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Degeneration of Sheaves on Fibered Surfaces

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Abstract

We construct moduli stacks of stable sheaves for surfaces fibered over marked nodal curves by using expanded degenerations. These moduli stacks carry a virtual class and therefore give rise to enumerative invariants. In the case of a surface with two irreducible components glued along a smooth divisor, we prove a degeneration formula that relates the moduli space associated to the surface with the relative spaces associated to the two components. For a smooth surface and no markings, our notion of stability agrees with slope stability with respect to a suitable choice of polarization. We apply our results to compute elliptic genera of moduli spaces of stable sheaves on some elliptic surfaces.

1. Introduction

Let $X_0 = Y_1 \cup_D Y_2$ be a projective surface that is the union of smooth surfaces Y_i along a common smooth divisor D. In [Don95, §§ 1 and 4], Donaldson raises the problem of constructing a good theory of stable sheaves on such an X_0 with the following properties.

I. It behaves well under smoothings. In other words, the numerical invariants of the theory on X_0 agree with the invariants of the usual moduli space of Gieseker-stable sheaves on a smoothing of X_0 when those are defined.

II. It behaves well under decomposition. More precisely, there should be spaces of 'relative stable sheaves' for a pair (Y, D) of a smooth surface Y with a smooth divisor D such that the moduli space of sheaves on X_0 can be related to the relative spaces for the pairs (Y_i, D).

Such a theory would enable one to compute sheaf-theoretic invariants of projective surfaces through degenerations to a reducible surface. This has been successfully implemented in other settings, notably in the Gromov–Witten theory ([Li01], [Li02], [ACGS20], [Ran22], [KLR23]) and the Donaldson–Thomas theory (see, for example, [LW15], [MR20]).

In this paper, we answer Donaldson's questions for fibered surfaces. Let $f : X \to C$ be a surface fibered over a marked nodal curve (see Definition 1.6 for the precise meaning). Then we propose the following definition of 'stability on the fiber'.

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Definition 1.1 (Cf. Definition 3.1). A coherent sheaf E on X is *f-stable* if it is torsion-free and:

(i) for every node or marked point $x \in C$, the restriction of E to $f^{-1}(x)$ is a slope-stable vector bundle, and
(ii) for every generic point η of an irreducible component of C, the restriction of E to $f^{-1}(\eta)$ is slope stable.

Our first main result is that this notion of stability leads to well-behaved moduli spaces. This gives an answer to (I) for fibered surfaces.

Theorem 1.2 (Cf. Theorem 3.25 and Proposition 4.4). *Let d and $r > 0$ be coprime integers. Let $c_1 \in H^2(X, \mathbb{Z})$ be a cohomology class with $c_1 \cap f^{-1}(x) = d[pt]$ for any $x \in C$. Let α be a generic stability condition on C and L_0 a line bundle on X of degree $d_0 > 0$ on fibers. Then for any $\Delta \in \mathbb{Z}$, there exists a proper Deligne–Mumford moduli stack*

$$M^{\alpha}_{X/C}(r, \overline{c_1}, \Delta)$$

parametrizing f-stable, α-balanced sheaves of rank r and first Chern class c_1 up to component twists and discriminant Δ on expansions of X. Moreover, this stack has a natural virtual fundamental class and the numerical invariants are invariant under deformations of X together with the fibration and choice of c_1.

Here, we introduced two additional subtleties: expansions of a degenerate surface (cf. Definition 2.1) and balancing with respect to a choice of L_0 and a stability condition α on a curve (cf. § 3.2).

Our construction automatically yields a notion of relative moduli space: putting markings $y_1, \dots, y_n$ on C corresponds to working relative to the divisor $f^{-1}(y_1) \cup \dots \cup f^{-1}(y_n)$.

Our second main result is a proof of II in the case that $X = Y_1 \cup_F Y_2$ is a union of two irreducible components which meet along a fiber F of f and so that we also have $C = C_1 \cup_x C_2$. It provides a *degeneration formula* for Donaldson-type invariants on fibered surfaces, directly analogous to the established ones in the Gromov–Witten and Donaldson–Thomas theories in the case of double point degenerations. We state this result somewhat informally (see Theorem 4.14 and Proposition 4.12 for more precise versions).

Theorem 1.3. *The enumerative invariants of moduli spaces of f-stable sheaves on X can be recovered from the relative invariants associated to the pairs (Y_i, F).*

For elliptic surfaces, the relative invariants can be determined from the absolute ones, which makes Theorem 1.3 more powerful in practice. To illustrate this, we give an application to elliptic genera. For a proper scheme M with perfect obstruction theory E_M, let $T_M := E_M^\vee$ denote the virtual tangent bundle. Let $K^0(M)$ denote the Grothendieck K-group of perfect complexes in M, and let y, q be formal variables. Define

$$\mathcal{E}_{q,y}(M) := \bigotimes_{n \geqslant 0} \bigwedge\nolimits_{-y^{-1}q^n} T_M^{\mathrm{vir}} \otimes \bigotimes_{n \geqslant 1} \bigwedge\nolimits_{-yq} (T_M^{\mathrm{vir}})^\vee \otimes \bigotimes_{n \geqslant 1} \mathrm{Sym}_{q^n}\left(T_M^{\mathrm{vir}} \oplus (T_M^{\mathrm{vir}})^\vee\right) \in K^0(M)[y, y^{-1}][\![q]\!].$$

Then, set

$$\mathrm{Ell}^{\mathrm{vir}}(M) := y^{-\frac{\mathrm{vdim}\, M}{2}} \int_{[M]^{\mathrm{vir}}} \mathrm{ch}(\mathcal{E}_{q,y}(M))\, \mathrm{td}(T_M^{\mathrm{vir}}).$$

This is the virtual elliptic genus as defined in [FG10]. In [GK19a], the authors predict an elegant formula for virtual elliptic genera of moduli spaces of sheaves on surfaces in the rank-two case. We also let $\phi_{0,1}(q,y)$ be the weak Jacobi form and $\mathbf{L}(\phi_{0,1},p)$ its Borcherds lift as presented in [GK19a]. We recall that if $f: X \to \mathbb{P}^1$ is an elliptic surface, its *degree* is the degree of the line bundle $(R^1 f_* \mathcal{O}_X)^\vee$.

Theorem 1.4. *Let X be a degree $e \geqslant 2$ elliptic surface over $\mathbb{P}^1$ without multiple or reducible fibers. Let $d \geqslant 1$ be minimal so that X has a d-section with divisor class D. Let H be an ample line bundle on X and $[F]$ the cohomology class of a fiber. Let $M_{X,H}(r,D,\Delta)$ denote the moduli space of H-Gieseker-stable sheaves on X of rank $r>0$, first Chern class D and discriminant Δ. Assume that d is coprime to r and that H is chosen so that stability equals semistability for sheaves of rank r and first Chern class c_1, and all Δ. Consider the generating series*

$$Z^{\mathrm{Ell}}_{X,r,c_1}(p) := \sum_{\substack{\Delta \in \mathbb{Z} \\ 0 \leqslant \ell < r}} \mathrm{Ell}^{\mathrm{vir}}(M_{X,H}(r, c_1 + \ell[F], \Delta))\, p^{\dim M_{X,H}(r, c_1+\ell[F],\Delta)}.$$

Then,

$$Z^{\mathrm{Ell}}_{X,r,c_1}(p) = \sum_{n \geqslant 0} \mathrm{Ell}(\mathrm{Hilb}_n(X))\, p^{2n} = \left(\frac{1}{\mathbf{L}(\phi_{0,1},p^2)}\right)^{\chi(\mathcal{O}_X)}.$$

In the case $e=2$, the surface X is a K3 surface and the statement reduces to the rank-one case, which was stated in [DMVV97] and has been proven in [BL03], [BL05].

Remark 1.5. The proof of Theorem 1.4 goes through with 'virtual elliptic genus' replaced by 'virtual cobordism' to give

$$Z^{\mathrm{cob}}_{X,r,c_1}(p) = \left(\sum_{n \geqslant 0} [K3^{[n]}] p^{2n}\right)^{\chi(\mathcal{O}_X)/2},$$

where the possible half-integer power is chosen to have constant coefficient one. In particular, this answers Göttsche and Kool's Conjecture 7.7 in [GK19a] affirmatively for this class of elliptic surfaces.

1.1. Background

Our approach is inspired by earlier constructions of Gieseker–Li [GL94] and Li [Li03], [Li18]. They consider moduli spaces of Simpson semistable sheaves on expansions of a degenerate surface X_0. Under good conditions, this does result in moduli spaces which behave well under deformations and carry a virtual fundamental class. However, their approach does not lead to a degeneration formula, since it is unclear how to relate the moduli space of sheaves on the degeneration with the relative spaces for the pairs (Y_i, D). The problem is that Simpson stability of a sheaf on X_0 cannot be determined by looking only at its restrictions to the Y_i, but also depends on how the sheaves are glued in a subtle manner.

By restricting to fibered surfaces, we get around this issue. The notion of f-stability is defined in terms of restriction on fibers, which can be checked on each component separately. The use of expansions is crucial for us; as in [GL94], it allows us to work only with sheaves that are locally free along the singular locus of X_0. Just as importantly, we can guarantee stability on the fibers over marked points and nodes, which gives us restriction maps to the moduli spaces of stable vector bundles on curves. This is central to the decomposition result of Theorem 1.3.

Since f-stability on $X \to C$ is unchanged with respect to tensoring by a line bundle pulled back from C, the moduli space of f-stable sheaves will not be separated in families when C degenerates from a smooth to a reducible curve, since the Picard scheme of C is not separated. Essentially, the issue is that in a one-parameter family of curves, the limit of the trivial line bundle does not need to be trivial, since one can twist by components of the special fiber. To deal with this issue, we need to restrict the possible twists of an f-stable sheaf. This is achieved by picking a stability condition α on C when the base curve becomes reducible and demanding a certain numerical balancing condition which singles out a unique choice of twist.

Although f-stability at first seems unrelated to Gieseker or slope stability, it turns out that these notions agree on a smooth fibered surface when one considers the latter with respect to a suitable choice of polarization. This is already present in the work of Yoshioka and is included here as Theorem 2.24. Moreover, due to results of Mochizuki [Moc09], for surfaces with $p_g(X) > 0$, virtual enumerative invariants are independent of choice of polarization whenever stability equals semistability. Thus, for such surfaces, there is no loss of generality in considering f-stability for the computation of invariants.

In the case that the surface $X = F \times C$ is a product of a smooth curve F and a marked nodal curve C, an f-stable sheaf over an expansion of C is the same thing as a prestable *quasi-map* [CFKM14] from C into the moduli space of stable sheaves on F. Hence, in this situation, our space $M^{\alpha}_{X/C}(r, c_1, \Delta)$ is related to a space of quasi-maps. If C is stable and of compact type, we expect that this identifies $M^{\alpha}_{X/C}$ with a product of the space of zero-stable quasi-maps whose domain curve is an expansion of C and a component of the Jacobian of C. If C is not of compact type, the situation is more subtle and one needs to consider a compactified Jacobian of C. The study of sheaves via quasi-map spaces has been succesfully pursued in [Nes24].

1.2. Structure of the paper

In § 3, we introduce the definitions and constructions that go into Theorem 1.2. Since we work over a general base B, we automatically obtain deformation invariance.

In § 4, we show the decomposition in the situation of Theorem 1.3. We include the proof of Theorem 1.4 in § 4.7.

In § 2, we collect some material that is needed for the main constructions in the later sections. For a first reading, we suggest only taking a look at Definition 2.1 and Lemma 2.2 in § 2.1, and otherwise to refer back to this section as needed.

1.3. Relation to other work

Since the modern mathematical definition of Vafa–Witten invariants on algebraic surfaces by Tanaka and Thomas ([TT20], [TT17]), there has been renewed interest in the enumerative geometry of moduli spaces of sheaves on surfaces. Göttsche and Kool, in a series of works, developed many conjectures for the structure of such invariants (see [GK19b] for an excellent overview). However, beyond the cases of Hilbert schemes and special classes of surfaces such as rational, elliptic or K3 surfaces, very little is known. Notable exceptions are results on Donaldson invariants ([GNY08], [GNY11]) and blowup formulas ([KT21], [KLT22]).

Recently, Dominic Joyce has announced results regarding deep structure theorems for enumerative invariants of surfaces with $p_g > 0$, building on his theory of wall-crossing in abelian categories and his version of Mochizuki's rank-reduction algorithm [Joy21]. His result shows that the generating series of invariants are determined in terms of a number of universal power series and universal constants, and of finitely many fundamental enumerative invariants of the surface (its *Seiberg–Witten invariants*). It would be interesting to investigate whether there exists a connection between his universal series and the degeneration formula of the present paper.

For rank-one sheaves, our results reduce to a theory of Hilbert schemes on degenerations, which is treated in [LW15] in the case of a smooth singular locus and was later generalized to arbitrary normal crossings degenerations in [MR20].

1.4. Further directions

One problem that is not addressed here is to generalize the results of § 4 to the case of a non-separating node, i.e. given a fibered surface $X \to C$ and a non-separating node x of C, to describe the enumerative invariants for the moduli space of f-stable sheaves on $X \to C$ in terms of those on $X' \to C'$, where $C' \to C$ is the partial normalization of C at x, and $X' = X \times_C C'$. This should be possible, but it requires a closer analysis of the combinatorics of the Picard scheme of C. An application of this, suggested by Jørgen Rennemo, would be to obtain a $(1+1)$-dimensional cohomological field theory by considering surfaces of the form $F \times C \to C$, where F is kept fixed and C is an arbitrary marked nodal curve.

In another direction, the constructions here should generalize beyond the case of surfaces, which we mostly require to obtain a good enumerative theory. We expect that the same methods generalize to give, for example, degeneration formulas for fibered Fano and Calabi–Yau threefolds. We thank Richard Thomas for pointing this out to us.

1.5. Notations and conventions

- All schemes and stacks we consider will be locally Noetherian over $\mathbb{C}$.
- By a curve over a base B, we mean a flat and finite type separated algebraic space over B with one-dimensional fibers. We drop reference to the base when $B = \operatorname{Spec}\mathbb{C}$.
- By a marked nodal curve over B we mean a curve over B with at worst nodal singularities and a finite number of sections which are disjoint and do not meet the singularities. Here, by 'at worst nodal', we mean étale locally of the form $Z(xy - f) \subseteq B \times \mathbb{A}^2$, where x, y are standard coordinates on $\mathbb{A}^2$ and f is a function on B.
- Unless noted otherwise, we will assume all curves to be proper over the base and to have geometrically connected fibers.
- For (numerical) divisor classes D_1, D_2 on a proper algebraic surface X, and more generally for classes in the second cohomology of X, we denote by $(D_1, D_2) \in \mathbb{Q}$ their intersection product.
- We will often indicate the base (resp., base changes) of a family of curves or surfaces simply by a subscript (resp., by the change thereof).
- We repeatedly use the *uniqueness part of the valuative criterion* and *the existence part of the valuative criterion* in the terminology of [Sta23, Definition 0CLG and Definition 0CLK] for morphisms of algebraic stacks. Since we are in the locally Noetherian case, we will consider only discrete valuation rings (DVRs) and finite field extensions in the valuative criteria, which is justified by [Sta23, Lemma 0E80 and Lemma 0CQM].

– We use $A_*(Y)$ to denote the Chow groups of a finite-type algebraic stack Y in the sense of [Kre99].
– We will sometimes work with Chow cycles on stacks Y which are only locally finite type and with projective pushforwards $Z \to Y$ between them; here we always implicitly restrict to some suitably large quasi-compact open subset of Y (and to its preimage in Z in the case of pushforward). As in [BS22, Appendix A], one can also *define* Chow groups in this case as a projective limit over all quasi-compact open substacks, although this will not be needed for us.

In this paper, we will use the following definition of a fibered surface.

Definition 1.6. We say $f : X \to C$ is a fibered surface if C is a marked nodal curve and:

(i) f is flat and proper of dimension one with geometrically connected fibers,
(ii) f is smooth over the nodes and marked points of C,
(iii) for every irreducible component $D \subseteq C$, the scheme-theoretic preimage $f^{-1}(D) \subseteq X$ is a smooth projective surface.

We say $f : X_B \to C_B$ is a family of fibered surfaces over a base B if C_B is a marked nodal curve over B and f is flat and proper such that $X_b \to C_b$ is a fibered surface for every geometric point b of B.

2. Preliminaries

2.1 Expansions and expanded degenerations

We define what we mean by a family of expansions of marked nodal curves and show some basic properties. Let $(C_B, \sigma_1, \ldots, \sigma_n)$ be a flat family of marked nodal curves over a base B. In this subsection we do not assume curves to be proper or geometrically connected.

Definition 2.1. Let T be a scheme. A family of *expansions of C_B over B* parametrized by T is given by a morphism $b : T \to B$, a flat family of marked nodal curves $(\widetilde{C}_T, \widetilde{\sigma}_1, \ldots, \widetilde{\sigma}_n)$ over T and a proper morphism of marked curves $c : \widetilde{C}_T \to C_T$ such that:

(i) the natural map $\mathcal{O}_{C_T} \to Rc_*\mathcal{O}_{\widetilde{C}_T}$ is an isomorphism,
(ii) for each $t \in T$, we have a fiberwise isomorphism of twisted dualizing sheaves $c^*\omega_{C_t}(\sigma_1 + \cdots + \sigma_n) \equiv \omega_{\widetilde{C}_t}(\widetilde{\sigma}_1 + \cdots + \widetilde{\sigma}_n)$.

We let $\mathrm{Exp}_{C/B}$ denote the stack parametrizing expansions over B.

One can give an alternative more explicit characterization, as in the following lemma.

Lemma 2.2. *Let $\widetilde{C}_T$ and C_T be flat families of marked nodal curves over a base T and let $c : \widetilde{C}_T \to C_T$ be a morphism of marked curves over B. Then condition (i) of Definition 2.1 is equivalent to i') below.*

(i') *The locus $\Sigma_c \subset C_T$ where the map c is not an isomorphism is quasi-finite over T. For each $x \in \Sigma_c$, the scheme-theoretic fiber $c^{-1}(x)$ is a connected nodal curve of arithmetic genus zero.*

Assuming this condition holds, then ii) of Definition 2.1 is equivalent to (ii') below.

(ii'') *If $x \in \Sigma_c$ is lying over $y \in T$, then x is either a node or a marked point in the fiber $C_y \subset C_T$ and $c^{-1}(x)$ is a chain of rational curves $R_1 \cup \cdots \cup R_j$ containing two distinguished points of $\widetilde{C}_y$, one of which lies on R_1 and one on R_j (these are two nodes if x is a node, or one node and one marked point if x is a marked point).*

Proof. We first address the first part. By properness, c is finite over the locus where c has zero-dimensional fibers. Then condition (i) holds over this locus if and only if c is an isomorphism there. Now let $x \in C_T$ be a point where c has a one-dimensional fiber. It follows that c has to be a contraction of components. Since formation of $R^1 c_*$ commutes with base change, the vanishing of $R^1 c_* \mathcal{O}_{\widetilde{C}}$ around x is equivalent to $c^{-1}(x)$ being of arithmetic genus zero.

We now address the second part. Assume (i) and (i') hold. If (ii) holds, then the twisted dualizing sheaf must be trivial on any components contracted by c. We already know each such component is rational, so they must have precisely two distinguished points. Since $c^{-1}(x)$ has arithmetic genus zero, its components must be arranged as a chain $R_1 \cup \cdots \cup R_j$. The distinguished points on the end of the chain must come from intersection with a remaining component of $\widetilde{C}$ or from marked points. If there is a marked point on $c^{-1}(x)$, it must map to a marked point of C (in particular, there can be only one). Otherwise, we must have that $c^{-1}(x)$ intersects the closure of $\widetilde{C}_y \setminus c^{-1}(x)$ at two points, lying on components S_0, S_1 (which might coincide) of $\widetilde{C}_y$ not inside $c^{-1}(x)$. Since c is an isomorphism over a punctured étale neighborhood of x, we must have a node at x with S_0, S_1 mapping to the two branches.

Conversely, assume that (ii') holds. Then use the following standard characterization of the dualizing sheaf ω_C of a nodal curve C: a section of ω_C is the same as a meromorphic section of the sheaf of differentials of the normalization C^ν with at most simple poles at the preimages of nodes of C and such that for a node $x \in C$, the residues at the two preimages of x in C^ν add to zero. Now, say $x \in \Sigma_c$ is a node. Then, working locally around x, we may assume that $\tilde{C}_y^\nu = C_y^\nu \coprod R_1 \coprod \cdots \coprod R_j$. We define a map $c^* \omega_{C_y} \to \omega_{\tilde{C}_y}$ by sending a section of ω_{C_y} to the unique section of $\omega_{\tilde{C}_y}$ with the same value on C_y^ν. This uses that $R_i \simeq \mathbb{P}^1$ and that for any two points z_1, z_2 on $\mathbb{P}^1$, there is a unique section of $\Omega_{\mathbb{P}^1}(z_1 + z_2)$ with prescribed residue r at z_1 (and which then has residue $-r$ at z_2). The resulting map $c^* \omega_{C_y} \to \omega_{\tilde{C}_y}$ sends a nowhere-vanishing section of ω_{C_y} to one of $\omega_{\tilde{C}_y}$ and hence defines an isomorphism. In the case of x being a marked point, one proceeds analogously. □

Lemma 2.3. *Let $\widetilde{C}_T$ and C_T be flat and proper families of nodal marked curves over a base T, and let $c : \widetilde{C}_T \to C_T$ be a morphism of marked curves over T. Then condition (i) of Definition 2.1 is an open condition. On the locus where (i) holds, condition (ii) is an open condition.*

Proof. Openness of (i) follows from Tor-independent cohomology and base change [Sta23, Lemma 08IB], using that $\widetilde{C}_T$ and C_T are flat over T, and from the fact that the support of a coherent sheaf is closed and commutes with any base change. Now assume that (i) holds for c. Then condition (ii') is equivalent to the vanishing of $R^1 c_* (\omega_{\widetilde{C}_B/B}(\sigma_1' + \ldots + \sigma_n'))^{\otimes 2}$, which is open on the base, since the formations of $\omega_{\widetilde{C}_B/B}$, of $R^1 c_*$ and of the support of a coherent sheaf all commute with base change in B. □

The following should be true without the assumption of properness, but we consider only that case for simplicity.

PROPOSITION 2.4. *Let $C_B \to B$ be a flat and proper family of nodal marked curves. The stack of expansions $\mathrm{Exp}_{C_B/B} \to B$ is algebraic. It is locally of finite presentation and flat over B of pure dimension zero.*

The following basic lemma lets us study expansions of curves locally on the curve.

LEMMA 2.5. *Let C_B, C'_B be flat families of marked nodal curves over B and suppose that we have an étale morphism $\gamma : C'_B \to C_B$ over B that induces isomorphisms of the singular and the marked loci. Then pullback along γ induces an isomorphism $\mathrm{Exp}_{C_B/B} \to \mathrm{Exp}_{C'_B/B}$.*

Proof. We construct a quasi-inverse using flat descent for algebraic spaces (e.g. [Sta23, Lemma 0ADV]). For this, we may assume that $C'_B \to C_B$ is surjective by replacing C'_B with $C'_B \coprod C^\circ_B$, where $C^\circ_B \subseteq C_B$ denotes the open subset obtained by removing the nodes and marked points (this leaves our assumptions as well as $\mathrm{Exp}_{C'_B/B}$ unchanged). Let T be a test scheme over B, and let $\widetilde{C}'_T \to C'_T$ be an expansion. Let $C''_T := C'_T \times_{C_T} C'_T$. Then the diagonal map $C'_T \to C''_T$ is an inclusion of a connected component. By assumption, the other components of C''_T do not contain any nodes or preimages of marked points. Thus, there is a unique structure of marked curve on C''_T such that the diagonal $C'_T \to C''_T$ and the two projections $\mathrm{pr}_i : C''_T \to C'_T$ preserve the markings. Moreover, we have that the expansions $\mathrm{pr}_1^* \widetilde{C}'_T$ and $\mathrm{pr}_2^* \widetilde{C}'_T$ of $\widetilde{C}''_T$ are canonically isomorphic: over the diagonal copy of C'_T inside C''_T they are identified with $\widetilde{C}'_T$, and over its complement they are trivial expansions. Using similar arguments, one checks that this isomorphism provides a canonical descent datum for $\widetilde{C}'_T$ and thus, by descent, an expansion $\widetilde{C}_T$ of C_T. In this way, we obtain a map $\mathrm{Exp}_{C'_B/B} \to \mathrm{Exp}_{C_B/B}$. A further routine check shows that this construction provides a quasi-inverse to pullback along γ. □

Proof of Proposition 2.4. Consider the stack $\mathscr{M}$ of all marked nodal curves [Sta23, Tag 0DSX] with universal family $\mathscr{C}$, and on $\mathscr{M} \times B$, consider the morphism space $\mathrm{Hom}_{\mathscr{M}\times B}(\mathscr{C}, C_B)$, which is an algebraic stack over B locally of finite presentation [Sta23, Tag 0DPN]. The locus that preserves the markings is closed. The locus in which (i) and (ii) of Definition 2.1 hold is then open in this closed substack. Thus, we get $\mathrm{Exp}_{C_B/B}$ as a locally closed substack, and, in particular, it is algebraic and locally of finite presentation over B.

For the remaining properties, we may work locally on B and assume without loss of generality that $B = \mathrm{Spec}\, R$ is the spectrum of a Henselian local ring with separably closed residue field. Then the result follows from Lemma 2.6. □

LEMMA 2.6. *Let $B = \mathrm{Spec}\, R$ be the spectrum of a Henselian local ring with separably closed residue field k, and let $C_B \to B$ be a proper flat family of nodal marked curves with markings $\sigma_1, \ldots, \sigma_n$ and with $q_1, \ldots, q_r$ the nodes of the special fiber C_k. Then $\mathrm{Exp}_{C_B/B} \to B$ is flat of relative dimension zero, and the singularities are products of pullbacks of the singularities of the form $\mathbb{A}^n \to \mathbb{A}^1$ given by multiplication of the coordinates.*

Proof. By the étale local structure of nodes combined with Lemma 2.5 and our assumptions on B, we may reduce to the case that C is a disjoint union of open sets of the following form: for each node q_i, there is a morphism $g_i : B \to \mathbb{A}^1$ such that the component U_i containing q_i is isomorphic to the pullback along g_i of a standard degeneration $\mathbb{A}^2 \to \mathbb{A}^1$ given by multiplication of the coordinates. Each marked point is contained in a component $V_j \simeq B \times \mathbb{A}^1$, with the marking given by the origin of $\mathbb{A}^1$. It follows that $\mathrm{Exp}_{C_B/B}$ is a fiber product over B of pullbacks of the stacks $\mathrm{Exp}_{\mathbb{A}^2/\mathbb{A}^1} \to \mathbb{A}^1$ and $\mathrm{Exp}_{(\mathbb{A}^1,0)} \to \mathrm{Spec}\, \mathbb{C}$. Following [Li01, § 1] and [ACFW13, § 1 + § 6], we can give explicit descriptions of $\mathrm{Exp}_{\mathbb{A}^2/\mathbb{A}^1}$ and $\mathrm{Exp}_{(\mathbb{A}^1,0)}$ from which the result follows.

Marked points. We first make the connection of our setting to theirs. Our Lemma 2.2 exhibits $\mathrm{Exp}_{(\mathbb{A}^1,0)}$ as equal to the stack $\mathcal{T}_{na\ddot{i}ve}(\mathbb{A}^1,0)$ of naïve expansions of Definitions 2.1.2 and 2.1.3 in [ACFW13]. By [ACFW13, Proposition 3.2.2 and Proposition 6.3.3], we further have an isomorphism $\mathcal{T}_{na\ddot{i}ve}(\mathbb{A}^1,0) \simeq \mathcal{T}_{Li}(\mathbb{A}^1,0)$, where the latter is the space of expanded pairs constructed in [Li02]. As explained in the proof of Proposition 6.3.3 in [ACFW13], the stack $\mathcal{T}_{Li}(\mathbb{A}^1,0)$ (and hence $\mathrm{Exp}_{(\mathbb{A}^1,0)}$) has a smooth cover by affine spaces $(\mathbb{A}^n)_{n\geqslant 0}$. Here, for each n, the map $\mathbb{A}^n \to \mathrm{Exp}_{(\mathbb{A}^1,0)}$ is induced by an explicit expansion $c[n]\colon \widetilde{C}[n]_{\mathbb{A}^n} \to \mathbb{A}^n \times \mathbb{A}^1$ in which the fiber over the origin of $\mathbb{A}^n$ is the expansion of $(\mathbb{A}^1,0)$ with the fiber over 0 being a chain of n copies of $\mathbb{P}^1$. Each coordinate axis of $\mathbb{A}^n$ corresponds to a deformation smoothing exactly one of the n nodes in the expansion. The automorphism group of the central fiber is $\mathbb{G}_m^n$. This group acts on the whole family $\widetilde{C}[n]_{\mathbb{A}^n}$ so that $c[n]$ becomes equivariant, where we let $\mathbb{G}_m^n$ act trivially on $\mathbb{A}^1$ and by scaling coordinates on $\mathbb{A}^n$. The induced map $[\mathbb{A}^n/\mathbb{G}_m^n] \to \mathrm{Exp}_{(\mathbb{A}^1,0)}$ is étale.

Nodes. Again by Lemma 2.2, we have an isomorphism $\mathrm{Exp}_{\mathbb{A}^2/\mathbb{A}^1} \simeq \mathfrak{T}_{na\ddot{i}ve}(\mathbb{A}^2/\mathbb{A}^1,0)$ where the latter is the stack of naïve expanded degenerations defined in [ACFW13, Definition 2.3.1 and Definition 2.3.2]. Combining their Proposition 3.2.3 and Proposition 6.3.3, we have an isomorphism $\mathfrak{T}_{na\ddot{i}ve}(\mathbb{A}^2/\mathbb{A}^1,0) \simeq \mathfrak{T}_{Li}(\mathbb{A}^2/\mathbb{A}^1,0)$, where the latter is the stack of expanded degenerations defined in [Li02]. As explained in [ACFW13] in the proof of Proposition 6.3.3 and in § 6.1, the stack $\mathfrak{T}_{Li}(\mathbb{A}^2/\mathbb{A}^1,0)$ (and hence $\mathrm{Exp}_{\mathbb{A}^2/\mathbb{A}^1}$) has a smooth cover by affine spaces $(\mathbb{A}^{n+1})_{n\geqslant 0}$. Again, the map $\mathbb{A}^{n+1} \to \mathrm{Exp}_{\mathbb{A}^2/\mathbb{A}^1}$ is obtained from an explicit standard model. In this case, this map is induced by a standard expanded degeneration $c[n]\colon \tilde{C}[n]_{\mathbb{A}^{n+1}} \to \mathbb{A}^{n+1} \times_{\mathbb{A}^1} \mathbb{A}^2$, where the map $\mathbb{A}^{n+1} \to \mathbb{A}^1$ is given by multiplication of all coordinates. The fiber over the origin of $\mathbb{A}^{n+1}$ is an expansion of the nodal curve $(xy=0) \subset \mathbb{A}^2$ with the preimage of the node being a chain of n copies of $\mathbb{P}^1$. Each coordinate direction of $\mathbb{A}^{n+1}$ corresponds to a smoothing of one of the $n+1$ nodes in the expansion. The automorphism group of the central fiber is $\mathbb{G}_m^n$, and the action extends to $\tilde{C}[n]$, which makes the expansion equivariant with respect to an action on $\mathbb{A}^{n+1}$ for which the map to $\mathbb{A}^1$ is invariant. The induced map $[\mathbb{A}^{n+1}/\mathbb{G}_m^n] \to \mathrm{Exp}_{\mathbb{A}^2/\mathbb{A}^1}$ is étale. □

From the proof, we may also conclude the following result.

Corollary 2.7. *Let $C \to B$ be a flat and proper family of marked nodal curves. Then the existence part of the valuative criterion for properness holds for $\mathrm{Exp}_{C/B} \to B$.*

Proof. As in the proof of Lemma 2.6, we may reduce the proof to showing the existence part of the valuative criterion for the standard models $\mathrm{Exp}_{(\mathbb{A}^1,0)} \to \operatorname{Spec} \mathbb{C}$ and $\mathrm{Exp}_{\mathbb{A}^2/\mathbb{A}^1 \to \mathbb{A}^1}$, which we have identified as stacks of naïve expansions. In [ACFW13, Lemma 3.1.3 and Lemma 3.1.6], these stacks are identified as relative stacks of semistable maps into a projective target. These stacks are well known to satisfy the existence part of the valuative criterion. For example, one may reduce to the case of stable maps by adding suitable sections (this may involve passing to an extension of the DVR) and then use the properness of the space of stable maps (see [AOV11, Theorem 4.3] for a version treating the case of families). □

2.2 Moduli spaces of vector bundles on curves

We recall some results regarding the existence of universal bundles and the structure of the Picard group for moduli spaces of stable vector bundles on curves. We also derive a version of the Bogomolov–Gieseker inequality for fiber-stable sheaves on a product with $\mathbb{P}^1$. Fix coprime integers r, d with $r > 0$. Let F be a smooth projective curve of genus g, and let $M_F(r,d)$ denote the moduli space of rank-r, degree-d stable vector bundles on F. For a degree-d line bundle L on F, let $M_F(r,L)$ denote the fiber of the determinant morphism $M_F(r,d) \to \mathrm{Pic}^d F$ over $[L]$.

Proposition 2.8. *Suppose that $g \geqslant 2$.*

(i) *For any $L \in \mathrm{Pic}^d F$, the Picard group of $M_F(r,L)$ is canonically isomorphic to $\mathbb{Z}$, identifying the ample generator with 1.*

(ii) *Let V be a vector bundle on F satisfying $\mathrm{rk}\, V = r$ and $\deg V = r(g-1) - d$. Then for any choice of universal sheaf $\mathcal{E}^u$ on $F \times M_F(r,L)$, the line bundle*

$$L_V := (\det R\pi_*(\mathcal{E}^u \otimes p^* V))^\vee$$

is an ample generator of $\mathrm{Pic} M_F(r,L)$.

(iii) *There is a unique choice of universal bundle on $M_F(r,L)$ such that we have $c_1(E|_{\{pt\}\times M_F(r,d)}) \simeq L_V^k$ for some integer $0 \leqslant k < r$. This k is the unique integer in these bounds that satisfies $dk - rk' = 1$ for some $k' \in \mathbb{Z}$. In particular, we have $(k,r) = 1$.*

Proof. The first two points are Theorem B in [DN89]. The last point is [Ram73, Remark 2.9]. ☐

Let L be a line bundle on $\mathbb{P}^1 \times F$ that is the pullback of a degree-$(d_0 \geqslant 1)$ line bundle from F. We have the projection $\pi : \mathbb{P}^1 \times F \to \mathbb{P}^1$. Let $V := L^{\otimes r(g-1)-d} \oplus \mathcal{O}_{\mathbb{P}^1 \times F}^{\oplus d_0 r - 1}$ so that $\mathrm{rk}\, V = r d_0$ and $\deg V|_F = d_0(r(g-1)-d)$. For a coherent sheaf E on $\mathbb{P}^1 \times F$, we define

$$L_V(E) := (\det R\pi_*(E \otimes V))^\vee.$$

Suppose that E has rank r and degree d on fibers of π. A computation using Grothendieck–Riemann–Roch gives

$$\deg L_V(E) = d_0(c_1(E)^2/2 - r\mathrm{ch}_2(E)) = \frac{d_0}{2}\Delta(E). \tag{1}$$

Here we list related results.

Lemma 2.9. *Let E be a rank-r coherent sheaf on $\mathbb{P}^1 \times F$ satisfying $(\det E, \mathcal{O}_{\mathbb{P}^1 \times F}(F)) = d$, and let N be a line bundle on $\mathbb{P}^1$. Then $L_V(E) \cong L_V(E \otimes \pi^* N)$.*

Proof. By Riemann–Roch, we have

$$\begin{aligned} \mathrm{rk}\, R\pi_*(E \otimes V) &= \deg E|_{\mathrm{pt}\times F}\, \mathrm{rk}\, V + \mathrm{rk}\, E \deg V|_{\mathrm{pt}\times F} + \mathrm{rk}\, E\, \mathrm{rk}\, V(1-g) \\ &= d_0\left(dr + r^2(g-1) - rd + r^2(1-g)\right) = 0. \end{aligned}$$

From the definition of $L_V(-)$, the result follows from the following chain of isomorphisms:

$$\begin{aligned} \det R\pi_*(E \otimes \pi^* N \otimes V) &\simeq \det\left(R\pi_*(E \otimes V) \otimes N\right) \simeq \det R\pi_*(E \otimes V) \otimes N^{\mathrm{rk}\, R\pi_*(E \otimes V)} \\ &\simeq \det R\pi_*(E \otimes V). \end{aligned}$$

Here, the first isomorphism is the projection formula and the second follows from the formula for the determinant of a tensor product of vector bundles after choosing a two-term resolution of $R\pi_*(E \otimes V)$. ☐

Let $0 \leqslant k < r$ be the unique integer in this range satisfying $dk - rk' = 1$ for some k'.

Lemma 2.10. *Let E be a degree-r coherent sheaf on $\mathbb{P}^1 \times F$ such that $c_1(E)$ has degree d on fibers over $\mathbb{P}^1$. Then*

$$(L, c_1(E)) \equiv k \deg L_V(E) \mod d_0 r. \tag{2}$$

Proof. We have that $c_1(E) = d[\mathbb{P}^1 \times y] + \ell[x \times F]$ for some $\ell \in \mathbb{Z}$. Thus, $c_1(E)^2 = 2\ell d$, and

$$k \deg L_V(E) = k d_0 r c_2(E) - k d_0 (r-1)\ell d \equiv d_0 \ell k d \equiv d_0 \ell = (L, c_1(E)) \mod d_0 r. \qquad \square$$

Lemma 2.11. *Suppose that E is a rank-r torsion-free coherent sheaf on $\mathbb{P}^1 \times F$ such that the restriction of E to the generic fiber over $\mathbb{P}^1$ is stable of degree d. Then $\Delta(E) \geqslant 0$, with equality if and only if E is a tensor product of the pullbacks of a stable sheaf on F and a line bundle on $\mathbb{P}^1$.*

Proof. Since E is torsion-free, it maps injectively to its double dual with zero-dimensional cokernel. We have $\Delta(E) \geqslant \Delta(E^{\vee\vee})$, with equality if and only if E is locally free. By replacing E with $E^{\vee\vee}$ if necessary, we may therefore assume that E is locally free.

By Langton's procedure of elementary modifications [Lan75], one may find a locally free subsheaf $E' \subset E$ whose restriction to every fiber over $\mathbb{P}^1$ is stable and such that E' is obtained from E through successive elementary modifications along maximally destabilizing quotients of fibers. One checks that $\Delta(E)$ strictly decreases after each such modification, so $\Delta(E') \leqslant \Delta(E)$, with equality if and only if E is already stable on every fiber. By replacing E with E' if necessary, we may assume that this is the case.

Then, after possibly tensoring E by a line bundle from $\mathbb{P}^1$, we may assume that E is a pullback of the universal sheaf along a morphism $\nu : \mathbb{P}^1 \to M_F(r, L)$ for some L. If $g = 1$, this implies that ν is constant. Otherwise, if $g \geqslant 2$, we have $\Delta(E) = 2k \deg \nu^* L_V$ for L_V as in Proposition 2.8. Since L_V is ample, this implies that $\deg \nu^* L_V \geqslant 0$, with equality if and only if ν is constant. $\square$

Remark 2.12. Without the statement about the case of equality, Lemma 2.11 also follows from the Bogomolov–Gieseker inequality for Gieseker-stable sheaves in view of Theorem 2.24.

2.3 Components of relative Picard schemes

In this subsection, we construct a stack parametrizing connected components of the relative Picard scheme for a family of fibered surfaces. We also construct a further quotient identifying line bundles which differ by component twists. This will be used for making precise the notion of 'fixing the first Chern class' in a family of fibered surfaces. In practice, one can do this by fixing a line bundle L on a family of fibered surfaces $X_B \to C_B$. This is then considered up to algebraic equivalence and up to twists coming from line bundles on C_B of total degree zero. We recommend skipping this subsection on a first reading.

Let $X_B \to C_B \to B$ be a family of fibered surfaces over B with structure morphism $\pi : X_B \to B$. We make the following technical assumption.

Assumption 2.13. *The sheaves $R^1\pi_*\mathcal{O}_{X_B}$ and $R^2\pi_*\mathcal{O}_{X_B}$ are locally free on B.*

This assumption guarantees that the dimensions of the cohomology groups $H^i(X_b, \mathcal{O}_{X_b})$ are locally constant on B and that cohomology commutes with base change for the structure sheaf. For a proper and smooth family of varieties (as for us in characteristic zero), this is a consequence of Deligne's theorem [Del68, Theorem (5.5) (i)]. In our case of semistable degenerations, it holds whenever π is representable by schemes (Remark 2.14), and we suspect that it holds unconditionally (Remark 2.15). This uses a version of Deligne's theorem in logarithmic geometry; a more direct argument would be desirable, but is not known to us.

Remark 2.14. Assumption 2.13 always holds if π is representable by schemes: the family of nodal marked curves $C_B \to B$ induces natural log-structures on B and C_B, with respect to which $C_B \to B$ is log-smooth [Kat00]. Pulling back along $X_B \to C_B$, we get a log-structure on X_B, with respect to which $X_B \to B$ is log-smooth, vertical and exact. It follows from [IKN05, Corollary 7.1] that $R^i\pi_*\mathcal{O}_{X_B}$ is locally free on B and commutes with any base change.

Remark 2.15. Assumption 2.13 is likely unnecessary. One way to see this would be if one had a generalization of [IKN05, Corollary 7.1] to algebraic spaces with a logarithmic structure. It has been pointed out to us by Luc Illusie that the proof there should work in the more general setting; see also [DI87, 4.2.5].

Since $X_B \to B$ is a flat and proper family with geometrically integral fibers, its relative Picard functor $\mathrm{Pic}_{X_B/B}$ exists as an algebraic space [Kle05, Theorem 4.18.6]. We collect some of its basic properties under the given assumptions.

Theorem 2.16. *Suppose that Assumption 2.13 holds: for example, if X_B is a relative scheme over B.*

(i) *The dimension of the identity component $\mathrm{Pic}^0_{X_b}$ of the Picard scheme of the fiber X_b is locally constant on B.*
(ii) *There is an open group subscheme $\mathrm{Pic}^0_{X_B/B} \subset \mathrm{Pic}_{X_B/B}$, of finite type over B, which restricts to the identity component of the Picard scheme over every point of B.*
(iii) *The morphism $\mathrm{Pic}^0_{X_B/B} \to B$ is smooth.*

Proof. By Assumption 2.13, the dimension of the tangent space of the Picard scheme Pic_{X_b} at the identity is locally constant. Since we are in characteristic zero, this implies (i). Then (ii) follows as in [Kle05, Proposition 5.20]. Finally, (iii) follows from the same argument as in [Kle05, Remark 5.21] using Assumption 2.13 in place of smoothness (the projectivity there is only used to invoke the GAGA theorem, which holds more generally for a proper morphism). □

Since the identity component $\mathrm{Pic}^0_{X_B/B}$ is smooth over B, we may take the free quotient algebraic space $\mathcal{NS}_{X_B/B} := [\mathrm{Pic}_{X_B/B}/\mathrm{Pic}^0_{X_B/B}]$. This is the *relative Néron–Severi group* and a quasi-separated group algebraic space. Similarly, we let $\mathcal{NS}_{C_B/B} := [\mathrm{Pic}_{C_B/B}/\mathrm{Pic}^0_{C_B/B}]$.

Proposition 2.17.

(i) *The algebraic space $\mathcal{NS}_{X_B/B}$ is unramified over B.*
(ii) *The algebraic space $\mathcal{NS}_{C_B/B}$ is étale over B.*
(iii) *The sub-algebraic space $\mathcal{NS}^0_{C_B/B}$ parametrizing line bundles with total degree zero is open and closed in $\mathcal{NS}_{C_B/B}$.*
(iv) *Pullback of line bundles induces an open and closed immersion*

$$\mathcal{NS}_{C_B/B} \to \mathcal{NS}_{X_B/B}.$$

Proof. By construction, the morphism $\mathcal{NS}_{X/B} \to B$ is locally of finite type and has everywhere-vanishing relative Kähler differentials, which implies (i). The same holds for $\mathcal{NS}_{C_B/B}$, which is moreover smooth over B. Hence, it is étale over B, which gives (ii). Point (iii) follows from the local constancy of the Euler characteristic. The morphism in point (iv) is well defined, since the morphism $\varphi : \mathrm{Pic}_{C_B/B} \to \mathrm{Pic}_{X_B/B}, L \mapsto f^*L$ preserves the identity components. By Lemma 2.18, the map φ is a closed embedding and $\varphi^{-1}\mathrm{Pic}^0_{X_B/B} = \mathrm{Pic}^0_{C_B/B}$. It follows that the induced morphism of quotient groups $\overline{\varphi} : \mathcal{NS}_{C_B/B} \to \mathcal{NS}_{X_B/B}$ is a monomorphism. We claim it is also proper. By the third isomorphism theorem, we identify

$$\mathcal{NS}_{X_B/B} = \left[\mathrm{Pic}_{X_B/B}\middle/\mathrm{Pic}^0_{X_B/B}\right] = \left[[\mathrm{Pic}_{X_B/B}/\mathrm{Pic}^0_{C_B/B}]\middle/[\mathrm{Pic}^0_{X_B/B}/\mathrm{Pic}^0_{C_B/B}]\right].$$

Hence, we may factor $\overline{\varphi}$ as

$$[\mathrm{Pic}_{C_B/B}/\mathrm{Pic}^0_{C_B/B}] \to [\mathrm{Pic}_{X_B/B}/\mathrm{Pic}^0_{C_B/B}] \to \left[[\mathrm{Pic}_{X_B/B}/\mathrm{Pic}^0_{C_B/B}]\middle/[\mathrm{Pic}^0_{X_B/B}/\mathrm{Pic}^0_{C_B/B}]\right].$$

The first of these is a closed embedding, since φ is a closed embedding. Properness of the second map follows from properness of the quotient $[\mathrm{Pic}^0_{X_B/B}/\mathrm{Pic}^0_{C_B/B}]$, which is Lemma 2.19 below. Finally, since $\mathcal{NS}_{C_B/B}$ is étale over B and $\mathcal{NS}_{X_B/B}$ is unramified over B, it follows that $\mathcal{NS}_{C_B/B}$ is formally étale, and hence étale over $\mathcal{NS}_{X_B/B}$. Any étale (even flat) monomorphism is, in particular, an open immersion [Sta23, Theorem 025G]. □

Lemma 2.18. *The morphism $\varphi : \mathrm{Pic}_{C_B/B} \to \mathrm{Pic}_{X_B/B}, L \mapsto f^*L$ induced by pullback is a closed immersion. Moreover, $\varphi^{-1}\mathrm{Pic}^0_{X_B/B} = \mathrm{Pic}^0_{C_B/B}$.*

Proof. A closed immersion is the same as a proper monomorphism. Since $f_*\mathcal{O}_{X_B} = \mathcal{O}_{C_B}$, we have for any line bundle L on C_B that $f_*f^*L = L$. The same holds after any base change on B, so pullback indeed gives a monomorphism. For properness, it suffices to check the existence part of the valuative criterion. (The uniqueness part is an immediate consequence of being a monomorphism.) Hence, we may assume $B = \mathrm{Spec}\, R$ for a DVR R with generic point η and that we are given a line bundle L on X_B so that L_η is isomorphic to a line bundle pulled back from C_η or, equivalently, so that f_*L is a line bundle on C_η and that $f^*f_*L \to L$ is an isomorphism over X_η. We want to exhibit a line bundle M on C_R so that $L \simeq f^*M$. Viewing $f : X_R \to C_R$ as a family of curves, then, at least over the locus where f is smooth, we can conclude that f_*L is free and $f^*f_*L \to L$ is an isomorphism, since the relative Picard scheme of a family of smooth curves is separated. Thus, these conditions hold, except possibly over a finite set of points $x_1, \dots, x_n \in C_\xi$ contained in the smooth locus of C_ξ. Since C_R is regular of dimension two around each of the x_i, the double dual $M := (f_*L)^{\vee\vee}$ is then locally free. But, since X_R is Cohen–Macaulay, any locally free sheaf is determined by its restriction away from any codimension-two locus. More precisely, this means that the isomorphism $f^*(f_*L)^{\vee\vee} \xleftarrow{\sim} f^*f_*L \xrightarrow{\sim} L$, defined away from the fibers over x_i, extends to an isomorphism $f^*M \xrightarrow{\sim} L$ on all of X_R.

To obtain properness, it remains to show that $\mathrm{Pic}_{C_B/B} \to \mathrm{Pic}_{X_B/B}$ is quasi-compact. For this, we may work locally on B, and even decompose B into locally closed strata [Sta23, Lemma 040W]. In particular, we may assume that C_B has constant topological type over B, that is, that C_B is a union of smooth curves over B. Moreover, we may assume that B is smooth, quasi-compact and irreducible, and that for each component C_i of C_B we have an effective Cartier divisor $D_i \subset f^{-1}C_i$ that is generically finite over C_i. Then the numbers $\chi(f^{-1}C_i, L|_{f^{-1}C_i} \otimes kD_i)$ are locally constant on $\mathrm{Pic}_{X_B/B}$ and hence define a decomposition into open and closed subsets. The preimage of each such subset is either empty or consists of line bundles on C_B of a fixed multi-degree which is a quasi-compact set.

For the statement about identity components, it is immediate that $\varphi^{-1}\mathrm{Pic}^0_{X_B/B}$ is an open subspace of $\mathrm{Pic}_{C_B/B}$ that contains $\mathrm{Pic}^0_{C_B/B}$. To show equality of these open subspaces, it is enough to work pointwise on B. Thus, we may reduce to the setting of the previous paragraph, where $C_B = \cup C_i$ is a union of smooth irreducible components and where we have chosen effective Cartier divisors $D_i \subset f^{-1}C_i$ generically finite over C_i. Then the identity component $\mathrm{Pic}^0_{X_B/B}$ is contained in the open subset defined by $\chi(f^{-1}C_i, L|_{f^{-1}C_i} \otimes kD_i) = 0$ for all i, k. The preimage of this subset under φ is exactly the subspace of $\mathrm{Pic}_{C_B/B}$ of line bundles of multi-degree zero, which is $\mathrm{Pic}^0_{C_B/B}$. ☐

Lemma 2.19. *The algebraic space* $[\mathrm{Pic}^0_{X_B/B}/\mathrm{Pic}^0_{C_B/B}]$ *is proper over* B.

Proof. By Lemma 2.18, the map $\mathrm{Pic}^0_{C_B/B} \to \mathrm{Pic}^0_{X_B/B}$ is a closed embedding. This implies that $Y_B := [\mathrm{Pic}^0_{X_B/B}/\mathrm{Pic}^0_{C_B/B}]$ is separated over B. By Theorem 2.16 (ii), it is also of finite type. It remains to check properness. Working locally on B, we may assume B is affine. By Chow's lemma [Sta23, Lemma 088U], we can find a proper birational map $g : Z \to Y_B$ with Z quasi-projective over B (in particular, a scheme). It suffices to show that Z is proper. By [Gro66, Corollary 15.7.11], a separated finite type morphism of schemes $Z \to B$ with geometrically connected fibers is proper provided that it has proper fibers and that – at least after base change along some proper surjective map $B' \to B$ – there exists a section. For the existence of a section, we may take B' to be the preimage of the identity section under g. Properness of the fibers of g is equivalent to properness of fibers of $Y_B \to B$. Hence, we are reduced to showing that $[\mathrm{Pic}^0_X/\mathrm{Pic}^0_C]$ is proper in the case that B is a point. For this, we can use the existence part of the valuative criterion. Let C_i and X_i be the irreducible components of C and X, respectively. Then, by writing any line bundle on C in terms of line bundles on its components and glueing data, we have an exact sequence

$$0 \to \mathbb{G}_m^b \to \mathrm{Pic}^0_C \to \prod \mathrm{Pic}^0_{C_i} \to 0,$$

where b is the first Betti number of the dual graph of C. For X in place of C, we analogously have

$$0 \to \mathbb{G}_m^b \to \mathrm{Pic}^0_X \to \prod \mathrm{Pic}^0_{X_i} \to \prod \mathrm{Pic}^0_{F_j},$$

where F_j runs through the components of the singular locus of X. This latter sequence shows that the quotient of Pic^0_X by $\mathbb{G}_m^b$ in $\prod \mathrm{Pic}^0_{X_i}$ is closed, and hence it is an abelian subvariety. Pullback along f induces the following commutative diagram.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{G}_m^b & \longrightarrow & \mathrm{Pic}^0_C & \longrightarrow & \prod \mathrm{Pic}^0_{C_i} & \longrightarrow & 0 \\
 & & \downarrow{\sim} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{G}_m^b & \longrightarrow & \mathrm{Pic}^0_X & \longrightarrow & \prod \mathrm{Pic}^0_{X_i} & \longrightarrow & \prod \mathrm{Pic}^0_{F_j}
\end{array}$$

It follows that we have an isomorphism

$$\left[\mathrm{Pic}^0_X/\mathrm{Pic}^0_C\right] \simeq \left[[\mathrm{Pic}^0_X/\mathbb{G}_m^b]\Big/[\mathrm{Pic}^0_C/\mathbb{G}_m^b]\right],$$

which is a quotient of an abelian variety by a subvariety, and hence it is an abelian variety. This shows the desired properness in the case of a point base. ☐

Definition 2.20. We let

$$\overline{\mathcal{NS}}_{X_B/B} := \mathcal{NS}_{X_B/B}/\mathcal{NS}^0_{C_B/B}$$

and

$$\overline{\mathcal{NS}}_{X_B/C_B} := \mathcal{NS}_{X_B/B}/\mathcal{NS}_{C_B/B}.$$

By Proposition 2.17, these are separated and unramified group algebraic spaces over B.

The following result describes how the relative Néron–Severi spaces interact with expansions.

Lemma 2.21. *Let $\widetilde{X}_T \to \widetilde{C}_T$ be an expansion of $X_T \to C_T$. Then pullback along $\widetilde{X}_T \to X_T$ induces isomorphisms $\overline{\mathcal{NS}}_{X_T/C_T} = \overline{\mathcal{NS}}_{X_B/C_B} \times_B T \xrightarrow{\sim} \overline{\mathcal{NS}}_{\widetilde{X}_T/\widetilde{C}_T}$ and $\overline{\mathcal{NS}}_{X_T/T} = \overline{\mathcal{NS}}_{X_B/B} \times_B T \xrightarrow{\sim} \overline{\mathcal{NS}}_{\widetilde{X}_T/T}$.*

Proof. We treat only the case of $\overline{\mathcal{NS}}_{X_T/T}$; the other is similar. Since the pullback map $\mathrm{Pic}_{X_T/T} \to \mathrm{Pic}_{\widetilde{X}_T/T}$ preserves the identity components, it descends to a morphism $\mathcal{NS}_{X_T/T} \to \mathcal{NS}_{\widetilde{X}_T/T}$. Since the total degree is preserved under pullback along $\widetilde{C}_T \to C_T$, this descends to a morphism $\overline{\mathcal{NS}}_{X_T/T} \to \overline{\mathcal{NS}}_{\widetilde{X}_T/T}$. We claim that this is an isomorphism. It is enough to show that each section of $\overline{\mathcal{NS}}_{\widetilde{X}_T/T}$ has a unique preimage. For this, we may work étale locally on T and assume that $\widetilde{C}_T \to T$ has sections meeting each irreducible component of each fiber. Suppose that $\overline{c_1} \in \overline{\mathcal{NS}}_{\widetilde{X}_T/T}(S)$ for some T-scheme S. Then, locally on S, we may assume that $\overline{c_1}$ is represented by some line bundle L on $\widetilde{X}_S$. Up to twisting L by a line bundle N pulled back from $\widetilde{C}_S$ of total degree zero, we may assume that L is pulled back from X_S so that $\overline{c_1}$ comes from an element $\overline{c_1'} \in \overline{\mathcal{NS}}_{X_T/T}(S)$. Moreover, we see that $\overline{c_1'}$ is uniquely determined, since any two possible choices of N differ by a line bundle pulled back from C_S. □

2.4 Stability on fibered surfaces

We collect some results of Yoshioka that allow us to compare f-stability on a smooth fibered surface with slope stability for a suitable polarization (see Theorem 2.24).

Let X be a smooth projective surface, and let $f: X \to C$ be a surjective morphism to a curve C with connected fibers. For a coherent sheaf E on X of rank $r > 0$, we define its *discriminant* as

$$\Delta(E) = 2rc_2(E) - (r-1)c_1(E)^2 = c_1(E)^2 - 2r\mathrm{ch}_2(E) \in \mathbb{Z}.$$

We recall the following result of Yoshioka [Yos96, Lemma 2.1].

Lemma 2.22. *Given an exact sequence $0 \to G_1 \to E \to G_2 \to 0$, where E_1 and E_2 have ranks $r_1 > 0$ and $r_2 > 0$ respectively, we have an equality*

$$\frac{1}{r}\Delta(E) = \frac{1}{r_1}\Delta(G_1) + \frac{1}{r_2}\Delta(G_2) - \frac{1}{rr_1r_2}\left(r_2c_1(G_1) - r_1c_1(G_2)\right)^2.$$

Proof. By additivity of the Chern character, we have

$$\begin{aligned}
&\Delta(E)/r - \Delta(G_1)/r_1 - \Delta(G_2)/r_2 \\
&\quad = (c_1(G_1) + c_1(G_2))^2/(r_1 + r_2) - c_1(G_1)^2/r_1 - c_1(G_2)^2/r_2 \\
&\quad = -\frac{1}{rr_1r_2}\left(r_2c_1(G_1) - r_1c_1(G_2)\right)^2.
\end{aligned}$$

□

Let F be the numerical divisor class of an arbitrary fiber of f, and let H be an ample line bundle on X. For a positive rational number t, we let $H_t := H + tF$.

PROPOSITION 2.23. *Let D be a divisor satisfying $(D,F)\neq 0$. Suppose that $(D,H_t)=0$ for some $t>0$. Then*

$$D^2 \leqslant -\frac{1}{(H,F)^2}(H^2+2t(H,F)).$$

Proof. This is [Yos99, Lemma 1.1]. □

Now we can prove an important consequence, which already appears in e.g. [Yos99], although it is stated there only for elliptic surfaces.

THEOREM 2.24. *Fix values $r,\Delta\in\mathbb{Z}$ with $r>0$ and $c_1\in H^2(X,\mathbb{Z})$ such that r and (F,c_1) are relatively prime.*

(1) *There exists a constant $C(r,c_1,\Delta)$ so that the collection of H_t-semistable sheaves with rank r, discriminant at most Δ and first Chern class c_1 is independent of H_t for all $t\geqslant C(r,c_1,\Delta)$.*
(2) *For any $t\geqslant C(r,c_1,\Delta)$, semistability with respect to H_t equals stability, and a sheaf is stable with respect to H_t if and only if its restriction to the generic fiber of f is stable as a sheaf on a curve.*

Proof. Let μ_t denote the slope with respect to H_t. Suppose that there is a change of stability condition at t_0, i.e. there is a sheaf E of the given invariants which is H_t-semistable for $t=t_0$ but not for bigger (resp., smaller) values of t. Then we may find an exact sequence

$$0\to E_1\to E\to E_2\to 0,$$

which is destabilizing for values of t slightly bigger (resp., smaller) than t_0, but which consists of semistable objects for $t=t_0$ (take part of a HN filtration). In particular, $\mu_{t_0}(E_1)=\mu_{t_0}(E_2)$. By Lemma 2.22 and the Bogomolov inequality applied to E_1, E_2, we have

$$\Delta(E)\geqslant -\frac{1}{rr_1r_2}(r_2c_1(E_1)-r_1c_1(E_2))^2=-D^2/(rr_1r_2),$$

where we set $D:=r_2c_1(E_1)-r_1c_2(E_1)$ and r_i is the rank of E_i. The assumption that r and (c_1,F) are coprime implies that $(D,F)=r_2(c_1(E_1),F)-r_1(c_1(E_2),F)\neq 0$. We also have $(D,H_{t_0})=r_1r_2(\mu_{t_0}(F_1)-\mu_{t_0}(F_2))=0$. Therefore, Proposition 2.23 applies to D, and we find that

$$\Delta(E)\geqslant \frac{1}{rr_1r_2(H,F)^2}(H^2+2t_0(H,F))\geqslant \frac{H^2+2t_0(H,F)}{r^3(H,F)^2}.$$

This shows that the values that t_0 can take are bounded above, so (1) follows.

To address (2), let $\eta\in C$ denote the generic point and $F_\eta:=f^{-1}(\eta)$. For any coherent sheaf G on X of rank $r_G>0$, the slope with respect to H_t is

$$\mu_t(G)=((c_1(G)),H)+t(c_1(G),F))/r_G.$$

Suppose that E is semistable beyond the last wall. Then for any subsheaf E', we have

$$\mu_t(E')\leqslant\mu_t(E)$$

for sufficiently large t. Dividing by t and taking the limit as $t\to\infty$ gives

$$\mu(E'|_{F_\eta})=(c_1(E'),F)/r'\leqslant(c_1(E),F)/r=\mu(E|_F).$$

By the coprimeness assumption, we have strict inequality, and therefore the restriction of E to F_η is stable.

Conversely, assume that the restriction of E to the generic fiber of f is stable. Then for any subsheaf E', we know that $(E',F)/r' < (E,F)/r$. Let $\mu_{0,max}(E)$ be the maximum value of μ_0 of a lower-rank subsheaf of E, and let $\mu_{max}(E|_F) < \mu(E|_F)$ be the least slope of a non-zero lower-rank subsheaf of $E|_F$. Let $E' \subset E$ be an arbitrary lower-rank subsheaf. Then we have

$$\mu_t(E') = (E',H)/r' + t(E',F)/r' \leqslant \mu_{0,max}(E) + t\mu_{max}(E)$$

and the right-hand side is strictly smaller than $\mu_t(E) = \mu_0(E) + t\mu(E|_F)$ for sufficiently large t, independent of E'. Therefore, there is no destabilizing subsheaf for sufficiently large t.

It follows from what we have shown that H_t-semistability of E for $t \gg 0$ is equivalent to H_t-stability and stability of $E|_{F_\eta}$, as desired. □

3. Moduli spaces of sheaves on fibrations

Throughout this section, we fix coprime integers r, d with $r > 0$. Recall that we assume all curves to be proper and connected.

3.1 Stacks of fiber-stable sheaves

Let $f: X \to C$ be a fibered surface over a nodal marked curve C as defined in Definition 1.6. In particular, the fibers of f over nodes and marked points, and over generic points of components of C, are smooth projective curves.

We define stability of sheaves relative to a fibration.

Definition 3.1. Let $\widetilde{C} \to C$ be an expansion of C, and let $\widetilde{X} := X \times_C \widetilde{C}$. A torsion-free coherent sheaf E of rank r and with fiber degree d on $\widetilde{X}$ is called *f-stable* if it satisfies the following conditions.

(i) The sheaf E is locally free at the fibers of $\widetilde{X} \to \widetilde{C}$ over singular and marked points.

(ii) For any generic point η of $\widetilde{C}$, the restriction of E to the fiber $\widetilde{X}_\eta$ over η is slope stable.

(iii) For any marked or singular point c of $\widetilde{C}$, the restriction of E to the fiber over c is slope stable.

We extend this notion to arbitrary families.

Definition 3.2. Let $f: X_B \to C_B$ be a family of fibered surfaces over some base B. Let T be a $\mathbb{C}$-scheme. A *family of f-stable sheaves on an expansion of X_B over B*, valued in T, is given by the following pieces of data:

(i) a morphism $T \to B$, with pullback family $X_T \to C_T$,

(ii) an expansion $c: \widetilde{C}_T \to C_T$ with associated fibered surface $\widetilde{X}_T := X_T \times_{C_T} \widetilde{C}_T \to \widetilde{C}_T$,

(iii) a T-flat coherent sheaf E_T on $\widetilde{X}$ such that, for every $t \in T$, the fiber E_t is f-stable for $X_t \to C_t$.

Let $f: X_B \to C_B$ be a family of fibered surfaces over some base B.

Proposition 3.3. *There is an algebraic stack $\mathcal{M}_{X_B/C_B}(r,d)$ over B parametrizing f-stable sheaves of rank r and fiber degree d on expansions of X_B over B.*

Remark 3.4. We make this explicit in the case $B = \operatorname{Spec}\mathbb{C}$. Say $X \to C$ is a fibered surface. Then a $\mathbb{C}$-point of $\mathcal{M}_{X_B/C_B}(r,d)$ is given by a pair (c, E), where $c: \widetilde{C} \to C$ is an expansion of C and

E an f-stable, rank-r sheaf on the induced $\widetilde{X} := X \times_C \widetilde{C}$ with fiber degree d. We will sometimes write this as $(\widetilde{X}, E)$.

An automorphism of the pair $(\widetilde{X}, E)$ is a pair (g, γ), where $g : \widetilde{X} \to \widetilde{X}$ is an automorphism that commutes with the contraction to X, and $\gamma : g^*E \to E$ is an isomorphism.

Proof. We work over the relative stack of expansions $\mathrm{Exp}_{C_B/B}$. Let $c : \mathcal{C} \to C_B$ denote the universal expansion, and let $\mathcal{X} := X_B \times_{C_B} \mathcal{C} \to \mathcal{C}$ denote the induced family of fibered surfaces over $\mathrm{Exp}_{C_B/B}$. Let $\mathscr{M}(r, d)$ denote the stack of all torsion-free coherent sheaves of rank r and fiber degree d on $\mathcal{X}$ over $\mathrm{Exp}_{C_B/B}$. Then, since being locally free and being slope stable are open conditions, one can see that the locus of those sheaves satisfying (i)-(iii) of Definition 3.1 is open. Thus, we get $\mathcal{M}_{X_B/C_B}(r, d) \subseteq \mathscr{M}(r, d)$ as the open locus of f-stable sheaves. □

Proposition 3.5. *The morphism $\mathcal{M}_{X_B/C_B}(r, d) \to B$ satisfies the existence part of the valuative criterion of properness.*

Proof. We may assume that $B = \mathrm{Spec}\, R$ for a DVR R with generic point η, that $\widetilde{C}_\eta \to C_\eta$ is a given expansion of C_B over η and that E_η is an f-stable sheaf on $\widetilde{X}_\eta := \widetilde{C}_\eta \times_C X$. Our goal is to show that we can extend this to a family of f-stable sheaves, possibly after passing to some extension of R. By Corollary 2.7, we can at least extend $\widetilde{C}_\eta$ to some expansion $\widetilde{C}_B$ over $B = \mathrm{Spec}\, R$. For the proof, we may as well replace C_B by $\widetilde{C}_B$ and hence assume that $\widetilde{C}_\eta \to C_\eta$ is an isomorphism. Let ξ denote the closed point of $\mathrm{Spec}\, R$.

Case 1: *C_η is smooth.* In this case, the desired result follows from Proposition 3.6 below.

Case 2: *C_η is of compact type.* We proceed by induction on the number of components of C_η. If there is only one component, we are in Case 1. Otherwise, we may decompose $C_B = C_B^1 \cup_q C_B^2$ along any chosen node q that extends over the generic fiber. We add a marked point q_i on each C_B^i where the node was. By inductive assumption, for each $i = 1, 2$, we can find an expansion $\widetilde{C}_B^i \to C_B^i$ that is an isomorphism over the generic fiber together with an f-stable sheaf E^i on $\widetilde{X}_B^i := X_B \times_{C_B} \widetilde{C}_B^i$ extending the restriction of E_η. We denote the marked point on $\widetilde{C}_B^i$ lying over q_i again by q_i. In order to glue the total family back together over $\mathrm{Spec}\, R$, we need to extend the isomorphism of $E^1|_{q_1}$ and $E^1|_{q_2}$ from η over all of $\mathrm{Spec}\, R$. But since these are fiberwise isomorphic families of stable sheaves, there exists such an extension, possibly after twisting one of E^1 or E^2 by a multiple of $\widetilde{X}_\xi^i$. Here, we use Knudsen's clutching construction [Knu83, Theorem 3.4] to glue the expansions $\widetilde{C}_B^i$ along the marked points.

Case 3: *C_η is arbitrary.* We do an induction on the first Betti number of the dual graph of C_η. If it is zero, we are in Case 2. Otherwise, we may choose a non-separating node q on C_B that extends over the generic fiber and take a partial normalization $\nu : C'_B \to C_B$ around q, while remembering the preimage of the node through adding in markings q_1 and q_2. Then, by our inductive hypothesis, we can find some expansion $\widetilde{C}'_B \to C'_B$ which is an isomorphism over η and an extension E' of $\nu^* E_\eta$ to $\tilde{X}'_B := X_B \times_{C_B} \widetilde{C}'_B$. We again denote the lifts of q_1, q_2 to $\widetilde{C}'_B$ by q_1, q_2. In order to glue together the total family along the markings q_1 and q_2, we need to produce an isomorphism $E'|q_1 \simeq E'|_{q_2}$ extending the given one over the generic fiber. As in the previous case, such an isomorphism exists after twisting, say, $E'|_{q_1}$ along the preimage of ξ. However, since $\widetilde{C}'_B$ is connected, it is a priori not clear how to do this without simultaneously changing $E'|_{q_2}$. To get around this issue, we may pass to a further expansion by blowing up $\widetilde{C}'_B$ along the intersection of q_1 and the special fiber $\widetilde{C}'_\xi$. Then we can achieve the desired modification by twisting the pullback of E' by an integer multiple of the exceptional component. □

Proposition 3.6. *Let R be a DVR with generic point η and let $X_R \to C_R \to \operatorname{Spec} R$ be a family of fibered surfaces over $\operatorname{Spec} R$ with C_η smooth over η. Let E_η be an f-stable sheaf of rank r and fiber degree d on X_η. Then, after possibly performing a base change on R, we can find an expansion $\widetilde{C}_R \to C_R$ which is an isomorphism over η and an f-stable sheaf E_R on $\widetilde{X}_R$.*

Proof. We may modify C_R by repeatedly blowing up singular points to obtain a family with regular total space [Sta23, Tag 0CDE], which will automatically be an expansion of C_R. Without loss of generality, we may therefore assume that C_R is nonsingular.

Let ξ denote the closed point of $\operatorname{Spec} R$. By the argument in [GL94, proof of the last statement of Proposition 3.3], after passing to some extension of R and further expanding C_R over the special fiber, we can assume that E_η extends to a torsion-free coherent sheaf E_R on X_R with the following properties.

(a) The restriction of E_R to X_ξ is torsion-free.

(b) The sheaf E_R is locally free along the fibers of $X_\xi \to C_\xi$ over singular and marked points.

Let $P(C_\xi)$ denote the collection of singular, marked and generic points of C_ξ. Unless E_R is f-stable, there exists $x \in P(C_\xi)$ such that the restriction E_x of E_R to the curve $f^{-1}(x)$ is unstable. Let a be the minimum of the values $\mu_{min}(E_x)$, where μ_{min} denotes the minimal slope in a Harder–Narasimhan filtration. Let ρ be the maximum rank of a maximally destabilizing quotient of E_x, ranging over those $x \in P(C_\xi)$ for which $\mu_{min}(E_x) = a$. We claim that after taking a base change on R and a further expansion of C_R, we may find a different completion of E_η such that either a increases or a stays the same and ρ decreases. Since there are only finitely many possible values for ρ, and since the set of possible values of a lies in $\mathbb{Z}/r$ and is bounded above by d/r, we find that after doing so finitely many times, we end up with an f-stable extension of E_η.

To prove this claim, consider the relative Quot-scheme

$$q : \operatorname{Quot}_{E_R, X_R/C_R}(\rho, a) \to C_R$$

parametrizing quotients of E_R of rank ρ and slope a on the fibers of $X_R \to C_R$. By assumption, the image of q contains some points of $P(C_\xi)$. From the minimal choice of a, ρ, it follows from an analysis of the relative deformation space that at each such point, the map q is locally a closed embedding (see the proof of Theorem 5 in [Nit11]).

We claim that, after passing to an extension of R and further expanding C_R along the special fiber, we can arrange that around each point in $P(C_\xi)$, the set-theoretic image of q is supported on C_ξ. This is equivalent to asking that the closure in C_R of the set-theoretic image $\operatorname{Im} q|_\eta$ is disjoint from $P(C_\xi)$. By f-stability, $\operatorname{Im} q|_\eta$ is a finite collection of closed points of C_η not meeting any nodes or marked points. After possibly passing to a finite extension of R, we may add new marked points $\rho_{i,\eta}$ so that $\operatorname{Im} q|_\eta$ is the union of the images of the $\rho_{i,\eta}$. Now the claim follows from the semistable reduction theorem for marked nodal curves. It follows that there is a closed subscheme $Z \subset C_R$ whose associated points are all in $P(C_\xi)$ and such that around each of its associated points it agrees with the closed subscheme defined by the Quot-scheme.

If Z contains a component C_i of C_ξ, we may replace E_R by the elementary modification of E_R along a maximally destabilizing quotient on that component. By the minimal choice of a and ρ, the resulting sheaf will still be locally free at fibers over marked points and nodes. This has the effect of dividing the ideal sheaf of Z by the uniformizer of that component. Thus, if Z is locally principal, one can do so until Z becomes empty, in which case there is no point left in $P(C_\xi)$ with maximally destabilizing subsheaf of slope a and rank ρ. If Z is not locally principal, one can use Lemma 3.7 to find an extension R' of R and an expansion $c: \widetilde{C}_{R'} \to C_{R'}$ over R' which is trivial over the generic fiber such that the scheme-theoretic preimage of Z in $\widetilde{C}_{R'}$ is principal. Since the relative Quot-scheme is compatible with pullback, this reduces to the case that Z is principal, which we already treated. □

Lemma 3.7. *Let $\pi: C_R \to \operatorname{Spec} R$ be a nodal marked curve over a DVR R with closed point ξ. Suppose that π has smooth generic fiber and that C_R is regular. Let $Z \subset C_R$ be a closed subscheme supported on C_ξ whose associated points are special, marked or generic points of components of the special fiber. Then there is an extension of DVRs $R \subset R'$ and an expansion $c: \widetilde{C}_{R'} \to C_{R'}$ which is trivial over the generic point of R' such that $c^{-1}(Z)$ has no marked points or nodes of $\widetilde{C}_\xi$ as associated points.*

Proof. We first argue that by repeatedly blowing up at marked points, one can achieve that the preimage of Z is principal around each marked point. Working étale locally around a marked point x, we may assume that Z is supported at x and that the family is locally given by $\operatorname{Spec} R[t] \to \operatorname{Spec} R$, with the section given by $t=0$. Let π be a uniformizer of R. Let Z_π and Z_t be the intersections of Z with the loci $(\pi=0)$ and $(t=0)$ respectively. We claim that the invariant $\ell(Z_\pi)+\ell(Z_t)$ decreases for the preimage of Z on the blowup. Indeed, the new marked point on the blowup is cut out by coordinates π, u, where $t=\pi u$. There exist elements $g_1 = t^a + \pi f_1$ and $g_2 = \pi^b + t f_2$ in the defining ideal I_Z of Z, where $a = \ell(Z_\pi)$ and $b = \ell(Z_t)$. Let also k be maximal so that $I_Z \subseteq (\pi, t)^k$. Thus, $q^{-1}Z$ contains the exceptional divisor to order at least k. Note that $1 \leqslant k \leqslant a, b$. Let $\widetilde{Z}$ be the non-principal part of $q^{-1}Z$ at the marked point. Then by a direct computation, one has

$$\begin{aligned} \ell(\widetilde{Z}_\pi) &\leqslant k, \\ \ell(\widetilde{Z}_u) &\leqslant b-k. \end{aligned}$$

In particular, $\ell(\widetilde{Z}_\pi)+\ell(\widetilde{Z}_\pi) \leqslant b < \ell(Z_\pi)+\ell(Z_t)$. Essentially the same argument works for x a node in C_ξ, where one has parameters s, t locally cutting out the components of C_ξ at x. Here, one needs to repeatedly blow up the nodes in the reduced preimage of C_ξ. This yields a modification $\widehat{C}_R \to C_R$, which principalizes Z and is an isomorphism over the generic fiber, but where the fiber $\widehat{C}_\xi$ may have non-reduced components. After taking a ramified extension R' of R with sufficiently divisible degree, taking the normalization of $\widehat{C}_{R'}$ and resolving the singularities through repeated blowups, we obtain an expansion of $\widetilde{C}_{R'} \to C_{R'}$ with the desired properties. □

3.2 Fixing twists from the base

In the last subsection, we saw that the moduli stack $\mathcal{M}_{X_B/C_B}(r,d)$ satisfies the existence part of the valuative criterion of properness. To get a proper moduli space, we need to introduce a further numerical stability condition which fixes twists by line bundles from C_B. For stability of line bundles on a curve, we heavily use ideas from [OS79] and [EP16].

Here, we fix $g \geqslant 0$ and consider only fibered surfaces $f: X \to C$ whose fibers have arithmetic genus g.

For a marked nodal curve C over a field, let $\mathrm{Irr}(C)$ denote the set of irreducible components of C. For a line bundle N on C, we use $\deg L$ to denote the total degree and $\underline{\deg} L$ to denote the component-wise degree, which is a function on $\mathrm{Irr}(C)$.

Definition 3.8.

(i) Let C be a marked nodal curve over a field. A *stability condition* on C is a map $\alpha : \mathrm{Irr}(C) \to \mathbb{R}$. We define the *degree* of α as $\sum_{D \in \mathrm{Irr}(C)} \alpha(D)$.

(ii) Let $C_B \to B$ be a family of marked nodal curves. A *stability condition* on C_B over B is given by a collection of stability conditions (α_x) for each field-valued point x of B, which are compatible in the following sense: if η specializes to ξ in B, there is an induced surjective morphism $\mathrm{Irr}(C_\xi) \to \mathrm{Irr}(C_\eta)$. We require that for each $D \in \mathrm{Irr}(C_\eta)$, we have that $\alpha_\eta(D)$ is equal to the sum of $\alpha_\xi(D')$ for D' mapping to D.

Remark 3.9. It follows that for a family of curves over a finite type base B, a stability condition is uniquely defined by its values on the most degenerate strata. For example, if B is the spectrum of a DVR, then giving a stability condition on C_B is the same as giving one over the special fiber.

Let $X \to C$ be a given fibration over a nodal marked curve, and let E be an f-stable sheaf on an expansion $\widetilde{X} \to \widetilde{C}$.

Definition 3.10.

(i) We say that a component of $\widetilde{C}$ (resp., of $\widetilde{X}$) is *exceptional* if it is contracted by $\widetilde{C} \to C$ (resp., by $\widetilde{X} \to X$).

(ii) We say that the expansion $\widetilde{X}$ is *minimal* if there is no intermediate expansion $\widetilde{X} \to \widetilde{X}' \to X$ such that E is isomorphic to a pullback from $\widetilde{X}'$.

We use the following abuse of notation: let $\widetilde{C} \to C$ be an expansion and α a stability condition on C. For $D \subset \widetilde{C}$ an irreducible component, we set

$$\alpha(D) := \begin{cases} 0, & \text{if } D \text{ is exceptional,} \\ \alpha(c(D)), & \text{if } D \text{ maps isomorphically to its image.} \end{cases}$$

For the rest of this subsection, let $f : X_B \to C_B$ be a family of fibered surfaces with genus g fibers over a base B. Let L_0 be a line bundle on X_B which has positive degree d_0 on each fiber over C_B. Let $0 \leqslant k < r$ be the unique integer such that $kd - rk' = 1$ for some $k' \in \mathbb{Z}$. Define $W := L_0^{\otimes(g-1)r-d} \oplus \mathcal{O}_X^{\oplus d_0 r - 1}$. For any coherent sheaf E on X of finite cohomological dimension, consider the line bundle

$$M(E) := \frac{\det Rf_*((\det E) \otimes L_0)}{\det Rf_*(\det E)} \otimes (\det Rf_*(E \otimes W))^{\otimes k}. \tag{3}$$

We similarly define $M(E)$ for any expansion of X_B by pulling back L_0.

Remark 3.11. This definition is chosen so that M has the following two properties whenever E has rank r and degree d on fibers of f, which is all that we will use in what follows.

(i) (Non-zero weight.) For any line bundle N on C_B, we have

$$M(E \otimes f^* N) = M(E) \otimes N^{d_0 r}.$$

(ii) (Normalizable on exceptional components.) Consider the base change $\widetilde{X} \to \widetilde{C}$ of f to any geometric point of B. Then for any exceptional component $D \subset \widetilde{C}$, we have that $M(E)|_D$ has degree an integer multiple of $d_0 r$.

The first property can be seen directly from Grothendieck–Riemann–Roch. The second is a consequence of Lemma 2.10.

We let α be a stability condition on C_B over B. For a subcurve $Z \subset C$ of a nodal marked curve, we let Z^c denote the 'complementary' subcurve formed by the union of components of C not contained in Z.

First, we consider the case where $X \to C$ is a fibered surface over a field base $B = \operatorname{Spec} k$. The following definition is essentially a version of Oda–Seshadri's stability for torsion-free rank-one sheaves [OS79, Equation $(***)$ on page 8], but phrased for line bundles on an expansion, following the ideas of [EP16]. We will make this connection explicit in Lemma 3.17 below.

Definition 3.12. Let $\widetilde{X} \to X$ be an expansion. We say that an f-stable sheaf E on $\widetilde{X}$ is *α-balanced* if:

(1) for each proper subcurve $\emptyset \subsetneq Z \subsetneq \widetilde{C}$ of $\widetilde{C}$ we have
$$\left| \frac{\deg M(E)|_Z}{r d_0} - \sum_{D \subset Z} \alpha(D) \right| \leqslant \frac{\#(Z \cap Z^c)}{2}, \tag{4}$$

(2) for each exceptional component $D \subset \widetilde{C}$ we have that $\deg M(E)|_D$ is non-negative.

Moreover, we say that E is *strictly* α-balanced if:

(3) whenever we have equality in (4), then one of Z or Z^c is a union of exceptional components.

One can see that (strict) α-balancedness is an open condition in families of f-stable sheaves, so this definition gives a well-behaved moduli functor for families over a general base B.

Example 3.13. In Definition 3.12, let $Z \subset \widetilde{C}$ be a chain of exceptional components. If Z maps to a marked point, (4) requires that $M(E)$ has degree zero on all components of Z. If Z maps to a node, then (4) requires that there is at most one component D of Z on which $M(E)$ has non-zero degree, in which case $\deg M(E)|_D = r d_0$.

Definition 3.14. We let $\mathcal{M}^{\alpha}_{X_B/C_B}(r,d) \subseteq \mathcal{M}_{X_B/C_B}(r,d)$ denote the open substack consisting of α-balanced sheaves on minimal expansions.

Proposition 3.15. *The stack $\mathcal{M}^{\alpha}_{X_B/C_B}(r,d)$ satisfies the existence part of the valuative criterion of properness.*

Proof. Let R be a DVR with generic and closed points η and ξ and with a given morphism $\operatorname{Spec} R \to B$. Let E_η be a sheaf on an expansion $\widetilde{X}_\eta \to \widetilde{C}_\eta$ of $X_\eta \to C_\eta$ such that E_η is f-stable and α-balanced. By Proposition 3.6, we can find some extension $\widetilde{X}_R \to \widetilde{C}_R$ of this data to an f-stable family E_R (possibly after replacing R by an extension). We claim that we can modify this data to obtain an α-balanced bundle on a minimal expansion.

For this, pick a line bundle N_0 on $\widetilde{C}_R$ that has degree zero on exceptional components of $\widetilde{C}_\xi$ and such that $M(E_R) \otimes N_0$ has degree a multiple of $r d_0$ on each component of C_ξ (this may require further extending R). This is possible by Remark 3.11 (ii). Then, further extending R if needed, we can pick N_1 on $\widetilde{C}_R$ such that $\underline{\deg}\, N_1|_{\widetilde{C}_\xi}^{\otimes rd_0} = \underline{\deg}\,(M(E|_Z)\otimes N_0)|_{\widetilde{C}_\xi}$. By changing N_0, we may in fact assume without loss of generality that $N_1^{\otimes rd_0} \cong M(E)\otimes N_0$ on $\widetilde{C}_R$.

Now consider the stability condition $\alpha' := \alpha + \underline{\deg}\, N_0/(rd_0)$. Then we have the following lemma.

Lemma 3.16. *A sheaf E is α-balanced if and only if N_1 is α'-semistable in the following sense: N_1 has non-negative degree on exceptional components, and for every subcurve $\emptyset \subsetneq Z \subsetneq \widetilde{C}$, we have*

$$\left|\deg N_1|_Z - \sum_{D\subset Z} \alpha'(D)\right| \leqslant \frac{\#(Z\cap Z^c)}{2}.$$

Proof. Let $\emptyset \subsetneq Z \subsetneq \widetilde{C}$ be a subcurve. Then

$$\deg N_1|_Z = \frac{\deg M(E)|_Z}{rd_0} + \frac{\deg N_0|_Z}{rd_0}$$

and

$$\sum_{D\subset Z} \alpha'(D) = \sum_{D\subset Z} \alpha(D) + \frac{\deg N_0|_D}{rd_0} = \sum_{D\subset Z} \alpha(D) + \frac{\deg N_0|_Z}{rd_0},$$

so

$$\left|\deg N_1|_Z - \sum_{D\subset Z} \alpha'(D)\right| = \left|\frac{\deg M(E)|_Z}{rd_0} - \sum_{D\subset Z} \alpha(D)\right|.$$

Since also the degree of N_1 on exceptional components is equal to a positive multiple of the degree of $M(E)$, the lemma follows. □

We consider the coherent sheaf $N_1' := c_* N_1$ on C_R with adjunction map $\psi : c^* N_1' \to N_1$. If this is surjective, we obtain an induced map $P : \widetilde{C}_R \to \mathbb{P}(N_1')$.

Lemma 3.17. *The following are equivalent (over the generic and closed point of R respectively).*

(i) *The line bundle N_1 is α'-semistable in the sense of Lemma 3.16.*

(ii) (a) *The sheaf N_1' is torsion-free, the morphism ψ is surjective and identifies N_1 with the pullback along P of the universal quotient of N_1', and*

(b) *N_1' is Oda–Seshadri α'-semistable in the sense of [KP19, Definition 4.1].*

Proof. By the arguments in [EP16, § 5], it follows that (ii) (a) is equivalent to the condition that N_1 has only degrees $0, 1$ on exceptional components and total degree at most 1 on each chain of exceptional components.

Then, one can check by hand that α'-stability for N_1 (in the sense of Lemma 3.16) and Oda–Seshadri α'-semistability for N_1' are equivalent by using a destabilizing subcurve for the one to construct one for the other. See also [EP16, Proposition] for the argument for a specific choice of stability condition. □

As in [KP19, Corollary 4.3], it follows from Simpson stability that any α'-semistable torsion-free sheaf on the generic fiber has an α'-semistable limit. Let N_2' be an α'-semistable limit that agrees with N_1' on the generic fiber. After possibly further expanding $\widetilde{C}_\xi$, we can assume that we have a morphism $P_2 : \widetilde{C}_R \to \mathbb{P}(N_2')$ and denote by N_2 the pullback of the universal line bundle along P_2. Note that this implies that N_1 and N_2 are isomorphic over η. We consider the line bundle $N_E := N_2 \otimes N_1^\vee$ on $\widetilde{C}_R$.

Claim: The sheaf $E_1 := E \otimes f^* N_E$ is α-balanced.

To see this, note that $M(E_1) = M(E) \otimes N_E^{rd_0}$. Therefore,

$$\underline{\deg}\, M(E_1) = \underline{\deg}\, M(E) + rd_0(\underline{\deg}\, N_2 - \underline{\deg}\, N_1) = -\underline{\deg}\, N_0 + rd_0\, \underline{\deg}\, N_2.$$

In particular, using the reverse direction of Lemma 3.17 and Lemma 3.16, we find that N_2 is α'-stable and that E_1 is α-balanced. Since N_E is trivial along $\widetilde{X}_\eta$, we find that E_1 is an α-balanced extension of E_η.

Finally, one can obtain a minimal expansion $\widetilde{C}$ by contracting the components D of $\widetilde{C}_\xi$ over which E_1 is isomorphic to a pullback along $F \times D \to F$, where F is the fiber of $X_\xi \to C_\xi$ over the image of D. Since we assumed that $\widetilde{X}_\eta \to \widetilde{C}_\eta$ was minimal, one can do this contraction without affecting the generic point. This uses that every component of $\widetilde{X}_\eta$ contains in its closure at least one component that is not contracted, which one can see, for example, using Lemma 2.11. □

Proposition 3.18. *Let E be a strictly α-balanced f-stable sheaf on an expansion $\widetilde{X} \to \widetilde{C}$. Then the subgroup of scalar automorphisms has finite index in the automorphism group of $(E, \widetilde{X})$.*

Proof. By f-stability, every automorphism of E as a sheaf on $\widetilde{X}$ must be scalar: since the restriction to each fiber over a generic point η of $\widetilde{C}$ is geometrically stable, any automorphism must be scalar over a dense open of $\widetilde{C}$, and therefore scalar, since $\widetilde{C}$ is reduced and E is flat over $\widetilde{C}$.

In particular, for every automorphism γ of $\widetilde{C}$ over C, there exists at most one isomorphism $\phi : \gamma^* E \to E$ up to scaling. On the other hand, each automorphism of $\widetilde{C}$ is given by scaling exceptional components. Let $D \subset \widetilde{C}$ be an exceptional component. By restricting to D, we may assume without loss of generality that $D = \mathbb{P}^1$ and that γ acts by multiplication with $a \in \mathbb{G}_m$. Then the restriction E_D of E to $X_D = \mathbb{P}^1 \times F$ is stable on the generic fiber over $\mathbb{P}^1$ (and over the fibers over $0, \infty$). If the map $\nu_E : \mathbb{P}^1 \to M_F(r,d)$ induced by E_D is non-trivial, then it is finite onto its image and a must preserve the fibers of ν_E. In particular, there are only finitely many possible values for a. If ν_E is constant, there might still be distinguished points in $\mathbb{P}^1$ over which the restriction of E_D to the fiber is not locally free or not stable; in this case again, a must permute the finite set of those points, so must belong to a finite set. If neither of these occur, then E_D is a pullback of a stable bundle from F twisted by the pullback of a degree-ℓ line bundle from $\mathbb{P}^1$. By α-balancedness, we have that ℓ is 0 or 1, and by minimality of $\widetilde{X}$, we must have $\ell = 1$. We claim that on any such component, a must be an rd_0th root of unity, which shows that there are only finitely many possible choices of γ.

We now show this last claim. Since scaling fixes the points $0, \infty \in \mathbb{P}^1$, we have that ϕ induces an automorphism of the restriction of E to the fibers over $0, \infty$, say ϕ_0, ϕ_∞, which are given by scalar multiplication. They are related by $\phi_\infty = a^{-1}\phi_0$. In particular, if a is not an rd_0th root of unity, then ϕ induces an automorphism of the pair $(\widetilde{C}, M(E))$ that is given by scaling $M(E)$ differently at different nodes which are fixed by γ. Without loss of generality, we may assume that there is one node of $\widetilde{C}$ at which this scaling is trivial. Then let $Z \subseteq \widetilde{C}$ be the maximal connected subcurve of $\widetilde{C}$ containing this node such that at all nodes in Z, the automorphism induced by ϕ is trivial. Then any irreducible component D' of $\widetilde{C}$ intersecting Z in a finite set must be exceptional, and we have $\deg M(E)|_{D'} = rd_0$. If Z is a chain of exceptional components, this means inequality (4) is violated. Otherwise, for this Z, the *strict* inequality in (4) is violated. In either case, this contradicts the assumption that E is strictly α-balanced. □

3.3 Boundedness and properness

Let $f: X_B \to C_B$ be a family of fibered surfaces over B. We assume here that B is connected. Let L_0 be a line bundle on X_B with degree $d_0 > 0$ on fibers over C_B, and let α be a stability condition on C_B. To get a bounded moduli space, we need to fix numerical invariants. For this, we will use the relative Néron–Severi scheme $\overline{\mathcal{NS}}_{X_B/B}$ constructed in § 2.3. In order for the results there to apply, we will impose Assumption 2.13 from here until the end of § 4.

We fix a section $\overline{c_1} \in \overline{\mathcal{NS}}_{X_B/B}(B)$ which has fiber degree d and $\Delta \in \mathbb{Z}$. In practice, one can often specify $\overline{c_1}$ in terms of a line bundle L on X_B, and this is always possible after passing to some étale cover of B. For a singular surface S over $\operatorname{Spec}\mathbb{C}$, we use the usual formula for the discriminant:

$$\Delta(E) = c_1(E)^2 - 2r\operatorname{ch}_2(E).$$

This makes sense whenever E has finite cohomological dimension on S.

Definition 3.19. We let $\mathcal{M}^{\alpha}_{X_B/C_B}(r, \overline{c_1}, \Delta) \subseteq \mathcal{M}^{\alpha}_{X_B/C_B}(r, d)$ denote the substack of sheaves whose fiberwise discriminant is Δ and for which the associated section of $\overline{\mathcal{NS}}_{X_B/B}$ defined by the determinant agrees with the pullback of $\overline{c_1}$.

If C_B has geometrically irreducible fibers over B, then $\mathcal{NS}_{X_B/B} = \overline{\mathcal{NS}}_{X_B/B}$, and we also use the notation $\mathcal{M}_{X_B/C_B}(r, c_1, \Delta)$ for $c_1 \in \mathcal{NS}_{X_B/B}$.

Note that this makes sense by Lemma 2.21.

Lemma 3.20. *The stack $\mathcal{M}^{\alpha}_{X_B/C_B}(r, c_1, \Delta)$ is an open and closed substack of $\mathcal{M}^{\alpha}_{X_B/C_B}(r, d)$.*

Proof. The condition that the degree of the cycle $2rc_2(E) - (r-1)c_1(E)^2$ equals Δ is an open and closed condition. Since $\overline{\mathcal{NS}}_{X_B/B}$ is separated and unramified over B, the section $\overline{c_1}$ determines an open and closed subspace. □

Proposition 3.21. *The stack $\mathcal{M}^{\alpha}_{X_B/C_B}(r, \overline{c_1}, \Delta)$ is of finite type over B.*

Proof. It only remains to show that it is quasi-compact over B. We may work locally on B and stratify B by the singularity type of C_B. In particular, we may assume that B is a finite type $\mathbb{C}$-scheme and that the singular locus of C_B is a disjoint union of copies of B. Let $C_1, \ldots, C_n$ denote the normalizations of the components of C_B, which are smooth over B, and $X_1, \ldots, X_n$ their preimages under f. We will also denote by X_i its lift to any expansion of C_B. Note that for any rank-r sheaf E on an expansion $\widetilde{X}_b \to X_b$ over a point $b \in B$, the discriminant of $E|_{\widetilde{X}_b}$ is equal to the sum over the discriminants of the components of the normalization of $\widetilde{X}_b$ (since this is true for $\operatorname{ch}_2(E)$ and $c_1(E)^2$). □

Claim 1*:* For every $b \in B$, every α-balanced f-stable sheaf E on an expansion of X_b and every i, we have $\Delta(E|_{X_i}) \geqslant 0$.

Proof. Since the discriminant is invariant under tensoring with a line bundle, we may assume that $c_1(E|_{X_i}) = c_1|_{X_i} + kF$ for a fixed lift c_1 of $\overline{c_1}$ and some $k \in \{0, \ldots r-1\}$. By Theorem 2.24, we can find a polarization H_i on X_i such that f-stability for $X_i \to C_i$ agrees with slope stability with respect to H_i for all sheaves of rank r, first Chern class of the form $c_1|_{X_i} + kF$ and with discriminant less or equal to -1. By the Bogomolov–Gieseker inequality [HL10, Theorem 3.4.1], there are no sheaves satisfying this condition, and hence $\Delta(E|_{X_i}) \geqslant 0$. □

Claim 2: There exists a number N_1 such that for an α-balanced f-stable sheaf on a minimal expansion $\widetilde{X}_b \to \widetilde{C}_b$, the number of exceptional components is at most N_1.

Proof. By Lemma 2.11, on each exceptional component Y of $\widetilde{X}_b$, E_Y is either a pullback tensored by a line bundle from $\widetilde{C}$, or $\Delta(E_Y) > 0$. By α-balancedness, there can be at most $g(C_b)$ components for which the first possibility occurs (and the line bundle has to be of degree one on the corresponding component). From this and Claim 1, it follows that the total number of exceptional components is bounded by $N_1 = \Delta + g(C_b)$. □

From Claim 2, it follows that $\mathcal{M}^{\alpha}_{X_B/C_B}(r, \overline{c_1}, \Delta)$ factors through a quasi-compact open subset of $\mathrm{Exp}_{C_B/B}$. Hence, it suffices to show that the preimage of each stratum of $\mathrm{Exp}_{C_B/B}$ is quasi-compact. This reduces to showing that the space of sheaves on a given expansion $\widetilde{C}_B \to C_B$ is quasi-compact. By α-stability, for each component there is only a finite choice of possible values that the first Chern class of a restriction can take.

In this case, an f-stable sheaf on $\widetilde{X}_B$ is the same as giving suitable f-stable sheaves on each component, together with isomorphisms. This reduces to the case where $\widetilde{X}_B$ has a single component. In this case, the space of f-stable sheaves with given $\overline{c_1}$ and Δ is open in the space of stable sheaves of fixed topological type with respect to a suitably chosen polarization, and hence it is quasi-compact [HL10, Theorem 3.3.7].

DEFINITION 3.22. Let $X_B \to C_B \to B$ be a family of fibered surfaces. We say that a stability condition α on C_B is generic if every α-balanced sheaf on a minimal expansion of X_B is in fact strictly α-balanced.

Remark 3.23. Suppose that $C_B \to B$ has a single most degenerate stratum over a closed point b. Then one can always choose a non-degenerate stability condition by choosing an $\alpha : \mathrm{Irr}(C_b) \to \mathbb{R}$ whose values form a $\mathbb{Q}$-vector space of dimension $|\,\mathrm{Irr}(C_b)| - 1$.

Properness.

DEFINITION 3.24. Let α be a generic stability condition on C_B. We denote the $\mathbb{G}_m$-rigidification of $\mathcal{M}^{\alpha}_{X_B/C_B}(r, \overline{c_1}, \Delta)$ along the scalar automorphisms by

$$M^{\alpha}_{X_B/C_B}(r, \overline{c_1}, \Delta).$$

By Proposition 3.18, for a choice of generic stability condition, the stack $M^{\alpha}_{X_B/C_B}(r, \overline{c_1}, \Delta)$ has finite stabilizer groups at every point and therefore is Deligne–Mumford.

THEOREM 3.25. *Let α be a generic stability condition. Then $M^{\alpha}_{X_B/C_B}(r, \overline{c_1}, \Delta)$ is a proper Deligne–Mumford stack over B.*

Proof. We already know that it is a finite type Deligne–Mumford stack. It satisfies the existence part of the valuative criterion of properness, since, by Proposition 3.15, this is true for $\mathcal{M}^{\alpha}_{X_B/C_B}(r, d)$. It only remains to address the uniqueness part of the valuative criterion. For this, we may assume that $B = \mathrm{Spec}\, R$ and that we are given expansions $\widetilde{X}_1 \to \widetilde{C}_1$ and $\widetilde{X}_2 \to \widetilde{C}_2$ of $X_B \to C_B$ together with respective α-balanced f-stable sheaves E_1 and E_2 and an isomorphism Ψ of the restrictions to the generic fibers. Let π be a uniformizer for R. Then we need to show that $\widetilde{C}_1 \simeq \widetilde{C}_2$ and that for some ℓ, the isomorphism $\pi^\ell \Psi$ can be extended to an isomorphism of E_1 and E_2 over the isomorphism of expansions.

We first choose a common further expansion $\widetilde{C}_1 \xleftarrow{c^1} \widetilde{C}_3 \xrightarrow{c^2} \widetilde{C}_2$ which is an isomorphism over η and is minimal in the sense that no component of $\widetilde{C}_3$ over ξ is contracted by both c^1 and c^2. Then, both $(c^1)^*E_1$ and $(c^2)^*E_2$ are α-balanced f-stable sheaves on $\widetilde{X}_3$, and we have a given isomorphism $\psi : (c^1)^*E_1|_{\widetilde{X}_{3,\eta}} \to (c^2)^*E_2|_{\widetilde{X}_{3,\eta}}$. Let π be a uniformizer of R. There is a unique choice of integer ℓ such that $\pi^\ell \psi$ extends to a morphism $E_1 \to E_2$ whose restriction to $\widetilde{X}_{3,\xi}$ is non-zero. This extension is then unique, and we will denote it again by $\pi^\ell \Psi$. By Lemma 3.26 below, $\pi^\ell \Psi$ is an isomorphism. In particular, any component of $\widetilde{C}_3$ which is contracted by c^1 is also contracted by c^2, so we have $\widetilde{C}_1 \cong \widetilde{C}_3 \cong \widetilde{C}_2$. This is precisely what we wanted to show. □

Lemma 3.26. *Let $f : X_R \to C_R$ be a family of fibered surfaces over a DVR, let α be a stability condition on C_R, and let $E_1, E_2 \in \mathcal{M}^\alpha_{X_R/C_R}(r, \overline{c_1}, \Delta)$ be strictly α-balanced f-stable sheaves on some expansion $\widetilde{X}_R \to \widetilde{C}_R$ of f (which is minimal for both). Let $\Psi : E_1 \to E_2$ be a morphism that is an isomorphism over the generic point of R and non-zero over the closed point. Then Ψ is an isomorphism.*

Proof. We consider E_1, E_2 as families of rank-r and degree-d sheaves on the family of curves given by f. Since E_1 has finite cohomological dimension and E_2 is flat over $\widetilde{C}_R$, it follows from [Sta23, Lemma 08IF] that the sheaf $N_0 := f_*\mathcal{H}om(E_1, E_2)$ is a reflexive rank-one sheaf on $\widetilde{C}_R$ (i.e. of the form $\mathcal{H}om(L, \mathcal{O}_{\widetilde{C}_R})$ for L coherent of rank one). As a reflexive sheaf, it is locally free away from the singular points of the two-dimensional scheme $\widetilde{C}_R$. On the other hand, let $x \in \widetilde{C}_R$ be a point where $f^{-1}(x)$ is smooth and both $E_1|_{f^{-1}(x)}$ and $E_2|_{f^{-}(x)}$ are stable. Then, since stability is an open condition, in a Zariski neighborhood U of x, E_1 and E_2 define two families of stable sheaves on U, which by assumption are generically isomorphic. It follows that N is locally free of rank one on U. By f-stability, this holds for (i) x any generic point of $\widetilde{C}_\eta$, so N has rank one, and (ii) x a node in either $\widetilde{C}_\xi$, so N is also locally free along the singular locus of $\widetilde{C}_R \to R$. We conclude that N is a line bundle on $\widetilde{C}_R$. The given morphism Ψ specifies a section of N. Since Ψ is an isomorphism on $\widetilde{C}_\eta$, this section is a generator of N on the generic fiber, and hence $N|_{\widetilde{C}_\eta} \simeq \mathcal{O}_{\widetilde{C}_\eta}$.

Moreover, we get a tautological morphism $E_1 \otimes f^*N \to E_2$, which is non-zero on each component of the special fiber. Therefore, its restriction to $\widetilde{X}_\xi$ is injective with cokernel Q supported on fibers of $\widetilde{X}_\xi \to \widetilde{C}_\xi$ away from the nodes. Since E_1 and E_2 have the same numerical invariants and N has total degree zero on $\widetilde{C}_\xi$, we find that $Q = 0$, so $E_1 \otimes f^*N \simeq E_2$. This implies that both E_1 and $E_1 \otimes f^*N$ are strictly α-balanced. Let $Z \subset \widetilde{C}_\xi$ be a connected component of the subcurve of components not contained in the Cartier divisor defined by Ψ. If $Z = \widetilde{C}_\xi$, then Ψ is an isomorphism, and we are done. So, suppose for the sake of contradiction that Z is a proper subcurve of $\widetilde{C}_\xi$. Since every component of its complement Z^c that meets Z has at least multiplicity one, we have $\deg N|_Z \geqslant \#(Z \cap Z^c) > 0$. Since $\deg M(E_1 \otimes f^*N)|_Z = \deg M(E_1)|_Z + rd_0 \deg N|_Z$, it follows that

$$\left| \frac{\deg M(E_1 \otimes f^*N)|_Z}{rd_0} - \sum_{D \subset Z} \alpha(D) \right| \geqslant \#(Z \cap Z^c) - \left| \frac{\deg M(E_1)|_Z}{rd_0} - \sum_{D \subset Z} \alpha(D) \right| \geqslant \frac{\#(Z \cap Z^c)}{2},$$

with equality if and only if

$$\frac{\deg M(E_1)|_Z}{rd_0} - \sum_{D\subset Z} \alpha(D) = -\frac{\#(Z\cap Z^c)}{2}.$$

But by α-balancedness of $E_1 \otimes f^*N$, equality must hold, so by strict α-balancedness, either Z or Z^c is a chain of exceptional components. It cannot be Z, since then $\deg M(E_1)|_Z$ is strictly negative, so it must be Z^c. Let $Z' \subset Z^c$ be a subchain of components along which the multiplicity of the Cartier divisor defined by Ψ, N is maximal. Then $\deg N|_{Z'} \leqslant -2$ (since Z' meets its complement in two points, and since the line bundle defined by the whole of $\widetilde{C}_\xi$ has vanishing degree on Z'). But, by α-balancedness,

$$0 \leqslant \frac{\deg M(E_1\otimes f^*N)|_{Z'}}{rd_0} \leqslant \frac{\deg M(E_1)|_{Z'}}{rd_0} - 2 \leqslant 1-2 = -1,$$

which is the desired contradiction. □

Remark 3.27. Note that by (3) and Grothendieck–Riemann–Roch, the total degree of $M(E)$ depends on E only through $c_1(E)$ and $\Delta(E)$. In particular, a formal application of Grothendieck–Riemann–Roch gives a unique number $\alpha(\overline{c_1}, \Delta)$ such that $M^\alpha_{X/C}(r, \overline{c_1}, \Delta) = \emptyset$ unless

$$\sum_{D\in \operatorname{Irr}(C)} \alpha(D) = \alpha(\overline{c_1}, \Delta). \tag{5}$$

When C is irreducible, a stability condition is just a scalar which determines whether the moduli space is (possibly) non-empty. In this case, we will abbreviate

$$M^b_{X/C}(r, \overline{c_1}, \Delta) := M^{\alpha(\overline{c_1},\Delta)}_{X/C}(r, \overline{c_1}, \Delta).$$

3.4 Perfect obstruction theories

We construct the perfect obstruction theory on the moduli stacks $\mathcal{M}_{X_B/C_B}(r,d)$ and their variants. The arguments here are relatively standard and we will not give all details. A *perfect obstruction theory* for a morphism $\mathcal{X}\to\mathcal{Y}$ is an object $E\in D(\mathcal{X})$ that is perfect with amplitude in $[-1,1]$ together with a morphism $E\to L_{\mathcal{X}/\mathcal{Y}}$ that is an isomorphism on h^1 and h^0 and surjective on h^{-1}. This coincides with the usual notion whenever $\mathcal{X}\to\mathcal{Y}$ is of DM-type.

Let $X_B\to C_B$ be a family of fibered surfaces. We abbreviate $\mathcal{M} := \mathcal{M}_{X_B/C_B}(r,d)$ and $\operatorname{Exp} := \operatorname{Exp}_{C_B/B}$. Consider the forgetful morphisms $\mathcal{M}\to\operatorname{Exp}\to B$. We let $\widetilde{X}\to\widetilde{C}$ denote the universal expansion on Exp and let $\mathcal{E}$ denote the universal sheaf on the pullback $\widetilde{X}_\mathcal{M}$ of $\widetilde{X}$ to $\mathcal{M}$. Let $\pi: \widetilde{X}_\mathcal{M}\to\mathcal{M}$ denote the projection. The Atiyah class defines a relative obstruction theory $(R\pi_* R\mathcal{H}om_0(\mathcal{E},\mathcal{E}))^\vee[-1]\to L_{\mathcal{M}/\operatorname{Exp}}$.

We have a factorization $\mathcal{M}\to\mathcal{P}ic_{\widetilde{X}/\operatorname{Exp}}\to\operatorname{Exp}$ of the forgetful map through the determinant morphism to the Picard stack. Let $\mathcal{L}$ denote the universal line bundle over $\widetilde{X}_{\mathcal{P}ic_{\widetilde{X}/\operatorname{Exp}}}$ and let π also denote the projection to $\mathcal{P}ic_{\widetilde{X}/\operatorname{Exp}}$. We have the relative obstruction theory

$$(R\pi_* R\mathcal{H}om(\mathcal{L},\mathcal{L}))^\vee[-1]\to L_{\mathcal{P}ic_{\widetilde{X}/\operatorname{Exp}}/\operatorname{Exp}}.$$

It is naturally compatible with the obstruction theory of $\mathcal{M}$ via the trace map.

Moreover, the trace-free part gives a canonical relative obstruction theory

$$(R\pi_* R\mathcal{H}om_0(\mathcal{E},\mathcal{E}))^\vee[-1]\to L_{\mathcal{M}/\mathcal{P}ic_{\widetilde{X}/\operatorname{Exp}}}.$$

We have a commutative diagram involving the $\mathbb{G}_m$-rigidifications of both stacks.

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\ r\ } & M := M_{X_B/C_B}(r,d) \\ \big\downarrow & & \big\downarrow \\ \mathcal{P}ic_{\widetilde{X}/\mathrm{Exp}} & \longrightarrow & Pic_{\widetilde{X}/Exp} \end{array}$$

The induced map $r^* L_{M/Pic_{\widetilde{X}/\mathrm{Exp}}} \to L_{\mathcal{M}/\mathcal{P}ic_{\widetilde{X}/\mathrm{Exp}}}$ is an isomorphism, so we may compose with its inverse to get a morphism $(R\pi_* R\mathcal{H}om_0(\mathcal{E},\mathcal{E}))^\vee[-1] \to r^* L_{M/Pic_{\widetilde{X}/Exp}}$. This last map descends to a canonical perfect obstruction theory for the determinant morphism $M \to \mathrm{Pic}_{\widetilde{X}/\mathrm{Exp}}$ over the relative Picard scheme. From this discussion, and the properties of virtual pullback [Man12, §§ 3 and 4], we immediately get the following proposition.

Proposition 3.28. *Suppose we are in the situation of Theorem 3.25. Then the stack* $M^\alpha_{X_B/C_B}(r, \overline{c_1}, \Delta)$ *has a relative perfect obstruction theory over* $\mathrm{Pic}_{\widetilde{X}/\mathrm{Exp}}$*. In particular, it has a natural virtual fundamental class*

$$\left[M^\alpha_{X_B/C_B}(r, \overline{c_1}, \Delta)\right]^{\mathrm{vir}} \in A_*(M^\alpha_{X_B/C_B}(r, \overline{c_1}, \Delta))$$

given by virtual pullback of the fundamental class of $\mathrm{Pic}_{\widetilde{X}/\mathrm{Exp}}$*. The formation of the virtual fundamental class is compatible with flat and lci pullbacks on* B*.*

It follows formally from this proposition that virtual integrals with respect to cohomology classes that are defined over the whole family B are deformation invariant. We will usually consider intersection numbers in homology rather than Chow groups, as explained in § 3.6 below.

3.5 Evaluation maps

Let $X_B \to C_B$ be a family of fibered surfaces and let $\sigma_1, \dots, \sigma_n : B \to C_B$ denote the markings of C_B. Let $F_i \to B$ denote the family of smooth curves obtained as the preimage of σ_i under f. For each i, we have a morphism of stacks $\mathcal{M}_{X_B/C_B}(r,d) \to \mathcal{M}_{F_i}(r,d)$. It fits into a commutative diagram

$$\begin{array}{ccc} \mathcal{M}_{X_B/C_B}(r,d) & \longrightarrow & \mathcal{M}_F(r,d) \\ \big\downarrow & & \big\downarrow \\ \mathcal{P}ic_{\widetilde{X}/B} & \longrightarrow & \mathcal{P}ic_{F_i/B} \end{array}$$

in which the horizontal maps are the restriction maps and the vertical maps are the determinant morphisms. For each map in this square, the obstruction theories for source and target are naturally compatible. We have an induced square of rigidifications, with induced relative obstruction theories, as shown by the following diagram.

$$\begin{array}{ccc} M_{X_B/C_B}(r,d) & \longrightarrow & M_F(r,d) \\ \big\downarrow & & \big\downarrow \\ \mathrm{Pic}_{\widetilde{X}/B} & \longrightarrow & \mathrm{Pic}_{F_i/B} \end{array}$$

3.6 Tautological classes

We want to study invariants which are defined by pairing certain tautological cohomology classes against the virtual fundamental class. We define here what we mean by tautological cohomology class. For a Deligne–Mumford stack $\mathcal{Y}$ over $\mathbb{C}$, we define its rational (co-)homology groups $H_*(\mathcal{Y},\mathbb{Q})$ (resp., $H^*(\mathcal{Y},\mathbb{Q})$) in terms of the simplicial scheme $Y_\bullet$ associated to an étale cover $Y_0 \to \mathcal{Y}$. When working with rational coefficients, as we do here, these are naturally isomorphic to the (co-) homology groups of the coarse moduli space of Y. This is nicely explained in the second half of [Beh04]. One also has a natural cycle class map $A_*(\mathcal{Y}) \to H_*^{BM}(\mathcal{Y},\mathbb{Q})$ into the Borel–Moore homology (cf. [AGV08, § 2]). When $\mathcal{Y}$ is proper, this equivalently gives a map $A_*(\mathcal{Y}) \to H_*(\mathcal{Y},\mathbb{Q})$.

Let $f: X \to C$ be a fibered surface over a fixed nodal marked curve C with markings $(x_1,\ldots,x_n)$, and let $F_1,\ldots,F_n$ denote the fibers over the markings. Let L_0 be a fixed line bundle of degree $d_0 > 0$ on X, and let $\overline{c_1} \in \overline{\mathcal{NS}}_X$ be a class of fiber degree d. Let also $\Delta \in \mathbb{Z}$. Let α be a generic stability condition on C. We consider the proper Deligne–Mumford stack $M := M^\alpha_{X/C}(r,\overline{c_1},\Delta)$. Let $\pi: \widetilde{X} \to M$ denote the universal expansion over M and $c: \widetilde{X} \to X$ the contraction map.

Lemma 3.29. *There is a natural map $\pi_!: H^*(\widetilde{X},\mathbb{Q}) \to H^{*-4}(M,\mathbb{Q})$.*

Proof. Since the morphism $\pi: \widetilde{X} \to M$ is flat, proper and representable of dimension two, any étale cover $M_0 \to M$ induces an étale cover $\widetilde{X}_0 \to \widetilde{X}$ by pullback, and we get an induced morphism of simplicial algebraic spaces $\pi_\bullet: \widetilde{X}_\bullet \to M_\bullet$, which is component-wise flat and proper of relative dimension two. We have a trace map $(R\pi_\bullet)_*\mathbb{Q} \to \mathbb{Q}[-4]$ (see [Ver76, 4.6] for the case of schemes, which carries over to our setting). In fact, fiberwise, $R^4\pi_*\mathbb{Q}$ is a $\mathbb{Q}$ vector space spanned by the orientation classes of irreducible components in the fiber, and the map $R^4\pi_*\mathbb{Q} \to \mathbb{Q}$ sends each generator to 1. This induces the desired morphism after passing to cohomology groups. □

Let $\gamma \in H^*(X,\mathbb{Q})$ be a cohomology class. Recall that M is the rigidification of the moduli stack $\mathcal{M}^\alpha_{X/C}(r,\overline{c_1},\Delta)$, and similarly, $\widetilde{X}$ is the rigidification of a family $\widetilde{\mathcal{X}}$. We denote by $\mathcal{E}$ the universal sheaf on $\widetilde{\mathcal{X}}$. While $\mathcal{E}$ does not descend to $\widetilde{X}$, the expression $\mathcal{E} \otimes (\det \mathcal{E})^{-(1/r)}$ makes sense as a rational K-theory class and does descend to $\widetilde{X}$. By abuse of notation, we denote it by $\widehat{\mathcal{E}}$. We make the following definition for $i \geqslant 0$:

$$\mathrm{ch}_i(\gamma) := \pi_! \left(\mathrm{ch}_i(\widehat{\mathcal{E}}) \cup c^*\gamma \right). \tag{6}$$

By abuse of notation, we denote by $T_M^{\mathrm{vir}} = -[R\mathrm{Hom}_0(\mathcal{E},\mathcal{E})]$ the K-theory class dual to the relative perfect obstruction theory of M over the relative Picard scheme. Here is an (incomplete) definition of tautological classes.

Definition 3.30. We say that a cohomology class in $H^*(M,\mathbb{Q})$ is *tautological* if it lies in the sub-ring generated by classes $\mathrm{ch}_i(\gamma)$ and classes $\mathrm{ch}_i(T_M^{\mathrm{vir}})$.

More generally, one can also consider classes defined in terms of K-theoretic objects, such as virtual Segre or Verlinde invariants (see, for example, [GK19b] for an overview).

4. The degeneration formula

In this section, we state and prove a special case of a degeneration formula for a one-parameter family of fibered surfaces where the general fiber has an unmarked smooth base curve, degenerating to a curve with one node and to irreducible components over the special fiber. To begin, we focus our attention on the special fiber of such a degeneration.

Let $f: X \to C$ be a fibered surface and suppose $C = D_1 \cup D_2$, where (D_1, x_1) and (D_2, x_2) are smooth curves with a single marking and the union is taken along the marked points. Let $Y_i := f^{-1}D_i$ and $F_i := f^{-1}(x_i)$ for $i = 1, 2$.

We fix some L_0 with fiber degree $d_0 > 0$ on X and a stability condition α on C. We assume that $\alpha(D_i) \notin \frac{1}{rd_0}\mathbb{Q}$ and, in particular, that the stability condition α is generic. We let $\alpha_i := \alpha(D_i)$. We further fix a section[1] $\overline{c_1}$ of $\overline{\mathcal{NS}}_X$ with degree d on fibers of f. We will use the following abuse of notation: if c_1' and c_1'' are points in $\mathcal{NS}_{Y_1}$ and $\mathcal{NS}_{Y_2}$ respectively, we write $c_1' + c_1'' = \overline{c_1}$ if there exists a lift c_1 of $\overline{c_1}$ to $\mathcal{NS}_X$ which restricts to c_1' and c_1'' on Y_1 and Y_2 respectively.

Finally, we write

$$\mathcal{M}^{\lfloor \alpha_1 \rceil}_{Y_1/D_1}(r, c_1', \Delta_1) := \coprod_{\substack{\beta \in \frac{1}{rd_0}\mathbb{Q} \\ |\beta - \alpha_i| < \frac{1}{2}}} \mathcal{M}^{\beta}_{Y_1/D_1}(r, c_1', \Delta_1),$$

and similarly for $\mathcal{M}^{\lfloor \alpha_2 \rceil}_{Y_2/D_2}(r, c_1'', \Delta_2)$ and the $\mathbb{G}_m$-rigidified versions of the stacks. Note that at most one term in the disjoint union is non-empty.

4.1 Glueing of sheaves

Let $c_1' \in \mathcal{NS}_{Y_1}$, $c_1'' \in \mathcal{NS}_{Y_2}$ such that $c_1' + c_1'' = \overline{c_1}$. Suppose that α satisfies (5). Let $\Delta_1, \Delta_2 \in \mathbb{Z}$ and $\Delta := \Delta_1 + \Delta_2$. There is an associated glueing morphism

$$\gamma: \mathcal{M}^{\lfloor \alpha_1 \rceil}_{Y_1/D_1}(r, c_1', \Delta_1) \times_{\mathcal{M}_F(r,d)} \mathcal{M}^{\lfloor \alpha_2 \rceil}_{Y_2/D_2}(r, c_1'', \Delta_2) \to \mathcal{M}^{\alpha}_{X/C}(r, \overline{c_1}, \Delta_1 + \Delta_2).$$

This induces a canonical morphism on $\mathbb{G}_m$-rigidifications

$$\Gamma: M^{\lfloor \alpha_1 \rceil}_{Y_1/D_1}(r, c_1', \Delta_1) \times_{M_F(r,d)} M^{\lfloor \alpha_2 \rceil}_{Y_2/D_2}(r, c_1'', \Delta_2) \to M^{\alpha}_{X/C}(r, \overline{c_1}, \Delta).$$

We similarly have a glueing morphism for Picard schemes

$$\mathrm{Pic}^{c_1'}_{\widetilde{Y}_1/\mathrm{Exp}_{D_1,x_1}} \times_{\mathrm{Pic}^d_F} \mathrm{Pic}^{c_1''}_{\widetilde{Y}_2/\mathrm{Exp}_{D_2,x_2}} \to \mathrm{Pic}^{\overline{c_1}}_{\widetilde{X}/\mathrm{Exp}_C}.$$

Here we use a superscript to denote the component of the Picard schemes mapping into the respective component of the Néron–Severi schemes. The glueing morphisms are compatible with taking the determinant, i.e. we have the folllowing commutative diagram.

$$\begin{array}{ccc} M^{\lfloor \alpha_1 \rceil}_{Y_1/D_1}(r, c_1', \Delta_1) \times_{M_F(r,d)} M^{\lfloor \alpha_2 \rceil}_{Y_2/D_2}(r, c_1'', \Delta_2) & \xrightarrow{\Gamma} & M^{\alpha}_{X/C}(r, \overline{c_1}, \Delta) \\ \downarrow & & \downarrow \\ \mathrm{Pic}^{c_1'}_{\widetilde{Y}_1/\,\mathrm{Exp}_{D_1}} \times_{\mathrm{Pic}^d_F} \mathrm{Pic}^{c_1''}_{\widetilde{Y}_2/\,\mathrm{Exp}_{D_2}} & \longrightarrow & \mathrm{Pic}^{\overline{c_1}}_{\widetilde{X}/\,\mathrm{Exp}_C} \end{array} \tag{7}$$

By taking the union over possible decompositions of the discriminant, we can obtain more.

Lemma 4.1. *The following natural diagram is Cartesian.*

---

[1]For example, by fixing a line bundle.

$$\begin{array}{ccc} \coprod_{\Delta_1+\Delta_2=\Delta} M^{\lfloor\alpha_1\rceil}_{Y_1/D_1}(r,c_1',\Delta_1)\times_{M_F(r,d)} M^{\lfloor\alpha_2\rceil}_{Y_2/D_2}(r,c_1'',\Delta_2) & \xrightarrow{\Gamma} & M^{\alpha}_{X/C}(r,\overline{c_1},\Delta) \\ \downarrow & & \downarrow \\ \mathrm{Pic}^{c_1'}_{\widetilde{Y}_1/\operatorname{Exp}_{D_1}}\times_{\mathrm{Pic}^d_F}\mathrm{Pic}^{c_1''}_{\widetilde{Y}_2/\operatorname{Exp}_{D_2}} & \xrightarrow{\Gamma'} & \mathrm{Pic}^{\overline{c_1}}_{\widetilde{X}/\operatorname{Exp}_C} \end{array} \tag{8}$$

Proof. This can be checked before passing to rigidifications. In that case, the fiber product corresponding to the lower right corner of the diagram has T-points given by an element of $\mathcal{M}^{\alpha}_{X/C}(r,\overline{c_1},\Delta)(T)$, i.e. a sheaf E_T on an expansion $\widetilde{X}_T$, together with a choice of decomposition $\widetilde{X}_T=\widetilde{Y}_{T,1}\cup\widetilde{Y}_{T,2}$ of the given expansion, such that we have $[\det E|_{\widetilde{Y}_{T,1}}]=c_1'$ and $[\det E|_{\widetilde{Y}_{T,2}}]=c_1''$. The stability condition on C implies that the restrictions will lie in the prescribed range of stabilities on the D_i. □

It follows from Lemma 4.1 that we have a natural relative obstruction theory on each product $M^{\lfloor\alpha_1\rceil}_{Y_1/D_1}(r,c_1',\Delta_1)\times_{M_F(r,d)} M^{\lfloor\alpha_2\rceil}_{Y_2/D_2}(r,c_1'',\Delta_2)$ over the product of relative Picard schemes $\mathrm{Pic}^{c_1'}_{\widetilde{Y}_1/\mathrm{Exp}_{D_1}}\times_{\mathrm{Pic}_F}\mathrm{Pic}^{c_1''}_{\widetilde{Y}_2/\mathrm{Exp}_{D_2}}$, given by pulling back the obstruction theory via Γ. In particular, we have a canonical virtual fundamental class on each product, obtained by pulling back the fundamental class of the base.

Proposition 4.2. *We have an equality in $A_*(M^{\alpha}_{X/C}(r,\overline{c_1},\Delta))$:*

$$[M^{\alpha}_{X/C}(r,\overline{c_1},\Delta)]^{\mathrm{vir}}=\sum_{\substack{c_1'+c_1''=\overline{c_1}\\ \Delta_1+\Delta_2=\Delta}}\Gamma_*\Big[M^{\lfloor\alpha_1\rceil}_{Y_1/D_1}(r,c_1',\Delta_1)\times_{M_F(r,d)} M^{\lfloor\alpha_2\rceil}_{Y_2/D_2}(r,c_1'',\Delta_2)\Big]^{\mathrm{vir}}.$$

Proof. By [Man12, Theorem 4.1] and Lemma 4.1, it is enough to show the analogous formula on the level of Picard schemes, i.e. that

$$\Big[\mathrm{Pic}_{\widetilde{X}/\mathrm{Exp}_C}\Big]=\sum_{c_1'+c_1''=\overline{c_1}}\Gamma'_*\Big[\mathrm{Pic}^{c_1'}_{\widetilde{Y}_1/\mathrm{Exp}_{D_1}}\times_{\mathrm{Pic}^d_F}\mathrm{Pic}^{c_1''}_{\widetilde{Y}_2/\mathrm{Exp}_{D_2}}\Big]$$

and that the glueing morphisms of relative Picard schemes are projective. This is shown in Lemma 4.3 below. □

Lemma 4.3.

(i) *The glueing map*

$$\Gamma':\mathrm{Pic}^{c_1'}_{\widetilde{Y}_1/\mathrm{Exp}_{D_1}}\times_{\mathrm{Pic}_F}\mathrm{Pic}^{c_1''}_{\widetilde{Y}_2/\mathrm{Exp}_{D_2}}\to\mathrm{Pic}^{\overline{c_1}}_{\widetilde{X}/\mathrm{Exp}_C}$$

is a proper and quasi-finite morphism, in particular it is projective.

(ii) *We have an equality of fundamental classes*[2]

$$\Big[\mathrm{Pic}^{\overline{c_1}}_{\widetilde{X}/\mathrm{Exp}_C}\Big]=\sum_{c_1'+c_2''=\overline{c_1}}\Gamma'_*\Big(\Big[\mathrm{Pic}^{c_1'}_{\widetilde{Y}_1/\mathrm{Exp}_{D_1}}\times_{\mathrm{Pic}_F}\mathrm{Pic}^{c_1''}_{\widetilde{Y}_2/\mathrm{Exp}_{D_2}}\Big]\Big).$$

Proof. Note that $\mathrm{Pic}^{c_1'}_{\widetilde{Y}_1/\mathrm{Exp}_{D_1}}\times_{\mathrm{Pic}_F}\mathrm{Pic}^{c_1''}_{\widetilde{Y}_2/\mathrm{Exp}_{D_2}}$ is smooth over $\mathrm{Exp}_{D_1}\times\mathrm{Exp}_{D_2}$. Indeed, smooth locally over the base, it is a union of translates of the identity component of $\mathrm{Pic}_{\widetilde{Y}_1\cup_F\widetilde{Y}_2/\mathrm{Exp}_{D_1}\times\mathrm{Exp}_{D_2}}$, which is known to be smooth by Theorem 2.16.

[2] As per our conventions, this is to be understood as an equality on any quasi-compact open substack of $\mathrm{Pic}^{\overline{c_1}}_{\widetilde{X}/\mathrm{Exp}_C}$.

Regarding i): One shows that Γ' is quasi-compact by an argument similar to the one used in the proof of Lemma 2.21. Then, it is straightforward to check the valuative criteria for properness. Quasi-finiteness follows, since for a given point L_1 of $\mathrm{Pic}^{\overline{c_1}}_{\widetilde{X}}$ defined on an expansion $\widetilde{X}_1 \to \widetilde{C}_1$, points in the preimage under Γ correspond to a choice of singular point in $\widetilde{C}_1$. Now ii) follows, since each glueing map is between schemes of the same dimension. Thus, to compute the image of the fundamental cycle $[\mathrm{Pic}^{c_1'}_{\widetilde{Y}_1/\mathrm{Exp}_{D_1}} \times_{\mathrm{Pic}_F} \mathrm{Pic}^{c_1''}_{\widetilde{Y}_2/\mathrm{Exp}_{D_2}}]$, it is enough to do so over a dense open on target and source. Hence, we may restrict to the generic points of Exp_{D_i} and Exp_C corresponding to trivial expansions. Then the result follows, since we have an isomorphism

$$\mathrm{Pic}^{\overline{c_1}}_{X} = \coprod_{c_1'+c_1''=\overline{c_1}} \mathrm{Pic}^{c_1'}_{Y_1} \times_{\mathrm{Pic}_F} \mathrm{Pic}^{c_1''}_{Y_2}.$$

□

4.2 Obstruction theories

Let $\Delta_1, \Delta_2 \in \mathbb{Z}$ with $\Delta = \Delta_1 + \Delta_2$. For convenience, we abbreviate

$$\begin{aligned} M_{Y_1/D_1} &:= M^{\lfloor \alpha_1 \rceil}_{Y_1/D_1}(r, c_1', \Delta_1), \\ \mathrm{Pic}_{\widetilde{Y}_1} &:= \mathrm{Pic}^{c_1'}_{\widetilde{Y}_1/\mathrm{Exp}_{D_1}}, \\ M_F &:= M_F(r, d), \end{aligned}$$

and similarly for $M_{Y_2/D_2}, \mathrm{Pic}_{\widetilde{Y}_2}, M^{\alpha}_{X/C}$ and $\mathrm{Pic}_{\widetilde{X}}$. In this subsection, we further analyze the virtual class on $M_{Y_1/D_1} \times_{M_F} M_{Y_2/D_2}$ that was constructed in § 4.1. Ultimately, we would like to write integrals of cohomology classes over this virtual class in an explicit way in terms of such integrals on M_{Y_1/D_1} and M_{Y_2/D_2} with their respective virtual classes. As a first step, the following result lets us reduce computations on $M_{Y_1/D_1} \times_{M_F} M_{Y_2/D_2}$ to the simpler $M_{Y_1/D_1} \times M_{Y_2/D_2}$ with its product virtual class.

Proposition 4.4.

(1) *For $i = 1, 2$, there is a relative perfect obstruction theory for the morphism $M_{Y_i/D_i} \to M_F$, which induces a canonical virtual pullback map.*

(2) *The following cycle classes on $M_{Y_1/D_1} \times_{M_F} M_{Y_2/D_2}$ agree:*
  - (i) *the virtual pullback of the fundamental class of $\mathrm{Pic}_{\widetilde{Y}_1} \times_{\mathrm{Pic}^d_F} \mathrm{Pic}_{\widetilde{Y}_2}$,*
  - (ii) *the Gysin-pullback of the product of virtual classes on $M_{Y_1/D_1} \times M_{Y_2/D_2}$ along the diagonal map $M_F \to M_F \times M_F$,*
  - (iii) *the virtual pullback of the fundamental class of M_F induced by the morphism $M_{Y_1/D_1} \to M_F$,*
  - (iv) *the virtual pullback of the fundamental class of M_F induced by the morphism $M_{Y_2/D_2} \to M_F$.*

Proof. We have a natural commutative diagram

$$\begin{array}{ccccc} M_{Y_1/D_1} & \xrightarrow{\varphi} & \mathrm{Pic}_{\widetilde{Y}_1} \times_{\mathrm{Pic}^d_F} M_F & \longrightarrow & M_F \\ & \searrow & \downarrow & & \downarrow \\ & & \mathrm{Pic}_{\widetilde{Y}_1} & \longrightarrow & \mathrm{Pic}^d_F \end{array}$$

in which the square is cartesian. The vertical maps are smooth, and the horizontal maps in the square are (smooth locally) local complete intersection morphisms (lci). Moreover, the vertical maps have natural obstruction theories given by the trace-free part of the Atiyah class of a universal sheaf over M_F, and these are naturally compatible with the obstruction theory of M_{Y_1/D_1} over $\mathrm{Pic}_{\widetilde{Y}_1}$ (also given by the trace-free part of the Atiyah class of the respective universal sheaf). It follows that we have a (non-canonical) relative perfect obstruction theory for the morphism $\varphi\colon M_{Y_1/D_1} \to \mathrm{Pic}_1 \times_{\mathrm{Pic}^d_F} M_F$. Since the horizontal maps in the square are lci, we may endow them with their canonical obstruction theories, which are then automatically compatible with the obstruction theory for φ. There is then an induced obstruction theory for the restriction map $M_{Y_1/D_1} \to M_F$, which has the property that the induced virtual pullback map factors through l.c.i pullback along $\mathrm{Pic}_{\widetilde{Y}_1} \to \mathrm{Pic}^d_F$ followed by the virtual pullback along φ. Since $\mathrm{Pic}_{\widetilde{Y}_1}$ is not Deligne–Mumford, the functoriality of obstruction theories here is not a consequence of the theory in [Man12]. Instead, we rely on Theorem B.5 of [AKL+22]. This proves the first point for $i=1$, and by symmetry for $i=2$.

Now, consider the following commutative diagram with Cartesian squares.

$$\begin{array}{ccccc}
M_{Y_1/D_1} \times_{M_F} M_{Y_2/D_2} & \longrightarrow & M_{Y_1/D_1} \times_{\mathrm{Pic}^d_F} M_{Y_2/D_2} & \longrightarrow & M_{Y_1/D_1} \times M_{Y_2/D_2} \\
& \searrow & \downarrow & & \downarrow \\
& & \mathrm{Pic}_{\widetilde{Y}_1} \times_{\mathrm{Pic}^d_F} \mathrm{Pic}_{\widetilde{Y}_2} & \longrightarrow & \mathrm{Pic}_{\widetilde{Y}_1} \times \mathrm{Pic}_{\widetilde{Y}_2} \\
& & \downarrow & & \downarrow \\
& & \mathrm{Pic}^d_F & \xrightarrow{\ \Delta\ } & \mathrm{Pic}^d_F \times \mathrm{Pic}^d_F
\end{array}$$

The vertical maps in the upper square have obstruction theories given by the sum of Atiyah classes. The obstruction theory of the left vertical map is compatible with the one of the diagonal map given by the Atiyah class. We get an induced obstruction theory on the map $M_{Y_1/D_1} \times_{M_F} M_{Y_2/D_2} \to M_{Y_1/D_1} \times_{\mathrm{Pic}^d_F} M_{Y_2/D_2}$ which is isomorphic to $\mathrm{Ext}^1_0(\mathcal{E}_D, \mathcal{E}_D)^\vee$ concentrated in degree -1. On the other hand, we have the following Cartesian diagram.

$$\begin{array}{ccc}
M_{Y_1/D_1} \times_{M_F} M_{Y_2/D_2} & \longrightarrow & M_{Y_1/D_1} \times_{\mathrm{Pic}^d_F} M_{Y_2/D_2} \\
\downarrow & & \downarrow \\
M_F & \xrightarrow{\ \Delta\ } & M_F \times_{\mathrm{Pic}^d_F} M_F
\end{array}$$

Since virtual pullback is independent of the precise choice of map in the obstruction theory, this shows that the virtual pullback map for the morphism $M_{Y_1/D_1} \times_{M_F} M_{Y_2/D_2} \to M_{Y_1/D_1} \times_{\mathrm{Pic}^d_F} M_{Y_2/D_2}$ is equal to the Gysin-pullback along the diagonal of M_F. Then, considering the diagram with Cartesian squares

$$\begin{array}{ccccc}
M_{Y_1/D_1} \times_{M_F} M_{Y_2/D_2} & \longrightarrow & M_{Y_1/D_1} \times_{\mathrm{Pic}^d_F} M_{Y_2/D_2} & \longrightarrow & M_{Y_1/D_1} \times M_{Y_2/D_2} \\
\downarrow & & \downarrow & & \downarrow \\
M_F & \xrightarrow{\ \Delta\ } & M_F \times_{\mathrm{Pic}^d_F} M_F & \longrightarrow & M_F \times M_F \\
& & \downarrow & & \downarrow \\
& & \mathrm{Pic}^d_F & \xrightarrow{\ \Delta\ } & \mathrm{Pic}^d_F \times \mathrm{Pic}^d_F
\end{array}$$

shows that the virtual class on $M_{Y_1/D_1} \times_{M_F} M_{Y_2/D_2}$ is the Gysin-pullback of the one on $M_{Y_1/D_1} \times M_{Y_2/D_2}$ along the diagonal morphism of $M_F \times M_F$. The last two equivalences follow, since virtual pullback commutes with lci pullback, and $M_{Y_1/D_1} \times_{M_F} M_{Y_2/D_2}$ is identified with the base change of $M_{Y_1/D_1} \times_{(M_F \times M_F)} M_{Y_2/D_2}$ along the diagonal $M_F \to M_F \times M_F$. □

4.3 Decomposition formulas

We show some basic results regarding how tautological classes interact with the glueing morphism. Let γ be a cohomology class on X, let γ_i be its restriction to Y_i for $i = 1, 2$, and let γ_F be its restriction to the singular fiber F. Consider the glueing map

$$\Gamma : M_{Y_1/D_1} \times_{M_F(r,d)} M_{Y_2/D_2} \to M_{X/C}.$$

Lemma 4.5. *We have* $\Gamma^*\mathrm{ch}_i(\gamma) = \mathrm{pr}_1^* \mathrm{ch}_i(\gamma_1) + \mathrm{pr}_2^* \mathrm{ch}_i(\gamma_2)$.

Proof. Let $\widetilde{X} \to M_{X/C}$ denote the universal expansion and $\Gamma^*\widetilde{X}$ its pullback to $M_{Y_1/D_1} \times_{M_F(r,d)} M_{Y_2/D_2}$ so that $\Gamma^*\widetilde{X} = \mathrm{pr}_1^* \widetilde{Y}_1 \cup_F \mathrm{pr}_2^* \widetilde{Y}_2$. Recall that $\mathrm{ch}_i(\gamma)$ is defined via (6) in terms of a Gysin map $\pi_!$, which commutes with the pullback along Γ. Then consider the following diagram of maps.

$$\begin{array}{ccc} \mathrm{pr}_1^* \widetilde{Y}_1 \coprod \mathrm{pr}_2^* \widetilde{Y}_2 & \xrightarrow{\sigma} & \mathrm{pr}_1^* \widetilde{Y}_1 \cup_F \mathrm{pr}_2^* \widetilde{Y}_2 \\ & \searrow^{\pi_{12}} & \downarrow{\scriptstyle \pi_\Gamma} \\ & & M_{Y_1/D_1} \times_{M_F(r,d)} M_{Y_2/D_2} \end{array}$$

Then one can check that we have an identity $(\pi_\Gamma)_! = (\pi_{12})_! \circ \sigma^*$ of maps $H^*(\mathrm{pr}_1^* \widetilde{Y}_1 \cup_F \mathrm{pr}_2^* \widetilde{Y}_2) \to H^{*-2}(M_{Y_1/D_1} \times_{M_F(r,d)} M_{Y_2/D_2})$. The lemma follows from this. □

We consider the restriction of classes derived from the virtual tangent bundle.

Lemma 4.6. *We have*

$$\Gamma^*\mathrm{ch}_i(T^{\mathrm{vir}}_{M_{X/C}}) = \mathrm{pr}_1^* \mathrm{ch}_i(T^{\mathrm{vir}}_{M_{Y_1/D_1}}) + \mathrm{pr}_2^* \mathrm{ch}_i(T^{\mathrm{vir}}_{M_{Y_2/D_2}}) - \mathrm{pr}_F^* \mathrm{ch}_i(T_{M_F/\mathrm{Pic}_F}).$$

Proof. The obstruction theory on $M_{X/C}$ is a descent of $R\mathrm{Hom}_0(\mathcal{E}, \mathcal{E})$, where $\mathcal{E}$ is the universal sheaf on the un-rigidified moduli stack. Since $[\Gamma^*\mathcal{E}] = [\mathrm{pr}_1^* \mathcal{E}_1] + [\mathrm{pr}_2^* \mathcal{E}_2] - [\mathrm{pr}_F^* \mathcal{E}_F]$, it follows from adjunction that

$$T^{\mathrm{vir}}_{M_{X/C}} = \mathrm{pr}_1^* T^{\mathrm{vir}}_{M_{Y_1/D_1}} + \mathrm{pr}_2^* T^{\mathrm{vir}}_{M_{Y_2/D_2}} - \mathrm{pr}_F^* T_{M_{Y_F}/\mathrm{Pic}_F},$$

and all of these are perfect objects. The result follows from this. □

4.4 Fixed determinant spaces

In order to give more precise statements for some of the invariants we consider, we want to work in some ‘fixed determinant’ theory. We make this precise here for the two cases we are interested in: a simple degeneration and a surface fibered over a smooth curve with a single marked point.

Simple degeneration. Let B be regular one-dimensional base and $X_B \to C_B$ a family of fibered surfaces. Assume the total space C_B is regular, that $C_B \to B$ is smooth outside $b_0 \in B$ and that C_{b_0} is a union of two components along a simple node. We say that $X_B \to C_B$ is a simple degeneration of fibered surfaces.

We consider the stack $\mathrm{Exp}_{C_B/B} \to B$ with universal expansions $\widetilde{X}_B \to \widetilde{C}_B$, and the relative Picard scheme $\mathrm{Pic}_{\widetilde{X}_B/\mathrm{Exp}_{C_B/B}}$.

For the following lemma, we introduce some notation: given an étale morphism $\beta : B \to \mathbb{A}^1$ such that $b_0 = \beta^{-1}(0)$, let $B[n] := B \times_{\mathbb{A}^1} \mathbb{A}^{n+1}$ and $C_B[n] \to B[n]$ be the standard degeneration as in [Li01, § 1.1]. Let also $X_B[n] := X_B \times_{C_B} C_B[n]$. This defines a smooth morphism $\beta_n : B[n] \to \mathrm{Exp}_{C_B/B}$. Say Y_1 and Y_2 are the irreducible components of X_{b_0}. Then let $Y_{1,k} \subset X_B[n]$ denote the divisor that corresponds to Y_1 over the kth coordinate hyperplane.

Lemma 4.7.

(a) *There is a minimal closed substack* $\overline{e} \subset \mathrm{Pic}_{\widetilde{X}_B/B}$ *through which the identity section* $\mathrm{Exp}_{C_B/B} \to \mathrm{Pic}_{\widetilde{X}_B/\mathrm{Exp}_{C_B/B}}$ *factors.*

(b) *The stack* $\overline{e}$ *is naturally a subgroup of* $\mathrm{Pic}_{\widetilde{X}_B/\mathrm{Exp}C_B/B}$*, and the structure map* $\overline{e} \to \mathrm{Exp}_{C_B/B}$ *is étale.*

(c) *For* $\beta : B \to \mathbb{A}^1$ *as above, we have that* $\beta[n]^{-1}\overline{e} \subseteq \mathrm{Pic}_{X_B[n]/B[n]}$ *is equal to the reduced subscheme supported on the union of sections defined by line bundles* $\mathcal{O}_{X_B[n]}(\sum_{i=1}^{n+1} a_k Y_{1,k})$ *for* $a_k \in \mathbb{Z}$*, which is a closed set.*

Proof. We will show that the union of sections $\mathcal{O}_{X_B[n]}(\sum_{i=1}^{n+1} a_k Y_{1,k})$ defines a closed subspace of $\mathrm{Pic}(X_B[n]/B[n])$, which is the closure of the identity section $B[n] \to \mathrm{Pic}_{X_B[n]/B[n]}$. It is then straightforward to see that the collection of such closed substacks for all n induces a closed substack of $\mathrm{Pic}_{\widetilde{X}_B/B}$, which is the minimal substack containing the identity section.

Since the pullback $\mathrm{Pic}_{C_B[n]/B[n]} \to \mathrm{Pic}_{X_B[n]/B[n]}$ is closed, it is enough to show the analogous statement for $\mathrm{Pic}_{C_B[n]/B[n]}$. Let $D_{1,k}$ denote the image of $Y_{1,k}$ in $C_B[n]$. Since the identity component $\mathrm{Pic}^0_{C_B[n]/B[n]}$ is separated, it follows that the identity section is closed in it. Then, for any other section ξ given by a line bundle $\mathcal{O}_{X_B[n]}(\sum a_k D_{1,k})$, we get a closed immersion $\xi \subseteq \xi \mathrm{Pic}^0_{C_B[n]/B[n]}$. It follows that the union over all such sections ξ is a closed subset in $\cup_\xi \xi \mathrm{Pic}^0_{X_B[n]/B[n]}$. But, in fact, $\cup_\xi \mathrm{Pic}^0_{C_B[n]/B[n]} = \mathrm{Pic}_{C_B[n]/B}$.

To see (b), it is enough to show that $\cup_\xi \xi \subset \mathrm{Pic}_{C_B[n]/B[n]}$ is a subgroup-space and étale over $B[n]$. The first point is clear, since the collection of sections ξ forms a group. To see that it is étale, we may work locally on the domain. But $\cup_\xi \xi \cap \xi_0 \mathrm{Pic}^0_{C_B[n]/B[n]} = \xi_0$, which is clearly étale over $B[n]$. □

Let L be a line bundle on X_B with degree d on fibers over C_B, let α be a generic stability condition on C_B, and let L_0 be a line bundle on X_B with fiber degree $d_0 > 0$. Let $\Delta \in \mathbb{Z}$.

Definition 4.8. We let $\mathcal{M}^\alpha_{X_B/C_B}(r, L, \Delta)$ denote the moduli stack of α-balanced f-stable sheaves on minimal expansions of X_B whose determinant map factors through $L\overline{e}$ and whose discriminant is Δ. This is naturally a closed substack of $\mathcal{M}^\alpha_{X_B/C_B}(r, \overline{c_1(L)}, \Delta)$, where $\overline{c_1(L)}$ is the section $\mathrm{Exp}_{C_B/B} \to \overline{\mathcal{NS}}_{X_B/B}$ induced by $c_1(L)$. We also let $M^\alpha_{X_B/C_B}(r, L, \Delta)$ denote the $\mathbb{G}_m$-rigidification of $\mathcal{M}^\alpha_{X_B/C_B}(r, L, \Delta)$.

Remark 4.9. We have an analogous result if $X \to C$ is a fibered surface with C a union of two smooth curves along a single node (e.g. if $X \to C$ is the central fiber of a simple degeneration, but without assuming a smoothing exists). We use the notation $\mathcal{M}^\alpha_{X/B}(r, L, \Delta)$ and $M^\alpha_{X/B}(r, L, \Delta)$ for the resulting moduli stacks. We leave the details to the reader, who may alternatively always assume that we are working with the central fiber of a simple degeneration.

Expansions. Let (C,x) be a smooth marked curve and $X\to C$ be a fibered surface. We consider the stack of expansions Exp_C and the relative Picard scheme $\operatorname{Pic}_{\widetilde{X}/\mathrm{Exp}_C}$. The stack Exp_C has a cover by affine spaces $\alpha_n:\mathbb{A}^n\to \mathrm{Exp}_C$, together with a standard expansion $X[n]\to C[n]$ [Li01, § 4.1]. We let Y_k denote the closure of the component induced by X over the kth coordinate hyperplane in $\mathbb{A}^n$. Then, we have the analogue to Lemma 4.7 with essentially the same proof, which we do not repeat here.

Lemma 4.10.

(a) *There is a minimal closed substack* $\overline{e}\subseteq \operatorname{Pic}_{\widetilde{X}/\mathrm{Exp}_C}$ *through which the identity section* $\mathrm{Exp}_{C,x}\to \operatorname{Pic}_{\widetilde{X}_B/\mathrm{Exp}_{C,x}}$ *factors.*

(b) *The stack* $\overline{e}$ *is naturally a subgroup of* $\operatorname{Pic}_{\widetilde{X}/\mathrm{Exp}_C}$, *and the structure map* $\overline{e}\to \mathrm{Exp}_{C,x}$ *is étale.*

(c) *Given an étale morphism* $\beta:B\to\mathbb{A}^1$, *the closed substack* $\alpha_n^{-1}\overline{e}\subseteq \operatorname{Pic}(X[n]/\mathbb{A}^n)$ *is equal to the reduced substack supported on the union of sections defined by line bundles* $\mathcal{O}_{X_B[n]}(\sum_{i=1}^{n+1}a_kY_k)$ *for* $a_k\in\mathbb{Z}$, *which is a closed set.*

Let L be a line bundle on X. Let $\alpha\in\mathbb{Q}$ be arbitrary.

Definition 4.11. We let $\mathcal{M}^{\alpha}_{X/C}(r,L,\Delta)$ denote the moduli stack of α-balanced f-stable sheaves on minimal expansions of X whose determinant map factors through $L\overline{e}$ and whose discriminant is Δ. This is naturally a closed substack of $\mathcal{M}^{\alpha}(r,\overline{c_1}(L),\Delta)$. We also denote by $M^{\alpha}_{X/C}(r,L,\Delta)$ the $\mathbb{G}_m$-rigidification.

The results of this section carry over to this setting in the obvious way. In particular, this applies to Lemmas 4.5 and 4.6 and to Propositions 4.2 and 4.4.

For clarity, we restate Proposition 4.2 for this setting explicitly. For this, suppose we are in the situation of Proposition 4.2 and that we have also fixed a line bundle L on X in class $\overline{c}_1$. By abuse of notation, write $L_1+L_2=\overline{L}$, if L_1 and L_2 are line bundles on Y_1 and Y_2 respectively, whose restrictions to F are isomorphic and such that there exists an integer ℓ with $L|_{Y_1}\simeq L_1(-\ell F)$ and $L|_{Y_2}\simeq L_2(\ell F)$. Let also $L_F:=L|_F$.

Proposition 4.12. *We have an equality in* $A_*(M^{\alpha}_{X/C}(r,L,\Delta))$:

$$[M^{\alpha}_{X/C}]^{\mathrm{vir}}=\sum_{\substack{L_1+L_2=\overline{L}\\ \Delta_1+\Delta_2=\Delta}}\Gamma_*\Big[M^{\lfloor\alpha_1\rceil}_{Y_1/D_1}(r,L_1,\Delta_1)\times_{M_F(r,L_F)}M^{\lfloor\alpha_2\rceil}_{Y_2/D_2}(r,L_2,\Delta_2)\Big]^{\mathrm{vir}}.$$

4.5 Invariants

We define the type of invariants that we want to consider in the degeneration formula. For simplicity, we restrict the discussion to a specific type of invariant and to the fixed determinant case. We expect that similar formulas hold for, say, Segre and (with some more work) Verlinde invariants. It also should not be essential to work with the fixed determinant version, but then one should consider insertions coming from the Picard scheme in order to get non-trivial invariants when the two theories differ.

Let A be a multiplicative genus (e.g. the Chern polynomial, χ_y-genus or elliptic genus). Let $X\to C$ be a fibered surface, where C is either smooth with possibly a marked point, or a union of two irreducible components along a single node. We assume we have fixed data α, L, L_0 as in § 4.4.

For $\mathcal{M} \to M$ a moduli stack and its $\mathbb{G}_m$-rigidification as considered throughout, we will consider cohomology classes of the form

$$\Phi(\mathcal{E}) = A(T^{\mathrm{vir}}) B(\mathcal{E}). \tag{9}$$

Here,

(i) $\mathcal{E}$ denotes the universal sheaf on some family of expansions over $\mathcal{M}$ of a given fibered surface $X \to C$,

(ii) A is a multiplicative transformation from the K-theory of perfect objects to cohomology with coefficients in some ring Λ containing $\mathbb{Q}$,

(iii) $B(\mathcal{E}) = \exp(\sum_{i,\gamma} \mathrm{ch}_i(\gamma) q_{\gamma,i})$, where i ranges through integers $\geqslant 2$ and γ ranges through a basis of cohomology of X.

Let $K := \Lambda[[(q_{\gamma,i})_{\gamma,i}]]$ denote the coefficient field of Φ. We let $\mathcal{E}$ denote the universal sheaf over $\mathcal{M}^{\alpha}_{X/C}(r, L, \Delta)$. We assume that α is generic and that it satisfies (5).

If $X \to C$ has no marked fiber (so either C is smooth or has two components and a single node), we define an invariant simply as

$$I^{\Phi}_{X/C}(r, L, \Delta) := \int \Phi(\mathcal{E}) \cap [M^{\alpha}_{X/C}(r, L, \Delta)]^{\mathrm{vir}} \in K.$$

Here, the left-hand side a priori implicitly depends on α, but it will follow from the decomposition formula that it is actually independent for any generic α satisfying (5).

If C is smooth with a single marked point, we let L_F denote the restriction of L to the marked fiber, and set

$$V := H_*(M_F(r, L_F), K).$$

Then we obtain invariants valued in V by pushing forward along the evaluation map

$$I^{\Phi}_{X/C}(r, L, \Delta) := \mathrm{ev}_* \left(\Phi(\mathcal{E}) \cap [M^{\alpha}_{X/C}(r, L, \Delta)]^{\mathrm{vir}}\right) \in V.$$

If C has a marked point, let F be the marked fiber. Otherwise, let F denote the fiber over an arbitrary smooth point of C. We set

$$Z_{X/C,\Phi}(q) := \sum_{\substack{\Delta \in \mathbb{Z} \\ 0 \leqslant \ell < r}} I^{\Phi}_{X/C}(r, L + \ell[F], \Delta)\, q^{\Delta - (r^2-1)\chi(\mathcal{O}_X)}.$$

This is valued in $K[[q]]$ or $V[[q]]$ respectively and depends implicitly on r, L and L_0. Here, we choose a different generic α satisfying (5) for each Δ and ℓ.

Remark 4.13. Note that for $c_1 = L + \ell F$, the discriminant and second Chern class are related by

$$\Delta = 2rc_2 - (r-1)c_1^2 = 2rc_2 - (r-1)L^{\cdot 2} - 2\ell(r-1)d.$$

As c_2 ranges through the integers and ℓ ranges in $[0, r-1]$, we find that the possible exponents of q for which the coefficient of $Z_{X/C,\Phi}$ is non-empty lie in $2\mathbb{Z} + (r-1)L^{\cdot 2} - (r^2-1)\chi(\mathcal{O}_X)$, and each such integer corresponds to a unique choice of ℓ and c_2. In other words, we have

$$Z_{X/C,\Phi}(q) = q^{-(r-1)L^{\cdot 2} - (r^2-1)\chi(\mathcal{O}_X)} \sum_{\substack{c_2 \in \mathbb{Z} \\ 0 \leqslant \ell < r}} I^{\Phi}_{X/C}\left(r, L + \ell[F], \Delta(c_2, \ell)\right)\, q^{2(rc_2 - \ell(r-1)d)}.$$

We see that the exponents of q in the sum range exactly through $2\mathbb{Z}$.

4.6 Degeneration formula for multiplicative classes

Let $X \to C$ be a fibered surface, where C is the union of two smooth curves along a single node, and let the set-up be as in § 4.5. We let $Y_i \to D_i$ be the surfaces fibered over a smooth curve with a single marking obtained as the components of $X \to C$ and let F denote the fiber of X over the node.

Let

$$V := H_*(M_F(r, L), K)$$

so that the invariants associated to Y_i/D_i for $i = 1, 2$ are valued in V. Note that V has a ring structure with respect to intersection product, which is graded commutative. We denote the intersection product of cycles by $\alpha \cdot \beta$. We define a bilinear pairing on V as

$$\begin{aligned} *_\Phi : V \times V &\to K \\ \alpha, \beta &\mapsto \int_{M_F(r,L)} A(T_{M_F(r,L)})^{-1} \cap (\alpha \cdot \beta)\,. \end{aligned}$$

We extend the multiplication $*_\Phi$ to Laurent series in q over V and K by applying it coefficientwise and dividing the final result by $q^{\dim M_F(r,L)}$.

THEOREM 4.14. *We have*

$$Z_{X/C,\Phi}(q) = Z_{X_1/C_1,\Phi}(q) *_\Phi Z_{X_2/C_2,\Phi}(q).$$

Proof. Comparing coefficients, and in view of Remark 4.13, we may consider c_2 and k, and therefore Δ, as fixed, and we need to show that, for some fixed chosen stability condition α on C, we have

$$I^\Phi_{X/C}(r, L + \ell[F], \Delta) = \sum I^\Phi_{Y_1/D_1}(r, L_1 + \ell_1[F], \Delta_1) *_\Phi I^\Phi_{Y_2/D_2}(r, L_2 + \ell_2[F], \Delta_2),$$

where the sum ranges over all $0 \leqslant \ell_1, \ell_2 < r$ and over all $\Delta_1, \Delta_2 \in \mathbb{Z}$ such that

$$\Delta - (r^2 - 1)\chi(\mathcal{O}_X) = \Delta_1 + \Delta_2 - (r^2 - 1)(\chi(\mathcal{O}_{Y_1}) + \chi(\mathcal{O}_{Y_2})) - \dim M_F(r, L_F),$$

or, equivalently, such that $\Delta = \Delta_1 + \Delta_2$. Writing out the definition of invariants, we have the equivalent formula

$$\int \Phi(\mathcal{E}) \cap [M^\alpha_{X/C}(r, L + \ell[F], \Delta)]^{\mathrm{vir}} = \sum (\mathrm{ev}_*(\Phi(\mathcal{E}_1) \cap [M^b_{Y_1/D_1}(r, L_1 + \ell_1 F, \Delta_1)]^{\mathrm{vir}}) *_\Phi \\ \mathrm{ev}_*(\Phi(\mathcal{E}_2) \cap [M^b_{Y_2/D_2}(r, L_2 + \ell_2 F, \Delta_2)]^{\mathrm{vir}})) \qquad (10)$$

for some choice of generic α satisfying (5) and where we use the notation of Remark 3.27.

Since $\Delta = \Delta_1 + \Delta_2$, and by the dependence of the discriminant on first and second Chern classes, we have, for each term of the sum for which the moduli spaces are non-empty, that $\ell \equiv \ell_1 + \ell_2 \mod r$. In particular, for each such term there exists unique representatives $\ell'_i \equiv \ell_i \mod r$ such that $|\alpha(c_1(L_i + \ell'_i F), \Delta_i) - \alpha_i| < 1/2$. It follows that $\ell = \ell'_1 + \ell'_2$. Letting $L'_i := L_i + \ell'_i F$ and $L' := L + \ell F$, we have in particular $L'_1 + L'_2 = \overline{L'}$ in the notation preceding Proposition 4.12. Since twisting by a line bundle induces an isomorphism between moduli spaces, we have

$$\begin{aligned} I^\Phi_{Y_i/D_i}(r, L_1 + \ell_1 F, \Delta_1) &= I^\Phi_{Y_i/D_i}(r, L_1 + \ell'_1 F, \Delta_1) \\ &= \mathrm{ev}_*\left(\Phi(\mathcal{E}_i) \cap \left[M^{\lfloor \alpha_i \rceil}_{Y_i/D_i}(r, L'_1, \Delta_i)\right]^{\mathrm{vir}}\right). \end{aligned}$$

In summary, we may rewrite the right-hand side of (10) as

$$\sum_{\substack{\Delta_1+\Delta_2=\Delta\\ L_1'+L_2'=\overline{L'}}} \mathrm{ev}_*\left(\Phi(\mathcal{E}_1)\cap\left[M^{\lfloor\alpha_1\rceil}_{Y_1/D_1}(r,L_1',\Delta_1)\right]^{\mathrm{vir}}\right) *_\Phi \mathrm{ev}_*\left(\Phi(\mathcal{E}_2)\cap[M^{\lfloor\alpha_2\rceil}_{Y_2/D_2}(r,L_2',\Delta_2)]^{\mathrm{vir}}\right).$$

We examine each term of this sum. Using the definition of $*_\Phi$, the projection formula and Proposition 4.4, we may rewrite a single term in this sum as the pushforward to a point of

$$\mathrm{pr}_F^* A(T_{M_F})^{-1}\cap\mathrm{pr}_1^*\Phi(\mathcal{E}_1)\cap\mathrm{pr}_2^*\Phi(\mathcal{E}_2)\cap\left[M^{\lfloor\alpha_1\rceil}_{Y_1/D_1}(r,L_1',\Delta_1)\times_{M_F}M^{\lfloor\alpha_2\rceil}_{Y_2/D_2}(r,L_2',\Delta_2)\right]^{\mathrm{vir}}.$$

Then, by Lemmas 4.5 and 4.6, this is equal to

$$\Gamma^*\Phi(\mathcal{E})\cap\left[M^{\lfloor\alpha_1\rceil}_{Y_1/D_1}(r,L_1',\Delta_1)\times_{M_F}M^{\lfloor\alpha_2\rceil}_{Y_2/D_2}(r,L_2',\Delta_2)\right]^{\mathrm{vir}},$$

where Γ is the glueing map to $M^\alpha_{X/C}(r,L',\Delta)$. Using this, we have reduced equation (10) to Proposition 4.2 and are done. □

4.7 Application to elliptic fibrations

Suppose that $X\to C$ has genus one fibers in the situation of Theorem 4.14. In this case, we have $V=H_*(M_F(r,L),K)=H_*(\mathrm{pt},K)\simeq K$, so the relative invariants are simply power series valued in the coefficient ring. In this case, the statement of Theorem 4.14 becomes especially simple.

Corollary 4.15. *If $g=1$, we have the following identity in $K[[q]]$:*

$$Z_{X/C,\Phi}(q)=Z_{Y_1/D_1,\Phi}(q)\,Z_{Y_2/D_2,\Phi}(q).$$

In the rest of this subsection, we give a proof of Theorem 1.4. We will consider the special case

$$\Phi(\mathcal{E})=A_{y,q}(T^{\mathrm{vir}})$$

of (9), obtained by setting $B=1$ and taking $A_{y,q}$ to be the insertion considered in [GL10, § 4.8] which defines the virtual elliptic genus [FG10]. By results of de Jong and Friedman, we have suitable degenerations.

Theorem 4.16. *Let X be an elliptic surface over $\mathbb{P}^1$ of degree $e\geqslant 2$ without multiple or reduced fibers. Let D be a d-section on X, and suppose that there exist no d'-sections for any $1\leqslant d'<d$. Then there exists a connected base B and a family of elliptic surfaces $X_B\to C_B\to B$ together with a family of n-sections $D_B\subset X_B$ such that the following hold.*

- (i) *For some $b_0\in B$, the triple $D_{b_0}\subset X_{b_0}\to C_{b_0}$ is isomorphic to $D\subset X\to\mathbb{P}^1$.*
- (ii) *For some $b_1\in B$, we have that:*
  - *$X_{b_1}\to C_{b_1}$ is obtained from glueing two elliptic surfaces $Y_1\to\mathbb{P}^1$ and $Y_2\to\mathbb{P}^1$ along an isomorphic fiber, where Y_1 is a degree $(e-1)$ elliptic surface and Y_2 is a rational elliptic surface, and*
  - *the divisor $D_{b_1}\subset X_{b_1}$ restricts to a d-section on Y_1 and to a smooth rational curve D satisfying $D^2=d-2$ on Y_2. Moreover, if $e\geqslant 2$, then Y_1 has no d'-sections for $1\leqslant d'<d$.*

Proof. For $e\geqslant 3$, this follows from Theorem 4.9 in [dJF11] together with constructions going into Claim 5.7 in [dJF11]. For $e=2$, it follows from similar arguments using the Torelli theorem for lattice polarized K3 surfaces. □

We will also need the following vanishing result.

Proposition 4.17. *Let E be an elliptic curve and consider $X = E \times \mathbb{P}^1 \to \mathbb{P}^1$, and let $x_1, \ldots, x_n$ be distinct points on $\mathbb{P}^1$. Let $r > 0$, and let L be a line bundle on $E \times \mathbb{P}^1$ that has degree d on fibers, with d coprime to r. Then we have*

$$Z^{\mathrm{Ell}}_{X/(\mathbb{P}^1,(x_1,\ldots,x_n))} = 1.$$

Proof. For any Δ for which the moduli space

$$M_{X/(\mathbb{P}^1,(x_1,\ldots,x_n))}(r, L, \Delta)$$

is non-empty, one can show that either it is a point, or it admits, up to a finite étale cover, an elliptic curve factor. In the latter case, all virtual Chern numbers vanish. The result follows from this. □

Proof of Theorem 1.4. Since enumerative invariants for surfaces with $p_g(X) > 0$ are independent of choice of polarization, we may use Theorem 2.24 and compute invariants using moduli spaces of f-stable sheaves.

In the case $e = 2$, the result follows from the DMVV formula and the fact that any moduli space of Gieseker-stable sheaves on a K3 surface is deformation invariant to a Hilbert scheme of points when stability equals semistability. Next, we show that for any rational elliptic surface $Y \to \mathbb{P}^1$ and any divisor D on Y of fiber degree coprime to r, we have (when taking invariants and generating series with respect to the moduli spaces of fiber-stable objects)

$$Z^{\mathrm{Ell}}_{Y/\mathbb{P}^1} = (Z^{\mathrm{Ell}}_{K3})^{1/2}.$$

By Proposition 4.17, the relative and absolute invariants agree. In particular, we may glue two identical copies of Y together along a fiber and deform the resulting surface to a smooth K3 surface while preserving the divisor class, see [Fri83, Theorem 5.10] and [Fri84, Proposition 4.3].

Now, we can argue inductively on e, with base case $e = 2$, by Theorem 4.16 and Corollary 4.15, to obtain an identity

$$Z^{\mathrm{Ell}}_{X/\mathbb{P}^1} = Z^{\mathrm{Ell}}_{X'/(\mathbb{P}^1,0)}\, Z^{\mathrm{Ell}}_{Y/(\mathbb{P}^1,0)},$$

where X' is an elliptic surface of degree $e-1$ with a chosen d-section D' (and which possesses no d'-sections for $1 \leqslant d' < d$) and where Y is a rational elliptic surface with a chosen rational curve D satisfying $D^{\cdot 2} = d - 2$. Using Proposition 4.17 again and by the inductive hypothesis,

$$Z^{\mathrm{Ell}}_{X/\mathbb{P}^1} = Z^{\mathrm{Ell}}_{X'/\mathbb{P}^1}\, Z^{\mathrm{Ell}}_{Y/\mathbb{P}^1} = (Z^{\mathrm{Ell}}_{K3})^{(e-1)/2} (Z^{\ell}_{K3})^{1/2}.$$

This finishes the proof. □

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Conflicts of Interest

None.

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REFERENCES

[ACFW13] D. Abramovich, C. Cadman, B. Fantechi and J. Wise, *Expanded degenerations and pairs*, Comm. Algebra **41** (2013), 2346–2386; MR 3225278.

[ACGS20] D. Abramovich, Q. Chen, M. Gross and B. Siebert, *Decomposition of degenerate Gromov–Witten invariants*, Compos. Math. **156** (2020), 2020–2075; MR 4177284.

[AOV11] D. Abramovich, M. Olsson and A. Vistoli, *Twisted stable maps to tame Artin stacks*, J. Algebr. Geom., **20** (2011), 399–477; MR 2786662.

[AKL$^+$22] D. Aranha, A. A. Khan, A. Latyntsev, H. Park and C. Ravi, *Localization theorems for algebraic stacks*, Preprint (2022), arXiv: 2207.01652v2

[AGV08] D. Abramovich, T. Graber and A. Vistoli, *Gromov–Witten theory of Deligne-Mumford stacks*, Amer. J. Math. **130** (2008), 1337–1398; MR 2450211.

[BS22] Y. Bae and J. Schmitt, *Chow rings of stacks of prestable curves I*, Forum Math. Sigma **10** (2022), e28, with an appendix by Bae, Schmitt and Jonathan Skowera; MR 4430955.

[Beh04] K. Behrend, *Cohomology of stacks*, in Intersection theory and moduli, ICTP Lect. Notes, XIX (Abdus Salam Int. Cent. Theoret. Phys., Trieste, 2004), 249–294; MR 2172499.

[BL03] L. Borisov and A. Libgober, *Elliptic genera of singular varieties*, Duke Math. J. **116** (2003), 319–351; MR 1953295.

[BL05] L. Borisov and A. Libgober, *McKay correspondence for elliptic genera*, Ann. of Math. (2) **161** (2005), 1521–1569; MR 2180406.

[CFKM14] I. t Ciocan-Fontanine, B. Kim and D. Maulik, *Stable quasimaps to GIT quotients*, J. Geom. Phys **75** (2014), 17–47; MR 3126932.

[Del68] P. Deligne*Théorème de Lefschetz et critères de dégénérescence de suites spectrales*, Inst. Hautes Études Sci. Publ. Math. **35** (1968), 259–278; MR 244265.

[DI87] P. Deligne and L. Illusie, *Relèvements modulo p^2 et décomposition du complexe de de Rham*, Invent. Math. **89** (1987), 247–270; MR 894379.

[dJF11] A. J. de Jong and R. Friedman, *On the geometry of principal homogeneous spaces*, Amer. J. Math. **133** (2011), 753–796; MR 2808331.

[DMVV97] R. Dijkgraaf, G. Moore, E. Verlinde and H. Verlinde, *Elliptic genera of symmetric products and second quantized strings*, Comm. Math. Phys **185** (1997), 197–209; MR 1463039.

[DN89] J.-M. Drezet and M. S. Narasimhan, *Groupe de Picard des variétés de modules de fibrés semi-stables sur les courbes algébriques*, Invent. Math. **97** (1989), 53–94; MR 999313.

[Don95] S. K. Donaldson, *Floer homology and algebraic geometry*, in Vector bundles in algebraic geometry (Durham, 1993), London Math. Soc. Lecture Note Ser., vol. 208 (Cambridge Univ. Press, Cambridge, 1995), 119–138; MR 1338415.

[EP16] E. Esteves and M. Pacini, *Semistable modifications of families of curves and compactified Jacobians*, Ark. Mat. **54** (2016), 55–83; MR 3475818.

[FG10] B. Fantechi and L. Göttsche, *Riemann–Roch theorems and elliptic genus for virtually smooth schemes*, Geom. Topol. **14** (2010), 83–115; MR 2578301.

[Fri83] R. Friedman, *Global smoothings of varieties with normal crossings*, Ann. of Math. (2) **118** (1983), 75–114; MR 707162.

[Fri84] R. Friedman, *A new proof of the global Torelli theorem for $K3$ surfaces*, Ann. of Math. (2) **120** (1984), 237–269; MR 763907.

[GK19a] L. Göttsche and M. Kool, *A rank 2 Dijkgraaf–Moore–Verlinde–Verlinde formula, Commun*, Number Theory Phys. **13** (2019), 165–201; MR 3951108.

[GK19b] L. Göttsche and M. Kool, *Sheaves on surfaces and virtual invariants*, in Surveys in differential geometry. Differential geometry, Calabi–Yau theory, and general relativity (Part 2), Surv. Differ. Geom., vol. 24 (Int. Press, Somerville, MA, 2019), 67–116.

[GL94] D. Gieseker and J. Li, *Irreducibility of moduli of rank-2 vector bundles on algebraic surfaces*, J. Differ. Geom. **40** (1994), 23–104; MR 1285529.

[GL10] E. Gasparim and C.-C. M. Liu, *The Nekrasov conjecture for toric surfaces*, Comm. Math. Phys. **293** (2010), 661–700; MR 2566160.

[GNY08] L. Göttsche, H. Nakajima and K. Yoshioka, *Instanton counting and Donaldson invariants*, J. Differ. Geom. **80** (2008), 343–390; MR 2472477.

[GNY11] L. Göttsche, H. Nakajima and K. Yoshioka, *Donaldson = Seiberg-Witten from Mochizuki's formula and instanton counting*, Publ. Res. Inst. Math. Sci. **47** (2011), 307–359; MR 2827729.

[Gro66] A. Grothendieck, *éléments de géométrie algébrique. IV. étude locale des schémas et des morphismes de schémas. III*, Inst. Hautes Études Sci. Publ. Math., **28** (1966), 255; MR 217086.

[HL10] D. Huybrechts and M. Lehn, *The geometry of moduli spaces of sheaves*, second edition, Cambridge Mathematical Library (Cambridge University Press, Cambridge, 2010); MR 2665168.

[IKN05] L. Illusie, K. Kato and C. Nakayama, *Quasi-unipotent logarithmic Riemann–Hilbert correspondences*, J. Math. Sci. Univ. Tokyo **12** (2005), 1–66; MR 2126784.

[Joy21] D. Joyce, *Enumerative invariants and wall-crossing formulae in abelian categories*, Preprint (2021), arXiv: 2111.04694.

[Kat00] F. Kato, *Log smooth deformation and moduli of log smooth curves*, Internat. J. Math. **11** (2000), 215–232; MR 1754621.

[KP19] J. L. Kass and N. Pagani, *The stability space of compactified universal Jacobians*, Trans. Amer. Math. Soc. **372** (2019), 4851–4887; MR 4009442.

[KLR23] B. Kim, H. Lho and H. Ruddat, *The degeneration formula for stable log maps*, Manuscripta Math. **170** (2023), 63–107; MR 4533479.

[Kle05] S. L. Kleiman, *The Picard scheme*, Preprint (2005), arXiv: math/0504020.

[Knu83] F. F. Knudsen, *The projectivity of the moduli space of stable curves. II. The stacks K3*, Math. Scand. **52** (1983), 161–199; MR 702953.

[Kre99] A. Kresch, *Cycle groups for Artin stacks*, Invent. Math. **138** (1999), 495–536; MR 1719823.

[KLT22] N. Kuhn, O. Leigh and Y. Tanaka, *The blowup formula for the instanton part of Vafa–Witten invariants on projective surfaces*, Preprint (2022), arXiv: 2205.12953

[KT21] N. Kuhn and Y. Tanaka, *A blowup formula for virtual enumerative invariants on projective surfaces*, Preprint (2021), arXiv: 2107.08155.

[Lan75] S. G. Langton, *Valuative criteria for families of vector bundles on algebraic varieties*, Ann. of Math. (2) **101** (1975), 88–110; MR 364255.

[Li01] J. Li, *Stable morphisms to singular schemes and relative stable morphisms*, J. Differential Geom. **57** (2001), 509–578; MR 1882667.

[Li02] J. Li, *A degeneration formula of GW-invariants*, J. Differential Geom. **60** (2002), 199–293; MR 1938113.

[Li03] J. Li, *Moduli spaces associated to a singular variety and the moduli of bundles over universal curves*, in Vector bundles and representation theory (Columbia, MO, 2002), Contemp. Math., vol. 322 (Amer. Math. Soc., Providence, RI, 2003), 57–74; MR 1987739.

[Li18] J. Li, *Toward an algebraic Donaldson–Floer theory*, in Modern geometry: A celebration of the work of Simon Donaldson, Proc. Sympos. Pure Math., vol. 99 (Amer. Math. Soc., Providence, RI, 2018), 161–191; MR 3838882.

[LW15] J. Li and B. Wu, *Good degeneration of Quot-schemes and coherent systems*, Comm. Anal. Geom. **23** (2015), 841–921; MR 3385781.

[Man12] C. Manolache, *Virtual pull-backs*, J. Algebraic Geom. **21** (2012), 201–245; MR 2877433.

[MR20] D. Maulik and D. Ranganathan, *Logarithmic Donaldson–Thomas theory*, Preprint (2020), arXiv: 2006.06603.

[Moc09] T. Mochizuki, *Donaldson type invariants for algebraic surfaces*, in Transition of moduli stacks, Lecture Notes in Mathematics, vol. 1972 (Springer-Verlag, Berlin, 1972), Transition of moduli stacks; MR 2508583.

[Nes24] D. Nesterov, *Quasimaps to moduli spaces of sheaves on a* $K3$ *surface*, Forum Math. Sigma **12** (2024), e61; MR 4746880.

[Nit11] N. Nitsure, *Schematic Harder–Narasimhan stratification*, Internat. J. Math. **22** (2011), 1365–1373; MR 2851213.

[OS79] T. Oda and C. S. Seshadri, *Compactifications of the generalized Jacobian variety*, Trans. Amer. Math. Soc. **253** (1979), 1–90; MR 536936.

[Ram73] S. Ramanan, *The moduli spaces of vector bundles over an algebraic curve*, Math. Ann. **200** (1973), 69–84; MR 325615.

[Ran22] D. Ranganathan, *Logarithmic Gromov–Witten theory with expansions*, Algebr. Geom. **9** (2022), 714–761; MR 4518245.

[Sta23] The Stacks Project Authors, *The Stacks Project* (2023), https://stacks.math.columbia.edu.

[TT17] Y. Tanaka and R. P. Thomas, *Vafa–Witten invariants for projective surfaces II: Semistable case*, Pure Appl. Math. Q. **13** (2017), 517–562; MR 3882207.

[TT20] Y. Tanaka and R. P. Thomas, *Vafa–Witten invariants for projective surfaces I: Stable case*, J. Algebraic Geom. **29** (2020), 603–668; MR 4158461.

[Ver76] J.-L. Verdier, *Le théorème de Riemann-Roch pour les variétés algébriques éventuellement singulières (d'après P. Baum, W. Fulton et R. MacPherson)*, in *Séminaire Bourbaki (1974/1975:Exposés Nos. 453-470)*, Lecture Notes in Math., vol. 514 (Springer, Berlin, 1976), 159; MR 0444656.

[Yos96] K. Yoshioka, *Chamber structure of polarizations and the moduli of stable sheaves on a ruled surface*, Internat. J. Math. **7** (1996), 411–431; MR 1395938.

[Yos99] K. Yoshioka, *Some notes on the moduli of stable sheaves on elliptic surfaces*, Nagoya Math. J. **154** (1999), 73–102; MR 1689173.

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