Available online at www.sciencedirect.com
ScienceDirect
Journal of Algebra 314 (2007) 693–748
JOURNAL OF Algebra
www.elsevier.com/locate/jalgebra

Fixed point ratios in actions of finite classical groups, III

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Received 27 January 2006
Available online 16 January 2007
Communicated by Jan Saxl

Abstract

This is the third in a series of four papers on fixed point ratios for non-subspace actions of finite classical groups. Our main result states that if G is a finite almost simple classical group and Ω is a faithful transitive non-subspace G-set then either $\mathrm{fpr}(x) \lesssim |x^G|^{-1/2}$ for all elements $x \in G$ of prime order, or (G, Ω) is one of a small number of known exceptions. In this paper we consider the case where G_ω is contained in one of the Aschbacher families $\mathscr{C}_2$ or $\mathscr{C}_3$.

Keywords: Finite classical group; Fixed point ratio; Primitive permutation group

1. Introduction

Let G be a finite almost simple classical group over $\mathbb{F}_q$, with socle G_0 and natural module V. Recall that a subgroup H of G is said to be a *non-subspace subgroup* if $H \cap G_0$ is contained in a maximal subgroup of G_0 which acts irreducibly on V, while a transitive action of G on a set Ω is a *non-subspace action* if the point stabilizer G_ω is a non-subspace subgroup of G (see [3, Definition 1]). Our main result, which we shall refer to as Theorem 1, states that if Ω is a faithful, transitive, non-subspace G-set then

$$\mathrm{fpr}(x) < \left|x^G\right|^{-\frac{1}{2}+\frac{1}{n}+\iota}$$

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0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2007.01.011

for all elements $x \in G$ of prime order, where either $\iota = 0$, or (G_0, Ω, ι) belongs to a short list of known exceptions (see [3, Table 1]). Here $\mathrm{fpr}(x)$ denotes the *fixed point ratio of* x, i.e. the proportion of points in Ω which are fixed by x. In general $n = \dim V$ (see Remark 1.2).

In order to prove Theorem 1, we may assume G is primitive and therefore apply Aschbacher's well-known result on the subgroup structure of finite classical groups. In [1], eight collections of subgroups of G are defined, labeled $\mathscr{C}_i$ for $1 \leqslant i \leqslant 8$, and it is shown that if H is a maximal subgroup of G not containing G_0 then either H is contained in one of the $\mathscr{C}_i$ collections, or it belongs to a family $\mathscr{S}$ of almost simple groups which act irreducibly on V (a small additional collection $\mathscr{N}$ arises if $G_0 = \mathrm{Sp}_4(q)'$ (q even) or $\mathrm{P}\Omega_8^+(q)$). A detailed description of these subgroup collections can be found in [9] (also see [4, §3.1]).

This is the third in a series of four papers. In the introductory note [3] we provided some background and motivation, stated our main results and we described two further applications of Theorem 1 to the study of primitive permutation groups. In [4] we established Theorem 1 in the case where the stabilizer G_ω is a non-subspace subgroup contained in one of the collections $\mathscr{C}_i$, where $4 \leqslant i \leqslant 8$. In this paper we assume G_ω belongs to $\mathscr{C}_2$ or $\mathscr{C}_3$. Roughly speaking, the subgroups in $\mathscr{C}_2$ are the stabilizers of decompositions $V = \bigoplus_i V_i$, where $\dim V_i = m$, while the stabilizers of prime degree field extensions of $\mathbb{F}_q$ comprise $\mathscr{C}_3$. Again, we refer the reader to [9, §§4.2–3] for further details. We complete the proof of Theorem 1 in [5] where we consider the remaining collections $\mathscr{S}$ and $\mathscr{N}$.

Theorem 1.1. *Let G be a finite almost simple classical group acting transitively and faithfully on a set Ω with point stabilizer $G_\omega \leqslant H$, where H is a maximal non-subspace subgroup of G in one of the Aschbacher collections $\mathscr{C}_2$ or $\mathscr{C}_3$. Then*

$$\mathrm{fpr}(x) < \left|x^G\right|^{-\frac{1}{2}+\frac{1}{n}+\iota}$$

for all elements $x \in G$ of prime order, where $\iota = 0$ or (G_0, H, ι) is listed in Table 1.1*, where G_0 denotes the socle of G.*

Remark 1.2. The integer $n = n(G)$ in the statement of Theorem 1.1 is defined as follows: if $G_0 \in \{\mathrm{Sp}_4(2)', \mathrm{SL}_3(2)\}$ then $n = 2$, otherwise n is defined to be the minimal degree of a nontrivial irreducible $K\widehat{G}_0$-module, where $\widehat{G}_0$ is a covering group of G_0 and K is the algebraic closure of $\mathbb{F}_q$. The *type of H* referred to in Table 1.1 provides an approximate group-theoretic structure for $H \cap \mathrm{PGL}(V)$.

Table 1.1
The exceptional cases with $\iota > 0$

G_0	Type of H	ι
$\mathrm{PSp}_n(q)$	$\mathrm{Sp}_{n/2}(q) \wr S_2$	$1/n$
$\mathrm{PSp}_n(q)$	$\mathrm{Sp}_{n/2}(q^2)$	$1/(n+2)$
$\mathrm{P}\Omega_n^\epsilon(q)$	$\mathrm{GL}_{n/2}^{\epsilon'}(q)$	$1/(n-2)$
$\mathrm{SU}_4(2)$	$\mathrm{GU}_1(2) \wr S_4$	0.010
$\Omega_8^+(2)$	$\mathrm{O}_4^-(2) \wr S_2$	0.001
$\mathrm{SL}_4(2)$	$\mathrm{GL}_2(4)$	0.020

Notation. We follow [9] in our notation for classical groups. In particular, we write $\mathrm{PSL}_n^\epsilon(q)$ for $\mathrm{PSL}_n(q)$ and $\mathrm{PSU}_n(q)$ when $\epsilon = +$ and $-$, respectively. Other notation and terminology is consistent with [3,4]. In particular, if $H \leqslant G$ and $x \in G$ then we define

$$f(x, H) := \frac{\log|x^G \cap H|}{\log|x^G|}$$

and thus Theorem 1 states that $f(x, H) < 1/2 + 1/n + \iota$ when H is a non-subspace subgroup and x has prime order (see [4, (1)]). We label representatives for conjugacy classes of unipotent involutions in symplectic and orthogonal groups as in [2], while our terminology for graph automorphisms is explained in [4, 3.47]. The *associated partition* of a unipotent element $x \in \mathrm{PGL}(V)$ is the partition of $\dim V$ which corresponds to the Jordan normal form of x on V (see [4, §3.3]). There is semisimple analogue, the *associated σ-tuple*, which is defined in [4, 3.27]. In addition, for $x \in \mathrm{PGL}(V)$ we define $\nu(x)$ to be the codimension of the largest eigenspace of x on $\bar{V} = V \otimes K$ (see [4, 3.16]). Finally, we write $i_r(S)$ for the number of elements of order r in S, where S is a subset of a finite group.

2. Proof of Theorem 1.1: $H \in \mathscr{C}_2$

The subgroups which comprise the collection $\mathscr{C}_2$ are the stabilizers of m-decompositions $V = V_1 \oplus \cdots \oplus V_t$ of the natural G_0-module V, where $t \geqslant 2$ and $\dim V_i = m$. The particular cases are listed in Table 2.1, where in the last column we record some necessary conditions for the existence and maximality of H in G (see [9, p. 100 and Tables 3.5.A–H]). For convenience, we postpone the analysis of totally singular $n/2$-decompositions to the next section (see cases (ii) and (vii)–(ix) in Table 3.1).

2.1. Preliminary results

Let $\bar{G}$ be a simple classical algebraic group over an algebraically closed field K of characteristic $p \geqslant 0$, with natural module $\bar{V}$ of dimension n. Let $\bar{H} \in \mathscr{C}_2$ be a maximal closed subgroup of $\bar{G}$, say $\bar{H}$ is the stabilizer in $\bar{G}$ of a decomposition $\bar{V} = V_1 \oplus \cdots \oplus V_t$ and assume each V_i is non-degenerate if $\bar{G}$ is a symplectic or orthogonal group. If $x \in \bar{G}$ has prime order then [6, Theorem 1] states that

$$\dim\bigl(x^{\bar{G}} \cap \bar{H}\bigr) \leqslant \left(\frac{1}{2} + \delta\right) \dim x^{\bar{G}},$$

Table 2.1
The collection $\mathscr{C}_2$

	G_0	Type of H	Conditions
(i)	$\mathrm{PSL}_n^\epsilon(q)$	$\mathrm{GL}_{n/t}^\epsilon(q) \wr S_t$	$(n, q) \neq (2t, 2)$, $q > 3$ if $(n, \epsilon) = (t, +)$
(ii)	$\mathrm{PSp}_n(q)$	$\mathrm{Sp}_{n/t}(q) \wr S_t$	$q > 2$ if $n = 2t$
(iii)	$\mathrm{P}\Omega_n^\epsilon(q)$	$\mathrm{O}_1(q) \wr S_n$	$q = p \geqslant 3$, $\epsilon = -$ if and only if $n \equiv 2\ (4)$ and $q \equiv 3\ (4)$
(iv)	$\mathrm{P}\Omega_n^\epsilon(q)$	$\mathrm{O}_{n/t}^{\epsilon'}(q) \wr S_t$	$n \geqslant 2t$, q odd if n/t odd, $(n/t, q) \neq (3, 3)$; if $\epsilon = +$: $q > 4$ if $(\epsilon', n/t) = (+, 2)$, $(\epsilon', n/t, q) \neq (+, 4, 2)$, $(\epsilon')^t = +$ if n/t even; if $\epsilon = -$: $\epsilon' \neq +$, t is odd if n/t even
(v)	$\mathrm{P}\Omega_n^\epsilon(q)$	$\mathrm{O}_{n/2}(q) \wr S_2$	$n/2$ odd, $q \equiv 2 + \epsilon\ (4)$

where $\delta = 1/n$ if $\bar{G} = \mathrm{Sp}_n(K)$ and $t = 2$, otherwise $\delta = 0$ (note that the entry '$1/(2n+2)$' appearing in the final column of [6, Table 1] should be '$1/2n$'). In fact, better bounds hold when $t > 2$.

Proposition 2.1. *Let $\bar{H} \in \mathscr{C}_2$ be the stabilizer in $\bar{G}$ of a decomposition $\bar{V} = V_1 \oplus \cdots \oplus V_t$ and assume that each V_i is non-degenerate if $\bar{G}$ is a symplectic or orthogonal group. Then*

$$\dim\bigl(x^{\bar{G}} \cap \bar{H}\bigr) \leqslant \left(\frac{1}{t} + \zeta\right) \dim x^{\bar{G}}$$

for all elements $x \in \bar{G}$ of prime order, where either $\zeta = 0$, or $\bar{G}$ is symplectic and $\zeta = (1+\alpha)/(n+2\alpha)$ with $\alpha = 1 - \delta_{2,t}$.

Proof. First assume $\bar{G} = \mathrm{PSp}_n(K)$, so $\bar{H} = (\mathrm{Sp}_m(K) \wr S_t) \cap \bar{G}$ for $m = n/t$. Let $x \in \bar{G}$ be an element of prime order r and note that $x^{\bar{G}} \cap \bar{H}$ is a finite union of $\bar{B}$-classes, where $\bar{B} = \mathrm{Sp}_m(K)^t \cap \bar{G} = \bar{H}^0$. In particular, by replacing x with a suitable $\bar{G}$-conjugate, we may assume that $\dim(x^{\bar{G}} \cap \bar{H}) = \dim x^{\bar{B}}$.

First suppose x is semisimple and r is odd. Then x is the image (modulo scalars) of an element $\hat{x} = (x_1, \ldots, x_t)\pi \in \mathrm{Sp}_m(K)^t\pi$, where $\pi \in S_t$ has cycle-shape $(r^h, 1^{t-hr})$ for some $h \geqslant 0$. We claim that

$$\dim\bigl(x^{\bar{G}} \cap \bar{H}\bigr) \leqslant \left(\frac{1}{t} + \frac{1+\alpha}{n+2}\right) \dim x^{\bar{G}}. \tag{1}$$

Now, if π induces the permutation $\prod_{i=1}^{h}((i-1)r+1 \ldots ir)$ on coordinates then the proof of [10, 4.5] implies that $\hat{x}$ is $\mathrm{Sp}_m(K)^t$-conjugate to $(I_m, \ldots, I_m, x_{hr+1}, \ldots, x_t)\pi$, where $x_i^r = 1$ for all $i > hr$, and thus

$$\dim\bigl(x^{\bar{G}} \cap \bar{H}\bigr) = \dim x^{\bar{B}} = h(r-1)\dim \mathrm{Sp}_m + \sum_{i=hr+1}^{t} \dim x_i^{\mathrm{Sp}_m}.$$

Let $\omega \in K$ be a primitive rth root of unity and suppose $\hat{x}$ admits the eigenvalue ω^i with multiplicity l_i on the natural $\mathrm{Sp}_n(K)$-module, where $0 \leqslant i \leqslant r-1$. We claim that there exist tr rational numbers $\{l_{ij}\colon 0 \leqslant i \leqslant r-1,\ 1 \leqslant j \leqslant t\}$ such that $\sum_j l_{ij} = l_i$ and

$$\dim\bigl(x^{\bar{G}} \cap \bar{H}\bigr) = \frac{1}{2}nm + \frac{1}{2}n - \frac{1}{2}l_0 - \frac{1}{2}\sum_{i=0}^{r-1}\left(\sum_{j=1}^{t} l_{ij}^2\right). \tag{2}$$

If $t = hr$ then $l_i = mh$ for each i, whence

$$\dim\bigl(x^{\bar{G}} \cap \bar{H}\bigr) = \dim x^{\bar{B}} = h(r-1)\dim \mathrm{Sp}_m = \frac{1}{2}n(m+1)\left(1 - \frac{1}{r}\right)$$

and thus (2) holds if we set $l_{ij} = m/r$ for all i, j. Now, if $t - hr = f > 0$ then we may assume x fixes each subspace in the set $\{V_j\colon 1 \leqslant j \leqslant f\}$. For $1 \leqslant j \leqslant f$, let y_{ij} be the multiplicity of ω^i

as an eigenvalue of x acting on V_j, so

$$\dim\big(x^{\bar{G}} \cap \bar{H}\big) = \frac{1}{2}(t-h)\big(m^2+m\big) - \frac{1}{2}\sum_{j=1}^{f}\left(y_{0j} + \sum_{i=0}^{r-1} y_{ij}^2\right)$$

and (2) follows if we set $l_{ij} = y_{ij}$ for $1 \leqslant j \leqslant f$, and $l_{ij} = m/r$ for $j > f$. Applying (2) we deduce that

$$\dim\big(x^{\bar{G}} \cap \bar{H}\big) \leqslant \frac{1}{2}nm + \frac{1}{2}n - \frac{1}{2}l_0 - \frac{1}{2t}\sum_{i=0}^{r-1} l_i^2 = \frac{1}{t}\dim x^{\bar{G}} + \frac{1}{2}(n-l_0)\left(1-\frac{1}{t}\right)$$

and (1) follows since

$$\dim x^{\bar{G}} \geqslant \frac{1}{4}(n+2)(n-l_0) \geqslant \frac{(n+2)(n-l_0)}{2(1+\alpha)}\left(1-\frac{1}{t}\right).$$

Next assume x is a semisimple involution. First suppose $C_{\bar{G}}(x)^0 = \mathrm{GL}_{n/2}$, so $\dim x^{\bar{G}} = \frac{1}{4}n(n+2)$. If $x \in \bar{B}\pi$ and π induces the permutation $(12)\dots(2h-1\,2h)$ on the coordinates then x lifts to an element $\hat{x} = (x_1,\dots,x_t)\pi \in \mathrm{Sp}_m(K) \wr S_t$ of order 4 and the proof of [10, 4.5] implies that $\hat{x}$ is $\mathrm{Sp}_m(K)^t$-conjugate to $(-I_m, I_m, \dots, -I_m, I_m, x_{2h+1}, \dots, x_t)\pi$, where $x_j = z = [-iI_{m/2}, iI_{m/2}]$ for all $j > 2h$ (here $i \in K$ satisfies $i^2 = -1$). In particular, the hypothesis $\dim(x^{\bar{G}} \cap \bar{H}) = \dim x^{\bar{B}}$ implies that $x \in \bar{B}$ since $2\dim z^{\mathrm{Sp}_m} > \dim \mathrm{Sp}_m$, and thus (1) follows since

$$\dim\big(x^{\bar{G}} \cap \bar{H}\big) = \frac{1}{4}nm + \frac{1}{2}n = \left(\frac{1}{t} + \frac{1+\alpha}{n+2}\right)\dim x^{\bar{G}} - n\left(\frac{\alpha}{4} + \frac{1}{2t} - \frac{1}{4}\right).$$

Next suppose x is $\bar{G}$-conjugate to $[-I_l, I_{n-l}]$, where $2 \leqslant l \leqslant n/2$ is even. Then $\dim x^{\bar{G}} = l(n-l)$ and the hypothesis $\dim(x^{\bar{G}} \cap \bar{H}) = \dim x^{\bar{B}}$ implies that $x \in \bar{B}\pi$, where $\pi \in S_t$ has cycle-shape $(2^a, 1^f)$ and $a = \lfloor l/m \rfloor$. If $f = 0$ then x is $\bar{G}$-conjugate to $[-I_{n/2}, I_{n/2}]$ and thus

$$\dim\big(x^{\bar{G}} \cap \bar{H}\big) = \frac{t}{2}\dim \mathrm{Sp}_m = \left(\frac{1}{t} + \frac{1}{n}\right)\dim x^{\bar{G}}.$$

On the other hand, if $f > 0$ then we may assume that x fixes each V_j, $1 \leqslant j \leqslant f$, and that the restriction of x to such a subspace V_j is Sp_m-conjugate to $[-I_{l_j}, I_{n-l_j}]$ for some even integer $l_j \geqslant 0$. Then

$$\dim\big(x^{\bar{G}} \cap \bar{H}\big) = a\dim \mathrm{Sp}_m + \sum_{j=1}^{f} l_j(m-l_j) \leqslant \frac{1}{2}am(m+1) + m(l-ma) - \frac{1}{f}(l-ma)^2$$

and we conclude that

$$\dim\big(x^{\bar{G}} \cap \bar{H}\big) \leqslant ml - \frac{l^2}{t} + \frac{1}{2}ma \leqslant ml - \frac{l^2}{t} + \frac{1}{2}l \leqslant \left(\frac{1}{t} + \frac{1}{n}\right)\dim x^{\bar{G}}.$$

Now assume x has order $r = p > 2$ and associated partition $\lambda = (r^{a_r}, \dots, 1^{a_1}) \vdash n$, so λ encodes the Jordan normal form of x on V (see [4, §3.3]). In analogy with the semisimple case, we can find tr rational numbers $\{a_{ij}\}$ such that $\sum_j a_{ij} = a_i$ for each $1 \leqslant i \leqslant r$ and

$$\dim\big(x^{\bar{G}} \cap \bar{H}\big) = \frac{1}{2}nm + \frac{1}{2}n - \frac{1}{2}\sum_{j=1}^{t}\left(\sum_{i=1}^{r}\left(\sum_{k=i}^{r} a_{kj}\right)^2\right) - \frac{1}{2}\sum_{i \text{ odd}} a_i$$

(see [6, 2.3]). This implies that

$$\begin{aligned}\dim\big(x^{\bar{G}} \cap \bar{H}\big) &\leqslant \frac{1}{2}nm + \frac{1}{2}n - \frac{1}{2t}\sum_{i=1}^{m}\left(\sum_{k=i}^{r} a_k\right)^2 - \frac{1}{2}\sum_{i \text{ odd}} a_i \\ &= \frac{1}{t}\dim x^{\bar{G}} + \frac{1}{2}\left(1 - \frac{1}{t}\right)\left(n - \sum_{i \text{ odd}} a_i\right)\end{aligned}$$

and (1) follows since $n = \sum_{i=1}^{r} i a_i$ and thus

$$\dim x^{\bar{G}} \geqslant \frac{n+2}{2(1+\alpha)}\left(n - \sum_{i \text{ odd}} a_i\right)\left(1 - \frac{1}{t}\right).$$

Finally, let us assume $r = p = 2$. Here we adopt the standard Aschbacher–Seitz [2] notation for representatives of the classes of unipotent involutions in $\bar{G}$. If x is $\bar{G}$-conjugate to b_l or c_l (according to the parity of l) then [6, 2.3(iv)] gives $\dim x^{\bar{G}} = l(n - l + 1)$ and the hypothesis $\dim(x^{\bar{G}} \cap \bar{H}) = \dim x^{\bar{B}}$ implies that $x \in \bar{B}$. In particular, if x acts on V_j with associated partition $(2^{l_j}, 1^{m-2l_j})$ then

$$\dim\big(x^{\bar{G}} \cap \bar{H}\big) \leqslant \sum_{j=1}^{t}\big((m+1)l_j - l_j^2\big) \leqslant (m+1)l - \frac{l^2}{t} = \frac{1}{t}\dim x^{\bar{G}} + l\left(1 - \frac{1}{t}\right)$$

and (1) quickly follows. On the other hand, if x is $\bar{G}$-conjugate to a_l then $\dim x^{\bar{G}} = l(n-l)$ and the definition of an a-type involution (see [2, §7]) implies that the restriction of each $y \in x^{\bar{G}} \cap \bar{H}$ to a fixed subspace V_j is Sp_m-conjugate to a_{l_j} for an even integer $l_j \geqslant 0$ (we set $a_0 = I_m$). Therefore,

$$\dim y^{\bar{B}} = \sum_j l_j\left(\frac{n}{t} - l_j\right) = \frac{nl}{t} - \sum_j l_j^2 \leqslant \frac{1}{t}\dim x^{\bar{G}}$$

for all $y \in x^{\bar{G}} \cap \bar{B}$. Now, if $x \in \mathrm{Sp}_m \times \mathrm{Sp}_m = \bar{J}$ is Sp_{2m}-conjugate to a_m then

$$\dim x^{\bar{J}} = 2\dim a_{m/2}^{\mathrm{Sp}_m} = \frac{1}{2}m^2 < \dim \mathrm{Sp}_m$$

and so the hypothesis $\dim(x^{\bar{G}} \cap \bar{H}) = \dim x^{\bar{B}}$ implies that $x \in \bar{B}\pi$, where $\pi \in S_t$ has cycle-shape $(2^a, 1^f)$ and $a = \lfloor l/m \rfloor$. In the usual manner we conclude that

$$\dim\left(x^{\bar{G}} \cap \bar{H}\right) \leqslant \left(\frac{1}{t} + \frac{1}{n}\right) \dim x^{\bar{G}}.$$

The argument for linear and orthogonal groups is very similar and left to the reader. □

Remark 2.2. The conclusion to Proposition 2.1 holds for arbitrary unipotent elements if $p = 0$.

Recall that if S is a subset of a finite group then we write $i_r(S)$ for the number of elements of order r in S. The following result is an easy exercise.

Lemma 2.3. *Let r be a prime and let $i_{r,k}(S_t)$ be the number of permutations in S_t with cycle shape $(r^k, 1^{t-rk})$, where S_t is the symmetric group on t letters. Then*

$$i_{r,k}(S_t) = \frac{t!}{k!(t-kr)!r^k} \quad \text{and} \quad i_r(S_t) = \sum_{k=1}^{\lfloor t/r \rfloor} i_{r,k}(S_t).$$

In Section 2.4 we will need the following technical result on orthogonal groups.

Lemma 2.4. *If q is odd and $l \geqslant 1$ then the following hold for all m:*

(i) $|\mathrm{O}_{2l}^{+}(q) : \mathrm{O}_{2m}^{+}(q)\mathrm{O}_{2l-2m}^{+}(q)| + |\mathrm{O}_{2l}^{+}(q) : \mathrm{O}_{2m}^{-}(q)\mathrm{O}_{2l-2m}^{-}(q)| < 2q^{2m(2l-2m)}$;
(ii) $|\mathrm{O}_{2l}^{-}(q) : \mathrm{O}_{2m}^{+}(q)\mathrm{O}_{2l-2m}^{-}(q)| + |\mathrm{O}_{2l}^{+}(q) : \mathrm{O}_{2m}^{-}(q)\mathrm{O}_{2l-2m}^{+}(q)| < 2q^{2m(2l-2m)}$;
(iii) $|\mathrm{O}_{2l+1}(q) : \mathrm{O}_{2m}^{+}(q)\mathrm{O}_{2l+1-2m}(q)| + |\mathrm{O}_{2l+1}(q) : \mathrm{O}_{2m}^{-}(q)\mathrm{O}_{2l+1-2m}(q)| < 2q^{2m(2l+1-2m)}$;
(iv) $|\mathrm{O}_{2l+1}(q) : \mathrm{O}_{2l}^{+}(q)\mathrm{O}_{1}(q)| + |\mathrm{O}_{2l+1}(q) : \mathrm{O}_{2l}^{-}(q)\mathrm{O}_{1}(q)| = q^{2l}$;
(v) $|\mathrm{O}_{2l}^{+}(q) : \mathrm{O}_{2m+1}(q)\mathrm{O}_{2l-2m-1}(q)| < |\mathrm{O}_{2l}^{-}(q) : \mathrm{O}_{2m+1}(q)\mathrm{O}_{2l-2m-1}(q)| < q^{(2m+1)(2l-2m-1)}$.

Proof. First consider (i). Without loss we may assume $m \geqslant l/2$ and thus

$$\left|\mathrm{O}_{2l}^{+}(q) : \mathrm{O}_{2m}^{+}(q)\mathrm{O}_{2l-2m}^{+}(q)\right| + \left|\mathrm{O}_{2l}^{+}(q) : \mathrm{O}_{2m}^{-}(q)\mathrm{O}_{2l-2m}^{-}(q)\right| = \frac{q^{2m(l-m)} \prod_{i=1}^{l-m} (q^{2m+2i} - 1)}{\prod_{i=1}^{l-m} (q^{2i} - 1)}.$$

The result now follows from [4, 3.8]. The other statements are derived in a similar fashion. □

Recall that in order to prove Theorem 1.1 it suffices to show that

$$f(x, H) := \frac{\log|x^G \cap H|}{\log|x^G|} < \frac{1}{2} + \frac{1}{n} + \iota$$

for all elements $x \in G$ of prime order. We start with the case $G_0 = \mathrm{PSL}_n^{\epsilon}(q)$.

2.2. Proof of Theorem 1.1: Case (i) of Table 2.1

Let σ be a Frobenius morphism of $\bar{G} = \mathrm{PSL}_n(K)$ such that $\bar{G}_\sigma$ has socle $G_0 = \mathrm{PSL}_n^\epsilon(q)$ and natural module V, where K is the algebraic closure of $\mathbb{F}_q$ and $q = p^f$ for a prime p. Let $\bar{B}$ denote the image of $\mathrm{GL}_{n/t}(K)^t$ in $\mathrm{PSL}_n(K)$ and observe that

$$H \cap \mathrm{PGL}(V) \leqslant \left(\left[(q-\epsilon)^{t-1}\right].\mathrm{PGL}^\epsilon_{\frac{n}{t}}(q)^t\right).S_t = B.S_t = \tilde{H},$$

where B is the image of $\mathrm{GL}^\epsilon_{n/t}(q)^t$ in $\mathrm{PGL}^\epsilon_n(q)$ and $[(q-\epsilon)^{t-1}]$ is a group of order $(q-\epsilon)^{t-1}$. We partition the proof into three parts: in Proposition 2.5 we assume $x \in H \cap \mathrm{PGL}(V)$ is semisimple, while we consider unipotent elements in Proposition 2.6. Finally, in Proposition 2.7, we deal with the outer automorphisms in $H - \mathrm{PGL}(V)$.

Proposition 2.5. *The conclusion to Theorem* 1.1 *holds in case* (i) *of Table* 2.1 *for semisimple elements in* $H \cap \mathrm{PGL}(V)$.

Proof. Let $x \in H \cap \mathrm{PGL}(V)$ be a semisimple element of prime order r. We partition the proof into several cases, where Case i.j.k is a subcase of Case i.j, which in turn is a subcase of Case i.

Case 1. $x^G \cap H \subseteq B$.

Let $E = C_{\bar{G}}(x)$ and let $H^1(\sigma, E/E^0)$ denote the set of σ-equivalence classes with respect to the induced action of σ on the finite group E/E^0 (see [4, 3.5]).

Case 1.1. $r > 2$, $|H^1(\sigma, E/E^0)| = 1$.

Let $i \geqslant 1$ be minimal such that r divides $q^i - 1$ and as in the statement of [4, 3.33] set

$$c = c(i,\epsilon) = \begin{cases} 2i & \text{if } \epsilon = - \text{ and } i \text{ is odd,} \\ i/2 & \text{if } \epsilon = - \text{ and } i \equiv 2\ (4), \\ i & \text{otherwise.} \end{cases}$$

According to [4, 3.35], the hypothesis $|H^1(\sigma, E/E^0)| = 1$ is equivalent to assuming that E is connected if $c = 1$. Furthermore, [4, 3.11] implies that each $y \in x^G \cap H$ lifts to an element $\hat{y} = (\hat{y}_1, \dots, \hat{y}_t) \in \hat{B}$ of order r, where $\hat{B} = \mathrm{GL}^\epsilon_{n/t}(q)^t$ and

$$\left|y^B\right| = \left|\hat{y}^{\hat{B}}\right| = \prod_j \left|\hat{y}_j{}^{\mathrm{GL}^\epsilon_{n/t}(q)}\right|.$$

Define the integers l and d as in [4, 3.32] and note that the hypothesis $x^G \cap H \subseteq B$ implies that $n \geqslant \max(tc, l + dc)$. Since $|x^{\bar{G}_\sigma}| = |x^{G_0}|$ (see [8, 4.2.2(j)]) we deduce that

$$\left|x^G\right| \geqslant \left|x^{\bar{G}_\sigma}\right| > \frac{1}{2}\left(\frac{q}{q+1}\right)^{d\alpha} q^{\dim x^{\bar{G}}} \tag{3}$$

(see [4, 3.30]) where

$$\alpha = \begin{cases} 1 & \text{if } \epsilon = - \text{ and } i \equiv 2\ (4), \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

and [4, 3.33] gives

$$\dim x^{\bar{G}} \geqslant n^2 - l^2 - \frac{1}{c}\big(n - l - c(d-1)\big)^2 - c(d-1). \tag{5}$$

Case 1.1.1. $c > 1$.

Let $\mu = (l, a_1, \dots, a_k)$ denote the associated σ-tuple of x (see [4, 3.27]) and write $\mathcal{E}_x$ for the multiset of eigenvalues of $\hat{x} \in \mathrm{GL}_n^{\epsilon}(q)$, where $\hat{x}$ is the unique lift of x of order r. Here l is the dimension of the 1-eigenspace of $\hat{x}$ and d is the number of non-zero a_j terms in μ. We claim that

$$\big|x^G \cap H\big| < 2\log_2 q . 2^{td(1-\alpha)}\left(\frac{q+1}{q}\right)^{\frac{t}{2}(1-\epsilon)}(d+1)^{\frac{n}{c}} q^{\frac{1}{t}\dim x^{\bar{G}}}. \tag{6}$$

To see this, first observe that Proposition 2.1 and [4, 3.30] imply that

$$\big|y^B\big| = \big|\hat{y}^{\hat{B}}\big| < 2^{td(1-\alpha)}\left(\frac{q+1}{q}\right)^{\frac{t}{2}(1-\epsilon)} q^{\frac{1}{t}\dim x^{\bar{G}}}$$

for all $y \in x^G \cap H$ and so it remains to show that the number of B-classes in $x^G \cap H$ is at most $2\log_2 q.(d+1)^{n/c}$. The term $2\log_2 q$ accounts for the effect of field and graph automorphisms of G_0 on $\mathcal{E}_x$ and so we just need to show that $M \leqslant (d+1)^{n/c}$, where M is the number of distinct ways the non-trivial σ-orbits in $\mathcal{E}_x$ can be distributed among the t direct factors in $\hat{B}$ (see [4, §3.4] for the definition of a σ-orbit). Now, if $\hat{x} = (\hat{x}_1, \dots, \hat{x}_t) \in \hat{B}$ and $n/t \equiv j\ (c)$ then $l_u \equiv j\ (c)$ for each $1 \leqslant u \leqslant t$, where l_u is the multiplicity of 1 in the eigenvalue set $\mathcal{E}_{\hat{x}_u}$. Therefore

$$M \leqslant \binom{\frac{n-tj}{c}}{\frac{l-tj}{c}\, a_{k_1} \dots a_{k_d}},$$

where $a_{k_v} > 0$ for all $1 \leqslant v \leqslant d$, and so the multinomial theorem implies that

$$M \leqslant (d+1)^{\frac{n-tj}{c}} \leqslant (d+1)^{\frac{n}{c}} \tag{7}$$

as required.

If we assume $t \geqslant 3$ then one can check that the bounds (3), (5) and (6) imply that $f(x,H) < 1/2 + 1/n$ unless $\epsilon = +$ and $(t,i,q) = (3,2,2)$. Here $(r,d) = (3,1)$ and it remains to deal with the cases $(n,l) \in \{(12,10),(9,7),(6,4),(6,0)\}$. If $(n,l) = (6,4)$ then x is $\bar{G}$-conjugate to $[I_4, \omega, \omega^2]$, where $\omega \in K$ is a primitive cube root of unity, and we calculate that $f(x,H) < 0.141$ since

$$\big|x^G \cap H\big| \leqslant 3\big|\mathrm{GL}_2(2) : \mathrm{GL}_1(4)\big| = 6, \qquad \big|x^G\big| \geqslant \big|\mathrm{GL}_6(2) : \mathrm{GL}_4(2)\mathrm{GL}_1(4)\big| = 333\,312.$$

The other cases are similar.

Now suppose $t = 2$. We claim that

$$\big|x^G \cap H\big| < 2\log_2 q . \left(\frac{n-l}{cd} + 1\right)^{d(t-1)} 2^{td(1-\alpha)}\left(\frac{q+1}{q}\right)^{\frac{t}{2}(1-\epsilon)} q^{\frac{1}{t}\dim x^{\bar{G}}} \tag{8}$$

for all $t \geqslant 2$, where α is defined as in (4). Arguing as before, it suffices to show that

$$M \leqslant \left(\frac{n-l}{cd}+1\right)^{d(t-1)},$$

where M is defined as above. If $a_{k_1}, \dots, a_{k_d}$ are non-zero then it is clear that $\mathcal{E}_{\hat{x}_1}$ is determined by a choice of d-tuple $(b_1, \dots, b_d)$, where $0 \leqslant b_j \leqslant a_{k_j}$. If N denotes the number of such d-tuples then $M \leqslant N^{t-1}$ since $\mathcal{E}_{\hat{x}_t}$ is uniquely determined once $\mathcal{E}_{\hat{x}_1}, \dots, \mathcal{E}_{\hat{x}_{t-1}}$ have been chosen. Therefore (8) holds since

$$N = \prod_{j=1}^{d}(a_{k_j}+1) \leqslant \left(\frac{\sum_j a_{k_j}}{d}+1\right)^d = \left(\frac{n-l}{cd}+1\right)^d. \tag{9}$$

If we set $t = 2$ then the bounds (3), (5) and (8) are always sufficient if $d \geqslant 3$, while we are left to deal with a handful of cases with $\epsilon = +$ when $d = 2$. Here the desired result quickly follows through direct calculation. For example, if $(n, l, i, q) = (8, 2, 3, 2)$ then $r = 7$ and $f(x, H) < 0.445$ since

$$\left|x^G \cap H\right| \leqslant 2\left|\mathrm{GL}_4(2) : \mathrm{GL}_1(2)\mathrm{GL}_1\left(2^3\right)\right|^2, \qquad \left|x^G\right| \geqslant \left|\mathrm{GL}_8(2) : \mathrm{GL}_2(2)\mathrm{GL}_1\left(2^3\right)^2\right|.$$

Now assume $(t, d, \epsilon) = (2, 1, +)$. We claim that

$$\left|x^G \cap H\right| < 2\log_2 q . 2^2\left(\frac{q^2+1}{q^2-1}\right)q^{\frac{1}{2}\dim x^{\bar{G}}}. \tag{10}$$

Without loss of generality, we may assume $a_1 > 0$. If $l = 0$ then $a_1 = n/i$ and [4, 3.30] implies that

$$\left|x^G \cap H\right| \leqslant 2\log_2 q . \left|\mathrm{GL}_{n/2}(q) : \mathrm{GL}_{n/2i}\left(q^i\right)\right|^2 < 2\log_2 q . q^{\frac{1}{2}\dim x^{\bar{G}}}$$

so let us assume $l > 0$. For all possible integers j in the range $0 \leqslant j \leqslant a_1$, choose $z_j = (y_1, y_2) \in x^G \cap H$ so that $\mathcal{E}_{y_1}$ contains precisely j copies of the non-trivial σ-orbit Ω_1 (recall that a_1 is defined to be the multiplicity of Ω_1 in $\mathcal{E}_x$). Then $|x^G \cap H| \leqslant 2\log_2 q . \sum_j |z_j^B|$, where

$$\left|z_j^B\right| = \frac{|\mathrm{GL}_{n/2}(q)|}{|\mathrm{GL}_{n/2-ji}(q)||\mathrm{GL}_j(q^i)|} . \frac{|\mathrm{GL}_{n/2}(q)|}{|\mathrm{GL}_{l+ji-n/2}(q)||\mathrm{GL}_{a_1-j}(q^i)|}$$

and thus

$$\left|x^G \cap H\right| < 2\log_2 q . 2^2 \sum_j q^{\dim z_j^{\bar{B}}},$$

where

$$\dim z_j^{\bar{B}} = -2i(i+1)j^2 + 2(n-l)(i+1)j + nl - l^2 - \frac{1}{i}(n-l)^2.$$

If $(n-l)/i$ is even then $\dim z_j^{\bar{B}} \leqslant \dim z_{(n-l)/2i}^{\bar{B}} = \frac{1}{2}\dim x^{\bar{G}}$ and (10) follows since

$$\sum_j q^{\dim z_j^{\bar{B}}} \leqslant 2\left(1+q^2+\cdots+q^{\frac{1}{2}\dim x^{\bar{G}}-2}\right)+q^{\frac{1}{2}\dim x^{\bar{G}}} \leqslant \left(\frac{q^2+1}{q^2-1}\right)q^{\frac{1}{2}\dim x^{\bar{G}}}.$$

Similarly, if $(n-l)/i$ is odd then

$$\sum_j q^{\dim z_j^{\bar{B}}} \leqslant 2\left(1+q^2+\cdots+q^{\frac{1}{2}\dim x^{\bar{G}}-\frac{1}{2}i(i+1)}\right)$$

and again the claim follows. With minor adjustments, the same argument applies when $(t,d,\epsilon) = (2,1,-)$ and it is easy to see that (10) holds.

Now, if $\epsilon = -$ then the bounds (3), (5) and (10) are sufficient unless $(n,l,i,q) = (4,0,1,4)$, where direct calculation yields $f(x,H) < 0.529$. If $(\epsilon,l) = (+,0)$ then the same bounds are almost always sufficient and the few cases which remain are easily dealt with. Finally, if $\epsilon = +$ and $l > 0$ then we quickly reduce to the case $(n,i,q) = (l+2,2,2)$, so x is $\bar{G}$-conjugate to $[I_{n-2},\omega,\omega^2]$ and $\omega \in K$ is a primitive cube root of unity. Here the reader can check that the bounds

$$\left|x^G \cap H\right| \leqslant 2\left(\frac{|\mathrm{GL}_{n/2}(2)|}{|\mathrm{GL}_{n/2-2}(2)||\mathrm{GL}_1(2^2)|}\right) = \frac{1}{3}2^{n-2}\left(2^{\frac{n}{2}-1}-1\right)\left(2^{\frac{n}{2}}-1\right)$$

and

$$\left|x^G\right| \geqslant \frac{|\mathrm{GL}_n(2)|}{|\mathrm{GL}_{n-2}(2)||\mathrm{GL}_1(2^2)|} = \frac{1}{3}2^{2n-3}\left(2^{n-1}-1\right)\left(2^n-1\right)$$

are good enough.

Case 1.1.2. $c = 1$.

Here $l > 0$ and $d+l \leqslant n \leqslant (d+1)l$ (see [4, 3.32(i)]). If $n = t \geqslant 3$ then $|x^B| = 1$ and arguing as before (see (6)) we deduce that $|x^G \cap H| \leqslant 2\log_2 q.(d+1)^n$. Then (3) and (5) are sufficient unless $(\epsilon,q) = (-,2)$. Here $r = 3$, so $d \leqslant 2$ and the desired result quickly follows through direct calculation. For the remainder we will assume $n \geqslant 2t$.

First consider the case $n = l+d$. If $d = 1$ then one can check that the bounds

$$\left|x^G \cap H\right| \leqslant 2\log_2 q.t\left(\frac{|\mathrm{GL}^{\epsilon}_{n/t}(q)|}{|\mathrm{GL}^{\epsilon}_{n/t-1}(q)||\mathrm{GL}^{\epsilon}_1(q)|}\right) = 2t\log_2 q.\left(\frac{q^{n/t-1}(q^{n/t}-\epsilon^{n/t})}{q-\epsilon}\right),$$

$$\left|x^G\right| \geqslant \frac{|\mathrm{GL}^{\epsilon}_n(q)|}{|\mathrm{GL}^{\epsilon}_{n-1}(q)||\mathrm{GL}^{\epsilon}_1(q)|} = \frac{q^{n-1}(q^n-\epsilon^n)}{q-\epsilon}$$

are always sufficient (note that $\epsilon = -$ if $q = 2$ since the hypothesis $c = 1$ implies that $r|(q-\epsilon)$). For $d \geqslant 2$ we claim that

$$\left|x^G \cap H\right| < 2\log_2 q.2^{\frac{d}{2}(1+\epsilon)}t^d q^{\frac{1}{t}\dim x^{\bar{G}}-d(1-\frac{1}{t})}. \tag{11}$$

To see this, let $\hat{y} = (\hat{y}_1, \dots, \hat{y}_t) \in \hat{B}$ be a lift of $y \in x^G \cap H$ of order r such that the 1-eigenspace of $\hat{y}$ has dimension l. Let l_k denote the multiplicity of 1 in $\mathcal{E}_{\hat{y}_k}$, so $\sum_k l_k = l$ and $|y^B| < 2^{\frac{d}{2}(1+\epsilon)} q^{\dim y^{\bar{B}}}$, where

$$\dim y^{\bar{B}} = \frac{n^2}{t} - \sum_{k=1}^{t} l_k^2 - n + l \leqslant \frac{n^2}{t} - \frac{l^2}{t} - n + l = \frac{1}{t}\dim x^{\bar{G}} - d\left(1 - \frac{1}{t}\right).$$

Then (11) follows since there are at most t^d distinct ways to distribute the d distinct eigenvalues $\lambda_i \neq 1$ among the t direct factors. Now, $\dim x^{\bar{G}} = 2ld + d^2 - d$ and applying (3) we find that (11) is sufficient unless $(\epsilon, t, d, q) = (-, 2, 2, 2)$. Here $n \geqslant 6$ (see Table 2.1) and x is $\bar{G}$-conjugate to $[I_{n-2}, \omega, \omega^2]$, where $\omega \in K$ is a primitive cube root of unity. Moreover,

$$\left|x^G \cap H\right| \leqslant 2\frac{|\mathrm{GU}_{n/2}(2)|}{|\mathrm{GU}_{n/2-2}(2)||\mathrm{GU}_1(2)|^2} + 2\left(\frac{|\mathrm{GU}_{n/2}(2)|}{|\mathrm{GU}_{n/2-1}(2)||\mathrm{GU}_1(2)|}\right)^2 < 10.2^{2n-6}$$

and the desired result follows since $|x^G| > \frac{1}{9}2^{4n-5}$.

Now suppose $n > l + d$ and $t \geqslant 3$, so (6) gives

$$\left|x^G \cap H\right| < 2\log_2 q.2^{\frac{1}{2}td(1+\epsilon)}(d+1)^n q^{\frac{1}{t}\dim x^{\bar{G}}}.$$

If $\epsilon = +$ then (3) and (5) are sufficient except for a handful of cases with $(t, q) = (3, 4)$ which are easily dealt with, while the same bounds are always sufficient if $\epsilon = -$, $t \geqslant 3$ and $q \geqslant 4$. Now assume $(\epsilon, q) = (-, 2)$ and $t \geqslant 3$, so $d \leqslant 2$. If $d = 1$ then $x = [I_l, \lambda I_{n-l}]$ for some $\lambda \neq 1$ and thus

$$\left|x^G \cap H\right| < 2\binom{t+n-l-1}{n-l} 2^{\frac{1}{t}\dim x^{\bar{G}}},$$

where $\dim x^{\bar{G}} = 2l(n-l)$ (the binomial coefficient can be interpreted combinatorially as the number of ways the $n-l$ eigenvalues equal to λ can be distributed among the t direct factors). Since $l + 1 < n \leqslant 2l$, it is easy to check that this bound with (3) is always sufficient whenever $t \geqslant 3$. Finally, if $d = 2$ then (8) gives

$$\left|x^G \cap H\right| < 2\left(\frac{n-l}{2} + 1\right)^{2(t-1)} 2^{\frac{1}{t}\dim x^{\bar{G}}}$$

and (3) and (5) are sufficient unless $(n, t, l) = (6, 3, 3)$. Here direct calculation yields $f(x, H) < 0.312$.

Next assume $n > l + d$ and $t = 2$. If $d = 1$ then $\dim x^{\bar{G}} = 2l(n-l)$ and an earlier argument (see (10)) implies that

$$\left|x^G \cap H\right| < 2\log_2 q.2^{1+\epsilon}\left(\frac{q^2+1}{q^2-1}\right) q^{l(n-l)}.$$

If $\epsilon = +$ then this bound with (3) is sufficient unless $(n, l, q) = (4, 2, 4)$, while it remains to deal with the case $(n, l, q) = (6, 4, 2)$ if $\epsilon = -$. In both cases, the desired result quickly follows through direct calculation. Next assume $d = 2$, so x is $\bar{G}$-conjugate to $[I_l, \alpha I_a, \beta I_{n-l-a}]$ for distinct $\alpha, \beta \in K - \{1\}$. We claim that

$$\left|x^G \cap H\right| < 2\log_2 q.\left(\frac{q^2+1}{q^2-1}\right)^2 2^{2(1+\epsilon)} q^{\frac{1}{2}\dim x^{\bar{G}}}. \tag{12}$$

To see this, first observe that $|x^G \cap H| \leqslant 2\log_2 q.\sum_{j,k} |\hat{x}_{jk}^{\hat{B}}|$, where $\hat{x}_{jk} = (\hat{y}_1, \hat{y}_2) \in \hat{B}$ and $\hat{y}_1 = [I_j, \alpha I_k, \beta I_{n/2-j-k}]$ up to $\mathrm{GL}_{n/2}(K)$-conjugacy. Then

$$\left|\hat{x}_{jk}^{\hat{B}}\right| = \frac{|\mathrm{GL}_{n/2}^{\epsilon}(q)|}{|\mathrm{GL}_j^{\epsilon}(q)||\mathrm{GL}_k^{\epsilon}(q)||\mathrm{GL}_{n/2-j-k}^{\epsilon}(q)|}.\frac{|\mathrm{GL}_{n/2}^{\epsilon}(q)|}{|\mathrm{GL}_{l-j}^{\epsilon}(q)||\mathrm{GL}_{a-k}^{\epsilon}(q)||\mathrm{GL}_{n/2-l+j-a+k}^{\epsilon}(q)|},$$

so $\sum_k |\hat{x}_{jk}^{\hat{B}}| < 2^{2(1+\epsilon)} \sum_k q^{f(j,k)}$, where $f(j,k) := \dim x_{jk}^{\bar{B}}$, and (12) quickly follows. If $\epsilon = +$ then the bounds (3), (5) and (12) are sufficient unless $(n,l,q) = (6,3,4)$, where direct calculation yields $f(x,H) < 0.499$. If $\epsilon = -$ then it remains to deal with the case $(n,q) = (l+3, 2)$. Here $\dim x^{\bar{G}} = 6n - 14$ and the desired result follows via (3) since $|x^G \cap H| < 2\log_2 q.2q^{3n-14}(1 + q^4 + q^6)$. Finally, let us assume $n > l + d$, $t = 2$ and $d \geqslant 3$, in which case $q \geqslant 8$ if $\epsilon = +$, while $q \geqslant 4$ if $\epsilon = -$. Arguing as before (see (8)) we deduce that

$$\left|x^G \cap H\right| < 2\log_2 q.\left(\frac{n-l}{d}+1\right)^d 2^{d(1+\epsilon)} q^{\frac{1}{2}\dim x^{\bar{G}}}$$

and the desired conclusion follows via (3) and (5).

Case 1.2. $r > 2$, $|H^1(\sigma, E/E^0)| = r$.

Here $c = 1$, r divides n and $E = C_{\bar{G}}(x)$ is non-connected (see [4, 3.34, 3.35]). Furthermore, $\dim x^{\bar{G}} = n^2(1 - 1/r)$ and the hypothesis $x^G \cap H \subseteq B$ implies that r does not divide $t!$, whence $r \geqslant 5$ if $t \geqslant 3$. In addition, the proof of [4, 3.35] implies that x lifts to an element $\hat{x} \in \mathrm{GL}_n^{\epsilon}(q)$ which is $\mathrm{GL}_n^{\epsilon}(q)$-conjugate to

$$\begin{pmatrix} & \lambda^j I_{n/r} \\ I_{n-n/r} & \end{pmatrix} \tag{13}$$

for some unique integer $0 \leqslant j \leqslant r-1$, where $Z(\mathrm{GL}_n^{\epsilon}(q)) = \langle \lambda I_n \rangle$. If $j = 0$ then $\hat{x}$ is $\mathrm{GL}_n^{\epsilon}(q)$-conjugate to the diagonal matrix $[I_{\frac{n}{r}}, \omega I_{\frac{n}{r}}, \ldots, \omega^{r-1} I_{\frac{n}{r}}] \in \mathrm{GL}_n^{\epsilon}(q)$, where ω is a primitive rth root of unity. In this case, [4, 3.35] gives

$$\left|x^G\right| \geqslant \frac{1}{r}\left|\mathrm{GL}_n^{\epsilon}(q) : \mathrm{GL}_{n/r}^{\epsilon}(q)^r\right| > \frac{1}{2r}\left(\frac{q}{q+1}\right)^{\frac{1}{2}(r-1)(1-\epsilon)} q^{n^2(1-\frac{1}{r})} \tag{14}$$

and we claim that

$$\left|x^G \cap H\right| < \left(\frac{n}{r}+1\right)^{(r-1)(t-1)} 2^{\frac{t}{2}(r-1)(1+\epsilon)} q^{\frac{1}{t}n^2(1-\frac{1}{r})}.$$

To see this, first observe that each $y \in x^G \cap H$ lifts to an element $\hat{y} \in \hat{B}$ of order r. Then appealing to Proposition 2.1 we deduce that

$$\left|y^B\right| \leqslant \left|\hat{y}^{\hat{B}}\right| < 2^{\frac{t}{2}(r-1)(1+\epsilon)} q^{\frac{1}{t}n^2(1-\frac{1}{r})}$$

and the claim follows since there are at most $(n/r+1)^{(r-1)(t-1)}$ distinct ways to partition the eigenvalue set $\mathcal{E}_{\hat{x}}$ into precisely t subsets (see (9), for example). These bounds are sufficient unless $(n,t,r) = (6,2,3)$ and $q = 3+\epsilon$, where direct calculation yields $f(x,H) < 0.537$.

Finally, if $1 \leqslant j \leqslant r-1$ then [4, 3.35] implies that

$$\left|x^G\right| \geqslant \frac{|\mathrm{GL}_n^{\epsilon}(q)|}{|\mathrm{GL}_{n/r}^{\epsilon}(q^r)|r} > \frac{1}{2r} q^{n^2(1-\frac{1}{r})} \tag{15}$$

and applying [4, 3.51] we deduce that

$$\begin{aligned}\left|x^G \cap H\right| &\leqslant \sum_{j=1}^{r-1} \left|\hat{z}_j^{\mathrm{GL}_{n/t}^{\epsilon}(q)}\right|^t = (r-1)\left(\frac{|\mathrm{GL}_{n/t}^{\epsilon}(q)|}{|\mathrm{GL}_{n/tr}^{\epsilon}(q^r)|}\right)^t \\ &< (r-1).2^{\frac{t}{2}(1+\epsilon)}\left(\frac{q+1}{q}\right)^{\frac{t}{2}(1-\epsilon)} q^{\frac{1}{t}n^2(1-\frac{1}{r})},\end{aligned}$$

where

$$\hat{z}_j = \begin{pmatrix} & \lambda^j I_{n/tr} \\ I_{n/t-n/tr} & \end{pmatrix} \in \mathrm{GL}_{n/t}^{\epsilon}(q). \tag{16}$$

These bounds are always sufficient.

Case 1.3. $r = 2$.

Write $s = \nu(x)$ for the codimension of the largest eigenspace of x on the natural $\bar{G}$-module (see [4, 3.16]) and note that the hypothesis $x^G \cap H \subseteq B$ implies that $s < n/t$. In particular, $C_{\bar{G}}(x)$ is connected and each $y \in x^G \cap H$ lifts to an involution $\hat{y} \in \hat{B}$. Now, $\dim x^{\bar{G}} = 2s(n-s)$ and applying Proposition 2.1 we deduce that

$$\left|x^G \cap H\right| < \binom{t+s-1}{s} 2^t q^{\frac{2s}{t}(n-s)}, \qquad \left|x^G\right| > \frac{1}{2}\left(\frac{q}{q+1}\right)^{\frac{1}{2}(1-\epsilon)} q^{2s(n-s)}.$$

If $t = 2$ then one can check that these bounds are sufficient unless $(s,q) = (1,3)$ or $(n,s,q) \in \{(6,2,3),(4,1,5)\}$. If $(s,q) = (1,3)$ then the bounds

$$\begin{aligned}\left|x^G \cap H\right| &\leqslant 2\left(\frac{|\mathrm{GL}_{n/2}^{\epsilon}(3)|}{|\mathrm{GL}_{n/2-1}^{\epsilon}(3)||\mathrm{GL}_1^{\epsilon}(3)|}\right) = 2\left(\frac{3^{n/2-1}(3^{n/2}-\epsilon^{n/2})}{3-\epsilon}\right), \\ \left|x^G\right| &\geqslant \frac{|\mathrm{GL}_n^{\epsilon}(3)|}{|\mathrm{GL}_{n-1}^{\epsilon}(3)||\mathrm{GL}_1^{\epsilon}(3)|} = \frac{3^{n-1}(3^n-\epsilon^n)}{3-\epsilon}\end{aligned}$$

are good enough, while the remaining two cases are easily dealt with through direct calculation. If $t \geqslant 3$ and $n \geqslant 2t$ then it remains to deal with the case $(n, t, s, q) = (6, 3, 1, 3)$. Here it is easy to check that $f(x, H) < 0.315$.

Case 2. $x^G \cap (H - B) \neq \emptyset$.

Here $x^G \cap B\pi$ is non-empty for some non-trivial permutation $\pi \in S_t$ of order r and cycle-shape $(r^{h(\pi)}, 1^{t-h(\pi)r})$. Set

$$h = \max\{h(\pi)\colon \pi \in S_t \text{ and } x^G \cap B\pi \neq \emptyset\} \tag{17}$$

and fix $\pi \in S_t$ such that $h(\pi) = h$. Referring to the decomposition $V = V_1 \oplus \cdots \oplus V_t$, we may assume π fixes each subspace V_j with $j \geqslant hr + 1$. If $|H^1(\sigma, E/E^0)| = 1$, where $E = C_{\bar{G}}(x)$, and $y \in B\rho$ is G-conjugate to x with $\rho \in S_t$ of cycle-shape $(r^k, 1^{t-kr})$, then [4, 3.11] implies that y lifts to an element $\hat{y} = (\hat{y}_1, \ldots, \hat{y}_t)\rho \in \mathrm{GL}_n^\epsilon(q)$ of order r and the proof of [10, 4.5] reveals that $\hat{y}$ is $\hat{B}$-conjugate to $(I_{n/t}, \ldots, I_{n/t}, \hat{y}_{kr+1}, \ldots, \hat{y}_t)\rho$. Therefore

$$|y^B| \leqslant |\hat{y}^{\hat{B}}| = |\mathrm{GL}_{n/t}^\epsilon(q)|^{k(r-1)} \prod_{j>kr} |\hat{y}_j^{\mathrm{GL}_{n/t}^\epsilon(q)}| \tag{18}$$

and

$$\dim x^{\bar{G}} \geqslant \dim \pi^{\bar{G}} = n^2 h(r-1)\frac{1}{t}\left(2 - \frac{hr}{t}\right). \tag{19}$$

Case 2.1. $r > 2$, $|H^1(\sigma, E/E^0)| = 1$, $c > 1$.

Define the integers i, c and α as in Case 1 and observe that

$$|x^G| > \frac{1}{2}\left(\frac{q}{q+1}\right)^{\alpha(r-1)} q^{\dim x^{\bar{G}}}. \tag{20}$$

If $n = t$ then the hypothesis $c > 1$ implies that $x^G \cap B\rho \neq \emptyset$ if and only if $\rho \in S_t$ has cycle-shape $(r^h, 1^{t-hr})$, whence $\dim x^{\bar{G}} = nh(r-1)(2 - hr/n)$ and applying (18) and Lemma 2.3 we deduce that

$$|x^G \cap H| = |x^G \cap (H - B)| < 2\log_2 q.\left(\frac{n!}{h!(n-hr)!r^h}\right)(q-\epsilon)^{h(r-1)}. \tag{21}$$

This bound with (20) is sufficient unless $(n, h, r, q, \epsilon) = (5, 1, 5, 2, -)$, where direct calculation yields $f(x, H) < 0.552$.

Now assume $n \geqslant 2t$. We claim that

$$|x^G \cap H| < 2\log_2 q.2\left(\frac{t^r}{r}\right)^h \left(\frac{r-1}{c} + 1\right)^{\frac{n}{c}} \left(\frac{q+1}{q}\right)^{\frac{t}{2}(1-\epsilon)} 2^{\frac{t}{c}(r-1)(1-\alpha)} q^{\frac{1}{t}\dim x^{\bar{G}}}. \tag{22}$$

To see this, first observe that $|x^G \cap H| \leqslant \sum_{k=0}^{h} |\rho_k^{S_t}| M_k N_k$, where $\rho_k \in S_t$ has cycle-shape $(r^k, 1^{t-rk})$, M_k is the size of the largest B-class in $x^G \cap B\rho_k$ and N_k is the number of distinct B-classes in $x^G \cap B\rho_k$. Applying Proposition 2.1 and [4, 3.9] we deduce that

$$M_k < M_k' = \left(\frac{q+1}{q}\right)^{\frac{1}{2}(t-rk)(1-\epsilon)} 2^{\frac{1}{c}(t-rk)(r-1)(1-\alpha)} q^{\frac{1}{t}\dim x^{\bar{G}}}$$

and arguing as before (see (7)) we have

$$N_k \leqslant N_k' = 2\log_2 q.\left(\frac{r-1}{c}+1\right)^{\frac{1}{c}(n-\frac{nrk}{t})}.$$

Therefore Lemma 2.3 gives

$$\left|x^G \cap H\right| < \sum_{k=0}^{h} \frac{t!}{k!(t-rk)!r^k} M_0' N_0' < \sum_{k=0}^{h} \left(\frac{t^r}{r}\right)^k M_0' N_0' < 2\left(\frac{t^r}{r}\right)^h M_0' N_0'$$

and (22) follows. It is easy to check that the bounds (19), (20) and (22) are always sufficient (note that we may assume $n \geqslant 3t$ if $(\epsilon, q) = (+, 2)$—see Table 2.1).

Case 2.2. $r > 2$, $|H^1(\sigma, E/E^0)| = 1$, $c = 1$.

Here $d = r-1$ and $r \leqslant n \leqslant rl$ since the σ-orbit of each rth root of unity in K is a singleton set. Also note that $t > hr$ (if $t = hr$ then $C_{\bar{G}}(x)$ is non-connected and [4, 3.35] implies that $|H^1(\sigma, E/E^0)| = r$). First suppose $n = t$. Then (19) and (20) hold and appealing to (21) and (22) we get

$$\left|x^G \cap H\right| < 2\log_2 q.2\left(\frac{t^r}{r}\right)^h (q-\epsilon)^{h(r-1)} r^t.$$

If $(\epsilon, q) \neq (-, 2)$ then these bounds are almost always sufficient and the exceptional cases are easily dealt with through direct calculation. If $(\epsilon, q) = (-, 2)$ then $r = 3$,

$$\left|x^G \cap H\right| \leqslant 2\sum_{k=0}^{h}\left[\left(\frac{n!}{k!(n-3k)!3^k}\right)3^{2k}.3^{n-3k}\right] < 4.3^{n-2h}n^{3h}$$

and thus (19) and (20) are sufficient for all $h \geqslant 3$, while we are left to deal with the cases $n \in \{7, 8, 9\}$ when $h = 2$. Here we calculate directly. Finally, if $h = 1$ then the maximality of h implies that x is $\bar{G}$-conjugate to $[I_l, \omega I_{n-l-1}, \omega^2]$ and thus

$$\left|x^G \cap H\right| \leqslant 2\left(\frac{n!}{l!(n-l-1)!} + \frac{n!}{(n-3)!3}3^2\binom{n-3}{l-1}\right), \qquad \left|x^G\right| > \frac{2}{9}2^{2nl+4n-2l^2-6l-6}.$$

These bounds are always sufficient if $n \geqslant 6$, while direct calculation gives $f(x, H) < 0.599$ when $n = 5$. Similarly, we get $f(x, H) < 0.718$ if $n = 4$.

Finally, if $n \geqslant 2t$ then (19) and (20) hold, and (22) is valid on substituting $c = 1$. The reader can check that these bounds suffice.

Case 2.3. $r > 2$, $|H^1(\sigma, E/E^0)| = r$.

First assume x lifts to an element $\hat{x} \in \mathrm{GL}_n^{\epsilon}(q)$ of order r. Then (14) holds and appealing to Proposition 2.1 and the proof of [10, 4.5] we deduce that

$$\begin{aligned}\left|x^G \cap H\right| &< \sum_{k=0}^{\lfloor t/r \rfloor}\left[\frac{t!}{k!(t-rk)!r^k}\left(\frac{q+1}{q}\right)^{\frac{1}{2}k(r-1)(1-\epsilon)} r^{n(1-\frac{rk}{t})}2^{\frac{1}{2}(r-1)(t-rk)(1+\epsilon)}\right]q^{\frac{1}{t}n^2(1-\frac{1}{r})}\\ &< 2\left(\frac{t^r}{r}\right)^{\frac{t}{r}}\left(\frac{q+1}{q}\right)^{\frac{t}{2r}(r-1)(1-\epsilon)} r^n 2^{\frac{t}{2}(r-1)(1+\epsilon)}q^{\frac{1}{t}n^2(1-\frac{1}{r})}.\end{aligned}$$

If $\epsilon = +$ and $n > t$ then these bounds are sufficient unless $(n, t, r, q) = (6, 3, 3, 4)$, where direct calculation yields $f(x, H) < 0.356$. Similarly, if $\epsilon = -$ and $n > t$ then we are left to deal with the case $(t, r, q) = (3, 3, 2)$ for $n \leqslant 12$. Here the more accurate bounds

$$\left|x^G \cap H\right| \leqslant \frac{n!}{(n/3)!^3}2^{\frac{2}{9}n^2} + 2\left|\mathrm{GU}_{n/3}(2)\right|^2, \qquad \left|x^G\right| \geqslant \frac{|\mathrm{GU}_n(2)|}{|\mathrm{GU}_{n/3}(2)|^3 3}$$

are good enough. If $n = t$ then it remains to deal with a handful of cases (n, r). Here the desired result quickly follows since

$$\left|x^G \cap H\right| \leqslant \sum_{k=0}^{\lfloor t/r \rfloor}\left[\frac{t!}{k!(t-rk)!r^k}(q-\epsilon)^{k(r-1)}\frac{(n-rk)!}{(n/r-k)!^r}\right], \qquad \left|x^G\right| \geqslant \frac{1}{r}\left|\mathrm{GL}_n^{\epsilon}(q) : \mathrm{GL}_{n/r}^{\epsilon}(q)^r\right|.$$

Now assume x lifts to an element $\hat{x} \in \mathrm{GL}_n^{\epsilon}(q)$ as in (13), with $j \geqslant 1$. Write $\hat{x} = (\hat{x}_1, \ldots, \hat{x}_t)\rho$, where $\hat{x}_i \in \mathrm{GL}_{n/t}^{\epsilon}(q)$ and $\rho \in S_t$ has cycle-shape $(r^k, 1^{t-rk})$. First assume tr divides n and note that if ρ induces the permutation $\prod_{i=1}^{k}((i-1)r+1 \ldots ir)$ on the coordinates and $i > kr$ then $\hat{x}_i$ is $\mathrm{GL}_{n/t}^{\epsilon}(q)$-conjugate to $\hat{z}_j$, where $\hat{z}_j$ is defined as in (16). Since $\hat{x}^r = \lambda^j I_n$ we have

$$\hat{x}_1 \ldots \hat{x}_r = \hat{x}_{r+1} \ldots \hat{x}_{2r} = \cdots = \hat{x}_{(k-1)r+1} \ldots \hat{x}_{kr} = \lambda^j I_{n/t}$$

and arguing as in the proof of [10, 4.5] we deduce that $\hat{x}$ is $\hat{B}$-conjugate to $b\rho$, where

$$b = \left(I_{n/t}, \ldots, I_{n/t}, \lambda^j I_{n/t}, \ldots, I_{n/t}, \ldots, I_{n/t}, \lambda^j I_{n/t}, \hat{x}_{kr+1}, \ldots, \hat{x}_t\right) \in \hat{B}.$$

Therefore

$$\begin{aligned}\left|x^G \cap H\right| &\leqslant (r-1)\sum_{k=0}^{\lfloor t/r \rfloor}\left[\frac{t!}{k!(t-kr)!r^k}\left|\mathrm{GL}_{n/t}^{\epsilon}(q)\right|^{k(r-1)}\left(\frac{|\mathrm{GL}_{n/t}^{\epsilon}(q)|}{|\mathrm{GL}_{n/tr}^{\epsilon}(q^r)|}\right)^{t-kr}\right]\\ &< (r-1).2\left(\frac{t^r}{r}\right)^{\frac{t}{r}}\left(\frac{q+1}{q}\right)^{\frac{t}{2}(1-\epsilon)}2^{\frac{t}{2}(1+\epsilon)}q^{\frac{1}{t}n^2(1-\frac{1}{r})} \qquad (23)\end{aligned}$$

and one can check that (15) is always sufficient.

Finally, let us assume n is not divisible by tr. Then r divides t and $x^G \cap B\rho$ is non-empty if and only if ρ has cycle-shape $(r^{t/r})$. Therefore

$$\begin{aligned}\left|x^G \cap H\right| &\leqslant (r-1)\frac{t!}{(t/r)!r^{t/r}}\left|\mathrm{GL}^{\epsilon}_{n/t}(q)\right|^{\frac{t}{r}(r-1)} \\ &< (r-1)\frac{t!}{(t/r)!r^{t/r}}\left(\frac{q+1}{q}\right)^{\frac{t}{2r}(r-1)(1-\epsilon)} q^{\frac{n^2}{tr}(r-1)} \qquad (24)\end{aligned}$$

and once again the result follows via (15).

Case 2.4. $r = 2$.

Write $s = \nu(x)$ and observe that $s = nh/t + j$ for a non-negative integer $j < n/t$. Let us first assume $s < n/2$. Then $C_{\bar{G}}(x)$ is connected, $t > 2h$ and

$$\left|x^G\right| > \frac{1}{2}\left(\frac{q}{q+1}\right)^{\frac{1}{2}(1-\epsilon)} q^{\dim x^{\bar{G}}}, \qquad (25)$$

where $\dim x^{\bar{G}} = 2s(n-s)$ (see [4, Table 3.8]). Arguing as before we deduce that

$$\left|x^G \cap H\right| < 2\binom{t^2}{2}^h 2^{\frac{t}{2}(1+\epsilon)} 2^n q^{\frac{1}{t}\dim x^{\bar{G}}} \qquad (26)$$

and we find that the bounds (19), (25) and (26) are always sufficient if $n \geqslant 2t$ and $h \geqslant 2$. If $h = 1$ then

$$\left|x^G \cap H\right| < \left(2^{\frac{1}{2}(1+\epsilon)}\binom{t}{2}\binom{t-3+j}{j} + \binom{n/t+j+t-1}{n/t+j}\right) 2^{\frac{1}{2}(t-2)(1+\epsilon)} q^{\frac{2s}{t}(n-s)}$$

and (25) is sufficient if $n \geqslant 2t$. If $n = t$ then the maximality of h implies that $s = h$, whence

$$\left|x^G \cap H\right| \leqslant \sum_{k=0}^{h}\left[\frac{n!}{k!(n-2k)!2^k}\binom{n-2k}{h-k}(q-\epsilon)^k\right] < \binom{n}{h} + \frac{n!}{(n-2h)!}(q-\epsilon)^h \qquad (27)$$

and again the desired result follows via (25).

For the remainder we may assume $s = n/2$, so $C_{\bar{G}}(x)$ is non-connected and

$$\left|x^G\right| > \frac{1}{4}\left(\frac{q}{q+1}\right)^{\frac{1}{2}(1-\epsilon)} q^{\frac{1}{2}n^2}. \qquad (28)$$

Suppose $C_G(x)$ is of type $\mathrm{GL}^{\epsilon}_{n/2}(q)^2$, so x lifts to an involution $\hat{x} \in \mathrm{GL}^{\epsilon}_n(q)$. If $n = t$ then (27) holds (with $h = n/2$) and this bound with (28) is always sufficient. Similarly, if $n \geqslant 2t$ then (26) holds (with $h = \lfloor t/2 \rfloor$) and the result quickly follows if $t \geqslant 3$. If $t = 2$ then

$$\left|x^G \cap (H-B)\right| \leqslant \left|\mathrm{GL}^{\epsilon}_{n/2}(q)\right| \leqslant (q+1)q^{\frac{1}{4}n^2-1},$$

$$\left|x^G \cap B\right| \leqslant \sum_{l=0}^{n/2}\left[\left(\frac{|\mathrm{GL}^{\epsilon}_{n/2}(q)|}{|\mathrm{GL}^{\epsilon}_{l}(q)||\mathrm{GL}^{\epsilon}_{n/2-l}(q)|}\right)^2\right] < 4\sum_{l=0}^{n/2} q^{2l(n-2l)} < 4\left(\frac{q^2+1}{q^2-1}\right)q^{\frac{1}{4}n^2}$$

and (28) is sufficient unless $(n,q)=(4,3)$, where direct calculation yields $f(x,H)<0.617$.

Finally, let us assume $C_G(x)$ is of type $\mathrm{GL}_{n/2}(q^2)$. If n/t is even then arguing as before (see (23)) we deduce that

$$\begin{aligned}\left|x^G\cap H\right| &\leqslant \sum_{k=0}^{\lfloor t/2\rfloor}\left[\frac{t!}{k!(t-2k)!2^k}\left|\mathrm{GL}^{\epsilon}_{n/t}(q)\right|^k\left(\frac{|\mathrm{GL}^{\epsilon}_{n/t}(q)|}{|\mathrm{GL}_{n/2t}(q^2)|}\right)^{t-2k}\right]\\ &< 2\left(\frac{t^2}{2}\right)^{\frac{t}{2}}\left(\frac{q+1}{q}\right)^{\frac{t}{2}(1-\epsilon)}2^t q^{\frac{n^2}{2t}}\end{aligned}$$

and (28) is sufficient unless $(n,t)=(4,2)$ or $(\epsilon,n,t)=(-,6,3)$. If $(n,t)=(4,2)$ then

$$\begin{aligned}\left|x^G\cap H\right| &\leqslant \left|\mathrm{GL}^{\epsilon}_2(q):\mathrm{GL}_1(q^2)\right|^2+\left|\mathrm{GL}^{\epsilon}_2(q)\right| = q^2(q-\epsilon)^2+q(q-\epsilon)(q^2-1),\\ \left|x^G\right| &\geqslant \frac{1}{2}\left|\mathrm{GL}^{\epsilon}_4(q):\mathrm{GL}_2(q^2)\right| = \frac{1}{2}q^4(q-\epsilon)(q^3-\epsilon)\end{aligned}$$

and we conclude that $f(x,H)<0.651$ for all $q\geqslant 3$. Similarly, we get $f(x,H)<0.550$ if $(\epsilon,n,t)=(-,6,3)$. Finally, if n/t is odd then t is even, (24) holds (with $r=2$) and the desired result follows via (28). □

Proposition 2.6. *The conclusion to Theorem* 1.1 *holds in case* (i) *of Table* 2.1 *for unipotent elements in* $H\cap\mathrm{PGL}(V)$.

Proof. Let $x\in H\cap\mathrm{PGL}(V)$ be an element of order p, with associated partition $\lambda\vdash n$ (see [4, §3.3]). Write $\hat{B}=\mathrm{GL}^{\epsilon}_{n/t}(q)^t$ and define h as in (17) (setting $r=p$), so $h=0$ if and only if $x^G\cap H\subseteq B$. Fix $\pi\in S_t$ with cycle-shape $(r^h,1^{t-hr})$ and suppose π induces the permutation $\prod_{i=1}^h((i-1)p+1,\dots,ip)$ on coordinates. If $y\in B\pi$ is G-conjugate to x then [4, 3.11] implies that y lifts to a unique element $\hat{y}=(\hat{y}_1,\dots,\hat{y}_t)\pi\in\hat{B}\pi$ of order p which is $\hat{B}$-conjugate to $(I_{n/t},\dots,I_{n/t},\hat{y}_{hp+1},\dots,\hat{y}_t)\pi$. Furthermore, (18) holds and

$$\lambda=\left(p^{\frac{nh}{t}+b_p},(p-1)^{b_{p-1}},\dots,1^{b_1}\right),\tag{29}$$

where the restriction of y to $V_{hp+1}\oplus\cdots\oplus V_t$ has associated partition $\lambda'=(p^{b_p},\dots,1^{b_1})\vdash n(t-hp)/t$.

Case 1. $x^G\cap H\subseteq B$.

We begin with two special cases; the general case will be considered in Case 1.3.

Case 1.1. $\lambda=(k^{n/k})$.

Here $2\leqslant k\leqslant p$ and k divides n/t since $x^G\cap H\subseteq B$. Furthermore, the hypotheses imply that p does not divide t if $k=p$. Applying Proposition 2.1 and [4, 3.18, 3.20(i)] we deduce that

$$\left|x^G\cap H\right|<\left(\frac{q+1}{q}\right)^{\frac{t}{2}(1-\epsilon)}q^{\frac{1}{t}\dim x^{\bar{G}}},\qquad \left|x^G\right|>\frac{1}{2}\left(\frac{q}{q+1}\right)^{\frac{1}{2}(1-\epsilon)}q^{\dim x^{\bar{G}}-1}$$

and [6, 2.4] implies that $\dim x^{\bar{G}}\geqslant\frac{1}{2}n^2$ (minimal if $k=2$). The result follows.

Case 1.2. $\lambda = (2^j, 1^{n-2j})$.

We may assume $j < n/2$ and thus [4, 3.20(i)] yields

$$\left|x^G\right| > \frac{1}{2}\left(\frac{q}{q+1}\right)^{\frac{1}{2}(1-\epsilon)} q^{\dim x^{\bar{G}}}, \tag{30}$$

where $\dim x^{\bar{G}} = 2j(n-j)$. Note that the prime order hypothesis on x implies that λ must have this form if $p = 2$, in which case $j < n/t$ since $x^G \cap H \subseteq B$. If $j = 1$ then

$$\left|x^G \cap H\right| < t.2^{\frac{1}{2}(1+\delta_{2,q})(1+\epsilon)} q^{2(\frac{n}{t}-1)}$$

and (30) is always sufficient if $t \geqslant 3$, while the bounds

$$\left|x^G \cap H\right| \leqslant 2(q-\epsilon)^{-1}\left(q^{n/2-1}-\epsilon\right)\left(q^{n/2}-1\right), \qquad \left|x^G\right| \geqslant (q-\epsilon)^{-1}\left(q^{n-1}-1\right)\left(q^n-\epsilon\right)$$

are good enough if $t = 2$. Now assume $j \geqslant 2$. Applying (18), Proposition 2.1 and [4, 3.18] we deduce that

$$\left|x^G \cap H\right| < \binom{t+j-1}{j} 2^{\frac{t}{2}(1+\delta_{2,q})(1+\epsilon)} \left(\frac{q+1}{q}\right)^{\frac{t}{2}(1-\epsilon)} q^{\frac{2j}{t}(n-j)}$$

since each B-class in $x^G \cap B$ is determined by a distribution of the j Jordan blocks of size two among the t direct factors in B. If $t \geqslant 3$ then this bound with (30) is sufficient unless $(n, t, j, q) = (9, 3, 2, 2)$, where direct calculation yields $f(x, H) < 0.354$. Now assume $t = 2$. Arguing as in the proof of Proposition 2.5 (see (10)) we deduce that

$$\left|x^G \cap H\right| < 2^{(1+\delta_{2,q})(1+\epsilon)} \left(\frac{q+1}{q}\right)^{1-\epsilon} \left(\frac{q^2+1}{q^2-1}\right) q^{j(n-j)}. \tag{31}$$

First assume $(q, \epsilon) = (2, +)$. Then $|x^G| > 2^{2j(n-j)-1}$ and if we assume $j \geqslant 4$ then (31) is sufficient unless $(n, j) = (10, 4)$, where direct calculation yields $f(x, H) < 0.514$. If $j = 3$ then there are at most two essentially distinct ways to write $(2^3, 1^{n-6})$ as a sum of two partitions of $n/2$ and we deduce that $|x^G \cap H| < 16.2^{3n-10} + 8a.2^{3n-18}$, where $a = 1$ if $n \geqslant 12$, otherwise $a = 0$. The result now follows since $|x^G| > 2^{6n-19}$. The case $j = 2$ is very similar. Finally, if $t = 2$ and $(q, \epsilon) \neq (2, +)$ then the bounds (30) and (31) are sufficient unless $(\epsilon, n, j, q) = (-, 6, 2, 2)$. Here direct calculation yields $f(x, H) < 0.399$.

Case 1.3. *General* λ.

Write $\lambda = (m^{a_m}, \ldots, 2^{a_2}, 1^l) \vdash n$, where $m = n/t$. In view of Case 1.2, we may assume $p > 2$. Let $d \geqslant 1$ be the number of non-zero terms a_j in λ and observe that

$$\left|x^G\right| > \frac{1}{2}\left(\frac{q}{q+1}\right)^{\frac{1}{2}(1-\epsilon)(d+1)} q^{\dim x^{\bar{G}}-1}. \tag{32}$$

If $d=1$ and $a_k>0$ then we may assume $k>2$ and $l>0$. Then Proposition 2.1 implies that

$$\left|x^G\cap H\right|<\binom{t-1+(n-l)/k}{(n-l)/k}2^t q^{\frac{1}{t}\dim x^{\bar{G}}},$$

where $n\geqslant \max(l+k,tk)$ and $\dim x^{\bar{G}}=(n^2-l^2)(1-1/k)$, and we find that (32) is sufficient unless $(n,t,l,k,q)=(6,2,3,3,3)$, where direct calculation yields $f(x,H)<0.370$. Now assume $d\geqslant 2$. We claim that

$$n\geqslant \max\left(t(d+1),\frac{1}{2}d^2+\frac{3}{2}d+l\right) \tag{33}$$

and

$$\dim x^{\bar{G}}\geqslant \frac{1}{2}n^2+\frac{1}{2}\left(d^2-d\right)n-\frac{1}{8}d^4-\frac{1}{12}d^3+\frac{3}{8}d^2-\frac{1}{6}d-\frac{1}{2}l^2. \tag{34}$$

To see this, suppose $r_1>\cdots>r_d\geqslant 2$ are the indices with $a_{r_k}>0$. Since $x^G\cap H\subseteq B$ we have $n/t\geqslant r_1\geqslant d+1$ and (33) follows since

$$n=l+\sum_{j=1}^{d}r_ja_{r_j}\geqslant l+\sum_{j=2}^{d+1}j=l+\frac{1}{2}(d+1)(d+2)-1.$$

The lower bound on $\dim x^{\bar{G}}$ follows from [6, 2.3, 2.4]. For example, if $\alpha=\frac{1}{2}(2n-2l-d^2-3d+4)$ is even and l and d are fixed then $(d+1,d,\dots,3,2^{\alpha/2},1^l)\vdash n$ is the least possible partition of n (with respect to the familiar dominance ordering on partitions) and the result follows via [6, 2.3, 2.4]. Next we claim that

$$\left|x^G\cap H\right|<2^{\frac{1}{2}td(1+\epsilon)}\left(\frac{q+1}{q}\right)^{\frac{t}{2}(1-\epsilon)}\left(\frac{n/2-d^2/4+d/4-l/2-1}{d}+1\right)^{d(t-1)}q^{\frac{1}{t}\dim x^{\bar{G}}}. \tag{35}$$

In view of (18), Proposition 2.1 and [4, 3.18] it is sufficient to show that the number N of B-classes in $x^G\cap B$ satisfies $N\leqslant Y^{d(t-1)}$, where

$$Y=\frac{n/2-d^2/4+d/4-l/2-1}{d}+1.$$

Such a B-class is determined by a choice of t partitions $\lambda_i\vdash n/t$ with $\lambda=\lambda_1\oplus\cdots\oplus\lambda_t$. It is clear that λ_t is uniquely determined once $\lambda_1,\dots,\lambda_{t-1}$ have been chosen, whence $N\leqslant M^{t-1}$, where M is the number of choices for λ_1. If $r_1>\cdots>r_d\geqslant 2$ are the indices with $a_{r_k}>0$ then λ_1 is uniquely determined by a choice of d-tuple $(x_1,\dots,x_d)$, where $0\leqslant x_j\leqslant a_{r_j}$ for each j. Of course, if M' denotes the number of all such d-tuples then

$$M\leqslant M'=\prod_{j=1}^{d}(a_{r_j}+1)\leqslant\left(\frac{\sum_j a_j}{d}+1\right)^d$$

and thus $M\leqslant Y^d$ since $\sum_j a_j$ is maximal when a_2 is as large as possible.

Calculating, we find that the bounds (32), (34) and (35) are sufficient unless $(\epsilon, t, d, q) = (+, 2, 2, 3)$ and $(n, l) \in \{(8, 3), (8, 1), (6, 1)\}$. These cases are easily settled via direct calculation.

Case 2. $x^G \cap (H - B) \neq \emptyset$.

Define $h > 0$ as in (17). Referring to (29), we observe that $\dim x^{\bar{G}}$ is minimal if $b_j = 0$ for all $j > 0$ and thus (19) holds (with $r = p$). Also note that [4, 3.20(i)] gives $|x^{\bar{G}_\sigma}| = |x^{G_0}|$.

Case 2.1. $n = t$.

Here $\lambda = (p^h, 1^{n-hp})$ so (30) holds and

$$\left|x^G \cap H\right| = \left|x^G \cap (H - B)\right| \leqslant \frac{n!}{h!(n - hp)!p^h}(q - \epsilon)^{h(p-1)}$$

since x^G meets $B\pi$ if and only if π has cycle-shape $(p^h, 1^{n-hp})$. If $\epsilon = +$ then we may assume $q \geqslant 4$ (see Table 2.1) and we find that the above bounds with (19) are always sufficient. The same is true if $\epsilon = -$ and $q > 2$. Finally, if $(\epsilon, q) = (-, 2)$ then we are left to deal with the following cases:

(n, h)	$\lvert x^G \cap H\rvert$	$\lvert x^G\rvert$	$f(x, H) <$
(5, 1)	30	165	0.667
(4, 1)	18	45	0.760*

These results are obtained through direct calculation. The asterisk appearing in the last row indicates that the case $(n, t, h, q) = (4, 4, 1, 2)$ is an exception to the main statement of Theorem 1.1 and is therefore recorded in Table 1.1.

Case 2.2. $n \geqslant 2t$, $p = 2$.

First assume $h = 1$, so $\lambda = (2^{n/t+j}, 1^{n-2n/t-2j})$ for some non-negative integer $j \leqslant n(1/2 - 1/t)$. Then arguing as before we deduce that

$$\left|x^G \cap B\right| < \binom{n/t + j + t - 1}{n/t + j} 2^{\frac{t}{2}(1+\delta_{2,q})(1+\epsilon)} \left(\frac{q+1}{q}\right)^{\frac{t}{2}(1-\epsilon)} q^{\frac{1}{t}\dim x^{\bar{G}}}$$

and

$$\left|x^G \cap (H - B)\right| < \binom{t}{2}\binom{t - 3 + j}{j} 2^{\frac{1}{2}(t-2)(1+\delta_{2,q})(1+\epsilon)} \left(\frac{q+1}{q}\right)^{\frac{1}{2}(t-1)(1-\epsilon)} q^{\frac{1}{t}\dim x^{\bar{G}}},$$

where $\dim x^{\bar{G}} = 2(n/t + j)(n - n/t - j)$. If $t \geqslant 3$ then the result follows via (30) so assume $t = 2$. Then

$$\left|x^G \cap (H - B)\right| \leqslant \left|\mathrm{GL}^{\epsilon}_{n/2}(q)\right| < \left(\frac{q+1}{q}\right)^{\frac{1}{2}(1-\epsilon)} q^{\frac{1}{4}n^2}, \qquad \left|x^G\right| > \frac{1}{2}\left(\frac{q}{q+1}\right)^{\frac{1}{2}(1-\epsilon)} q^{\frac{1}{2}n^2}$$

and either $x^G \cap B$ is empty, or $n \equiv 0\,(4)$ and

$$\left|x^G \cap B\right| < \left(\frac{|\mathrm{GL}^{\epsilon}_{n/2}(q)|}{|\mathrm{GL}^{\epsilon}_{n/4}(q)|q^{n^2/16}}\right)^2 < \left(\frac{q+1}{q}\right)^{1-\epsilon} q^{\frac{1}{4}n^2}.$$

These bounds are always sufficient.

Now assume $h > 1$. Arguing as in the proof of Proposition 2.5 (see (22)) we deduce that

$$\left|x^G \cap H\right| < 2\left(\frac{t^2}{2}\right)^h 2^{\frac{n}{2}} 2^{\frac{t}{2}(1+\delta_{2,q})(1+\epsilon)} \left(\frac{q+1}{q}\right)^{\frac{t}{2}(1-\epsilon)} q^{\frac{1}{t}\dim x^{\bar{G}}}.$$

(Note that if $\rho \in S_t$ has cycle-shape $(2^{nk/t}, 1^{n-2nk/t})$ then the number of B-classes in $x^G \cap B\rho$ is at most N, where N is the number of distinct ways one can distribute $n(h-k)/t$ Jordan 2-blocks among $t-2k$ direct factors. This accounts for the $2^{n/2}$ factor in the above bound since $N \leqslant 2^{n/2-nk/t}$.) The reader can check that this bound with (30) and (19) is sufficient unless $(n,q) = (2t,2)$ and (t,h,ϵ) is one of a handful of possibilities. These exceptional cases are easily dealt with.

Case 2.3. $n \geqslant 2t$, $p > 2$.

Here

$$\left|x^G\right| > \frac{1}{2}\left(\frac{q}{q+1}\right)^{\frac{1}{2}(p-1)(1-\epsilon)} q^{\dim x^{\bar{G}}}$$

and in the usual manner we deduce that

$$\left|x^G \cap H\right| < 2\left(\frac{t^p}{p}\right)^h p^{n+\frac{n}{t}h(1-p)} 2^{\frac{t}{2}(p-1)(1+\epsilon)} \left(\frac{q+1}{q}\right)^{\frac{t}{2}(1-\epsilon)} q^{\frac{1}{t}\dim x^{\bar{G}}}. \tag{36}$$

Applying the lower bound on $\dim x^{\bar{G}}$ given in (19) we find that these bounds are sufficient unless $(n,t,h,q,\epsilon) = (6,3,1,3,+)$. In this case direct calculation yields $f(x,H) < 0.321$. □

Proposition 2.7. *The conclusion to Theorem* 1.1 *holds in case* (i) *of Table* 2.1 *for elements in* $H - \mathrm{PGL}(V)$.

Proof. First assume $x \in G$ is a field automorphism of prime order r, in which case r is odd if $\epsilon = -$ (see [4, 3.42]). Then $q = q_0^r$ and

$$\left|x^G\right| \geqslant \frac{|\mathrm{PSL}^{\epsilon}_n(q)|}{|\mathrm{PGL}^{\epsilon}_n(q^{1/r})|} > \frac{1}{2}(q+1)^{-1} q^{(n^2-1)(1-\frac{1}{r})}. \tag{37}$$

By [4, 3.50] we have $x^G \cap H \subseteq \tilde{H}x$, where $\tilde{H} = B.S_t$, and applying [4, 3.43] we deduce that

$$\left|x^G \cap H\right| \leqslant (q-\epsilon)^{-1} \sum_{j=0}^{\lfloor t/r \rfloor} \left[\left|\rho_j^{S_t}\right| \left|\mathrm{GL}^{\epsilon}_{n/t}(q)\right|^{j(r-1)} \left(\frac{|\mathrm{GL}^{\epsilon}_{n/t}(q)|}{|\mathrm{PGL}^{\epsilon}_{n/t}(q^{1/r})|}\right)^{t-jr}\right], \tag{38}$$

where $\rho_j \in S_t$ has cycle-shape $(r^j, 1^{t-jr})$. In particular, if $n = t$ then

$$\left|x^G \cap H\right| \leqslant (q-\epsilon)^{t-1}\big(i_r(S_t)+1\big) \leqslant (q-\epsilon)^{t-1}t!$$

(where $i_r(S_t)$ denotes the number of elements of order r in S_t) and one can check that (37) is sufficient unless $(r,\epsilon) = (2,+)$ and $t = 2$ or 3. If $t = 2$ then $|x^G \cap H| \leqslant 10(q-1)^3$ since $i_2(S_4) = 9$ and the result follows via (37). Similarly, if $t = 3$ then (38) gives $|x^G \cap H| \leqslant q^2+q-2$ and the bound $|x^G| > \frac{1}{6}q^4$ is always sufficient. Now assume $n \geqslant 2t$. Then (38) implies that

$$\left|x^G \cap H\right| < (q-\epsilon)^{t-1}t!2^t q^{(\frac{n^2}{t}-t)(1-\frac{1}{r})}$$

and we are left to deal with the case $(n,t,r,\epsilon) = (4,2,2,+)$ for $q \in \{4,9,16\}$. Here the result is easily established through direct calculation. For example, if $q = 4$ then $f(x,H) < 0.546$ since

$$\left|x^G \cap H\right| \leqslant \frac{1}{3}\left(\frac{|\mathrm{GL}_2(4)|}{|\mathrm{PGL}_2(2)|}\right)^2 + \left|\mathrm{PGL}_2(4)\right| = 360, \qquad \left|x^G\right| \geqslant \frac{|\mathrm{SL}_4(4)|}{|\mathrm{PGL}_4(2)|} = 48\,960.$$

The argument for involutory graph-field automorphisms is very similar.

Finally, let us assume x is an involutory graph automorphism and assume for now that $n \geqslant 3t$. Then x permutes the t direct factors in B, inducing an involutory graph automorphism on any factor which is fixed. Recall from [4] that x is said to be a *symplectic type* graph automorphism if $C_{G_0}(x)$ has socle $\mathrm{PSp}_n(q)$, otherwise x is *non-symplectic* (see [4, 3.47]). By [4, 3.48] we have

$$\left|x^G\right| > \frac{1}{2}(q+1)^{-1}q^{\frac{1}{2}(n^2+\alpha n-2)}, \tag{39}$$

where $\alpha = 1$ if x is non-symplectic, otherwise $\alpha = -1$. We claim that the following two conditions hold:

(I) If x is symplectic then x induces a symplectic-type graph automorphism on each factor in B which is fixed.
(II) If x is non-symplectic and $p \neq 2$ then x induces a non-symplectic graph automorphism on each fixed factor in B; if $p = 2$ then at least one factor must be fixed and acted on as a non-symplectic graph automorphism.

An easy way to see this is to view the algebraic group $\mathrm{GL}_n(K)$ (where K is the algebraic closure of $\mathbb{F}_q$) as the stabilizer in $\mathrm{Sp}_{2n}(K)$ of a maximal totally singular subspace of the natural $\mathrm{Sp}_{2n}(K)$-module $\bar{V}$ and then calculate the action of x on $\bar{V}$. Then $\nu(x) = n$ (with respect to $\bar{V}$) and it is easy to see that $C_{G_0}(x)$ is symplectic if and only if n is even and x is $\mathrm{Sp}_{2n}(K)$-conjugate to $[-I_n, I_n]$ or a_n, according to the parity of p. Set $\delta = +$ if x is non-symplectic, otherwise $\delta = -$, and suppose x permutes the t factors in B with cycle-shape $(2^j, 1^{t-2j})$ and induces a non-symplectic graph automorphism on precisely $0 \leqslant k \leqslant t-2j$ of the fixed factors. Then x is $\mathrm{Sp}_{2n}(K)$-conjugate to the block-diagonal matrix $[X_\delta^j, Y^k, Z^{t-2j-k}] \in \mathrm{Sp}_{2n}(K)$, where the elements $X_\delta \in \mathrm{Sp}_{4n/t}(K)$ and $Y, Z \in \mathrm{Sp}_{2n/t}(K)$ are given as follows up to conjugacy (here $i \in K$ satisfies $i^2 = -1$ and we adopt the notation of [2] for unipotent involutions in symplectic groups):

	$p \neq 2$	$p = 2$
X_+	$[-iI_{2n/t}, iI_{2n/t}]$	$a_{2n/t}$
X_-	$[-I_{2n/t}, I_{2n/t}]$	$a_{2n/t}$
Y	$[-iI_{n/t}, iI_{n/t}]$	$b_{n/t}$ or $c_{n/t}$
Z	$[-I_{n/t}, I_{n/t}]$	$a_{n/t}$

The conditions (I) and (II) follow immediately.

If $n = 2t$ then x induces a non-trivial automorphism on each fixed direct factor $\mathrm{GL}_2^\epsilon(q)$ in $\hat{B} = \mathrm{GL}_2^\epsilon(q)^t$ which restricts to an inner automorphism i_x of $\mathrm{SL}_2(q)$. In analogy with the case $n \geqslant 3t$, we say that x induces a *symplectic-type* automorphism on a fixed factor if and only if i_x centralizes $\mathrm{SL}_2(q)$, otherwise the action of x on the fixed factor is said to be *non-symplectic*. With this terminology, it is easy to see that conditions (I) and (II) are valid if we omit each occurrence of the term 'graph'. Finally, if $n = t$ then x acts by inversion on each fixed factor and it is easy to see that x does not fix a factor if $C_{G_0}(x)$ is symplectic, while at least one factor is fixed if $C_{G_0}(x)$ is non-symplectic and $p = 2$.

First assume $C_{G_0}(x)$ is symplectic, so n is even. Now, if n/t is odd then t is even and our above comments imply that x permutes the t factors with cycle-shape $(2^{t/2})$. Therefore

$$\left|x^G \cap H\right| \leqslant (q-\epsilon)^{-1} \frac{t!}{(t/2)!2^{t/2}} \left|\mathrm{GL}_{n/t}^\epsilon(q)\right|^{\frac{t}{2}} \leqslant \frac{t!}{(t/2)!2^{t/2}} (q-\epsilon)^{\frac{t}{2}-1} q^{\frac{n^2}{2t}-\frac{t}{2}}$$

and (39) is sufficient unless $q = 2$ and $(n,t) \in \{(6,2),(4,4)\}$. These cases are easily settled through direct calculation. Next assume $C_{G_0}(x)$ is symplectic and n/t is even. In view of condition (I) we deduce that

$$\left|x^G \cap H\right| \leqslant (q-\epsilon)^{-1} \sum_{j=0}^{\lfloor t/2 \rfloor} \left[\left|\rho_j^{S_t}\right| \left|\mathrm{GL}_{n/t}^\epsilon(q)\right|^j \left(\frac{|\mathrm{GL}_{n/t}^\epsilon(q)|}{|\mathrm{Sp}_{n/t}(q)|}\right)^{t-2j}\right] < (q-\epsilon)^{t-1} t! 2^t q^{\frac{n^2}{2t}-\frac{t}{2}}$$

and thus (39) is sufficient if $t \geqslant 3$. If $t = 2$ and $n \geqslant 8$ then

$$\left|x^G \cap H\right| \leqslant (q-\epsilon)\left(\frac{|\mathrm{PGL}_{n/2}^\epsilon(q)|}{|\mathrm{Sp}_{n/2}(q)|}\right)^2 + \left|\mathrm{PGL}_{n/2}^\epsilon(q)\right| < q^{\frac{1}{4}n^2-\frac{1}{2}n-2}\left(4(q-\epsilon)+q^{\frac{1}{2}n+1}\right)$$

and it remains to deal with the case $(n,q,\epsilon) = (8,2,-)$, where a direct calculation yields $f(x,H) < 0.439$. Finally, if $(n,t) = (4,2)$ then $q > 2$ (see Table 2.1) and the bounds $|x^G \cap H| \leqslant (q-\epsilon) + |\mathrm{PGL}_2(q)|$ and $|x^G| \geqslant (2, q-\epsilon)^{-1} q^2(q^3-\epsilon)$ (see [9, 4.5.6, 4.8.2]) are good enough.

Finally, let us assume $C_{G_0}(x)$ is non-symplectic. If $n = t$ then

$$\left|x^G \cap H\right| \leqslant (q-\epsilon)^{-1} \sum_{j=0}^{\alpha} \left[\left|\rho_j^{S_t}\right| (q-\epsilon)^{t-j}\right] \leqslant (q-\epsilon)^{t-1}\left(i_2(S_t)+1\right) \leqslant t!(q-\epsilon)^{t-1},$$

where $\alpha = t/2 - 1$ if $(t,p) = 2$, otherwise $\alpha = \lfloor t/2 \rfloor$. Now if $\epsilon = +$ then (39) is sufficient unless $t = 3$ and $q < 5$; if $\epsilon = -$ then the same bounds are sufficient if $t \geqslant 14$, or if $q \geqslant 8$. For these outstanding cases, the desired result follows by applying (39) and the more accurate upper bound

$|x^G \cap H| \leqslant (q-\epsilon)^{t-1}(i_2(S_t)+1)$. Now assume $n \geqslant 2t$. If $p=2$ and n/t is even then (II) implies that

$$\left|x^G \cap H\right| \leqslant (q-\epsilon)^{-1}\sum_{j=0}^{\alpha}\left[\left|\rho_j^{S_t}\right|\left|\mathrm{GL}_{n/t}^{\epsilon}(q)\right|^j g(j)\right],$$

where

$$g(j) = \sum_{k=1}^{t-2j}\left[\binom{t-2j}{k}\left(\frac{|\mathrm{GL}_{n/t}^{\epsilon}(q)|}{|\mathrm{Sp}_{n/t-2}(q)|q^{n/t-1}}\right)^k\left(\frac{|\mathrm{GL}_{n/t}^{\epsilon}(q)|}{|\mathrm{Sp}_{n/t}(q)|}\right)^{t-2j-k}\right]$$

and α is defined as before. Therefore

$$\left|x^G \cap H\right| < (q-\epsilon)^{t-1}t!2^t q^{\frac{n^2}{2t}+\frac{n}{2}-t}$$

and it is easy to see that this bound also holds if $\mathrm{hcf}(n/t, p, 2) = 1$. If $t \geqslant 3$ then the desired result follows via (39) so assume $t = 2$ and $n \geqslant 6$. If p is odd then

$$\left|x^G \cap H\right| \leqslant (q-\epsilon)\left(\frac{|\mathrm{PGL}_{n/2}^{\epsilon}(q)|}{|\mathrm{SO}_{n/2}^{\epsilon'}(q)|}\right)^2 + \left|\mathrm{PGL}_{n/2}^{\epsilon}(q)\right| < q^{\frac{1}{4}n^2-1}\left(4(q-\epsilon)q^{\frac{1}{2}n-1}+1\right)$$

and (39) is sufficient unless $(n, q, \epsilon) = (6, 3, -)$, where direct calculation yields $f(x, H) < 0.586$. Similarly, if $p = 2$ then

$$\left|x^G \cap H\right| < 4(q-\epsilon)q^{\frac{1}{4}n^2-2}\left(q^{\frac{1}{2}n}+2\beta\right),$$

where $\beta = 1$ if $n/2$ is even, zero otherwise. By applying (39), we reduce to a handful of cases which are easily settled. Finally, let us assume $(n, t) = (4, 2)$. If $p \neq 2$ then

$$\left|x^G \cap H\right| \leqslant (q-\epsilon)\left(\frac{|\mathrm{PGL}_2(q)|}{|\mathrm{PGO}_2^{+}(q)|}+\frac{|\mathrm{PGL}_2(q)|}{|\mathrm{PGO}_2^{-}(q)|}\right)^2 + \left|\mathrm{PGL}_2(q)\right| = q^4(q-\epsilon)+q\left(q^2-1\right)$$

and the desired result follows since

$$\left|x^G\right| \geqslant \frac{|\mathrm{PSL}_4^{\epsilon}(q)|}{|\mathrm{SO}_4^{-}(q)|} = (4, q-\epsilon)^{-1}q^4\left(q^2-1\right)\left(q^3-\epsilon\right).$$

Similarly, if $p = 2$ then $|x^G \cap H| \leqslant (q^4-1)(q-\epsilon)$, $|x^G| \geqslant q^2(q^3-\epsilon)(q^4-1)$ and again these bounds are always sufficient. □

2.3. *Proof of Theorem 1.1: Case (ii) of Table 2.1*

Let σ be a Frobenius morphism of $\bar{G} = \mathrm{PSp}_n(K)$ such that $\bar{G}_\sigma$ has socle $G_0 = \mathrm{PSp}_n(q)$. If $t = 2$ then $\iota = 1/n$ and so we may assume $n \geqslant 8$. Observe that

$$H \cap \mathrm{PGL}(V) \leqslant \left(\left((2, q-1)^{t-1}.\mathrm{PSp}_{n/t}(q)^t\right).(2, q-1)\right).S_t = B.S_t,$$

where B is the image of $\mathrm{GSp}_{n/t}(q)^t$ in $\mathrm{PGSp}_n(q) = \bar{G}_\sigma$. If q is odd then $B = \tilde{B}.\langle\delta\rangle$, where $\tilde{B}$ is the image of $\mathrm{Sp}_{n/t}(q)^t$ in $\mathrm{PSp}_n(q)$ and δ is an involutory diagonal automorphism of $\mathrm{PSp}_n(q)$.

Let $x \in H \cap \mathrm{PGL}(V)$ be an element of prime order r and suppose $y \in B\rho$ is G-conjugate to x, where $\rho \in S_t$ has cycle-shape $(r^k, 1^{t-kr})$ for some $k \geqslant 0$. Without loss, we may assume y fixes each subspace V_j with $j > kr$ in the decomposition $V = V_1 \oplus \cdots \oplus V_t$. If r is odd then [4, 3.11] implies that y lifts to a unique element $\hat{y} = (\hat{y}_1, \ldots, \hat{y}_t)\pi \in \mathrm{Sp}_n(q)$ of order r, and the proof of [10, 4.5] reveals that $\hat{y}$ is $\hat{B}$-conjugate to $(I_{n/t}, \ldots, I_{n/t}, \hat{y}_{kr+1}, \ldots, \hat{y}_t)\pi$, where $\hat{B} = \mathrm{Sp}_{n/t}(q)^t$ and each $\hat{y}_j \in \mathrm{Sp}_{n/t}(q)$ with $j > kr$ satisfies $\hat{y}_j^r = I_{n/t}$. Therefore

$$\left|y^B\right| \leqslant \left|\hat{y}^{\hat{B}}\right| = \left|\mathrm{Sp}_{n/t}(q)\right|^{k(r-1)} \prod_{j>kr} \left|\hat{y}_j^{\mathrm{Sp}_{n/t}(q)}\right| \tag{40}$$

and it is easy to see that the same bound holds if $r = 2$ and $C_{\bar{G}}(x)$ is connected. We also note that if $p = 2$ then each involution $\rho \in S_t$ with cycle-shape $(2^k, 1^{t-2k})$ is G-conjugate to $a_{nk/t}$, where we label involutions as in [2].

The case $x \in H - \mathrm{PGL}(V)$ is very straightforward. Indeed, if $x \in G$ is a field automorphism of prime order r, then $q = q_0^r$ and [4, 3.48] gives $|x^G| > \frac{1}{4}q^{(n^2+n)(1-1/r)/2}$. Furthermore, we have

$$\left|x^G \cap H\right| \leqslant \sum_{j=0}^{\lfloor t/r \rfloor} \left[\left|\rho_j^{S_t}\right|\left|\mathrm{Sp}_{n/t}(q)\right|^{j(r-1)}\left(\frac{|\mathrm{Sp}_{n/t}(q)|}{|\mathrm{Sp}_{n/t}(q^{1/r})|}\right)^{t-jr}\right] < 2^t t! q^{\frac{1}{2}(\frac{n^2}{t}+n)(1-\frac{1}{r})},$$

where $\rho_j \in S_t$ has cycle-shape $(r^j, 1^{t-jr})$, and one can easily check that these bounds are sufficient unless $(n, t, r, q) = (6, 3, 2, 4)$. Here direct calculation yields $f(x, H) < 0.535$.

Proposition 2.8. *The conclusion to Theorem* 1.1 *holds in case* (ii) *of Table* 2.1 *for semisimple elements in* $H \cap \mathrm{PGL}(V)$.

Proof. Let $x \in H \cap \mathrm{PGL}(V)$ be a semisimple element of prime order r. We prove the proposition in two parts, starting with the case $x^G \cap H \subseteq B$.

Case 1. $x^G \cap H \subseteq B$.

First assume $r > 2$. Let $i \geqslant 1$ be minimal such that $r \mid (q^i - 1)$ and let $\mu = (l, a_1, \ldots, a_k)$ denote the associated σ-tuple of x, where $k = (r-1)/i$ (see [4, 3.27]). Let d be the number of non-zero terms a_j in μ and note that d is even if i is odd. From the proof of Proposition 2.1 we have

$$\dim x^{\bar{B}} \leqslant \frac{1}{t}\dim x^{\bar{G}} + \frac{1}{2}(n-l)\left(1 - \frac{1}{t}\right)$$

and [4, 3.30] implies that

$$\left|x^G\right| > \frac{1}{2}\left(\frac{q}{q+1}\right)^{d(2-e)} q^{\dim x^{\bar{G}}}, \tag{41}$$

where $e=2$ if i is odd, otherwise $e=1$. Appealing to (40) and arguing as in the proof of Proposition 2.5 (see (8)) we deduce that

$$\left|x^G \cap H\right| < \log_2 q.\left(\frac{n-l}{di}+1\right)^{\frac{d}{e}(t-1)} 2^{\frac{1}{2}(e-1)dt} q^{\frac{1}{t}\dim x^{\bar{G}}+\frac{1}{2}(n-l)(1-\frac{1}{t})}.$$

Now $n \geqslant \max(l+di, eti)$ and [4, 3.33] gives a lower bound for $\dim x^{\bar{G}}$. Then one can check that these bounds are sufficient except for a small number of cases with $i<3$ which are easily settled via direct calculation.

Now assume $r=2$. Write $s=\nu(x)$ and observe that the hypothesis $x^G \cap H \subseteq B$ implies that $s<n/t$. In particular, s is even and x lifts to an involution $\hat{x}=(\hat{x}_1,\dots,\hat{x}_t) \in \hat{B}$. If $\nu(\hat{x}_i)=s_i$ (with respect to the natural $\mathrm{Sp}_{n/t}(q)$-module) then

$$\dim x^{\bar{B}} = \sum_{i=1}^{t} s_i\left(\frac{n}{t}-s_i\right) = \frac{ns}{t} - \sum_{i=1}^{t} s_i^2 \leqslant \frac{1}{t}s(n-s) = \frac{1}{t}\dim x^{\bar{G}}$$

and thus

$$\left|x^G \cap H\right| < \binom{t+s/2-1}{s/2} 2^t q^{\frac{1}{t}s(n-s)}, \qquad \left|x^G\right| > \frac{1}{2}q^{s(n-s)}.$$

It is easy to check that these bounds are sufficient for all $t \geqslant 3$. Finally, if $t=2$ then

$$\begin{aligned}\left|x^G \cap H\right| &\leqslant \sum_{j=0}^{s/2}\left[\frac{|\mathrm{Sp}_{n/2}(q)|}{|\mathrm{Sp}_{2j}(q)||\mathrm{Sp}_{n/2-2j}(q)|}.\frac{|\mathrm{Sp}_{n/2}(q)|}{|\mathrm{Sp}_{s-2j}(q)||\mathrm{Sp}_{n/2-s+2j}(q)|}\right] \\ &< 4\sum_{j=0}^{s/2} q^{\frac{1}{2}(ns+8sj-2s^2-16j^2)} < 4\left(\frac{q^2+1}{q^2-1}\right) q^{\frac{1}{2}s(n-s)}\end{aligned}$$

and the bound $|x^G| > \frac{1}{2}q^{s(n-s)}$ is good enough.

Case 2. $x^G \cap (H-B) \neq \emptyset$.

Write $x=b\pi$, where $b \in B$ and $\pi \in S_t$ has cycle-shape $(2^h, 1^{t-2h})$ with h defined as in (17). If $C_{\bar{G}}(x)$ is connected then x lifts to an element $\hat{x}=(\hat{x}_1,\dots,\hat{x}_t)\pi \in \mathrm{Sp}_n(q)$ of order r which is $\hat{B}$-conjugate to $(I_{n/t},\dots,I_{n/t},\hat{x}_{hr+1},\dots,\hat{x}_t)\pi$ and it is easy to see that $\dim x^{\bar{G}}$ is minimal if $\hat{x}_j = I_{n/t}$ for each j, i.e.

$$\dim x^{\bar{G}} \geqslant \dim \pi^{\bar{G}} = \begin{cases} \frac{nh}{t}\left(n-\frac{nh}{t}\right) & \text{if } r=2, \\ \frac{nh}{2t}(r-1)(2n+1-nhr/t) & \text{otherwise.}\end{cases} \tag{42}$$

Case 2.1. $r>2$.

Let $i \geqslant 1$ be minimal such that $r \mid (q^i - 1)$ and define e as in Case 1. Then [4, 3.30] gives

$$\left|x^G\right| > \frac{1}{2}\left(\frac{q}{q+1}\right)^{\frac{1}{2}(2-e)(r-1)} q^{\dim x^{\bar{G}}} \tag{43}$$

and we claim that

$$\left|x^G \cap H\right| < \log_2 q.2\left(\frac{t^r}{r}\right)^h 2^{\frac{1}{2}(e-1)(r-1)t}\left(\frac{1}{2}(r+1)\right)^{\frac{n}{2}} q^{(\frac{1}{t}+\frac{2}{n+2})\dim x^{\bar{G}}}. \tag{44}$$

If $y \in x^G \cap H$ then using (40) and the proof of Proposition 2.1 we deduce that

$$\left|y^B\right| < 2^{\frac{1}{2}(e-1)(r-1)t} q^{(\frac{1}{t}+\frac{2}{n+2})\dim x^{\bar{G}}}.$$

The claim now follows because the number of distinct B-classes in $x^G \cap H$ is at most

$$\log_2 q.\sum_{k=0}^{h}\left[\frac{t!}{k!(t-kr)!r^k}\left(\frac{r-1}{ei}+1\right)^{\frac{1}{ei}(n-\frac{nrk}{t})}\right] < \log_2 q.2\left(\frac{t^r}{r}\right)^h\left(\frac{1}{2}(r+1)\right)^{\frac{n}{2}}$$

(see (7) and (22), for example). The reader can check that the bounds (42), (43) and (44) are sufficient with the exception of a small number of cases with $h = 1$ and $r \in \{3, 5\}$. Here the desired result quickly follows through direct calculation.

Case 2.2. $r = 2$.

First suppose $C_{\bar{G}}(x)$ is connected, so $t > 2h$ since $\nu(x) < n/2$. Appealing to the proof of Proposition 2.1 we deduce that

$$\left|x^G \cap H\right| < 2\left(\frac{t^2}{2}\right)^h 2^{\frac{n}{2}+t} q^{(\frac{1}{t}+\frac{1}{n})\dim x^{\bar{G}}}$$

(see (44)) where $\dim x^{\bar{G}} \geqslant (nh/t)(n - nh/t)$. Now $|x^G| > \frac{1}{2}q^{\dim x^{\bar{G}}}$ and these bounds are sufficient when $n > 2t$ unless $(h, n, t) = (1, 12, 3)$ and $q \in \{3, 5\}$, where direct calculation yields $f(x, H) < 0.343$. If $n = 2t$ then the maximality of h implies that $\nu(x) = 2h$ and the result follows since $|x^G| > \frac{1}{2}q^{4h(t-h)}$ and

$$\left|x^G \cap H\right| \leqslant \sum_{k=0}^{h}\left[\frac{t!}{k!(t-2k)!2^k}\binom{t-2k}{h-k}\left|\mathrm{Sp}_2(q)\right|^k\right] < t^{2h} q^h \left(q^2 - 1\right)^h.$$

Now assume $C_{\bar{G}}(x)$ is non-connected. There are four cases to consider. If $C_G(x)$ is of type $\mathrm{Sp}_{n/2}(q)^2$ then $|x^G| > \frac{1}{4}q^{n^2/4}$ and our earlier arguments apply since each $y \in x^G \cap B$ lifts to an involution in $\hat{B}$. We leave the details to the reader. Next assume $C_G(x)$ is of type $\mathrm{Sp}_{n/2}(q^2)$. If

$n/2t$ is odd then t is even and $x^G \cap B\rho$ is non-empty if and only if $\rho \in S_t$ has cycle-shape $(2^{t/2})$. Therefore

$$\left|x^G \cap H\right| \leqslant \frac{t!}{(t/2)!2^{t/2}} \frac{1}{2} \left|\mathrm{Sp}_{n/t}(q)\right|^{\frac{t}{2}} < \frac{t!}{(t/2)!2^{t/2+1}} q^{\frac{n^2}{4t}+\frac{n}{4}}$$

and the desired result follows since $|x^G| > \frac{1}{4}q^{n^2/4}$. On the other hand, if $n/2t$ is even then

$$\left|x^G \cap H\right| \leqslant \sum_{k=0}^{\lfloor t/2 \rfloor} \left[\frac{t!}{k!(t-2k)!2^k} \left|\mathrm{Sp}_{n/t}(q)\right|^k \left(\frac{|\mathrm{Sp}_{n/t}(q)|}{|\mathrm{Sp}_{n/2t}(q^2)|}\right)^{t-2k}\right] < 2\left(\frac{t^2}{2}\right)^{\lfloor \frac{t}{2} \rfloor} q^{\frac{n^2}{4t}+\frac{n}{4}}$$

and one can check that the bound $|x^G| > \frac{1}{4}q^{n^2/4}$ is sufficient unless $(n,t,q) = (8,2,3)$, where direct calculation yields $f(x,H) < 0.618$. Finally, if $C_G(x)$ is of type $\mathrm{GL}^{\epsilon}_{n/2}(q)$ then

$$\left|x^G \cap H\right| \leqslant \sum_{k=0}^{\lfloor t/2 \rfloor} \left[\frac{t!}{k!(t-2k)!2^k} \left|\mathrm{Sp}_{n/t}(q)\right|^k \left(\frac{|\mathrm{Sp}_{n/t}(q)|}{|\mathrm{GL}^{\epsilon}_{n/2t}(q)|}\right)^{t-2k}\right] < 2\left(\frac{t^2}{2}\right)^{\lfloor \frac{t}{2} \rfloor} 2^t q^{\frac{n^2}{4t}+\frac{n}{2}},$$

$$\left|x^G\right| = \frac{|\mathrm{Sp}_n(q)|}{|\mathrm{GL}^{\epsilon}_{n/2}(q)|2} > \frac{1}{4}\left(\frac{q}{q+1}\right) q^{\frac{1}{4}n(n+2)}$$

and we are left to deal with a handful of cases which are easily settled. For example, if $t = 3$ then the above bounds are sufficient unless $n = 6$, where $f(x,H) < 0.609$ since

$$\left|x^G \cap H\right| \leqslant \left(\frac{|\mathrm{Sp}_2(q)|}{|\mathrm{GL}^{\epsilon}_1(q)|}\right)^3 + 3\left|\mathrm{Sp}_2(q)\right| \frac{|\mathrm{Sp}_2(q)|}{|\mathrm{GL}^{\epsilon}_1(q)|}, \qquad \left|x^G\right| = \frac{|\mathrm{Sp}_6(q)|}{|\mathrm{GL}^{\epsilon}_3(q)|2}. \qquad \square$$

Proposition 2.9. *The conclusion to Theorem* 1.1 *holds in case* (ii) *of Table* 2.1 *for unipotent elements in* $H \cap \mathrm{PGL}(V)$.

Proof. Let $x \in H \cap \mathrm{PGL}(V)$ be a unipotent element of order p, with associated partition $\lambda \vdash n$. Note that any odd parts in λ must occur with an even multiplicity (see [4, §3.3]).

Case 1. $x^G \cap H \subseteq B$, $p > 2$.

According to the proof of Proposition 2.1 we have

$$\dim x^{\bar{B}} \leqslant \frac{1}{t}\dim x^{\bar{G}} + \frac{1}{2}(n-e)\left(1-\frac{1}{t}\right), \tag{45}$$

where e is the number of odd parts in λ. If $\lambda = (k^{n/k})$ for some $k \geqslant 2$, then the hypothesis $x^G \cap H \subseteq B$ implies that $k \leqslant n/t$ and applying (40) and (45) we deduce that

$$\left|x^G \cap H\right| < 2^t q^{\frac{1}{t}\dim x^{\bar{G}} + \frac{1}{2}(1-\frac{1}{t})n}, \qquad \left|x^G\right| > \frac{1}{4}(q+1)^{-1} q^{\dim x^{\bar{G}}+1},$$

where $\dim x^{\bar{G}} \geqslant \frac{1}{4}n(n+2)$. These bounds are sufficient unless $(n,t)=(6,3)$, where direct calculation yields $f(x,H)<0.501$. Next assume $\lambda=(2^j,1^{n-2j})$ for some $1 \leqslant j < n/2$, so $\dim x^{\bar{G}} = j(n-j+1)$. If $j=1$ then the desired result follows from the bounds $|x^G \cap H| < t.q^{n/t}$ and $|x^G| > \frac{1}{4}q^n$. Now assume $j \geqslant 2$. If $n=2t$ then $t \geqslant 3$ and the bounds $|x^G \cap H| < \binom{t}{j}q^{2j}$ and $|x^G| > \frac{1}{4}(q+1)^{-1}q^{j(2t-j+1)+1}$ suffice so we can assume $n \geqslant 4t$. If $j=2$ then $|x^G \cap H| < \binom{t}{2}q^{2n/t} + 2tq^{2n/t-2}$, $|x^G| > \frac{1}{4}(q+1)^{-1}q^{2n-1}$ and the result follows. For $j \geqslant 3$ the bounds

$$\left|x^G \cap H\right| < \binom{t+j-1}{j} 2^t q^{\frac{1}{t}\dim x^{\bar{G}} + j(1-\frac{1}{t})}, \qquad \left|x^G\right| > \frac{1}{4}q^{\dim x^{\bar{G}}}$$

are sufficient unless $(n,t,j,q)=(8,2,3,3)$. Here a more accurate calculation yields $f(x,H) < 0.423$.

Now assume $\lambda = (m^{a_m},\dots,2^{a_2},1^l) \vdash n$, where $m=n/t \geqslant 2$, and let $d \geqslant 1$ denote the number of non-zero terms a_j. (Note that the prime order hypothesis implies that $a_j=0$ if $j>p$.) The case $d=1$ is straightforward so let us assume $d \geqslant 2$. Then arguing as in the proof of Proposition 2.6 (see (34)), and using the fact that odd parts in λ have an even multiplicity, we deduce that

$$\dim x^{\bar{G}} \geqslant \frac{1}{4}n^2 + \frac{1}{4}\left(d^2-d+2\right)n - \frac{1}{16}d^4 - \frac{1}{24}d^3 + \frac{3}{16}d^2 - \frac{1}{3}d - \frac{1}{4}l^2 - \frac{1}{2}l$$

and $n \geqslant \max(t(4d+2)/3, l+2d+2d^2/3-2/3)$. Furthermore, from (45) we get

$$\left|x^G \cap H\right| < 2^{td}\left(\frac{n/2-d^2/4+d/4-l/2-1}{d}+1\right)^{d(t-1)} q^{\frac{1}{t}\dim x^{\bar{G}} + \frac{1}{2}(n-l)(1-\frac{1}{t})}$$

(see (35)). Now [4, 3.18] implies that

$$\left|x^G\right| > \left(\frac{1}{2}\right)^{d+1}\left(\frac{q}{q+1}\right)^d q^{\dim x^{\bar{G}}}$$

and one can check that these bounds are sufficient with the exception of a small number of cases with $(t,q)=(2,3)$. These remaining cases are easily dealt with by computing more accurate bounds. For instance, if $(n,l)=(12,0)$ then $\lambda=(3^2,2^3)$ and we deduce that $f(x,H)<0.547$ since $|x^G \cap H| < 2.3^{26}$ and $|x^G| > \frac{1}{4}3^{50}$.

Case 2. $x^G \cap H \subseteq B$, $p=2$.

First assume x is G-conjugate to a_l for some even integer l. Then the hypothesis $x^G \cap H \subseteq B$ implies that $l < n/t$ since every element of order two in S_t is an a-type involution. Now, if $(\,,\,)$ is a non-degenerate G-invariant symmetric bilinear form on V then $(vx,v)=0$ for all $v \in V$. In particular, if $y=(y_1,\dots,y_t) \in x^G \cap B$ then each non-trivial y_i must be an a-type involution, hence $n \geqslant 4t$. Now, if $l=2$ then the bounds $|x^G \cap H| < 2tq^{2n/t-4}$ and $|x^G| > \frac{1}{2}q^{2n-4}$ are always sufficient. If $l \geqslant 4$ then using Proposition 2.1 we deduce that

$$\left|x^G \cap H\right| < \binom{t+l/2-1}{l/2} 2^t q^{\frac{1}{t}l(n-l)}, \qquad \left|x^G\right| > \frac{1}{2}q^{l(n-l)}$$

and the reader can check that these bounds are good enough.

For the remainder we may assume that x is G-conjugate to either b_l or c_l, with the precise type depending on the parity of l. Then the hypothesis $x^G \cap H \subseteq B$ implies that $l \leqslant n/t$ and we note that if $y = (y_1, \dots, y_t) \in x^G \cap B$ then at least one y_j is a b- or c-type involution. Now, if $n = 2t$ then each non-trivial y_i must be $\mathrm{Sp}_2(q)$-conjugate to b_1 and the subsequent bounds $|x^G \cap H| < \binom{t}{l} q^{2l}$ and $|x^G| > \frac{1}{2} q^{l(2t-l+1)}$ are always sufficient. Assume for the remainder that $n \geqslant 4t$. If $l = 1$ then $|x^G \cap H| < tq^{n/t}$, $|x^G| > \frac{1}{2}q^n$ and the result follows. Similarly, if $l = 2$ then the bounds $|x^G \cap H| < \binom{t}{2} q^{2n/t} + 2tq^{2n/t-2}$ and $|x^G| > \frac{1}{2}q^{2n-2}$ suffice. Now assume $l \geqslant 3$. Using the proof of Proposition 2.1 we deduce that

$$\left|x^G \cap H\right| < \binom{t+l-1}{l} 2^{2t} q^{\frac{1}{t}\dim x^{\bar{G}} + (1-\frac{1}{t})l}, \qquad \left|x^G\right| > \frac{1}{2} q^{\dim x^{\bar{G}}},$$

where $\dim x^{\bar{G}} = l(n-l+1)$. (Note that the number of B-classes in $x^G \cap B$ is determined by the number of ways l Jordan 2-blocks can be distributed among the t direct factors, together with one of two choices (either a- or c-type) for each factor which is assigned an even number of blocks. This number is at most $\binom{t+l-1}{l} 2^t$.) If we assume $t \geqslant 3$ then these bounds are sufficient with the exception of a handful of cases with which are easily settled through direct calculation.

For $t = 2$ we require more accurate bounds. Let N_1 (respectively N_2) denote the number of elements $(y_1, y_2) \in x^G \cap B$ such that one (respectively neither) of the y_i is an a-type involution. Then $|x^G \cap H| = N_1 + N_2$ and we claim that

$$N_1 < 2^3 \left(\frac{q^2+1}{q^2-1}\right) q^{\frac{1}{2}\dim x^{\bar{G}}}, \qquad N_2 < 2^2 \left(\frac{q^2+1}{q^2-1}\right) q^{\frac{1}{2}(\dim x^{\bar{G}} + l)}. \tag{46}$$

First consider N_1. For all possible even integers $j \geqslant 0$, choose $x_j = (y_1, y_2) \in x^G \cap B$ such that y_1 is $\mathrm{Sp}_{n/2}(q)$-conjugate to a_j (set $a_0 = I_{n/2}$). Then using [4, 3.22] we calculate that

$$N_1 = 2\sum_j \left|x_j^B\right| < 2^3 \sum_j q^{j(n/2-j)+(l-j)(n/2-l+j+1)} = 2^3 \sum_j q^{f(j)}.$$

Evidently $\max_{j\in\mathbb{Z}} f(j) \leqslant f(l/2) = \frac{1}{2}\dim x^{\bar{G}}$ and the claim for N_1 now follows since $f(j)$ is even and $|f(j+1) - f(j)| \geqslant 2$ for all j. Similar reasoning establishes the upper bound for N_2. Now $|x^G| > \frac{1}{2}q^{\dim x^{\bar{G}}}$ and we find that the upper bound on $|x^G \cap H|$ derived from (46) is always sufficient if $l \geqslant 3$ and $q \geqslant 4$. Similarly, if $q = 2$ and $l \geqslant 4$ then it remains to deal with the case $(n, l) = (8, 4)$, where direct calculation gives $f(x, H) < 0.590$. Finally, if $(l, q) = (3, 2)$ then using the proof of [4, 3.22] we calculate that

$$\left|x^G \cap H\right| \leqslant 2\left|b_1^{\mathrm{Sp}_{n/2}(2)}\right|\left(\left|a_2^{\mathrm{Sp}_{n/2}(2)}\right| + \left|c_2^{\mathrm{Sp}_{n/2}(2)}\right|\right) = \frac{8}{3}\left(2^{\frac{n}{2}-2} - 1\right)\left(2^{\frac{n}{2}} - 1\right)^2,$$

$$\left|x^G\right| \geqslant \frac{4}{3}\left(2^{n-4} - 1\right)\left(2^{n-2} - 1\right)\left(2^n - 1\right)$$

and the reader can check that these bounds are always sufficient.

Case 3. $x^G \cap (H - B) \neq \emptyset$.

Fix $\pi \in S_t$ with cycle-shape $(2^h, 1^{t-2h})$, where h is defined as in (17), and note that (42) holds (with $r = p$). First assume $p > 2$. Then $|x^G|$ is minimal if $\lambda = (p^{nh/t}, 1^{n-nhp/t}) \vdash n$ and thus

$$\left|x^G\right| > \frac{1}{2}q^{\frac{nh}{2t}(p-1)(2n+1-nhp/t)}. \tag{47}$$

If $n = 2t$ then the maximality of h implies that $\lambda = (p^{2h}, 1^{n-2hp})$, so

$$\left|x^G \cap H\right| = \left|x^G \cap (H - B)\right| \leqslant \frac{t!}{h!(t-hp)!p^h}\left(q\left(q^2-1\right)\right)^{h(p-1)}$$

and the result follows via (47). Now assume $n \geqslant 4t$. Using Proposition 2.1 and arguing as in the proof of Proposition 2.6 (see (36)) we deduce that

$$\left|x^G \cap H\right| < 2\left(\frac{t^p}{p}\right)^h p^{n+\frac{n}{t}h(1-p)}2^{pt}q^{(\frac{1}{t}+\frac{2}{n+2})\dim x^{\bar{G}}}.$$

In view of (42), we calculate that this bound with (47) is sufficient unless $(h, p) = (1, 3)$ and (n, t) is one of a handful of cases. As usual, these exceptional cases are easily settled through direct calculation.

Next assume $p = 2$ and x is an a-type involution, say x is G-conjugate to $a_{nh/t+j}$, where $0 \leqslant j < n/t$ is even. If $t = 2$ then $(h, j) = (1, 0)$ and

$$\left|x^G \cap (H - B)\right| \leqslant \left|\mathrm{Sp}_{n/2}(q)\right| < q^{\frac{1}{8}n(n+2)}, \qquad \left|x^G\right| > \frac{1}{2}q^{\frac{1}{4}n^2}.$$

Clearly, either $x^G \cap B$ is empty, or $n \equiv 0\ (8)$ and $|x^G \cap B| < 4q^{n^2/8}$ since each $y \in x^G \cap B$ must act on both V_1 and V_2 as an $a_{n/4}$ involution. We leave the reader to check that these bounds are always sufficient. Now assume $t \geqslant 3$. Evidently, each B-class in $x^G \cap B\pi$ is determined by a choice of elements $x_{2h+1}, \ldots, x_t$ in $\mathrm{Sp}_{n/t}(q)$ (up to conjugacy) such that each non-trivial x_k is conjugate to a_{l_k} for some even integer l_k and $\sum_k l_k = j$. If $n = 2t$ then $q \geqslant 4$ (see Table 2.1), $j = 0$ and the result follows since

$$\left|x^G \cap H\right| = \left|x^G \cap (H - B)\right| < \left(\frac{t!}{h!(t-2h)!2^h}\right)q^{3h}, \qquad \left|x^G\right| > \frac{1}{2}q^{4h(t-h)}.$$

Now assume $n \geqslant 4t$. Then [4, 3.22] gives $|x^G| > \frac{1}{2}q^{\dim x^{\bar{G}}}$ and from the proof of Proposition 2.1 we deduce that

$$\left|x^G \cap H\right| < 2\left(\frac{t^2}{2}\right)^h 2^{\frac{n}{4}+t}q^{(\frac{1}{t}+\frac{1}{n})\dim x^{\bar{G}}}.$$

(Note that if $x^G \cap B\rho$ is non-empty then the number of B-classes in $x^G \cap B\rho$ is at most $2^{n/4}$.) Applying (42) (with $r = 2$) we find that the above bounds are always sufficient if $h \geqslant 2$. If $h = 1$

then $|x^G| > \frac{1}{2}q^{\dim x^{\bar{G}}}$, where $\dim x^{\bar{G}} = (n/t + j)(n - n/t - j)$, and the desired result follows since the proof of Proposition 2.1 implies that

$$\left|x^G \cap B\right| < \binom{t + n/2t + j/2 - 1}{n/2t + j/2} 2^t q^{\frac{1}{t}\dim x^{\bar{G}}}$$

and

$$\left|x^G \cap (H - B)\right| < \binom{t}{2}\binom{t - 3 + j/2}{j/2} 2^{t-2} q^{(\frac{1}{t}+\frac{1}{n})\dim x^{\bar{G}}}.$$

Finally, let us assume $p = 2$ and x is conjugate to b_l or c_l, where $l = nh/t + j$ and $1 \leqslant j \leqslant n/t$. In particular, we note that $t > 2h$. Now, if $n = 2t$ then

$$\left|x^G \cap H\right| < \sum_{k=0}^{h}\left[\left|\rho_k^{S_t}\right|\binom{t - 2k}{2h + j - 2k} q^{4h+2j-k}\right] < \left(\binom{t}{2h + j} + \frac{t!}{(t - 2h - j)!j!}\right) q^{4h+2j},$$

where $\rho_k \in S_t$ has cycle-shape $(2^k, 1^{t-2k})$, and the desired result follows since we have $|x^G| > \frac{1}{2}q^{(2h+j)(2t-2h-j+1)}$. Now assume $n \geqslant 4t$. Arguing as before we deduce that

$$\left|x^G \cap H\right| < 2\left(\frac{t^2}{2}\right)^h 2^{\frac{n}{2}+t} 2^t q^{(\frac{1}{t}+\frac{1}{n})\dim x^{\bar{G}}}, \qquad \left|x^G\right| > \frac{1}{2}q^{\dim x^{\bar{G}}},$$

where $\dim x^{\bar{G}} \geqslant (nh/t + 1)(n - nh/t)$, and we quickly reduce to the case $h = 1$. Here

$$\left|x^G \cap B\right| < 2^t \binom{t + n/t + j - 1}{n/t + j} 2^t q^{(\frac{1}{t}+\frac{2}{n+2})\dim x^{\bar{G}}}$$

and

$$\left|x^G \cap (H - B)\right| < 2^{t-2}\binom{t}{2}\binom{t - 3 + j}{j} 2^{t-2} q^{(\frac{1}{t}+\frac{2}{n+2})\dim x^{\bar{G}}},$$

where $\dim x^{\bar{G}} \geqslant (n/t + j)(n - n/t - j + 1)$. Now $|x^G| > \frac{1}{2}q^{\dim x^{\bar{G}}}$ and we find that these bounds are sufficient except for a small number of cases (n, t, j) with which we can calculate directly. For example, if $(n, t, j) = (12, 3, 2)$ then x is G-conjugate to c_6, so $|x^G| > \frac{1}{2}q^{42}$. If $y = (y_1, y_2, y_3) \in x^G \cap B$ then at least one y_i is $\mathrm{Sp}_4(q)$-conjugate to c_2 and so [4, 3.22] implies that $|x^G \cap B| < 2^3(q^{18} + 3q^{16} + 3q^{14})$. If $\pi = (12) \in S_3$ and $z \in B\pi$ then z is B-conjugate to $[I_4, I_4, c_2]\pi$, so $|x^G \cap (H - B)| < 3.2q^{16}$. We conclude that $f(x, H) < 0.539$ for all $q \geqslant 2$. □

2.4. Proof of Theorem 1.1: Cases (iii), (iv) and (v) of Table 2.1

Fix a Frobenius morphism σ of $\bar{G} = \mathrm{PSO}_n(K)$ such that $\bar{G}_\sigma$ has socle $\mathrm{P}\Omega_n^\epsilon(q)$. Let $(\triangle)$ denote the hypothesis "$(n, \epsilon) = (8, +)$ *and G contains triality automorphisms*," and note that if $(\triangle)$ holds then we may assume H is of type $\mathrm{O}_4^+(q) \wr S_2$ or $\mathrm{O}_2^{\epsilon'}(q) \wr S_4$ (see [4, 3.3]).

Proposition 2.10. *The conclusion to Theorem* 1.1 *holds in case* (iii) *of Table* 2.1.

Proof. Here $q = p \geqslant 3$, $n \geqslant 7$ and $H \leqslant 2^{n-1}.S_n = B.S_n \leqslant \mathrm{PGL}(V)$. Let $x \in H$ be an element of prime order r and note that $x^G \cap (H - B)$ is non-empty. First assume r is odd. Then $x^G \cap B\pi$ is non-empty if and only if $\pi \in S_t$ has cycle-shape $(r^h, 1^{n-hr})$ for a uniquely determined integer $h \geqslant 1$, hence

$$\left|x^G \cap H\right| = \left|x^G \cap (H - B)\right| \leqslant \left(\frac{n!}{h!(n-hr)!r^h}\right)2^{h(r-1)}$$

and

$$\left|x^G\right| > \begin{cases} \frac{1}{2}\left(\frac{q}{q+1}\right)^{\frac{1}{2}(r-1)} q^{\dim x^{\bar{G}}} & \text{if } r \neq p, \\ \frac{1}{8}\left(\frac{q}{q+1}\right)^2 q^{\dim x^{\bar{G}}} & \text{if } r = p, \end{cases}$$

where $\dim x^{\bar{G}} = \frac{1}{2}(r-1)(2nh - h^2 r - h)$. These bounds are sufficient unless $(n, r, q) = (7, 3, 3)$, where direct calculation yields $f(x, H) < 0.590$.

Now assume $r = 2$. Define $h \geqslant 1$ as in (17) and observe that the maximality of h implies that $\nu(x) = h$. If $n = 2h$ and $C_G(x)$ is of type $\mathrm{GL}^{\epsilon}_{n/2}(q)$ then the result follows since

$$\left|x^G \cap H\right| = \left|x^G \cap (H - B)\right| \leqslant \frac{n!}{(n/2)!}, \qquad \left|x^G\right| > \frac{1}{4}(q+1)^{-1} q^{\frac{1}{4}n(n-2)+1}.$$

For the remainder we may assume x lifts to an involution in $\mathrm{O}_1(q) \wr S_n$ and thus

$$\left|x^G \cap H\right| \leqslant \sum_{j=0}^{h} \left[\frac{n!}{j!(n-2j)!2^j} \left|\mathrm{O}_1(q)\right|^j \binom{n-2j}{h-j}\right] \leqslant \frac{n!}{(n-2h)!}. \tag{48}$$

If $n = 2h$ then $|x^G| > \frac{1}{8}q^{n^2/4}$ and we are left to deal with the case $(n, q) = (8, 3)$. Here (48) gives $|x^G \cap H| \leqslant 14\,630$ and we conclude that $f(x, H) < 0.619 < 5/8$ since $|x^G| > \frac{1}{8}3^{16}$. On the other hand, if $n > 2h$ then $|x^G| > \frac{1}{4}(q+1)^{-1}q^{h(n-h)+1}$ and (48) is sufficient if $h \geqslant 2$, with the exception of a handful of cases which are easy to deal with. For instance, if $(n, h, q) = (7, 2, 3)$ then $H \cap G_0 \cong 2^6.A_7$ (see [9, 4.2.15]) and therefore $f(x, H) < 0.609$ if $x \in G_0$ since

$$\left|x^G \cap H\right| \leqslant \binom{7}{2} + \frac{7!}{2!3!} = 441, \qquad \left|x^G\right| = \left|\mathrm{O}_7(3) : \mathrm{O}_5(3)\mathrm{O}_2^-(3)\right| = 22\,113.$$

Similarly, we calculate that $f(x, H) < 0.500$ if $x \notin G_0$. Finally, if $h = 1$ then $|x^G| > \frac{1}{4}q^{n-1}$, $|x^G \cap H| \leqslant n + \binom{n}{2}|\mathrm{O}_1(q)| = n^2$ and we are left to deal with the cases $(n, q) = (8, 3)$ and $(7, 3)$. These are easily settled through direct calculation. □

Proposition 2.11. *The conclusion to Theorem* 1.1 *holds for cases* (iv) *and* (v) *of Table* 2.1.

Proof. We deal with both cases simultaneously. Let $\bar{B} = \mathrm{PSO}_{n/t}(K)^t$ and observe that

$$H \cap \mathrm{PGL}(V) \leqslant \left(\left((2, q-1)^{t-1}.\mathrm{PO}^{\epsilon'}_{n/t}(q)^t\right).(2, n/t, q-1)\right).S_t = B.S_t,$$

where B is the image of $\mathrm{GO}^{\epsilon'}_{n/t}(q)^t$ in $\mathrm{PGO}^{\epsilon}_n(q)$ and $(2, n/t, q-1)$ is a cyclic group of order $\mathrm{hcf}(2, n/t, q-1)$. Let $x \in H \cap \mathrm{PGL}(V)$ be an element of prime order r and suppose $y \in x^G \cap B\rho$, where $\rho \in S_t$ has cycle-shape $(r^k, 1^{t-kr})$ for some $k \geqslant 0$. Without loss, we may assume y fixes each subspace V_j with $j > kr$ in the decomposition $V = V_1 \oplus \cdots \oplus V_t$. If we assume $\nu(x) < n/2$ if $r = 2 < p$ then y lifts to an element $\hat{y} = (\hat{y}_1, \ldots, \hat{y}_t)\rho \in \hat{B}\rho$ of order r which is $\hat{B}$-conjugate to $(I_{n/t}, \ldots, I_{n/t}, \hat{y}_{kr+1}, \ldots, \hat{y}_t)\rho$, where $\hat{B} = \mathrm{O}^{\epsilon'}_{n/t}(q)^t$ and

$$\left|y^B\right| \leqslant \left|\hat{y}^{\hat{B}}\right| = \left|\mathrm{O}^{\epsilon'}_{n/t}(q)\right|^{k(r-1)} \prod_{j>kr} \left|\hat{y}_j^{\mathrm{O}^{\epsilon'}_{n/t}(q)}\right|. \tag{49}$$

We note that if $p = 2$ then every element of order two in S_t acts on V as an a-type involution.

Case 1. $x \in H \cap \mathrm{PGL}(V)$.

This is similar to the proof of Proposition 2.9 and for brevity we will assume x is a semisimple involution.

Assume for now that ($\triangle$) does not hold. Write $s = \nu(x)$, define h as in (17) and let us start by assuming $s < n/2$ and $h = 0$, so $x^G \cap H \subseteq B$ and $\dim x^{\bar{G}} = s(n-s)$. If $s = 1$ then Lemma 2.4 yields $|x^G \cap H| < t(q+1)q^{n/t-2}$, $|x^G| > \frac{1}{4}q^{n-1}$ and we are left to deal with the case $(t, q) = (2, 3)$. Here $\epsilon = +$ (see Table 2.1) and the result follows if $n \equiv 0\ (4)$ since

$$\left|x^G \cap H\right| \leqslant 2\left|\mathrm{O}^-_{\frac{n}{2}}(3) : \mathrm{O}_{\frac{n}{2}-1}(3)\right|, \qquad \left|x^G\right| \geqslant \frac{1}{2}\left|\mathrm{O}^+_n(3) : \mathrm{O}_{n-1}(3)\right|.$$

The case $n \equiv 2\,(4)$ is similar. If $s = 2$ then the desired result follows since

$$\left|x^G \cap H\right| < \binom{t}{2}\left(\frac{q+1}{q}\right)^2 q^{2(\frac{n}{t}-1)} + t.2q^{2\frac{n}{t}-4}, \qquad \left|x^G\right| > \frac{1}{4}(q+1)^{-1}q^{2n-3}$$

so let us assume $s \geqslant 3$. Then applying Proposition 2.1 and Lemma 2.4 we see that

$$\left|x^G \cap H\right| < \binom{t+s-1}{s} 2^t q^{\frac{1}{t}s(n-s)}, \qquad \left|x^G\right| > \frac{1}{4}q^{s(n-s)}$$

and we quickly reduce to the case $t = 2$. Here

$$\left|x^G \cap H\right| < 2^2\left(\frac{q^2+1}{q^2-1}\right)q^{\frac{1}{2}s(n-s)} \tag{50}$$

and if we assume $s \geqslant 3$ then the bound $|x^G| > \frac{1}{4}q^{s(n-s)}$ is sufficient unless $s = q = 3$ and $n \in \{8, 10\}$. These cases are easily dealt with through direct calculation.

Next assume $s < n/2$ and $h > 0$. Then $t > 2h$,

$$\left|x^G\right| > \frac{1}{4}(q+1)^{-1}q^{s(n-s)+1} \tag{51}$$

and applying Proposition 2.1, Lemma 2.4 and (49) we deduce that

$$\left|x^G \cap H\right| < 2\left(\frac{t^2}{2}\right)^h 2^{t+n} q^{\frac{1}{t}s(n-s)}. \tag{52}$$

If we assume $h \geqslant 3$ then these bounds are sufficient unless $(n,t,h,q) = (14,7,3,3)$. Here the hypothesis $s < n/2$ implies that $s = 6$ and direct calculation yields $f(x,H) < 0.260$. If $h = 2$ then we are left to deal with a handful of cases with $n = 2t$. Here the maximality of h implies that $s \in \{4,5\}$ and the desired result follows by applying (51) and a more accurate upper bound for $|x^G \cap H|$. For instance, if $s = 4$ then

$$\begin{aligned}\left|x^G \cap H\right| \leqslant{}& \binom{t}{4}(q+1)^4 + 12\binom{t}{4}(q+1)^3 + \left(12\binom{t}{4} + 3\binom{t}{3}\right)(q+1)^2\\ &+ 6\binom{t}{3}(q+1) + \binom{t}{2}\end{aligned}$$

and the result follows via (51). Now assume $h = 1$. If $n = 2t$ and $s = 2$ then $|x^G \cap H| \leqslant \binom{t}{2}(q+1)^2 + 2\binom{t}{2}(q+1) + t$ and (51) is good enough. The case $s = 3$ is similar. If $n > 2t$ then (51) is always sufficient since

$$\left|x^G \cap H\right| < \left(2\binom{t+s-1}{s} + \binom{t}{2}\binom{t+j-3}{j}\right)2^{t-1} q^{\frac{1}{t}s(n-s)}.$$

Next assume $s = n/2$. If $C_G(x)$ is of type $\mathrm{O}^{\epsilon''}_{n/2}(q) \times \mathrm{O}^{\epsilon'''}_{n/2}(q)$ then our earlier work applies since each $y \in x^G \cap B\rho$ lifts to an involution $\hat{y} = (\hat{y}_1, \dots, \hat{y}_t)\rho \in \hat{B}\rho$. In particular, if $t = 2$ then appealing to (49) and (50) we deduce that

$$\left|x^G \cap B\right| < 2^2\left(\frac{q^2+1}{q^2-1}\right)q^{\frac{1}{8}n^2}, \qquad \left|x^G \cap (H-B)\right| \leqslant \frac{1}{2}\left|\mathrm{O}^{\epsilon'}_{n/2}(q)\right| < q^{\frac{1}{8}n(n-2)}$$

and the bound $|x^G| > \frac{1}{8}q^{n^2/4}$ is sufficient unless $(n,q) \in \{(10,3),(8,3)\}$; these cases are easily settled. Now assume $t \geqslant 3$. Applying Proposition 2.1, Lemma 2.4 and (49) we deduce that

$$\left|x^G \cap H\right| < 2\left(\frac{t^2}{2}\right)^{\lfloor \frac{t}{2} \rfloor} 2^{t+n} q^{\frac{n^2}{4t}}$$

(see (52)) and if we assume $n > 2t$ then the bound $|x^G| > \frac{1}{8}q^{n^2/4}$ is sufficient unless $(n,t,q) = (12,3,3)$, where direct calculation yields $f(x,H) < 0.370$. Finally, suppose $n = 2t$. If $t \geqslant 5$ then the previous bounds are sufficient with the exception of the cases

$$(t,q) \in \big\{(8,3),(7,3),(6,3),(5,5),(5,3)\big\}.$$

Here we apply the more accurate upper bound

$$\left|x^G \cap H\right| \leqslant \sum_{k=0}^{t'/2}\left[\frac{t!}{k!(t-2k)!}(q-\epsilon')^{k+\beta}\sum_{l=0}^{t'/2-k}\left(\binom{t'-2k}{2l}(q-\epsilon')^{2l}\binom{t'-2k-2l}{t'/2-k-l}\right)\right], \tag{53}$$

where $t' = t - \beta$ and $\beta = 1$ is t is odd, zero otherwise. For instance, if $(t, q) = (5, 3)$ then (53) gives $|x^G \cap H| \leqslant 11\,416$ and thus $f(x, H) < 0.348$ since $|x^G| \geqslant |\mathrm{O}_{10}^+(3) : \mathrm{O}_5(3)^2|$. Finally, if $(n, t) = (8, 4)$ then (53) yields

$$\left|x^G \cap H\right| \leqslant (q-\epsilon')^4 + 12(q-\epsilon')^3 + 24(q-\epsilon')^2 + 24(q-\epsilon') + 6$$

and we conclude that $f(x, H) < 0.456$ since $|x^G| \geqslant \frac{1}{2}|\mathrm{O}_8^+(q) : \mathrm{O}_4^-(q)^2|$.

Next assume $C_G(x)$ is of type $\mathrm{O}_{n/2}^{\epsilon''}(q^2)$, where $\epsilon'' = \epsilon$ if $n/2$ is even. Then $|x^G| > \frac{1}{8}q^{n^2/4}$ and if we assume n/t is odd then

$$\left|x^G \cap H\right| \leqslant \frac{t!}{(t/2)!2^{t/2}} \frac{1}{2}\left|\mathrm{O}_{n/t}^{\epsilon'}(q)\right|^{\frac{t}{2}} < \frac{t!}{(t/2)!2} q^{\frac{n^2}{4t} - \frac{n}{4}}$$

since t is even and $x^G \cap B\rho$ is non-empty if and only if $\rho \in S_t$ has cycle-shape $(2^{t/2})$. These bounds are always sufficient. On the other hand, if n/t is even then

$$\left|x^G \cap H\right| \leqslant \sum_{k=0}^{\lfloor t/2 \rfloor} \left[\frac{t!}{k!(t-2k)!2^k}\left|\mathrm{O}_{n/t}^{\epsilon'}(q)\right|^k \left(\frac{|\mathrm{O}_{n/t}^{\epsilon'}(q)|}{|\mathrm{O}_{n/2t}^{\zeta}(q^2)|}\right)^{t-2k}\right] < 2\left(\frac{t^2}{2}\right)^{\lfloor \frac{t}{2} \rfloor}\left(\frac{q+1}{q}\right)^t 2^t q^{\frac{n^2}{4t}},$$

where $\zeta = \epsilon'$ if $n/2t$ is even. Since $|x^G| > \frac{1}{8}q^{n^2/4}$ we quickly reduce to the case $t = 2$. Here the more accurate bounds

$$\left|x^G \cap H\right| \leqslant \left(\frac{|\mathrm{O}_{n/2}^{\epsilon'}(q)|}{|\mathrm{O}_{n/4}^{\zeta}(q^2)|}\right)^2 + \frac{1}{2}\left|\mathrm{O}_{n/2}^{\epsilon'}(q)\right| < \left(q^{\frac{n}{4}} + 1\right)q^{\frac{1}{8}n(n-2)}, \qquad \left|x^G\right| > \frac{1}{8}q^{\frac{1}{4}n^2}$$

are sufficient unless $(n, q) = (8, 3)$, where direct calculation yields $f(x, H) < 0.541$.

Finally, let us assume $C_G(x)$ is of type $\mathrm{GL}_{n/2}^{\epsilon''}(q)$, so

$$\left|x^G\right| > \frac{1}{4}(q+1)^{-1}q^{\frac{1}{4}n(n-2)+1}. \tag{54}$$

Assume for now that n/t is even and let the symbol (†) represent the following conditions on ϵ'' and n/t with respect to ϵ':

ϵ'	Conditions
$+$	$\epsilon'' = +$ if $n/t \equiv 2\ (4)$
$-$	$\epsilon'' = -$ and $n/t \equiv 2\ (4)$

If (†) holds then from [8, Table 4.5.1] we deduce that

$$\left|x^G \cap H\right| \leqslant \sum_{k=0}^{\lfloor t/2 \rfloor} \left[\frac{t!}{k!(t-2k)!2^k}\left|\mathrm{O}_{n/t}^{\epsilon'}(q)\right|^k \left(\frac{|\mathrm{O}_{n/t}^{\epsilon'}(q)|}{|\mathrm{GL}_{n/2t}^{\epsilon''}(q)|}\right)^{t-2k}\right] < 2\left(\frac{t^2}{2}\right)^{\lfloor \frac{t}{2} \rfloor} 2^{2t} q^{\frac{n^2}{4t} - \frac{n}{4}} \tag{55}$$

and if we assume $t \geqslant 3$ then (54) is sufficient unless $(n,t)=(8,4)$ or $(n,t,q) \in \{(12,6,3), (10,5,3)\}$. If $(n,t)=(8,4)$ then the result follows via (54) since (55) gives

$$\left|x^G \cap H\right| \leqslant 2^4 + 2^3\binom{4}{2}(q+1) + 3.2^2(q+1)^2.$$

The other two outstanding cases are similar. If $t=2$ then $\epsilon=+$ (see Table 2.1), $n \equiv 0$ (4) and the bounds

$$\left|x^G \cap H\right| \leqslant \left(\frac{|\mathrm{O}^{\epsilon'}_{n/2}(q)|}{|\mathrm{GL}^{\epsilon''}_{n/4}(q)|}\right)^2 + \frac{1}{2}\left|\mathrm{O}^{\epsilon'}_{n/2}(q)\right| < \left(2^4 + q^{\frac{n}{4}}\right)q^{\frac{1}{8}n(n-4)}$$

and (54) are sufficient unless $n=8$ and $q \in \{3,5\}$. Here $x^G \cap B$ is empty if $\epsilon'=-$ (see [4, Table 3.8]) and direct calculation yields $f(x,H) < 0.546$.

Now assume that either n/t is odd or (†) does not hold if n/t is even. Then t is even and $x^G \cap B\rho$ is non-empty if and only if $\rho \in S_t$ has cycle-shape $(2^{t/2})$. Therefore

$$\left|x^G \cap H\right| \leqslant \frac{t!}{(t/2)!2^{t/2}}\frac{1}{2}\left|\mathrm{O}^{\epsilon'}_{n/t}(q)\right|^{\frac{t}{2}} < \frac{t!}{(t/2)!}\frac{1}{2}\left(\frac{q+1}{q}\right)^{\frac{t}{2}} q^{\frac{1}{4t}n(n-t)}$$

and the reader can check that this bound with (54) is always sufficient.

To complete the proof, let us suppose (△) holds and recall that we may assume H is of type $\mathrm{O}^+_4(q) \wr S_2$ or $\mathrm{O}^{\epsilon'}_2(q) \wr S_4$. In view of [4, 3.55(iii)], we can also assume that $C_G(x)$ is not of type $\mathrm{O}^+_4(q)^2$. If $\nu(x)=1$ then $x^G \cap H \subseteq B$ and the bounds $|x^G| \geqslant \frac{3}{2}q^3(q^4-1)$ and

$$\left|x^G \cap H\right| \leqslant \begin{cases} 6q(q^2-1) & \text{if } t=2, \\ 12(q+1) & \text{if } t=4 \end{cases}$$

are always sufficient. Similarly, if $\nu(x)=3$ and $t=2$ then $x^G \cap H \subseteq B$ and the bounds

$$\left|x^G \cap H\right| \leqslant 12\frac{|\mathrm{O}^+_4(q)|}{|\mathrm{O}_3(q)||\mathrm{O}_1(q)|}\left(1 + \frac{|\mathrm{O}^+_4(q)|}{|\mathrm{O}^+_2(q)|^2} + \frac{|\mathrm{O}^+_4(q)|}{|\mathrm{O}^-_2(q)|^2}\right) < 24q^7 + 12q^3$$

and $|x^G| > \frac{3}{4}q^{15}$ are sufficient unless $q=3$, where direct calculation yields $f(x,H) < 0.559$. If $t=4$ then $|x^G| > \frac{3}{4}q^{15}$ and the bounds

$$\begin{aligned} \left|x^G \cap B\right| &\leqslant 12(q-\epsilon')^3 + 36(q-\epsilon'), \\ \left|x^G \cap (H-B)\right| &\leqslant 3\binom{4}{2}\left|\mathrm{O}^{\epsilon'}_2(q)\right|2(q-\epsilon') = 72(q-\epsilon')^2 \end{aligned}$$

are good enough.

Next assume $C_G(x)$ is of type $\mathrm{GL}^{\epsilon''}_4(q)$. If $t=2$ then the bounds

$$\begin{aligned} \left|x^G \cap H\right| &\leqslant 2\left(\frac{|\mathrm{O}^+_4(q)|}{|\mathrm{O}^+_2(q)|^2} + \frac{|\mathrm{O}^+_4(q)|}{|\mathrm{O}^-_2(q)|^2}\right) + \left(\frac{2|\mathrm{O}^+_4(q)|}{|\mathrm{O}_3(q)||\mathrm{O}_1(q)|}\right)^2 + \left(\frac{|\mathrm{O}^+_4(q)|}{|\mathrm{GL}^{\epsilon''}_2(q)|}\right)^2 + \frac{1}{2}\left|\mathrm{O}^+_4(q)\right| \\ &= 2q^6 + 4q^2(q+\epsilon'')^2 - 2q^4 + 4q^2 \end{aligned}$$

(see [4, 3.55(iii)]) and

$$\left|x^G\right| \geqslant 3\frac{|\mathrm{SO}_8^+(q)|}{|\mathrm{GL}_4^{\epsilon''}(q)|2} = \frac{3}{2}q^6(q+\epsilon'')\left(q^2+1\right)\left(q^3+\epsilon''\right) \tag{56}$$

are always sufficient. Similarly, if $t=4$ then

$$\begin{aligned}\left|x^G \cap B\right| &\leqslant 4+\binom{4}{2}(q-\epsilon')^2+\left|\mathrm{O}_2^{\epsilon'}(q):\mathrm{GL}_1^{\epsilon'}(q)\right|^4 = 6(q-\epsilon')^2+20,\\ \left|x^G\cap(H-B)\right| &\leqslant \binom{4}{2}\left|\mathrm{O}_2^{\epsilon'}(q)\right|\left(1+\left|\mathrm{O}_2^{\epsilon'}(q):\mathrm{GL}_1^{\epsilon'}(q)\right|^2\right)+\frac{3}{2}\left|\mathrm{O}_2^{\epsilon'}(q)\right|^2\\ &= 60(q-\epsilon')+6(q-\epsilon')^2\end{aligned}$$

and the desired result follows via (56). Finally, let us assume x is conjugate to $[-I_4, I_4]$. If $t=2$ then the bounds

$$\begin{aligned}\left|x^G\cap H\right| \leqslant{}& \left(\frac{2|\mathrm{O}_4^+(q)|}{|\mathrm{O}_3(q)||\mathrm{O}_1(q)|}\right)^2+\left(\frac{|\mathrm{O}_4^+(q)|}{|\mathrm{O}_2^+(q)|^2}+\frac{|\mathrm{O}_4^+(q)|}{|\mathrm{O}_2^-(q)|^2}\right)^2+2\left(\frac{|\mathrm{O}_4^+(q)|}{|\mathrm{O}_2^+(q^2)|}\right)^2\\ &+1+\frac{3}{2}\left|\mathrm{O}_4^+(q)\right|\\ ={}& 4q^2\left(q^2-1\right)^2+\left(q^4+q^2\right)^2+2q^4\left(q^2-1\right)^2+1\end{aligned}$$

and

$$\left|x^G\right| \geqslant 3\frac{|\mathrm{SO}_8^+(q)|}{|\mathrm{SO}_4^-(q)|^2 4} = \frac{3}{4}q^8\left(q^2-1\right)\left(q^6-1\right)$$

suffice. The case $t=4$ is very similar.

Case 2. $x \in H - \mathrm{PGL}(V)$.

First assume $x\in G$ is a field automorphism of prime order r, so $q=q_0^r$ and [4, 3.48] gives

$$\left|x^G\right| > \frac{1}{4}q^{\frac{1}{2}(n^2-n)(1-\frac{1}{r})}. \tag{57}$$

Now, if r is odd then

$$\begin{aligned}\left|x^G\cap H\right| &\leqslant \sum_{j=0}^{\lfloor t/r\rfloor}\left[\frac{t!}{j!(t-jr)!r^j}\left|\mathrm{O}_{n/t}^{\epsilon'}(q)\right|^{j(r-1)}\left(\frac{|\mathrm{O}_{n/t}^{\epsilon'}(q)|}{|\mathrm{O}_{n/t}^{\epsilon'}(q^{1/r})|}\right)^{t-jr}\right]\\ &< 2^t t! q^{\frac{1}{2}(\frac{n^2}{t}-n)(1-\frac{1}{r})}\end{aligned} \tag{58}$$

and the desired result follows via (57). Next assume $q = q_0^2$ and x is an involutory field or graph-field automorphism, so $\epsilon \neq -$ and (57) holds (with $r = 2$). If n/t is odd then (58) is valid (with $r = 2$) and we find that (57) is always sufficient. If $\epsilon' = +$ then

$$\left|x^G \cap H\right| \leqslant \sum_{j=0}^{\lfloor t/2 \rfloor} \left[\frac{t!}{j!(t-2j)!2^j} \left|\mathrm{O}^+_{n/t}(q)\right|^j \left(\frac{|\mathrm{O}^+_{n/t}(q)|}{|\mathrm{O}^+_{n/t}(q^{1/2})|} + \frac{|\mathrm{O}^+_{n/t}(q)|}{|\mathrm{O}^-_{n/t}(q^{1/2})|} \right)^{t-2j} \right] < 2^{2t} t! q^{\frac{n^2}{4t} - \frac{n}{4}}$$

and (57) is sufficient unless $(n,t,q) = (8,2,4)$ (note that $(n,t,q) \neq (8,4,4)$—see Table 2.1). Here direct calculation yields $f(x,H) < 0.530$. Finally, if $\epsilon' = -$ then t is even (see Table 2.1) and $x^G \cap B\rho$ is non-empty if and only if $\rho \in S_t$ has cycle-shape $(2^{t/2})$. Therefore

$$\left|x^G \cap H\right| \leqslant \frac{t!}{(t/2)!2^{t/2}} \left|\mathrm{O}^-_{n/t}(q)\right|^{\frac{t}{2}} < \frac{t!}{(t/2)!} \left(\frac{q+1}{q} \right)^{\frac{t}{2}} q^{\frac{n^2}{4t} - \frac{n}{4}}$$

and the desired result follows via (57).

Now assume (△) holds. If x is a triality graph-field automorphism then $q = q_0^3$ and [4, 3.48] gives $|x^G| > \frac{1}{4} q^{56/3}$. Further, if $t = 4$ then the trivial bound

$$\left|x^G \cap H\right| < |H| \leqslant 3 \log_2 q.4!2^4(q+1)^4 \tag{59}$$

is always sufficient. On the other hand, if $t = 2$ then we may assume $\epsilon' = +$. Since $\Omega_4^+(q) \cong \mathrm{SL}_2(q) \circ \mathrm{SL}_2(q)$ (central product) we deduce that

$$\left|x^G \cap H\right| \leqslant \frac{4!}{3} \left|\mathrm{SL}_2(q)\right|^2 \frac{|\mathrm{SL}_2(q)|}{|\mathrm{SL}_2(q^{1/3})|} < 16 q^8$$

and the desired result follows since $|x^G| > \frac{1}{4} q^{56/3}$.

Finally, let us assume x is a triality graph automorphism. We begin with the case $t = 4$. If x is a non-G_2 triality (see [4, 3.47]) then $|x^G| > \frac{1}{8} q^{20}$ (see [4, Table 3.10]) and we find that (59) is sufficient for all $q \geqslant 4$. If $q = 3$ then a calculation using GAP [7] yields $f(x,H) < 0.405$. Similarly, if $q = 2$ then $\epsilon' = -$ (see Table 2.1) and using GAP we deduce that $f(x,H) < 0.555$. If x is a G_2-type triality then $|x^G| > \frac{1}{8} q^{14}$ and (59) is sufficient for all $q \geqslant 9$. The cases $5 \leqslant q \leqslant 8$ are easily settled. For example, if $q = 5$ then $|H \cap G_0| \leqslant 62\,208$ (see [9, 4.2.11]), $|x^{G_0}| = 1\,521\,000\,000$ and thus

$$f(x,H) \leqslant \frac{\log(24.62208)}{\log(8.1521000000)} < 0.613.$$

Finally, if $q < 5$ then $\epsilon' = -$ (see Table 2.1) and we compute the following results using GAP [7,11]:

q	$\lvert x^G \cap H\rvert \leqslant$	$\lvert x^G\rvert \geqslant$	$f(x,H) <$
4	800	266 342 400	0.345
3	512	1 166 400	0.447
2	288	14 400	0.592

For the remainder we may assume $t = 2$, so $\epsilon' = +$ and $q \geqslant 3$ (see Table 2.1). As previously remarked, there is an isomorphism $\Omega_4^+(q) \cong \mathrm{SL}_2(q) \circ \mathrm{SL}_2(q)$ and it is helpful to consider the situation at the level of algebraic groups. We have $A_1^4.S_4 \leqslant D_4.S_3$ and a triality graph automorphism τ acts as a 3-cycle on the A_1-factors and centralizes the fixed factor if $C_{D_4}(\tau) = G_2$, otherwise τ induces an inner automorphism of order three on the fixed factor. Therefore, if x is a G_2-type triality then

$$\left|x^G \cap H\right| \leqslant \frac{4!}{3}\left|\mathrm{SL}_2(q)\right|^2 < 8q^6, \qquad \left|x^G\right| > \frac{1}{8}q^{14}$$

and we are left to deal with the case $q = 3$, where direct calculation yields $f(x, H) < 0.604$. Likewise, if x is a non-G_2 triality then

$$\left|x^G \cap H\right| \leqslant \frac{4!}{3}\left|\mathrm{SL}_2(q)\right|^2 \frac{|\mathrm{SL}_2(q)|}{q-1} < 16q^8, \qquad \left|x^G\right| > \frac{1}{8}q^{20}$$

and we conclude that $f(x, H) < 0.582$ for all $q \geqslant 3$. □

3. Proof of Theorem 1.1: $H \in \mathscr{C}_3$

The subgroups in $\mathscr{C}_3$ arise from prime degree field extensions of $\mathbb{F}_q$, where the degree k divides the dimension n of the natural G_0-module V. As advertised in Section 2, here we also deal with the $\mathscr{C}_2$-subgroups of unitary, symplectic and orthogonal groups which stabilize a totally singular $n/2$-decomposition of V. The cases we shall consider in this section are listed in Table 3.1 (see [9, Tables 4.2.A, 4.3.A]).

Proposition 3.1. *The conclusion to Theorem* 1.1 *holds in cases* (i) *and* (ii) *of Table* 3.1.

Proof. We may assume $n \geqslant 3$. Let $\bar{G} = \mathrm{PSL}_n(K)$, $\bar{B} = \mathrm{PSL}_{n/k}(K)$ and let σ be a Frobenius morphism of $\bar{G}$ such that $\bar{G}_\sigma$ has socle $\mathrm{PSL}_n^\epsilon(q)$. Let V denote the natural G_0-module. We only give details for case (i) since a very similar argument applies in case (ii) and the reader can easily make the necessary minor adjustments. We partition the proof into a number of separate cases, where Case i.j is a subcase of Case i.

Table 3.1
The collection $\mathscr{C}_3$

	G_0	Type of H	Conditions
(i)	$\mathrm{PSL}_n^\epsilon(q)$	$\mathrm{GL}_{n/k}^\epsilon(q^k)$	k odd if $\epsilon = -$
(ii)	$\mathrm{PSU}_n(q)$	$\mathrm{GL}_{n/2}(q^2)$	n even
(iii)	$\mathrm{PSp}_n(q)$	$\mathrm{Sp}_{n/k}(q^k)$	n/k even
(iv)	$\mathrm{P}\Omega_n^\epsilon(q)$	$\mathrm{O}_{n/k}^\epsilon(q^k)$	$n/k \geqslant 4$ even
(v)	$\Omega_n(q)$	$\mathrm{O}_{n/k}(q^k)$	nkq odd, $n/k \geqslant 3$
(vi)	$\mathrm{P}\Omega_n^\epsilon(q)$	$\mathrm{O}_{n/2}(q^2)$	$n/2$ odd, $q \equiv -\epsilon$ (4)
(vii)	$\mathrm{PSp}_n(q)$	$\mathrm{GL}_{n/2}^\epsilon(q).2$	q odd
(viii)	$\mathrm{P}\Omega_n^+(q)$	$\mathrm{GL}_{n/2}^{\epsilon'}(q).2$	$n \equiv 0$ (4) if $\epsilon' = -$
(ix)	$\mathrm{P}\Omega_n^-(q)$	$\mathrm{GU}_{n/2}(q).2$	$n \equiv 2$ (4)

Case 1. $x \in H \cap \mathrm{PGL}(V)$.

According to [9, (4.3.10)] we have

$$H \cap \mathrm{PGL}(V) \leqslant \left(\left(\frac{q^k-\epsilon}{q-\epsilon}\right).\mathrm{PGL}^{\epsilon}_{\frac{n}{k}}\left(q^k\right)\right).\langle\phi\rangle = B.k,$$

where ϕ acts on $\mathrm{PGL}^{\epsilon}_{n/k}(q^k)$ as a field automorphism of order k and B is the image of $\mathrm{GL}^{\epsilon}_{n/k}(q^k)$ in $\mathrm{PGL}^{\epsilon}_n(q)$. Let $x \in H \cap \mathrm{PGL}(V)$ be an element of prime order r and write $B = \hat{B}/Z$, where $\hat{B} = \mathrm{GL}^{\epsilon}_{n/k}(q^k)$ and Z is a cyclic group of order $q-\epsilon$. If $x \in B$ then the proof of [4, 3.11] implies that either there exists an element $\hat{x} \in \hat{B}$ of order r such that $|x^B| = |\hat{x}^{\hat{B}}|$, or $r \mid (q-\epsilon)$ and $C_{\bar{G}}(x)$ is non-connected. Set $s = \nu(x)$ with respect to V and note that

$$\left|x^G \cap H\right| \leqslant \left|H \cap \mathrm{PGL}(V)\right| < 2kq^{\frac{n^2}{k}-1}. \tag{60}$$

Case 1.1. $k \geqslant 5$.

If $C_{\bar{G}}(x)$ is non-connected then r divides n and the bounds (14) and (60) are sufficient for all $k \geqslant 5$. Now assume $C_{\bar{G}}(x)$ is connected. If $s \geqslant n/2$ then [4, 3.38] implies that $|x^G| > \frac{1}{2}(q+1)^{-n}q^{(n^2+2n-2)/2}$ and thus (60) is sufficient unless $(n,k,q) = (5,5,2)$. Here r is odd and [4, 3.36] gives $|x^G| > (1/2)(2/3)^4 2^{15}$ since $s \geqslant 3$. We conclude that $f(x,H) < 0.625$ since $|x^G \cap H| \leqslant |H \cap \mathrm{PGL}(V)| \leqslant 155$.

Next suppose $s < n/2$. If $x^G \cap (H - B) \neq \emptyset$ then $r = k$ and

$$x = \begin{cases} [I_{n/k}, \omega I_{n/k}, \ldots, \omega^{k-1} I_{n/k}] & \text{if } p \neq k, \\ [J_k^{n/k}] & \text{if } p = k \end{cases} \tag{61}$$

(up to $\bar{G}$-conjugacy), where $\omega \in K$ is a primitive kth root of unity and J_k is a standard Jordan block of size k. Therefore $s = n(1-1/k) \geqslant n/2$ and so the hypothesis $s < n/2$ implies that $x^G \cap H \subseteq B$. Consequently, we may define $s_0 = \nu(x)$ with respect to the action of x on the natural B-module and we note that $s_0 > 0$. Therefore $s \geqslant k$ and $n \geqslant 3k$ since $s \geqslant ks_0$ (see the proof of [10, 4.2]). Now, if x is unipotent then $|x^G| > \frac{1}{2}(q+1)^{-1}q^{2s(n-s)+1}$ and appealing to [4, 3.15, 3.24, 3.38] we deduce that

$$\begin{aligned}\left|x^G \cap H\right| &< \left(\frac{q^k-\epsilon}{q-\epsilon}\right).2\left(\frac{q^k}{q^k-1}\right)^{\frac{s}{k}} q^{\frac{1}{k}(2ns-s^2-sk)}.\sum_{s_0=1}^{\lfloor s/k \rfloor} k_{s_0,p,u}\left(\mathrm{PGL}^{\epsilon}_{\frac{n}{k}}\left(q^k\right)\right) \\ &< 4\left(\frac{q^k}{q^k-1}\right)^{\frac{s}{k}+1} q^{\frac{1}{k}(2ns-s^2+k^2-k)},\end{aligned}$$

where $k_{s_0,p,u}(\mathrm{PGL}^{\epsilon}_{\frac{n}{k}}(q^k))$ denotes the number of distinct classes of elements y in $\mathrm{PGL}^{\epsilon}_{\frac{n}{k}}(q^k)$ of order p with $\nu(y) = s_0$. These bounds also hold if x is semisimple (see [4, 3.40]) and the desired result follows since $5 \leqslant k \leqslant s \leqslant \frac{1}{2}(n-1)$.

Case 1.2. $k < 5$, $x^G \cap (H - B) \neq \emptyset$.

Here $r = k$ and (61) holds (up to $\bar{G}$-conjugacy). If $k = 3$ then the desired result quickly follows via (60). Now assume $k = 2$, so $\epsilon = +$ (see Table 3.1). Applying [4, 3.43] we deduce that

$$\left|x^G \cap (H - B)\right| \leqslant (q+1)\left|\phi^{\mathrm{PGL}_{n/2}(q^2)}\right| < 2(q+1)q^{\frac{1}{4}n^2-1}. \tag{62}$$

If $p = 2$ then $|x^G| > \frac{1}{2}q^{n^2/2}$ and (62) is sufficient if $n \equiv 2$ (4) since $x^G \cap B$ is empty. Otherwise, if $n \equiv 0$ (4), then any element in $x^G \cap B$ is $\bar{B}$-conjugate to $[J_2^{n/4}]$, whence $|x^G \cap B| < q^{n^2/4}$ and we are left to deal with the case $(n, q) = (4, 2)$, where direct calculation yields $f(x, H) < 0.712$. Now assume $p \neq 2$. If $n \equiv 2$ (4) then either $x^G \cap B$ is empty, or $C_G(x)$ is of type $\mathrm{GL}_{n/2}(q^2)$ and $x^G \cap B = \{z\}$, where z is the unique central involution in B. In this case, the desired result follows via (62) since $|x^G| > \frac{1}{4}q^{n^2/2}$. On the other hand, if $n \equiv 0$ (4) and $C_G(x)$ is of type $\mathrm{GL}_{n/2}(q)^2$ then $x^G \cap B = y_1^B$, while we have $x^G \cap B = \{z\} \cup y_2^B$ if $C_G(x)$ is of type $\mathrm{GL}_{n/2}(q^2)$,

$$y_1 = \begin{pmatrix} & I_{n/4} \\ I_{n/4} & \end{pmatrix}, \qquad y_2 = \begin{pmatrix} & \omega^{q+1} I_{n/4} \\ I_{n/4} & \end{pmatrix}$$

and $\mathbb{F}_{q^2}^* = \langle \omega \rangle$. In either case we deduce that

$$\left|x^G \cap B\right| \leqslant 1 + \frac{1}{2}\left|\mathrm{GL}_{n/2}\left(q^2\right) : \mathrm{GL}_{n/4}\left(q^2\right)^2\right| < q^{\frac{1}{4}n^2}, \qquad \left|x^G\right| > \frac{1}{4}q^{\frac{1}{2}n^2}$$

and (62) is sufficient unless $(n, q) = (4, 3)$, where direct calculation gives $f(x, H) < 0.661$.

Case 1.3. $k < 5$, $x^G \cap H \subseteq B$, $r = p$.

Suppose $x \in B$ has associated partition $\lambda' = (m^{a_m}, \ldots, 1^{a_1}) \vdash m$, where $m = n/k$. Then the Jordan form of x on V corresponds to the partition $\lambda = (m^{ka_m}, \ldots, 1^{ka_1}) \vdash n$. In particular, the corresponding $\bar{B}$- and $\bar{G}$-classes are uniquely determined by λ and [6, 2.3] implies that $\dim x^{\bar{G}} = k^2 \dim x^{\bar{B}}$. Therefore

$$\left|x^G \cap H\right| \leqslant \left|x^{\mathrm{PGL}_{n/k}^{\epsilon}(q^k)}\right| < 2^t q^{\frac{1}{k}\dim x^{\bar{G}}}, \qquad \left|x^G\right| > \frac{1}{2}\left(\frac{q}{q+1}\right)^t q^{\dim x^{\bar{G}}-1} \tag{63}$$

where t is the number of non-zero terms a_j in λ. If $t = 1$ then $n \geqslant 2k$, [6, 2.4] implies that $\dim x^{\bar{G}} \geqslant \frac{1}{2}n^2$ and one can check that the bounds in (63) are sufficient unless $(n, k, q) = (4, 2, 3)$, where direct calculation yields $f(x, H) < 0.545$ (note that $x^G \cap (H - B)$ is non-empty if $(n, k, q) = (4, 2, 2)$). Now assume $t \geqslant 2$. Then $n \geqslant \frac{1}{2}kt(t+1)$ and [4, 3.25] yields

$$\dim x^{\bar{G}} \geqslant k^2\left(\frac{n}{k}\left(t^2 - t\right) - \frac{1}{4}t^4 + \frac{1}{6}t^3 + \frac{1}{4}t^2 - \frac{1}{6}t\right).$$

Therefore (63) is sufficient unless $(k, t) = (2, 2)$ and $q \leqslant 3$. In these cases the bounds $|x^G \cap H| < 2q^{\frac{1}{2}\dim x^{\bar{G}}}$ and $|x^G| > \frac{1}{2}q^{\dim x^{\bar{G}}}$ are good enough since $\dim x^{\bar{G}} \geqslant 4n - 8$.

Case 1.4. $k < 5$, $x^G \cap H \subseteq B$, $r \neq p$.

Suppose $r = 2$. If $C_{\bar{G}}(x)$ is connected then $|x^G \cap H| < 2q^{\frac{1}{k}\dim x^{\bar{G}}}$, $|x^G| > \frac{1}{2}(q+1)^{-1}q^{\dim x^{\bar{G}}+1}$ and the result follows since $\dim x^{\bar{G}} \geqslant 2k(n-k)$. On the other hand, if $C_{\bar{G}}(x)$ is non-connected then the hypotheses imply that $k = 3$ and $n \equiv 0\,(6)$, in which case the subsequent bounds $|x^G \cap H| < 2q^{n^2/6}$ and $|x^G| > \frac{1}{4}(q+1)^{-1}q^{n^2/2+1}$ are always sufficient.

Now assume $r > 2$. For now we will assume $C_{\bar{G}}(x)$ is connected. Let $i \geqslant 1$ be minimal such that $r \mid (q^i - 1)$ and $i_0 \geqslant 1$ minimal such that $r \mid (q^{ki_0} - 1)$, so

$$i_0 = \begin{cases} i/k & \text{if } k \text{ divides } i, \\ i & \text{otherwise.} \end{cases} \tag{64}$$

Define the integers l and d as in [4, 3.32] and define $c = c(i, \epsilon)$ as in the statement of [4, 3.33] (see Case 1.1 in the proof of Proposition 2.5).

Suppose k does not divide i. Then $i = i_0$ and σ- and σ^k-orbits coincide (see [4, 3.26]). In particular, if $c > 1$ and $x \in G$ has associated σ-tuple $\mu = (l, a_1, \ldots, a_t)$ then each non-zero term in μ must be multiple of k. Indeed, x acts on the natural B-module with associated σ^k-tuple $\mu' = (l/k, a_1/k, \ldots, a_t/k)$ and thus $\dim x^{\bar{G}} = k^2 \dim x^{\bar{B}}$. Now

$$\left|x^G\right| > \frac{1}{2}\left(\frac{q}{q+1}\right)^{\alpha d} q^{\dim x^{\bar{G}}} \tag{65}$$

and $|x^G \cap H| < 2\log_2 q.2^{d(1-\alpha)+\alpha}q^{\frac{1}{k}\dim x^{\bar{G}}}$, where α is defined in (4) and

$$\dim x^{\bar{G}} \geqslant n^2 - l^2 - \frac{1}{c}\big(n - l - kc(d-1)\big)^2 - ck^2(d-1).$$

The same bounds hold if $c = 1$ and the result follows since $n \geqslant l + kdc$.

Now assume k does divide i, so $i_0 = i/k$ and each non-trivial σ-orbit is a union of k distinct σ^k-orbits. In particular, we note that $c > 1$ and thus we may assume x has associated σ-tuple $\mu = (l, a_1, \ldots, a_t)$. For $k \leqslant 3$ we claim that

$$\left|x^G \cap H\right| < 2\log_2 q.2^{kd(1-\alpha)+\alpha}\left(\frac{q^k}{q^k - 1}\right)^d q^{\frac{1}{k}\dim x^{\bar{G}}}, \tag{66}$$

where α is defined as before. Applying (65) and the lower bound on $\dim x^{\bar{G}}$ given in [4, 3.33], we find that (66) is sufficient except for a handful of cases for which we can calculate more accurate bounds. For example, if $(k, i, q) = (2, 2, 2)$ and $n = l + 2$ then the reader can check that the bounds

$$\left|x^G \cap H\right| \leqslant 2\frac{|\mathrm{GL}_{n/2}(4)|}{|\mathrm{GL}_{n/2-1}(4)||\mathrm{GL}_1(4)|}, \qquad \left|x^G\right| \geqslant \frac{|\mathrm{GL}_n(2)|}{|\mathrm{GL}_{n-2}(2)||\mathrm{GL}_1(2^2)|}$$

are sufficient for all $n \geqslant 4$. It remains to justify (66).

Proof of (66). Modulo field and graph automorphisms, each B-class in $x^G \cap B$ is determined by a tk-tuple of the form

$$(b_{11}, \ldots, b_{1k}, b_{21}, \ldots, b_{2k}, \ldots, b_{tk}),$$

where the b_{ij} are non-negative integers such that $\sum_j b_{ij} = a_i$ for each $1 \leqslant i \leqslant t$. Let $\mathscr{B}$ denote the set of all such tk-tuples. For each $b \in \mathscr{B}$, let $x_b \in B$ represent the corresponding B-class in $x^G \cap B$ and fix $\hat{x}_b \in \hat{B}$ of order r such that $|x_b^B| = |\hat{x}_b^{\hat{B}}|$. Accounting for the possible effect of field and graph automorphisms, it follows that

$$\left|x^G \cap H\right| \leqslant 2\log_2 q . \sum_{b \in \mathscr{B}} \left|\hat{x}_b^{\hat{B}}\right|.$$

Assume for now that $k = 3$, so $|\hat{x}_b^{\hat{B}}| < 2^{3d(1-\alpha)+\alpha} q^{3\dim x_b^{\bar{B}}}$ for all $b \in \mathscr{B}$. Let a be the number of terms a_j in μ which are not divisible by 3, and note that if

$$\Sigma := \sum_{b \in \mathscr{B}} q^{3\dim x_b^{\bar{B}}} < 3^a \left(\frac{q^3}{q^3 - 1}\right)^d q^{3\zeta} \tag{67}$$

holds, where $\zeta = \max_{b \in \mathscr{B}} \dim x_b^{\bar{B}} = \frac{1}{9}\dim x^{\bar{G}} - \frac{2}{9}ac$, then

$$\left|x^G \cap H\right| < 2\log_2 q . 2^{3d(1-\alpha)+\alpha} \left(\frac{3}{q^{2c/3}}\right)^a \left(\frac{q^3}{q^3 - 1}\right)^d q^{\frac{1}{3}\dim x^{\bar{G}}}$$

and (66) follows. To establish (67) we argue by induction on d. Without loss of generality we may assume that $a_1 > 0$.

Suppose $d = 1$. For $0 \leqslant j \leqslant a_1$ define

$$\Sigma_j = \sum_{b \in \mathscr{B}_j} q^{3\dim x_b^{\bar{B}}}, \qquad \zeta_j = \max_{b \in \mathscr{B}_j} \dim x_b^{\bar{B}},$$

where a tuple $b \in \mathscr{B}$ belongs to $\mathscr{B}_j \subseteq \mathscr{B}$ if and only if $b_{11} = j$. Clearly $\Sigma = \Sigma_0 + \cdots + \Sigma_{a_1}$. Next fix j and observe that if $b \in \mathscr{B}_j$ and $b_{12} = v$ then $\dim x_b^{\bar{B}} = f(v) = c_1 v^2 + c_2 v + c_3$ for some constants c_i, with $c_1 < 0$. Now $f(v)$ is even (it is the dimension of a $\bar{B}$-class) and thus

$$\Sigma_j < 2\left(1 + \left(q^3\right)^2 + \left(q^3\right)^4 + \cdots + \left(q^3\right)^{\zeta_j - 2}\right) + \eta q^{3\zeta_j} < \eta\left(\frac{q^6 + 2 - \eta}{q^6 - 1}\right) q^{3\zeta_j},$$

where $\eta = 2$ if $a_1 - j$ is odd, otherwise $\eta = 1$. Now, if $a = 0$ then $\zeta = \zeta_j$ if and only if $j = \frac{1}{3}a_1$ and it follows that

$$\Sigma < 2^2\left(\frac{q^6}{q^6 - 1}\right)\left(1 + \left(q^3\right)^2 + \left(q^3\right)^4 + \cdots + \left(q^3\right)^{\zeta - 2}\right) + \left(\frac{q^6 + 1}{q^6 - 1}\right) q^{3\zeta} < \left(\frac{q^3}{q^3 - 1}\right) q^{3\zeta}$$

since $\frac{2}{3}a_1$ is even. Similarly, if $a = 1$ then

$$\Sigma < 2^2\left(\frac{q^6}{q^6-1}\right)\left(1+\left(q^3\right)^2+\left(q^3\right)^4+\cdots+\left(q^3\right)^{\zeta-2}\right)+\left(2\left(\frac{q^6}{q^6-1}\right)+\frac{q^6+1}{q^6-1}\right)q^{3\zeta}$$
$$< 3\left(\frac{q^3}{q^3-1}\right)q^{3\zeta}$$

and we conclude that (67) holds when $d=1$.

Now assume $d>1$. For $0\leqslant j\leqslant a_1$ define $\mathscr{B}_j$, Σ_j and ζ_j as before. If $a_1\equiv 0$ (3) then the inductive hypothesis implies that

$$\Sigma_j < 3^a\left(\frac{q^3}{q^3-1}\right)^{d-1}q^{3\zeta_j}$$

and thus

$$\Sigma < 3^a\left(\frac{q^3}{q^3-1}\right)^{d-1}\left(2\left(1+\left(q^3\right)^2+\cdots+\left(q^3\right)^{\zeta-2}\right)+q^{3\zeta}\right) < 3^a\left(\frac{q^3}{q^3-1}\right)^{d}q^{3\zeta}.$$

The case $a_1\not\equiv 0$ (3) is very similar. This establishes (67) and thus (66) holds when $k=3$. The argument when $k=2$ is similar (and shorter).

Finally, let us assume r is odd and $C_{\bar{G}}(x)$ is non-connected. Then the hypothesis $x^G\cap H\subseteq B$ implies that $r\neq k$, so rk divides n. If $k=3$ then the bounds (14) and (60) are sufficient, so assume $k=2$ and $\epsilon=+$. Define $i\geqslant 1$ as before and observe that our earlier arguments apply if $i>1$. For example, if i is even then $|x^G|>\frac{1}{2}q^{n^2(1-1/r)}$ and (66) implies that

$$\left|x^G\cap H\right| < 2^{r-1}\left(\frac{q^2}{q^2-1}\right)^{\frac{1}{2}(r-1)}q^{\frac{1}{2}n^2(1-\frac{1}{r})}$$

since $d\leqslant\frac{1}{2}(r-1)$ and $\mathcal{E}_x$, the multiset of eigenvalues of $\hat{x}\in \mathrm{GL}_n(q)$, is $\mathrm{Aut}(G_0)$-invariant. One can check that this bound with (14) is always sufficient. If $i=1$ and x is $\bar{G}_\sigma$-conjugate to x_0 (in the notation of [4, 3.35]) then $x^G\cap H=x^H$ and the result follows via (14) since

$$\left|x^G\cap H\right|\leqslant\frac{|\mathrm{GL}_{n/2}(q^2)|}{|\mathrm{GL}_{n/2r}(q^2)|^r r} < 2^{r-1}q^{\frac{1}{2}n^2(1-\frac{1}{r})}.$$

On the other hand, if x is not $\bar{G}_\sigma$-conjugate to x_0 then

$$\left|x^G\cap H\right|\leqslant(r-1)\frac{|\mathrm{GL}_{n/2}(q^2)|}{|\mathrm{GL}_{n/2r}(q^{2r})|r} < 2q^{\frac{1}{2}n^2(1-\frac{1}{r})}$$

and once again (14) is always sufficient.

Case 2. $x\in H-\mathrm{PGL}(V)$.

Let us begin by assuming $x\in G$ is a field automorphism of prime order r, so $q=q_0^r$ and $r\neq k$ since every element of order k in $H\cap \mathrm{P}\Gamma\mathrm{L}(V)$ lies in $B.\langle\phi\rangle\leqslant \mathrm{PGL}(V)$, where $\mathrm{P}\Gamma\mathrm{L}(V)$ is the projective general semilinear group on V. Applying [4, 3.15, 3.43] we deduce that

$$\left|x^G \cap H\right| \leqslant \left(\frac{q^k-\epsilon}{q-\epsilon}\right)\frac{|\mathrm{PGL}^{\epsilon}_{n/k}(q^k)|}{|\mathrm{PGL}^{\epsilon}_{n/k}(q^{k/r})|} < 2\left(\frac{q^k-1}{q-1}\right)q^{k(\frac{n^2}{k^2}-1)(1-\frac{1}{r})}$$

and thus (37) is sufficient unless $(n,k,r)=(3,3,2)$. Here $\epsilon=+$ and $f(x,H)<0.812$ since $|x^G \cap H| \leqslant q^2+q+1$ and $|x^G| > \frac{1}{6}q^4$. Similar reasoning applies if $x \in G$ is an involutory graph-field automorphism. (Note that k is odd in this case since every involution in H lies in $\mathrm{PGL}(V).\langle\gamma\rangle$ if $k=2$, where γ is an involutory graph automorphism.)

To complete the proof, let us assume $x \in G$ is an involutory graph automorphism. First suppose $n \geqslant 3k$ and k is odd. Then $x^G \cap H \subseteq Bx$ and x induces an involutory graph automorphism on B such that $C_B(x)$ and $C_{G_0}(x)$ are of the same type. In particular, if n is even and $C_{G_0}(x)$ is symplectic then

$$\left|x^G \cap H\right| \leqslant \left(\frac{q^k-\epsilon}{q-\epsilon}\right)\frac{|\mathrm{PGL}^{\epsilon}_{n/k}(q^k)|}{|\mathrm{Sp}_{n/k}(q^k)|} < 2\left(\frac{q^k-1}{q-1}\right)q^{\frac{n^2}{2k}-\frac{n}{2}-k}$$

and the result follows via (39). The non-symplectic case is very similar. Next assume k is odd and $n<3k$. If $n=k$ then the bounds $|x^G \cap H| \leqslant (q-1)^{-1}(q^n-1)$ and $|x^G| > \frac{1}{2}(q+1)^{-1}q^{(n^2+n-2)/2}$ are always sufficient. If $n=2k$ then x induces an automorphism on $\hat{B}=\mathrm{GL}^{\epsilon}_2(q^k)$ which restricts to an inner automorphism i_x of $\mathrm{SL}_2(q^k)$. Now, if i_x is non-trivial then $C_{G_0}(x)$ is non-symplectic and (39) is sufficient since

$$\left|x^G \cap H\right| \leqslant \left(\frac{q^k-\epsilon}{q-\epsilon}\right)\left(\frac{|\mathrm{PGL}_2(q^k)|}{|\mathrm{PGO}^+_2(q^k)|}+\frac{|\mathrm{PGL}_2(q^k)|}{|\mathrm{PGO}^-_2(q^k)|}\right) \leqslant \left(\frac{q^k-1}{q-1}\right)q^{2k}.$$

On the other hand, if i_x centralizes $\mathrm{SL}_2(q^k)$ then $C_{G_0}(x)$ is symplectic and again the result follows via (39) since $|x^G \cap H| \leqslant (q-1)^{-1}(q^k-1)$.

Next assume $k=2$ and $n \neq 4$. Then $\epsilon=+$ (see Table 3.1) and we observe that $C_{G_0}(x)$ is non-symplectic if $n \equiv 2\ (4)$. Therefore, for any n, we have $x^G \cap H \subseteq Bx \cup Bx\phi$ where x acts on B as an involutory graph automorphism such that $C_B(x)$ and $C_{G_0}(x)$ are of the same type, while $x\phi$ induces an involutory graph-field automorphism on B. Now, if $n \equiv 0\ (4)$ and $C_{G_0}(x)$ is symplectic then [4, 3.43] gives

$$\left|x^G \cap H\right| \leqslant (q+1)\frac{|\mathrm{PGL}_{n/2}(q^2)|}{|\mathrm{Sp}_{n/2}(q^2)|}+\frac{|\mathrm{PGL}_{n/2}(q^2)|}{|\mathrm{PGU}_{n/2}(q)|} < \left(2q^{\frac{n}{2}+1}+q+1\right)q^{\frac{1}{4}n^2-\frac{n}{2}-2}$$

and $|x^G| > \frac{1}{2}q^{(n^2-n-4)/2}$. If $n \geqslant 8$ then these bounds are sufficient unless $(n,q)=(8,2)$, where $B=\mathrm{GL}_4(4)$ and direct calculation yields $f(x,H)<0.573$. The non-symplectic case is similar. Finally, if $n=4$ and $C_{G_0}(x)$ is non-symplectic then $f(x,H)<0.699$ since

$$\left|x^G \cap H\right| \leqslant (q+1)\left(\frac{|\mathrm{PGL}_2(q^2)|}{|\mathrm{PGO}^+_2(q^2)|}+\frac{|\mathrm{PGL}_2(q^2)|}{|\mathrm{PGO}^-_2(q^2)|}\right)+\frac{|\mathrm{PGL}_2(q^2)|}{|\mathrm{PGU}_2(q)|} = q\left(q^4+q^3+q^2+1\right)$$

and $|x^G| \geqslant (4, q-1)^{-1}q^4(q^2-1)(q^3-1)$. On the other hand, if $C_{G_0}(x)$ is symplectic then

$$\left|x^G \cap H\right| \leqslant (q+1) + \frac{|\mathrm{PGL}_2(q^2)|}{|\mathrm{PGU}_2(q)|} = q^3+2q+1, \qquad \left|x^G\right| \geqslant (4, q-1)^{-1}q^2\left(q^3-1\right)$$

and thus $f(x, H) < 3/4$ unless $q = 2$. Here $f(x, H) = (\log 13)/(\log 28) \approx 0.770^*$ and this exceptional case is recorded in Table 1.1. □

Proposition 3.2. *The conclusion to Theorem* 1.1 *holds in cases* (iii)–(vi) *of Table* 3.1.

Proof. These cases are all very similar and we only give details for (iii). Here the statement of Theorem 1.1 gives $\iota = 1/(n+2)$ if $k = 2$. Define $\bar{G} = \mathrm{PSp}_n(K)$, $\bar{B} = \mathrm{PSp}_{n/k}(K)$ and let σ be a Frobenius morphism of $\bar{G}$ such that $\bar{G}_\sigma$ has socle $G_0 = \mathrm{PSp}_n(q)$. According to [4, 3.3], if $(n, p) = (4, 2)$ then we may assume G does not contain any graph-field automorphisms. (Similarly, in cases (iv) and (vi) we are free to assume G does not contain a triality automorphism if $G_0 = \mathrm{P}\Omega_8^+(q)$—see [4, 3.3].)

According to [9, p.116] we have $H \cap \mathrm{PGL}(V) \leqslant B.\langle\phi\rangle = \tilde{H}$, where ϕ acts on B as a field automorphism of order k and

$$B \cong \begin{cases} \langle z\rangle \times \mathrm{PSp}_{n/2}(q^2) & \text{if } k = 2 \text{ and } p \neq 2, \\ \mathrm{PGSp}_{n/k}(q^k) & \text{otherwise.} \end{cases}$$

Here $z \in \bar{G}_\sigma - G_0$ is an involution with $C_G(z)$ of type $\mathrm{Sp}_{n/2}(q^2)$.

Now, if $x \in H - \mathrm{PGL}(V)$ has prime order r then x is a field automorphism, so $q = q_0^r$, $r \neq k$ and the result follows since [4, 3.43, 3.48] imply that

$$\left|x^G \cap H\right| \leqslant \left|\mathrm{Sp}_{n/k}\left(q^k\right) : \mathrm{Sp}_{n/k}\left(q^{k/r}\right)\right| < 2q^{\frac{n}{2k}(n+k)(1-\frac{1}{r})}, \qquad \left|x^G\right| > \frac{1}{4}q^{\frac{1}{2}n(n+1)(1-\frac{1}{r})}.$$

For the remainder, let us assume $x \in H \cap \mathrm{PGL}(V)$ has prime order r. Arguing as in the proof of the previous proposition, applying [4, 3.24, 3.38, 3.40], we quickly reduce to the case $k < 5$.

Case 1. $k < 5$, $x^G \cap (H - B) \neq \emptyset$.

Here $r = k$ and applying [4, 3.43] we deduce that

$$\left|x^G \cap (H - B)\right| \leqslant (k-1)\left|\mathrm{Sp}_{n/k}\left(q^k\right) : \mathrm{Sp}_{n/k}(q)\right| < 2(k-1)q^{\frac{n}{2k}(k-1)(\frac{n}{k}+1)}. \tag{68}$$

Let us start by assuming $k = 3$. If $p = 3$ then x has associated partition $\lambda = (3^{n/3})$ (see (61)) and thus $|x^G| > \frac{1}{2}q^{n(n+1)/3}$. Furthermore, if $x^G \cap B$ is non-empty then $n \equiv 0$ (18) and $|x^G \cap B| < q^{n(n+3)/9}$ since each $y \in x^G \cap B$ is $\bar{B}$-conjugate to $[J_3^{n/9}]$. These bounds with (68) are always sufficient. The case $p \neq 3$ is very similar. Next assume $(k, p) = (2, 2)$, so $|x^G| > \frac{1}{2}q^{n^2/4}$ since x is $\bar{G}$-conjugate to $a_{n/2}$. Furthermore, either $x^G \cap B$ is empty, or $n \equiv 0$ (8) and $|x^G \cap B| < 2q^{n^2/8}$ since each $y \in x^G \cap B$ is $\bar{B}$-conjugate to $a_{n/4}$ (see (69) below). Therefore, if $n \geqslant 8$ then (68) is sufficient unless $(n, q) = (8, 2)$, where direct calculation yields $f(x, H) < 0.668$. If $n = 4$

then $|x^G \cap H| = |x^G \cap (H - B)| = q(q^2+1)$, $|x^G| = q^4 - 1$ and thus $f(x, H) < 0.851$ for all $q \geqslant 2$.

Finally, let us assume $k = 2$ and p is odd. Here $C_{\bar{G}}(x)$ is non-connected and there are four cases to consider. If $C_G(x)$ is of type $\mathrm{Sp}_{n/2}(q^2)$ then $|x^G| > \frac{1}{4}q^{n^2/4}$, $x^G \cap B = \{z\}$ and we find that (68) is always sufficient if $n \geqslant 8$, while $f(x, H) < 0.774$ if $n = 4$ since

$$|x^G \cap H| \leqslant 1 + \frac{1}{2}|\mathrm{Sp}_2(q^2) : \mathrm{Sp}_2(q)| = 1 + \frac{1}{2}q(q^2+1), \qquad |x^G| = \frac{1}{2}q^2(q^2-1).$$

A similar argument applies when $C_G(x)$ is of type $\mathrm{Sp}_{n/2}(q)^2$ so let us assume $C_G(x)$ is of type $\mathrm{GL}^{\epsilon}_{n/2}(q)$. If $q \equiv \epsilon\ (4)$ then $x^G \cap B = (1, t)^B$, where $t \in \mathrm{PSp}_{n/2}(q^2)$ is an involution with centralizer of type $\mathrm{GL}_{n/4}(q^2)$ (note that $t \in \mathrm{PSp}_{n/2}(q^2)$ since $q^2 \equiv 1\ (4)$—see [8, Table 4.5.1]). Therefore

$$|x^G \cap B| \leqslant \frac{|\mathrm{Sp}_{n/2}(q^2)|}{|\mathrm{GL}_{n/4}(q^2)|2} < q^{\frac{1}{8}n(n+4)}, \qquad |x^G| > \frac{1}{4}(q+1)^{-1}q^{\frac{1}{4}(n^2+2n+4)}$$

and (68) is sufficient unless $(n, q) = (4, 3)$, where direct calculation gives $f(x, H) < 0.732$. Similarly, if $q \equiv -\epsilon\ (4)$ then $x^G \cap B = (z, t)^B$ so the previous bounds hold and again it remains to deal with the case $(n, q) = (4, 3)$. This time we calculate that $f(x, H) < 0.651$.

Case 2. $k < 5$, $x^G \cap H \subseteq B$, $r = p$.

First assume $p = 2$ and observe that the natural embedding $\mathrm{Sp}_{n/k}(q^k) \hookrightarrow \mathrm{Sp}_n(q)$ induces the following maps on involution class representatives:

$$a_l \mapsto a_{kl}, \qquad c_l \mapsto c_{kl}, \qquad b_l \mapsto \begin{cases} b_{kl} & \text{if } k \text{ is odd,} \\ c_{kl} & \text{if } k = 2. \end{cases} \tag{69}$$

If x is G-conjugate to a_{kl} then $\dim x^{\bar{G}} = k^2 \dim x^{\bar{B}}$, [4, 3.22] implies that $|x^G \cap H| < 2q^{\dim x^{\bar{B}}}$, $|x^G| > \frac{1}{2}q^{\dim x^{\bar{G}}}$ and the desired result follows since $\dim x^{\bar{G}} \geqslant 2n - 4$ and $n \geqslant 4k$. Similarly, if x is G-conjugate to b_{kl} or c_{kl} then the bounds $|x^G \cap H| < 2q^{l(n-kl+k)}$ and $|x^G| > \frac{1}{2}q^{kl(n-kl+1)}$ are sufficient unless $(k, n, l, q) = (2, 4, 1, 2)$. Here direct calculation yields $f(x, H) < 0.712$ since $|x^G \cap H| = 15$ and $|x^G| = 45$.

Now assume p is odd. Let $\lambda = (m^{ka_m}, \ldots, 1^{ka_1}) \vdash n$ be the associated partition of x, where $m = n/k$. We claim that

$$|x^G \cap H| < 2^t q^{(\frac{1}{2} + \frac{\delta_{2,k}}{n+2})\dim x^{\bar{G}}}, \qquad |x^G| > \left(\frac{1}{2}\right)^{t+1}\left(\frac{q}{q+1}\right)^{t\delta_{2,k}} q^{\dim x^{\bar{G}}}, \tag{70}$$

where t is the number of non-zero terms a_j in λ. In view of [4, 3.18], it is sufficient to show that

$$\dim x^{\bar{B}} \leqslant \left(\frac{1}{2k} + \frac{\delta_{2,k}}{2n+4}\right)\dim x^{\bar{G}}. \tag{71}$$

Let $d = \sum_{i \text{ odd}} a_i$ and observe that $\dim x^{\bar{G}} = k^2 \dim x^{\bar{B}} - \frac{1}{2}(k-1)(n-kd)$. If $k = 3$ then [6, 2.3, 2.4] imply that $\dim x^{\bar{G}} \geqslant n^2/4 + n/2 - 9d^2/4 + 9d/2$ (minimal if $\lambda = (2^{n/2-3d/2}, 1^{3d})$)

and (71) quickly follows since $n \geqslant 3d+6$. Similar reasoning applies when $k=2$ (note that we have equality in (71) if $\lambda = (2^{n/2})$).

Let us now apply (70). If $t=1$ then $\dim x^{\bar{G}} \geqslant \frac{1}{4}n(n+2)$ and we are left to deal with the case $(n,k,q)=(4,2,3)$, where direct calculation yields $f(x,H) < 0.800$. Now assume $t \geqslant 2$. We claim that

$$\dim x^{\bar{G}} \geqslant g(n,t) = \begin{cases} \frac{3}{2}nt(t-1) - \frac{9}{8}t^4 + \frac{3}{4}t^3 + \frac{15}{8}t^2 - \frac{3}{4}t - \frac{3}{4} & \text{if } k=3, \\ (t^2-t)n - \frac{1}{2}t^4 + \frac{1}{3}t^3 + t^2 - \frac{1}{3}t - \frac{1}{2} & \text{if } k=2. \end{cases} \tag{72}$$

If $k=2$ then [6, 2.4] implies that $\dim x^{\bar{G}} \geqslant \dim y^{\bar{G}}$, where $y \in \bar{G}$ is unipotent with associated partition $(t^2, \dots, 2^2, 1^{n-t^2-t+2}) \vdash n$. In this case (72) follows from [6, 2.3]. Now assume $k=3$ and define

$$f(\rho) = \frac{1}{2}n^2 + \frac{1}{2}n - \sum_{i<j} i a_i a_j - \frac{1}{2}\sum_i i a_i^2 - \frac{1}{2}\sum_{i \text{ odd}} a_i,$$

where $\rho = (n^{a_n}, \dots, 1^{a_1}) \vdash n$ is an arbitrary partition of n. Then $f(\rho) = \dim y^{\bar{G}}$ if ρ corresponds to a unipotent element $y \in \bar{G}$, while $g(n,t) = f(\rho')$ for

$$\rho' = \left(t^3, (t-1)^3, \dots, 2^3, 1^{n-\frac{3}{2}t^2-\frac{3}{2}t+3}\right) \vdash n.$$

The claim now follows by arguing as in the proof of [4, 3.25].

If $k=3$ then $n \geqslant \frac{3}{2}t(t+1)$ and (72) implies that the bounds in (70) are sufficient unless $(t,q)=(2,3)$. Here $n \geqslant 12$ and (70) is good enough since $\dim x^{\bar{G}} \geqslant 3n-6$ (minimal if $\lambda = (2^3, 1^{n-6})$). Now assume $k=2$. Then $n \geqslant t(t+1)$ and if we assume $t \geqslant 3$ then (70) (with (72)) is sufficient unless $(t,q)=(3,3)$. In this case $n \geqslant 20$ (since a_1 and a_3 must be positive multiples of 4) and the result follows via (70) since (72) gives $\dim x^{\bar{G}} \geqslant 6n-24$. Finally, if $t=2$ and $\lambda \neq (2^2, 1^{n-4})$ then $\dim x^{\bar{G}} \geqslant 4n-12$ (minimal if $\lambda = (2^4, 1^{n-8})$) and (70) is sufficient. If $\lambda = (2^2, 1^{n-4})$ then $|x^G \cap H| < q^n$, $|x^G| > \frac{1}{4}(q+1)^{-1}q^{2n-1}$ and the desired result follows.

Case 3. $k<5$, $x^G \cap H \subseteq B$, $r \neq p$.

If $r=2$ then the hypothesis $x^G \cap H \subseteq B$ implies that x is $\bar{G}$-conjugate to $[-I_{2ka}, I_{n-2ka}]$ for some positive integer $a < n/4k$. In this case, the subsequent bounds $|x^G \cap H| < 2q^{\frac{1}{k}\dim x^{\bar{G}}}$ and $|x^G| > \frac{1}{2}q^{\dim x^{\bar{G}}}$ are always sufficient since $\dim x^{\bar{G}} \geqslant 2k(n-2k)$. Now assume $r>2$. Define the integers i and i_0 as in the proof of the previous proposition and observe that (64) holds. Let $\mu = (l, a_1, \dots, a_t)$ denote the associated σ-tuple of $x \in G$ and let d be the number of non-zero a_j terms in μ. Note that (41) holds and that d is even if i is odd.

If k does not divide i then $i=i_0$ and thus σ- and σ^k-orbits coincide. Therefore each term in μ is divisible by k and we calculate that $k^2 \dim x^{\bar{B}} = \dim x^{\bar{G}} + \frac{1}{2}(n-l)(k-1)$, whence

$$\left|x^G \cap H\right| < \log_2 q . 2^{\frac{d}{2}(e-1)} q^{\frac{1}{k}\dim x^{\bar{G}} + \frac{1}{2k}(n-l)(k-1)},$$

where $e=2$ if i is odd, otherwise $e=1$. The result now follows from (41) since

$$\dim x^{\bar{G}} \geqslant \frac{1}{2}\left(n^2+n-l^2-l-\frac{1}{ei}\big(n-l-ki(d-e)\big)^2-k^2i(d-e)\right). \tag{73}$$

Now assume k divides i, so $i_0=i/k$ and each non-trivial σ-orbit is a union of k distinct σ^k-orbits. If $k=2$ and $i \equiv 2$ (4) then each term in μ must be even and we deduce that

$$\left|x^G \cap H\right| < \log_2 q.2^d q^{\frac{1}{2}\dim x^{\bar{G}}+\frac{1}{4}(n-l)},$$

where (73) holds and $n \geqslant l+2di$. Then (41) is sufficient unless $(n,l,i,d)=(4,0,2,1)$ and $q \in \{2,4\}$. Here direct calculation yields $f(x,H)<0.813$. Now assume $(k, i \bmod 4) \neq (2,2)$. Then arguing as in the proof of Proposition 3.1 (in particular, the proof of (66)) we deduce that

$$\left|x^G \cap H\right| < \log_2 q.2^{\frac{1}{2}kd(e-1)}\left(\frac{q^k}{q^k-1}\right)^{\frac{d}{e}} q^{\frac{1}{k}\dim x^{\bar{G}}+\frac{1}{2k}(n-l)(k-1)}$$

and the desired result follows via (41) and the lower bound on $\dim x^{\bar{G}}$ given in [4, 3.33]. □

Proposition 3.3. *The conclusion to Theorem* 1.1 *holds in cases* (vii)–(ix) *of Table* 3.1.

Proof. All three cases are very similar and we only give details for (viii) and (ix), which we deal with simultaneously; say H is of type $\mathrm{GL}^{\epsilon'}_{n/2}(q).2$. Define $\bar{G}=\mathrm{PSO}_n(K)$, $\bar{B}=\mathrm{PSL}_{n/2}(K)$, where $n \geqslant 8$, and let σ be a Frobenius morphism of $\bar{G}$ such that $\bar{G}_\sigma$ has socle $G_0=\mathrm{P}\Omega^{\epsilon}_n(q)$. In addition, let σ' be a Frobenius morphism of $\bar{B}$ such that $\bar{B}_{\sigma'} \cong \mathrm{PGL}^{\epsilon'}_{n/2}(q)$. Recall from the statement of Theorem 1.1 that $\iota=1/(n-2)$ and note that we may assume G is without triality if $(n,\epsilon)=(8,+)$ (see [4, 3.3]). Also observe that $H \cap \mathrm{PGL}(V) \leqslant \tilde{H}$, where $\tilde{H}=C_{\tilde{G}}(z)$ for a suitable involution $z \in \bar{G}_\sigma$ if p is odd, while $\tilde{H}=\mathrm{GL}^{\epsilon'}_{n/2}(q).\langle\psi\rangle=B.2$ if $p=2$, with ψ inducing an involutory graph automorphism on B (see [9, 4.2.7, 4.3.18], for example).

If $x \in H-\mathrm{PGL}(V)$ has odd prime order r then x is a field automorphism, $q=q_0^r$ and [4, 3.48] states that $|x^G|>\frac{1}{4}q^{n(n-1)(1-1/r)/2}$. Moreover, [4, 3.15, 3.38] imply that

$$\left|x^G \cap H\right| \leqslant \frac{1}{2}(q-\epsilon')\frac{|\mathrm{PGL}^{\epsilon'}_{n/2}(q)|}{|\mathrm{PGL}^{\epsilon'}_{n/2}(q^{1/r})|} < (q+1)q^{\frac{1}{4}(n^2-4)(1-\frac{1}{r})}$$

and the result follows. If x is an involution then $q=q_0^2$ and $\epsilon'=+$ since every involution in H lies in $\mathrm{PGL}(V)$ if $\epsilon'=-$. Again, applying [4, 3.15, 3.38] we deduce that

$$\left|x^G \cap H\right| \leqslant \frac{1}{2}(q-1)\left(\frac{|\mathrm{PGL}_{n/2}(q)|}{|\mathrm{PGL}_{n/2}(q^{1/2})|}+\frac{|\mathrm{PGL}_{n/2}(q)|}{|\mathrm{PGU}_{n/2}(q^{1/2})|}\right) < 2(q-1)q^{\frac{1}{8}n^2-\frac{1}{2}}$$

and the bound $|x^G|>\frac{1}{4}q^{n(n-1)/4}$ is always sufficient. Now assume $x \in H \cap \mathrm{PGL}(V)$. For the reader's convenience, we partition the proof into three cases.

Case 1. $r = p$.

First assume $p = 2$. If $x^G \cap H \subseteq B$ then x is $\bar{G}$-conjugate to a_{2l} for some $1 \leqslant l < n/4$ and the desired result follows since $|x^G \cap H| < 2q^{l(n-2l)}$ and $|x^G| > \frac{1}{2}q^{2l(n-2l-1)}$. Now assume $x^G \cap (H - B) \neq \emptyset$, so $\nu(x) = n/2$. If $n \equiv 0$ (4) and x is $\tilde{G}$-conjugate to $a_{n/2}$ then [4, 3.22] implies that $|x^G \cap B| < 2q^{n^2/8}$, $|x^G| > \frac{1}{2}q^{n(n-2)/4}$ and we have

$$\left|x^G \cap (H - B)\right| \leqslant \left|\mathrm{GL}^{\epsilon'}_{n/2}(q) : \mathrm{Sp}_{n/2}(q)\right| < 2(q+1)q^{\frac{1}{8}(n^2-2n-8)}$$

since each $y \in x^G \cap (H - B)$ acts on B as a symplectic-type graph automorphism. These bounds are sufficient unless $(n, q) = (8, 2)$, where direct calculation yields $f(x, H) < 0.721$. On the other hand, if x is G-conjugate to $b_{n/2}$ or $c_{n/2}$ (according to the parity of $n/2$) then $|x^G| > \frac{1}{2}q^{n^2/4}$ and $|x^G \cap H| < 2(q+1)q^{(n^2+2n-8)/8}$ since $x^G \cap B$ is empty and each $y \in x^G \cap (H - B)$ induces a non-symplectic graph automorphism on B. These bounds are always sufficient.

Now assume $p > 2$. Let $\lambda = (m^{2a_m}, \dots, 1^{2a_1}) \vdash n$ denote the associated partition of $x \in G$, where $m = n/2$, and write t for the number of non-zero terms a_j. Then applying [6, Theorem 1] and [4, 3.21] we deduce that

$$\left|x^G \cap H\right| < 2^t q^{(\frac{1}{2}+\frac{1}{n-2})\dim x^{\bar{G}}}, \qquad \left|x^G\right| > \left(\frac{1}{2}\right)^{t+1}\left(\frac{q}{q+1}\right)^t q^{\dim x^{\bar{G}}}. \tag{74}$$

If $t = 1$ then [6, 2.4] implies that $\dim x^{\bar{G}} \geqslant \frac{1}{4}n(n-2)$ and thus (74) is sufficient unless $(n, q) = (8, 3)$, where direct calculation yields $f(x, H) < 0.707$. Now assume $t \geqslant 2$ and observe that $n \geqslant t(t+1)$ and $\dim x^{\bar{G}} \geqslant n(t^2 - t) - t^4/2 + t^3/3 - t/3$ (minimal if $\lambda = (t^2, \dots, 2^2, 1^{n-t^2-t+2})$). If $t \geqslant 3$ then these bounds imply that (74) is sufficient unless $(t, q) = (3, 3)$ and $12 \leqslant n \leqslant 16$. These cases are easily settled through direct calculation. Now assume $t = 2$ and set $d = \sum_{i \text{ odd}} a_i$. If $d = 0$ then there exists a non-zero a_j with $j \geqslant 4$, hence $p \geqslant 5$, $\dim x^{\bar{G}} \geqslant n^2/4 + 3n/2 - 8$ (minimal if $\lambda = (4^2, 2^{n/2-4})$) and it is easy to check that (74) is always sufficient. Now assume $d > 0$ and observe that $n \equiv 0$ (4) if and only if d is even. Applying [6, 2.3, 2.4] we deduce that

$$\dim x^{\bar{B}} = \frac{1}{2}\dim x^{\bar{G}} + \frac{1}{4}n - \frac{1}{2}d, \qquad \dim x^{\bar{G}} \geqslant \frac{1}{4}n^2 - \frac{1}{2}n - d^2 + d$$

(minimal if $\lambda = (2^{n/2-d}, 1^{2d})$). In particular, if $d = 1$ then $n \geqslant 10$ and the bounds

$$\left|x^G \cap H\right| < 2q^{\frac{1}{2}\dim x^{\bar{G}}+\frac{1}{4}n-\frac{1}{2}}, \qquad \left|x^G\right| > \frac{1}{4}(q+1)^{-1}q^{\dim x^{\bar{G}}+1}$$

are always sufficient. Finally, if $d \geqslant 2$ then $n \geqslant 2d + 4$ and the desired result follows since $|x^G \cap H| < 2q^{\frac{1}{4}(2\dim x^{\bar{G}}+n-2d)}$ and $|x^G| > \frac{1}{4}q^{\dim x^{\bar{G}}}$.

Case 2. $r \neq p$, $r = 2$.

If $\nu(x) < n/2$ then x is $\bar{G}$-conjugate to $[-I_{2a}, I_{n-2a}]$, for some positive integer $a < n/4$, and the bounds $|x^G \cap H| < 2q^{a(n-2a)}$ and $|x^G| > \frac{1}{4}(q+1)^{-1}q^{2a(n-2a)+1}$ are sufficient without exception. Now assume $\nu(x)=n/2$. If $n\equiv 2\ (4)$ and x is an involutory graph automorphism of G_0 then $|x^G|>\frac{1}{4}q^{n^2/4}$ and the result follows since

$$\left|x^G\cap H\right|\leqslant \frac{1}{2}(q-\epsilon')\frac{|\mathrm{PGL}^{\epsilon'}_{n/2}(q)|}{|\mathrm{SO}_{n/2}(q)|}<(q+1)q^{\frac{1}{8}(n^2+2n-8)}.$$

On the other hand, if $n\equiv 2\ (4)$ and $C_G(x)$ is of type $\mathrm{GL}^{\epsilon''}_{n/2}(q)$ then (54) holds and $\epsilon=\epsilon'=\epsilon''$ (see Table 3.1 and [4, Table 3.8]). Moreover, we have

$$\left|x^G\cap H\right|\leqslant \sum_{j=0}^{\frac{1}{4}(n-2)}\frac{|\mathrm{GL}^{\epsilon'}_{n/2}(q)|}{|\mathrm{GL}^{\epsilon'}_{j}(q)||\mathrm{GL}^{\epsilon'}_{n/2-j}(q)|}<2\left(\frac{q^2}{q^2-1}\right)q^{\frac{1}{8}n^2-\frac{1}{2}}$$

and the desired result follows. Now assume $n\equiv 0\,(4)$, so $\epsilon=+$ (see Table 3.1). If $C_G(x)$ is of type $\mathrm{O}^{+}_{n/2}(q^2)$ or $\mathrm{O}^{\epsilon''}_{n/2}(q)^2$ then

$$\begin{aligned}\left|x^G\cap H\right|&\leqslant \frac{1}{2}(q-\epsilon')\left(\frac{|\mathrm{PGL}^{\epsilon'}_{n/2}(q)|}{|\mathrm{PGO}^{+}_{n/2}(q)|}+\frac{|\mathrm{PGL}^{\epsilon'}_{n/2}(q)|}{|\mathrm{PGO}^{-}_{n/2}(q)|}\right)+\frac{|\mathrm{GL}^{\epsilon'}_{n/2}(q)|}{|\mathrm{GL}^{\epsilon'}_{n/4}(q)|^2 2}+\frac{|\mathrm{GL}^{\epsilon'}_{n/2}(q)|}{|\mathrm{GL}_{n/4}(q^2)|2}\\&<\left((q+1)q^{\frac{n}{4}-1}+2\right)q^{\frac{1}{8}n^2}\end{aligned}$$

and the bound $|x^G|>\frac{1}{8}q^{n^2/4}$ is always sufficient. Finally, let us assume $n\equiv 0\ (4)$ and $C_G(x)$ is of type $\mathrm{GL}^{\epsilon''}_{n/2}(q)$. If $\epsilon'=\epsilon''$ then

$$\begin{aligned}\left|x^G\cap H\right|&\leqslant \frac{1}{2}(q-\epsilon')\frac{|\mathrm{PGL}^{\epsilon'}_{n/2}(q)|}{|\mathrm{Sp}_{n/2}(q)|}+\frac{|\mathrm{GL}^{\epsilon'}_{n/2}(q)|}{|\mathrm{GL}^{\epsilon'}_{n/4}(q)|^2 2}+\sum_{j=0}^{\frac{1}{4}n-1}\frac{|\mathrm{GL}^{\epsilon'}_{n/2}(q)|}{|\mathrm{GL}^{\epsilon'}_{j}(q)||\mathrm{GL}^{\epsilon'}_{n/2-j}(q)|}\\&<\left(q^{\frac{n}{4}+1}+q+1\right)q^{\frac{1}{8}n^2-\frac{1}{4}n-1}+2\left(q^2-1\right)^{-1}q^{\frac{1}{8}n^2}\end{aligned}$$

and (54) is sufficient unless $(n,q)=(8,3)$, where direct calculation yields $f(x,H)<0.681$. Similarly, if $\epsilon'=-\epsilon''$ then

$$\left|x^G\cap H\right|\leqslant \frac{1}{2}(q-\epsilon')\frac{|\mathrm{PGL}^{\epsilon'}_{n/2}(q)|}{|\mathrm{Sp}_{n/2}(q)|}+\frac{|\mathrm{GL}^{\epsilon'}_{n/2}(q)|}{|\mathrm{GL}_{n/4}(q^2)|2}<\left(q^{\frac{n}{4}+1}+q+1\right)q^{\frac{1}{8}n^2-\frac{1}{4}n-1}$$

and (54) is always sufficient.

Case 3. $r\neq p$, $r>2$.

Since r is odd, each $y\in x^G\cap H$ lifts to an element $\hat{y}\in \mathrm{GL}^{\epsilon'}_{n/2}(q)$ of order r. Let $i\geqslant 1$ be minimal such that $r\mid (q^i-1)$, let $\mu=(l,a_1,\ldots,a_t)$ denote the associated σ-tuple of x and let d be the number of non-zero terms a_j in μ, so d is even if i is odd. Define the integer $c=c(i,\epsilon')$ as in the statement of [4, 3.33] (see Case 1.1 in the proof of Proposition 2.5).

First assume c is even. Then each non-zero term in μ is even and

$$\dim x^{\bar{B}} = \frac{1}{4}\left(n^2 - l^2 - c\sum_j a_j^2\right) = \frac{1}{2}\dim x^{\bar{G}} + \frac{1}{4}(n-l).$$

Furthermore, we deduce that

$$\left|x^G \cap H\right| < \log_2 q.2^{\frac{d}{e}}\left(\frac{q+1}{q}\right)^{\frac{1}{2}(1-\epsilon')} q^{\frac{1}{2}\dim x^{\bar{G}} + \frac{1}{4}(n-l)}$$

and

$$\left|x^G\right| > \frac{1}{2}\left(\frac{q}{q+1}\right)^{d(2-e)+1} q^{\dim x^{\bar{G}}}, \tag{75}$$

where

$$\dim x^{\bar{G}} \geqslant \frac{1}{2}\left(n^2 - n - l^2 + l - \frac{1}{ei}\big(n - l - 2i(d-e)\big)^2 - 4i(d-e)\right)$$

and $e = 2$ if i is odd, otherwise $e = 1$. Now $n \geqslant l + 2di$ and these bounds are sufficient except for a handful of cases with which we can calculate directly.

Now assume c is odd. Then $\dim x^{\bar{B}} \leqslant \frac{1}{2}\dim x^{\bar{G}} + \frac{1}{4}(n-l)$ and we claim that

$$\left|x^G \cap H\right| < \log_2 q.2^{\frac{d}{2}(1+\epsilon')}\left(\frac{q+1}{q}\right)^{\frac{1}{2}(1-\epsilon')}\left(\frac{q}{q-1}\right)^{\frac{d}{e}} q^{\frac{1}{2}\dim x^{\bar{G}} + \frac{1}{4}(n-l)}. \tag{76}$$

To see this, suppose $\epsilon' = +$, in which case i is odd and d is even. Then modulo field automorphisms, each B-class in $x^G \cap B$ is determined by a choice of s-tuple $(b_1, \ldots, b_s)$, where each $b_j \leqslant a_j$ is a non-negative integer and $s = (r-1)/2c = t/2$. Let $\mathscr{B}$ denote the set of all such s-tuples and for each $b \in \mathscr{B}$ let $x_b \in B$ represent the B-class corresponding to b. Then

$$\left|x^G \cap H\right| \leqslant \log_2 q.\sum_{b\in\mathscr{B}} \left|\hat{x}_b^{\mathrm{GL}_{n/2}(q)}\right|,$$

where $\hat{x}_b \in \mathrm{GL}_{n/2}(q)$ has order r and $|x_b^B| = |\hat{x}_b^{\mathrm{GL}_{n/2}(q)}|$, and thus (76) holds if

$$\Sigma := \sum_{b\in\mathscr{B}} q^{\dim x_b^{\bar{B}}} \leqslant \left(\frac{q}{q-1}\right)^{\frac{d}{2}} q^{\frac{1}{2}\dim x^{\bar{G}} + \frac{1}{4}(n-l)}.$$

If $a \geqslant 0$ is the number of terms a_j in μ which are odd then

$$\alpha := \max_{b\in\mathscr{B}} \dim x_b^{\bar{B}} = \frac{1}{2}\dim x^{\bar{G}} + \frac{1}{4}(n-l) - \frac{1}{4}ai$$

and so it suffices to show that

$$\Sigma \leqslant 2^{\frac{a}{2}}\left(\frac{q}{q-1}\right)^{\frac{d}{2}} q^{\alpha}.$$

We now proceed by induction on d. The argument is similar to the proof of (66) and we leave the details to the reader. The case $\epsilon' = -$ is very similar.

Now $n \geqslant l + di$ and if we apply (76), together with (75) and the lower bound on $\dim x^{\bar{G}}$ given in [4, 3.33], we find that we are left to deal with a handful of exceptional cases. For example, if $\epsilon' = +$ then it remains to consider the cases $(n,q) \in \{(10,4),(8,8),(8,7),(8,4)\}$ for $(i,l,d) = (1,0,2)$. These are easily settled. For instance, if $(n,q) = (10,4)$ then $r = 3$ and $f(x,H) < 0.631$ since $|x^G| \geqslant |\Omega_{10}^{+}(4) : \mathrm{GL}_5(4)|$ and

$$\left|x^G \cap H\right| \leqslant 2 + 2\frac{|\mathrm{GL}_5(4)|}{|\mathrm{GL}_4(4)||\mathrm{GL}_1(4)|} + 2\frac{|\mathrm{GL}_5(4)|}{|\mathrm{GL}_3(4)||\mathrm{GL}_2(4)|}. \qquad \square$$

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