# MINITWISTORS

A thesis submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy at the University of Oxford

P.E. Jones
Wadham College, Oxford.

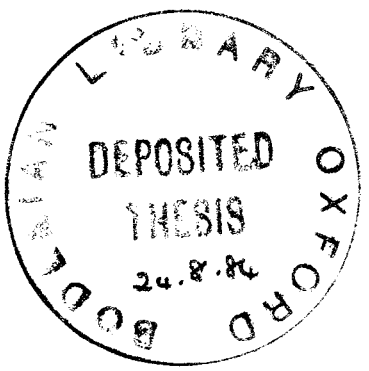
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Abstract

The relationship between complex three-manifolds and complex surfaces admitting rational curves of normal bundle $\mathcal{O}(2)$ is discussed. These surfaces are called "minitwistor spaces". For two particular minitwistor spaces (the two-quadric and the total space of the bundle $\mathcal{O}(2)$ on $\mathbb{P}_1$), a transform from holomorphic cohomology on the surface to solutions of certain differential equations on the three-manifold is constructed and conditions under which it is an isomorphism are found. The relationship to the Penrose transform in twistor theory is investigated.

A method for constructing solutions of the Einstein-Weyl equations on the factor space of an anti self dual space-time by a conformal Killing vector is given. It is shown how this reflects the relationship between the appropriate twistor space and its factor space by the corresponding vector field.

Finally, certain classes of twistor space are given the structure of holomorphic bundles over a minitwistor space. These bundles encode solutions of the $\mathbb{C}^*$ Bogomolny equations on a three-space. These solutions are shown to be the ones appearing in the expression for the metric on the space-time corresponding to the twistor space.



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## Introduction

Solutions of the zero rest mass equations

$$\nabla_A{}^{A'} \phi_{A' \ldots L'} = 0 \qquad (0.1)$$

on complex Minkowski space can be seen to be related to functions of three complex variables by means of the Kirchoff integral formula [10]. A solution is given in terms of an integral involving a function on a given complex null cone. Analogously one might expect to be able to express solutions of the analogues of (0.1) on a three-space in terms of functions of <u>two</u> complex variables. A major part of this thesis will be concerned with establishing a correspondence between these solutions and certain sheaf cohomology classes on a complex two-manifold which will be referred to as a "<u>minitwistor space</u>".

The simplest case arises from looking at fields on the space obtained by factoring out complex affine Minkowski space by time translations. The minitwistor space, $\mathbb{T}$, appropriate for this has already been introduced by Hitchin [6] as a tool for studying solutions of the Bogomolny equations

$$*F = \nabla \Phi$$

on $\mathbb{R}^3$ (here $F$ is the curvature form of an $SU(2)$ connection $\nabla$ on $\mathbb{R}^3$, $\Phi$ is the Higgs field and "$*$" is the duality operator).

Chapter one will describe the geometry of $\mathbb{T}$ and its relation to twistor geometry, then will proceed to

treat the cohomological description of fields on three-space. The zero homogeneity cohomology will be shown to describe solutions of the Bogomolny equations for a $\mathbb{C}^*$ connection. There will follow a discussion of how cohomology on $\mathbb{T}$ is related to twistor cohomology. Finally the correspondence between anti self dual Dirac monopoles and certain line bundles on $\mathbb{T}$ will be described. Chapter II will repeat the analysis of chapter I for another important example of a minitwistor space - the two-quadric. Chapter III will open with a review of Hitchin's definition of a generalized minitwistor space. This will be followed by a discussion of how these arise (as factor spaces of "non linear graviton" twistor spaces by vector fields) and their relation to the factor spaces of half-flat space-times by conformal Killing vectors.

Chapter IV will describe how various twistor spaces arise as bundles over a minitwistor space and will give examples of how the solutions to the $\mathbb{C}^*$ Bogomolny equations encoded in these bundle structures appear in the half-flat space-time metrics.

Familiarity with twistor geometry including the non linear graviton construction will be assumed. Abstract index notation [11] will be adopted.

## CHAPTER 1 The Time Translation Minitwistor Space

### §1.1 Three-space Geometry and the Definition of $\mathbb{T}$.

Let $\mathbb{M}^{\mathrm{I}}$ denote complex affine Minkowski space, which will be given coordinates $x^{\underline{A}\underline{A}'}$ where

$$x^{\underline{A}\underline{A}'} = \frac{1}{\sqrt{2}} \begin{pmatrix} t+z & x-iy \\ x+iy & t-z \end{pmatrix}.$$

$\mathbb{M}^{\mathrm{I}}$ has metric $ds^2 = dx_{\underline{A}\underline{A}'} dx^{\underline{A}\underline{A}'} = 2\det\left[dx^{\underline{A}\underline{A}'}\right]$. For t, x, y, z real, we obtain <u>real</u> affine Minkowski space $M^{\mathrm{I}} \subset \mathbb{M}^{\mathrm{I}}$. The abstract index correspondence [11] between vectors and spinors will be written as

$$V^a = V^{AA'}.$$

The unit time-translation Killing vector $t^{AA'}$ has coordinates

$$t^{\underline{A}\underline{A}'} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

$t^{AA'}$ gives an isomorphism between the primed and unprimed spin spaces at each point in space-time. Given any space-time point $x^{AA'}$, we may pick out its three-space components by forming

$$x^{A'B'} = \sqrt{2}\, x_{\underline{A}}{}^{(\underline{A}'} t^{\underline{B}')\underline{A}} = \frac{1}{\sqrt{2}} \begin{pmatrix} -(x+iy) & z \\ z & x-iy \end{pmatrix}.$$

These serve as coordinates on the factor space, $\mathbb{E}$, of integral curves of $t^{AA'}\nabla_{AA'}$ $\left(= \frac{\partial}{\partial t}\right)$. $\mathbb{E}$ is a complex affine three-space with complex Euclidean metric given by

$$ds^2 = dx^{A'B'} dx_{A'B'} = 2\det\left[dx^{A'B'}\right].$$

For x, y, z real we obtain the subset $E \subset \mathbb{E}$, which is a real affine three-space with (negative definite) Euclidean metric.

A complex three-vector factorises into a pair of spinors which are defined up to proportionality by the vector, i.e.

$$x^{A'B'} = \alpha^{(A'}\beta^{B')}.$$

The vector is null if and only if it can be written as

$$x^{A'B'} = \alpha^{A'}\alpha^{B'}$$

for some spinor $\alpha^{A'}$. A <u>real</u> three-vector can be written in the form

$$x^{A'B'} = \alpha^{(A'}\hat{\alpha}^{B')}$$

for some spinor $\alpha^{A'}$, which will be defined up to multiplication by a phase $e^{i\theta}$ ($\theta$ real). Here "$\wedge$" denotes the spinor conjugation appropriate to $SU(2)$, namely

$$\hat{\alpha}^{0'} = -\overline{\alpha}^{1'} \quad ; \quad \hat{\alpha}^{1'} = \overline{\alpha}^{0'}$$

(the bar denotes normal conjugation).

Now, $\mathbb{M}^{\mathrm{I}}$ can be "completed" to give a compact manifold $\mathbb{M}$ by the addition of a null cone "$\mathcal{I}$" at infinity. Let $\mathbb{PT}$ denote the projective twistor space corresponding to $\mathbb{M}$. Denote by $\mathrm{I}$ the (projective) line in $\mathbb{PT}$ corresponding to the vertex, i, of $\mathcal{I}$, and let $\mathbb{PT}^{\mathrm{I}}$ denote the complement of $\mathrm{I}$ in $\mathbb{PT}$. Lines in $\mathbb{PT}^{\mathrm{I}}$ are then in one to one correspondence with the points of $\mathbb{M}^{\mathrm{I}}$.

Consider the trajectories of the Killing vector $\frac{\partial}{\partial t}$

on $\mathbb{M}$ . These all meet at the vertex of $\mathcal{I}$ (fig. 1.1).

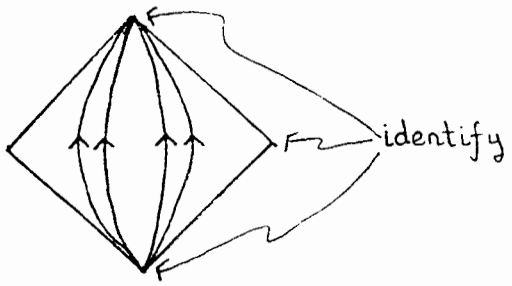



Fig. 1.1. Trajectories of $\frac{\partial}{\partial t}$ on $\mathbb{M}$ .

Each trajectory gives rise to a one-parameter family of lines in $\mathbb{PT}$ which has the line $I$ as one of its members. This family of lines generates a non singular quadric. To describe the quadrics, focus attention on one trajectory. Suppose it passes through a point $x^{AA'} \in \mathbb{M}^I$ , then it can be thought of as the world line of a free particle of unit mass, momentum $t^{AA'}$ , and angular momentum $2x^{[a}t^{b]}$ about the coordinate origin. The quadric Q corresponding to that trajectory is then given by

$$A_{\alpha\beta} Z^\alpha Z^\beta = 0$$

where $Z^\alpha$ are homogeneous coordinates on twistor space and $A_{\alpha\beta}$ is the kinematic twistor of the particle i.e.

$$A_{\alpha\beta} = \begin{pmatrix} 0 & t_A{}^{B'} \\ t^{A'}{}_B & 2ix_A^{(A'}t^{B')A} \end{pmatrix} = \begin{pmatrix} 0 & t_A{}^{B'} \\ t^{A'}{}_B & \sqrt{2}\, i\, x^{A'B'} \end{pmatrix}.$$

The zero in the top left hand corner reflects the fact that the quadric includes the line $I$ .

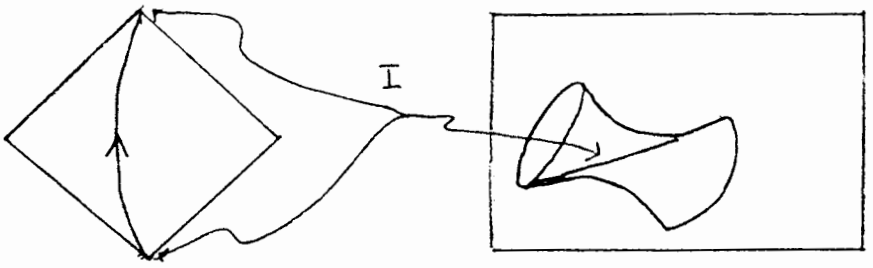



Fig. 1.2

Motion along the trajectories of $\frac{\partial}{\partial t}$ , i.e.

$$x^a \longmapsto x^a(\lambda) = x^a + \lambda t^a$$

preserves $\alpha$ -planes and hence gives a motion of twistor space, namely

$$\begin{pmatrix} \omega^A(\lambda) \\ \pi_{A'}(\lambda) \end{pmatrix} = \begin{pmatrix} \varepsilon_B{}^A & i\lambda t^{AB'} \\ 0 & \varepsilon_{A'}{}^{B'} \end{pmatrix} \begin{pmatrix} \omega^B \\ \pi_B \end{pmatrix}$$

In this way, $\frac{\partial}{\partial t}$ give rise to a vector field

$$\frac{\partial Z^\alpha}{\partial \lambda}\bigg|_{\lambda=0} \frac{\partial}{\partial Z^\alpha} = i t^{AA'} \pi_{A'} \frac{\partial}{\partial \omega^A}$$

on $\mathbb{PT}$ which vanishes on the line $I$ (time translation leaves the $\alpha$ -planes at infinity fixed). The integral curves of this vector field form the other system of generators of the quadrics.

Removing $I$ from $\mathbb{PT}$ corresponds to removing $\mathcal{I}$ from $\mathbb{M}$ and, after doing this, we can define two factor spaces:- the space $\mathbb{E}$ of trajectories of $\frac{\partial}{\partial t}$ in $\mathbb{M}^I$ , and the space $\mathbb{T}$ of integral curves of $i t^{AA'} \pi_{A'} \frac{\partial}{\partial \omega^A}$ in $\mathbb{PT}^I$.

We will now determine the complex manifold structure of $\mathbb{T}$ . $\mathbb{PT}^I$ is covered by two coordinate patches:

$U_0 = \{\pi_{0'} \neq 0\}$ with coordinates $\left(\frac{\pi_{1'}}{\pi_{0'}}, \frac{\omega^0}{\pi_{0'}}, \frac{\omega^{1'}}{\pi_{0'}}\right)$,

$U_1 = \{\pi_{1'} \neq 0\}$ with coordinates $\left(\frac{\pi_{0'}}{\pi_{1'}}, \frac{\omega^0}{\pi_{1'}}, \frac{\omega^{1'}}{\pi_{1'}}\right)$.

These project onto corresponding patches of $\mathbb{T}$ :

$$V_0 \quad \text{with coordinates} \quad \left(\frac{\pi_{1'}}{\pi_{0'}}, \frac{\pi_{1'}\omega^0}{\pi_{0'}^2} - \frac{\omega^{1'}}{\pi_{0'}}\right) := (\zeta, Z^0) \tag{1.1}$$

$$V_1 \quad \text{with coordinates} \quad \left(\frac{\pi_{0'}}{\pi_{1'}}, \frac{\omega^0}{\pi_{1'}} - \frac{\pi_{0'}\omega^{1'}}{\pi_{1'}^2}\right) := (\tilde{\zeta}, Z^1).$$

The transition relation is

$$\tilde{\zeta} = 1/\zeta \quad ; \quad Z^1 = \tilde{\zeta}^2 Z^0.$$

Hence, $\mathbb{T}$ is the total space of the holomorphic line bundle $\mathcal{O}(2)$ over $\mathbb{P}_1$. Between the spaces $\mathbb{T}$ and $\mathbb{E}$ we have an analogue of the twistor correspondence. Let $x^{A'B'} \in \mathbb{E}$ and choose any point $x^{AA'}$ in the trajectory over $x^{A'B'}$. This gives rise to a line $\omega^A = i x^{AA'} \pi_{A'}$ in $P\mathbb{T}^I$. Projecting this onto $\mathbb{T}$ we obtain the curve

$$Z^0(\zeta) = \frac{\pi_{1'}\omega^0(x,\pi)}{\pi_{0'}^2} - \frac{\omega^{1'}(x,\pi)}{\pi_{0'}} = \frac{i}{\sqrt{2}}\left(-(x+iy)\pi_{0'}^2 + 2z\pi_{0'}\pi_{1'} + (x-iy)\pi_{1'}^2\right)/\pi_{0'}^2$$

$$= i x^{A'B'} \frac{\pi_{A'}}{\pi_{0'}} \frac{\pi_{B'}}{\pi_{0'}}$$

$$Z^1(\tilde{\zeta}) = i x^{A'B'} \frac{\pi_{A'}}{\pi_{1'}} \frac{\pi_{B'}}{\pi_{1'}}.$$

Thus the curve depends only on the trajectory $x^{A'B'}$ and not on the choice of point in the trajectory.

$\{Z^0(x,\zeta), Z^1(x,\tilde{\zeta})\}$ is a section of $\mathbb{T}$.

Conversely, every section is defined by a quadratic polynomial $i x^{A'B'} \pi_{A'} \pi_{B'}$ and hence determines a point of $\mathbb{E}$. Consequently we have a one to one correspondence between points, $x$, of $\mathbb{E}$ and sections $C_x$ of $\mathbb{T}$. $\mathbb{T}$ will be referred to as a <u>minitwistor space</u>.

Fixing a minitwistor $Z \in \mathbb{T}$ with coordinates $(\zeta, Z^0)$ defines the subset

$$\gamma_Z = \{ x^{A'B'} \mid Z^0 = i x^{A'B'} \pi_{A'} \pi_{B'} / \pi_{0'}^2 \}$$

of $\mathbb{E}$. Any vector tangent to $\gamma_Z$ must be of the form $\pi^{(A'} \alpha^{B')}$ for some $\alpha^{A'}$, where $\pi^{A'}$ is defined up to proportionality by $Z$. The normal to $\gamma_Z$ must therefore be proportional to $\pi^{A'} \pi^{B'}$ and hence $\gamma_Z$ is a <u>null</u> two-plane in $\mathbb{E}$.

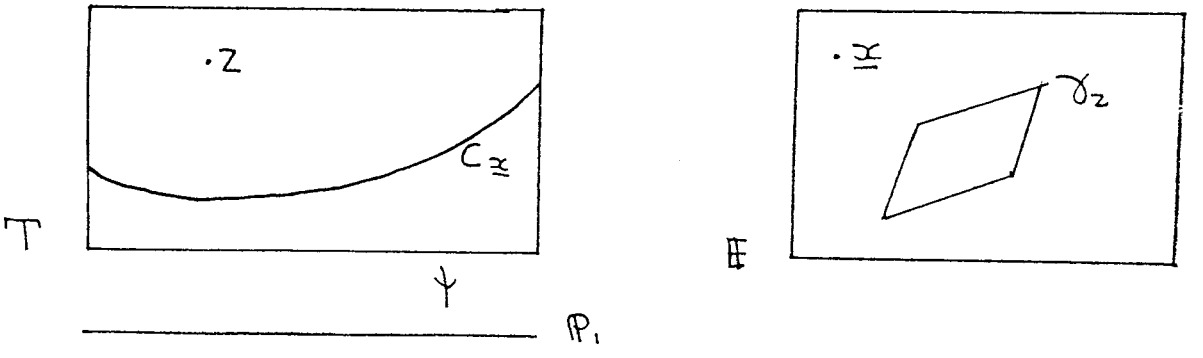



Fig. 1.3

The curve $C_{\underline{x}}$ corresponding to $\underline{x} \in \mathbb{E}$ is the $\mathbb{P}_1$'s worth of two planes tangent to the null cone of $\underline{x}$.

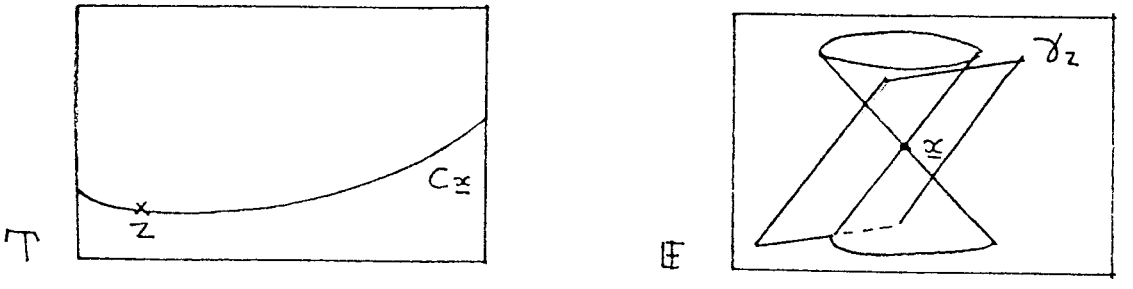



Fig. 1.4

Fixing two minitwistors $Z$ & $W$, there is a one-parameter family of sections of $\mathbb{T}$ that pass through both

of them. These correspond to the points of the complex geodesic given by the intersection of the planes $\gamma_z$ & $\gamma_w$ .

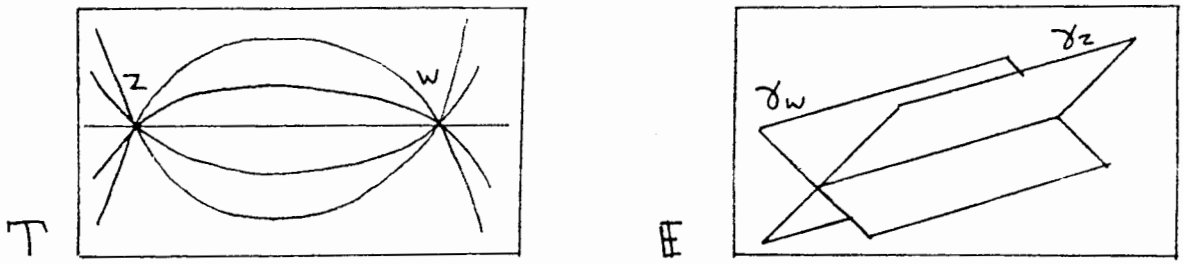
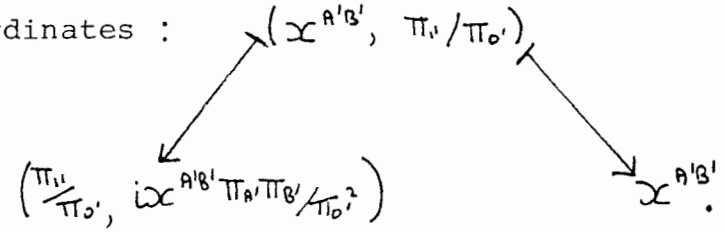



Fig. 1.5

The correspondence between $\mathbb{E}$ and $\mathbb{T}$ can be exhibited by means of the double fibration

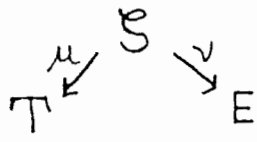



where $\mathcal{S}$ is the projective spin bundle of $\mathbb{E}$ . In local coordinates :




The two-plane $\gamma_z \subset \mathbb{E}$ corresponding to $z \in \mathbb{T}$ is $\nu\mu^{-1}z$ and the curve $C_x \subset \mathbb{T}$ corresponding to $x \in \mathbb{E}$ is $\mu\nu^{-1}x$ .

It is instructive to see what happens if we consider <u>real</u> Euclidean three-space $E \subset \mathbb{E}$ . Restricting the correspondence diagram to



$\mathcal{S}$ is now five real-dimensional and the fibres of $\mu$ are one real-dimensional. They are the integral curves of the geodesic flow on $\mathcal{S} = E \times \mathbb{P}_1$ . In this decomposition of $\mathcal{S}$ , the $\mathbb{P}_1$ is the sphere's worth of directions in real three-space. The geodesic flow is obtained by lifting each geodesic $x(\lambda)$ to the curve $(x(\lambda), \text{direction of } x(\lambda))$ in $\mathcal{S}$ .

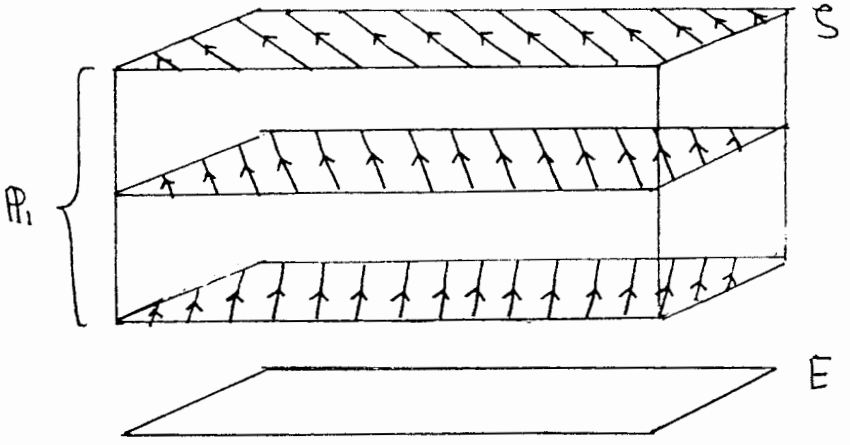



Fig. 1.6 Projective spin bundle of $E$ fibred by geodesic flow

Explicitly, the flow is

$$\frac{1}{(\sqrt{2}\,\pi^{c'}\hat{\pi}_{c'})}\,\pi^{A'}\hat{\pi}^{B'}\nabla_{A'B'} \qquad (\text{"}\wedge\text{"} = SU(2) \text{ conjugation})$$

$$= Y^j \frac{\partial}{\partial x^j} \qquad j = 1,2,3$$

where $Y^j$ are the three spin-zero spherical harmonics

$$\left(\frac{\zeta+\bar{\zeta}}{1+\zeta\bar{\zeta}}\,,\ \frac{\zeta-\bar{\zeta}}{i(1+\zeta\bar{\zeta})}\,,\ \frac{1-\zeta\bar{\zeta}}{1+\zeta\bar{\zeta}}\right).$$

The image of a minitwistor $Z$ under $\nu\mu^{-1}$ is an oriented geodesic in $E$ . As a point set this is just the intersection with $E$ of the two-plane $\gamma_Z$ . The orientation is determined by the complex structure on $\gamma_Z$ .

## §1.2 Sheaves on $\mathbb{T}$

To define the relevant sheaves, it will be useful to think of $\mathbb{T}$ as a weighted projective space. Let $(\omega, \pi_{A'})$ be coordinates on $\mathbb{C}^3$, then $\mathbb{T}$ is the factor space of $\mathbb{C}^3 - \{\pi_{A'} = 0\}$ by the relation

$$(\omega, \pi_{A'}) \sim (\lambda^2\omega, \lambda\pi_{A'}) \qquad \lambda \in \mathbb{C}^*$$

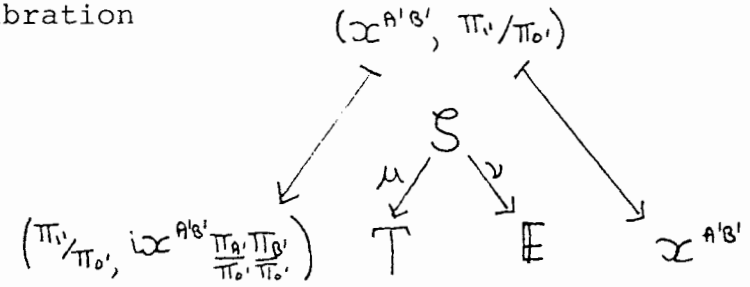
Thus a function on $\mathbb{T}$ is a function of three complex variables $f(\omega, \pi_{A'})$ with the homogeneity property $f(\lambda^2\omega, \lambda\pi_{A'}) = f(\omega, \pi_{A'})$. Define the sheaf $\mathcal{O}(n)$ on $\mathbb{T}$ to be the one whose sections are functions with the homogeneity property

$$f(\lambda^2\omega, \lambda\pi_{A'}) = \lambda^n f(\omega, \pi_{A'}) \tag{1.2}$$

This is just the pull-back of the sheaf $\mathcal{O}(n)$ on $\mathbb{P}_1$ along the projection $\pi : \mathbb{T} \to \mathbb{P}_1$.

## §1.3 Negative Homogeneity Minitwistor Cohomology

We will now make extensive use of the double fibration




to derive an analogue of the correspondence described in [2] between cohomology on $\mathbb{PT}$ and fields on $\mathbb{M}$. In this section we will treat the case of cohomology in the

sheaves $\mathcal{O}(-n-2)$ $n \geqslant 0$ . Since this is almost identical to the twistor case, only the barest outline will be given. The notation for spinor sheaves will follow closely that in [2].

We first identify the relative de-Rham sequences for $\mu : \mathcal{S} \to \mathcal{T}$ . The sheaves of relative forms can easily be seen to be given by

$$\Omega^0_\mu = \mathcal{O} \quad \text{(sheaf of holomorphic functions on } \mathcal{S})$$

$$\Omega^1_\mu = \mathcal{O}_{A'}(1)\,[-1]'$$

$$\Omega^2_\mu = \mathcal{O}_{[A'B']}(2)\,[-2]' = \mathcal{O}(2)\,[-3]'.$$

The relative exterior derivative $d_\mu$ is simply $\pi^{B'}\nabla_{B'A'}$ and the relative de Rham sequence

$$\Omega^0_\mu \xrightarrow{d_\mu} \Omega^1_\mu \xrightarrow{d_\mu} \Omega^2_\mu$$

becomes, in notation appropriate for taking $\mathcal{V}$- direct images:

$$\mathcal{O} \longrightarrow \mathcal{O}_{A'}(1)\,[-1]' \longrightarrow \mathcal{O}_{[A'B']}(2)\,[-2]'$$

$$f \longmapsto \pi^{B'}\nabla_{B'A'} f$$

$$g_{A'} \longmapsto \pi^{C'}\nabla_{C'[B'} g_{A']}$$

On tensoring with $\mathcal{O}(-n-2)$ and taking first direct images we obtain

$$\begin{array}{ccccc}
U'\subset\mathcal{S}: & \mathcal{O}(-n-2) & \xrightarrow{\pi^{B'}\nabla_{B'A'}} & \mathcal{O}_{A'}(-n-1)\,[-1]' & \xrightarrow{\pi^{B'}\nabla_{B'[A'}} \quad \mathcal{O}(-n)\,[-3]' \\
 & \downarrow & & \downarrow & \downarrow \\
U\subset\mathbb{E}: & \mathcal{O}_{(L'B'\ldots D')}\,[-1]' & \xrightarrow{\nabla_{A'L'}} & \mathcal{O}_{A'(B'\ldots D')}\,[-2]' & \xrightarrow{\nabla^{A'B'}} \quad \mathcal{O}_{(C'\ldots D')}\,[-4]' \\
 & n \text{ indices} & & n-1 \text{ indices} & n-2 \text{ indices.}
\end{array}$$

Let $U$ be an open subset of $\mathbb{E}$. Set $U' = \nu^{-1}U$ and $U'' = \mu\nu^{-1}U$. Then, taking sections of the bottom row over $U$, and using isomorphisms obtained from the Leray spectral sequence, we have

$$\begin{array}{ccccc} H^1(U', \Omega^0_\mu(-n-2)) & \longrightarrow & H^1(U', \Omega^1_\mu(-n-2)) & \longrightarrow & H^1(U', \Omega^2_\mu(-n-2)) \\ \| & & \| & & \| \\ \Gamma(U, \mathcal{O}_{(A'B'\ldots D')}[-1]') & \longrightarrow & \Gamma(U, \mathcal{O}_{A'(B'\ldots D')}[-2]') & \longrightarrow & \Gamma(U, \mathcal{O}_{(C'\ldots D')}[-4]') \end{array} \qquad (1.3)$$

Now, the exact sequence (where $\mu^{-1}$ means topological inverse

$$\mathcal{U}'\mathcal{CS}: \quad 0 \to \mu^{-1}\mathcal{O}(-n-2) \to \Omega^0_\mu(-n-2) \to \Omega^1_\mu(-n-2) \to \Omega^2_\mu(-n-2) \to 0$$

image) is a resolution of $\mu^{-1}\mathcal{O}(-n-2)$, and, as in [2] we have a spectral sequence

$$E_1^{p,q} = H^q(U', \Omega^p_\mu(-n-2)) \Rightarrow H^{p+q}(U', \mu^{-1}\mathcal{O}(-n-2))$$

where $d_1 : E_1^{p,q} \to E_1^{p+1,q}$ is induced by $d_\mu : \Omega^p_\mu(-n-2) \to \Omega^{p+1}_\mu(-n-2)$

Extracting information from the spectral sequence in the usual way tells us that, for $n \geq 1$

$$\begin{aligned} H^1(U', \mu^{-1}\mathcal{O}(-n-2)) &\cong \ker d_1 : E_1^{0,1} \to E_1^{1,1} \\ &\cong \ker d_\mu : H^1(U', \Omega^0_\mu(-n-2)) \to H^1(U', \Omega^1_\mu(-n-2)) \\ \text{by (1.3)} \quad &\cong \ker \nabla_{A'}{}^{L'} : \Gamma(U, \mathcal{O}_{(L'B'\ldots D')}[-1]') \to \Gamma(U, \mathcal{O}_{A'(B'\ldots D')}[-2]'). \end{aligned}$$

A slight modification of the argument gives us the result that:

$$H^1(U', \mu^{-1}\mathcal{O}(-2)) \cong \ker \nabla^2 : \Gamma(U, \mathcal{O}[-1]') \to \Gamma(U, \mathcal{O}[-3]').$$

The remaining step is the result that

$$H^1(U', \mu^{-1}\mathcal{O}(-n-2)) \cong H^1(U'', \mathcal{O}(-n-2))$$

for $U$ connected and $H^1(U \cap \gamma_2, \mathbb{C}) = 0$ $\forall$ null two-planes $\gamma_2$.

This is proved in the same way as its analogue in [2]. We will refer to a set $U$ which satisfies the conditions above as regular.

We can now state:

Proposition 1.1

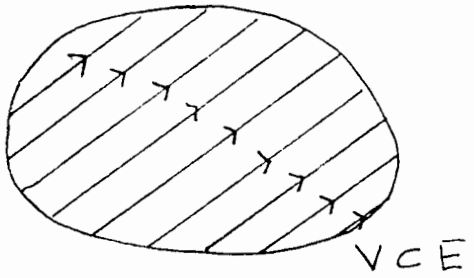
If $U \subset \mathbb{E}$ is regular, then solutions of $\nabla_{A'}{}^{L'} \phi_{L'B'\ldots D'} = 0$ where $\phi_{L'B'\ldots D'} \in \Gamma(U, \mathcal{O}(L'B'\ldots D')\,[-1]')$ (n indices on $\phi$), are in one to one correspondence with elements of $H^1(U'', \mathcal{O}(-n-2))$ for $n \geq 1$. Moreover, solutions of $\nabla^2 \phi = 0$, where $\phi \in \Gamma(U, \mathcal{O}\,[-1]')$ are in one to one correspondence with elements of $H^1(U'', \mathcal{O}(-2))$.

---

The equations $\nabla_{A'}{}^{L'} \phi_{L'B'\ldots D'} = 0$ will be called the "linear field equations" and by analogy with standard twistor theory, the map $H^1(\mathcal{O}(-n-2)) \longrightarrow \{\text{fields}\}$ will be called the $\mathcal{P}$ -transform. Solutions of the linear field equations correspond to solutions of the zero-rest mass equations on space-time which are Lie-dragged along the time-translation vector field.

## §1.4 Fields on Real Space

Let $V \subset E \subset \mathbb{E}$ be a convex open set. The corresponding set $V'' = \mu\nu^{-1}V$ in $\mathbb{T}$ is also open. To see this, pick any direction $\zeta$, then the fibre of $V''$ over $\zeta$ is the space of geodesics in $V$ in the direction $\zeta$ and is clearly open.



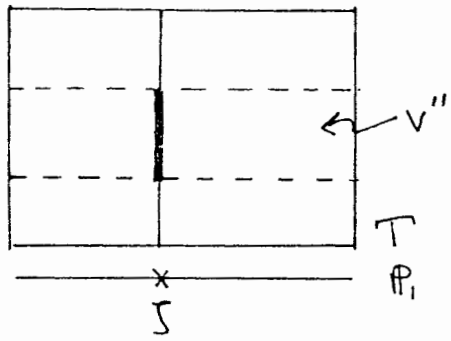




Fig. 1.7

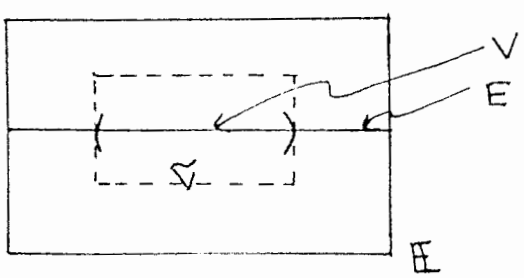
Let $\phi_{L'B'\ldots D'}$ (n indices) be an analytic solution of the linear field equations on $V$. These can be written in the form

$$\begin{pmatrix} -\frac{\partial}{\partial x} + i\frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial z} & \frac{\partial}{\partial x} + i\frac{\partial}{\partial y} \end{pmatrix} \begin{pmatrix} \phi^{0'B'\ldots D'} \\ \phi^{1'B'\ldots D'} \end{pmatrix} = 0$$

Their symbol is therefore

$$\begin{pmatrix} -\xi_x + i\xi_y & \xi_z \\ \xi_z & \xi_x + i\xi_y \end{pmatrix}$$

where $(\xi_x, \xi_y, \xi_z)$ denotes an indeterminate non-zero cotangent vector defined on $E$. This is invertible provided $\xi_x^2 + \xi_y^2 + \xi_z^2 \neq 0$, which certainly holds on $E$. The equations are thus elliptic on $V$ and the solution $\phi_{L'B'\ldots D'}$ extends to some neighbourhood $\tilde{V}$ of $V$ in $E$.



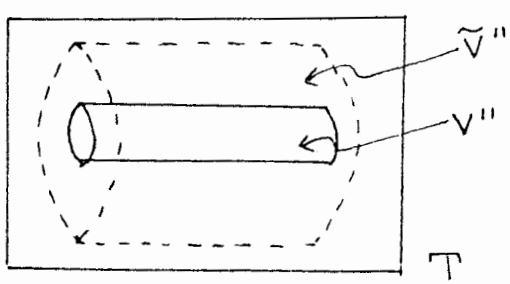




Fig. 1.8

By proposition 1.1 this gives rise to a cohomology class on $\tilde{V}''$, which restricts to a cohomology class on $V''$. The inverse procedure is got by taking a cohomology class on $V''$. This gives rise to a solution of the linear field-equations on a neighbourhood of $V$ in $\mathbb{E}$, which restricts to an analytic field on $V$. A similar procedure works for Laplace's equation and we arrive at

Proposition 1.2

If $V \subset E$ is a convex open set, then solutions of $\nabla_{A'}{}^{L'} \phi_{L'B'\ldots D'} = 0$, where $\phi_{L'B'\ldots D'} \in \Gamma(V, \mathcal{A}_{(L'B'\ldots D')}[-1]')$ ($\mathcal{A}_{(L'B'\ldots D')}$ is the sheaf of real-analytic, conformal weight $[-1]'$ spinors fields with n indices) are in one to one correspondence with elements of $H^1(V'', \mathcal{O}(-n-2))$ for $n > -1$. Moreover, solutions of $\nabla^2 \phi = 0$ where $\phi \in \Gamma(V, \mathcal{A}[-1]')$ are in one to one correspondence with elements of $H^1(V'', \mathcal{O}(-2))$.

§1.5 Contour Integral Formulae (negative homogeneity)

Let $U \subset \mathbb{E}$ be regular, and suppose that we have a minitwistor function $f(\omega, \pi_{A'})$ with its singularities in two disconnected regions of $U''$ defining an element of $H^1(U'', \mathcal{O}(-n-2))$. Then we may evaluate the corresponding field on $U$ by setting

$$\phi_{L'B'\ldots D'}(x) = \frac{1}{2\pi i} \oint \pi_{L'} \pi_{B'} \ldots \pi_{D'} f(ix^{M'N'} \pi_{M'} \pi_{N'}, \pi_{P'}) \pi_{E'} d\pi^{E'}$$

where, for each $x$ the contour threads between the two singular regions on the curve $C_x$. This agrees with the formula given by Ward [13], and his expression for an

axisymmetric field can be seen to arise as follows:

Recall that the expression for the minitwistor curve corresponding to $(x,y,z) \in \mathbb{E}$ is $Z^\circ = \frac{i}{\sqrt{2}}\{-(x+iy) + 2z\zeta + (x-iy)\zeta^2\}$. Therefore, if we keep the expression

$$\eta := \zeta Z^\circ = \frac{i}{\sqrt{2}}\{-(x+iy)\zeta + 2z + (x-iy)/\zeta\}$$

constant and multiply $\zeta$ by $e^{i\theta}$, we obtain a motion of $\mathbb{T}$ which takes the curve corresonding to $(x,y,z)$ into the one corresponding to the point $(x,y,z)$ rotated through angle $\theta$ about the $z$ axis. Thus, if a minitwistor function $f(\eta,\zeta)$ is to produce an axisymmetric field, it must satisfy

$$f(\eta,\zeta) = f(\eta, e^{i\theta}\zeta)$$

but, since $f$ is also to be holomorphic in $\zeta$, it must be independent of $\zeta$. The conclusion then, is that minitwistor functions independent of the $\mathbb{P}_1$ coordinate produce axisymmetric fields.

## §1.6 Positive Homogeneity Minitwistor cohomology

We now consider elements of $H^1(\mathbb{U}'', \mathcal{O}(n-2))$ $n \geq 1$. As in the twistor case, evaluation of the field at some point $x^{A'B'}$ has something to do with restricting the cohomology class to the various formal neighbourhoods of the curve $C_x$. We first set up the formal neighbourhood machinery.

Let $\mathcal{I}$ denote the ideal sheaf of holomorphic functions on $\mathbb{U}''$ which vanish on $C_x$. Then $\mathcal{O}/\mathcal{I}$ will be the sheaf of holomorphic functions on $C_x$. We will write

this as $\mathcal{O}_{C_x}$ . The $m^{th}$ formal neighbourhood $C_x^{(m)}$ of $C_x$ in $U''$ is defined to be the ringed space with the same underlying point set as $C_x$ but with the structure sheaf $\mathcal{O}/\mathcal{I}^{(m+1)}$ . $\mathcal{I}/\mathcal{I}^2$ is the sheaf of sections of the co-normal bundle of $C_x$ in $U''$ . To see this, consider a Taylor expansion normal to $C_x$ . The first term drops out since the function vanishes on $C_x$ and the third and higher ones go since we are working mod $\mathcal{I}^2$. Thus we are left with the normal derivative as required.

Since a section of $\mathcal{O}_{C_x}^{(1)}$ (the sheaf of functions on the first formal neighbourhood) consists of a function on $C_x$ together with its normal derivative, there is a short exact sequence

$$0 \longrightarrow \mathcal{I}/\mathcal{I}^2 \longrightarrow \mathcal{O}_{C_x}^{(1)} \longrightarrow \mathcal{O}_{C_x} \longrightarrow 0 \ .$$

This generalises to

$$0 \longrightarrow \mathcal{I}^{m+1}/\mathcal{I}^{m+2} \longrightarrow \mathcal{O}_{C_x}^{(m+1)} \longrightarrow \mathcal{O}_{C_x}^{(m)} \longrightarrow 0 \ .$$

Using a Taylor expansion, it is easy to see that $\mathcal{I}^{(m+1)}/\mathcal{I}^{(m+2)}$ is the $(m+1)^{th}$ symmetric tensor power of the conormal bundle of $C_x$ in $U''$ .

The normal bundle of $C_x$ is $\mathcal{O}(2)$ , hence the conormal bundle is $\mathcal{O}(-2)$ and the exact sequence becomes

$$0 \longrightarrow \mathcal{O}(-2m-2) \longrightarrow \mathcal{O}_{C_x}^{(m+1)} \longrightarrow \mathcal{O}_{C_x}^{(m)} \longrightarrow 0 \qquad (1.5).$$

Given any holomorphic line bundle $L$ on $U''$ , we can Taylor-expand its transition functions in the coordinate "normal" to $C_x$ . Truncating the expression at the $m^{th}$ derivative term we obtain an object in $H^1\left(C_x^{(m)}, \mathcal{O}^*_{C_x^{(m)}}\right)$ called the restriction of $L$ to $C_x^{(m)}$ and denoted $L^{(m)}$ . (1.5)

gives us an exact sequence

$$0 \to \mathcal{O}(-2m-2) \otimes L \to L^{(m+1)} \to L^{(m)} \to 0$$

In particular for the bundle $L = \mathcal{O}(n-2)$ we get

$$0 \to \mathcal{O}(n-2(m+2)) \to \mathcal{O}^{(m+1)}(n-2) \to \mathcal{O}^{(m)}(n-2) \to 0 \qquad (1.6)$$

For $m = 0$, this is

$$0 \to \mathcal{O}(n-4) \to \mathcal{O}^{(1)}(n-2) \to \mathcal{O}(n-2) \to 0$$

and gives a long exact sequence of cohomology groups for $n \geq 1$ :

$$0 \to \mathbb{S}^{n-4} \to H^0(C_x, \mathcal{O}^{(1)}(n-2)) \xrightarrow{\beta} \mathbb{S}^{n-2} \xrightarrow{\alpha} \mathbb{S}_{2-n} \to H^1(C_x, \mathcal{O}^{(1)}(n-2)) \to 0$$

where $\mathbb{S}_p$ is the space of symmetric spinors with $p$ lower indices (taken to be zero if $p<0$ and $\mathbb{C}$ if $p=0$ ) and $\mathbb{S}^p$ the same with upper indices. Since $\beta$ is surjective, $\alpha$ is the zero map and we obtain an exact sequence

$$0 \to \mathbb{S}_{2-n} \to H^1(C_x, \mathcal{O}^{(1)}(n-2)) \to 0$$

For $m$ a positive integer, we similarly obtain the sequence

$$0 \to \mathbb{S}_{2m+2-n} \to H^1(C_x, \mathcal{O}^{(m+1)}(n-2)) \to H^1(C_x, \mathcal{O}^{(m)}(n-2)) \to 0 \qquad (1.7)$$

Now, the sequence (1.6) splits (map $\mathcal{O}^{(m)}(n-2)$ to $\mathcal{O}^{(m+1)}(n-2)$ by taking the $(m+1)^{th}$ normal derivative to be zero), consequently so does the sequence (1.7). In general therefore, we have

$$H^1(C_x, \mathcal{O}^{(m+1)}(n-2)) \cong H^1(C_x, \mathcal{O}^{(m)}(n-2)) \oplus \mathbb{S}_{2m+2-n}$$

Restricting an element of $H^1(U'', \mathcal{O}(n-2))$ to the $m^{th}$ formal

neighbourhood of $C_x$ in $U''$ thus gives us a collection of $m$ spinors at $x \in U$ :

$1^{st}$ nbd : $\phi_{\underbrace{(A'\ldots B')}_{2-n}}$

$2^{nd}$ nbd : $\phi_{(A'\ldots B')}$ , $\phi_{\underbrace{(A'\ldots B'C'D')}_{4-n}}$

$m^{th}$ nbd : $\phi_{(A'\ldots B')}$ , $\phi_{(A'\ldots B'C'D')}$ , ..... , $\phi_{\underbrace{(A'\ldots B'C'D'\ldots L'M')}_{2m-n}}$

and as the curve $C_x$ varies these become holomorphic spinor fields on $U$ .

Suppose that $f(\omega, \pi_{A'})$ is a function representing the cohomology class (so that $f(\lambda^2\omega, \lambda\pi_{A'}) = \lambda^{n-2} f(\omega, \pi_{A'})$ ) then the isomorphism

$$\$_{\underbrace{(A'\ldots B')}_{2-n}} \cong H^1(C_x, \mathcal{O}^{(1)}(n-2))$$

is given by

$$\phi_{A'\ldots B'} = \frac{1}{2\pi i} \oint \pi_{A'} \ldots \pi_{B'} \frac{\partial f}{\partial \omega} \Delta\pi$$

Similarly, the $\$_{(A'\ldots B'C'D')}$ part of the element of $H^1(C_x, \mathcal{O}^{(2)}(n-2))$ is given by

$$\phi_{A'\ldots B'C'D'} = \frac{1}{2\pi i} \oint \pi_{A'} \ldots \pi_{B'} \pi_{C'} \pi_{D'} \frac{\partial^2 f}{\partial \omega^2} \Delta\pi$$

etc. The successive integrands are related by the operator $\pi_{C'} \pi_{D'} \frac{\partial}{\partial \omega}$ . On the three-space, this operator simply gives the gradient. Thus the collection of spinor fields obtained by restricting the element of $H^1(U'', \mathcal{O}(n-2))$ to the various neighbourhoods is

$1^{st}$ nbd : $\phi_{(A'\ldots B')}$

$2^{nd}$ nbd : $\phi_{(A'\ldots B')}$ , $\nabla_{(C'D'}\phi_{A'\ldots B')}$

$m^{th}$ nbd : $\phi_{(A'\ldots B')}$, $\nabla_{(C'D'}\phi_{A'\ldots B')}$, $\ldots\ldots$ $\nabla_{(L'M'}\ldots\nabla_{C'D'}\phi_{A'\ldots B')}$

In fact the symmetrizations are unnecessary since taking the gradient produces a field which is already symmetric e.g.

$$\begin{aligned}\nabla_{C'[D'}\phi_{A']\ldots B'} &= \tfrac{1}{2}\,\varepsilon_{D'A'}\,\nabla_{C'}{}^{L'}\phi_{L'\ldots B'} \\ &= \tfrac{1}{2}\,\varepsilon_{D'A'}\cdot\frac{1}{2\pi i}\,\nabla_{C'}{}^{L'}\oint \pi_{L'}\ldots\pi_{B'}\,\frac{\partial^{n}f}{\partial\omega^{n}}\,\Delta\pi \\ &= 0\end{aligned}$$

on substituting $\omega = ix^{L'M'}\pi_{L'}\pi_{M'}$.

To summarize then, given an element of $H^1(U'', \mathcal{O}(n-2))$ $n \geq 1$, if $n$ is even, restricting to the $n/2^{th}$ neighbourhoods gives us a scalar field

$$\phi = \frac{1}{2\pi i}\oint \frac{\partial^{n}f}{\partial\omega^{n/2}}\,\Delta\pi \qquad (1.8)$$

satisfying $\nabla^2\phi = 0$ and restricting to higher neighbourhoods gives us successive gradients of $\phi$. If $n$ is odd, restricting to the $(n+1)/2^{th}$ neighbourhoods gives us a spinor field

$$\phi_{A'} = \frac{1}{2\pi i}\oint \pi_{A'}\,\frac{\partial^{(n+1)}f}{\partial\omega^{(n+1)}}\,\Delta\pi \qquad (1.9)$$

satisfying $\nabla_{B'}{}^{A'}\phi_{A'} = 0$, and again higher neighbourhoods give us successive gradients.

We see that if $n$ is even all the information is contained in a solution of the Laplace equation and if $n$ is odd, it is contained in a solution of the "neutrino equation". We

can see this in terms of cohomology as follows: First, consider the short exact sequence

$$0 \longrightarrow \rho^{-1}\mathcal{O}(n-2) \xrightarrow{i} \mathcal{O}(n-2) \xrightarrow{\frac{\partial}{\partial\omega}} \mathcal{O}(n-4) \longrightarrow 0$$

where $\rho : T \to \mathbb{P}_1$ is projection and $i$ is inclusion. Sections of the topological inverse image, $\rho^{-1}\mathcal{O}(n-2)$ are just functions of $\pi_{A'}$ homogeneous of degree $(n-2)$. Part of the corresponding long exact sequence is

$$\to H^1(U'', \rho^{-1}\mathcal{O}(n-2)) \to H^1(U'', \mathcal{O}(n-2)) \to H^1(U'', \mathcal{O}(n-4)) \to H^2(U'', \rho^{-1}\mathcal{O}(n-2)) \to$$

Suppose that $U$ is topologically trivial. Then the fibres of $\rho| : U'' \to \mathbb{P}_1$ are also topologically trivial, and we can invoke a theorem of Buchdahl [1] to deduce that $H^p(U'', \rho^{-1}\mathcal{O}(n-2)) \cong H^p(\mathbb{P}_1, \mathcal{O}(n-2))$ $p=1,2$. This vanishes for $n \geqslant 1$ and we thus get

$$H^1(U'', \mathcal{O}(n-2)) \cong H^1(U'', \mathcal{O}(n-4)).$$

We can iterate this isomorphism and deduce

$$H^1(U'', \mathcal{O}(n-2)) \cong \begin{cases} H^1(U'', \mathcal{O}(-2)) & n \text{ even} \\ H^1(U'', \mathcal{O}(-1)) & n \text{ odd}. \end{cases}$$

We have already seen that $H^1(U'', \mathcal{O}(-2))$ is isomorphic to the space of solutions of $\nabla^2\phi = 0$ on $U$. Moreover, the same techniques that were used to prove proposition 1.1 tell us that $H^1(U'', \mathcal{O}(-1))$ is isomoprhic to the space of solutions of $\nabla_{A'}{}^{B'}\phi_{B'} = 0$ on $U$. We summarize our results in

Proposition 1.3

If $U \subset \mathbb{H}$ is regular, then for $n \geqslant 0$ :

$$H^1(U'', \mathcal{O}(n-2)) \cong \begin{cases} \text{holomorphic solutions to } \nabla^2\phi = 0 \text{ on } U & n \text{ even} \\ \text{holomorphic solutions to } \nabla_{A'}{}^{B'}\phi_{B'} = 0 \text{ on } U & n \text{ odd}. \end{cases}$$

these have a contour integral description

$$\phi = \frac{1}{2\pi i} \oint \frac{\partial^{n/2} f}{\partial \omega^{n/2}} \Delta\pi \qquad n \text{ even}$$

$$\phi_{B'} = \frac{1}{2\pi i} \oint \frac{\partial^{(n+1)/2} f}{\partial \omega^{(n+1)/2}} \pi_{B'} \Delta\pi \qquad n \text{ odd.}$$

---

It is interesting to see how Maxwell fields fit into this framework. Suppose we have a function $f$ representing an element of $H^1(U'', \mathcal{O})$. The exact sequence (1.7) becomes (for $n=2$, $m=1$)

$$0 \longrightarrow \mathbb{S}_{(A'B')} \longrightarrow H^1(\mathcal{C}x, \mathcal{O}^{(2)}) \longrightarrow \mathbb{C} \longrightarrow 0$$

The second formal neighbourhood thus gives us two objects, a vector

$$V_{A'B'} = \frac{1}{2\pi i} \oint \pi_{A'} \pi_{B'} \frac{\partial^2 f}{\partial \omega^2} \Delta\pi$$

and a scalar

$$\phi = \frac{1}{2\pi i} \oint \frac{\partial f}{\partial \omega} \Delta\pi$$

where

$$V_{A'B'} = -i \nabla_{A'B'} \phi \qquad (1.10)$$

Now, the de Rahm sequence on $U \subset \mathbb{E}$ :

$$0 \longrightarrow \mathbb{C} \longrightarrow \Omega^0 \longrightarrow \Omega^1 \longrightarrow \Omega^2 \longrightarrow \Omega^3 \longrightarrow 0$$

can be written in terms of spinors (taking duals where necessary) in the form

$$0 \longrightarrow \mathbb{C} \longrightarrow \mathcal{O} \xrightarrow[\nabla_{A'B'}]{\text{grad}} \mathcal{O}_{(A'B')} \xrightarrow[\nabla^{A'}_{\ (C'}]{\text{curl}} \mathcal{O}_{(B'C')} \xrightarrow[\nabla^{B'C'}]{\text{div}} \mathcal{O} \longrightarrow 0$$

Suppose that $\mathcal{U}$ is Stein. Then this is an acyclic resolution of $\mathbb{C}$, therefore

$$H^2(\mathcal{U},\mathbb{C}) \cong \frac{\ker \operatorname{div} : \Gamma(\mathcal{U},\mathcal{O}_{(B'C')}) \to \Gamma(\mathcal{U},\mathcal{O})}{\operatorname{Im} \operatorname{curl} : \Gamma(\mathcal{U},\mathcal{O}_{(A'B')}) \to \Gamma(\mathcal{U},\mathcal{O}_{(B'C')})}$$

Suppose now that $H^2(\mathcal{U},\mathbb{C}) = 0$, then given $V_{A'B'}$ of the form (1.10), there exists a vector potential $A_{A'B'}$ such that $\underline{V} = \operatorname{curl} \underline{A}$.

We conclude that if $\mathcal{U}$ is Stein and $H^2(\mathcal{U},\mathbb{C}) = 0$ in addition to being regular, then

$$H^1(\mathcal{U}'',\mathcal{O}) \cong \left\{ (\underline{A},\phi) \;\middle|\; \operatorname{curl} \underline{A} = -i \underline{\nabla} \phi \right\}$$

where $\underline{A}$ is defined up to gauge transformations.

$$\underline{A} \to \underline{A} + \underline{\nabla} \lambda .$$

The equations $\operatorname{curl} \underline{A} = -i \underline{\nabla} \phi$ will be referred to as the $\mathbb{C}^*$ Bogomolny equations.

## §1.7 Relationship with Twistor Cohomology

We have seen how analytic cohomology on a subset $\mathcal{U}'' \subset \mathcal{T}$ encodes solutions of the linear field equations $\nabla^{A'B'} \phi_{B'\ldots L'} = 0$ and $\nabla^2 \phi = 0$ on $\mathcal{U} \subset \mathbb{E}$. These solutions can in turn be identified with stationary solutions of the zero rest mass equations on $\tau^{-1}\mathcal{U} \subset \mathbb{M}^{\mathrm{I}}$ where $\tau$ denotes the projection $\mathbb{M}^{\mathrm{I}} \to \mathbb{E}$ (we will also use $\tau$ to denote the corresponding projection $\mathbb{PT}^{\mathrm{I}} \to \mathcal{T}$). The time-translation vector field $t^{AA'} \pi_{A'} \frac{\partial}{\partial \omega^A}$ maps homogeneous twistor functions to functions of the same homogeneity and hence induces a map

$$\tau : H^1(\tau^{-1}\mathcal{U}'', \mathcal{O}(-n-2)) \to H^1(\tau^{-1}\mathcal{U}'', \mathcal{O}(-n-2)).$$

on cohomology. Stationary zero rest mass fields on $\mathcal{T}^{-1}U$ then correspond to cohomology classes (of the appropriate homogeneity) which are eliminated by $\mathcal{T}$. The situation is illustrated in fig. 1.9

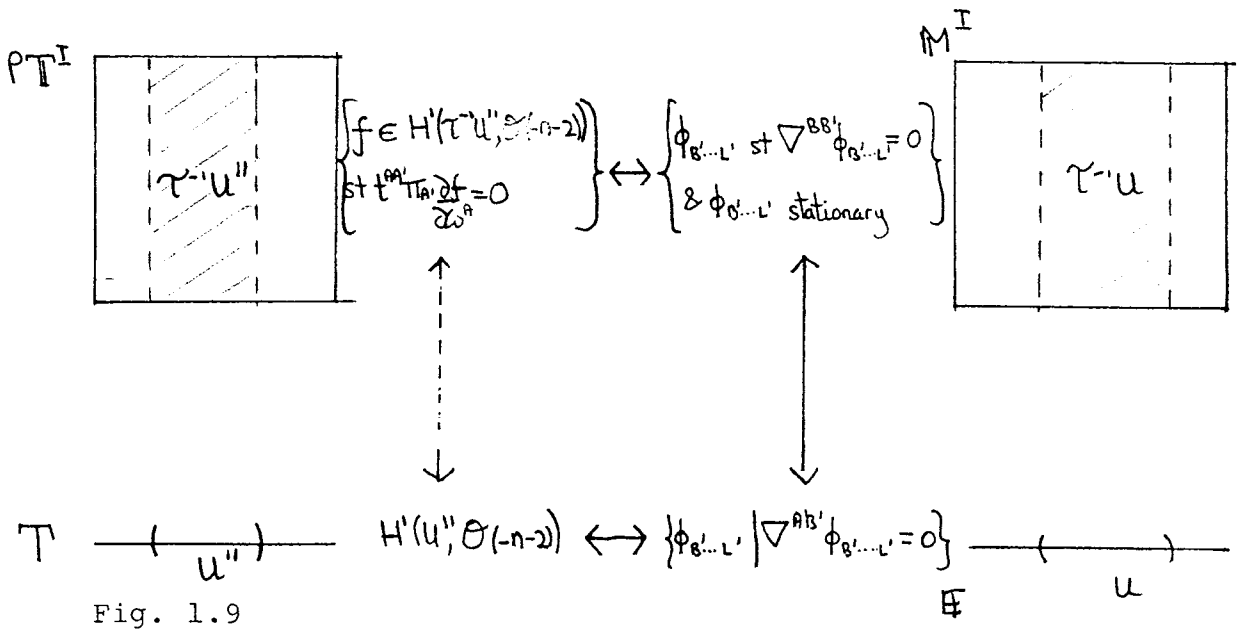



Fig. 1.9

We now show how to establish the isomorphism given by the dotted arrow without going via space-time. Let $\mathcal{T}^{-1}\mathcal{O}(-n-2)$ be the topological inverse image under $\mathcal{T}$, of the sheaf $\mathcal{O}(-n-2)$ on $\mathbb{T}$. We have a short exact sequence on $\mathcal{T}^{-1}U''$

$$0 \longrightarrow \mathcal{T}^{-1}\mathcal{O}(-n-2) \xrightarrow{i} \mathcal{O}(-n-2) \xrightarrow{t^{AA'}\pi_{A'}\frac{\partial}{\partial\omega^A}} \mathcal{O}(-n-2) \longrightarrow 0$$

where $i$ is inclusion. In checking exactness, the only non-obvious point is surjectivity of $t^{AA'}\pi_{A'}\frac{\partial}{\partial\omega^A}$ but this follows on trivializing $\mathcal{O}(-n-2)$ and using the Cauchy-Kowalewsky theorem. The long exact sequence on cohomology contains the part

$$\to H^0(\mathcal{O}(-n-2)) \xrightarrow{t^{AA'}\pi_{A'}\frac{\partial}{\partial\omega^A}} H^0(\mathcal{O}(-n-2)) \to H^1(\mathcal{T}^{-1}\mathcal{O}(-n-2)) \xrightarrow{i} H^1(\mathcal{O}(-n-2)) \xrightarrow{t^{AA'}\pi_{A'}\frac{\partial}{\partial\omega^A}} H^1(\mathcal{O}(-n-2))$$

For $n \geq -1$ , $H^0(\mathcal{O}(-n-2))$ vanishes, so $i$ is injective and cohomology classes eliminated by $t^{AA'}\pi_{A'}\frac{\partial}{\partial\omega^A}$ are in one to one correspondence with elements of $H^1(\mathcal{T}^{-1}\mathcal{O}(-n-2))$ . Since the fibres of $\mathcal{T}$ are connected and simply connected, we have (see [2])

$$H^1(\mathcal{U}, \mathcal{O}(-n-2)) \cong H^1(\mathcal{T}^{-1}\mathcal{U}, \mathcal{T}^{-1}\mathcal{O}(-n-2)).$$

Thus, for $n \geq -1$ we conclude that

$$H^1(\mathcal{U}, \mathcal{O}(-n-2)) \cong \ker t^{AA'}\pi_{A'}\frac{\partial}{\partial\omega^A} : H^1(\mathcal{T}^{-1}\mathcal{U}, \mathcal{O}(-n-2)) \to H^1(\mathcal{T}^{-1}\mathcal{U}, \mathcal{O}(-n-2))$$

$$\cong \{\text{stationary Z.R.M. fields with } n \text{ indices}\}$$

For $n < -1$ , the situation is slightly different. Consider the case $n = -2$ . The above long exact sequence gives (since $t^{AA'}\pi_{A'}\frac{\partial}{\partial\omega^A}$ is the zero map on $H^0(\mathcal{O})$ )

$$0 \to \mathbb{C} \to H^1(\mathcal{T}^{-1}\mathcal{U}, \mathcal{T}^{-1}\mathcal{O}) \to \left(\ker t^{AA'}\pi_{A'}\frac{\partial}{\partial\omega^A} : H^1(\mathcal{O}) \to H^1(\mathcal{O})\right) \to 0$$

It is not too difficult to see that the map on potentials induced by $t^{AA'}\pi_{A'}\frac{\partial}{\partial\omega^A}$ on cohomology is

$$\psi_{AA'} \longrightarrow t_{A'}^{\ B}\nabla_{C'(A}\psi_{B)}{}^{C'}$$

and hence $\ker t^{AA'}\pi_{A'}\frac{\partial}{\partial\omega^A} = \left\{\psi_{AA'} \middle| \begin{array}{l}\psi_{AA'} \text{ an A.S.D. pot}^{\text{l}} \\ \& \ t_{A'}^{\ B}\nabla_{C'(A}\psi_{B)}{}^{C'} = \nabla_{AA'}\chi \ \text{ some } \chi\end{array}\right\}/\text{gauge}$

From an element of $\ker t^{AA'}\pi_{A'}\frac{\partial}{\partial\omega^A}$ , one can therefore construct up to a constant, a scalar field $\chi$ which satisfies

$$t^{CC'}\nabla_{CC'}\chi = 0 \quad ; \quad \Box\chi = 0$$

i.e. we have a solution of the Laplace equation on $\mathcal{U}$ up

to a constant. Recalling that solutions of the Laplace equation are representable by the minitwistor cohomology group $H^1(\mathbb{U}'', \mathcal{O})$ (see proposition 1.3), we have a diagram

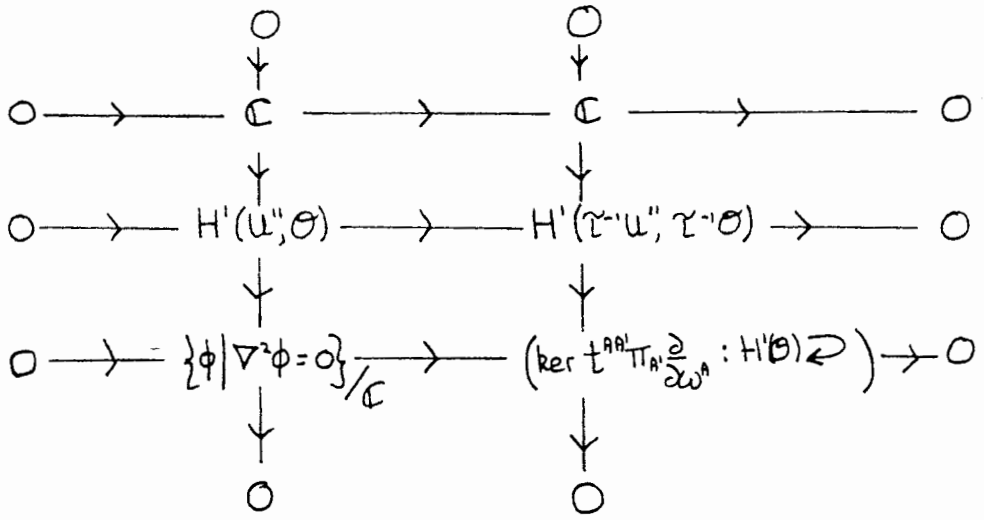



Fig. 1.10

As a matter of interest, we can see how a stationary A.S.D. Maxwell field $F_{ab}$ gives rise to a solution of the $\mathbb{C}^*$ Bogomolny equations by writing the anti-self-duality relation $\frac{1}{2}\varepsilon^{abcd} F_{cd} = -i F^{ab}$ in terms of a stationary four-potential $A_a = (A_0, A_i)$ as

$$\varepsilon^{ijk} \nabla_{[j} A_{k]} = -i \left(\nabla_0 A^i - \nabla^i A_0\right) = -i \nabla_i A_0$$

$$\text{i.e.} \quad \text{curl } \underline{A} = -i \underline{\nabla} \phi \qquad \phi := A_0$$

In the choice of four-potential for $F_{ab}$ we have the freedom $\phi \to \phi + \text{const}$, reflecting the fact that twistor cohomology annihilated by $t^{AA'} \pi_{A'} \frac{\partial}{\partial \omega^A}$ encodes solutions of the Bogomolny equations up to a constant.

§1.8 An Inverse Minitwistor Function

Suppose we have a solution $(\underline{A}, \phi)$ of the $\mathbb{C}^*$ Bogomolny equations on $\mathbb{E}$. Then $\underline{A}$ and $\phi$ can be built respectively into the symmetric and skew parts of a two index spinor

$$\Phi^{A'B'} = \frac{1}{\sqrt{2}} \begin{pmatrix} -(A^x + iA^y) & A^z - \phi \\ A^z + \phi & (A^x - iA^y) \end{pmatrix} \qquad (1.11)$$

Thus

$$\Phi^{(A'B')} = \frac{1}{\sqrt{2}} \begin{pmatrix} -(A^x + iA^y) & A^z \\ A^z & (A^x - iA^y) \end{pmatrix}; \quad \phi = -\frac{1}{\sqrt{2}} \Phi_{D'}{}^{D'}$$

The Bogomolny equations then become

$$\nabla_{C'}{}^{(A'} \Phi^{B')C'} = 0 \qquad (1.12)$$

We may form the object $\Phi_{A'} := \Phi_{A'B'} \pi^{B'}$. This is a section of $\Omega^1_\mu = \mathcal{O}_{A'}(1)[-1]'$ i.e. a relative one form on the spin bundle $\mathcal{S}$ with respect to the fibration $\mu : \mathcal{S} \to \mathbb{T}$. $\Phi_{A'}$ is closed with respect to the exterior derivative $d_\mu = \pi^{B'} \nabla_{B'A'}$ since

$$\pi^{B'} \nabla_{B'A'} \Phi^{A'}$$

$$= \pi^{B'} \pi^{C'} \nabla_{A'(B'} \Phi^{A'}{}_{C')}$$

which vanishes by the Bogomolny equations. We thus have a closed one-form on each two-plane $\gamma_z$, $z \in \mathbb{T}$. Pick two skew null geodesics and label them $N_0$ and $N_1$. Each null geodesic defines a unique null two-plane (suppose the direction of the geodesic to be given by $\pi_{A'} \pi_{B'}$, pick a point $x^{A'B'}$ on it, then the two-plane is the one defined by the minitwistor $(ix^{A'B'} \pi_{A'} \pi_{B'}, \pi_{C'})$).

Suppose the two-planes defined by $N_0$ , $N_1$ (call them $\gamma_0$ , $\gamma_1$ ) correspond to minitwistors living in the fibres $\pi_{0'} = 0$ and $\pi_{1'} = 0$ of $T$ respectively. A minitwistor $Z$ in the complement of these fibres then corresponds to a two-plane $\gamma_Z$ which is not parallel to $\gamma_0$ or $\gamma_1$ , and which intersects $N_0$ and $N_1$ in points $P_0$ , $P_1$ respectively.

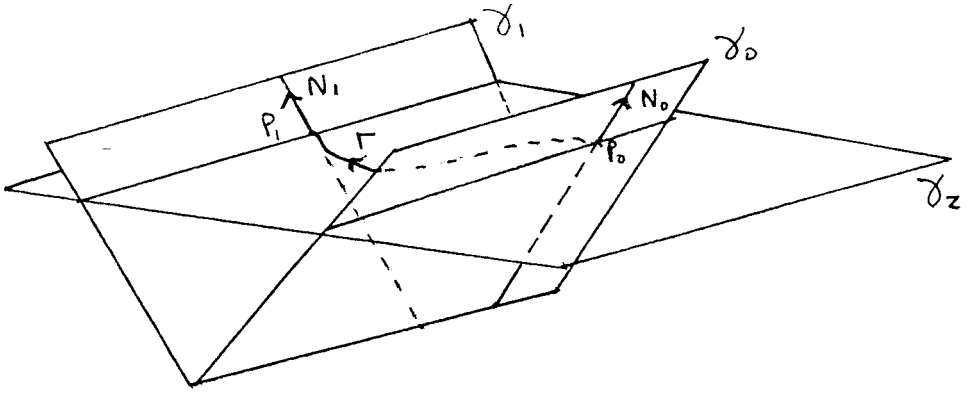



Fig. 1.11

- A general point on $\gamma_Z$ can be written as

$$x^{A'B'} = \overset{\circ}{x}{}^{A'B'} + s^{(A'}\pi^{B')}/\pi_{0'}$$

where, for a minitwistor $(\zeta, Z^\circ)$ , $\overset{\circ}{x}{}^{A'B'}$ is defined to be

$$\overset{\circ}{x}{}^{A'B'} = \begin{pmatrix} Z^\circ/i & 0 \\ 0 & 0 \end{pmatrix}$$

Thus $s^{A'}$ serve as a pair of coordinates on the two-plane $\gamma_Z$ . Now, we may form

$$f(\omega, \pi_{A'}) = \int_{P_0}^{P_1} \Omega \qquad (1.13)$$

where $\Omega$ is the relative one-form on the null two-planes (the fibres of $\mu : \mathbb{S} \to T'$) which is given locally by $\Phi_{A'B'C'D'}\pi^{B'}\frac{ds^{A'}}{\pi_{0'}}$.

The corresponding cohomology class is:

1) Independent of the choice of geodesics $N_0$ and $N_1$ in $\gamma_0, \gamma_1$ respectively and independent of the choice of path $\Gamma$ from $P_0$ to $P_1$ since the integrand is closed on $\gamma_Z$ .

2) Undefined for a minitwistor $(\omega, \pi_{A'})$ where $\pi_{0'} = 0$ or $\pi_{1'} = 0$.

We have thus defined an element of $H^1(T, \mathcal{O})$ with respect to the two set Stein cover $\{\pi_{0'} \neq 0\}; \{\pi_{1'} \neq 0\}$ of $T$. We must show that $\mathcal{P}$ -transforming $f(\omega, \pi_{A'})$ gives the solution of the Bogomolny equations that we started with. To do this, we can proceed as follows:

To evaluate the field at $x^{A'B'}$ restrict $f(\omega, \pi_{A'})$ to the curve $C_x$, giving a function $f(ix^{A'B'}\pi_{A'}\pi_{B'}, \pi_{C'})$. Since $H^1(C_x, \mathcal{O}) = 0$ this splits i.e. $f(x, \pi) = f_1(x, \pi) - f_0(x, \pi)$ where $f_1$ & $f_0$ are defined on the $\pi_{1'} \neq 0$ and $\pi_{0'} \neq 0$ subsets of $C_x$ respectively. The equation $\pi^{A'} \nabla_{A'B'} f(x, \pi) = 0$ tells us that

$$\pi^{A'} \nabla_{A'B'} f_1 = \pi^{A'} \nabla_{A'B'} f_0 = \psi_{A'B'}(x) \pi^{B'}$$

for some spinor field $\psi_{A'B'}$, where $\psi_{A'B'}$ encodes (by (1.11)) the solution of the Bogomolny equations given by $\mathcal{P}$ -transforming $f$. We must show that $\psi_{A'B'}$ is the same as the solution $\Phi_{A'B'}$ from which $f$ was formed by (1.13). To do this, note that a suitable splitting of $f(ix^{A'B'}\pi_{A'}\pi_{B'}, \pi_{C'})$ is

$$f_1(x, \pi) = \int_{P_0}^{x} \Omega$$

$$f_0(x, \pi) = \int_{P_1}^{x} \Omega$$

and on applying $\pi^{A'} \nabla_{A'B'}$ we recover $\Phi_{A'B'} \pi^{B'}$.

§1.9 The Mini-Twisted Photon

Elements of $H^1(\mathcal{U}'', \mathcal{O})$ encode, for suitable $\mathcal{U}$, solutions $(\underline{A}, \phi)$ of the $\mathbb{C}^*$ Bogomolny equations ( $\underline{A}$ is defined up to gauge transformations). As in the twistor case, use of the exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}^* \longrightarrow 0$$

enables us to think of these as line bundles on $\mathcal{U}''$. Explicitly, given $(\underline{A}, \phi)$, and a minitwistor $Z = (Z^\circ, \zeta) \in \mathcal{U}''$ we define the fibre over $Z$ to be the space of solutions $\pi_{L'}$ of

$$\left( \pi^{A'} \nabla_{A'B'} - \pi^{A'} A_{A'B'} + \phi / R \, \pi_{B'} \right) \pi_{L'} = 0 \qquad (1.14)$$

on $\gamma_Z$, where $[\pi_{L'}] = \zeta$. This can be written

$$\pi^{A'} \left( \nabla_{A'B'} - \Phi_{A'B'} \right) \pi_{L'} = 0 \qquad (1.15)$$

(see (1.11)). The condition for this parallel transport of spinors to be integrable on $\gamma_Z$ is then,

$$\pi^{A'} (\nabla_{A'B'} - \Phi_{A'B'}) \pi^{C'} (\nabla_{C'D'} - \Phi_{C'D'}) \pi_{L'} - (B' \leftrightarrow D') = 0$$

Using (1.15) we can easily see that this reduces to

$$\pi^{A'} \pi^{C'} \nabla_{D'(A'} \Phi_{C')}{}^{D'} = 0$$

which holds as a consequence of the $\mathbb{C}^*$ Bogomolny equations (1.12). The parallel transport equation (1.15) on $\gamma_Z$ can be integrated as follows:

Given a spinor $\overset{\circ}{\pi}_{A'}$ at $\overset{\circ}{x}{}^{A'B'} \in \gamma_Z$ (where $\overset{\circ}{\pi}_{A'}$ is in the projective equivalence class determined by $Z$), transporting it to $x^{A'B'}$ along a curve $\Gamma$ in $\gamma_Z$ gives

the spinor

$$\pi_{c'}(x) = \overset{\circ}{\pi}_{c'} \exp\left( \int_\Gamma \Phi_{A'B'} \frac{\Pi^{B'}}{\pi_{o'}} \, dS^{A'} \right) \tag{1.16}$$

where of course we are assuming that $Z$ is not in the fibre of $T$ over $\pi_{o'} = 0$. The Bogomolny equations guarantee closure of the integrand and hence independence of the integral on variations of the path $\Gamma$ between $\overset{\circ}{x}^{A'B'}$ and $x^{A'B'}$, provided $U'' \cap \gamma_z$ is simply connected.

The line bundle constructed in this way will be trivial on the holomorphic curves. If $x^{A'B'} \in U$, we can trivialise the bundle over $C_x$ by picking a non zero spinor at $x^{A'B'}$ and using (1.16) to transport it over the null planes through $x^{A'B'}$.

Conversely, given a bundle trivial on holomorphic curves we can extract the potential: suppose $U''$ has a two-set Stein cover $\{V, \hat{V}\}$ with transition function $f(\zeta, Z^\circ)$. $f|_{C_x}$ must split in the form $f(x,\pi) = G(x,\pi) / \hat{G}(x,\pi)$. We then extract a spinor field according to

$$G^{-1} \pi^{A'} \nabla_{A'B'} G = \Phi_{A'B'}(x) \pi^{A'} \tag{1.17}$$

The scalar and vector parts of the solution to the Bogomolny equations are then respectively the skew and symmetric parts of $\Phi_{A'B'}$.

## §1.10 The Coulomb Field

In this section we construct the mini-twisted photon for the anti self-dual Coulomb field with source at the

origin of $E$, and show how the description of the potential as a connection on a non trivial line bundle over $E-\{o\}$ arises cohomologically.

Recall that in the real picture, the minitwistor space $T$ is the space of oriented geodesics in $E$. Removing the origin from $E$ (as we must if a source is sitting there) causes the geodesics through it to split into two. The space of oriented geodesics for $E-\{o\}$ is non-Hausdorff and is described by taking two copies $T_\alpha$ and $T_\beta$ of $T$ and identifying them everywhere except along their zero sections. We will call this minitwistor space $T'$.

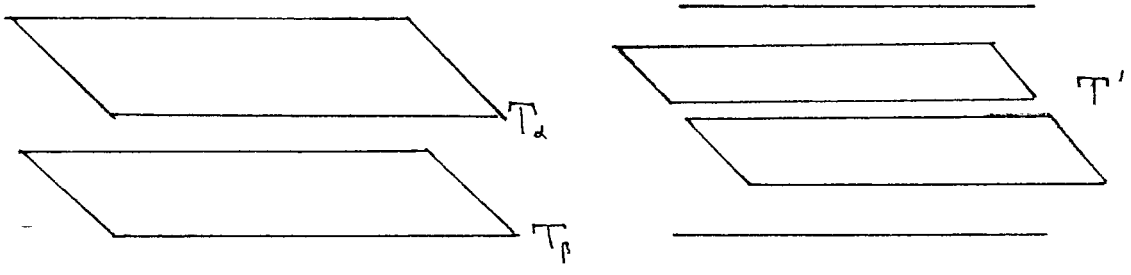



Fig. 1.11

The two zero sections in $T'$ correspond to the sphere's worth of incoming geodesics at $O$ and to the sphere's worth of outgoing ones. $T'$ has a four set Stein cover

$$U_{\alpha 0} = \{\pi_{0'} \neq 0\} \subset T_\alpha \qquad U_{\beta 0} = \{\pi_{0'} \neq 0\} \subset T_\beta$$

$$U_{\alpha 1} = \{\pi_{1'} \neq 0\} \subset T_\alpha \qquad U_{\beta 1} = \{\pi_{1'} \neq 0\} \subset T_\beta.$$

Take a point $x^{A'B'} \neq O$ in $E$. $x^{A'B'}$ gives rise to a holomorphic curve $C_x$ in $T'$, corresponding to the sphere's worth of oriented geodesics through $x^{A'B'}$. $C_x$ intersects

the zero section twice, one intersection point being taken from the $T_\alpha$ copy and the other from the $T_\beta$ copy, since one of the geodesics is incoming at $O$ and the other outgoing.

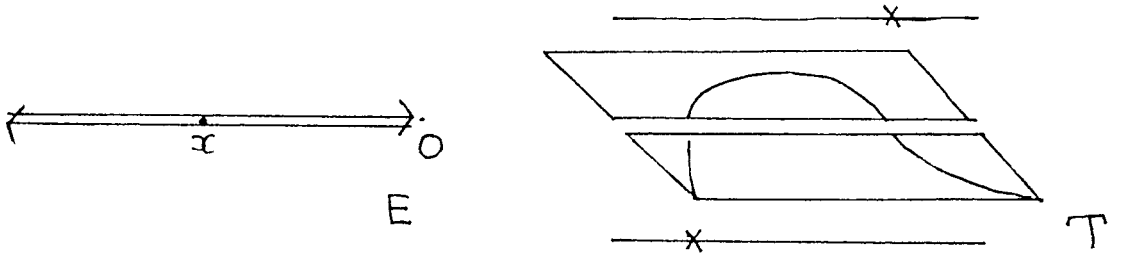



Fig. 1.12

We now proceed to construct a holomorphic line bundle $L$ over $T'$. Let the fibre coordinates be

$$\alpha^\circ \text{ on } U_{\alpha 0} \qquad \beta^\circ \text{ on } U_{\beta 0}$$
$$\alpha^\prime \text{ on } U_{\alpha 1} \qquad \beta^\prime \text{ on } U_{\beta 1}.$$

For the transition functions, take

$$\left\{\begin{matrix}\alpha^\circ = \alpha^\prime \\ \beta^\circ = \zeta^{-2}\beta^\prime\end{matrix}\right\} \text{ on } \left\{\begin{matrix}U_{\alpha 0} \cap U_{\alpha 1} \\ U_{\beta 0} \cap U_{\beta 1}\end{matrix}\right\} \quad (\text{where } \zeta = \pi_{1'}/\pi_{0'}) \qquad (1.18)$$

so that we have, on restricting to $T_\alpha$ and $T_\beta$ the bundles $\mathcal{O}$ and $\mathcal{O}(-2)$ respectively. Patch then together on $T'$ according to

$$\alpha^\circ = Z^\circ \beta^\circ \text{ on } U_{\alpha 0} \cap U_{\beta 0} \; ; \; \alpha^\circ = Z^\prime \beta^\prime \text{ on } U_{\alpha 0} \cap U_{\beta 1} \qquad (1.19)$$
$$\alpha^\prime = Z^\prime \beta^\prime \text{ on } U_{\alpha 1} \cap U_{\beta 1} \; ; \; \alpha^\prime = Z^\circ \beta^\circ \text{ on } U_{\alpha 1} \cap U_{\beta 0}$$

where $Z^\circ$, $Z^\prime$ are the fibre coordinates on minitwistor space fibred over $\mathbb{P}_1$. This defines the line bundle $L$. Now, consider $L|_{C_x}$. Since one point of $C_x$ (say at $\zeta = A$) is from the "$\alpha$ zero section" and one (say at $\zeta = B$) is

from the "$\beta$ zero section" (fig. 1.12) then we may use as a cover of $C_x$ the intersection of $C_x$ with the two sets $U_{\alpha 0}$ and $U_{\beta 1}$.

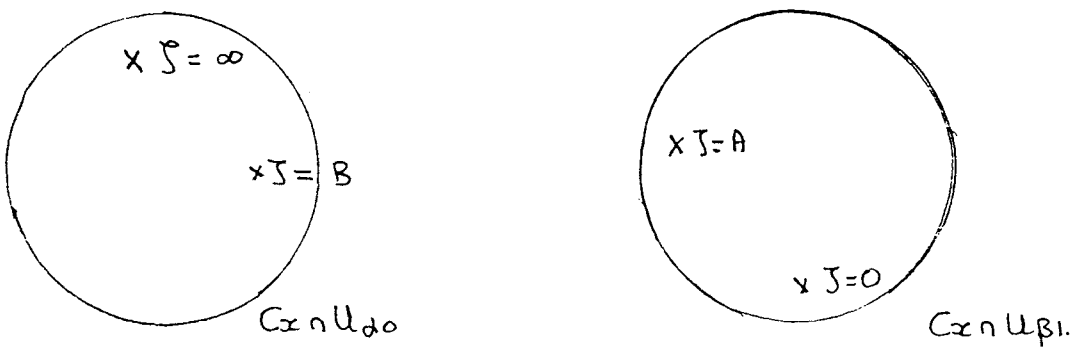



Fig. 1.13

Each of these sets is $\cong \mathbb{C}^*$, and they give us a two set Stein cover of $C_x$. $C_x$ is given by

$$Z^1 = i/\sqrt{2}\{-(x+iy)\zeta^{-2} + 2z\zeta^{-1} + (x-iy)\}$$

so the transition function for $L|_{C_x}$ is

$$\alpha^\circ = i/\sqrt{2}\{-(x+iy)\zeta^{-2} + 2z\zeta^{-1} + (x-iy)\}\beta^1$$

$$= a/\zeta^2(\zeta - A)(\zeta - B)\beta^1$$

where

$$a = i/\sqrt{2}(x-iy) \quad ; \quad A = \frac{(x+iy)}{(z-r)} \quad ; \quad B = \frac{(x+iy)}{(z+r)}$$

$$r^2 = x^2 + y^2 + z^2 \quad , \quad (r+z)(r-z) = (x+iy)(x-iy)$$

Now $\frac{1}{(\zeta - B)}$ is holomorphic and non zero on $C_x \cap U_{\alpha 0}$ and $a(1 - A/\zeta)$ is holomorphic and non zero on $C_x \cap U_{\beta 1}$, thus we may multiply by these functions and reduce the transition function for $L|_{C_x}$ to $1/\zeta$. Hence we deduce $L|_{C_x} \cong \mathcal{O}(-1)$. For a minitwisted photon, however, we need a line bundle which is trivial on the curves, so instead we consider

$\mathcal{C} = L \otimes \mathcal{O}(1)$ . Restricted to $C_x$ this has the transition function

$$\alpha^\circ = \frac{a}{\zeta}(\zeta - A)(\zeta - B)\beta^1$$

$$= \frac{i}{\sqrt{2}}\frac{(x-iy)}{\zeta}\left(\zeta - \frac{(x+iy)}{(z-r)}\right)\left(\zeta - \frac{(x+iy)}{(z+r)}\right)$$

This transition function splits in two ways:

$$\frac{i}{\sqrt{2}}\left[(x-iy)+\frac{(r+z)}{\zeta}\right]\left[\frac{1}{\zeta - \frac{(x+iy)}{z+r}}\right]^{-1} := G_+ \hat{G}_+^{-1}$$

and

$$\frac{i}{\sqrt{2}}\left[(x-iy)+\frac{(z-r)}{\zeta}\right]\left[\frac{1}{\zeta - \frac{(x+iy)}{z-r}}\right]^{-1} := G_- \hat{G}_-^{-1}$$

The first splitting is valid for curves $C_x$ where $x$ is away from $z+r=0$ i.e. away from the negative $z$ -axis and the second for $C_x$ where $x$ is away from the positive $z$ -axis. From these splittings, we can extract two spinor fields $\Phi^+_{A'B'}$ and $\Phi^-_{A'B'}$ where $\Phi^+_{A'B'}$ is defined for points off the negative $z$ axis, and $\Phi^-_{A'B'}$ for points off the positive $z$ axis, according to

$$G_+^{-1}\pi^{A'}\nabla_{A'B'}G_+ = \Phi^+_{A'B'}\pi^{A'}$$

$$G_-^{-1}\pi^{A'}\nabla_{A'B'}G_- = \Phi^-_{A'B'}\pi^{A'}$$

Explicitly, these turn out to be

$$\Phi^+_{A'B'} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 \\ \frac{1}{r} & \frac{(x+iy)}{r(z+r)} \end{pmatrix} \qquad \Phi^-_{A'B'} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 \\ \frac{1}{r} & \frac{(x+iy)}{r(z-r)} \end{pmatrix}$$

From their skew and symmetric parts we extract scalar and vector potentials using (1.12):

$$\phi^+ = \frac{1}{2r} \qquad \underline{A}^+ = -\frac{1}{2}\left(\frac{x+iy}{r(z+r)}, \frac{x+iy}{ir(z+r)}, \frac{1}{r}\right)$$

$$\phi^- = \frac{1}{2r} \qquad \underline{A}^- = -\frac{1}{2}\left(\frac{x+iy}{r(z-r)}, \frac{x+iy}{ir(z-r)}, \frac{1}{r}\right)$$

These satisfy

$$\text{curl } \underline{A}^+ = -i\,\underline{\nabla}\,\phi^+$$

$$\text{curl } \underline{A}^- = -i\,\underline{\nabla}\,\phi^- .$$

The local vector potentials are related by

$$\underline{A}^- = \underline{A}^+ + \frac{1}{(x-iy)}\,\underline{\nabla}\,(x-iy)$$

and therefore can be interpreted as defining a connection on a non trivial complex line bundle $\ell$ over $\mathbb{R}^3-\{0\}$. $\mathbb{R}^3-\{0\}$ is covered by the two contractible open sets

$$U_+ = \{(x,y,z) \in \mathbb{R}^3-\{0\} \mid z \geqslant 0\}$$

$$U_- = \{(x,y,z) \in \mathbb{R}^3-\{0\} \mid z \leqslant 0\}$$

These intersect in the $xy$ plane (minus the origin) on which we have the transition function $x-iy$ defining $\ell$. This is the usual topological set up for describing a Dirac monopole of unit (magnetic) charge. Of course the field we have described has a scalar potential $\phi$ corresponding to unit electric charge as well. This is because it is obtained from a stationary anti self-dual field on space-time. The electric and magnetic fields

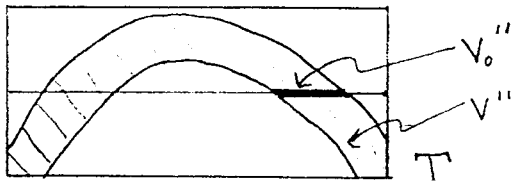
$$\underline{E} = -\underline{\nabla}\,\phi \quad ; \quad \underline{B} = \text{curl } \underline{A}$$

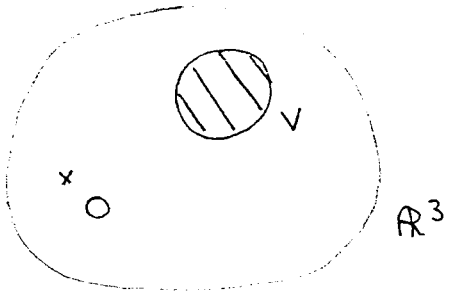
are related by $\underline{E} = -i\,\underline{B}$.

To obtain a monopole with $n$ times this charge ($n \in \mathbb{Z}$) one uses the bundle $\mathcal{C}^n$ on $T'$ (the $n^{th}$ tensor

power of $\mathcal{C}$ ). The transition function relating $\underline{A}^+$ and $\underline{A}^-$ becomes $(x-iy)^n$ and gives rise to the bundle $\ell^n$ on $\mathbb{R}^3-\{0\}$.

If one wanted to describe the bundle $\mathcal{C}$ only over some set $V''$ in $T'$ corresponding to a convex set $V$ in $\mathbb{R}^3$ away from the origin, then a simple way is to use divisors [14]:




$\mathcal{C}|_{V''}$ is the bundle corresponding to the divisor given by one component $V_0''$ of the intersection of $V''$ with the zero section.

# CHAPTER 2 The Quadric as a Minitwistor Space

## §2.1 The Dilation Vector Field

Dilations on $\mathbb{M}$ are represented on $\mathbb{PT}$ by the one-parameter subgroup of $SU(2,2)$ consisting of matrices of the form

$$\begin{pmatrix} \lambda & & & \\ & \lambda & & \\ & & \lambda^{-1} & \\ & & & \lambda^{-1} \end{pmatrix}$$

Differentiating, we see that this gives rise to the vector field

$$V = \omega^A \frac{\partial}{\partial \omega^A} - \pi_{A'} \frac{\partial}{\partial \pi_{A'}} .$$

This has zeroes where it is proportional to the Euler vector field $Z^\alpha \frac{\partial}{\partial Z^\alpha}$ . This happens on a pair of skew lines in $\mathbb{PT}$ :

The line "$\mathbb{I}$" , given by $\pi_{A'} = 0$

The line "$O$" , given by $\omega^A = 0$.

These correspond in space-time to the vertex of $\mathcal{I}$ and to the origin respectively. The integral curves of $V$ are the transversals to $O$ and $\mathbb{I}$ . They represent points on the intersection of $\mathcal{I}$ with the null cone of the origin. To obtain a minitwistor set up, remove $O$ and $\mathbb{I}$ from $\mathbb{PT}$ and the corresponding null cones from $\mathbb{M}$ , and factor out $\mathbb{PT}$ by the integral curves of $V$ and $\mathbb{M}$ – {null cones} by the dilation trajectories.

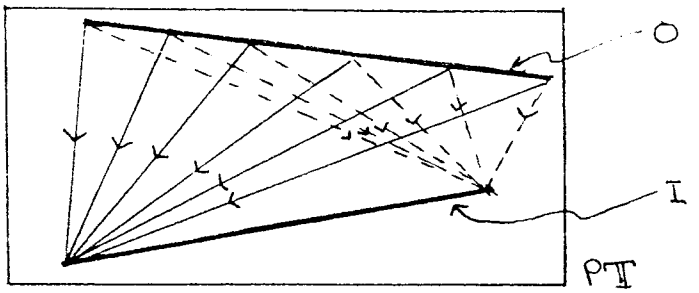



Fig. 2.1

Denote the complement of $O$ and $I$ in $PT$ by $PT^{OI}$ and the complement of the corresponding null cones in $M$ by $M^{OI}$. The integral curves of $V$ (each biholomorphic to $\mathbb{C}^*$) are labelled by their points of intersection with $O$ and $I$, and hence $V$ factors $PT^{OI}$ onto $\mathbb{P}_1 \times \mathbb{P}_1$. This is the minitwistor space appropriate for dilation-invariant fields and will be denoted by $Q$ (for quadric).

To see what the space of dilation trajectories on $M^{OI}$ is, note that $M^{OI}$ is simply $\mathbb{C}^4$ minus the null cone of the origin. The space of trajectories is thus $\mathbb{P}_3$ minus a quadric. The quadric is the space of generators of the null cone of the origin. Denote the space of trajectories by $H$. As a complex manifold, H is biholomorphic to the group $PL(2;\mathbb{C})$ of conformal automorphisms of the Riemann sphere.

Let $x^{AA'} \in M^{OI}$. This corresponds to a line $L_x \subset PT$ which doesn't intersect $O$ or $I$. Projecting along the integral curves of $V$ thus defines a $\mathbb{P}_1$ submanifold of $Q$ corresponding to the trajectory $[x] = \{\lambda x \mid \lambda \in \mathbb{C}^*\}$ through $x$.

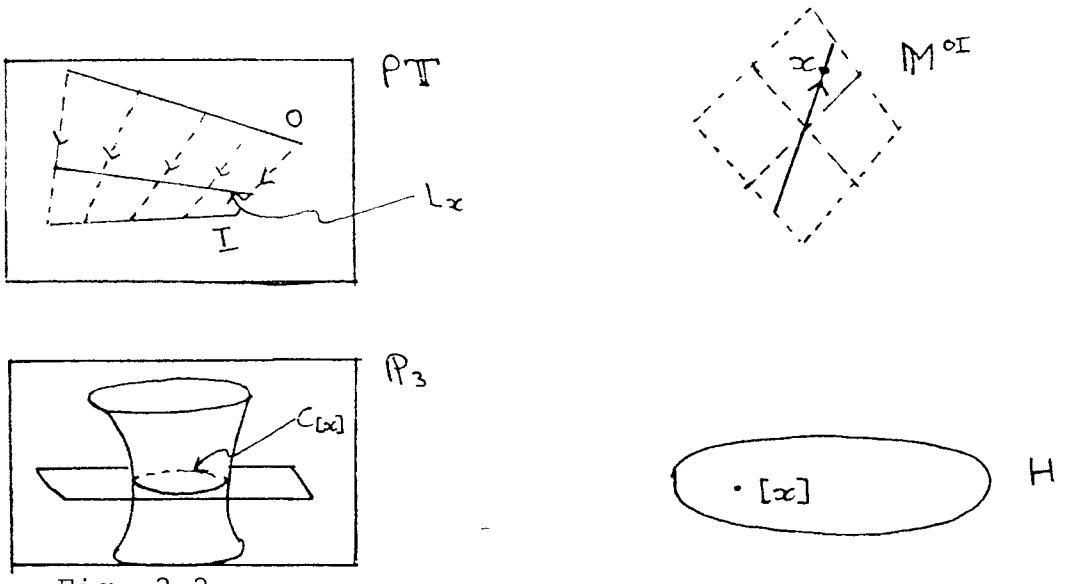



Fig. 2.2

$L_x$ is given by $\omega^A = i x^{AA'} \pi_{A'}$ . $\omega^A$ and $\pi_{A'}$ serve as homogeneous coordinates on $I$ and $O$ respectively. The relation $\omega^A = i x^{AA'} \pi_{A'}$ defines the $P_1$ submanifold $C_{[x]}$ of $Q$ as a plane section of $Q$ . Thus points in $H$ correspond to plane sections of $Q$ which aren't tangent planes, since $x^{AA'}$ is not null.

## §2.2 The Double Fibration

We now seek an analogue for $Q$ , of the double fibration we used in chapter 1 to investigate the cohomology of $T$ . $Q$ Can be thought of as embedded in $P_3$ using the inclusion

$$\begin{array}{ccc} P(\mathbb{S}_A) \times P(\mathbb{S}_{A'}) & \longrightarrow & P(\mathbb{S}_{AA'}) \\ \omega_A \ , \quad \pi_{A'} & \longmapsto & \omega_A \pi_{A'} \end{array}$$

and is given by the relation $\det(x_{AA'}) = 0$ . The pair $(\omega_A, \pi_{A'})$ subject to $(\omega_A, \pi_{A'}) \mapsto (\lambda \omega_A, \mu \pi_{A'})$ will be used

as homogeneous coordinates on $Q$. The space $H$ of plane sections is $\mathbb{P}_3{}^* - Q^*$ (where $Q^*$ is the dual quadric consisting of planes in $\mathbb{P}_3$ tangent to $Q$) and has homogeneous coordinates $x^{AA'}$.

Take $x \in H$ and consider a generic infinitesimal perturbation to $x + dx$. The corresponding planes in $\mathbb{P}_3$ intersect in a line, which meets $Q$ twice. The plane sections $C_x$ and $C_{x+dx}$ thus meet in two points and hence the normal bundle of $C_x$ is $\mathcal{O}(2)$ (as is the case for the minitwistor curves in $\mathbb{T}$). Define a conformal metric on $H$ by specifying that $x$ and $x + dx$ are null separated when $C_x$ and $C_{x+dx}$ meet in <u>one</u> point.

Denote the bundle of tangent planes to $Q^*$ in $H$ by $C$. This is our candidate for the correspondence space which sits at the top of the double fibration. The fibre of $C$ over $q^* \in Q^*$ is then $\mathbb{P}_2$ minus a pair of lines. We have a rank 2 fibration

$$\mu : C \to Q$$

given by mapping $c \in C$ to the point $q \in Q$ represented by the tangent plane to $Q^*$ in which $c$ sits. Since an embedding is given, tangent planes to $Q^*$ are planes in the embedding space $\mathbb{P}_3{}^*$. Given $h \in \mathbb{P}_3{}^* - Q^*$, there is a $\mathbb{P}_1$'s worth of planes through $h$ and tangent to $Q^*$. We thus have a fibration

$$\nu : C \to H$$

with fibre $\mathbb{P}_1$, given by mapping $c \in C$ to itself (since the fibre of $\mu : C \to Q$ through $c$ is realised as a plane in $H$).

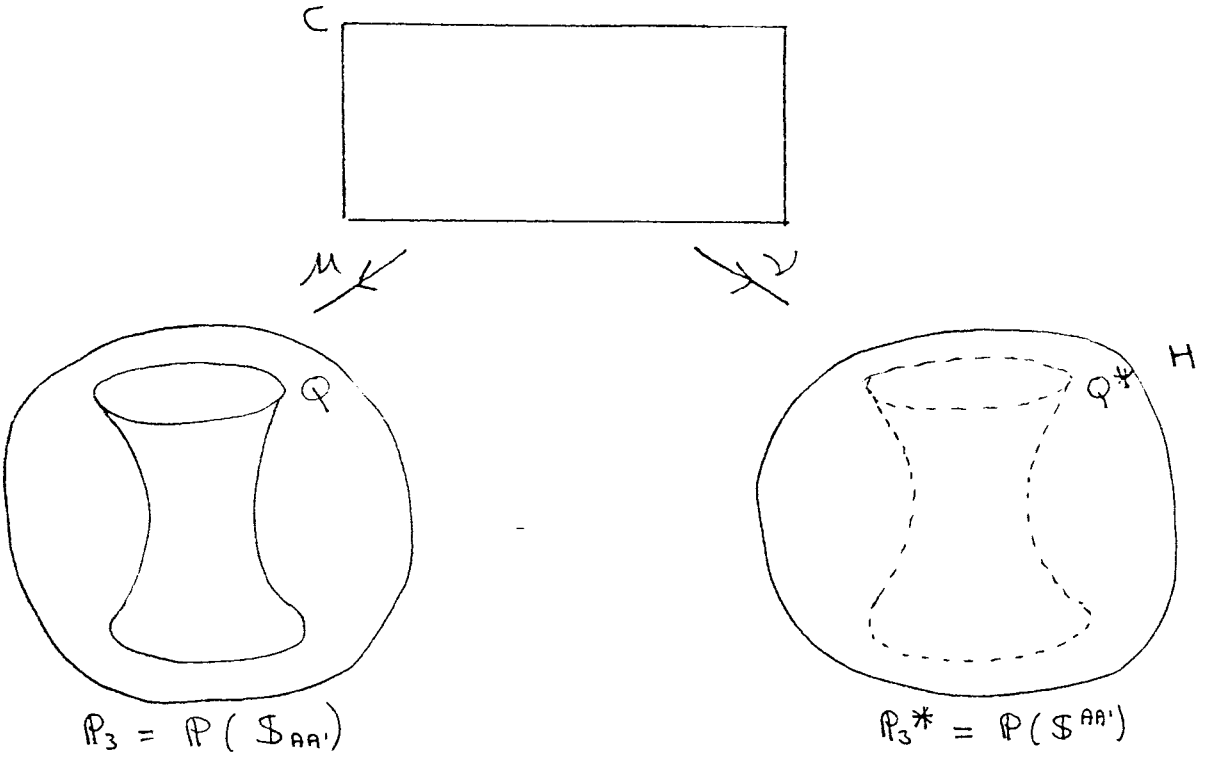



Fig. 2.3

§2.3 The E.P.W. Correspondence

We now derive a correspondence (which we name the "E.P.W. correspondence" since it is analogous to the twistor case worked out by Eastwood, Penrose and Wells [2]) between cohomology of $Q$ and fields on $H$. The natural bundle on $Q$ to investigate is $\mathcal{O}(m,n)$ (sections of $\mathcal{O}(m,n)$ are functions $f(\omega_A, \pi_{A'})$ homogeneous of degree $m$ in $\omega_A$ and $n$ in $\pi_{A'}$). The first step is to pull $\mathcal{O}(m,n)$ back to $C$, and express it in coordinates adapted to the $\nu$ fibration. The mapping $\nu$ decomposes $C$ as $H \times P_1$. There are then two obvious choices of coordinates on $C$, namely $(x^{AA'}, \pi_{B'})$ and $(x^{AA'}, \omega_A)$. Let $f(\omega_A, \pi_{A'})$ be a section of $\mathcal{O}(m,n) \to Q$ thus

$$f(\lambda\omega_A, \mu\pi_{A'}) = \lambda^m \mu^n f(\omega_A, \pi_{A'})$$

Pulling back using coordinates $(x^{AA'}, \pi_{B'})$ on $C$ ; put

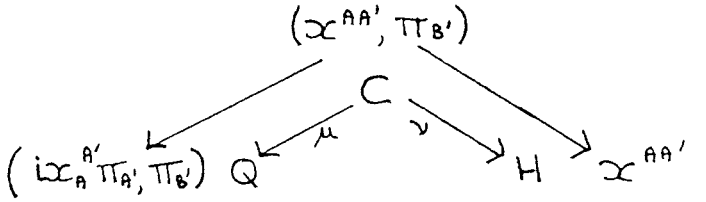
$$g(x^{AA'}, \pi_{B'}) = f(x_A{}^{A'} \pi_{A'}, \pi_{B'})$$

Then $g(\lambda x^{AA'}, \mu \pi_{B'}) = \lambda^m \mu^{m+n} g(x^{AA'}, \pi_{B'})$, so $g$ is homogeneous of degree $m+n$ in the $\mathbb{P}_1$ variable and $m$ in the $\mathbb{P}_3{}^*$ variable. Denote the sheaf of such functions on $C$ by $\mathcal{O}^{\pi}(m, m+n)$ where the $\pi$ is to remind us that the homogeneity statement is made using $\pi_{A'}$ as $\mathbb{P}_1$ coordinate on $C$ .

Similarly, pulling back using $(x^{AA'}, \omega_B)$ we get a function homogeneous of degree $m+n$ in the $\mathbb{P}_1$ variable and $n$ in the $\mathbb{P}_3{}^*$ variable. Denote the sheaf of such functions by $\mathcal{O}^{\omega}(n, m+n)$ . We thus have

$$\mu^* \mathcal{O}(m,n) = \mathcal{O}^{\pi}(m, m+n) = \mathcal{O}^{\omega}(n, m+n)$$

In setting up the correspondence, we will fix $(x^{AA'}, \pi_{B'})$ as the coordinates on $C$ , so we have




Let $U \subset H$ , and set $U' = \nu^{-1} U$ , $U'' = \mu \nu^{-1} U$ . We call $U$ <u>regular</u> if its intersection, $I_q$ , with each tangent plane $q$ to $Q^*$ is connected and $H^1(I_q, \mathbb{C}) = 0$ .

Let $f(x^{AA'}, \pi_{B'})$ be homogeneous of degree zero in each variable i.e. a function on $C$ . $f$ will be constant along the fibres of $\mu$ provided

$$\pi^{A'} \nabla_{AA'} f = 0.$$

thus
We identify the relative exterior derivative as $d_\mu = \pi^{A'} \nabla_{AA'}$

The sheaf $\Omega^1_\mu$ of relative one-forms then becomes

$$\Omega^1_\mu = \mathcal{O}^{\pi}_{A}(-1, 1)[-1]'$$

Similarly, we have $\Omega^2_\mu = \mathcal{O}^{\pi}(-2,2)[-2]'[-1]$.

The relative De Rham sequence on $\mathcal{C}$ is then

$$\Omega^0_\mu \longrightarrow \Omega^1_\mu \longrightarrow \Omega^2_\mu$$

$$\mathcal{O} \longrightarrow \mathcal{O}^{\pi}_{A}(-1,1)[-1]' \longrightarrow \mathcal{O}^{\pi}(-2,2)[-2]'[-1]$$

$$f \longmapsto \pi^{A'} \nabla_{AA'} f$$

$$g_A \longmapsto \pi^{B'} \nabla^{A}{}_{B'} g_A$$

Tensoring with $\mu^* \mathcal{O}(-m,-n)$ gives

$$\mathcal{O}^{\pi}(-m,-m-n) \longrightarrow \mathcal{O}^{\pi}_{A}(-1-m,1-m-n)[-1]' \longrightarrow \mathcal{O}^{\pi}(-2-m,2-m-n)[-2]'[-1]$$

Write $\mathcal{O}_{A'}(p)$ for the sheaf of spinor valued functions on $H$ homogeneous of degree $p$ in $x^{AA'}$. A simple calculation gives $\mathcal{V}^1_* \mu^* \mathcal{O}(-m,-n) = \mathcal{O}_{\underbrace{(A' \ldots D')}_{m+n-2}}(-m)[-1]'$

and the first direct image of the relative De Rahm sequence is

$$\mathcal{O}_{(A' \ldots D')}(-m)[-1]' \xrightarrow{\nabla_A{}^{D'}} \mathcal{O}_{A(A' \ldots C')}(-1-m)[-2]' \xrightarrow{\nabla^{AA'}} \mathcal{O}_{(B' \ldots C')}(-2-m)[-3]'[-1]$$

Now, running through the spectral sequence arguments of [2], we arrive at the conclusion that, for $\mathcal{U}$ regular and $m+n>2$

$$H^1(\mathcal{U}'', \mathcal{O}(-m,-n)) \cong \ker \nabla_A{}^{D'} : \Gamma(\mathcal{U}, \mathcal{O}_{(A' \ldots D')}(-m)[-1]') \to \Gamma(\mathcal{U}, \mathcal{O}_{A(A' \ldots C')}(-1-m)[-2]')$$

- (2.1)

As in chapter 1, the field has a contour integral description.

Explicitly, if $f(\omega_A, \pi_{A'})$ is a function homogeneous of degree $-m, -n$ in $\omega_A$, $\pi_{A'}$ representing the cohomology class, then the contour integral for the field on $H$ is

$$\phi_{\underbrace{A'\ldots D'}_{m+n-2}}(x) = \oint \pi_{A'}\ldots\pi_{D'} f(x_A{}^{A'}\pi_{A'}, \pi_{B'})\, \pi^{E'} d\pi_{E'}$$

Now consider what happens if, instead we use $\omega_A$ as the $\mathbb{P}_1$ coordinate on $C$. $d\mu$ becomes $\omega^A \nabla_{AA'}$ and we have $\Omega^1_\mu = \mathcal{O}^\omega_{A'}(-1,1)[-1]$, $\Omega^2_\mu = \mathcal{O}^\omega(-2,2)[-2][-1]'$. Note that there are isomorphisms with the previous expressions e.g.

$$\mathcal{O}^\omega_{A'}(-1,1)[-1] \cong \mathcal{O}^\pi_A(-1,1)[-1]'$$

$$f_{A'}(x^{BB'}, \omega_B) \longmapsto \frac{\varepsilon_{DA} x^{DA'} f_{A'}(x^{BB'}, \varepsilon_{CB} x^{CB'} \pi_{B'})}{\varepsilon_{EF}\varepsilon_{E'F'} x^{EE'} x^{FF'}}$$

The relative De Rahm sequence looks like

$$\mathcal{O} \xrightarrow{\omega^A \nabla_{AA'}} \mathcal{O}^\omega_{A'}(-1,1)[-1] \xrightarrow{\omega^A \nabla_A{}^{A'}} \mathcal{O}^\omega(-2,2)[-2][-1]'$$

and the E.P.W. arguments give us the result that, provided $m+n > 2$ and $U$ is regular

$$H^1(U'', \mathcal{O}(-m,-n)) \cong \ker \nabla^D_{A'} : \Gamma(U, \mathcal{O}_{(A\ldots D)}(-n)[-1]) \to \Gamma(U, \mathcal{O}_{A'(A\ldots C)}(-1-n)[-2])$$

The field is given by a contour integral expression:

$$\phi_{A\ldots D}(x) = \oint \omega_A \ldots \omega_D f(\omega_B, x^{C}{}_{C'}\omega_C)\, \omega^E d\omega_E .$$

For $m+n = 2$, as in the twistor case, the situation is slightly different and we have

$$H^1(U'', \mathcal{O}(-m,-n)) \cong \ker \Box : \Gamma(U, \mathcal{O}(-m)[-1]') \longrightarrow \Gamma(U, \mathcal{O}(-m-2)[-2]'[-1]) \qquad (2.2)$$

$$\cong \ker \Box : \Gamma(U, \mathcal{O}(-n)[1]) \longrightarrow \Gamma(U, \mathcal{O}(-n-2)[2][-1]')$$

For $m+n=1$ , we have

$$H^1(U'', \mathcal{O}(-m,-n)) \cong \ker \nabla^{AA'} : \Gamma(U, \mathcal{O}_A(-1-m)[-1]') \longrightarrow \Gamma(U, \mathcal{O}^{A'}(-2-m)[-2]'[1]) \qquad (2.3)$$

For $m+n \geq 0$ , the conclusion is

$$\cong \ker \nabla^{AA'} : \Gamma(U, \mathcal{O}_{A'}(-1-n)[-1]) \longrightarrow \Gamma(U, \mathcal{O}^{A}(-2-n)[-2][-1]')$$

$$H^1(U'', \mathcal{O}(m,n)) \cong \frac{\ker \nabla^{AN'} : \Gamma(U, \mathcal{O}_A^{(A'\ldots L'M')}(m-1)[-1]') \longrightarrow \Gamma(U, \mathcal{O}^{(A'\ldots M'N')}(m-2)[-2]'[1])}{\operatorname{Im} \nabla_A^{M'} : \Gamma(U, \mathcal{O}^{(A'\ldots L')}(m)) \longrightarrow \Gamma(U, \mathcal{O}_A^{(A'\ldots L'M')}(m-1)[-1]')} \qquad (2.4)$$

( provided $\nabla_A^{M'} : H^1(U, \mathcal{O}^{(A'\ldots L')}(m)) \longrightarrow H^1(U, \mathcal{O}_A^{(A'\ldots L'M')}[-1]')$ is injective )

$$\cong \frac{\ker \nabla^{NA'} : \Gamma(U, \mathcal{O}_{A'}^{(A\ldots LM)}(n-1)[-1]) \longrightarrow \Gamma(U, \mathcal{O}^{(A'\ldots L'M')}(n-2)[-2][1]')}{\operatorname{Im} \nabla^{M}{}_{A'} : \Gamma(U, \mathcal{O}^{(A\ldots L)}(n)) \longrightarrow \Gamma(U, \mathcal{O}_{A'}^{(A\ldots LM)}(n-1)[-1])}$$

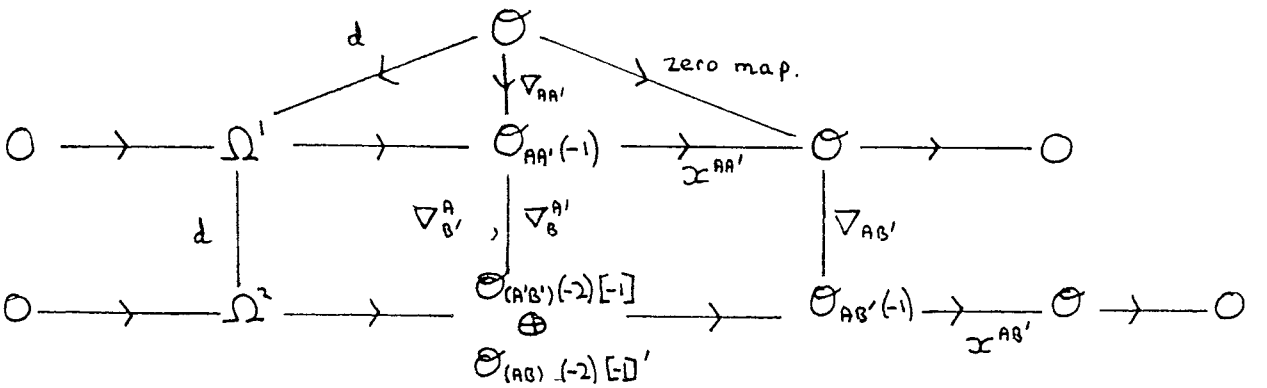
( provided $\nabla_{A'}^{M} : H^1(U, \mathcal{O}^{(A\ldots L)}(n)) \longrightarrow H^1(U, \mathcal{O}_{A'}^{(A\ldots LM)}[-1]')$ is injective )

## §2.4 The Bogomolny Equations

Recall that in the time translation case (§1.6), the group $H^1(U'', \mathcal{O})$ was shown to correspond to solutions $(\underline{A}, \phi)$ of the Bogomolny equations $\operatorname{curl} \underline{A} = -i \underline{\nabla} \phi$ on the subset $U$ of three-space. The analogues of these equations on $H$ will be

$$* d\omega = d\phi \qquad (2.5)$$

where $\omega$ & $\phi$ are respectively a one-form and scalar on $H$. Forms on $H$ are defined in terms of homogeneous forms on $M$ by the following diagram




Thus $\omega$ is a one form $\omega_{AA'}(x)$ on $M$, homogeneous of degree $-1$ and satisfying $x^{AA'}\omega_{AA'} = 0$, and $\phi$ is a function on $M$ homogeneous of degree zero.

The two form $d\omega$ is represented by the pair

$$(\nabla^C_{(A'}\omega_{B')C}\ ,\ \nabla_{(A}{}^{C'}\omega_{B)C'})$$

which satisfy $x_A{}^{A'}\nabla^C_{(A'}\omega_{B')C} = x_B{}^{B'}\nabla_{(A}{}^{C'}\omega_{B)C'}$.

* $d\omega$ is a one form, which is given by

$$2i\, x_A{}^{A'}\nabla^C_{(A'}\omega_{B')C} \quad \text{or} \quad 2i\, x_B{}^{B'}\nabla_{(A}{}^{C'}\omega_{B)C'}.$$

The Bogomolny equations (2.5) on $H$ can therefore be written as

$$2i\, x_A{}^{A'}\nabla^C_{(A'}\omega_{B')C} = \nabla_{AB'}\phi \qquad (2.6)$$

$$\text{or} \quad 2i\, x_B{}^{B'}\nabla_{(A}{}^{C'}\omega_{B)C'} = \nabla_{AB'}\phi \qquad (2.7)$$

and a solution of them is a pair $(\omega_{AA'}, \phi)$ up to the gauge transformation $\omega_{AA'} \mapsto \omega_{AA'} + \nabla_{AA'}f$ where $f$ is homogeneous

of degree zero. Each of (2.6) and (2.7) looks like four equations, but in reality there are only three since contracting with $x^{AB'}$ gives the identity zero equals zero.

Suppose now we are given an element of $H^1(U', \mathcal{O}(0,0))$ so, according to (2.4) we have a potential $\psi_{AA'}$ homogeneous of degree $-1$ and satisfying $\nabla_{C(A'}\psi_{B')}{}^{C} = 0$ up to gauge. We construct a solution $(\omega, \phi)$ of the Bogomolny equations as follows:

Set $$\phi = -i x^{AA'}\psi_{AA'} = -i(x.\psi)$$

$$\omega_{AA'} = \psi_{AA'} - \frac{(x.\psi)}{(x.x)} x_{AA'} .$$

Then $\phi$ is homogeneous of degree zero, $\omega_{AA'}$ is a one-form on $H$, and

$$\begin{aligned} 2i x_A{}^{A'} \nabla_{(A'}{}^{C} \omega_{B')C} &= -2i x_A{}^{A'} x_{C(A'} \nabla_{B')}{}^{C} \Theta \qquad \left(\Theta = \frac{(x.\psi)}{(x.x)}\right) \\ &= -2i x_A{}^{A'} (x_{CA'} \nabla^{C}_{B'} \Theta - x_{C[A'} \nabla_{B']}{}^{C} \Theta) \\ &= -i(x.x)\nabla_{AB'}\Theta + i x_A{}^{A'} (x.\nabla)\Theta\, \varepsilon_{A'B'} \\ &= -i(x.x)\nabla_{AB'}\Theta - 2i x_{AB'}\Theta \\ &= \nabla_{AB'}\phi . \end{aligned}$$

as required. Conversely, one can construct a homogeneous potential modulo gauge from a solution of the Bogomolny equations by setting

$$\psi_{AA'} = \omega_{AA'} + i\frac{\phi\, x_{AA'}}{(x.x)} .$$

## §2.5 Some Useful Isomorphisms

Recall that in the time-translation minitwistor case we were able to identify the positive homogeneity cohomology with either $H^1(\mathcal{O}(-1))$ or $H^1(\mathcal{O}(-2))$ depending on whether

the homogeneity was odd or even. The analogous procedure here gives a slightly different result:

We first define an operator $\eth^k$ in the $\pi_{A'}$ variables on $Q$ by requiring that

$$0 \to \mathcal{O}(m,-1-k)[-k]' \xrightarrow{\pi^{A'}\dots\pi^{K'}} \mathcal{O}^{(A'\dots K')}(m,-1) \xrightarrow{\pi_{A'}} \mathcal{O}^{(B'\dots K')}(m,0) \to 0$$

$$\eth^k \nwarrow \quad \nearrow \frac{\partial}{\partial\pi_{A'}}\dots\frac{\partial}{\partial\pi_{K'}}$$

$$\mathcal{O}(m,k-1)$$

commutes. $\eth^k$ then gives rise to a short exact sequence

$$0 \to \rho^{-1}\mathcal{O}^{(A'\dots J')}(m) \xrightarrow{\pi_{A'}\dots\pi_{J'}} \mathcal{O}(m,k-1) \xrightarrow{\eth^k} \mathcal{O}(m,-1-k)[k]' \to 0$$

(Here $\rho : Q \to \mathbb{P}_1$ is the projection $(\omega_A, \pi_{A'}) \to \omega_A$). This leads to a long exact sequence

$$\to H^1(U'', \rho^{-1}\mathcal{O}^{(A'\dots J')}(m)) \to H^1(U'', \mathcal{O}(m,k-1)) \to H^1(U'', \mathcal{O}(m,-1-k)[-k]') \to H^2(U'', \rho^{-1}\mathcal{O}^{(A'\dots J')}(m))$$

Now, assuming $\mathcal{U}$ is topologically trivial, then the fibres of $\rho| : U'' \to \mathbb{P}_1$ will also be topologically trivial and we can use Buchdahl's theorem [1] to deduce that $H^p(U'', \rho^{-1}\mathcal{O}^{(A'\dots J')}(m)) \cong H^p(\mathbb{P}_1, \mathcal{O}^{(A'\dots J')}(m))$ $p=1,2$. Since this vanishes for $m \geqslant -1$, we have

$$H^1(U'', \mathcal{O}(m,k-1)) \cong H^1(U'', \mathcal{O}(m,-1-k)[k]') \qquad m \geqslant -1 \qquad (2.9)$$

Doing a similar trick with the $\omega_A$ variables we deduce

$$H^1(U'', \mathcal{O}(k-1,n)) \cong H^1(U'', \mathcal{O}(-1-k,n)[k]) \qquad n \geqslant -1.$$

## §2.6 Relationship with Twistor Cohomology

Consider now the situation with which we started out i.e. $\mathbb{PT}^{0I}$ fibred by the dilation vector field over

the quadric. Denote this $\mathbb{C}^*$ fibration by $\tau : \mathbb{PT}^{or} \to Q$. We seek a relation between $H^1(U'', \mathcal{O}(V))$ and $H^1(\tau^{-1}U'', \tau^{-1}\mathcal{O}(V))$ where $V$ is some holomorphic vector bundle on $U'' \subset Q$. The tool we use to investigate this is the hypercohomology spectral sequence. If $0 \to \mathcal{S} \to \mathcal{R}^\cdot$ is a resolution of the sheaf $\mathcal{S}$ on $\tau^{-1}U''$ then there is a spectral sequence

$$E_1^{p,q} = H^q(\tau^{-1}U'', \mathcal{R}^p) \Rightarrow H^{p+q}(\tau^{-1}U'', \mathcal{S})$$

From this, we deduce a short exact sequence

$$0 \longrightarrow E_\infty^{1,0} \longrightarrow H^1(\tau^{-1}U'', \mathcal{S}) \longrightarrow E_\infty^{0,1} \longrightarrow 0$$

which, since the spectral sequence is first quadrant, is given by

$$0 \longrightarrow E_2^{1,0} \longrightarrow H^1(\tau^{-1}U'', \mathcal{S}) \longrightarrow E_3^{0,1} \longrightarrow 0$$

Chasing through the definitions, we see that this means the sequence

$$0 \longrightarrow \frac{\ker\left[\Gamma(\tau^{-1}U'', R^1) \to \Gamma(\tau^{-1}U'', R^2)\right]}{\mathrm{Im}\left[\Gamma(\tau^{-1}U'', R^0) \to \Gamma(\tau^{-1}U'', R^1)\right]} \longrightarrow H^1(\tau^{-1}U'', \mathcal{S}) \longrightarrow \tag{2.10}$$

$$\ker\left[\left(\ker\left[H^1(\tau^{-1}U'', R^0) \to H^1(\tau^{-1}U'', R^1)\right]\right) \to \left(\frac{\ker\left[\Gamma(\tau^{-1}U'', R^2) \to \Gamma(\tau^{-1}U'', R^3)\right]}{\mathrm{Im}\left[\Gamma(\tau^{-1}U'', R^1) \to \Gamma(\tau^{-1}U'', R^2)\right]}\right)\right] \to 0$$

is exact. We now apply this to the Dolbeault resolution

$$0 \to \tau^{-1}\mathcal{O}(V) \to \tau^{-1}\mathcal{E}^{0,0}(V) \to \tau^{-1}\mathcal{E}^{0,1}(V) \to \ldots$$

of $\tau^{-1}\mathcal{O}(V)$. The fibres of $\tau$ are connected, so

$$\Gamma(\tau^{-1}U'', \tau^{-1}\mathcal{E}^{0,0}(V)) \cong \Gamma(U'', \mathcal{E}^{0,p}(V)).$$

The first term in (2.10) thus becomes

$$\frac{\ker\left[\Gamma(U'', \Xi^{0,1}(V)) \to \Gamma(U'', \Xi^{0,2}(V))\right]}{\mathrm{Im}\left[\Gamma(U'', \Xi^{0,0}(V)) \to \Gamma(U'', \Xi^{0,1}(V))\right]} . \qquad (2.11)$$

However, since the resolution

$$0 \longrightarrow \mathcal{O}(V) \longrightarrow \Xi^{0,0}(V) \longrightarrow \Xi^{0,1}(V) \longrightarrow \cdots$$

is acyclic, (2.11) is precisely $H^1(U'', \mathcal{O}(V))$ . Doing a similar thing to the last term in (2.10), the sequence becomes

$$0 \longrightarrow H^1(U'', \mathcal{O}(V)) \longrightarrow H^1(\tau^{-1}U'', \tau^{-1}\mathcal{O}(V)) \qquad (2.12)$$

$$\longrightarrow \ker\left[\left(\ker\left[H^1(\tau^{-1}U'', \tau^{-1}\Xi^{0,0}(V)) \to H^1(\tau^{-1}U'', \tau^{-1}\Xi^{0,1}(V))\right] \to H^2(U'', \mathcal{O}(V))\right] \to 0$$

The last term can be tidied up. First, we identify $H^1(\tau^{-1}U'', \tau^{-1}\Xi^{0,0}(V))$ . Write $\Xi^{0,0}(V) = \Xi(V)$ . There is a resolution

$$0 \longrightarrow \tau^{-1}\Xi(V) \longrightarrow \Xi(\tau^*V) \longrightarrow \Xi^1_\tau(\tau^*V) \longrightarrow 0$$

and we have $H^1(\tau^{-1}U'', \tau^{-1}\Xi(V)) = \dfrac{\Gamma(\tau^{-1}U'', \Xi^1_\tau(\tau^*V))}{\mathrm{Im}\, d_\tau : \Gamma(\tau^{-1}U'', \Xi(\tau^*V)) \to \Gamma(\tau^{-1}U'', \Xi^1_\tau(\tau^*V))}$

If we consider ordinary (i.e. not bundle-valued) relative forms for the moment, then we can write down an exact sequence:

$$0 \longrightarrow \left\{\begin{matrix} d_\tau\text{-exact relative} \\ \text{one-forms}\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{relative one} \\ \text{-forms}\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{functions on} \\ U''\end{matrix}\right\} \longrightarrow 0$$

$$0 \longrightarrow \mathrm{Im}\, d_\tau : \left(\Gamma(\tau^{-1}U'', \Xi) \to \Gamma(\tau^{-1}U'', \Xi^1_\tau)\right) \longrightarrow \Gamma(\tau^{-1}U'', \Xi^1_\tau) \longrightarrow \Gamma(U'', \Xi) \longrightarrow 0$$

$$\omega \longrightarrow \oint \omega$$

where the integral is taken round a closed path encircling the origin in the fibres of $\tau$ . We therefore deduce

$$H^1(\tau^{-1}U'', \tau^{-1}\Xi(V)) = \Gamma(U'', \Xi).$$

The same arguments work for $V$- valued forms and we have

$$H^1(\tau^{-1}U'', \tau^{-1}\mathcal{E}(V)) = \Gamma(U'', \mathcal{E}(V))$$

This immediately suggests a way of identifying $H^1(\tau^{-1}U'', \tau^{-1}\mathcal{E}^{0,1}(V))$ also:

$$\begin{aligned} H^1(\tau^{-1}U'', \tau^{-1}\mathcal{E}^{0,1}(V)) &= H^1(\tau^{-1}U'', \tau^{-1}\mathcal{E}(V\otimes T^{0,1})) \\ &= \Gamma(U'', \mathcal{E}(V\otimes T^{0,1})) \\ &= \Gamma(U'', \mathcal{E}^{0,1}(V)). \end{aligned}$$

In fact, in (2.12), the map $H^1(\tau^{-1}U'', \tau^{-1}\mathcal{E}^{0,0}(V)) \to H^1(\tau^{-1}U'', \tau^{-1}\mathcal{E}^{0,1}(V))$ is precisely

$$\Gamma(U'', \mathcal{E}(V)) \xrightarrow{\bar\partial} \Gamma(U'', \mathcal{E}^{0,1}(V))$$

and its kernel is $\Gamma(U'', \mathcal{O}(V))$ . The last term in the exact sequence (2.12) is thus $\ker[\Gamma(U'', \mathcal{O}(V)) \to H^2(U'', \mathcal{O}(V))]$ . Now $U''$ is a non-compact subset of the two-manifold $Q$ , hence by a result of Malgrange [9], we have $H^2(U'', \mathcal{O}(V)) = 0$. The sequence (2.12) is now:

$$0 \to H^1(U'', \mathcal{O}(V)) \to H^1(\tau^{-1}U'', \tau^{-1}\mathcal{O}(V)) \to \Gamma(U'', \mathcal{O}(V)) \to 0 \qquad (2.13)$$

and this provides the relationship between minitwistor and twistor cohomology.

## §2.7 Interpretation of the Exact Sequence

We now attempt to interpret (2.13) in terms of fields on space-time, for the homogeneous bundles $V = \mathcal{O}(-n,-n)$. §2.3 has already provided the interpretation of $H^1(U'', \mathcal{O}(-n,-n))$ in terms of fields which are sections of the homogeneous bundles $\mathcal{O}_{(A'\ldots D')}(-n)[-1]'$ etc. on $U \subset H$ . $U$ is the projectivisation of some region $\tau^{-1}U \subset \mathbb{M}^{oI}$ by the dilation vector field, and the fields on $U$ correspond exactly to zero rest

mass fields $\phi_{A'\ldots D'} \in \Gamma(\mathbb{T}^{-1}\mathcal{U}, \mathcal{O}(A'\ldots D')[-1]')$ which have the homogeneity property

$$\phi_{A'\ldots D'}(\lambda x^{CC'}) = \lambda^{-m}\phi_{A'\ldots D'}(x^{CC'}).$$

To interpret $H^1(\mathbb{T}^{-1}\mathcal{U}'', \mathbb{T}^{-1}\mathcal{O}(-m,-n))$ we use the short exact sequence on $\mathbb{T}^{-1}\mathcal{U}''$ : $0 \to \mathbb{T}^{-1}\mathcal{O}(-m,-n) \to \mathcal{O}(-m,-n) \to \mathcal{O}(-m,-n) \to 0$ (2.14)

The topological inverse image, $\mathbb{T}^{-1}\mathcal{O}(-m,-n)$ is exactly the sheaf of functions on $\mathbb{T}^{-1}\mathcal{U}''$ with the homogeneity property $f(\lambda\omega_A, \mu\pi_{A'}) = \lambda^{-m}\mu^{-n} f(\omega_A, \pi_{A'})$ . There are two obvious possibilities for the map $\mathcal{O}(-m,-n) \to \mathcal{O}(-m,-n)$ which make the sequence exact, namely

$$d_\omega := \omega_A \frac{\partial}{\partial \omega_A} + m \quad ; \quad d_\pi := \pi_{A'} \frac{\partial}{\partial \pi_{A'}} + n$$

A portion of the long exact sequence on cohomology is, using $d_\omega$ :

$$H^0(\mathcal{O}(-m,-n)) \xrightarrow{d_\omega} H^0(\mathcal{O}(-m,-n)) \to H^1(\mathbb{T}^{-1}\mathcal{O}(-m,-n)) \to H^1(\mathcal{O}(-m,-n)) \xrightarrow{d_\omega} H^1(\mathcal{O}(-m,-n))$$

For $-m-n<0$ , the $H^0$'s vanish and we obtain

$$H^1(\mathbb{T}^{-1}\mathcal{U}'', \mathbb{T}^{-1}\mathcal{O}(-m,-n)) \cong \ker d_\omega : H^1(\mathbb{T}^{-1}\mathcal{U}'', \mathcal{O}(-m,-n)) \to H^1(\mathbb{T}^{-1}\mathcal{U}'', \mathcal{O}(-m,-n)).$$

This map on twistor cohomology gives rise to the following map on space-time fields [4]:

$$\phi_{\underbrace{B'\ldots D'}_{m+n-2}} \to (x^{AA'}\nabla_{AA'} + m)\,\phi_{B'\ldots D'} \qquad m+n \geq 2.$$

Thus $H^1(\mathbb{T}^{-1}\mathcal{U}'', \mathbb{T}^{-1}\mathcal{O}(-m,-n))$ is exactly the set of zero rest mass fields with $m+n-2$ indices, homogeneous of degree $m$ in the space-time coordinates, and is therefore isomorphic to $H^1(\mathcal{U}'', \mathcal{O}(-m,-n))$ . This agrees with the exact sequence (2.13) since $\Gamma(\mathcal{U}'', \mathcal{O}(-m,-n))$ certainly vanishes for $m+n \geq 2$ .

The situation is summarised in the following diagram;

for $m+n \geq 2$

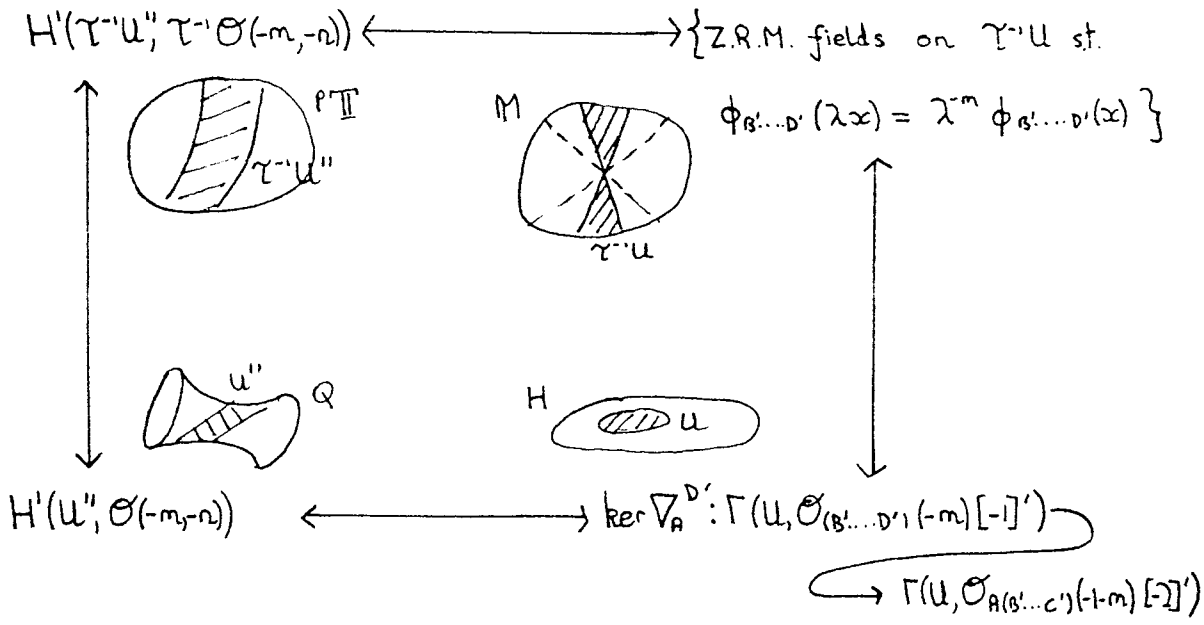



Fig. 2.3

For $m+n = 1$, we have $H^1(\mathbb{T}^{-1}U'', \mathbb{T}^{-1}\mathcal{O}(m,n))$ given by $\ker d_\omega$: $H^1(\mathbb{T}^{-1}U'', \mathcal{O}(-1)) \to H^1(\mathbb{T}^{-1}U'', \mathcal{O}(-1))$, which, in space-time terms is neutrino fields satisfying

$$\left(x^{AA'}\nabla_{AA'} + (m+1)\right)\phi_B = 0$$

i.e. homogeneous of degree $-m-1$ ([4]). The diagram still commutes since $H^1(U'', \mathcal{O}(-m,-n))$ is now given by (2.3).

For the positive homogeneity cases, the situation is slightly different. For simplicity, we will consider the case $m=n=0$. The exact sequence (2.13) becomes

$$0 \to H^1(U'', \mathcal{O}(0,0)) \to H^1(\mathbb{T}^{-1}U'', \mathbb{T}^{-1}\mathcal{O}(0,0)) \to \Gamma(U'', \mathcal{O}(0,0)) \to 0$$

i.e.

$$0 \to H^1(U'', \mathcal{O}(0,0)) \to H^1(\mathbb{T}^{-1}U'', \mathbb{T}^{-1}\mathcal{O}(0,0)) \to \mathbb{C} \to 0 \qquad (2.15)$$

We can deduce another short exact sequence containing $H^1(\mathcal{T}^{-1}\mathcal{U}, \mathcal{T}^{-1}\mathcal{O}(0,0))$ by considering (2.14), which becomes

$$0 \longrightarrow \mathcal{T}^{-1}\mathcal{O}(0,0) \longrightarrow \mathcal{O} \xrightarrow{\omega_A \frac{\partial}{\partial \omega_A}} \mathcal{O} \longrightarrow 0$$

From the corresponding long exact sequence, we deduce exactness of

$$0 \longrightarrow \mathbb{C} \longrightarrow H^1(\mathcal{T}^{-1}\mathcal{U}, \mathcal{T}^{-1}\mathcal{O}(0,0)) \longrightarrow \left(\ker \omega_A \frac{\partial}{\partial \omega_A} : H^1(\mathcal{T}^{-1}\mathcal{U}, \mathcal{O}) \circlearrowright\right) \longrightarrow 0 \qquad (2.16)$$

Elements of $H^1(\mathcal{T}^{-1}\mathcal{U}, \mathcal{O})$ are anti self-dual Maxwell potentials modulo gauge on $\mathcal{T}^{-1}\mathcal{U} \subset \mathbb{M}^{OI}$, and the effect of $\omega_A \frac{\partial}{\partial \omega_A}$ is

$$H^1(\mathcal{T}^{-1}\mathcal{U}, \mathcal{O}) \longrightarrow H^1(\mathcal{T}^{-1}\mathcal{U}, \mathcal{O})$$
$$\psi_{AA'} \longmapsto x^B{}_{A'} \nabla_{C'(A} \psi_{B)}{}^{C'}$$

Hence

$$\ker \omega_A \frac{\partial}{\partial \omega_A} = \left\{ \psi_{AA'} \;\middle|\; \begin{array}{l} \psi_{AA'} \text{ an A.S.D. potential,} \\ x^B{}_{A'} \nabla_{C'(A} \psi_{B)}{}^{C'} = \nabla_{AA'} \chi \\ \text{for some } \chi \end{array} \right\} / \text{gauge.}$$

N.B. the relation

$$x^B{}_{A'} \nabla_{C'(A} \psi_{B)}{}^{C'} = \nabla_{AA'} \chi$$

implies that $\chi$ is :

1) homogeneous of degree zero

2) a solution of the wave equation $\Box \chi = 0$

Given $\psi_{AA'}$, $\chi$ is determined up to a constant and we have another short exact sequence

$$0 \longrightarrow \mathbb{C} \longrightarrow \left\{ \psi_{AA'} \;\middle|\; \begin{array}{l} \psi_{AA'} \text{ an A.S.D. pot}^{\text{l}}, \\ x^B{}_{A'} \nabla_{C'(A} \psi_{B)}{}^{C'} = \nabla_{AA'} \chi \\ \text{for some } \chi \end{array} \right\} / \text{gauge} \xrightarrow{\gamma} \left\{ \phi \;\middle|\; \begin{array}{l} \phi \text{ hom (0)}, \\ \Box \phi = 0 \end{array} \right\} / \mathbb{C} \longrightarrow 0$$

The kernel of $\gamma$ consists of potentials (modulo gauge) which are constant multiples of

$$x^{AA'}/(x.x) .$$

Such potentials would be pure gauge ($\nabla_{AA'} \ln(x.x)$) were it not for the fact that the logarithm does not make global sense. To check surjectivity of $\gamma$, suppose we are given $\phi$ st $\Box\phi = 0$ and $\phi$ is homogeneous of degree zero. Form

$$x_B^{\ C'}/(x.x)\, \nabla_{AC'}\phi$$

This is an anti self-dual Maxwell field on $\mathcal{T}^{-1}\mathcal{U}$. Such fields are related to potentials modulo gauge by

$$0 \to H^1(\mathcal{T}^{-1}\mathcal{U}, \mathbb{C}) \to \{\text{potentials}\}/\text{gauge} \to \{\text{fields}\} \to H^2(\mathcal{T}^{-1}\mathcal{U}, \mathcal{C}) \to 0$$

(see [2]) i.e.

$$0 \to \mathbb{C} \to \{\text{potentials}\}/\text{gauge} \to \{\text{fields}\} \to 0$$

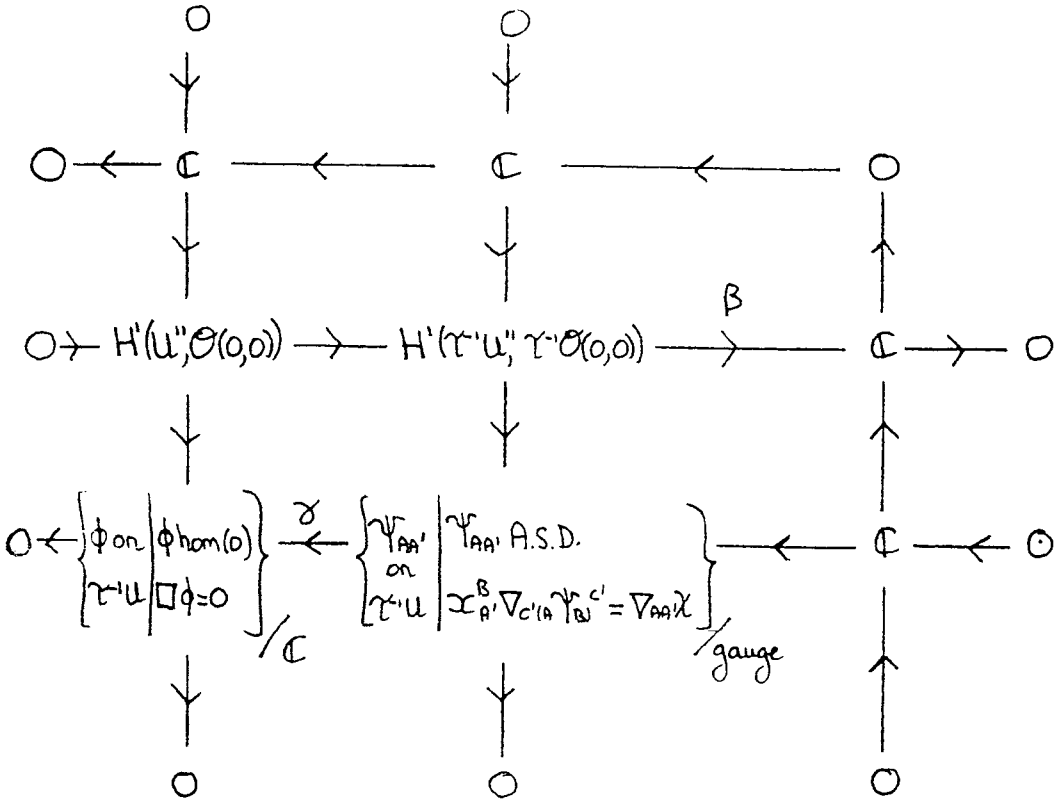
Therefore, there exists $\Psi_{AA'}$ s.t.

$$\nabla_{C'(A}\Psi_{B)}^{\ C'} = x_B^{\ C'}/(x.x)\, \nabla_{AC'}\phi$$

i.e.

$$x_{A'}^{\ B}\, \nabla_{C'(A}\Psi_{B)}^{\ C'} = \nabla_{AC'}\phi$$

as required. The exact sequences (2.15), (2.16) & (2.17) fit together in a diagram




To describe the map $\beta$ , take $\omega \in H^1(\tau^{-1}U'', \tau^{-1}\mathcal{O}(0,0))$ , then, restricting to a fibre of $\tau$ we get an element of $H^1(\mathbb{C}^*, \mathbb{C}) \cong \mathbb{C}$ , and as the fibre varies, we get a function on $U''$ , which must be constant since $U''$ contains rational curves.

# CHAPTER 3 General Minitwistor Spaces

## §3.1 The Hitchin Definition and the Einstein-Weyl Equations

Chapters one and two extensively discussed two minitwistor spaces, each of which was a complex two-manifold with a three dimensional family of rational curves of normal bundle $\mathcal{O}(2)$ . We now briefly review Hitchin's discussion [7] of the general complex two-manifold with this property.

Let $\mathcal{Y}$ be such a manifold and let $\mathcal{H}$ denote the parameter space of the curves. The curve corresponding to $x \in \mathcal{H}$ will be denoted $C_x$ . A tangent vector at $x$ can be thought of as joining $x$ to an infinitesimally perturbed point $x + dx$ . Then $C_x$ is perturbed to $C_{x+dx}$ by a section of the normal bundle of $C_x$ , so

$$T_x \mathcal{H} \cong H^0(C_x, \mathcal{O}(2)) \cong \mathbb{C}^3.$$

Provided the normal bundle of the curves $C_x$ stays fixed, $\mathcal{H}$ will be non-singular. Generically, a global section of the normal bundle of $C_x$ vanishes twice, and the corresponding perturbed curve intersects the original one twice, hence $C_x$ is said to have self intersection number two. There will, however, be elements of $H^0(C_x, \mathcal{O}(2))$ which vanish at one point on $C_x$ to order two. We define a conformal structure on $\mathcal{H}$ by calling the corresponding tangent vectors null.

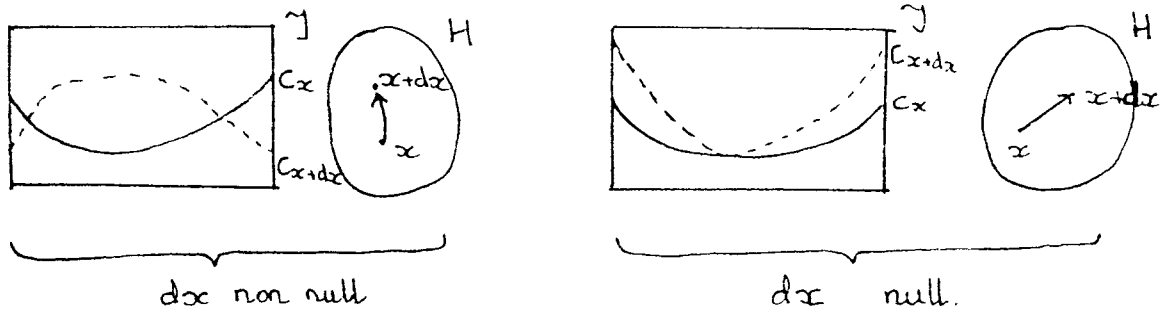



Fig. 3.1

Let $V$ be a tangent vector at $x$. $V$ defines a pair of points on $C_x$ (the ones at which the corresponding section of the normal bundle vanishes), which encode the direction of $V$. There is a one parameter family of curves through these points. This gives rise to a curve in $H$ which starts off in the direction $V$. This procedure singles out a family of curves in $H$ which we call geodesics, thus defining a projective connection on $H$.

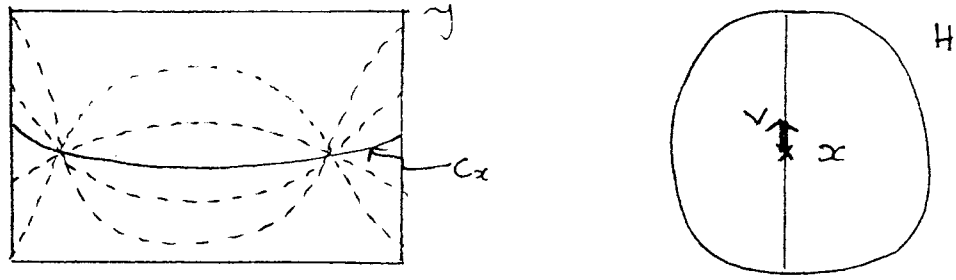



Fig. 3.2

If $V$ was null, the geodesic defined by this procedure clearly remains null, and in this sense, the projective connection is compatible with the conformal structure. Hitchin shows that, since $H$ is three dimensional, one can then produce a unique affine connection which is compatible with the projective structure (i.e. has the same geodesics) and with the conformal structure

(in the sense that $\nabla_i g_{jk} = \omega_i g_{jk}$ for some $\omega_i$ where $g_{jk}$ is any representative metric in the conformal class). A space with such a structure (conformal metric, compatible affine connection) is called a Weyl space.

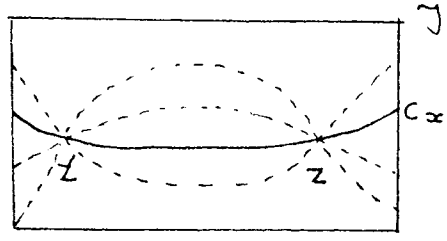
The Weyl spaces $H$ obtained from minitwistor spaces by the above procedure have two extra properties [7]. The affine connection is:

1) Torsion free

2) Such that the Einstein-Weyl equations

$$R_{(ij)} = \Lambda g_{ij} \qquad (3.1)$$

hold, where $g_{ij}$ is any representative metric in the conformal class. Fixing a minitwistor $Z \in \mathcal{T}$ and considering all the curves through it defines for us a two-surface $\gamma_Z \subset H$. Let $X$ be on $\gamma_Z$, then $C_X$ goes through $Z$. Any other point $Z'$ on $C_X$ defines a direction tangent to $\gamma_Z$, through which there passes a geodesic contained within $\gamma_Z$, given by the one parameter



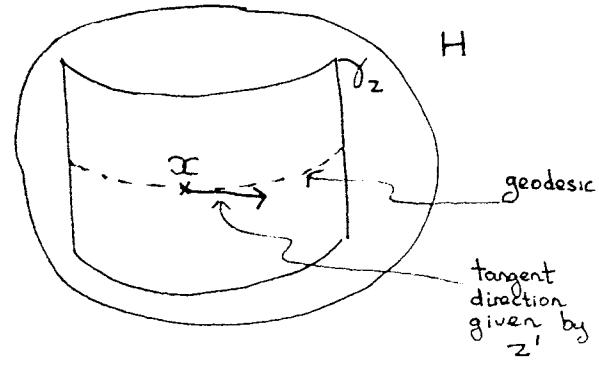




Fig. 3.3

family of curves through $Z$ and $Z'$. Thus, $\gamma_Z$ is totally geodesic. Moreover, there is a null direction tangent to $\gamma_Z$, given when $Z$ and $Z'$ coincide. This gives us a

null geodesic in $\gamma_z$ through $x$ . In this way $\gamma_z$ is ruled by a family of null geodesics.

Minitwistors $Z$ thus correspond to totally geodesic null two surfaces $\gamma_z \subset H$ . Hitchin shows that the Einstein-Weyl equations are the integrability conditions for the distribution which defines them. We thus have an inverse construction - given a solution of the Einstein-Weyl equations, the relevant minitwistor space is simply the space of totally geodesic null two-surfaces.

## §3.2 Conformal Killing Vectors and Weyl Spaces

In this section, we relate a step by step construction which defines a Weyl structure on the factor space of a complex space-time by a conformal Killing vector. Thus let $\mathcal{M}$ be a complex space-time with metric $g_{ab}$ and connection $\nabla_a$ . Suppose $K^a$ is a non-null conformal Killing vector on $\mathcal{M}$ and suppose that $\mathcal{M}$ has been suitably restricted, if necessary, so that $K^a$ does not vanish on $\mathcal{M}_\Lambda$ and the factor space of $\mathcal{M}$ by $K^a$ is non-pathological. Denote by $H$ the space of trajectories of $K^a$ . We now define a Weyl structure on $H$ , starting with the conformal metric.

Let $x \in H$ . Denote the trajectory of $K^a$ over $x$ by $t_x$ . A tangent vector $X \in T_x H$ then corresponds to an orthogonal lie-propagated Jacobi field $J_x^a$ along $t_x$ , i.e.

$$\begin{cases} K_a J_x^a = 0 \\ \mathcal{L}_K J_x^a = 0 \end{cases}$$

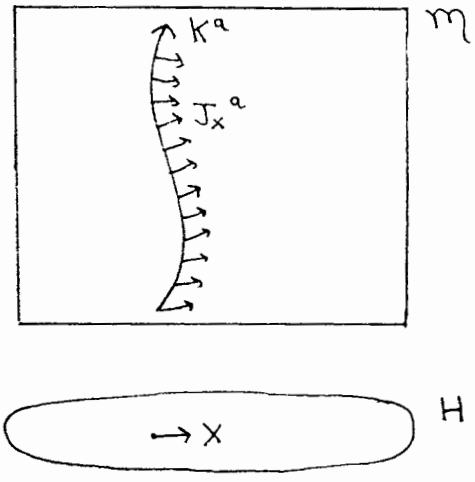



Fig. 3.4

Definition 3.1

Define a conformal metric on $H$ by saying that $X$ is null

$$\Leftrightarrow \quad g_{ab} J_x^a J_x^b = 0$$

N.B. We have $\mathcal{L}_K(g_{ab} J_x^a J_x^b) \propto g_{ab} J_x^a J_x^b$ , so the condition holds everywhere along $t_x$ if it holds at one point.

The next structure we need is a connection on $H$ , i.e. a vector field $D_X Y$ on $H$ , defined for any pair of vector fields $X$ and $Y$ , and satisfying the usual conditions. The first thing one might try is

$$(D_X Y)^b = J_x^a \nabla_a J_Y{}^b$$

For this to give a well defined vector field on $H$ , it should be lie-propagated along $k^a$ . We check to see if this is the case:

$$\begin{aligned}\mathcal{L}_K (D_X Y)^b &= J_x^a \mathcal{L}_K (\nabla_a J_Y)^b \\ &= J_x^a K^c \nabla_c \nabla_a J_Y{}^b + J_x^a (\nabla_c J_Y{}^b)(\nabla_a K^c) \\ &\quad - J_x^a (\nabla_a J_Y{}^c)(\nabla_c K^b) .\end{aligned}$$

Now, $\mathcal{L}_K (J_Y^b) = 0$, $\therefore\ J_X^a \nabla_a (\mathcal{L}_K J_Y^b) = 0$

i.e. $J_X^a \nabla_a (K^c \nabla_c J_Y^b - J_Y^c \nabla_c K^b) = 0$

$$\therefore\ J_X^a K^c \nabla_a \nabla_c J_Y^b + J_X^a (\nabla_a K^c)(\nabla_c J_Y^b) - J_X^a (\nabla_a J_Y^c)(\nabla_c K^b) - J_X^a J_Y^c \nabla_a \nabla_c K^b = 0$$

On subtracting, we get

$$\mathcal{L}_K (D_X Y)^b = J_X^a K^c (\nabla_c \nabla_a - \nabla_a \nabla_c) J_Y^b + J_X^a J_Y^c \nabla_a \nabla_c K^b$$
$$= J_X^a K^c R_{cad}{}^b J_Y^d + J_X^a J_Y^c \nabla_a \nabla_c K^b$$

To clarify the last term, we use

Lemma 3.1

If $K^a$ is a conformal Killing vector, then

$$\nabla_a \nabla_c K_b = R_{dabc} K^d + \frac{1}{2}\{ g_{ab} \nabla_c \lambda + g_{bc} \nabla_a \lambda - g_{ac} \nabla_b \lambda \}$$

where $\lambda = \frac{1}{2} \nabla_c K^c$.

Proof Differentiating the definition of conformal Killing vector in three different index combinations, we have

$$\nabla_c \nabla_a K_b + \nabla_c \nabla_b K_a = \nabla_c (\lambda g_{ab}) \qquad (a)$$
$$\nabla_a \nabla_b K_c + \nabla_a \nabla_c K_b = \nabla_a (\lambda g_{bc}) \qquad (b)$$
$$\nabla_b \nabla_c K_a + \nabla_b \nabla_a K_c = \nabla_b (\lambda g_{ac}) \qquad (c)$$

Then, (a) + (b) - (c) tells us the symmetric derivative of $K_b$ :

$$\nabla_a \nabla_c K_b + \nabla_c \nabla_a K_b = \nabla_c (\lambda g_{ab}) + \nabla_a (\lambda g_{bc}) - \nabla_b (\lambda g_{ac}) - R_{cbda} K^d - R_{abdc} K^d$$

Hence, we have

$$\nabla_a\nabla_c K_b = \nabla_{[a}\nabla_{c]} K_b + \nabla_{(a}\nabla_{c)} K_b$$

$$= \tfrac{1}{2}\{R_{acdb}K^d - R_{cbda}K^d - R_{abdc}K^d + \nabla_a(\lambda g_{bc}) + \nabla_c(\lambda g_{ab}) - \nabla_b(\lambda g_{ac})\}$$

$$= R_{dabc}K^d + \tfrac{1}{2}\{g_{ab}\nabla_c\lambda + g_{bc}\nabla_a\lambda - g_{ac}\nabla_b\lambda\}.$$

as required.

---

Using this lemma, we obtain

$$\mathcal{L}_K(D_XY)^b = J_X^a(R_{cad}{}^b K^c J_Y^d + R_{da}{}^b{}_c K^d J_Y^c) + \tfrac{1}{2}J_X^a J_Y^c(\delta_a^b\nabla_c\lambda + \delta_c^b\nabla_a\lambda - g_{ac}\nabla^b\lambda).$$

The curvature terms cancel and we are left with

$$\mathcal{L}_K(D_XY)^b = \tfrac{1}{2}J_X^a J_Y^c(\delta_a^b\nabla_c\lambda + \delta_c^b\nabla_a\lambda - g_{ac}\nabla^b\lambda).$$

Since this does not vanish, we must modify our definition of $D_XY$ to

$$(D_XY)^b = J_X^a\nabla_a J_Y^b - \gamma_{ac}^b J_X^a J_Y^c \tag{3.2}$$

where $\gamma_{ac}^b$ has been chosen so that

$$\mathcal{L}_K(J_X^a\nabla_a J_Y^b - \gamma_{ac}^b J_X^a J_Y^c) = 0. \tag{3.3}$$

Therefore, we seek $\gamma_{ac}^b$ such that

$$\mathcal{L}_K(\gamma_{ac}^b) = \tfrac{1}{2}(\delta_a^b\nabla_c\lambda + \delta_c^b\nabla_a\lambda - g_{ca}\nabla^b\lambda).$$

This will hold if we put

$$\gamma_{ac}^b = \tfrac{1}{2}\{\delta_a^b\omega_c + \delta_c^b\omega_a - g_{ac}g^{bd}\omega_d\}$$

where $\omega_a$ has been chosen so that

$$\mathcal{L}_K\omega_a = \nabla_a\lambda. \tag{3.4}$$

One choice of $\omega_a$ which works is

$$\omega_a = \nabla_a \ln (K^c K_c).$$

Thus, with these choices, the vector field $D_X Y$ defined by (3.2) is lie-propagated along $K^a$. However to define a unique tangent vector field on $H$, it should also be orthogonal to $K^a$. To ensure this, modify the definition (3.2) by multiplying by the projection operator

$$h^c{}_b = \left(\delta^c_b - K^c K_b/(K.K)\right).$$

which projects onto the subspace orthogonal to $K^a$. This does not affect the property (3.3), since

$$\begin{aligned} \mathcal{L}_K h^c{}_b &= -\mathcal{L}_K \left(K^c K_b/(K.K)\right) \\ &= -K^c \mathcal{L}_K \left(K_b/(K.K)\right) \\ &= -K^c \left\{ 1/(K.K)\, \mathcal{L}_K K_b - K_b/(K.K)^2\, K^c \mathcal{L}_K K_c \right\} \\ &= 0 \quad \text{since } \mathcal{L}_K K_b = \bar{\lambda} K_b. \end{aligned}$$

We now have a well-defined connection $D$ on $H$. We must check that it is compatible with the conformal metric. This amounts to showing that a geodesic which starts out in a null direction remains null. Thus, let $J_X^a$ be the Jacobi field on $\mathcal{M}$ corresponding to the tangent vector field $X$ of a geodesic on $H$. Suppose that the geodesic is affinely parameterised, so that

$$J_X^a D_a J_X^b = 0.$$

We then check how the norm of $J_X$ changes along the geodesic:

$$
\begin{aligned}
& J_x^a D_a (J_x . J_x) \\
&= J_x^a J_x^b J_x^c D_a g_{bc} \\
&= J_x^a J_x^b J_x^c h_a^e h_b^f h_c^g (\nabla_e g_{fg} + \gamma^d_{ef} g_{dg} + \gamma^d_{eg} g_{fd}) \\
&= J_x^a J_x^b J_x^c h_a^e h_b^f h_c^g \omega_e g_{fg} \\
&= (J_x^a h_a^e \omega_e) J_x^b J_x^c \left(\delta_b^f - \frac{K^f K_b}{(K.K)}\right)\left(\delta_c^g - \frac{K^g K_c}{(K.K)}\right) g_{fg} \\
&= (J_x^a h_a^e \omega_e)(J_x . J_x) \qquad (\text{since } K . J_x = 0)
\end{aligned}
$$

i.e. the derivative of the norm along the geodesic is proportional to the norm. Therefore, if the norm vanishes at one point on the geodesic it vanishes everywhere and the result is proven.

In summary we have proved,

Lemma 3.2

Let $\mathcal{M}$ be a complex space-time with metric $g_{ab}$, connection $\nabla_a$ and conformal Killing vector $K^a$. Then the space of trajectories $H$ can be given the structure of a Weyl space as follows:

Tangent vectors $X$ to $H$ correspond to orthogonal lie propagated Jacobi fields $J_X$ along $K^a$. Define a conformal metric by saying that $X$ is null $\Leftrightarrow$ $g_{ab} J_x^a J_x^b = 0$.

Define a connection on $H$ by

$$J_x^a D_a J_y^b = h_c^b (J_x^a \nabla_a J_y^c - \gamma^c_{ad} J_x^a J_y^d) \qquad (3.5)$$

where

$$h^b_c = (\delta^b_c - K^b K_c/(K.K))$$

$$\gamma^b_{ac} = \tfrac{1}{2}(\delta^c_a \omega_d + \delta^c_d \omega_a - g_{ad}\omega^c) \qquad (3.6)$$

and

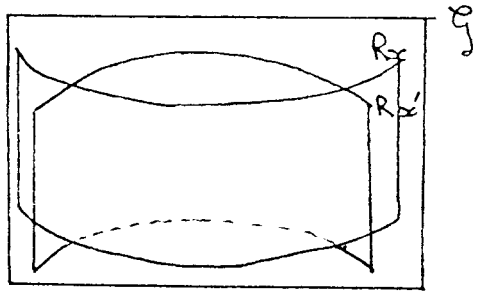
$$\omega_a = \nabla_a \ln(K.K).$$

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§3.3 Relationship with Twistor Theory - the case where $K^a$ is Hypersurface-Orthogonal

Suppose now that our space-time $\mathcal{M}$ is right-flat. By Penrose's "non linear graviton" construction [12], then, $\mathcal{M}$ correspond to a twistor space $\mathcal{Q}$. Since the conformal structure of $\mathcal{M}$ is encoded into the complex manifold structure of $\mathcal{Q}$, then, since $K^a$ is an infinitesimal conformal motion of $\mathcal{M}$, it induces a vector field $V$ on $\mathcal{Q}$. Denote by $\mathcal{Y}$ the space of integral curves of $V$.

We now show that $\mathcal{Y}$ is a minitwistor space, i.e. has a family of rational curves with normal bundle $\mathcal{O}(2)$. Let $x$ and $x'$ be infinitesimally separated points in $H$, corresponding to trajectories $t_x$ and $t_{x'}$ in $\mathcal{M}$. $t_x$ and $t_{x'}$ correspond to a pair of ruled surfaces $R_x$ and $R_{x'}$ in the twistor space $\mathcal{Q}$.



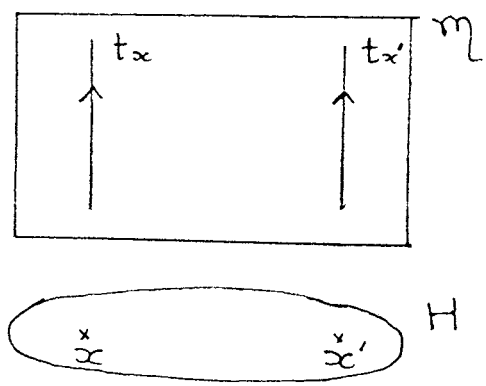
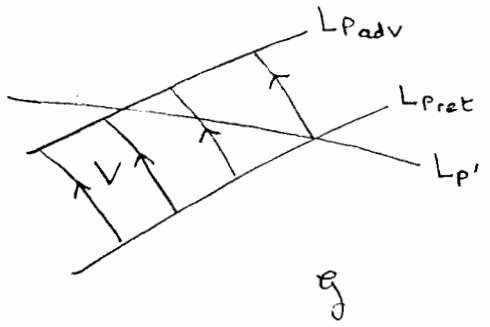




Fig 3.5

The way $R_x$ and $R_{x'}$ intersect is determined as follows: Suppose $x$ and $x'$ are not null separated. Take a point $p'$ on $t_{x'}$, then $p'$ is null separated from two points $p_{adv}$ and $p_{ret}$ on $t_x$. Therefore, in twistor space, the line $L_{p'}$ on $R_{x'}$ intersects the two lines $L_{p_{adv}}$ and $L_{p_{ret}}$ on $R_x$.



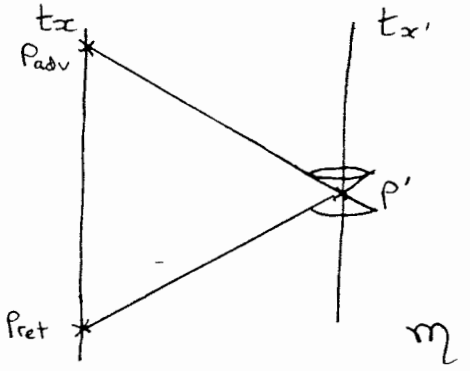




Fig. 3.6

This happens for each $p'$ on $t_x$ and the intersection points line up along a pair of integral curves of $V$. One trajectory corresponds to taking the $\alpha$-plane corresponding to the connecting vector $p' \to p'_{ret}$ and sliding it up the trajectories, and the other corresponds to doing the same with $p' \to p_{adv}$.

Thus a generic infinitesimal perturbation of $x \in H$ to $x'$ perturbs the ruled surface $R_x$ to another ruled surface $R_{x'}$ which intersects it in two integral curves of $V$. $R_x$ is ruled not only by the lines corresponding to points of $t_x$, but also by the integral curves of $V$. Therefore $R_x$ projects onto a rational curve $C_x$ in $\mathcal{J}$. The perturbation $x \to x'$ takes $C_x$ into $C_{x'}$ where $C_{x'}$ intersects $C_x$ in a pair of points by the above reasoning. $C_x$ thus has self intersection number two, and $\mathcal{J}$ is a minitwistor space, with $H$ as the parameter space of normal bundle $\mathcal{O}(2)$ curves.

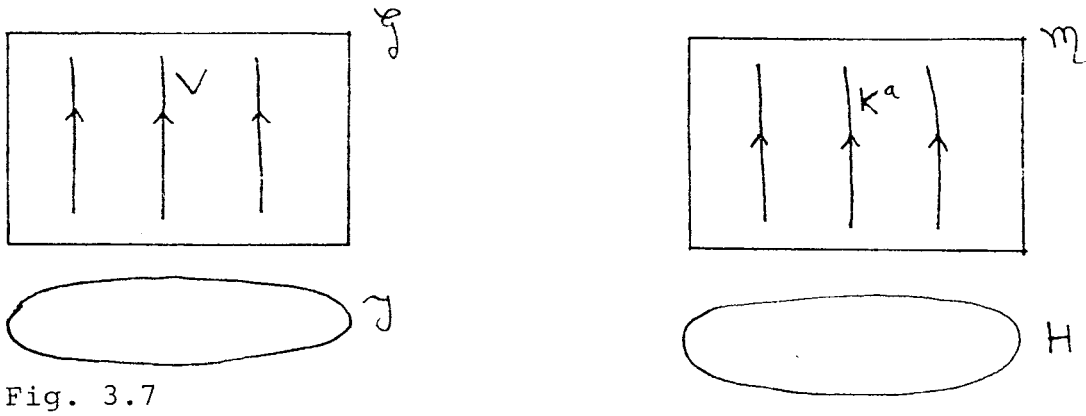



Fig. 3.7

There are thus two Weyl structures on $H$ - the one defined by $\mathcal{J}$ using the Hitchin procedure of § 3.1, and the one defined by $\mathcal{M}$ using the procedure of § 3.2. We now determine the conditions under which these structures agree.

Firstly, we compare the conformal metrics. Let $x, x' \in H$ be infinitesimally null separated with respect to the Weyl structure inherited from $\mathcal{M}$. Thus the trajectories $t_x$ and $t_{x'}$ in $\mathcal{M}$ are connected by a Jacobi field which is null and orthogonal to $K^a$. A point $p'$ on $t_{x'}$ is now null separated from only <u>one</u> point on $t_x$.

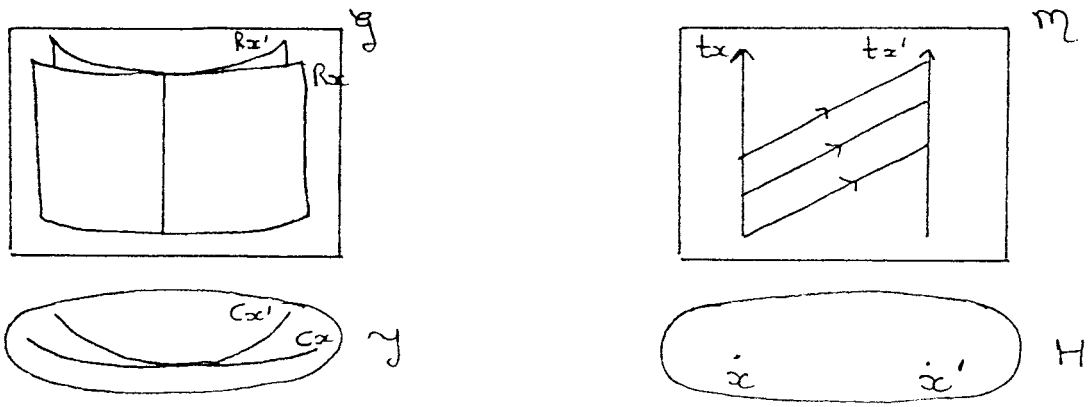



Fig. 3.8

The ruled surfaces $R_x$ and $R_{x'}$ intersect in <u>one</u> integral curve of $V$, and the projected curves $C_x$ and $C_{x'}$ intersect in one point. This means that $x$ and $x'$ are also null separated with respect to the Hitchin Weyl structure. The argument also works in reverse and hence the two Weyl structures agree as far as the conformal metrics are concerned. We now compare the connections:

Fix a pair of minitwistors $Z, Z' \in \mathcal{Y}$. These lift to two integral curves of $V$ in $\mathcal{Q}$, which in turn correspond to a pair of one-parameter families of $\alpha$-planes in $\mathcal{M}$. Each pair of $\alpha$- planes intersect in one point and the totality of such intersection points is a two-surface $S$ in $\mathcal{M}$. Since the families of $\alpha$- planes were generated by translating twistors up integral curves of $V$, $S$ is ruled by trajectories of the conformal Killing vector $K^a$ and hence projects down to a curve $L$ in $H$.

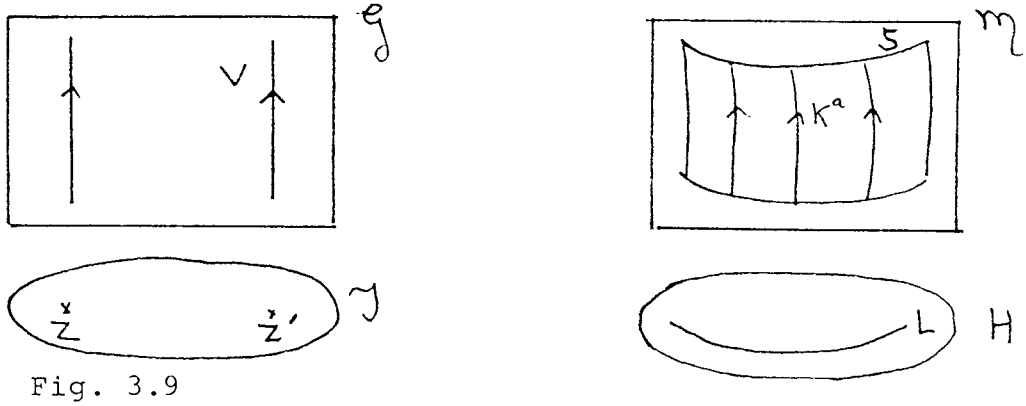



Fig. 3.9

$L$ is the geodesic (with respect to the Hitchin Weyl structure) corresponding to the minitwistors $Z, Z'$. We now determine the condition for it also to be a geodesic

with respect to the Weyl structure inherited from $\mathcal{M}$. To do this, we write out the tangent vector field to $L$ in terms of Jacobi fields on $\mathcal{M}$, impose the geodesic equations for the connection of lemma 3.2, and see what the implications are for $K^a$.

$S$ is the set of intersection points of two one-parameter families of $\alpha$- planes. Fix an $\alpha$- plane in one of the families and intersect it with all the $\alpha$- planes of the other family. This gives us a null curve in $S$ whose tangent vector is $\ell^a$ say. Doing this for each $\alpha$- plane in the first family in turn gives us a ruling of $S$ by null curves. Alternatively, we may repeat the procedure, but starting with an $\alpha$- plane of the second family. This gives another family of null curves, with tangent vector $n^a$ say, ruling $S$.

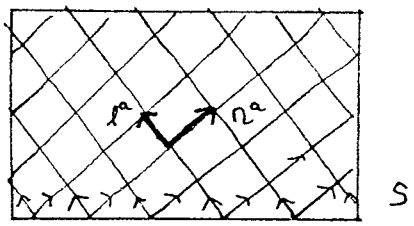



Fig. 3.10

Suppose that the first family of $\alpha$- planes has tangent vectors of the form $\ell^a = \alpha^A \pi^{A'}$ and the second family has ones of the form $n^a = \beta^A \rho^{A'}$. At this stage we need :

Lemma 3.3

Consider the one-parameter family of $\alpha$- planes obtained by translating an $\alpha$- plane along the trajectories

of a conformal Killing vector $K^a$. Then, if $[\pi_{A'}]$ is the projective spinor field of this family of $\alpha$- planes, the scale can be chosen so that $\mathcal{L}_K \pi_{A'} = 0$.

Proof

Translation along the trajectories of $K^a$ preserves $\alpha$- planes, so in particular, the two-plane element $F_{ab} := k_{[a} l_{b]}$ must be preserved (where $k^a$ and $l^a$ are two independent null directions tangent to the $\alpha$- plane). Thus,

$$\mathcal{L}_K F_{ab} \propto F_{ab} \Rightarrow \mathcal{L}_K (\pi_{A'} \pi_{B'} \varepsilon_{AB}) = \alpha \pi_{A'} \pi_{B'} \varepsilon_{AB} \quad *$$

$$\therefore \pi_{A'} (\mathcal{L}_K \pi_{B'}) \varepsilon_{AB} + (\mathcal{L}_K \pi_{A'}) \pi_{B'} \varepsilon_{AB} + \pi_{A'} \pi_{B'} (\mathcal{L}_K \varepsilon_{AB}) = \alpha \pi_{A'} \pi_{B'} \varepsilon_{AB}$$

Hitting with $\pi^{B'}$:

$$\pi^{B'} (\mathcal{L}_K \pi_{B'}) = 0$$

Thus, $(\mathcal{L}_K \pi_{B'}) \propto \pi_{B'}$ and the result follows.

Now, the vectors $l^a$ and $n^a$ spanning $S$ can be written as

$$l^a = K^{AB'} \rho_{B'} \pi_{C'} \varepsilon^{A'C'} \Theta := o^A \pi^{A'}$$

$$n^a = K^{AB'} \pi_{B'} \rho_{C'} \varepsilon^{A'C'} \Theta := \iota^A \rho^{A'}.$$

where $\pi^{A'}$ and $\rho^{A'}$ have been chosen as in lemma 3.3, and the factor $\Theta = (K.K)^{\frac{1}{2}}$ is included to ensure that

$$\mathcal{L}_K l^a = \mathcal{L}_K n^a = 0. \tag{3.8}$$

Now, we have

$$l^a - n^a = K^{AB'} \varepsilon^{A'C'} (\rho_{B'} \pi_{C'} - \pi_{B'} \rho_{C'}) \Theta$$

$$= \Theta (\rho_{D'} \pi^{D'}) K^{AA'}. \tag{3.9}$$

* where the Lie - derivative of a spinor is given by

$\mathcal{L}_K \pi_{A'} = K^{BB'} \nabla_{BB'} \pi_{A'} - \frac{1}{2} \phi_{A'B'} \pi^{B'} + \frac{1}{8} (\nabla_a K^a) \pi_{A'}$ ; $\nabla_{[a} K_{b]} = \frac{1}{2} (\phi_{AB} \varepsilon_{A'B'} + \phi_{A'B'} \varepsilon_{AB})$

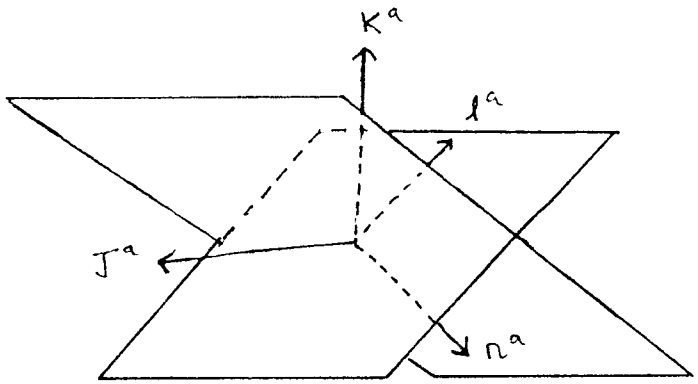



Fig. 3.11

Put $$J^a = l^a + n^a.$$

Then

$$J^a K_a = \mathcal{O} K^{AB'}(\rho_{B'}\pi^{A'} + \pi_{B'}\rho^{A'})K_{AA'} = 0$$

Thus $J^a$ is orthogonal to $K^a$. Moreover, it is lie-propagated along $K^a$ by (3.8). This ensures that it defines a vector field on the factor space $H$ of trajectories of $K^a$. In fact since $J^a$ is tangent to the surface $S$, the vector field it defines on $H$ is tangent to the projected curve $L$. We must now see under what conditions $L$ is a geodesic with respect to the connection of lemma 3.2 i.e. under what conditions

$$J^a D_a J^b \quad \alpha \quad J^b. \tag{3.10}$$

(3.10) means

$$h^b{}_c (J^a \nabla_a J^c - \gamma^c_{ad} J^a J^d) \quad \alpha J^b$$

i.e.

$$J^a \nabla_a J^b - \gamma^b_{ad} J^a J^d = \alpha J^b + \beta K^b \qquad \text{for some } \alpha, \beta,$$

since $h$ projects onto the subspace orthogonal to $K^a$. Substituting (3.6) for $\gamma^b_{ad}$, we are interested in the conditions under which

$$J^a \nabla_a J^b + \tfrac{1}{2} J^2 \omega^b - (J.\omega) J^b = \alpha J^b + \beta K^b \qquad (3.11)$$

for some $\alpha, \beta$.

We can absorb the $(J.\omega)$ term into $\alpha J^b$ and we are left with

$$J^a \nabla_a J^b + \tfrac{1}{2} J^2 \omega^b = \alpha J^b + \beta K^b \qquad \text{for some } \alpha, \beta.$$

From (3.7) we have $\omega^b = \frac{2}{K^2} K_c \nabla^b K^c$

Therefore we arrive at

$$J^a \nabla_a J^b + \frac{J^2}{K^2} K_c \nabla^b K^c = \alpha J^b + \beta K^b \quad \text{for some } \alpha, \beta.$$

We now use the conformal Killing equation

$$\nabla^b K^c + \nabla^c K^b = \lambda g^{bc}$$

to substitute for $\nabla^b K^c$ and absorb the extra term in $\beta K^b$. This gives

$$J^a \nabla_a J^b - \frac{J^2}{K^2} K^c \nabla_c K^b = \alpha J^b + \beta K^b \qquad \text{for some } \alpha, \beta.$$

Now, $J^2 = 2 l^a n_a$

But, $-2 l^a n_a = (l^a - n^a)^2 = \Theta^2 (\rho_{D'} \pi^{D'})^2 K^2$ by (3.9)

Therefore $\frac{J^2}{K^2} = -\Theta^2 (\rho_{D'} \pi^{D'})^2$ and we get

$$J^a \nabla_a J^b + \Theta^2 (\rho_{D'} \pi^{D'})^2 K^c \nabla_c K^b = \alpha J^b + \beta K^b \quad \text{for some } \alpha, \beta.$$

Substituting for $J^a$ & $K^a$ in terms of $l^a$ & $n^a$, we get

$$(l+n)^a \nabla_a (l+n)^b + \Theta^2 (\rho_{D'} \pi^{D'})^2 \frac{1}{\Theta(\rho_{D'} \pi^{D'})} (l^a - n^a) \nabla_a \left( \frac{1}{\Theta \rho_{D'} \pi^{D'}} (l-n)^b \right) = \alpha l^b + \beta n^b$$

for some $\alpha, \beta$.

When the $\nabla_a$ acts on $1/\theta(\rho_{D'}\pi^{D'})$ we have a term which we can absorb on the RHS. Thus we get

$$(\ell+n)^a\nabla_a(\ell+n)^b + (\ell-n)^a\nabla_a(\ell-n)^b = \alpha\ell^b + \beta n^b$$

for some $\alpha, \beta$.

Expanding, we obtain the condition for $L$ to be geodesic (with respect to the $D$ -connection) in the form

$$\ell^a\nabla_a\ell^b + n^a\nabla_a n^b = \alpha\ell^b + \beta n^b \qquad (3.12)$$

for some $\alpha, \beta$.

To determine when this actually holds, proceed as follows. Complete $\ell^a$ and $n^a$ to make a null tetrad

$$\left.\begin{aligned} \ell^a &= k^{AB'}\rho_{B'}\pi^{A'}\theta &:= o^A\pi^{A'} \\ n^a &= k^{AB'}\pi_{B'}\rho^{A'}\theta &:= \iota^A\rho^{A'} \\ m^a &= k^{AB'}\pi_{B'}\pi^{A'}\theta &:= \iota^A\pi^{A'} \\ \tilde{m}^a &= k^{AB'}\rho_{B'}\rho^{A'}\theta &:= o^A\rho^{A'} \end{aligned}\right\} \qquad (3.13)$$

Then, to show that the LHS of (3.12) does not contain any component in the $\tilde{m}^b$ direction, we must show that its contraction with $m^b$ vanishes, i.e.

$$m_b(\ell^a\nabla_a\ell^b + n^a\nabla_a n^b) = 0$$

i.e. $\iota_B\pi_{B'}(o^A\pi^{A'}\nabla_{AA'}(o^B\pi^{B'}) + \iota^A\rho^{A'}\nabla_{AA'}(\iota^B\rho^{B'})) = 0$

Now, we have $\pi^{A'}\nabla_{AA'}\pi^{B'} = \rho^{A'}\nabla_{AA'}\rho^{B'} = 0$ since $\pi^{B'}$ and $\rho^{B'}$ are the spinor fields associated with $\alpha$- planes, so our condition becomes

$$\iota_B\pi_{B'}(\pi^{B'}o^A\pi^{A'}\nabla_{AA'}o^B + \rho^{B'}\iota^A\rho^{A'}\nabla_{AA'}\iota^B) = 0$$

i.e. $$l^B l^A \rho^{A'} \nabla_{AA'} l_B = 0$$

Substituting $l_B = K_B{}^{B'} \pi_{B'} \Theta$, we arrive at

$$l^B l^A \rho^{A'} \nabla_{AA'} (K_B{}^{B'} \pi_{B'}) = 0$$

i.e. $$l^B l^A \rho^{A'} \left( -\pi^{B'} \nabla_{(a} K_{b)} - \pi^{B'} \nabla_{[a} K_{b]} + K_B{}^{B'} \nabla_{AA'} \pi_{B'} \right) = 0$$

i.e. $$l^B l^A \rho^{A'} \left( -\pi^{B'} \frac{\lambda}{2} \varepsilon_{AB} \varepsilon_{A'B'} - \frac{1}{2} \pi^{B'} \tilde{\Phi}_{AB} \varepsilon_{A'B'} - \frac{1}{2} \pi^{B'} \phi_{A'B'} \varepsilon_{AB} + K_B{}^{B'} \nabla_{AA'} \pi_{B'} \right) = 0$$

where

$$\nabla_{AA'} K_{BB'} - \nabla_{BB'} K_{AA'} = \tilde{\Phi}_{AB} \varepsilon_{A'B'} + \phi_{A'B'} \varepsilon_{AB} \qquad (3.14)$$

Our condition is now

$$K^{BC'} \pi_{C'} K^{AD'} \pi_{D'} \rho^{A'} \left( \frac{1}{2} \tilde{\Phi}_{AB} \pi_{A'} + K_B{}^{B'} \nabla_{AA'} \pi_{B'} \right) = 0$$

i.e. $$\frac{1}{2} K^{BC'} \pi_{C'} K^{AD'} \pi_{D'} \tilde{\Phi}_{AB} (\rho^{A'} \pi_{A'}) + \frac{1}{2} K^2 K^{AD'} \pi_{D'} \rho^{A'} \pi^{B'} \nabla_{AA'} \pi_{B'} = 0.$$

Now, $\pi^{A'} \nabla_{AA'} \pi_{B'} = 0$ so we can write

$$\nabla_{AA'} \pi_{B'} = \xi_{AB'} \pi_{A'}$$

and our condition becomes

$$K^{BC'} \pi_{C'} K^{AD'} \pi_{D'} \tilde{\Phi}_{AB} (\rho^{A'} \pi_{A'}) + K^2 K^{AD'} \pi_{D'} \rho^{A'} \pi^{B'} \xi_{AB'} \pi_{A'} = 0$$

Now recall we have chosen $\pi_{B'}$ so that

$$\mathcal{L}_K \pi_{B'} = 0$$

i.e. $$K^{AD'} \nabla_{AD'} \pi_{B'} - \frac{1}{2} \phi_{B'A'} \pi^{A'} + \frac{1}{8} (\nabla . K) \pi_{B'} = 0$$

i.e. $$K^{AD'} \xi_{AB'} \pi_{D'} = \frac{1}{2} \phi_{B'A'} \pi^{A'} - \frac{1}{8} (\nabla . K) \pi_{B'}$$

Using this, the condition becomes

$$K^{BC'} \pi_{C'} K^{AD'} \pi_{D'} \tilde{\Phi}_{AB} (\rho^{A'} \pi_{A'}) + K^2 \pi^{B'} \left( \frac{1}{2} \phi_{B'D'} \pi^{D'} - \frac{1}{8} (\nabla . K) \pi_{B'} \right) (\rho^{A'} \pi_{A'}) = 0$$

ie. $$\pi^{C'} \pi^{D'} \left( K^B{}_{C'} K^A{}_{D'} \tilde{\Phi}_{AB} + \frac{1}{2} K^2 \phi_{C'D'} \right) = 0 \qquad (3.15).$$

Therefore if Hitchin geodesics are to be geodesic with respect to the $\mathbb{D}$- connection in general, we must have

$$K^{B}{}_{C'} K^{A}{}_{D'} \tilde{\Phi}_{AB} + \tfrac{1}{2} K^{2} \Phi_{C'D'} = 0 \qquad (3.16)$$

Now, the condition that $K^{a}$ be a hypersurface-orthogonal vector is

$$K^{a} (\nabla K)^{*}_{ab} = 0$$

i.e. $$K^{a} (\tilde{\Phi}_{AB} \varepsilon_{A'B'} - \Phi_{A'B'} \varepsilon_{AB}) = 0$$

i.e. $$K^{AB'} \tilde{\Phi}_{AB} - K_{B}{}^{A'} \Phi_{A'}{}^{B'} = 0 \qquad (3.17)$$

and contracting with $K^{BC'}$ gives (3.16). Thus if all Hitchin geodesics are also to be geodesic with respect to the connection $\mathbb{D}$ of lemma 3.2, the conformal Killing vector must be hypersurface-orthogonal (H.S.O.). One can reverse the arguments and deduce that hypersurface-orthogonality of $K^{a}$ is also the condition required for all curves geodesic with respect to $\mathbb{D}$ to be also geodesic with respect to the Hitchin connection. Thus, hypersurface-orthogonality is the condition for the projective connections to agree, and since in three dimensions [7] a projective connection preserving the conformal structure determines the affine connection, we come to the following conclusion:

Proposition 3.1

If $K^{a}$ is a H.S.O. conformal Killing vector on a right-flat space-time $\mathcal{M}$ and $\mathcal{J}$ is the minitwistor space obtained from the corresponding twistor space, then the

Hitchin Weyl structure on the space of trajectories of $K^a$ is given by lemma 3.2.

Unfortunately, the requirement that a right-flat space-time admit a H.S.O. non-null conformal Killing vector is rather restrictive, as is evidenced by the following result:

Proposition 3.2

If $\mathcal{M}$ is a right-flat space-time admitting a H.S.O. non null conformal Killing vector $K^a$, then $\mathcal{M}$ is necessarily flat.

Proof

We have $2\nabla_a K_b = 2\nabla_{(a}K_{b)} + 2\nabla_{[a}K_{b]}$

i.e. $$2\nabla_a K_b = \lambda \varepsilon_{AB}\varepsilon_{A'B'} + \tilde{\phi}_{AB}\varepsilon_{A'B'} + \phi_{A'B'}\varepsilon_{AB} \qquad (3.18)$$

In addition, differentiating (3.17) gives us

$$(\nabla_{CC'}K^{A}{}_{B'})\tilde{\phi}_{AB} + K^{A}{}_{B'}(\nabla_{CC'}\tilde{\phi}_{AB}) = (\nabla_{CC'}K_{B}{}^{A'})\phi_{A'B'} + K_{B}{}^{A'}(\nabla_{CC'}\phi_{A'B'}) \qquad (3.19)$$

Since $K^a$ is a conformal Killing vector and $\mathcal{M}$ is right-flat, lemma 3.1 gives us

$$\nabla_{AA'}\nabla_{BB'}K_{CC'} = \Psi_{DACB}\varepsilon_{D'A'}\varepsilon_{C'B'}K^{DD'} + \tfrac{1}{2}\{\varepsilon_{AC}\varepsilon_{A'C'}\nabla_{BB'}\lambda + \varepsilon_{CB}\varepsilon_{C'B'}\nabla_{AA'}\lambda - \varepsilon_{AB}\varepsilon_{A'B'}\nabla_{CC'}\lambda\}.$$

Substituting for $\nabla_{BB'}K_{CC'}$ from (3.18), the $\varepsilon_{CB}\varepsilon_{C'B'}\nabla_{AA'}\lambda$ term cancels and we get

$$\nabla_{AA'}(\tilde{\phi}_{BC}\varepsilon_{B'C'} + \phi_{B'C'}\varepsilon_{BC}) = 2\Psi_{DACB}\varepsilon_{D'A'}\varepsilon_{C'B'}K^{DD'} + \varepsilon_{AC}\varepsilon_{A'C'}\nabla_{BB'}\lambda - \varepsilon_{AB}\varepsilon_{A'B'}\nabla_{CC'}\lambda.$$

Skewing on $B'C'$ gives

$$\nabla_{AA'}\tilde{\phi}_{BC} = -2\Psi_{DACB}\varepsilon_{D'A'}K^{DD'} + \frac{1}{2}\left(\varepsilon_{AC}\nabla_{BA'}\lambda + \varepsilon_{AB}\nabla_{CA'}\lambda\right)$$

Symmetrizing on $B'C'$ and contracting with $\varepsilon^{BC}$ gives

$$\nabla_{AA'}\phi_{B'C'} = \frac{1}{2}\left(\varepsilon_{A'B'}\nabla_{AC'}\lambda + \varepsilon_{A'C'}\nabla_{AB'}\lambda\right)$$

Substituting these expressions for the derivatives of $\tilde{\phi}_{BC}$, $\phi_{B'C'}$ and $K_{BB'}$ into (3.19) we obtain

$$\frac{1}{2}\left(\lambda\varepsilon_C{}^A\varepsilon_{C'B'} + \tilde{\phi}_C{}^A\varepsilon_{C'B'} + \phi_{C'B'}\varepsilon_C{}^A\right)\tilde{\phi}_{AB} + K^A{}_{B'}\left(-2\Psi_{DCBA}\varepsilon_{D'C'}K^{DD'}\right.$$

$$\left.+\frac{1}{2}\varepsilon_{CB}\nabla_{AC'}\lambda + \frac{1}{2}\varepsilon_{CA}\nabla_{BC'}\lambda\right) = \frac{1}{2}\left(\lambda\varepsilon_{CB}\varepsilon_{C'}{}^{A'} + \tilde{\phi}_{CB}\varepsilon_{C'}{}^{A'} + \phi_{C'}{}^{A'}\varepsilon_{CB}\right)\phi_{A'B'}$$

$$+\frac{K_B{}^{A'}}{2}\left(\varepsilon_{C'A'}\nabla_{CB'}\lambda + \varepsilon_{C'B'}\nabla_{CA'}\lambda\right)$$

i.e. $\lambda\tilde{\phi}_{CB}\varepsilon_{C'B'} + \tilde{\phi}_C{}^A\tilde{\phi}_{AB}\varepsilon_{C'B'} - 4K^A{}_{B'}K^D{}_{C'}\Psi_{DCBA} + \varepsilon_{CB}\nabla_{AC'}\lambda$

$- K_{CB'}\nabla_{BC'}\lambda = \lambda\varepsilon_{CB}\phi_{C'B'} + \phi_{C'}{}^{A'}\phi_{A'B'}\varepsilon_{CB} + \varepsilon_{C'B'}K_B{}^{A'}\nabla_{CA'}\lambda$

$$- K_{BC'}\nabla_{CB'}\lambda.$$

Now symmetrise on $BC$ and $B'C'$, so that any term involving $\varepsilon_{BC}$ or $\varepsilon_{B'C'}$ drops out. (N.B. the combination $K_{CB'}\nabla_{BC'}\lambda - K_{BC'}\nabla_{CB'}\lambda$ also drops out since it can be written as a combination of $\varepsilon$'s). We are left with

$$K^A{}_{B'}K^D{}_{C'}\Psi_{DCBA} = 0$$

and since $K^a$ is non null we deduce that

$$\Psi_{DCBA} = 0$$

i.e. $\mathcal{M}$ is flat as required.

## §3.4 Checking the Einstein-Weyl Equations in the H.S.O. case

In the case when $K^a$ is H.S.O., the parameter space $H$ of curves in the minitwistor space $\mathcal{J}$ can be realised as a hypersurface in (flat) space-time. The Hitchin Weyl structure on $H$ obtained from $\mathcal{J}$ is then given, according to proposition 3.1, by

1) A conformal metric given by the conformal class of the induced metric.

2) A connection

$$X^a D_a Y^b = h^b{}_c \left( X^a \nabla_a Y^c - \gamma^c_{ad} X^a Y^d \right)$$

where $X$ and $Y$ are tangent to $H$. In fact there is a simple characterisation of this connection - it is the Levi-Civita connection of the three-metric $\frac{1}{K^2} h_{bc}$ on $H$, where $h_{bc}$ is the induced metric given by

$$h_{bc} = \left( \eta_{bc} - K_b K_c / K^2 \right)$$

($\eta_{bc}$ is the Minkowski metric). To check this, differentiate with respect to the Weyl connection:

$$h^a{}_d h^b{}_e h^c{}_f \left[ \nabla_a \left( \frac{1}{K^2} h_{bc} \right) + \gamma^l_{ba} \frac{h_{lc}}{K^2} - \gamma^l_{ca} \frac{h_{bl}}{K^2} \right]$$

$$= h^a{}_d h^b{}_e h^c{}_f \left\{ h_{bc} \nabla_a \left( 1/K^2 \right) + 1/K^2 \left[ \nabla_a \left( \eta_{bc} - K_b K_c / K^2 \right) \right.\right.$$

$$\left.\left. + \tfrac{1}{2} \left( \delta^l_b \omega_a + \delta^l_a \omega_b - \eta_{ab} \omega^l \right) h_{lc} + \tfrac{1}{2} \left( \delta^l_c \omega_a + \delta^l_a \omega_c - \eta_{ca} \omega^l \right) h_{bl} \right] \right\}$$

Since $h$ projects onto the subspace orthogonal to $K^a$, $h^a{}_d h^b{}_e h^c{}_f$ kills $-\frac{K_b K_c}{K^2}$ and we are left with

$$h^a{}_d h_{ef} \nabla_a(\tfrac{1}{K^2}) + \tfrac{1}{K^2} h^a{}_d h^b{}_e h^c{}_f \Big\{ (\eta_{bc} - K_b K_c/K^2)\,\omega_a \cdot$$
$$+\tfrac{1}{2}(\eta_{ac} - \tfrac{K_a K_c}{K^2})\omega_b + \tfrac{1}{2}(\eta_{ab} - \tfrac{K_a K_b}{K^2})\omega_c - \tfrac{1}{2}\eta_{ab}\omega^l(\eta_{lc} - \tfrac{K_l K_c}{K^2})$$
$$-\tfrac{1}{2}\eta_{ca}\omega^l(\eta_{bl} - K_b K_l/K^2)$$

$$= h^a{}_d h_{ef} \nabla_a(1/K^2) + 1/K^2\, h^a{}_d h^b{}_e h^c{}_f \Big\{ \eta_{bc}\omega_a + \tfrac{1}{2}\eta_{ac}\omega_b + \tfrac{1}{2}\eta_{ab}\omega_c$$
$$- \tfrac{1}{2}\eta_{ab}\omega_c - \tfrac{1}{2}\eta_{ac}\omega_b \Big\}$$

$$= h^a{}_d h_{ef} \left( \nabla_a(1/K^2) + \omega_a/K^2 \right)$$

$$= 0 \quad \text{since} \quad \omega_a = \nabla_a \ln(K^2) = 1/K^2\, \nabla_a(K^2).$$

Therefore, the Weyl connection $D_a$ on $H$ is indeed the levi Civita connection of $1/K^2\, h_{bc}$. We use this to check that it satisfies the Einstein-Weyl equations (3.1). The Ricci tensor of the Weyl connection is related to the Ricci tensor ${}^3R^b{}_d$ of the Levi Civita connection $\tilde{\nabla}_a$ for $h_{bc}$ by a conformal rescaling formula [5] and is given by

$$K^2({}^3R^b{}_d) - (K^2)^{\frac{1}{2}} h^{bc} \tilde{\nabla}_c \tilde{\nabla}_d (K^2)^{\frac{1}{2}} - (K^2)^{3/2} \left( \tilde{\nabla}^a \tilde{\nabla}_a (K^2)^{\frac{1}{2}} \right) h^b{}_d$$

Therefore, to check that the Einstein-Weyl equations hold, we must verify that

$$K^2\, {}^3R^b{}_d - (K^2)^{\frac{1}{2}} h^{bc} \tilde{\nabla}_c \tilde{\nabla}_d (K^2)^{\frac{1}{2}} \;\alpha\; h^b{}_d \qquad (3.20)$$

We check the first term first. The Ricci tensor ${}^3R_{bd}$ is given in terms of the Ricci tensor of the metric on $\eta$ and the second fundamental form $\chi_{ab}$ of $H$ by a simple formula [5], which reduces, since $\eta$ is flat, to

$${}^3R_{bd} = \chi^a{}_a \chi_{bd} - \chi^a{}_d \chi_{ba}$$

Now, $\chi_{ab} = h_{(a}{}^{c} h_{b)}{}^{d} \nabla_c n_d$ , $n_a = K_a/(K^2)^{\frac{1}{2}}$

$= h_a{}^c h_b{}^d \nabla_{(c} n_{d)}$

$= h_a{}^c h_b{}^d \nabla_{(c} \frac{1}{(K^2)^{\frac{1}{2}}} K_{d)}$

$= \frac{1}{2} \frac{\lambda}{(K^2)^{\frac{1}{2}}} h_{ab}$ (3.21)

Since $K^a$ is a conformal Killing vector. Therefore, we conclude that ${}^3R_{bd} \propto h_{bd}$ . Thus to verify (3.20) we have to prove that

$$h^{bc} \tilde{\nabla}_c \tilde{\nabla}_d (K^2)^{\frac{1}{2}} \propto h^b{}_d$$

i.e. $$h^{bc} h_c{}^f h_d{}^g \nabla_f \nabla_g (K^2)^{\frac{1}{2}} \propto h^b{}_d$$

i.e. $$-\frac{1}{2(K^2)^{3/2}} h^{bf} h_d{}^g (\nabla_f K^2)(\nabla_g K^2) + \frac{1}{(K^2)^{\frac{1}{2}}} h^{bf} h_d{}^g \nabla_f \nabla_g K^2 \propto h^b{}_d$$

i.e. $$-\frac{2}{(K^2)^{3/2}} h^{bf} h_d{}^g K_\ell K_m (\nabla_f K^\ell)(\nabla_g K^m) + \frac{2}{(K^2)^{\frac{1}{2}}} h^{bf} h_d{}^g (\nabla_f K_c)(\nabla_g K^c)$$

$$+ \frac{1}{(K^2)^{\frac{1}{2}}} h^{bf} h_d{}^g K^c \nabla_f \nabla_g K_c \propto h^b{}_d$$

Combining the first two terms on the LHS we get

$$\frac{2}{(K^2)^{\frac{1}{2}}} h^{bf} h_d{}^g \left(\eta_{\ell m} - \frac{K_\ell K_m}{K^2}\right)(\nabla_f K^\ell)(\nabla_g K^m)$$

$$= \frac{2}{(K^2)^{\frac{1}{2}}} h^{bf} h_d{}^g h^{\ell m} (\nabla_f K^\ell)(\nabla_g K_m)$$

$$= \frac{2}{(K^2)^{\frac{1}{2}}} h^{bf} h_c{}^\ell (\nabla_f K_\ell) h_d{}^g h^{cm} (\nabla_g K_m)$$

$$= \frac{2}{(K^2)^{\frac{1}{2}}} \chi^b{}_c \chi_d{}^c$$

$$= \frac{\lambda^2}{2(K^2)^{3/2}} h^b{}_d \quad \text{by (3.21)}$$

The third term on the LHS is

$$\tfrac{1}{(K^2)^{\frac{1}{2}}}\, h^{bf} h_d{}^g K^c \nabla_f \nabla_g K_c$$

$$= \tfrac{1}{2(K^2)^{\frac{1}{2}}}\, h^{bf} h_d{}^g K^c \left[\eta_{fc}\nabla_g\lambda + \eta_{cg}\nabla_f\lambda - \eta_{fg}\nabla_c\lambda\right] \quad \text{by lemma (3.1)}$$

$$= -\frac{1}{2(K^2)^{\frac{1}{2}}} \left(K^c \nabla_c \lambda\right) h^b{}_d$$

Thus (3.20) is established and the Ricci tensor $R_{ab}$ for the connection $D_a$ satisfies

$$R_{ab} \propto h_{ab}$$

Therefore $H$ with metric $\frac{1}{K^2} h_{ab}$ and Levi-Civita connection $D_a$ is a constant-curvature three-manifold.

## §3.5 The case where $K^a$ is not Hypersurface-Orthogonal

We now return to the situation where $H$ is the space of trajectories of a general non null conformal Killing vector $K^a$ on a right flat space-time and modify the definition (lemma 3.2) of the connection $D_a$ so that it agrees with the connection defined by the minitwistor space $\mathcal{J}$.

The one form $\omega_a$ which appears in the connection coefficients was chosen to satisfy (3.4) i.e.

$$\mathcal{L}_K \omega_a = \nabla_a \lambda$$

and this ensured that $D_a$ gave a well defined connection on $H$. Up to now, we have made the choice

$$\omega_a = \nabla_a \ln(K.K).$$

However, (3.4) is still satisfied if we replace $\omega_a$ by $\omega_a + \nu_a$ where

$$\mathcal{L}_K \nu_a = 0.$$

Thus, the expression

$$\gamma^c_{ad} = \tfrac{1}{2}\left(\delta^c_a(\omega_d+\nu_d) + \delta^c_d(\omega_a+\nu_a) - g_{ad}(\omega^c+\nu^c)\right)$$

gives well-defined connection on $H$ given by (3.5). The problem is to choose $\nu_a$ such that the connection agrees with the Hitchin minitwistor connection for arbitrary (i.e. not necessarily H.S.O.) non null $K^a$.

To determine $\nu_a$ we substitute $\omega_a+\nu_a$ for $\omega_a$ in (3.11) and run through the calculation which led up to (3.15). The calculation is identical except for an extra term involving $\nu_a$ at each step, and (3.15) becomes

$$\Pi^{C'}\Pi^{D'}\left(K^B{}_{C'}K^A{}_{D'}\tilde{\Phi}_{AB} + \tfrac{1}{2}K^2\phi_{C'D'}\right) + K^2\nu_b m^b/\Theta = 0.$$

Recalling (3.13), we have $m^b = K^{BC'}\pi_{C'}\Pi^{B'}\Theta$, and we see that

$$\nu_b = \frac{1}{K^2}\left(K^A{}_{B'}\tilde{\Phi}_{AB} - K_B{}^{A'}\phi_{A'B'}\right)$$

gives the desired result. $\nu_a$ is perhaps best expressed as

$$\nu_a = \frac{-2i}{K^2}\varepsilon_{ab}{}^{cd}K^b\nabla_c K_d$$

In this form it is relatively easy to check that $\mathcal{L}_K\nu_a = 0$. Also, $\varepsilon_{ab}{}^{cd}K^b\nabla_c K_d$ is the expression whose vanishing characterises hypersurface-orthogonality, so our new connection agrees with the one of lemma 3.2 when $K^a$ is H.S.O. The one-form that we are using in the connection coefficients is now

$$\omega_a+\nu_a = \nabla_a\ln(K^2) - 2i/K^2\,\varepsilon_{ab}{}^{cd}K^b\nabla_c K_d$$

$$= \frac{2}{(K^2)}K^b\left(\nabla_a K_b - 2i\,(*\nabla K)_{ab}\right)$$

$((*\nabla K)_{ab}$ is the dual of the derivative of $K^a$ )

$$= \frac{2K^b}{K^2}\left(\nabla_{[a}K_{b]} - \tfrac{1}{2}i\,(*\nabla K)_{ab} + \nabla_{(a}K_{b)}\right)$$

$$= \frac{2}{K^2}K^b\left((\nabla K)^+_{ab} + \frac{\lambda}{2}g_{ab} - i(*\nabla K)_{ab}\right)$$

where $(\nabla K)^+_{ab}$ is the self dual part of the derivative of $K_a$ .

We can summarise the main result of this chapter in

Proposition 3.2

Let $\mathcal{M}$ be a right-flat space-time with non null conformal Killing vector $K^a$ and $H$ be the factor space. Then the twistor space $\mathcal{Q}$ for $\mathcal{M}$ is factored onto a minitwistor space $\mathcal{Y}$ by the corresponding vector field. $\mathcal{Y}$ is the minitwistor space for $H$ and the Hitchin Weyl structure on $H$ is given in terms of structures on $\mathcal{M}$ by :

1) The conformal metric of definition 3.1

2) The connection given by

$$J_x^a D_a J_y^b = h^b{}_c\left(J_x^a \nabla_a J_y^b - \gamma^c_{ad} J_x^a J_y^d\right)$$

where (i) $J_x^a$ is the Jacobi field (orthogonal to and lie propagated along $K^a$ ) corresponding to a vector field $X$ on $H$ .

(ii) $h^b{}_c = (\delta^b{}_c - K^b K_c / K^2)$

(iii) $\gamma^c_{ad} = \frac{1}{2}(\delta^c{}_a \mu_d + \delta^c{}_d \mu_a - g_{ad}\mu^c)$

(iv) $\mu_a = \frac{2}{(K^4)}K^b\left((\nabla K)^+_{ab} + \frac{\lambda}{2}g_{ab} - i(*\nabla K)_{ab}\right)$

$\lambda = \frac{1}{2}\nabla_c K^c$

Example

In [12] is described a family of twistor spaces corresponding to right-flat space-times of the form

$$ds^2 = \phi^{-1}(dt - A_i dx^i)^2 + \phi\, dx_i\, dx^i \qquad i=1,2,3$$

where $A^i(x^j)$ and $\phi(x^j)$ are respectively a three-vector and scalar satisfying $\operatorname{curl} \underline{A} = \operatorname{grad} \phi$ . They all have a Killing vector $\frac{\partial}{\partial t}$ , whose derivative is A.S.D. Factoring out by $\frac{\partial}{\partial t}$ , what is the Weyl structure on the factor space $H$ according to proposition 3.2?

We can use $x^i$ as coordinates on $H$ . Tangent vectors to $H$ correspond to Jacobi fields on $\mathcal{M}$ orthogonal to $\frac{\partial}{\partial t}$ i.e. of the form $(A_i V^i, V^i)$ . Plugging these into $ds^2$ we see that the conformal metric on $H$ is simply the Euclidean one $[\delta_{ij}]$ .

Since $\frac{\partial}{\partial t}$ is a Killing vector, its divergence $\lambda$ vanishes. Moreover, since its derivative is ASD, $(\nabla K)^+_{ab} = 0$ . Therefore $\mu_a = -\nabla_a \ln K^2 = \nabla_a \ln \phi$ . The Weyl connection $D_a$ is given in terms of the space-time connection $\nabla_a$ by:

$$J_X{}^a D_a J_Y{}^b = h^b{}_c\, J_X{}^a (\nabla_a J_Y{}^c - \gamma_{ad}{}^c J_Y{}^d)$$

It satisfies $h_a{}^e h_b{}^f h_c{}^g D_g (K^{-2} h_{ef}) = 0$

where $h_{ef} = g_{ef} - K_e K_f / K^2$.

$K^{-2} h_{ef}$ is the tensor on $\mathcal{M}$ (orthogonal to $K^a = \frac{\partial}{\partial t}$ and lie-propagated) corresponding to the Euclidean metric $\delta_{ij}$ on $H$ . Therefore the Weyl connection is simply the levi-Civita connection of the flat metric. Thus, as a Weyl space, $H$ is the same as Euclidean space $\mathbb{E}$ .

# CHAPTER 4 Constructing Anti Self-dual Space-Times from Solutions of the $\mathbb{C}^*$ Bogomolny Equations

## §4.1 Affine Bundles

Let $U''$ be an open subset of the minitwistor space $T$ and let $[f] \in H^1(U'', \mathcal{O}(n))$ be a cohomology class representing a solution of the appropriate equation on the corresponding set $U \subset \mathbb{E}$ as in chapter 1. We can encode this information into the structure of a holomorphic affine line bundle on $U''$. One way to construct this is the following: Suppose $\{U_i\}$ is a Stein cover of $U''$ and $\{f_{ij}\}$ is a cocycle representing $[f]$ with respect to $\{U_i\}$. Let $T_{ij}$ be the transition function for the bundle $\mathcal{O}(1)$ on $T$ with respect to $\{U_i\}$, so that fibre coordinates $\zeta_i$ for $\mathcal{O}(n)$ over $U_i$ will satisfy

$$\zeta_i = (T_{ij})^n \zeta_j$$

If $S$ is some section of $\mathcal{O}(n)$, write $\zeta_i(S)$ for the local function describing it over $U_i$. We can now use $\{f_{ij}\}$ to construct a holomorphic affine line bundle on $U''$ by patching together fibre coordinates $\xi_i$ according to

$$\begin{pmatrix} \xi_i \\ 1 \end{pmatrix} = \begin{pmatrix} (T_{ij})^n & \zeta_i(f_{ij}) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \xi_j \\ 1 \end{pmatrix} \qquad (4.1)$$

Thinking of the column vectors as homogeneous coordinates on $\mathbb{P}_1$, we see that points $\begin{pmatrix} \xi_i \\ 1 \end{pmatrix}$ belong to the complement of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ in $\mathbb{P}_1$, and this complement is naturally an affine space. Given an affine line bundle we can associate to

each fibre its vector space of translations, giving us a line bundle. For the affine bundle given by (4.1), this translation bundle is simply $\mathcal{O}(n)$. In constructing the affine bundle, the information contained in $\{f_{ij}\}$ is recorded in the way that the local zero sections over $U_i$ and $U_j$ fail to match up over $U_i \cap U_j$.

In this chapter, we will be particularly interested in the case $n=0$, where solutions of the Bogomolny equations on $\mathbb{E}$ correspond to affine line bundles on $T$ with <u>trivial</u> translation bundle. An example of such a bundle occurs in the fibration $PT^I \to T$ given by time-translations. The integral curves of the vector field are affine spaces. To describe the fibration, instead of using coordinates

$$\left(\pi_1/\pi_0, \omega^0/\pi_0, \omega^1/\pi_0\right) ; \left(\pi_0/\pi_1, \omega^0/\pi_1, \omega^1/\pi_1\right)$$

on $PT^I$, we use

$$\left(\pi_1/\pi_0, \pi_1\omega^0/\pi_0^2 - \omega^1/\pi_0, \omega^0/\pi_0\right) := \left(\zeta, Z^0, \omega^0/\pi_0\right)$$

$$\left(\pi_0/\pi_1, \omega^0/\pi_1 - \pi_0\omega^1/\pi_1^2, \omega^1/\pi_1\right) := \left(\tilde{\zeta}, Z^1, \omega^1/\pi_1\right)$$

The transition function is

$$\begin{pmatrix} \omega^1/\pi_1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & \tilde{\zeta}Z^0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \omega^0/\pi_0 \\ 1 \end{pmatrix}$$

with respect to the two set cover $\{\pi_0 \neq 0\}, \{\pi_1 \neq 0\}$ of $T$. Now, in the notation of chapter 1, $\tilde{\zeta}Z^0$ is just $\omega/\pi_0\pi_1$, so by proposition 1.3, we see that the affine bundle $PT^I \to T$ represents the scalar field

$$\phi(x) = \frac{1}{2\pi i}\oint \frac{\partial}{\partial\omega}\left(\frac{\omega}{\pi_0\pi_{1'}}\right)\Delta\pi \qquad = 1$$

and hence corresponds to the solution $(\underline{A},\phi) = (\underline{0},1)$ (up to gauge in $\underline{A}$ ) of the $\mathbb{C}^*$ Bogomolny equations.

## §4.2 The Twistor Spaces of Tod and Ward

In [12] some examples of deformations of flat twistor space are discussed. The deformed twistor spaces all have the property that they admit a global quadratic polynomial function $q$ . $q$ defines for us a vector field

$$V = \frac{\partial q}{\partial\omega_A}\frac{\partial}{\partial\omega^A}$$

on the curved twistor space $\mathcal{Q}$ . This gives rise to a Killing vector $K^a$ on the space-time. To obtain $K^a$, set $q(\omega^A(x),\pi_{A'}) = \omega_{A'B'}(x)\pi^{A'}\pi^{B'}$ then, $K^a$ is given by

$$\nabla_{AA'}\omega_{B'C'} = \varepsilon_{A'(B'}K_{C')A}.$$

Tod and Ward make various choices of $q$ on flat twistor space, then define deformations which preserve them in the sense that $q(\hat{\omega}^A,\hat{\pi}_{A'}) = q(\omega^A,\pi_{A'})$ , where $(\hat{\omega}^A,\hat{\pi}_{A'})$, $(\omega^A,\pi_{A'})$ are coordinates on the deformed twistor space. We will be concerned with the two families of deformation that they refer to as types A and B. We now show how, for these cases factoring out $\mathcal{Q}$ by $V$ gives rise to a minitwistor space.

Type A $\qquad q = \omega^0\omega^1$

transition function
$$\begin{aligned}\hat{\omega}^0 &= e^{-\vartheta}\omega^0\\ \hat{\omega}^1 &= e^{\vartheta}\omega^1\\ \hat{\pi}_{A'} &= \pi_{A'}\end{aligned} \qquad (4.2)$$

where $g$ depends on $\omega^A$ only through the combination $\omega^0\omega^1$, i.e. $g = g(q, \pi_{A'})$, and $g$ is to satisfy $g(\lambda^2 q, \lambda\pi_{A'}) = g(q, \pi_{A'})$ (compare this with (1.2)).

$q$ gives rise to the vector field

$$V = \omega^1 \frac{\partial}{\partial \omega^1} - \omega^0 \frac{\partial}{\partial \omega^0}$$

on flat twistor space, corresponding to the Killing vector

$$K_{AA'} = \phi_A{}^B x_{BA'} \qquad (\phi_{AB} = O_{(AB)})$$

which is an anti self-dual rotation on Minkowski space.

The integral curves of $V$ are given on $PT$ by

$$\left\{ \begin{array}{l} \omega^0\omega^1 = \text{const} \\ \pi_{A'} = \text{const} \end{array} \right\}$$

Fig (4.1) illustrates them

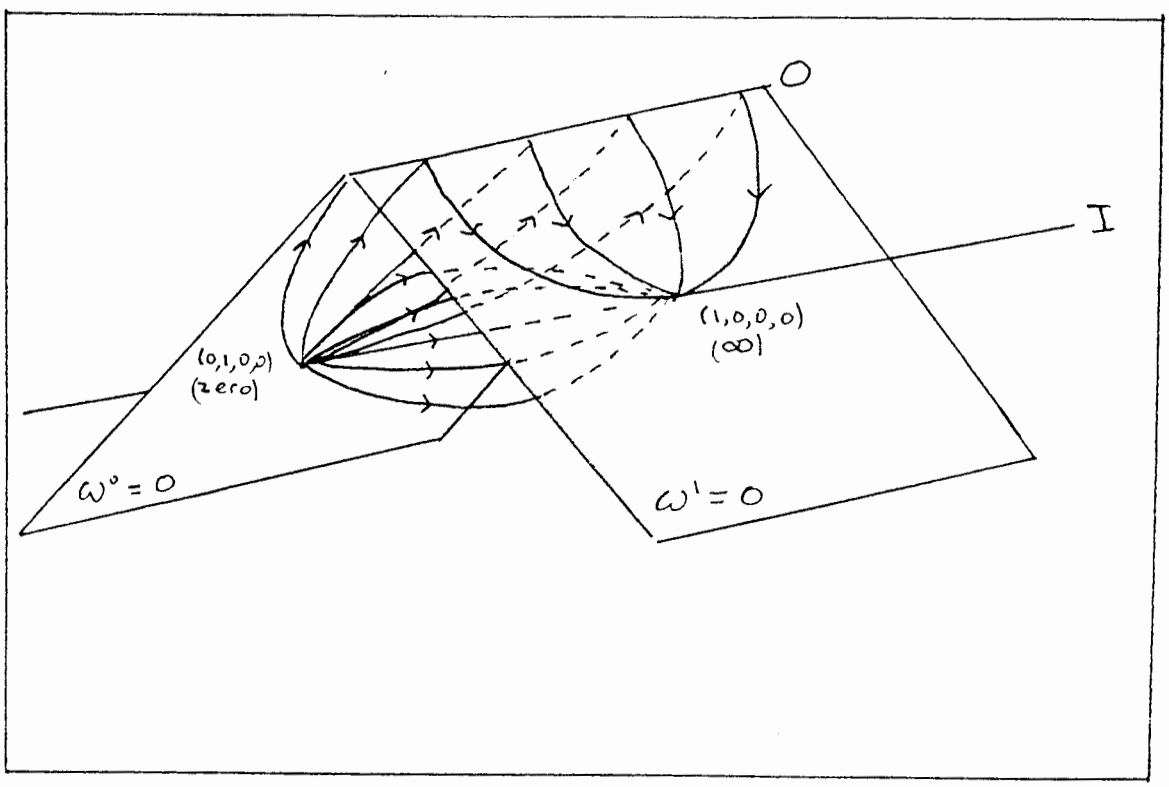



Fig. 4.1

All trajectories join the points $(1,0,0,0)$ and $(0,1,0,0)$ except for the ones lying in $\omega_1 = 0$ or in $\omega_0 = 0$, which join points on the line $O$ to $(1,0,0,0)$ and $(0,1,0,0)$ respectively. If we now remove the line $O$ and the two singular points on $I$, we have twistor space fibred by the integral curves of $V$, each of which is biholomorphic to $\mathbb{C}^*$. The space of integral curves has as parameters

$$\left(\pi_{1'}/\pi_{0'},\ \omega^0\omega^1/\pi_{0'}^2\right) := (\zeta, Z^0) \qquad \text{on} \quad \pi_{0'} \neq 0$$

$$\left(\pi_{0'}/\pi_{1'},\ \omega^0\omega^1/\pi_{1'}^2\right) := (\tilde{\zeta}, Z^1) \qquad \text{on} \quad \pi_{1'} \neq 0.$$

These coordinates uniquely specify the integral curves apart from where $\omega^0\omega^1 = 0$. If $\omega^0\omega^1 = 0$, then for each $\zeta$ there are <u>two</u> integral curves with parameter $(\zeta, 0)$. The patching relations on the space of curves are

$$\tilde{\zeta} = 1/\zeta \quad ; \quad Z^1 = \tilde{\zeta}^2 Z^0.$$

The space of integral curves is therefore the non-Hausdorff space $T'$ ($T$ with its zero section doubled up) discussed in §1.10. $T'$ is obtained by identifying two copies $T_0, T_1$ of $T$ everywhere except along their zero sections. $T_0$ comes from factoring out $\left(PT - \{\omega^0 = 0\}\right) \cong \mathbb{C}^3$ by $V$ and $T_1$ from $PT - \{\omega^1 = 0\}$.

Now, we want to describe $PT$ (minus $O$ and the two singular points on $I$) as a $\mathbb{C}^*$ bundle over $T'$. Along the fibres over $T_0 \subset T'$ (i.e. the integral curves of $V$ for which $\omega^0 \neq 0$) we may use coordinates $\lambda^0 := \omega^0/\pi_{0'}$ on $\pi_{0'} \neq 0$ and $\lambda^1 := \omega^1/\pi_{1'}$ on $\pi_{1'} \neq 0$. The transition function is then

$$\lambda' = \frac{\pi_{0'}}{\pi_{1'}}\,\lambda^0 \tag{4.3}$$

i.e. restricted to $T_0 \subset T'$ the fibration is just the $\mathbb{C}^*$ version of $\mathcal{O}(1)$. To identify it with $\mathcal{O}(1)$ we must say which point on the integral curve is zero and which is infinity. As we have set it up, the point $(0,1,0,0)$ is zero and $(1,0,0,0)$ infinity (see fig. 4.1). Now, if we set up fibre coordinates on the integral curves over $T_1$ (i.e. the curves for which $\omega' \neq 0$), for consistency we must use $\xi^0 := \pi_{0'}/\omega'$ on $\pi_{0'} \neq 0$ and $\xi' := \pi_{1'}/\omega'$ on $\pi_{1'} \neq 0$. The transition function is then

$$\xi' = \pi_{1'}/\pi_{0'}\,\xi^0$$

so, restricted to $T_1 \subset T'$ the fibration is $\mathcal{O}(-1)$.

$T'$ has a four set cover $T_0 \cap \{\pi_{0'} \neq 0\}$ $T_0 \cap \{\pi_{1'} \neq 0\}$ $T_1 \cap \{\pi_{0'} \neq 0\}$ $T_1 \cap \{\pi_{1'} \neq 0\}$.

and with respect to this, the transition functions for flat twistor space fibred by the A.S.D. rotation trajectories are (cf. (1.18) and (1.19):

$$\lambda' = \tilde{\zeta}\,\lambda^0 \qquad \xi' = \zeta\,\xi^0 \tag{4.4}$$

$$\lambda' = z'\xi' \qquad \lambda^0 = z^0\xi^0$$

This is the bundle on $T$ that encodes the Coulomb field on $\mathbb{R}^3 - \{0\}$, provided that, when extracting the field at a point $x$, one point is taken from the $T_0$ zero section and one from the $T_1$ zero section (fig. 1.12).

Now if we look at the <u>deformed</u> twistor space $\mathcal{Q}$ as

a $\mathbb{C}^*$ bundle over $\mathbb{T}'$, the transition functions (4.4) become

$$\lambda' = \tilde{f} e^{-\vartheta} \lambda^\circ \qquad \xi' = f e^{-\vartheta} \xi^\circ \qquad (4.5)$$
$$\lambda' = Z' \xi' \qquad \lambda^\circ = Z^\circ \xi^\circ$$

(where $\lambda^\circ = \omega^\circ / \pi_{0'}$, $\lambda' = \hat{\omega}^\circ / \pi_{0'}$, $\xi^\circ = \pi_{0'} / \omega'$, $\xi' = \pi_{1'} / \hat{\omega}'$ ).

Now, we can think of $g$ as a twistor function, split and obtain an A.S.D. Maxwell potential $\Lambda_a$ which is lie-dragged along the A.S.D. rotation trajectories $K^a$ on Minkowski space. From this, we construct an A.S.D. metric in the form

$$ds^2 = B^{-1}(A\vartheta - R\Lambda)^2 + B\left(4dR^2 + R^2(d\theta^2 + \sin^2\theta d\phi^2)\right) \qquad (4.6)$$
$$\vartheta = R(d\Psi + \cos\theta d\phi) \quad ; \quad B = 1 + K^a \Lambda_a$$

As $\Lambda_a \to 0$ the metric becomes

$$R^2 d\Psi^2 + 2R^2 \cos\theta \, d\phi d\Psi + 4dR^2 + R^2(d\theta^2 + d\phi^2)$$

which is just flat space (set $U = R \exp \frac{i}{2}(\Psi + \phi) \cos\theta/2$ $Y = R \exp -\frac{i}{2}(\Psi - \phi) \sin\theta/2$, the metric is then $2(du d\bar{u} + dY d\bar{Y})$ ).

Alternatively, we can think of $g$ as a minitwistor function, split it and obtain a solution $(\underline{A}, V)$ of the $\mathbb{C}^*$ Bogomolny equations on Euclidean three space. Then one can construct an A.S.D. metric

$$ds^2 = V^{-1}(dt - A_i dx^i)^2 + V dx^i dx^i \qquad (4.7)$$

The $\mathbb{C}^*$ bundle given by (4.5) (i.e. the twistor space $\mathcal{G}$ )

represents the solution of the Bogomolny equations given by adding $(\underline{A}, V)$ to the Coulomb field. However, the Coulomb field doesn't contribute to the metric in the sense that, if $(\underline{\omega}, \phi)$ is the Coulomb field then

$$ds^2 = \phi^{-1}(dt + \omega_i dx^i)^2 + \phi\, dx^i dx^i$$

is just flat space [3]. The two metric forms (4.6) and (4.7) describe the same space-time. To see this use the coordinate transformation suggested by Tod and Ward:

$$R^2 = 2mr$$

$$\Psi = t/2m - \phi$$

$$\Lambda = VR^2/4m^2\,(d\Psi + \cos\theta\, d\phi) + A_i dx^i/2m - (d\Psi + d\phi).$$

Type B $\quad q = t_A{}^{A'} \omega^A \pi_{A'} \qquad (t_A{}^{A'}$ non null).

The deformation is given by

$$\hat{\omega}^A = \omega^A - i t^{AA'} \pi_{A'} g \tag{4.8}$$

where $g$ satisfies the same conditions as for type A.
The vector field on flat twistor space is

$$V = t^{AA'} \pi_{A'} \frac{\partial}{\partial \omega^A}$$

corresponding to the (constant) translation Killing vector $t^{AA'}$ on Minkowski space. $V$ vanishes on the line $I$ and gives rise to a fibration of $\mathbb{P}\mathbb{T}^I$ where the fibres are affine spaces (each is the complement of a point in $\mathbb{P}_1$).
The space of integral curves can be given the coordinates

$$(\zeta, Z^0) := (\pi_{1'}/\pi_{0'},\ t_A{}^{A'} \omega^A \pi_{A'}/\pi_{0'}{}^2)$$

$$(\tilde{\zeta}, Z^1) := (\pi_{0'}/\pi_{1'},\ t_A{}^{A'} \omega^A \pi_{A'}/\pi_{1'}{}^2) \tag{4.9}$$

and is therefore the minitwistor space $\mathbb{T}$ again. As coordinates along the integral curves we can use

$$\xi = \frac{t_{A}^{1'}\omega^{A}}{\pi_{0'}} \qquad \text{on} \quad \pi_{0'} \neq 0$$

$$\hat{\xi} = \frac{t_{A}^{0'}\omega^{A}}{\pi_{1'}} \qquad \text{on} \quad \pi_{1'} \neq 0$$

These are related by

$$\begin{pmatrix} \hat{\xi} \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & \tilde{\xi} Z^{0} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \xi \\ 1 \end{pmatrix}$$

Thus $V$ makes $P\mathbb{T}^{I} \to \mathbb{T}$ into the affine bundle discussed in §4.1, which represents the solution $(\underline{A}, \phi) = (\underline{0}, 1)$ of the Bogomolny equations on $\mathbb{C}^{3}$. Now, as we deform $P\mathbb{T}^{I}$ according to (4.8), we find that (4.9) still defines coordinates on the minitwistor space (since $t_{A}^{A'}\omega^{A}\pi_{A'}$ is preserved). Thus (4.8) simply deforms the affine bundle structure of $P\mathbb{T}^{I} \to \mathbb{T}$. In the deformed space we may use

$$\xi := \frac{t_{A}^{1'}\omega^{A}}{\pi_{0'}} \quad ; \quad \hat{\xi} := \frac{t_{A}^{0'}\hat{\omega}^{A}}{\pi_{1'}}$$

as fibre coordinates. Using the fact that

$$Z^{0} = \frac{t_{A}^{A'}\omega^{A}\pi_{A'}}{\pi_{0'}^{2}} = \frac{t_{A}^{A'}\hat{\omega}^{A}\pi_{A'}}{\pi_{0'}^{2}}$$

we obtain the transition function

$$\begin{pmatrix} \hat{\xi} \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & ig - \tilde{\xi} Z^{0} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \xi \\ 1 \end{pmatrix}$$

for the deformed bundle. Tod and Ward show that the metric corresponding to the twistor space $\mathcal{Y}$ is

$$ds^2 = \phi^{-1}(dt - A_i dx^i)^2 - \phi\, dx^i dx^i \qquad (4.10)$$

where $\underline{A}$ and $\phi$ are obtained either by

(i) treating $g$ as a twistor function, extracting an A.S.D. Maxwell potential $\Lambda_a$ and setting $\phi = 1 + \Lambda_0$ ; $A_i = \Lambda_i$

or (ii) taking the ministwistor function $ig - \tilde{\zeta} Z^0$ and extracting the solution $(\underline{A}, \phi)$ of the Bogomolny equations.

For vanishing deformation, $g$ becomes a coboundary, so $\underline{A} \to 0$ (or pure gauge), $\phi \to 1$ , thus (4.10) becomes the Minkowski metric.

In summary then, the Tod and Ward twistor spaces of types A and B are respectively the $\mathbb{C}^*$ and affine bundles over $\mathbb{T}$ representing the solution of the Bogomolny equations appearing in the expression for the metric. Moreover the bundles are deformations of the fibrations obtained in the flat case by considering A.S.D. rotation and translation Killing vectors on Minkowski space.

## §4.3 The Eguchi-Hanson Solution

We now show how the Eguchi-Hanson twistor space fits into this picture. Following [8] we describe the twistor space as the subspace $\mathcal{Q}$ of the total space of the bundle $\mathcal{O}(2) \oplus \mathcal{O}(2) \oplus \mathcal{O}(2) \to \mathbb{P}_1$ given by

$$xy = (z - P_1(\zeta))(z - P_2(\zeta))$$

where $x, y, z \in \mathcal{O}(2)$ and $P_1$ and $P_2$ are quadratic polynomials. For fixed $\zeta$ , this is a non-singular quadric

in $\mathbb{C}^3$ except for the two values of $\zeta$ for which $p_1 = p_2$ over which the quadric degenerates to a cone.

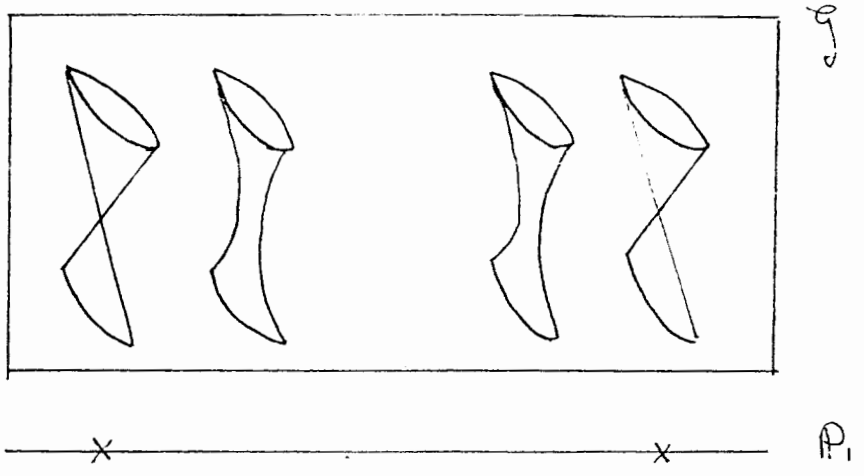



Fig. 4.2

The vector field $V = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$ on $\mathcal{Q}$ gives rise to a Killing vector on the positive definite space-time $\mathcal{M}$. The factor space of $\mathcal{M}$ by this Killing vector is $\mathbb{R}^3$. We now examine the corresponding factor space of $\mathcal{Q}$ by $V$:

The integral curves of $V$ project onto the copy of $\mathcal{O}(2)$ given by $(\zeta, Z)$. Over a generic point $(\zeta, Z)$ the integral curve is

$$xy = (Z - p_1(\zeta))(Z - p_2(\zeta)) = \text{const}$$

and is biholomorphic to $\mathbb{C}^*$. Over the points $(\zeta, p_i(\zeta))$ $i = 1$ or $2$, the curve is $xy = 0$ i.e. consists of the $x$ and $y$ axis in $\mathbb{C}^2$. We make this into two disjoint copies of $\mathbb{C}^*$ by removing the origin, i.e. removing from $\mathcal{Q}$ points of the form

$$(\zeta, x, y, Z) = (\zeta, 0, 0, p_i(\zeta)) \qquad i = 1 \text{ or } 2.$$

We then have a fibration over $T = \mathcal{O}(2)$ where each fibre is biholomorphic to $\mathbb{C}^*$ except over the two sections $Z = \rho_i(\zeta)$ of $T$, over which the fibre is a pair of copies of $\mathbb{C}^*$. The correct factor space is thus a non Hausdorff version, $T''$, of $T$ with the two sections $Z = \rho_i(\zeta)$ doubled up.

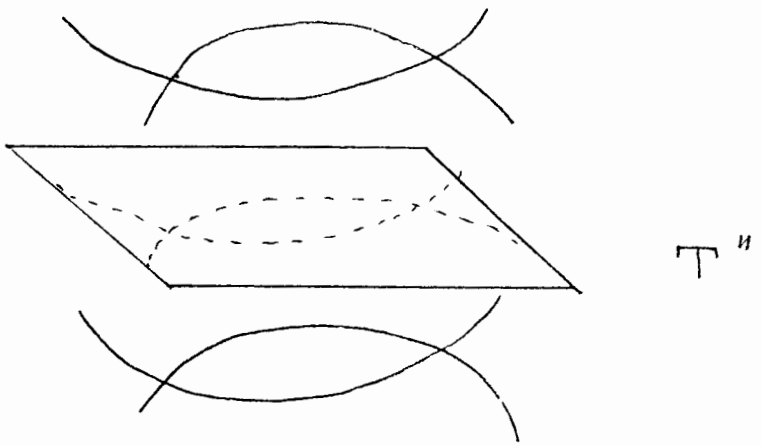



Fig. 4.3

We have remarked that the fibres of $\mathcal{G} \to T''$ are <u>biholomorphic</u> to $\mathbb{C}^*$. To make $\mathcal{G} \to T''$ into a $\mathbb{C}^*$ bundle we must give each of them the structure of $\mathbb{C}^*$. To do this, note that $\mathcal{G}$ can be completed by adding a hyperplane at infinity common to each fibre of $\mathcal{O}(2) \oplus \mathcal{O}(2) \oplus \mathcal{O}(2) \to \mathbb{P}_1$. This adds a plane section $I$ at infinity, common to each quadric in $\mathcal{G}$. (fig 4.4)

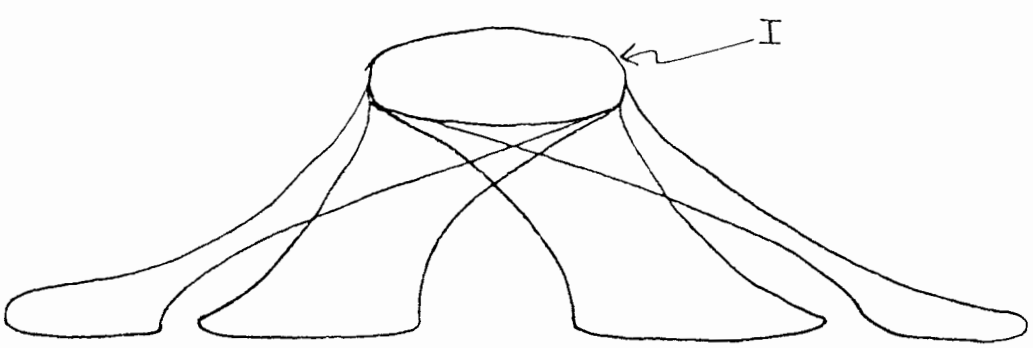



Fig. 4.4

The integral curves of $V$ are plane sections of quadrics $\subset \mathbb{C}^3$ and hence extend to plane sections of the completed quadrics $\subset \mathbb{P}^3$ . Thus each integral curve is a rational curve which meets the plane section at infinity, $I$ , in two points, except for the ones corresponding to planes tangent to the quadric which consist of a pair of $P_1$'s each meeting $I$ in one point.

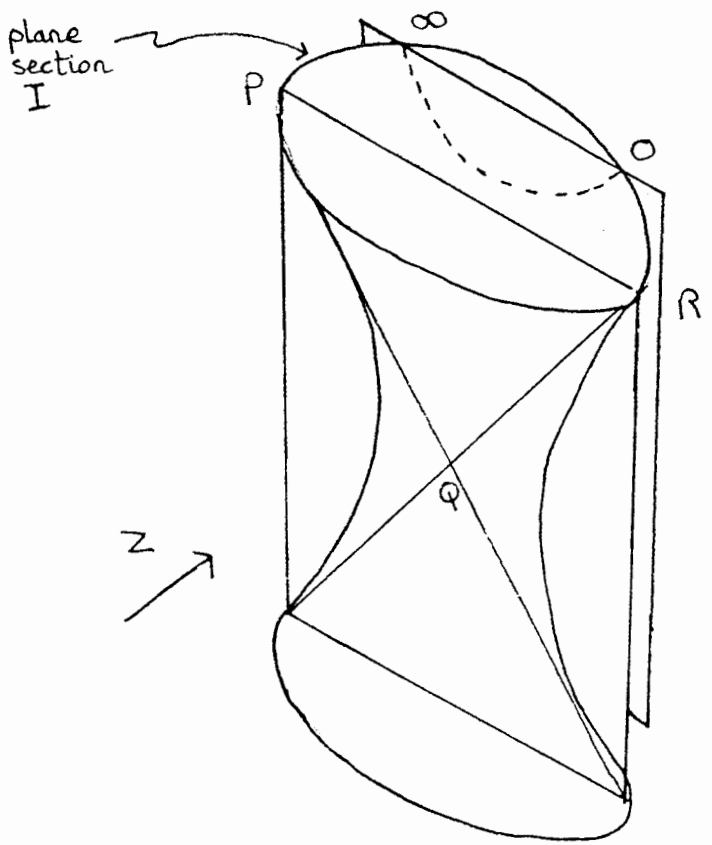



Fig. 4.5 As $z$ varies, the plane $z = \text{const}$ sweeps across the quadric. For two values of $z$ , namely $P_1(J)$ , $P_2(J)$ it is a tangent plane.

On the generic integral curve, its two intersection points with $I$ can be used to define zero and infinity, thus giving a $\mathbb{C}^*$ structure. If zero and infinity are assigned to the (non tangent) plane section as shown in fig. 4.5, then on the tangent plane we must assign infinity to $P$ and

zero to $R$ . Therefore, if we use $x$ as the fibre coordinate along $QP$ , we must use $\frac{1}{y}$ rather than $y$ as the fibre coordinate along $QR$ .

If we consider $T''$ as obtained from two copies $T_0, T_1$ of $T$ identified away from $z = P_1(\zeta)$ or $P_2(\zeta)$ , then $\mathcal{O} \to T''$ has fibre coordinates $x$ over $T_0$ (i.e. restricted to $T_0$ it is $\mathcal{O}(2)$ ) and $\frac{1}{y}$ over $T_1$ (i.e. restricted to $T_1$ it is $\mathcal{O}(-2)$ ). The transition function is

$$x = (z - P_1(\zeta))(z - P_2(\zeta)) \cdot \frac{1}{y} \tag{4.11}$$

Consider a minitwistor curve $C \subset T''$ . $C$ intersects $P_1(\zeta)$ and $P_2(\zeta)$ in two points each. Take a pair from $T_0$ and a pair from $T_1$ as in fig. 4.6

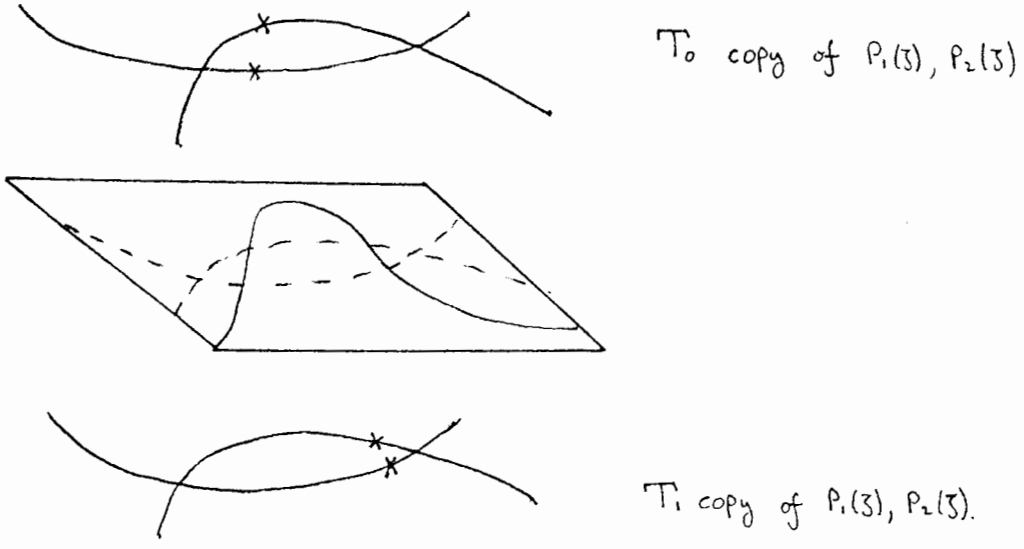



Fig. 4.6

Restricting the bundle to $C$ and splitting the transition functions (4.11) as in §1.10 we obtain the solution $(\phi, \underline{A})$

of the Bogomolny equations corresponding to unit charges at $\underline{x}_1$ and $\underline{x}_2$ in $\mathbb{R}^3$ ( where these are the points represented by the minitwistor curves $P_1(\zeta)$ and $P_2(\zeta)$ ).

Hitchin shows the metric corresponding to $\mathcal{G}$ to be

$$\phi^{-1}(dt - A_i dx^i)^2 - \phi\, dx^i dx^i \qquad x^i \in \mathbb{R}^3$$

Of course $\mathbb{T}''$ is precisely the minitwistor space for the complement of $\underline{x}^1$ and $\underline{x}^2$ in $\mathbb{R}^3$ . We again have a picture where the twistor space is the $\mathbb{C}^*$ bundle over $\mathbb{T}$ representing the solution of the Bogomolny equations appearing in the metric.

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