular, the collection of quotient bundles $(B, \breve{\pi}, \hat{B})$, $(B, \breve{\pi}', \hat{B}')$ of some smooth fibre bundle B satisfying $\breve{\pi}' = \Phi \circ \breve{\pi}$ for some bundle isomorphism $(\Phi, \mathrm{id}_M) : \hat{B} \to \hat{B}'$ is not simply a special case of the quotient object construction in $\mathcal{FB}$ (unlike in $\mathcal{NB}_n$), because the condition that $\breve{\pi}$ is an epimorphism covering the identity on M is not preserved under isomorphism in $\mathcal{FB}$ (whereas it holds automatically in $\mathcal{NB}_n$). Now admittedly, this bare technical problem can be overcome (essentially by defining one's way out of it), but the resulting definition is not particularly transparent. For this reason, I will retain Geroch's definition 2.5.1 for the not-necessarily-natural case, though it should be borne in mind that to recover that definition from the case of a natural quotient $(X, \mathfrak{Y})$ one must choose an element of $\mathfrak{Y}$ as well as an object of $\mathcal{M}_n$.

Finally, it is worth remarking briefly on an important special case of (natural) quotient bundles. Let $B \xrightarrow{\pi} M$ be a bundle, and suppose that there exist bundles $\hat{B} \xrightarrow{\hat{\pi}} M$, $\hat{B}' \xrightarrow{\hat{\pi}'} M$ and smooth maps $\check{\pi} : B \to \hat{B}$, $\check{\pi}' : B \to \hat{B}'$ such that the following diagram is a pullback (in $\mathcal{FB}$):

$$\begin{tikzcd}
B \arrow[r, "(\check{\pi}, \mathrm{id}_M)"] \arrow[d, "(\check{\pi}', \mathrm{id}_M)"'] & \hat{B} \arrow[d, "(\hat{\pi}, \mathrm{id}_M)"] \\
\hat{B}' \arrow[r, "(\hat{\pi}', \mathrm{id}_M)"] & 1_n M
\end{tikzcd}$$

In this case, B is just the ordinary fibred product $\hat{B}' \times_M \hat{B}$, $(B, \check{\pi}, \hat{B})$ is a quotient bundle, and by construction, for any section $\psi : M \to \hat{B}$, the bundle $B_{|\psi}$ is (canonically) isomorphic (as bundles) to $\hat{B}'$. The situation simplifies further in the case of natural bundles. Let $X : \mathcal{M}_n \to \mathcal{FB}$ be a natural bundle, and suppose that there exist natural bundles $Y : \mathcal{M}_n \to \mathcal{FB}$, $Z : \mathcal{M}_n \to \mathcal{FB}$ and natural transformations $y : X \to Y$, $z : X \to Z$ such that the following diagram is a pullback (in $\mathcal{NB}_n$):

$$\begin{tikzcd}
X \arrow[r, "y"] \arrow[d, "z"'] & Y \arrow[d, "\pi_Y"] \\
Z \arrow[r, "\pi_Z"] & 1_n
\end{tikzcd}$$

In this case, note that since 1_n is terminal, X is just the ordinary product $Z \times Y$ in $\mathcal{NB}_n$, $(X, [y])$ is a natural quotient, and for any $M \in \mathrm{ob}(\mathcal{M}_n)$ and any section $\psi : M \to YM$, the bundle $XM_{|\psi}$ is (canonically) isomorphic (as bundles) to ZM.

Before going into more detail on quotient bundles, let's see why they matter. Given our definition of a natural equation in §2.3, one might reasonably ask: are standard examples of field theories considered in physics natural? In general, the answer to this question is 'no'. (Here, I closely follow Weatherall and March

(2025).) Consider the vacuum Maxwell's equations. The bundle of two-forms $F : \mathcal{M}_n \to \mathcal{FB}$ is a natural bundle, thus given a relativistic spacetime (M, g_{ab}), the source-free Maxwell equations $\nabla_n F^{na} = \mathbf{0}$, $\nabla_{[a}F_{bc]} = \mathbf{0}$ determined by g_{ab} and its associated Levi-Civita connection ∇ is a (vector) subbundle E of $J^1FM \to FM \to M$. But E is not preserved by arbitrary diffeomorphisms (or rather, their lifts to J^1FM); it is straightforward to see that if $\varphi : M \to M$ is a diffeomorphism, then $J^1F\varphi(E) = E$ iff $\varphi_* g_{ab} = g_{ab}$. Hence, Maxwell's equations cannot be natural.

There is a simple intuition behind this result: Maxwell's equations don't just depend on the Faraday two-form, they also depend on the metric and its associated Levi-Civita connection. But this also gives us an idea how to 'naturalize' Maxwell's equations, by making the dependence on the metric and Levi-Civita connection explicit. That is, whilst $J^1F\varphi(E)$ may not coincide with E, it will coincide with the equation one would get by expressing Maxwell's equations relative to $\varphi_* g_{ab}$.

(Natural) quotient bundles can help us make this idea precise. To this end, we define:

Definition 2.5.3 (Naturalization)**.** *Let $B \xrightarrow{\pi} M$ be a fibre bundle and let $E \subset J^kB$ be a kth-order equation on B. An ℓth-order, $\ell \geq k$, naturalization of E consists of the following data: an ℓth-order natural equation $(\mathfrak{E}, X)$, a natural quotient bundle $(X, \mathfrak{Y})$, a section $\psi : M \to \mathfrak{Y}M$, and a bundle isomorphism $(\Phi, \varphi) : B \to XM_{|\psi}$, such that $j^\ell\Phi((\pi_B^{(\ell,k)})^{-1}(E)) = \mathfrak{E}M_{|\psi}$, where $\mathfrak{E}M_{|\psi}$ denotes the submanifold of $J^\ell XM_{|\psi}$ defined via $\mathfrak{E}M_{|\psi} = e_M(EM)_{|\psi}$, where $e : E \to J^\ell X$ is any representative of $\mathfrak{E}$.*

In this case, if $y : X \to Y$ is any representative of $\mathfrak{Y}$, we will say that we have an ℓth-order naturalization of E over Y.

In other words, a naturalization of an equation E on a bundle $B \xrightarrow{\pi} M$ consists of a natural equation $(\mathfrak{E}, X)$ on a natural quotient bundle $(X, \mathfrak{Y})$, which gives rise to the equation E relative to some section of $\mathfrak{Y}M$. (If we have $\ell > k$, it gives rise to the ℓth-order equation obtained from E via the pullback with respect to the projection map $\pi_B^{(\ell,k)}$.) Now suppose that $X = Z \times Y$, where $y : X \to Y$ is in $\mathfrak{Y}$. Then, in particular, $(\mathfrak{E}, X)$ is a naturalization of E over Y, and for any $M \in \mathrm{ob}(\mathcal{M}_n)$, sections of YM can be thought of as giving rise to different (but related) equations on ZM (including E). Note that, since naturalization involves a choice about how to extend E to different n-manifolds (and sections of $\mathfrak{Y}$ over those n-manifolds), one should not expect a general prescription for naturalizations (though, as we will show in the next section, these do exist in certain special cases).

The concept of a naturalization can be used to explicate what it is for an equation to 'depend' on some background structure:

Definition 2.5.4 (Dependence)**.** *Let $B \xrightarrow{\pi} M$ be a fibre bundle and let $E \subset J^k B$ be a kth-order equation on B. Let $Y : \mathcal{M}_n \to \mathcal{FB}$ be a natural bundle. Then E depends only on the background structure Y, to at most ℓth order, $\ell \geq k$, iff there exists an ℓth-order naturalization $(\mathfrak{E}, X)$ of E over Y such that, for any $M \in \mathrm{ob}(\mathcal{M}_n)$, any sections $\psi, \chi : M \to YM$, and any $p \in M$, the condition $j^\ell_p \psi = j^\ell_p \chi$ implies $\mathfrak{E}M_{|\psi} \cap \pi^{-1}_{J^\ell XM}(p) = \mathfrak{E}M_{|\chi} \cap \pi^{-1}_{J^\ell XM}(p)$.*

This deserves some comment. First, the condition that there exists an ℓth-order naturalization $(\mathfrak{E}, X)$ of E over Y captures the requirement that the equation E explicitly depends on only the background structure Y to ℓth order.[17] The second condition, meanwhile, says that this explicit ℓth-order dependence on Y exhausts the order of dependence on Y in the equation $\mathfrak{E}$, in the sense that for any $M \in \mathrm{ob}(\mathcal{M}_n)$, sections of YM which are ℓ-equivalent at some $p \in M$ give rise to equations which agree at p. To get a handle on how these conditions can come apart, consider, e.g., the natural bundle $X : \overline{\mathcal{M}}_n \to \mathcal{FB}$ whose fibres are manifolds of pairs (∇, g_{ab}) of (torsion-free) affine connections and Lorentzian metrics. Let $G : \overline{\mathcal{M}}_n \to \mathcal{FB}$ be the bundle whose fibres are manifolds of Lorentzian metrics. We can construct a natural quotient bundle $(X, \mathfrak{G})$ by letting $\mathfrak{G} = [g]$, where $g : X \to G$ is the natural transformation which

[17] Of course, it might depend on less than this—in general, one can always construct further naturalizations by building in additional fields to the quotient bundle X or by pulling back $\mathfrak{E}$ to a higher-order jet bundle over X.

take pairs (∇, g_{ab}) to metrics g_{ab}. But in general, given a first-order natural equation $(\mathfrak{E}, X)$ and some $M \in \mathrm{ob}(\overline{\mathcal{M}}_n)$, sections $\psi, \chi : M \to GM$ satisfying the condition $j^\ell_p\psi = j^\ell_p\chi$ at some $p \in M$ will not give rise to equations which satisfy $\mathfrak{E}M_{|\psi} \cap \pi^{-1}_{J^\ell XM}(p) = \mathfrak{E}M_{|\chi} \cap \pi^{-1}_{J^\ell XM}(p)$ (this happens, e.g. if we let $\ell = 1$ and $(\mathfrak{E}, X)$ be the (first-order) equation imposing metric compatibility and Einstein's equation, or the equation imposing metric compatibility and that $R^a{}_{bcd} = \mathbf{0}$). The same remarks apply if we build additional fields into the bundle X, e.g. one-form or two-form fields.

Again, it is helpful to see how naturalization and dependence work with an example. Consider, again, Maxwell's equations. Let $F \times G : \overline{\mathcal{M}}_n \to \mathcal{FB}$ be the natural bundle whose fibres, over each n-manifold, are manifolds of pairs (F_{ab}, g_{ab}), where F_{ab} is a two-form and g_{ab} is a Lorentzian metric. We construct a natural quotient $(F \times G, \mathfrak{G})$ by letting $\mathfrak{G} = [g]$, where $g : F \times G \to G$ is the natural transformation which take pairs (F_{ab}, g_{ab}) to metrics g_{ab}, so that our original spacetime corresponds to a section $\psi : M \to GM$. Then we can define a first-order, natural equation $(\mathfrak{E}, F \times G)$ on $F \times G$ by taking the source-free Maxwell's equation at every point relative to the metric and its Levi-Civita derivative operator at that point.[18] The result is a naturalization of E, and E depends only on the background fields G (i.e., the metric). Finally, observe that for any $M \in \mathrm{ob}(\overline{\mathcal{M}}_n)$, any sections $\psi, \chi : M \to GM$, and any $p \in M$, the condition $j^1_p\psi = j^1_p\chi$ implies $\mathfrak{E}M_{|\psi} \cap \pi^{-1}_{J^1FM}(p) = \mathfrak{E}M_{|\chi} \cap \pi^{-1}_{J^1FM}(p)$, so the dependence of E on the background fields G is at most first order.

As a second example, consider the theories which Pooley (2017) and Read (2023) call SR1 and SR2. SR1 is a second-order theory formulated on the natural bundle K of real scalar fields, for a (fixed) Minkowski spacetime background (M, η_{ab}), with $E \subset J^2KM$ given by the source-free Klein–Gordon equation $\nabla_n\nabla^n\varphi = 0$ determined by η_{ab} and its associated Levi-Civita connection ∇.

[18] This makes sense, since the Levi-Civita derivative operator is the unique first-order natural operator which takes G to the bundle of affine connections (Kolář et al. 1993, §29.7). (The same result holds for Riemannian metrics with $n = 2$ or $n > 3$; for Riemannian metrics with $n = 1$ or $n = 3$, the Levi-Civita derivative operator is still natural, but is not the unique such operator—see Kolář et al. (1993, §33.19).)

This theory is not natural, for exactly the same reason as the Maxwell example discussed above. SR2, meanwhile, is a second-order natural theory $(\mathfrak{E}, K \times G)$ on the bundle $K \times G : \overline{\mathcal{M}}_n \to \mathcal{FB}$, with $\mathfrak{E}$ the natural equation defined by taking the source-free Klein–Gordon equation, at each point, relative to the metric and its Levi-Civita derivative operator at that point, whenever that Levi-Civita derivative operator satisfies $R^a{}_{bcd} = \mathbf{0}$ there. It is straightforward to see that $(\mathfrak{E}, K \times G)$ is a (second-order) naturalization of E over G. But this also illustrates my above point about naturalization not being a unique prescription, at least not in general. One could equally well drop the condition $R^a{}_{bcd} = \mathbf{0}$ (or indeed, replace it with some other suitable condition, e.g., the vacuum Einstein's equation $R_{ab} = \mathbf{0}$, thus yielding the theory Pooley (2017) and Read (2023) call GR2).[19] The result would then be a different naturalization of E over G.

I can now introduce Weatherall and March's (2025) definition of minimal coupling. The idea, in brief (though see Weatherall and March (2025) for further motivation), is that we have now given a precise definition of what it is for an equation to depend on some background structure, to some order, in terms of naturalization. Minimal coupling then just says that an equation depends only on the metric, to at most first order. This gives us two (closely related) definitions. In what follows, let $G : \overline{\mathcal{M}}_n \to \mathcal{FB}$ be the (first-order, natural) bundle whose fibres are manifolds of Lorentzian metrics. Our first definition concerns what it is for a natural equation to be minimally coupled to the metric:

Definition 2.5.5 (Minimally coupled equation)**.** *Let* $(\mathfrak{E}, X)$ *be a first-order natural equation, where* $X : \overline{\mathcal{M}}_n \to \mathcal{FB}$ *is a natural bundle, and suppose that there exists a natural quotient bundle* $(X, \mathfrak{G})$*, where* $g : X \to G$ *is in* $\mathfrak{G}$*. Then* $(\mathfrak{E}, X)$ *is minimally coupled iff, for any* $M \in \mathrm{ob}(\overline{\mathcal{M}}_n)$*, any sections* $\psi, \chi : M \to GM$*, and any* $p \in M$ *the condition* $j^1_p\psi = j^1_p\chi$ *implies* $\mathfrak{E}M_{|\psi} \cap \pi^{-1}_{J^1XM}(p) =$

[19] Note that the theory Pooley (2017) and Read (2023) call GR1—which replaces the condition $R^a{}_{bcd} = \mathbf{0}$ with Einstein's equation, as sourced by the stress-energy for φ at each point, would not quite be a naturalization of SR1 in the present sense, because if $\psi : M \to GM$ is a section corresponding to the Minkowski metric on M, then one does not recover SR1, but rather the sector of SR1 in which the stress-energy for the matter fields φ vanishes.

$\mathfrak{E}M_{|\chi} \cap \pi^{-1}_{J^1XM}(p)$.

In other words, a first-order natural equation is minimally coupled iff, relative to any $M \in \mathrm{ob}(\overline{\mathcal{M}}_n)$ and any section $\psi : M \to GM$, it gives rise to 'denaturalized' equations which depend only on the metric, to at most first order. Similarly, we can also say what it is to have a minimal coupling of a not-necessarily natural equation. Namely:

Definition 2.5.6 (Minimal coupling)**.** *Let $E \subset J^1B \to B \xrightarrow{\pi} M$ be a first-order natural equation on B. A minimal coupling of E consists of a first-order naturalization $(\mathfrak{E}, X)$ of E over G such that for any $M \in \mathrm{ob}(\overline{\mathcal{M}}_n)$, any sections $\psi, \chi : M \to GM$, and any $p \in M$ the condition $j^1_p\psi = j^1_p\chi$ implies $\mathfrak{E}M_{|\psi} \cap \pi^{-1}_{J^1XM}(p) = \mathfrak{E}M_{|\chi} \cap \pi^{-1}_{J^1XM}(p)$.*

2.6 Minimal coupling is unique

Definitions 2.5.5 and 2.5.6 represent a substantial improvement on previous attempts to explicate **MC** in the literature. For one, they are precise and unambiguous,[20] and capture something 'intrinsic' about a system of partial differential equation—*viz.* dependence only on the metric to at most first order, in the sense of definition 2.5.4. They also give us the resources to understand the sense in which minimal coupling is 'ambiguous' (or "hyperintensional" (Fletcher 2020, p. 266)) for second- or higher-order systems of equations—in that it only *makes sense* to ask whether there exists a first-order naturalization of a system of equations over the bundle of Lorentzian metrics when that system of equations is first-order (see Weatherall and March (2025, p. 16, esp. fn. 9) for detailed discussion).[21]

[20] As a further application of this point, albeit one which would take us a little too far afield here, note that these definitions also give us the resources to resolve questions about the 'correct' application of minimal coupling in the presence of non-metricity and torsion—on which, see e.g. Delhom (2020). In particular, they vindicate—and make unsurprising—Delhom's conclusion that minimal coupling in the presence of non-metricity and torsion is just the usual coupling to the Levi-Civita connection of the metric, rather than to the non-metric or torsionful connection.

[21] But note that minimal coupling *does* apply to any system of equations which can be transformed into a first-order system via an appropriate choice of auxiliarly variables—on which, see §2.4 for details—so the restriction to first-order systems is less narrow than it might appear.

Still, one might ask for more. In particular—and as noted in §1—minimal coupling is often thought of as a 'prescription' for constructing general-relativistic systems of equations from special-relativistic ones. Now, as Weatherall and March (2025) note, given a (first-order) special-relativistic system of equations E on a natural bundle XM, $X : \overline{\mathcal{M}}_n \to \mathcal{FB}$, $M \in \mathrm{ob}(\overline{\mathcal{M}}_n)$, any minimal coupling of E in the sense of definition 2.5.6 will induce a minimally-coupled natural system of equations, in the sense of definition 2.5.5, which, by construction is defined over any $M \in \mathrm{ob}(\overline{\mathcal{M}}_n)$ and any section $\psi : M \to GM$. However, whether it is sensible to think of this as a prescription for constructing general-relativistic systems of equations from special-relativistic equations will depend on whether there is a unique minimal coupling, i.e. first-order naturalization over G, of E.[22] In this direction, Weatherall and March (2025) argue that, among the various (in fact, continuum many) formulations of the vector potential Maxwell's equations that are usually taken to be minimally coupled according to the 'comma goes to semicolon' rule (Linnemann and Read 2021; Misner et al. 1973; Read et al. 2018; Trautman 1965), only one of these is minimally coupled in the sense of definition 2.5.5 (and, moreover, is precisely the one which is standardly taken to be the 'correct' version of the vector potential Maxwell's equations).[23]

I now show that this situation is, in fact, completely general, in the sense that, given a first-order equation E on a bundle $B \xrightarrow{\pi} M$, if there exists a minimal coupling $(\mathfrak{E}, X)$ of E, for some section of $\psi : M \to GM$ and some bundle isomorphism of B and $XM_{|\psi}$, in the sense of definition 2.5.5, then $(\mathfrak{E}, X)$ is the unique equation on the bundle X which is a minimal coupling of E for that section and that bundle isomorphism. It follows that the only 'ambiguity' involved in thinking of minimal coupling as a 'prescription' for constructing general-relativistic systems of equations from special-relativistic ones is in the choice of the the bundle X—essentially, in the global topology of X, which must be locally a fibred product—and so is unique insofar as one has in mind a 'canonical' such choice (e.g. when $B = YM$ for some natural bundle Y, the

[22] Weatherall and March (2025) also give an informal argument to this effect.

[23] Weatherall and March (2025) only discuss two such candidates explicitly, but the same arguments carry over straightforwardly to any of the others.

bundle $X = Y \times G$, given the bundles Y and G). The proof relies on the following result due to Fletcher and Weatherall (2023a) (which I have translated into the notation used in this thesis):[24]

Theorem 2.6.1 (Fletcher and Weatherall, 2023)**.** *Let $M, N \in \mathrm{ob}(\overline{\mathcal{M}}_n)$, and suppose that N is diffeomorphic to $\mathbb{R}^n$. Then for any sections $\psi : M \to GM$, $\chi : N \to GN$, any $p \in M$, and $q \in N$, there exists an open neighbourhood O of p and a diffeomorphism $\varphi : N \to O$ such that $\varphi(q) = p$ and $j^1_q(G\varphi^*\psi) = j^1_q\chi$.*

Proof. See Fletcher and Weatherall (2023a, theorem 5). □

We begin with a couple of lemmas. The first result will be relevant to much of what comes later:

Lemma 2.6.2. *Let $(X, \mathfrak{G})$ be a natural quotient bundle of $X : \overline{\mathcal{M}}_n \to \mathcal{FB}$ over G, and $(\mathfrak{E}, X)$ a first-order natural equation which is minimally coupled. Then for any sections $\psi : M \to GM$, $\psi' : M' \to GM'$ and any $p \in M$, $p' \in M'$, there exist open neighborhoods O, O' of p, p' and a diffeomorphism $\varphi : O \to O'$ such that $j^1_{p'}(G\varphi_*\psi) = j^1_{p'}\psi'$ and $J^1X\varphi_*(\mathfrak{E}M_{|\psi} \cap \pi^{-1}_{J^1XM}(p)) = \mathfrak{E}M'_{|\psi'} \cap \pi^{-1}_{J^1XM'}(p')$.*

Proof. Let $N \in \mathrm{ob}(\mathcal{M}_n)$ be diffeomorphic to $\mathbb{R}^n$. Now let $\psi : M \to GM$ be a section of GM, $\chi : N \to GN$ a section of GN, and take any points $p \in M$, $q \in N$. By Theorem 2.6.1, there exists an open neighborhood O of p and a diffeomorphism $\varphi : N \to O$ such that $\varphi(q) = p$ and $j^1_q(G\varphi^*\psi) = j^1_q\chi$. Moreover, since $(\mathfrak{E}, X)$ is natural, we know that $J^1X\varphi^*(\mathfrak{E}M) = \mathfrak{E}N$. But then $J^1X\varphi^*(\mathfrak{E}M_{|\psi} \cap \pi^{-1}_{J^1XM}(p)) = J^1X\varphi^*(\mathfrak{E}M_{|\psi}) \cap \pi^{-1}_{J^1XN}(q) = \mathfrak{E}N_{|J^1X\varphi^*\psi} \cap \pi^{-1}_{J^1XN}(q) = \mathfrak{E}N_{|G\varphi^*\psi} \cap \pi^{-1}_{J^1XN}(q) = \mathfrak{E}N_{|\chi} \cap \pi^{-1}_{J^1XN}(q)$, where we have used that smooth bundle morphisms are fibre preserving in the first equality, that $J^1X\varphi^*(\mathfrak{E}M) = \mathfrak{E}N$ in the second, that $(X, \mathfrak{G})$ is a natural quotient in the third, and the final equality follows from the fact that $G\varphi^*\psi$ and χ coincide to first order at q. By the same reasoning, for any section $\psi' : M' \to GM'$, and

[24] Actually, and as Fletcher and Weatherall (2023a) show, something stronger is true: given curve (segments) $\gamma : I \to M$, $\gamma' : I' \to M'$ whose images contain p, p' respectively, one can choose V' and φ such that, in addition, $\varphi^*\gamma' = \gamma$, and $G\varphi^*(\psi')(x) = \psi(x)$, and $j^1_x G\varphi^*(\psi') = j^1_x\psi$ for all $x \in \gamma[I]$. We will not need this in what follows, however.

any $p' \in M'$, there exists an open neighborhood O' of p' and a diffeomorphism $\varphi' : N \to O'$ such that $J^1X\varphi'^*(\mathfrak{E}M'_{|\psi'} \cap \pi^{-1}_{J^1XM'}(p')) = \mathfrak{E}N_{|\chi} \cap \pi^{-1}_{J^1XN}(q)$. But then $J^1X\varphi^*(\mathfrak{E}M_{|\psi} \cap \pi^{-1}_{J^1XM}(p)) = J^1X\varphi'^*(\mathfrak{E}M'_{|\psi'} \cap \pi^{-1}_{J^1XM'}(p'))$, so $\varphi' \circ \varphi^{-1} : O \to O'$ is a diffeomorphism satisfying the conditions of the proposition. $\square$

Lemma 2.6.3. *Let $(X, \mathfrak{Y})$ be a natural quotient bundle of X over Y. Then for any $M \in \mathrm{ob}(\mathcal{M}_n)$ and any submanifolds $S, S' \subset J^1XM$, if for all sections $\psi : M \to YM$, $S_{|\psi} = S'_{|\psi}$, then $S = S'$.*

Proof. Since $(X, \mathfrak{Y})$ is a natural quotient bundle of X over Y, there is some epic natural transformation $y : X \to Y$. Thus $j^1y : J^1X \to J^1Y$ is also an epic natural transformation. It follows that, given any $x \in J^1XM$, there exists some section $\psi : M \to YM$ such that $x \in J^1FM_{|\psi}$. Now suppose that $x \in S$. Then there exists some $\psi : M \to YM$ such that $x \in S_{|\psi}$ and hence $x \in S'_{|\psi}$ and $x \in S'$, and conversely for S'. But then for any $x \in J^1XM$ we have $x \in S$ iff $x \in S'$. So $S = S'$. $\square$

We then have the following uniqueness result for minimal coupling:

Theorem 2.6.4 (Uniqueness of minimal coupling)**.** *Let E be a first-order equation on $B \xrightarrow{\pi} M$, where $M \in \mathrm{ob}(\overline{\mathcal{M}}_n)$. Suppose that there exist minimal couplings $(\mathfrak{E}, X)$, $(\mathfrak{E}', X)$ on $X : \overline{\mathcal{M}}_n \to \mathcal{FB}$, such that there exist a section $\psi : M \to GM$ and a bundle isomorphism $(\Phi, \varphi) : B \to XM_{|\psi}$ such that the restrictions $\mathfrak{E}M_{|\psi}$, $\mathfrak{E}'M_{|\psi}$ of $\mathfrak{E}M$, $\mathfrak{E}'M$ to ψ coincide with $j^1\Phi(E)$. Then:*

- *(i) For any $M \in \mathrm{ob}(\overline{\mathcal{M}}_n)$, any section $\psi : M \to GM$, and any $p \in M$, $\mathfrak{E}M_{|\psi} \cap \pi^{-1}_{J^1XM}(p) = \mathfrak{E}'M_{|\psi} \cap \pi^{-1}_{J^1XM}(p)$.*
- *(ii) For any $M \in \mathrm{ob}(\overline{\mathcal{M}}_n)$ and any section $\psi : M \to GM$, $\mathfrak{E}M_{|\psi} = \mathfrak{E}'M_{|\psi}$.*
- *(iii) For any $M \in \mathrm{ob}(\overline{\mathcal{M}}_n)$, $\mathfrak{E}M = \mathfrak{E}'M$.*
- *(iv) $\mathfrak{E} = \mathfrak{E}'$.*

Proof. Since $\mathfrak{E}M_\chi = j^1\Phi(E) = \mathfrak{E}'M_{|\chi}$, (i) follows from Lemma 2.6.2. But (i) implies (ii), and (ii) implies (iii) by Lemma 2.6.3. So we just need to show that (iii) implies (iv). But since $\mathfrak{E}M = e_M(EM)$ and $\mathfrak{E}'M = e'_M(E'M)$, where $e : E \to J^1X$, $e' : E' \to J^1X$ are arbitrary representatives of $\mathfrak{E}$, $\mathfrak{E}'$ respectively, and for each $M \in \mathcal{M}_n$, e_M and e'_M are diffeomorphisms onto their images, (iii) implies that ${e'_M}^{-1} \circ e_M : \mathfrak{E}M \to \mathfrak{E}'M$ is a diffeomorphism. So we can define a natural isomorphism $i : E \to E'$ whose component, at each $M \in \mathrm{ob}(\overline{\mathcal{M}}_n)$, is ${e'_M}^{-1} \circ e_M$, and clearly we have $e = e' \circ i$, from which it follows that $e' : E' \to J^1X$ is in $\mathfrak{E}$. So we are done. $\square$

2.7 Quasilinearity

Having discussed natural equations and minimal coupling in some detail, the remainder of this section will concern symmetric hyperbolicity, and its interaction with naturality. For this, we need to begin by discussing quasilinearity. We have the following definition:

Definition 2.7.1 (Quasilinearity)**.** *Let $E \subset J^kB \to J^{k-1}B \to ... \to J^1B \to B \to M$ be a kth-order equation on B. Then E is quasilinear iff $\pi^{(k,k-1)}_{B|E} : E \to J^{k-1}B$ is an affine subbundle of $\pi^{(k,k-1)}_B : J^kB \to J^{k-1}B$.*

In other words, to say that an equation is quasilinear is just to say that if $\psi, \psi' : M \to B$ are both solutions of the equation, i.e. $j^k\psi, j^k\psi' \in E$, then we also have $j^k\psi + \alpha(j^k\psi - j^k\psi') \in E$ for any number α. (This is just a way of stating with jets the more usual notion of quasilinearity as 'linearity of an equation in its highest derivatives'—see below.) There is an obvious extension of the notion of quasilinearity to natural equations:

Definition 2.7.2 (Natural quasilinearity)**.** *Let $(\mathfrak{E}, X)$ be a kth-order natural equation on $X : \mathcal{M}_n \to \mathcal{FB}$. Then $(\mathfrak{E}, X)$ is quasilinear iff, for each $e : E \to J^kX$ in $\mathfrak{E}$, (i) $\pi^{(k,k-1)}_X \circ e$ is epic, and (ii), for each $M \in \mathrm{ob}(\mathcal{M}_n)$, and each $p \in J^{k-1}XM$, $(e_M)_{|(\pi^{(k,k-1)}_{XM})^{-1}(p)}$ is an affine map.*

Here, condition (i) asserts that, for each $M \in \mathrm{ob}(\mathcal{M}_n)$, $\pi^{(k,k-1)}_{XM} : \mathfrak{E}M \to J^{k-1}XM$ is a smooth fibre bundle, whilst condition (ii) asserts that this bundle is an affine subbundle of $\pi^{(k,k-1)}_{XM|\mathfrak{E}\mathfrak{M}} : J^kXM \to J^{k-1}XM$.

These definitions have the following straightforward consequence. Let E be a kth-order quasilinear equation on $B \xrightarrow{\pi} M$. Since E is an affine subbundle of $\pi_B^{(k,k-1)} : J^k B \to J^{k-1}B$, it gives rise to an affine bundle morphism k—in effect, a quotient mapping—from $\pi_B^{(k,k-1)} : J^k B \to J^{k-1}B$ to a vector bundle over $J^{k-1}B$ (naturally considered as an affine bundle), whose fibre, at each point in $J^{k-1}B$, is the quotient space of the fibre of $J^k B$ at that point by the fibre of E at that point, i.e. the quotient bundle $(VB \otimes S^k T^* M)/U$, where U is the modelling vector bundle for $\pi_{B|E}^{(k,k-1)} : E \to J^{k-1}B$. (Recall that $VB \otimes S^k T^* M$ is the modelling vector bundle for $\pi_B^{(k,k-1)} : J^k B \to J^{k-1}B$.) In other words, and letting i denote the inclusion of E into $J^k B$, the following diagram is fibrewise exact:

$$\begin{tikzcd}
E \arrow[r, "i"] \arrow[d, "\pi_{B|E}^{(k,k-1)}"] & J^k B \arrow[r, "k"] \arrow[d, "\pi_B^{(k,k-1)}"] & (VB \otimes S^k T^* M)/U \arrow[d] \\
J^{k-1}B \arrow[r, "\mathrm{id}_{J^{k-1}B}"] & J^{k-1}B \arrow[r, "\mathrm{id}_{J^{k-1}B}"] & J^{k-1}B
\end{tikzcd}$$

(In fact, the existence of an affine bundle morphism k to some vector bundle completing such a diagram is an alternative way of characterising a quasilinear equation.) In what follows, we will call the bundle $(VB \otimes S^k T^* M)/U$ the equation space, k the equation map, and denote vectors in the equation space with capital Latin indices. Then given a section $\psi : M \to B$, E can be expressed as an 'equation' as follows: $k^A(j^k\psi) = \mathbf{0}$. (One then recovers the more usual definition of quasilinearity as 'linearity of an equation in its highest derivatives' by considering the expression of this 'equation' in local fibred coordinates, see e.g. Khavkine (2012, pp. 11–12).)

In the above construction, we introduced a bundle U, which we defined as the modelling vector bundle for $\pi_{B|E}^{(k,k-1)} : E \to J^{k-1}B$. We will call this bundle the presystem:

Definition 2.7.3 (Presystem)**.** *Let E be a kth-order quasilinear equation on $B \xrightarrow{\pi} M$. The presystem $U \to J^{k-1}B$ for E is the modelling vector bundle for*

$\pi^{(k,k-1)}_{B|E} : E \to J^{k-1}B$.

Note in particular that if $(\mathfrak{E}, X)$ is a natural equation, then its presystem $U_{\mathfrak{E}}$ is a natural bundle (Kolář et al. 1993, §14.10).

The presystem of a system of (first-order) quasilinear equations will be important when we come to discuss symmetric hyperbolic systems in the next section. Here, the point is that the presystem of a quasilinear equation gives us a useful way of thinking about sources for an equation. As noted above, if E is a kth-order quasilinear equation on $B \xrightarrow{\pi} M$, then given a section $\psi : M \to B$, E can be expressed as $k^A(j^k\psi) = \mathbf{0}$. But now suppose that we move to a different (kth-order, quasilinear) equation, with the same presystem. Then we would have $k^A(j^k\psi) = j^A$, where j^A is any section of $(VB \otimes S^kT^*M)/U \to J^{k-1}B$. Thus we can think of sections of $(VB \otimes S^kT^*M)/U \to J^{k-1}B$ as giving rise to a family of 'sourced' equations, relative to our original 'unsourced' equation E. Note that this notion of 'sourcedness' *only* makes sense relative to a choice for the original equation E.

Amongst other things, this means that given a vertical vector field $v^{\alpha'}$ on B, and in the presence of a connection ∇ on B, we can say what it is for $v^{\alpha'}$ to be (or not to be) a source for E (relative to that connection). Namely, we say that $v^{\alpha'}$ does not source E whenever $\nabla_{a_1}...\nabla_{a_k}v^{\alpha'} \in U$ (or in other words, whenever $k^A(\nabla_{a_1}...\nabla_{a_k}v^{\alpha'}) = \mathbf{0}$), and does otherwise. (If B is a natural, first-order bundle, then we can also consider the case where ∇ is induced by a connection on M.)

To close this section, I will briefly remark on how quasilinearity interacts with minimal coupling and (de)naturalization. To this end, note that if $(\mathfrak{E}, X)$ is a first-order, quasilinear, minimally coupled natural equation, then given any $M \in \mathrm{ob}(\overline{\mathcal{M}}_n)$ and any section $\psi : M \to GM$, the denaturalized system of equations $\mathfrak{E}M_{|\psi}$ is also quasilinear:

Theorem 2.7.4. *Let $(\mathfrak{E}, X)$ be a first-order, quasilinear, minimally coupled natural equation on X. Then for any $M \in \mathrm{ob}(\overline{\mathcal{M}}_n)$ and any section $\psi : M \to GM$, $\mathfrak{E}M_{|\psi}$ is quasilinear.*

Proof. We have the following commuting diagram:

$$\begin{tikzcd}
\mathfrak{E}M_{|\psi} \arrow[r, "{(i', \mathrm{id}_M)}"] \arrow[d, "\pi^{(1,0)}_{XM|\mathfrak{E}M_{|\psi}}"] & \mathfrak{E}M \arrow[d, "\pi^{(1,0)}_{XM|\mathfrak{E}M}"] \\
XM_{|\psi} \arrow[r, "{(i, \mathrm{id}_M)}"] \arrow[d, "\pi_{XM|\psi}"] & XM \arrow[d, "g_M"] \\
1_n M \arrow[r, "{(\psi, \mathrm{id}_M)}"] & GM
\end{tikzcd}$$

Here, i' is the inclusion of $\mathfrak{E}M_{|\psi}$ into $\mathfrak{E}M$, i is the inclusion of $XM_{|\psi}$ into XM, and all other maps are defined as usual. Since the lower and outer squares are pullbacks, so is the upper one. But since $(\mathfrak{E}, X)$ is quasilinear, this immediately implies that $\pi^{(1,0)}_{XM|\mathfrak{E}M_{|\psi}} : \mathfrak{E}M_{|\psi} \to XM_{|\psi}$ is an affine subbundle of $\pi^{(1,0)}_{XM|\psi} : J^1XM_{|\psi} \to XM_{|\psi}$. $\square$

It is worth noting that nothing in this argument depends on the choice of natural quotient bundle, so one actually has a slightly more general result: for any natural quotient $(X, \mathfrak{Y})$ of X, the same is true.

A second point does depend on the details of the natural quotient bundle. One has the following obvious consequence of lemma 2.6.2 (the basic idea of which is just that if two quasilinear equations coincide over some point, then so do their presystems):

Theorem 2.7.5. *Let $(\mathfrak{E}, X)$ be a first-order, quasilinear, minimally coupled equation. Then for any M, $M' \in \mathrm{ob}(\overline{\mathcal{M}}_n)$, sections $\psi : M \to GM$, $\psi' : M' \to GM'$, and any $p \in M$, $p' \in M'$, there exist open neighborhoods O, O' of p, p' and a diffeomorphism $\varphi : O \to O'$ such that $j^1_{p'}(G\varphi_*\psi) = j^1_{p'}\psi'$ and $J^1X\varphi_*(U_{\mathfrak{E}M_{|\psi}} \cap \pi^{-1}_{J^1XM}(p)) = U_{\mathfrak{E}M'_{|\psi'}} \cap \pi^{-1}_{J^1XM'}(p')$.*

Proof. By lemma 2.6.2, for any M, $M' \in \mathrm{ob}(\overline{\mathcal{M}}_n)$, sections $\psi : M \to GM$, $\psi' : M' \to GM'$, and any $p \in M$, $p' \in M'$, there exist open neighborhoods O, O' of p, p' and a diffeomorphism $\chi : O \to O'$ such that $J^1X\varphi_*(\mathfrak{E}M_{|\psi} \cap$

$\pi^{-1}_{J^1XM}(p)) = \mathfrak{E}M'_{|\psi'} \cap \pi^{-1}_{J^1XM'}(p')$. But since the fibre of $U_{\mathfrak{E}M_{|\psi}} \cap \pi^{-1}_{J^1XM}(p)$ over each point in $\pi^{-1}_{XM_{|\psi}}(p)$ consists of the modelling vector space for $\mathfrak{E}M_{|\psi}$ at that point, the condition $J^1X\varphi_*(\mathfrak{E}M_{|\psi} \cap \pi^{-1}_{J^1XM}(p)) = \mathfrak{E}M'_{|\psi'} \cap \pi^{-1}_{J^1XM'}(p')$ implies $J^1X\varphi_*(U_{\mathfrak{E}M_{|\psi}} \cap \pi^{-1}_{J^1XM}(p)) = U_{\mathfrak{E}M'_{|\psi'}} \cap \pi^{-1}_{J^1XM'}(p')$. □

2.8 Symmetric hyperbolic equations

With this background on quasilinearity in hand, we can finally discuss symmetric hyperbolic equations. I will begin with a series of definitions:[25]

Definition 2.8.1 (Symmetrizer). *Let $E \subset J^1B \to B \to M$ be a first-order quasilinear equation, with presystem $U \to B$. A symmetrizer is a tensor field $h^a{}_{\alpha'\beta'}$ on B, symmetric in α' and β', which annihilates every tensor $\theta^{\alpha'}{}_a$ in U at each $p \in B$.*

Definition 2.8.2 (Causal covector). *Let $E \subset J^1B \to B \to M$ be a first-order quasilinear equation, with presystem $U \to B$, and let $h^a{}_{\alpha'\beta'}$ be a symmetrizer. A covector n_a at some $p \in M$ is causal for $h^a{}_{\alpha'\beta'}$ at some $x \in \pi^{-1}(p)$ iff $n_a h^a{}_{\alpha'\beta'}$ is positive definite at x.*

Definition 2.8.3 (Characteristic covector). *Let $E \subset J^1B \to B \to M$ be a first-order quasilinear equation, with presystem $U \to B$. A covector n_a at some $p \in M$ is characteristic at some $x \in \pi^{-1}(p)$ iff there exists a non-zero vertical vector $v^{\alpha'}$ at x such that $n_a v^{\alpha'} \in U$.*

Definition 2.8.4 (Hyperbolization). *Let $E \subset J^1B \to B \to M$ be a first-order quasilinear equation, with presystem $U \to B$, and let $h^a{}_{\alpha'\beta'}$ be a symmetrizer. Then $h^a{}_{\alpha'\beta'}$ is a hyperbolization iff it admits causal covectors at each $x \in B$.*

Definition 2.8.5 (Maximal hyperbolization). *Let $E \subset J^1B \to B \to M$ be a first-order quasilinear equation, with presystem $U \to B$, and let $h^a{}_{\alpha'\beta'}$ be a hyperbolization. Then $h^a{}_{\alpha'\beta'}$ is maximal iff at each $x \in B$, the boundary of the collection of causal covectors for $h^a{}_{\alpha'\beta'}$ contains only characteristics.*

[25] The beginning of this section closely follows the presentations in Geroch (1996), Khavkine (2012), and Weatherall (2014).

Definition 2.8.6 (Symmetric hyperbolic system)**.** *Let $E \subset J^1B \to B \to M$ be a first-order quasilinear equation. Then E is symmetric hyperbolic iff it admits a maximal hyperbolization.*

To understand these definitions, note first that the causal covectors associated with a symmetriser at some point p form an open convex cone at p. Now say that a vector ξ^a is causal for $h^a{}_{\alpha'\beta'}$ at some $x \in B$ iff $\xi^a n_a \geq 0$ for every causal covector n_a for $h^a{}_{\alpha'\beta'}$ at p. The causal vectors at a point form a closed convex cone. One can think of the causal covectors (for a given symmetriser) at a point as representing possible initial value surfaces, and the causal vectors (for a given symmetriser) at a point as representing possible 'signal propagation' velocities off of those surfaces. (The justification for this way of thinking about causal (co)vectors at a point will become clear in what follows.)

So let E be a quasilinear, symmetric hyperbolic equation on a bundle $B \to M$, and fix a hyperbolization $h^a{}_{\alpha'\beta'}$ of E. By initial data for E (relative to the fixed choice of hyperbolization $h^a{}_{\alpha'\beta'}$), we mean a codimension 1 submanifold S of M together with a section $\phi : S \to B$ such that at each $p \in S$ the normal n_a to S at p is causal for $h^a{}_{\alpha'\beta'}$ at $\phi(p)$. Given initial data (S, ϕ) for E, by a solution of E for that initial data we mean a solution $\psi : O \to B$ of E (on some O containing S) such that ψ agrees with ϕ on S. Given a solution $\psi : O \to B$, a smooth curve $\gamma : I \to M$ is causal relative to ψ if, whenever $\gamma(s) \in O$, its tangent vector at $\gamma(s)$ is causal for $h^a{}_{\alpha'\beta'}$ at $\psi(p)$. A point p is an endpoint of a smooth curve $\gamma : I \to M$ iff for any open set O containing p, there is a parameter $s_0 \in I$ such that either for all $s \geq s_0$ or all $s \leq s_0$, $\gamma(s) \in O$. Then, given initial data (S, ϕ), the domain of dependence $D(S)$ of S is the collection of points $p \in M$ such that (i) there exists a solution $\psi : O \to B$ of E for (S, ϕ) with $p \in O$, and (ii) for any smooth curve $\gamma : I \to M$ whose image contains p, and which is causal relative to solution $\psi : O \to B$ of E for (S, ϕ) with $p \in O$, and without endpoint in O, then $\gamma(I)$ intersects S.

We can now state the following (classic, fundamental) uniqueness result for symmetric hyperbolic systems, essentially due to Friedrichs (1954):

Theorem 2.8.7. *E be a quasilinear, symmetric hyperbolic equation on a bundle $B \to M$, and fix a hyperbolization $h^a{}_{\alpha'\beta'}$ of E. Let (S, ϕ), (S, ϕ') be initial data for this system. Suppose that there is some open (in the submanifold topology on S) $S' \subset S$ on which ϕ and ϕ' agree. Then given any solutions $\psi : O \to B$, $\psi' : O' \to B$ of E for (S, ϕ), (S, ϕ'), ψ and ψ' agree on $D(S')$.*

(There is, of course, also the corresponding existence result—but that would require a discussion of constraints and integrability, which would take us too far afield here.) Thus, initial data on S suffice to fix solutions throughout $D(S)$. It is for this reason that one can think of causal vectors as representing possible signal propagation velocities.

In setting up this theorem, we did not mention either characteristics or maximality. So what role do they play? For this, note that the uniqueness result above applies to equations *with a particular hyperbolization*. In general, equations admit multiple hyperbolizations, with different initial value surfaces associated with them. But we are really interested in the initial value formulation for the full equation, not the hyperbolized version. In other words, one would like to know what the possible initial value surfaces are for the full equation—i.e. what the largest collection of causal covectors is.

This is where characteristics become important. To see this, note that characteristic covectors cannot be causal. This is because if n_a is characteristic (i.e. $n_a v^{\alpha'} \in U$ for some non-zero $v^{\alpha'}$, and $n_a h^a{}_{\alpha'\beta'}$ annihilates U, then $n_a h^a{}_{\alpha'\beta'}$ cannot be positive definite, because there exists a non-zero vector $v^{\alpha'}$ such that $n_a h^a{}_{\alpha'\beta'} v^{\alpha'} = \mathbf{0}$—i.e., the vector $v^{\alpha'}$ such that $n_a v^{\alpha'} \in U$. Thus, if the boundary of the collection of causal covectors for some hyperbolization contains only characteristics, we know there cannot be another hyperbolization with wider causal cones. Thus, the maximal hyperbolizations reflect the full causal structure of the equation with which we began.

To see how hyperbolizations work with an example, consider again the Maxwell system. The presystem U for this system has, as fibres, vector spaces of tensors u_{abc}, antisymmetric in b and c, that satisfy $g^{ab} u_{abc} = \mathbf{0}$ and $u_{[abc]} = \mathbf{0}$.

So let $p \in FM$. We are looking for symmetric tensors $h^{a}{}_{\alpha'\beta'}$—i.e. tensors $h^{amnxy} = h^{a[mn][xy]} = h^{axymn}$—with the property that $h^{amnxy}u_{amn} = \mathbf{0}$ for all $u_{amn} \in U$. This can happen in two ways: either $h^{a}{}_{mnxy}$ contracts the first two indices of U with g^{ab} or else antisymmetrizes on all three indices. So we can get hyperbolizations for the Maxwell system by considering tensors of the form $T^{abmnxy} = k(g^{b[m}g^{n][x}g^{y]a} + g^{b[x}g^{y][n}g^{m]a}) + lg^{ab}g^{m[x]g^{y]n}}$. (Here, the first term annihilates U by antisymmetrizing on all three indices and the second term annihilates U by taking its trace; k and l are numbers which are chosen to ensure positive definiteness—which we get if $k = 2l$. We take $k = 1/2$ by convention.) We get a hyperbolization from T^{abmnxy} by contracting with any timelike covector t_b, so that the possible hyperbolizations are tensors $h^{amnxy} = T^{abmnxy}t_b = 1/2(t^{[m}g^{n][x}g^{y]a} + t^{[x}g^{y][n}g^{m]a}) + 1/4t^{a}g^{m[x}g^{y]n}$. This is a symmetrizer by construction, and is a hyperbolization since $n_a h^{amnxy}$ is positive definite for all n_a (co-oriented with t_a) timelike. (This follows from the fact that the Maxwell stress–energy tensor satisfies the DEC.) Finally, note that the characteristic covectors for this system are precisely the null covectors: for any null n_a, we have $n_a n_{[b}x_{c]} \in U$ for any x_a orthogonal to n_a. So these hyperbolizations are also maximal.

At this point, it is worth pausing to note an important feature of the definitions we have encountered thus far. I have now introduced two important kinds of systems of partial differential equations: natural equations, and symmetric hyperbolic equations. But in general—and as pointed out by Geroch (1996) and Weatherall and March (2025)—natural theories cannot be symmetric hyperbolic. There is an easy way to see this.[26] Say that a natural equation $(\mathfrak{E}, X)$ is 'sufficiently rich' iff it admits solutions $\psi : M \to XM$ such that there exists a diffeomorphism $\varphi : M \to M$ for which $X\varphi \circ \psi \neq \psi \circ \varphi$. Meanwhile, a theory 'admits a well-posed initial value problem' iff, given suitable 'initial data' on an $(n-1)$-submanifold of M, there exists an open set O containing that submanifold and a smooth solution on O, agreeing with the initial data on that

[26] A somewhat different argument is given by Geroch (1996), though he does not explicitly discuss this in terms of naturality. Geroch's 'combined system' would be a natural theory, in the present sense.

submanifold, which is unique in the sense that any other such solution agreeing with the initial data coincides with that solution throughout O.[27] We have the following result due to Weatherall and March (2025):

Theorem 2.8.8 (Weatherall and March, 2025)**.** *No sufficiently rich natural theory admits a well-posed initial value problem.*

Now observe that, by the above uniqueness theorem for quasilinear symmetric hyperbolic systems of partial differential equations, such theories do admit a well-posed initial value problem, in the above sense.[28] Thus no sufficiently rich natural theory can be symmetric hyperbolic.

Still, natural equations may still give rise to symmetric hyperbolic systems, relative to a fixed choice of background structure. In fact, the Maxwell example we have been considering has exactly this character. So let $(\mathfrak{E}, X)$ be a natural, quasilinear equation, and $(X, \mathfrak{Y})$ a natural quotient bundle. Suppose that there exists some $M \in \mathrm{ob}(\mathcal{M}_n)$ and some section $\psi : M \to \mathfrak{Y}M$ such that $\mathfrak{E}M_{|\psi}$ is symmetric hyperbolic. When does this also occur for every other choice of background structure (on every $M \in \mathrm{ob}(\mathcal{M}_n)$)?[29] In what follows, we will be especially interested in the case of minimally coupled equations, i.e. when $(X, \mathfrak{G})$ is a natural quotient of X over G satisfying the conditions of definition 2.5.5.

Before I discuss a strategy which does work, let me begin by mentioning a (closely related) one which does not, at least in this case. Suppose that for every $M \in \mathrm{ob}(\overline{\mathcal{M}}_n)$ and every section $\psi : M \to GM$, $XM_{|\psi}$ is isomorphic to YM for some natural bundle, Y. Suppose further that we could find a first-order natural operator $H : G \to T \otimes S^2V^*Y$ with the property that $H(M, \psi)$ is a (maximal) hyperbolization of $\mathfrak{E}M_{|\psi}$. (One could think of such an operator as

[27] See Geroch (1996) for more precise definitions and discussion.

[28] Again, I am glossing over subtleties here, primarily having to do with completeness and integrability of constraints for such systems, which e.g. enter into the characterisation of 'suitable' initial data—see Geroch (1996)—but these details will not matter for our purposes.

[29] An alternative approach would be to develop a theory of 'almost symmetric hyperbolicity' which is appropriate for natural equations, and show that whenever a natural equation is 'almost symmetric hyperbolic' in this sense, it gives rise to symmetric hyperbolic equations relative to any (appropriate) choice of background structure. This approach would take us too far afield, here, but I suggest that it would be a worthwhile (and philosophically rich) one to pursue. For suggestions in this direction, see Geroch (1996).

an assignment of a hyperbolization to every choice of background structure.) Then we would have that $H(M, \psi)$ is a (maximal) hyperbolization of $\mathfrak{E}M_{|\psi}$ for every $M \in \mathrm{ob}(\overline{\mathcal{M}}_n)$ and every section $\psi : M \to GM$. This follows immediately from the definition of a maximal hyperbolization, the definition of a natural operator, and theorems 2.6.1, 2.7.4, and 2.7.5.

Unfortunately, this does not work, because the only such operator is the zero operator (Kolář et al. 1993, §33.2). However, it does suggest an alternative strategy, which is to find a first-order natural operator acting on G which returns not maximal hyperbolizations, but mixed index tensor fields which encode a space of maximal hyperbolizations, relative to some further choices. Indeed, as we saw above, this was exactly how we defined maximal hyperbolizations for the Maxwell example—by introducing a tensor $T^{a}{}_{\alpha'}{}^{b}{}_{\beta'}$, depending only on the metric to first order, with the property that for any timelike t_a, $t_n T^{n}{}_{\alpha'}{}^{b}{}_{\beta'}$ is a hyperbolization. The following result establishes that the existence of a first-order natural operator $H : G \to S^2(T \otimes V^*Y)$ coinciding with such a tensor field at (M, ψ) is sufficient for every denaturalized system of equations to be symmetric hyperbolic:

Theorem 2.8.9. *Let $(\mathfrak{E}, X)$ be a minimally coupled, quasilinear system. Suppose that for every $M \in \mathrm{ob}(\overline{\mathcal{M}}_n)$ and every section $\psi : M \to GM$, $XM_{|\psi}$ is isomorphic to YM for some natural bundle, Y. Further suppose that $\mathfrak{E}M_{|\psi}$ is symmetric hyperbolic for some $M \in \mathrm{ob}(\overline{\mathcal{M}}_n)$ and $\psi : M \to GM$. Let $T^{a}{}_{\alpha'}{}^{b}{}_{\beta'}$ be a tensor on $XM_{|\psi}$ such that for any timelike t_a, $t_n T^{n}{}_{\alpha'}{}^{b}{}_{\beta'}$ is a maximal hyperbolization of $\mathfrak{E}M_{|\psi}$. Suppose further that there exists a first-order natural operator $H : G \to S^2(T \otimes V^*Y)$ such that $H(M, \psi) = T^{a}{}_{\alpha'}{}^{b}{}_{\beta'}$. Then for any $M' \in \mathrm{ob}(\mathcal{M}_n)$ and any section $\psi' : M' \to GM'$, $\mathfrak{E}M'_{|\psi'}$ is symmetric hyperbolic.*

Proof. By lemma 2.7.5, we know that for any $M, M' \in \mathrm{ob}(\mathcal{M}_n)$, any sections $\psi : M \to GM$, $\psi' : M' \to GM'$, and any $p \in M$, $p' \in M'$, there exist open neighbourhoods O, O' of p, p' respectively and a diffeomorphism $\varphi : O \to O'$ such that $j^1_{p'}(G\varphi_*\psi) = j^1_{p'}\psi'$ and $J^1X\varphi_*(U_{\mathfrak{E}M_{|\psi}} \cap \pi^{-1}_{J^1XM}(p)) = U_{\mathfrak{E}M'_{|\psi'}} \cap \pi^{-1}_{J^1XM'}(p')$. Since H is first order, we know that for any such diffeomorphism $H(M',\psi')(p') = S^2(T\otimes V^*Y)\varphi_* H(M,\psi)(p)$. Now let t_a be any timelike covector relative to ψ at p. (Since $j^1_{p'}(G\varphi_*\psi) = j^1_{p'}\psi'$, this implies that $\chi_* t_a$ is timelike relative to ψ' at p'.) Then $\varphi_* t_a H(M',\psi')(p) = \varphi_* t_a T'^{a}{}_{\alpha'}{}^{b}{}_{\beta'} = S^2(T\otimes V^*Y)\varphi_*(t_a T^{a}{}_{\alpha'}{}^{b}{}_{\beta'})$, so $\varphi_* t_a T'^{a}{}_{\alpha'}{}^{b}{}_{\beta'}$ annihilates $U_{\mathfrak{E}M'_{|\psi'}} \cap \pi^{-1}_{J^1XM'}(p')$. Moreover, since $J^1X\varphi_*(U_{\mathfrak{E}M_{|\psi}} \cap \pi^{-1}_{J^1XM}(p)) = U_{\mathfrak{E}M'_{|\psi'}} \cap \pi^{-1}_{J^1XM'}(p')$, it follows that for any (other) covector n_a at p, if n_a is characteristic for $\mathfrak{E}M_{|\psi}$ at p, then $\varphi_* n_a$ is characteristic for $\mathfrak{E}M'_{|\psi'}$ at p', and likewise if n_a is causal for $t_a T^{a}{}_{\alpha'}{}^{b}{}_{\beta'}$ at p then $\varphi_* n_a$ is causal for $\varphi_* t_a T'^{a}{}_{\alpha'}{}^{b}{}_{\beta'}$ at p'. So for any timelike t'_a at p', we must have that $t'_a T'^{a}{}_{\alpha'}{}^{b}{}_{\beta'}$ is a maximal hyperbolisation of $\mathfrak{E}M'_{|\psi'}$ at p'. Since smooth timelike vector fields can always be found locally, it follows that $\mathfrak{E}M'_{|\psi'}$ is symmetric hyperbolic. □

In summary: whilst a natural, minimally coupled, quasilinear system of equations cannot be symmetric hyperbolic, we can make sense of what it is for such a system of equations to be symmetric hyperbolic relative to any choice of background structure—i.e., relative to any Lorentzian metric. Sufficient for this is the existence of a first-order natural operator $H : G \to S^2(T\otimes V^*Y)$—i.e. an assignment, to every denaturalized equation, of a mixed index tensor field $T^{a}{}_{\alpha'}{}^{b}{}_{\beta'}$—such that $t_a T^{a}{}_{\alpha'}{}^{b}{}_{\beta'}$ is a hyperbolization of the equation for one such choice of metric, for any timelike t_a (relative to that metric). We will return to these mixed index tensor fields $T^{a}{}_{\alpha'}{}^{b}{}_{\beta'}$ in §4.

3 The adaptation of matter to spacetime geometry

With this technical background on minimally coupled, quasilinear, symmetric hyperbolic theories in hand, I will now move on to discuss **motion** and **causality**. The task of this section will be to isolate sufficient conditions for a system of partial differential equations, i.e. a candidate dynamics for matter fields, to satisfy these two conditions. As mentioned in §1, I will focus on the Geroch–Earman causality condition discussed by Earman (2014), Geroch (2011), and Weatherall (2014) and the Geroch–Weatherall theorem (Geroch and Weatherall 2018) as approaches to explicating these conditions. My presentation here aims only to isolate and motivate the main results, and the sense in which they capture **motion** and **causality**—further details and philosophical discussion can be found in the above references, as well as Weatherall (2020).

3.1 Causality

I will begin with **causality**, since this requires somewhat less setup, given what we have already done. Suppose we have a first-order, quasilinear, symmetric hyperbolic system of equations, and a hyperbolization of that system. Then, as we have seen in §2.8, causal vectors for that hyperbolization can be thought of as representing possible 'signal propagation velocities' associated with the (hyperbolized) system of equations. This forms the basis for our discussion of **causality**. One would like to know what the maximal possible signal propagation velocities associated with a system of partial differential equations are. To this end, we define:

Definition 3.1.1 (Characteristic surface)**.** *Let $E \subset J^1B \to B \to M$ be a first-order quasilinear symmetric hyperbolic equation. A characteristic surface is a codimension-1 embedded submanifold Σ of M whose normal, at every $p \in \Sigma$, is a characteristic covector for E.*

Then for any maximal hyperbolization $h^a{}_{\alpha'\beta'}$ of E, the boundary of the collection of causal vectors at any $p \in M$ are tangents to characteristic surfaces. This means that tangents to characteristic surfaces can be thought of as representing maximal signal propagation velocities. One can elevate this to a sufficient condition for superluminal propagation (Earman 2014; Geroch 2011; Weatherall 2014):

Condition 3.1.2 (Superluminal propagation, Geroch–Earman)**.** *Let $E \subset J^1B \to B \to (M, g_{ab})$ be a first-order, quasilinear, symmetric hyperbolic system. Then solutions may propagate superluminally iff there exists a hyperbolization $h^a{}_{\alpha'\beta'}$ of E and some open region $O \subset M$ such that for all $p \in O$, the collection of causal vectors for $h^a{}_{\alpha'\beta'}$ has as a proper subset one lobe of the metric lightcone.*

Or, as Weatherall (2014, p. 116) notes, one can also turn this round into a sufficient condition for no superluminal propagation (i.e. **causality**):

Condition 3.1.3 (Geroch–Earman causality)**.** *Let $E \subset J^1B \to B \to (M, g_{ab})$ be a first-order, quasilinear, symmetric hyperbolic system. Then solutions may not propagate superluminally if for any hyperbolization $h^a{}_{\alpha'\beta'}$ of E and any $p \in M$, the collection of causal vectors for $h^a{}_{\alpha'\beta'}$ is a subset of one lobe of the metric lightcone at p.*

3.2 Motion

I will now move on to discuss **motion**. Here, we begin from a rather different place. As noted in §1, one way of getting a handle on **motion** would be to take small-body motion (the geodesic principle) as an explication of **motion**—i.e. the principle that test particles follow only timelike or null geodesics. However, it is not entirely clear what it means to speak of test bodies in general relativity.[30] I will now, following Geroch and Weatherall (2018), outline one approach to making this precise.

Let (M, g_{ab}) be a relativistic spacetime. For the purposes of this section, we will focus our attention on matter fields on (M, g_{ab}) which can be associated with a rank-two, symmetric, tensor field T^{ab} which is non-vanishing only where the matter fields are. In what follows, we will consider collections C of

[30]See Weatherall (2020) for further discussion of this point.

such symmetric fields T^{ab}, each of which satisfy the dominant energy condition (DEC). The DEC, recall, is as follows:

Dominant energy condition (DEC): for any $p \in M$, and any two cooriented timelike covectors t_a, ξ_a at p, either $T^{ab} = \mathbf{0}$, or $t_n \xi_m T^{nm} > \mathbf{0}$.

So let C be such a collection. Each such T^{ab} is naturally associated with a distribution $\mathbf{T}^{ab}$,[31] whose action on test fields x_{ab} is defined by $\mathbf{T}^{nm}\{x_{nm}\} = \int_M T^{nm} x_{nm}$. We will say that a test field x_{ab} satisfies the dual energy condition at some $p \in M$ iff x_{ab} can be written as a sum of symmetrized outer products of co-oriented causal (i.e. timelike or null, note the unfortunate clash of terminology with §2.8) covectors. Meanwhile, a test field x_{ab} is generic at some $p \in M$ iff for any non-vanishing symmetric tensor T^{ab} satisfying the dominant energy condition at p, $T^{nm} x_{nm} > 0$. Geroch and Weatherall (2018) then introduce the following definition: C tracks a curve γ iff, for any smooth test field x_{ab} satisfying the dual energy condition on some neighbourhood of γ and generic at some $p \in \gamma$, there is some T^{ab} in C such that $\mathbf{T}^{nm}\{x_{nm}\} > 0$.

To understand this definition, note that since the fields T^{ab} in C satisfy the DEC, the result of contracting it with a test field x_{ab} which satisfies the dual energy condition at some $p \in M$ is a field which is non-negative at p. This means that for any T^{ab} in C, if x_{ab} is a test field satisfying the dual energy condition everywhere, then $\mathbf{T}^{nm}\{x_{nm}\}$ is non-negative. As such, test fields x_{ab} satisfying the dual energy condition can be thought of as providing a standard for measuring the 'amount of T^{ab} in the region where x_{ab} has support' (relative to the choice of test field x_{ab}).

Now, observe that the definition of tracking states that for any test field x_{ab}, satisfying the dual energy condition in some neighborhood of γ, there is a field T^{ab} in C whose action on x_{ab} (as a distribution $\mathbf{T}^{ab}$) is strictly positive. As such, tracking can be thought of as capturing the idea that, relative to any standard for measuring 'amount of T^{ab} in the neighbourhood of γ' induced by test fields x_{ab}, there are fields T^{ab} in C which are as concentrated as one likes

[31] I will use boldface lettering to denote distributions. For details on the definition of distributions, see Geroch and Weatherall (2018) or Weatherall (2020)

near γ. Moreover, since the test fields x_{ab} always are compactly supported, this captures a sense in which a collection C of symmetric fields T^{ab} which tracks a curve γ includes elements which follow γ arbitrarily tightly, for arbitrarily long times.

So far, we have given a definition of what it is for a collection of symmetric fields to track a curve, γ. Now we get to the payoff. First, we define a further condition on the fields T^{ab}:

Conservation condition (CC): $\nabla_n T^{na} = \mathbf{0}$.

We then have the following theorem, due to Geroch and Weatherall (2018):

Theorem 3.2.1 (Geroch and Weatherall, 2018)**.** *Let (M, g_{ab}) be a relativistic spacetime, γ a curve therein, and C a collection of symmetric fields T^{ab}, each satisfying the DEC, which tracks γ. Further suppose that each T^{ab} satisfies the CC. Then γ is a timelike or null geodesic.*

(In fact, and as discussed by Geroch and Weatherall (2018) and Weatherall (2020), the antecedent of this theorem is sufficient to establish a variety of stronger results. In particular, it is sufficient for the existence of a sequence of fields $T_1^{ab}, T_2^{ab}, \ldots$, each a positive multiple of some element of C, that converges, in the sense of distributions, to $\delta_\gamma u^a u^b$.)

For our purposes, the significance of theorem 3.2.1 is that it establishes that any matter field which can be associated with a rank-two, symmetric tensor field, which is non-vanishing only where the matter fields are, and satisfies the DEC and CC, will obey the geodesic principle in the small-body limit. Note, in particular, that nothing in the statement of theorem 3.2.1 requires the collection C of symmetric fields T^{ab} to be a collection of Einstein–Hilbert stress–energies. It is in this sense that DEC and CC are sufficient for a system of partial differential equations to satisfy **motion**.

4 Minimally coupled, quasilinear, symmetric hyperbolic theories and the adaptation of matter to spacetime geometry

In the previous section, we have isolated sufficient conditions for a system of quasilinear, symmetric hyperbolic partial differential equations to be adapted to a relativistic spacetime geometry per **motion** and **causality**. To reiterate, these are:

- The system is Geroch–Earman causal (i.e. satisfies condition 3.1.3).
- Solutions of the equation representing matter fields can be associated with a rank-2 symmetric tensor field, which is non-vanishing only where the matter fields are, and satisfies (i) the DEC, and (ii) the CC.

The task of this section is to show that these conditions are necessarily satisfied by minimally coupled, quasilinear, symmetric hyperbolic theories. In doing so, I substantiate the link between minimal coupling and adapted dynamics proposed by Brown (2005), Knox (2013), Read (2020a), and Read et al. (2018). It turns out that whilst **MC** by itself is not sufficient for **motion** and **causality**, it *is* so sufficient for an important general class of physical systems—namely, those described by quasilinear symmetric hyperbolic partial differential equations on sections of a first-order natural bundle, and which can be associated with a natural operator $H : G \to S^2(T \otimes V^* Y)$ which defines maximal hyperbolizations relative to any appropriate choice of covector (field), as described in §2.8.[32]

I have separated this section into three parts—dealing, respectively, with the Geroch–Earman causality condition, the DEC, and the CC—for ease of reading. However, it is worth noting at the outset that this risks obscuring an important feature of these results, which is that the Geroch–Earman causality condition in fact comes out as a lemma on the way to the DEC. This substantially closes the gap between Geroch–Earman causality and the DEC—whilst in general, Geroch–Earman causality is neither necessary or sufficient for the DEC, it is so

[32]In fact, as we will see, the existence of such a natural operator and the restriction to first-order natural bundles are not needed for either the Geroch–Earman causality condition or the DEC; they enter only into the proof of the CC.

sufficient for systems when the rank-two symmetric tensor field which features in the DEC is related to the existence of maximal hyperbolizations, in a sense I make precise below.

Whilst the arguments of this section are somewhat technical, the important point to keep in mind is that they are all, essentially, definability proofs. For Geroch–Earman causality and the DEC, the basic idea is just that if a quasilinear, symmetric hyperbolic system of partial differential equations depends only on the metric to first order, then its spaces of characteristic covectors—and hence the causal cones of its maximal hyperbolizations—must depend only on the metric to first order. One then shows, in effect, that the only open convex sets of covectors which depend only on the metric to first order are the sets of (future-directed or past-directed) timelike covectors at each point. For the CC, the basic idea is to show that any tensor which depends only on the metric to first-order must be covariantly constant with respect to that metric. These claims are informal; they will become clear in what follows.

4.1 Geroch–Earman causality

We begin with the following lemma:

Lemma 4.1.1. *Let $(\mathfrak{E}, X)$ be a first-order, minimally coupled, quasilinear system. Then for any $M \in \mathrm{ob}(\overline{\mathcal{M}}_n)$, any section $\psi : M \to GM$, and any $p \in M$, if any timelike (spacelike, non-zero null) covector at p with respect to ψ is characteristic for $\mathfrak{E}M_{|\psi}$ at some $x \in \pi^{-1}_{XM_{|\psi}}(p)$, then all timelike (spacelike, null) covectors at p are characteristic for $\mathfrak{E}M_{|\psi}$ at x.*

Proof. Recall that a covector n_a at p is characteristic for $\mathfrak{E}M_{|\psi}$ at some $x \in \pi^{-1}_{XM_{|\psi}}(p)$ iff there exists a non-zero vertical vector $v^{\alpha'}$ at x such that $n_a v^{\alpha'}$ is in $U_{\mathfrak{E}M_{|\psi}}$. We know, by lemma 2.7.5, that for any M, $M' \in \mathrm{ob}(\overline{\mathcal{M}}_n)$, sections $\psi : M \to GM$, $\psi' : M' \to GM'$ and any $p \in M$, $p' \in M'$, there exist open neighborhoods O, O' of p, p' and a diffeomorphism $\varphi : O \to O'$ such that $j^1_{p'}(G\varphi_*\psi) = j^1_{p'}\psi'$ and $J^1X\varphi_*(U_{\mathfrak{E}M_{|\psi}} \cap \pi^{-1}_{J^1XM}(p)) = U_{\mathfrak{E}M'_{|\psi'}} \cap \pi^{-1}_{J^1XM'}(p')$, and hence that for any such diffeomorphism φ, $n_a v^{\alpha'} \in U$ iff $X\varphi_*(n_a v^{\alpha'}) =$

$\varphi_*(n_a)X\varphi_*(v^{\alpha'}) \in U'$. Now consider the case where $M \simeq \mathbb{R}^n$ and ψ is the Minkowski metric on M. Then given any two unit timelike (spacelike, non-zero null) covectors n_a, n'_a at p, there exists a diffeomorphism $\varphi : M \to M$ such that $G\varphi(\psi) = \psi$ and $\varphi_* n_a = n'_a$. In particular, and using that the elements of U at each point form a vector space and that the zero covector is always characteristic, this implies that if any timelike (spacelike, non-zero null) covector at p with respect to ψ is characteristic for $\mathfrak{E}M_{|\psi}$ at some $x \in \pi^{-1}_{XM_{|\psi}}(p)$, then every timelike (spacelike, null) covector with respect to ψ is. But then (since $M \simeq \mathbb{R}^n$), theorem 2.6.1, lemma 2.6.2, and theorem 2.7.5 imply that for any other M', ψ', p' the same is true. $\square$

Note that for an arbitrary Lorentzian manifold, one does not have, for any two unit timelike (spacelike, non-zero null) covectors n_a, n'_a at some $p \in M$, that there exists an isometry $\varphi : M \to M$ such that $\varphi_* n_a = n'_a$. This is true only if the spacetime is isotropic. The point here is that naturality and theorem 2.6.1 ensure that we can always refer the situation to some isotropic spacetime, e.g. Minkowski spacetime. Lemma 2.6.2 and theorem 2.7.5 then ensure that any pointwise properties of the equation expressed relative to this isotropic spacetime must also hold for the equation expressed relative to the spacetime with which we began.

The next lemma specializes the situation slightly. If $\mathfrak{E}M_{|\psi}$ is, in addition, symmetric hyperbolic (for some $M \in \mathrm{ob}(\overline{\mathcal{M}}_n)$ and some section $\psi : M \to GM$), then lemma 4.1.1 in fact implies that every null covector must be characteristic at every point in $XM_{|\psi}$:

Lemma 4.1.2. *Let $(\mathfrak{E}, X)$ be a first-order, minimally coupled, quasilinear system, and suppose that $\mathfrak{E}M_{|\psi}$ is symmetric hyperbolic for some $M \in \mathrm{ob}(\overline{\mathcal{M}}_n)$ and some section $\psi : M \to GM$. Then for any $p \in M$, if n_a is null at p with respect to ψ then n_a is characteristic for $\mathfrak{E}M_{|\psi}$ at every $x \in \pi^{-1}_{XM_{|\psi}}(p)$.*

Proof. Suppose otherwise, i.e. there is some null n_a at some $p \in M$ which is not characteristic for some $x \in \pi^{-1}_{XM_{|\psi}}(p)$. Since the zero covector is always characteristic, we know that $n_a \neq \mathbf{0}$, so by lemma 4.1.1, no no-zero null covectors at p are characteristic at x. Since $\mathfrak{E}M_{|\psi}$ is symmetric hyperbolic, there exists a hyperbolization $h^a{}_{\alpha'\beta'}$ and an open convex cone of causal covectors for $h^a{}_{\alpha'\beta'}$ at x whose boundary contains only characteristics. Hence there must be some characteristic covector n'_a for $\mathfrak{E}M_{|\psi}$ at x (otherwise, by maximality, all covectors at p would be causal for $h^a{}_{\alpha'\beta'}$ at x, but this is impossible, since if $n'_n h^n{}_{\alpha'\beta'}$ is positive definite then $-n'_n h^n{}_{\alpha'\beta'}$ is negative definite). From the above, we know that n'_a must be either timelike or spacelike, so by lemma 4.1.1, either all timelike covectors at p are characteristic at x or all spacelike ones are. But then there can be no open convex subsets of $T^*_p M$ which contain no characteristics and whose boundary contains only characteristics, since both the spaces of timelike and spacelike covectors at p are bounded by the space of null covectors at p, and from the above, no non-zero null covector can be characteristic. This implies that $\mathfrak{E}M_{|\psi}$ cannot admit a maximal hyperbolization at x, and so cannot be symmetric hyperbolic. So we have our contradiction. □

Lemma 4.1.2 leaves it open whether there are any non-null covectors with respect to ψ which are characteristic for $\mathfrak{E}M_{|\psi}$ at some point in $XM_{|\psi}$. The following result establishes that, if there are, they cannot be timelike:

Lemma 4.1.3. *Let $(\mathfrak{E}, X)$ be a first-order, minimally coupled, quasilinear system, and suppose that $\mathfrak{E}M_{|\psi}$ is symmetric hyperbolic for some $M \in \mathrm{ob}(\overline{\mathcal{M}}_n)$ and some section $\psi : M \to GM$. Then for any $p \in M$, and any $x \in \pi^{-1}_{XM_{|\psi}}(p)$, no timelike covector at p with respect to ψ is characteristic for $\mathfrak{E}M_{\psi}$ at x.*

Proof. Suppose otherwise, i.e. there exists some timelike n_a at some $p \in M$ which is characteristic for $\mathfrak{E}M_{|\psi}$ at some $x \in \pi^{-1}_{XM_{|\psi}}(p)$. By lemma 4.1.1, this implies that all timelike n_a are characteristic for $\mathfrak{E}M_{|\psi}$ at x. By lemma 4.1.2, we know that all null covectors are characteristic for $\mathfrak{E}M_{|\psi}$ at x. In particular, this implies that no spacelike covectors can be characteristic at x, since if any spacelike covector is characteristic at x then by lemma 4.1.1 all spacelike covectors must be, but if all covectors are characteristic at x then $\mathfrak{E}M_{|\psi}$ cannot be symmetric hyperbolic. It follows that any maximal hyperbolization $h^a{}_{\alpha'\beta'}$ of $\mathfrak{E}M_{|\psi}$ must have, as its collection of causal covectors at x, the set of all spacelike covectors at p. But since the set of causal covectors at x must be convex, this is impossible. $\square$

This takes us immediately to our first result:

Theorem 4.1.4 (Geroch–Earman causality theorem)**.** *Let $(\mathfrak{E}, X)$ be a first-order, minimally coupled, quasilinear system, and suppose that $\mathfrak{E}M_{|\psi}$ is symmetric hyperbolic for some $M \in \mathrm{ob}(\overline{\mathcal{M}}_n)$ and some section $\psi : M \to GM$. Then for any $p \in M$, and any $x \in \pi^{-1}_{XM_{|\psi}}(p)$, if $h^a{}_{\alpha'\beta'}$ is a maximal hyperbolization, then a vector is causal for $h^a{}_{\alpha'\beta'}$ at x iff it is (directed) timelike.*

Proof. Let $p \in M$ and let $x \in \pi^{-1}_{XM_{|\psi}}(p)$. We know, by lemmas 4.1.2 and 4.1.3, that every null covector and no timelike covector with respect to ψ at p is characteristic for $\mathfrak{E}M_{|\psi}$ at x. If any spacelike covector is characteristic for $\mathfrak{E}M_{|\psi}$ at x, then we are done, since then the only open convex sets of (non-characteristic) covectors at p whose boundary contains only characteristics are the sets of future-directed and past-directed timelike covectors at p (recall from §2.8 that characteristics cannot be causal). Conversely, suppose that no spacelike covectors are characteristic for $\mathfrak{E}M_{|\psi}$ at x. Then the only sets of covectors at p whose boundaries contain only characteristics are the sets of future-directed and past-directed timelike covectors, and the set of spacelike covectors. But the set of spacelike covectors at p is not convex. $\square$

4.2 The DEC

Having established the Geroch–Earman causality theorem, the derivation of the DEC is straightforward. We have the following result:

Theorem 4.2.1 (DEC theorem, a.k.a. Weatherall's conjecture)**.** *Let $(\mathfrak{E}, X)$ be a first-order, minimally coupled, quasilinear system, and suppose that $\mathfrak{E}M_{|\psi}$ is symmetric hyperbolic for some $M \in \mathrm{ob}(\overline{\mathcal{M}}_n)$ and some section $\psi : M \to GM$. Let $T^a{}_{\alpha'}{}^b{}_{\beta'}$ be a tensor on $XM_{|\psi}$ such that for any timelike t_a, $t_n T^n{}_{\alpha'}{}^b{}_{\beta'}$ is a maximal hyperbolization of $\mathfrak{E}M_{|\psi}$. Then either $T^a{}_{\mu'}{}^b{}_{\nu'} v^{\mu'} v^{\nu'}$ satisfies the DEC for all $v^{\alpha'}$, or $-T^a{}_{\mu'}{}^b{}_{\nu'} v^{\mu'} v^{\nu'}$ satisfies the DEC for all $v^{\alpha'}$.*

Proof. By lemma 4.1.4, if $h^a{}_{\alpha'\beta'}$ is any maximal hyperbolization of $\mathfrak{E}M_{|\psi}$, then n_a is causal for $h^a{}_{\alpha'\beta'}$ iff it is (past- or future-directed) timelike. Hence, by the definition of a maximal hyperbolization, if t_a, ξ_a are any two cooriented timelike covectors at some $p \in M$, then if $T^a{}_{\alpha'}{}^b{}_{\beta'}$ is a tensor on $XM_{|\psi}$ satisfying the conditions of the proposition, either $t_n \xi_m T^n{}_{\alpha'}{}^m{}_{\beta'}$ or $-t_n \xi_m T^n{}_{\alpha'}{}^m{}_{\beta'}$ is positive definite, and hence for all $v^{\alpha'}$, either $v^{\alpha'} \neq \mathbf{0}$ and $t_n \xi_m T^n{}_{\mu'}{}^m{}_{\nu'} v^{\mu'} v^{\nu'} > 0$ (or similarly in the case $-t_n \xi_m T^n{}_{\alpha'}{}^m{}_{\beta'}$ is positive definite), or $v^{\alpha'} = \mathbf{0}$ and $t_n \xi_m T^n{}_{\mu'}{}^m{}_{\nu'} v^{\mu'} v^{\nu'} = 0$. But the latter case, of course, obtains only if $T^a{}_{\mu'}{}^b{}_{\nu'} v^{\mu'} v^{\nu'} = \mathbf{0}$. □

Of course, this derivation of the DEC does not mention the stress–energy associated with matter fields, i.e. with solutions of $\mathfrak{E}M_\psi$. As emphasized in §3.2, this does not matter, at least for the purposes of establishing **motion**.[33] What matters is that we have found a rank-two symmetric tensor field—namely $T^a{}_{\mu'}{}^b{}_{\nu'} v^{\mu'} v^{\nu'}$ (or its negative)—which is non-vanishing only where the matter fields are, and satisfies the DEC. To see that the first condition is satisfied, note that if $XM_{|\psi}$ is a vector bundle (which, of course, is the only case in which one can make sense of what it is for matter fields, i.e. sections of $XM_{|\psi}$ to vanish at a point), then since $VXM_{|\psi}$ is canonically isomorphic to $XM_{|\psi} \times_M XM_{|\psi}$, any section $\psi : M \to XM_{|\psi}$ (i.e. a candidate solution of $\mathfrak{E}M_\psi$) canonically induces a section $\psi \times_M \psi : M \to VXM_\psi$, i.e. an assignment, to each point in the image of ψ, of a vertical vector at that point, which is non-zero at that point only when ψ is. Note that nothing in this, or in the derivation of the DEC, depends on ψ being a solution of $\mathfrak{E}M_\psi$.

4.3 The CC

Having established the Geroch–Earman causality theorem and DEC theorem, we can finally turn to the CC. We have the following:

[33] That said, in light of what I have shown here, under what conditions e.g. the Einstein–Hilbert stress energy for some matter field coincides with a tensor $T^a{}_{\mu'}{}^b{}_{\nu'} v^{\mu'} v^{\nu'}$ with the properties described above is evidently a question of substantial (technical and philosophical) interest. I will postpone this for future work, see §6.

Theorem 4.3.1 (CC theorem)**.** *Let $(\mathfrak{E}, X)$ be a first-order, minimally coupled, quasilinear system, and suppose that $\mathfrak{E}M_{|\psi}$ is symmetric hyperbolic for some $M \in \mathrm{ob}(\overline{\mathcal{M}}_n)$ and some section $\psi : M \to GM$. Let $T^{a}{}_{\alpha'}{}^{b}{}_{\beta'}$ be a tensor on $XM_{|\psi}$ such that for any timelike t_a, $t_n T^{n}{}_{\alpha'}{}^{b}{}_{\beta'}$ is a maximal hyperbolization of $\mathfrak{E}M_{|\psi}$. Suppose further that (i) for any $M' \in \mathrm{ob}(\overline{\mathcal{M}}_n)$ and any section $\psi' : M' \to GM'$, $XM'_{|\psi'} = YM'$ for some first-order natural bundle Y, and (ii) there exists a first-order natural operator $H : G \to S^2(T \otimes V^*Y)$ such that $H(M, \psi) = T^{a}{}_{\alpha'}{}^{b}{}_{\beta'}$. Then for all $v^{\alpha'}$, $\nabla_n(T^{n}{}_{\mu'}{}^{a}{}_{\nu'} v^{\mu'} v^{\nu'}) = T^{n}{}_{\mu'}{}^{a}{}_{\nu'} v^{\mu'} j^{\nu'}{}_{n} + T^{n}{}_{\mu'}{}^{a}{}_{\nu'} v^{\nu'} j^{\mu'}{}_{n}$, where $j^{\alpha'}{}_{a}$ is any representative of $[\nabla_a v^{\alpha'}] \in (VY \otimes T^*)M/U_{\mathfrak{E}M_{|\psi}}$.*

Proof. We know that $\nabla_n(T^{n}{}_{\mu'}{}^{a}{}_{\nu'} v^{\mu'} v^{\nu'}) = v^{\mu'} v^{\nu'} \nabla_n T^{n}{}_{\mu'}{}^{a}{}_{\nu'} + T^{n}{}_{\mu'}{}^{a}{}_{\nu'} v^{\mu'} \nabla_n v^{\nu'} + T^{n}{}_{\mu'}{}^{a}{}_{\nu'} v^{\nu'} \nabla_n v^{\mu'}$ (this makes sense, since Y is first order). Since $\nabla_a v^{\alpha'} \in (VY \otimes T^*)M$, and $T^{a}{}_{\alpha'}{}^{b}{}_{\beta'}$ annihilates every element of $U_{\mathfrak{E}M_{|\psi}} \subset (VY \otimes T^*)M$, this implies that $\nabla_n(T^{n}{}_{\mu'}{}^{a}{}_{\nu'} v^{\mu'} v^{\nu'}) = v^{\mu'} v^{\nu'} \nabla_n T^{n}{}_{\mu'}{}^{a}{}_{\nu'} + T^{n}{}_{\mu'}{}^{a}{}_{\nu'} v^{\mu'} j^{\nu'}{}_{n} + T^{n}{}_{\mu'}{}^{a}{}_{\nu'} v^{\nu'} j^{\mu'}{}_{n}$, where $j^{\alpha'}{}_{a}$ is any representative of $[\nabla_a v^{\alpha'}] \in (VY \otimes T^*)M/U_{\mathfrak{E}M_{|\psi}}$.

But now consider $\nabla_n T^{n}{}_{\alpha'}{}^{a}{}_{\beta'}$. Let $M' \simeq \mathbb{R}^n$, let $\psi' : M' \to GM'$ be the Minkowski metric on M', let $T'^{a}{}_{\alpha'}{}^{b}{}_{\beta'} = H(M', \psi')$, and consider any $p \in M$, $p' \in M'$. By theorem 2.6.1 and lemma 2.7.5, there exist open neighborhoods O, O' of p, p' and a diffeomorphism $\varphi : O \to O'$ such that $j^1_{p'}(G\varphi_*\psi) = j^1_{p'}\psi'$, $J^1X\varphi_*(U_{\mathfrak{E}M_{|\psi}} \cap \pi^{-1}_{J^1XM}(p)) = U_{\mathfrak{E}M'_{|\psi'}} \cap \pi^{-1}_{J^1XM'}(p')$ and, since H is first order, $H(M', \psi')(p') = S^2(T \otimes V^*Y)\varphi_* H(M, \psi)(p)$. But now, for any other $q' \in M'$, there is some diffeomorphism $\varphi' : M' \to M'$ such that (i) $G\varphi'(\chi') = \chi'$, (ii) $\varphi'(p') = q'$, and (iii) $\varphi' = \varphi'_t$ for some one-parameter family of diffeomorphisms φ'_t generated by a vector field ξ^a on M' such that $\nabla'_a \xi^b = \mathbf{0}$, where ∇' is the Levi-Civita connection for ψ'. (This follows from the fact that Minkowski spacetime is homogeneous and flat.) Since $H : G \to S^2(T \otimes V^*X)$ is natural, we know for any such diffeomorphism that $H(M', \psi') = S^2(T \otimes V^*Y)\varphi'_* H(M', \psi')$. But then we must have that $\nabla'_c T'^{a}{}_{\alpha'}{}^{b}{}_{\beta'} = \mathbf{0}$. (This follows from the corresponding fact about Lie derivatives and the fact that, since such a diffeomorphism exists for each $q' \in M'$, one can always find a basis of such vector fields $\overset{i}{\xi}{}^a$, $i = 1, ..., n$ at p'.)

Finally, letting ∇ denote the Levi-Civita connection on M for ψ, since $j^1_{p'}(G\varphi_*\psi) = j^1_{p'}\psi'$, we have $\chi_*\nabla = \nabla'$ at p' and hence $S^2(T\otimes V^*Y)\varphi_*(\nabla_n T^n{}_{\alpha'}{}^b{}_{\beta'}) = \nabla'_n T'^n{}_{\alpha'}{}^b{}_{\beta'} = \mathbf{0}$ at p'. Thus $\nabla_n T^n{}_{\alpha'}{}^b{}_{\beta'} = \mathbf{0}$ at p. But since p was arbitrary, this implies $\nabla_n T^n{}_{\alpha'}{}^b{}_{\beta'} = \mathbf{0}$ throughout M. So we are done. $\square$

This theorem is, of course, not quite the CC. Nor should we expect it to be. As it stands, we have not imposed any conditions on the vertical vector field $v^{\alpha'}$ we are considering (the association of sections of $XM_{|\psi}$ with vertical vectors on $XM_{|\psi}$ proceeds in exactly the same way as described in the discussion of the DEC theorem in the preceding section), and it is well-known that the stress-energies associated with matter fields are not, in general, covariantly conserved except in the absence of sources. This is where the discussion in §2.7 comes in. There, we articulated what it is for a vertical vector field to be (or not to be) a source for a quasilinear equation, relative to some choice of connection on the bundle (or on M, in the case of a first-order natural bundle over M). The following corollary of theorem 4.3.1 states that, when $v^{\alpha'}$ does not source $\mathfrak{E}M_{|\psi}$ in this sense, relative to the Levi-Civita connection for ψ, then the CC holds exactly:

Corollary 4.3.2. *Under the same conditions as theorem 4.3.1, for any $v^{\alpha'}$ such that $\nabla_a v^{\alpha'} \in U_{\mathfrak{E}M_{|\psi}}$, $T^a{}_{\mu'}{}^b{}_{\nu'} v^{\mu'} v^{\nu'}$ satisfies the CC.*

5 The strong equivalence principle, local (approximate) Poincaré symmetry, and the local validity of special relativity

The previous section concerned the relationship between **MC** and **motion** and **causality**. There, I showed that minimally coupled, quasilinear, symmetric hyperbolic theories necessarily satisfy **motion** and **causality**, in the sense that they satisfy the Geroch–Earman causality condition, and their solutions can be associated with a rank-2 symmetric tensor field, which is non-vanishing only where those solutions are, and satisfies the (i) DEC, and (ii) in the absence of sources, the CC.

However, as noted in §1, **MC** is not the only condition which is taken to be relevant to the adaptation of matter to spacetime geometry—one also has **SEP**, **symmetry**, and **LSR**. This leaves it open in what sense, if any, **SEP**, **symmetry**, or **LSR** are sufficient for the dynamics of matter fields to be adapted to spacetime geometry *per* **motion** and **causality**. Moreover, and again as noted in §1, **SEP**, **symmetry**, and **LSR**, along with **MC**, are often taken to be related to one another in various ways. It would be desirable to get clear on what, exactly, the relationships betweeen these conditions are supposed to be. The task of this final section is to provide precise statements of **SEP**, **symmetry**, and **LSR**, and in doing so, to address these two questions.

5.1 The strong equivalence principle

I will begin with **SEP**. As noted in §1, **SEP** is often claimed by defenders of the dynamical approach to be necessary (or sufficient) for matter to be adapted to spacetime geometry, in the senses specified by **motion**, **causality**, and **time**:

> Why should one restrict to those solutions of GR in which [**SEP**] is satisfied? The reason is that this principle [...] is typically regarded to constitute an important condition for the chronogeometricity of the metric field—that is, for intervals as given by the metric field to be read off by stable rods and clocks built from matter fields. (Read 2020a, pp. 180–181)

> That light rays trace out null geodesics of the field is again a consequence of the strong equivalence principle, which asserts that locally Maxwell's equations of electrodynamics are valid. (Brown 2005, p. 176)

What, then, is **SEP**? I will begin with the following representative statement, given by Brown (2005):

> There exists in a neighbourhood of each event preferred coordinates, called locally inertial at that event. For each fundamental non-gravitational interaction, to the extent that tidal gravitational forces can be ignored, the laws governing the interaction find their simplest form in these coordinates. This is their special relativistic form, independent of spacetime location. (Brown 2005, p. 169)

Whilst it is not my aim here to rehearse the various criticisms of this kind of statement of **SEP** which have been made by e.g. Fletcher (2020), Fletcher and Weatherall (2023b), and Weatherall (2021), it is helpful to begin by noting that the above-quoted passage from Brown raises an immediate (and obvious) worry—namely, that it is not clear how to state formally what it means to say "to the extent that tidal gravitational forces can be ignored".[34] This worry is sometimes taken to motivate what are known as 'pointy' versions of **SEP** (Ghins and Budden 2001; Knox 2013), which make 'taking their special relativistic form in some coordinate system' a property of systems of partial differential equations which holds only at a point, rather than a neighbourhood of that point. For example Ghins and Budden (2001) state **SEP** as the principle that

> [For any $p \in M$,] there exists a local chart x^μ of a neighbourhood of p such that the fundamental dynamical and curvature-free special

[34]This point is acknowledged and discussed by Brown and Read (2016, 2022), Ghins and Budden (2001), Knox (2013), and Read et al. (2018), as well as by Weatherall (2021). Of course, this is not to say that one might not have some other handle on what this means—perhaps in terms of, e.g. the empirical negligibility of tidal gravitational forces in appropriate predictive or explanatory contexts (Brown 2005; Knox 2013; Wallace 2017)—the point is that it is not clear how to state this as a precise, mathematical condition on a system of partial differential equations.

relativistic laws hold in their standard vectorial form in x^μ at p. (Ghins and Budden 2001, p. 43)

Likewise, Read et al. (2018) take the "basic idea" of **SEP** to be that[35]

> [The] dynamical laws of [special relativity]—i.e. dynamical laws governing matter fields in a fixed Minkowski background, featuring no curvature terms—are recovered in GR at any point. (Read et al. 2018, p. 14)

For the time being, I will restrict attention to these 'pointy' statements of **SEP**, and return to their 'un-pointy' variants at the end of this section and in §5.3.1.

Let us focus on getting clear on what is intended by these statements of **SEP**. For our purposes, there are two salient ideas here. The first is the notion of a system of equations taking its 'special relativistic form'. Our discussion of background structure and naturalization in §2.5 provides an obvious route to making this idea precise. Let $(\mathfrak{E}, X)$ be a kth-order natural equation, and suppose that there exists a natural quotient bundle $(X, \mathfrak{G})$ of X over G. (G, recall, is the bundle whose fibres are manifolds of Lorentzian metrics.) Let $M \in \mathrm{ob}(\overline{\mathcal{M}}_n)$, let $\psi : M \to GM$ be a section, and consider the equation $\mathfrak{E}M_{|\psi}$. There is a straightforward notion available of what it is to have this 'same equation, expressed relative to a flat metric' (or the 'special-relativistic version of this equation')—namely, the equation $\mathfrak{E}M'_{|\psi'}$, where $M' \simeq \mathbb{R}^n$ and $\psi' : M' \to GM'$ is any section whose induced metric is flat (e.g. the Minkowski metric on M').[36]

The second salient idea in the above statements of **SEP** is that this 'special-relativistic form' of an equation is to be recovered in coordinates at a point. In our current framework, we can understand this as follows. We know, from theorem 2.6.1, that given any $M \in \mathrm{ob}(\overline{\mathcal{M}}_n)$, any $\psi : M \to GM$, any $p \in M$, there exists an open neighbourhood O of p diffeomorphic to $\mathbb{R}^n$ and a local section $\chi : O \to GM$ whose induced metric is flat and such that $j^1_p\chi = j^1_p\psi$.

[35] Though note that Read et al. (2018) also go on to discuss 'un-pointy' statements of **SEP**.

[36] Note that this construction relies crucially on having specified the natural equation $(\mathfrak{E}, X)$.

So we can give a precise statement of what it is for $\mathfrak{E}M_{|\psi}$ to agree with the 'same equation, expressed relative to a flat metric' at p—namely, that $\mathfrak{E}M_{|\psi} \cap \pi^{-1}_{J^k XM}(p) = \mathfrak{E}M_{|\chi} \cap \pi^{-1}_{J^k XM}(p)$. Putting this all together, we can now state our first proposed explication of **SEP**:

Definition 5.1.1 (Strong equivalence principle (first pass))**.** *Let $(\mathfrak{E}, X)$ be a kth-order natural equation, and suppose that there exists a natural quotient bundle $(X, \mathfrak{G})$ of X over G. Then $(\mathfrak{E}, X)$ satisfies the strong equivalence principle iff for any $M \in \mathrm{ob}(\overline{\mathcal{M}}_n)$, any section $\psi : M \to GM$, and any $p \in M$, there exists an open neighborhood O of p and a section $\chi : O \to GM$ whose induced metric is flat and such that $j^1_p\chi = j^1_p\psi$ and $\mathfrak{E}M_{|\psi} \cap \pi^{-1}_{J^k XM}(p) = \mathfrak{E}M_{|\chi} \cap \pi^{-1}_{J^k XM}(p)$.*

Definition 5.1.1 says that a system of equations formulated relative to an arbitrary Lorentzian metric satisfies **SEP** just in case in the neighbourhood of any point, there exists a flat metric coinciding with the metric (to first order) at that point, and such that the same equation, formulated relative to that flat metric, agrees with the original equation at that same point.[37] Definition 5.1.1 is thus closely related to the 'pointy' statements of **SEP** mentioned above, with the notion of 'same equation, expressed relative to a flat metric' (made precise through naturality) replacing the notion of an equation taking its 'special relativistic form' in coordinates at a point.

Still, in light of theorem 2.6.1, one might worry about the significance of the restriction to flat metrics in definition 5.1.1—for recall, theorem 2.6.1 says that *any* two metrics can be made to agree, to first order, at some point, by the action of a local diffeomorphism. This leads to a second possible explication of **SEP**:

[37] Note that whilst it might be tempting to gloss definition 5.1.1 (and likewise definition 5.1.2, which we will come to in a moment) as saying that the solution spaces of $\mathfrak{E}M_{|\psi}$ and $\mathfrak{E}M_{|\chi}$ coincide at p, this is slightly misleading, since the solution properties of $\mathfrak{E}M_{|\psi}$ and $\mathfrak{E}M_{|\chi}$ properly concern when n-dimensional subdistributions of the Cartan distribution on $J^k XM_{|\psi}$, $J^k XM_{|\chi}$ everywhere tangent to $\mathfrak{E}M_{|\psi}$, $\mathfrak{E}M_{|\chi}$ can be integrated over M, and absent further regularity conditions, there is no guarantee that for every point in $\mathfrak{E}M_{|\psi} \cap \pi^{-1}_{J^k XM}(p)$, $\mathfrak{E}M_{|\chi} \cap \pi^{-1}_{J^k XM}(p)$, there exist such integral submanifolds of $\mathfrak{E}M_{|\psi}$, $\mathfrak{E}M_{|\chi}$ containing that point. We will return to this issue in §5.3.

Definition 5.1.2 (Strong equivalence principle). *Let $(\mathfrak{E}, X)$ be a kth-order natural equation, and suppose that there exists a natural quotient bundle $(X, \mathfrak{G})$ of X over G. Then $(\mathfrak{E}, X)$ satisfies the strong equivalence principle iff for any M, $M' \in \mathrm{ob}(\overline{\mathcal{M}}_n)$, any sections $\psi : M \to GM$, $\psi' : M' \to GM'$, and any $p \in M$, $p' \in M'$, there exist open neighborhoods O, O' of p, p' and a diffeomorphism $\varphi : O \to O'$ such that $j^1_{p'}(G\varphi_*\psi) = j^1_{p'}\psi'$ and $J^kX\varphi_*(\mathfrak{E}M_{|\psi} \cap \pi^{-1}_{J^kXM}(p)) = \mathfrak{E}M'_{|\psi'} \cap \pi^{-1}_{J^kXM'}(p')$.*

In other words, a system of equations formulated relative to an arbitrary Lorentzian metric satisfies **SEP**, in the sense of definition 5.1.2, just in case the same equation, formulated relative to any other Lorentzian metric, can be made to agree with that equation at any point by the action of some diffeomorphism, under which the metrics coincide to first-order there. Or, even more informally, the system of equations, formulated relative to an arbitrary Lorentzian metric, is locally (i.e. at any point) 'the same as' for any other choice of Lorentzian metric.

Whilst definition 5.1.1 is perhaps closer to statements of **SEP** that one finds in the literature, definition 5.1.2 is my preferred statement of **SEP**. This is for several reasons. First, and as already noted, the significance of the restriction to flat metrics (and metrics on M) in definition 5.1.1 is opaque—as established in theorem 2.6.1, given *any* two metrics (on any M, M') and any $p \in M$, $p' \in M'$, there exists a local diffeomorphism such that they coincide, to first order, at p. Definition 5.1.2 does away with this restriction. Second, and relatedly, clearly any system of equations which satisfies **SEP** per definition 5.1.2 also satisfies **SEP** per definition 5.1.1. But the converse is also true, as can be seen straightforwardly using the arguments of Fletcher and Weatherall (2023a, theorem 5). Thus there is an obvious sense in which nothing turns on the choice to restrict the statement of **SEP** to flat metrics. Third, definition 5.1.2 captures a clear sense in which systems of equations formulated relative to arbitrary Lorentzian manifolds may be regarded as equivalent—namely that any two such equations can always be made to agree at some point by the action of some local diffeomorphism.

With this explication of **SEP** in hand, let us consider its relationship to **MC**. First, lemma 2.6.2 immediately establishes that **MC**, in the sense of definition 2.5.5, entails **SEP**, in the sense of definition 5.1.2. Second, and perhaps a little more surprisingly, theorem 2.6.1 and lemma 2.6.2 also show that the converse entailment holds for first-order systems, i.e. any first-order system of equations satisfying **SEP** satisfies **MC**.

Still, definition 5.1.2 does not only apply to first-order systems, and so one might wonder about the relationship between **SEP** and **MC** for systems which are not first order. Now, as noted in §2.6, definition 2.5.5 does not directly apply to higher-order systems of equations; however, these systems of equations can often be reduced to first-order systems (recall §2.4), to which the definition *does* apply. The following pair of theorems establish that, under certain additional assumptions, **SEP** is also necessary and sufficient for any such reduction to satisfy **MC**:

Theorem 5.1.3. *Let $(\mathfrak{E}, X)$ be a kth-order natural equation, $(X, \mathfrak{G})$ a natural quotient bundle of X over G, and suppose there exists a natural reduction $(\mathfrak{E}', X', f)$ of $(\mathfrak{E}, X)$ to order 1. Suppose further that there exists a natural quotient bundle $(X', \mathfrak{G}')$ of X' over G satisfying $\pi_G^{(k,1)} \circ j^k g = j^1 g' \circ f$, where $g : X \to G$, $g' : X' \to G$ are in $\mathfrak{G}$, $\mathfrak{G}'$ respectively. Suppose that $(\mathfrak{E}', X')$ is minimally coupled. Then $(\mathfrak{E}, X)$ satisfies the strong equivalence principle.*

Proof. Given theorem 2.6.1 and lemma 2.6.2, it suffices to show that for any $M \in \mathrm{ob}(\overline{\mathcal{M}}_n)$, any $p \in M$, and any sections $\psi, \chi : M \to GM$, the condition $j^1_p\psi = j^1_p\chi$ implies $\mathfrak{E}M_{|\psi} \cap \pi^{-1}_{J^kXM}(p) = \mathfrak{E}M_{|\chi} \cap \pi^{-1}_{J^kXM}(p)$. So let $\psi, \chi : M \to GM$ be sections and suppose that $j^1_p\psi = j^1_p\chi$. Since $(\mathfrak{E}', X')$ is minimally coupled, we know that $\mathfrak{E}'M_{|\psi} \cap \pi^{-1}_{J^1X'M}(p) = \mathfrak{E}'M_{|\chi} \cap \pi^{-1}_{J^1X'M}(p)$. Moreover, $\mathfrak{E}M = f^*_M(\mathfrak{E}'M)$. In particular, and using that the component f_M is a smooth bundle morphism covering the identity on M, this means that $\mathfrak{E}M_{|f^*_M\psi} \cap \pi^{-1}_{J^kXM}(p) = f^*_M(\mathfrak{E}'M_{|\psi} \cap \pi^{-1}_{J^1X'M}(p)) = f^*_M(\mathfrak{E}'M_{|\chi} \cap \pi^{-1}_{J^1X'M}(p)) = \mathfrak{E}M_{|f^*_M\chi} \cap \pi^{-1}_{J^kXM}(p)$. But since $\pi_G^{(k,1)} \circ j^k g = j^1 g' \circ f$, we know that $\mathfrak{E}M_{|f^*_M\psi} = \mathfrak{E}M_{|\psi}$, and likewise

$\mathfrak{E}M_{|f^*_M\chi} = \mathfrak{E}M_{|\chi}$. So we must have that $\mathfrak{E}M_{|\psi} \cap \pi^{-1}_{J^kXM}(p) = \mathfrak{E}M_{|\chi} \cap \pi^{-1}_{J^kXM}(p)$. $\square$

Theorem 5.1.4. *Let $(\mathfrak{E}, X)$ be a kth-order natural equation, $(X, \mathfrak{G})$ a natural quotient bundle of X over G, and suppose that $(\mathfrak{E}, X)$ satisfies the strong equivalence principle. Then for any natural reduction $(\mathfrak{E}', X', f)$ of $(\mathfrak{E}, X)$ to order 1 such that there exists a natural quotient bundle $(X', \mathfrak{G}')$ of X' over G satisfying $\pi^{(k,1)}_G \circ j^k g = j^1 g' \circ f$, where $g : X \to G$, $g' : X' \to G$ are in $\mathfrak{G}$, $\mathfrak{G}'$ respectively, $(\mathfrak{E}', X')$ is minimally coupled.*

Proof. Let $M \in \mathrm{ob}(\overline{\mathcal{M}}_n)$, let $p \in M$, let $\psi, \chi : M \to GM$ be sections, and suppose that $j^1_p\psi = j^1_p\chi$. Since $(\mathfrak{E}, X)$ satisfies the strong equivalence principle, we know that for any two such sections, $\mathfrak{E}M_{|\psi} \cap \pi^{-1}_{J^kXM}(p) = \mathfrak{E}M_{|\chi} \cap \pi^{-1}_{J^kXM}(p)$. Moreover, $\mathfrak{E}'M = f_M(\mathfrak{E}M)$. Again, using that the component f_M is a smooth bundle morphism covering the identity on M, this means that $\mathfrak{E}'M_{|\psi} \cap \pi^{-1}_{J^1X'M}(p) = f_M(\mathfrak{E}M_{|f^*_M\psi} \cap \pi^{-1}_{J^kXM}(p))$, and likewise $f_M(\mathfrak{E}M_{|f^*_M\chi} \cap \pi^{-1}_{J^kXM}(p)) = \mathfrak{E}'M_{|\chi} \cap \pi^{-1}_{J^1X'M}(p)$. But since $\pi^{(k,1)}_G \circ j^k g = j^1 g' \circ f$, we know that $\mathfrak{E}M_{|f^*_M\psi} = \mathfrak{E}M_{|\psi}$, and likewise $\mathfrak{E}M_{|f^*_M\chi} = \mathfrak{E}M_{|\chi}$. So since $j^1_p\psi = j^1_p\chi$, we must have $\mathfrak{E}'M_{|\psi} \cap \pi^{-1}_{J^1X'M}(p) = \mathfrak{E}'M_{|\chi} \cap \pi^{-1}_{J^1X'M}(p)$. $\square$

In both theorems 5.1.3 and 5.1.4, we have assumed that $\pi^{(k,1)}_G \circ j^k g = j^1 g' \circ f$. This condition says that the (natural) quotient bundle structure $(X, \mathfrak{G})$, $(X', \mathfrak{G}')$ on X and X' 'factors through' the order reduction $(\mathfrak{E}', X', f)$ of $(\mathfrak{E}, X)$ via the canonical projection $\pi^{(k,1)}_G$, in the sense that we have the following commuting diagram:

$$\begin{tikzcd}
J^kX \arrow[r, "j^kg"] \arrow[d, "f"] & J^kG \arrow[d, "\pi^{(k,1)}_G"] \\
J^1X' \arrow[r, "j^1g'"] & J^1G
\end{tikzcd}$$

Several comments are in order. First, theorems 5.1.3 and 5.1.4 suffice to substantiate Brown's (2005) claim that **MC** is sufficient for **SEP**, and Knox's

(2013) claim that **MC** is necessary for **SEP**.[38] Second, given the results of §4, this also substantiates the link between quasilinear, symmetric hyperbolic systems satisfying **SEP** (or their reductions to order 1) and adapted dynamics in the sense of **motion** and **causality** proposed by Brown (2005), Knox (2013, 2019), Read (2020a), and Read et al. (2018). Third, as alluded to earlier, it is worth noting the statement of **SEP** given in definition 5.1.2 concerns a property of systems of partial differential equations which holds only at a point. It seems to be an (often tacit) assumption in much of the dynamical relativity literature that such a 'pointy' statement of **SEP** either could not be adequate (at least for the work that defenders of the dynamical approach want **SEP** to do), or is in some other way unsatisfactory—thus, e.g. Read et al. (2018), after introducing two 'pointy' statements of **SEP**, quickly move on to discuss 'local approximate' versions, whilst Knox (2013, §2.3) suggests that 'pointy' statements of **SEP** are "unphysical". This has led to a literature (Ghins and Budden 2001; Knox 2013; Teh et al. forthcoming; Wallace 2017) attempting to isolate a contentful and viable statement of **SEP** which captures a property of extended spacetime regions (and matter fields on those regions). I do not want to suggest that these approaches are not of (technical or philosophical) value. However, from the current perspective, we can see that they may not be necessary, at least for the purposes of ensuring that conditions such as **motion** and **causality** hold of a system of partial differential equations. 'Pointy' statements of **SEP**, such as definition 5.1.1 or 5.1.2, are perfectly adequate for this task.

5.2 Local (approximate) Poincaré symmetry

I will now move on to consider a second condition which is often paired with discussions of **SEP**—*viz.* the invariance of a system of partial differential equations under local (approximate) Poincaré symmetries (i.e. **symmetry**). Here is

[38] That said, these results also give pause to the claim in Read et al. (2018, p. 14) that "minimal coupling [...] leads to (apparent) violations of [a version of] the strong equivalence principle"—though Read et al. are working here with 'syntactic' approaches to explicating criteria such as **MC** and **SEP** etc. mentioned in §1, so the results here should not be thought of as in direct tension with this claim.

Read (2020a) again:

> [The] essential aspect of the SEP is the imposition that, in the neighbourhood of any $p \in M$ in GR, laws of physics recover their 'special relativistic form'—where I shall understand this to mean: *a Poincaré invariant form.* (Read 2020a, p. 179, emphasis in original)

Similarly, Read et al. (2018), after introducing the "basic idea" of **SEP** as the requirement that "the dynamical laws of SR [...] are recovered in GR at any point" (Read et al. 2018, p. 14), continue almost without comment to formulate **SEP** as the condition that the "dynamical equations for non-gravitational fields reduce to a Poincaré invariant form at any $p \in M$" (possibly subject to further requirements) (Read et al. 2018, p. 17).

As with **SEP**, my aim here is not to rehearse the critiques of these kind of statements of **symmetry** that have been made by e.g. Dewar (2020), Fletcher (2020), Fletcher and Weatherall (2023a,b), and Weatherall (2021) (as mentioned in §1), but to get clear on what is intended by them. For our purposes, there are two salient features of the above statements of **symmetry** worth noting. The first is the connection to **SEP**—**symmetry** is intended as an explication of what is meant by **SEP**. The second is the *reason* that **symmetry** is supposed to be an explication of what is meant by **SEP**—because Poincaré invariance of a system of partial differential equations is supposed to be an explication of what it is for that system of equations to be special relativistic.

At this point, it is helpful to introduce two different kinds of symmetries of a system of equations, relativized to a choice of background structure, discussed by Weatherall and March (2025). Let $(\mathfrak{E}, X)$ be a natural equation, let $(X, \mathfrak{Y})$ be a natural quotient bundle of X over Y, let $M \in \mathrm{ob}(\mathcal{M}_n)$, and let $\psi : M \to YM$ be a section. A *spacetime symmetry* is a diffeomorphism $\varphi : M \to M$ such that $Y\varphi^*\psi = \psi$. A *dynamical symmetry*, meanwhile, is a diffeomorphism $\varphi : M \to M$ such that $X\varphi^*\mathfrak{E}M_{|\psi} = \mathfrak{E}M_{|\psi}$. In other words, a spacetime symmetry is a diffeomorphism that preserves the choice of background structure, whereas a dynamical symmetry is a diffeomorphism that preserves the equation relative to that background structure, irrespective of whether it preserves the background structure itself.

With things set up in this way, and as Weatherall and March (2025) note, it is straightforward to see that every spacetime symmetry of an equation, relative to some fixed choice of background structure, is also a dynamical symmetry.[39] Indeed, this follows immediately from the fact that the full system of equations $(\mathfrak{E}, X)$ is natural. In particular, letting $Y = G$, $M \simeq \mathbb{R}^n$, and $\psi : M \to GM$ be the Minkowski metric on M, we have that any Poincaré transformation is a dynamical symmetry of $\mathfrak{E}M_{|\psi}$—or in other words, Poincaré symmetry of a system of equations (relative to a choice of background structure) is a necessary condition for that system of equations to be special relativistic in the sense discussed in §5.1 (i.e. for the background structure to be the Minkowski metric).

Putting these ideas together with the discussion of **SEP** in §5.1, we are now in a position to give a precise statement of **symmetry**:

Definition 5.2.1 (Local Poincaré invariance, equation version)**.** *Let* $(\mathfrak{E}, X)$ *be a kth-order natural equation, and suppose that there exists a natural quotient bundle* $(X, \mathfrak{G})$ *of* X *over* G*. Then* $(\mathfrak{E}, X)$ *is locally Poincaré invariant iff, for any* $M \in \mathrm{ob}(\overline{\mathcal{M}}_n)$ *any section* $\psi : M \to GM$*, any* $p \in M$*, any* $N \simeq \mathbb{R}^n$*, any section* $\chi : N \to GN$ *whose induced metric is flat, and any* $q \in N$*, there exists a smooth embedding* $\varphi : N \to M$ *such that* $\varphi(q) = p$*,* $j^1_q(G\varphi^*\psi) = j^1_q\chi$*,* $J^1X\varphi_*(\mathfrak{E}N_{|\chi} \cap \pi^{-1}_{J^1XN}(q)) = \mathfrak{E}M_{|\psi} \cap \pi^{-1}_{J^1XM}(p)$*, and* $J^kX\varphi'^*(\mathfrak{E}N)_{|\chi}) = \mathfrak{E}N_{|\chi}$ *for any diffeomorphism* $\varphi' : N \to N$ *such that* $G\varphi'(\chi) = \chi$*.*

Definition 5.2.1 says that an equation is locally Poincaré invariant iff at any point, it can be made to agree with the same equation expressed relative to the Minkowski metric via the action of some diffeomorphism, and expressed relative to the Minkowski metric, its dynamical symmetries have the Poincaré group as a subgroup. Lemma 2.6.2 immediately establishes that any minimally coupled

[39] See Weatherall and March (2025) for discussion of this point, in relation to Earman's (1989) famous 'symmetry matching' principles. Note that this provides an alternative (and precise) way of recovering Earman's (1989, p. 47) argument that 'general covariance' ensures that every spacetime symmetry of a theory will also be a dynamical symmetry.

system of equations is locally Poincaré invariant, in this sense. Likewise, it is straightforward to see that any system of equations satisfying **SEP** is locally Poincaré invariant, and *vice versa*. It is in this sense that one might, following Read (2020a) and Read et al. (2018), take **symmetry** as an explication of what is meant by **SEP**.

Given the results of §4, this suffices to substantiate the link between **symmetry** and adapted dynamics proposed by Read (2020a) and Read et al. (2018). But in fact, we can say more. For this, it is helpful to note first that there is nothing special about flat spacetimes (or Poincaré symmetries) in definition 5.2.1; one could equally well replace the flat metric χ that features in definition 5.2.1 with an arbitrary Lorentzian metric ((anti-)de Sitter, Schwarzschild, Kerr, etc.). The result would be a definition that was equivalent to **SEP** in the same sense as definition 5.2.1, i.e. any system of equations satisfying it would also satisfy **SEP**, and *vice versa*. In other words, any system of equations which is 'locally Poincaré invariant' is also 'locally (anti-)de Sitter invariant', 'locally Schwarzschild invariant', etc. (see also Fletcher and Weatherall (2023a)).

In this light, one might ask: what, if anything, is special about local Poincaré invariance (rather than, say, (anti-)de Sitter, Schwarzschild, Kerr, etc.)? For this, we need to return to §4. Whilst definition 5.2.1 is equivalent to definition 5.1.2 (and definition 2.5.5, for first-order systems), it does draw attention to an important *property* of such systems. In fact, we made use of exactly this property in the proof of lemma 4.1.1. There, the argument relied on the fact that given any two covectors n_a, n'_a at some p in Minkowski spacetime (M, η_{ab}) with $\eta_{ab}n^a n^b = \eta_{ab}n'^a n'^b$, there exists an isometry (i.e. Poincaré transformation) $\varphi : M \to M$ such that $\varphi_* n_a = n'_a$. Moreover if one returns to the proofs of lemmas 4.1.2-4.1.3 and theorems 4.1.4 and 4.2.1 (Geroch–Earman causality and the DEC), one sees that they rely only on lemma 4.1.1. In other words, what is special about the Poincaré symmetries of Minkowski spacetime (in the context of lemma 4.1.1, and thus in relation to Geroch–Earman causality and the DEC) is that Minkowski spacetime is isotropic.

That said, of course Minkowski spacetime is not the only Lorentzian geometry which is isotropic, so one might wonder about the significance of Poincaré symmetries as compared with other isotropic spacetimes ((anti-)de Sitter, say). Similarly, the isotropy of Minkowski spacetime involves only the Lorentz subgroup of the Poincaré group, and so one might wonder about the significance of local *Poincaré* symmetries (as opposed to local Lorentz symmetries). For this, we need our second explication of local Poincaré invariance, which applies not to systems of partial differential equations but to geometric object fields:

Definition 5.2.2 (Local Poincaré invariance, geometric object field version)**.** *Let $F : G \to X$ be a natural operator. Then $F(M, \psi)$ is locally Poincaré invariant iff for any $M \in \mathrm{ob}(\overline{\mathcal{M}}_n)$, any section $\psi : M \to GM$, any $p \in M$, any $N \simeq \mathbb{R}^n$, any section $\chi : N \to GN$ whose induced metric is flat, and any $q \in N$, there exists a smooth embedding $\varphi : N \to M$ such that $\varphi(q) = p$, $j^1_q(G\varphi^*\psi) = j^1_q\chi$, $X\varphi^* F(M, \psi)(p) = F(N, \chi)(q)$, and for any diffeomorphism $\varphi' : N \to N$ such that $G\varphi'(\chi) = \chi$, $X\varphi'^* F(N, \chi) = F(N, \chi)$.*

It follows immediately that for any first-order natural operator $F : G \to X$, any $M \in \mathrm{ob}(\overline{\mathcal{M}}_n)$, and any section $\psi : M \to GM$, $F(M, \psi)$ is locally Poincaré invariant. In fact, we made use of exactly this property of $T^a{}_{\alpha'}{}^b{}_{\beta'}$ in the proof of theorem 4.3.1 (the CC), along with local Poincaré invariance of $\mathfrak{E}$. And for that theorem, we did need the full Poincaré group—for recall, theorem 4.3.1 relies on the fact that any point in Minkowski spacetime can be mapped onto any other by some isometry (i.e. diffeomorphism enacting a Poincaré transformation), i.e. that it is homogeneous. But the proof of theorem 4.3.1 made use not just of the fact that Minkowski spacetime is homogeneous, but also that it is flat—for it required that at each point, a basis of Killing fields $\overset{i}{\xi}{}^a$, $i = 1, ..., n$ could be chosen such that $\nabla_a \xi^b = \mathbf{0}$ for each $\overset{i}{\xi}{}^a$. This is not true except in flat spacetime.

This provides a precise way of making sense of the idea expressed by Read (2020a) and Read et al. (2018) that there is something particularly salient about the Poincaré symmetries of natural equations on Minkowski spacetime for understanding how matter might be adapted to spacetime geometry, in the senses specified by **motion** and **causality**. It also substantially clarifies (and defuses) a recent debate between Fletcher and Weatherall (2023a), Gomes (2022), and Teh et al. (forthcoming) about the sense in which Minkowski spacetime is (or is not) 'special' for the purposes of understanding how matter comes to be adapted to a relativistic spacetime.[40] Lemma 4.1.1 and theorem 4.3.1 show precisely how (and why) one might take Minkowski spacetime in particular to be relevant to this issue—and that not just any Lorentzian manifold can be used in the same way—only flat, maximally symmetric, spacetime will do. In that sense, Minkowski spacetime is a salient one to consider. But Minkowski spacetime is not unique among Lorentzian geometries in sense that it is not the only spacetime (with associated group of spacetime symmetries) which could be substituted without loss into definition 5.2.1—and so, any other sense in which Minkowski spacetime is 'special' compared with other Lorentzian geometries, say, (anti-)de Sitter or Schwarzschild spacetime, is probably better captured by the pragmatic considerations mentioned by Fletcher and Weatherall (2023a, §4).

5.3 The local validity of special relativity

Finally, we turn to **LSR**. We have now considered three criteria—**MC**, **SEP**, and **symmetry**—for when a system of partial differential equations might be said to be adapted to a relativistic spacetime geometry. But criteria such as **MC**, **SEP**, and **symmetry** are also often claimed to be relevant to quite a different task—namely, that of capturing a sense in which special relativity is 'locally valid' in general relativity, and so explaining, e.g. how general relativity might be said to subsume special relativity, or why, from the perspective of general relativity, special relativity has been successful in the domains in which it has.

Now, granted, there is an obvious sense in which the first-pass statement of **SEP** given in definition 5.1.1, or **symmetry**, per definition 5.2.1, capture ways in which special relativity (or, better, special-relativistic matter theories)

[40] That said, of course this does not touch on the discussion of Teh et al. (forthcoming) about 'scale-relative reasoning' (though I will return to this briefly in §5.3).

might be thought of as 'locally valid' in general relativity—at least at a point. However, this sense of 'local validity' is rather weak—at least for the purposes of the two explanatory tasks mentioned above. For the first: special relativity is a theory which makes claims about extended spacetime regions, not just matter fields at a point. And likewise for the second: special relativity, in the domains in which it is successful, is not just applied at a point, but to extended spacetime regions.

One approach to closing this gap—and explicating **LSR**—would be to isolate a precise condition, akin to the 'un-pointy' variants of **SEP** mentioned above, which captures an analogue of **SEP** or **symmetry** which holds approximately on extended spacetime regions, in the neighbourhood of a point. In fact, we already have the tools to do just this, and I explore this option in §5.3.1. A second approach, which requires an importantly different kind of information about systems of partial differential equations than that which we have been considering thus far, would be to focus on when the solutions of a general-relativistic system of equations 'locally approximate' the solutions of their special-relativistic counterparts on some extended spacetime regions. I will consider this option in §5.3.2.

5.3.1 The strong equivalence principle, reprise

Our first criterion for when a system of equations might be said to satisfy **LSR** is a natural continuation of some of the ideas developed in §5.1. The idea is that we have now given a precise definition of both a system of equations, and what it is to have the same system of equations, expressed relative to two different metrics. One can then make rigorous sense of what it is for two such equations to 'locally approximate' one another (in the neighbourhood of some point) by introducing an appropriate distance function, which measures 'how close one equation is to the other'.

So let $(\mathfrak{E}, X)$ be a kth-order natural equation on $X : \overline{\mathcal{M}}_n \to \mathcal{FB}$, and suppose that there exists a natural quotient bundle $(X, \mathfrak{G})$ of X over G. Let

$M \in \mathrm{ob}(\overline{\mathcal{M}}_n)$, let $\psi : M \to GM$ be a section, and consider a point $p \in M$, a neighbourhood O of p diffeomorphic to $\mathbb{R}^n$, and a local section $\chi : O \to GM$ whose induced metric is flat. Then, as we have seen, $\mathfrak{E}M_{|\chi}$ can be thought of as the 'same equation' as $\mathfrak{E}M_{|\psi}$, expressed relative to a (local) flat metric. Further assume that $X = Y \times G$ for some natural bundle Y. (This ensures that we have a canonical way of thinking of $\mathfrak{E}M_{|\psi}$, $\mathfrak{E}M_{|\chi}$ as submanifolds of the same bundle J^kYM.) We can now introduce the following criterion for when a theory might be said to be 'locally special relativistic', in the sense that it 'satisfies **SEP** approximately' in the neighbourhood of some point:

Definition 5.3.1 (LSR schema, SEP-style)**.** *Let $(\mathfrak{E}, X)$ be a kth-order natural equation on a natural bundle $X : \overline{\mathcal{M}}_n \to \mathcal{FB}$, and suppose that there exists a natural quotient bundle $(X, \mathfrak{G})$ of X over G with $X = Y \times G$ for some natural bundle Y. Then $(\mathfrak{E}, X)$ is locally special relativistic iff for any $M \in \mathrm{ob}(\overline{\mathcal{M}}_n)$, any section $\psi : M \to GM$, any $p \in M$, any open neighborhood O of p and section $\chi : O \to GM$ whose induced metric is flat and such that $j^1_p\chi = j^1_p\psi$, and any $\varepsilon > 0$, there exists a sub-neighbourhood $O' \subset O$ of p on which $d_{O'}(\mathfrak{E}M_{|\psi}, \mathfrak{E}M_{|\chi}) < \varepsilon$, where $d_{O'}$ is some suitable, context-dependent distance function on pairs of (closed) submanifolds of J^kXO'.*

The SEP-style **LSR** schema invokes a distance function d_O on pairs of (closed) submanifolds of J^kXO. We have not specified what this distance function should be, and one finds different criteria for different choices of d_O.[41] That said, there are some obvious desiderata on the distance functions one considers. This is perhaps clearest when one considers, e.g. the subtleties involved in specifying a notion of 'distance' between two subspaces of a metric space. Let X be a metric space and A a closed subspace thereof. Then given a point $x \in X$, there is a well-defined distance function $d_X(x, A) = \inf\{|x - y|, y \in A\}$. This extends to a distance function $d_X(B, A) = \inf\{|x - y|, y \in A, x \in B\}$ whenever A is closed and B is compact. But an analogous distance function is not well-defined when both A and B are closed but not necessarily compact. Moreover, even when it

[41] I take the freedom to choose such a distance function to reflect the context-sensitivity associated with talk of approximation—see Fletcher (2016) for discussion on this point.

is, there is an obvious sense in which this does not give us what we want—for $\inf\{|x-y|, y \in A, x \in B\} = 0$ whenever $A \cap B \neq \varnothing$, whereas, presumably, what one would like for the relevant notion of two equations 'locally approximating' one another is that they count as 'approximating one another exactly' on some region iff they coincide on that region.[42] This already highlights two conditions we should expect the distance function d_O in definition 5.3.1 to satisfy. First, clearly d_O must be well-defined on pairs of closed fibred submanifolds of J^kXO. Second, d_O should be separating on pairs of closed fibred submanifolds of J^kXO, and, in fact, fibrewise separating—that is, restricted to any point $p \in O$, d_p should vanish on any two closed submanifolds of $\pi^{-1}_{J^kXM}(p)$ only if they coincide.

Without some choice of d_O, it is unclear how to evaluate the relationship between the **LSR** schema given in definition 5.3.1 and criteria such as **SEP**, **MC**, **symmetry**. However, I will presently show that on at least one concrete instantiation of this schema meeting the conditions outlined above, and for a large class of natural theories, **SEP** is sufficient for **LSR**. For this purpose, we will consider the following class of distance functions, which are best introduced in steps, but might be described as 'supremum distance functions on O of Hausdorff distances between the fibres of $\mathfrak{E}M_{|\psi}$, $\mathfrak{E}M_{|\chi}$ induced by the Frobenius norm on tensor fields induced by a metric on O'—though I do not claim that these are somehow the 'canonical' choice.

For this, assume first that Y is a natural affine bundle, whose modelling natural vector bundle is first-order. (This implies that J^kY is a natural affine bundle, modelled on some first-order natural vector bundle.) Let h_{ab} be a Riemannian metric on some neighbourhood $O' \subset O$ of p. Since Y is a natural affine bundle, h_{ab} induces a norm $|x-x'|_h$ on pairs x, x' of points on the fibre of J^kYM at each $p' \in O'$. (Namely, one takes $|x-x'|_h$ to be the Frobenius norm on the tensor $(x-x')$ induced by h_{ab}. That $(x-x')$ can be identified with a tensor on M at p is a consequence of the fact that Y is a natural affine bundle

[42] Somewhat more technically, the point is that $\inf\{|x-y|, y \in A, x \in B\}$ defines a (symmetric) premetric, but not a semimetric, on the space of compact subsets of X, and so is not separating on the space of compact subsets of X.

whose modelling vector bundle is first-order.)

The distance function $|x - x'|_h$ on each fibre $\pi^{-1}_{J^kYM}(p)$ induces the Hausdorff distance function between pairs of (closed) submanifolds of that fibre. That is, given any (closed) submanifold A of $\pi^{-1}_{J^kYM}(p')$, and any $x \in \pi^{-1}_{J^kYM}(p')$, one defines the distance $d_h(x, A) = \inf\{|x - x'|_h, x' \in A\}$. For any other (closed) submanifold B of $\pi^{-1}_{J^kYM}(p')$, one then takes $d_h(B, A) = \sup\{d_h(x, A), x \in B\}$. (Note that $d_h(B, A)$ is not symmetric.) One then takes $d(B, A; h) = \max\{d_h(A, B), d_h(B, A)\}$. In particular, this means that given the equations $\mathfrak{E}M_\psi$, $\mathfrak{E}M_\chi$, one has, for each $p' \in O'$, $d(\mathfrak{E}M_{|\chi} \cap \pi^{-1}_{J^kX}(p'), \mathfrak{E}M_{|\psi} \cap \pi^{-1}_{J^kX}(p'); h)$. Finally, given any (relatively compact) subregion O'' of O', one has $d_{O''}(\mathfrak{E}M_{|\chi}, \mathfrak{E}M_{|\psi}; h) := \sup\{d(\mathfrak{E}M_{|\chi} \cap \pi^{-1}_{J^kX}(p'), \mathfrak{E}M_{|\psi} \cap \pi^{-1}_{J^kX}(p'); h), p' \in O''\}$.

One immediately has the following result:

Theorem 5.3.2. *Let $(\mathfrak{E}, X)$ be a kth-order natural equation on $X : \overline{\mathcal{M}}_n \to \mathcal{FB}$, and suppose that there exists a natural quotient bundle $(X, \mathfrak{G})$ of X over G, with $X = Y \times G$ for some natural bundle Y. Suppose that Y is a natural affine bundle, with the modelling natural vector bundle U for Y first-order. Further suppose that $(\mathfrak{E}, X)$ satisfies the strong equivalence principle. Then for any $M \in \mathrm{ob}(\overline{\mathcal{M}}_n)$, any section $\psi : M \to GM$, any $p \in M$, any open neighborhood O of p and section $\chi : O \to GM$ whose induced metric is flat and such that $j^1_p\chi = j^1_p\psi$, any $\varepsilon > 0$, and any Riemannian metric h on a subneighbourhood $O' \subset O$ of p, there exists a sub-sub-neighbourhood $O'' \subset O'$ of p on which $d_{O''}(\mathfrak{E}M_{|\psi}, \mathfrak{E}M_{|\chi}; h) < \varepsilon$, where $d_{O''}(\cdot, \cdot; h)$ is as above.*

(This follows from the fact that, since $(\mathfrak{E}, X)$ satisfies **SEP**, $\mathfrak{E}M_{|\chi} \cap \pi^{-1}_{J^kYM}(p) = \mathfrak{E}M_{|\psi} \cap \pi^{-1}_{J^kYM}(p)$, so that one has $d_p(\mathfrak{E}M_{|\psi}, \mathfrak{E}M_{|\chi}; h) = 0$ for any h. The result then follows by continuity.) In other words, any (suitable) system of equations which is 'special relativistic at a point' (in the sense of **SEP**) is also 'approximately special relativistic' in sufficiently small neighbourhoods of that point, according to the standard of approximation given by the distance function $d_O(\cdot, \cdot; h)$. The results of section §5.1 then suffice to establish analogous results if one substitutes **SEP** for **MC**, in cases where **MC** applies.

(As an aside, I take the discussion of Teh et al. (forthcoming) about the importance of 'scale-relative' reasoning for making sense of criteria such as **LSR** to be consistent with—and in fact complementary to—the perspective offered here. Indeed, I take these kind of discussions to concern precisely such questions as, e.g. the relationship between the (maximum) size of the neighbourhood O' and the bound ε, for different systems of partial differential equations and different choices of section ψ in the **LSR** schema, given some distance function. That such a neighbourhood O' exists, for each ε, may well be a consequence of e.g. **SEP** and mathematical facts about continuity, as argued above, but figuring out exactly what it should be is a matter of deep physical significance which surely demands consideration of questions about relative lengthscales of e.g. curvature as compared with some neighbourhood of a point (Teh et al. forthcoming; Wallace 2017). The same point carries over, *mutatis mutandis*, to the Ehlers-style criteria for **LSR** which feature in the next section.)

5.3.2 Minimally coupled quasilinear symmetric hyperbolic theories are LSR, Chez Ehlers

The previous discussion of **LSR** concerned when two equations, formulated relative to curved and flat Lorentzian metrics, might be said to 'locally approximate' one another. But one might also be interested in a slightly different sense of **LSR**, concerning when the solutions of those two equations 'locally approximate' one another. In general, these are not equivalent—for the solution properties of a system of equations properly concern when 'pointwise' solutions of the system can be integrated over M to form a consistent submanifold.

Whilst it is not my aim here to develop systematically the theory of integrability of partial differential equations, I will briefly recall some definitions which are relevant.[43] Let $E \subset J^1 B \to B \to M$ be a first-order, quasilinear system of equations, with presystem U. A constraint is a tensor field $c^{ab}{}_{\alpha'}$ on B, antisymmetric in a and b which annihilates every element of U at each point.

[43]I take this presentation of integrability from Jim Weatherall.

That an equation is integrable means that it admits (local) smooth solutions through every point in B, i.e. smooth sections $\psi : O \to B$ whose image $j^1\psi$ lies in E. This will occur iff every section $B \to E$—i.e. every connection ω on B whose image lies in E—is flat, up to translation by elements of U; or in other words, iff its curvature $\Omega^{\alpha'}{}_{ab}$ lies in the kernel of $c^{ab}{}_{\alpha'}$ for every constraint.

We can now introduce two criteria, suggested by Fletcher and Weatherall (2023b), for when a system of equations might be said to be 'locally special relativistic' in the sense that curved spacetime solutions 'locally approximate' flat spacetime solutions or *vice versa* (which I have translated into the notation of this thesis):

Condition 5.3.3 (LSR schema, chez Ehlers, g-approximating version)**.** *Let $(\mathfrak{E}, X)$ be a kth-order natural equation, and suppose that there exists a natural quotient bundle $(X, \mathfrak{G})$ of X over G. Then for any $M \in \mathrm{ob}(\overline{\mathcal{M}}_n)$, any section $\psi : M \to GM$, any $p \in M$, any flat local section $\psi' : O \to GM$ on some neighborhood O of p such that $j^1_p\psi = j^1_p\psi'$, any solution $\chi' : O \to XM_{|\psi'}$ of $\mathfrak{E}M_{|\psi'}$ and any $\epsilon > 0$, there exists, on a subneighborhood $O' \subset O$ of p, a solution $\chi : O' \to XM_{|\psi}$ of $\mathfrak{E}M_{|\psi}$ and a sub-subneighborhood $O'' \subset O'$ of p on which $||\chi - \chi'|| < \epsilon$, where $|| \cdot ||$ is some suitable, context-dependent norm.*

Condition 5.3.4 (LSR schema, chez Ehlers, η-approximating version)**.** *Let $(\mathfrak{E}, X)$ be a kth-order natural equation, and suppose that there exists a natural quotient bundle $(X, \mathfrak{G})$ of X over G. Then for any $M \in \mathrm{ob}(\overline{\mathcal{M}}_n)$, any section $\psi : M \to GM$, any $p \in M$, any flat local section $\psi' : O \to GM$ on some neighborhood O of p such that $j^1_p\psi = j^1_p\psi'$, any solution $\chi : O \to XM_{|\psi}$ of $\mathfrak{E}M_{|\psi}$ and any $\epsilon > 0$, there exists, on a subneighborhood $O' \subset O$ of p, a solution $\chi' : O' \to XM_{|\psi'}$ of $\mathfrak{E}M_{|\psi'}$ and a sub-subneighborhood $O'' \subset O'$ of p on which $||\chi - \chi'|| < \epsilon$, where $|| \cdot ||$ is some suitable, context-dependent norm.*

(See Fletcher and Weatherall (2023b) for further discussion of and motivation for these criteria.) As in the SEP-style schema for **LSR** given in definition 5.3.1, both the g-approximating and η-approximating Ehlers schemata invoke a norm $|| \cdot ||$. Again, we do not specify what this norm should be. However, following

Fletcher and Weatherall (2023b), I will now show that, for at least one plausible choice of norm, both the g-approximating and η-approximating Ehlers criteria are satisfied by a large class of (first order) minimally coupled theories. The only additional assumptions needed are quasilinearity and integrability. Following Fletcher and Weatherall (2023b), I will focus on the C^1 supremum norm on tensor fields relative to some (fiducial) Riemannian metric h on M as an example (see Fletcher and Weatherall (2023a,b) for details). Fletcher and Weatherall show that two important examples of minimally coupled, quasilinear, symmetric hyperbolic theories will be **LSR** on both the Ehlers criteria—*viz.* minimally coupled Maxwell and Klein–Gordon theory. We have the following slightly more general result:

Theorem 5.3.5 (g-approximating LSR)**.** *Let $(\mathfrak{E}, X)$ be a first-order, minimally coupled, quasilinear equation, with $X = Y \times G$,*[44] *where $Y : \mathcal{M}_n \to \mathcal{FB}$ is a natural affine bundle modelled on a first-order natural vector bundle. Then for any $M \in \mathrm{ob}(\overline{\mathcal{M}}_n)$, any section $\psi : M \to GM$, any $p \in M$, any flat local section $\psi' : O \to GM$ on some neighborhood of p such that $j^1_p\psi = j^1_p\psi'$ and $\mathfrak{E}M_{|\psi}$ is integrable, any solution $\chi' : O \to XM_{|\psi'}$ of $\mathfrak{E}M_{|\psi'}$ and any $\epsilon > 0$, there exists a subneighborhood $O' \subset O$ of p, a solution $\chi : O' \to XM_{|\psi}$ of $\mathfrak{E}M_{|\psi}$ and a sub-subneighborhood $O'' \subset O'$ of p on which $||\chi - \chi'|| = d_{O''}(\chi, \chi', h, 1) < \epsilon$, where $||\cdot||$ denotes the C^1 supremum norm.*

Proof. It suffices to show that there exists a neighbourhood O' of p and a solution $\chi : O' \to XM_{|\psi}$ of $\mathfrak{E}M_{|\psi}$ such that $\varphi(p) = \varphi'(p)$ and $j^1_p\varphi = j^1_p\varphi'$. By lemmas 2.6.2 and 2.7.5, we know that $\mathfrak{E}M_{|\psi} \cap \pi^{-1}_{J^1XM}(p) = \mathfrak{E}M_{|\psi'} \cap \pi^{-1}_{J^1XM}(p)$ and $U_{\mathfrak{E}M_{|\psi}} \cap \pi^{-1}_{J^1XM}(p) = U_{\mathfrak{E}M_{|\psi'}} \cap \pi^{-1}_{J^1XM}(p)$. So it suffices to show that for any $p \in M$ and any $x \in \mathfrak{E}M_{|\psi} \cap \pi^{-1}_{J^1XM}(p)$, there exists a neighbourhood O' of p and a section $\chi : O' \to XM_{|\psi}$ such that $j^1_p\chi' = j^1_p\chi$. Since $j^1\chi' \in \mathfrak{E}M_{|\psi} \cap \pi^{-1}_{J^1XM}(p)$, for this, it suffices that $\mathfrak{E}M_{|\psi}$ be quasilinear and (locally) integrable. (For then, by quasilinearity and integrability, there exists a flat

[44] Again, this condition is needed to ensure that we have a canonical way of thinking about solutions of two different denaturalized systems of equations on some $M \in \mathrm{ob}(\overline{\mathcal{M}}_n)$ as sections of the same bundle.

connection $\omega : XM_{|\psi} \to \mathfrak{E}M_{|\psi}$ on some neigbourhood of $\pi^{-1}_{XM_{|\psi}}(p)$ which lies in $\mathfrak{E}M_{|\psi}$ and with $\omega(\chi(p)) = j^1_p\chi$. Any (local) constant section of $XM_{|\psi}$ with respect to ω through $\chi(p)$ yields a solution of $\mathfrak{E}M_{|\psi}$ with the desired properties, and such exist, since ω is flat.) $\square$

Exactly the same argument suffices to establish an analogous result for the η-approximating Ehlers criterion, with the C^1 supremum norm:

Theorem 5.3.6 (η-approximating LSR)**.** *Let $(\mathfrak{E}, X)$ be a first-order, minimally coupled, quasilinear equation, with $X = Y \times G$, where $Y : \mathcal{M}_n \to \mathcal{FB}$ is a natural affine bundle modelled on a first-order natural vector bundle. Then for any $M \in \mathrm{ob}(\overline{\mathcal{M}}_n)$, any section $\psi : M \to GM$, any $p \in M$, any flat local section $\psi' : O \to GM$ on some neighborhood of p such that $j^1_p\psi = j^1_p\psi'$ and $\mathfrak{E}M_{|\psi}$ is integrable, any solution $\chi : O \to XM_{|\psi}$ of $\mathfrak{E}M_{|\psi}$ and any $\epsilon > 0$, there exists a subneighborhood $O' \subset O$ of p, a solution $\chi : O' \to XM_{|\psi'}$ of $\mathfrak{E}M_{|\psi'}$ and a sub-subneighborhood $O'' \subset O'$ of p on which $||\chi - \chi'|| = d_{O''}(\chi, \chi', h, 1) < \epsilon$, where $||\cdot||$ denotes the C^1 supremum norm.*

In other words, any minimally coupled quasilinear system of partial differential equations $(\mathfrak{E}, X)$ which admits, for each $M \in \mathrm{ob}(\overline{\mathcal{M}}_n)$ and each section $\psi : M \to GM$, solutions through each point in $\mathfrak{E}M_\psi$ is **LSR** on both the g-approximating and η-approximating Ehlers-style criteria. §5.1 then suffices to establish analogous results for systems of partial differential equations satisfying **SEP**, when their reductions to order 1 satisfy the above conditions. Together with the results of §5.3.1, this gives us a two clear and precise senses in which **MC** or **SEP** are sufficient for something like the 'local validity' of special relativity, at least for the choices of norms considered here.

6 Close

In his “23 open problems in the philosophy of physics”, James Read (2025) identifies problem number 4 as the following:

4. Develop a clear understanding of the technical conditions under which material fields come to be ‘adapted’ to a given spacetime geometry.

I take the work of this thesis to be a significant contribution to addressing precisely this kind of problem.

Nonetheless, I do not take it to be dispositive, nor exhaustive. Indeed, there are several remaining issues which I have barely touched on here, which—to my mind at least—any full solution to Read’s problem 4 would ultimately have to address. To end this thesis, I want to briefly say something about what I take the most important of these issues (and other possible extensions of this work) to be.

First, **time**. The technical results about ‘adapted’ dynamics in this thesis have focused exclusively on **motion** and **causality**. This leaves open the question of sufficient conditions for matter fields to exhibit the third feature of ‘adapted’ dynamics identified in §1, namely **time**. Now, as noted in §1, in the literature, there exist various constructions for explicating **time**—pendulum clocks, harmonic oscillators, muons, light clocks, etc. But these are somewhat heterogeneous, and so one might ask for something more general—e.g. a theorem to the effect that any ‘suitably periodic’ solution, when confined to sufficiently small neighbourhoods of a (timelike) curve, will measure arclength along that curve arbitrarily closely.

Second, the Lagrangian formalism. This is a natural extension of the derivations of the DEC and CC in §4. As emphasised in §§3 and 4, nothing in **motion** requires that the rank-two symmetric tensor associated to matter fields which satisfies the DEC and CC be the Einstein–Hilbert stress–energy for those matter fields. But, the DEC and CC are probably most often thought about in connection with the Einstein–Hilbert stress–energy associated with matter—and in fact, it is well-known that there are general arguments to the effect that, providing the metric plays certain roles in the Lagrangian associated with matter fields, then the Einstein–Hilbert stress–energy for those matter fields will necessarily satisfy the CC (see e.g. Weatherall (2019)). It would be of interest to establish under what conditions the Einstein–Hilbert stress–energy for matter fields coincides with a tensor $T^{a}{}_{\mu'}{}^{b}{}_{\nu'}v^{\mu'}v^{\nu'}$ defining maximal hyperbolizations of the (order-reduced) Euler-Lagrange equations for those matter fields, as described in the DEC theorem, and so satisfies the DEC.

Third, the hyperbolicity of Einstein's equation (and other gauge field equations, e.g. the Yang–Mills equation). This builds on some of the ideas discussed in §2.8. The idea here is that we have now provided a 'natural' way of thinking about the existence of (maximal) hyperbolizations of a system of partial differential equations, in terms of mixed index symmetric tensor fields $T^{a}{}_{\mu'}{}^{b}{}_{\nu'}$ satisfying various conditions. As such, one might wonder if these mixed index tensor fields could be used to give an 'intrinsic' characterization of the sense in which e.g. Einstein's equation or the Yang–Mills equation is 'almost symmetric hyperbolic', i.e. admits a well-posed initial value problem up to (unique) isomorphism.

Of course, to say something systematic on any of these questions would take us far beyond the scope of this thesis. But at the very least, I hope that the tools developed here will prove useful for answering them—and that this thesis has showcased the value in seeking to do so, for our understanding of space, time, and matter.

A Fibre bundles

A.1 Useful constructions with fibre bundles

Let $B \stackrel{\pi}{\to} M$ be a smooth fibre bundle. We are interested in ways in which we can construct further fibre bundles from B.

First, let $B \stackrel{\pi}{\to} M$ be a smooth fibre bundle, and let $f : N \to M$ be a smooth map. The *pullback bundle* f^*B of B by f is the bundle $f^*B \stackrel{f^*\pi}{\to} N$ with total space f^*B consisting of all pairs (x, p), $x \in B$, $p \in N$ such that $f(p) = \pi(x)$, and with projection map $f^*\pi$ given by $f^*\pi(x, p) = p$. One can think of the fibred product as attaching, to each $p \in N$, a copy of the fibre $\pi^{-1}(f(p))$ of B at $f(p)$.

A special case of the pullback bundle construction comes when we have another smooth fibre bundle $B' \stackrel{\pi'}{\to} M$ over M. In that case, the total space of the bundle π'^*B (or equivalently π^*B') is denoted $B \times_M B'$. This bundle comes equipped with three projection maps: $\pi'^*\pi$ (making it a bundle over B'), $\pi^*\pi'$ (making it a bundle over B), and $\pi' \circ \pi'^*\pi = \pi \circ \pi^*\pi'$ (making it a bundle over M). The bundle $B \times_M B'$ is called the *fibred product*. The fibred product can be thought of as the bundle whose fibre, at each $p \in M$, is the Cartesian product of the fibres $\pi^{-1}(p)$, $\pi'^{-1}(p)$ of B and B' at p.

If B, B' are vector bundles over M, then further constructions are available. In particular, one has the bundle $B \oplus B'$, whose fibre, at each $p \in M$, is the direct sum $\pi^{-1}(p) \oplus \pi'^{-1}(p)$;[45] likewise, one has the bundle $B \otimes B'$, whose fibre, at each $p \in M$, is the tensor product $\pi^{-1}(p) \otimes \pi'^{-1}(p)$. One can also iterate these constructions. So, for instance, one has the bundle $\otimes^k B$ (the kth tensor power of B), and, (anti)symmetrising, $\wedge^k B$ (the kth exterior power of B) and $S^k B$ (the k-times symmetric product of B).

[45] Note that, given the obvious linear structure on $B \times_M B'$, this bundle in fact coincides with $B \oplus B'$.

A.2 Principal and associated bundles

Principal and associated bundles are perhaps most familiar from the 'geometric' formulation of Yang–Mills theory and other gauge theories. Since they are also relevant to the theory of the order of natural bundles, I will briefly recall the key definitions here.

Let G be a Lie group. A *principal bundle* $G \to P \xrightarrow{\pi} M$ with structure group G is a smooth fibre bundle equipped with a smooth, free, fibre-preserving (right) action $P \times G \to P$, such that, for any $p \in M$, there exists a neighbourhood O of p and a local trivialization $\varphi : O \times G \to \pi^{-1}(O)$ such that, for any $p' \in O$ and any $g, g' \in G$, $\varphi(p', g)g' = \varphi(p', gg')$. A smooth principal bundle morphism is a smooth bundle morphism $(\Phi, \varphi) : P \to P'$ together with a Lie group homomorphism $h : G \to G'$ such that, for any $x \in P$, $g \in G$, $\Phi(xg) = \Phi(x)h(g)$.

Let $G \to P \xrightarrow{\pi} M$ be a principal fibre bundle, and let S be any smooth manifold equipped with a smooth (left) G action. The *associated bundle* $P \times_G S \xrightarrow{\bar{\pi}} M$ has, as its total space $P \times_G S$, the space of equivalence classes of pairs $[(x, s)]$, $x \in P$, $s \in S$ under the equivalence relation $(x, s) \sim (xg, g^{-1}s)$ for all $g \in G$. Its projection map $\bar{\pi}$ is given by $\bar{\pi}([x, s]) = \pi(x)$. Any smooth principal bundle morphism $(\Phi, \varphi) : P \to P'$ of principal bundles with structure group G induces a smooth bundle morphism of associated bundles $(\bar{\Phi}, \varphi) : P \times_G S \to P' \times_G S$ via $\bar{\Phi}([x, s]) = [\Phi(x), s]$.

A.3 Connections on bundles

Let $B \xrightarrow{\pi} M$ be a smooth fibre bundle. A connection on B is, equivalently:

- A smooth assignment, to each $x \in B$, of vector space $H_x B$ such that $T_x B = V_x B \oplus H_x B$.
- A smooth section ${H^\alpha}_a : B \to J^1 B$.
- A smooth vertical-valued one-form ${\omega^{\alpha'}}_\beta$ on B.

(The equivalence of the first and second definitions is immediate from the definition of $J^1 B$. The equivalence of the second and third definitions follows from the fact that, given a section $H^{\alpha}{}_{a} : B \to J^1B$, there exists a unique vertical-valued one-form on B which annihilates $H^{\alpha}{}_{a}$ at every point, and conversely, given a smooth vertical-valued one-form $\omega^{\alpha'}{}_{\beta}$ on B, there exists a unique section $H^{\alpha}{}_{a} : B \to J^1B$ such that $H^{\alpha}{}_{a}(\mathrm{d}\pi)^{a}{}_{\beta} = \delta^{\alpha}{}_{\beta} - \omega^{\alpha'}{}_{\beta}$.) For a principal connection on a principal bundle $G \to P \xrightarrow{\pi} M$ with structure group G, we require in addition that the connection be equivariant with respect to the (right) G action on P, in the sense that for any $x \in P$ and any $g \in G$, $H_{xg}P$ coincides with the pushforward of H_xP along the (right) action of g.

Any principal connection $\omega^{\alpha'}{}_{\beta}$ on a principal bundle $G \to P \xrightarrow{\pi} M$ determines, uniquely, a connection $\overset{\omega}{\nabla}$ on each associated bundle $P \times_G S$, as follows. Let $q : P \times S \to P \times_G S$ denote the quotient mapping $q(x,s) = [x,s]$ under the equivalence relation $\sim$. Recalling that every vertical vector is, in particular, a vector, we can identify $\omega^{\alpha'}{}_{\beta}$ with a smooth vector bundle morphism $\omega : TP \to TP$. Then there exists a unique smooth vector bundle morphism $\bar{\omega} : TP \times_{TG} TS \to TP \times_{TG} TS$ such that the following diagram commutes:

$$\begin{tikzcd}
TP \times TS \arrow[r, "\omega \times \mathrm{id}_{TS}"] \arrow[d, "\mathrm{d}q"] & TP \times TS \arrow[d, "\mathrm{d}q"] \\
TP \times_{TG} TS \arrow[r, "\bar{\omega}"] & TP \times_{TG} TS
\end{tikzcd}$$

Then, recalling that $T(P \times_G S) = TP \times_{TG} TS$ and $\mathrm{d}q(VP \times TS) = P \times_G TS = V(P \times_G S)$, it follows that we have $\bar{\omega} : T(P \times_G S) \to V(P \times_G S)$, so $\bar{\omega}$ is a connection on $P \times_G S$.

In general, not every connection on an associated bundle is induced by some principal connection on P. An important special case where there is a one-to-one correspondence between principal connections on P and connections on $P \times_G S$ is the case where $P \times_G S$ is a vector bundle and P is the (tangent) frame bundle for $P \times_G S$. Say that a connection on a vector bundle $B \xrightarrow{\pi} M$ is *linear* iff the map $\omega^{\alpha'}{}_{\beta} : TB \to VB$ is fibrewise linear. Then, given any linear connection $\bar{\omega}^{\alpha'}{}_{\beta}$ on a vector bundle B, there exists a unique linear connection $\omega^{\alpha'}{}_{\beta}$ on the principal bundle $\mathrm{GL}(\mathbb{R}^n, B)$ whose induced connection on B is $\bar{\omega}^{\alpha'}{}_{\beta}$

(Kolář et al. 1993, §11.9), where n is the fibre dimension of B. In particular, this applies when $B = TM$, the tangent bundle of M, and $P = LM$, the bundle of (linear) frames on M.

Finally, note that linear connections on TM are precisely the covariant derivative operators on M, since any covariant derivative operator on M is uniquely determined by its action on vectors tangent to M. As a result, if X is any first-order natural bundle over n-manifolds, then any covariant derivative operator on M induces, uniquely, a connection on XM.

B Category theory

B.1 Categories, functors, and natural transformations

In general, a category $\mathcal{C}$ consists of a collection of objects $\text{ob}(\mathcal{C})$, and a collection of arrows (or morphisms) $\text{mor}(\mathcal{C})$, satisfying certain axioms. Formally, one has:

- A collection of objects $\text{ob}(\mathcal{C})$;
- For all a, b in $\text{ob}(\mathcal{C})$, a function $(a, b) \to \text{hom}_{\mathcal{C}}(a, b)$;
- (Composition of arrows) For all a, b, c in $\text{ob}(\mathcal{C})$, all g in $\text{hom}_{\mathcal{C}}(b, c)$, and all f in $\text{hom}_{\mathcal{C}}(a, b)$, a function $(g, f) \to g \circ f$;
- (Identity arrow) For all a in $\text{ob}(\mathcal{C})$, an arrow id_a in $\text{hom}_{\mathcal{C}}(a, a)$;

satisfying

(i) (Associativity) For all a, b, c, d in $\text{ob}(\mathcal{C})$, all h in $\text{hom}_{\mathcal{C}}(c, d)$, all g in $\text{hom}_{\mathcal{C}}(b, c)$, and all f in $\text{hom}_{\mathcal{C}}(a, b)$, $h \circ (g \circ f) = (h \circ g) \circ f$.

(ii) (Unity) For all a, b, c in $\text{ob}(\mathcal{C})$, all g in $\text{hom}_{\mathcal{C}}(b, c)$, and all f in $\text{hom}_{\mathcal{C}}(a, b)$, $\text{id}_b \circ f = f$ and $g \circ \text{id}_b = g$.

(iii) (Disjointness) If $(a, b) \neq (a', b')$ then $\text{hom}_{\mathcal{C}}(a, b)$ and $\text{hom}_{\mathcal{C}}(a', b')$ are disjoint.

Let $\mathcal{C}$ be a category, and $f : a \to b$ an arrow therein. We call a the source of f and b the target of f. We say that f is an *epimorphism* (or, is an epi, or, is epic) iff, for any arrows $g, h : b \to c$, the condition $g \circ f = h \circ f$ implies $g = h$. (The concept of an epimorphism generalises the notion of a surjective map between sets.) Similarly, we say that f is a *monomorphism* (or, is a mono, or, is monic) iff, for any arrows $g, h : c \to a$, the condition $f \circ g = f \circ h$ implies $g = h$. (This generalises the notion of an injective map between sets.) Finally, we say that f is an *isomorphism* (or, is an iso, or, is iso) iff there exists an arrow $f^{-1} : b \to a$ such that $f^{-1} \circ f = \text{id}_a$ and $f \circ f^{-1} = \text{id}_b$. (Note that any isomorphism is necessarily both epic and monic, but in general, the converse does not hold, i.e. an arrow which is both epic and monic need not be an isomorphism.)

Note that, given a category $\mathcal{C}$, we can construct its opposite category, $\mathcal{C}^{\mathrm{op}}$, in which we reverse the direction of every arrow in $\mathcal{C}$. It is straightforward to see that every epimorphism in $\mathcal{C}$ is a monomorphism in $\mathcal{C}^{\mathrm{op}}$, and conversely every monomorphism in $\mathcal{C}$ is an epimorphism in $\mathcal{C}^{\mathrm{op}}$. Thus, the concepts of an epimorphism and monomorphism are one example of a duality in category theory.

Given categories $\mathcal{C}$, $\mathcal{D}$, a *functor* $F : \mathcal{C} \to \mathcal{D}$ is a map which takes each object a in $\mathcal{C}$ to an object $F(a)$ in $\mathcal{D}$, and each arrow $f : a \to b$ in $\mathcal{C}$ to a morphism $F(f) : F(a) \to F(b)$ in $\mathcal{D}$, in a way which preserves composition and identity, i.e. such that:

(i) For any arrows $f : a \to b$, $g : b \to c$ in $\mathcal{C}$, $F(g \circ f) = F(g) \circ F(f)$.

(ii) For every object a in $\mathcal{C}$, $F(\mathrm{id}_a) = \mathrm{id}_{F(a)}$.

Now let $\mathcal{C}$, $\mathcal{D}$ be categories, and let $F, G : \mathcal{C} \to \mathcal{D}$ be functors. A *natural transformation* $\alpha : F \to G$ is an assignment, to each object a in $\mathcal{C}$, of an arrow $\alpha_a : F(a) \to G(a)$ in $\mathcal{D}$, called the *component* of α at a, such that for any arrow $f : a \to b$ in $\mathcal{C}$, the following diagram commutes:

$$\begin{tikzcd}
F(a) \arrow[r, "F(f)"] \arrow[d, "\alpha_a"] & F(b) \arrow[d, "\alpha_b"] \\
G(a) \arrow[r, "G(f)"] & G(b)
\end{tikzcd}$$

In this way, given categories $\mathcal{C}$, $\mathcal{D}$, one can construct a category, known as the functor category from $\mathcal{C}$ to $\mathcal{D}$, whose objects are functors $F : \mathcal{C} \to \mathcal{D}$ and whose arrows are natural transformations.

We can also talk about what it is for a natural transformation between functors $F, G : \mathcal{C} \to \mathcal{D}$ to be an epimorphism (monomorphism, isomorphism). In fact, there are two (in general inequivalent) ways of doing this. First, we could say that a natural transformation $\alpha : F \to G$ is an epimorphism (monomorphism, isomorphism) iff it is an epimorphism (monomorphism, isomorphism) in the functor category from $\mathcal{C}$ to $\mathcal{D}$. Alternatively, we could say that a natural transformation $\alpha : F \to G$ is a pointwise epimorphism (monomorphism, isomorphism) iff the component α_a for every object a in $\mathcal{C}$ is an epimorphism (monomorphism, isomorphism) in $\mathcal{D}$. These two notions coincide for isomorphisms, but not for epimorphisms or monomorphisms—in particular, every natural transformation which is pointwise epic is epic, and likewise every natural transformation which is pointwise monic is monic, but the converses need not hold. In this thesis, I will use the terms epic and monic for natural transformations to refer to natural transformations which are pointwise epic and monic, in the above sense.

B.2 Subobjects and quotient objects

Given a category $\mathcal{C}$, one can also define subobjects and quotient objects, which generalise e.g. the notions of a subset and quotient set, or a subgroup and quotient group. Let a be an object in $\mathcal{C}$. We define an equivalence relation $\sim$ on monomorphisms $f : b \to a$, $g : c \to a$ with target a as follows: $f \sim g$ iff there exists an isomorphism $i : b \to c$ such that the following diagram commutes:

$$\begin{tikzcd}
b \arrow[r, "f"] \arrow[d, "i"] & a \\
c \arrow[ur, "g"'] &
\end{tikzcd}$$

Similarly, we can also define an equivalence relation $\sim$ on epimorphisms $f : a \to b$, $g : a \to c$ with source a: $f \sim g$ iff there exists an isomorphism $i : b \to c$ such that the following diagram commutes:

$$\begin{tikzcd}
a \arrow[r, "f"] \arrow[dr, "g"'] & b \arrow[d, "i"] \\
& c
\end{tikzcd}$$

A subobject of a in $\mathcal{C}$ is then an equivalence class of monomorphisms with target a under $\sim$. Likewise, a quotient object of a in $\mathcal{C}$ is an equivalence class of epimorphisms with source a under $\sim$.

To understand these definitions, note that the equivalence relation $\sim$ on monomorphisms generalises the notion of an equivalence class of injective functions into some set which have the same image. In this case, a subobject provides a way of picking out this image, without worrying about the choice of domain used to define the injection. Similarly, the equivalence relation $\sim$ on epimorphisms generalises the notion of an equivalence class of surjective functions from some set which agree on which elements of the domain get mapped to the same element of the codomain. In this case, a quotient object provides a way of picking out the relation 'gets mapped to the same element of the codomain', without worrying about the particular choice of codomain used to define the surjection.

Finally, note that the concepts of a subobject and quotient object are dual, i.e. every subobject of a in $\mathcal{C}$ is a quotient object of a in $\mathcal{C}^{\mathrm{op}}$, and *vice versa*.

B.3 Useful limits and colimits

A thorough treatment of limits and colimits would require a discussion of representable functors and/or cones, which lie beyond the scope of this appendix. Here, I will confine myself to giving explicit definitions of the limits and colimits which feature in this thesis.

Let $\mathcal{C}$ be a category, a, b, c objects therein, and $f : a \to b$, $g : c \to b$ arrows. Given a diagram

$$\begin{tikzcd} & c \arrow[d, "g"] \\ a \arrow[r, "f"] & b \end{tikzcd}$$

called a *pullback diagram* (or cospan), the *pullback* of this diagram, if it exists, is an object d in $\mathcal{C}$ along with arrows $p : d \to a$, $q : d \to c$ such that the following diagram commutes:

$$\begin{tikzcd}
d \arrow[r, "q"] \arrow[d, "p"] & c \arrow[d, "g"] \\
a \arrow[r, "f"] & b
\end{tikzcd}$$

Moreover, the object d must be universal with respect to this diagram. That is, given any other object d' in $\mathcal{C}$ and arrows $p' : d' \to a$, $q' : d' \to c$ such that $f \circ p = g \circ q$ there must exist a unique arrow $h : d' \to d$ such that $p' = p \circ h$ and $q' = q \circ h$. The situation is illustrated in the following diagram:

$$\begin{tikzcd}
d' \arrow[dr, dashed, "h"] \arrow[drr, dashed, bend left, "q'"] \arrow[ddr, dashed, bend right, "p'"'] & & \\
& d \arrow[r, "q"] \arrow[d, "p"] & c \arrow[d, "g"] \\
& a \arrow[r, "f"] & b
\end{tikzcd}$$

The pullback is sometimes denoted $a \times_{fbg} c$, or simply $a \times_b c$, and the arrows p, q as f^*g, g^*f respectively.

In the category of sets, the pullback is the subset of the Cartesian product $a \times c$ consisting of all pairs (x, y), $x \in a$, $y \in c$, such that $f(x) = g(y)$. If a, c are smooth fibre bundles over b, then the pullback is simply the fibred product. The dual concept is that of a *pushout*, which is a pullback in $\mathcal{C}^{\text{op}}$.

Again, let $\mathcal{C}$ be a category, and a, b objects therein. Given a diagram

$$a \qquad\qquad\qquad\qquad b$$

the *product*, if it exists, is an object c in $\mathcal{C}$ along with arrows $f : c \to a$, $g : c \to b$, which is universal with respect to this diagram, in the sense that given any other object c' in $\mathcal{C}$ and arrows $f' : c' \to a$, $g' : c' \to b$, there exists a unique arrow

$h : c' \to c$ such that the following diagram commutes:

$$\begin{tikzcd}
 & c' \arrow[dl, dashed, "f'"'] \arrow[d, dashed, "h"] \arrow[dr, dashed, "g'"] & \\
a & c \arrow[l, "f"'] \arrow[r, "g"] & b
\end{tikzcd}$$

The product is sometimes denoted $a \times b$. In the category of sets, the product is just the ordinary Cartesian product of two sets. The dual concept is that of a *coproduct*. Note that in any category with a terminal object (which I will describe presently), the product of two objects, if it exists, is just their pullback over the terminal object.

Let $\mathcal{C}$ be a category. The *terminal object*, if it exists, is the limit of the empty diagram. That is, it is an object 1 in $\mathcal{C}$ such that, for any object a in $\mathcal{C}$, there exists a unique arrow $f : a \to b$. The terminal object in the category of sets is the set $\{1\}$. The dual concept is that of an *initial object*.

C Notation

A The natural bundle of one-forms

B A smooth fibre bundle.

$(B, \check{\pi}, \hat{B})$ A quotient bundle.

E An equation.

$(\mathfrak{E}, X)$ A natural equation on X.

F Usually the natural bundle of two-forms. Occasionally a natural operator.

$\mathcal{FB}$ The category of smooth fibre bundles whose arrows are smooth bundle morphisms.

G The natural bundle over $\overline{\mathcal{M}}_n$ of Lorentzian metrics.

H A first-order natural operator from G to $S^2(T^* \otimes VY)$, for some natural bundle Y.

J^k The functor from $\mathcal{FB}$ to itself (or $\mathcal{NB}_n$ to itself) taking every bundle to its kth jet prolongation.

k The equation map of a system of quasilinear equations.

K The natural bundle of real scalar fields.

L^r The natural bundle of r-frames. For $r = 1$, sometimes denoted L.

M A smooth n-manifold.

$\mathcal{M}_n$ The category of smooth n-manifolds whose arrows are smooth embeddings.

$\overline{\mathcal{M}}_n$ The full subcategory of $\mathcal{M}_n$ whose objects are smooth n-manifolds admitting a Lorentzian metrics.

N Usually a smooth n-manifold.

$\mathcal{NB}_n$ The category of natural bundles over n-manifolds whose arrows are natural transformations.

O Usually an open neighbourhood of some point.

S Usually a submanifold of some manifold. Occasionally the typical fibre of a bundle.

S^k The k-times symmetric tensor product of a vector bundle with itself.

T The tangent bundle functor.

T^* The cotangent bundle functor.

U Usually the presystem of a system of quasilinear equations. Occasionally the modelling vector bundle of an affine bundle.

V The functor from $\mathcal{FB}$ to itself (or $\mathcal{NB}_n$ to itself) taking bundles to their bundles of vertical vectors.

V^* The functor from $\mathcal{FB}$ to itself (or $\mathcal{NB}_n$ to itself) taking bundles to their bundles of vertical covectors.

X A natural bundle.

$(X, \mathfrak{Y})$ A natural quotient bundle of X over Y.

Y A natural bundle.

$\iota^{(\ell_1,\ell_2,\ldots,\ell_r)}$ The natural transformation from J^k to $J^{\ell_1}J^{\ell_2}...J^{\ell_r}$, for $\sum_i \ell_i = k$.

π The projection map of a smooth bundle.

$\pi^{(k,\ell)}$ The natural transformation from J^k to J^ℓ whose components are given by the canonical projection.

φ A smooth embedding.

(Φ, φ) A smooth bundle morphism. Sometimes abbreviated to Φ.

χ A smooth (local) section.

ψ A smooth (local) section.

1_n The terminal object of $\mathcal{NB}_n$.

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