

## ***hp*-version interior penalty DGFEMs for the biharmonic equation**

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We construct *hp*-version interior penalty discontinuous Galerkin finite element methods (DGFEMs) for the biharmonic equation, including symmetric and nonsymmetric interior penalty discontinuous Galerkin methods and their combinations: semisymmetric methods. Our main concern is to establish the stability and to develop the *a priori* error analysis of these methods. We establish error bounds that are optimal in  $h$  and slightly suboptimal in  $p$ . The theoretical results are confirmed by numerical experiments.

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# 1 Introduction

Conforming finite element methods for the numerical solution of boundary value problems for the biharmonic equation require that the approximate solution lie in a finite-dimensional subspace of the Sobolev space  $H^2(\Omega)$ . In particular, this necessitates the use of  $C^1$  finite elements; i.e., the basis functions of the finite element space, together with their first partial derivatives, need to be continuous over  $\bar{\Omega}$ . Because the construction of such finite element spaces is fairly involved,  $H^2(\Omega)$ -conforming finite elements are rarely used in practical computations. One way to relax these regularity requirements is to use nonconforming methods which rely on continuous finite element basis functions that do not belong to  $H^2(\Omega)$  (and are, therefore, not included in  $C^1(\bar{\Omega})$  either). For details, see [12], [13] and references therein.

Other approaches to avoid the use of  $C^1$  finite elements include hybrid and mixed finite element methods. The literature on mixed methods is extensive, and we refer to the survey paper [27] and the monograph of Brezzi and Fortin [10] for general results concerning the construction and the analysis of these methods. Some applications of mixed and hybrid methods for the biharmonic problem are presented in [30], [21], [19] and [11].

In recent years discontinuous Galerkin finite element methods (DGFEMs) have been widely used for the numerical solution of a large range of computational problems for partial differential equations, including linear and nonlinear hyperbolic problems, convection-dominated diffusion problems and second-order elliptic problems. For an excellent historical survey of the subject, a summary of various DGFEMs, and an extensive list of references, we refer to the special volume [14].

Already in some of the early papers on DGFEMs for second-order elliptic equations the bilinear form of the method included interior penalty terms to penalise jumps across element faces in the numerical solution (cf. [1], [33] and [22]) and to ensure the coercivity of the bilinear form. More recently, starting from the same basic premise, several authors introduced DGFEMs for second-order elliptic problems, such as the nonsymmetric interior penalty DGFEM (NIPG) (cf. Rivi re, Wheeler, Girault [26], [25], and Houston, Schwab and S li [32], [17]), and the symmetric interior penalty DGFEM (SIPG) (cf. Arnold [1] and Wheeler [33]). A discontinuous Galerkin finite element method without interior penalty terms was proposed by Baumann and Oden [7], [23]. A detailed study and a unified error analysis of DGFEMs for second-order elliptic problems is given in the paper by Arnold *et al.* [2].

An interior-penalty finite element method for fourth-order elliptic equations was proposed in [15] and in [5]. These methods used classical continuous finite element spaces with penalization of discontinuities of derivatives at element faces. Recently this approach was further developed in [16], where a continuous/discontinuous Galerkin method, which combines concepts from the theory of continuous and discontinuous Galerkin methods with ideas from the theory of stabilised methods, was proposed for fourth-order elliptic partial differential equations.

The purpose of this paper is to extend to higher-order elliptic equations the  $hp$ -version of the interior penalty DGFEM in symmetric and nonsymmetric formulations. Our main

concern is to establish the stability of these methods and to derive *a priori* error bounds for the methods. For reasons of clarity of exposition, we consider the simple case of the Dirichlet problem for the biharmonic equation, although the basic ideas developed here are readily extendable to linear elliptic operators of order  $2m$  for any  $m \geq 1$ .

The paper is structured as follows. In Section 2 we introduce finite element spaces consisting of discontinuous piecewise polynomials and broken Sobolev spaces. Then we formulate, in Section 3, the model boundary value problem for the biharmonic equation; we consider the weak discontinuous formulation of the problem, show the consistency of this formulation leading to a Galerkin orthogonality property, and demonstrate the boundedness of the associated bilinear form. In Section 4 we present a family of Discontinuous Galerkin methods for the biharmonic equation, which includes NIPG and SIPG methods and their combinations: the semisymmetric methods SSIPG1 and SSIPG2. In this section we also prove the coercivity of the general bilinear form and deduce from this result the coercivity of these four methods, for suitable choices of the penalty parameters. In Section 5 we prove *hp*-version *a priori* error bounds in the energy norm for the interior penalty Galerkin methods introduced in Section 4. First, using the coercivity results from Section 4, we prove *hp*-version error bounds for each of NIPG, SIPG, SSIPG1 and SSIPG2, following the ideas from [24]. Then, in the case of the NIPG method, we present an alternative *hp*-version error analysis, inspired by the results of [17]; thus we obtain the same order of convergence but with a weaker restriction on the size of the penalty parameters with respect to the polynomial degree  $p$ . In particular, we establish error bounds that are optimal in  $h$  and suboptimal in  $p$ . Section 6 presents a series of numerical experiments which confirm the theoretically predicted convergence rates.

## 2 Finite element spaces

Suppose that  $\Omega$  is a bounded, open, convex polyhedral domain in  $\mathbb{R}^d$ ,  $d \geq 2$ , with boundary  $\partial\Omega$  which is the union of its open  $(d-1)$ -dimensional faces. Let us consider a family of partitions  $\{\mathcal{K}_h\}$  of  $\Omega$ , parametrised by  $h > 0$ , into disjoint open and convex element domains  $K = K_j$  such that

$$\bar{\Omega} = \bigcup_{K \in \mathcal{K}_h} \bar{K} \quad \text{and} \quad K_i \cap K_j = \emptyset \quad \text{for } i \neq j.$$

We define a piecewise constant mesh function  $h_{\mathcal{K}}$  by

$$h_{\mathcal{K}}(x) = h_K = \text{diam}(K), \quad x \in K, \quad K \in \mathcal{K},$$

and put

$$h = \max_{K \in \mathcal{K}_h} h_K.$$

Let  $\hat{K}$  be a fixed master element in  $\mathbb{R}^d$ ; here we shall suppose that  $\hat{K}$  is the open unit hypercube in  $\mathbb{R}^d$ . We shall further assume that each  $K \in \mathcal{K}_h$  is an affine image of the

master element  $\widehat{K}$ :

$$K = F_K(\widehat{K}), \quad K \in \mathcal{K}_h.$$

Let  $\mathcal{E}$  be the set of all open  $(d-1)$ -dimensional faces of all elements  $K \in \mathcal{K}_h$ . We also define a piecewise constant face-function on  $\mathcal{E}$ :

$$h_{\mathcal{E}}(x) = h_e = \text{diam}(e), \quad x \in e.$$

Let us assume that the family of partitions  $\{\mathcal{K}_h\}_{h>0}$  is shape-regular (cf. Remark 2.2, p.114, in [9]). We note that for a shape-regular family there exists a positive constant  $c$  (shape-regularity constant), independent of  $h$ , such that

$$ch_K \leq h_e \leq h_K \quad \forall K \in \bigcup_{h>0} \mathcal{K}_h \quad \forall e \in \partial K; \quad (2.1)$$

hence, for any element  $K \in \mathcal{K}_h$ ,  $h_K$  and  $h_e$  are equal to within a constant.

For a nonnegative integer  $m$ , we denote by  $\mathcal{Q}_m(\widehat{K})$  the set of all tensor-product polynomials of degree  $m$  or less in each coordinate direction. Then, to each  $K \in \mathcal{K}_h$  we assign a nonnegative integer  $p_K$  (the local polynomial degree) and a nonnegative integer  $s_K$  (the local Sobolev space index). Collecting the  $p_K, s_K$  and  $F_K$  in the vectors  $\mathbf{p} = (p_K : K \in \mathcal{K}_h)$ ,  $\mathbf{s} = (s_K : K \in \mathcal{K}_h)$  and  $\mathbf{F} = (F_K : K \in \mathcal{K}_h)$ , respectively, we introduce the finite element space

$$S^{\mathbf{p}}(\Omega, \mathcal{K}_h, \mathbf{F}) = \left\{ u \in L^2(\Omega) : u|_K \circ F_K \in \mathcal{Q}_{p_K}(\widehat{K}) \quad \forall K \in \mathcal{K}_h \right\}.$$

Moreover we define, for the partition  $\mathcal{K}_h$ , the broken Sobolev space

$$H^s(\Omega, \mathcal{K}_h) = \left\{ u \in L^2(\Omega) : u|_K \in H^{s_K}(K) \quad \forall K \in \mathcal{K}_h \right\}$$

with the broken Sobolev norm and seminorm

$$\|u\|_{\mathbf{s}, \mathcal{K}_h} = \left( \sum_{K \in \mathcal{K}_h} \|u\|_{H^{s_K}(K)}^2 \right)^{\frac{1}{2}}, \quad |u|_{\mathbf{s}, \mathcal{K}_h} = \left( \sum_{K \in \mathcal{K}_h} |u|_{H^{s_K}(K)}^2 \right)^{\frac{1}{2}}.$$

We shall write  $H^s(\Omega, \mathcal{K}_h)$ ,  $\|u\|_{s, \mathcal{K}_h}$  and  $|u|_{s, \mathcal{K}_h}$  when  $s_K = s$ ,  $K \in \mathcal{K}_h$ . For  $u \in H^2(\Omega, \mathcal{K}_h)$  let us write  $u_K = u|_K$  and define the broken gradient  $\nabla_{\mathcal{K}_h}$  by  $(\nabla_{\mathcal{K}_h} u)|_K = (\nabla u|_K)$  and the broken Laplacian  $\Delta_{\mathcal{K}_h}$  by  $(\Delta_{\mathcal{K}_h} u)|_K = (\Delta u|_K)$ ,  $K \in \mathcal{K}_h$ .

Let us further introduce the set  $\mathcal{E}_{\text{int}}$  of all interior faces

$$\mathcal{E}_{\text{int}} = \{e \in \mathcal{E} : e \subset \Omega\}$$

and the set  $\mathcal{E}_{\Gamma}$  of all boundary faces

$$\mathcal{E}_{\partial} = \{e \in \mathcal{E} : e \subset \partial\Omega\}.$$

For an integer  $m$  we define

$$\{p^m\}_{\mathcal{E}}(x) = \{p^m\}_e = \frac{p_K^m + p_{K'}^m}{2}, \quad x \in e,$$

where  $e \in \mathcal{E}_{\text{int}}$  and the elements  $K$  and  $K'$  share the face  $e$ ; for  $e \in \mathcal{E}_{\partial}$ ,  $e \subset \partial K$ , we let  $\{p^m\}_{\mathcal{E}}(x) = p_K^m$ .

Further, let

$$\Gamma_{\text{int}} = \{x \in \Omega : x \in e \text{ for some } e \in \mathcal{E}_{\text{int}}\}$$

and  $\Gamma = \Gamma_{\text{int}} \cup \partial\Omega$ . We define, for  $u, v \in L^2(\Gamma)$ , the inner product

$$\langle u, v \rangle_{L^2(\Gamma)} = \int_{\Gamma_{\text{int}}} uv \, ds + \int_{\partial\Omega} uv \, ds$$

with associated norm  $\|\cdot\|_{L^2(\Gamma)}$ .

### 3 Weak formulation on broken Sobolev spaces

We consider the following boundary value problem:

$$\begin{aligned} \Delta^2 u &= f \quad \text{in } \Omega \\ u &= g_0 \quad \text{on } \partial\Omega \\ \eta \cdot \nabla u &= g_1 \quad \text{on } \partial\Omega \end{aligned} \tag{3.1}$$

where the operator  $\Delta$  is defined by

$$\Delta u = \sum_{i=1}^d \frac{\partial^2 u}{\partial x_i^2} \quad \text{and} \quad \Delta^2 u = \Delta(\Delta u),$$

$\eta$  is the external unit normal vector to  $\partial\Omega$  and  $f$ ,  $g_0$  and  $g_1$  are given functions defined in  $\Omega$  and on  $\partial\Omega$ , respectively. Under suitable conditions (cf. [8]) on  $\Omega$  and on the data  $f$ ,  $g_0$  and  $g_1$ , the boundary value problem (3.1) possesses a unique solution  $u \in H^4(\Omega)$ , that depends continuously on the data of the problem.

We now proceed with the derivation of the weak formulation of the boundary value problem (3.1) for the biharmonic equation. We shall suppose for the moment that the solution  $u$  of the problem is a sufficiently smooth function (the regularity of  $u$  shall be discussed later).

For each face  $e \in \mathcal{E}_{\text{int}}$  let  $i$  and  $j$  be such indices that  $i > j$  and the elements  $K_i$  and  $K_j$  share the face  $e$ . Let us define the (element-numbering-dependent) jump across  $e$  and the mean value on  $e$  of  $u \in H^1(\Omega, \mathcal{K}_h)$  by

$$[u]_e = u|_{\partial K_i \cap e} - u|_{\partial K_j \cap e} \quad \text{and} \quad \{u\}_e = \frac{1}{2} \left( u|_{\partial K_i \cap e} + u|_{\partial K_j \cap e} \right),$$

respectively.

For the sake of convenience, we extend the definitions of the jump and the mean value to faces  $e \in \mathcal{E}_{\partial}$  by letting

$$[u]_e = u|_e \quad \text{and} \quad \{u\}_e = \frac{1}{2} u|_e.$$

With each face  $e \in \mathcal{E}_{\text{int}}$  we associate the unit normal vector  $\nu = \eta_{K_i}$  to  $e$  which points from  $K_i$  to  $K_j$ , and with each  $e \in \mathcal{E}_{\partial}$  we associate the external unit normal vector  $\nu = \eta_K$  where  $e \subset \partial K$ .

Let us note that, for a given face  $e \in \mathcal{E}_{\text{int}}$  shared by two adjacent elements  $K_i$  and  $K_j$  ( $i > j$ ), we can write

$$(\eta_{K_i} \cdot \nabla u_{K_i}) v_{K_i} + (\eta_{K_j} \cdot \nabla u_{K_j}) v_{K_j} = (\nu \cdot \nabla u_{K_i}) v_{K_i} - (\nu \cdot \nabla u_{K_j}) v_{K_j}.$$

Hence, by analogy with the formula

$$ac - bd = \frac{1}{2} (a + b) (c - d) + \frac{1}{2} (a - b) (c + d) \quad \forall a, b, c, d \in \mathbb{R},$$

we get

$$\begin{aligned} (\eta_{K_i} \cdot \nabla u_{K_i}) v_{K_i} + (\eta_{K_j} \cdot \nabla u_{K_j}) v_{K_j} &= \{\nu \cdot \nabla u\} [v] + [\nu \cdot \nabla u] \{v\} \\ &\quad \forall u, v \in H^1(\Omega, \mathcal{K}_h). \end{aligned} \quad (3.2)$$

Using integration by parts over some element  $K \in \mathcal{K}_h$ , for any  $u \in H^4(K)$  and  $v \in H^2(K)$  we obtain

$$\int_K \Delta u \Delta v \, dx = \int_K (\Delta^2 u) v \, dx - \int_{\partial K} \eta_K \cdot \nabla (\Delta u) v \, ds + \int_{\partial K} \Delta u (\eta_K \cdot \nabla v) \, ds \quad \forall K \in \mathcal{K}_h. \quad (3.3)$$

On summing the equation (3.3) over all  $K \in \mathcal{K}_h$  and using the relation (3.2), we deduce that

$$\begin{aligned} &\sum_{K \in \mathcal{K}_h} \int_K \Delta u \Delta v \, dx \\ &= \sum_{K \in \mathcal{K}_h} \int_K (\Delta^2 u) v \, dx - \int_{\Gamma_{\text{int}}} (\{\nu \cdot \nabla (\Delta u)\} [v] + [\nu \cdot \nabla (\Delta u)] \{v\}) \, ds \\ &\quad + \int_{\Gamma_{\text{int}}} (\{\Delta u\} [\nu \cdot \nabla v] + [\Delta u] \{\nu \cdot \nabla v\}) \, ds \\ &\quad + \int_{\partial \Omega} (-\nu \cdot \nabla (\Delta u) v + \Delta u (\nu \cdot \nabla v)) \, ds \end{aligned} \quad (3.4)$$

for any functions  $u \in H^4(\Omega, \mathcal{K}_h)$  and  $v \in H^2(\Omega, \mathcal{K}_h)$ . Note that using the extended definition of the jump and the mean value, in order to include faces of elements that lie on  $\partial \Omega$ , we may rewrite this formula as

$$\begin{aligned} &\sum_{K \in \mathcal{K}_h} \langle \Delta u, \Delta v \rangle_{L^2(K)} \\ &= \sum_{K \in \mathcal{K}_h} \langle \Delta^2 u, v \rangle_{L^2(K)} - \langle \{\nu \cdot \nabla (\Delta u)\}, [v] \rangle_{L^2(\Gamma)} - \langle [\nu \cdot \nabla (\Delta u)], \{v\} \rangle_{L^2(\Gamma)} \\ &\quad + \langle \{\Delta u\}, [\nu \cdot \nabla v] \rangle_{L^2(\Gamma)} + \langle [\Delta u], \{\nu \cdot \nabla v\} \rangle_{L^2(\Gamma)}. \end{aligned} \quad (3.5)$$

Let us introduce the bilinear form

$$B_{\text{DG}}(u, v) = B_{\mathcal{K}_h}(u, v) + B_{\Gamma}(u, v) + B_s(u, v), \quad (3.6)$$

where

$$B_{\mathcal{K}_h}(u, v) = \sum_{K \in \mathcal{K}_h} \langle \Delta u, \Delta v \rangle_{L^2(K)} \quad (3.7)$$

$$\begin{aligned} B_{\Gamma}(u, v) &= J_1(u, v) + \lambda_1 J_1(v, u) - (J_2(u, v) + \lambda_2 J_2(v, u)), \\ J_1(u, v) &= \langle \{\nu \cdot \nabla (\Delta u)\}, [v] \rangle_{L^2(\Gamma)}, \\ J_2(u, v) &= \langle \{\Delta u\}, [\nu \cdot \nabla v] \rangle_{L^2(\Gamma)}. \end{aligned} \quad (3.8)$$

$$B_s(u, v) = \langle \alpha [u], [v] \rangle_{L^2(\Gamma)} + \langle \beta [\nu \cdot \nabla u], [\nu \cdot \nabla v] \rangle_{L^2(\Gamma)}. \quad (3.9)$$

Here  $\lambda_1, \lambda_2 \in [-1, 1]$  are real numbers, whose values are chosen so as to ensure that  $B_{\text{DG}}(\cdot, \cdot)$  has certain desirable properties (such as symmetry and coercivity). The functions  $\alpha, \beta \geq 0$  defined on  $\Gamma$  by

$$\alpha|_e = \alpha_e, \quad \beta|_e = \beta_e \quad \forall e \in \mathcal{E},$$

are called the discontinuity-penalisation parameters; the nonnegative constants  $\alpha_e$  and  $\beta_e$  depend on the discretisation parameters  $h$  and  $p$  in a manner that will be specified later on in the text.

We consider the linear functional  $l(\cdot)$  on  $H^4(\Omega, \mathcal{K}_h)$ , defined by

$$l(v) = l_{\Delta}(v) + l_s(v),$$

$$l_{\Delta}(v) = \sum_{K \in \mathcal{K}_h} \langle f, v \rangle_{L^2(K)} + \lambda_1 \langle g_0, \nu \cdot \nabla (\Delta v) \rangle_{L^2(\partial\Omega)} - \lambda_2 \langle g_1, \Delta v \rangle_{L^2(\partial\Omega)}, \quad (3.10)$$

$$l_s(v) = \langle \alpha g_0, v \rangle_{L^2(\partial\Omega)} + \langle \beta g_1, \nu \cdot \nabla v \rangle_{L^2(\partial\Omega)}. \quad (3.11)$$

Then, the broken weak formulation of the boundary value problem for the biharmonic equation reads as follows: find  $u \in H^4(\Omega, \mathcal{K}_h)$  such that

$$B_{\text{DG}}(u, v) = l(v) \quad \forall v \in H^4(\Omega, \mathcal{K}_h). \quad (3.12)$$

We shall associate with the bilinear form  $B_{\text{DG}}(\cdot, \cdot)$  the energy (semi)norm  $\|\cdot\|_{\Delta}$  defined by

$$\|u\|_{\Delta}^2 = B_{\mathcal{K}_h}(u, u) = \sum_{K \in \mathcal{K}_h} \langle \Delta u, \Delta u \rangle_{L^2(K)}, \quad u \in H^2(\Omega, \mathcal{K}_h), \quad (3.13)$$

and the norms  $\|\cdot\|_{\text{DG}}$  and  $|||\cdot|||_{\text{DG}}$ , defined by

$$\|u\|_{\text{DG}}^2 = \|u\|_{\Delta}^2 + \|\sqrt{\alpha}[u]\|_{L^2(\Gamma)}^2 + \left\| \sqrt{\beta}[\nu \cdot \nabla u] \right\|_{L^2(\Gamma)}^2, \quad u \in H^2(\Omega, \mathcal{K}_h), \quad (3.14)$$

and

$$|||u|||_{\text{DG}}^2 = \|u\|_{\text{DG}}^2 + \left\| \frac{1}{\sqrt{\alpha}} \{\nu \cdot \nabla (\Delta u)\} \right\|_{L^2(\Gamma)}^2 + \left\| \frac{1}{\sqrt{\beta}} \{\Delta u\} \right\|_{L^2(\Gamma)}^2, \quad u \in H^4(\Omega, \mathcal{K}_h). \quad (3.15)$$

The norm  $|||\cdot|||_{\text{DG}}^2$  represents an extension, to the biharmonic equation, of the norm introduced in Houston *et al.* in [32] and [18] for second-order elliptic equations, and the norm  $|||\cdot|||_{\text{DG}}^2$  is analogous to the norm introduced by Baumann *et al.* in [7], [23] and by Baker *et al.* in [6].

**Lemma 1** *If  $\alpha > 0$  and  $\beta > 0$  on  $\mathcal{E}$ , then  $\|u\|_{\text{DG}}$  is a norm on  $H^2(\Omega, \mathcal{K}_h)$ .*

*Proof.* If  $\|u\|_{\text{DG}} = 0$  for some  $u \in H^2(\Omega, \mathcal{K}_h)$ , then  $u$  is a solution to the following interface problem

$$\begin{aligned} \Delta u &= 0, & \text{in } K \text{ for all } K \in \mathcal{K}_h, \\ [u]_e &= 0, \quad [\nu \cdot \nabla u]_e = 0 & \text{for all } e \in \mathcal{E}_{\text{int}}, \\ u &= 0 & \text{for all } e \in \mathcal{E}_{\partial}. \end{aligned}$$

Then, it follows from the theory of elliptic interface problems (e.g. [29], [20]) that  $u = 0$  on the whole of  $\bar{\Omega}$ . The other axioms of norm (homogeneity and the triangle inequality) are easily verified.  $\square$

We note in passing that since  $H^4(\Omega, \mathcal{K}_h) \subset H^2(\Omega, \mathcal{K}_h)$ , then  $\|\cdot\|_{\text{DG}}$  is also a norm on  $H^4(\Omega, \mathcal{K}_h)$  and, therefore, so is  $|||\cdot|||$ .

With these definitions of DG-norms, we have the following continuity result for the bilinear form (3.6), based on the Cauchy–Schwarz inequality.

**Theorem 2** *Let  $B_{\text{DG}}(\cdot, \cdot)$  be the bilinear form defined in (3.6) with  $\lambda_1, \lambda_2 \in [-1, 1]$  and  $\alpha, \beta \geq 0$ . Then, there exists a positive constant  $C$ , such that*

$$|B_{\text{DG}}(u, v)| \leq C |||u|||_{\text{DG}} |||v|||_{\text{DG}}.$$

*Proof.* This result follows by a simple extension of the continuity estimates derived in [24].  $\square$

Let us note that for  $\lambda_1 = \lambda_2 = -1$  the bilinear form  $B_{\text{DG}}(\cdot, \cdot)$  is coercive on  $H^4(\Omega, \mathcal{K}_h)$  with respect to the norm  $|||\cdot|||_{\text{DG}}$ , but is not continuous with respect to this norm; this complicates the proof of the existence and uniqueness of solution to the problem (3.12). To date, even for second-order elliptic problems, the stability of DGM, SIPG and NIPG has not been proved for  $d \geq 2$ ; for  $d = 1$ , the stability of DGM was shown in [3]. Recently Romkes, Oden and Prudhomme [28] introduced a new stabilised DGM (SDGM) formulation for second-order elliptic boundary value problems and proved existence and uniqueness of a solution to SDGM through establishing an inf-sup condition in an appropriate function space.



We shall now show that a strong solution to the boundary value problem for the biharmonic equation, which is smooth enough at the interelement boundaries, is the solution to the problem in the broken weak formulation. Let us start by demonstrating weak continuity of fluxes across the element faces  $e \in \mathcal{E}_{\text{int}}$ .

**Lemma 3** *Suppose that  $u \in H^4(\Omega)$ ; then, for any  $e \in \mathcal{E}_{\text{int}}$ , we have*

$$\int_e [u] v \, ds = \int_e [\nu \cdot \nabla u] v \, ds = \int_e [\Delta u] v \, ds = \int_e [\nu \cdot \nabla (\Delta u)] v \, ds = 0 \quad \forall v \in L^2(e).$$

*Proof.* We follow the ideas of [28], where the weak continuity of  $u$  and  $\nu \cdot \nabla u$  was proved. Let  $e \in \mathcal{E}_{\text{int}}$  and let  $K$  and  $K'$  be the elements sharing the face  $e$ . Let  $\tilde{K} = \text{int}(\overline{K \cup K'})$ . Then, for any  $\varphi \in \mathcal{D}(\tilde{K})$ , after integrating by parts, we have

$$\begin{aligned} \int_{\tilde{K}} \Delta(\Delta u) \varphi \, dx &= - \int_{\tilde{K}} \nabla(\Delta u) \cdot \nabla \varphi \, dx + \int_{\partial \tilde{K}} \eta \cdot \nabla(\Delta u) \varphi \, ds \\ &= - \int_{\tilde{K}} \nabla(\Delta u) \cdot \nabla \varphi \, dx. \end{aligned}$$

If we split the left-hand side integral and perform integration by parts in each  $K$  and  $K'$ , we obtain

$$\begin{aligned} \int_{\tilde{K}} \Delta(\Delta u) \varphi \, dx &= \int_K \Delta(\Delta u) \varphi \, dx + \int_{K'} \Delta(\Delta u) \varphi \, dx \\ &\quad - \int_K \nabla(\Delta u) \cdot \nabla \varphi \, dx - \int_{K'} \nabla(\Delta u) \cdot \nabla \varphi \, dx + \int_e \eta \cdot [\nabla(\Delta u)] \varphi \, ds \\ &= - \int_{\tilde{K}} \nabla(\Delta u) \cdot \nabla \varphi \, dx + \int_e \eta \cdot [\nabla(\Delta u)] \varphi \, ds. \end{aligned}$$

From these two identities it follows that

$$\int_e \eta \cdot [\nabla(\Delta u)] \varphi \, ds = 0 \quad \forall \varphi \in \mathcal{D}(\tilde{K}). \quad (3.16)$$

Hence,

$$\int_e \eta \cdot [\nabla(\Delta u)] \varphi \, ds = 0 \quad \forall \varphi \in \mathcal{D}(e).$$

As  $\mathcal{D}(e)$  is dense in  $L^2(e)$ , it follows that

$$\int_e \eta \cdot [\nabla(\Delta u)] \varphi \, ds = 0 \quad \forall \varphi \in L^2(e),$$

as required.

Similarly, using the integration-by-parts formula (3.3), we have

$$\begin{aligned}
\int_{\tilde{K}} \Delta(\Delta u) \varphi \, dx &= - \int_{\tilde{K}} \Delta u \Delta \varphi \, dx + \int_{\partial \tilde{K}} \eta \cdot \nabla(\Delta u) \varphi \, ds - \int_{\partial \tilde{K}} \Delta u (\eta \cdot \nabla \varphi) \, ds \\
&= - \int_{\tilde{K}} \Delta u \Delta \varphi \, dx
\end{aligned}$$

and

$$\begin{aligned}
\int_{\tilde{K}} \Delta(\Delta u) \varphi \, dx &= \int_K \Delta u \Delta \varphi \, dx + \int_{K'} \Delta u \Delta \varphi \, dx \\
&\quad + \int_e \eta \cdot [\nabla(\Delta u)] \varphi \, dx - \int_e [\Delta u] (\eta \cdot \nabla \varphi) \, ds \\
&= - \int_{\tilde{K}} \nabla(\Delta u) \cdot \nabla \varphi \, dx + \int_e \eta \cdot [\nabla(\Delta u)] \varphi \, dx - \int_e [\Delta u] (\eta \cdot \nabla \varphi) \, ds,
\end{aligned}$$

which, together with (3.16), implies that

$$\int_e [\Delta u] (\eta \cdot \nabla \varphi) \, ds = 0 \quad \forall \varphi \in \mathcal{D}(\tilde{K}). \quad (3.17)$$

That completes the proof.  $\square$

**Theorem 4** *The broken weak formulation (3.12) of the boundary value problem (3.1) is consistent in the space  $H^4(\Omega)$  in the sense that any solution  $u$  to the boundary value problem, such that  $u \in H^4(\Omega)$ , solves (3.12) as well.*

*Proof.* Using the integration-by-parts formula (3.3) and the defining expressions for  $B_{\text{DG}}$ ,  $l_\Delta$  and  $l_s$ , for  $u \in H^4(\Omega, \mathcal{K}_h)$  we have that

$$\begin{aligned}
B_{\text{DG}}(u, v) - l(v) &= \sum_{K \in \mathcal{K}_h} \int_K (\Delta^2 u - f) v \, dx \\
&\quad - \int_{\Gamma_{\text{int}}} [\nu \cdot \nabla(\Delta u)] \{v\} \, ds + \int_{\Gamma_{\text{int}}} [\Delta u] \{\nu \cdot \nabla v\} \, ds \\
&\quad + \lambda_1 \int_{\Gamma_{\text{int}}} [u] \{\nu \cdot \nabla(\Delta v)\} \, ds - \lambda_2 \int_{\Gamma_{\text{int}}} [\nu \cdot \nabla u] \{\Delta v\} \, ds \\
&\quad + \int_{\Gamma} \lambda_1 (u - g_0) \mu \cdot \nabla(\Delta v) \, ds - \lambda_2 \int_{\Gamma} (\mu \cdot \nabla u - g_1) \Delta v \, ds \\
&\quad - \int_{\Gamma} \alpha (u - g_0) v \, ds - \int_{\Gamma} \beta (\mu \cdot \nabla u - g_1) [\nu \cdot \nabla v] \, ds \\
&\quad - \int_{\Gamma_{\text{int}}} \alpha [u] [v] \, ds - \int_{\Gamma_{\text{int}}} \beta [\nu \cdot \nabla u] [\nu \cdot \nabla v] \, ds \quad \forall v \in H^4(\Omega, \mathcal{K}_h).
\end{aligned}$$

Now, using Lemma 3 it follows that any solution  $u \in H^4(\Omega)$  to the boundary value problem (3.16) is a weak discontinuous solution of (3.12).  $\square$

An immediate consequence of consistency is the Galerkin orthogonality property

$$B_{\text{DG}}(u - u_{\text{DG}}, v) = 0 \quad \forall v \in H^4(\Omega, \mathcal{K}_h), \quad (3.18)$$

where  $u \in H^4(\Omega)$  is a strong solution to the boundary value problem (3.16) and  $u_{\text{DG}} \in H^4(\Omega, \mathcal{K}_h)$  is a solution to the broken weak formulation.

In what follows we shall suppose that the solution  $u$  to the boundary value problem (3.16) is sufficiently smooth, that is  $u \in H^4(\Omega)$ , and that, therefore, the broken weak formulation (3.12) of the boundary value problem admits a (unique) solution.

## 4 DGFEM formulations and the coercivity of the bilinear forms over finite element spaces

We can associate now with the broken weak formulation considered above the following Discontinuous Galerkin Finite Element Method (DGFEM): find  $u_{\text{DG}} \in S^{\mathbf{P}}(\Omega, \mathcal{K}_h, \mathbf{F})$  such that

$$B_{\text{DG}}(u_{\text{DG}}, v) = l(v) \quad \forall v \in S^{\mathbf{P}}(\Omega, \mathcal{K}_h, \mathbf{F}). \quad (4.1)$$

In order to ensure that (4.1) is meaningful, we shall assume that  $p_K \geq 3$  for all  $K \in \mathcal{K}_h$ .

The choice  $\lambda_1 = \lambda_2 = -1$  gives rise to the nonsymmetric interior penalty Galerkin (NIPG) formulation, which was introduced by Rivière, Wheeler and Girault [26], [25] and Houston, Schwab and Süli [32], [17] for elliptic equations of second order. It is straightforward to show that the bilinear form is coercive for this formulation.

**Theorem 5** *Let  $\lambda_1 = \lambda_2 = -1$ ,  $\alpha > 0$ ,  $\beta > 0$ ; then the NIPG method has a unique solution  $u_{\text{DG}} \in S(\Omega, \mathcal{K}_h, \mathbf{F})$ .*

*Proof.* As it is easy to see from (3.8), for  $\lambda_1 = \lambda_2 = -1$  we have

$$\begin{aligned} B_{\text{DG}}(u, u) &= \sum_{K \in \mathcal{K}_h} \|\Delta u\|_{L^2(K)}^2 + \|\sqrt{\alpha}[u]\|_{L^2(\Gamma)}^2 + \left\| \sqrt{\beta}[\nu \cdot \nabla u] \right\|_{L^2(\Gamma)}^2 \\ &\equiv \|u\|_{\text{DG}}^2 \quad \forall u \in S^{\mathbf{P}}(\Omega, \mathcal{K}_h, \mathbf{F}). \end{aligned} \quad (4.2)$$

Since we have shown that  $\|\cdot\|_{\text{DG}}$  is a norm on the space  $H^4(\Omega, \mathcal{K}_h)$  and  $S^{\mathbf{P}}(\Omega, \mathcal{K}_h, \mathbf{F}) \subset H^4(\Omega, \mathcal{K}_h)$  we have that  $\|\cdot\|_{\text{DG}}$  is also a norm on  $S^{\mathbf{P}}(\Omega, \mathcal{K}_h, \mathbf{F})$ .

Hence,  $B(\cdot, \cdot)$  is a coercive bilinear form on the finite-dimensional space  $S^{\mathbf{P}}(\Omega, \mathcal{K}_h, \mathbf{F})$ , and therefore the problem (4.1) has a unique solution in this space.  $\square$

Setting  $\lambda_1 = \lambda_2 = 1$  yields the symmetric interior penalty Galerkin (SIPG) formulation with a symmetric bilinear form; unfortunately, the bilinear form is noncoercive unless the penalty parameters are chosen sufficiently large. This formulation was introduced by Arnold [1] and Wheeler [33] for second-order elliptic equations. The NIPG formulation

without penalty terms, that is with  $\alpha = \beta = 0$ , corresponds to the Discontinuous Galerkin Method (DGM) by Baumann and Oden [7], [23] in the case of second-order elliptic problems. In this formulation the bilinear form is nonnegative. For the biharmonic equation we shall introduce here two semisymmetric interior penalty Galerkin methods:SSIPG1 and SSIPG2, which correspond to  $\lambda_1 = -1, \lambda_2 = 1$  and  $\lambda_1 = 1, \lambda_2 = -1$ , respectively.

We shall suppose throughout that the strong solution  $u$  to the boundary value problem satisfies the smoothness assumption  $u \in H^4(\Omega)$ , so as to ensure that  $u$  is a solution to (3.12) and therefore to (4.1). Consequently the Galerkin orthogonality property

$$B_{\text{DG}}(u - u_{\text{DG}}, v) = 0 \quad (4.3)$$

will hold for all  $v \in S^{\mathbf{P}}(\Omega, \mathcal{K}_h, \mathbf{F})$ .

**Lemma 6** *Let  $B_{\text{DG}}(\cdot, \cdot)$  be the bilinear form defined in (3.6) with  $\lambda_1, \lambda_2 \in [-1, 1]$  and with*

$$\alpha_e = \sigma_\alpha \frac{\{p^6\}_e}{h_e^3}, \quad \beta_e = \sigma_\beta \frac{\{p^2\}_e}{h_e}$$

*on  $\mathcal{E}$ . Let  $c_\alpha$  and  $c_\beta$  be positive constants whose values are specified in the proof of the lemma and assume that  $\underline{\sigma}_\alpha > 0$  and  $\underline{\sigma}_\beta > 0$  are such that*

$$\max \left( \frac{1 + \lambda_1}{2}, \frac{1 + \lambda_2}{2} \right) \left( \frac{c_\alpha}{\underline{\sigma}_\alpha} \left( \frac{1 + \lambda_1}{2} \right) + \frac{c_\beta}{\underline{\sigma}_\beta} \left( \frac{1 + \lambda_2}{2} \right) \right) < 1.$$

*Suppose that  $\sigma_\alpha \geq \underline{\sigma}_\alpha$  and  $\sigma_\beta \geq \underline{\sigma}_\beta$ . Then, there exists a positive constant  $\theta$  such that*

$$B_{\text{DG}}(u, u) \geq \theta \|u\|_{\text{DG}}^2 \quad \forall u \in S^{\mathbf{P}}(\Omega, \mathcal{K}_h, \mathbf{F}).$$

*Proof.* Suppose that  $\lambda_1, \lambda_2 \in [-1, 1]$  and

$$\max \left( \frac{1 + \lambda_1}{2}, \frac{1 + \lambda_2}{2} \right) \left( \frac{c_\alpha}{\underline{\sigma}_\alpha} \left( \frac{1 + \lambda_1}{2} \right) + \frac{c_\beta}{\underline{\sigma}_\beta} \left( \frac{1 + \lambda_2}{2} \right) \right) < 1.$$

Then, on adopting the notational convention  $0^{-1} = +\infty$ , we have that

$$\max \left( \frac{1 + \lambda_1}{2}, \frac{1 + \lambda_2}{2} \right) < \left( \frac{c_\alpha}{\underline{\sigma}_\alpha} \left( \frac{1 + \lambda_1}{2} \right) + \frac{c_\beta}{\underline{\sigma}_\beta} \left( \frac{1 + \lambda_2}{2} \right) \right)^{-1}.$$

Hence there exists  $\varepsilon > 0$  such that

$$\max \left( \frac{1 + \lambda_1}{2}, \frac{1 + \lambda_2}{2} \right) < \varepsilon < \left( \frac{c_\alpha}{\underline{\sigma}_\alpha} \left( \frac{1 + \lambda_1}{2} \right) + \frac{c_\beta}{\underline{\sigma}_\beta} \left( \frac{1 + \lambda_2}{2} \right) \right)^{-1}.$$

Consequently,

$$1 - \frac{1 + \lambda_1}{2\varepsilon} > 0, \quad 1 - \frac{1 + \lambda_2}{2\varepsilon} > 0,$$

and

$$1 - \varepsilon \left( \frac{c_\alpha}{\underline{\sigma}_\alpha} \left( \frac{1 + \lambda_1}{2} \right) + \frac{c_\beta}{\underline{\sigma}_\beta} \left( \frac{1 + \lambda_2}{2} \right) \right) > 0.$$

For  $\lambda_1, \lambda_2 \in [-1, 1]$  let us choose  $\theta$  such that, simultaneously,

$$0 < \theta < 1 - \frac{1 + \lambda_1}{2\varepsilon}, \quad 0 < \theta < 1 - \frac{1 + \lambda_2}{2\varepsilon},$$

and

$$0 < \theta < \frac{1 - \varepsilon \left( \frac{c_\alpha}{\underline{\sigma}_\alpha} \left( \frac{1 + \lambda_1}{2} \right) + \frac{c_\beta}{\underline{\sigma}_\beta} \left( \frac{1 + \lambda_2}{2} \right) \right)}{1 + \frac{c_\alpha}{\underline{\sigma}_\alpha} + \frac{c_\beta}{\underline{\sigma}_\beta}}.$$

Then,

$$1 - \theta - \frac{1 + \lambda_1}{2\varepsilon} > 0, \quad 1 - \theta - \frac{1 + \lambda_2}{2\varepsilon} > 0,$$

and

$$1 - \theta - \frac{c_\alpha}{\underline{\sigma}_\alpha} \left( \theta + \frac{1 + \lambda_1}{2} \varepsilon \right) - \frac{c_\beta}{\underline{\sigma}_\beta} \left( \theta + \frac{1 + \lambda_2}{2} \varepsilon \right) > 0.$$

We shall suppose in what follows that  $\varepsilon$  and  $\theta$  are defined as indicated above.

For an arbitrary  $u \in S^{\mathbf{P}}(\Omega, \mathcal{K}, \mathbf{F})$ , we then have

$$\begin{aligned} B_{\text{DG}}(u, u) &= \theta \|u\|_{\text{DG}}^2 \\ &= (1 - \theta) B_{\mathcal{K}_h}(u, u) + (1 + \lambda_1) J_1(u, u) - (1 + \lambda_2) J_2(u, u) + (1 - \theta) B_s(u, u) \\ &\quad - \theta \left( \left\| \frac{1}{\sqrt{\alpha}} \{ \nu \cdot \nabla (\Delta u) \} \right\|_{L^2(\Gamma)}^2 + \left\| \frac{1}{\sqrt{\beta}} \{ \Delta u \} \right\|_{L^2(\Gamma)}^2 \right) \end{aligned}$$

Further, we have

$$|J_1(u, u)| \leq \frac{1}{2} \left( \varepsilon \left\| \frac{1}{\sqrt{\alpha}} \{ \nu \cdot \nabla (\Delta u) \} \right\|_{L^2(\Gamma)}^2 + \frac{1}{\varepsilon} \left\| \sqrt{\alpha} [\Delta u] \right\|_{L^2(\Gamma)}^2 \right)$$

and

$$|J_2(u, u)| \leq \frac{1}{2} \left( \varepsilon \left\| \frac{1}{\sqrt{\beta}} \Delta u \right\|_{L^2(\Gamma)}^2 + \frac{1}{\varepsilon} \left\| \sqrt{\beta} [\nu \cdot \nabla u] \right\|_{L^2(\Gamma)}^2 \right),$$

and therefore

$$\begin{aligned} B_{\text{DG}}(u, u) &= \theta \|u\|_{\text{DG}}^2 \\ &\geq (1 - \theta) B_{\mathcal{K}_h}(u, u) - \left( \theta + \frac{1 + \lambda_1}{2} \varepsilon \right) \left\| \frac{1}{\sqrt{\alpha}} \{ \nu \cdot \nabla (\Delta u) \} \right\|_{L^2(\Gamma)}^2 \\ &\quad - \left( \theta + \frac{1 + \lambda_2}{2} \varepsilon \right) \left\| \frac{1}{\sqrt{\beta}} \Delta u \right\|_{L^2(\Gamma)}^2 \end{aligned}$$

$$\begin{aligned}
& + \left(1 - \theta - \frac{1 + \lambda_1}{2\varepsilon}\right) \|\sqrt{\alpha} [\Delta u]\|_{L^2(\Gamma)}^2 \\
& + \left(1 - \theta - \frac{1 + \lambda_2}{2\varepsilon}\right) \|\sqrt{\beta} [\nu \cdot \nabla u]\|_{L^2(\Gamma)}^2.
\end{aligned}$$

Given a face  $e \in \mathcal{E}_{\text{int}}$ , let  $K$  and  $K'$  be the elements sharing the face  $e$ . Then, using the inverse inequalities

$$\|\xi\|_{L^2(\partial K)}^2 \leq c_0 \frac{p_K^2}{h_K} \|\xi\|_{L^2(K)}^2 \quad \text{and} \quad \|\nabla \xi\|_{L^2(\partial K)}^2 \leq c_1 \frac{p_K^6}{h_K^3} \|\xi\|_{L^2(K)}^2 \quad \forall \xi \in \mathcal{Q}_{p_K}(K) \quad (4.4)$$

with the constants  $c_0, c_1$  depending only on the shape-regularity constant (see Theorem 4.76 in [31]), and by applying (2.1), we obtain

$$\begin{aligned}
\left\| \frac{1}{\sqrt{\beta}} \Delta u \right\|_{L^2(e)}^2 & \leq c_\beta \frac{h_e}{\{p^2\}_e} \left( \frac{p_K^2}{2h_K} \|\Delta u\|_{L^2(K)}^2 + \frac{p_{K'}^2}{2h_{K'}} \|\Delta u\|_{L^2(K')}^2 \right) \\
& \leq c_\beta \left( \|\Delta u\|_{L^2(K)}^2 + \|\Delta u\|_{L^2(K')}^2 \right),
\end{aligned}$$

with some constant  $c_\beta$  that depends on  $c_0$ . Since the faces  $e \in \mathcal{E}_\partial$  can be handled quite similarly, we obtain

$$\left\| \frac{1}{\sqrt{\beta}} \Delta u \right\|_{L^2(\Gamma)}^2 \leq c_\beta B_{\mathcal{K}_h}(u, u).$$

Analogously we can prove that

$$\left\| \frac{1}{\sqrt{\alpha}} \{\nu \cdot \nabla (\Delta u)\} \right\|_{L^2(\Gamma)}^2 \leq c_\alpha B_{\mathcal{K}_h}(u, u),$$

where the constant  $c_\alpha$  depends only on  $c_1$ .

Combining the above results, we obtain

$$\begin{aligned}
B_{\text{DG}}(u, u) & - \theta \|u\|_{\text{DG}}^2 \\
& \geq \left(1 - \theta - \frac{c_\alpha}{\sigma_\alpha} \left(\theta + \frac{1 + \lambda_1}{2} \varepsilon\right) - \frac{c_\beta}{\sigma_\beta} \left(\theta + \frac{1 + \lambda_2}{2} \varepsilon\right)\right) B_{\mathcal{K}_h}(u, u) \\
& + \left(1 - \theta - \frac{1 + \lambda_1}{2\varepsilon}\right) \|\sqrt{\alpha} [\Delta u]\|_{L^2(\Gamma)}^2 \\
& + \left(1 - \theta - \frac{1 + \lambda_2}{2\varepsilon}\right) \|\sqrt{\beta} [\nu \cdot \nabla u]\|_{L^2(\Gamma)}^2.
\end{aligned}$$

Now, our choice of  $\varepsilon$  and  $\theta$  ensures that the three constants multiplying the expressions  $B_{\mathcal{K}_h}(u, u)$ ,  $\|\sqrt{\alpha} [\Delta u]\|_{L^2(\Gamma)}^2$  and  $\|\sqrt{\beta} [\nu \cdot \nabla u]\|_{L^2(\Gamma)}^2$ , respectively, are all positive. Hence, it follows that

$$B_{\text{DG}}(u, u) - \theta \|u\|_{\text{DG}}^2 \geq 0,$$

which is the desired result.  $\square$

We can now easily deduce the coercivity of the bilinear forms associated with the methods defined above.

**Theorem 7** *Let  $B_{\text{DG}}(\cdot, \cdot)$  be the bilinear form defined in (3.6) with*

$$\alpha_e = \sigma_\alpha \frac{\{p^6\}_e}{h_e^3} \quad \text{and} \quad \beta_e = \sigma_\beta \frac{\{p^2\}_e}{h_e} \quad \text{on } \mathcal{E},$$

*and let  $c_\alpha$  and  $c_\beta$  be positive constants as in Lemma 6. Let us suppose that:*

- (i) *for  $\lambda_1 = \lambda_2 = -1$  (NIPG method)  $\sigma_\alpha$  and  $\sigma_\beta$  are arbitrary positive numbers;*
- (ii) *for  $\lambda_1 = \lambda_2 = 1$  (SIPG method)  $\sigma_\alpha \geq \underline{\sigma}_\alpha$ ,  $\sigma_\beta \geq \underline{\sigma}_\beta$ , where  $\underline{\sigma}_\alpha$  and  $\underline{\sigma}_\beta$  are some positive constants such that  $\frac{c_\alpha}{\underline{\sigma}_\alpha} + \frac{c_\beta}{\underline{\sigma}_\beta} < 1$ ;*
- (iii) *for  $\lambda_1 = -1$ ,  $\lambda_2 = 1$  (SSIPG1 method)  $\sigma_\alpha > 0$  and  $\sigma_\beta \geq \underline{\sigma}_\beta$ , where  $\underline{\sigma}_\beta$  is some positive constant such that  $\underline{\sigma}_\beta > c_\beta$ ;*
- (iv) *for  $\lambda_1 = 1$ ,  $\lambda_2 = -1$  (SSIPG2 method)  $\sigma_\alpha \geq \underline{\sigma}_\alpha$  and  $\sigma_\beta > 0$ , where  $\underline{\sigma}_\alpha$  is some positive constant such that  $\underline{\sigma}_\alpha > c_\alpha$ .*

*Then, there exists a constant  $\theta > 0$  such that*

$$B_{\text{DG}}(u, u) \geq \theta \|u\|_{\text{DG}}^2 \quad \forall u \in S^{\mathbf{p}}(\Omega, \mathcal{K}_h, \mathbf{F}).$$

## 5 Error estimation

In this section we prove  $hp$ -version *a priori* error estimates in the  $\|\cdot\|_{\text{DG}}$  norm for the interior penalty Galerkin methods introduced above. Given that the norm  $\|\cdot\|_{\text{DG}}$  is subordinate to the norm  $|||\cdot|||_{\text{DG}}$ , it suffices to estimate the error with respect to this latter norm.

Let  $\Pi_{\mathbf{p}}$  denote any (linear) projection operator in  $H^s(\Omega, \mathcal{K}_h)$  onto the finite element space  $S^{\mathbf{p}}(\Omega, \mathcal{K}_h, \mathbf{F})$ . We may then decompose the global error  $u - u_{\text{DG}}$  as follows:

$$u - u_{\text{DG}} = (u - \Pi_{\mathbf{p}}u) + (\Pi_{\mathbf{p}}u - u_{\text{DG}}) \equiv \eta + \xi;$$

so we have

$$|||u - u_{\text{DG}}|||_{\text{DG}} \leq |||\eta|||_{\text{DG}} + |||\xi|||_{\text{DG}}. \quad (5.1)$$

Our error analysis below will provide a bound on  $|||\xi|||_{\text{DG}}$  in terms of suitable norms of  $\eta$ . Thereby, we shall obtain a bound on  $|||u - u_{\text{DG}}|||_{\text{DG}}$  in terms of various Sobolev norms of  $\eta$ . Hence, to complete the error analysis we shall need to quantify Sobolev norms of  $\eta$ . To this end, in what follows, we shall consider particular projectors in  $H^s(\Omega, \mathcal{K}_h)$ . We shall rely on the following result from approximation theory (cf. [4]).

**Lemma 8** *Suppose that a partition  $\mathcal{K}_h$  of  $\Omega$  consists of  $d$ -dimensional simplexes or parallelepipeds. Then, for every  $u \in H^{\mathbf{t}}(\Omega, \mathcal{K}_h)$ ,  $\mathbf{t} = (t_K : K \in \mathcal{K}_h)$ , and for each  $\mathbf{p} = (p_K : K \in \mathcal{K}_h, p_K \in \mathbb{N})$ , there exists a projector*

$$\Pi_{\mathbf{p}}^{\mathbf{h}} : H^{\mathbf{t}}(\Omega, \mathcal{K}_h) \rightarrow S^{\mathbf{p}}(\Omega, \mathcal{K}_h, \mathbf{F}), \quad (\Pi_{\mathbf{p}}^{\mathbf{h}} u)|_K = \Pi_{p_K}^{h_K}(u|_K)$$

such that, for  $0 \leq q \leq t_K$ ,

$$\|u - \Pi_{p_K}^{h_K} u\|_{H^q(K)} \leq C \frac{h_K^{s_K - q}}{p_K^{t_K - q}} \|u\|_{H^{t_K}(K)} \quad \forall K \in \mathcal{K}_h, \quad (5.2)$$

and for  $0 \leq q \leq t_K - 1$

$$\|D^\alpha(u - \Pi_{p_K}^{h_K} u)\|_{L_2(\partial K)} \leq C \frac{h_K^{s_K - q - \frac{1}{2}}}{p_K^{t_K - q - \frac{1}{2}}} \|u\|_{H^{t_K}(K)}, \quad |\alpha| = q, \quad \forall K \in \mathcal{K}_h, \quad (5.3)$$

where  $s_K = \min(p_K + 1, t_K)$  and  $C$  is a constant independent of  $u, h_K$  and  $p_K$ , but dependent on  $t = \max_{K \in \mathcal{K}_h} t_K$ .

As in [17], we shall assume that the polynomial degree vector  $\mathbf{p}$ , with  $p_K \in \mathbb{N}$ , has bounded local variation, that is, there exists a constant  $\rho > 0$  such that for any pair of elements  $K$  and  $K'$  which share some face  $e \in \mathcal{E}$ , one has

$$\rho^{-1} p_{K'} \leq p_K \leq \rho p_{K'}.$$

**Theorem 9** *Suppose that  $\Omega$  is a bounded polyhedral domain in  $\mathbb{R}^d$  and that  $\{\mathcal{K}_h\}_{h>0}$  is a shape-regular family of partitions, formed by  $d$ -dimensional parallelepipeds. Let  $\mathbf{p} = (p_K : K \in \mathcal{K}_h, p_K \in \mathbb{N}, p_K \geq 3, K \in \mathcal{K}_h)$  be any polynomial degree vector of bounded local variation. For each face  $e \in \mathcal{E}$ , we define positive, real, piecewise constant face-functions  $\alpha$  and  $\beta$ , by*

$$\alpha_e = \sigma_\alpha \frac{\{p^6\}_e}{h_e^3} \quad \text{and} \quad \beta_e = \sigma_\beta \frac{\{p^2\}_e}{h_e}.$$

Let us also suppose that the parameters  $\sigma_\alpha$  and  $\sigma_\beta$  are such that the bilinear form  $B_{\text{DG}}(\cdot, \cdot)$  is coercive (see Lemma 6 and Theorem 7). Then, if the exact solution  $u$  to the problem (3.12) belongs to  $H^{\mathbf{t}}(\Omega, \mathcal{K}_h)$ ,  $\mathbf{t} = (t_K : K \in \mathcal{K}_h, t_K \geq 4, K \in \mathcal{K}_h)$ , then the solution  $u_{\text{DG}} \in S^{\mathbf{p}}(\Omega, \mathcal{K}_h, \mathbf{F})$  of the problem (4.1) satisfies the following error bound:

$$\|u - u_{\text{DG}}\|_{\text{DG}}^2 \leq C \sum_{K \in \mathcal{K}_h} \frac{h_K^{2s_K - 4}}{p_K^{2t_K - 7}} \|u\|_{H^{t_K}(K)}^2, \quad (5.4)$$

where  $1 \leq s_K \leq \min(p_K + 1, t_K)$ , and  $C$  is a constant dependent only on the space dimension  $d$ , the shape-regularity constant  $c$ , the local polynomial degree variation constant  $\rho$  and on  $t = \max_{K \in \mathcal{K}_h} t_K$ .



*Proof.* From the Galerkin orthogonality property we have that

$$B_{\text{DG}}(u - u_{\text{DG}}, \xi) = 0$$

since  $\xi = \Pi_{\mathbf{P}} u - u_{\text{DG}} \in S^{\mathbf{P}}(\Omega, \mathcal{K}_h, \mathbf{F})$ . Therefore, we get

$$B_{\text{DG}}(u - u_{\text{DG}}, \xi) = B_{\text{DG}}(\eta + \xi, \xi) = B_{\text{DG}}(\eta, \xi) + B_{\text{DG}}(\xi, \xi) = 0,$$

that is

$$B_{\text{DG}}(\xi, \xi) = -B_{\text{DG}}(\eta, \xi) \quad \text{and} \quad B_{\text{DG}}(\xi, \xi) = |B_{\text{DG}}(\eta, \xi)|.$$

Hence from the coercivity and continuity of the bilinear form  $B_{\text{DG}}(\cdot, \cdot)$  we have

$$|||\xi|||_{\text{DG}}^2 \leq \theta^{-1} B_{\text{DG}}(\xi, \xi) = \theta^{-1} |B_{\text{DG}}(\eta, \xi)| \leq C |||\eta|||_{\text{DG}} |||\xi|||_{\text{DG}},$$

which implies that

$$|||\xi|||_{\text{DG}} \leq C |||\eta|||_{\text{DG}},$$

and from (5.1) we then get

$$|||u - u_{\text{DG}}|||_{\text{DG}} \leq C |||\eta|||_{\text{DG}},$$

Thus, to complete the proof it only remains to estimate each of the terms that enter the definition (3.15) of the norm  $|||\bullet|||_{\text{DG}}$ .

For the first norm, using the inequality (5.2) with  $q = 2$  we get

$$\sum_{K \in \mathcal{K}_h} \|\Delta \eta\|_{L^2(K)}^2 \leq C \sum_{K \in \mathcal{K}_h} \|\eta\|_{H^2(K)}^2 \leq C \sum_{K \in \mathcal{K}_h} \frac{h_K^{2s_K-4}}{p_K^{2t_K-4}} \|u\|_{H^{t_K}(K)}^2.$$

For a face  $e \in \mathcal{E}_{\partial}$  let  $K$  be the element such that  $e \subset \partial K$ , and for a face  $e \in \mathcal{E}_{\text{int}}$  let  $K$  and  $K'$  be the elements sharing the face  $e$ . We can then write

$$\begin{aligned} \left\| \frac{1}{\sqrt{\alpha}} \{\nu \cdot \nabla(\Delta \eta)\} \right\|_{L^2(\Gamma)}^2 &\leq \sum_{e \in \mathcal{E}_{\partial}} \frac{1}{\alpha_e} \|\nabla(\Delta \eta_K)\|_{L^2(\partial K)}^2 \\ &\quad + \sum_{e \in \mathcal{E}_{\text{int}}} \frac{1}{\alpha_e} \left( \frac{1}{2} \|\nabla(\Delta \eta_K)\|_{L^2(\partial K)}^2 + \frac{1}{2} \|\nabla(\Delta \eta_{K'})\|_{L^2(\partial K')}^2 \right). \end{aligned}$$

Applying in these inequalities (5.3) with  $q = 3$  we get

$$\begin{aligned} \left\| \frac{1}{\sqrt{\alpha}} \{\nu \cdot \nabla(\Delta \eta)\} \right\|_{L^2(\Gamma)}^2 &\leq C \sum_{e \in \mathcal{E}_{\partial}} \frac{h_e^3}{\{p^6\}_e} \frac{h_K^{2s_K-7}}{p_K^{2t_K-7}} \|u\|_{H^{t_K}(K)}^2 \\ &\quad + C \sum_{e \in \mathcal{E}_{\text{int}}} \frac{h_e^3}{\{p^6\}_e} \left( \frac{h_K^{2s_K-7}}{p_K^{2t_K-7}} \|u\|_{H^{t_K}(K)}^2 + \frac{h_{K'}^{2s_{K'}-7}}{p_{K'}^{2t_{K'}-7}} \|u\|_{H^{t_{K'}}(K')}^2 \right) \\ &\leq C \sum_{K \in \mathcal{K}_h} \frac{h_K^{2s_K-4}}{p_K^{2t_K-1}} \|u\|_{H^{t_K}(K)}^2. \end{aligned}$$

Analogously we deduce that

$$\begin{aligned} \left\| \frac{1}{\sqrt{\beta}} \{\Delta\eta\} \right\|_{L^2(\Gamma)}^2 &\leq C \sum_{K \in \mathcal{K}_h} \frac{h_K^{2s_K-4}}{p_K^{2t_K-3}} \|u\|_{H^{t_K}(K)}^2, \\ \|\alpha[\eta]\|_{L^2(\Gamma)}^2 &\leq C \sum_{K \in \mathcal{K}_h} \frac{h_K^{2s_K-4}}{p_K^{2t_K-7}} \|u\|_{H^{t_K}(K)}^2 \end{aligned}$$

and

$$\|\beta[\nu \cdot \nabla \eta]\|_{L^2(\Gamma)}^2 \leq C \sum_{K \in \mathcal{K}_h} \frac{h_K^{2s_K-4}}{p_K^{2t_K-5}} \|u\|_{H^{t_K}(K)}^2.$$

Combining these inequalities, we have

$$\begin{aligned} \|u - u_{\text{DG}}\|_{\text{DG}}^2 &\leq C \|\eta\|_{\text{DG}}^2 \\ &\leq C \sum_{K \in \mathcal{K}_h} \left( \frac{h_K^{2s_K-4}}{p_K^{2t_K-4}} + \frac{h_K^{2s_K-4}}{p_K^{2t_K-1}} + \frac{h_K^{2s_K-4}}{p_K^{2t_K-3}} + \frac{h_K^{2s_K-4}}{p_K^{2t_K-7}} + \frac{h_K^{2s_K-4}}{p_K^{2t_K-5}} \right) \|u\|_{H^{t_K}(K)}^2 \\ &\leq C \sum_{K \in \mathcal{K}_h} \frac{h_K^{2s_K-4}}{p_K^{2t_K-7}} \|u\|_{H^{t_K}(K)}^2, \end{aligned}$$

hence (5.4) is proved.  $\square$

It is worth noting that the resulting *a priori* error estimate is optimal in  $h$  but is  $p$ -suboptimal by  $\frac{3}{2}$  orders of  $p$ .

Next we shall present an alternative proof of the *a priori* error bound for the NIPG method, following the argument given in [17]. While we cannot recover the optimality of the error bound in  $p$ , we can reduce the power of  $p$  in the penalty terms. We begin by proving the following lemma, based on the fact that the bilinear form  $B_{\text{DG}}(\cdot, \cdot)$  is weakly coercive.

**Lemma 10** *Let  $B_{\text{DG}}(\cdot, \cdot)$  be the bilinear form defined in (3.6) with  $\lambda_1 = \lambda_1 = -1$  and  $\alpha > 0$ ,  $\beta > 0$  on  $\Gamma$ , and let  $\mathcal{K}_h$  be a shape-regular partition of  $\Omega$  consisting of  $d$ -dimensional parallelepipeds. Then, there exists a positive constant  $C$  which depends only on the dimension  $d$  and the shape-regularity constant, such that the following inequality holds:*

$$\|\xi\|_{\text{DG}} \leq C \|\eta\|_*, \quad (5.5)$$

where

$$\begin{aligned} \|\eta\|_*^2 &= \sum_{K \in \mathcal{K}_h} \|\Delta\eta\|_{L^2(K)}^2 + \left\| \frac{1}{\sqrt{\alpha}} \{\nu \cdot \nabla(\Delta\eta)\} \right\|_{L^2(\Gamma)}^2 \\ &\quad + \left\| \frac{1}{\sqrt{\beta}} \{\Delta\eta\} \right\|_{L^2(\Gamma)}^2 + \|\sqrt{\alpha + \gamma}[\eta]\|_{L^2(\Gamma)}^2 + \|\sqrt{\beta + \delta}[\nu \cdot \nabla \eta]\|_{L^2(\Gamma)}^2, \end{aligned} \quad (5.6)$$

with  $\gamma = \{p^6\}_{\mathcal{E}} h_{\mathcal{E}}^{-3}$  and  $\delta = \{p^2\}_{\mathcal{E}} h_{\mathcal{E}}^{-1}$ .

*Proof.* From the Galerkin orthogonality property we can see that to prove (5.5) it is sufficient to show that

$$\|\xi\|_{\text{DG}}^2 = B_{\text{DG}}(\xi, \xi) = |B_{\text{DG}}(\eta, \xi)| \leq C \|\eta\|_* \|\xi\|_{\text{DG}},$$

so we now need to estimate each of the terms in (3.6).

For the first term we have

$$|B_{\mathcal{K}_h}(\eta, \xi)| \leq \left( \sum_{K \in \mathcal{K}_h} \|\Delta \eta\|_{L^2(K)}^2 \right)^{\frac{1}{2}} \left( \sum_{K \in \mathcal{K}_h} \|\Delta \xi\|_{L^2(K)}^2 \right)^{\frac{1}{2}} \leq \left( \sum_{K \in \mathcal{K}_h} \|\Delta \eta\|_{L^2(K)}^2 \right)^{\frac{1}{2}} \|\xi\|_{\text{DG}}.$$

For the next term, we can write

$$|B_{\Gamma}(\eta, \xi)| \leq |J_1(\eta, \xi)| + |J_1(\xi, \eta)| + |J_2(\eta, \xi)| + |J_2(\xi, \eta)|.$$

Now,

$$\begin{aligned} |J_1(\eta, \xi)| &= \left| \left\langle \frac{1}{\sqrt{\alpha}} \{\nu \cdot \nabla(\Delta \eta)\}, \sqrt{\alpha} [\xi] \right\rangle_{L^2(\Gamma)} \right| \\ &\leq \left\| \frac{1}{\sqrt{\alpha}} \{\nu \cdot \nabla(\Delta \eta)\} \right\|_{L^2(\Gamma)} \|\sqrt{\alpha} [\xi]\|_{L^2(\Gamma)} \leq \left\| \frac{1}{\sqrt{\alpha}} \{\nu \cdot \nabla(\Delta \eta)\} \right\|_{L^2(\Gamma)} \|\xi\|_{\text{DG}} \end{aligned}$$

and

$$|J_2(\eta, \xi)| \leq \left\| \frac{1}{\sqrt{\beta}} \{\Delta \eta\} \right\|_{L^2(\Gamma)} \|\sqrt{\beta} [\nu \cdot \nabla \xi]\|_{L^2(\Gamma)} \leq \left\| \frac{1}{\sqrt{\beta}} \{\Delta \eta\} \right\|_{L^2(\Gamma)} \|\xi\|_{\text{DG}}.$$

The term  $J_1(\xi, \eta)$  is bounded as follows:

$$\begin{aligned} |J_1(\xi, \eta)| &= \left| \left\langle \sqrt{\gamma} [\eta], \frac{1}{\sqrt{\gamma}} \{\nu \cdot \nabla(\Delta \xi)\} \right\rangle_{L^2(\Gamma)} \right| \\ &\leq \|\sqrt{\gamma} [\eta]\|_{L^2(\Gamma)} \left\| \frac{1}{\sqrt{\gamma}} \{\nu \cdot \nabla(\Delta \xi)\} \right\|_{L^2(\Gamma)}. \end{aligned}$$

To bound the second factor on the right-hand side of the last inequality, as in the proof of Lemma 6, we use the inverse inequalities (4.4); for example, for a face  $e \in \mathcal{E}_{\text{int}}$  we have

$$\begin{aligned} \left\| \frac{1}{\sqrt{\gamma_e}} \{\nu \cdot \nabla(\Delta \xi)\} \right\|_{L^2(e)}^2 &\leq C \frac{h_e^3}{\{p^6\}_e} \left( \frac{p_K^6}{2h_K^3} \|\Delta \xi\|_{L^2(K)}^2 + \frac{p_{K'}^6}{2h_{K'}^3} \|\Delta \xi\|_{L^2(K')}^2 \right) \\ &\leq C \left( \|\Delta \xi\|_{L^2(K)}^2 + \|\Delta \xi\|_{L^2(K')}^2 \right), \end{aligned}$$

where  $K$  and  $K'$  are the elements sharing the face  $e$ . Therefore, we get

$$|J_1(\xi, \eta)| \leq C \|\sqrt{\gamma} [\eta]\|_{L^2(\Gamma)} \|\xi\|_{\text{DG}}.$$

Similarly, we have

$$|J_2(\xi, \eta)| \leq C \left\| \sqrt{\delta} [\nu \cdot \nabla \eta] \right\|_{L^2(\Gamma)} \|\xi\|_{\text{DG}}.$$

Finally, let us note that

$$\begin{aligned} |B_s(\eta, \xi)| &\leq \left\| \sqrt{\alpha} [\eta] \right\|_{L^2(\Gamma)} \left\| \sqrt{\alpha} [\xi] \right\|_{L^2(\Gamma)} + \left\| \sqrt{\beta} [\nu \cdot \nabla \eta] \right\|_{L^2(\Gamma)} \left\| \sqrt{\beta} [\nu \cdot \nabla \xi] \right\|_{L^2(\Gamma)} \\ &\leq \left( \left\| \sqrt{\alpha} [\eta] \right\|_{L^2(\Gamma)} + \left\| \sqrt{\beta} [\nu \cdot \nabla \eta] \right\|_{L^2(\Gamma)} \right) \|\xi\|_{\text{DG}}. \end{aligned}$$

Collecting these bounds gives the desired result.  $\square$

Let us return, once again, to the *a priori* error analysis of the *hp*-NIPG method.

**Theorem 11** *Suppose that  $\Omega$  is a bounded polyhedral domain in  $\mathbb{R}^d$  and  $\{\mathcal{K}_h\}_{h>0}$  is a shape-regular family of partitions, consisting of  $d$ -dimensional parallelepipeds. Let  $\mathbf{p} = (p_K : K \in \mathcal{K}_h, p_K \in \mathbb{N}, p_K \geq 3, K \in \mathcal{K}_h)$  be any polynomial degree vector of bounded local variation. To each face  $e \in \mathcal{E}$  we assign the positive real constant face functions  $\alpha$  and  $\beta$ :*

$$\alpha_e = \sigma_\alpha \frac{\{p^{l_\alpha}\}_e}{h_e^3} \text{ and } \beta_e = \sigma_\beta \frac{\{p^{l_\beta}\}_e}{h_e},$$

where  $\sigma_\alpha, \sigma_\beta$  are arbitrary positive real numbers and

$$0 \leq l_\alpha \leq 6, \quad -2 \leq l_\beta \leq 4.$$

Let us suppose that the exact solution  $u$  to the problem (3.12) belongs to  $H^{\mathbf{t}}(\Omega, \mathcal{K}_h)$ ,  $\mathbf{t} = (t_K : K \in \mathcal{K}_h, t_K \geq 4, K \in \mathcal{K}_h)$ . Then, the solution  $u_{\text{DG}} \in S^{\mathbf{p}}(\Omega, \mathcal{K}_h, \mathbf{F})$  obtained from the NIPG method (4.1) satisfies the following error bound:

$$\|u - u_{\text{DG}}\|_{\text{DG}}^2 \leq C \sum_{K \in \mathcal{K}_h} \frac{h_K^{2s_K-4}}{p_K^{2t_K-7}} \|u\|_{H^{t_K}(K)}^2, \quad (5.7)$$

where  $1 \leq s_K \leq \min(p_K + 1, t_K)$  and  $C$  is a constant dependent only on the space dimension  $d$ , the shape-regularity constant  $c$ , the local polynomial degree variation constant  $\rho$  and  $t = \max_{K \in \mathcal{K}_h} t_K$ .

*Proof.* Comparing the expressions for the norms (4.2) and (5.5), we can see that

$$\|\eta\|_{\text{DG}} \leq \|\eta\|_* \quad \forall \eta \in H^{\mathbf{s}}(\Omega, \mathcal{K}_h),$$

so from Lemma 10 we get

$$\|u - u_{\text{DG}}\|_{\text{DG}} \leq \|\xi\|_{\text{DG}} + \|\eta\|_{\text{DG}} \leq 2 \|\eta\|_*.$$

Consequently, estimating each of the terms on the right-hand side of (5.5), as in proof of Theorem 9, we get

$$\begin{aligned} \|u - u_{\text{DG}}\|_{\text{DG}} &\leq C \sum_{K \in \mathcal{K}_h} \left( \frac{h_K^{2s_K-4}}{p_K^{2t_K-4}} + \frac{h_K^{2s_K-4}}{p_K^{2t_K+l_\alpha-7}} + \frac{h_K^{2s_K-4}}{p_K^{2t_K+l_\beta-5}} \right) \|u\|_{H^{t_K}(K)}^2 \\ &\quad + C \sum_{K \in \mathcal{K}_h} \left[ \left( \frac{p_K^{l_\alpha}}{h_K^3} + \frac{p_K^6}{h_K^3} \right) \frac{h_K^{2s_K-1}}{p_K^{2t_K-1}} + \left( \frac{p_K^{l_\beta}}{h_K} + \frac{p_K^2}{h_K} \right) \frac{h_K^{2s_K-3}}{p_K^{2t_K-3}} \right] \|u\|_{H^{t_K}(K)}^2. \end{aligned}$$

□

## 6 Numerical experiments

In this section we present some numerical experiments to confirm the *a priori* error estimates stated in the Theorems 9 and 11.

### 6.1 Example 1

In this example we consider the boundary value problem (3.16) with  $\Omega = (0, 1)^2$ ,  $g_0 = g_1 = 0$  and with  $f$  chosen so that the analytical solution to the problem is given by

$$u(x, y) = [xy(1-x)(1-y)]^2.$$

As one would expect, our numerical experiments show that the DGFEM solution with  $p \geq 4$  coincides with the exact solution.

### 6.2 Example 2

In our second example we consider the same homogeneous Dirichlet problem as in Example 1, but now with right-hand side  $f$  corresponding to the exact solution

$$u(x, y) = \sin^2(\pi x) \sin^2(\pi y).$$

Of course, in this case  $u_{\text{DG}} \neq u$  on all meshes and for all values of the polynomial degree  $p$ . Here we study the dependence of the orders of convergence of  $\|u - u_{\text{DG}}\|_{L^2(\Omega, \mathcal{K}_h)}$ ,  $\|\nabla(u - u_{\text{DG}})\|_{L^2(\Omega, \mathcal{K}_h)}$  and the energy norm  $\|u - u_{\text{DG}}\|_\Delta$  on the choice of the discontinuity-penalisation parameters

$$\alpha_e = \sigma_\alpha \frac{\{p^{l_\alpha}\}_e}{h_e^3} \quad \text{and} \quad \beta_e = \sigma_\beta \frac{\{p^{l_\beta}\}_e}{h_e}, \quad e \in \mathcal{E}.$$

For reasons of consistency, we use the energy norm rather than the DG-norm, given that the energy norm  $\|\cdot\|_\Delta$  is independent of the choice of the of discontinuity-penalisation

parameters; also, for the uniformly elliptic boundary value problem under consideration the energy norm is equivalent to the broken  $H^2$ -norm.

We emphasise the fact that in order for the *a priori* error estimates which we proved in the previous section to be valid in case of the symmetric and the semisymmetric methods, the penalty parameters  $\sigma_\alpha$  and  $\sigma_\beta$  must be larger than the constants  $\underline{\sigma}_\alpha$  and  $\underline{\sigma}_\beta$ , respectively. These constants depend on the constants in the inverse inequalities, the shape-regularity of the mesh and the degree of the maximum local polynomial degree  $\max_{\kappa \in \mathcal{T}} p_\kappa$  used, and are not easy to determine accurately. For the present model problem, for reasons of consistency, for each of the methods considered we used the value  $\underline{\sigma}_\alpha = \underline{\sigma}_\beta = 10$  (which was found through numerical experiments).

In order to confirm the theoretical order of convergence, we employed a sequence of structured quadrilateral meshes  $\{\mathcal{K}_{h_i}\}$  where  $h_i = 0.5h_{i-1}$ ,  $i \geq 1$ . The numerical order of convergence  $m_i$  was calculated from the formula

$$m_i = \frac{\log_{10} \left( \frac{\text{Err}_{i+1}}{\text{Err}_i} \right)}{\log_{10} (0.5)},$$

where  $\text{Err}_i$  denotes a norm of the error on the mesh  $\mathcal{K}_{h_i}$ .

We first consider the  $h$ -convergence of all four methods with  $l_\alpha = 6$  and  $l_\beta = 4$ . In Tables 1, 2 and 3 we show the numerically observed orders of convergence of the energy norm of the error, the  $L^2$ -norm of the gradient of the error and the  $L^2$ -norm of the error, respectively, for polynomials of degree 2 to 6. As one can see from Table 1, all four methods exhibit the optimal  $\mathcal{O}(h^{p-1})$  convergence in the energy norm. Let us note that we have only proved convergence for  $p \geq 3$ ; nevertheless, we observe convergence of all methods in the energy norm for  $p = 2$  as well, and all four methods exhibit an optimal convergence rate in this case too. With respect to the other norms the situation is quite different. For the SIPG method, which is adjoint consistent, we observe (cf. Tables 2 and 3) that the gradient of the error and the error itself still converge with the optimal rates  $\mathcal{O}(h^p)$  and  $\mathcal{O}(h^{p+1})$ , correspondingly. None of the other methods is adjoint consistent, and therefore their  $h$ -convergence is of an inferior rate. Nevertheless, the SSIPG1 method, which is symmetric with respect to each of the terms involving derivatives up to and including second order, exhibits optimal convergence rates in each of the three norms as well. The  $L^2$ -norm of the gradient of the error of the NIPG method behaves like  $\mathcal{O}(h^p)$  for odd  $p$  (which is the optimal convergence rate) and is suboptimal for even  $p$ , while the  $L^2$ -norm of the error converges with suboptimal rate which is better for odd  $p$  and worse for even  $p$ . A similar behaviour of the  $L^2$ -norm of the error has been reported by Baumann, Oden and Babuška [23] for second-order elliptic problems on quadrilateral meshes. The SSIP2 method exhibits the same behaviour for the error and for the gradient of the error.

In the last columns of the Tables 1-3 we present convergence results for NIPG methods with  $l_\alpha = 0$  and  $l_\beta = -2$ , cf. Theorem 11. We observe that the error converges with optimal rate  $p - 1$  in energy norm and convergence is better for large values of  $p$ . The rate of convergence of all of the other norms of the error is suboptimal and is worse than the convergence rate of the error of the method with  $l_\alpha = 6$  and  $l_\beta = 4$ .

Thus, for the elliptic problems of fourth order, we numerically observed  $h$ -optimal convergence in the energy norm for the SIPG, SSIPG1, SSIPG2 and NIPG methods. We observed optimal convergence of the error in the SIPG and SSIPG1 methods in the  $L^2$ -norm of the gradient of the error and in the  $L^2$ -norm of the error for all values of  $p$  too. Convergence of the  $L^2$ -norm of the gradient of the error for the SSIPG2 and NIPG methods is only optimal for  $p$  odd, while convergence of the  $L^2$ -norm of the error for these two methods is suboptimal for all values of  $p$ .

Table 1.  
Orders of convergence for  $\|u - u_{\text{DG}}\|_{\Delta}$

Method	SIPG	SSIPG1	SSIPG2	NIPG	NIPG
	$l_{\alpha} = 6, l_{\beta} = 4$			$l_{\alpha} = 0, l_{\beta} = -2$	
$p = 2$	1,34506	1,34625	1,34321	1,34784	1,24714
	0,99852	0,99848	0,99706	0,99722	1,11189
	0,99987	0,99983	0,99954	0,99956	1,08656
$p = 3$	0,46813	0,46272	0,46324	0,46264	1,38958
	1,98228	1,98191	1,98236	1,98228	2,1018
	1,99584	1,99568	1,99578	1,99573	2,09398
$p = 4$	3,38414	3,38354	3,38291	3,34613	2,52516
	2,97853	2,97891	2,97783	2,96924	2,59447
	2,99686	2,99607	2,99686	2,99293	2,8664
$p = 5$	1,75198	1,7485	2,84983	1,75029	1,72059
	3,97925	3,97889	4,27051	3,9789	4,42657
	3,99511	3,99482	4,15477	3,99497	4,24959
$p = 6$	5,41404	5,41254	5,41224	5,41314	5,04371
	4,97765	4,97778	4,97775	4,97784	4,90484
	4,99531	4,99528	4,99522	4,99524	4,92453

Table 2.  
Orders of convergence for  $\|\nabla(u - u_{\text{DG}})\|_{L^2(\Omega, \mathcal{K}_h)}$

Method	SIPG	SSIPG1	SSIPG2	NIPG	NIPG
	$l_{\alpha} = 6, l_{\beta} = 4$			$l_{\alpha} = 0, l_{\beta} = -2$	
$p = 2$	2,30792	2,31622	2,42038	2,42745	2,42016
	2,01225	2,0159	2,04613	2,0507	1,80982
	2,0038	2,0045	2,00476	2,00609	1,76959
$p = 3$	0,8711	0,89521	0,99281	1,03424	1,9135
	2,93462	2,932	2,81191	2,81191	2,05382
	2,98257	2,98261	2,71454	2,71465	1,82807
$p = 4$	4,3231	4,31861	4,36413	4,36073	2,40315
	3,94334	3,94415	3,97853	3,97951	2,69413
	3,98555	3,98516	3,99644	3,99614	3,51232
$p = 5$	2,3936	2,42041	2,46272	2,49483	2,44798
	4,97671	4,98171	4,87523	4,88217	4,94904
	4,99072	4,99421	4,75988	4,76452	4,35182
$p = 6$	6,37182	6,37643	6,41637	6,41932	5,84764
	5,96236	5,96326	5,99611	5,9964	5,70101
	5,99103	5,99049	6,00071	5,99951	5,54671



Table 3.  
Orders of convergence for  $\|u - u_{\text{DG}}\|_{L^2(\Omega, \mathcal{K}_h)}$

Method	SIPG	SSIPG1	SSIPG2	NIPG	NIPG
	$l_\alpha = 6, l_\beta = 4$			$l_\alpha = 0, l_\beta = -2$	
$p = 2$	2,41577	2,4271	2,41737	2,58643	1,13575
	2,09885	2,10438	1,94087	2,16257	1,22046
	2,01886	2,01807	1,8045	2,02454	1,74164
$p = 3$	0,83429	0,94915	1,3146	1,4291	2,71055
	3,94158	3,96729	2,55163	2,55237	1,8365
	3,97525	3,99599	2,1728	2,18075	1,60348
$p = 4$	5,19749	5,17607	4,87001	4,82593	2,44499
	4,90448	4,88257	4,49589	4,4629	2,71621
	4,97582	4,92418	4,30752	4,27498	3,49241
$p = 5$	3,01595	3,09147	2,84983	2,93078	2,56052
	5,95887	5,96873	4,27051	4,2903	5,61809
	5,98334	5,99164	4,15477	4,16822	3,96086
$p = 6$	7,34539	7,3398	6,81645	6,81137	5,63235
	6,95115	6,94587	6,42796	6,41393	5,32673
	6,9871	6,94132	6,26071	6,21905	4,94726

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