




What Bordism-Theoretic Anomaly Cancellation Can Do for U

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Abstract: We perform a bordism computation to show that the $E_{7(7)}(\mathbb{R})$ U-duality symmetry of 4d $\mathcal{N} = 8$ supergravity could have an anomaly invisible to perturbative methods; then we show that this anomaly is trivial. We compute the relevant bordism group using the Adams and Atiyah–Hirzebruch spectral sequences, and we show the anomaly vanishes by computing η -invariants on the Wu manifold, which generates the bordism group.

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1. Introduction

One of the most surprising discoveries in the field of string theory is the existence of duality symmetries. These symmetries show that the same theory can be described in superficially different ways. In some cases, this can be seen via a transformation of the parameters of the theory, or even the spacetime itself. One such symmetry is U-duality, given by the group $E_{n(n)}(\mathbb{Z})$. By starting with an 11-dimensional theory which encompasses the type IIA string theory, and compactifying on an n -torus, we gain an $SL_n(\mathbb{Z})$ symmetry from the mapping class group on the n -torus. We arrive at the same theory by compactifying 10d type IIB on an $n - 1$ -torus, and obtain an $O_{n-1,n-1}(\mathbb{Z})$ symmetry related to T-duality. The group $E_{n(n)}(\mathbb{Z})$ is then generated by the two aforementioned groups.

In the low energy regime of the 11d theory, which is 11d supergravity, we have an embedding of $E_{n(n)}(\mathbb{Z}) \hookrightarrow E_{n(n)}$ upon applying the torus compactification procedure. The latter group is the U-duality of supergravity. One finds a maximally noncompact form of E_n after the compactification, and this is denoted $E_{n(n)}(\mathbb{R})$. The maximally noncompact form of a Lie group of rank n contains n more noncompact generators than compact generators. For the purpose of this paper, we reduce 11-dimensional supergravity on a 7-dimensional torus. This gives a maximal supergravity theory, i.e. 4d $\mathcal{N} = 8$ supergravity, with an E_7 symmetry.¹ The noncompact form is $E_{7(7)}$ which is 133-dimensional and is diffeomorphic, but not isomorphic, to $SU_8/\{\pm 1\} \times \mathbb{R}^{70}$.

Because this is a symmetry of the theory, one can ask if it is anomalous, and in particular if there are any global anomalies. Since 4d $\mathcal{N} = 8$ supergravity arises as the low energy effective theory of string theory, then a strong theorem of quantum gravity saying that there are no global symmetries implies that the U-duality symmetry must be gaugeable. Therefore, the existence of any global anomaly would require a mechanism for its cancellation. It would therefore be an interesting question to consider if additional topological terms need to be added to cancel the nonperturbative anomaly as in [DDHM22], but we show that with the matter content of 4d maximal supergravity is sufficient to cancel the anomaly on the nose.

The purpose of this paper is to answer:

Question 1.1. Can 4d $\mathcal{N} = 8$ supergravity with an $E_{7(7)}$ symmetry have a nontrivial anomaly topological field theory (TFT)? If it can, how do we show that the anomaly vanishes?

We find that theories with this symmetry type can have a nontrivial anomaly, so we have to check whether 4d $\mathcal{N} = 8$ supergravity carries this nontrivial anomaly.

Theorem 1.2. *The group of deformation classes of 5d reflection-positive, invertible TFTs on spin-SU₈ manifolds is isomorphic to $\mathbb{Z}/2$. In this group, the anomaly field theory of 4d $\mathcal{N} = 8$ supergravity is trivial.*

Remark 1.3. We also show that the nontrivial deformation class contains the invertible field theory whose partition function on a closed 5-manifold M is $\int_M w_2(M)w_3(M)$, which is nonzero on the Wu manifold. This theory, and its appearance as the anomaly theory of some 4d QFTs, are discussed in detail by Wang–Wen–Witten [WWW19], who describe how this anomaly can be canceled by tensoring with a TFT carrying the same anomaly.

¹ Dimensional reduction of IIB supergravity on an 6-dimensional torus also yields the same symmetry.

The order of the global anomaly is equal to the order of a bordism group in degree 5 that can be computed from the Adams spectral sequence. We find that the global anomaly is $\mathbb{Z}/2$ valued, but nonetheless is trivial when we take into account the matter content of 4d $\mathcal{N} = 8$ supergravity. In order to see the cancellation we first find the manifold generator of the bordism group, which happens to be the Wu manifold, and compute η -invariants on it. This bordism computation is also mathematically intriguing because we find ourselves working over the entire Steenrod algebra, however the specific properties of the problem we are interested in make this tractable.

This work only focuses on U-duality as the group $E_{7(7)}$ rather than $E_{7(7)}(\mathbb{Z})$, because the cohomology of the discrete group that arises in string theory is not known, and a strategy we employ of taking the maximal compact subgroup will not work. But one could imagine running a similar Adams computation for the group $E_7(\mathbb{Z})$ and checking that the anomaly vanishes. There are also a plethora of dualities that arise from compactifying 11d supergravity that one can also compute anomalies of, among them are the U-dualities that arise from compactifying on lower dimensional tori.

The structure of the paper is as follows: in Sect. 2 we present the symmetries and tangential structure for the maximal 4d supergravity theory with U-duality symmetry and turn it into a bordism computation. We also give details on the field content of the theory and how it is compatible with the type of manifold we are considering. In Sect. 3 we review the possibility of global anomalies, and invertible field theories. In Sect. 4 we perform the spectral sequence computation and give the manifold generator for the bordism group in question. In Sect. 5 we show that the anomaly vanishes by considering the field content on the manifold generator.

2. Placing the U-Duality Symmetry on Manifolds

In this section, we review how the $E_{7(7)}$ U-duality symmetry acts on the fields of 4d $\mathcal{N} = 8$ supergravity; then we discuss what kinds of manifolds are valid backgrounds in the presence of this symmetry. We assume that we have already Wick-rotated into Euclidean signature. We determine a Lie group H_4 with a map $\rho_4: H_4 \rightarrow O_4$ such that 4d $\mathcal{N} = 8$ supergravity can be formulated on 4-manifolds M equipped with a metric and an H_4 -connection $P, \Theta \rightarrow M$, such that $\rho_4(\Theta)$ is the Levi-Civita connection. As we review in Sect. 3, anomalies are classified in terms of bordism; once we know H_4 and ρ_4 , Freed–Hopkins’ work [FH21b] tells us what bordism groups to compute.

The field content of 4d $\mathcal{N} = 8$ supergravity coincides with the spectrum of type IIB closed string theory compactified on T^6 and consists of the following fields:

- 70 scalar fields,
- 56 gauginos (spin 1/2),
- 28 vector bosons (spin 1),
- 8 gravitinos (spin 3/2), and
- 1 graviton (spin 2).

Cremmer–Julia [CJ79] exhibited an $e_{7(7)}$ symmetry of this theory, meaning an action on the fields for which the Lagrangian is invariant. Here, $e_{7(7)}$ is the Lie algebra of the real, noncompact Lie group $E_{7(7)}$, which is the split form of the complex Lie group $E_7(\mathbb{C})$. Cartan [Car14, §VIII] constructed $E_{7(7)}$ explicitly as follows: the 56-dimensional vector space

$$V := \Lambda^2(\mathbb{R}^8) \oplus \Lambda^2((\mathbb{R}^8)^*) \tag{2.1}$$

has a canonical symplectic form coming from the duality pairing. $E_{7(7)}$ is defined to be the subgroup of $\text{Sp}(V)$ preserving the quartic form

$$q(x^{ab}, y_{cd}) = x^{ad} y_{bc} x^{cd} y_{da} - \frac{1}{4} x^{ab} y_{ab} x^{cd} y_{cd} + \frac{1}{96} \left(\epsilon_{abc\dots h} x^{ab} x^{cd} x^{ef} x^{gh} + \epsilon^{abc\dots h} y_{ab} y_{cd} y_{ef} y_{gh} \right). \tag{2.2}$$

Thus, by construction, $E_{7(7)}$ comes with a 56-dimensional representation, which we denote **56**.

$E_{7(7)}$ is noncompact; its maximal compact is $\text{SU}_8/\{\pm 1\}$, giving us an embedding $\mathfrak{su}_8 \subset \mathfrak{e}_{7(7)}$. Thus $\pi_1(E_{7(7)}) \cong \mathbb{Z}/2$; let $\tilde{E}_{7(7)}$ denote the universal cover, which is a double cover.

There is an action of $\mathfrak{e}_{7(7)}$ on the fields of 4d $\mathcal{N} = 8$ supergravity, but in this paper we only need to know how $\mathfrak{su}_8 \subset \mathfrak{e}_{7(7)}$ acts: we will see in Sect. 3.2 that the anomaly calculation factors through the maximal compact subgroup of $E_{7(7)}$. For the full $\mathfrak{e}_{7(7)}$ story, see [FM13, §2]; the $\mathfrak{e}_{7(7)}$ -action exponentiates to an $\tilde{E}_{7(7)}$ -action on the fields. The \mathfrak{su}_8 -action is:

- (1) The 70 scalar fields can be repackaged into a single field valued in $E_{7(7)}/(\text{SU}_8/\{\pm 1\})$ with trivial \mathfrak{su}_8 -action.
- (2) The gauginos transform in the representation **56** := $\Lambda^3(\mathbb{C}^8)$.
- (3) The vector bosons transform in the 28-dimensional representation $\Lambda^2(\mathbb{C}^8)$, which we call **28**.
- (4) The gravitinos transform in the defining representation of \mathfrak{su}_8 , which we denote **8**.
- (5) The graviton transforms in the trivial representation.

The presence of fermions (the gauginos and gravitinos) means that we must have data of a spin structure, or something like it, to formulate this theory. In quantum physics, a strong form of G -symmetry is to couple to a G -connection, suggesting that we should formulate 4d $\mathcal{N} = 8$ supergravity on Riemannian spin 4-manifolds M together with an $\tilde{E}_{7(7)}$ -bundle $P \rightarrow M$ and a connection on P . The spin of each field tells us which representation of Spin_4 it transforms as, and we just learned how the fields transform under the $\tilde{E}_{7(7)}$ -symmetry, so we can place this theory on manifolds M with a *geometric* $\text{Spin}_4 \times \tilde{E}_{7(7)}$ -structure, i.e. a metric and a principal $\text{Spin}_4 \times \tilde{E}_{7(7)}$ -bundle $P \rightarrow M$ with connection whose induced O_4 -connection is the Levi-Civita connection. The fields are sections of associated bundles to P and the representations they transform in. The Lagrangian is invariant under the $\text{Spin}_4 \times \tilde{E}_{7(7)}$ -symmetry, so defines a functional on the space of fields, and we can study this field theory as usual.

However, we can do better! We will see that the representations above factor through a quotient H_4 of $\text{Spin}_4 \times \tilde{E}_{7(7)}$, which we define below in (2.4), so the same procedure above works with H_4 in place of $\text{Spin}_4 \times \tilde{E}_{7(7)}$. A lift of the structure group to H_4 is less data than a lift all the way to $\text{Spin}_4 \times \tilde{E}_{7(7)}$, so we expect to be able to define 4d $\mathcal{N} = 8$ supergravity on more manifolds. This is similar to the way that the $\text{SL}_2(\mathbb{Z})$ duality symmetry in type IIB string theory can be placed not just on manifolds with a $\text{Spin}_{10} \times \text{Mp}_2(\mathbb{Z})$ -structure,² but on the larger class of manifolds with a $\text{Spin}_{10} \times_{\{\pm 1\}} \text{Mp}_2(\mathbb{Z})$ -structure [PS16, §5],

² Here $\text{Mp}_2(\mathbb{Z})$ is the *metaplectic group*, a central extension of $\text{SL}_2(\mathbb{Z})$ of the form

$$1 \longrightarrow \{\pm 1\} \longrightarrow \text{Mp}_2(\mathbb{Z}) \longrightarrow \text{SL}_2(\mathbb{Z}) \longrightarrow 1. \tag{2.3}$$

or how certain SU_2 gauge theories can be defined on manifolds with a $\text{Spin}_n \times_{\{\pm 1\}} SU_2$ structure [WWW19, AM21, Law23].

Let $-1 \in \text{Spin}_4$ be the nonidentity element of the kernel of $\text{Spin}_4 \rightarrow SO_4$ and let x be the nonidentity element of the kernel of $\tilde{E}_{7(7)} \rightarrow E_{7(7)}$. The key fact allowing us to descend to a quotient is that -1 acts nontrivially on the representations of $\text{Spin}_4 \times \tilde{E}_{7(7)}$ above, and x acts nontrivially, but on a given representation, these two elements both act by 1 or they both act by -1 . We can check this even though we have not specified the entire $e_{7(7)}$ -action on the fields, because $-1 \in \tilde{E}_{7(7)}$ is contained in the copy of SU_8 in $\tilde{E}_{7(7)}$, and we have specified the \mathfrak{su}_8 -action. Therefore the $\mathbb{Z}/2$ subgroup of $\text{Spin}_4 \times \tilde{E}_{7(7)}$ generated by $(-1, x)$ acts trivially, and we can form the quotient

$$H_4 := \text{Spin}_4 \times_{\{\pm 1\}} \tilde{E}_{7(7)} = (\text{Spin}_4 \times \tilde{E}_{7(7)}) / \langle (-1, x) \rangle. \tag{2.4}$$

The representations that the fields transform in all descend to representations of H_4 , so following the procedure above, we can define 4d $\mathcal{N} = 8$ supergravity on manifolds M with a *geometric H_4 -structure*, i.e. a metric, an H_4 -bundle $P \rightarrow M$, and a connection on P whose induced O_4 -connection is the Levi-Civita connection.

Remark 2.5. As a check to determine that we have the correct symmetry group, we can compare with other string dualities. The U-duality group contains the S-duality group for type IIB string theory, which comes geometrically from the fact that 4d $\mathcal{N} = 8$ supergravity can be constructed by compactifying type IIB string theory on T^6 . Therefore the ways in which the duality groups mix with the spin structure must be compatible. As explained by Pantev-Sharpe [PS16, §5], the $SL_2(\mathbb{Z})$ duality symmetry of type IIB string theory mixes with the spin structure to form the group $\text{Spin}_{10} \times_{\{\pm 1\}} \text{Mp}_2(\mathbb{Z})$, where $\text{Mp}_2(\mathbb{Z})$ is the metaplectic group from Footnote 2.

Therefore the way in which the U-duality group mixes with $\{\pm 1\} \subset \text{Spin}_4$ must also be nontrivial. Extensions of a group G by $\{\pm 1\}$ are classified by $H^2(BG; \{\pm 1\})$. If G is connected, BG is simply connected and the Hurewicz and universal coefficient theorems together provide a natural identification

$$H^2(BG; \{\pm 1\}) \xrightarrow{\cong} \text{Hom}(\pi_2(BG), \{\pm 1\}) = \text{Hom}(\pi_1(G), \{\pm 1\}). \tag{2.6}$$

As $\pi_1(E_{7(7)}) \cong \mathbb{Z}/2$, there is only one nontrivial extension of $E_{7(7)}$ by $\{\pm 1\}$, namely the universal cover $\tilde{E}_{7(7)} \rightarrow E_{7(7)}$. That is, by comparing with S-duality, we again obtain the group H_4 , providing a useful double-check on our calculation above.

3. Anomalies, Invertible Field Theories, and Bordism

3.1. Generalities on anomalies and invertible field theories. It is sometimes said that in mathematical physics, if you ask four people what an anomaly is, you will get five answers. The goal of this section is to explain our perspective on anomalies, due to Freed–Teleman [FT14], and how to reduce the determination of the anomaly to a question in algebraic topology, an approach due to Freed–Hopkins–Teleman [FHT10] and Freed–Hopkins [FH21b].

Whatever an anomaly is, it signals a mild inconsistency in the definition of a quantum field theory. For example, if a quantum field theory Z is n -dimensional, one ought to be able to evaluate it on a closed n -manifold M , possibly equipped with some geometric structure, to obtain a complex number $Z(M)$, called the *partition function* of M . If Z has an anomaly, $Z(M)$ might only be defined after some additional choices, and in the

absence of those choices $Z(M)$ is merely an element of a one-dimensional complex vector space $\alpha(M)$.

The theory Z is local in M , so $\alpha(M)$ should also be local in M . One way to express this locality is to ask that $\alpha(M)$ is the state space of M for some $(n + 1)$ -dimensional quantum field theory α , called the *anomaly field theory* α of Z . The condition that the state spaces of α are one-dimensional follows from the fact that α is an *invertible field theory* [FM06, Definition 5.7], meaning that there is some other field theory α^{-1} such that $\alpha \otimes \alpha^{-1}$ is isomorphic to the trivial field theory $\mathbf{1}$.^{3,4} This approach to anomalies is due to Freed–Teleman [FT14]; see also Freed [Fre14, Fre19].

We can therefore understand the possible anomalies associated to a given n -dimensional quantum field theory Z by classifying the $(n + 1)$ -dimensional invertible field theories with the same symmetry type as Z . The classification of invertible *topological* field theories is due to Freed–Hopkins–Teleman [FHT10], who lift the question into stable homotopy theory; see Grady–Pavlov [GP21, §5] for a recent generalization to the nontopological setting.

Supergravity with its U-duality symmetry is a unitary quantum field theory, and therefore its anomaly theory satisfies the Wick-rotated analogue of unitarity: *reflection positivity*. Freed–Hopkins [FH21b] classify reflection-positive invertible field theories, again using stable homotopy theory. Let $\mathbf{O} := \lim_{n \rightarrow \infty} \mathbf{O}_n$ be the infinite orthogonal group.

Theorem 3.1 (Freed–Hopkins [FH21b, Theorem 2.19]). *Let $n \geq 3$, H_n be a compact Lie group, and $\rho_n: H_n \rightarrow \mathbf{O}_n$ be a homomorphism whose image contains \mathbf{SO}_n . Then there is canonical data of a topological group H and a continuous homomorphism $\rho: H \rightarrow \mathbf{O}$ such that the pullback of ρ along $\mathbf{O}_n \hookrightarrow \mathbf{O}$ is ρ_n .*

In other words, when the hypotheses of this theorem hold, we have more than just H_n -structures on n -manifolds; we can define H -structures on manifolds of any dimension, by asking for a lift of the classifying map of the stable tangent bundle $M \rightarrow \mathbf{BO}$ to BH ; a manifold equipped with such a lift is called an *H -manifold*. Following Lashof [Las63], this allows us to define bordism groups Ω_k^H and a homotopy-theoretic object called the *Thom spectrum MTH* , whose homotopy groups are the H -bordism groups.⁵ See [BC18, §2] for more on the definition of MTH and its context in stable homotopy theory.

Theorem 3.2 (Freed–Hopkins [FH21b]). *With H_n as in Theorem 3.1, the abelian group of deformation classes of n -dimensional reflection-positive invertible topological field theories on H_n -manifolds is naturally isomorphic to the torsion subgroup of $[MTH, \Sigma^{n+1} I_{\mathbb{Z}}]$.*

³ The relationship between invertibility and one-dimensional state spaces is that $\alpha \otimes \alpha^{-1} \simeq \mathbf{1}$ means that on any closed, n -manifold M , there is an isomorphism of complex vector spaces $\alpha(M) \otimes \alpha^{-1}(M) \cong \mathbf{1}(M) = \mathbb{C}$. This forces $\alpha(M)$ and $\alpha^{-1}(M)$ to be one-dimensional. Often the converse is also true: see Schommer-Pries [SP18].

⁴ In some cases, we do not want to assume α extends to closed n -manifolds; see Freed–Teleman [FT14] for more information. But the U-duality anomaly we investigate in this paper does extend.

⁵ In homotopy theory, it is common to work with H -structures on the stable normal bundle instead of on the tangent bundle; the Thom spectrum whose homotopy groups are bordism groups of manifolds with H -structures on their stable normal bundle is called MH . For all groups H we discuss in this paper, an H -structure on the tangent bundle of M induces an H -structure on the stable normal bundle of M and vice versa, and $MTH \simeq MH$; thus this distinction will not matter for us. However, in some other situations, it does matter; for example, $MTPin^+ \not\simeq MPin^+$.

Freed–Hopkins then conjecture (*ibid.*, Conjecture 8.37) that the whole group $[MTH, \Sigma^{n+1}I_{\mathbb{Z}}]$ classifies all reflection-positive invertible field theories, topological or not.

The notation $[MTH, \Sigma^{n+1}I_{\mathbb{Z}}]$ means the abelian group of homotopy classes of maps between MTH and an object $\Sigma^{n+1}I_{\mathbb{Z}}$ belonging to stable homotopy theory; see [FH21b, §6.1] for a brief introduction in a mathematical physics context. We mentioned MTH above; $I_{\mathbb{Z}}$ is the *Anderson dual of the sphere spectrum* [And69, Yos75], characterized up to homotopy equivalence by its universal property, which says that there is a natural short exact sequence

$$0 \longrightarrow \text{Ext}(\pi_{n-1}(E), \mathbb{Z}) \longrightarrow [E, \Sigma^n I_{\mathbb{Z}}] \longrightarrow \text{Hom}(\pi_n(E), \mathbb{Z}) \longrightarrow 0. \quad (3.3)$$

Applying this when $E = MTH$, we obtain a short exact sequence

$$0 \longrightarrow \text{Ext}(\Omega_{n+1}^H, \mathbb{Z}) \xrightarrow{\varphi} [MTH, \Sigma^{n+2}I_{\mathbb{Z}}] \xrightarrow{\psi} \text{Hom}(\Omega_{n+2}^H, \mathbb{Z}) \longrightarrow 0 \quad (3.4)$$

decomposing the group of possible anomalies of unitary QFTs on H_n -manifolds. These two factors admit interpretations in terms of anomalies.

- (1) The quotient $\text{Hom}(\Omega_{n+2}^H, \mathbb{Z})$ is a free abelian group of degree- $(n + 2)$ characteristic classes of H -manifolds. The map ψ sends an anomaly field theory to its anomaly polynomial. This is the part of the anomaly visible to perturbative methods, and sometimes is called the *local anomaly*.
- (2) The subgroup $\text{Ext}(\Omega_{n+1}^H, \mathbb{Z})$ is isomorphic to the abelian group of torsion bordism invariants $f: \Omega_{n+1}^H \rightarrow \mathbb{C}^\times$. These classify the reflection-positive invertible *topological* field theories α_f : the correspondence is that the bordism invariant f is the partition function of α_f . This part of an anomaly field theory is generally invisible to perturbative methods and is called the *global anomaly*.

Work of Yamashita–Yonekura [YY23] and Yamashita [Yam23] relates this short exact sequence to a differential generalized cohomology theory extending $\text{Map}(MTH, \Sigma^{n+1}I_{\mathbb{Z}})$.

3.2. Specializing to the U -duality symmetry type. For us, $n = 4$ and the symmetry type is $H_4 = \text{Spin} \times_{\{\pm 1\}} \tilde{E}_{7(7)}$. This group is not compact, so Theorems 3.1 and 3.2 above do not apply. However, we can work around this obstacle: Marcus [Mar85] proved that the anomaly polynomial of the $E_{7(7)}$ symmetry vanishes,⁶ meaning that the anomaly field theory is a *topological* field theory. Thinking of topological field theories as symmetric monoidal functors $\text{Bord}_n^{H_n} \rightarrow \mathcal{C}$, we can freely adjust the structure we put on manifolds in these theories as long as the induced map on bordism categories is an equivalence. We make two adjustments.

- (1) First, forget the metric and connection in the definition of a geometric H_4 -structure. The space of such data is contractible and therefore can be ignored for topological field theories.

⁶ Marcus’ analysis does not discuss the question of H_4 versus $\text{Spin}_4 \times \tilde{E}_{7(7)}$, but this does not matter: in many cases including the one we study, the anomaly polynomial for a d -dimensional field theory on G -manifolds is an element of $H^{d+2}(BG; \mathbb{Q})$, and rational cohomology is insensitive to finite covers such as $\text{Spin}_4 \times \tilde{E}_{7(7)} \rightarrow H_4$. Thus Marcus’ computation applies in our case too.

- (2) We can then replace H_4 with its maximal compact subgroup: for any Lie group G with $\pi_0(G)$ finite, inclusion of the maximal compact subgroup $K \hookrightarrow G$ is a homotopy equivalence [Mal45, Iwa49] and defines a natural equivalence of groupoids $Bun_K(X) \xrightarrow{\sim} Bun_G(X)$ on spaces X , hence a symmetric monoidal equivalence of bordism categories of manifolds with these kinds of bundles.

$Spin_4$ is compact, and the maximal compact subgroup of $\tilde{E}_{7(7)}$ is SU_8 , so the maximal compact of H_4 is the group $Spin_4 \times_{\{\pm 1\}} SU_8$. Now Theorems 3.1 and 3.2 apply: the stabilization of $Spin_4 \times_{\{\pm 1\}} SU_8$ is $Spin\text{-}SU_8 := Spin \times_{\{\pm 1\}} SU_8$, and the anomaly field theory is classified by the torsion subgroup of $[MT(Spin\text{-}SU_8), \Sigma^6 I_{\mathbb{Z}}]$, which is determined by $\Omega_5^{Spin\text{-}SU_8}$.

In Theorem 4.23, we prove $\Omega_5^{Spin\text{-}SU_8} \cong \mathbb{Z}/2$, so there is potential for the anomaly field theory to be nontrivial. The particular anomaly we study is similar to the anomaly classified by $\Omega_5^{Spin\text{-}SU_2} = (\mathbb{Z}/2)^2$ where one of the $\mathbb{Z}/2$ is the new SU_2 anomaly in [WWW19], as well as $\Omega_5^{Spin\text{-}Spin_{10}} = \mathbb{Z}/2$. The latter group is applicable to $Spin_{10}$ GUT-anomalies studied in [WY22, WY21, YW22], with a more field theoretic point of view.

The quotient map $Spin\text{-}SU_8 \rightarrow SO \times SU_8/\{\pm 1\}$ by the central $\{\pm 1\}$ subgroup means that a spin- SU_8 structure on a manifold M induces an orientation on M and a principal $SU_8/\{\pm 1\}$ -bundle $P \rightarrow M$, similar to the way a spin^c structure induces an orientation and a principal U_1 -bundle. To lift from an orientation and a U_1 -bundle to a spin^c structure one must trivialize a characteristic class, and likewise for spin- SU_8 structures, as we now prove.

Lemma 3.5. $H^1(B(SU_8/\{\pm 1\}); \mathbb{Z}/2) \cong 0$ and $H^2(B(SU_8/\{\pm 1\}); \mathbb{Z}/2) \cong \mathbb{Z}/2$.

Proof. Since $SU_8/\{\pm 1\}$ is connected, $B(SU_8/\{\pm 1\})$ is simply connected, and the Hurewicz and universal coefficient theorems then tell us $H^1(B(SU_8/\{\pm 1\}); \mathbb{Z}/2) = 0$. For H^2 , use that $\pi_2(B(SU_8/\{\pm 1\})) \cong \pi_1(SU_8/\{\pm 1\}) \cong \mathbb{Z}/2$, because $SU_8 \rightarrow SU_8/\{\pm 1\}$ is a double cover by a simply connected space; then apply the Hurewicz and universal coefficient theorems again. \square

Lemma 3.6. *The central extension*

$$1 \longrightarrow \{\pm 1\} \longrightarrow Spin \times_{\{\pm 1\}} SU_8 \longrightarrow SO \times SU_8/\{\pm 1\} \longrightarrow 1 \tag{3.7}$$

is classified by $w_2 + a \in H^2(B(SO \times SU_8/\{\pm 1\}); \mathbb{Z}/2)$, where a is the nonzero element of $H^2(BG_8; \mathbb{Z}/2)$.

Proof. Since $B(G \times H) \simeq BG \times BH$ for topological groups G and H , and since $H^1(BSO; \mathbb{Z}/2)$ and $H^1(BSU_8/\{\pm 1\}; \mathbb{Z}/2)$ both vanish, the Künneth formula identifies

$$H^2(B(SO \times SU_8/\{\pm 1\}); \mathbb{Z}/2) \cong H^2(BSO; \mathbb{Z}/2) \oplus H^2(BSU_8/\{\pm 1\}; \mathbb{Z}/2). \tag{3.8}$$

In view of the Künneth splitting (3.8) and the fact that both $H^2(BSO; \mathbb{Z}/2)$ and $H^2(BSU_8/\{\pm 1\}; \mathbb{Z}/2)$ are isomorphic to $\mathbb{Z}/2$, to show $x = w_2 + a$ it suffices to pull the extension (3.7) back along the inclusions $i_1 : SO \rightarrow SO \times SU_8/\{\pm 1\}$ and $i_2 : SU_8/\{\pm 1\} \rightarrow SO \times SU_8/\{\pm 1\}$ and see that both pullbacks do not split.

After pulling back by i_1 , we get the extension $1 \rightarrow \{\pm 1\} \rightarrow Spin \rightarrow SO \rightarrow 1$, which is not split, as it is classified by w_2 ; after pulling back by i_2 , we get the extension $1 \rightarrow \{\pm 1\} \rightarrow SU_8 \rightarrow SU_8/\{\pm 1\} \rightarrow 1$; because SU_8 is connected but $\{\pm 1\}$ is not connected, this extension is also not split. Therefore $x = w_2 + a$. \square

Lemma 3.9. *Let M be an oriented manifold and $P \rightarrow M$ be a principal $SU_8/\{\pm 1\}$ -bundle. The data of a spin- SU_8 -structure inducing this orientation and principal $SU_8/\{\pm 1\}$ -bundle is equivalent to a trivialization of $w_2(M) + a(P)$, where a is the unique nonzero element of $H^2(B(SU_8/\{\pm 1\}); \mathbb{Z}/2)$.⁷*

Proof. Apply the classifying space functor to the short exact sequence (3.7) to obtain a fibration

$$\begin{CD} B(\text{Spin} \times_{\{\pm 1\}} SU_8) @. \\ @VV\pi V @. \\ B(SO \times SU_8/\{\pm 1\}) @>w_2+a>> K(\mathbb{Z}/2, 2), \end{CD} \tag{3.10}$$

implying that for any space X and map $f : X \rightarrow B(SO \times SU_8/\{\pm 1\})$, the data of a lift of f to a map $\tilde{f} : X \rightarrow B(\text{Spin} \times_{\{\pm 1\}} SU_8)$ such that $\pi \circ \tilde{f} = f$ is equivalent data of a null-homotopy of $(w_2 + a) \circ f$. Now let $X = M$ and f be the map classifying TM with its orientation and P to conclude. \square

Remark 3.11. If we knew a was w_2 of the associated vector bundle to a representation ρ of $SU_8/\{\pm 1\}$, then Lemma 3.9 would follow from [DDHM23, Corollary 10.23]. However, no such ρ exists, as we discuss in Theorem 4.2.

Remark 3.12. Computing bordism groups to determine whether an anomaly is trivial is a well-established technique in the mathematical physics literature: other papers taking this approach include [Wit86, Kil88, Mon15, Cam17, Mon17, GPW18, Hsi18, STY18, DGL20, GEM19, MM19, TY19, WW19b, WW19a, WWZ20, WY19, BLT20, DL21b, DL20, DL21a, GOP+20, HKT20, HTY22, JF20, KPMT20, Tho20, WW20b, WW20a, WW20c, FH21a, FH21b, DDHM22, DGG21, GP23, Gri21, Koi23, LOT21b, LOT21a, LT21, TY23, Yu21, WNC21, DGL22, Deb23, LY22, Tac22, Yon22].

4. Spectral Sequence Computation

The E_2 page for U-duality in the Adams spectral sequence is [Ada58, Theorem 2.1, 2.2]

$$\text{Ext}_{\mathcal{A}}^{s,t}(H^*(MT(\text{Spin-SU}_8); \mathbb{Z}/2), \mathbb{Z}/2) \Rightarrow \pi_{s-t}(MT(\text{Spin-SU}_8))_2^\wedge \cong (\Omega_{s-t}^{\text{Spin-SU}_8})_2^\wedge, \tag{4.1}$$

which converges to the 2-completion of the desired bordism group via the Pontrjagin-Thom construction.

Let $G_8 := SU_8/\{\pm 1\}$. The standard way to tackle Adams spectral sequence questions such as (4.1) would be to re-express a spin- SU_8 structure on a vector bundle $E \rightarrow M$ as data of a principal G_8 -bundle $P \rightarrow M$ and a spin structure on $E \oplus \rho_P$, where ρ_P is the associated bundle to P and some representation ρ of G_8 . Once this is done, one invokes a change-of-rings theorem that makes calculating the E_2 -page of (4.1) much easier. For several great examples of this technique, see [Cam17, BC18].

Unfortunately, this strategy is not available for spin- SU_8 bordism. The reason is that ρ , thought of as a map $\rho : G_8 \rightarrow O_n$ for some n , cannot lift to a map $G_8 \rightarrow \text{Spin}_n$; if it does, a spin structure on $E \oplus \rho_P$ is equivalent to a spin structure on E by the two-out-of-three property. However, G_8 does not have any non-spin representations.

⁷ By a *trivialization* of a characteristic class $a \in H^\ell(X; A)$, we mean data of a null-homotopy of the corresponding map $X \rightarrow K(A, \ell)$. Thus a trivialization of $w_1(M)$ on a manifold is equivalent data to an orientation of M , a trivialization of $w_2(M)$ on an oriented manifold is equivalent data to a spin structure on M , etc.

Theorem 4.2 (Speyer [Spe22]). *All representations $\rho: G_8 \rightarrow O_n$ lift to Spin_n .*

We give another proof of this fact using $H^*(BG_8; \mathbb{Z}/2)$, which we calculate in Theorem 4.4 in low degrees. Theorem 4.4 appears later in this paper, but does not use Theorem 4.2, so this argument is not circular.

Proof. Suppose that a representation ρ of G_8 admitting no lift to Spin_n exists, and let $V \rightarrow BG_8$ be its associated vector bundle; since ρ does not admit a lift to Spin_n , V has no spin structure, so $w_2(V) \neq 0$. The \mathcal{A} -action on the cohomology of the Thom spectrum $(BG_8)^V$ is determined by $H^*(BG_8; \mathbb{Z}/2)$ and $w_2(V)$ by a formula; we will show below that there is no nonzero choice of $w_2(V)$ such that this formula satisfies the Adem relations, so no such ρ can exist.

In Lemma 3.5, we saw that $H^1(BG_8; \mathbb{Z}/2) = 0$ and $H^2(BG_8; \mathbb{Z}/2) \cong \mathbb{Z}/2$. Let a be the nonzero element of $H^2(BG_8; \mathbb{Z}/2)$. Since $H^1(BG_8; \mathbb{Z}/2)$ vanishes, $w_1(V) = 0$, and since $H^2(BG_8; \mathbb{Z}/2) = \{0, a\}$ and $w_2(V) \neq 0$, $w_2(V) = a$. Let b and d be the classes we construct define in $H^*(BG_8; \mathbb{Z}/2)$ in Theorem 4.4; then, using the Thom isomorphism and how it affects the \mathcal{A} -module structure on cohomology (see, e.g., [BC18, §3.3, §3.4]), we can compute that if U is the Thom class in the mod 2 cohomology of $(BG_8)^V$, then $\text{Sq}^2\text{Sq}^1\text{Sq}^2U = U(ab + d)$, $\text{Sq}^4\text{Sq}^1U = 0$, and there is no class x with $\text{Sq}^1(Ux) = U(ab + d)$. This contradicts the Adem relation $\text{Sq}^2\text{Sq}^1\text{Sq}^2 = \text{Sq}^4\text{Sq}^1 + \text{Sq}^1\text{Sq}^4$ that must hold in the cohomology of any space or spectrum. \square

Thus we cannot proceed via the usual change-of-rings simplification, and we must run the Adams spectral sequence over the entire mod 2 Steenrod algebra \mathcal{A} , which is harder. Similar problems occur in a few other places in the mathematical physics literature, including [FH21a, Deb23]. It would be interesting to find more problems where similar complications occur when trying to work with twisted spin bordism.

In order to set up the Adams computation, a necessary step is to establish the two theorems in Sect. 4.2 with the goal to give the Steenrod actions on $H^*(B(\text{Spin-SU}_8); \mathbb{Z}/2)$. Applying the Thom isomorphism takes care of the rest. We also detail the simplifications that make working over the entire Steenrod algebra accessible. We refer the reader to [BC18] which highlights many of the computational details of the Adams spectral sequence, but mainly employs a change of rings to work over $\mathcal{A}(1)$. We start by showing that computing the 2-completion is sufficient for the tangential structure we are considering.

4.1. Nothing interesting at odd primes. We will show that the Adams spectral sequence computation that we run which only gives the two torsion part of the anomaly is sufficient for our purposes.

Proposition 4.3. $\Omega_*^{\text{Spin-SU}_8}$ has no p -torsion when p is an odd prime.

Proof. The quotient $\text{Spin} \times \text{SU}_8 \rightarrow \text{Spin-SU}_8$ is a double cover, hence on classifying spaces is a fiber bundle with fiber $B\mathbb{Z}/2$. $H^*(B\mathbb{Z}/2; \mathbb{Z}/p) = \mathbb{Z}/p$ concentrated in degree 0, so $B(\text{Spin} \times \text{SU}_8) \rightarrow B(\text{Spin-SU}_8)$ is an isomorphism on \mathbb{Z}/p cohomology (e.g. look at the Serre spectral sequence for this fiber bundle). The Thom isomorphism lifts this to an isomorphism of cohomology of the relevant Thom spectra, and then the stable Whitehead theorem implies that the forgetful map $\Omega_*^{\text{Spin}}(BSU_8) \rightarrow \Omega_*^{\text{Spin-SU}_8}$ is an isomorphism on p -torsion.

The same argument applies to the double cover $\text{Spin} \times \text{SU}_8 \rightarrow \text{SO} \times \text{SU}_8$, so the p -torsion in $\Omega_*^{\text{Spin-SU}_8}$ is isomorphic to the p -torsion in $\Omega_*^{\text{SO}}(BSU_8)$. Now apply the

Atiyah–Hirzebruch spectral sequence. Averbuh [Ave59] and Milnor [Mil60, Theorem 5] prove there is no p -torsion in Ω_*^{SO} , and Borel [Bor51, Proposition 29.2] shows there is no p -torsion in $H_*(BSU_8; \mathbb{Z})$ and $H_*(BSU_8; \mathbb{Z}/2)$. Therefore the only way to obtain p -torsion in $\Omega_*^{SO}(BSU_8)$ would be from a differential between free summands, but all free summands in Ω_*^{SO} and $H_*(BSU_8; \mathbb{Z})$ are contained in even degrees, so there are no differentials between free summands, and therefore no p -torsion. \square

4.2. *Computing the cohomology of $B(\text{Spin-SU}_8)$.* We first prove Theorem 4.4, where we compute $H^*(BG_8; \mathbb{Z}/2)$ and its \mathcal{A} -module structure in low degrees. We then use this to compute $H^*(B(\text{Spin-SU}_8); \mathbb{Z}/2)$ as an \mathcal{A} -module in low degrees in Theorem 4.14, allowing us to run the Adams spectral sequence in Sect. 4.3. Our computations make heavy use of the Serre spectral sequence; for more on the Serre spectral sequence and its application to physical problems see [GEM19, Yu21, Yu22, LY22, LT21, DL21b, DGL22]. See also Manjunath-Calvera-Barkeshli [MCB23, §D.6], who perform a related Serre spectral sequence computation to determine some integral cohomology groups of BG_8 .

Theorem 4.4. $H^*(BG_8; \mathbb{Z}/2) \cong \mathbb{Z}/2[a, b, c, d, e, \dots]/(\dots)$ with $|a| = 2, |b| = 3, |c| = 4, |d| = 5,$ and $|e| = 6,$ and there are no other generators or relations below degree 7. The Steenrod squares are

$$\begin{aligned} \text{Sq}(a) &= a + b + a^2 \\ \text{Sq}(b) &= b + d + b^2 \\ \text{Sq}(c) &= c + e + \text{Sq}^3(c) + c^2 \\ \text{Sq}(d) &= d + b^2 + \text{Sq}^3(d) + \text{Sq}^4(d) + d^2. \end{aligned} \tag{4.5}$$

Proof. We consider the Serre spectral sequence for the fibration $B\mathbb{Z}/2 \rightarrow BSU_8 \rightarrow BG_8$ with integer coefficients, which has signature

$$E_2^{*,*} = H^*(BG_8; H^*(B\mathbb{Z}/2; \mathbb{Z})) \implies H^*(BSU_8; \mathbb{Z}). \tag{4.6}$$

Since BG_8 is simply connected, we do not need to worry about local coefficients. We know that $H^*(B\mathbb{Z}/2; \mathbb{Z}) \cong \mathbb{Z}[z]/2z$, where $|z| = 2$, and Borel [Bor53, §29] computed $H^*(BSU_8; \mathbb{Z}) \cong \mathbb{Z}[c_2, \dots, c_8]$, with $|c_i| = 2i$, so we may run the spectral sequence in reverse. The E_2 -page for (4.6) is:

6	z^3	0	0	αz^3	$c_2 z^3$	βz^3	$(c_3 z^3, \alpha z^3)$	
5	0	0	0	0	0	0	0	
4	z^2	0	0	αz^2	$c_2 z^2$	βz^2	$(c_3 z^2, \alpha^2 z^2)$	
3	0	0	0	0	0	0	0	
2	z	0	0	αz	$c_2 z$	βz	$(c_3 z, \alpha^2 z)$	
1	0	0	0	0	0	0	0	
0	1	0	0	α	c_2	β	(c_3, α^2)	
		0	1	2	3	4	5	6.

(4.7)

As $H^2(BSU_8; \mathbb{Z}) = 0, z \in E_2^{0,2} = H^2(B\mathbb{Z}/2; \mathbb{Z})$ admits a differential. The only option is a transgressing d_3 ; let $\alpha := d_3(z)$. Since $2z = 0, 2\alpha = 0$. The Leibniz rule (now with signs) tells us

$$d_2(z^2) = zd_2(z) + d_2(z)z = 2\alpha z = 0. \tag{4.8}$$

Therefore if z^2 admits a differential, the differential must be the transgressing $d_5: E_4^{0,4} \rightarrow E_4^{5,0}$, see (4.7). But z^2 does admit a differential. One way to see this is to compute the pullback $H^4(BSU_8; \mathbb{Z}) \rightarrow H^4(B\mathbb{Z}/2; \mathbb{Z})$. Since $H^4(BSU_8; \mathbb{Z})$ is generated by c_2 of the defining representation \mathbb{C}^8 , we can restrict that representation to $\mathbb{Z}/2$ and compute its second Chern class to compute the pullback map. As a representation of $\mathbb{Z}/2$, \mathbb{C}^8 is a direct sum of 8 copies of the sign representation, so its total Chern class is $c(8\sigma) = (1+z)^8$ by the Whitney sum rule, and the z^2 term is $\binom{8}{2}z^2$, which is even. Since $2z^2 = 0$, this implies c_2 pulls back to 0. If z^2 did not support a differential, then it would be in the image of this pullback map, so we have discovered that z^2 admits a differential, specifically d_5 . Let $\beta := d_5(z^2)$. From the spectral sequence we see that $H^4(BG_8; \mathbb{Z})$ is isomorphic to $H^4(BSU_8; \mathbb{Z})$, and c_2 is an element in this cohomology. By using the mod 2 reduction map from $H^4(BSU_8; \mathbb{Z}) \rightarrow H^4(BSU_8; \mathbb{Z}/2)$, and the pullback map induced from $SU_8 \xrightarrow{f} G_8$ we see that c is the mod 2 reduction of c_2 in $H^4(BG_8; \mathbb{Z})$. This is summarized in the following diagram:

$$\begin{array}{ccc}
 c_2 \in H^4(BG_8; \mathbb{Z}) & \xrightarrow{\text{mod } 2} & H^4(BG_8; \mathbb{Z}/2) \\
 \downarrow \cong & & \downarrow f^* \\
 H^4(BSU_8; \mathbb{Z}) & \longrightarrow & H^4(BSU_8; \mathbb{Z}/2).
 \end{array} \tag{4.9}$$

Let \tilde{c}_2 be the class in $H^4(BSU_8; \mathbb{Z})$ given by the equivalence in the above diagram, and \bar{c}_2 be the mod 2 reduction with $f^*: c \mapsto \bar{c}_2$. We see that $\text{Sq}^2(\bar{c}_2) = \bar{c}_3$ in $H^6(BSU_8; \mathbb{Z}/2)$ and therefore get that $e := \text{Sq}^2(c)$ maps to \bar{c}_3 . We now consider the Serre spectral sequence for the same fibration but with $\mathbb{Z}/2$ coefficients with signature

$$E_2^{*,*} = H^*(BG_8; H^*(B\mathbb{Z}/2; \mathbb{Z}/2)) \implies H^*(BSU_8; \mathbb{Z}/2). \tag{4.10}$$

The E_2 -page for (4.10) is:

$$\begin{array}{cccccccc}
 5 & w_1^5 & & & & & & \\
 4 & w_1^4 & & & & & & \\
 3 & w_1^3 & & & & & & \\
 2 & w_1^2 & & & & & & \\
 1 & w_1 & & & & & & \\
 0 & 1 & 0 & a & b & \bar{c}_2 & d & \bar{c}_3 \\
 \hline
 & 0 & 1 & 2 & 3 & 4 & 5 & 6.
 \end{array} \tag{4.11}$$

The classes \bar{c}_2 and \bar{c}_3 must be detected on the $q = 0$ row. As for the rest of the classes in the $p = 0$ column we start with w_1 which supports a differential so that $d_2(w_1) := a$. Furthermore, w_1^2 must also transgress so that $d_3(w_1^2) := b$. We now apply Kudo’s transgression theorem [Kud56] which states that Steenrod squares commute with transgression in the Serre spectral sequence to furthermore see that $d_3(w_1^2) = d_3(\text{Sq}^1(w_1)) = \text{Sq}^1(a)$. Since we formerly showed that \bar{c}_2 already exists, w_1^4 must also transgress so that $d_4(w_1^4) = d_3(\text{Sq}^2(w_1^2)) = \text{Sq}^2 \text{Sq}^1(a) = \text{Sq}^2(b) := d$. Then using the Adem relation we get that $\text{Sq}^1(d) = b^2$, and this concludes the proof. \square

We now compute $H^*(B(\text{Spin-SU}_8); \mathbb{Z}/2)$, which is what we actually need for U-duality. Recall from Lemma 3.6 that the central extension

$$0 \longrightarrow \mathbb{Z}/2 \longrightarrow \text{Spin-SU}_8 \longrightarrow \text{SO} \times G_8 \longrightarrow 0, \tag{4.12}$$

which we think of physically as “quotienting by fermion parity,” is classified by $w_2 + a \in H^2(B(\text{SO} \times G_8); \mathbb{Z}/2)$.

Taking classifying spaces, we obtain a fibration

$$B\mathbb{Z}/2 \longrightarrow B(\text{Spin-SU}_8) \longrightarrow B(\text{SO} \times G_8), \tag{4.13}$$

and we apply the Serre spectral sequence to this fibration using knowledge of the cohomology of BG_8 .

Theorem 4.14. *The cohomology ring for $H^*(B(\text{Spin-SU}_8); \mathbb{Z}/2)$ considering generators in degree 6 and below is given by $H^*(B(\text{Spin-SU}_8); \mathbb{Z}/2) \cong \mathbb{Z}/2[a, b, c, w_4, d, e, \dots]$ with $|a| = 2, |b| = 3, |c| = 4, |w_4| = 4, |d| = 5,$ and $|e| = 6$. The map $\text{Spin-SU}_8 \rightarrow \text{SO} \times G_8$ induces a quotient map on cohomology with the identification $a = w_2, b = w_3, ab + d = w_5$ where w_i are the Stiefel-Whitney classes of BSO . The Steenrod squares of $a, b, c, d,$ and e are given in (4.5) along with*

$$\begin{aligned} \text{Sq}^1 w_4 &= ab + d, \\ \text{Sq}^2 w_4 &= aw_4 + x, \end{aligned} \tag{4.15}$$

where x is a class in $H^6(B(\text{Spin-SU}_8); \mathbb{Z}/2)$ that is linearly independent from aw_4 .

Proof. We run the Serre spectral sequence with signature

$$E_2^{*,*} = H^*(B(\text{SO} \times G_8); H^*(B\mathbb{Z}/2; \mathbb{Z}/2)) \implies H^*(B(\text{Spin-SU}_8); \mathbb{Z}/2), \tag{4.16}$$

where the E_2 -page is given by:

5	t^5	0						
4	t^4	0						
3	t^3	0						
2	t^2	0	$(t^2a, t^2a + t^2w_2)$	\dots				
1	t	0	$(ta, ta + tw_2)$	$(tb, tb + tw_3)$	\dots			
0	1	0	$(a, a + w_2)$	$(b, b + w_3)$	$\left(\begin{matrix} a^2, c, w_4, a^2 + w_2^2 \\ a(a + w_2) \end{matrix} \right)$	$\left(\begin{matrix} ab + d + w_5, (a + w_2)(b + w_3) \\ a(b + w_3), b(a + w_2) \end{matrix} \right)$		
			0	1	2	3	4	5.

(4.17)

The w_i are the Stiefel-Whitney classes of BSO , and t is the generator of the cohomology $H^*(B\mathbb{Z}/2; \mathbb{Z}/2)$. The differential $d_2 : E_2^{0,1} \rightarrow E_2^{2,0}$ hits the class for the extension (4.13) that gives Spin-SU_8 , which is $a + w_2$, and identifies $a = w_2$. Applying the Leibniz rule shows $d_2(t^{2n+1}) = t^{2n}a$, and that $d_2(t^{2n}) = 0$. We apply Kudo’s transgression theorem to see therefore $d_3 : E_3^{0,2} \rightarrow E_3^{3,0}$ sends $t^2 \mapsto b + w_3$, since the transgressing d_3 sends $\text{Sq}^1 t = t^2$ to $\text{Sq}^1(a + w_1)$. In total degree 4, there is a likewise d_5 differential that takes t^4 to $ab + d + w_5$, i.e. this differential takes $\text{Sq}^2 t^2$ to $\text{Sq}^2(b + w_3)$.⁸ We see that there is a new class w_4 which pulled back from BSO . Applying the Wu formula then establishes the formulas in (4.15), where x is the pullback of w_6 by the map $B(\text{Spin-SU}_8) \rightarrow BSO$. We still need to show that x and aw_4 are linearly independent. In $H^6(BSO; \mathbb{Z}/2)$, w_2w_4 and w_6 are linearly independent, so the only way their images in $H^6(B(\text{Spin-SU}_8); \mathbb{Z}/2)$ could be linearly dependent is if a differential in the Serre spectral sequence (4.16) hits w_6 or $w_2w_4 + w_6$. As we showed above, the Leibniz rule means all differentials in

⁸ Another way to see that d_5 identifies w_5 with $ab + d$ is from the point of view of \mathcal{A} -modules. As we will see in the proof of Theorem 4.23, the only way to have the \mathcal{A} -module in Fig. 2 is if $\text{Sq}^1(w_4U) = w_5U$ agrees with $\text{Sq}^2(bU) = (ab + d)U$

this spectral sequence are determined by the differentials out of $E_r^{0,q}$, so it suffices to understand these differentials for $0 \leq q \leq 5$, so as to land in total degree 6. We have already calculated all of these differentials in this proof, and using their values and the Leibniz rule, we see that both w_6 and $w_2w_4 + w_6$ survive to the E_∞ -page, implying as claimed that x and aw_4 are linearly independent. \square

Corollary 4.18. *The composition*

$$H^k(BSO; \mathbb{Z}/2) \rightarrow H^k(BSO \times BG_8; \mathbb{Z}/2) \rightarrow H^k(B(\text{Spin-SU}_8); \mathbb{Z}/2) \quad (4.19)$$

is injective for $k \leq 6$, and an isomorphism for $0 \leq k \leq 3$ and $k = 5$; for $k = 4$, a complementary subspace to the image of (4.19) is spanned by c , and for degree 6, a complementary subspace is spanned by $\{e, ac\}$.

Proof. We saw in Theorem 4.14 that if we quotient by all elements in degrees 7 and above, the pullback map

$$H^*(BSO \times BG_8; \mathbb{Z}/2) \rightarrow H^*(B(\text{Spin-SU}_8); \mathbb{Z}/2) \quad (4.20)$$

is surjective with kernel the ideal generated by $\{w_2 + a, w_3 + b, w_5 + ab + d\}$. Therefore the images of the classes w_2, w_3, w_4, w_5 , and w_6 , together with all products of these classes in degrees 6 and below, are linearly independent in $H^*(B(\text{Spin-SU}_8); \mathbb{Z}/2)$. As these classes span the subspace of $H^*(BSO; \mathbb{Z}/2)$ in degrees 6 and below, we have proven the lemma. \square

Corollary 4.21. *The pullback map*

$$H^k(MTSO; \mathbb{Z}/2) \rightarrow H^k(MT(\text{Spin-SU}_8); \mathbb{Z}/2), \quad (4.22)$$

is also injective in degrees 6 and below and an isomorphism in degrees $0 \leq k \leq 3$ and $k = 5$; complementary subspaces to the image of (4.22) are spanned by Uc for $k = 4$ and $\{Uac, Ue\}$ for $k = 6$.

Proof. This follows from Corollary 4.18 using the naturality of the Thom isomorphism. \square

4.3. The Adams computation. In this section, we run the Adams spectral sequence for Spin-SU_8 bordism.

Theorem 4.23. *Up to degree 5, the first few groups of Spin-SU_8 bordism are*

$$\begin{aligned} \Omega_0^{\text{Spin-SU}_8} &\cong \mathbb{Z} \\ \Omega_1^{\text{Spin-SU}_8} &\cong 0 \\ \Omega_2^{\text{Spin-SU}_8} &\cong 0 \\ \Omega_3^{\text{Spin-SU}_8} &\cong 0 \\ \Omega_4^{\text{Spin-SU}_8} &\cong \mathbb{Z}^2 \\ \Omega_5^{\text{Spin-SU}_8} &\cong \mathbb{Z}/2. \end{aligned} \quad (4.24)$$

Treating $d \in H^5(BG_8; \mathbb{Z}/2)$ as a characteristic class, the bordism invariant $(M, P) \mapsto \int_M d(P) \in \mathbb{Z}/2$ realizes the isomorphism $\Omega_5^{\text{Spin-SU}_8} \rightarrow \mathbb{Z}/2$.

Before we begin the proof of Theorem 4.23, we need a few lemmas about \mathcal{A} -modules.

Definition 4.25. Let $C\eta$ denote the \mathcal{A} -module $\Sigma^{-2}\tilde{H}^*(\mathbb{C}\mathbb{P}^2; \mathbb{Z}/2)$,⁹

Lemma 4.26. *If one quotients by all elements in degrees 7 and above, there is an isomorphism of \mathcal{A} -modules*

$$H^*(MT(\text{Spin-SU}_8); \mathbb{Z}/2) \cong \mathcal{A} \otimes_{\mathcal{A}(0)} \mathbb{Z}/2 \oplus \Sigma^4(\mathcal{A} \otimes_{\mathcal{A}(0)} \mathbb{Z}/2) \oplus \Sigma^4 C\eta \oplus \Sigma^5 \mathcal{A} \oplus P, \tag{4.27}$$

where P contains no nonzero elements in degrees 5 and below. U generates $\mathcal{A} \otimes_{\mathcal{A}(0)} \mathbb{Z}/2$, Ua^2 generates $\Sigma^4(\mathcal{A} \otimes_{\mathcal{A}(0)} \mathbb{Z}/2)$, Uc generates $\Sigma^4 C\eta$.

Proof. In this proof, we implicitly quotient graded rings by their ideals of elements in degrees 7 and above.

Because (4.22) is so close to an isomorphism, it tells us almost all of the \mathcal{A} -module structure we need in terms of the \mathcal{A} -module structure on $H^*(MTSO; \mathbb{Z}/2)$. Wall [Wal60] showed that, localized at $p = 2$, $MTSO$ is a direct sum of Eilenberg-Mac Lane spectra

$$MTSO_{(2)} \xrightarrow{\cong} H\mathbb{Z} \vee \Sigma^4 H\mathbb{Z} \vee \Sigma^5 H\mathbb{Z}/2 \vee \dots \tag{4.28}$$

where all unlisted summands are at least 7-connected, hence do not matter in our calculation. Therefore on cohomology we have

$$H^*(MTSO; \mathbb{Z}/2) \cong H^*(H\mathbb{Z}) \oplus \Sigma^4 H^*(H\mathbb{Z}) \oplus \Sigma^5 H^*(H\mathbb{Z}/2) \oplus \dots \tag{4.29}$$

Serre [Ser53] showed that $H^*(H\mathbb{Z}) = \mathcal{A} \otimes_{\mathcal{A}(0)} \mathbb{Z}/2$ and $H^*(H\mathbb{Z}/2) = \mathcal{A}$.

By Corollary 4.21, $\mathcal{S} := \mathcal{A} \otimes_{\mathcal{A}(0)} \mathbb{Z}/2 \oplus \Sigma^4(\mathcal{A} \otimes_{\mathcal{A}(0)} \mathbb{Z}/2) \oplus \Sigma^5 \mathcal{A}$ is an \mathcal{A} -submodule of $H^*(MT(\text{Spin-SU}_8); \mathbb{Z}/2)$; we want to show that \mathcal{S} is a direct summand, and compute Steenrod squares on a complementary subspace.

To show that \mathcal{S} is a direct summand, it suffices to compute Steenrod squares on a generating set for a complementary subspace, such as $\{Uc, Ue, Uac\}$, and show that those Steenrod squares do not land in $\mathcal{S} \setminus 0$. For Ue and Uac this is vacuous: their squares land in degrees 7 or higher, which we have killed. Similarly, for Uc , we only need to compute Sq^1 and Sq^2 . We can use the formula for Steenrod squares in a Thom spectrum [BC18, Remark 3.3.5]¹⁰

$$Sq^k(Ux) = \sum_{i=0}^k Sq^i(U)Sq^{k-i}(x) = \sum_{i=0}^k U w_i(-V) Sq^{k-i}(x), \tag{4.30}$$

where $-V \rightarrow B(\text{Spin-SU}_8)$ is the pullback of the inverse of the tautological vector bundle on BSO . Since $w_1(-V) = 0$ and $Sq^1(c) = 0$, $Sq^1(Uc) = 0$; since $w_2(-V) = a$ and $Sq^2(c) = e$, $Sq^2(Uc) = U(ac + e) \neq 0$. Therefore \mathcal{S} is indeed a direct summand, and the subspace spanned by $\{Uc, U(ac + e)\}$ spans the promised $\Sigma^4 C\eta$ and Ue spans P . □

Lemma 4.31. *Ext $_{\mathcal{A}}(C\eta)$ is isomorphic to $\mathbb{Z}/2[h_0]$ in topological degree 0 and vanishes in topological degree 1.*

⁹ This \mathcal{A} -module is called $C\eta$ because it is the cohomology of the cofiber of the Hopf map $\eta: \Sigma\mathbb{S} \rightarrow \mathbb{S}$, the nontrivial element of $\pi_1(\mathbb{S})$.

¹⁰ The appearance of $-V$, rather than V , is because we are studying bordism of spin-SU₈ structures on tangent bundles, rather than on stable normal bundles. See Footnote 5.

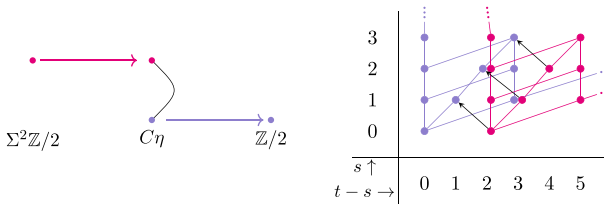


Fig. 1. Left: the short exact sequence (4.32). Right: the induced long exact sequence in Ext groups. These diagrams are part of the proof of Lemma 4.31

Proof. We use a standard technique: $C\eta$ is part of a short exact sequence of \mathcal{A} -modules

$$0 \longrightarrow \Sigma^2\mathbb{Z}/2 \longrightarrow C\eta \longrightarrow \mathbb{Z}/2 \longrightarrow 0, \tag{4.32}$$

and a short exact sequence of \mathcal{A} -modules induces a long exact sequence of Ext groups. It is conventional to draw this as if on the E_1 -page of an Adams-graded spectral sequence; see [BC18, §4.6] for more information and some additional examples. We draw the short exact sequence (4.32) in Fig. 1, left, and we draw the induced long exact sequence in Ext in Fig. 1, right. Looking at this long exact sequence, there are three boundary maps that could be nonzero in the range displayed; because boundary maps commute with the $\text{Ext}_{\mathcal{A}}(\mathbb{Z}/2)$ -action, these boundary maps are all determined by

$$\partial : \text{Ext}_{\mathcal{A}}^{0,2}(\Sigma^2\mathbb{Z}/2) \rightarrow \text{Ext}_{\mathcal{A}}^{1,2}(\mathbb{Z}/2). \tag{4.33}$$

This boundary map is either 0 or an isomorphism, and it must be an isomorphism, because

$$\text{Ext}_{\mathcal{A}}^{0,2}(C\eta) = \text{Hom}_{\mathcal{A}}(C\eta, \Sigma^2\mathbb{Z}/2) = 0, \tag{4.34}$$

and if the boundary map vanished, we would obtain $\mathbb{Z}/2$ for this Ext group. Thus we know $\text{Ext}_{\mathcal{A}}(C\eta)$ in the range we need. \square

Proof of Theorem 4.23. To compute the E_2 -page of the Adams spectral sequence we need to know Ext of each summand in (4.27) ($\text{Ext}(-)$ means $\text{Ext}_{\mathcal{A}}^{*,*}(-; \mathbb{Z}/2)$.) Quotienting by elements in degrees 7 and above does not change the Ext groups in topological degrees less than 6; one way to see this is to run the long exact sequence in Ext groups induced by the short exact sequence

$$0 \longrightarrow \tau_{\geq 7}M \longrightarrow M \longrightarrow M/\tau_{\geq 7}M \longrightarrow 0, \tag{4.35}$$

where M is an \mathcal{A} -module and $\tau_{\geq 7}M$ is the submodule of M of classes in degree 7 and above. Therefore, since we are only interested in topological degrees 5 and below, we need not distinguish M from $M/\tau_{\geq 7}M$ for the purpose of computing Ext groups.

By using the change of rings theorem [BC18, Section 4.5], we get $\text{Ext}_{\mathcal{A}}(\mathcal{A} \otimes_{\mathcal{A}(0)} \mathbb{Z}/2, \mathbb{Z}/2) = \text{Ext}_{\mathcal{A}(0)}(\mathbb{Z}/2, \mathbb{Z}/2)$, and since $\mathcal{A}(0)$ only includes Sq^1 , this just gives $\mathbb{Z}/2[h_0]$ [BC18, Remark 4.5.4], where $h_0 \in \text{Ext}^{1,1}$. The same logic applies to $\text{Ext}_{\mathcal{A}}(\Sigma^4(\mathcal{A} \otimes_{\mathcal{A}(0)} \mathbb{Z}/2))$, and $\text{Ext}_{\mathcal{A}}(\Sigma^5\mathcal{A})$ contributes a $\mathbb{Z}/2$ in degree 5.

Compiling the information of Ext on (4.27) we draw the E_2 -page of the Adams spectral sequence through topological degree 5 in Fig. 3.

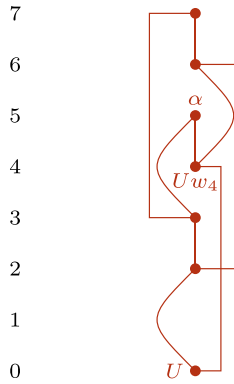


Fig. 2. The only relevant higher Steenrod operation in this degree is Sq^4 , which acts on U to give Uw_4 . This is connected to $\alpha = (ab + d)U$ by Sq^1 . (This module continues past degree 7, but we do not need any higher-degree information.)

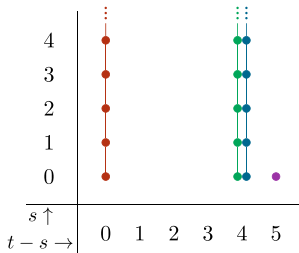


Fig. 3. The E_2 -page of the Adams spectral sequence computing $\Omega_*^{\text{Spin-SU}_8}$

In this range, the only differentials that could be nonzero go from the 5-line to the 4-line. Usually we would need to know the 6-line in order to determine if there are any differentials from the 6-line to the 5-line, so that we could evaluate $\Omega_5^{\text{Spin-SU}_8}$, but the 5-line is concentrated in filtration zero, and all Adams differentials land in filtration 2 or higher, so what we have computed is good enough.

Returning to the differentials from the 5-line to the 4-line: Adams differentials must commute with the action of h_0 on the E_r -page, and h_0 acts by 0 on the 5-line but injectively on the 4-line, so these differentials must also vanish. Thus the spectral sequence collapses giving the bordism groups in the theorem statement. The fact that $\Omega_5^{\text{Spin-SU}_8} \cong \mathbb{Z}/2$ is detected by $\int d$ follows from the fact that its image in the E_∞ -page is in Adams filtration zero, corresponding to Ext of the free $\Sigma^5 \mathcal{A}$ summand generated by Ud ; see [FH21a, §8.4]. \square

4.4. Determining the manifold generator. We now determine the generator of $\Omega_5^{\text{Spin-SU}_8} \cong \mathbb{Z}/2$. We start by considering a map $\tilde{\Phi}: \text{SU}_2 \rightarrow \text{SU}_8$ sending a matrix A to its fourfold block sum $A \oplus A \oplus A \oplus A$. This sends $-1 \mapsto -1$, so $\tilde{\Phi}$ descends to a map

$$\Phi: \text{SO}_3 = \text{SU}_2/\{\pm 1\} \longrightarrow \text{SU}_8/\{\pm 1\} = G_8. \tag{4.36}$$

Recall that $H^*(\text{BSO}_3; \mathbb{Z}/2) \cong \mathbb{Z}/2[w_2, w_3]$ and that there are three classes a, b , and d in $H^*(B(\text{SU}_8/\{\pm 1\}); \mathbb{Z}/2)$.

Lemma 4.37. $\Phi^*(a) = w_2$, $\Phi^*(b) = w_3$, and $\Phi^*(d) = w_2w_3$.

This will imply that to find a generator, all we have to do is find a closed, oriented 5-manifold M with a principal SO_3 -bundle $P \rightarrow M$ with $w_2(M) = w_2(P)$ and $w_2(P)w_3(P) \neq 0$. This is easier than directly working with G_8 !

Proof of Lemma 4.37. Once we show $\Phi^*(a) = w_2$, we're done:

$$\Phi^*(b) = \Phi^*(Sq^1(a)) = Sq^1(\Phi^*(a)) = Sq^1(w_2) = w_3, \tag{4.38a}$$

where the last equal sign follows by the Wu formula. In a similar way

$$\Phi^*(d) = \Phi^*(Sq^2(b)) = Sq^2(\Phi^*(b)) = Sq^2(w_3) = w_2w_3, \tag{4.38b}$$

again using the Wu formula. So all we have to do is pull back a .

Consider the commutative diagram of short exact sequences

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathbb{Z}/2 & \longrightarrow & SU_2 & \longrightarrow & SO_3 & \longrightarrow & 1 \\ & & \parallel & & \downarrow \tilde{\Phi} & & \downarrow \Phi & & \\ 1 & \longrightarrow & \mathbb{Z}/2 & \longrightarrow & SU_8 & \longrightarrow & G_8 & \longrightarrow & 1. \end{array} \tag{4.39}$$

Taking classifying spaces, this shows that the pullback of the fiber bundle $B\mathbb{Z}/2 \rightarrow BSU_8 \rightarrow BG_8$ along the map $\Phi: BSO_3 \rightarrow BG_8$ is the fiber bundle $B\mathbb{Z}/2 \rightarrow BSU_2 \rightarrow BSO_3$. We therefore obtain a map between the Serre spectral sequences computing the mod 2 cohomology rings of BSU_2 and BSU_8 , and it is an isomorphism on $E_2^{0,*}$, i.e. on the cohomology of the fiber.

Both BSU_2 and BSU_8 are simply connected, so $H^1(-; \mathbb{Z}/2)$ vanishes for both spaces. Therefore in both of these Serre spectral sequences, the generator of $E_2^{0,1} = H^1(B\mathbb{Z}/2; \mathbb{Z}/2)$ must admit a differential. The only differential that can be nonzero is the transgressing $d_2: E_2^{0,1} \rightarrow E_2^{2,0}$; in $E_2(SU_8)$, we saw in (4.11) that $d_2(w_1) = a$, and in $E_2(SU_2)$, $d_2(x) = w_2$, because w_2 is the only nonzero element of $E_2^{2,0} = H^2(BSO_3; \mathbb{Z}/2)$. Since the pullback map of spectral sequences commutes with differentials, this means $\Phi^*(a) = w_2$ as desired. \square

Now let $W := SU_3/SO_3$, which is a closed, oriented 5-manifold called the *Wu manifold*, and let $P \rightarrow W$ be the quotient $SU_3 \rightarrow SU_3/SO_3$, which is a principal SO_3 -bundle.

Remark 4.40. The name ‘‘Wu manifold’’ for W is due to Barden [Bar65, §1]. The name refers to work of Wu [Wu50] (see also Dold [Dol56]), though the manifold Wu studied is bordant to, but not diffeomorphic to, W . See [LOT21b, Footnote 9] for more on the history of W .

We will need to know $H^*(W; \mathbb{Z}/2)$ and the Stiefel-Whitney classes $w(TM)$ and $w(P)$. Most of this is in the literature, but scattered, so we give a complete account in Lemma 4.41, Corollary 4.44 and Proposition 4.45.

Lemma 4.41 (Barden [Bar65, Lemma 1.1]). *W is connected and simply connected, and $H^2(W; \mathbb{Z}/2) \cong \mathbb{Z}/2$.*

Proof. Connectivity follows from connectivity of SU_3 .

In view of the Hurewicz and universal coefficient theorems, it suffices to show $\pi_1(W) \cong 0$ and $\pi_2(W) \cong \mathbb{Z}/2$. To show this, use the long exact sequence in homotopy groups associated to the fiber bundle $SO_3 \rightarrow SU_3 \rightarrow W$: for $\pi_1(W) = 0$, use exactness at

$$\cdots \longrightarrow \pi_1(SU_3) \longrightarrow \pi_1(W) \longrightarrow \pi_0(SO_3) \longrightarrow \cdots, \tag{4.42}$$

together with the fact that SO_3 is connected and SU_3 is simply connected. To show $\pi_2(W) \cong \mathbb{Z}/2$, use exactness at

$$\cdots \longrightarrow \pi_2(SU_3) \longrightarrow \pi_2(W) \longrightarrow \pi_1(SO_3) \longrightarrow \pi_1(SU_3) \longrightarrow \cdots \tag{4.43}$$

and the isomorphisms $\pi_1(SO_3) \cong \mathbb{Z}/2$, $\pi_1(SU_3) \cong 0$, and $\pi_2(SU_3) \cong 0$; the fact that $\pi_2(G)$ vanishes for G a connected, simply connected, simple Lie group is due to Bott [Bot56, Theorem A]. \square

Corollary 4.44. $H^*(W; \mathbb{Z}/2) \cong \mathbb{Z}/2[z_2, z_3]/(z_2^2, z_3^2)$ with $|z_2| = 2$ and $|z_3| = 3$.

Proof. This is the unique cohomology ring satisfying both Lemma 4.41 and Poincaré duality. \square

Proposition 4.45. *In $H^*(W; \mathbb{Z}/2)$, the Steenrod squares are*

$$\begin{aligned} Sq(z_2) &= z_2 + z_3 \\ Sq(z_3) &= z_3 + z_2z_3, \end{aligned} \tag{4.46}$$

and the Stiefel-Whitney class is $w(W) = 1 + z_2 + z_3$. Moreover, $w(P) = 1 + z_2 + z_3$. Thus $w_2(M) = w_2(P)$, so W with G_8 -bundle induced from P has a spin- SU_8 structure, and $w_2(P)w_3(P) \neq 0$, meaning (W, P) is our sought-after generator of $\Omega_5^{\text{Spin-SU}_8}$.

The class $w_2(W)$ was computed by Barden [Bar65, Lemma 1.1(v)]. The rest of $w(TM)$ was found by Landweber-Stong (see [LM89, Appendix D]), and the Steenrod squares in $H^*(W; \mathbb{Z}/2)$ by Calabi (see [Flo73, §3]). Finally, Chen [Che17, §6] computes $w(P)$.

Proof. Once we know the Steenrod squares are as claimed, the total Stiefel-Whitney class of W follows from Wu’s theorem as follows. The second Wu class v_2 is defined to be the Poincaré dual of the map

$$x \mapsto \int_W Sq^2(x): H^3(W; \mathbb{Z}/2) \rightarrow H^5(W; \mathbb{Z}/2) \rightarrow \mathbb{Z}/2 \tag{4.47}$$

via the Poincaré duality identification $H^2(W; \mathbb{Z}/2) \cong (H^3(W; \mathbb{Z}/2))^\vee$. Wu’s theorem shows that $v_2 = w_2 + w_1^2$, so since $H^1(W; \mathbb{Z}/2) = 0$ (Lemma 4.41), $w_1 = 0$ and $w_2 = v_2$. Since $Sq^2(z_3) = z_2z_3$, $w_2 \neq 0$, so it must be z_2 . Then $w_3 = Sq^1(w_2) = z_3$; w_4 is trivial for degree reasons (Corollary 4.44); and $w_5 = 0$ follows from the Wu formula calculating $Sq^1(w_4)$.

Consider the Serre spectral sequence for the fiber bundle

$$\begin{array}{ccc} SO_3 & \longrightarrow & SU_3 \\ & & \downarrow \\ & & W, \end{array} \tag{4.48}$$

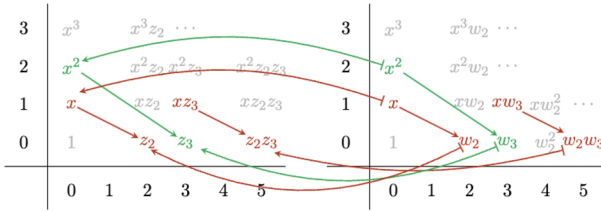


Fig. 4. The fiber bundle $SO_3 \rightarrow SU_3 \rightarrow W$ pulls back from the universal SO_3 -bundle $SO_3 \rightarrow ESO_3 \rightarrow BSO_3$, inducing a map of Serre spectral sequences. This map commutes with differentials and is the identity on $E_2^{0,*} = H^*(SO_3; \mathbb{Z}/2)$, allowing us to conclude that w_2 pulls back to z_2 , w_3 pulls back to z_3 , and w_2w_3 pulls back to z_2z_3 . This is a picture proof of part of Proposition 4.45

which has signature

$$E_2^{*,*} = H^*(W; H^*(SO_3; \mathbb{Z}/2)) \implies H^*(SU_3; \mathbb{Z}/2). \tag{4.49}$$

A priori we must account for the action of $\pi_1(W)$ on $H^*(SO_3; \mathbb{Z}/2)$, but by Lemma 4.41, W is simply connected; therefore we do not have to worry about this. As manifolds, $SO_3 \cong \mathbb{R}P^3$, so $H^*(SO_3; \mathbb{Z}/2) \cong \mathbb{Z}/2[x]/(x^4)$. Also, $H^*(SU_3; \mathbb{Z}/2) \cong \mathbb{Z}/2[b_2, b_3]/(b_2^2, b_3^2)$, with $|b_2| = 3$ and $|b_3| = 5$ [Bor54, §8], allowing us to draw the E_2 -page of the Serre spectral sequence (4.49), which appears in Fig. 4, left.

The fibration (4.48) pulls back from the universal SO_3 -bundle $SO_3 \rightarrow ESO_3 \rightarrow BSO_3$ via the classifying map f_P for P , inducing a map of Serre spectral sequences that commutes with the differentials. We draw this map in Fig. 4. This map is an isomorphism on the line $E_2^{0,*}$, so $x \in E_2^{0,1}(SU_3)$ pulls back from the generator $x \in E_2^{0,1}(ESO_3)$ —and therefore $d_2(x) = z_2$ pulls back from a class in $E_2^{2,0} = H^2(BSO_3; \mathbb{Z}/2)$. The only nonzero class in that degree is w_2 , so $f_P^*(w_2) = z_2$, i.e. $w_2(P) = z_2$.

The Leibniz rule that in the Serre spectral sequence for SU_3 , $d_2(x^2) = 2xd_2(x) = 0$. But because $H^2(SU_2; \mathbb{Z}/2) = 0$, x^2 must support some nontrivial differential. Apart from d_2 , the only option is the transgressing $d_3: E_3^{0,2} \rightarrow E_3^{3,0}$, forcing $d_3(x^2) = z_3$. A similar argument in the Serre spectral sequence for ESO_3 shows that in that spectral sequence, $d_3(x^2) = w_3$; therefore $f_P^*(w_3) = z_3$ and $w_3(P) = z_3$. Pullback commutes with Steenrod squares and $Sq^1(w_2) = w_3$, so $Sq^1(z_2) = z_3$. Finally, $f_P^*(w_2w_3) = z_2z_3$, and the Wu formula implies $Sq^2(w_3) = w_2w_3$, so $Sq^2(z_3) = z_2z_3$. We have computed all the Steenrod squares that could be nonzero for degree reasons, and along the way shown $w(P) = 1 + z_2 + z_3$: the higher-degree Stiefel-Whitney classes of a principal SO_3 -bundle vanish. \square

5. Evaluating on the Anomaly

With the knowledge of the generating manifold for the $\mathbb{Z}/2$ in degree 5 as the Wu manifold, we can consider evaluating the anomaly of the theory with the field content given in Sect. 2. Since G_8 acts trivially on the scalars and the graviton only the remaining three fields could have anomalies.

Definition 5.1. The global anomaly for a fermion on a Riemannian manifold M in a representation R coupled to background G gauge field is given by an invertible field theory with partition function the exponential of an η -invariant of the Dirac operator, $\eta_{M,R}(\mathcal{D}_{\text{Dirac}})$ [WY19, Section 4.3].

- For gauginos it is given by $\alpha_{1/2} = \exp(\pi i \eta_{M,R}(\mathcal{D}_{\text{Dirac}})/2)$ [Wit16, FH21a].
- For gravitinos it is given by $\alpha_{3/2} = \exp(\pi i \eta_{\text{gravitino}}/2)$ where

$$\eta_{\text{gravitino}} = \eta_{M,R}(\mathcal{D}_{\text{Dirac} \times TM}) - 2\eta_{M,R}(\mathcal{D}_{\text{Dirac}}), \tag{5.2}$$

and $\eta_{M,R}(\mathcal{D}_{\text{Dirac} \times TW})$ is the Dirac operator acting on the spinor bundle tensored with the tangent bundle.¹¹

For the remainder of the paper we will sometimes drop the M subscript label when there is no confusion that we are considering the generating manifold. We also replace TM by TW where W is the Wu manifold.

Lemma 5.3. *If $R = \sum_i R_i$ then $\eta_{\sum_i R_i}(\mathcal{D}_{\text{Dirac}}) = \sum_i \eta_{R_i}(\mathcal{D}_{\text{Dirac}})$.*

The anomaly for the vector boson is not given in terms of an η -invariant, but we assume that it is also an invertible theory, and we show that it also vanishes. The next section is dedicated to showing:

Theorem 5.4. *The total anomaly (global and perturbative) of $4d \mathcal{N} = 8$ supergravity arising from the gaugino, vector boson, and gravitino, vanishes on the Wu manifold.*

5.1. Evaluating on the Wu manifold. The full anomaly denoted by α can be written schematically as

$$“\alpha = \alpha_{1/2}^{\text{pert}} \otimes \alpha_1^{\text{pert}} \otimes \alpha_{3/2}^{\text{pert}} \otimes \alpha_{1/2}^{\text{np}} \otimes \alpha_1^{\text{np}} \otimes \alpha_{3/2}^{\text{np}}” \tag{5.5}$$

where we have split up each part of the perturbative and nonperturbative anomaly coming from the gaugino, vector boson, and gravitino. Technically speaking, separating the anomaly in this way is not something that can be done canonically. By (3.4) the non-topological part arises as a quotient of the invertible theory by the topological theories. We write the anomaly in such a way in order to make it organizationally more clear.

The vector bosons can be defined without choosing a spin structure, and therefore the partition function of their anomaly field theory factors through the quotient by fermion parity. That is, the tangential structure is

$$\text{SO} \times G_8 = (\text{Spin-SU}_8)/\{\pm 1\}. \tag{5.6}$$

We will proceed in understanding the perturbative anomalies by isolating α_1^{pert} .

Lemma 5.7. *The perturbative anomaly for the vector bosons independently vanishes.*

Proof. With the knowledge that the manifold generator for the anomaly is the Wu manifold, we will further restrict to the SO_3 inside of G_8 ; we are left to computing $\Omega_6^{\text{SO}}(\text{BSO}_3) \otimes \mathbb{Q}$, which isolates the free summand. For the degree we are after, we can compute the bordism group via the AHSS. We take the E^2 page of

$$E_{p,q}^2 = H_p(\text{BSO}_3, \Omega_q^{\text{SO}}(\text{pt})) \implies \Omega_6^{\text{SO}}(\text{BSO}_3), \tag{5.8}$$

¹¹ Different sources in the literature give slightly different formulas for (5.2), all of the form $\eta_{\text{gravitino}} = \eta_{M,R}(\mathcal{D}_{\text{Dirac} \times TM}) - k\eta_{M,R}(\mathcal{D}_{\text{Dirac}})$, where k is an even integer depending on the dimension of the theory. For a four-dimensional theory, $k = 2$: this is the unique value of k consistent with perturbative anomaly cancellation for the U-duality symmetry as in [Mar85] and agrees with the value given in [FH21a, §3.2] following [Fre02, Appendix A].

where

$$\Omega_*^{SO}(\text{pt}) = \{ \mathbb{Z}, 0, 0, 0, \mathbb{Z}, \mathbb{Z}/2, 0, \dots \}, \tag{5.9}$$

and tensor with \mathbb{Q} . This will kill the torsion parts that arise on the line $q = 5$, and the result is equivalent to the E_∞ page, as all differentials vanish. The E_∞ page of $\Omega_6^{SO}(BSO_3) \otimes \mathbb{Q}$ is given by

6	0						
5	0	0					
4	\mathbb{Q}	0	0	0	\mathbb{Q}	0	
3	0	0	0	0	0	0	
2	0	0	0	0	0	0	
1	0	0	0	0	0	0	
0	\mathbb{Q}	0	0	0	\mathbb{Q}	0	
	0	1	2	3	4	5	6

(5.10)

We see that the perturbative anomaly of the vector boson vanishes. □

Corollary 5.11. *The perturbative anomalies from the fractional spin particles vanish on their own.*

Having established this corollary, we may now pullback the anomaly in (5.5) to the nonperturbative part, and the equation becomes literally true.

The η -invariant for the contributions in $\alpha_{1/2}^{np} \otimes \alpha_{3/2}^{np}$ is therefore a bordism invariant.

Lemma 5.12. *The spin-SU₈ bordism invariant $\alpha_{1/2}^{np} \otimes \alpha_{3/2}^{np} : \Omega_5^{\text{Spin-SU}_8} \rightarrow \mathbb{C}^\times$ vanishes.*

Proof. It suffices to check on a generating set for $\Omega_5^{\text{Spin-SU}_8}$, and in Sect. 4.4 we showed that the Wu manifold W generates. In particular, the spin-SU₈ structure on W is induced from a spin-SU₂ structure, so the η -invariants that make up the partition functions of $\alpha_{1/2}^{np}$ and $\alpha_{3/2}^{np}$ can be computed by pulling back representations from SU₈ to SU₂, then evaluating the corresponding SU₂ η -invariants on W .

We will show that there are representations V_1 and V_2 of SU₂ such that $\alpha_{1/2}^{np}(W) = \eta_{W, 2V_1}(\mathcal{D}_{\text{Dirac}})$ and $\alpha_{3/2}^{np}(W) = \eta_{W, 2V_2}(\mathcal{D}_{\text{Dirac}})$. By Lemma 5.3, this means

$$(\alpha_{1/2}^{np} \otimes \alpha_{3/2}^{np})(W) = \exp\left(\frac{i\pi}{2} \eta_{W, V_1 \oplus V_2}(\mathcal{D}_{\text{Dirac}})\right)^2. \tag{5.13}$$

The quantity $\exp((i\pi/2)\eta_{W, V_1 \oplus V_2}(\mathcal{D}_{\text{Dirac}}))$ is a bordism invariant of spin-SU₂ manifolds: its perturbative part (i.e. its image in $\text{Hom}(\Omega_6^{\text{Spin-SU}_2}, \mathbb{Z})$) vanishes, because that invariant is 1/2 of the perturbative part of $\alpha_{1/2} \otimes \alpha_{3/2}$, which we showed vanishes in Corollary 5.11 (and is \mathbb{Z} -valued, so 1/2 of it is uniquely defined if it exists). The equation $2[W] = 0$ in $\Omega_5^{\text{Spin-SU}_8}$ pulls back to imply $2[W] = 0$ in $\Omega_5^{\text{Spin-SU}_2}$,¹² so since $\alpha_{1/2}^{np} \otimes \alpha_{3/2}^{np}(W)$ is the square of another spin-SU₂ bordism invariant on W by (5.13), we conclude $\alpha_{1/2}^{np} \otimes \alpha_{3/2}^{np}(W) = 0$.

The rest of this proof is devoted to finding V_1 and V_2 . We consider how 56, 28, and 8 split via our fourfold embedding of SU₂ into G_8 for the Wu manifold. We see that 56

¹² Alternatively, the equation $2[W] = 0$ in spin-SU₂ bordism follows from calculations of Freed–Hopkins [FH21b, Figure 5, case $s = 4$].

gives the dimension of the alternating three forms in 8-dimensions, **28** the dimension of alternating two forms, and **8** is the defining representation. The branchings are given by

$$\mathbf{56} \rightarrow 2(10 \times \mathbf{2} + 2 \times \mathbf{4}), \tag{5.14}$$

$$\mathbf{28} \rightarrow 2(3 \times \mathbf{3} + 5 \times \mathbf{1}), \tag{5.15}$$

$$\mathbf{8} \rightarrow 4 \times \mathbf{2}, \tag{5.16}$$

where the right hand side is in terms of \mathfrak{su}_2 representations. In increasing numerical order, they are the trivial, defining, adjoint, and Sym^4 representation. To show this, notice that **8** splits as $V^{\oplus 4}$ when we pull back to SU_2 , where $V = \mathbf{2}$. We then consider the ways of splitting the alternating three forms. This can be done as

$$\Lambda^2 V \otimes \Lambda^1 V \otimes \Lambda^0 V \otimes \Lambda^0 V = \mathbb{C} \otimes V \otimes \mathbb{C} \otimes \mathbb{C} \tag{5.17}$$

in 12 ways, essentially partitioning 3 into a sum of length 2. The \mathbb{C} for both $\Lambda^2 V$ and $\Lambda^0 V$ show that they are isomorphic as representations. It can also split into

$$\Lambda^1 V \otimes \Lambda^1 V \otimes \Lambda^1 V \otimes \Lambda^0 V = V \otimes V \otimes V \otimes \mathbb{C} \tag{5.18}$$

in 4 ways. The fact that the third tensor product of the defining representation decomposes as $\mathbf{2} \otimes \mathbf{2} \otimes \mathbf{2} = \mathbf{2} + \mathbf{2} + \mathbf{4}$, gives us (5.14). Similarly, the two forms can be split into

$$\Lambda^2 V \otimes \Lambda^0 V \otimes \Lambda^0 V \otimes \Lambda^0 V \quad \text{and} \quad \Lambda^1 V \otimes \Lambda^1 V \otimes \Lambda^0 V \otimes \Lambda^0 V \tag{5.19}$$

in 4 ways and 6 ways, respectively. The fact that $\mathbf{2} \otimes \mathbf{2} = \mathbf{1} + \mathbf{3}$, establishes (5.15).

As a spin $3/2$ particle, the gravitino contains a spinor index as well as a Lorentz index, therefore in order to use (5.2) for the anomaly, we need to use the fact that the tangent bundle of the Wu manifold is an associated bundle.

Lemma 5.20. *The tangent bundle of the Wu manifold W is given by*

$$TW = \text{SU}_3 \times_{\text{SO}_3} \frac{\mathfrak{su}_3}{\mathfrak{so}_3}.$$

Proof. The fact that the Wu manifold is a homogeneous space allows us to use the following general procedure to construct its tangent bundle. For $H \subset G$ a closed subgroup of a Lie group G , we have the following exact sequence of adjoint representations of H :

$$1 \longrightarrow \mathfrak{h} \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{g}/\mathfrak{h} \longrightarrow 1. \tag{5.21}$$

The canonical principal H -bundle $H \rightarrow G/H$ gives an exact functor from representations of H to vector bundles over G/H . This gives a corresponding sequence of vector bundles:

$$1 \longrightarrow G \times_H \mathfrak{h} \longrightarrow G \times_H \mathfrak{g} \longrightarrow G \times_H \mathfrak{g}/\mathfrak{h} \longrightarrow 1. \tag{5.22}$$

There is an isomorphism $G \times_H \mathfrak{g}/\mathfrak{h} \rightarrow T(G/H)$ [BH58, §7.4]. Let $p: G \rightarrow G/H$ and L_X be the left invariant vector field generated by $X \in \mathfrak{h}$. Then the mapping of $(g, X + \mathfrak{h}) \in G \times (\mathfrak{g}/\mathfrak{h})$ to $T_g p \cdot L_X(g) \in T_{gH}(G/H)$ gives the isomorphism. Specifically for our problem, we have the SO_3 -bundle $\text{SU}_3 \rightarrow W$, which by the present construction gives the desired result. \square

Remark 5.23. This is an example of the “mixing construction”: for a principal G -bundle $P \rightarrow M$ and a G -representation V , the space $P \times_G V$ is a vector bundle over M with rank equal to the dimension of V .

Back to $\eta_{W,R}(\mathcal{D}_{\text{Dirac} \times TW})$ in $\eta_{\text{gravitino}}$. In terms of $\eta_R(\mathcal{D}_{\text{Dirac}})$ this term in (5.2) unpacks to

$$\eta_R(\mathcal{D}_{\text{Dirac} \times TW}) = \eta_{W,TW \otimes \mathbf{8}}(\mathcal{D}_{\text{Dirac}}). \tag{5.24}$$

We want to show this η -invariant is for a representation of the form $V_2 \oplus V_2$. We therefore need to study the representation $TW \otimes \mathbf{8}$. Since TW is the associated bundle to $\mathfrak{su}_3/\mathfrak{so}_3$, $\eta_{W,TW \otimes \mathbf{8}}(\mathcal{D}_{\text{Dirac}})$ is the η -invariant for the Dirac operator associated to the SO_3 -representation $(\mathfrak{su}_3/\mathfrak{so}_3) \otimes \mathbf{8}$. The \mathfrak{su}_8 -representation $\mathbf{8}$ factors as $(\mathbf{2} \oplus \mathbf{2}) \oplus (\mathbf{2} \oplus \mathbf{2})$ as an \mathfrak{su}_2 -representation by (5.16), so we can set $V_2 := (\mathfrak{su}_3/\mathfrak{so}_3) \otimes (\mathbf{2} \oplus \mathbf{2})$ and conclude. \square

We now move onto the nonperturbative anomaly from the vector bosons, which is accessible from $\Omega_5^{\text{SO}}(B\text{SO}_3)$. By applying (5.9) to the AHSS, we only need to consider $H_5(B\text{SO}_3, \mathbb{Z})$ as well as the $\mathbb{Z}/2$ element in bidegree $(0, 5)$. One can evaluate the torsion part of $H_5(B\text{SO}_3; \mathbb{Z})$ by the universal coefficient theorem; this will involve looking at $H^6(B\text{SO}_3; \mathbb{Z})$. We find that this is given by $w_2 w_3$ of the SO_3 bundle and is nontrivial on the Wu manifold. Then the AHSS says $\Omega_5^{\text{SO}}(B\text{SO}_3) = \mathbb{Z}/2 \times \mathbb{Z}/2$ detected by the bordism invariants $\int w_2(TM)w_3(TM)$ and $\int w_2(P)w_3(P)$; these are generated by W with the trivial bundle, and W with the principal SO_3 -bundle. We see that while the bosonic anomaly is in principle $\mathbb{Z}/2 \times \mathbb{Z}/2$ valued, and coupling to spin structure eliminates one of the $\mathbb{Z}/2$. Using (5.15), for the representation of the vector boson, the anomaly is also twice of something as a bordism invariant. This is reasonable since the anomaly of multiple particles is the tensor product of their anomalies¹³. The anomaly for the vector bosons is 2 times something as a bordism invariant, since the perturbative part vanished, and considering that we have argued that everything else in (5.5) vanishes aside from α_1^{np} , we have that $\alpha = \alpha_1^{\text{np}}$. But α is $\mathbb{Z}/2$ valued, and with α_1^{np} equating to 0 mod 2, the full anomaly vanishes, thus establishing Theorem 5.4.

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¹³ For the gaugino and gravitino we could employ the decomposition of representations directly to the η -invariant. In the case of the vector boson, we use the fact that direct sums of representations goes to tensor products of anomalies.

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