Multiscale Gyrokinetics for Rotating Tokamak Plasmas

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To Deborah Abel, William Abel, and in memory of Alan Keith Abel
1944-1998, R.I.P.

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Abstract

This thesis presents a complete theoretical framework for turbulence and transport in tokamak plasmas. The fundamental scale separations present in plasma turbulence are codified as an asymptotic expansion in the ratio of the gyroradius to the equilibrium scale length. Proceeding order-by-order in this expansion, a framework for plasma turbulence is developed. It comprises an instantaneous equilibrium, the fluctuations driven by gradients in the equilibrium quantities, and the transport-timescale evolution of mean profiles of these quantities driven by the fluctuations. The equilibrium distribution functions are local Maxwellians with each flux surface rotating toroidally as a rigid body. Large-scale deviations of the distribution function from a Maxwellian are given by neoclassical theory. The fluctuations are determined by the high-flow gyrokinetic equation, from which we derive the governing principle for gyrokinetic turbulence in tokamaks: the conservation and local cascade of free energy. Transport equations for the evolution of the mean density, temperature and flow velocity profiles are derived. These transport equations show how the neoclassical corrections and the fluctuations act back upon the mean profiles through fluxes and heating.

This framework is further developed by exploiting the scale separation between ions and the electrons. The gyrokinetic equation is expanded in powers of the electron to ion mass ratio, which provides a rigorous method for deriving the electron response to ion-scale turbulence. We prove that such turbulence cannot change the magnetic topology, and argue that, therefore, the magnetic field lies on fluctuating flux surfaces. These flux surfaces are used to construct magnetic coordinates, and in these coordinates a closed system of equations for the electron response is derived. All fast electron timescales have been eliminated from these equations. Simplified transport equations for electrons in this limit are also derived.

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Preface

This thesis is the result of 5 years of research. The results of some of the chapters contained herein are already published online. Chapter 2 has been submitted to Reports on Progress in Physics and is available online at http://arxiv.org/pdf/1209.4782v1. Chapter 3 is in preparation for the New Journal of Physics and is available online at http://arxiv.org/pdf/1210.1417v1. During the course of the research projects that comprise this thesis other results, which do not fit neatly into this narrative, have been obtained. These results produced three other publications: a discussion of model collision operators within the framework of gyrokinetics [1, 2] and a study of the linear destabilization of Alfvén eigenmodes in tokamaks [3]. Other related publications include a report on the numerical implementation of the transport equations of Ch. 2 [4] and an experimental study of plasma turbulence in MAST [5]. Another piece of research that has not been included is an extension of the framework derived in Ch. 2 to energetic non-Maxwellian species in a burning tokamak plasma. A paper on this is currently in preparation [6].

CHAPTER 1

Introduction

Deriving clean energy from thermonuclear reactions has been the dream of scientists and engineers ever since the dying days of the second world war. Many strands of research have pursued this goal, the most promising of which is magnetically-confined fusion – the idea of creating a reactor that traps ionized hydrogen isotopes with magnetic fields. Due to the extreme temperatures and pressures involved, enormous gradients must be set up between the relatively cold edge of such a device, where the temperature must be less than 100 eV, to the centre, where the fuel temperature can reach many tens of thousands of electron volts. How quickly particles and energy slip down these gradients and leak out of the reactor is called their confinement – better confinement means a slower leakage of particles and heat, and thus a more efficient reactor.

The energy, particle, and momentum confinement of present-day fusion experiments and proposed future devices, such as ITER, is limited by transport that is induced by small-scale rapid fluctuations within the plasma – turbulent transport. Thus, having a framework in which we can analyse and simulate plasma turbulence is of paramount importance to the fusion programme.

1.1 Turbulence in Tokamak Plasmas

Plasma turbulence in fusion devices is a fundamentally multiscale problem both in space and in time. The turbulent fluctuations driven by background gradients typically occur at scales associated with the ion Larmor radius, ρ_i (or smaller), whilst the mean temperature, density, and bulk velocity profiles vary smoothly over the system scale (in tokamaks, the minor radius a). Similarly, the fluctuation frequency ω is much larger than the rate at which the mean profiles evolve, $\sim \tau_E^{-1}$, where τ_E is the energy confinement time. Table. 1.1 gives approximate values for these space and time scales in some large tokamaks – it is manifest that the separation of scales is very strong in such plasmas. Because of this scale separation, it is possible to average over the time and space scales associated with the turbulent fluctuations and consider mean fields that slowly evolve due to turbulent and collisional transport, with the turbulence in turn driven by gradients in the same mean fields [7, 8, 9, 10, 11, 12].

There are also scale separations associated with the turbulence itself. Turbulence in a strongly magnetized plasma occurs at frequencies ω which are much smaller than the cyclotron frequency of the ions, Ω_i (see Table. 1.1). The turbulence is also strongly anisotropic, viz., correlation lengths along the mean magnetic field are much longer than correlation lengths across the field; particles can stream rapidly along field lines but only drift slowly across them. These two properties of the turbulence are the foundation of the gyrokinetic theory [13, 14, 15, 16], in which the fast cyclotron time scales are averaged out and the full 6D kinetics reduced to a simpler 5D formulation, the kinetics of charged rings.

In this thesis, we unify this hierarchy of timescales and spatial scales in one formulation. We use the physical scale separations inherent in plasma dynamics to determine how the mean fields influence the evolution of the small-scale turbulence, and how the turbulent fluctuations react back upon the mean fields. To quantify this back reaction, we derive the transport equations, in which it becomes manifest by what physical mechanisms the turbulence can affect the mean distribution functions and fields.

This procedure for tackling the challenging theoretical problem of turbulence in tokamak plasmas is known as "Multiscale Gyrokinetics".

1.2 Multiscale Gyrokinetics

Chapter 2 is devoted to the general derivation of the multiscale gyrokinetic framework. Sections 2.1–2.9 present a pedagogical derivation all the way from the Vlasov-Landau-Maxwell system of equations to the non-equilibrium mean-field thermodynamics of the system. Formally, we will use the small parameter $\epsilon = \rho_i/a$ (known elsewhere as ρ^*) to quantify the scale separations discussed above and then perform an asymptotic expansion in $\epsilon \ll 1$. The order of presentation is the order of this asymptotic expansion, in the spirit of asymptotology [22]. However, there are three conceptual strands interwoven in the derivation that deserve to be highlighted separately.

Firstly, there is the equilibrium – the instantaneous solution for the mean fields. First we recover the toroidally rotating Maxwellian for the mean distribution function

Parameter	JET	D-IIID	K-STAR	TFTR	ITER (projected)
B, T	3.5	2	3.5	5-6	5
n_e, cm^{-3}	5×10^{13}	5×10^{13}	5×10^{13}	10^{14}	10^{14}
T_i, keV	5–15	7	5–10	5–32	25
u, km/s	500	200	0.1–100	100	50
$M = u/v_{\mathrm{th}_i}$	0.3–0.5	0.35	0.01–0.2	0.1–0.2	0.05
λ_{De}, cm	1.3×10^{-2}	8.8×10^{-3}	8.2×10^{-3}	4×10^{-3}	1.2×10^{-2}
ρ_i, cm	0.051	0.06	0.045	0.032	0.032
ρ_e, cm	8.5×10^{-4}	1×10^{-3}	7.5×10^{-4}	5.3×10^{-4}	5.3×10^{-4}
a, cm	100	50	50	87	200
ν_{ee}, s^{-1}	1.6×10^3	4.5×10^3	7.2×10^3	1.3×10^3	1.3×10^3
$\omega \sim v_{\mathrm{th}_i}/a$, s^{-1}	2.0×10^4	1.6×10^5	1.5×10^5	1.4×10^5	5.5×10^4
Ω_i, s^{-1}	1.7×10^8	9.6×10^7	1.7×10^8	2.6×10^8	2.4×10^8
τ_E, s	0.5	0.1	0.3	0.3	3.5
$\epsilon = \rho_i/a$	5×10^{-3}	1.21×10^{-2}	6.9×10^{-3}	5.2×10^{-3}	2.28×10^{-3}
β_i	2.5%	3.5%	1–4%	4.5%	4%

Table 1.1: Typical length and time scales in selected fusion devices (approximate). JET parameters estimated from [17], DIII-D from [18], TFTR from [19], and ITER from TRANSP studies[20, 21].

(§2.5.1) and find that the toroidal rotation is a rigid-body motion of nested flux surfaces (§2.3.1 and §2.5.1). This solution has an arbitrary density and temperature for each species and an arbitrary angular velocity on each flux surface. Then, in §2.5.3, we find that the the poloidal variation of the density is determined by the balance between centrifugal forces and electrostatic fields set up within a flux surface. With these results in hand, the poloidal flux function is determined, in §2.6.2, by a generalized Grad-Shafranov equation (2.136). Finally, the first-order correction to the mean distribution function is given by the solution of the neoclassical drift-kinetic equation, which is also derived within our unified formalism (§2.6.1).

Secondly, there are the fluctuations that feed on the free-energy sources (gradients) present in this (local) equilibrium. The fluctuating distribution function splits into the Boltzmann response to the fluctuating electromagnetic fields and a distribution of charged rings (§2.5.4). This non-Boltzmann part of the distribution function is governed by the gyrokinetic equation (§2.6.4). This system of equations is closed by Maxwell's equations for the fluctuating electromagnetic fields in §2.6.5. Finally, in §2.8.2, we conclude this strand by deriving the conservation law that governs the nature of the fluctuations: the conservation of free energy.

Thirdly, there is the long-time evolution of the mean fields. This starts in §2.6.3, where we determine the evolution of the mean magnetic field. This turns out to be completely independent of the fluctuations. The back-reaction of the small-scale turbulence on the mean profiles is found in the transport equations of §2.7. Examining particle, momentum and energy conservation, we find (2.166), (2.179), and (2.194), which determine the transport-timescale evolution of the density, angular velocity and temperature profiles in terms of fluxes and sources, which in turn are given as functions of the turbulent and neoclassical distribution functions and fields. This strand concludes with the results of §2.8.1 and §2.9: that the total energy is conserved on the transport timescale and that the increase of mean entropy can be written in the usual way as a combination of heating terms and the product of fluxes and thermodynamic gradients. The results of Ch. 2 are summarised in §2.11.

This thesis (and Ch. 2 in particular) is written in a way that presents both red-erivations of many known results, cast in forms suitable for our unified framework, and a number of new results – we have, throughout the exposition, striven to give credit where credit is due without attempting to provide a fully exhaustive literature review. To guide a reader encountering gyrokinetic theory for the first time, it is perhaps useful to put our approach into the context of other, alternative, approaches. Our exposition falls very much within what might be termed "traditional gyrokinetics," in which the theory is viewed as an order-by-order asymptotic expansion. We have made no attempt to adjust our equations to achieve a Hamiltonian structure or exact energy conservation *within* each asymptotic order (see [23] and references therein) – indeed, the salient point of §2.8 is that energy in a multiscale system flows between fluctuations and mean fields and so "between different orders" of the asymptotic expansion. The expansion terminates at the transport order (§2.7) in the sense that the equations are closed and energy is conserved overall. Similarly, we make no attempt to formulate "global" equations that simultaneously describe the long and short scales – in fact, we take scale separation to be a virtue and consistently enforce it within our formalism. This said, alternative approaches (sometimes termed "modern gyrokinetics") have many virtues of their own to recommend them – not least some fascinating mathematics – and we refer the curious reader to recent reviews [23, 24] where they are presented.

1.3 Simplifying the Electron Dynamics

In the treatment of Chapter 2, all species are considered equal. However, there is a second scale separation that is important, the scale separation between the plasma ions and the electrons, characterised by the small parameter $\delta = \sqrt{m_e/m_i}$, where m_e and m_i are the electron and ion masses, respectively. For a deuterium plasma,

$\delta \approx 1/60$.

Ion-scale fluctuations (e.g., Ion Temperature Gradient (ITG) [25, 26, 27], Parallel Velocity Gradient (PVG) [28, 29] or Trapped Electron Mode (TEM) [30] driven turbulence) have length scales perpendicular to the magnetic field comparable to the ion gyroradius and have frequencies comparable to the ion transit or bounce frequencies. In contrast, electron-scale fluctuations have perpendicular length scales comparable to the electron gyroradius, and frequencies comparable to the electron transit or bounce frequency. Unless they are suppressed (e.g., by strongly sheared flows), ion-scale fluctuations with their larger "mixing length" usually dominate the transport – and are therefore considered to be more important. In Ch. 3, we focus on the electron response to such fluctuations.

The electrons are much lighter than the ions, hence ion-scale fluctuations are long-wavelength compared to the electron gyroradius and low-frequency compared to the electron transit or bounce frequencies. It is clear that in simulating ion-scale turbulence one does not wish to resolve the short electron timescales and in deriving theories of such turbulence one is compelled to find ways of effectively averaging over the electron dynamics. This has motivated the development of models for the electron response to ion-scale turbulence that contrive to eliminate fast processes due to electrons [31, 32, 33, 34]. These models have either been physically motivated rather than derived from first principles or limited to low-β (the ratio of thermal to magnetic pressure), non-rotating plasmas. In Ch. 3, we derive for the first time a fully general (i.e., including $\beta \sim 1$ effects, sonic toroidal rotation, and arbitrary axisymmetric geometries) and rigorous set of equations governing the electron behaviour in the presence of ion-scale fluctuations – we postpone a detailed comparison with previous models to §3.7. These equations eliminate the fast electron timescale via an expansion of the gyrokinetic equations in $\delta \ll 1$, subsidiary to the expansion in $\epsilon \ll 1$.

The fast processes arise from the streaming of electrons along magnetic field lines. In electrostatic turbulence, this streaming is along the unperturbed mean field and easily handled. The electrostatic approximation only holds at very low β. However, β can be viewed as a crude measure of the efficiency of a tokamak – how much plasma pressure can be confined by a given magnetic field. High plasma pressure gives rise to an increased fusion reaction rate, whereas producing a high magnetic field requires increased investment (monetarily and energetically) in coils and power supplies. Thus, any future reactor design will operate with as high a β as possible – one of the goals of the ITER experiment is to investigate high-β configurations [35]. Indeed, spherical tokamaks (such as MAST [36]) already operate at relatively high β. Thus, we require an electron response model that incorporates the fluctuations of the magnetic field in a fully general way. As we shall see, if we consider only ion-scale fluctuations, then the topology of the magnetic field is preserved by the fluctuations (§3.2.2). However, to determine the electron behaviour, we will need to make assumptions about the initial structure of the magnetic field and the effect of other scales. In particular, whether the field is regular, with nested flux surfaces, or stochastic. We show in §3.2.3 that if the magnetic field were significantly stochastic (i.e., if the ion heat transport due to the stochastic field were comparable to the transport due to the fluctuating $\boldsymbol{E} \times \boldsymbol{B}$ drifts) then the electron heat transport would be larger than the ion transport by a factor of $\delta^{-1} \gg 1$. Such poor electron heat confinement is not observed in experiments, indeed it is common for the level of ion heat transport to exceed the level of electron heat transport in conventional tokamaks. Thus, we assume that any stochastic component of the magnetic field is small enough to be ignored (§3.2.3) and, therefore, that the field lines lie on fluctuating magnetic surfaces.

The remainder of Ch. 3 is dedicated to the precise derivation of the equations governing the electron response to ion-scale turbulence (§3.3). We also go on to discuss simplified transport equations for the electrons which are derived relative to the exact magnetic field (§3.6).

The summaries of our findings are presented, for each of the two main chapters, in §2.11 and §3.8. Finally, in Ch. 4, we conclude this thesis by discussing the key results of Chapters 2 & 3: how they fit into the broader framework of fusion plasma physics and where they may lead next.

Chapter 2

Fluctuations, Transport and Energy Flows

The work in this chapter is drawn entirely from [37], written by the Author. References to Appendices made in this chapter refer to the Appendices of this work – which has been submitted to Reports on Progress in Physics and is available online at `http://arxiv.org/pdf/1209.4782v1`.

In this chapter we provide our promised derivation of multiscale gyrokinetics (see §1.2). We start by recapitulating the fundamental equations of plasma physics in §2.1 and splitting all quantities into mean and fluctuating parts in §2.2. We then impose order on our multiple small parameters and scale separations by introducing the fundamental gyrokinetic ordering in §2.2.4. This allows us to formulate the entire problem as a systematic expansion in the small ratio $\epsilon = \rho_i/a$. In subsequent sections, we proceed to expand the equations of §2.1 order by order in ϵ, zeroth order in §2.3, first in §2.5, second in §2.6 and finally, third (transport order) in §2.7; pausing in §2.4 to introduce rotating gyrokinetic variables, which are convenient for handling the turbulent kinetics in toroidally rotating plasmas.

Sections 2.1 to 2.7 are, necessarily, quite technical. Readers who believe themselves to be already familiar with the formalism presented in these sections may wish to skip directly to §2.8. Sections 2.8 and 2.9 attempt to explain the physical content of multiscale gyrokinetics by combining all our earlier results and examining how energy and entropy flow between the various constituent parts of our multiscale system. In §2.8.1, we prove that our transport equations conserve the total energy, and that the fluctuations can do no net work on the mean fields. Explaining this result leads us in §2.8.2 to consider the balance of free energy of the fluctuations, as this is the fundamental conserved quantity of kinetic turbulence [38, 39, 40, 41]. The equation derived for the conservation of free energy clearly demonstrates a local (to a given flux surface) turbulent cascade that takes the energy injected by instabilities and dissipates it via collisions. In §2.9, we link the free-energy cascade and the mean-field transport through the evolution of the mean entropy of the system.

In §2.10, we present the low-Mach-number limit of the system of equations given in this chapter. This limit removes many cumbersome technical complications and so Sections 2.10.2–2.10.6 can be used as a concise summary of the basic structure of our multiscale hierarchy. It is this set of equations which is implemented in current linked-flux-tube transport codes [4, 42].

Finally, in §2.11, we summarize the key results of this chapter.

2.1 Fundamental Equations

As our starting point we take the Fokker-Planck kinetic equation for f_s, the distribution function of species s,

$$\frac{df_s}{dt} = \frac{\partial f_s}{\partial t} + \boldsymbol{v}\cdot\nabla f_s + \frac{Z_s e}{m_s}\left(\widetilde{\boldsymbol{E}} + \frac{1}{c}\boldsymbol{v}\times\widetilde{\boldsymbol{B}}\right)\cdot\frac{\partial f_s}{\partial \boldsymbol{v}} = C[f_s] + S_s, \tag{2.1}$$

where Z_s is the charge of the particles of species s as a multiple of the fundamental charge e, m_s their mass and $\boldsymbol{v}$ their velocity. We will work in Gaussian units throughout with c the speed of light, $\widetilde{\boldsymbol{E}}$ the electric field and $\widetilde{\boldsymbol{B}}$ the magnetic field (throughout this work, tildes denote exact fields containing both the mean and the fluctuating parts; see §2.2.1). On the right hand side, $C[f]$ is the Landau collision operator and S_s an arbitrary source term that stands in for all physical processes not yet accounted for, e.g. atomic physics, fusion reactions, Bremsstrahlung, radio frequency heating and current drive.

The electric and magnetic fields obey Maxwell's equations:

$$\nabla\cdot\widetilde{\boldsymbol{E}} = 4\pi\widetilde{\varrho}, \tag{2.2}$$

$$\nabla\cdot\widetilde{\boldsymbol{B}} = 0, \tag{2.3}$$

$$\frac{\partial\widetilde{\boldsymbol{B}}}{\partial t} = -c\nabla\times\widetilde{\boldsymbol{E}}, \tag{2.4}$$

$$\nabla\times\widetilde{\boldsymbol{B}} = \frac{4\pi}{c}\widetilde{\boldsymbol{j}} + \frac{1}{c}\frac{\partial\widetilde{\boldsymbol{E}}}{\partial t}, \tag{2.5}$$

where the charge density $\widetilde{\varrho}$ and current $\widetilde{\boldsymbol{j}}$ are

$$\widetilde{\varrho} = \sum_s Z_s e\int d^3\boldsymbol{v} f_s, \tag{2.6}$$

$$\widetilde{\boldsymbol{j}} = \sum_s Z_s e\int d^3\boldsymbol{v}\,\boldsymbol{v} f_s. \tag{2.7}$$

For the purposes of this work, we neglect Debye-scale effects and relativistic effects i.e.

$$k_\perp^2 \lambda_{\mathrm{De}}^2 \ll 1, \tag{2.8}$$

$$\frac{v_{\mathrm{th}_s}^2}{c^2} \ll 1, \tag{2.9}$$

where $\lambda_{\mathrm{De}} = \sqrt{T_e/4\pi n_e e^2}$ is the Debye length, $v_{\mathrm{th}_s} = \sqrt{2T_s/m_s}$ the thermal speed and T_s and n_s are the mean temperature and density of species s, where the notion of mean will be rigorously defined in §2.2.1. The assumptions (2.8) and (2.9) are well satisfied in most modern fusion experiments and will be in future ones.

We use (2.8) to replace (2.2) with the quasineutrality constraint [1]

$$\widetilde{\varrho} = 0, \tag{2.10}$$

and (2.9) to drop the displacement current in (2.5) (the Darwin, or quasi-static, approximation), giving Ampère's Law,

$$\nabla \times \widetilde{\boldsymbol{B}} = \frac{4\pi}{c}\widetilde{\boldsymbol{j}}. \tag{2.11}$$

We also satisfy (2.3) and (2.4) by introducing the scalar potential $\widetilde{\varphi}$ and the vector potential $\widetilde{\boldsymbol{A}}$,

$$\widetilde{\boldsymbol{E}} = -\nabla\widetilde{\varphi} - \frac{1}{c}\frac{\partial \widetilde{\boldsymbol{A}}}{\partial t}, \tag{2.12}$$

$$\widetilde{\boldsymbol{B}} = \nabla \times \widetilde{\boldsymbol{A}}. \tag{2.13}$$

We will work in the Coulomb gauge $\nabla \cdot \widetilde{\boldsymbol{A}} = 0$. Thus, the four Maxwell equations (2.2)–(2.5) are replaced by (2.10), (2.11), (2.12) and (2.13).

2.2 Fluctuations and Mean Fields

2.2.1 Small-Scale Averaging

In order to analyse the mean and fluctuating quantities separately, we introduce the concept of an average over fluctuations, denoted by $\langle\cdot\rangle_{\mathrm{turb}}$. Formally, we demand that it separate any arbitrary physical quantity g into an averaged part $\langle g\rangle_{\mathrm{turb}}$ and a fluctuating part $\delta g = g - \langle g\rangle_{\mathrm{turb}}$, which by construction vanishes under the average, $\langle \delta g\rangle_{\mathrm{turb}} = 0$.

[1]The quasineutrality constraint implicitly defines the fluctuating electrostatic potential and the mean electric field within a flux surface but cannot be used to determine the mean radial electric field [43], so we derive an equation for the radial electric field from momentum conservation in §2.7.2 .

As there is a separation of scales, we can interpret this average as an average over the fluctuating temporal and spatial scales. The equilibrium length scale, denoted a (and understood to be, e.g., the tokamak minor radius), is well separated from the fluctuation length scale, taken to be the Larmor radius $\rho_s = v_{\mathrm{th}_s}/\Omega_s$, where $\Omega_s = Z_s eB/m_s c$. Therefore, we can pick an intermediate scale λ that satisfies

$$a \gg \lambda \gg \rho_s \tag{2.14}$$

and define the perpendicular spatial average $\langle\cdot\rangle_\perp$ of a function $g(\boldsymbol{r}, \boldsymbol{v}, t)$ by

$$\langle g(\boldsymbol{r}, \boldsymbol{v}, t)\rangle_\perp = \int_{\lambda_\perp^2} d^2\boldsymbol{r}'_\perp g(\boldsymbol{r}'_\perp, l, \boldsymbol{v}, t) \Bigg/ \int_{\lambda_\perp^2} d^2\boldsymbol{r}_\perp, \tag{2.15}$$

where $\lambda_\perp^2$ is a small surface which is everywhere normal to the magnetic field and has spatial extent of the order of λ in both perpendicular directions. The integrals are taken at constant $\boldsymbol{v}$, t and l, where l is the distance along a given field line, or any other field-aligned coordinate – see Fig. 2.1. Clearly, an averaged function cannot vary on the small length scales $\sim \rho_s$, and any function that varies only on the equilibrium length scale $\sim a$ is unaffected by the average. Similarly, the typical fluctuation time scale ω^{-1} and the timescale of the evolution of the mean profiles, taken to be the transport time τ_E, are also well separated. Therefore, we can pick an intermediate time T that satisfies

$$\tau_E \gg T \gg \omega^{-1} \tag{2.16}$$

and use it to define the temporal average $\langle\cdot\rangle_T$ by

$$\langle g(\boldsymbol{r}, \boldsymbol{v}, t)\rangle_T = \frac{1}{T}\int_{t-T/2}^{t+T/2} dt' g(\boldsymbol{r}, \boldsymbol{v}, t'), \tag{2.17}$$

with the integral taken at constant $\boldsymbol{r}$ and $\boldsymbol{v}$. Once again, we see that averaged functions cannot vary on the short timescale ω^{-1}, and that functions that vary on the transport timescale are unchanged.

We now define the full average over fluctuations by

$$\langle g(\boldsymbol{r}, \boldsymbol{v}, t)\rangle_{\mathrm{turb}} = \langle\langle g\rangle_\perp\rangle_T\,. \tag{2.18}$$

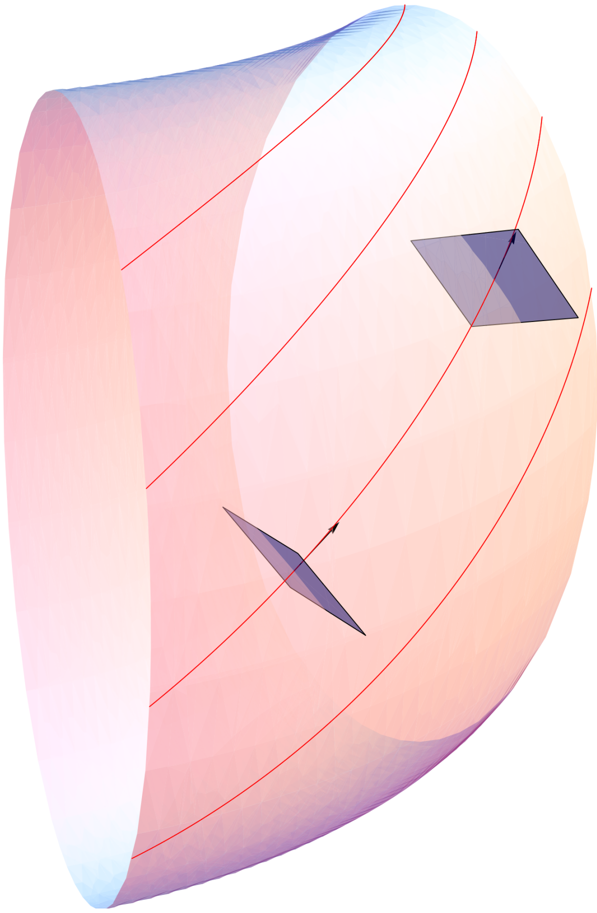

Figure 2.1: A section of a toroidal flux surface showing field lines and, in blue, the small perpendicular patches $\lambda_\perp^2$ over which fluctuations are averaged in (2.15). Black arrows denote the normals to the patches, and are aligned with the magnetic field.

We can split the distribution function f_s and all fields into mean and fluctuating parts:

$$f_s = F_s + \delta f_s, \qquad F_s = \langle f_s \rangle_{\mathrm{turb}}, \tag{2.19}$$

$$\widetilde{\boldsymbol{E}} = \boldsymbol{E} + \delta \boldsymbol{E}, \qquad \boldsymbol{E} = \left\langle \widetilde{\boldsymbol{E}} \right\rangle_{\mathrm{turb}}, \tag{2.20}$$

$$\widetilde{\boldsymbol{B}} = \boldsymbol{B} + \delta \boldsymbol{B}, \qquad \boldsymbol{B} = \left\langle \widetilde{\boldsymbol{B}} \right\rangle_{\mathrm{turb}}, \tag{2.21}$$

$$\widetilde{\boldsymbol{A}} = \boldsymbol{A} + \delta \boldsymbol{A}, \qquad \boldsymbol{A} = \left\langle \widetilde{\boldsymbol{A}} \right\rangle_{\mathrm{turb}}, \tag{2.22}$$

$$\widetilde{\varphi} = \varphi + \delta \varphi, \qquad \varphi = \langle \widetilde{\varphi} \rangle_{\mathrm{turb}}. \tag{2.23}$$

Let us use the average (2.18) to restate the formal assumptions about temporal and spatial variation of the mean and fluctuating quantities ($\langle g \rangle_{\mathrm{turb}}$ and δg):

$$\frac{\partial}{\partial t} \ln \langle g \rangle_{\mathrm{turb}} \sim \tau_E^{-1}, \tag{2.24}$$

$$\frac{\partial}{\partial t} \ln \delta g \sim \omega, \tag{2.25}$$

$$\nabla \ln \langle g \rangle_{\mathrm{turb}} \sim \boldsymbol{b} \cdot \nabla \ln \delta g \sim a^{-1}, \tag{2.26}$$

$$\nabla_\perp \ln \delta g \sim k_\perp \sim \rho_s^{-1}, \tag{2.27}$$

where $\boldsymbol{b}$ is the unit vector in the direction of the averaged magnetic field, and $\perp$ and $\|$ denote components of vectors or operators perpendicular and parallel to the averaged magnetic field, respectively. In §2.2.4, we will formally order all time scales, spatial scales and fluctuation amplitudes with respect to the small parameter $\epsilon = \rho_s / a$.

2.2.2 Axisymmetry and Magnetic Geometry

We will assume the axisymmetry of all mean quantities. In the case of modern tokamaks, the deviation from axisymmetry in the magnetic field is extremely small, much smaller than ρ_s/a in such devices. Let us introduce the cylindrical coordinate system: major radius R, vertical position z and toroidal angle ϕ. We pick this system such that (R, ϕ, z) is a right-handed coordinate system – see Fig. 2.2. Assuming that the toroidal magnetic field variation is the same as the toroidal variation of any averaged quantity, we formally demand that for any quantity $g(\boldsymbol{r})$,

$$\frac{\partial \ln \langle g \rangle_{\mathrm{turb}}}{\partial \phi} = 0. \tag{2.28}$$

At the end of §2.4.2, we will extend the notion of axisymmetry to distribution functions, which depend upon velocity space as well as the position $\boldsymbol{r}$. The averaged form of (2.13) is

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} = \left(\frac{\partial A_R}{\partial z} - \frac{\partial A_z}{\partial R} \right) R \nabla \phi + \frac{1}{R} \frac{\partial \left(R A_\phi \right)}{\partial R} \nabla z - \frac{\partial A_\phi}{\partial z} \nabla R, \tag{2.29}$$

where

$$\boldsymbol{A} = A_\phi \nabla \phi + A_R \nabla R + A_z \nabla z \tag{2.30}$$

and we have used (2.28) to drop all ϕ derivatives. Therefore, the magnetic field can be written in the usual toroidal decomposition [44]:

$$\boldsymbol{B} = I \nabla \phi + \nabla \psi \times \nabla \phi, \tag{2.31}$$

where

$$\psi(R, z) = A_\phi = R^2 \boldsymbol{A} \cdot \nabla \phi \tag{2.32}$$

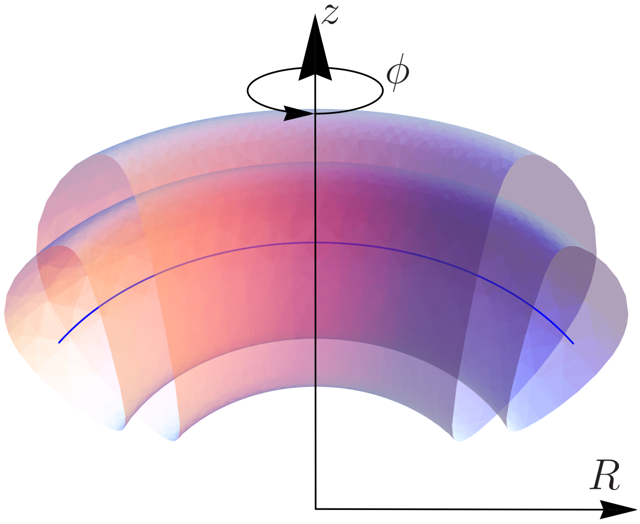

Figure 2.2: The toroidal coordinate system (R,z,ϕ) showing the magnetic axis and flux surfaces

is the poloidal flux function and

$$I(R,z) = R\left(\frac{\partial A_R}{\partial z} - \frac{\partial A_z}{\partial R}\right) = R^2\boldsymbol{B}\cdot\nabla\phi. \tag{2.33}$$

The toroidal symmetry guarantees the existence of well-defined flux surfaces [45, 46]. Topologically, these are nested tori. Since $\boldsymbol{B}\cdot\nabla\psi = 0$, these surfaces can be labelled by ψ. We will see that many mean quantities will only depend on R and z through $\psi(R,z)$.

2.2.3 Flux-Surface Averaging and the Motion of Flux Surfaces

It will be convenient to define an average over the surface labelled by ψ, which we do as follows. For an arbitrary function $g(\boldsymbol{r})$,

$$\langle g(\boldsymbol{r})\rangle_\psi(\psi) = \lim_{\Delta\psi\to 0}\left\{\int_{\Delta(\psi)} d^3\boldsymbol{r}\, g(\boldsymbol{r}) \bigg/ \int_{\Delta(\psi)} d^3\boldsymbol{r}\right\}, \tag{2.34}$$

where the domain of integration $\Delta(\psi)$ is the annulus between the flux surface labelled by ψ and that labelled by $\psi+\Delta\psi$ (see Fig. 2.3).

Note that

$$\begin{aligned}\frac{\partial}{\partial\psi}\int_{D(\psi,t)} d^3\boldsymbol{r}g &= \lim_{\Delta\psi\to 0}\frac{1}{\Delta\psi}\left(\int_{D(\psi+\Delta\psi,t)} d^3\boldsymbol{r}g - \int_{D(\psi,t)} d^3\boldsymbol{r}g\right)\\ &= \lim_{\Delta\psi\to 0}\frac{1}{\Delta\psi}\int_\Delta d^3\boldsymbol{r}g = V'\langle g\rangle_\psi,\end{aligned} \tag{2.35}$$

where $D(\psi, t)$ is the volume enclosed by the flux surface labelled by ψ (i.e., the volume between that surface and the magnetic axis; see Fig. 2.3), $V = \int_{D(\psi,t)} d^3\boldsymbol{r}$ is the volume of this region and

$$V' = \lim_{\Delta\psi\to 0} \frac{1}{\Delta\psi} \int_{\Delta} d^3\boldsymbol{r} = \frac{\partial V}{\partial \psi}. \tag{2.36}$$

Thus, an alternate definition of the flux-surface average is

$$\langle g \rangle_\psi = \frac{1}{V'} \frac{\partial}{\partial \psi} \int_{D(\psi,t)} d^3\boldsymbol{r}\, g = \frac{1}{V'} \frac{\partial}{\partial \psi} \int_0^\psi d\psi' \int_{\partial D(\psi',t)} \frac{dS}{|\nabla\psi|} g = \frac{1}{V'} \int_{\partial D(\psi,t)} \frac{dS}{|\nabla\psi|} g, \tag{2.37}$$

where the surface integral is taken over the boundary of D, $\partial D(\psi, t)$, and we have defined ψ in such a way that $\psi = 0$ at the magnetic axis.[2]

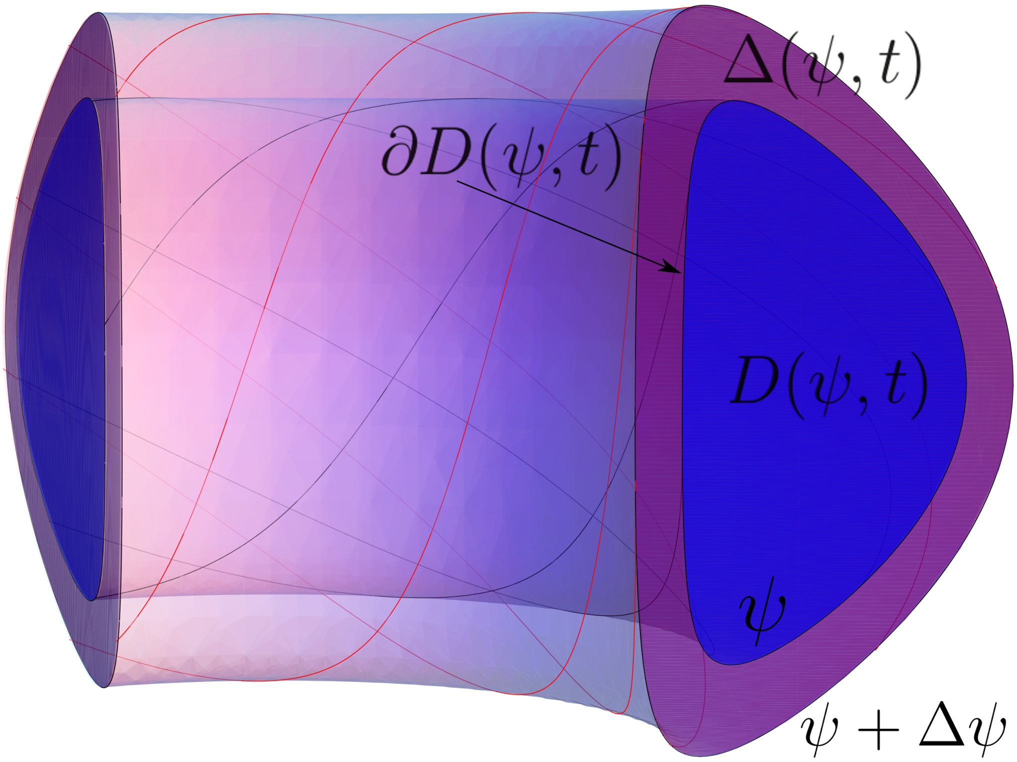

Figure 2.3: A section of the torus showing the regions $D(\psi, t)$ and $\Delta(\psi, t)$ used in defining the flux-surface average (2.34), (2.37) and the surface ∂D separating these regions.

We will discover in the subsequent sections (Sections 2.6.3, 2.7 and 2.8) that flux-surface averaging allows us to close evolution equations for the mean fields. Two

[2]In our choice of coordinates, ψ increases away from the magnetic axis if the toroidal current flows in the negative ϕ direction and the toroidal field is in the positive ϕ direction. Similarly, if the toroidal field and current are both in the positive ϕ direction, ψ will decrease away from the magnetic axis. If the positive ϕ direction is fixed to be opposite to that of the toroidal current then ψ will always increase outwards.

mathematical identities will be useful in those derivations: the formula for the flux-surface average of a divergence and that for the flux-surface average of a time derivative.

First, using (2.37), we find for any vector field $\boldsymbol{A}(\boldsymbol{r})$,

$$\langle \nabla \cdot \boldsymbol{A} \rangle_\psi = \frac{1}{V'} \frac{\partial}{\partial \psi} \int_{D(\psi,t)} d^3\boldsymbol{r} \nabla \cdot \boldsymbol{A} = \frac{1}{V'} \frac{\partial}{\partial \psi} \int_{\partial D(\psi,t)} \frac{dS}{|\nabla \psi|} \boldsymbol{A} \cdot \nabla \psi. \tag{2.38}$$

Therefore,

$$\langle \nabla \cdot \boldsymbol{A} \rangle_\psi = \frac{1}{V'} \frac{\partial}{\partial \psi} V' \langle \boldsymbol{A} \cdot \nabla \psi \rangle_\psi , \tag{2.39}$$

which is the first of the two identities we will require later. This identity also implies that

$$\langle \boldsymbol{B} \cdot \nabla g \rangle_\psi = \langle \nabla \cdot (g\boldsymbol{B}) \rangle_\psi = \frac{1}{V'} \frac{\partial}{\partial \psi} V' \langle g\boldsymbol{B} \cdot \nabla \psi \rangle_\psi = 0. \tag{2.40}$$

Thus, the flux-surface average annihilates the operator $\boldsymbol{B} \cdot \nabla$ [47].

Since the constant-ψ surfaces change in time, flux-surface averages do not commute with time derivatives. Let us consider

$$\begin{aligned} \left.\frac{\partial}{\partial t}\right|_\psi \int_{D(\psi,t)} d^3\boldsymbol{r} g &= \int_{D(\psi,t)} d^3\boldsymbol{r} \frac{\partial g}{\partial t} + \int_{\partial D(\psi,t)} \frac{dS}{|\nabla \psi|} g\boldsymbol{V}_\psi \cdot \nabla \psi \\ &= \int_{D(\psi,t)} d^3\boldsymbol{r} \frac{\partial g}{\partial t} + V' \langle g\boldsymbol{V}_\psi \cdot \nabla \psi \rangle_\psi , \end{aligned} \tag{2.41}$$

where $\boldsymbol{V}_\psi$ is the velocity with which the boundary of $D(\psi, t)$ moves. The time derivative in the left-hand side of (2.41) is taken at constant flux label ψ. Taking the derivative of (2.41) with respect to ψ and using (2.37), we find

$$\left\langle \frac{\partial g}{\partial t} \right\rangle_\psi = \frac{1}{V'} \frac{\partial}{\partial \psi} \int_{D(\psi,t)} d^3\boldsymbol{r} \frac{\partial g}{\partial t} = \frac{1}{V'} \left.\frac{\partial}{\partial t}\right|_\psi V' \langle g \rangle_\psi - \frac{1}{V'} \frac{\partial}{\partial \psi} V' \langle g\boldsymbol{V}_\psi \cdot \nabla \psi \rangle_\psi . \tag{2.42}$$

Finally, as the boundary of $D(\psi, t)$ is defined to be a constant-ψ surface, we have

$$\frac{\partial \psi}{\partial t} + \boldsymbol{V}_\psi \cdot \nabla \psi = 0. \tag{2.43}$$

As $\boldsymbol{V}_\psi$ is defined to be the velocity of a flux surface, we can demand that $\boldsymbol{V}_\psi$ has no component in the surface and so

$$\boldsymbol{V}_\psi = -\frac{\partial \psi}{\partial t} \frac{\nabla \psi}{|\nabla \psi|^2}. \tag{2.44}$$

Therefore,

$$\left\langle \frac{\partial g}{\partial t} \right\rangle_\psi = \frac{1}{V'} \left.\frac{\partial}{\partial t}\right|_\psi V' \langle g \rangle_\psi + \frac{1}{V'} \frac{\partial}{\partial \psi} V' \left\langle g \frac{\partial \psi}{\partial t} \right\rangle_\psi , \tag{2.45}$$

which is the second of the identities we were seeking.

2.2.4 The Gyrokinetic Ordering

To proceed further we need to impose order on the three small parameters we have, $\epsilon = \rho_s/a$, ω/Ω_s and the anisotropy of the turbulence $k_\parallel/k_\perp$, as well as on the amplitudes of the fluctuations. Then each term in the kinetic equation (2.1) will have a well defined order in terms of ϵ when compared to $\Omega_s f_s$.

We postulate the following standard ordering [16] of the time and space scales with respect to ϵ[3],

$$\frac{|\delta \boldsymbol{B}|}{|\boldsymbol{B}|} \sim \frac{|\delta \boldsymbol{E}|}{|\boldsymbol{E}|} \sim \frac{\delta f_s}{f_s} \sim \frac{k_\parallel}{k_\perp} \sim \frac{\omega}{\Omega_s} \sim \frac{\rho_s}{a} = \epsilon \tag{2.46}$$

Note that when treating the electromagnetic fields, we order the electric field $\boldsymbol{E}$ and the magnetic field $\boldsymbol{B}$ as

$$\boldsymbol{E} \sim \frac{v_{\mathrm{th}_s}}{c}\boldsymbol{B}, \tag{2.47}$$

which is equivalent to assuming that the $\boldsymbol{E} \times \boldsymbol{B}$ flows are at most sonic and not relativistic.

From these assumptions, we can make a simple random-walk estimate of the turbulent thermal diffusivity $\chi_{T_s} \sim \rho_s^2 \omega$ (gyro-Bohm diffusion [48]) and then order the transport time on the basis of this estimate:

$$\frac{1}{\tau_E} \sim \frac{\chi_T}{a^2} \sim \frac{\omega}{\Omega_s}\left(\frac{\rho_s}{a}\right)^2 \Omega_s \sim \epsilon^3 \Omega_s. \tag{2.48}$$

For the collision operator $C[f_s]$, we choose an ordering that allows the plasma to be either collisional or collisionless, namely $C[f_s] \sim \omega f_s$. Thus, all collision frequencies ν are ordered

$$\frac{\nu}{\Omega_s} \sim \epsilon, \tag{2.49}$$

and any further assumption about the collisionality will be handled as a subsidiary expansion. Even though the collision frequency ν is often smaller than ω, the collision operator $C[\delta f_s]$ must be retained as the turbulence will otherwise generate arbitrarily small scales in velocity space [38, 40, 41, 1, 49] [4]. We will return to this point in §2.8.2 when discussing free-energy balance.

Finally, we order the source term as $S_s \sim F_s/\tau_E$ which restricts us to sources of particles, energy and momentum that do not alter the Maxwellian form of the equilibrium distribution function (found in §2.5.1). In [6], this restriction is relaxed to consider sources that produce high-energy tails and other non-Maxwellian distributions.

[3] In this thesis, we use the symbol $\sim$ to mean "is the same order as" rather than the more usual "is asymptotically equivalent to".

[4] We choose not to order velocity-space derivatives, which would allow the introduction of smaller collision frequencies whilst retaining $C[f_s] \sim \omega f_s$.

We can now expand the mean (large-scale) distribution function F_s and the fluctuating (small-scale) distribution function δf_s as follows

$$F_s = F_{0s} + F_{1s} + F_{2s} + \cdots, \tag{2.50}$$

$$\delta f_s = \delta f_{1s} + \delta f_{2s} + \cdots, \tag{2.51}$$

where $F_{0s} \sim f_s$, $F_{1s} \sim \delta f_{1s} \sim \epsilon f_s$, $F_{2s} \sim \delta f_{2s} \sim \epsilon^2 f_s$, etc.

In the following sections, we insert (2.50) and (2.51) into the Fokker-Planck equation (2.1) and expand order by order in ϵ.

2.3 Zeroth Order $\mathcal{O}\left(\Omega_s f_s\right)$

In this section, we derive the lowest-order implications of imposing the ordering of §2.2.4 on the equations of §2.1.

We start by taking the lowest-order components of (2.10) and (2.11). The former just states that the densities n_s must satisfy the quasineutrality constraint,

$$\sum_s Z_s e n_s = 0, \qquad n_s = \int d^3\boldsymbol{v} F_{0s}. \tag{2.52}$$

Substituting the toroidal decomposition of the magnetic field, (2.31), in Ampère's law, (2.11), we merely find that the mean current $\boldsymbol{j} = \left\langle \widetilde{\boldsymbol{j}} \right\rangle_{\text{turb}}$ must be zero to this order,

$$\boldsymbol{j} = \sum_s Z_s e n_s \boldsymbol{u}_s = 0, \qquad \boldsymbol{u}_s = \frac{1}{n_s}\int d^3\boldsymbol{v}\boldsymbol{v} F_{0s}. \tag{2.53}$$

By taking the lowest-order component of (2.12), we also learn that the mean electric field is predominantly electrostatic:

$$\boldsymbol{E} = -\nabla\varphi + \mathcal{O}\left(\epsilon^2 \frac{v_{\text{th}_s}}{c} B\right). \tag{2.54}$$

2.3.1 Sonic Flows

We now prove that if the plasma has any perpendicular flow faster than the drift velocity then it is a purely toroidal $\boldsymbol{E} \times \boldsymbol{B}$ flow. This restriction on the flow has previously been found in the context of neoclassical collisional transport [50, 51, 52, 53].

From (2.1), we find to lowest order in ϵ,

$$\left(\boldsymbol{E} + \frac{1}{c}\boldsymbol{v} \times \boldsymbol{B}\right) \cdot \frac{\partial F_s}{\partial \boldsymbol{v}} = 0. \tag{2.55}$$

Multiplying by $\boldsymbol{v}$ and integrating over all velocities, we obtain

$$\boldsymbol{E} + \frac{1}{c}\boldsymbol{u}_s \times \boldsymbol{B} = 0. \tag{2.56}$$

This implies that the perpendicular part of $\boldsymbol{u}_s$ is species independent. Using (2.56) and (2.54) we have

$$\boldsymbol{E} \cdot \boldsymbol{B} = \boldsymbol{B} \cdot \nabla\varphi = 0. \tag{2.57}$$

Therefore, to lowest order, the electrostatic potential is a flux function and we can partially solve for φ as follows:

$$\varphi = \Phi\left(\psi\right) + \varphi_0, \tag{2.58}$$

where $\varphi_0 \sim \epsilon\Phi$. There is some arbitrariness in the definition of φ_0 by (2.58) as we can add any function of ψ to it. We resolve this by requiring that φ_0 vanish when averaged over a flux surface:

$$\left\langle\varphi_0\right\rangle_\psi = 0, \qquad \left\langle\varphi\right\rangle_\psi = \Phi(\psi). \tag{2.59}$$

Solving (2.56) for $\boldsymbol{u}_s$ shows that any sonic flow that is present is a $\boldsymbol{E}\times\boldsymbol{B}$ flow plus some arbitrary parallel flow

$$\boldsymbol{u}_s = \frac{c}{B}\boldsymbol{b}\times\nabla\varphi + u_{\parallel s}\boldsymbol{b}, \tag{2.60}$$

where $\boldsymbol{b} = \boldsymbol{B}\,/|\boldsymbol{B}|$. Using the solution (2.58) for φ and the toroidal decomposition of the magnetic field, (2.31), to expand $\boldsymbol{b}$, we have

$$\boldsymbol{u}_s = \omega(\psi)R^2\nabla\phi + \left[u_{\parallel s} - \omega(\psi)\frac{I}{B}\right]\boldsymbol{b}, \qquad \omega(\psi) = c\frac{d\Phi}{d\psi}. \tag{2.61}$$

The first term in (2.61) is a species-independent rigid-body toroidal rotation of each individual flux surface with an angular velocity $\omega(\psi)$. This will be denoted by

$$\boldsymbol{u} = \omega(\psi)R^2\nabla\phi. \tag{2.62}$$

The second term is a purely parallel flow, which in §2.5.1 will be shown to vanish to lowest order.

2.4 Rotating Gyrokinetic Variables

Before continuing to expand the kinetic equation (2.1), it will be convenient to introduce the following new variables [7, 16, 54, 55, 15]: the guiding-centre position

$\boldsymbol{R}_s$, particle energy ε_s, magnetic moment μ_s, gyrophase ϑ, and the sign of the parallel velocity σ. The variable transformation of the phase space is

$$(\boldsymbol{r}, \boldsymbol{v}) \rightarrow (\boldsymbol{R}_s, \varepsilon_s, \mu_s, \vartheta, \sigma) \tag{2.63}$$

and the new variables are defined by

$$\boldsymbol{R}_s = \boldsymbol{r} - \frac{\boldsymbol{b} \times \boldsymbol{w}}{\Omega_s}, \tag{2.64}$$

$$\varepsilon_s = \frac{1}{2} m_s v^2 + Z_s e \left[\Phi(\psi) + \varphi_0\right] - Z_s e \Phi\left(\psi_s^*\right), \tag{2.65}$$

$$\mu_s = \frac{m_s w_\perp^2}{2B}, \tag{2.66}$$

$$\sigma = \frac{w_\parallel}{|w_\parallel|}, \tag{2.67}$$

where $\boldsymbol{w}$ is the peculiar velocity of the particles with respect to the toroidal rotation:

$$\boldsymbol{w} = \boldsymbol{v} - \boldsymbol{u} = w_\parallel \boldsymbol{b} + w_\perp \left(\cos\vartheta\, \boldsymbol{e}_2 - \sin\vartheta\, \boldsymbol{e}_1\right), \tag{2.68}$$

with the toroidal velocity determined by (2.62), $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ arbitrary orthogonal unit vectors perpendicular to the magnetic field (with $\boldsymbol{b} = \boldsymbol{e}_2 \times \boldsymbol{e}_1$)[5], and ψ^* is a flux-like quantity proportional to the toroidal canonical angular momentum:

$$\psi_s^*(\boldsymbol{r}, \boldsymbol{v}) = \psi(\boldsymbol{r}) + \frac{m_s c}{Z_s e}\left(\boldsymbol{v} \cdot \nabla\phi\right) R^2 = \psi + \frac{m_s c}{Z_s e}\left(\boldsymbol{w} \cdot \nabla\phi\right) R^2 + \frac{B R^2 \omega(\psi)}{\Omega_s}. \tag{2.69}$$

The new velocity variables ε_s and μ_s are closely related to conserved quantities of the particle motion in the mean electromagnetic fields. The magnetic moment μ_s is the first adiabatic invariant of the particle gyromotion [57, 58, 59]. The energy variable, ε_s, is constructed from the conserved total energy and the conserved quantity ψ_s^* so that it is the energy in the rotating frame to lowest order. This can be seen by expanding $\Phi(\psi_s^*)$ around $\Phi(\psi)$, yielding to lowest order

$$\varepsilon_s = \frac{1}{2} m_s w^2 - \frac{1}{2} m_s \omega^2(\psi) R^2 + Z_s e \varphi_0 + \mathcal{O}(\epsilon T_s). \tag{2.70}$$

The kinetic equation (2.1) can be written in these new variables as

$$\frac{df_s}{dt} = \frac{\partial f_s}{\partial t} + \dot{\boldsymbol{R}}_s \cdot \frac{\partial f_s}{\partial \boldsymbol{R}_s} + \dot{\mu}_s \frac{\partial f_s}{\partial \mu_s} + \dot{\varepsilon}_s \frac{\partial f_s}{\partial \varepsilon_s} + \dot{\vartheta} \frac{\partial f_s}{\partial \vartheta} = C[f_s] + S_s, \tag{2.71}$$

where $\dot{g} = dg/dt$ denotes the time derivative of g along a particle orbit. The expressions for $\dot{\boldsymbol{R}}_s$, $\dot{\mu}_s$, $\dot{\varepsilon}_s$, and $\dot{\vartheta}$ are derived in Appendix A, see (A.9), (A.25), (A.33), and (A.39).

[5]There might be a concern that we cannot define such an $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$, this is resolved in [56] where it is proved that for toroidal confinement devices we can always make a globally-valid choice of these vectors

2.4.1 The Gyroaverage

Transformation from the spatial coordinate $\boldsymbol{r}$ to the guiding-centre position $\boldsymbol{R}_s$ (known as the Catto transformation [15]) allows us to introduce gyroaveraging — a crucial mathematical device that will enable us to close the equations for F_s and δf_s. We define the gyroaverage of a quantity g by

$$\langle g \rangle_{\boldsymbol{R}} = \frac{1}{2\pi} \oint d\vartheta g(\boldsymbol{R}_s, \varepsilon_s, \mu_s, \vartheta, \sigma), \tag{2.72}$$

where the integral is performed holding $\boldsymbol{R}_s$, ε_s, μ_s constant.

2.4.2 Derivatives and Integrals

All quantities that have velocity-space dependence will be functions of $\boldsymbol{R}_s$, ε_s, μ_s, ϑ and σ. The quantities that are only functions of space will be evaluated at the spatial position $\boldsymbol{r}$, unless explicitly stated otherwise, with $\boldsymbol{r}$ considered as a function of $\boldsymbol{R}_s, \mu_s, \vartheta$. In what follows, ∇ is reserved for derivatives with respect to $\boldsymbol{r}$, while derivatives with respect to $\boldsymbol{R}_s$ will be written explicitly as $\partial / \partial \boldsymbol{R}_s$.

As the variables we have chosen are adapted to the particle motion, we will not transform the field equations into these variables. The fields are functions of space but not velocity so they remain functions of $\boldsymbol{r}$ and t in accordance with the principle stated above. Therefore, the integrals over velocity in the definitions of density, (2.6), and current, (2.7), are to be taken at constant $\boldsymbol{r}$, so functions of $\boldsymbol{R}_s$, ε_s and μ_s appearing in the integrand are to be considered as functions of $\boldsymbol{r}$ and $\boldsymbol{v}$ via the definitions (2.64), (2.65) and (2.66). Namely, for any function g,

$$\int d^3\boldsymbol{w} g(\boldsymbol{R}_s, \varepsilon_s, \mu_s, \vartheta, \sigma) = \int d^3\boldsymbol{w} g(\boldsymbol{R}_s(\boldsymbol{r}, \boldsymbol{w}), \varepsilon_s(\boldsymbol{r}, \boldsymbol{w}), \mu_s(\boldsymbol{r}, \boldsymbol{w}), \vartheta(\boldsymbol{r}, \boldsymbol{w}), \sigma(\boldsymbol{r}, \boldsymbol{w})). \tag{2.73}$$

This is how all velocity-space integrals will be performed in this thesis unless explicitly stated otherwise. With this definition, integrals over $\boldsymbol{v}$ and $\boldsymbol{w}$ at constant $\boldsymbol{r}$ are equivalent.

Finally, as promised in §2.2.2, we now give a precise mathematical formulation of the assumption of axisymmetry of mean fields as applied to the mean distribution function:

$$(\nabla\phi) \cdot \left.\frac{\partial}{\partial \boldsymbol{R}_s}\right|_{\varepsilon_s, \mu_s, \vartheta} F_s = 0, \tag{2.74}$$

which, to lowest order in ϵ, is equivalent to

$$(\nabla\phi) \cdot \nabla|_{w_\parallel, w_\perp, \vartheta} F_s = 0. \tag{2.75}$$

However, this is not equivalent to $(\nabla\phi) \cdot \nabla|_{\boldsymbol{w}} F_s = 0$: indeed in (2.68), the basis vectors $\boldsymbol{b}$, $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ possess non-zero variation in the ϕ direction.

2.4.3 Gyrotropy of F_{0s}

To zeroth order, the kinetic equation (2.71) turns out to be just

$$\Omega_s \left.\frac{\partial F_{0s}}{\partial \vartheta}\right|_{\boldsymbol{R}_s,\mu_s,\varepsilon_s} = 0. \tag{2.76}$$

Thus, F_{0s} is gyrophase independent. Note that this means that, to lowest order,

$$\int d^3\boldsymbol{w}\boldsymbol{w}_\perp F_{0s} = 0, \tag{2.77}$$

confirming the earlier result that the lowest-order perpendicular velocity is completely contained in $\boldsymbol{u}_s$ as defined by (2.61).

2.5 First Order $\mathcal{O}(\epsilon\Omega_s f_s)$: The Maxwell-Boltzmann Equilibrium

In this section, we start from the kinetic equation (2.71), and expand it to first order. Using the lowest-order expressions for $\dot{\boldsymbol{R}}_s$, $\dot{\varepsilon}_s$ and $\dot{\vartheta}$ given by (A.10), (A.34), and (A.39) respectively we find

$$\left(w_\parallel \boldsymbol{b} + \boldsymbol{u}\right)\cdot\frac{\partial F_{0s}}{\partial \boldsymbol{R}_s} + \dot{\mu}_s\frac{\partial F_{0s}}{\partial \mu_s} - Z_s e\boldsymbol{w}_\perp\cdot\nabla\delta\varphi'\frac{\partial F_{0s}}{\partial \varepsilon_s} + \Omega_s\frac{\partial}{\partial\vartheta}\left(F_{1s}+\delta f_{1s}\right) = C\left[F_{0s}\right], \tag{2.78}$$

where $\delta\varphi'$ is the fluctuating electrostatic potential in the toroidally-rotating frame, given by

$$\delta\varphi' = \delta\varphi - \frac{1}{c}\boldsymbol{u}\cdot\delta\boldsymbol{A}, \tag{2.79}$$

and $\boldsymbol{u}$ without the species index is the toroidal-rotation component of $\boldsymbol{u}_s$ defined by (2.62).

In the following subsections, we analyse (2.78) to discover that: the lowest-order mean distribution function F_{0s} is Maxwellian; the bulk motion is the purely toroidal rotation of flux surfaces, i.e. the second term in (2.61) vanishes and $\boldsymbol{u}_s = \boldsymbol{u}$; and δf_s consists of the Boltzmann response to $\delta\varphi'$ and a gyrophase-independent distribution of charged rings $h_s(\boldsymbol{R}_s, \mu_s, \varepsilon_s, \sigma, t)$.

2.5.1 Maxwellian Equilibrium

We first prove that F_{0s} is Maxwellian. Gyroaverging (2.78) and using the fact that, to lowest order, $\langle\dot{\mu}_s\rangle_{\boldsymbol{R}} = 0$ (see (A.29)), $\langle\boldsymbol{w}_\perp\cdot\nabla\delta\varphi'\rangle_{\boldsymbol{R}} = 0$ (see (A.4)), and $\boldsymbol{u}\cdot\partial F_{0s}/\partial\boldsymbol{R}_s = 0$ (axisymmetry), we obtain

$$w_\parallel \boldsymbol{b}\cdot\frac{\partial F_{0s}}{\partial \boldsymbol{R}_s} = \langle C[F_{0s}]\rangle_{\boldsymbol{R}}. \tag{2.80}$$

Multiplying this by $1 + \ln F_{0s}$, integrating over all velocities and averaging over the flux surface, we find

$$\left\langle \int d^3\boldsymbol{w} w_\parallel \boldsymbol{b} \cdot \frac{\partial}{\partial \boldsymbol{R}_s} \left(F_{0s} \ln F_{0s} \right) \right\rangle_\psi = \left\langle \int d^3\boldsymbol{w} \ln F_{0s}\, C[F_{0s}] \right\rangle_\psi . \tag{2.81}$$

To lowest order, we can replace all instances of $\boldsymbol{R}_s$ on the left-hand side of this equation with $\boldsymbol{r}$ and write the velocity-space integral in terms of integrals over ε_s, μ_s, ϑ and σ at constant $\boldsymbol{r}$ using

$$\int d^3\boldsymbol{w} = \sum_\sigma \int \frac{B(\boldsymbol{r}) d\varepsilon_s d\mu_s d\vartheta}{m_s^2 \left| w_\parallel \right|} + \mathcal{O}(\epsilon v_{\mathrm{th}_s}^3). \tag{2.82}$$

This gives

$$\begin{aligned} \left\langle \int d^3\boldsymbol{w} w_\parallel \boldsymbol{b} \cdot \frac{\partial}{\partial \boldsymbol{R}_s} \left(F_{0s} \ln F_{0s} \right) \right\rangle_\psi &= \left\langle 2\pi \sum_\sigma \int \frac{B d\varepsilon_s d\mu_s}{m_s^2 |w_\parallel|} w_\parallel \boldsymbol{b} \cdot \nabla \left(F_{0s} \ln F_{0s} \right) \right\rangle_\psi \\ &= \left\langle 2\pi \sum_\sigma \sigma \int \frac{d\varepsilon_s d\mu_s}{m_s^2} \boldsymbol{B} \cdot \nabla \left(F_{0s} \ln F_{0s} \right) \right\rangle_\psi \\ &= \left\langle \nabla \cdot \left(\boldsymbol{b} \int d^3\boldsymbol{w} w_\parallel F_{0s} \ln F_{0s} \right) \right\rangle_\psi . \end{aligned} \tag{2.83}$$

By using (2.39) to express the action of the flux-surface average on a divergence and the fact that $\boldsymbol{b} \cdot \nabla \psi = 0$, we conclude that the above expression vanishes. Therefore, from (2.81),

$$\left\langle \int d^3\boldsymbol{w} \ln F_{0s}\, C[F_{0s}] \right\rangle_\psi = 0. \tag{2.84}$$

By Boltzmann's H-Theorem, (2.84) implies that F_{0s} is a local Maxwellian [60]. We know that, by definition, this Maxwellian has density n_s, temperature T_s and velocity $\boldsymbol{u}_s$, where $\boldsymbol{u}_s$ is given by (2.61).[6] Thus, F_{0s} can be written as

$$F_{0s} = n_s(\boldsymbol{r}) \left[\frac{m_s}{2\pi T_s(\boldsymbol{r})} \right]^{3/2} \exp \left\{ - \frac{m_s \left[w^2 - 2 m_s w_\parallel \hat{u}_{\parallel s}(\boldsymbol{r}) + \hat{u}_{\parallel s}^2(\boldsymbol{r}) \right]}{2 T_s(\boldsymbol{r})} \right\}, \tag{2.85}$$

where

$$\hat{u}_{\parallel s} = u_{\parallel s} - \frac{\omega(\psi) I}{B}. \tag{2.86}$$

This form does not contradict the condition (2.76) that F_{0s} must be gyrophase-independent whilst holding $\boldsymbol{R}_s$, ε_s and μ_s fixed because the difference between

[6] Formally, (2.84) implies that all species have the same temperature and mean velocity. We will show that they do indeed have the same mean velocity, but we will gratuitously retain the species dependence of the temperature. This will only be of importance when we come to discuss heat transport and so we defer the discussion of interspecies temperature differences to the end of §2.7.3.

$\boldsymbol{R}_s$ and $\boldsymbol{r}$ is higher order. However, we still wish to have F_{0s} expressed in the $(\boldsymbol{R}_s, \varepsilon_s, \mu_s, \sigma)$ variables. We accomplish this by using (2.70) to express $m_s w^2$ in terms of ε_s and find

$$F_{0s} = N_s(\boldsymbol{R}_s)\left[\frac{m_s}{2\pi T_s(\boldsymbol{R}_s)}\right]^{3/2} \exp\left[-\frac{\varepsilon_s}{T_s(\boldsymbol{R}_s)} - \frac{m_s w_\parallel \hat{u}_{\parallel s}(\boldsymbol{R}_s)}{T_s(\boldsymbol{R}_s)}\right] + \mathcal{O}(\epsilon F_s), \tag{2.87}$$

where

$$N_s = n_s \exp\left[-\frac{m_s\omega^2(\psi)R^2}{2T_s} + \frac{Z_s e\varphi_0}{T_s} + \frac{m_s\hat{u}_{\parallel s}^2}{2T_s}\right], \tag{2.88}$$

and we have used the fact that to lowest order the mean fields taken at the guiding-centre position $\boldsymbol{R}_s$ are the same as when taken at the particle position $\boldsymbol{r}$.

Inserting F_{0s}, given by (2.85), back into (2.80) and dividing through by F_{0s} we get

$$w_\parallel \boldsymbol{b}\cdot\nabla\left(\ln N_s - \frac{3}{2}\ln T_s\right) + \frac{w_\parallel \varepsilon_s}{T_s^2}\boldsymbol{b}\cdot\nabla T_s - w_\parallel \boldsymbol{b}\cdot\nabla\left(\frac{m_s w_\parallel \hat{u}_{\parallel s}}{T_s}\right) = 0. \tag{2.89}$$

This equation must hold for all velocities $\boldsymbol{w}$, so each term in this equation must vanish independently. Tackling the second term first, we see that the temperature T_s must be a flux function to lowest order:

$$\boldsymbol{b}\cdot\nabla T_s = 0, \tag{2.90}$$

so $T_s = T_s(\psi)$. For the first term to vanish the same must be the case for N_s:

$$\boldsymbol{b}\cdot\nabla N_s = 0, \tag{2.91}$$

so $N_s = N_s(\psi)$. Turning now to the third term of (2.89), we see that

$$2w_\parallel^2 \boldsymbol{b}\cdot\nabla\hat{u}_{\parallel s} + \hat{u}_{\parallel s}\boldsymbol{b}\cdot\nabla w_\parallel^2 = 0, \tag{2.92}$$

which can only be solved for arbitrary $w_\parallel$ by $\hat{u}_{\parallel s} = 0$. Thus, the background Maxwellian only depends on ε_s and so is isotropic in $\boldsymbol{w}$ and the-lowest order (sonic) flow is a pure toroidal rotation: $\boldsymbol{u}_s = \boldsymbol{u} = \omega(\psi)R^2\nabla\phi$. As this flow is species-independent there are no currents in F_{0s}, since the plasma is quasineutral (2.52). This is consistent with the lowest-order Ampère's Law (2.53). The vanishing of the parallel component of (2.61) is in accord with the well known result [52, 60, 61] that poloidal flow is strongly damped on the ion-ion collision time, which, in our ordering is, indeed, $1/\epsilon^2$ shorter than the timescale of the evolution of the bulk flow.

We can finally write the complete solution for F_{0s} as

$$F_{0s} = N_s(\psi(\boldsymbol{R}_s))\left[\frac{m_s}{2\pi T_s(\psi(\boldsymbol{R}_s))}\right]^{3/2} e^{-\varepsilon_s/T_s(\psi(\boldsymbol{R}_s))}. \tag{2.93}$$

This form of the distribution function is manifestly gyrophase independent holding $\boldsymbol{R}_s$ and ε_s constant and so is consistent with (2.76).

2.5.2 Gyrotropy of F_{1s}

Substituting (2.93) back into the first-order kinetic equation (2.78), we find

$$\Omega_s \frac{\partial}{\partial \vartheta}\left(F_{1s} + \delta f_{1s}\right) = -\frac{Z_s e}{T_s} \boldsymbol{w}_\perp \cdot \nabla \delta\varphi' F_{0s}. \tag{2.94}$$

Averaging this over the fluctuations gives

$$\Omega_s \left.\frac{\partial F_{1s}}{\partial \vartheta}\right|_{\boldsymbol{R}_s, \mu_s, \varepsilon_s} = 0. \tag{2.95}$$

Thus, F_{1s} is gyrophase independent.

2.5.3 Poloidal Density Variation

Rearranging (2.88) and using $\hat{u}_{\parallel s} = 0$, we see that the physical density of a species is given by

$$n_s = N_s\left(\psi\right) \exp\left[\frac{m_s \omega^2(\psi) R^2}{2T_s} - \frac{Z_s e \varphi_0}{T_s}\right] + \mathcal{O}(\epsilon n_s), \tag{2.96}$$

which is not a flux function, unless the rotation is subsonic (see §2.10) [52, 60]. We can insert this into the lowest-order quasineutrality condition (2.52) to determine φ_0, subject to the constraint that $\langle\varphi_0\rangle_\psi = 0$ (by definition, see (2.59)):

$$\sum_s Z_s N_s(\psi) \exp\left[\frac{m_s \omega^2(\psi) R^2}{2T_s} - \frac{Z_s e \varphi_0}{T_s}\right] = 0. \tag{2.97}$$

Physically, this demonstrates that the poloidal variation of the density on a flux surface is governed by the balance between centrifugal forces which sweep heavy particles (i.e. ions) to the outboard side (higher R) and the electrostatic potential that is set up within the flux surface to maintain quasineutrality.

2.5.4 The Boltzmann Response

Taking the lowest-order fluctuating components of (2.12) and (2.13), we see that the fluctuating electric field in the toroidally rotating frame is electrostatic,

$$\delta \boldsymbol{E} + \frac{1}{c}\boldsymbol{u} \times \delta \boldsymbol{B} = -\nabla_\perp \delta\varphi' + \mathcal{O}(\epsilon^2 \boldsymbol{E}), \tag{2.98}$$

where $\delta\varphi'$ is defined in (2.79). Using (2.95), the first order kinetic equation (2.94) is

$$\Omega_s \frac{\partial \delta f_{1s}}{\partial \vartheta} = -\frac{Z_s e}{T_s} \boldsymbol{w}_\perp \cdot \nabla \delta\varphi' F_{0s}. \tag{2.99}$$

Using (A.3) for $\delta\varphi'$, we can integrate this with respect to ϑ:

$$\boldsymbol{w}_\perp \cdot \nabla \delta\varphi' = \Omega_s \left.\frac{\partial \delta\varphi'}{\partial \vartheta}\right|_{\boldsymbol{R}_s, \varepsilon_s, \mu_s}, \tag{2.100}$$

and obtain

$$\delta f_{1s} = -\frac{Z_s e \delta\varphi'(\boldsymbol{r})}{T_s} F_{0s} + h_s\left(\boldsymbol{R}_s, \mu_s, \varepsilon_s, \sigma, t\right), \tag{2.101}$$

where the constant of gyrophase integration h_s is the gyrophase-independent distribution of Larmor rings, which will completely describe the kinetics of the small scale turbulence. The gyrophase-dependent part of δf_{1s} is just the Boltzmann response in the moving frame to the electrostatic field (2.98).

2.5.5 Summary: The Lowest-Order Solution

Collating the results of the previous subsections we have the solution for f_s to first order in ϵ:

$$\begin{aligned} f_s &= F_s + \delta f_s, && (2.102)\\ F_s &= F_{0s}\left(\psi(\boldsymbol{R}_s), \varepsilon_s\right) + F_{1s}\left(\boldsymbol{R}_s, \varepsilon_s, \mu_s, \sigma\right) + \mathcal{O}\left(\epsilon^2 f\right), && (2.103)\\ \delta f_s &= -\frac{Z_s e}{T_s}\delta\varphi'(\boldsymbol{r}) F_{0s} + h_s\left(\boldsymbol{R}_s, \varepsilon_s, \mu_s, \sigma\right) + \mathcal{O}\left(\epsilon^2 f\right), && (2.104)\\ F_{0s} &= N_s(\psi(\boldsymbol{R}_s))\left[\frac{m_s}{2\pi T_s(\psi(\boldsymbol{R}_s))}\right]^{3/2} e^{-\varepsilon_s/T_s(\psi(\boldsymbol{R}_s))}, && (2.105) \end{aligned}$$

In the next section we find equations that govern the gyrophase-independent functions F_{1s} and h_s.

2.6 Second Order $\mathcal{O}\left(\epsilon^2\Omega_s f_s\right)$: Neoclassical Theory and Gyrokinetics

In order to completely determine the distribution function f_s given by (2.102), we need equations for F_{1s} and h_s. In order to find them, let us substitute the form of f_s summarized in §2.5.5 into the exact kinetic equation (2.71) and keep only terms up to second order in ϵ:

$$\begin{aligned} &\frac{\partial h_s}{\partial t} + \left(\dot{\boldsymbol{R}}_s \cdot \frac{\partial}{\partial \boldsymbol{R}_s} + \dot{\mu}_s \frac{\partial}{\partial \mu_s} + \dot{\varepsilon}_s \frac{\partial}{\partial \varepsilon_s}\right)\left(F_{0s} + F_{1s} + h_s\right) = \\ &\quad -\Omega_s \frac{\partial}{\partial \vartheta}\left(F_{2s} + \delta f_{2s}\right) + \frac{d}{dt}\left(\frac{Z_s e \delta\varphi'}{T_s} F_{0s}\right) + C[F_{0s} + F_{1s} + h_s], \end{aligned} \tag{2.106}$$

where d/dt is the full derivative along a particle orbit as in (2.71) and we have used the gyrophase independence of F_{0s} (2.76), F_{1s} (2.95) and h_s (2.101). Equation (2.106) contains F_{2s} and δf_{2s}, about which we have no information. It is thus crucial that F_{0s}, F_{1s} and h_s are gyrophase-independent, as this allows us to gyroaverage (2.106) to find an equation that does not contain any second-order distribution functions:

$$\begin{aligned}\frac{\partial h_s}{\partial t} + \left\langle \dot{\boldsymbol{R}}_s \right\rangle_{\boldsymbol{R}} \cdot \frac{\partial}{\partial \boldsymbol{R}_s} (F_{0s} + F_{1s} + h_s) = \\ \langle \dot{\varepsilon}_s \rangle_{\boldsymbol{R}} \frac{F_{0s}}{T_s} + \left\langle \frac{d}{dt} \left(\frac{Z_s e \delta\varphi'}{T_s} F_{0s} \right) \right\rangle_{\boldsymbol{R}} + \langle C[F_{0s} + F_{1s} + h_s] \rangle_{\boldsymbol{R}},\end{aligned} \tag{2.107}$$

where we have used F_{0s} given by (2.105) and the fact, derived in Appendix A, that $\langle \dot{\mu}_s \rangle_{\boldsymbol{R}} = \mathcal{O}(\epsilon^2 \Omega_s \mu_s)$ (see (A.29)) and $\langle \dot{\varepsilon}_s \rangle_{\boldsymbol{R}} = \mathcal{O}(\epsilon^2 \Omega_s T_s)$ (see (A.35)).

The gyroaverages appearing in (2.107) are calculated in Appendix A: we use (A.23) for $\left\langle \dot{\boldsymbol{R}}_s \right\rangle_{\boldsymbol{R}}$ and (A.55) for the combination of gyroaverages appearing on the right-hand side:

$$\begin{aligned}&\left[\frac{\partial}{\partial t} + \boldsymbol{u}(\boldsymbol{R}_s) \cdot \frac{\partial}{\partial \boldsymbol{R}_s} \right] h_s + w_\parallel \boldsymbol{b} \cdot \frac{\partial}{\partial \boldsymbol{R}_s} (F_{1s} + h_s) + \left(\boldsymbol{V}_{\mathrm{D}s} + \langle \boldsymbol{V}_\chi \rangle_{\boldsymbol{R}} \right) \cdot \frac{\partial}{\partial \boldsymbol{R}_s} (F_{0s} + h_s) \\ &= \frac{Z_s e F_{0s}}{T_s} \left[\frac{\partial}{\partial t} + \boldsymbol{u}(\boldsymbol{R}_s) \cdot \frac{\partial}{\partial \boldsymbol{R}_s} \right] \langle \chi \rangle_{\boldsymbol{R}} - \frac{m_s F_{0s}}{T_s} \left[\frac{I w_\parallel}{B} + \omega(\psi) R^2 \right] \frac{d\omega}{d\psi} \langle \boldsymbol{V}_\chi \rangle_{\boldsymbol{R}} \cdot \nabla\psi \\ &\quad - \frac{Z_s e}{T_s c} w_\parallel F_{0s} \frac{\partial \boldsymbol{A}}{\partial t} \cdot \boldsymbol{b} + \langle C[F_{0s} + F_{1s} + h_s] \rangle_{\boldsymbol{R}},\end{aligned} \tag{2.108}$$

where we have defined the gyrokinetic potential

$$\chi = \delta\varphi - \frac{1}{c} \boldsymbol{v} \cdot \delta\boldsymbol{A} = \delta\varphi' - \frac{1}{c} \boldsymbol{w} \cdot \delta\boldsymbol{A}, \tag{2.109}$$

the associated fluctuating velocity field[7]

$$\boldsymbol{V}_\chi = \frac{c}{B} \boldsymbol{b} \times \nabla\chi, \qquad \langle \boldsymbol{V}_\chi \rangle_{\boldsymbol{R}} = \frac{c}{B} \boldsymbol{b} \times \frac{\partial \langle \chi \rangle_{\boldsymbol{R}}}{\partial \boldsymbol{R}_s} + \mathcal{O}(\epsilon^2 v_{\mathrm{th}_s}), \tag{2.110}$$

and the guiding-centre drift velocity

$$\boldsymbol{V}_{\mathrm{D}s} = \frac{\boldsymbol{b}}{\Omega_s} \times \left[w_\parallel^2 \boldsymbol{b} \cdot \nabla \boldsymbol{b} + \frac{1}{2} w_\perp^2 \nabla \ln B \; - \omega^2(\psi) R \nabla R - 2 w_\parallel \omega(\psi) \boldsymbol{b} \times \nabla z + \frac{Z_s e}{m_s} \nabla \varphi_0 \right], \tag{2.111}$$

which consists of, in order of appearance in (2.111), the curvature drift, the ∇B drift, the centrifugal drift, the Coriolis drift, and the mean first-order $\boldsymbol{E} \times \boldsymbol{B}$ drift.

The detailed derivation of (2.108) is given in Appendix A.6. In the rest of this section, we will use (2.108) to derive a closed solution for F_{1s} and h_s, accurate to first order in ϵ, in terms of the unknown functions $N_s(\psi)$, T_s, $\omega(\psi)$, I and ψ, which parametrise the equilibrium. Equation (2.108) can be split into a mean equation (see

[7] This can be shown to consist, physically, of the fluctuating $\boldsymbol{E} \times \boldsymbol{B}$ drift in the rotating frame, the motion of guiding centres along fluctuating field lines and the fluctuating ∇B drift. This is proved in detail for gyrokinetics in a non-rotating slab in [62].

(2.112)), which will determine F_{1s}, and a fluctuating part (see (2.144)), which will determine h_s. We will find that the mean equation is precisely the neoclassical drift-kinetic equation [54] and the fluctuating equation is the gyrokinetic equation for a rotating plasma [7, 63]. These two equations are closed by the mean and fluctuating Maxwell equations, which relate ψ and I to F_{1s} (Sections 2.6.2 and 2.6.3) and the fluctuating fields to h_s (§2.6.5).

2.6.1 Neoclassical Distribution Function

Averaging (2.108) over the fluctuations, we obtain

$$w_\parallel \boldsymbol{b}\cdot\frac{\partial F_{1s}}{\partial \boldsymbol{R}_s} - \langle C[F_{1s}]\rangle_{\boldsymbol{R}} = -\boldsymbol{V}_{\mathrm{D}s}\cdot\frac{\partial F_{0s}}{\partial \boldsymbol{R}_s} - \frac{Z_s e}{T_s c} w_\parallel F_{0s}\frac{\partial \boldsymbol{A}}{\partial t}\cdot\boldsymbol{b} + \langle C[F_{0s}]\rangle_{\boldsymbol{R}}. \tag{2.112}$$

This is just the usual neoclassical drift-kinetic equation [54, 60, 52]. The collisional term involving F_{1s} can be split from that on F_{0s} because the collision operator is linearised around the Maxwellian part of F_{0s}. Note that $C[F_{0s}]$ is only zero to lowest order, so in (2.112) it represents collisions on the first-order departures of F_{0s} from a pure Maxwellian (having to do with the absorption into F_{0s} of some non-Maxwellian velocity dependence via $\psi(\boldsymbol{R}_s)$ and ε_s; see §2.5.1 and Appendix D).

We now wish to solve (2.112) by first separating particular solutions corresponding to some of the source terms in its right-hand side. To deal with the first of these terms, we let

$$F_{1s} = \widehat{F}_{1s}(\boldsymbol{R}_s, \varepsilon_s, \mu_s, \sigma) + F_{1s}^*, \tag{2.113}$$

$$F_{1s}^* = \frac{m_s c}{Z_s e}\left[\frac{I w_\parallel}{B} + \omega(\psi) R^2\right]\frac{\partial F_{0s}}{\partial \psi}, \tag{2.114}$$

where

$$\begin{aligned} w_\parallel \boldsymbol{b}\cdot\frac{\partial F_{1s}^*}{\partial \boldsymbol{R}_s} &= w_\parallel \boldsymbol{b}\cdot\frac{\partial}{\partial \boldsymbol{R}_s}\left[\frac{I w_\parallel + \omega(\psi)R^2 B}{\Omega_s}\right]\frac{\partial F_{0s}}{\partial \psi} \\ &= -\left(\boldsymbol{V}_{\mathrm{D}s}\cdot\nabla\psi\right)\frac{\partial F_{0s}}{\partial \psi} = -\boldsymbol{V}_{\mathrm{D}s}\cdot\frac{\partial F_{0s}}{\partial \boldsymbol{R}_s}, \end{aligned} \tag{2.115}$$

where we have used the well-known identity [52]

$$\boldsymbol{V}_{\mathrm{D}s}\cdot\nabla\psi = -w_\parallel \boldsymbol{b}\cdot\nabla|_{\varepsilon_s,\mu_s,\vartheta}\left[\frac{I w_\parallel}{\Omega_s} + \frac{BR^2}{\Omega_s}\omega(\psi)\right]. \tag{2.116}$$

Therefore,

$$w_\parallel \boldsymbol{b}\cdot\frac{\partial \widehat{F}_{1s}}{\partial \boldsymbol{R}_s} - \left\langle C\left[\widehat{F}_{1s}\right]\right\rangle_{\boldsymbol{R}} = -\frac{Z_s e}{T_s c} w_\parallel F_{0s}\frac{\partial \boldsymbol{A}}{\partial t}\cdot\boldsymbol{b} + \langle C[F_{0s} + F_{1s}^*]\rangle_{\boldsymbol{R}}. \tag{2.117}$$

We now wish to replace the term depending on $\boldsymbol{A}$ with a source term that depends on $\langle \boldsymbol{E}\cdot\boldsymbol{B}\rangle_\psi$, which will later turn out to be opportune (see Sections 2.6.2 and 2.6.3). To do this, we follow [10, 11, 64] and introduce the ansatz

$$\widehat{F}_{1s} = F_s^{(\mathrm{nc})}(\boldsymbol{R}_s, \varepsilon_s, \mu_s, \sigma) - \frac{Z_s e}{T_s} F_{0s} \int^l dl' \left(B \frac{\langle \boldsymbol{E}\cdot\boldsymbol{B}\rangle_\psi}{\langle B^2\rangle_\psi} + \frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t}\cdot \boldsymbol{b} \right), \tag{2.118}$$

where the integral is performed along a fixed field line, and l' is the distance along that field line. Note that the second term in (2.118) carries no current. Inserting (2.118) into (2.117), we find

$$w_\parallel \boldsymbol{b}\cdot \frac{\partial F_s^{(\mathrm{nc})}}{\partial \boldsymbol{R}_s} - \left\langle C\left[F_s^{(\mathrm{nc})}\right]\right\rangle_{\boldsymbol{R}} = 2\Omega_s \frac{w_\parallel c}{v_{\mathrm{th}_s}^2} \frac{\langle \boldsymbol{E}\cdot\boldsymbol{B}\rangle_\psi}{\langle B^2\rangle_\psi} F_{0s} + \langle C[F_{0s} + F_{1s}^*]\rangle_{\boldsymbol{R}}. \tag{2.119}$$

Finally, we split

$$F_s^{(\mathrm{nc})} = F_s^{(\mathrm{E})} \frac{c}{v_{\mathrm{th}_s}} \frac{\langle \boldsymbol{E}\cdot\boldsymbol{B}\rangle_\psi}{\langle B^2\rangle_\psi} + \tilde{F}_s^{(\mathrm{nc})}, \tag{2.120}$$

where $F_s^{(\mathrm{E})}$ and $\tilde{F}_s^{(\mathrm{nc})}$ satisfy

$$w_\parallel \boldsymbol{b}\cdot \frac{\partial F_s^{(\mathrm{E})}}{\partial \boldsymbol{R}_s} - C\left[F_s^{(\mathrm{E})}\right] = 2\Omega_s \frac{w_\parallel}{v_{\mathrm{th}_s}} F_{0s}, \tag{2.121}$$

$$w_\parallel \boldsymbol{b}\cdot \frac{\partial \tilde{F}_s^{(\mathrm{nc})}}{\partial \boldsymbol{R}_s} - C\left[\tilde{F}_s^{(\mathrm{nc})}\right] = \langle C[F_{0s} + F_{1s}^*]\rangle_{\boldsymbol{R}}. \tag{2.122}$$

This pair of equations can now be solved without knowing $\langle \boldsymbol{E}\cdot\boldsymbol{B}\rangle_\psi$ and then $F_s^{(\mathrm{nc})}$ is given as a function of $\langle \boldsymbol{E}\cdot\boldsymbol{B}\rangle_\psi$ by (2.120). This will be used in solving for the evolution of the magnetic field in §2.6.2 and §2.6.3.

2.6.2 Ampère's Law and the Magnetic Equilibrium

The average of Ampère's Law (2.11) over the fluctuations is

$$\boldsymbol{j} = \frac{c}{4\pi}\nabla\times\boldsymbol{B}, \qquad \boldsymbol{j} = \sum_s Z_s e \int d^3\boldsymbol{w}\,\boldsymbol{w} F_s, \tag{2.123}$$

where $\boldsymbol{j}$ is first-order (to lowest order, $\boldsymbol{j}=0$; see (2.53)). Using the axisymmetric form for $\boldsymbol{B}$ (2.31), we can express $\nabla\times\boldsymbol{B}$ in terms of derivatives of I and ψ:

$$\boldsymbol{j} = \frac{c}{4\pi}\left[\nabla I \times \nabla\phi - (\Delta^*\psi)\nabla\phi\right], \tag{2.124}$$

where Δ^* is the Grad-Shafranov operator

$$\Delta^*\psi = \left(\frac{\partial^2}{\partial R^2} - \frac{1}{R}\frac{\partial}{\partial R} + \frac{\partial^2}{\partial z^2}\right)\psi. \tag{2.125}$$

We can now use the complete solution for F_s given by (2.103) to determine $\boldsymbol{j}$. This is done in Appendix B.1 (see (B.17)):

$$\begin{aligned}\boldsymbol{j} = cR^2 \sum_s n_s \left\{ T_s \frac{d \ln N_s}{d\psi} + \left[Z_s e \varphi_0 - \frac{1}{2} m_s \omega^2(\psi) R^2 + T_s \right] \frac{d \ln T_s}{d\psi} \right\} \nabla\phi \\ + cR^2 \left(\sum_s m_s n_s R^2 \right) \omega(\psi) \frac{d\omega}{d\psi} \nabla\phi + K(\psi) \boldsymbol{B},\end{aligned} \tag{2.126}$$

and we have defined

$$K(\psi) = \sum_s \frac{Z_s e}{B} \int d^3\boldsymbol{w} w_\parallel F_s^{(\mathrm{nc})}, \tag{2.127}$$

with $F_s^{(\mathrm{nc})}$ given by the neoclassical drift-kinetic equation (2.119). The fact that K is a flux function follows directly from (2.119), and is derived in Appendix B.1 (see (B.15)). Equation (2.126) is consistent with this, as can be seen by taking the divergence of both sides. Note that the perpendicular part of (2.126) is a statement of force balance for the mean quantities. Indeed, taking the cross product of it with $\boldsymbol{B}$ and using $R^2 \nabla\phi \times \boldsymbol{B} = \nabla\psi$, we get

$$\begin{aligned}\frac{1}{c} \boldsymbol{j} \times \boldsymbol{B} &= \sum_s n_s \left\{ T_s \frac{d \ln N_s}{d\psi} + \left[Z_s e \varphi_0 - \frac{1}{2} m_s \omega^2(\psi) R^2 + T_s \right] \frac{d \ln T_s}{d\psi} \right\} \nabla\psi \\ &\quad + \left(\sum_s m_s n_s R^2 \right) \omega(\psi) \frac{d\omega}{d\psi} \nabla\psi \\ &= \sum_s \left(\nabla p_s + m_s n_s \boldsymbol{u} \cdot \nabla \boldsymbol{u} \right),\end{aligned} \tag{2.128}$$

where we have defined the pressure $p_s = n_s T_s$ and used (2.96) for $N_s(\psi)$ and (2.62) for $\boldsymbol{u}$. This is just the balance between the Lorentz force, pressure and inertia.

We will now take three projections of (2.124): onto $\nabla\psi$, onto $\nabla\psi \times \nabla\phi$, and onto $\nabla\phi$. Taking the projection onto $\nabla\psi$ first, we have, from (2.124),

$$\boldsymbol{j} \cdot \nabla\psi = \frac{c}{4\pi} \left(\nabla I \times \nabla\phi \right) \cdot \nabla\psi \tag{2.129}$$

and from (2.126)

$$\boldsymbol{j} \cdot \nabla\psi = 0, \tag{2.130}$$

whence

$$\left(\nabla\psi \times \nabla\phi \right) \cdot \nabla I = \boldsymbol{B} \cdot \nabla I = 0. \tag{2.131}$$

Therefore, $I = I(\psi)$ is a flux function.

Taking the projection onto $\nabla\psi \times \nabla\phi$, we have from (2.124)

$$\boldsymbol{j} \cdot \left(\nabla\psi \times \nabla\phi \right) = \frac{c}{4\pi R^2} \frac{dI}{d\psi} |\nabla\psi|^2 \tag{2.132}$$

and from (2.126)

$$\boldsymbol{j}\cdot(\nabla\psi\times\nabla\phi) = K(\psi)\boldsymbol{B}\cdot(\nabla\psi\times\nabla\phi) = K(\psi)|\nabla\psi|^2R^{-2}. \tag{2.133}$$

This gives us the following equation for $I(\psi)$

$$\frac{dI}{d\psi} = \frac{4\pi}{c}K(\psi). \tag{2.134}$$

Finally, the toroidal component of (2.124) is just

$$\boldsymbol{j}\cdot\nabla\phi = -\frac{c}{4\pi R^2}\Delta^*\psi. \tag{2.135}$$

Using the toroidal current from (2.126) and expressing $K(\psi)$ via (2.134), we obtain

$$\begin{aligned}\Delta^*\psi = &-4\pi R^2\sum_s n_s\left\{T_s\frac{d\ln N_s}{d\psi} + \left[Z_se\varphi_0 - \frac{1}{2}m_s\omega^2(\psi)R^2 + T_s\right]\frac{d\ln T_s}{d\psi}\right\}\\ &-4\pi R^2\left(\sum_s m_sn_sR^2\right)\omega(\psi)\frac{d\omega}{d\psi} - I(\psi)\frac{dI}{d\psi},\end{aligned} \tag{2.136}$$

which allows us to determine ψ as a function of R and z if we know N_s, T_s, ω and I. This is the generalization of the Grad-Shafranov equation for a rotating plasma.

2.6.3 Evolution of the Magnetic Field

The results of the previous section complete the solution, including time dependence, for the magnetic field. We can solve (2.119) for $F_s^{(\mathrm{nc})}$ to find $K(\psi)$ in terms of $\langle\boldsymbol{E}\cdot\boldsymbol{B}\rangle_\psi$, and then use (2.134) to find $\langle\boldsymbol{E}\cdot\boldsymbol{B}\rangle_\psi$ in terms of $I(\psi)$. We can then use (2.33) to find $I(\psi)$ in terms of $\boldsymbol{A}$ and so our expression for

$$\langle\boldsymbol{E}\cdot\boldsymbol{B}\rangle_\psi = -\frac{1}{c}\left\langle\frac{\partial\boldsymbol{A}}{\partial t}\cdot\boldsymbol{B}\right\rangle_\psi \tag{2.137}$$

is an evolution equation for $\boldsymbol{A}$ (and thus $\boldsymbol{B}$). Indeed, taking the time derivative of (2.33) divided by R^2, we find

$$\frac{\partial}{\partial t}\frac{I(\psi)}{R^2} = \frac{\partial\boldsymbol{B}}{\partial t}\cdot\nabla\phi = -c\nabla\cdot(\boldsymbol{E}\times\nabla\phi)\,. \tag{2.138}$$

Flux-surface averaging and using (2.39) and (2.45) we have

$$\left.\frac{\partial}{\partial t}\right|_\psi V'I(\psi)\left\langle R^{-2}\right\rangle_\psi = c\frac{\partial}{\partial\psi}V'\left\langle\boldsymbol{E}\cdot\boldsymbol{B}\right\rangle_\psi, \tag{2.139}$$

where we have used (2.31) to rewrite the right-hand side in terms of $\boldsymbol{B}$.

In the study of tokamak plasmas, it is conventional to work with the safety factor $q(\psi)$ rather than $I(\psi)$. The safety factor is defined in terms of the toroidal flux Ψ and the poloidal flux ψ by

$$q(\psi) = \frac{1}{2\pi}\frac{d\Psi}{d\psi}, \qquad \Psi = \frac{1}{2\pi}\int_{D(\psi,t)} d^3\boldsymbol{r}\boldsymbol{B}\cdot\nabla\phi, \tag{2.140}$$

where the integration is carried out over the volume interior to the flux surface labelled by ψ (see §2.2.3). Geometrically, $q(\psi)$ is the number of times a field line on a given flux surface toroidally winds round the vertical symmetry axis for each poloidal transit around the magnetic axis. From (2.140), the definition of the flux-surface average (2.37), and the axisymmetric form of the magnetic field (2.31), we can express Ψ as

$$\Psi = \frac{1}{2\pi}\int_0^{\psi} d\psi' \int_{\partial D(\psi,t)} \frac{dS}{|\nabla\psi|} I(\psi')R^{-2} = \frac{1}{2\pi}\int_0^{\psi} d\psi' V'(\psi') I(\psi') \left\langle R^{-2}\right\rangle_\psi(\psi'), \tag{2.141}$$

where V' is defined by (2.36). Therefore,

$$q(\psi) = \frac{1}{4\pi^2} V' I(\psi)\left\langle R^{-2}\right\rangle_\psi . \tag{2.142}$$

From (2.139) and (2.142) we immediately find

$$\left.\frac{\partial}{\partial t}\right|_\psi q = \frac{c}{4\pi^2}\frac{\partial}{\partial\psi} V' \left\langle \boldsymbol{E}\cdot\boldsymbol{B}\right\rangle_\psi . \tag{2.143}$$

Thus, $q(\psi)$ evolves via (2.143), $I(\psi)$ is determined by (2.142), and $\psi(R, z)$ is then instantaneously determined from the generalized Grad-Shafranov equation (2.136). This completes our solution for the evolution of the magnetic field.

It turns out that $q(\psi)$ can be shown to evolve on a timescale that is even slower than the transport timescale, the resistive timescale, i.e., $c\left\langle \boldsymbol{E}\cdot\boldsymbol{B}\right\rangle_\psi$ turns out to be of the order of $v_{\mathrm{th}_i}(m_e/m_i)^{1/2}\epsilon^2 B^2$. This result is proved in Appendix C of [65], where we also relate the evolution of $q(\psi)$ to the conservation of magnetic flux. Indeed, the geometric interpretation of $q(\psi)$ is consistent with the notion that this longer timescale is the timescale on which the topology of the magnetic field changes, even if other properties of the field may change more rapidly.

2.6.4 The Gyrokinetic Equation

Turning to the fluctuating component of the second-order kinetic equation (2.108), we find[8]

[8] This is in agreement with the nonlinear gyrokinetic equation of [7], but not with that of [55]. The discrepancy arises from a subtlety in the gyroaveraging of $\boldsymbol{u}\cdot\nabla\chi$ that is explained in detail in Appendix A.6 (see (A.54) and Appendix A.6.2.)

$$\begin{aligned}\left[\frac{\partial}{\partial t}+\boldsymbol{u}(\boldsymbol{R}_s)\cdot\frac{\partial}{\partial\boldsymbol{R}_s}\right]h_s+\left(w_\parallel\boldsymbol{b}+\boldsymbol{V}_{\mathrm{D}s}+\langle\boldsymbol{V}_\chi\rangle_{\boldsymbol{R}}\right)\cdot\frac{\partial h_s}{\partial\boldsymbol{R}_s}-\langle C[h_s]\rangle_{\boldsymbol{R}}\\ =\frac{Z_seF_{0s}}{T_s}\left[\frac{\partial}{\partial t}+\boldsymbol{u}(\boldsymbol{R}_s)\cdot\frac{\partial}{\partial\boldsymbol{R}_s}\right]\langle\chi\rangle_{\boldsymbol{R}}\\ -\left\{\frac{\partial F_{0s}}{\partial\psi}+\frac{m_sF_{0s}}{T_s}\left[\frac{I(\psi)w_\parallel}{B}+\omega(\psi)R^2\right]\frac{d\omega}{d\psi}\right\}\langle\boldsymbol{V}_\chi\rangle_{\boldsymbol{R}}\cdot\nabla\psi\end{aligned} \tag{2.144}$$

with $\boldsymbol{V}_\chi$ and $\boldsymbol{V}_{\mathrm{D}s}$ given by (2.110) and (2.111), respectively. Explicit forms for the collision term $\langle C[h_s]\rangle_{\boldsymbol{R}}$ are proposed in [66, 1, 40] for various model collision operators, however, only the properties common to all such operators will be needed in this work so we leave this term in its general form. Finally, we reiterate that this equation is written in terms of the variables $\boldsymbol{R}_s$, ε_s, μ_s and ϑ; thus $w_\parallel$ is given by

$$w_\parallel=\sqrt{\frac{2}{m_s}\left[\varepsilon_s-\mu_sB-Z_se\varphi_0+\frac{1}{2}m_s\omega^2(\psi)R^2\right]}+\mathcal{O}(\epsilon v_{\mathrm{th}_s}). \tag{2.145}$$

It is the gyrokinetic equation (2.144) which is solved by the gyrokinetic turbulence simulation codes [67, 68, 69, 70] and has formed the basis of a large body of theoretical analysis of tokamak microinstabilites and turbulence.[9] The turbulence is driven by the density and temperature gradients in the equilibrium (through the $\partial F_{0s}/\partial\psi$ term on the right-hand side of (2.144)) and by the velocity gradient (through the $d\omega/d\psi$ term). It is important to note that this equation is only coupled to the neoclassical distribution function $F_s^{(\mathrm{nc})}$ through the slow evolution of N_s, T_s, ω and I.

2.6.5 The Fluctuating Maxwell's Equations

To solve (2.144), we require a way of obtaining $\langle\chi\rangle_{\boldsymbol{R}}$ from h_s. This is provided by the fluctuating parts of Maxwell's equations.

[9] In the gyrokinetic literature, there is much discussion of the so-called "parallel nonlinearity" [71, 72], which is proportional to $\delta E_\parallel\,(\partial\delta f_s/\partial v_\parallel)$. Some derivations of gyrokinetics include this term in the gyrokinetic equation, claiming that it has beneficial numerical properties and an important role in turbulent heating and conservation of energy. The gyrokinetic equation (2.144) derived here does not contain the parallel nonlinearity because the latter is higher order than the terms we retain and only appears in the derivation of the transport equations. We will (in §2.8) derive conservation laws that prove that our equations nevertheless conserve, in the mean, energy and satisfy Boltzmann's H-theorem. It is true that in our formalism there is no formal *energy* conservation law for gyrokinetics on the fluctuating timescale (in contrast to versions of gyrokinetics derived from Hamiltonians; see, e.g., [73]). This is because energy is exchanged between the fluctuations and the mean fields. We will show in §2.8.2 that, on fluctuating timescales, our gyrokinetic equation conserves the *free energy*, which is the relevant conserved quantity for kinetic turbulence [38, 39, 40, 41] and, moreover, the conservation of which is crucial for the mean energy to be conserved on the transport timescale. Note that numerical evidence [71] has shown that the inclusion of the parallel nonlinearity in the gyrokinetic equation does not affect the solutions, which is as expected for a term that is smaller than all the others in the equation.

The fluctuating component of the quasineutrality condition (2.10) is

$$\sum_s \frac{Z_s^2 e^2 n_s \delta\varphi'}{T_s} = \sum_s Z_s e \int d^3\boldsymbol{w} \langle h_s \rangle_{\boldsymbol{r}}, \tag{2.146}$$

where $\delta\varphi'$ is given by (2.79). The integral over velocities is performed at constant $\boldsymbol{r}$ as discussed in §2.4.2. We have now made this explicit by introducing the gyroaverage at constant $\boldsymbol{r}$, $w_\parallel$, and $w_\perp$:

$$\langle h_s \rangle_{\boldsymbol{r}} = \frac{1}{2\pi} \oint d\vartheta h_s(\boldsymbol{R}_s(\boldsymbol{r}, w_\parallel, w_\perp, \vartheta), \varepsilon_s(\boldsymbol{r}, w_\parallel, w_\perp, \vartheta), \mu_s(\boldsymbol{r}, w_\perp), \sigma). \tag{2.147}$$

The fluctuating component of Ampère's Law (2.11) is

$$\nabla \times \delta\boldsymbol{B} = -\nabla^2 \delta\boldsymbol{A} = \frac{4\pi}{c} \sum_s Z_s e \int d^3\boldsymbol{w} \langle \boldsymbol{w} h_s \rangle_{\boldsymbol{r}}. \tag{2.148}$$

The parallel component of this equation gives us the following equation for $\delta A_\parallel$:

$$-\nabla_\perp^2 \delta A_\parallel = \frac{4\pi}{c} \sum_s Z_s e \int d^3\boldsymbol{w} w_\parallel \langle h_s \rangle_{\boldsymbol{r}}. \tag{2.149}$$

It turns out to be convenient for various calculations to work with $\delta B_\parallel$ rather than $\delta\boldsymbol{A}_\perp$. Thus, we take the divergence of $\boldsymbol{b} \times$ (2.148) to find

$$\nabla \cdot [\boldsymbol{b} \times (\nabla \times \delta\boldsymbol{B})] = \nabla_\perp^2 \delta B_\parallel = \frac{4\pi}{c} \sum_s \int d^3\boldsymbol{w} \nabla_\perp \cdot \langle \boldsymbol{b} \times \boldsymbol{w}_\perp h_s \rangle_{\boldsymbol{r}}. \tag{2.150}$$

We now use

$$\boldsymbol{b} \times \boldsymbol{w} = -\frac{\partial \boldsymbol{w}}{\partial \vartheta}, \tag{2.151}$$

integrate by parts inside the gyroaverage, and use (see (A.3))

$$\boldsymbol{w}_\perp \cdot \nabla h_s = -\Omega_s \left.\frac{\partial h_s}{\partial \vartheta}\right|_{\boldsymbol{r}, \varepsilon_s, \mu_s} + \mathcal{O}(\epsilon \Omega_s h_s) \tag{2.152}$$

to obtain

$$\nabla_\perp^2 \frac{\delta B_\parallel B}{4\pi} + \boldsymbol{\nabla}_\perp \boldsymbol{\nabla}_\perp : \sum_s \int d^3\boldsymbol{w} \langle m_s \boldsymbol{w}_\perp \boldsymbol{w}_\perp h_s \rangle_{\boldsymbol{r}} = 0. \tag{2.153}$$

Once again (cf. (2.128)), perpendicular Ampère's law has become the equation for perpendicular force balance. This eliminates fluctuations such as the compressional Alfvén wave (see the appendix of [74]), which are not force-balanced.

This completes the description of the fluctuations, which are now determined by a closed set of equations: (2.104), (2.144), (2.146), (2.149) and (2.153).

2.7 Third Order $\mathcal{O}(\epsilon^3\Omega_s f_s)$: Transport Equations

So far we have derived equations for the fast evolution of the fluctuations, instantaneous relations between equilibrium quantities, and an equation for the slow evolution of the magnetic field. These hold for a given set of three flux functions: N_s, ω and T_s. In this section, we derive equations for the slow (transport-timescale) evolution of these functions. We refer to these as the transport equations.

To derive such equations, we will need to go to the next order in our expansion in ϵ, namely $\mathcal{O}(\epsilon^3\Omega_s f_s)$. As we wish to maintain the physical interpretation of evolution equations for n_s, ω and T_s in terms of the transport of particles, momentum, and heat, we return to the original $\boldsymbol{r}$ and $\boldsymbol{w}$ variables to begin our derivation. The averaged form of (2.1), written in these variables, is

$$\begin{aligned}\frac{\partial F_s}{\partial t} + (\boldsymbol{u}+\boldsymbol{w})\cdot\nabla F_s + \left[\boldsymbol{a}_s - \frac{\partial \boldsymbol{u}}{\partial t} - (\boldsymbol{u}+\boldsymbol{w})\cdot\nabla\boldsymbol{u}\right]\cdot\frac{\partial F_s}{\partial \boldsymbol{w}} + \left\langle \delta\boldsymbol{a}_s\cdot\frac{\partial \delta f_s}{\partial \boldsymbol{w}}\right\rangle_{\mathrm{turb}} \\ = C[F_s] + S_s,\end{aligned} \tag{2.154}$$

where we have naturally split the particle acceleration into its mean part $\boldsymbol{a}_s$ and its fluctuating part $\delta\boldsymbol{a}_s$. The mean acceleration is

$$\boldsymbol{a}_s = \frac{Z_s e}{m_s}\left[\boldsymbol{E} + \frac{1}{c}(\boldsymbol{u}+\boldsymbol{w})\times\boldsymbol{B}\right] = -\frac{Z_s e}{m_s}\left(\nabla\varphi_0 + \frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t}\right) + \Omega_s\boldsymbol{w}\times\boldsymbol{b}, \tag{2.155}$$

where we have used the results of §2.3.1. The fluctuating acceleration is

$$\begin{aligned}\delta\boldsymbol{a}_s &= \frac{Z_s e}{m_s}\left[\delta\boldsymbol{E} + \frac{1}{c}(\boldsymbol{u}+\boldsymbol{w})\times\delta\boldsymbol{B}\right] \\ &= -\frac{Z_s e}{m_s}\left[\nabla\chi + \frac{1}{c}\boldsymbol{w}\cdot\nabla\delta\boldsymbol{A} + \frac{1}{c}\left(\frac{\partial}{\partial t} + \boldsymbol{u}\cdot\nabla\right)\delta\boldsymbol{A}\right] \\ &= -\frac{Z_s e}{m_s}\left[\nabla\left(\delta\varphi' - \frac{1}{c}\boldsymbol{w}\cdot\delta\boldsymbol{A}\right) + \frac{1}{c}\boldsymbol{w}_\perp\cdot\nabla\delta\boldsymbol{A}\right] + \mathcal{O}(\epsilon^2 v_{\mathrm{th}_s}\Omega_s),\end{aligned} \tag{2.156}$$

where χ is defined by (2.109) and $\delta\varphi'$ by (2.79). Equation (2.154) will correctly describe the transport-timescale evolution of F_s if we keep all terms up to order $\mathcal{O}(\epsilon^3\Omega_s f_s)$. This clearly requires knowledge of the distribution function including $\mathcal{O}(\epsilon^2 f_s)$ corrections.

From the previous two sections, the particle distribution function is

$$\begin{aligned}f_s = F_{0s}(\psi(\boldsymbol{R}_s),\varepsilon_s) + F_{1s}(\boldsymbol{R}_s,\varepsilon_s,\mu_s,\sigma) + F_{2s}(\boldsymbol{r},\boldsymbol{v}) \\ - \frac{Z_s e}{T_s}\delta\varphi'(\boldsymbol{r})F_{0s} + h_s(\boldsymbol{R}_s,\varepsilon_s,\mu_s,\sigma) + \delta f_{2s}(\boldsymbol{r},\boldsymbol{v}) + \cdots,\end{aligned} \tag{2.157}$$

where the equilibrium distribution function F_{0s} is given by (2.93), F_{1s} and h_s are obtained from the neoclassical drift-kinetic equation (2.112), and the gyrokinetic equation (2.144), respectively, and we have absorbed all (as yet unknown) higher-order terms into F_{2s} and δf_{2s}. The exact electric and magnetic fields are

$$\widetilde{\boldsymbol{E}} = \boldsymbol{E} + \delta\boldsymbol{E} = -\nabla\Phi - \nabla\varphi_0 - \nabla\delta\varphi - \frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t} - \frac{1}{c}\frac{\partial \delta\boldsymbol{A}}{\partial t} \tag{2.158}$$

and

$$\widetilde{\boldsymbol{B}} = \boldsymbol{B} + \delta\boldsymbol{B} = I(\psi)\nabla\phi + \nabla\psi\times\nabla\phi + \nabla\times\delta\boldsymbol{A}. \tag{2.159}$$

In the expressions for the fields, $I(\psi)$ and $\psi(R,z)$ are determined as explained in Sections 2.6.2 and 2.6.3, φ_0 is obtained from the quasineutrality condition (2.97), and $\delta\varphi'$ and $\delta\boldsymbol{A}$ are given by the fluctuating Maxwell's equations (2.146), (2.149) and (2.153).

The solution (2.157) for f_s will need to be completed with information about F_{2s} and δf_{2s}. We can obtain the gyrophase-dependent part of F_s to second order by expanding (2.154) to second order in ϵ:

$$\begin{aligned}\Omega_s\frac{\partial F_s}{\partial\vartheta} =& -(\boldsymbol{u}+\boldsymbol{w})\cdot\nabla F_s + \left[\frac{Z_s e}{m_s}\left(\nabla\varphi_0 + \frac{1}{c}\frac{\partial\boldsymbol{A}}{\partial t}\right) + (\boldsymbol{u}+\boldsymbol{w})\cdot\nabla\boldsymbol{u}\right]\cdot\frac{\partial F_s}{\partial\boldsymbol{w}}\\ &- \left\langle\delta\boldsymbol{a}_s\cdot\frac{\partial\delta f_s}{\partial\boldsymbol{w}}\right\rangle_{\rm turb} + C[F_s] + \mathcal{O}(\epsilon^3\Omega_s f_s),\end{aligned} \tag{2.160}$$

where we have used

$$(\boldsymbol{w}\times\boldsymbol{b})\cdot\left.\frac{\partial}{\partial\boldsymbol{w}}\right|_{\boldsymbol{r}} = \left.\frac{\partial}{\partial\vartheta}\right|_{\boldsymbol{r},w_\parallel,w_\perp}. \tag{2.161}$$

Similarly, the fluctuating part of (2.1) can be expanded to obtain the gyrophase dependence of δf_s to second order:

$$\begin{aligned}\left(\boldsymbol{w}_\perp\cdot\nabla + \Omega_s\frac{\partial}{\partial\vartheta}\right)\delta f_s =& -\left(\frac{\partial}{\partial t} + \boldsymbol{u}\cdot\nabla\right)\delta f_s - w_\parallel\boldsymbol{b}\cdot\nabla\delta f_s - \delta\boldsymbol{a}_s\cdot\frac{\partial F_s}{\partial\boldsymbol{w}}\\ &+ \left[\frac{Z_s e}{m_s}\left(\nabla\varphi_0 + \frac{1}{c}\frac{\partial\boldsymbol{A}}{\partial t}\right) + (\boldsymbol{u}+\boldsymbol{w})\cdot\nabla\boldsymbol{u}\right]\cdot\frac{\partial\delta f_s}{\partial\boldsymbol{w}} - \delta\boldsymbol{a}_s\cdot\frac{\partial\delta f_s}{\partial\boldsymbol{w}}\\ &+ \left\langle\delta\boldsymbol{a}_s\cdot\frac{\partial\delta}{\partial\boldsymbol{w}}\right\rangle_{\rm turb} + C[\delta f_s] + \mathcal{O}(\epsilon^3\Omega_s f_s).\end{aligned} \tag{2.162}$$

By examining the order of the terms on the right-hand side of (2.160) and (2.162), we see that we only need the particle distribution function correct to $\mathcal{O}(\epsilon f_s)$ in order to evaluate the left-hand side correctly to order $\mathcal{O}(\Omega_s\epsilon^2 f_s)$.

In the subsequent sections, we will take moments of (2.154) and show that (2.160) and (2.162) contain sufficient information about the second-order parts of the distribution function to close the resulting transport equations in terms of known quantities – i.e., the gyrophase-independent parts of F_{2s} and δf_{2s} do not affect transport.

2.7.1 Particle Transport

To derive the particle transport equation, we integrate (2.154) over all velocities to find

$$\frac{\partial n_s}{\partial t} + \nabla \cdot (n_s \boldsymbol{U}_s) = S_s^{(n)}, \tag{2.163}$$

where

$$S_s^{(n)} = \int d^3\boldsymbol{w} S_s, \tag{2.164}$$

$$n_s \boldsymbol{U}_s = \int d^3\boldsymbol{w}\boldsymbol{w} F_s. \tag{2.165}$$

Equation (2.163) clearly describes all particle transport: n_s is created or destroyed by the source $S_s^{(n)}$ and advected by the mean flux $n_s\boldsymbol{U}_s$. As the first and last terms in this equation are $\mathcal{O}(\epsilon^3 \Omega_s n_s)$, we conclude that we require $n_s\boldsymbol{U}_s$ correct to order $\mathcal{O}(\epsilon^2 n_s v_{\mathrm{th}_s})$ to evaluate the divergence. It is clear that the parallel part of the flux will depend on the gyrophase-independent part of F_{2s}, which we cannot easily find. In contrast, the perpendicular flux will only depend on the gyrophase-dependent piece of F_{2s}, which we can find via (2.160).

To reduce (2.163) to an equation only containing the perpendicular flux, we apply the flux-surface average, defined by (2.34). This results in

$$\frac{1}{V'} \left.\frac{\partial}{\partial t}\right|_{\psi} V' \left\langle n_s \right\rangle_\psi + \frac{1}{V'}\frac{\partial}{\partial \psi} V' \left\langle \Gamma_s \right\rangle_\psi = \left\langle S_s^{(n)} \right\rangle_\psi, \tag{2.166}$$

where we have used (2.39) and (2.45) to simplify the flux-surface averages of the divergence and the time derivative. We have combined the radial particle flux[10] and the terms due to the motion of the flux surface into

$$\Gamma_s = n_s \boldsymbol{U}_s \cdot \nabla\psi + n_s \frac{\partial \psi}{\partial t}. \tag{2.167}$$

We can write the first term of (2.167) as

$$n_s \boldsymbol{U}_s \cdot \nabla \psi = \int d^3\boldsymbol{w} \left(\boldsymbol{w}_\perp \cdot \nabla\psi\right) F_s = \int d^3\boldsymbol{w} R^2 B \left(\boldsymbol{v}\cdot\nabla\phi\right) \left.\frac{\partial F_s}{\partial \vartheta}\right|_{\boldsymbol{r}, w_\parallel, w_\perp}, \tag{2.168}$$

where $\boldsymbol{v}$ is just shorthand for $\boldsymbol{u} + \boldsymbol{w}$ and we have used

$$\boldsymbol{w}_\perp \cdot \nabla\psi = -R^2 B \boldsymbol{w} \cdot \left(\boldsymbol{b} \times \nabla\phi\right) = R^2 B \left(\boldsymbol{b}\times\boldsymbol{w}\right)\cdot\nabla\phi = -R^2 B \left.\frac{\partial \boldsymbol{v}}{\partial\vartheta}\right|_{\boldsymbol{r}, w_\parallel, w_\perp} \cdot \nabla\phi \tag{2.169}$$

[10] Whilst this flux is not in the geometrically radial direction, but is in fact the cross-flux-surface flux, it is both convenient and conventional to refer to it as the radial flux.

and integrated by parts with respect to ϑ. So, in order to calculate the radial flux to second order, it is sufficient to know $\partial F_s/\partial\vartheta$ up to second order only.

In Appendix C.1, we perform the explicit evaluation of $\langle n_s \boldsymbol{U}_s \cdot \nabla\psi\rangle_\psi$ via the kinetic equation (2.160), resulting in (C.10), which we substitute into (2.167) to find:

$$\begin{aligned}\langle \Gamma_s\rangle_\psi = &\left\langle \int d^3\boldsymbol{w} \left(\frac{\boldsymbol{w}\times\boldsymbol{b}}{\Omega_s}\cdot\nabla\psi\right) C[F_{0s}]\right\rangle_\psi + \left\langle \int d^3\boldsymbol{w} F_s^{(\mathrm{nc})}\boldsymbol{V}_{\mathrm{D}s}\cdot\nabla\psi\right\rangle_\psi \\ &- \langle n_s\rangle_\psi I(\psi)\frac{\langle \boldsymbol{E}\cdot\boldsymbol{B}\rangle_\psi}{\langle B^2\rangle_\psi} + \left\langle\left\langle \int d^3\boldsymbol{w} \langle h_s\, \boldsymbol{V}_\chi\rangle_{\boldsymbol{r}}\cdot\nabla\psi\right\rangle_{\mathrm{turb}}\right\rangle_\psi .\end{aligned} \tag{2.170}$$

The first term in the above expression is due to classical collisional transport with F_{0s} given by (2.93) [11] , the second and third terms are due to neoclassical transport with $F_s^{(\mathrm{nc})}$ and $\langle \boldsymbol{E}\cdot\boldsymbol{B}\rangle_\psi$ calculated as explained in Sections 2.6.1-2.6.3[12], and the final term gives the turbulent contribution to the particle flux in terms of h_s, which is the solution of the gyrokinetic equation (2.144)[13]. We note that the composition of a flux-surface average (defined by (2.34)) with the fluctuation average (defined by (2.18)) is

$$\langle\langle g(\boldsymbol{r},t)\rangle_{\mathrm{turb}}\rangle_\psi = \frac{1}{V'}\frac{1}{T}\int\limits_{t-T/2}^{t+T/2} dt' \int\limits_{\Delta_\lambda} d^3\boldsymbol{r}' V' g(\boldsymbol{r}',t'), \tag{2.171}$$

where Δ_λ is the annulus bounded by two flux surfaces a distance λ apart centered on $\psi(\boldsymbol{r})$. This is exactly the average over a flux tube that is implemented in current gyrokinetic linked-flux-tube codes [4, 42].

Thus, we now have an equation, (2.166), that determines $\langle n_s\rangle_\psi$, but we still need to use this to find N_s and hence n_s via (2.96). This is done by flux-surface averaging (2.96) to find

$$N_s = \langle n_s\rangle_\psi \bigg/ \left\langle \exp\left[\frac{m_s\omega^2(\psi)R^2}{2T_s} - \frac{Z_s e\varphi_0}{T_s}\right]\right\rangle_\psi , \tag{2.172}$$

which allows us to express N_s in terms of $\langle n_s\rangle_\psi$ and known quantities. In particular, we can solve (2.172) simultaneously with the quasineutrality condition (2.97) to find N_s and φ_0 from $\langle n_s\rangle_\psi$.

[11] Note that F_{0s} is only Maxwellian to lowest order, so $C[F_{0s}] = \mathcal{O}(\epsilon^2\Omega_s F_{0s})$ is non-zero.

[12] Explicit expressions for the collisional fluxes can be found in [75, 47] and in [52, 53, 54] for the case of non-zero $\omega(\psi)$.

[13] Having determined the particle flux, we note that the classical, neoclassical and turbulent contributions are all independently ambipolar [7]. This is explicitly proved in Appendix C.2.

2.7.2 Momentum Transport

To find an evolution equation for $\boldsymbol{u} = \omega(\psi)R^2\nabla\phi$, we multiply (2.154) by $m_s(\boldsymbol{v}\cdot\nabla\phi)R^2$, where $\boldsymbol{v} = \boldsymbol{w}+\boldsymbol{u}$, integrate over $\boldsymbol{w}$, and sum over species to find

$$\begin{aligned}&\frac{\partial}{\partial t}\sum_s m_s n_s R^2\omega(\psi) + \left[\nabla\cdot\sum_s\left(\mathbf{\Pi}_s + m_s n_s\boldsymbol{U}_s\boldsymbol{u} + m_s n_s\boldsymbol{u}\boldsymbol{U}_s\right)\right]\cdot(\nabla\phi)R^2\\&-\sum_s m_s\left\langle\int d^3\boldsymbol{w}\,(\delta\boldsymbol{a}_s\cdot\nabla\phi)R^2\delta f_s\right\rangle_{\rm turb} - \frac{1}{c}(\boldsymbol{j}\times\boldsymbol{B})\cdot(\nabla\phi)R^2 = S^{(\omega)},\end{aligned} \tag{2.173}$$

where the viscous stress tensor $\mathbf{\Pi}_s$ is

$$\mathbf{\Pi}_s = \int d^3\boldsymbol{w}\, m_s\boldsymbol{w}\boldsymbol{w}F_s, \tag{2.174}$$

the source of toroidal angular momentum is

$$S^{(\omega)} = \sum_s\left[\int d^3\boldsymbol{w}\, m_s(\boldsymbol{w}\cdot\nabla\phi)R^2 S_s + m_s\omega(\psi)S_s^{(n)}\right], \tag{2.175}$$

where $S_s^{(n)}$ is given by (2.164) and we have used the quasineutrality condition (2.52). The second term in (2.175) is due to injected particles joining the mean flow, while the first term contains also contributions from particle-conserving momentum injection mechanisms, e.g., momentum injection by waves [76].

We now wish to write all terms except the time derivative and the source as full divergences. The second term on the left-hand side of (2.173) can be written so because, for any symmetric tensor field $\mathbf{A}$,

$$(\nabla\cdot\mathbf{A})\cdot(\nabla\phi)R^2 = \nabla\cdot\left(R^2\mathbf{A}\cdot\nabla\phi\right). \tag{2.176}$$

The third term (due to the fluctuations) can also be written as a divergence:

$$\begin{aligned}&-\sum_s m_s\left\langle\int d^3\boldsymbol{w}\,(\delta\boldsymbol{a}_s\cdot\nabla\phi)R^2\delta f_s\right\rangle_{\rm turb}\\&= -\frac{1}{c}\langle\delta\boldsymbol{j}\times\delta\boldsymbol{B}\rangle_{\rm turb}\cdot(\nabla\phi)R^2\\&= \nabla\cdot\left[\left\langle\frac{\delta B^2}{8\pi}\mathbf{I} - \frac{\delta\boldsymbol{B}\delta\boldsymbol{B}}{4\pi}\right\rangle_{\rm turb}\cdot(\nabla\phi)R^2\right],\end{aligned} \tag{2.177}$$

where $\mathbf{I}$ is the unit dyadic and we have used (2.156) for $\delta\boldsymbol{a}_s$, the fluctuating quasineutrality condition (2.146), expressed the Lorentz force as the divergence of the Maxwell stress in the usual way and used (2.176). The final term on the left-hand side can similarly be written as the divergence of the Maxwell stress associated with the mean magnetic field:

$$-\frac{1}{c}(\boldsymbol{j}\times\boldsymbol{B})\cdot(\nabla\phi)R^2 = \nabla\cdot\left[\left(\frac{B^2}{8\pi}\mathbf{I} - \frac{\boldsymbol{B}\boldsymbol{B}}{4\pi}\right)\cdot(\nabla\phi)R^2\right]. \tag{2.178}$$

We now flux-surface average (2.173), using the identities (2.39) and (2.45) for the flux-surface average of a divergence and of a time derivative. This gives

$$\frac{1}{V'}\left.\frac{\partial}{\partial t}\right|_{\psi} V' J\omega(\psi) + \frac{1}{V'}\frac{\partial}{\partial\psi}V'\left\langle \pi_{\text{tot}}^{(\psi\phi)}\right\rangle_{\psi} = \left\langle S^{(\omega)}\right\rangle_{\psi}, \tag{2.179}$$

where

$$J = \sum_s m_s \left\langle n_s R^2\right\rangle_{\psi} \tag{2.180}$$

is the flux-surface-averaged moment of inertia and the total flux of angular momentum is given by

$$\begin{aligned}\pi_{\text{tot}}^{(\psi\phi)} &= \sum_s (\nabla\psi)\cdot\mathbf{\Pi}_s\cdot(\nabla\phi)\,R^2 + \sum_s m_s\omega(\psi)R^2\Gamma_s \\ &\quad - \frac{1}{4\pi}(\nabla\psi)\cdot\langle\delta\boldsymbol{B}\delta\boldsymbol{B}\rangle_{\text{turb}}\cdot(\nabla\phi)\,R^2,\end{aligned} \tag{2.181}$$

where Γ_s is given by (2.167).

Proceeding analogously to the derivation of the particle transport (see (2.168)), we can write the radial angular momentum flux due to the particles of species s as

$$(\nabla\psi)\cdot\mathbf{\Pi}_s\cdot(\nabla\phi)\,R^2 + m_s n_s\omega(\psi)R^2\boldsymbol{U}_s\cdot\nabla\psi = \int d^3\boldsymbol{w} B m_s \frac{1}{2}\left(R^2\boldsymbol{v}\cdot\nabla\phi\right)^2\frac{\partial F_s}{\partial\vartheta}, \tag{2.182}$$

where $\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{w}$. We evaluate this explicitly in Appendix C.3 to find

$$\left\langle \pi_{\text{tot}}^{(\psi\phi)}\right\rangle_{\psi} = \sum_s\left[\left\langle \pi_s^{(\psi\phi)}\right\rangle_{\psi} + m_s\omega(\psi)\left\langle R^2\Gamma_s\right\rangle_{\psi}\right] + \left\langle \pi_{\text{EM}}^{(\psi\phi)}\right\rangle_{\psi}, \tag{2.183}$$

where $\pi_s^{(\psi\phi)}$ contains the classical, neoclassical and turbulent viscous stresses:

$$\left\langle \pi_s^{(\psi\phi)}\right\rangle_{\psi} = \left\langle \pi_s^{(\text{cl})}\right\rangle_{\psi} + \left\langle \pi_s^{(\text{nc})}\right\rangle_{\psi} + \left\langle\left\langle\int d^3\boldsymbol{w} m_s R^2 \langle(\boldsymbol{w}\cdot\nabla\phi)\,h_s\boldsymbol{V}_\chi\rangle_{\boldsymbol{r}}\cdot\nabla\psi\right\rangle_{\text{turb}}\right\rangle_{\psi} \tag{2.184}$$

with

$$\begin{aligned}&\left\langle \pi_s^{(\text{cl})}\right\rangle_{\psi} = \\ &\quad\frac{c}{Z_s e}\left\langle\left(\int d^3\boldsymbol{w}\frac{m_s^2}{2}\boldsymbol{w}_\perp\boldsymbol{w}_\perp\, C[F_{0s}]\right) : \left\{R^4(\nabla\phi)(\nabla\phi) - \frac{1}{2}\left[R^2 - \frac{I^2(\psi)}{B^2}\right]\mathbf{I}\right\}\right\rangle_{\psi} \\ &\quad + \left\langle\int d^3\boldsymbol{w} m_s\frac{I(\psi)w_\parallel}{B}\left(\frac{\boldsymbol{w}\times\boldsymbol{b}}{\Omega_s}\cdot\nabla\psi\right)C[F_{0s}]\right\rangle_{\psi}\end{aligned} \tag{2.185}$$

and

$$\left\langle \pi_s^{(\text{nc})}\right\rangle_{\psi} = -\frac{m_s c}{Z_s e}\left\langle\int d^3\boldsymbol{w} F_s^{(\text{nc})} w_\parallel \boldsymbol{b}\cdot\nabla\left\{m_s\frac{I^2(\psi)w_\parallel^2}{B^2} + \mu_s\frac{|\nabla\psi|^2}{B^2}\right\}\right\rangle_{\psi}, \tag{2.186}$$

$m_s\omega(\psi)\left\langle R^2\Gamma_s\right\rangle_\psi$ is the convective flux of angular momentum with $\langle R^2\Gamma_s\rangle_\psi$ given by (cf. (2.170)):

$$\begin{aligned}\left\langle R^2\Gamma_s\right\rangle_\psi =& \left\langle R^2\int d^3\boldsymbol{w}\left(\frac{\boldsymbol{w}\times\boldsymbol{b}}{\Omega_s}\cdot\nabla\psi\right)C[F_{0s}]\right\rangle_\psi \\ &+\left\langle\int d^3\boldsymbol{w}F_s^{(\mathrm{nc})}w_\parallel\boldsymbol{b}\cdot\nabla\left[R^2\frac{I(\psi)w_\parallel}{\Omega_s}+R^2\frac{BR^2\omega(\psi)}{\Omega_s}\right]\right\rangle_\psi \\ &-\left\langle R^2n_s\right\rangle_\psi I(\psi)\frac{\langle\boldsymbol{E}\cdot\boldsymbol{B}\rangle_\psi}{\langle B^2\rangle_\psi}+\left\langle R^2\left\langle\int d^3\boldsymbol{w}\langle h_s\boldsymbol{V}_\chi\rangle_{\boldsymbol{r}}\cdot\nabla\psi\right\rangle_{\mathrm{turb}}\right\rangle_\psi,\end{aligned} \tag{2.187}$$

and the electromagnetic angular momentum flux $\pi_{\mathrm{EM}}^{(\psi\phi)}$ is

$$\pi_{\mathrm{EM}}^{(\psi\phi)}=-\left(\nabla\psi\right)\cdot\left\langle\frac{1}{4\pi}\delta\boldsymbol{B}\delta\boldsymbol{B}+\frac{1}{c}\delta\boldsymbol{j}\delta\boldsymbol{A}\right\rangle_{\mathrm{turb}}\cdot\left(\nabla\phi\right)R^2. \tag{2.188}$$

The two terms in the last expression are, respectively, the Maxwell stress and the advection of electromagnetic momentum by the particles[14]

$$\frac{1}{c}\delta\boldsymbol{j}\delta\boldsymbol{A}=\sum_s\int d^3\boldsymbol{w}\langle h_s\boldsymbol{w}\rangle_{\boldsymbol{r}}\frac{Z_se}{c}\delta\boldsymbol{A}. \tag{2.189}$$

2.7.3 Heat Transport and Heating

Turning to heat transport next, we multiply (2.154) by $m_sw^2/2$ and integrate over all velocities to obtain

$$\begin{aligned}\frac{3}{2}\frac{\partial}{\partial t}n_sT_s+\nabla\cdot\boldsymbol{Q}_s+Z_se\left(\nabla\varphi_0+\frac{1}{c}\frac{\partial\boldsymbol{A}}{\partial t}\right)\cdot(n_s\boldsymbol{U}_s)+m_sn_s\boldsymbol{u}\cdot(\nabla\boldsymbol{u})\cdot\boldsymbol{U}_s+\boldsymbol{\Pi}_s:\nabla\boldsymbol{u} \\ -\left\langle\int d^3\boldsymbol{w}m_s\boldsymbol{w}\cdot\delta\boldsymbol{a}_s\delta f_s\right\rangle_{\mathrm{turb}}=C_s^{(E)}+S_s^{(E)},\end{aligned} \tag{2.190}$$

where we have defined the heat flux

$$\boldsymbol{Q}_s=\int d^3\boldsymbol{w}\frac{1}{2}m_sw^2\boldsymbol{w}F_s, \tag{2.191}$$

the collisional energy transfer to (or from) species s

$$C_s^{(E)}=\int d^3\boldsymbol{w}\frac{1}{2}m_sw^2\,C[F_s]\,, \tag{2.192}$$

and the energy source

$$S_s^{(E)}=\int d^3\boldsymbol{w}\frac{1}{2}m_sw^2S_s. \tag{2.193}$$

[14] In the long-wavelength limit, this second term is small as the fluctuating velocity is dominated by the $\boldsymbol{E}\times\boldsymbol{B}$ velocity, which carries no current.

Acting in the same vein as for the particle and momentum transport equations, we now flux-surface average (2.190). This flux-surface average is carried out in Appendix C.4, where we obtain:

$$\begin{aligned} &\frac{3}{2}\frac{1}{V'}\frac{\partial}{\partial t}\bigg|_\psi V' \langle n_s\rangle_\psi T_s + \frac{1}{V'}\frac{\partial}{\partial\psi}V'\langle q_s\rangle_\psi \\ &\quad = P_s^{\mathrm{visc}} + P_s^{\mathrm{Ohm}} + P_s^{\mathrm{comp}} + P_s^{\mathrm{pot}} + P_s^{\mathrm{turb}} + \left\langle C_s^{(E)}\right\rangle_\psi + \left\langle S_s^{(E)}\right\rangle_\psi . \end{aligned} \tag{2.194}$$

Let us detail the terms in this equation.

The Heat Flux.

The "heat flux" q_s in (2.194) is defined by the following formula, whose origin is explained in Appendix C.4.6:

$$\begin{aligned} \langle q_s\rangle_\psi = &\left\langle \left[\boldsymbol{Q}_s + \left(Z_s e\varphi_0 - \frac{1}{2}m_s\omega^2(\psi)R^2\right) n_s\boldsymbol{U}_s + Z_s e\left\langle \int d^3\boldsymbol{w}\,\delta\varphi'\langle h_s\boldsymbol{w}\rangle_{\boldsymbol{r}}\right\rangle_{\mathrm{turb}}\right]\cdot\nabla\psi\right\rangle_\psi \\ &+ \left\langle n_s\left[\frac{5}{2}T_s + Z_s e\varphi_0 - \frac{1}{2}m_s\omega^2(\psi)R^2\right]\frac{\partial\psi}{\partial t}\right\rangle_\psi . \end{aligned} \tag{2.195}$$

This form of the heat flux is most useful for interpreting its constituent parts. The first term in the first line of (2.195) is the usual flux of thermal energy, the second term is the flux of the mean potential energy (electrostatic and centrifugal), and the third is the flux of the fluctuating electrostatic potential energy. The second line of (2.195) is minus the flow of heat and energy carried by the motion of the flux surfaces (remembering that $\boldsymbol{V}_\psi\cdot\nabla\psi = -\partial\psi/\partial t$ and observing that the expression multiplying $\partial\psi/\partial t$ is the enthalpy carried by the plasma[15]). As we include the potential energy fluxes in (2.195), q_s is not, strictly speaking, the heat flux in the usual sense. We will elaborate upon this in §2.7.3 where we discuss the potential energy exchange term P_s^{pot}. For simplicity, we will continue to refer to $\langle q_s\rangle_\psi$ as the radial heat flux. This usage will be vindicated on physical grounds by the appearance of $\langle q_s\rangle_\psi$ multiplying the temperature gradient in the expression (2.235) for the entropy production.

Unfortunately, (2.195) is not a useful form for actually evaluating the heat flux. An explicit form for it is calculated in terms of known quantities in Appendix C.5:

$$\begin{aligned} \langle q_s\rangle_\psi = &\left\langle \int d^3\boldsymbol{w}\,\varepsilon_s\left(\frac{\boldsymbol{w}\times\boldsymbol{b}}{\Omega_s}\cdot\nabla\psi\right) C[F_{0s}]\right\rangle_\psi + \left\langle \int d^3\boldsymbol{w}\,\varepsilon_s F_s^{(\mathrm{nc})}\boldsymbol{V}_{\mathrm{D}s}\cdot\nabla\psi\right\rangle_\psi \\ &- \left\langle n_s\left[\frac{5T_s}{2} + Z_s e\varphi_0 - \frac{1}{2}m_s\omega^2(\psi)R^2\right]\right\rangle_\psi I(\psi)\frac{\langle \boldsymbol{E}\cdot\boldsymbol{B}\rangle_\psi}{\langle B^2\rangle_\psi} \\ &+ \left\langle\left\langle \int d^3\boldsymbol{w}\,\varepsilon_s\langle h_s\boldsymbol{V}_\chi\rangle_{\boldsymbol{r}}\cdot\nabla\psi\right\rangle_{\mathrm{turb}}\right\rangle_\psi . \end{aligned} \tag{2.196}$$

[15] The first term is the enthalpy of the ideal gas of species s [77] and the second and third terms are the potential energy contributions to the internal energy.

Let us now examine the heating terms on the right-hand side of (2.194).

Viscous Heating.

This is (see Appendix C.4.2)

$$P_s^{\text{visc}} = -\left[\left\langle \pi_s^{(\psi\phi)} \right\rangle_\psi + m_s\omega(\psi)\left\langle R^2\Gamma_s \right\rangle_\psi\right]\frac{d\omega}{d\psi}, \tag{2.197}$$

where $\left\langle \pi_s^{(\psi\phi)} \right\rangle_\psi$ is given by (2.184) and $\left\langle R^2\Gamma_s \right\rangle_\psi$ by (2.187). Usually, the heating due to the mass flow (the second term inside the square brackets in (2.197)) is not included in the viscous heating. We group them together here because the term in the square brackets in (2.197) is precisely the species-dependent term in the total momentum flux (2.183). We will come back to this point when interpreting the rotational part of P_s^{pot} in §2.7.3. The combination of momentum fluxes seen in (2.197) will appear multiplying $d\omega/d\psi$ in the expression (2.235) for the entropy production, vindicating its interpretation as the total viscous heating of species s.

Ohmic Heating.

The mean Ohmic heating due to the induced parallel electric field is (see Appendix C.4.3)

$$P_s^{\text{Ohm}} = K_s(\psi)\left\langle \boldsymbol{E}\cdot\boldsymbol{B} \right\rangle_\psi, \tag{2.198}$$

where

$$K_s(\psi) = \frac{Z_s e}{B}\int d^3\boldsymbol{w}\, w_\parallel \widehat{F}_{1s}. \tag{2.199}$$

Compressional Heating.

This is due to the motion of the flux surfaces and is given by (see Appendix C.4.4)

$$P_s^{\text{comp}} = -T_s\left\langle n_s\nabla\cdot\boldsymbol{V}_\psi \right\rangle_\psi = T_s\left\langle n_s\nabla\cdot\left(\frac{\partial\psi}{\partial t}\frac{\nabla\psi}{|\nabla\psi|^2}\right)\right\rangle_\psi, \tag{2.200}$$

where we have used the expression (2.44) for the velocity $\boldsymbol{V}_\psi$ of a flux surface.

Exchange between Potential and Thermal Energy.

The change in thermal energy due to exchange with the electrostatic and rotational potential energy is (see Appendix C.4.5)

$$
\begin{aligned}
P_s^{\mathrm{pot}} = &- \left\langle Z_s e\varphi_0 \left(\frac{\partial n_s}{\partial t} + \boldsymbol{V}_\psi \cdot \nabla n_s\right)\right\rangle_\psi - \left\langle Z_s e n_s \varphi_0 \nabla \cdot \boldsymbol{V}_\psi \right\rangle_\psi \\
&+ \frac{\omega^2(\psi)}{2V'} \left.\frac{\partial}{\partial t}\right|_\psi V' m_s \left\langle R^2 n_s \right\rangle_\psi - \frac{1}{2} m_s \omega^2(\psi) \left\langle n_s \boldsymbol{V}_\psi \cdot \nabla R^2 \right\rangle_\psi \\
&+ \left\langle \left(Z_s e\varphi_0 - \frac{1}{2} m_s \omega^2(\psi) R^2\right) S_s^{(n)} \right\rangle_\psi .
\end{aligned}
\tag{2.201}
$$

The first line is the energy exchange with the electrostatic potential energy, the second the exchange with the rotational energy. More transparently, we can group the terms in (2.201) involving φ_0 together with the terms involving φ_0 in our definition (2.195) of the heat flux to write the total as

$$
\begin{aligned}
&- \left\langle Z_s e\varphi_0 \left(\frac{\partial n_s}{\partial t} + \boldsymbol{V}_\psi \cdot \nabla n_s\right)\right\rangle_\psi - \left\langle Z_s e n_s \varphi_0 \nabla \cdot \boldsymbol{V}_\psi \right\rangle_\psi + \left\langle Z_s \varphi_0 S_s^{(n)} \right\rangle_\psi \\
&- \frac{1}{V'} \frac{\partial}{\partial \psi} V' \left\langle Z_s e\varphi_0 n_s \boldsymbol{U}_s \cdot \nabla\psi + Z_s e \varphi_0 n_s \frac{\partial \psi}{\partial t} \right\rangle_\psi \\
&\qquad = -Z_s e \left\langle n_s \left(\boldsymbol{U}_s - \boldsymbol{V}_\psi\right) \cdot \nabla \varphi_0 \right\rangle_\psi ,
\end{aligned}
\tag{2.202}
$$

the work done by the mean electric field due to φ_0.[16] Unfortunately, we do not know the small (i.e., of order $\epsilon^2 v_{\mathrm{th}_s} n_s$) poloidal component of $n_s \boldsymbol{U}_s$; the only way of calculating the right-hand side of (2.202) is to calculate the left-hand side explicitly. This was the reason for splitting the work done into the "heating" and "flux" terms and distributing them between (2.201) and (2.195), respectively.

Similarly, gathering the terms involving the rotation in (2.201) and (2.195), as well as the second term in (2.197) (see discussion in §2.7.3), we can rewrite them as

$$
\begin{aligned}
&\frac{\omega^2(\psi)}{2V'} \left.\frac{\partial}{\partial t}\right|_\psi V' m_s \left\langle R^2 n_s \right\rangle_\psi - \frac{1}{2} m_s \omega^2(\psi) \left\langle n_s \boldsymbol{V}_\psi \cdot \nabla R^2 \right\rangle_\psi - \frac{1}{2} m_s \omega^2(\psi) \left\langle R^2 S_s^{(n)} \right\rangle_\psi \\
&+ \frac{1}{2V'} \frac{\partial}{\partial \psi} V' m_s \omega^2(\psi) \left\langle n_s R^2 \boldsymbol{U}_s \cdot \nabla\psi - n_s R^2 \frac{\partial \psi}{\partial t} \right\rangle_\psi - m_s \omega(\psi) \left\langle R^2 \Gamma_s \right\rangle_\psi \frac{d\omega}{d\psi} \\
&= \frac{1}{2} m_s \omega^2(\psi) \left\langle n_s \left(\boldsymbol{U}_s - \boldsymbol{V}_\psi\right) \cdot \nabla R^2 \right\rangle_\psi .
\end{aligned}
\tag{2.203}
$$

which is the work done by the centrifugal force (the Coriolis force does no work). Again, as with the work done by the electric field (2.202), we cannot explicitly evaluate the right-hand side of (2.203).

[16] Here, $\boldsymbol{V}_\psi$ appears because we measure all velocities relative to $\boldsymbol{V}_\psi$ and so, for a force $\boldsymbol{F}$, we consider $n_s \boldsymbol{F} \cdot \boldsymbol{V}_\psi$ to be work done accelerating the plasma to $\boldsymbol{V}_\psi$, not heating.

Turbulent Heating.

Finally, the energy exchange with the fluctuations is (see Appendix C.4.1)

$$\begin{aligned} P_s^{\mathrm{turb}} &= Z_s e \left\langle \left\langle \int d^3\boldsymbol{w} \left\langle h_s \left(\frac{\partial}{\partial t} + \boldsymbol{u} \cdot \nabla \right) \chi \right\rangle_{\boldsymbol{r}} \right\rangle_{\mathrm{turb}} \right\rangle_{\psi} \\ &\quad - \frac{Z_s e}{c} \omega(\psi) \left\langle \left\langle \int d^3\boldsymbol{w} \langle h_s \boldsymbol{w} \rangle_{\boldsymbol{r}} \cdot (\delta \boldsymbol{A} \times \nabla z) \right\rangle_{\mathrm{turb}} \right\rangle_{\psi} \\ &= P_s^{\mathrm{diss}} - P_s^{\mathrm{drive}}. \end{aligned} \tag{2.204}$$

The last expression reflects the fact that P_s^{turb} consists of two parts (see (E.28) in Appendix E.4): the turbulent heating (dissipation of fluctuations on collisions):

$$P_s^{\mathrm{diss}} = - \left\langle \left\langle \int d^3\boldsymbol{w} \frac{T_s h_s}{F_{0s}} C[h_s] \right\rangle_{\mathrm{turb}} \right\rangle_{\psi} \tag{2.205}$$

and the energy injection into the fluctuations due to the mean density, temperature and velocity gradients:

$$\begin{aligned} P_s^{\mathrm{drive}} &= - \left\langle \left\langle \int d^3\boldsymbol{w} \langle h_s \boldsymbol{V}_\chi \rangle_{\boldsymbol{r}} \cdot \nabla \psi \right\rangle_{\mathrm{turb}} \right\rangle_{\psi} T_s \frac{d \ln N_s}{d\psi} \\ &\quad - \left\langle \left\langle \int d^3\boldsymbol{w} \left(\varepsilon_s - \frac{3}{2} T_s \right) \langle h_s \boldsymbol{V}_\chi \rangle_{\boldsymbol{r}} \cdot \nabla \psi \right\rangle_{\mathrm{turb}} \right\rangle_{\psi} \frac{d \ln T_s}{d\psi} \\ &\quad + \left\langle \left\langle \int d^3\boldsymbol{w} m_s R^2 (\nabla \phi) \cdot \langle \boldsymbol{v} h_s \boldsymbol{V}_\chi \rangle_{\boldsymbol{r}} \cdot \nabla \psi \right\rangle_{\mathrm{turb}} \right\rangle_{\psi} \frac{d\omega}{d\psi}. \end{aligned} \tag{2.206}$$

This is the energy borrowed from the internal energy of the equilibrium and transferred into turbulence by such mechanisms as the ITG, PVG, ETG, and other gradient-driven instabilities [68, 25, 26, 28]. In §2.8, we will show that all of this energy is eventually returned to the equilibrium via the turbulent heating term: namely, after summation over species, net turbulent heating – the difference between $\sum_s P_s^{\mathrm{drive}}$ and $\sum_s P_s^{\mathrm{diss}}$ – is either viscous heating (conversion of rotational energy into heat at constant internal energy) or due to turbulent electromagnetic energy injected by antenna-like mechanisms, whose dissipation is the only piece of $\sum_s P_s^{\mathrm{diss}}$ not cancelled by $\sum_s P_s^{\mathrm{drive}}$.

Collisional Heating.

Formally, the collisional heating term $C_s^{(E)}$ is both too large – $\mathcal{O}(\epsilon \Omega_s n_s T_s)$ or $\mathcal{O}(\epsilon^2 \Omega_s n_s T_s)$ if all mean temperatures are equal, as $C[F_{0s}] \sim \epsilon \nu F_{0s}$ if all temperatures are equal – and contains contributions from the gyrophase-independent part of F_{2s}, which we have not calculated. This problem has arisen from our decision to retain different temperatures for different species, a choice formally inconsistent with our orderings, as we acknowledged in the footnote preceding (2.85). There are two naïve solutions to this problem. The first is to respect our ordering and assume that all temperatures are equal (which follows from (2.84)). In this case we can sum the heat transport equation (2.194) over all species and, as collisions conserve energy, the collisional heating term will vanish

$$\sum_s C_s^{(E)} = 0. \tag{2.207}$$

Alternatively, if, completely formally, we order the interspecies collision frequency to be small $\nu_{s\,s'} \sim \epsilon^2 \nu_{s\,s}$, then this heating term will simply be the temperature equilibration between the Maxwellian equilibrium distributions of the species [60]:

$$C_s^{(E)} = \sum_{s'} \nu_{s\,s'}^{(\mathrm{E})} \left(T_{s'} - T_s\right), \tag{2.208}$$

where

$$\nu_{s\,s'}^{(\mathrm{E})} = 2^{5/2}\pi^{1/2} e^4 \frac{m_s^{1/2} m_{s'}^{1/2} n_s n_{s'} Z_s^2 Z_{s'}^2 \ln \Lambda_{s\,s'}}{\left(m_s T_{s'} + m_{s'} T_s\right)^{3/2}}, \tag{2.209}$$

and $\ln \Lambda_{s\,s'}$ is the Coulomb logarithm. However, this stretches the credibility of our ordering as there is no physical reason for $\nu_{s\,s'}$ to be smaller than $\nu_{s\,s}$ in general. Only in the special case of a very light species (electrons) or a very highly charged species (large-Z_s impurities) is it reasonable to so order the temperature equilibration time. Thus, we elect to treat electrons separately but sum (2.194) over all other thermal species (ions), which should all have the same temperature.

2.8 Energy Conservation in Multiscale Gyrokinetics

In this section, we bring together the results of Sections 2.6 and 2.7 to determine how energy and entropy flow between mean and fluctuating quantities.

Firstly, in §2.8.1, we discuss energy conservation on the transport timescale. We show that the conserved quantity is the total energy of the plasma – the sum of the thermal energy of each species, the rotational energy and the magnetic energy. We also show that the fluctuations can do no net work upon the plasma, i.e., there is no net turbulent heating. [17]

Secondly, in §2.8.2, we derive the conservation laws that hold on the fluctuation timescale. We show that the conserved quantity is the free energy of the fluctuations. We then show that its conservation is local, i.e., neighbouring flux surfaces do not exchange free energy. We also use the steady-state version of this conservation law to

[17] This holds only in the absence of direct injection of energy into the fluctuating scales – as is (usually) the case in tokamaks.

interpret the absence of work done by the fluctuations (demonstrated in §2.8.1) as a precise balance between two energy flows: kinetic energy of the thermal particle motion and of the bulk plasma is converted into turbulent fluctuation energy via instabilities driven by mean-field gradients (e.g., ITG [25, 26], ETG [68], PVG [28, 29, 78]); it is then cascaded to small scales in phase space, where it is dissipated (converted back into thermal energy) by collisions.

Finally, in §2.8.3, we discuss why the gradients of the magnetic field cannot drive fluctuations on their own and note the impossibility of anomalous resistivity in our formalism.

2.8.1 Energy Conservation on the Transport Timescale

Thermal Energy.

We begin by considering the evolution of the total thermal energy. In Appendix E.1, we sum (2.194) over all species and simplify the result to obtain:

$$\frac{3}{2}\frac{1}{V'}\left.\frac{\partial}{\partial t}\right|_{\psi} V'\sum_s \langle n_s\rangle_{\psi} T_s + \frac{1}{V'}\frac{\partial}{\partial\psi}V'\left[\sum_s\left(\langle q_s\rangle_{\psi} - T_s\left\langle n_s\frac{\partial\psi}{\partial t}\right\rangle_{\psi}\right) - \langle\varphi_0\boldsymbol{j}\cdot\nabla\psi\rangle_{\psi}\right]$$
$$= -\left\langle\pi^{(\psi\phi)}_{\rm tot}\right\rangle_{\psi}\frac{d\omega}{d\psi} + \frac{\omega^2(\psi)}{2V'}\left.\frac{\partial}{\partial t}\right|_{\psi} V'J + \langle\boldsymbol{E}\cdot\boldsymbol{j}\rangle_{\psi}$$
$$+\left\langle\pi^{(\psi\phi)}_{\rm EM}\right\rangle_{\psi}\frac{d\omega}{d\psi} + \sum_s P^{\rm turb}_s + \sum_s\left\langle S^{(E)}_s - \frac{1}{2}m_s\omega^2(\psi)R^2 S^{(n)}_s\right\rangle_{\psi}, \tag{2.210}$$

where J is the plasma's moment of inertia, defined in (2.180), $\pi^{(\psi\phi)}_{\rm tot}$ is defined in (2.181), $\pi^{(\psi\phi)}_{\rm EM}$ in (2.188), and $P^{\rm turb}_s$ in (2.204). Examining the terms on the right-hand side of the above equation, we identify the first and second as exchange between thermal and rotational energy, the third as the Ohmic heating associated with the mean fields, and the last term as the energy injected by the heat and particle sources. We show in Appendix E.2 (and, by a different route, in §2.8.2) that the fourth and fifth terms on the right-hand side of (2.210), due to the fluctuations, cancel exactly (see (E.15)):

$$\sum_s P^{\rm turb}_s + \left\langle\pi^{(\psi\phi)}_{\rm EM}\right\rangle_{\psi}\frac{d\omega}{d\psi} = 0. \tag{2.211}$$

Thus, the heating due to turbulence arises entirely out of the turbulent contribution to $\pi^{(\psi\phi)}_{\rm tot}$, which results in viscous heating in (2.210) (the first term on the right-hand side) – converting rotational energy into thermal energy or vice versa.

Rotational Energy.

The last point is made manifest by considering the evolution of the rotational kinetic energy. Multiplying (2.179) by $\omega(\psi)$, we obtain

$$\begin{aligned}\frac{1}{2}\frac{1}{V'}\left.\frac{\partial}{\partial t}\right|_{\psi} V'J\omega^2(\psi) + \frac{1}{V'}\frac{\partial}{\partial\psi}V'\omega(\psi)\left\langle \pi_{\rm tot}^{(\psi\phi)}\right\rangle_{\psi} \\ = \left\langle \pi_{\rm tot}^{(\psi\phi)}\right\rangle_{\psi}\frac{d\omega}{d\psi} - \frac{\omega^2(\psi)}{2V'}\left.\frac{\partial}{\partial t}\right|_{\psi} V'J + \omega(\psi)\left\langle S^{(\omega)}\right\rangle_{\psi},\end{aligned} \tag{2.212}$$

where we have written the left-hand side as an exact time derivative and an exact spatial derivative and collected the terms arising from this manipulation on the right-hand side so as to parallel (2.210). We now add (2.212) to (2.210) to find

$$\begin{aligned}\frac{1}{V'}\left.\frac{\partial}{\partial t}\right|_{\psi} V'U + \frac{1}{V'}\frac{\partial}{\partial\psi}V'\left\langle \Gamma^{(U)}\right\rangle_{\psi} = \langle \boldsymbol{E}\cdot\boldsymbol{j}\rangle_{\psi} \\ + \sum_s \left\langle S_s^{(E)} + \omega(\psi)S^{(\omega)} - \frac{1}{2}m_s\omega^2(\psi)R^2 S_s^{(n)}\right\rangle_{\psi},\end{aligned} \tag{2.213}$$

where the total kinetic energy of the plasma (thermal energy and rotational kinetic energy) is

$$U = \sum_s \frac{3}{2}\langle n_s\rangle_{\psi} T_s + \frac{1}{2}J\omega^2(\psi), \tag{2.214}$$

and its flux is

$$\begin{aligned}\left\langle \Gamma^{(U)}\right\rangle_{\psi} &= \sum_s\left(\langle q_s\rangle_{\psi} - T_s\left\langle n_s\frac{\partial\psi}{\partial t}\right\rangle_{\psi}\right) + \omega(\psi)\left\langle \pi_{\rm tot}^{(\psi\phi)}\right\rangle_{\psi} - \langle\varphi_0\boldsymbol{j}\cdot\nabla\psi\rangle_{\psi} \\ &= \left\langle\left\{\sum_s\left[\boldsymbol{Q}_s + \frac{1}{2}m_s\omega^2(\psi)R^2 n_s\boldsymbol{U}_s + \boldsymbol{u}\cdot\boldsymbol{\Pi}_s\right] + \frac{c}{4\pi}\langle\delta\boldsymbol{E}\times\delta\boldsymbol{B}\rangle_{\rm turb}\right\}\cdot\nabla\psi\right\rangle_{\psi} \\ &\quad + \left\langle U\frac{\partial\psi}{\partial t}\right\rangle_{\psi}.\end{aligned} \tag{2.215}$$

The two forms of $\left\langle\Gamma^{(U)}\right\rangle_{\psi}$ written above are shown to be equivalent in Appendix E.3. They serve two different purposes. The former is easier to evaluate for a specific practical implementation of the transport equations, while the latter is easier to interpret physically. In the second form of $\Gamma^{(U)}$, the terms are identified as: the heat flux (see footnote in §2.7.3 before (2.196)), the flux of rotational energy, the energy flux due to the viscous stress, the Poynting flux due to the fluctuations and a term due to the motion of flux surfaces.

Magnetic Energy.

The evolution equation (2.213) for U has two sources on the right-hand side: Ohmic heating and energy injection due to heat and particle sources. Let us now express the Ohmic heating term (the first term on the right-hand side of (2.213)) by using Poynting's theorem for the mean fields:

$$\frac{\partial}{\partial t}\frac{B^2}{8\pi} + \frac{c}{4\pi}\nabla\cdot(\boldsymbol{E}\times\boldsymbol{B}) = -\boldsymbol{E}\cdot\boldsymbol{j}. \tag{2.216}$$

Averaging this equation over a flux surface and adding it to (2.213), we obtain

$$\begin{aligned}\frac{1}{V'}\left.\frac{\partial}{\partial t}\right|_{\psi} V'\left(U + \frac{\langle B^2\rangle_\psi}{8\pi}\right) + \frac{1}{V'}\frac{\partial}{\partial\psi}V'\left(\left\langle\Gamma^{(U)}\right\rangle_\psi - \frac{c}{4\pi}I(\psi)\left\langle\boldsymbol{E}\cdot\boldsymbol{B}\right\rangle_\psi + \left\langle\frac{3B^2}{8\pi}\frac{\partial\psi}{\partial t}\right\rangle_\psi\right)\\ = \sum_s\left\langle S_s^{(E)} + \omega(\psi)S^{(\omega)} - \frac{1}{2}m_s\omega^2(\psi)R^2S_s^{(n)}\right\rangle_\psi,\end{aligned} \tag{2.217}$$

where we have used the flux-surface-average identities (2.39) and (2.45) and $\boldsymbol{B}\times\nabla\psi = -I(\psi)\boldsymbol{B} + R^2B^2\nabla\phi$ to rewrite the radial Poynting flux in the following way:

$$(\boldsymbol{E}\times\boldsymbol{B})\cdot\nabla\psi = -I(\psi)\boldsymbol{E}\cdot\boldsymbol{B} + R^2B^2\boldsymbol{E}\cdot\nabla\phi = -I(\psi)\boldsymbol{E}\cdot\boldsymbol{B} + \frac{B^2}{c}\frac{\partial\psi}{\partial t}. \tag{2.218}$$

Equation (2.217) is the desired energy conservation law on the transport timescale: there are no sources save those explicitly contained in the kinetic equation.

Note that, as (2.216) contains no contribution from the fluctuations, turbulence cannot cause any exchange between the total kinetic energy U and the energy of the mean magnetic field. These results, surprising at first glance, in fact follow directly from Poynting's theorem for the fluctuating fields:

$$\frac{\partial}{\partial t}\frac{\delta B^2}{8\pi} + \frac{c}{4\pi}\nabla\cdot(\delta\boldsymbol{E}\times\delta\boldsymbol{B}) = -\delta\boldsymbol{E}\cdot\delta\boldsymbol{j}, \tag{2.219}$$

where we have dropped the energy in the electric field as it is negligible for non-relativistic plasmas. Averaging (2.219) over the fluctuations gives

$$\frac{c}{4\pi}\nabla\cdot\langle\delta\boldsymbol{E}\times\delta\boldsymbol{B}\rangle_{\mathrm{turb}} = -\langle\delta\boldsymbol{E}\cdot\delta\boldsymbol{j}\rangle_{\mathrm{turb}}, \tag{2.220}$$

as the time derivative of the averaged fluctuating magnetic energy is ϵ^2 smaller than the other two terms. This clearly demonstrates that the work done on the particles by the fluctuating fields (the right-hand side) is balanced by the Poynting flux of fluctuating electromagnetic energy (the left-hand side), so the fluctuations cannot change the total kinetic energy U. The Poynting flux associated with the fluctuations appears explicitly in the second line of (2.215). If we wished to consider electromagnetic fluctuations driven by an external source, this would enter the energy-conservation equation not through a source term but through the fluctuating Poynting flux at the plasma boundary. We will return to the physical interpretation of this result in §2.8.3, after discussing the conservation of free energy in the next section.

2.8.2 Free Energy Conservation and the Turbulent Cascade

In this section, we derive the fluctuating counterpart to the energy conservation law of the previous section. The conserved (cascading) quantity for kinetic turbulence is not the energy of the fluctuations, but is in fact a combination of the entropy and the energy known as the free energy [38, 39, 40, 41]. In our notation, this quantity is

$$W(\psi) = \sum_s \left\langle \left\langle \int d^3\boldsymbol{w} \frac{T_s \delta f_s^2}{2F_{0s}} \right\rangle_\perp \right\rangle_\psi + \left\langle \left\langle \frac{\delta B^2}{8\pi} \right\rangle_\perp \right\rangle_\psi , \tag{2.221}$$

where we have averaged over an annular region of space centered on the flux surface labelled by ψ (the composition of the flux-surface average and the perpendicular spatial average)[18], but *not* over time. The first term in (2.221) is $-\sum_s T_s \delta S_s$, where δS_s is the part of the mean entropy density of species s due to the fluctuations (note that this is not the same as the fluctuating part of the entropy density because δS_s has a nonzero average; see §2.9.1). Thus, (2.221) agrees with the usual thermodynamic definition of Helmholtz free energy.

In Appendix E.4, we derive an evolution equation for W from the gyrokinetic equation (2.144):

$$\begin{aligned}
\left.\frac{\partial}{\partial t}\right|_\psi W = &\sum_s \left\langle \left\langle \int d^3\boldsymbol{w} \left\langle \frac{T_s h_s}{F_{0s}} C[h_s] \right\rangle_{\boldsymbol{r}} \right\rangle_\perp \right\rangle_\psi \\
&- \sum_s \left\langle \left\langle \int d^3\boldsymbol{w} \langle h_s \boldsymbol{V}_\chi \rangle_{\boldsymbol{r}} \right\rangle_\perp \cdot \nabla\psi \right\rangle_\psi T_s \left(\frac{d\ln N_s}{d\psi} - \frac{3}{2}\frac{d\ln T_s}{d\psi} \right) \\
&- \sum_s \left\langle \left\langle \int d^3\boldsymbol{w} \varepsilon_s \langle h_s \boldsymbol{V}_\chi \rangle_{\boldsymbol{r}} \right\rangle_\perp \cdot \nabla\psi \right\rangle_\psi \frac{d\ln T_s}{d\psi} \\
&- \left\langle R^2 \left(\nabla\phi\right) \cdot \left[\sum_s \left\langle \int d^3\boldsymbol{w} \langle \boldsymbol{v} h_s \boldsymbol{V}_\chi \rangle_{\boldsymbol{r}} \right\rangle_\perp - \left\langle \frac{1}{4\pi}\delta\boldsymbol{B}\delta\boldsymbol{B} + \frac{1}{c}\delta\boldsymbol{A}\delta\boldsymbol{j} \right\rangle_\perp \right] \cdot \nabla\psi \right\rangle_\psi \frac{d\omega}{d\psi} .
\end{aligned} \tag{2.222}$$

This is the free-energy balance on a given flux surface.

The Local Cascade Law.

Before interpreting the sources and sinks in this equation, we would like to highlight the fact that there is no flux of free energy – the turbulent cascade of free energy is essentially local and does not couple neighbouring flux surfaces. This means that turbulence excited on a given flux surface must dissipate on that surface and,

[18] For the purposes of this section, we can visualise a flux surface as having a finite width (comparable to the intermediate length scale λ introduced in §2.2.1) and so in our discussion we will drop the distinction between a flux surface and the annulus centered upon it.

moreover, that the turbulence is excited by the local gradients and cannot be excited by turbulence spreading from neighbouring flux surfaces. Succinctly: *that which is stirred up in the flux surface, stays in the flux surface* – the local (in space) cascade law of tokamak turbulence.

Sources and Sinks of Free Energy.

The first term on the right-hand side of (2.222) is the sink of free energy due to collisional dissipation[19]. The remaining source terms are the entropy generated by the turbulent fluxes transporting mean quantities along their gradients. As the fluxes are generally dominated by the largest turbulent scales and the collisional dissipation is most important when acting on the smallest scales (in phase space), we can interpret (2.222) in terms of a cascade in phase space analogous to the Kolmogorov cascade of energy in hydrodynamic turbulence [40, 41]: fluctuations are excited at large scales, then nonlinearly cascaded, whilst conserving W, to small scales, where they are destroyed by collisional dissipation.

It is important to note that only gradients in the thermodynamic and mechanical quantities (density, temperature and angular velocity) appear in the right-hand side of (2.222). Only gradients in those quantities represent deviations from the global thermodynamic equilibrium and can, therefore, drive fluctuations (as the presence of fluctuations implies non-zero W). Gradients in the magnetic field strength or magnetic curvature do not have this property; the relaxation of the mean magnetic field to a minimum energy state proceeds independently of the fluctuations (i.e., there is no fluctuating contribution to (2.216); see further discussion in §2.8.3).

Statistically Steady-State Turbulence.

Time averaging (2.222), we see that a balance between free-energy injection and dissipation is set up on a timescale much shorter than the intermediate time T (introduced for the purposes of time averaging in §2.2.1). This balance can be written as

$$\sum_s P_s^{\rm drive} - \left\langle \pi_{\rm EM}^{(\psi\phi)} \right\rangle_\psi \frac{d\omega}{d\psi} = \sum_s P_s^{\rm diss}, \tag{2.223}$$

where $P_s^{\rm drive}$ and $P_s^{\rm diss}$ are defined by (2.206) and (2.205), respectively, and $\pi_{\rm EM}^{(\psi\phi)}$ by (2.188). This is another derivation of (2.211). This clearly demonstrates that the steady-state turbulence extracts all its energy from the equilibrium, but it also returns exactly the same amount of energy via collisions. This explains how the fluctu-

[19]The fact that it is a sink and not a source follows from the H-Theorem for the linearised collision operator $C[h_s]$ (see §2.9.2).

ations can be continuously driven and dissipated without changing the total kinetic energy U. Turbulence can, however, move energy between various constituent parts of U, notably energy extracted from one species does not have to be returned to that species, and so fluctuations can redistribute energy between species (subject to the caveats about temperature equilibration at the end of §2.7.3). This can lead to systematic dependences of equilibrium temperatures on the nature of the species, e.g., in the case of effective heating of heavy minority ions [79].

Similarly, turbulence can convert rotational energy into thermal energy (or even vice versa, e.g., if there is an angular momentum pinch or intrinsic rotation [80, 81]).

Equation (2.222) implies that in order to sustain a non-zero steady-state flux of particles, heat or angular momentum, collisional dissipation is required. This is the reason, alluded to in §2.2.4, why it was crucial to retain $C[\delta f_s]$ at the same order as $\partial \delta f_s / \partial t$. Similarly, this is why all simulations of the gyrokinetic equation (2.144) have to include some dissipation in order to reach a steady state [38, 82, 83].

2.8.3 The Special Status of the Mean Magnetic Field

At several points in the above discussion, we have noted that the mean magnetic field behaves very differently from the other mean fields. In this section, we collect these individual points together to further elucidate the special role that the mean magnetic field plays in multiscale gyrokinetics.

The mean magnetic field is determined from the Grad-Shafranov equation (2.136) and the evolution equation for the safety factor $q(\psi)$ (2.143) (see the detailed discussion in §2.6.3). These equations contain no direct contribution from the fluctuations. In this sense they are very different from the transport equations determining the mean density, velocity, and temperature profiles. This separation is also reflected in the energetics of the magnetic field. The fluctuations cannot directly convert magnetic energy into thermal energy (note the absence of any fluctuation-dependent term in (2.216)). Correspondingly, the magnetic field is not a source of turbulent free energy (there are no source terms in (2.222) arising from gradients in $\boldsymbol{B}$), i.e., energy cannot be borrowed from the magnetic field to fuel fluctuations in the same way that it can be from the thermal or kinetic energy of the plasma.

Ultimately, these results imply that there is no turbulent contribution to the diffusion of mean magnetic flux. The fluctuations considered here cannot reconnect the mean magnetic field, i.e., they can change the topology of the total magnetic field $\widetilde{\boldsymbol{B}} = \boldsymbol{B} + \delta\boldsymbol{B}$ but *only* on small spatial and temporal scales, while the topology of $\boldsymbol{B}$ is preserved – indeed this is true automatically as we assumed axisymmetry of the equilibrium (see §2.2.2). Equivalently, we can say that there is no anomalous resistivity due to gyrokinetic turbulence (i.e., the evolution of $q(\psi)$, given by (2.143), cannot be sped up by turbulence).

2.9 Entropy Flows and the Thermodynamics of Multiscale Gyrokinetics

In the preceeding section, we have established the nature of the energy flows, both between the equilibrium fields and the fluctuations, and between neighbouring flux surfaces. In this section, we discuss these results in explicitly thermodynamic language.

In §2.9.1, we derive an evolution equation for the mean entropy and show that this is not only consistent with the above results but also provides further evidence that flux surfaces interact only through particle, momentum and heat fluxes (as opposed to turbulence spreading). In §2.9.2, we relate the evolution of the mean entropy to notions of dissipation and equilibration through Boltzmann's H-Theorem. In §2.9.3, we relate the sources of mean entropy to sinks of free energy discussed previously. We discover that dissipation increases mean entropy via three routes: collisional and turbulent temperature equilibration, Ohmic heating due to the induced electric field, and fluxes that strive to flatten gradients in the thermodynamic quantities n_s, T_s and $\omega(\psi)$. We also discuss the effects that can occur in systems where two or more equilibration mechanisms compete.

2.9.1 Entropy Balance

The (Boltzmann) entropy density of species s is given by

$$\widetilde{H}_s = -\int d^3\boldsymbol{w} f_s \ln\left(\frac{8\pi^3\hbar^3 f_s}{m_s^3}\right), \qquad \widetilde{H} = \sum_s \widetilde{H}_s, \tag{2.224}$$

where $\hbar$ is Planck's constant.[20] To lowest order in ϵ, $f_s = F_{0s}$ and so the mean entropy density of the plasma is

$$H = \left\langle \widetilde{H} \right\rangle_{\text{turb}} = -\sum_s n_s \left[\ln\left(\frac{n_s}{n_{Qs}}\right) - \frac{3}{2}\right] + \mathcal{O}(\epsilon H), \tag{2.225}$$

where $n_{Qs} = (m_s T_s / 2\pi\hbar^2)^{3/2}$ is the quantum density of states for the ideal gas of species s. This is just the standard result for a mixture of ideal gases in local thermodynamic equilibrium [77], as expected. We are in local thermodynamic equilibrium:

[20]For an explanation of this suprise appearance of $\hbar$ in a classical formula see §7 of [77].

f_s is Maxwellian to lowest order (the equilibrium is local, not global, as n_s and T_s are functions of space). Because we are considering a given flux surface, the presence of the rigid rotation $\omega(\psi)$ does not contribute to the entropy density at this surface. Expanding $f_s = F_s + \delta f_s$, we find an expression for the mean entropy density that retains terms up to order ϵ^2:

$$H = -\sum_s \int d^3\boldsymbol{w} F_s \ln\left(\frac{8\pi^3\hbar^3 F_s}{m_s^3}\right) - \sum_s \left\langle \int d^3\boldsymbol{w} \frac{\delta f_s^2}{2F_s} \right\rangle_{\text{turb}} + \mathcal{O}(\epsilon^3 H), \tag{2.226}$$

where the first term is the entropy of the mean distribution F_s and the second term is the contribution from the fluctuations, δS_s, which was introduced in the discussion of free-energy conservation (see (2.221); in (2.226), δS_s is also time-averaged).

We now derive an evolution equation for H. Multiplying the kinetic equation (2.1) by $-\left[1 + \ln\left(8\pi^3\hbar^3 f_s/m_s^3\right)\right]$, integrating over all velocities, summing over species, averaging over the fluctuations and over a flux surface, we obtain

$$\frac{1}{V'}\frac{\partial}{\partial t}\bigg|_\psi V'\left\langle H\right\rangle_\psi + \frac{1}{V'}\frac{\partial}{\partial\psi}V'\left\langle \Gamma^{(H)}\right\rangle_\psi = \left\langle C^{(H)}\right\rangle_\psi + \left\langle S^{(H)}\right\rangle_\psi, \tag{2.227}$$

where we have used (2.45) to simplify the time derivative, (2.39) for the flux-surface average of a divergence, and defined the radial entropy flux

$$\Gamma^{(H)} = -\sum_s \left\langle \int d^3\boldsymbol{w}\boldsymbol{v} f_s \ln\left(\frac{8\pi^3\hbar^3 f_s}{m_s^3}\right)\right\rangle_{\text{turb}} \cdot \nabla\psi + H\frac{\partial\psi}{\partial t}, \tag{2.228}$$

the collisional entropy production

$$C^{(H)} = -\sum_s \left\langle \int d^3\boldsymbol{w} \ln\left(\frac{8\pi^3\hbar^3 f_s}{m_s^3}\right) C[f_s]\right\rangle_{\text{turb}}, \tag{2.229}$$

and the explicit entropy source

$$\begin{aligned} S^{(H)} &= -\sum_s \left\langle \int d^3\boldsymbol{w}\left[1 + \ln\left(\frac{8\pi^3\hbar^3 f_s}{m_s^3}\right)\right] S_s \right\rangle_{\text{turb}} \\ &= -\sum_s \left[\ln\left(\frac{n_s}{n_{Qs}}\right) S_s^{(n)} - \frac{3}{2}\frac{S_s^{(E)}}{T_s}\right]. \end{aligned} \tag{2.230}$$

In Appendix E.5, we show that the entropy flux (2.228) can be written in terms of the particle and heat fluxes through a flux surface as follows

$$\left\langle \Gamma^{(H)}\right\rangle_\psi = -\sum_s \left[\left(1 + \frac{\Upsilon_s}{T_s}\right)\left\langle \Gamma_s\right\rangle_\psi + \frac{\left\langle q_s\right\rangle_\psi}{T_s}\right], \tag{2.231}$$

where we have introduced the chemical potential of an ideal gas of particles of species s:

$$\Upsilon_s(\psi) = T_s \ln\left(\frac{n_s}{n_{Qs}}\right) + Z_s e\varphi_0 - \frac{1}{2}m_s\omega^2(\psi)R^2 = T_s \ln\left(\frac{N_s}{n_{Qs}}\right), \tag{2.232}$$

(we used (2.96) to obtain the last equality; note that the last two terms of the first equality are the potential energy per particle). Equation (2.231) is analogous to the same result for a mixture of ideal gases [84, 85]. We have already seen, in §2.7, that each annulus interacts with neighbouring annuli via the particle, momentum, and heat fluxes. The expression (2.231) for the entropy flux shows that the annuli exchange entropy only through the particle and heat fluxes (but not momentum, as we should expect, given that the mean entropy density does not depend upon $\omega(\psi)$).

By expanding f_s in terms of F_s and δf_s, we can write the mean (collisional) entropy production term as

$$\begin{aligned} \left\langle C^{(\mathrm{H})}\right\rangle_\psi = &-\sum_s \int d^3\boldsymbol{w} \left\langle \ln\left(\frac{8\pi^3\hbar^3 F_s}{m_s^3}\right) C[F_s]\right\rangle_\psi \\ &-\sum_s \int d^3\boldsymbol{w} \left\langle \left\langle \frac{h_s}{F_{0s}} C[h_s]\right\rangle_{\mathrm{turb}}\right\rangle_\psi + \mathcal{O}(\epsilon^4\Omega_s H), \end{aligned} \tag{2.233}$$

where we have used (2.104) for δf_s. In this expression, we can clearly identify the separate contributions from collisional dissipation of F_s and collisional dissipation of δf_s. It is important to note that the second term in (2.233) is minus the time average of the first term on the right hand side of (2.222) – the rate of entropy production due to the fluctuations is precisely the rate at which free energy is dissipated.

We determined in §2.8 that the energy budget for the fluctuations consists of borrowing some energy from the mean thermal or bulk kinetic energy, cascading it to small scales, and then returning it. We also showed that in this process no net work was done on the plasma. We seem to have arrived at a contradictory result: we know that, in the absence of viscous heating, there is no local heating, but (2.233) shows that we have a monotonically increasing entropy (the right-hand side of (2.233) is positive-definite if we are away from thermodynamic equilibrium). This apparent contradiction is resolved in the next two sections, where we consider the precise nature of the thermodynamic equilibration of our system.

2.9.2 Boltzmann's H-Theorem and Global Equilibrium

Boltzmann's H-Theorem [86] states that for a homogenous dilute gas of hard spheres, the entropy $\widetilde{H}_s$, as defined by (2.224), is non-decreasing and the only steady-state distribution is a Maxwellian. Let us examine the corresponding theorem for the plasmas under consideration here. The general form of the H-Theorem for a dilute plasma is: if $f_s(\boldsymbol{r},\boldsymbol{v})$ is the distribution function for species s, then [60]

$$\sum_s \int d^3\boldsymbol{v} \ln f_s\, C[f_s] \le 0, \tag{2.234}$$

with equality achieved if and only if for each species f_s is a Maxwellian distribution and all their temperatures and mean velocities are equal. We have already called upon this result in §2.5.1 to prove that F_{0s} is Maxwellian. We now examine its implications for the thermodynamics of the plasma.

First of all, we see that $C^{(H)} \geq 0$, i.e., collisions generate entropy. Moreover, the two terms in the expression (2.233) for $C^{(H)}$ are separately non-negative. Thus, the relaxation of the equilibrium F_s and the dissipation of the turbulent free energy are separate net sources of entropy.

It is instructive to discuss the equilibrium towards which these relaxation process are driving the system. In the absence of sources or fluxes of entropy connecting the plasma to the outside world, the system is in a global equilibrium if $C^{(H)}$ vanishes when integrated over the plasma volume. However, as $\left\langle C^{(H)} \right\rangle_\psi$ is positive definite on each flux surface, for its volume integral to vanish, it must vanish locally on each surface. From the H-Theorem, we know that this is the case only if f_s is a Maxwellian up to possible errors of $\mathcal{O}(\epsilon^2 f_s)$ (these smaller deviations from a Maxwellian generate an evolution of the entropy on timescales longer than τ_E and so are negligible). By applying the H-theorem to the first term of (2.233), we see that F_s must be Maxwellian for $C^{(H)}$ to vanish. From the explicit expression (B.7) for F_s (Appendix B), we conclude that for F_s to be Maxwellian, all the gradients in F_{0s} (i.e. those of n_s, T_s and ω) must vanish and the neoclassical distribution function $F_s^{(\mathrm{nc})}$ must also be Maxwellian (in the absence of gradients in F_{0s}, (2.119), which determines $F_s^{(\mathrm{nc})}$, admits purely Maxwellian solutions).[21] Thus, collisions, which drove the distribution on each flux surface to be Maxwellian on shorter timescales, now strive to smooth out the gradients between the flux surfaces.

The H-theorem has determined the form of the background distribution function, the direction of the free-energy cascade *from* injection scales *to* collisional scales in phase space, and it has determined the ultimate equilibrium to which the system wishes, in some sense, to arrive at. There remains but one issue to clarify: how the relaxation towards this global equilibrium is achieved.

2.9.3 Approach to Equilibrium and Multichannel Transport.

The precise nature of the relaxation process is hidden inside the entropy production terms in (2.233). In Appendix E.6, we show that the collisional dissipation in (2.233) can be expressed entirely in terms of fluxes through a flux surface and local

[21] Even though we do not have an evolution equation for it, $F_s^{(\mathrm{nc})}$ evolves in time due to the time evolution of the right-hand side of (2.119) and the time evolution of the collision frequency.

relaxation terms[22] [7, 8]:

$$\left\langle C^{(\mathrm{H})}\right\rangle_\psi = \sum_s \frac{1}{T_s}\left(\left\langle C_s^{(E)}\right\rangle_\psi + P_s^{\mathrm{turb}} + P_s^{\mathrm{Ohm}}\right)$$
$$- \sum_s \left\{ \langle \Gamma_s \rangle_\psi \frac{d}{d\psi}\frac{\Upsilon_s}{T_s} + \frac{\langle q_s \rangle_\psi}{T_s}\frac{d\ln T_s}{d\psi} + \frac{1}{T_s}\left[\left\langle \pi_s^{(\psi\phi)}\right\rangle_\psi + m_s\omega(\psi)\left\langle R^2\Gamma_s\right\rangle_\psi\right]\frac{d\omega}{d\psi}\right\}, \tag{2.235}$$

where $C_s^{(E)}$ is defined in (2.192), P_s^{turb} in (2.204), P_s^{Ohm} in (2.198), $\langle \Gamma_s \rangle_\psi$ in (2.170), Υ_s in (2.232), $\langle q_s \rangle_\psi$ in (2.196), $\left\langle \pi_s^{(\psi\phi)}\right\rangle_\psi$ in (2.184), and $\langle R^2\Gamma_s \rangle_\psi$ in (2.187). Note that

$$\frac{d}{d\psi}\frac{\Upsilon_s}{T_s} = \frac{d\ln N_s}{d\psi} - \frac{3}{2}\frac{d\ln T_s}{d\psi}. \tag{2.236}$$

We now interpret (2.235), term by term.

The first term is just the collisional equilibration of mean temperatures; it vanishes if all temperatures are equal.

The second term contains two effects. Firstly, if all temperatures are equal ($T_s = T$), then we can use (2.211) to write this term as a viscous heating term due to $\left\langle \pi_{\mathrm{EM}}^{(\psi\phi)}\right\rangle_\psi$ and combine it with the last term in (2.235) to form the total viscous heating $-(1/T)\left\langle \pi_{\mathrm{tot}}^{(\psi\phi)}\right\rangle_\psi d\omega/d\psi$. Secondly, if there is no rotation ($\omega = 0$) then we find that, from (2.211), $\sum_s P_s^{\mathrm{turb}} = 0$. Therefore, in the absence of rotation, this term represents a change in entropy not due to net heating but due to the *differential* heating of different species by the turbulence. Hence, in the absence of rotation, this term is a source of entropy if it equilibrates temperatures but a sink of entropy if it drives them apart. That this term can generate entropy without net heating is part of the resolution of the apparent contradiction noted at the end of §2.9.1.

The third term in (2.235) is the Ohmic heating due to the induced parallel electric field.

The remaining terms in (2.235) (the second sum) can be interpreted as fluxes multiplying their corresponding gradients.

The first term contains the particle flux and the gradient of the local chemical potential. The appearance of the chemical potential is natural as we can consider the flux surface to be a thermodynamic system that can gain or lose particles.

The second term is just the heat flux (in the extended definition adopted in §2.7.3) multiplying the temperature gradient.

[22] We note that this is a somewhat arbitrary distinction. For example, the viscous heating (the last term in (2.235)) can be considered as a flux-gradient term for the flux of angular momentum, or a local heating term.

Finally, the term multiplying the angular velocity gradient is not precisely the momentum flux as the electromagnetic momentum flux $\pi^{(\psi\phi)}_{\mathrm{EM}}$ is hidden within P^{turb}_s as discussed above. This ambiguity arises because, if the temperatures are unequal, it is impossible to separate uniquely the temperature equilibration due to turbulence driven by angular velocity shear and viscous heating due to the angular momentum flux driven by the turbulence.

Individually, these flux-gradient terms are positive for any flux that acts to flatten the corresponding gradient, and negative if the flux steepens the gradient.[23] The presence of turbulent contributions to the fluxes in these terms is the second part of the resolution of the contradiction that was raised at the end of §2.9.1 – the fluctuations can increase entropy without heating the plasma by flattening gradients. From (2.234), we know that $C^{(H)}$ is positive. However, it is clearly possible to have some negative terms in (2.235) without violating this constraint – up-gradient fluxes of particles, heat, or momentum are examples of just such negative terms. It is the multichannel nature of the transport that allows such phenomena: offsetting a pinch in one channel against a larger outward flux in another to observe a net increase in entropy.

Stating that the transport is multi-channel simply means that gradients in any one thermodynamic quantity (n_s, T_s, or ω) can drive fluxes of any other quantity. For example, trapped-electron-mode turbulence can be driven predominantly by the electron density gradient but produces both a particle flux and an electron heat flux [87]. Combining this multi-channel property and the fact that the terms due to the fluctuations in (2.235) can have differing signs and are only constrained to be positive upon summation can result in suprising effects – such as the spontaneous steepening of a gradient in ω merely from temperature-gradient driven turbulence [81].

Finally, we note that, as expected from the discussions in §2.8.3, gradients of the mean magnetic field do not appear as sources of entropy.

2.10 Multiscale Gyrokinetics at Low Mach Number

In Sections 2.3–2.7, a multiscale hierarchy of equations is derived from the Fokker-Planck kinetic equation by a systematic expansion in $\epsilon = \rho/a$. The full generality of these equations is, in fact, unnecessary for a large class of plasma conditions, namely

[23] Take, for example, the first term in the sum, which is the entropy production arising from the particle flux $\langle \Gamma_s \rangle_\psi$ and the density gradient $d \ln N_s / d\psi$; assume all other gradients are zero. If ψ increases away from the magnetic axis, $\langle \Gamma_s \rangle_\psi$ will be positive for a flux or particles that is going away from the magnetic axis and $d \ln N_s / d\psi$ will be negative for a density profile that is peaked on-axis. Thus, their contribution to the entropy production $-\sum_s \langle \Gamma_s \rangle_\psi \left(d \ln N_s / d\psi \right)$ will be positive for a flux that relaxes the gradient, and negative for one that steepens it. This is independent of the definition of ψ as it enters in both terms and the effect of flipping the sign of ψ cancels between the two.

those where the Mach number $M = R\omega(\psi)/v_{\text{th}_s}$ of the toroidal rotation is low, i.e., $M \ll 1$. In this section, we investigate the low-Mach-number limit: formulating the expansion in $M \ll 1$ precisely in §2.10.1 before presenting the results of this expansion in Sections 2.10.2–2.10.6. It is not necessary to read the detailed derivations in Sections 2.3–2.8 and their attendant appendices to use the results presented this section; however familiarity with the notation of Sections 2.1 and 2.2 will be assumed. Because of the considerable simplifications that the low-Mach-number expansion brings, the exposition in this section can be read as an easier-to-grasp summary of the basic structure of multiscale gyrokinetics.

2.10.1 The Low-Mach-Number Ordering

There are two distinct low-Mach-number regimes. In the extreme limit of $M \sim \epsilon$, all dependence on the angular velocity ω drops out of the equations for both the mean and fluctuating fields, and (2.179) is no longer sufficient to calculate angular momentum transport. This is the "low-flow regime" discussed in detail in [88, 43]. In this chapter, we deal instead with the intermediate limit where $\epsilon \ll M \ll 1$. This is the expected parameter regime for many current and future fusion experiments (see Table. 1.1 in §1.1).

There are two physical effects of plasma rotation that survive in this limit. They are, firstly, the suppression of small-scale turbulence by the perpendicular flow shear [89, 90] [24] (the $\boldsymbol{u} \cdot \partial h_s/\partial \boldsymbol{R}_s$ term in (2.144)) and, secondly, the instability drive due to the parallel flow shear that can itself excite turbulence [28, 29, 78] (the $d\omega/d\psi$ term on the right-hand-side of (2.144)). We retain the first of these terms by ordering the fluctuating frequency in the plasma frame

$$\left(\frac{\partial}{\partial t} + \boldsymbol{u} \cdot \nabla\right) \sim \frac{v_{\text{th}_s}}{a} \tag{2.237}$$

independently of the Mach number. The shear in the flow is hidden within the spatial dependence of the velocity. To make the presence of perpendicular flow shear more transparent, we assume that the domain of interest is shorter than the length scale of the flow and then expand the velocity around its value $\boldsymbol{u}_0$ at the centre of the domain

[24] The physics of the suppression of turbulence by sheared flows was originally discussed in these papers in the context of poloidal flows. The crucial fact is not that the flow shear is poloidal, but that it is perpendicular to the magnetic field. Thus, the same physical mechanisms (shearing apart of eddies, tilting of eddies due to the flow shear) apply here, despite the fact that the perpendicular shear comes from a purely toroidal flow.

$\boldsymbol{R}_{s0}$:

$$\left(\frac{\partial}{\partial t} + \boldsymbol{u}(\boldsymbol{R}_s) \cdot \frac{\partial}{\partial \boldsymbol{R}_s}\right) h_s = \left(\frac{\partial}{\partial t} + \boldsymbol{u}_0 \cdot \frac{\partial}{\partial \boldsymbol{R}_s}\right) h_s + (\boldsymbol{R}_s - \boldsymbol{R}_{s0}) \cdot [\nabla \boldsymbol{u}(\boldsymbol{R}_{s0})] \cdot \frac{\partial h_s}{\partial \boldsymbol{R}_s} + \cdots . \tag{2.238}$$

The second term on the right-hand side of this expression is now explicitly the action of the perpendicular flow shear upon the distribution function h_s. To retain only the part of this term due to shear in ω, we formally order the gradients of $\omega(\psi)$ to be sharp: $a\nabla \ln \omega(\psi) \sim M^{-1}$ so that $d\omega/d\psi$ is zeroth-order in M. For conciseness, we will continue to write the advection term unexpanded as $\boldsymbol{u}(\boldsymbol{R}_s)\partial/\partial\boldsymbol{R}_s$, but the reader should keep in mind the idea of solving these equations in a small domain and Taylor-expanding the mean velocity field.

This ordering for the gradients of ω is also precisely the ordering required to retain the parallel-velocity-gradient drive term on the right-hand side of the gyrokinetic equation. We also order the timescale of variation of $\omega(\psi, t)$ as $M\tau_E$ rather than τ_E, in order to retain the transport-time variation of the velocity. With these orderings, we will be able to keep the two effects we wish to examine and neglect all other effects of rapid toroidal rotation, e.g., Coriolis and centrifugal forces. In the following sections, we will apply this low-Mach-number expansion to the multiscale hierarchy derived in the preceding sections and present the ensuing equations to leading order in M.

2.10.2 Mean Fields

First we derive the equations for the mean distribution function F_s and the mean fields $\boldsymbol{E}$ and $\boldsymbol{B}$.

In the low-Mach-number limit, the mean distribution function is given by (see §2.10.3 for details)

$$F_s = F_{0s}\left(\psi(\boldsymbol{R}_s), \varepsilon_s\right) + \widehat{F}_{1s}\left(\boldsymbol{R}_s, \varepsilon_s, \mu_s, \sigma\right) + \frac{I(\psi) w_\parallel}{\Omega_s} \frac{\partial F_{0s}}{\partial \psi} + \mathcal{O}\left(\epsilon M f_s\right), \tag{2.239}$$

$$F_{0s} = n_s(\psi(\boldsymbol{R}_s)) \left[\frac{m_s}{2\pi T_s(\psi(\boldsymbol{R}_s))}\right]^{3/2} e^{-\varepsilon_s/T_s(\psi(\boldsymbol{R}_s))}, \tag{2.240}$$

where n_s and T_s are the density and temperature of species s, the energy variable is now given by (cf. (2.70))

$$\varepsilon_s = \frac{1}{2} m_s w^2 + \mathcal{O}(M^2 T_s), \tag{2.241}$$

and $\boldsymbol{R}_s$ and μ_s are still given by (2.64) and (2.66), respectively. The distinction between the particle velocity $\boldsymbol{v}$ and the peculiar velocity $\boldsymbol{w} = \boldsymbol{v} - \boldsymbol{u}$ can now be dropped, and so we use $\boldsymbol{w}$ as the particle velocity throughout this section. The neoclassical part of the mean distribution function, $\widehat{F}_{1s}$, is found from (2.118) and (2.120)–(2.122), as before.

The mean electric field is

$$\boldsymbol{E} = -\nabla\Phi(\psi) + \mathcal{O}\left(\epsilon M^2 \frac{v_{\mathrm{th}_s}}{c} B\right), \tag{2.242}$$

where Φ is related to the angular velocity $\omega(\psi)$ through (2.61). The mean magnetic field is given by the two functions $\psi(R, z)$ and $I(\psi)$ via (2.31). In the low-Mach-number limit, (2.136) becomes the usual Grad-Shafranov equation:

$$\Delta^*\psi = -4\pi R^2 \sum_s \frac{dp_s}{d\psi} - I(\psi)\frac{dI}{d\psi}, \tag{2.243}$$

where $p_s = n_s T_s$ is the pressure of species s, which is now a flux function (see §2.10.3). The second component of the magnetic field, $I(\psi)$, is determined by the same method as in the high-flow regime – this is detailed in §2.6.3.

2.10.3 Vanishing of Centrifugal Effects

As an example of how these results have been obtained, let us prove explicitly that φ_0 and the poloidal variation of the density (see §2.5.3) vanish in the low-Mach-number limit. First, we expand the lowest-order quasineutrality condition (2.97) in powers of M to find

$$\sum_s Z_s N_s(\psi) \exp\left[-\frac{Z_s e \varphi_0}{T_s(\psi)}\right] = \mathcal{O}\left(M^2 n_s\right). \tag{2.244}$$

Since the only spatial dependence in (2.244) is via ψ, we know that the solution (for φ_0, given N_s) will be of the form $\varphi_0 = \varphi_0(\psi) + \mathcal{O}(T_s M^2/Z_s e)$. However, we have defined φ_0 so that it has no flux-surface average (see (2.58) and (2.59)). Thus, φ_0 must vanish to lowest order in M^2. Therefore, from (2.96), we find $n_s = N_s(\psi) + \mathcal{O}(M^2 n_s)$. Thus, n_s is a flux function to the required order in M and we can drop the distinction between N_s and n_s. The quasineutrality constraint (2.244) then reduces to

$$\sum_s Z_s n_s(\psi) = 0. \tag{2.245}$$

2.10.4 Fluctuations

In the low-Mach-number limit, the fluctuating distribution function is (cf. (2.104))

$$\delta f_s = -\frac{Z_s e}{T_s}\delta\varphi(\boldsymbol{r})F_{0s} + h_s\left(\boldsymbol{R}_s, \varepsilon_s, \mu_s, \sigma\right) + \mathcal{O}\left(\epsilon M f\right), \tag{2.246}$$

where $\delta\varphi$ is the fluctuating electrostatic potential, which determines the fluctuating electric field to the required order:

$$\delta\boldsymbol{E} = -\nabla\delta\varphi + \mathcal{O}\left(\frac{v_{\mathrm{th}_s}}{c}\epsilon MB\right). \tag{2.247}$$

The gyrokinetic distribution function h_s is given by the low-Mach-number gyrokinetic equation, which is the same as (2.144) but with centrifugal and Coriolis effects neglected:

$$\begin{aligned}\left[\frac{\partial}{\partial t} + \boldsymbol{u}(\boldsymbol{R}_s)\cdot\frac{\partial}{\partial \boldsymbol{R}_s}\right] h_s + \left(w_\parallel \boldsymbol{b} + \boldsymbol{V}_{\mathrm{D}s} + \langle \boldsymbol{V}_\chi\rangle_{\boldsymbol{R}}\right)\cdot\frac{\partial h_s}{\partial \boldsymbol{R}_s} - \langle C[h_s]\rangle_{\boldsymbol{R}} \\ = \frac{Z_s e F_{0s}}{T_s}\left[\frac{\partial}{\partial t} + \boldsymbol{u}(\boldsymbol{R}_s)\cdot\frac{\partial}{\partial \boldsymbol{R}_s}\right]\langle\chi\rangle_{\boldsymbol{R}} - \left[\frac{\partial F_{0s}}{\partial\psi} + \frac{m_s I(\psi) w_\parallel F_{0s}}{T_s B}\frac{d\omega}{d\psi}\right]\langle \boldsymbol{V}_\chi\rangle_{\boldsymbol{R}}\cdot\nabla\psi,\end{aligned} \tag{2.248}$$

where the guiding-centre drift velocity is now (cf. (2.111))

$$\boldsymbol{V}_{\mathrm{D}s} = \frac{\boldsymbol{b}}{\Omega_s}\times\left(w_\parallel^2 \boldsymbol{b}\cdot\nabla\boldsymbol{b} + \frac{1}{2}w_\perp^2\nabla\ln B\right) + \mathcal{O}(\epsilon M v_{\mathrm{th}_s}). \tag{2.249}$$

The definition of χ is now

$$\chi = \delta\varphi - \frac{1}{c}\boldsymbol{w}\cdot\delta\boldsymbol{A}, \tag{2.250}$$

because $\delta\varphi' \approx \delta\varphi$ in the low-Mach-number limit. The definitions (2.110) of $\boldsymbol{V}_\chi$ and (2.72) of the gyroaverage remain unchanged.

The equation for h_s is closed through constitutive relations for the fluctuating fields. The fluctuating potential $\delta\varphi$ obeys the quasineutrality condition (cf. (2.146)):

$$\sum_s \frac{Z_s^2 e^2 n_s \delta\varphi}{T_s} = \sum_s Z_s e \int d^3\boldsymbol{w}\langle h_s\rangle_{\boldsymbol{r}} \tag{2.251}$$

and the fluctuating magnetic field is $\delta\boldsymbol{B} = \delta B_\parallel \boldsymbol{b} + \boldsymbol{b}\times\nabla\delta A_\parallel$, with $\delta A_\parallel$ and $\delta B_\parallel$ determined from the parallel (2.149) and perpendicular (2.153) components of Ampère's law (2.153), respectively.

2.10.5 Transport

The long-time evolution of the densities n_s and temperatures T_s are given by the low-Mach-number limits of the transport equations (2.166), (2.179), and (2.194) of §2.7:[25]

$$\frac{1}{V'}\frac{\partial}{\partial t}\bigg|_\psi V' n_s + \frac{1}{V'}\frac{\partial}{\partial\psi}V'\left\langle\Gamma_s\right\rangle_\psi = \left\langle S_s^{(n)}\right\rangle_\psi, \tag{2.252}$$

[25] With the transport-time variation of $\omega(\psi)$ ordered as M^{-1} as discussed in §2.10.1, all terms in (2.179) are the same order.

$$\frac{1}{V'}\left.\frac{\partial}{\partial t}\right|_{\psi} V' J\omega + \frac{1}{V'}\frac{\partial}{\partial \psi} V' \left\langle \pi_{\text{tot}}^{(\psi\phi)} \right\rangle_{\psi} = \left\langle S^{(\omega)} \right\rangle_{\psi}, \tag{2.253}$$

$$\frac{3}{2}\frac{1}{V'}\left.\frac{\partial}{\partial t}\right|_{\psi} V' p_s + \frac{1}{V'}\frac{\partial}{\partial \psi} V' \left\langle q_s \right\rangle_{\psi} = P_s^{\text{visc}} + P_s^{\text{Ohm}} + P_s^{\text{comp}} + P_s^{\text{turb}} + \left\langle C_s^{(E)} \right\rangle_{\psi} + \left\langle S_s^{(E)} \right\rangle_{\psi}. \tag{2.254}$$

The expressions (2.170) for the particle flux $\langle \Gamma_s \rangle_{\psi}$, (2.164) for the particle source $S_s^{(n)}$, (2.180) for the moment of inertia J, and (2.193) for the energy source $S_s^{(E)}$ are all unchanged in the low-Mach-number limit. The momentum flux (2.183), source (2.175), and the heat flux (2.196) lose some terms that are higher-order in M:

$$\left\langle \pi_{\text{tot}}^{(\psi\phi)} \right\rangle_{\psi} = \sum_s \left\langle \pi_s^{(\psi\phi)} \right\rangle_{\psi} + \left\langle \pi_{\text{EM}}^{(\psi\phi)} \right\rangle_{\psi}, \tag{2.255}$$

where $\left\langle \pi_s^{(\psi\phi)} \right\rangle_{\psi}$ and $\pi_{\text{EM}}^{(\psi\phi)}$ are given by (2.184) and (2.188), respectively,

$$S^{(\omega)} = \sum_s \int d^3\boldsymbol{w} m_s \left(\boldsymbol{w}\cdot\nabla\phi\right) R^2 S_s, \tag{2.256}$$

$$\langle q_s \rangle_{\psi} = \left\langle \int d^3\boldsymbol{w} \varepsilon_s \left(\frac{\boldsymbol{w}\times\boldsymbol{b}}{\Omega_s}\cdot\nabla\psi\right) C[F_{0s}] \right\rangle_{\psi} + \left\langle \int d^3\boldsymbol{w} \varepsilon_s F_s^{(\text{nc})} \boldsymbol{V}_{\text{D}s}\cdot\nabla\psi \right\rangle_{\psi} - \frac{5}{2} p_s(\psi) I(\psi) \frac{\langle \boldsymbol{E}\cdot\boldsymbol{B}\rangle_{\psi}}{\langle B^2\rangle_{\psi}} + \left\langle \left\langle \int d^3\boldsymbol{w} \varepsilon_s \langle h_s \boldsymbol{V}_{\chi}\rangle_{\boldsymbol{r}}\cdot\nabla\psi \right\rangle_{\text{turb}} \right\rangle_{\psi}. \tag{2.257}$$

The Ohmic heating P_s^{Ohm} is given by (2.198), the compressional heating P_s^{comp} by (2.200); they are unchanged in the low-Mach-number limit. The other two heating terms, viz., viscous (2.197) and turbulent (2.204), are simplified:

$$P_s^{\text{visc}} = -\left\langle \pi_s^{(\psi\phi)} \right\rangle_{\psi} \frac{d\omega}{d\psi}, \tag{2.258}$$

$$P_s^{\text{turb}} = Z_s e \left\langle \left\langle \int d^3\boldsymbol{w} \left\langle h_s \left(\frac{\partial}{\partial t} + \boldsymbol{u}\cdot\nabla\right)\chi \right\rangle_{\boldsymbol{r}} \right\rangle_{\text{turb}} \right\rangle_{\psi} = P_s^{\text{diss}} - P_s^{\text{drive}}, \tag{2.259}$$

where $\left\langle \pi_s^{(\psi\phi)} \right\rangle_{\psi}$ is given by (2.184) and P_s^{diss} and P_s^{drive} are given by (2.205) and (2.206), respectively.

The system of equations (2.252)–(2.259) with the solutions for the neoclassical (§2.10.2) and fluctuating (§2.10.4) distribution functions and the mean magnetic field (§2.10.2) is a closed system for coupled turbulence and transport in the low-Mach-number limit.

2.10.6 Energy and Entropy

In the low-Mach-number limit, we are able to neglect the rotational energy compared to the thermal energy of the system (despite the viscous heating appearing in the temperature evolution equation (2.254) – because we allow the turbulence to generate a large momentum flux from the sharp gradients in $\omega(\psi)$). Thus, the energy conservation law (2.217) reads

$$\begin{aligned} &\frac{1}{V'}\left.\frac{\partial}{\partial t}\right|_\psi V'\left(\sum_s \frac{3}{2}n_s T_s + \frac{\langle B^2\rangle_\psi}{8\pi}\right) \\ &\quad + \frac{1}{V'}\frac{\partial}{\partial \psi}V'\left(\left\langle \Gamma^{(U)}\right\rangle_\psi - \frac{c}{4\pi}I(\psi)\left\langle \boldsymbol{E}\cdot\boldsymbol{B}\right\rangle_\psi + \left\langle \frac{3B^2}{8\pi}\frac{\partial\psi}{\partial t}\right\rangle_\psi\right) = \sum_s \left\langle S_s^{(E)}\right\rangle_\psi , \end{aligned} \tag{2.260}$$

where the flux of kinetic energy (in the low-Mach-number limit, this is purely thermal energy), previously given by (2.215), is

$$\begin{aligned} \left\langle \Gamma^{(U)}\right\rangle_\psi = \sum_s \langle q_s\rangle_\psi = &\left\langle \left(\sum_s \boldsymbol{Q}_s + \frac{c}{4\pi}\langle \delta\boldsymbol{E}\times\delta\boldsymbol{B}\rangle_{\text{turb}}\right)\cdot\nabla\psi\right\rangle_\psi \\ &+ \sum_s \frac{3}{2}n_s T_s\left\langle \frac{\partial\psi}{\partial t}\right\rangle_\psi . \end{aligned} \tag{2.261}$$

Here $\boldsymbol{Q}_s$ is the usual heat flux defined by (2.191) and $\langle q_s\rangle_\psi$ is given by (2.257).

The free energy (2.221) of the fluctuations continues to satisfy (2.222) in the low-Mach-number limit, but with all instances of N_s replaced by n_s. Note that because the gradients of $\omega(\psi)$ have been retained in the low-Mach-number ordering, they contribute to the free-energy balance, and hence can drive turbulence. Equations (2.260) and (2.222) should be considered as constraints on the low-Mach-number system of Sections 2.10.2–2.10.5: any solution of these equations should identically satisfy these constraints.

The entropy balance of §2.9 is unaffected by the low-Mach-number expansion (up to changes in the chemical potential (2.232) and the transport fluxes). Importantly, the entropy generation and source of free energy due to the angular velocity gradient remains. This is understandable as we have constructed our ordering so that the momentum flux is finite and the gradient in the flow is finite, and thus the entropy generation due to viscous heating remains finite and comparable to all other forms of entropy generation.

2.11 Summary

In this chapter, we have presented a complete and self-consistent derivation of multiscale gyrokinetics from first principles, drawing on and in some cases correcting previous work [7, 8, 9, 10, 11, 12, 55], as well as incorporating the evolution of the equilibrium magnetic field into this theoretical framework. We have shown how, in this framework, the gyrokinetic equation for fluctuations can be derived [8, 16, 55, 91] (§2.6.4), in parallel with the neoclassical drift kinetic equation (§2.6.1) more usually derived in a different setting [54]. This system of equations describes fast, small-scale and slow, large-scale perturbations of the local Maxwellian equilibrium driven by the gradients in the local equilibrium. We close this system on the long (transport) time scale by deriving evolution equations for the equilibrium magnetic field (§2.6.3) and equations governing the transport of particles (§2.7.1), momentum (§2.7.2) and energy (§2.7.3).

We are now in a position to draw together all of these results to form a clear picture of the evolution of a rotating turbulent plasma in a tokamak. The summary of our work can be reduced to three key points.

Firstly, flux surfaces form effectively isolated systems on timescales shorter than the transport time. They rotate toroidally as rigid bodies (2.62), they are in local thermodynamic equilibrium (2.93) and the turbulence within them is established via a local balance between energy injection and dissipation (2.223). Thus, we can conceptually think of a tokamak plasma as being made up of annular regions with a radial width given by some intermediate length (2.14) – these regions are independent on the intermediate time scale (2.16). It is this decomposition that underlies analytical and numerical considerations of the gyrokinetic system (2.144), (2.146), (2.149), and (2.153) in any one such annular region, the so-called "local approximation" [92].

Secondly, on long timescales (2.48), these independent annular regions are coupled by the transport equations of §2.7. These are local equations for the plasma density (2.166), angular velocity (2.179), and temperature (2.194), linking each annular region to its neighbours via transport fluxes that can be computed from averages over local solutions to the small-scale equations. Despite the dramatic simplification that has occurred in going from the full kinetic equation (2.1) to evolving only a few flux functions, we have retained all the critical information about the plasma. This linked-flux-surface approach to transport provides both a transparent theoretical framework and a basis for practically feasible numerical simulations of global tokamak dynamics on confinement timescales [12, 4, 42].

Thirdly, the (global) mean fields and the (local) fluctuations are also linked via energy and entropy flows (and so are individual flux surfaces). §2.8 is dedicated to the flows of energy, which express the basic conservation properties of the system. The results of this section are consequences of the multiscale representation of the plasma dynamics that is derived in the preceding sections. No extra information was used in their derivation. Thus, the energy balance (2.217) is a statement that if the mean quantities are evolved according to the transport equations of §2.7, then the mean energy of the system is conserved. Similarly, if the gyrokinetic equation (2.144) and the field equations (2.146), (2.149), and (2.153) are solved to determine the fluctuating distribution function and fluctuating electromagnetic fields, then the free energy (2.221) is the conserved quantity in the sense that (2.222) is satisfied by the solutions. These two statements are what is meant by energy conservation in our multiscale system.

A direct consequence of the multiscale plasma dynamics is the evolution of our multiscale system towards thermodynamic equilibrium (§2.9) – the entropy of the system evolves according to (2.227). As a consequence of writing the entropy production in a convenient form (2.235), we are able to diagnose how our system proceeds towards a global thermodynamic equilibrium and what that equilibrium is. As the system approaches a global equilibrium, the collisional and (relative) turbulent heating equilibrate the temperatures and the Ohmic heating dissipates the parallel induced electric field; the fluxes flatten gradients. Therefore, the global equilibrium is a state with no temperature differences, no parallel induced electric field, and no spatial gradients in n_s, T_s or ω (as a consequence of this, there are no fluctuations). However, this simple diagnosis is complicated by the multichannel nature of the transport in our system: far from equilibrium, it may be thermodynamically favourable for the system to initially *steepen* gradients in some of n_s, T_s or ω in order to flatten other gradients more rapidly.

In the course of our presentation of this general framework, we have elucidated a number of issues that have the potential to be confusing and indeed have been so in the past. It is useful to itemize some of them here:

- Turbulence is local to a given flux surface and does not spread – §2.8.2.
- There is no turbulent heating – §2.8.1 (except in the sense of turbulent redistribution of energy between species).
- The evolution of the mean magnetic field is determined only by the mean profiles, and not directly by the turbulence – §2.6.3.
- Gradients in the magnetic field do not drive fluctuations – §2.8.3
- The absence of the parallel nonlinearity in the gyrokinetic equation (2.144) does not upset the energy or entropy conservation properties of the system – see footnote on page 32.

Let us now discuss some further directions and natural extensions following from this work.

We have accomplished the construction of multiscale gyrokinetics by exploiting the scale separations inherent to plasma turbulence: between the fluctuations and the equilibrium and between the particle motion and the fluctuations (§2.2). However, these are not the only scale separations available to us. In Ch. 3, this framework is extended by exploiting the scale separation between ion- and electron-scale fluctuations, where this scale separation is found to have profound implications for the structure of the fluctuating magnetic field.

In this chapter, we have restricted ourselves to situations where the equilibrium distribution functions for all species are Maxwellian. In a burning plasma, we expect there to be at least one non-Maxwellian species – the fusion-produced α-particles. In [6], we will consider precisely this situation – demonstrating how such a species can be incorporated into the theoretical framework developed here.

We have also restricted ourselves to the so-called "high-flow ordering," where the mean rotation velocity is ordered to be comparable to the sound speed. The "low-flow" ordering, where the mean velocity is comparable to the drift velocities, requires a much more involved calculation to determine the transport of toroidal angular momentum correctly [43, 88, 93]. It is precisely this regime that must be considered if a detailed and complete study of the so-called "intrinsic" rotation is to be undertaken [94]. This alternative ordering may hold the key to how a slowly-moving or stationary plasma can transition into a rotating one [95] or how L-H transitions occur in the edge of a slowly-rotating plasma [96].

Finally, it is impossible to present a set of equations such as the ones found in this chapter without discussing how to solve them numerically. Currently, the linked-flux-tube code `TRINITY` [4] solves the low-Mach-number system of equations presented in §2.10. It is anticipated that this code will be extended to solve the complete set of equations derived here. With the advent of ever faster supercomputers, the combination of multiscale theoretical approaches and advanced numerical algorithms may allow us, for the first time, to simulate a burning plasma, from the small scales of kinetic turbulence, through the scales associated with energetic fusion products, to the plasma equilibrium scales and even the timescales of the resistive evolution of the plasma. As a tool to study potential reactor designs, this would be invaluable. Hitherto, experiments have been optimised for MHD stability. Henceforth, it is possible to envision a time when they will be optimised not only for macro-stability but also for micro-stability and designed to achieve dramatically reduced turbulent transport (e.g., [97]).

In the next chapter we will see how our system simplifies when we make use of the fact that electrons are much lighter than ions.

Chapter 3

Reduced Models for Electron Dynamics

The work in this chapter is drawn entirely from [65], written by the Author. References to Appendices made in this chapter refer to the Appendices of this work – which is in preparation for the New Journal of Physics and is available online at `http://arxiv.org/pdf/1210.1417v1`.

In this chapter, we perform the systematic elimination of the fast electron timescales discussed in §1.3. In §3.1.1, we introduce the electron gyrokinetic equation and its attendant notation. In §3.1.2, we develop the formal expansion in $\delta = \sqrt{m_e/m_i} \ll 1$ and order all physical quantities with respect to this small parameter. The key result of this chapter – the gyrokinetic system with reduced electron equations – is derived via a systematic expansion in δ. This derivation is performed in Sections 3.2 and 3.3 with the results concisely summarized in §3.4.

In §3.2.2, we use the expansion in $\delta \ll 1$ to prove that long-wavelength low-frequency fluctuations preserve the topology of magnetic field lines, generalising the results of [98, 99, 100, 101]. With the assumption of an initially regular magnetic field, this result enables us to introduce a coordinate system aligned to the exact magnetic field in §3.2.4. It is this set of coordinates that allows us, in §3.3.2 and §3.3.3, to average over the fast electron motion along exact magnetic field lines and so close our equations for the electron distribution function. These equations are particularly simple in the limit of strong collisions ($\nu_e \gg \omega$), which we derive in §3.5.

The second major result of this chapter is a simplified system of transport equations for electrons. This is presented in §3.6 and derived in Appendix H. These equations emphasise the particularly simple form that the transport fluxes take in the low-mass-ratio limit. In parallel, we present the transport equations for the collisional electron model of §3.5, in which case the transport equations take on an even simpler form.

We make a detailed comparison of our equations to previously derived models in §3.7, emphasising both the differences and the similarities. Finally, we summarise the key results in §3.8.

3.1 Electron Gyrokinetics and the Mass-Ratio Expansion

We study the response of electrons to ion-scale fluctuations within the framework of multiscale gyrokinetics – see Chapter 2 and [37]. First, we will introduce this framework in §3.1.1 along with all attendant notation. We choose to recap this information from Chapter 2 in order to facilitate reading this chapter in a self-contained fashion. Then, in §3.1.2, we formally order all quantities in the small parameter $\delta = \sqrt{m_e/m_i}$.

3.1.1 Gyrokinetics in a Rotating Tokamak Plasma

In this section, we introduce the assumptions and resulting equations of this formalism. The detailed derivation of these results can be found in Ch. 2.

First, we split all physical quantities into mean and fluctuating parts:

$$\widetilde{\boldsymbol{B}} = \boldsymbol{B} + \delta\boldsymbol{B}, \qquad \boldsymbol{B} = \left\langle \widetilde{\boldsymbol{B}} \right\rangle_{\text{turb}}, \tag{3.1}$$

$$\widetilde{\boldsymbol{E}} = \boldsymbol{E} + \delta\boldsymbol{E}, \qquad \boldsymbol{E} = \left\langle \widetilde{\boldsymbol{E}} \right\rangle_{\text{turb}}, \tag{3.2}$$

$$f_s = F_s + \delta f_s, \qquad F_s = \langle f_s \rangle_{\text{turb}}, \tag{3.3}$$

where $\widetilde{\boldsymbol{B}}$ is the magnetic field, $\widetilde{\boldsymbol{E}}$ the electric field, f_s the distribution function of species s and $\langle \cdot \rangle_{\text{turb}}$ is some average over the fluctuations.

We assume that the fluctuations obey the standard gyrokinetic ordering[1] :

$$\frac{\omega}{\Omega_s} \sim \frac{k_\parallel}{k_\perp} \sim \frac{|\delta\boldsymbol{E}|}{|\boldsymbol{E}|} \sim \frac{|\delta\boldsymbol{B}|}{|\boldsymbol{B}|} \sim \frac{\delta f_s}{F_s} \sim \frac{\nu_s}{\Omega_s} \sim \frac{\rho_s}{a} = \epsilon \ll 1, \tag{3.4}$$

where ω is the typical fluctuation frequency in the frame rotating with the plasma, $\Omega_s = Z_s eB/m_s c$ is the cyclotron frequency of species s with charge $Z_s e$ and mass m_s in the mean magnetic field $B = |\boldsymbol{B}|$, $k_\parallel$ and $k_\perp$ are typical wavenumbers parallel and perpendicular to the mean magnetic field, ν_s is a typical collision frequency, $\rho_s = v_{\text{th}_s}/\Omega_s$ is the thermal Larmor radius, $v_{\text{th}_s} = \sqrt{2T_s/m_s}$ is the thermal speed of species

[1] In this chapter, as in Ch. 2, the symbol $\sim$ is used to mean "is the same order as" (with respect to the appropriate expansion) rather than the more usual "is asymptotically equivalent to".

s with mean temperature T_s and a is a typical equilibrium length scale. The mean quantities ($\boldsymbol{B}$, $\boldsymbol{E}$, F_s) are assumed to vary on the long length scale a and the long (transport) timescale τ_E, the energy confinement time. This long timescale is ordered so that

$$\tau_E^{-1} \sim \epsilon^3 \Omega_s. \tag{3.5}$$

Typical values for the timescales and length scales in a tokamak are given in Table (1.1), which shows the very large scale separation, characterised by the small parameter ϵ. This scale separation in time and space of the mean and fluctuating quantities motivates us to define the average $\langle \cdot \rangle_{\mathrm{turb}}$ in terms of temporal and spatial averages over intermediate time and length scales. This is done precisely in equations (2.14)–(2.18) of §2.2.1.

It is shown in §2.2.2 of Ch. 2 that if we introduce the cylindrical coordinate system (R, z, ϕ) and assume axisymmetry with respect to ϕ, the mean magnetic field can be written (to lowest order in ϵ) as

$$\boldsymbol{B} = I(\psi, t)\nabla\phi + \nabla\psi \times \nabla\phi, \tag{3.6}$$

where the poloidal flux function ψ is given in terms of the mean vector potential $\boldsymbol{A}$ by

$$\psi(R, z, t) = A_\phi = R^2 \boldsymbol{A} \cdot \nabla\phi, \tag{3.7}$$

and $I(\psi, t) = R^2 \boldsymbol{B} \cdot \nabla\phi$ is the toroidal component of the mean magnetic field. We will also need the following alternate form of the magnetic field [102, 103]:

$$\boldsymbol{B} = \nabla\psi \times \nabla\alpha, \tag{3.8}$$

where α is a variable that labels different field lines within a flux surface. We cut the toroidal surface on the inboard side to force α to be single-valued. This cut forms an axisymmetric ribbon with one edge the magnetic axis and the other the plasma boundary or separatrix. Letting l be the distance along a field line, (ψ, α, l) is a set of field-aligned coordinates for $\boldsymbol{B}$. There are many different forms of field aligned coordinates and none of our results depend on a particular choice. Nevertheless, we will use (ψ, α, l) for specificity. Axisymmetric functions are functions of ψ and l but not α. Note that both $\psi(R, z)$ and $I(\psi)$ are mean quantities and therefore vary on the transport timescale.

We assume that the mean electric field is large, $\boldsymbol{E} \sim (v_{\mathrm{th}_s}/c)\,\boldsymbol{B}$, which can be shown to imply that the plasma rotates toroidally with a velocity that is species-independent and whose angular velocity only depends on the flux label ψ [51] and the slow transport timescale:

$$\boldsymbol{u} = \omega(\psi, t) R^2 \nabla\phi. \tag{3.9}$$

If we express the mean electric field in terms of potentials using Gaussian units and Coulomb gauge,

$$\boldsymbol{E} = -\nabla\varphi - \frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t}, \tag{3.10}$$

then the scalar potential takes the form

$$\varphi = \Phi(\psi, t) + \varphi_0, \qquad \varphi_0 \sim \epsilon\Phi, \tag{3.11}$$

and we have the following relationship between $\omega(\psi, t)$ and $\Phi(\psi, t)$:

$$\omega(\psi, t) = c\frac{d\Phi}{d\psi}. \tag{3.12}$$

With these results for the fields, our orderings imply that the distribution function f_s of species s takes the following form:

$$f_s = F_s + \delta f_s, \tag{3.13}$$

$$F_s = F_{0s}\left(\psi(\boldsymbol{R}_s), \varepsilon_s, t\right) + F_{1s}\left(\boldsymbol{R}_s, \varepsilon_s, \mu_s, \sigma, t\right) + \mathcal{O}\left(\epsilon^2 f\right), \tag{3.14}$$

$$\delta f_s = -\frac{Z_s e}{T_s}\delta\varphi'(\boldsymbol{r}, t)F_{0s} + h_s\left(\boldsymbol{R}_s, \varepsilon_s, \mu_s, \sigma, t\right) + \mathcal{O}\left(\epsilon^2 f\right). \tag{3.15}$$

Let us explain this notation. We have followed §2.4 and changed phase-space variables from $(\boldsymbol{r}, \boldsymbol{v})$ to $(\boldsymbol{R}_s, \varepsilon_s, \mu_s, \vartheta, \sigma)$:

$$\boldsymbol{R}_s = \boldsymbol{r} - \frac{1}{\Omega_s}\boldsymbol{b} \times \boldsymbol{w}, \tag{3.16}$$

$$\varepsilon_s = \frac{1}{2}m_s v^2 + Z_s e\left(\Phi(\psi, t) + \varphi_0\right) - Z_s e\Phi(\psi_s^*, t), \tag{3.17}$$

$$\mu_s = \frac{m_s w_\perp^2}{2B}, \tag{3.18}$$

where $\boldsymbol{b} = \boldsymbol{B}/B$ is the unit vector along the mean magnetic field, the peculiar velocity $\boldsymbol{w}$ is

$$\boldsymbol{w} = \boldsymbol{v} - \boldsymbol{u} = w_\parallel \boldsymbol{b} + w_\perp\left(\cos\vartheta\, \boldsymbol{e}_2 - \sin\vartheta\, \boldsymbol{e}_1\right), \tag{3.19}$$

where $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ are two arbitrary orthogonal unit vectors perpendicular to the magnetic field, the direction of the parallel motion is $\sigma = w_\parallel/|w_\parallel|$, and we have defined

$$\psi_s^*(\boldsymbol{r}, \boldsymbol{v}, t) = \psi(\boldsymbol{r}, t) + \frac{m_s c}{Z_s e}R^2\boldsymbol{v}\cdot\nabla\phi. \tag{3.20}$$

The quantity $\psi_s^*(\boldsymbol{r}, \boldsymbol{v}, t)$ is equal to $c/(Z_s e)$ times the canonical toroidal angular momentum and is thus conserved in an exactly axisymmetric system – i.e., in the absence of fluctuations. In these variables, (3.14) splits the mean distribution function into the near Maxwellian equilibrium

$$F_{0s} = N_s(\psi(\boldsymbol{R}_s, t), t)\left[\frac{m_s}{2\pi T_s(\psi(\boldsymbol{R}_s, t), t)}\right]^{3/2} e^{-\varepsilon_s/T_s(\psi(\boldsymbol{R}_s,t),t)}, \tag{3.21}$$

and the neoclassical distribution function F_{1s}, which describes large-scale $\mathcal{O}(\epsilon)$ deviations from a Maxwellian. The function N_s is related to the mean density n_s by

$$n_s = N_s(\psi(\boldsymbol{r},t),t)\exp\left[-\frac{Z_s e\varphi_0(\boldsymbol{r},t)}{T_s(\psi(\boldsymbol{r},t),t)} + \frac{m_s\omega^2(\psi(\boldsymbol{r},t),t)R^2}{2T_s(\psi(\boldsymbol{r},t),t)}\right]. \tag{3.22}$$

We now drop the explicit slow time dependence of all quantities – this is for notational clarity only and is not an assumption that the mean quantities are constant.

The fluctuating distribution function, given by (3.15), is composed of the Boltzmann response (the first term of (3.15)), where the fluctuating potentials $\delta\varphi$ and $\delta\boldsymbol{A}$ have been combined to form the electrostatic potential in the frame rotating with the plasma

$$\delta\varphi' = \delta\varphi - \frac{1}{c}\boldsymbol{u}\cdot\delta\boldsymbol{A}, \tag{3.23}$$

and the gyrokinetic distribution function h_s. Assuming that all mean quantities are known, then the gyrokinetic distribution function h_s is given by the gyrokinetic equation (see (2.144) in §2.6.4):

$$\begin{aligned}\left[\frac{\partial}{\partial t} + \boldsymbol{u}(\boldsymbol{R}_s)\cdot\frac{\partial}{\partial\boldsymbol{R}_s}\right]h_s + \left(w_\parallel\boldsymbol{b} + \boldsymbol{V}_{\mathrm{D}s} + \langle\boldsymbol{V}_\chi\rangle_{\boldsymbol{R}}\right)\cdot\frac{\partial h_s}{\partial\boldsymbol{R}_s} - \langle C[h_s]\rangle_{\boldsymbol{R}} \\ = \frac{Z_s e F_{0s}}{T_s}\left[\frac{\partial}{\partial t} + \boldsymbol{u}(\boldsymbol{R}_s)\cdot\frac{\partial}{\partial\boldsymbol{R}_s}\right]\langle\chi\rangle_{\boldsymbol{R}} \\ - \left\{\frac{\partial F_{0s}}{\partial\psi} + \frac{m_s F_{0s}}{T_s}\left[\frac{I(\psi)w_\parallel}{B} + \omega(\psi)R^2\right]\frac{d\omega}{d\psi}\right\}\langle\boldsymbol{V}_\chi\rangle_{\boldsymbol{R}}\cdot\nabla\psi,\end{aligned} \tag{3.24}$$

where the guiding-centre drift velocity is

$$\boldsymbol{V}_{\mathrm{D}s} = \frac{\boldsymbol{b}}{\Omega_s}\times\left[w_\parallel^2\boldsymbol{b}\cdot\nabla\boldsymbol{b} + \frac{1}{2}w_\perp^2\nabla\ln B - \omega^2(\psi,t)R\nabla R - 2w_\parallel\omega(\psi,t)\boldsymbol{b}\times\nabla z + \frac{Z_s e}{m_s}\nabla\varphi_0\right], \tag{3.25}$$

and where we have defined the gyrokinetic potential

$$\chi = \delta\varphi - \frac{1}{c}\boldsymbol{v}\cdot\delta\boldsymbol{A} = \delta\varphi' - \frac{1}{c}\boldsymbol{w}\cdot\delta\boldsymbol{A}, \tag{3.26}$$

and the fluctuating velocity due to χ[2]

$$\boldsymbol{V}_\chi = \frac{c}{B}\boldsymbol{b}\times\nabla\chi, \qquad \langle\boldsymbol{V}_\chi\rangle_{\boldsymbol{R}} = \frac{c}{B}\boldsymbol{b}\times\frac{\partial\langle\chi\rangle_{\boldsymbol{R}}}{\partial\boldsymbol{R}_s} + \mathcal{O}(\epsilon^2 v_{\mathrm{th}_s}). \tag{3.27}$$

We have also introduced the gyroaverage at constant $\boldsymbol{R}_s$, ε_s and μ_s:

$$\langle\chi(\boldsymbol{r},\boldsymbol{w},t)\rangle_{\boldsymbol{R}} = \frac{1}{2\pi}\oint d\vartheta\,\chi\left(\boldsymbol{r}(\boldsymbol{R}_s,\varepsilon_s,\mu_s,\vartheta),\boldsymbol{w}(\boldsymbol{R}_s,\varepsilon_s,\mu_s,\vartheta,\sigma),t\right). \tag{3.28}$$

[2] This can be shown to consist, physically, of the fluctuating $\boldsymbol{E}\times\boldsymbol{B}$ drift in the rotating frame, the motion of guiding centres along perturbed field lines and the fluctuating ∇B drift.

The equation for h_s is closed through Maxwell's equations for the fluctuating fields. The fluctuating potential $\delta\varphi'$ obeys the quasineutrality condition:

$$\sum_s \frac{Z_s^2 e^2 n_s \delta\varphi'}{T_s} = \sum_s Z_s e \int d^3\boldsymbol{w} \langle h_s \rangle_{\boldsymbol{r}}, \tag{3.29}$$

where the integral over velocities is performed at constant $\boldsymbol{r}$ and the gyroaverage at constant $\boldsymbol{r}$, $w_\parallel$, and $w_\perp$ is defined by

$$\langle h_s \rangle_{\boldsymbol{r}} = \frac{1}{2\pi} \oint d\vartheta h_s(\boldsymbol{R}_s(\boldsymbol{r}, w_\parallel, w_\perp, \vartheta), \varepsilon_s(\boldsymbol{r}, w_\parallel, w_\perp, \vartheta), \mu_s(\boldsymbol{r}, w_\perp), \sigma, t). \tag{3.30}$$

The fluctuating magnetic field is determined from $\delta A_\parallel = \boldsymbol{b} \cdot \delta\boldsymbol{A}$ and $\delta B_\parallel = \boldsymbol{b} \cdot \delta\boldsymbol{B}$, which obey the parallel component of Ampére's law

$$-\nabla_\perp^2 \delta A_\parallel = \frac{4\pi}{c} \sum_s Z_s e \int d^3\boldsymbol{w} w_\parallel \langle h_s \rangle_{\boldsymbol{r}}, \tag{3.31}$$

and the perpendicular part of Ampére's law (see the discussion in §2.6.5)

$$\nabla_\perp^2 \frac{\delta B_\parallel B}{4\pi} + \boldsymbol{\nabla}_\perp \boldsymbol{\nabla}_\perp : \sum_s \int d^3\boldsymbol{w} \langle m_s \boldsymbol{w}_\perp \boldsymbol{w}_\perp h_s \rangle_{\boldsymbol{r}} = 0, \tag{3.32}$$

respectively. The evolution equation (3.24) and constituent relations (3.29), (3.31) and (3.32) form a closed system of equations for determining h_s, $\delta\varphi'$, $\delta A_\parallel$, $\delta B_\parallel$ on the turbulent timescale. Typically, the system (3.24), (3.29), (3.31), and (3.32) is solved until statistical equilibrium is reached (a few nonlinear turnover times). The slowly-varying equilibrium is treated as constant during this evolution. The slow time dependence of F_{0s} is determined by transport equations for $\omega(\psi, t)$, $N_s(\psi, t)$, $T_s(\psi, t)$ and $I(\psi, t)$ and the equilibrium condition determining $\psi(\boldsymbol{r}, t)$. Such equations are given in §2.6.2 for $\psi(R, z, t)$, in §2.6.3 for $I(\psi, t)$ and in §2.7 for $N_s(\psi, t)$, $\omega(\psi, t)$, and $T_s(\psi, t)$.

3.1.2 The Mass-Ratio Expansion

In order to express the scale separation of the electron and ion scales, we introduce the secondary small parameter $\delta = \sqrt{m_e/m_i} \ll 1$. We will use a subscript i to denote any ion species, of which there may be many. We consider the expansion in $\delta \ll 1$ to be a subsidiary expansion of the gyrokinetic system (equations (3.24), (3.29), (3.31), and (3.32))[3]. Thus, we assume that $\epsilon = \rho_i/a$, i.e., that the fundamental gyrokinetic expansion parameter is that of the ions, and that

$$\epsilon \ll \delta \ll 1. \tag{3.33}$$

[3]This is in contrast to the approach taken by [34], see §3.7 for details.

All other dimensionless parameters are treated as finite – i.e., independent of δ. Therefore, we are assuming that $T_i \sim T_e$ and $\beta = 8\pi p/B^2 \sim 1$, where p is the plasma pressure. In typical fusion plasmas, β is often small and one might worry about this assumption. A more detailed analysis shows that our expansion in δ requires that [40]

$$\beta \gg \frac{m_e}{m_i} = \delta^2, \tag{3.34}$$

which is equivalent to requiring that the Alfvén speed is smaller than the electron thermal speed.[4] In plasmas of interest, this is indeed satisfied and we are justified in treating β as finite.

We assume that fluctuating quantities vary on ion rather than electron scales. Thus, we order the timescales of fluctuating quantities in the rotating frame and the size of the fluctuations by

$$\begin{aligned} \frac{\partial}{\partial t} + \boldsymbol{u}\cdot\nabla \sim v_{\mathrm{th}_i}\boldsymbol{b}\cdot\frac{\partial}{\partial \boldsymbol{R}_s} \sim \frac{c}{B}\left|\left(\nabla\delta\varphi' \times \boldsymbol{b}\right)\cdot\nabla_\perp\right| \sim \\ \frac{v_{\mathrm{th}_i}}{B}\left|\left(\nabla\delta A_\parallel \times \boldsymbol{b}\right)\cdot\nabla_\perp\right| \sim \frac{cT}{eB^2}\left|\left(\nabla\delta B_\parallel \times \boldsymbol{b}\right)\cdot\nabla_\perp\right| \sim \frac{v_{\mathrm{th}_i}}{a} \end{aligned} \tag{3.35}$$

so that the typical fluctuation timescale, the ion parallel streaming time, and the ion nonlinear timescale due to the fluctuating fields are all comparable. We assume that the toroidal rotation velocity $\boldsymbol{u}$ is comparable to the ion thermal speed $\boldsymbol{u} \sim v_{\mathrm{th}_i}$, which implies that the electric field is ordered as

$$\boldsymbol{E} \sim \frac{v_{\mathrm{th}_i}}{c}\boldsymbol{B}. \tag{3.36}$$

The perpendicular length scales are of course ordered as

$$|\nabla_\perp| \sim k_\perp \sim \rho_i^{-1}. \tag{3.37}$$

Clearly, the turbulent dynamics of the ions are not simplified by the subsidiary expansion in δ and hence h_i is determined by the gyrokinetic equation (3.24).

The electron dynamics, however, are simplified because

$$k_\parallel v_{\mathrm{th}_e} \sim \delta^{-1}\frac{v_{\mathrm{th}_i}}{a} \sim \frac{v_{\mathrm{th}_e}}{B}\left|\left(\nabla\delta A_\parallel \times \boldsymbol{b}\right)\cdot\nabla_\perp\right| \gg \frac{\partial}{\partial t} + \boldsymbol{u}\cdot\nabla, \tag{3.38}$$

so the parallel electron motion is much faster than the evolution of the fluctuations and also

$$k_\perp \rho_e \sim \delta \ll 1, \tag{3.39}$$

[4] For cases where β is so low that this does not hold, an expansion of slab gyrokinetics for very low β plasmas has been carried out in [104].

so we can neglect finite-electron-gyroradius effects. We choose the electron collision rate ν_e to be comparable to the fluctuation frequency, i.e.,

$$\nu_e \sim \frac{v_{\mathrm{th}_i}}{a} \sim \frac{1}{\delta}\nu_i. \tag{3.40}$$

Note that although electron collisions are comparable to the fluctuation frequency, and, therefore, important in the electron fluctuation dynamics, this would conventionally be considered a low collisionality plasma since $\nu_{\mathrm{eff}} = \nu_i a/v_{\mathrm{th}_i} \sim \nu_e a/v_{\mathrm{th}_e} \sim \delta \ll 1$. Under these assumptions, we expand h_e, $\delta\varphi'$, $\delta A_\parallel$ and $\delta B_\parallel$ as

$$\begin{aligned} h_e &= h_e^{(0)} + h_e^{(1)} + \cdots, & \quad \delta\varphi' &= \delta\varphi'^{(0)} + \delta\varphi'^{(1)} + \cdots, \\ \delta A_\parallel &= \delta A_\parallel{}^{(0)} + \delta A_\parallel{}^{(1)} + \cdots, \quad \text{and} & \quad \delta B_\parallel &= \delta B_\parallel{}^{(0)} + \delta B_\parallel{}^{(1)} + \cdots, \end{aligned} \tag{3.41}$$

where $h_e^{(1)} \sim \delta\, h_e^{(0)}$ *etc.*[5] In Sections 3.2 and 3.3, we systematically expand the gyrokinetic system of equations, (3.24), (3.29), (3.31) and (3.32) in powers of δ to obtain $h_e^{(0)}$ and close the system at first order in δ.

3.2 Zeroth Order: Flux Conservation and Field-Aligned Coordinates

In this section, we apply the orderings of §3.1.2 to the gyrokinetic equation for electrons and examine the consequences to lowest order.

In order to expand the gyrokinetic equation, in Appendix A, we express $\langle\chi\rangle_{\boldsymbol{R}}$ explicitly in terms of the fields and find, (A.3):

$$\langle\chi\rangle_{\boldsymbol{R}} = \underbrace{\delta\varphi'(\boldsymbol{R}_e)}_{\mathcal{O}(\epsilon\frac{T}{e})} - \underbrace{\frac{w_\parallel}{c}\delta A_\parallel(\boldsymbol{R}_e)}_{\mathcal{O}(\frac{\epsilon}{\delta}\frac{T}{e})} - \underbrace{\frac{m_e w_\perp^2}{2e}\frac{\delta B_\parallel(\boldsymbol{R}_e)}{B}}_{\mathcal{O}(\epsilon\frac{T}{e})} + \mathcal{O}\left(\epsilon\delta\frac{T_e}{e}\right). \tag{3.42}$$

Substituting this result and the expansion (3.41) into the gyrokinetic equation (3.24), we find, to lowest order in δ:

$$\begin{aligned} w_\parallel \boldsymbol{b}\cdot\frac{\partial h_e^{(0)}}{\partial \boldsymbol{R}_e} - \frac{w_\parallel}{B}\left(\boldsymbol{b}\times\frac{\partial \delta A_\parallel{}^{(0)}}{\partial \boldsymbol{R}_e}\right)\cdot\frac{\partial h_e^{(0)}}{\partial \boldsymbol{R}_e} = \\ \frac{e w_\parallel F_{0e}}{cT_e}\left[\frac{\partial}{\partial t} + \boldsymbol{u}(\boldsymbol{R}_e)\cdot\frac{\partial}{\partial \boldsymbol{R}_e}\right]\delta A_\parallel{}^{(0)} + \frac{w_\parallel}{B}\left(\boldsymbol{b}\times\frac{\partial \delta A_\parallel{}^{(0)}}{\partial \boldsymbol{R}_e}\right)\cdot\frac{\partial F_{0e}}{\partial \boldsymbol{R}_e}, \end{aligned} \tag{3.43}$$

where $\delta A_\parallel$ is evaluated at $\boldsymbol{R}_e$ and we have used $\boldsymbol{u}(\boldsymbol{R}_e)\cdot(\partial w_\parallel/\partial \boldsymbol{R}_e) = 0$ (axisymmetry). These equations are written in terms of the variables $\boldsymbol{R}_e$, ε_e, μ_e and ϑ. It will be

[5] In fact, we will only require $\delta\varphi'$ or $\delta B_\parallel$ to lowest order in δ. Thus, we will drop the superscripts on these fields.

convenient to instead convert back to using $\boldsymbol{r}$ as our spatial variable. Since (3.43) is only accurate up to corrections of order $\mathcal{O}(\delta\epsilon^2\Omega_i F_{0e})$ and all functions in (3.43) are evaluated at $\boldsymbol{R}_e$, we can use the fact that $k_\perp \rho_e \sim \delta$ to replace $\boldsymbol{R}_e$ by $\boldsymbol{r}$ everywhere. The function ε_e is now given by the simplified expression:

$$\varepsilon_e = \frac{1}{2} m_e w_\parallel^2 + \mu_e B - e\varphi_0 + \mathcal{O}\left(\delta^2 T_e\right). \tag{3.44}$$

Defining $\widetilde{\boldsymbol{b}} = \widetilde{\boldsymbol{B}}/|\widetilde{\boldsymbol{B}}|$ to be the unit vector along the exact magnetic field, we have

$$\widetilde{\boldsymbol{b}} = \boldsymbol{b} + \frac{\delta \boldsymbol{B}_\perp}{B} + \mathcal{O}(\epsilon^2) = \boldsymbol{b} - \frac{1}{B}\left(\boldsymbol{b} \times \nabla \delta {A_\parallel}^{(0)}\right) + \mathcal{O}(\delta\epsilon). \tag{3.45}$$

Thus, after a little algebra (using (3.45)), we can rewrite all but one term of (3.43) as derivatives along the exact field line:

$$\begin{aligned} w_\parallel \widetilde{\boldsymbol{b}} \cdot \nabla \left(\frac{\delta f_e}{F_{0e}} - \frac{e\delta\varphi'}{T_e}\right) =& \frac{e w_\parallel}{c T_e}\left(\frac{\partial}{\partial t} + \boldsymbol{u}\cdot\nabla\right)\delta {A_\parallel}^{(0)} \\ & - w_\parallel \widetilde{\boldsymbol{b}} \cdot \nabla \ln N_e - w_\parallel \left(\frac{\varepsilon_e}{T_e} - \frac{3}{2}\right)\widetilde{\boldsymbol{b}} \cdot \nabla \ln T_e, \end{aligned} \tag{3.46}$$

where we have used

$$\delta f_e = \frac{e\delta\varphi'}{T_e} F_{0e} + h_e^{(0)} + \mathcal{O}\left(\epsilon\delta F_{0e}\right), \tag{3.47}$$

the Maxwellian form (3.21) of F_{0e}, and divided through by F_{0e}. Equation (3.46) is an inhomogeneous ordinary differential equation along the exact field lines for δf_e.

In the remainder of this section, we will find the general solution of (3.46). The particular solution to the inhomogeneous problem is constructed in §3.2.1. In order to solve the homogenous part of (3.46), it is necessary to discover the structure of the exact magnetic field. Thus, in §3.2.2, we prove that magnetic flux is conserved to lowest order in δ and hence that the topology of the magnetic field is fixed. This result then allows us to introduce (in §3.2.4) field-aligned coordinates in which it is simple to solve the homogenous part of (3.46). This general solution will provide us with an equation for $\delta A_\parallel$ and place constraints on the form of $h_e^{(0)}$.

3.2.1 Particular Solutions of (3.46)

We can now find particular solutions of (3.46). To do this, we write δf_e in the following form:

$$\delta f_e = \lambda(\boldsymbol{r}, \varepsilon_e, \mu_e, \sigma) F_{0e} + g_e(\boldsymbol{r}, \varepsilon_e, \mu_e, \sigma) + \mathcal{O}\left(\epsilon\delta F_{0e}\right), \tag{3.48}$$

where g_e satisfies the homogenous equation

$$w_\parallel \widetilde{\boldsymbol{b}} \cdot \nabla g_e = 0. \tag{3.49}$$

This can clearly be done for any δf_e by suitable choice of λ. Substituting (3.48) into (3.46) we obtain

$$\widetilde{\boldsymbol{b}}\cdot\nabla\left(\lambda-\frac{e\delta\varphi'}{T_e}\right)=\frac{e}{cT_e}\left(\frac{\partial}{\partial t}+\boldsymbol{u}\cdot\nabla\right)\delta A_{\|}{}^{(0)}-\widetilde{\boldsymbol{b}}\cdot\nabla\ln N_e-\left(\frac{\varepsilon_e}{T_e}-\frac{3}{2}\right)\widetilde{\boldsymbol{b}}\cdot\nabla\ln T_e. \quad (3.50)$$

To find the form of λ, we take the derivative of this equation with respect to μ_e and the second derivative with respect to ε_e to obtain

$$\widetilde{\boldsymbol{b}}\cdot\nabla\frac{\partial\lambda}{\partial\mu_e}=0 \text{ and } \widetilde{\boldsymbol{b}}\cdot\nabla\frac{\partial^2\lambda}{\partial{\varepsilon_e}^2}=0, \quad (3.51)$$

respectively. Clearly, any solution to these equations is either independent of μ_e and a linear function of ε_e or satisfies $\widetilde{\boldsymbol{b}}\cdot\nabla\lambda=0$. Any solution of the second kind can be absorbed into the function g_e, leaving us with the general solution for λ given by

$$\lambda=\frac{\overline{\delta n}_e(\boldsymbol{r})}{n_e}+\left(\frac{\varepsilon_e}{T_e}-\frac{3}{2}\right)\frac{\overline{\delta T}_e(\boldsymbol{r})}{T_e}, \quad (3.52)$$

where we have written the linear function of ε_e in the natural way as a perturbed Maxwellian. However, it should be noted that $\overline{\delta n}_e(\boldsymbol{r})$ is not the perturbed electron density and $\overline{\delta T}_e(\boldsymbol{r})$ is not the perturbed electron temperature because g_e can have both density and energy moments. Inserting (3.52) into (3.50) and taking a derivative with respect to ε_e, we find the following equation for $\overline{\delta T}_e$:

$$\widetilde{\boldsymbol{b}}\cdot\nabla\left(\ln T_e+\frac{\overline{\delta T}_e}{T_e}\right)=0. \quad (3.53)$$

Finally, substituting both (3.52) and (3.53) into (3.50), we obtain:

$$\widetilde{\boldsymbol{b}}\cdot\nabla\left(\frac{\overline{\delta n}_e}{n_e}-\frac{e\delta\varphi'}{T_e}+\ln N_e\right)=\frac{e}{cT_e}\left(\frac{\partial}{\partial t}+\boldsymbol{u}\cdot\nabla\right)\delta A_{\|}. \quad (3.54)$$

Familiar physics is contained in these two equations. It will be shown in §3.2.5 that the contribution to g_e from passing electrons is a function only of $\tilde{\psi}$ and not of $\tilde{\alpha}$ or $\tilde{l}$, and, therefore, so are its density and temperature. Thus, for the purposes of interpreting (3.54) and (3.53), we can add the density and temperature of the passing part of g_e to $\overline{\delta n}_e$ and $\overline{\delta T}_e$ respectively. Hence, (3.53) describes how (part of) the temperature of the electrons is equalised along field lines, due to rapid thermal conduction. The only part of the temperature that does not equalize along the field line, as we shall see when we solve (3.49), is the part of the temperature of the trapped particles contained in g_e. This is to be expected; the trapped particles do not traverse the entire field line and so cannot equalise their temperature along it.

Equation (3.54) describes how the pressure gradient of the electrons (again, not including the trapped particle pressure) along the field line is balanced by a parallel electric field. As we shall demonstrate, it is useful to think of (3.54) as an evolution equation for $\delta A_{\|}$ where $\delta\varphi'$ and $\overline{\delta n}_e$ are given.

3.2.2 Flux Conservation

To solve (3.49), (3.53) and (3.54) we will need to know the spatial structure of the total magnetic field. As (3.54) determines $\delta A_{\|}$, it determines $\delta \boldsymbol{B}_{\perp}$ and thence $\widetilde{\boldsymbol{b}}$. In this section, we will prove that (3.54) implies that magnetic flux is conserved and consequently that the evolution of the magnetic perturbation cannot change the structure of the magnetic field lines.

Magnetic flux is said to be conserved if there exists an effective velocity field $\boldsymbol{u}_{\text{eff}}$ such that closed curves moving with this velocity always enclose the same amount of magnetic flux. It is proved by Newcomb in [105] that if such a $\boldsymbol{u}_{\text{eff}}$ exists, it also preserves magnetic field lines and thus the magnetic topology is fixed. Now, if the parallel electric field satisfies

$$\widetilde{\boldsymbol{E}} \cdot \widetilde{\boldsymbol{b}} = -\widetilde{\boldsymbol{b}} \cdot \nabla \xi, \tag{3.55}$$

where ξ is a single-valued scalar function, then the velocity field

$$\boldsymbol{u}_{\text{eff}} = \frac{c}{\widetilde{B}^2} \left(\widetilde{\boldsymbol{E}} + \nabla \xi \right) \times \widetilde{\boldsymbol{B}} + U(\boldsymbol{r}) \widetilde{\boldsymbol{b}}, \tag{3.56}$$

(where $\widetilde{B} = |\widetilde{\boldsymbol{B}}|$ and $U(\boldsymbol{r})$ is an arbitrary single-valued function) satisfies $\widetilde{\boldsymbol{E}} + \boldsymbol{u}_{\text{eff}} \times \widetilde{\boldsymbol{B}}/c = -\nabla \xi$. Thus Faraday's law becomes $\partial \widetilde{\boldsymbol{B}}/\partial t = \nabla \times (\boldsymbol{u}_{\text{eff}} \times \widetilde{\boldsymbol{B}})$. The familiar proof of the frozen-in theorem (that is reproduced in almost all MHD textbooks) clearly applies if we recognise $\boldsymbol{u}_{\text{eff}}$ as the flux- and field-line-preserving velocity. The condition (3.55) is a special case of the general conditions for the existence of frozen flux that were investigated in [105].

In Appendix B, we prove that we can write $\widetilde{\boldsymbol{E}} \cdot \widetilde{\boldsymbol{b}}$ in terms of potentials as

$$\widetilde{\boldsymbol{E}} \cdot \widetilde{\boldsymbol{b}} = -\widetilde{\boldsymbol{b}} \cdot \nabla \varphi_0 - \widetilde{\boldsymbol{b}} \cdot \nabla \delta \varphi' - \frac{1}{c} \frac{\partial \boldsymbol{A}}{\partial t} \cdot \boldsymbol{b} - \frac{1}{c} \left(\frac{\partial}{\partial t} + \boldsymbol{u} \cdot \nabla \right) \delta A_{\|} + \mathcal{O}\left(\epsilon^3 \frac{v_{\text{th}_i}}{c} B \right). \tag{3.57}$$

In Appendix C, we prove that the mean magnetic flux is conserved irrespective of the fluctuations, and so (see (C.7))

$$\frac{\partial \boldsymbol{A}}{\partial t} \cdot \boldsymbol{b} = \boldsymbol{b} \cdot \nabla \frac{T_e n_{1e}}{e n_e}, \tag{3.58}$$

where n_{1e} is the $\mathcal{O}(\epsilon)$ correction to the mean electron density (see discussion before (C.7)). Finally, using (3.54) and (3.58) in (3.57), we obtain

$$\widetilde{\boldsymbol{E}} \cdot \widetilde{\boldsymbol{b}} = -\widetilde{\boldsymbol{b}} \cdot \nabla \left[\frac{T_e \left(\delta \overline{n}_e + n_{1e} \right)}{e n_e} + \frac{T_e}{e} \ln N_e + \varphi_0 \right], \tag{3.59}$$

which is indeed in the form (3.55).[6] Thus, we can define

$$\xi = \frac{T_e}{e}\left(\ln N_e + \frac{\overline{\delta n_e} + n_{1e}}{n_e}\right) + \varphi_0 + \tilde{\xi}, \tag{3.60}$$

where $\tilde{\xi}$ is an arbitrary single-valued function that satisfies

$$\widetilde{\boldsymbol{b}} \cdot \nabla\tilde{\xi} = 0, \tag{3.61}$$

and with this definition, (3.56) defines the field-line-preserving velocity $\boldsymbol{u}_{\text{eff}}$. In (3.56), we are free to choose $U(\boldsymbol{r})$ and $\tilde{\xi}$ subject to the constraint $\widetilde{\boldsymbol{b}}\cdot\nabla\tilde{\xi} = 0$. We postpone making choices for these parameters until §3.2.5. Since the field is frozen to $\boldsymbol{u}_{\text{eff}}$ the ion-scale turbulence cannot alter the magnetic field structure on the turbulent timescale.

3.2.3 Structure of the Magnetic Field

In the previous section we have demonstrated that ion-scale turbulence preserves field lines. The topology of the field is thus fixed. But what is this topology? Electron-scale fluctuations *can* alter the topology and so can generate a stochastic field[7] from an initially regular one. Such fluctuations may arise "locally" from electron-scale instabilities (e.g. Electron Temperature Gradient instabilities or Microtearing), but electron-scale fluctuations may also be driven "non-locally" by a cascade from ion-scale fluctuations. In both cases, some tangling of the magnetic field is possible, indeed probable. Thus, at some intuitive level, the magnetic field is expected to be stochastic. A stochastic field will be advected by $\boldsymbol{u}_{\text{eff}}$ and remain stochastic.

If the magnetic field were stochastic and stationary then the heat diffusivity of the electrons would be given by the well-known Rechester-Rosenbluth formula [106]:

$$\chi_{T_e} = D_{st} v_{\text{th}_e} \sim v_{\text{th}_e} l_c \left(\frac{\delta B_{\text{stoch}}}{B}\right)^2, \tag{3.62}$$

where D_{st} is a species-independent diffusion coefficient characterising the stochasticity of the field[8] and l_c is the parallel correlation length of the stochastic component of

[6] Strictly speaking, we have proven that $\widetilde{\boldsymbol{E}} \cdot \widetilde{\boldsymbol{b}} = -\widetilde{\boldsymbol{b}} \cdot \nabla\xi + \tilde{E}_{\|}\widetilde{\boldsymbol{b}}$ where $\tilde{E}_{\|} \sim \epsilon^3 v_{\text{th}_i} B/c$. Thus the electric field is precisely $\widetilde{\boldsymbol{E}} = -\boldsymbol{u}_{\text{eff}} \times \widetilde{\boldsymbol{B}}/c - \nabla\xi + \tilde{E}_{\|}\widetilde{\boldsymbol{b}}$. Substituting this into Faraday's law, we find that

$$\frac{\partial \widetilde{\boldsymbol{B}}}{\partial t} = \nabla \times \left(\boldsymbol{u}_{\text{eff}} \times \widetilde{\boldsymbol{B}}\right) - c\nabla \times \left(\tilde{E}_{\|}\widetilde{\boldsymbol{b}}\right).$$

Estimating the size of the left hand side and the second term on the right hand side, we see that the correction $\tilde{E}_{\|}$ will affect neither the lowest-order mean field $\boldsymbol{B}$ nor the lowest-order fluctuation $\delta\boldsymbol{B}$. Thus, we can safely ignore it.

[7] A stochastic field is one in which points on neighbouring field lines become exponentially separated as they travel along field lines [106].

[8] The field-line diffusivity D_{st} is defined as [106]

$$D_{st} = \lim_{l\to\infty} \overline{(\delta r(l) - \delta r(0))^2}/2l,$$

the field $\delta B_{\mathrm{stoch}}$. Let us assume the stochastic component of the field is comparable to the ion-scale field perturbations (i.e., $\delta B_{\mathrm{stoch}}/B \sim \epsilon$) and the correlation length is the system size (i.e., $l_c \sim a$). Then, (3.62) predicts that the electron heat diffusivity in a tokamak should be δ^{-1} larger than the ion heat diffusivity (both the electrostatic and electromagnetic parts of the ion heat diffusivity). This is not backed up by experimental observations; ion and electron heat diffusivities are observed to be of the same order of magnitude. Thus, we take this as experimental justification for assuming that the amount of magnetic field stochasticity is limited. [9]

We therefore assume good flux surfaces in solving Eqs. (3.49), (3.53) and (3.54) – these surfaces will be approximations to the exact field that we will refer to as the "ion-scale field". The ion-scale turbulence will distort the ion-scale field but the surfaces will remain intact. A smaller stochastic field component with $l_c \sim a$ and $\delta B_{stoch}/B \sim \epsilon\delta^{1/2}$ would cause observable levels of transport and we cannot rule out a component of this size. Indeed, such a field would change the final electron equations in this chapter. However, we will assume for simplicity that the stochastic component of the field is zero to order $\mathcal{O}(\epsilon\delta)$ (i.e., $\delta B_{\mathrm{stoch}}/B \ll \epsilon\delta$) in this thesis.

3.2.4 Field-Aligned Coordinates

If we write the ion-scale magnetic field in the Clebsch form

$$\widetilde{\boldsymbol{B}} = \nabla\tilde{\psi} \times \nabla\tilde{\alpha}, \tag{3.63}$$

then $\tilde{\psi}$ is a single-valued function that labels the flux surfaces and $\tilde{\alpha}$ is a function that labels different field lines within a given flux surface. To ensure that $\tilde{\alpha}$ is single-valued, we make a cut in the poloidal domain on the inboard side of the tokamak. This cut makes a ribbon forming a toroidal loop with one edge of the ribbon being the magnetic axis and the other the plasma boundary or separatrix. We allow the

where $\delta r(l)$ is the radial displacement of a point after following a field line for a distance l and the overbar denotes statistical averaging over field lines (the factor of 2 in the denominator is chosen to agree with the definition in [106]).

[9] We can formulate this condition precisely as a condition on the field-line diffusivity (defined in the footnote on page 78) . The limit on stochasticity can be expressed as

$$D_{st} \ll \epsilon\rho_e.$$

where we obtain this estimate by insisting that the change in a function g following the field line is dominated by explicit variation along a field line ($k_\parallel g$) rather than the field line connecting neighbouring points in the perpendicular plane ($D_{st}k_\perp^2 g$). The existence of a stochastic field giving rise to a field-line diffusivity of size $D_{st} \lesssim \epsilon\rho_e$ would not invalidate the existence of $\tilde{\psi}$ and $\tilde{\alpha}$, as (for the purposes of studying ion-scale turbulence) we could define $\tilde{\psi}$ surfaces by the mean radial location of a field line averaged along a magnetic correlation length l_c. This introduces a displacement $\lesssim \rho_e$ of the $\tilde{\psi}$ surface from the exact position of the field line, but for turbulence where $k_\perp\rho_e \ll 1$ this correction is negligible.

cut to move with the velocity $\boldsymbol{u}_{\text{eff}}$. This is legitimate as $\boldsymbol{u}_{\text{eff}}$ is continuous across the cut. Since the cut is perturbed by the turbulence, it will not now be axisymmetric. The expression (3.63) for the magnetic field allows us (see, e.g., [103]) to introduce a

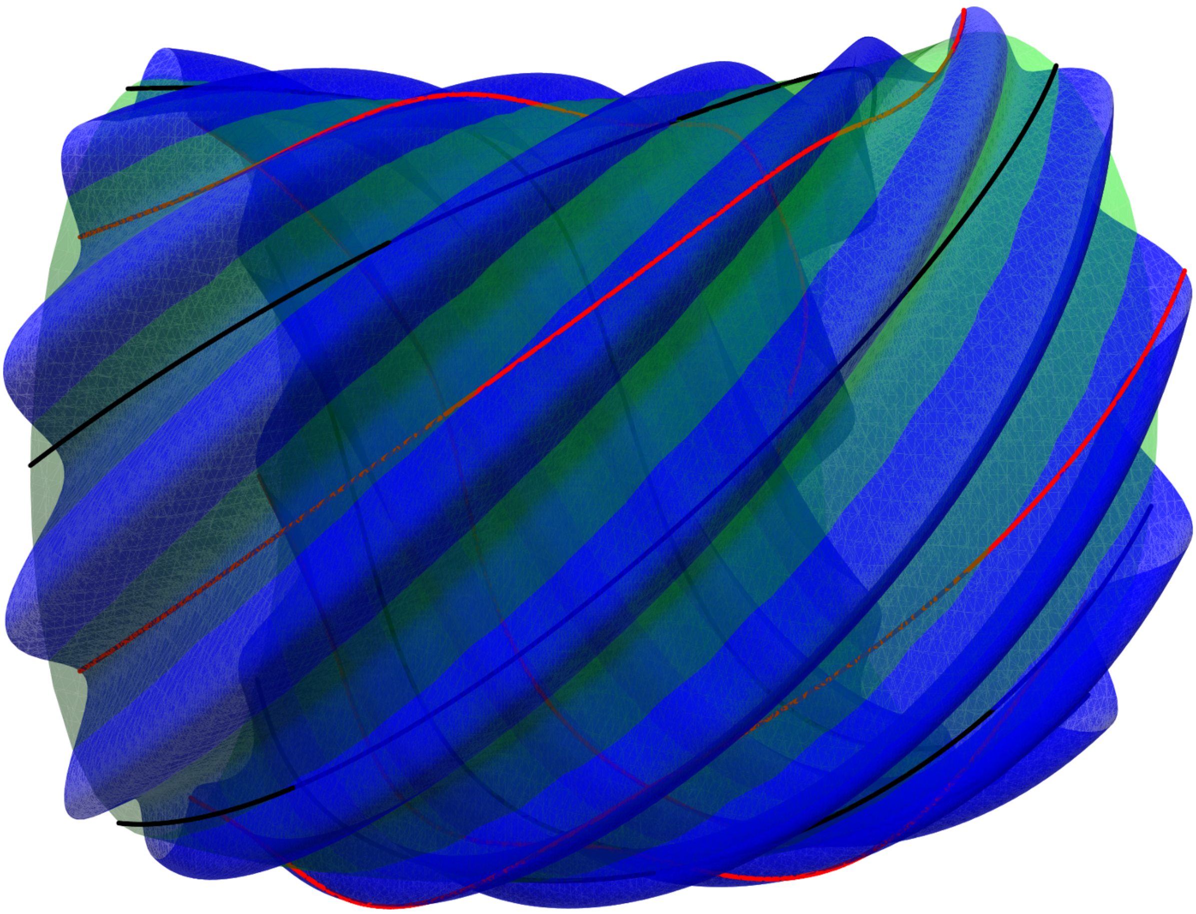

Figure 3.1: The exact field $\widetilde{\boldsymbol{B}}$ and its $\tilde{\psi} = \text{const.}$ surfaces (red lines, blue surface) plotted atop the mean field $\boldsymbol{B}$ and a $\psi = \text{const.}$ surface (black lines, green surface).

system of field-aligned coordinates ($\tilde{\psi}$, $\tilde{\alpha}$, $\tilde{l}$) where $\tilde{l}$ measures distance along a given field line. By definition, $\tilde{l}$ runs from one side of the cut in the poloidal plane to the other side of the cut. As $\delta\boldsymbol{B} \ll \boldsymbol{B}$, the variables aligned to the exact field only deviate slightly from those aligned to the background field:

$$|\tilde{\psi} - \psi| \sim \epsilon\psi, \qquad |\tilde{\alpha} - \alpha| \sim \epsilon\alpha, \qquad |\tilde{l} - l| \sim \epsilon l. \tag{3.64}$$

Importantly, this means that all axisymmetric mean quantities are, to lowest order in ϵ, functions of $\tilde{\psi}$ and $\tilde{l}$ but not of $\tilde{\alpha}$.

We need expressions for the spatial derivative operators $\widetilde{\boldsymbol{b}} \cdot \nabla$ and the Poisson bracket $\{a, b\} = \boldsymbol{b} \cdot (\nabla a \times \nabla b)$ in our new coordinates. By using the fact that $\widetilde{\boldsymbol{b}} \cdot \nabla\tilde{\psi} = \widetilde{\boldsymbol{b}} \cdot \nabla\tilde{\alpha} = 0$, we discover that

$$\widetilde{\boldsymbol{b}} \cdot \nabla a = \widetilde{\boldsymbol{b}} \cdot \nabla\tilde{l}\, \left.\frac{\partial a}{\partial \tilde{l}}\right|_{\tilde{\psi},\tilde{\alpha}} = \left.\frac{\partial a}{\partial \tilde{l}}\right|_{\tilde{\psi},\tilde{\alpha}} \tag{3.65}$$

for any function $a(\boldsymbol{r})$. Similarly, by using the fact that $\widetilde{\boldsymbol{b}} \cdot (\nabla\tilde{\alpha} \times \nabla\tilde{\psi}) = B$ to lowest order, we see that the Poisson bracket takes the form

$$\{a, b\} = \widetilde{\boldsymbol{b}} \cdot (\nabla a \times \nabla b) + \mathcal{O}(\epsilon|\nabla a||\nabla b|) = B\left(\frac{\partial a}{\partial\tilde{\alpha}}\frac{\partial b}{\partial\tilde{\psi}} - \frac{\partial a}{\partial\tilde{\psi}}\frac{\partial b}{\partial\tilde{\alpha}}\right) + \mathcal{O}(\epsilon|\nabla a||\nabla b|). \quad (3.66)$$

Since $\boldsymbol{u}_{\text{eff}}$ is a field-line-preserving velocity, we have that

$$\left(\frac{\partial}{\partial t} + \boldsymbol{u}_{\text{eff}} \cdot \nabla\right)\tilde{\psi} = 0, \quad (3.67)$$

and

$$\left(\frac{\partial}{\partial t} + \boldsymbol{u}_{\text{eff}} \cdot \nabla\right)\tilde{\alpha} = 0. \quad (3.68)$$

As $\tilde{l} \sim l$ and $\nabla \cdot \boldsymbol{u}_{\text{eff}} \sim \epsilon\omega$, $\boldsymbol{u}_{\text{eff}}$ also approximately preserves the length of field lines and we have, to lowest order in ϵ,

$$\left(\frac{\partial}{\partial t} + \boldsymbol{u}_{\text{eff}} \cdot \nabla\right)\tilde{l} = 0. \quad (3.69)$$

It is of course convenient to choose the cut in the definition of $\tilde{\alpha}$ such that it is convected with $\boldsymbol{u}_{\text{eff}}$.

Finally, the time derivative at constant $\tilde{\psi}$, $\tilde{\alpha}$ and $\tilde{l}$ is related to the time derivative at constant $\boldsymbol{r}$ by

$$\left.\frac{\partial}{\partial t}\right|_{\tilde{\psi},\tilde{\alpha},\tilde{l}} = \frac{\partial}{\partial t} + \boldsymbol{u}_{\text{eff}} \cdot \nabla. \quad (3.70)$$

3.2.5 Solution of the Lowest-Order Equations

We now use the magnetic coordinates introduced in §3.2.4 to solve (3.49) and (3.53). We split velocity space (ε_e and μ_e) into two regions: the passing region where, for a given $\tilde{\psi}$, the equation

$$w_\parallel(\tilde{l}, \tilde{\psi}, \varepsilon_e, \mu_e) = \sqrt{\frac{2}{m_e}\left[\varepsilon_e - \mu_e B + e\varphi_0\right]} = 0 \quad (3.71)$$

has no solutions for any $\tilde{l}$ (i.e., $\varepsilon_e > \mu_e B - e\varphi_0$ for all $\tilde{l}$) and the trapped region in which there are two values of $\tilde{l}$ at which $w_\parallel = 0$. The points where $w_\parallel$ vanishes are called the bounce points of the electron's orbit. In the passing region, (3.49) implies that

$$\frac{\partial g_e}{\partial\tilde{l}} = 0, \quad (3.72)$$

so g_e is constant along field lines. As most magnetic surfaces are irrational, one field line spans the entire surface and so, for passing electrons, g_e cannot depend on $\tilde{\alpha}$:

$$g_e = g_{pe}(\tilde{\psi}, \varepsilon_e, \mu_e, \sigma, t). \quad (3.73)$$

In the trapped region of velocity space, (3.49) implies that

$$\text{either}\quad w_\parallel(\tilde{l},\tilde{\psi},\varepsilon_e,\mu_e)=0,\quad \text{or}\quad \frac{\partial g_e}{\partial \tilde{l}}=0. \tag{3.74}$$

Thus, g_e must be constant along magnetic field lines, but only between the bounce points. So g_e can depend on the location of the bounce points, and, therefore, on $\tilde{\alpha}$:

$$g_e = g_{te}(\tilde{\psi},\tilde{\alpha},\varepsilon_e,\mu_e,t), \tag{3.75}$$

where the dependence on σ has been dropped as g_e must be continuous as $w_\parallel$ passes through zero and changes sign. For convenience, we define g_{pe} to be zero in the trapped region and g_{te} to be zero in the passing region, so the complete solution is just the sum of the two functions:

$$g_e = g_{pe}(\tilde{\psi},\varepsilon_e,\mu_e,\sigma,t)+g_{te}(\tilde{\psi},\tilde{\alpha},\varepsilon_e,\mu_e,t). \tag{3.76}$$

The solution of (3.53) in our new coordinates is

$$T_e(\psi)+\overline{\delta T}_e=\tilde{T}_e(\tilde{\psi}), \tag{3.77}$$

for some function $\tilde{T}_e$. As we can absorb any part of $\overline{\delta T}_e$ that is only a function of $\tilde{\psi}$ into g_e, we can pick $\tilde{T}_e(\tilde{\psi})=T_e(\tilde{\psi})$ and so

$$\overline{\delta T}_e=\left(\tilde{\psi}-\psi\right)\frac{dT_e}{d\psi}. \tag{3.78}$$

We can also now pick the arbitrary function $\tilde{\xi}$ in the definition (3.60) of ξ to be any function of $\tilde{\psi}$. In Appendix E, we find convenient forms for the arbitrary functions $\tilde{\xi}(\tilde{\psi})$ and $U(\boldsymbol{r})$ to obtain (see (E.6))

$$\boldsymbol{u}_{\text{eff}}=\boldsymbol{u}+\frac{c}{B}\widetilde{\boldsymbol{b}}\times\nabla\left(\delta\varphi'-\zeta\right), \tag{3.79}$$

where we have defined

$$\zeta=\frac{T_e}{e}\left[\frac{\overline{\delta n}_e}{n_e}-\left(\tilde{\psi}-\psi\right)\frac{d\ln N_e}{d\psi}\right]. \tag{3.80}$$

With this definition, we can express the general solution of (3.46) as

$$\begin{aligned}\delta f_e=&\frac{e\zeta}{T_e}F_{0e}+\left(\tilde{\psi}-\psi\right)\frac{\partial F_{0e}}{\partial\psi}+g_{pe}(\tilde{\psi},\varepsilon_e,\mu_e,\sigma)+g_{te}(\tilde{\psi},\tilde{\alpha},\varepsilon_e,\mu_e)\\&+\mathcal{O}\left(\epsilon\delta F_{0e}\right),\end{aligned} \tag{3.81}$$

and the evolution equation (3.54) for $\delta A_\parallel$ becomes

$$\left(\frac{\partial}{\partial t}+\boldsymbol{u}\cdot\nabla\right)\delta A_\parallel=c\widetilde{\boldsymbol{b}}\cdot\nabla\left(\zeta-\delta\varphi'\right)+\mathcal{O}\left(\epsilon\delta\Omega_i\delta A_\parallel\right). \tag{3.82}$$

This equation allows us to solve for $\delta A_\parallel$ correctly to lowest order in δ. Equations for the evolution of ζ, g_{pe} and g_{te} at the next order in our expansion are obtained in §3.3.

3.2.6 Solution for $\delta \boldsymbol{B}_\perp$

At this juncture, we have enough information to determine $\delta \boldsymbol{B}_\perp$ in two ways. Firstly, we have the evolution equations (3.67) and (3.68) for $\tilde{\psi}$ and $\tilde{\alpha}$, with $\boldsymbol{u}_{\text{eff}}$ given by (3.79). This enables us to calculate $\tilde{\psi}$ and $\tilde{\alpha}$ correctly to first order in ϵ. In Appendix D, we demonstrate that this gives us enough information to construct $\delta A_\parallel$ from the equations

$$\frac{\partial \delta A_\parallel}{\partial \tilde{\psi}} = -\frac{\partial \tilde{\alpha}}{\partial l} \quad \text{and} \quad \frac{\partial \delta A_\parallel}{\partial \tilde{\alpha}} = \frac{\partial \tilde{\psi}}{\partial l}. \tag{3.83}$$

Secondly, we could solve for $\delta A_\parallel$ from the evolution equation (3.82) and then use (3.83) to determine $\tilde{\psi}$ and $\tilde{\alpha}$. Either of these methods is valid, and provides a solution for $\delta A_\parallel$ correct to the requisite order. For the purposes of simulating our equations, a third option presents itself: determine $\delta A_\parallel$ from (3.82), $\tilde{\psi}$ and $\tilde{\alpha}$ from (3.67) and (3.68), and then use (3.83) as consistency checks upon the solutions thus obtained. We choose the first of these options, i.e., to eliminate $\delta A_\parallel$ in favour of the fluctuating fields $\tilde{\psi} - \psi$ and $\tilde{\alpha} - \alpha$.

One point remains to be resolved. It would appear that knowing $\tilde{\psi}$ and $\tilde{\alpha}$ determines $\delta \boldsymbol{B}$ completely (via (3.63)). However, to determine $\delta B_\parallel$ from $\tilde{\psi}$ and $\tilde{\alpha}$, one must know $\tilde{\psi}$ and $\tilde{\alpha}$ to $\mathcal{O}(\epsilon^2)$ or, equivalently, the compressible part of $\boldsymbol{u}_{\text{eff}}$ to higher order ($\mathcal{O}(\epsilon^2 v_{\text{th}_i})$; see Appendix D), but we do not. Therefore, we determine $\delta B_\parallel$ from (3.32) which involves only lowest-order quantities – quantities that we do calculate.

3.2.7 Parallel Electron Currents

As explained in the previous section, we have obtained a solution for $\delta A_\parallel$. However, $\delta A_\parallel$ is usually determined from the parallel component of Ampère's Law, (3.31). Substituting our expansion for h_e (3.41) into the parallel component of Ampére's law (3.31), we find

$$\nabla^2 \delta A_\parallel = \frac{4\pi e}{c}\left(\int d^3\boldsymbol{w} w_\parallel \langle h_e^{(0)} + h_e^{(1)} \rangle_{\boldsymbol{r}}\right) - \frac{4\pi}{c}\sum_{s=i} Z_s e \int d^3\boldsymbol{w} w_\parallel \langle h_s \rangle_{\boldsymbol{r}}, \tag{3.84}$$

where a sum over $s = i$ denotes a sum over all ion species. In this equation, the term due to $h_e^{(0)}$ is larger than the rest by δ^{-1}. Thus, to lowest order,

$$\int d^3\boldsymbol{w} w_\parallel h_e^{(0)} = \int d^3\boldsymbol{w} w_\parallel g_{pe} = 0. \tag{3.85}$$

This is a *constraint* on g_{pe} in the lowest-order solution given by (3.81). Note that the trapped particles do not contribute to the current constraint. The parallel electron flow (and current) is comparable to that of the ions and contained in $h_e^{(1)}$. Defining the parallel electron flow

$$\delta u_{\|e} = \int d^3\boldsymbol{w} w_\| h_e^{(1)} \tag{3.86}$$

reduces (3.84) to an equation for $\delta u_{\|e}$:

$$\delta u_{\|e} = \frac{c}{4\pi e}\nabla^2 \delta A_\| + \sum_{s=i} Z_s \int d^3\boldsymbol{w} w_\| \langle h_s \rangle_{\boldsymbol{r}}. \tag{3.87}$$

Thus, instead of determining $\delta A_\|$ from the electron current, Ampère's law determines the electron flow from $\delta A_\|$ [40]. As we are no longer using $\delta A_\|$ to describe $\delta \boldsymbol{B}_\perp$ in our system of equations, we consider $\nabla_\perp^2 \delta A_\|$ to be a shorthand for a complicated expression involving derivatives of $\tilde{\psi} - \psi$ and $\tilde{\alpha} - \alpha$ via the relations (D.10) or (D.11). We require $\delta u_{\|e}$ to complete the solution in the next order, but we do not need any information about $h_e^{(1)}$ other than $\delta u_{\|e}$.

3.3 First Order: Dynamical Equations for Electrons

In this section, we return to the gyrokinetic equation (3.24) and proceed to the next order in the mass-ratio expansion in order to find evolution equations for g_e and ζ.

Subtracting (3.43) from (3.24) and using the solutions for $h_e^{(0)}$ and $\delta A_\|$ from §3.2.5, we obtain the first-order part of the gyrokinetic equation:

$$\begin{aligned}
&\left(\frac{\partial}{\partial t} + \boldsymbol{u}\cdot\nabla\right) h_e^{(0)} + w_\| \widetilde{\boldsymbol{b}}\cdot\nabla h_e^{(1)} - \frac{w_\|}{B}\left\{\delta A_\|^{(1)}, h_e^{(0)}\right\} + \boldsymbol{V}_{\mathrm{De}}\cdot\nabla h_e^{(0)} \\
&\quad + \frac{c}{B}\left\{\delta\varphi' - \frac{m_e w_\perp^2}{2e}\frac{\delta B_\|}{B}, h_e^{(0)}\right\} - \left\langle C\left[h_e^{(0)}\right]\right\rangle_{\boldsymbol{R}} \\
&= -\frac{eF_{0e}}{T_e}\left(\frac{\partial}{\partial t} + \boldsymbol{u}\cdot\nabla\right)\left(\delta\varphi' - \frac{m_e w_\perp^2}{2e}\frac{\delta B_\|}{B}\right) - \frac{c}{B}\left\{\delta\varphi' - \frac{m_e w_\perp^2}{2e}\frac{\delta B_\|}{B}, F_{0e}\right\} \\
&\quad + \frac{I(\psi) m_e w_\|^2 F_{0e}}{B^2 T_e}\frac{d\omega}{d\psi}\left(\boldsymbol{b}\times\nabla\delta A_\|\right)\cdot\nabla\psi \\
&\quad + w_\| F_{0e}\left[\frac{e}{cT_e}\left(\frac{\partial}{\partial t} + \boldsymbol{u}\cdot\nabla\right)\delta A_\|^{(1)} + \frac{1}{B}\left\{\delta A_\|^{(1)}, \ln F_{0e}\right\}\right],
\end{aligned} \tag{3.88}$$

where we have used (3.45), rewritten terms involving $\langle \boldsymbol{V}_\chi \rangle_{\boldsymbol{R}}$ as Poisson brackets using (3.27) and (3.66), and, as in §3.2, replaced $\boldsymbol{R}_e$ with $\boldsymbol{r}$ as our spatial variable. Thus, the collision operator in (3.88) is evaluated at $\boldsymbol{R}_e = \boldsymbol{r}$. The drift velocity in (3.88) now contains only the leading-order part:

$$\boldsymbol{V}_{\mathrm{De}} = -\frac{c}{eB}\boldsymbol{b}\times\left(m_e w_\|^2 \boldsymbol{b}\cdot\nabla\boldsymbol{b} + \frac{1}{2}m_e w_\perp^2 \nabla \ln B - e\nabla\varphi_0\right) + \mathcal{O}(\epsilon\delta v_{\mathrm{th}_i}). \tag{3.89}$$

We now rewrite (3.46) in terms of δf_e and ζ:

$$\begin{aligned}
&\left(\frac{\partial}{\partial t}+\boldsymbol{u}_{\text{eff}}\cdot\nabla\right)\delta f_e+w_\parallel\widetilde{\boldsymbol{b}}\cdot\nabla h_e^{(1)}-\frac{w_\parallel}{B}\left\{\delta A_\parallel^{(1)},h_e^{(0)}\right\}+\boldsymbol{V}_{\text{De}}\cdot\nabla\left(\delta f_e-\frac{e\delta\varphi'}{T_e}F_{0e}\right)\\
&\quad+\frac{c}{B}\left\{\zeta-\frac{m_e w_\perp^2}{2e}\frac{\delta B_\parallel}{B},\delta f_e\right\}-\langle C[\delta f_e]\rangle_{\boldsymbol{R}}\\
&=\left(\frac{\partial}{\partial t}+\boldsymbol{u}_{\text{eff}}\cdot\nabla\right)\frac{w_\perp^2}{v_{\text{th}_e}^2}\frac{\delta B_\parallel}{B}F_{0e}+\left\{\zeta,\frac{w_\perp^2}{v_{\text{th}_e}^2}\frac{\delta B_\parallel}{B}F_{0e}\right\}-\frac{c}{B}\left\{\delta\varphi'-\frac{m_e w_\perp^2}{2e}\frac{\delta B_\parallel}{B},F_{0e}\right\}\\
&\quad-\frac{I(\psi)m_e w_\parallel^2F_{0e}}{B^2T_e}\frac{d\omega}{d\psi}\left(\boldsymbol{b}\times\nabla\delta A_\parallel\right)\cdot\nabla\psi\\
&\quad+w_\parallel F_{0e}\left[\frac{e}{cT_e}\left(\frac{\partial}{\partial t}+\boldsymbol{u}\cdot\nabla\right)\delta A_\parallel^{(1)}+\frac{1}{B}\left\{\delta A_\parallel^{(1)},\ln F_{0e}\right\}\right],
\end{aligned}\tag{3.90}$$

where we have used (3.47), (3.80) and the fact that

$$\boldsymbol{u}_{\text{eff}}\cdot\nabla=\boldsymbol{u}\cdot\nabla+\frac{c}{B}\left\{\delta\varphi'-\zeta,\cdot\right\},\tag{3.91}$$

which follows directly from (3.79) and (3.66).

Finally, we substitute the form of δf_e from (3.81), to obtain

$$\begin{aligned}
&\left.\frac{\partial g_e}{\partial t}\right|_{\tilde\psi,\tilde\alpha,\tilde l}+w_\parallel\widetilde{\boldsymbol{b}}\cdot\nabla h_e^{(1)}+\boldsymbol{V}_{\text{De}}\cdot\nabla g_e+\frac{c}{eB}\left\{e\zeta-\mu_e\delta B_\parallel,g_e\right\}-\langle C[g_e]\rangle_{\boldsymbol{R}}\\
&=-\frac{F_{0e}}{T_e}\left.\frac{\partial}{\partial t}\right|_{\tilde\psi,\tilde\alpha,\tilde l}\left(e\zeta-\mu_e\delta B_\parallel\right)-\boldsymbol{V}_{\text{De}}\cdot\nabla(\tilde\psi-\psi)\frac{\partial F_{0e}}{\partial\psi}+\boldsymbol{V}_{\text{De}}\cdot\nabla\left(\delta\varphi'-\zeta\right)\frac{eF_{0e}}{T_e}\\
&\quad+\frac{c}{e}\frac{\partial}{\partial\tilde\alpha}\left(e\zeta-\mu_e\delta B_\parallel\right)\frac{\partial F_{0e}}{\partial\tilde\psi}+\frac{I(\psi)m_e w_\parallel^2F_{0e}}{B^2T_e}\frac{d\omega}{d\psi}\left(\boldsymbol{b}\times\nabla\delta A_\parallel\right)\cdot\nabla\psi\\
&\quad+w_\parallel F_{0e}\left[\frac{e}{cT_e}\left(\frac{\partial}{\partial t}+\boldsymbol{u}\cdot\nabla\right)\delta A_\parallel^{(1)}+\frac{1}{B}\left\{\delta A_\parallel^{(1)},\ln F_{0e}\right\}\right]+\frac{w_\parallel}{B}\left\{\delta A_\parallel^{(1)},h_e^{(0)}\right\},
\end{aligned}\tag{3.92}$$

where we have gathered all terms involving g_e together on the left-hand side and all terms involving $\delta A_\parallel^{(1)}$ on the right. Notationally, (3.92) can be confusing as we have made some abbreviations for the sake of conciseness. Firstly, all spatial derivatives are taken at constant t, ε_e and μ_e. Thus, for equations that are solved in the moving coordinate system

$$\nabla=\nabla\tilde\psi\left.\frac{\partial}{\partial\tilde\psi}\right|_{\tilde\alpha,\tilde l,t,\varepsilon_e,\mu_e}+\nabla\tilde\alpha\left.\frac{\partial}{\partial\tilde\alpha}\right|_{\tilde\psi,\tilde l,t,\varepsilon_e,\mu_e}+\nabla\tilde l\left.\frac{\partial}{\partial\tilde l}\right|_{\tilde\psi,\tilde\alpha,t,\varepsilon_e,\mu_e}.\tag{3.93}$$

As this notation is cumbersome, we will continue to use ∇ where appropriate (i.e., where it would expand to more than one derivative). Secondly, all the Poisson brackets in (3.92) are shorthand for the expression (3.66) in terms of $\tilde\psi$ and $\tilde\alpha$ derivatives. Finally, this equation is, intrinsically, an equation *in the frame moving with the magnetic* *field*, and all functions should be interpreted as functions of t, $\tilde{\psi}$, $\tilde{\alpha}$, and $\tilde{l}$ rather than t and $\boldsymbol{r}$.

Equation (3.92) describes how g_e, and thus δf_e, evolves dynamically in response to given perturbed fields. As $\boldsymbol{u}_{\text{eff}}$ is also considered known, (3.67)–(3.69) can be solved to give the complete transformation from the fixed to the moving coordinate system. Thus, we consider all the quantities in (3.92) to be known in both the fixed coordinates (which will be needed when we come to the field equations) (ψ, α, l), and the moving coordinates, $(\tilde{\psi}, \tilde{\alpha}, \tilde{l})$. However, in its current form, (3.92) is not closed, as we know neither $h_e^{(1)}$ nor $\delta A_\parallel^{(1)}$. In the following sections we derive evolution equations for ζ, g_{te} and g_{pe} as solubility constraints of (3.92).

3.3.1 Fluctuating Continuity Equation

The one piece of information that is known about $h_e^{(1)}$ is its parallel velocity moment $\delta u_{\parallel e}$, determined by (3.87). Thus, we integrate (3.92) over all velocities and find

$$\begin{aligned}
&\left.\frac{\partial}{\partial t}\right|_{\tilde{\psi},\tilde{\alpha},\tilde{l}} \left[\frac{e\zeta}{T_e} n_e + \int d^3\boldsymbol{w} g_e - \frac{\delta B_\parallel}{B} n_e\right] + B\frac{\partial}{\partial \tilde{l}}\frac{\delta u_{\parallel e}}{B} + \int d^3\boldsymbol{w} \boldsymbol{V}_{\mathrm{D}e}\cdot\nabla g_e \\
&\quad + \frac{c}{B}\left\{\zeta, \int d^3\boldsymbol{w} g_e\right\} - \frac{c}{e}\left\{\frac{\delta B_\parallel}{B}, \int d^3\boldsymbol{w}\mu_e g_e\right\} \\
&= -\int d^3\boldsymbol{w}\boldsymbol{V}_D\cdot\nabla(\tilde{\psi}-\psi)\frac{\partial F_{0e}}{\partial\psi} + \left(\widehat{\boldsymbol{V}}_D + \boldsymbol{V}_\varphi\right)\cdot\nabla\left(\delta\varphi' - \zeta\right)\frac{en_e}{T_e} \\
&\quad + cn_e\frac{\partial}{\partial\tilde{\alpha}}\left(\zeta - \delta B_\parallel\frac{T_e}{eB}\right)\left(\frac{d\ln N_e}{d\psi} - e\varphi_0\frac{d\ln T_e}{d\psi}\right) - c\frac{\partial \delta B_\parallel}{\partial\tilde{\alpha}}\frac{n_e T_e}{eB}\frac{d\ln T_e}{d\psi} \\
&\quad - \frac{In_e}{B}\frac{d\omega}{d\psi}\frac{\partial}{\partial\tilde{l}}(\tilde{\psi}-\psi),
\end{aligned} \tag{3.94}$$

where we have used $B\widetilde{\boldsymbol{b}} = \widetilde{\boldsymbol{B}} + \mathcal{O}(\epsilon B)$, the fact that $h_e^{(0)}$ carries no current, and

$$\int d^3\boldsymbol{w} = \sum_\sigma \int \frac{B d\varepsilon_e d\mu_e d\vartheta}{m_e^2|w_\parallel|}. \tag{3.95}$$

We have also defined

$$\widehat{\boldsymbol{V}}_D = -\frac{cT_e}{eB}\boldsymbol{b}\times(\boldsymbol{b}\cdot\nabla\boldsymbol{b} + \nabla\ln B) \tag{3.96}$$

and

$$\boldsymbol{V}_\varphi = \frac{c}{B}\boldsymbol{b}\times\nabla\varphi_0 \tag{3.97}$$

so that

$$\int d^3\boldsymbol{w}\boldsymbol{V}_{\mathrm{D}e}F_{0e} = \left(\widehat{\boldsymbol{V}}_D + \boldsymbol{V}_\varphi\right)n_e. \tag{3.98}$$

We have also used $\nabla\cdot\widetilde{\boldsymbol{B}} = 0$ and (D.11) to eliminate $\widetilde{\boldsymbol{b}}$ and $\delta A_\parallel$ from our equation. It is important to note that this equation is written in the moving coordinate system.

It relates $\zeta(\tilde{\psi}, \tilde{\alpha}, \tilde{l}, t)$ to g_e and the other fluctuating fields, also given as functions of $\tilde{\psi}$, $\tilde{\alpha}$ and $\tilde{l}$. In this context, $\tilde{\psi} - \psi$ should be viewed as just another fluctuating field. Therefore, all gradients should be thought of in terms of derivatives with respect to $\tilde{\psi}$ and $\tilde{\alpha}$, for the Poisson bracket this is done via (3.66).

That (3.94) is essentially the fluctuating continuity equation is not immediately obvious. Let us attempt to make this more transparent. Clearly, the first two terms under the time derivative in (3.94) are the perturbed electron density due to the fluctuations; the time derivative of $\delta B_\parallel$ adds to this the fluctuation in the electron density due to the compression of the flux surfaces.[10] This total density is changed by various flows – the compression in the parallel flow $\delta u_{\parallel e}$, the flow of electrons past the flux surface given by $(c/B)\widetilde{\boldsymbol{b}} \times \nabla\zeta$, the grad-$B$ drift due to $\delta B_\parallel$, and the magnetic drifts. Indeed, even the final term in (3.94) can be interpreted as a compression – it is the compression of the electron fluid due to the perturbed flux surfaces "sticking out" into regions with different toroidal angular velocities.

Equation (3.94) provides us with an evolution equation for ζ if we know g_e and the perturbed fields. The evolution equations for g_e given ζ are found in the next two sections.

3.3.2 Trapped Electrons and the Bounce Average

In order to derive an equation for g_e in the trapped region of velocity space, we need to define an averaging operator that eliminates the fast electron bounce motion (i.e., annhilates $w_\parallel \widetilde{\boldsymbol{b}} \cdot \nabla$ and thereby eliminates $h_e^{(1)}$). The appropriate average is the bounce average along the perturbed field *i.e., at fixed* $\tilde{\psi}$ *and* $\tilde{\alpha}$. For an arbitrary function $a(\tilde{\psi}, \tilde{\alpha}, \tilde{l}, \varepsilon_e, \mu_e, \sigma)$, we define the bounce average to be

$$\langle a \rangle_\parallel = \oint \frac{d\tilde{l}}{w_\parallel} a \Bigg/ \oint \frac{d\tilde{l}}{w_\parallel}\,, \tag{3.99}$$

where the integration is carried out from one bounce point where $w_\parallel = 0$, along the magnetic field, to the second bounce point where $w_\parallel$ passes through zero and changes sign, and then back to the first bounce point. This integral is carried out at constant $\tilde{\psi}$, $\tilde{\alpha}$, ε_e, μ_e and ϑ. Thus, equivalently, the bounce average is

$$\langle a \rangle_\parallel = \int_{l_1}^{l_2} \frac{d\tilde{l}}{|w_\parallel|} \left[a(\sigma = 1) + a(\sigma = -1)\right] \Bigg/ 2\int_{l_1}^{l_2} \frac{d\tilde{l}}{|w_\parallel|}, \tag{3.100}$$

[10] In the derivation of the transport equations we show that

$$\left.\frac{\partial}{\partial t}\right|_{\tilde{\psi},\tilde{\alpha},\tilde{l}} \frac{\delta B_\parallel}{B} = \left(\mathsf{I} - \widetilde{\boldsymbol{b}}\widetilde{\boldsymbol{b}}\right) : \nabla \boldsymbol{u}_{\text{eff}},$$

so $\partial \delta B_\parallel / \partial t$ is equal to the perpenicular compression of the flux surfaces.

where l_1 and l_2 are the bounce points of the particle orbit.

Clearly, the bounce average commutes with the time derivative at fixed moving coordinates $\tilde{\psi}$, $\tilde{\alpha}$ and $\tilde{l}$. In Appendix F, we prove that

$$\left\langle w_{\parallel}\widetilde{\boldsymbol{b}}\cdot\nabla h_e^{(1)}\right\rangle_{\parallel}=0 \tag{3.101}$$

and that, to lowest order in ϵ,

$$\left\langle \boldsymbol{V}_{\mathrm{De}}\cdot\nabla\tilde{\psi}\right\rangle_{\parallel}=0. \tag{3.102}$$

Using these results, we take (3.92) in the trapped region of velocity space and perform the bounce average to find

$$\begin{aligned}
&\left.\frac{\partial}{\partial t}\right|_{\tilde{\psi},\tilde{\alpha},\tilde{l}} g_{te}+\langle\boldsymbol{V}_{\mathrm{De}}\cdot\nabla g_{te}\rangle_{\parallel}+\frac{c}{eB}\left\{e\,\langle\zeta\rangle_{\parallel}-\mu_e\left\langle\delta B_{\parallel}\right\rangle_{\parallel},g_{te}\right\}-\langle\langle C[g_{te}]\rangle_{\boldsymbol{R}}\rangle_{\parallel}\\
&=-\left.\frac{\partial}{\partial t}\right|_{\tilde{\psi},\tilde{\alpha},\tilde{l}}\left(\langle\zeta\rangle_{\parallel}-\mu_e\left\langle\delta B_{\parallel}\right\rangle_{\parallel}\right)\frac{F_{0e}}{T_e}-\left\langle\boldsymbol{V}_{\mathrm{De}}\cdot\nabla\left(\tilde{\psi}-\psi\right)\right\rangle_{\parallel}\frac{\partial F_{0e}}{\partial\psi}\\
&\quad+\langle\boldsymbol{V}_{\mathrm{De}}\cdot\nabla\left(\delta\varphi'-\zeta\right)\rangle_{\parallel}\frac{eF_{0e}}{T_e}+\frac{c}{e}\frac{\partial}{\partial\tilde{\alpha}}\left(e\,\langle\zeta\rangle_{\parallel}-\mu_e\left\langle\delta B_{\parallel}\right\rangle_{\parallel}\right)\frac{\partial F_{0e}}{\partial\psi}\\
&\quad+\left\langle\frac{m_e w_{\parallel}^2}{B}\frac{\partial}{\partial\tilde{l}}(\tilde{\psi}-\psi)\right\rangle_{\parallel}\frac{I(\psi)}{T_e}\frac{d\omega}{d\psi}F_{0e},
\end{aligned} \tag{3.103}$$

where the last line in (3.92) has vanished under the bounce average because it is antisymmetric in σ (the sign of the parallel velocity). This equation is a closed equation for g_{te}, once the fields and ζ are given. Note that all quantities must be expressed in the moving coordinates, in order to perform the bounce average. Again, as with (3.94), all gradients should be interpreted in terms of derivatives with respect to the moving coordinates as should the Poisson bracket (via (3.66)).

3.3.3 Passing Electrons and the Flux Surface Average

In the passing region of velocity space, the fast streaming of electrons along perturbed field lines causes a passing electron to sample an entire perturbed flux surface ($\tilde{\psi}=$ const. surface). Thus, the averaging procedure we will need is the average over a perturbed flux surface. For any function a, this is defined by

$$\langle a(\boldsymbol{r},\varepsilon_e,\mu_e,\vartheta,\sigma)\rangle_{\tilde{\psi}}=\lim_{\Delta\psi\to 0}\left\{\int_{\Delta}d^3\boldsymbol{r}\,a(\boldsymbol{r},\varepsilon_e,\mu_e,\vartheta,\sigma)\Big/\int_{\Delta}d^3\boldsymbol{r}\right\}, \tag{3.104}$$

where the region of integration Δ is the volume between the surface labelled by $\tilde{\psi}$ and that labelled by $\tilde{\psi}+\Delta\psi$, and the integral is taken at constant ε_e, μ_e and ϑ. In Appendix G, we prove that

$$\left\langle\widetilde{\boldsymbol{B}}\cdot\nabla h_e^{(1)}\right\rangle_{\tilde{\psi}}=0, \tag{3.105}$$

that the flux surface average commutes with the time derivative following the magnetic field, that

$$\left\langle \frac{B}{|w_\parallel|} \boldsymbol{V}_{\mathrm{De}} \cdot \nabla\psi \right\rangle_{\tilde{\psi}} = 0, \tag{3.106}$$

and that, for any function $a(\boldsymbol{r})$,

$$\left\langle \frac{B}{|w_\parallel|} \left(\widetilde{\boldsymbol{b}} \times \nabla a\right) \cdot \nabla\tilde{\psi} \right\rangle_{\tilde{\psi}} = 0. \tag{3.107}$$

We also prove that, for any fluctuating function λ,

$$\left\langle \frac{B}{|w_\parallel|} \boldsymbol{V}_{\mathrm{De}} \cdot \nabla\lambda \right\rangle_{\tilde{\psi}} = 0. \tag{3.108}$$

Taking (3.92) in the passing region of velocity space, multiplying by $B/|w_\parallel|$, and flux-surface averaging (to eliminate $w_\parallel \widetilde{\boldsymbol{b}} \cdot \nabla h_e^{(1)}$), we obtain:

$$\begin{aligned}
&\left\langle \frac{B}{|w_\parallel|} \right\rangle_{\tilde{\psi}} \left.\frac{\partial g_{pe}}{\partial t}\right|_{\tilde{\psi},\tilde{\alpha},\tilde{l}} - \left\langle \frac{B}{|w_\parallel|} \langle C[g_{pe}] \rangle_{\boldsymbol{R}} \right\rangle_{\tilde{\psi}} \\
&\quad = -\frac{F_{0e}}{T_e} \left.\frac{\partial}{\partial t}\right|_{\tilde{\psi},\tilde{\alpha},\tilde{l}} \left\langle \frac{B}{|w_\parallel|} \left(e\zeta - \mu_e \delta B_\parallel\right) \right\rangle_{\tilde{\psi}} + \left\langle \frac{m_e |w_\parallel|}{B} \left(\boldsymbol{b} \times \nabla\delta A_\parallel\right) \cdot \nabla\psi \right\rangle_{\tilde{\psi}} \frac{I(\psi)}{T_e} \frac{d\omega}{d\psi} F_{0e} \\
&\qquad + \sigma \left\langle B F_{0e} \left(\frac{e}{cT_e} \frac{\partial \delta A_\parallel^{(1)}}{\partial t} + \frac{1}{B} \left\{ \delta A_\parallel^{(1)}, \ln F_{0e} \right\} \right) \right\rangle_{\tilde{\psi}},
\end{aligned} \tag{3.109}$$

where the final term in (3.92) vanishes because in the passing region g_e is only a function of $\tilde{\psi}$ and we can use (3.107). We still have to eliminate the final line of (3.109). We note that we only require the part of g_{pe} that is even in σ – we require its density in the quasineutrality condition (3.29) and its pressure in perpendicular Ampère's law (3.32). Thus, we sum (3.109) over σ and henceforth g_{pe} denotes only the even part. Hence we obtain

$$\begin{aligned}
&\left\langle \frac{B}{|w_\parallel|} \right\rangle_{\tilde{\psi}} \left.\frac{\partial g_{pe}}{\partial t}\right|_{\tilde{\psi},\tilde{\alpha},\tilde{l}} - \left\langle \frac{B}{|w_\parallel|} \langle C[g_{pe}] \rangle_{\boldsymbol{R}} \right\rangle_{\tilde{\psi}} \\
&\quad = -\frac{F_{0e}}{T_e} \left.\frac{\partial}{\partial t}\right|_{\tilde{\psi},\tilde{\alpha},\tilde{l}} \left\langle \frac{B}{|w_\parallel|} \left(e\zeta - \mu_e \delta B_\parallel\right) \right\rangle_{\tilde{\psi}} - \left\langle m_e |w_\parallel| \frac{\partial}{\partial \tilde{l}} (\tilde{\psi} - \psi) \right\rangle_{\tilde{\psi}} \frac{I(\psi)}{T_e} \frac{d\omega}{d\psi} F_{0e}.
\end{aligned} \tag{3.110}$$

This completes our solution for g_e, as (3.110) is a closed equation for the part of g_{pe} that is even in σ. It is useful to note that only the zonal (i.e., $\tilde{\alpha}$-independent) parts of ζ and $\delta B_\parallel$ will give a significant drive for g_{pe} – the $\tilde{\alpha}$-dependent parts will tend to cancel under the flux-surface average.

3.3.4 Field Equations

In this section, we finally close our system of equations by writing the field equations in terms of our solution for δf_e and ion quantities.

We find $\delta\varphi'$ by using the solution (3.81) for δf_e in the quasineutrality condition (3.29):

$$\frac{1}{n_e}\sum_{s=i}\frac{Z_s^2 e n_s \delta\varphi'}{T_s} = \frac{e\zeta}{T_e} + \left(\tilde{\psi} - \psi\right)\frac{d\ln N_e}{d\psi} + \frac{1}{n_e}\int d^3\boldsymbol{w}\, g_e + \sum_{s=i}\frac{Z_s}{n_e}\int d^3\boldsymbol{w}\langle h_s\rangle_{\boldsymbol{r}}. \tag{3.111}$$

Similarly, we find $\delta B_\parallel$ by using (3.81) in (3.32) to find

$$\begin{aligned} -\nabla_\perp^2\frac{\delta B_\parallel B}{4\pi} = &\sum_{s=i}\nabla_\perp\nabla_\perp : \int m_s\langle \boldsymbol{w}_\perp\boldsymbol{w}_\perp h_s\rangle_{\boldsymbol{r}} + n_e\nabla_\perp^2\left(\zeta - \delta\varphi'\right) \\ &+ n_e\nabla_\perp^2\left(\tilde{\psi} - \psi\right)\left[T_e\frac{d\ln N_e}{d\psi} + \left(T_e - e\varphi_0\right)\frac{d\ln T_e}{d\psi}\right] \\ &+ \nabla_\perp^2\int d^3\boldsymbol{w}\frac{m_e w_\perp^2}{2}g_e. \end{aligned} \tag{3.112}$$

These two equations determine $\delta\varphi'$ and $\delta B_\parallel$.

It is important to note that (3.111) and (3.112) are *not* naturally solved in the moving coordinate system. It is because of this coupling to the fixed coordinates (and the ion physics) that we require the mapping between the coordinate systems (from the solution of (3.67)–(3.69)).

3.4 Summary of the Reduced Electron Model

The set of electron equations we have derived is complicated: the electron timescale has been removed at the expense of introducing more dependent variables and more equations, some integro-differential. In these equations, we have the nine unknowns $\tilde{\psi}$, $\tilde{\alpha}$, $\delta u_{\parallel e}$, ζ, g_{te}, g_{pe}, h_i, $\delta\varphi'$, and $\delta B_\parallel$. We have provided a complete set of nine equations to determine these quantities. These are:

- (3.67) and (3.68) to evolve $\tilde{\psi}$ and $\tilde{\alpha}$ which define the moving coordinate system that moves with velocity $\boldsymbol{u}_{\text{eff}}$ given by (3.79),
- (3.87) to determine $\delta u_{\parallel e}$,
- and the coupled set (3.94), (3.103), (3.110), and (3.24) to evolve ζ, g_{te}, g_{pe}, and h_i with the constituent relations (3.111) and (3.112) determining $\delta\varphi'$ and $\delta B_\parallel$.

Together, these equations comprise a rigorously-derived model for ion-scale turbulence, accurate up to corrections of order δ. The only assumption we have had to make, beyond our orderings, is that any stochastic component of the magnetic field (if present) satisfies $\delta B_{\mathrm{stoch}}/B \ll \epsilon\delta$.

3.5 The Collisional Limit

The equations for δf_e in §3.2.5 and §3.3.4 are derived under the assumption that $\nu_{ee} \sim \omega \sim k_\parallel v_{\mathrm{th}_i}$. In many cases of interest, the collision frequency is in fact much larger than this: $\nu_{ee} \gg \omega$. We can take this limit as a subsidiary expansion, subsidiary to our expansion in the mass ratio.[11]

Formally, we introduce the parameter $\nu_e^* = \nu_{ee} q R / v_{\mathrm{th}_e}$, where $q(\psi)$ is the safety factor (see §2.6.3) and R the cylindrical radial coordinate, also known as the major radius of the tokamak (see §2.2.2). We then let $\nu_e^* \gg \delta$ and expand in

$$\frac{\delta}{\nu_e^*} \ll 1. \tag{3.113}$$

The results of this expansion will be presented in the following sections.

3.5.1 Maxwellian Electrons

In our collisional expansion, $C[g_e]$ is the dominant term in (3.92). Thus, up to corrections that are small in $\omega/\nu_{ee} \ll 1$, g_e is given by

$$C[g_e] = 0. \tag{3.114}$$

Hence, g_e is a perturbed Maxwellian to leading order:

$$g_e = \left\{ \frac{\delta n_H}{N_e} + \left(\frac{\varepsilon_e}{T_e} - \frac{3}{2} \right) \frac{\delta T_e}{T_e} \right\} F_{0e} + \mathcal{O}\left(\frac{\delta}{\nu_e^*} \epsilon f_e \right). \tag{3.115}$$

where δn_H and δT_e are arbitrary functions of $\boldsymbol{r}$ – δn_H will shortly be shown to be the homogenous (i.e., a function only of $\tilde{\psi}$) part of the electron density perturbation and δT_e the perturbed electron temperature.

Now, we know from §3.2.1 that g_e is a solution of (3.49). Substituting this into (3.49), we see that $\delta n_H/N_e$ and $\delta T_e/T_e$ are functions of $\tilde{\psi}$ only (the derivatives in (3.49) being taken at constant ε_e). Thus, in the this limit, collisions trap and de-trap electrons

[11]Strictly speaking, to treat this as a subsidiary expansion we also require that $\nu_{ee} \ll k_\parallel v_{\mathrm{th}_e}$, so that it does not change the form of (3.43) and (3.46). However, if ν_{ee} is large enough that $\nu_{ee} \sim k_\parallel v_{\mathrm{th}_e}$, then the homogenous equation for g_e is $w_\parallel \tilde{\boldsymbol{b}} \cdot \nabla g_e = C[g_e]$. The only solution of this equation is a perturbed Maxwellian with perturbed densities and temperatures that are functions of $\tilde{\psi}$ only. Thus, the results of this section still hold.

rapidly enough that there is no longer a distinction between passing and trapped electrons ($\tilde{\alpha}$-dependence being the signature of kinetic trapped electrons) – all electrons sample the entire flux surface on the fluctuation timescale.

Let us now further simplify our solution for g_e. Substituting (3.115) into (3.81), we obtain

$$\delta f_e = \left(\frac{e\zeta}{T_e} + \frac{\delta n_H}{N_e}\right) F_{0e} + \left(\tilde{\psi} - \psi\right) \frac{\partial F_{0e}}{\partial \psi} + \left(\frac{\varepsilon_e}{T_e} - \frac{3}{2}\right) \frac{\delta T_e(\tilde{\psi})}{T_e} F_{0e} + \mathcal{O}\left(\epsilon \delta F_{0e}\right). \quad (3.116)$$

From this expression, we observe that we were justified in our notation for δT_e – it really is the perturbed electron temperature relative to the moving surfaces. We also note that ζ and δn_H both contribute to the perturbed density. As we have the freedom to add an arbitrary function of $\tilde{\psi}$ to the definition of ζ – it does not change the velocity of the flux surface (see the end of §3.2.2), we redefine ζ to be $\zeta - T_e \delta n_H / e N_e$ and so eliminate δn_H entirely. Thus, our solution for δf_e becomes

$$\delta f_e = \frac{e\zeta}{T_e} F_{0e} + \left(\tilde{\psi} - \psi\right) \frac{\partial F_{0e}}{\partial \psi} + \left(\frac{\varepsilon_e}{T_e} - \frac{3}{2}\right) \frac{\delta T_e(\tilde{\psi})}{T_e} F_{0e} + \mathcal{O}\left(\frac{\delta}{\nu_e^*} \epsilon F_{0e}\right), \quad (3.117)$$

where the entire density perturbation is contained in the Boltzmann-like first term.

3.5.2 Fluctuating Continuity Equation

We now derive an equation for ζ in the collisional limit.

Using the solution (3.115) for g_e in (3.94), we obtain, to lowest order in δ/ν_e^*,

$$\begin{aligned}
&\left.\frac{\partial}{\partial t}\right|_{\tilde{\psi},\tilde{\alpha},\tilde{l}} \left(\frac{e\zeta}{T_e} - \frac{e\varphi_0}{T_e}\frac{\delta T_e}{T_e} - \frac{\delta B_\parallel}{B}\right) n_e + B \frac{\partial}{\partial \tilde{l}} \frac{\delta u_{\parallel e}}{B} - \frac{cT_e}{eB}\left\{\frac{\delta B_\parallel}{B}, \frac{\delta T_e}{T_e}\right\} n_e \left(1 - \frac{e\varphi_0}{T_e}\right) \\
&= -\widehat{\boldsymbol{V}}_D \cdot \nabla\left(\tilde{\psi} - \psi\right)\left(\frac{d\ln N_e}{d\psi} + \frac{d\ln T_e}{d\psi}\right) n_e - \boldsymbol{V}_\varphi \cdot \nabla\left(\tilde{\psi} - \psi\right)\frac{d\ln N_e}{d\psi} n_e \\
&\quad + \frac{ec\varphi_0}{BT_e}\left\{\zeta, \frac{\delta T_e}{T_e}\right\} - \widehat{\boldsymbol{V}}_D \cdot \nabla \frac{\delta T_e}{T_e} n_e + \left(\widehat{\boldsymbol{V}}_D + \boldsymbol{V}_\varphi\right) \cdot \nabla\left(\delta\varphi' - \zeta\right)\frac{en_e}{T_e} \\
&\quad + c\frac{\partial}{\partial\tilde{\alpha}}\left(\zeta - \delta B_\parallel \frac{T_e}{eB}\right)\frac{\partial \ln N_e}{\partial \psi} n_e - c\frac{\partial \delta B_\parallel}{\partial \tilde{\alpha}}\frac{n_e T_e}{eB}\frac{\partial \ln T_e}{\partial \psi} - \frac{In_e}{B}\frac{d\omega}{d\psi}\frac{\partial}{\partial \tilde{l}}(\tilde{\psi} - \psi),
\end{aligned} \quad (3.118)$$

where we have used the fact that, as we have absorbed the density perturbation into ζ, $\int d^3\boldsymbol{w} g_e = \mathcal{O}((\delta/\nu_e^*)\epsilon n_e)$. As in §3.3.1, this is just the fluctuating continuity equation for electrons.

3.5.3 Fluctuating Energy Equation

To close our system of collisional equations, we now need the fluctuating energy equation from which we will determine δT_e. Thus, we multiply (3.92) by $m_e w^2/(2T_e) -$

3/2 and integrate over all velocities to find

$$\begin{aligned}
&\left.\frac{\partial}{\partial t}\right|_{\tilde{\psi},\tilde{\alpha},\tilde{l}}\left(\frac{3}{2}\frac{\delta T_e}{T_e}-\frac{\delta B_\parallel}{B}\right)n_e+\nabla\cdot\left(\widetilde{\boldsymbol{b}}q_{\parallel e}\right)+\frac{c}{B}\left\{\zeta-\frac{T_e}{e}\frac{\delta B_\parallel}{B},\frac{\delta T_e}{T_e}\right\}n_e\\
&\quad=-\frac{1}{2}\left(7\widehat{\boldsymbol{V}}_D+3\boldsymbol{V}_\varphi\right)\cdot\nabla\left(\tilde{\psi}-\psi\right)\frac{\partial}{\partial\tilde{\psi}}\left(\frac{\delta T_e}{T_e}+\ln T_e\right)n_e\\
&\qquad-\widehat{\boldsymbol{V}}_D\cdot\nabla\left(\tilde{\psi}-\psi\right)\frac{\partial\ln N_e}{\partial\tilde{\psi}}n_e+\widehat{\boldsymbol{V}}_D\cdot\nabla\left(\delta\varphi'-\zeta\right)\frac{en_e}{T_e}\\
&\qquad-\frac{\partial\delta B_\parallel}{\partial\tilde{\alpha}}\frac{cT_en_e}{eB}\left(\frac{\partial\ln N_e}{\partial\psi}+\frac{\partial\ln T_e}{\partial\psi}\right)-\frac{In_e}{B}\frac{d\omega}{d\psi}\frac{\partial}{\partial\tilde{l}}(\tilde{\psi}-\psi),
\end{aligned}\tag{3.119}$$

where $q_{\parallel e}=\int d^3\boldsymbol{w}\,(m_ew^2/2T_e-3/2)\,h_e^{(1)}$. As we do not know $q_{\parallel e}$, we wish to eliminate it from this equation. Therefore, we average (3.119) over the perturbed flux surfaces and obtain

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(\frac{\delta T_e}{T_e}\left\langle n_e\right\rangle_{\tilde{\psi}}+\left\langle\frac{2\delta B_\parallel}{3B}n_e\right\rangle_{\tilde{\psi}}\right)+\left\langle\boldsymbol{V}_\varphi\cdot\nabla\left(\frac{\delta T_e}{T_e}n_e\right)\right\rangle_{\tilde{\psi}}=\\
&-\left\langle\left(\frac{7}{3}\widehat{\boldsymbol{V}}_D+\boldsymbol{V}_\varphi\right)\cdot\nabla(\tilde{\psi}-\psi)\frac{\partial\ln T_e}{\partial\tilde{\psi}}n_e\right\rangle_{\tilde{\psi}}-\frac{2}{3}\left\langle\widehat{\boldsymbol{V}}_D\cdot\nabla(\tilde{\psi}-\psi)\frac{\partial\ln N_e}{\partial\tilde{\psi}}n_e\right\rangle_{\tilde{\psi}}\\
&-\frac{2}{3}\left\langle\frac{n_e}{B}\frac{\partial}{\partial\tilde{l}}(\tilde{\psi}-\psi)\right\rangle_{\tilde{\psi}}I(\psi)\frac{d\omega}{d\psi},
\end{aligned}\tag{3.120}$$

where we have used (G.6) to eliminate some terms.

Let us examine the terms in (3.120). Other than the last term on the right-hand side, which is due to the viscous heating of electrons, these terms seem fairly opaque. However, they all vanish if we apply the low-Mach-number ordering of §2.10.1 – we can move n_e outside the flux-surface averages and then use the properties of the flux-surface average and the drift velocities to eliminate the terms in question. Thus, we conclude that these terms, which disappear if the Mach number is small, are to do with the exchange of potential energy of the electrons with thermal energy.

Equation (3.120) completes our solution for δf_e. In the next section, we close the system by working out the electron contributions to the field equations in the collisional limit.

3.5.4 Field Equations

We now use our solution for δf_e to simplify the field equations. Substituting the solution (3.117) into (3.111) gives the following equation for $\delta\varphi'$:

$$\frac{1}{n_e}\sum_{s=i}\frac{Z_s^2en_s\delta\varphi'}{T_s}=\frac{e\zeta}{T_e}+\left(\tilde{\psi}-\psi\right)\frac{d\ln N_e}{d\psi}-\frac{e\varphi_0}{T_e}\frac{\delta T_e}{T_e}+\sum_{s=i}\frac{Z_s}{n_e}\int d^3\boldsymbol{w}\langle h_s\rangle_{\boldsymbol{r}}.\tag{3.121}$$

Similarly, by substituting (3.117) into (3.112), we obtain an equation for $\delta B_\parallel$ in the collisional limit:

$$-\nabla_\perp^2 \frac{\delta B_\parallel B}{4\pi} = \sum_{s=i} \nabla_\perp \nabla_\perp : \int m_s \langle \boldsymbol{w}_\perp \boldsymbol{w}_\perp h_s \rangle_{\boldsymbol{r}} + n_e \nabla_\perp^2 \left(\zeta - \delta\varphi'\right) + n_e \nabla_\perp^2 \left(\tilde{\psi} - \psi\right) \left[T_e \frac{d\ln N_e}{d\psi} + (T_e - e\varphi_0) \frac{d\ln T_e}{d\psi}\right] + \left(1 - \frac{e\varphi_0}{T_e}\right) n_e \nabla_\perp^2 \delta T_e. \tag{3.122}$$

The field equations (3.121) and (3.122) close the collisional electron model, which we now summarise.

3.5.5 The Collisional Electron Model

The collisional electron model is a simplified version of the model given in §3.4. We have replaced the distribution functions g_{pe} and g_{te} by a single perturbed electron density and temperature. The complete set of equations describing this model is:

- (3.67) and (3.68) to evolve $\tilde{\psi}$ and $\tilde{\alpha}$ which define the moving coordinate system,
- (3.87) to determine $\delta u_{\parallel e}$,
- ζ is given by (3.118),
- δT_e is given by (3.119),
- (3.121) and (3.122) determine the fields $\delta\varphi'$ and $\delta B_\parallel$ respectively,
- and the ion dynamics (h_i) are given by (3.24).

3.6 The Transport Timescale

The procedure of §3.4 is sufficient to calculate the evolution of our plasma on the fast (turbulent) timescale. Our expansion in δ also has consequences for the evolution of n_e, T_e and ω on the slow (transport) timescale. In §2.7 of Ch. 2, transport equations for n_e and T_e were derived by considering the transport of particles and energy across $\psi = \mathrm{const.}$ surfaces. It is clear from the results of §3.2.2 and §3.2.3 that, for electron transport, it is more appropriate to consider $\tilde{\psi} = \mathrm{const.}$ surfaces.

To this end, in Appendix H, we rewrite the electron kinetic equation relative to the moving flux surfaces. We then average this equation over the turbulence and over the fluctuating flux surfaces. Finally, we integrate over velocity space to find transport equations for particles and multiply by ε_e and integrate over velocities to obtain the electron heat transport equation. This procedure is entirely analogous to that undertaken in §2.7 save for the extra complexity introduced by the rapid fluctuation of the $\tilde{\psi} = \mathrm{const.}$ surfaces. We summarize the results of these calculations in the following sections.

3.6.1 Particle Transport

In Appendix H.3, we derive the electron particle transport equation, correct to lowest order in δ:

$$\frac{1}{V'} \frac{\partial}{\partial t}\bigg|_{\tilde{\psi}} V' \langle n_e \rangle_{\tilde{\psi}} + \frac{1}{V'} \frac{\partial}{\partial \tilde{\psi}} V' \langle \Gamma_e \rangle_{\tilde{\psi}} = \left\langle S_e^{(n)} \right\rangle_{\tilde{\psi}}, \tag{3.123}$$

where $S_e^{(n)}$ is defined by (2.164) and the particle flux is given by

$$\langle \Gamma_e \rangle_{\tilde{\psi}} = \left\langle \left\langle c \int d^3\boldsymbol{w} g_{te} \frac{\partial}{\partial \tilde{\alpha}} \left(\zeta + \frac{w_\perp^2}{2\Omega_e} \frac{\delta B_\parallel}{B} \right) \right\rangle_{\mathrm{turb}} \right\rangle_{\tilde{\psi}}. \tag{3.124}$$

Note that whilst we use the same notation for the particle flux Γ_e as in Ch. 2, it is not the same quantity. In this chapter the transport equations are derived for transport across a $\tilde{\psi} = \mathrm{const.}$ surface not a $\psi = \mathrm{const.}$ surface.

Several important features of this flux immediately present themselves. Firstly, only the trapped electrons contribute – the passing particles play no part in the transport. Secondly, there is no "flutter" transport despite the fact that the contribution to (2.170) from $\delta A_\parallel$ is clearly not zero. This is, in fact, the primary advantage of writing our transport equations with respect to the moving surfaces, we observe this cancellation analytically rather than having to recover it via a very careful evaluation of (2.166) and (2.170). Finally, we see that drifts that do cause transport are the drift due to ζ and the ∇B drift due to $\delta B_\parallel$. This supports our interpretation of the drift due to ζ being the drift of electrons across the moving surfaces.

Electron Transport in the Collisional Model

As we proved in §3.115, in the collisional limit the electrons are trapped and detrapped fast enough to remove the distinction between trapped and passing particles – in effect all electrons are passing. This immediately implies that if the electrons are collisional, there is *no* transport of electrons. Thus, the electron transport time is $\mathcal{O}(\delta^{-1} (\omega/\nu_{ee}))$ longer than the ion transport time – collisions are actually beneficial. Naturally, with sufficiently large collisions, neoclassical and classical electron transport reappear in the fluxes and the confinement worsens again.

3.6.2 Electron Heat Transport

Similarly, in §Appendix I.4, we derive the electron heat transport equation, to lowest order in δ:

$$\frac{1}{V'}\frac{3}{2}\left.\frac{\partial}{\partial t}\right|_{\tilde{\psi}} V' \langle n_e \rangle_{\tilde{\psi}} T_e + \frac{1}{V'}\frac{\partial}{\partial \tilde{\psi}} V' \langle q_e \rangle_{\tilde{\psi}} = P_e^{\text{turb}} + P_s^{\text{comp}} + P_s^{\text{pot}} + C_s^{(E)} + S_s^{(E)}, \tag{3.125}$$

where $C_s^{(E)}$ and $S_s^{(E)}$ are defined by (2.192) and (2.193) respectively, the heat flux is given by

$$\langle q_e \rangle_{\tilde{\psi}} = \left\langle \left\langle c \int d^3\boldsymbol{w} \left(\frac{1}{2} m_e w^2 - e\varphi_0\right) g_{te} \frac{\partial}{\partial \tilde{\alpha}} \left(\zeta + \frac{w_\perp^2}{2\Omega_e}\frac{\delta B_\parallel}{B}\right)\right\rangle_{\text{turb}} \right\rangle_{\tilde{\psi}}, \tag{3.126}$$

and the heat sources are given by

$$P_s^{\text{comp}} = -T_e \langle n_e \nabla \cdot \boldsymbol{V}_\psi \rangle_{\tilde{\psi}}, \tag{3.127}$$

$$\begin{aligned} P_s^{\text{pot}} = &\left\langle e\varphi_0 \left(\left.\frac{\partial}{\partial t}\right|_{\tilde{\psi},\tilde{\alpha},\tilde{l}} n_e - S_e^{(n)} \right) \right\rangle_{\tilde{\psi}} + \langle e\varphi_0 n_e \nabla \cdot \boldsymbol{V}_\psi \rangle_{\tilde{\psi}} \\ &+ \left\langle e\varphi_0 \left\langle \left(\frac{e\zeta}{T_e} n_e + \int d^3\boldsymbol{w}\, g_e \right) \nabla \cdot \boldsymbol{u}_{\text{eff}} \right\rangle_{\text{turb}} \right\rangle_{\tilde{\psi}}, \end{aligned} \tag{3.128}$$

and

$$P_e^{\text{turb}} = -\left\langle \int d^3\boldsymbol{w} \left\langle g_e \left.\frac{\partial}{\partial t}\right|_{\tilde{\psi},\tilde{\alpha},\tilde{l}} \left(e\zeta - \mu_e \delta B_\parallel\right) \right\rangle_{\text{turb}} \right\rangle_{\tilde{\psi}} + e \left\langle n_e \langle \zeta \nabla \cdot \boldsymbol{u}_{\text{eff}} \rangle_{\text{turb}} \right\rangle_{\tilde{\psi}}. \tag{3.129}$$

In these expressions for the heating, the divergence of $\boldsymbol{u}_{\text{eff}}$ is given by

$$\nabla \cdot \boldsymbol{u}_{\text{eff}} = \left.\frac{\partial}{\partial t}\right|_{\tilde{\psi},\tilde{\alpha},\tilde{l}} \frac{\delta B_\parallel}{\widetilde{B}} + \left(\widetilde{\boldsymbol{b}} \cdot \nabla \psi\right) \frac{I(\psi)}{B}\frac{d\omega}{d\psi} - \frac{c}{B}\left(\boldsymbol{b} \cdot \nabla \boldsymbol{b}\right) \cdot \boldsymbol{b} \times \nabla \left(\zeta - \delta\varphi'\right). \tag{3.130}$$

This expression is correct and closed to the required order in ϵ and δ. Again, it is crucial to note that whilst we use the same notation for q_e, P_s^{comp}, P_s^{pot}, and P_e^{turb} as in Ch. 2, these are not the same quantities.

Let us now interpret the heat flux and the heating terms. The heat flux parallels the particle flux: there is no flutter transport, only trapped particles contribute to the flux, and they contribute via the $\boldsymbol{E} \times \boldsymbol{B}$-drift across flux surfaces and the ∇B-drift due to $\delta B_\parallel$.

Of the heating terms, the first, P_s^{comp} is just the compressional heating due to the mean motion of the flux surfaces. The second, P_s^{pot}, is the change in the thermal energy of the electrons due to the change in their electrostatic potential energy.[12] Finally, we have the turbulent heat source P_e^{turb}. We interpret the terms in the expression (3.129) for P_e^{turb} as follows: the first term is heating due to work done against the parallel electric field (through ζ); the term involving the time-derivative of $\delta B_\parallel$ is in fact composed of the electron viscous heating, compressional heating due to the fluctuating motion of the $\tilde{\psi} = \mathrm{const.}$ surfaces and work done bending the magnetic field. This can be seen by substituting for $\delta B_\parallel$ from (3.130) – the term on the left-hand side of (3.130) is the compressional heating, the second term on the right is part of the electron viscous heating. The final term is the potential energy contribution to the compressional heating due to the fluctuating motion of the exact flux surfaces.

Electron Heating in the Collisional Model

Again, in the collisional limit, the flux in (3.125) vanishes. We are just left with local heating terms, of which only the turbulent heating simplifies:

$$P_e^{\mathrm{turb}} = \left\langle \left\langle n_e \delta T_e \left.\frac{\partial}{\partial t}\right|_{\tilde{\psi},\tilde{\alpha},\tilde{l}} \frac{\delta B_\parallel}{B} \right\rangle_{\mathrm{turb}} \right\rangle_{\tilde{\psi}} + e \left\langle n_e \langle \zeta \nabla \cdot \boldsymbol{u}_{\mathrm{eff}} \rangle_{\mathrm{turb}} \right\rangle_{\tilde{\psi}}, \tag{3.131}$$

where $\nabla \cdot \boldsymbol{u}_{\mathrm{eff}}$ is given by (3.130) as before. Thus, despite the lack of any transport, the electron response to ion-scale turbulence can still dissipate free energy. All other terms in (3.125) remain unchanged.

3.6.3 Momentum Transport

As the momentum transport involves all species, not merely electrons, we do not write a new equation for it in the moving frame. The transport equation for toroidal angular momentum is (Equation (2.179) of Ch. 2)

$$\frac{1}{V'} \left.\frac{\partial}{\partial t}\right|_{\psi} V' J \omega(\psi) + \frac{1}{V'}\frac{\partial}{\partial \psi} V' \left\langle \pi_{\mathrm{tot}}^{(\psi\phi)} \right\rangle_{\psi} = \left\langle S^{(\omega)} \right\rangle_{\psi}, \tag{3.132}$$

where the moment of inertia of the plasma is now given by

$$J = \sum_{s=i} m_s \left\langle n_s R^2 \right\rangle_{\psi}, \tag{3.133}$$

and the total momentum flux is

$$\begin{aligned} \left\langle \pi_{\mathrm{tot}}^{(\psi\phi)} \right\rangle_{\psi} = & \left\langle \pi_e^{(\psi\phi)} \right\rangle_{\psi} + \sum_{s=i} \left(\left\langle \pi_s^{(\psi\phi)} \right\rangle_{\psi} + m_s \omega(\psi) \left\langle R^2 \Gamma_s \right\rangle_{\psi} \right) \\ & - \sum_{s=i} \frac{Z_s e}{c} \left\langle \int d^3\boldsymbol{w} \langle \langle h_s \boldsymbol{w} \cdot \nabla\psi \rangle_{\boldsymbol{r}} \delta \boldsymbol{A} \rangle_{\mathrm{turb}} \cdot R^2 \nabla\phi \right\rangle_{\psi} \\ & - \frac{1}{4\pi} \left\langle \nabla\psi \cdot \langle \delta\boldsymbol{B} \delta\boldsymbol{B} \rangle_{\mathrm{turb}} \cdot R^2 \nabla\phi \right\rangle_{\psi}, \end{aligned} \tag{3.134}$$

[12] For further discussion of the nature of this potential-energy-exchange term, see §2.7.3 of Ch. 2.

where the sums over $s = i$ are taken over all ion species and fluxes $\pi_s^{(\psi\phi)}$ and Γ_s for the ions are given by (2.184) and (2.170) of Ch. 2, respectively. The electron momentum flux is now (see Appendix H.5)

$$\left\langle \pi_e^{(\psi\phi)} \right\rangle_\psi = \left\langle I(\psi) \left\langle \int d^3\boldsymbol{w} \left(m_e w_\parallel^2 - \frac{1}{2} m_e w_\perp^2 \right) g_e \frac{\partial}{\partial \tilde{l}} (\tilde{\psi} - \psi) \right\rangle_{\mathrm{turb}} \right\rangle_\psi . \tag{3.135}$$

The three evolution equations (3.123), (3.125) and (3.132) close our system on the transport timescale. The fluxes are written entirely in terms of fluctuating quantities that our reduced electron model provides (note that nowhere do we require the part of g_{pe} that is odd in σ).

The Collisional Limit

In the collisional limit, g_e is Maxwellian, and so isotropic in velocity space – see §3.115. Therefore, the velocity integral in (3.135) vanishes and so there is no angular momentum transport due to the electrons.

3.7 Comparison to Previous Models

Many previous works have introduced physically-motivated model responses for the electrons. Some are based on the linear electron response [32] and others use fluid moments of the electron distribution function to try and capture the passing particle response [33, 98, 107]. Our model improves upon this by being a rigorous consequence of the electron gyrokinetic equation, clearly demonstrating how the electron response fits into the full turbulent transport framework. Also, by retaining the bounce- and flux-surface-averaged kinetics of the electrons, we retain the distinction between the passing and trapped populations and retain drift resonances that may be lost in a fluid picture. We recover some of the results of the fluid models in the collisional limit, where there are no trapped electrons. In particular, the fact that δT_e is only a function of $\tilde{\psi}$ means that the temperature has equilibrated along perturbed field lines as expected. Even in this limit, the previous models did not allow for a rapid toroidal rotation (comparable to the ion thermal speed). Also all these models neglect $\delta B_\parallel$ and only retain $\delta A_\parallel$ in the perturbed magnetic field.

One of the original and still widely used models for the electron response is the adiabatic electron response corrected for zonal perturbations [108]. It can be shown that this is a rigorous limit of our model in the limit of vanishing β (the electrostatic limit) and collisional electrons. For collisionless electrostatic turbulence, similar equations have been derived rigorously in [31]. The electrostatic model of [31] also neither handles general magnetic geometry nor discusses the behaviour of passing electrons.

This limit is useful as an easily-implemented stepping stone between the adiabatic electron model (strictly only valid for quite collisional plasmas) and resolving the full physics by solving the electron gyrokinetic equation.

The most complete previous model is that of [34]. However, their model is derived under two restrictive assumptions. Firstly, the mass ratio is assumed to be much smaller than we assume here ($\delta \sim \epsilon$ rather than $1 \gg \delta \gg \epsilon$) and secondly $\delta B_{\parallel}$ is assumed to be identically zero. We note that (3.81) corresponds directly to equation (54) of [34], but that the further assumption made in [34] of (in our notation) $g_{pe} = 0$ is unjustified ($g_{pe} = 0$ is not a solution of (3.110)) and hence neglects the electron response that is zonal with respect to the perturbed flux surfaces. The model of [34] also does not include the effects of sonic toroidal rotation.

(which is either computationally prohibitive or formally invalid depending on whether one chooses to resolve the electron scales or not).

Finally, we must discuss how our model compares to the common practice of simulating the full electron gyrokinetic equation at ion scales (so-called "kinetic electrons"). Firstly, the electron gyrokinetic equation clearly contains electron-gyroscale and electron-bounce-frequency physics, so simulating only the ion scales is formally invalid – deliberately leaving the simulation unresolved in some sense. Secondly, how an unresolved simulation of the electron gyrokinetic equation behaves will be sensitive to the numerical algorithm used – it is possible to design a numerical algorithm that numerically bounce-averages the kinetic equation in the long-timestep limit, but of the commonly used gyrokinetic codes only `GS2` [109] does this. Indeed, the only way to confirm that simulating "kinetic electrons" gives the correct physics is either to compare with full two-scale simulations that resolve the electron dynamics as well as the ion dynamics or to compare with simulations of the electron model presented in this chapter. As well resolved two-scale simulations are generally beyond current computational resources (but see [110, 111] for the latest advances in this direction), simulations of our model are the only way to confirm the validity of this practice.

3.8 Conclusions

In this chapter, we have derived equations describing the electron response to ion-scale turbulence. The fast electron timescales have been eliminated from this system. The scale separation between electron and ion scales both motivates and enables the derivation of such a model. Starting from electron gyrokinetics (Ch. 2), we systematically expanded our equations in the square root of the electron-to-ion mass ratio to obtain our model. As discussed in the introduction, as studies of turbulence at non-zero β become more and more important – both for conventional and spherical tokamaks – having an electron response model that handles this rigorously will be necessary. We have derived full ($\beta \sim 1$) collisional and arbitrary-collisionality models – summarised in §3.5 and §3.4 respectively.

The procedure by which we constructed this model, a rigorous expansions in $\sqrt{m_e/m_i} \ll 1$, allowed us to delineate clearly which effects are due to electron-scale physics and which to ion-scale physics. Most importantly, the elimination of the electron scales has a profound impact on the structure of the fluctuating magnetic field. We prove in §3.2.2 that purely ion-scale turbulence conserves the total magnetic flux, and, moreover, that a velocity field can be found into which the magnetic field is frozen. This generalization of the usual MHD frozen-in theorem demonstrates that it is *only* electron-scale effects that can change the magnetic topology. The inclusion of $\beta \sim 1$, trapped particles, sonic rotation, and arbitrary axisymmetric geometry does *not* change this fundamental fact.

Similarly, whilst deriving transport equations across the exact flux surfaces, we have obtained another general property of ion-scale turbulence: passing electrons do not contribute to transport fluxes (see §3.6). This result could prove useful in designing magnetic geometries that have beneficial trapped-particle properties, safe in knowledge that the passing particles never contribute to the transport.

In addition to these fundamental general results, two other ancillary results have been proved in the course of this work. Firstly, in Appendix C, we proved that the topology of the mean magnetic field (i.e., the $q(\psi)$ profile) evolves on the resistive rather than the transport timescale. Importantly, this result holds independently of the nature of the turbulence. Secondly, in Appendix J, we proved that the commonly used adiabatic electron model is, in fact, the correct model for collisional ($\nu \gg \omega$) and electrostatic ($\beta \ll 1$) electrons.

Throughout our derivation, we have simply assumed that the magnetic stochasticity produced by electron-scale fluctuations is sufficiently small (see §3.2.3 for a detailed discussion of how small the field needs to be) that we can neglect it. This assumption must be investigated in detail and will be explored in future work. Similarly, the potential effects of small resonant regions around rational surfaces (which would give a finite width to each $\tilde{\psi} = \mathrm{const.}$ surface) should be examined closely. Indeed, a natural extension of this work would be to include the reaction of a small amount of purely electron-scale turbulence on the ion scales. There is increasing numerical evidence that electron-scale turbulence (such as Microtearing [112]) may generate stochastic magnetic fields [113, 114, 115]. Some simulations even suggest that purely ion-scale turbulence, such as ITG, can generate such fields [116, 117, 118] – this result would be in contradiction of the results of this paper and a detailed analysis must be conducted to resolve the discrepancy.

As always, the final arbiter of the usefulness of these models will be their applications, e.g., in nonlinear studies of finite-β ITG turbulence or kinetic ballooning mode turbulence.

Chapter 4

Future Directions

In Ch. 2, we derived, from first principles, the theory of multiscale gyrokinetics. Our formulation included, for the first time, the effects of sonic toroidal rotation, the effects of collisional and turbulent transport on the mean profiles, and the evolution of the magnetic field. This theoretical framework enables us to tackle many of the most exciting and important questions in plasma physics.

Firstly, this framework forms a solid foundation from which reinvigorated theoretical study of turbulence can spring. This includes the study of basic properties of turbulence: indeed, models based on simple physics assumptions coupled with the local cascade law (see §2.8.2) have already met with remarkable numerical [119] and experimental [5] success. This also provides a promising avenue of further research, including the possibility of a theoretical model of the "Dimits shift" [48, 120] and the transition to strong turbulence.

Secondly, the inclusion of the transport-timescale behaviour of the plasma in our framework may be the key to understanding the feedback mechanisms in plasma turbulence that lead to such important phenomena as transport barriers [121, 122, 17, 123, 124]. Especially important is the feedback provided by the evolution of the magnetic field, which is known to be important in Shafranov-shift stabilisation of turbulence in ITBs [125]. Generally speaking, detailed examination of the feedback mechanisms present in the equations of Ch. 2 will be fundamental to furthering our knowledge of basic plasma physics in multiscale systems.

In Ch. 3, we used the framework of Ch. 2 to explore the properties of ion-scale turbulence where the electron scales are not important. We derived the first rigorous model for the electron response to such turbulence that includes $\beta \sim 1$ effects, sonic toroidal rotation, and arbitrary magnetic geometry. As β is a rough measure of the efficiency of a fusion reactor (roughly-speaking it quantifies how much plasma a given magnetic field can confine), any advances that shed light upon how plasma turbulence and transport behave at increasing β are critical. As discussed in §1.3, for non-zero β, the electron dynamics in ion-scale turbulence become more and more important to handle correctly. Hence, the results of Ch. 3 will have great impact on the future studies of (necessarily electromagnetic) turbulence. One of the crucial questions that such studies will address is the question of a microstability-induced β limit, i.e., does the small-scale turbulence in a fusion reactor impose a more restrictive limit on the plasma pressure than, e.g., MHD or Neoclassical Tearing Modes. Similarly, whilst electrostatic ITG turbulence is well-studied [25, 26, 126, 48, 127, 27, 119, 120, 123] very little is known about the nonlinear properties of ITG turbulence at non-zero β [128, 129, 130]. The equations presented in Ch. 3 would be ideal for performing such nonlinear studies – being both more efficient and more physically relevant than "kinetic-electron" models in which the full electron gyrokinetic equation is simulated on a coarse grid.

Looking forward, the most glaring omission from our framework is any discussion of energetic particles. Future experiments and proposed fusion reactors will not operate with a plasma containing only bulk ions and electrons – they will be burning plasmas, containing energetic fusion products. There are several important questions to answer regarding the interaction of these energetic particles with turbulence and transport. How does the turbulence that is driven by the gradients in the fuel ions transport the energetic particles [131, 132]? Do the energetic particles affect the stability and saturation amplitudes of the ion-scale turbulence [133]? Do the energetic particles drive fluctuations themselves [134], and how do these fluctuations saturate [135, 136, 137, 138]? An important open question is whether we can answer such questions by an extension of the framework presented in this thesis (on which a paper [6] is in preparation).

It is also necessary to consider some of the open questions discussed in §3.8. The most important of these is the effect of electron-scale turbulence on ion scales. Extending the theory of Ch. 3 to include some amount of electron-scale turbulence will enable the study of such questions as: how magnetic stochasticity arises; how the resulting small-scale braiding of the magnetic field saturates; and, possibly most importantly, how the transition from ion-scale turbulence to electron-scale turbulence occurs when the ion-scale turbulence is suppressed by an appropriate choice of plasma configuration (e.g., strong flow shear [123, 97], strong negative magnetic shear [139, 140], or strong Shafranov shift [125]).

Ultimately, the theoretical, numerical and experimental work stemming from these foundations will lead to three core goals. Firstly, a more fundamental understanding of the physics of the interplay between large-scale fields and small-scale fluctuations is of paramount importance. Secondly, even with advances in modern supercomputing technology, numerical simulations need to exploit every scale separation and physically-relevant approximation available if they are to have any hope of encompassing the complex dynamics of tokamak plasma turbulence and transport. The theoretical framework presented here provides some of these simplifications and, moreover, is the starting point for further analytical work in identifying the key physics that needs to be simulated. Finally, the synthesis of these two goals would be efficient, physics-based models of how reactor plasmas perform; such models would allow us to optimise plasma parameters over the enormous space of possible configurations to find efficient and commercially viable designs for future fusion reactors. The job of theoretical plasma physics in making fusion power a reality is far from complete, but further physical understanding and considered analytical work, based on rigorous theoretical frameworks such as those presented here, will be central to making progress.

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