

# Fine-grained Dichotomies for the Tutte Plane and Boolean $\#$ CSP

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**Abstract** Jaeger, Vertigan, and Welsh (Math. Proc. Cambridge Philos. Soc., 1990) proved a dichotomy for the complexity of evaluating the Tutte polynomial at fixed points: The evaluation is  $\#$ P-hard almost everywhere, and the remaining points admit polynomial-time algorithms. Dell, Husfeldt, and Wahlén (ICALP, 2010) and Husfeldt and Taslaman (IPEC, 2010), in combination with the results of Curticapean (ICALP, 2015), extended the  $\#$ P-hardness results to tight lower bounds under the counting exponential time hypothesis  $\#$ ETH, with the exception of the line  $y = 1$ , which was left open. We complete the dichotomy theorem for the Tutte polynomial under  $\#$ ETH by proving that the number of all acyclic subgraphs of a given  $n$ -vertex graph cannot be determined in time  $\exp(o(n))$  unless  $\#$ ETH fails.

Another dichotomy theorem we strengthen is the one of Creignou and Hermann (Inf. Comput., 1996) for counting the number of satisfying assignments to a constraint satisfaction problem instance over the Boolean domain. We prove that the  $\#$ P-hard cases cannot be solved in time  $\exp(o(n))$  unless  $\#$ ETH fails. The main ingredient is to prove that the number of independent sets in bipartite graphs with  $n$  vertices cannot be computed in time  $\exp(o(n))$  unless  $\#$ ETH fails.

In order to prove our results, we use the block interpolation idea by Curticapean and transfer it to systems of linear equations that might not directly correspond to interpolation.

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## 1 Introduction

Counting combinatorial objects is at least as hard as detecting their existence, and often it is harder. Valiant [18] introduced the complexity class  $\#P$  to study the complexity of counting problems and proved that counting the number of perfect matchings in a given bipartite graph is  $\#P$ -complete. By a theorem of Toda [17], we know that  $PH \subseteq P^{\#P}$  holds; in particular, for every problem in the entire polynomial-time hierarchy, there is a polynomial-time algorithm that is given access to an oracle for counting perfect matchings. This theorem suggests that counting is much harder than decision.

When faced with a problem that is NP-hard or  $\#P$ -hard, the area of exact algorithms strives to find the fastest exponential-time algorithm for a problem, or find reasons why faster algorithms might not exist. For example, the fastest known algorithm for counting perfect matchings in  $n$ -vertex graphs [1] runs in time  $2^{n/2} \text{poly}(n)$ . It has been hypothesized that no  $1.99^{n/2}$ -time algorithm for the problem exists. But we do not know whether such an algorithm has implications for the strong exponential time hypothesis, which states: For all  $\varepsilon > 0$ , there is some  $k$  such that the problem of deciding satisfiability of boolean formulas in  $k$ -CNF on  $n$  variables does not have an algorithm running in time  $(2 - \varepsilon)^n$ . However, we know by [8] that the term  $O(n)$  in the exponent is asymptotically tight, in the sense that a  $\exp(o(n))$ -time algorithm for counting perfect matchings would violate the (randomized) exponential time hypothesis (ETH) by Impagliazzo and Paturi [11]. Using the idea of block interpolation, Curticapean [7] strengthened the hardness by showing that a  $\exp(o(n))$ -time algorithm for counting perfect matchings would violate the (deterministic) counting exponential time hypothesis ( $\#ETH$ ).

Our main results are hardness results under  $\#ETH$  for 1) the problem of counting all forests in a graph, that is, its acyclic subgraphs, and 2) the problem of counting the number of independent sets in a bipartite graph. In particular, we show that, if  $\#ETH$  holds, then neither of these problems has an algorithm running in time  $\exp(o(n))$  even in simple  $n$ -vertex graphs of bounded average or maximum degree, respectively. We use these results to lift two known “FP vs.  $\#P$ -hard” dichotomy theorems to their more refined and asymptotically tight “FP vs.  $\#ETH$ -hard” variants. Here FP is the class of functions computable in polynomial time. Since ETH implies  $\#ETH$ , our results could also be stated under ETH.

### 1.1 The Tutte Polynomial under $\#ETH$

The Tutte polynomial of a graph  $G$  with vertices  $V$  and edges  $E$  is the bivariate polynomial  $T(G; x, y)$  defined via

$$T(G; x, y) = \sum_{A \subseteq E} (x - 1)^{k(A) - k(E)} (y - 1)^{k(A) + |A| - |V|}, \quad (1)$$

where  $k(A)$  is the number of connected components of the graph  $(V, A)$ . The Tutte polynomial captures many combinatorial properties of a graph in a common framework, such as the number of spanning trees, forests, proper colorings, and certain flows and orientations, but also less obvious connections to other fields, such as link polynomials from knot theory, reliability polynomials from network theory, and (perhaps most importantly) the Ising and Potts models from statistical physics. We make no attempt to survey the literature or the different applications for the Tutte polynomial, and instead refer to the upcoming CRC handbook on the Tutte polynomial [9].

Since  $T(G; -2, 0)$  corresponds to the number of proper 3-colorings of  $G$ , we cannot hope to compute all coefficients of  $T(G; x, y)$  in polynomial time, unless  $\#P = P$ . Instead, the literature and this paper focus on the complexity of evaluating the Tutte polynomial at fixed evaluation points. That is, for each  $(x, y) \in \mathbf{Q}^2$ , we consider the function  $T_{x,y}$  defined as  $G \mapsto T(G; x, y)$ . Jaeger, Vertigan, and Welsh [13] proved that this function is either  $\#P$ -hard to compute or has a polynomial-time algorithm. In particular, if  $(x, y)$  satisfies  $(x - 1)(y - 1) = 1$ , then  $T_{x,y}$  corresponds to the 1-state Potts model and has a polynomial-time algorithm. If  $(x, y)$  is one of the four points  $(1, 1)$ ,  $(-1, -1)$ ,  $(0, -1)$ , or  $(-1, 0)$ , it also has a polynomial-time algorithm. The most interesting point here is  $T(G; 1, 1)$ , which corresponds to the number of spanning trees in  $G$ .

A naïve algorithm to compute the Tutte polynomial of a graph with  $m$  edges runs in time  $\exp(O(m))$ . Björklund et al. [2] gave an algorithm running in time  $\exp(O(n))$ , where  $n$  is the number of vertices. Dell et al. [8] proved for all hard points, except for points with  $y = 1$ , that an  $\exp(o(n/\log^3 n))$ -time algorithm for  $T_{x,y}$  on simple graphs would violate  $\#ETH$ . Distressingly, this result not only left open one line, but also left a gap in the running time. Curticapean [7] introduced the technique of block interpolation to close the running time gap: Under  $\#ETH$ , there is no  $\exp(o(n))$ -time algorithm for  $T_{x,y}$  on simple graphs at any hard point  $(x, y)$  with  $y \neq 1$ .<sup>1</sup>

*Our contribution.* We resolve the complexity of the missing line  $y = 1$  under  $\#ETH$ . On this line, the Tutte polynomial counts forests weighted in some way, and the main result is the following theorem.

**Theorem 1 (Forest counting is hard under  $\#ETH$ ).** *If  $\#ETH$  holds, then there exist constants  $\varepsilon, C > 0$  such that no  $\exp(\varepsilon n)$ -time algorithm can compute the number of forests in a given simple  $n$ -vertex graph with at most  $Cn$  edges.*

The fact that the problem remains hard even on simple sparse graphs makes the theorem stronger. The previously best known lower bound under  $\#ETH$  was that forests cannot be counted in time  $\exp(n^\delta)$  where  $\delta > 0$  is some constant depending on the instance blow-up caused by the known  $\#P$ -hardness reductions for forest counting; the thesis of Taslaman [16] shows a

<sup>1</sup> The conference version of [7] does not handle the case  $y = 0$ , but the full version (to appear) does.

detailed proof for  $\delta = \frac{1}{8}$ . Our approach also yields a #P-hardness proof for forest counting that is simpler than the proofs we found in the literature, such as the proof appearing in “Complexity of Graph Polynomials” by Steven D. Noble, chapter 13 of [10].<sup>2</sup>

Combined with all previous results [13, 8, 7], we can now formally state a complete #ETH dichotomy theorem for the Tutte polynomial over the reals.

**Corollary 2 (Dichotomy for the real Tutte plane under #ETH).**

Let  $(x, y) \in \mathbb{Q}^2$ . If  $(x, y)$  satisfies

$$(x - 1)(y - 1) = 1 \text{ or } (x, y) \in \{(1, 1), (-1, -1), (0, -1), (-1, 0)\},$$

then  $T_{x,y}$  can be computed in polynomial time. Otherwise  $T_{x,y}$  is #P-hard and, if #ETH is true, then there exists  $\varepsilon > 0$  such that  $T_{x,y}$  cannot be computed in time  $\exp(\varepsilon n)$ , even for simple graphs.

The result also holds for sparse simple graphs. We stated the results only for rational numbers in order to avoid discussions about how real numbers should be represented.

For the proof of Theorem 1, we establish a reduction chain that starts with the problem of counting perfect matchings on sparse graphs, which is known to be hard under #ETH. As an intermediate step, we find it convenient to work with the multivariate forest polynomial as defined, for example, by Sokal [15]. After a simple transformation of the graph, we are able to extract the number of perfect matchings of the original graph from the multivariate forest polynomial of the transformed graph, even when only two different variables are used. Subsequently, we use Curticapean’s idea of block interpolation [7] to reduce the problem of computing all coefficients of the bivariate forest polynomial to the problem of evaluating the univariate forest polynomial on multigraphs where all edge multiplicities are bounded by a constant. Finally, we replace parallel edges with parallel paths of constant length to reduce to the problem of evaluating the univariate forest polynomial on simple graphs.

## 1.2 #CSP over the Boolean Domain under #ETH

In the second part of this paper, we consider constraint satisfaction problems (CSPs) over the Boolean domain, which are a natural generalization of the satisfiability problem for  $k$ -CNF formulas. A *constraint* is a relation  $R \subseteq \{0, 1\}^k$  for some  $k \in \mathbb{N}$ , and a set  $\Gamma$  of constraints is a *constraint language*.  $\text{CSP}(\Gamma)$  is the constraint satisfaction problem where all constraints occurring in the instances are of a type contained in  $\Gamma$ , and  $\#\text{CSP}(\Gamma)$  is the corresponding counting version, which asks the number of satisfying assignments. If all constraints happen to be *affine*, that is, they are linear equations over  $\text{GF}(2)$ , then the number of solutions can be determined in polynomial time by applying Gaussian elimination and determining the size of the solution space.

<sup>2</sup> For a different and not fully published proof, Jaeger, Vertigan and Welsh [13] refer to private communication with Mark Jerrum as well as the PhD thesis of Vertigan [19].

Creignou and Hermann [6] prove that, as soon as you allow even just one constraint type that is not affine, the resulting CSP problem is #P-hard.

*Our contribution.* We prove that all Boolean #CSPs that are #P-hard are in fact hard under #ETH. The #P-hardness is established in [6] by reductions from counting independent sets in bipartite graphs, which we prove to be hard in the following theorem.

**Theorem 3 (Counting bipartite independent sets is hard).** *If #ETH holds, then there exist constants  $\varepsilon > 0$  and  $D \in \mathbf{N}$  such that no  $\exp(\varepsilon n)$ -time algorithm can compute the number of independent sets in bipartite  $n$ -vertex graphs of maximum degree at most  $D$ .*

The fact that the problem is hard even on graphs of bounded degree makes the theorem stronger.

The number of independent sets in bipartite graphs has a prominent role in counting complexity. Currently, the complexity of *approximating* this number is unknown, and many problems in approximate counting turn out to be polynomial-time equivalent to approximately counting independent sets in bipartite graphs. Theorem 3 shows that this mysterious situation does not occur for the exact counting problem in the exponential-time setting: it is just as hard as counting satisfying assignments of 3-CNFs.

The #P-hardness of counting independent sets in bipartite graphs was established by Provan and Ball [14]. The main ingredient in their proof is a system of linear equations that does not seem to correspond to polynomial interpolation directly. We prove Theorem 3 by transferring the block interpolation idea from [7] to this system of linear equations, which we do using a Kronecker power of the original system.

Theorem 3 combined with existing reductions in [6] yields the fine-grained dichotomy.

**Corollary 4 (Creignou and Hermann under #ETH).** *Let  $\Gamma$  be a finite constraint language. If every constraint in  $\Gamma$  is affine, then  $\#CSP(\Gamma)$  has a polynomial-time algorithm. Otherwise  $\#CSP(\Gamma)$  is #P-complete, and if #ETH holds, it cannot be computed in time  $\exp(o(n))$  where  $n$  is the number of variables.*

We consider Corollary 4 to be a first step towards understanding the fine-grained complexity of technically much more challenging dichotomies, such as the ones for counting CSPs with complex weights of Cai and Chen [3], or the dichotomy for Holant problems with symmetric signatures over the Boolean domain of Cai, Lu and Cia [4].

## 2 Preliminaries

Given a matrix  $A$  of size  $m_1 \times n_1$  and a matrix  $B$  of size  $m_2 \times n_2$  their *Kronecker product*  $A \otimes B$  is a matrix of size  $m_1 m_2 \times n_1 n_2$  given by

$$A \otimes B = \begin{bmatrix} a_{11}B & \dots & a_{1n}B \\ \vdots & & \vdots \\ a_{m1}B & \dots & a_{mn}B \end{bmatrix}.$$

Let  $A^{\otimes n}$  be the matrix defined by  $A^{\otimes 1} = A$  and  $A^{\otimes n+1} = A \otimes A^{\otimes n}$ . If  $A$  and  $B$  are quadratic matrices of size  $n_a$  and  $n_b$ , respectively,  $\det(A \otimes B) = \det(A)^{n_b} \cdot \det(B)^{n_a}$ .

The exponential time hypothesis (ETH) by Impagliazzo and Paturi [11] is that satisfiability of 3-CNF formulas cannot be computed substantially faster than by trying all possible assignments. The counting version of this hypothesis [8], which is a weaker assumption (clearly, counting the number of solutions entails deciding existence of a solution), reads as follows:

(#ETH) *There is a constant  $\varepsilon > 0$  such that no deterministic algorithm can compute #3-SAT in time  $\exp(\varepsilon n)$ , where  $n$  is the number of variables.*

A different way of formulating #ETH is to say no algorithm can compute #3-SAT in time  $\exp(o(n))$ . The latter statement is clearly implied by the formal statement, and it will be more convenient for discussion to use this form.

The sparsification lemma by Impagliazzo, Paturi, and Zane [12] is that every  $k$ -CNF formula  $\varphi$  can be written as the disjunction of  $2^{\varepsilon n}$  formulas in  $k$ -CNF, each of which has at most  $c(k, \varepsilon)n$  clauses. Moreover, this disjunction of sparse formulas can be computed from  $\varphi$  and  $\varepsilon$  in time  $2^{\varepsilon n} \text{poly}(m)$ . The density  $c = c(k, \varepsilon)$  is the *sparsification constant*, and the best known bound is  $c(k, \varepsilon) = (k/\varepsilon)^{3k}$  [5]. It was observed [8] that the disjunction can be made so that every assignment satisfies at most one of the sparse formulas in the disjunction, and so the sparsification lemma applies to #ETH as well. In particular, #ETH implies that #3-SAT cannot be computed in time  $\exp(o(m))$ , where  $m$  is the number of clauses.

We also make use of the following result, whose proof is based on block interpolation.

**Theorem 5 (Curticapean [7]).** *If #ETH holds, then there are constants  $\varepsilon, D > 0$  such that neither of the following problems have  $\exp(\varepsilon n)$ -time algorithms on  $n$ -vertex graphs  $G$ , even if  $G$  is simple and of maximum degree at most  $D$ :*

- *Computing the number of perfect matchings of  $G$ .*
- *Computing the number of independent sets of  $G$ .*

### 3 Counting Forests is #ETH-hard

Let  $\mathcal{F}(G)$  be the set of all *forests* of  $G$ , that is, edge subsets  $A \subseteq E(G)$  such that the graph  $(V(G), A)$  is acyclic. Consider now the definition of the Tutte polynomial in (1). For  $y = 1$ , the expression  $(y - 1)$  is of course zero. Conventionally,  $0^t = 1$  if and only if  $t = 0$ , and 0 otherwise. Therefore, the only non-vanishing summands on the right side of (1) are those where  $(y - 1)$  appears with an exponent of zero. For a single summand, this is the case precisely if  $k(A) + |A| - |V| = 0$ . This, in turn, is equivalent to  $A$  being a forest, which allows us to conclude:

$$T(G; x, 1) = \sum_{A \in \mathcal{F}(G)} (x - 1)^{k(A) - k(E)}. \quad (2)$$

The goal of this section is prove that, for every fixed  $x \neq 1$ , computing the value  $T(G; x, 1)$  for a given graph  $G$  is hard under #ETH. More formally, we show the following theorem.

**Theorem 6.** *Let  $x \in \mathbf{R} \setminus \{1\}$ . If #ETH holds, then there exist  $\varepsilon, C > 0$  such that the function that maps simple  $n$ -vertex graphs  $G$  with at most  $Cn$  edges to the value  $T(G; x, 1)$  cannot be computed in time  $2^{\varepsilon n}$ .*

In particular, this is true for  $T(G; 2, 1)$ , which is the number of forests in  $G$ . Thus, Theorem 6 yields Theorem 1 as its special case with  $x = 2$ .

#### 3.1 The Multivariate Forest Polynomial

A *weighted graph*  $(G, w)$  is an undirected graph  $G$  in which every edge  $e \in E(G)$  has a weight  $w_e$ , which is an element of some ring. We use the *multivariate forest polynomial*, defined e.g. by Sokal [15, (2.14)] as follows:

$$F(G; w) = \sum_{A \in \mathcal{F}(G)} \prod_{e \in A} w_e. \quad (3)$$

Setting all weights  $w_e$  to the single variable  $x$  yields the *univariate forest polynomial*:

$$F(G; x) = \sum_{A \in \mathcal{F}(G)} x^{|A|} = \sum_{k=0}^{|E(G)|} a_k(G) x^k,$$

where  $a_k(G)$  is the number of forests with  $k$  edges in  $G$ . For all  $x \in \mathbf{R} \setminus \{1\}$ , the formal relation between  $T(G; x, 1)$  and the univariate forest polynomial is given by the identity

$$\begin{aligned} T(G; x, 1) &= (x - 1)^{|V| - k(E)} \sum_{A \in \mathcal{F}(G)} (x - 1)^{-|A|} \\ &= (x - 1)^{|V| - k(E)} \cdot F\left(G; \frac{1}{x - 1}\right). \end{aligned} \quad (4)$$

The first equality follows from (2) and the discussion preceding it. Thus, evaluating the forest polynomial and evaluating the Tutte polynomial for  $y = 1$  are polynomial-time equivalent.

For a forest  $A \in \mathcal{F}(G)$ , let  $\mathcal{C}(A)$  be the family its connected components. A connected component of  $A$  is a maximal set  $U \subseteq V(G)$  that is connected in  $(V(G), A)$ . Clearly  $\mathcal{C}(A)$  is a partition of  $V(G)$ , each element is a (maximal) tree of  $A$ , and trees  $U$  with  $|U| = 1$  are allowed.

**Lemma 7 (Adding an apex).** *Let  $(G, w)$  be a weighted graph. Let  $(G', w')$  be obtained from it by adding a new vertex  $a$  and joining it with each vertex  $v \in V(G)$  using edges of weight  $z_v$ . Then*

$$F(G'; w') = \sum_{A \in \mathcal{F}(G)} \left( \prod_{e \in A} w_e \prod_{U \in \mathcal{C}(A)} \left( 1 + \sum_{u \in U} z_u \right) \right). \quad (5)$$

Moreover, if we set  $z_v = -1$  for all  $v \in V(G)$  and  $w'_e = w$  for all  $e \in E(G)$ , then the coefficient of  $w^{n/2}$  in  $F(G'; w')$  equals the number of perfect matchings of  $G$  in absolute value.

We first define a projection  $\phi$  that maps any forest  $A'$  in the graph  $G'$  to a forest  $\phi(A')$  in the original graph  $G$ . In particular,  $\phi$  simply removes all edges added in the construction of  $G'$ , that is, we define  $\phi(A') = E(G) \cap A'$  for all  $A' \in \mathcal{F}(G')$ . Clearly,  $\phi(A')$  is a forest in  $G$ .

Next we characterize the forests  $A'$  that map to the same  $\phi(A')$ . Let  $A$  be a fixed forest in  $G$ . Then a forest  $A'$  in  $G'$  maps to  $A$  under  $\phi$  if and only if the set  $X := A' \setminus A$  satisfies the following property:

(P) For every tree  $U \in \mathcal{C}(A)$ , at most one edge of  $X$  is incident on  $U$ .

The forward direction of this claim follows from the fact that  $A'$  is a forest, and so in addition to the trees  $U \in \mathcal{C}(A)$  it can contain at most one edge connecting each  $U$  to  $a$ ; otherwise the tree and the two edges to  $a$  would contain a cycle in  $A'$ . For the backward direction of the claim, observe that adding a set  $X$  with the property (P) to  $A$  cannot introduce a cycle.

Finally, we calculate the weight contribution of all  $A'$  that map to the same  $A$ . Let  $A'$  be a forest in  $G'$ , let  $A = \phi(A')$  and  $X = A' \setminus A$ . The weight contribution of  $A'$  in the definition of  $F(G')$  is  $\prod_{e \in A'} w'_e$ . For all  $e \in A$ , we have  $w'_e = w_e$ . For each  $e \in X$ , we have  $e = \{a, v_e\}$  for some  $v_e \in V(G)$ , and thus  $w'_e = z_{v_e}$ . Summing over all weight terms for  $A'$  with image  $A$  yields

$$\sum_{\substack{A' \in \mathcal{F}(G') \\ \phi(A') = A}} \prod_{e \in A'} w'_e = \prod_{e \in A} w_e \cdot \sum_X \prod_{e \in X} z_{v_e} = \prod_{e \in A} w_e \cdot \prod_{U \in \mathcal{C}(A)} \left( 1 + \sum_{u \in U} z_u \right). \quad (6)$$

The sum in the middle is over all  $X$  with the property (P), and the first equality follows from the bijection between forests  $A'$  and sets  $X$  with property (P). For the second equality, we use property (P) and the distributive law. We obtain (5) by taking the sum of equations (6) over all  $A \in \mathcal{F}(G)$ .



For the moreover part of the lemma, we set  $w'_e = w$  for all  $e \in E(G)$  and  $z_v = -1$  for all  $v \in V(G)$ , and observe

$$F(G'; w') = \sum_{A \in \mathcal{F}(G)} w^{|A|} \prod_{U \in \mathcal{C}(A)} (1 - |U|).$$

The coefficient of  $w^{n/2}$  in  $F(G')$  satisfies

$$[w^{n/2}]F(G') = \sum_{\substack{A \in \mathcal{F}(G) \\ |A|=n/2}} \prod_{U \in \mathcal{C}(A)} (1 - |U|). \quad (7)$$

If  $(V(G), A)$  is an acyclic graph with exactly  $n/2$  edges, then either it is a perfect matching or it contains an isolated vertex. If it contains an isolated vertex  $v$ , then  $\{v\} \in \mathcal{C}(A)$  and thus the product in (7) is equal to zero for this particular  $A$ . It follows that  $A$  does not contribute to the sum if it is not a perfect matching. On the other hand, if  $A$  is a perfect matching, then  $|U| = 2$  holds for all  $U \in \mathcal{C}(A)$ , so the product in (7) is equal to 1 or  $-1$ , depending on the parity of  $n/2$ . Overall, we obtain that  $[w^{n/2}]F(G')$  is equal in absolute value to the number of perfect matchings of  $G$ . ■

Lemma 7 shows that computing the coefficients of the multivariate forest polynomial is at least as hard as counting perfect matchings; moreover, this is true even if at most two different edge weights are used. Next we reduce from the multivariate forest polynomial with at most two distinct weights to the problem of evaluating the univariate polynomial in multigraphs. We do so via a so-called oracle self-reduction, whose queries are sparse multigraphs in which each edge has at most a constant number of parallel edges.

**Lemma 8 (From two weights to small weights).**

Let  $x$  and  $y$  be two variables, and let  $z \in \mathbf{R} \setminus \{0\}$  be fixed. There is an algorithm as follows:

1. Its input is a weighted graph  $(G, w)$  with  $w_e \in \{x, y\}$  for all  $e \in E(G)$ , and a real  $\varepsilon > 0$ .
2. It outputs all coefficients of  $F(G; w)$ , which is a bivariate polynomial in  $x$  and  $y$ .
3. It runs in time  $\exp(\varepsilon |E(G)|)$ .
4. It has access to an oracle that, given  $w'$ , returns the real number  $F(G; w')$ .
5. There is a constant  $C_\varepsilon \in \mathbf{N}$  that only depends on  $\varepsilon$  such that all oracle queries  $w'$  made by the algorithm satisfy  $w'_e \in \{0 \cdot z, \dots, C_\varepsilon \cdot z\}$  for every edge  $e \in E(G)$ .

Let  $(G, w)$  with  $w_e \in \{x, y\}$  for all  $e \in E(G)$  and  $\varepsilon > 0$  be given as input. Define  $C_\varepsilon \in \mathbf{N}$  as a large enough constant to be determined later, and let  $m$  be the smallest multiple of  $C_\varepsilon$  that is at least  $|E(G)|$ . This implies  $|E(G)| \leq m < |E(G)| + C_\varepsilon$ .

By the definition (3) of the multivariate forest polynomial,  $F(G; w)$  is a bivariate polynomial over the variables  $x$  and  $y$  and with total degree at most  $m$ .

We partition the edge set  $E(G)$  into blocks of size at most  $C_\varepsilon$  in such a way that edges  $e$  and  $e'$  with  $w_e \neq w_{e'}$  never belong to the same block. Since  $x$  and  $y$  occur at most  $m$  times, the number of blocks is at most  $m/C_\varepsilon$ . For each block, we introduce new variables  $x_i$  and  $y_i$ , and based on the edge partition, we obtain a new weight function  $v$  with the following properties: If  $w_e = x$ , then  $v_e \in \{x_1, \dots, x_{m/C_\varepsilon}\}$ . If  $w_e = y$ , then  $v_e \in \{y_1, \dots, y_{m/C_\varepsilon}\}$ . And each  $x_i$  and  $y_i$  occurs as  $v_e$  for at most  $C_\varepsilon$  different edges  $e$ . With these weights, the multivariate forest polynomial  $F(G; v)$  is a polynomial  $p$  over the  $2m/C_\varepsilon$  variables  $\{x_i, y_i : 1 \leq i \leq m/C_\varepsilon\}$  and each variable has individual degree at most  $C_\varepsilon$ . Moreover, the polynomial  $F(G; w)$  can be recovered from  $F(G; v)$  by replacing each  $x_i$  with  $x$  and each  $y_i$  with  $y$ .

The goal of the desired algorithm is to compute the *coefficients* of the bivariate polynomial  $F(G; w)$ , and it is able to query the *values*  $F(G; w')$  for any real vector  $w'$ . Since  $p$  is a polynomial with

$$p(x_1, \dots, x_{m/C_\varepsilon}, y_1, \dots, y_{m/C_\varepsilon}) = F(G; v),$$

the oracle allows us to query values  $p(\xi)$  for real vectors  $\xi \in \mathbf{R}^{2m/C_\varepsilon}$ . The resulting vectors  $w'$  that we query satisfy  $w'_e = \xi_j$  if  $e$  belongs to block  $j$  of the partition. The algorithm is as follows:

1. Given:  $(G, w)$  and  $\varepsilon > 0$ .
2. Construct a vector  $v$  over the variables  $x_i$  and  $y_i$  as discussed. [Now  $F(G; v)$  is a polynomial  $p$  in  $2m/C_\varepsilon$  variables and with individual degree at most  $C_\varepsilon$ .]
3. For all points  $\xi \in \{0 \cdot z, \dots, C_\varepsilon \cdot z\}^{2m/C_\varepsilon}$ , use the oracle to obtain the value  $p(\xi)$ .
4. Use multivariate Lagrange interpolation to compute the coefficients of the polynomial  $p$ .
5. Replace each occurrence of  $x_i$  with  $x$  and each occurrence of  $y_i$  with  $y$  to recover the coefficients of  $F(G; w)$ .

First note that all resulting oracle queries  $F(G; w')$  are indeed of the form that is required since each entry of  $\xi$  and thus  $w'$  is in  $\{0 \cdot z, \dots, C_\varepsilon \cdot z\}$ . In step 3, we evaluate the polynomial  $F(G; v)$  on all points of a  $(2m/C_\varepsilon)$ -dimensional grid dilated by  $z$  and with side-length  $C_\varepsilon + 1$  in each dimension. The evaluations on such a grid are sufficient to perform multivariate Lagrange interpolation (see, e.g., [7]) for  $F(G; v)$ , which yields the coefficients of the multivariate polynomial  $F(G; v)$  and thus of the bivariate polynomial  $F(G; w)$ .

The running time of the algorithm is polynomial in the size  $(C_\varepsilon + 1)^{2m/C_\varepsilon}$  of the grid. Now, choose  $C_\varepsilon$  large enough, depending on  $\varepsilon$ , such that this is at most  $\exp(\varepsilon m)$ . ■

The graphs queried by the algorithm can be turned into multigraphs where each edge has weight exactly  $z$  as follows. If an edge  $e$  of a graph  $G$  has weight  $w'_e$  with  $w'_e = \mu_e \cdot z$  for some  $\mu_e \in \mathbf{N}$ , then we can simulate this weight by replacing  $e$  with  $\mu_e$  parallel edges of weight  $z$ . Doing this for all edges yields a multigraph  $G_\mu$  such that  $F(G_\mu; z) = F(G; w')$  holds. In particular,

$F(G_\mu; 1)$  is the number of forests in  $G_\mu$ , so Lemma 8 for  $z = 1$  can be seen as a reduction from two weights to forest counting in multigraphs where each edge has at most  $O(1)$  parallel copies.

Thus, the combination of Lemma 7 and Lemma 8 shows, for all fixed  $x \neq 0$ , that it is hard to evaluate  $F(G; x)$  for multigraphs with at most a constant number of parallel edges. Next we apply a stretch to make the graphs simple. To this end, we calculate the effect of a  $k$ -stretch on the univariate forest polynomial of a graph.

**Lemma 9 (The forest polynomial under a  $k$ -stretch).** *Let  $G$  be an arbitrary multigraph on  $m$  edges, all having the same weight  $w \in \mathbf{R}$ . Let  $k \geq 2$  be an integer such that the number  $g_k(w)$  with*

$$g_k(w) = \frac{w^k}{(w+1)^k - w^k}$$

*is well-defined. Let  $G'$  be the simple graph obtained from  $G$  by replacing every edge by a path of  $k$  edges. Then,*

$$F(G'; w) = ((w+1)^k - w^k)^m \cdot F(G; g_k(w)).$$

Define a mapping  $\phi$  that maps forests in  $G'$  to forests in  $G$  as follows: Given a forest  $A'$  of  $G'$ , the image  $\phi(A')$  contains the edge  $e \in E(G)$  if and only if  $A'$  contains all the  $k$  edges of  $G'$  that  $e$  got stretched into. These edges then form a forest in  $G$ . That is, subgraphs  $A'$  that only differ by edges in “incomplete paths” are mapped to the same multigraph by  $\phi$ .

Clearly,  $\phi$  partitions  $\mathcal{F}(G')$  into sets of forests with the same image under  $\phi$ . Let  $A$  be a forest in  $G$ , and let us describe a way to generate all  $A'$  with  $\phi(A') = A$ . First, for each  $e \in A$ , add its corresponding path in  $G'$  of length  $k$  to  $A'$ . Moreover, for each edge  $e \in E(G) \setminus A$ , we can add to  $A'$  any proper subset of edges from the  $k$ -path in  $G'$  that corresponds to  $e$ . Therefore, at each  $e \in E(G) \setminus A$  independently, there are  $\binom{k}{i}$  ways to extend  $A'$  by  $i$  edges to a forest in  $G'$ , for  $i \in \{0, \dots, k-1\}$ . A forest  $A'$  can be obtained in this fashion if and only if  $\phi(A') = A$  holds.

For a fixed  $A$ , let us consider all summands  $w^{|A'|}$  in  $F(G'; w)$  with  $\phi(A') = A$ . By the above considerations, the total weight contribution of these summands is

$$w^{k|A|} \cdot \left( \sum_{i=0}^{k-1} \binom{k}{i} w^i \right)^{m-|A|} = w^{k|A|} \cdot ((w+1)^k - w^k)^{m-|A|}$$

by the binomial theorem. These remarks justify the following calculation for the forest polynomial:

$$\begin{aligned} F(G'; w) &= \sum_{A \in \mathcal{F}(G)} \sum_{\substack{A' \in \mathcal{F}(G') \\ \phi(A') = A}} w^{|A'|} = \sum_{A \in \mathcal{F}(G)} w^{k|A|} \cdot ((w+1)^k - w^k)^{m-|A|} \\ &= ((w+1)^k - w^k)^m \cdot \sum_{A \in \mathcal{F}(G)} \left( \frac{w^k}{(w+1)^k - w^k} \right)^{|A|}. \end{aligned}$$

Since the sum in the last line is equal to  $F(G; g_k(w))$ , this concludes the proof.  $\blacksquare$

We are now in position to formally prove the main theorem of this section.

*Proof of Theorem 6* Let  $x \in \mathbf{R} \setminus \{1\}$ , and let  $t = (x - 1)^{-1}$ . Suppose that, for all  $\varepsilon > 0$ , there exists an algorithm  $B_\varepsilon$  to compute the mapping  $G \mapsto T(G; x, 1)$  in time  $2^{\varepsilon n}$  for given simple graphs  $G$  with at most  $C'_\varepsilon n$  edges, where  $C'_\varepsilon$  will be chosen later. By (4), algorithm  $B_\varepsilon$  can be used to compute values  $F(G; (x - 1)^{-1})$  with no relevant overhead in the running time. Given such an algorithm (or family of algorithms), we devise a similar algorithm for counting perfect matchings, which together with Theorem 5 implies that #ETH is false.

Let  $G$  be a simple  $n$ -vertex graph with at most  $Cn$  edges. Let  $G'$  be the graph obtained from  $G$  as in Lemma 7 by adding an apex, labeling the edges incident to the apex with the indeterminate  $z$ , and all other edges with the indeterminate  $w$ . By Lemma 7, the coefficients of the corresponding bivariate forest polynomial of  $G'$  are sufficient to extract the number of perfect matchings of  $G$ , so it remains to compute these coefficients.

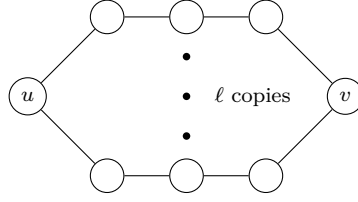
To obtain the coefficients, we use Lemma 8, keeping in mind the remark following its proof. The reduction guaranteed by the lemma produces  $2^{\varepsilon m}$  multigraphs  $H$ , all with the same vertex set  $V(G')$ . Moreover, each  $H$  has at most  $C_\varepsilon |E(G')| = C_\varepsilon (|E(G)| + n) \leq O(C_\varepsilon n)$  edges, and the multiplicity of each edge is at most  $C_\varepsilon$ . Finally, each edge of each  $H$  is assigned the same weight  $z$ , which we will choose later.

For each  $H$ , the reduction makes only one query, where it asks for the value  $F(H; z)$ . Our assumed algorithm however only works for simple graphs, so we perform a 3-stretch to obtain a simple graph  $H'$  with at most  $3|E(H)| \leq O(C_\varepsilon n)$  edges. Lemma 9 allows us to efficiently compute the value  $F(H; z)$  when we are given the value  $F(H'; t)$  and  $z = g_3(t)$  holds. Since  $g_k$  is a total function whenever  $k$  is a positive odd integer, and 3 is indeed odd, the value  $g_3(t)$  is well-defined, and we set  $z = g_3(t)$ .

Set  $C'_\varepsilon$  large enough so that  $E(H') \leq C'_\varepsilon \cdot n$  holds. Tracing back the reduction chain, we can use algorithm  $B_\varepsilon$  to compute  $T(H'; x, 1)$  in time  $2^{\varepsilon n}$  any  $\varepsilon > 0$ . Using (4), we get the value of  $F(H'; t)$  since  $x \neq 1$ . This, in turn, yields the value of  $F(H; z)$  since  $(z + 1)^k - z^k \neq 0$  and  $g_3(t) = z$ . We do this for each of the  $2^{\varepsilon m}$  queries  $H$  that the reduction in Lemma 8 makes. Finally, the latter reduction outputs the coefficients of the bivariate forest polynomial of  $G'$ , which contains the information on the number of perfect matchings of  $G$ .

To conclude, assuming the existence of the algorithm family  $(B_\varepsilon)_{\varepsilon > 0}$ , we are able to count perfect matchings in time  $\text{poly}(2^{\varepsilon m})$  for all  $\varepsilon > 0$ , which implies via Theorem 5 that #ETH is false.  $\blacksquare$

Note that the construction from the proof of Theorem 1 implies hardness of  $T(G; x, 1)$  for tripartite  $G$ , and also in the bipartite case whenever  $x \neq -1$ .



**Fig. 1** The gadget  $H_\ell$  of Provan and Ball [14] as used in the proof of Theorem 3. It corresponds to an  $\ell$ -thickening of the edge  $\{u, v\}$ , followed by a 4-stretch of each of the  $\ell$  parallel edges.

#### 4 Counting Solutions to Boolean CSPs under #ETH

In this section, we prove that the #P-hard cases of the dichotomy theorem for Boolean CSPs by Creignou and Hermann [6] are also hard under #ETH. The main difficulty is to establish #ETH-hardness of counting independent sets in bipartite graphs. We do so first, and afterwards observe that all other reductions in [6] can be used without modification.

##### 4.1 Counting Independent Sets in Bipartite Graphs is #ETH-hard

We prove that the problem of counting independent sets in bipartite graphs admits no subexponential algorithm under #ETH, even for sparse and simple graphs.

*Proof of Theorem 3* We reduce from the problem of counting independent sets in graphs of bounded degree; by Theorem 5, this problem does not have a subexponential-time algorithm. First we note that a set is an independent set if and only its complement is a vertex cover. Hence their numbers are equal. We devise a subexponential-time oracle reduction family to reduce counting vertex covers in general to counting them in bipartite graphs.

Given a graph  $G$  with  $n$  vertices and  $m$  edges, and a running time parameter  $d \in \mathbb{N}$ , the reduction works as follows. We partition the edges into  $|E|/d$  blocks of size at most  $d$  each. We denote the blocks by  $B_1, \dots, B_{m/d}$ . Next, for each  $\vec{\ell} = (\ell_1, \dots, \ell_{m/d}) \in \mathbb{N}^{m/d}$ , we denote  $G_{\vec{\ell}}$  as the graph obtained from  $G$  by replacing each edge  $e \in B_i$  with a copy of the gadget  $H_{\ell_i}$  shown in Figure 1. Note that  $G_{\vec{\ell}}$  is bipartite.

**Observation 10 (Provan and Ball).** *The number of vertex covers of  $H_\ell$  containing neither  $u$  nor  $v$  is  $2^\ell$ , the number of vertex covers containing a particular one of  $u$  or  $v$  is  $3^\ell$ , and the number of vertex covers containing both  $u$  and  $v$  is  $5^\ell$ .*

We follow the proof of Provan and Ball, but do so in a block-wise fashion. To this end, let  $T$  be the set of all  $(m/d) \times 3$  matrices with entries from  $\{0, \dots, d\}$ . The *type* of a set  $S \subseteq V(G)$  is the matrix  $t \in T$  such that, for all  $i \in \{1, \dots, m/d\}$  and  $j \in \{1, 2, 3\}$ ,  $t_{ij}$  equals the number of edges in  $B_i$  that

have exactly  $j - 1$  endpoints in  $S$ . Every set  $S \subseteq V(G)$  has exactly one type. Let  $x_t$  be the number of all sets  $S \subseteq V(G)$  that have type  $t$ .

We classify vertex covers  $C \subseteq V(G_{\vec{\ell}})$  of  $G_{\vec{\ell}}$  by their intersection with  $V(G)$ , so let  $S = C \cap V(G)$  and let  $t$  be the type of  $S$ . By Observation 10 and the fact that all inserted gadgets act independently after conditioning on the intersection of the vertex covers of  $G_{\vec{\ell}}$  with  $V(G)$ , there are exactly  $\prod_{i=1}^{m/\ell} (2^{t_{i1}} 3^{t_{i2}} 5^{t_{i3}})^{\ell_i}$  vertex covers  $C'$  whose intersection with  $V(G)$  is  $S$ . Moreover, the number of sets  $S$  of type  $t$  is equal to  $x_t$ . Hence the number  $N_{\vec{\ell}}$  of vertex covers of  $G_{\vec{\ell}}$  satisfies

$$N_{\vec{\ell}} = \sum_{t \in T} x_t \cdot \prod_{i=1}^{m/d} (2^{t_{i1}} 3^{t_{i2}} 5^{t_{i3}})^{\ell_i} \quad (8)$$

Since  $G_{\vec{\ell}}$  is bipartite, our reduction can query the oracle to obtain the numbers  $N_{\vec{\ell}}$  for all  $\vec{\ell} \in [(d+1)^3]^{n/d}$ . This yields a system of linear equations of type (8), where the  $x_t$  for  $t \in T$  are the unknowns; note that we have exactly  $|T|$  equations and unknowns. Let  $M$  be the corresponding  $|T| \times |T|$  matrix, so that the system can be written as  $N = M \cdot x$ .

It remains to prove that  $M$  is invertible. For this, we observe that  $M$  can be decomposed into a tensor product of smaller matrices as follows. Let  $A$  be the matrix of dimension  $(d+1)^3 \times (d+1)^3$ , where the row indices  $\ell$  are from  $[(d+1)^3]$ , the column indices  $\tau$  are from  $\{0, \dots, d\}^3$ , and the entries are defined via  $A_{\ell\tau} = (2^{\tau_1} 3^{\tau_2} 5^{\tau_3})^{\ell}$ . Provan and Ball, as well as the reader, observe that  $A$  is the transpose of a Vandermonde matrix. Due to the uniqueness of the prime factorization, the evaluation points  $2^{\tau_1} 3^{\tau_2} 5^{\tau_3}$  are distinct for distinct  $\tau$ , and thus  $\det(A) \neq 0$ . Furthermore, we observe that  $M = A^{\otimes n/d}$  holds, which implies  $\det(M) \neq 0$  and that  $M$  is invertible.

Since  $M$  is invertible, we can solve the equation system  $N = M \cdot x$  in time polynomial in its size, and compute  $x_t$  for all  $t \in T$ . Finally, we compute the sum of  $x_t$  over all matrices  $t$  whose first column contains only zeros. This yields the number of all sets  $S \subseteq V(G)$  that intersect every edge of  $G$  at least once, that is, the number of vertex covers of  $G$  which equals, as mentioned above, the number of independent sets of  $G$ .

Assume that #ETH holds, and let  $\varepsilon, D > 0$  be the constants from Theorem 5, which are such that no algorithm can count independent sets in general graphs of maximum degree  $D$  in time  $2^{\varepsilon n}$ . We apply our reduction to such a graph; it makes at most  $d^{O(m/d)}$  queries to the oracle. Since  $m \leq Dn$  holds, and the running time for solving the linear equation system is polynomial in the number of queries, we can choose  $d \in \mathbb{N}$  to be a large enough constant depending on  $\varepsilon > 0$  to achieve an overall running time of  $O(2^{\varepsilon n/2})$  for the reduction. Also note that the queries to the oracle for bipartite graphs have degree at most  $(d+1)^3 \cdot D$ , which is a constant that only depends on  $\varepsilon$ . If there was an algorithm for counting independent sets in bipartite graphs that ran in time  $O(2^{\varepsilon n/2})$ , we would get a combined algorithm for counting independent sets in general graphs that would be faster than the choice of  $\varepsilon$  and  $D$  would allow. Hence, under #ETH, there are constants  $\varepsilon', D' > 0$  such that no

$O(2^{\varepsilon' n})$ -time algorithm can count all independent sets on graphs of maximum degree at most  $D'$ . ■

## 4.2 The Boolean CSP dichotomy

Instances of the constraint satisfaction problem  $\#CSP(\Gamma)$  are conjunctions of relations in  $\Gamma$  applied to variables over the Boolean domain and the goal is to compute the number of satisfying assignments. A satisfying assignment is an assignment to the variables such that the formula evaluates to true, that is, every relation in the conjunction evaluates to true. A more detailed description of the problem can be found in the paper of Creignou and Hermann [6].

Creignou and Hermann prove Theorem 4 by reducing from one of two problems: Either from  $\#Pos2Sat$ , the problem of counting satisfying assignments of a 2-CNF where every literal is positive, or from  $\#Imp2Sat$ , the problem of counting satisfying assignments of a 2-CNF where every clause contains exactly one positive and one negative literal. A straightforward analysis of the construction reveals that the reductions only lead to a linear overhead. More precisely:

**Observation 11.** *Given an instance of  $\#Pos2Sat$  or  $\#Imp2Sat$  with  $n$  variables and a set  $\Gamma$  of logical relations such that at least one of the relations is not affine, the construction of Creignou and Hermann results in an instance of  $\#CSP(\Gamma)$  of size  $cn$  where  $c$  only depends on the size of the largest non-affine relation in  $\Gamma$ .*

Therefore it suffices to establish that neither  $\#Pos2Sat$  nor  $\#Imp2Sat$  have a  $\exp(o(n))$ -time algorithm. Since  $\#Pos2Sat$  is identical to counting vertex covers in (general) graphs, Theorem 5 applies here. The  $\#ETH$ -hardness of  $\#Imp2Sat$  follows by a known reduction from counting independent sets in bipartite graphs, which we include here for completeness.

**Lemma 12.** *Assuming  $\#ETH$ , there is no algorithm that solves  $\#Imp2Sat$  in time  $\exp(o(n))$  where  $n$  is the number of variables.*

Given a bipartite graph  $G = (V \dot{\cup} U, E)$  with constant degree we construct a 2-CNF  $F$  by adding a clause  $(v \rightarrow u)$  for every edge  $\{v, u\} \in E$ . Now the number of independent sets in  $G$  equals the number of satisfying assignments of  $F$ . Furthermore the existence of an algorithm that solves  $\#Imp2Sat$  in time  $\exp(o(n))$  would imply the existence of an algorithm that solves  $\#BIS$  in time  $\exp(o(n))$ . Applying Theorem 3 we obtain that such an algorithm would refute  $\#ETH$ . ■

We sketch how to obtain the  $\#ETH$  dichotomy theorem for Boolean CSPs.

*Proof of Corollary 4* If every relation in  $\Gamma$  is affine then we can solve  $\#CSP(\Gamma)$  in polynomial time using Gaussian elimination as in [6]. Otherwise, the problem is  $\#P$ -hard by [6]. If, in addition,  $\#ETH$  holds,  $\#CSP(\Gamma)$  cannot be solved in time  $\exp(o(n))$  as a subexponential algorithm could also be used to solve  $\#Pos2Sat$  or  $\#Imp2Sat$  (see Observation 11) in time  $\exp(o(n))$  which is not possible assuming  $\#ETH$  (by Theorem 5 and Lemma 12). ■

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