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The Avrunin–Scott theorem for quantum complete intersections

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ABSTRACT

We prove the Avrunin–Scott theorem for quantum complete intersections, relating the rank variety of a module to its support variety.

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1. Introduction

Inspired by the impact of the theories of varieties for modules over group algebras, similar theories have been studied for other classes of algebras. For example, using Hochschild cohomology, Snashall and Solberg developed a theory of support varieties for finite-dimensional algebras in [SnS]. As shown in [EHSST], this theory is very powerful when the cohomology of the algebra satisfies sufficient finite generation, and for selfinjective such algebras the theory shares many of the properties of that for group algebras. However, support varieties are difficult to compute. In [Car], Carlson introduced rank varieties for modules over group algebras of elementary abelian groups, varieties defined without using cohomology. Given a module over such an algebra, its rank variety is very explicit and easy to compute. Moreover, Avrunin and Scott proved in [AvS] that the support variety of a module is in fact isomorphic to its rank variety. Needless to say, this result has had important consequences (see, for example, the introduction in [AvS] or [ErH]).

Motivated by this, the second author and Holloway introduced in [ErH] rank varieties for truncated polynomial algebras in which the generators square to zero. Such algebras also have support

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varieties, and it was shown that these two varieties are related by an analogue of the Avrunin–Scott theorem.

In this paper, we study rank varieties for quantum complete intersections, a class of algebras originating from work by Manin and Avramov, Gasharov and Peeva (cf. [Man] and [AGP]). When the defining parameters are roots of unity, then these algebras have support varieties; it was shown in [BeO] that finite generation of cohomology holds. However, certain quantum complete intersections also have rank varieties, and one would therefore like to know if and how the support and rank varieties are related. This is the purpose of this paper; we prove an analogue of the Avrunin–Scott theorem, relating rank varieties to support varieties.

2. Varieties

Throughout this section, let k be an algebraically closed field. All modules considered are assumed to be left modules and finitely generated. We start by recalling the definitions and some results on support varieties; details can be found in [EHSST] and [SnS].

Let Λ be a finite-dimensional k -algebra with Jacobson radical τ . We denote by Λ^e the enveloping algebra $\Lambda \otimes_k \Lambda^{\text{op}}$ of Λ . The n th Hochschild cohomology group of Λ , denoted $\text{HH}^n(\Lambda)$, is the vector space $\text{Ext}_{\Lambda^e}^n(\Lambda, \Lambda)$ of n -fold bimodule extensions of Λ with itself. The Yoneda product turns $\text{HH}^*(\Lambda) = \bigoplus_{n=0}^{\infty} \text{HH}^n(\Lambda)$ into a graded k -algebra, the Hochschild cohomology ring of Λ . This algebra is graded commutative, that is, given two homogeneous elements $\eta, \theta \in \text{HH}^*(\Lambda)$, the equality $\eta\theta = (-1)^{|\eta||\theta|}\theta\eta$ holds. In particular, the even part $\text{HH}^{2*}(\Lambda) = \bigoplus_{n=0}^{\infty} \text{HH}^{2n}(\Lambda)$ of $\text{HH}^*(\Lambda)$ is a commutative ring.

Every homogeneous element $\eta \in \text{HH}^*(\Lambda)$ can be represented by a bimodule extension

$$\eta: 0 \rightarrow \Lambda \rightarrow B_{|\eta|} \rightarrow \cdots \rightarrow B_1 \rightarrow \Lambda \rightarrow 0.$$

Given a Λ -module M , the complex

$$\eta \otimes_{\Lambda} M: 0 \rightarrow M \rightarrow B_{|\eta|} \otimes_{\Lambda} M \rightarrow \cdots \rightarrow B_1 \otimes_{\Lambda} M \rightarrow M \rightarrow 0$$

is exact, hence the tensor product $- \otimes_{\Lambda} M$ induces a homomorphism

$$\text{HH}^*(\Lambda) \xrightarrow{\varphi_M} \text{Ext}_{\Lambda}^*(M, M)$$

of graded k -algebras. Thus, given two Λ -modules M and N , the graded vector space $\text{Ext}_{\Lambda}^*(M, N)$ becomes a graded module over $\text{HH}^*(\Lambda)$ in two ways; either via φ_M or via φ_N . These two module structures are equal up to a graded sign, that is, given homogeneous elements $\eta \in \text{HH}^*(\Lambda)$ and $\theta \in \text{Ext}_{\Lambda}^*(M, N)$, the equality $\varphi_N(\eta) \circ \theta = (-1)^{|\eta||\theta|} \theta \circ \varphi_M(\eta)$ holds, where “ \circ ” denotes the Yoneda product.

Let H be a commutative graded subalgebra of $\text{HH}^*(\Lambda)$. The support variety of an ordered pair (M, N) of Λ -modules (with respect to H), denoted $V_H(M, N)$, is defined as

$$V_H(M, N) \stackrel{\text{def}}{=} \{ \mathfrak{m} \in \text{MaxSpec } H \mid \text{Ann}_H \text{Ext}_{\Lambda}^*(M, N) \subseteq \mathfrak{m} \},$$

where $\text{MaxSpec } H$ is the set of maximal ideals of H . There are equalities

$$V_H(M, \Lambda/\tau) = V_H(M, M) = V_H(\Lambda/\tau, M),$$

and this set is defined to be the support variety $V_H(M)$ of M .

In general, support varieties do not contain any homological information on the modules involved. For example, if $\text{HH}^*(\Lambda)$ is finite-dimensional, that is, if $\text{HH}^n(\Lambda) = 0$ for $n \gg 0$, then the support variety of any pair of modules is trivial. However, under certain finiteness conditions, the situation is quite different. Suppose H is Noetherian and $\text{Ext}_{\Lambda}^*(M, N)$ is a finitely generated H -module for all Λ -modules M and N (this is equivalent to $\text{Ext}_{\Lambda}^*(\Lambda/\tau, \Lambda/\tau)$ being finitely generated over H). In this case,

the dimension of $V_H(M, N)$ equals the polynomial rate of growth of $\text{Ext}_A^*(M, N)$. In particular, the dimension of the support variety of a module equals its complexity, hence a module has finite projective dimension if and only if its support variety is trivial. Moreover, the dimension of the support variety of a module is one if and only if its minimal projective resolution is bounded. Under the finite generation hypothesis given, this happens precisely when the module is eventually periodic, that is, when its minimal projective resolution becomes periodic from some step on.

When the algebra A , in addition to satisfying the finite generation hypothesis, is also selfinjective, then the projective support variety of an indecomposable module is connected. Namely, let M be a A -module whose support variety decomposes as $V_H(M) = V_1 \cup V_2$, where V_1 and V_2 are closed homogeneous varieties such that $V_1 \cap V_2$ is trivial. Then there are submodules M_1 and M_2 of M , with the property that $M = M_1 \oplus M_2$ and $V_H(M_i) = V_i$. In particular, this implies that the support variety of an indecomposable periodic module is a single line.

We now turn to the class of algebras for which we will prove the Avrunin–Scott theorem. These are analogues of truncated polynomial rings. Let $c \geq 1$ be an integer, and let $\mathbf{q} = (q_{ij})$ be a $c \times c$ commutation matrix with entries in k . That is, the diagonal entries q_{ii} are all 1, and $q_{ij}q_{ji} = 1$ for $i \neq j$. Furthermore, let $\mathbf{a}_c = (a_1, \dots, a_c)$ be an ordered sequence of c integers with $a_i \geq 2$. The *quantum complete intersection* $A_{\mathbf{q}}^{\mathbf{a}_c}$ determined by these data is the algebra

$$A_{\mathbf{q}}^{\mathbf{a}_c} \stackrel{\text{def}}{=} k\langle x_1, \dots, x_c \rangle / (x_i^{a_i}, x_i x_j - q_{ij} x_j x_i),$$

which is selfinjective and finite-dimensional of dimension $\prod a_i$. It was proved in [BeO] that if all the commutators q_{ij} are roots of unity, then $\text{HH}^{2*}(A_{\mathbf{q}}^{\mathbf{a}_c})$ is Noetherian, and $\text{Ext}_{A_{\mathbf{q}}^{\mathbf{a}_c}}^*(M, N)$ is a finitely generated $\text{HH}^{2*}(A_{\mathbf{q}}^{\mathbf{a}_c})$ -module for all $A_{\mathbf{q}}^{\mathbf{a}_c}$ -modules M and N . Thus, in this case, the support varieties with respect to $\text{HH}^{2*}(A_{\mathbf{q}}^{\mathbf{a}_c})$ detect projective and periodic modules, as we saw above. The Krull dimension of $\text{HH}^{2*}(A_{\mathbf{q}}^{\mathbf{a}_c})$ is c , that is, the number of generators defining the quantum complete intersection. Therefore, the support varieties are homogeneous affine subsets of k^c .

Now fix an integer $a \geq 2$, and define a' by

$$a' \stackrel{\text{def}}{=} \begin{cases} a / \gcd(a, \text{char } k) & \text{if } \text{char } k > 0, \\ a & \text{if } \text{char } k = 0. \end{cases}$$

Let $q \in k$ be a primitive a' th root of unity, let \mathbf{q} be the commutation matrix with $q_{ij} = q$ for $i < j$, and let \mathbf{a}_c be the c -tuple (a, \dots, a) . Then we denote the corresponding quantum complete intersection $A_{\mathbf{q}}^{\mathbf{a}_c}$ by A_q^c , i.e.

$$A_q^c = k\langle x_1, \dots, x_c \rangle / (\{x_i^a\}, \{x_i x_j - q x_j x_i\}_{i < j}).$$

Given any c -tuple $\lambda = (\lambda_1, \dots, \lambda_c) \in k^c$, denote the element $\lambda_1 x_1 + \dots + \lambda_c x_c \in A_q^c$ by u_λ , and let $k[u_\lambda]$ be the subalgebra of A_q^c generated by this element. Then $u_\lambda^a = 0$ by [BEH, Lemma 2.3], and the *rank variety* of an A_q^c -module M , denoted $V_{A_q^c}^r(M)$, is defined as

$$V_{A_q^c}^r(M) \stackrel{\text{def}}{=} \{0\} \cup \{0 \neq \lambda \in k^c \mid M \text{ is not a projective } k[u_\lambda]\text{-module}\}.$$

When λ is nonzero, then since $u_\lambda^a = 0$, the subalgebra $k[u_\lambda]$ is isomorphic to the truncated polynomial ring $k[x]/(x^a)$. Therefore, the requirement that an A_q^c -module M is not $k[u_\lambda]$ -projective is equivalent to the requirement that the k -linear map $M \xrightarrow{u_\lambda} M$ satisfies

$$\dim \text{Im}(\cdot u_\lambda) < ((n-1)/n) \dim M.$$

This explains the choice of terminology and shows that, like support varieties, rank varieties are homogeneous affine subsets of k^c .

Our aim is to relate the rank variety $V_{A_q^c}^r(M)$ of an A_q^c -module M to its support variety $V_{\mathrm{HH}^{2*}(A_q^c)}(M)$. We do this by exploiting the structure of the Ext-algebra of the simple A_q^c -module k . These algebras were determined for all quantum complete intersections in [BeO, Theorem 5.3]. For A_q^c , it is given by

$$\mathrm{Ext}_{A_q^c}^*(k, k) = k\langle y_1, \dots, y_c, z_1, \dots, z_c \rangle / I,$$

where I is the ideal generated by the relations

$$\left(\begin{array}{ll} y_j y_i + q y_i y_j & \text{for all } i < j \\ y_i z_j - z_j y_i & \text{for all } i, j \\ z_i z_j - z_j z_i & \text{for all } i, j \\ y_i^2 - z_i & \text{for all } i \text{ if } a = 2 \\ y_i^2 & \text{for all } i \text{ if } a \neq 2 \end{array} \right)$$

and where the homological degrees of y_i and z_i are $|y_i| = 1$ and $|z_i| = 2$. We see that the commutative subalgebra of $\mathrm{Ext}_{A_q^c}^*(k, k)$ generated by z_1, \dots, z_c is the polynomial ring $k[z_1, \dots, z_c]$. Now our algebra A_q^c is \mathbb{Z}^c -graded, with the generator x_i in degree e_i , where e_i denotes the i th unit vector $(0, \dots, 0, 1, 0, \dots, 0)$. This grading induces an internal grading on $\mathrm{Ext}_{A_q^c}^n(k, k)$ for all n , and from the paragraph preceding [Opp, Corollary 3.5] it follows that the internal degree of z_i in $\mathrm{Ext}_{A_q^c}^2(k, k)$ is $-ae_i$. By [Opp, Corollary 3.5], for each $1 \leq i \leq c$, every element in $\mathrm{Ext}_{A_q^c}^2(k, k)$ of internal degree $-ae_i$ is contained in the image of the ring homomorphism $\mathrm{HH}^*(A_q^c) \xrightarrow{\varphi_k} \mathrm{Ext}_{A_q^c}^*(k, k)$. Therefore there exists a polynomial subalgebra $H = k[\eta_1, \dots, \eta_c]$ of $\mathrm{HH}^{2*}(A_q^c)$, with the property that $|\eta_i| = 2$ and $\varphi_k(\eta_i) = z_i$ for all i .

Let H be such a polynomial subalgebra of $\mathrm{HH}^{2*}(A_q^c)$. Then for all A_q^c -modules M , the support varieties $V_H(M)$ and $V_{\mathrm{HH}^{2*}(A_q^c)}(M)$ are isomorphic. To see this, denote by R the polynomial subalgebra of $\mathrm{Ext}_A^*(k, k)$ generated by the z_i . For each A -module M , the set

$$\{\mathfrak{m} \in \mathrm{MaxSpec} R \mid \mathrm{Ann}_R \mathrm{Ext}_A^*(M, k) \subseteq \mathfrak{m}\}$$

is an affine subset of $\mathrm{MaxSpec} R$, namely the *projective relative support variety* of M with respect to R (introduced in [BeS]). Since $\varphi_k(H) = \varphi_k(\mathrm{HH}^{2*}(A)) = R$, it follows from [BeS, Proposition 3.4] that the above affine subset of $\mathrm{MaxSpec} R$ is isomorphic to both $V_H(M)$ and $V_{\mathrm{HH}^{2*}(A)}(M)$. This shows that $V_H(M)$ and $V_{\mathrm{HH}^{2*}(A)}(M)$ are isomorphic.

3. The Avrunin–Scott theorem

Throughout this section, we let k, a, c and q be as in the previous section, and we denote the algebra A_q^c by A . We fix a polynomial subalgebra $H = k[\eta_1, \dots, \eta_c]$ of $\mathrm{HH}^{2*}(A)$, with $\eta_i \in \mathrm{HH}^2(A)$ and $\varphi_k(\eta_i) = z_i \in \mathrm{Ext}_A^2(k, k)$. Moreover, we identify the maximal ideals of H with the points of k^c . Finally, given a nonzero point $\lambda = (\lambda_1, \dots, \lambda_c) \in k^c$, we denote the corresponding line in k^c by ℓ_λ , and the element $\lambda_1 x_1 + \dots + \lambda_c x_c \in A$ by u_λ .

As we saw in the previous section, the support varieties $V_H(M)$ and $V_{\mathrm{HH}^{2*}(A)}(M)$ are isomorphic. The analogue of the Avrunin–Scott theorem we prove relates $V_A^r(M)$ to $V_H(M)$ for all A -modules M , and therefore also $V_A^r(M)$ to $V_{\mathrm{HH}^{2*}(A)}(M)$. We do this in a number of steps, the first of which is the following result. It provides a stable map description of rank varieties.

Proposition 3.1. *Given a nonzero point $\lambda \in k^c$, the implications*

$$\ell_\lambda \subseteq V_A^r(M) \Leftrightarrow \underline{\mathrm{Hom}}_A(Au_\lambda, M) \neq 0 \Leftrightarrow \underline{\mathrm{Hom}}_A(Au_\lambda^{a-1}, M) \neq 0$$

hold for every A -module M .

Proof. Applying $\text{Hom}_A(-, M)$ to the right exact sequence

$$A \xrightarrow{\cdot u_\lambda^{a-1}} A \xrightarrow{\cdot u_\lambda} Au_\lambda \rightarrow 0$$

gives the left exact sequence

$$0 \rightarrow \text{Hom}_A(Au_\lambda, M) \rightarrow M \xrightarrow{\cdot u_\lambda^{a-1}} M$$

of vector spaces. From this sequence we obtain the isomorphism $\text{Hom}_A(Au_\lambda, M) \simeq \text{Ker}(M \xrightarrow{\cdot u_\lambda^{a-1}} M)$, and interchanging the roles of the two elements u_λ and u_λ^{a-1} , we obtain the isomorphism $\text{Hom}_A(Au_\lambda^{a-1}, M) \simeq \text{Ker}(M \xrightarrow{\cdot u_\lambda} M)$. Next, consider the two exact sequences

$$0 \rightarrow Au_\lambda^{a-1} \hookrightarrow A \xrightarrow{\cdot u_\lambda} Au_\lambda \rightarrow 0,$$

$$0 \rightarrow Au_\lambda \hookrightarrow A \xrightarrow{\cdot u_\lambda^{a-1}} Au_\lambda^{a-1} \rightarrow 0,$$

which show that the modules Au_λ^{a-1} and Au_λ are syzygies of each other. Applying $\text{Hom}_A(-, M)$ to the first sequence, we obtain the long exact sequence

$$0 \rightarrow \text{Hom}_A(Au_\lambda, M) \rightarrow M \rightarrow \text{Hom}_A(Au_\lambda^{a-1}, M) \rightarrow \text{Ext}_A^1(Au_\lambda, M) \rightarrow 0$$

of vector spaces. The isomorphisms we obtained above, together with the isomorphisms $\text{Ext}_A^1(Au_\lambda, M) \simeq \underline{\text{Hom}}_A(\Omega_A^1(Au_\lambda), M) \simeq \underline{\text{Hom}}_A(Au_\lambda^{a-1}, M)$, give the equality

$$\dim \text{Ker}(\cdot u_\lambda^{a-1}) + \dim \text{Ker}(\cdot u_\lambda) = \dim M + \dim \underline{\text{Hom}}_A(Au_\lambda^{a-1}, M).$$

Similarly, by applying $\text{Hom}_A(-, M)$ to the other short exact sequence, we obtain the equality

$$\dim \text{Ker}(\cdot u_\lambda) + \dim \text{Ker}(\cdot u_\lambda^{a-1}) = \dim M + \dim \underline{\text{Hom}}_A(Au_\lambda, M).$$

By [ErH, Remark 3.2(ii)], the requirement that M is not a projective $k[u_\lambda]$ -module is equivalent to the requirement

$$\dim \text{Ker}(\cdot u_\lambda) + \dim \text{Ker}(\cdot u_\lambda^{a-1}) > \dim M,$$

and so the result follows. \square

Our next aim is to determine the support variety $V_H(Au_\lambda)$ for all nonzero points $\lambda \in k^c$. We start with the following general result. Recall that a k -algebra Λ is *Frobenius* if, as a left module over itself, it is isomorphic to $D(\Lambda)$, where D denotes the vector space dual $\text{Hom}_k(-, k)$.

Lemma 3.2. *Let Λ and Γ be two finite-dimensional k -algebras, with Λ selfinjective and Γ Frobenius. Furthermore, let B be a Γ - Λ -bimodule which is projective both as a Γ -module and as a Λ -module. Then for every Λ -module X and every Γ -module Y , there is a natural isomorphism*

$$\text{Ext}_\Lambda^n(X, D(B) \otimes_\Gamma Y) \simeq \text{Ext}_\Gamma^n(B \otimes_\Lambda X, Y)$$

for each $n \geq 0$.

Proof. Adjointness gives a natural isomorphism

$$\mathrm{Hom}_\Lambda(X, \mathrm{Hom}_\Gamma(B, Y)) \simeq \mathrm{Hom}_\Gamma(B \otimes_\Lambda X, Y)$$

for every Λ -module X and every Γ -module Y . The functor $\mathrm{Hom}_\Gamma(B, -)$ is isomorphic to $\mathrm{Hom}_\Gamma(B, \Gamma \otimes_\Gamma -)$, which in turn is isomorphic to $\mathrm{Hom}_\Gamma(B, \Gamma) \otimes_\Gamma -$ since B is a projective Γ -module. Now Γ , being Frobenius, is isomorphic as a left Γ -module to $D(\Gamma)$, and so adjointness gives

$$\mathrm{Hom}_\Gamma(B, \Gamma) \simeq \mathrm{Hom}_\Gamma(B, D(\Gamma)) \simeq D(\Gamma \otimes_\Gamma B) \simeq D(B).$$

Therefore, the functor $\mathrm{Hom}_\Gamma(B, -)$ is isomorphic to $D(B) \otimes_\Gamma -$, proving the case $n = 0$. For the case $n > 0$, note that since B is a projective Λ -module, the Λ -module $D(B) \otimes_\Gamma P$ is projective for every projective Γ -module P . However, since both Λ and Γ are selfinjective, this means that $D(B) \otimes_\Gamma I$ is an injective Λ -module for every injective Γ -module I . Moreover, given a Γ -module Y with an injective resolution \mathbf{I} , the complex $D(B) \otimes_\Gamma \mathbf{I}$ is an injective resolution of the Λ -module $D(B) \otimes_\Gamma Y$, since $D(B)$ is a projective right Γ -module. Therefore the isomorphisms

$$\begin{aligned} \mathrm{Ext}_\Gamma^n(B \otimes_\Lambda X, Y) &\simeq H^n(\mathrm{Hom}_\Gamma(B \otimes_\Lambda X, \mathbf{I})) \\ &\simeq H^n(\mathrm{Hom}_\Lambda(X, D(B) \otimes_\Gamma \mathbf{I})) \\ &\simeq \mathrm{Ext}_\Lambda^n(X, D(B) \otimes_\Gamma Y) \end{aligned}$$

hold for every Λ -module X . \square

In order to determine the support variety of Au_λ , we exploit some nice properties of certain bi-modules arising from elements of the Hochschild cohomology ring. Let ζ be a homogeneous element of $\mathrm{HH}^*(A)$, represented by a map $\Omega_{A^e}^{|\zeta|}(A) \xrightarrow{f_\zeta} A$, say. Then ζ corresponds to the bottom short exact sequence in the exact commutative pushout diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Omega_{A^e}^{|\zeta|}(A) & \longrightarrow & P_{|\zeta|-1} & \longrightarrow & \Omega_{A^e}^{|\zeta|-1}(A) \longrightarrow 0 \\ & & \downarrow f_\zeta & & \downarrow & & \parallel \\ 0 & \longrightarrow & A & \longrightarrow & K_\zeta & \longrightarrow & \Omega_{A^e}^{|\zeta|-1}(A) \longrightarrow 0 \end{array}$$

where $P_{|\zeta|-1}$ is the projective cover of $\Omega_{A^e}^{|\zeta|-1}(A)$. If ζ is an element of H and M is an A -module, then by [EHSST, Proposition 4.3] the support variety of the A -module $K_\zeta \otimes_A M$ is given by

$$\mathrm{V}_H(K_\zeta \otimes_A M) = \mathrm{V}_H(\zeta) \cap \mathrm{V}_H(M),$$

where $\mathrm{V}_H(\zeta) = \{\alpha \in k^c \mid \zeta(\alpha) = 0\}$. In the following result, we use the pushout bimodule K_ζ to give a criterion for when two lines in k^c are perpendicular. Given a nonzero point $\mu \in k^c$, we denote the set $\{\alpha \in k^c \mid \sum \alpha_i \mu_i = 0\}$ by ℓ_μ^\perp ; this is the hyperplane perpendicular to the line ℓ_μ .

Lemma 3.3. *Let M be a periodic A -module of period 1, and suppose $\mathrm{V}_H(M)$ is a single line ℓ_α , where $\alpha \in k^c$ is a nonzero point. Then, given any nonzero point $\mu \in k^c$, the implication*

$$\underline{\mathrm{Hom}}_A(M, K_\zeta \otimes_A k) \neq 0 \quad \Rightarrow \quad \ell_\alpha \subseteq \ell_\mu^\perp$$

holds, where $\zeta = \sum \mu_i \eta_i \in H^2$.

Proof. Since the bimodule K_ζ is projective both as a left and as a right A -module, we see from the previous lemma that $\underline{\text{Hom}}_A(M, K_\zeta \otimes_A k)$ is naturally isomorphic to $\underline{\text{Hom}}_A(D(K_\zeta) \otimes_A M, k)$. Thus $\underline{\text{Hom}}_A(M, K_\zeta \otimes_A k)$ is nonzero if and only if the same holds for $\underline{\text{Hom}}_A(D(K_\zeta) \otimes_A M, k)$, and this is equivalent to $D(K_\zeta) \otimes_A M$ not being a projective A -module. The latter happens if and only if $V_H(D(K_\zeta) \otimes_A M) \neq 0$.

Using the previous lemma once more, we see that the H -modules $\text{Ext}_A^*(D(K_\zeta) \otimes_A M, k)$ and $\text{Ext}_A^*(M, K_\zeta \otimes_A k)$ are isomorphic. Therefore

$$\begin{aligned} V_H(D(K_\zeta) \otimes_A M) &= V_H(M, K_\zeta \otimes_A k) \\ &\subseteq V_H(M) \cap V_H(K_\zeta \otimes_A k) \\ &= V_H(M) \cap V_H(\zeta) \\ &= \ell_\alpha \cap \ell_\mu^\perp, \end{aligned}$$

and so if $\underline{\text{Hom}}_A(M, K_\zeta \otimes_A k)$ is nonzero then $\ell_\alpha \subseteq \ell_\mu^\perp$. \square

Next, define a map $k^c \xrightarrow{F} k^c$ of affine spaces by

$$(\alpha_1, \dots, \alpha_c) \mapsto (\alpha_1^a, \dots, \alpha_c^a).$$

Our aim now is to show that $V_H(Au_\lambda) = \ell_{F(\lambda)}$ for every nonzero point $\lambda \in k^c$. In order to prove this, we need the following lemma.

Lemma 3.4. *Let μ and λ be nonzero points in k^c with $\ell_\mu \subseteq \ell_{F(\lambda)}^\perp$, and denote the element $\sum \mu_i \eta_i$ in H by ζ . Then there exists a monomorphism*

$$Au_\lambda \rightarrow K_\zeta \otimes_A k,$$

and consequently $\underline{\text{Hom}}_A(Au_\lambda, K_\zeta \otimes_A k)$ is nonzero.

Proof. Note that we have implicitly assumed that c is at least 2. Denote the radical of A by τ . The second syzygy $\Omega_A^2(k)$ is the kernel of the projective cover $A^c \rightarrow \tau$, which maps the i th generator ε_i of the free A -module A^c to x_i . The generators of $\Omega_A^2(k)$ are the two sets

$$\{x_i^{a-1} \varepsilon_i\}_{i=1}^c, \quad \{qx_j \varepsilon_i - x_i \varepsilon_j\}_{i < j},$$

and the element $z_i \in \text{Ext}_A^2(k, k)$ is represented by the homomorphism $\Omega_A^2(k) \xrightarrow{f_{z_i}} k$ mapping $x_i^{a-1} \varepsilon_i$ to 1 and all the other generators to zero. In the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Omega_A^2(k) & \longrightarrow & A^c & \longrightarrow & \Omega_A^1(k) \longrightarrow 0 \\ & & \downarrow f_{z_i} & & \downarrow f_i & & \downarrow \Omega_A^{-1}(f_{z_i}) \\ 0 & \longrightarrow & k & \xrightarrow{\cdot(\prod x_i^{a-1})} & A & \xrightarrow{\pi} & \Omega_A^{-1}(k) \longrightarrow 0 \end{array}$$

the map f_i , given by

$$e_j \mapsto \begin{cases} q^{i-1} \prod_{n \neq i} x_n^{a-1} & \text{when } j = i, \\ 0 & \text{when } j \neq i, \end{cases}$$

provides the first step in a lifting of f_{z_i} . Thus the map $\Omega_A^{-1}(f_{z_i})$ factorizes as $\Omega_A^{-1}(f_{z_i}) = \pi g_i$, where $A \xrightarrow{\pi} \Omega_A^{-1}(k)$ is the natural quotient map and $\Omega_A^1(k) \xrightarrow{g_i} A$ is the map given by

$$x_j \mapsto \begin{cases} q^{i-1} \prod_{n \neq i} x_n^{a-1} & \text{when } j = i, \\ 0 & \text{when } j \neq i. \end{cases}$$

Now we use the fact that $K_\zeta \otimes_A k$ is the pullback of

$$\begin{array}{ccc} & \Omega_A^1(k) & \\ & \downarrow \Omega_A^{-1}(\sum \mu_i f_{z_i}) & \\ A & \xrightarrow{\pi} & \Omega_A^{-1}(k) \end{array}$$

to obtain a monomorphism $Au_\lambda \rightarrow K_\zeta \otimes_A k$. We do this by constructing a monomorphism $Au_\lambda \xrightarrow{f} A \oplus \Omega_A^1(k)$ whose composition with $A \oplus \Omega_A^1(k) \xrightarrow{(\pi, \Omega_A^{-1}(\sum \mu_i f_{z_i}))} \Omega_A^{-1}(k)$ is zero. Namely, define f by

$$u_\lambda \mapsto \left(\sum_{i=1}^c \mu_i g_i(u_\lambda), -u_\lambda \right).$$

To show that this map is well defined, we need to check that $f(u_\lambda^a) = 0$, that is, that the element $u_\lambda^{a-1} f(u_\lambda)$ is zero. Computing directly, we see that

$$\begin{aligned} u_\lambda^{a-1} \sum_{i=1}^c \mu_i g_i(u_\lambda) &= u_\lambda^{a-1} \sum_{i=1}^c \left(\mu_i \lambda_i q^{i-1} \prod_{n \neq i} x_n^{a-1} \right) \\ &= \sum_{i=1}^c \lambda_i^{a-1} x_i^{a-1} \left(\mu_i \lambda_i q^{i-1} \prod_{n \neq i} x_n^{a-1} \right) \\ &= \left(\sum_{i=1}^c \mu_i \lambda_i^a \right) \prod_{j=1}^c x_j^{a-1} \\ &= 0 \end{aligned}$$

since $\ell_\mu \subseteq \ell_{F(\lambda)}^\perp$. Thus the map f is well defined, and it is obviously a monomorphism. \square

We are now ready to prove that $V_H(Au_\lambda) = \ell_{F(\lambda)}$ for every nonzero point $\lambda \in k^c$.

Proposition 3.5. *If $\lambda \in k^c$ is nonzero, then $V_H(Au_\lambda) = \ell_{F(\lambda)}$.*

Proof. Denote by T the A -module $Au_\lambda \oplus Au_\lambda^{a-1}$. This is a periodic module of period 1, and its support variety equals that of Au_λ since Au_λ and Au_λ^{a-1} are syzygies of each other. Moreover, the module Au_λ is indecomposable, hence its support variety is a single line. Let therefore $\alpha \in k^c$ be a nonzero point such that $V_H(Au_\lambda) = \ell_\alpha$. Furthermore, let $\mu \in k^c$ be a nonzero point such that $\ell_\mu \subseteq \ell_{F(\lambda)}^\perp$, and denote the element $\mu_1 \eta_1 + \cdots + \mu_c \eta_c \in H$ by ζ . By the previous lemma, the space $\text{Hom}_A(T, K_\zeta \otimes_A k)$ is nonzero, and so from Lemma 3.3 we obtain the inclusion $\ell_\alpha \subseteq \ell_\mu^\perp$. Therefore $\ell_\alpha = \ell_{F(\lambda)}$. \square

Using Propositions 3.1 and 3.5, we now prove our main result, namely the quantum complete intersection analogue of the Avrunin–Scott theorem. It shows that the map $k^c \xrightarrow{F} k^c$ maps the rank variety of a module onto its support variety.

Theorem 3.6. $F(V_A^r(M)) = V_H(M)$ for every A -module M .

Proof. The result is clearly true if M is projective. If M is not projective, let $\lambda \in k^c$ be a nonzero point whose corresponding line ℓ_λ is contained in $V_A^r(M)$. Then $\underline{\text{Hom}}_A(Au_\lambda, M)$ and $\underline{\text{Hom}}_A(Au_\lambda^{a-1}, M)$ are both nonzero by Proposition 3.1, and so $\text{Ext}_A^i(Au_\lambda, M) \neq 0$ for $i > 0$. This implies that $V_H(Au_\lambda, M)$ is nontrivial, and since this variety is contained in $V_H(Au_\lambda) \cap V_H(M)$, we see from Proposition 3.5 that the line $\ell_{F(\lambda)}$ must be contained in $V_H(M)$. Consequently, the inclusion $F(V_A^r(M)) \subseteq V_H(M)$ holds.

To prove the reverse inclusion, we argue by induction on the dimension of the support variety of M . We may assume that M is indecomposable, since the variety (rank or support) of a direct sum is the union of the varieties of the summands. Suppose first that the dimension of $V_H(M)$ is one, i.e. that M is periodic. Then $V_H(M)$ is a line, and we proved above that this variety contains $F(V_A^r(M))$. If $F(V_A^r(M)) = 0$, then $V_A^r(M) = 0$, which is not the case since M is not projective. Hence $F(V_A^r(M))$ is nontrivial, and so $F(V_A^r(M)) = V_H(M)$ in this case. Next, suppose that $\dim V_H(M) > 1$, and let $\mu \in k^c$ be a nonzero point such that $\dim(\ell_\mu^\perp \cap V_H(M)) < \dim V_H(M)$. Denote the element $\mu_1\eta_1 + \cdots + \mu_c\eta_c \in H$ by ζ , and let $\Omega_{A^e}^2(A) \xrightarrow{f_\zeta} A$ be a bimodule map representing this element. Since the bottom exact sequence in the pushout diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Omega_{A^e}^2(A) & \longrightarrow & P_1 & \longrightarrow & \Omega_{A^e}^1(A) \longrightarrow 0 \\ & & \downarrow f_\zeta & & \downarrow & & \parallel \\ 0 & \longrightarrow & A & \longrightarrow & K_\zeta & \longrightarrow & \Omega_{A^e}^1(A) \longrightarrow 0 \end{array}$$

splits as right A -modules, it stays exact after tensoring with M . This gives a short exact sequence

$$0 \rightarrow M \rightarrow K_\zeta \otimes_A M \rightarrow \Omega_A^1(M) \oplus P \rightarrow 0,$$

in which P is projective, and so

$$V_A^r(K_\zeta \otimes_A M) \subseteq V_A^r(M) \cup V_A^r(\Omega_A^1(M) \oplus P) = V_A^r(M).$$

Now since $V_H(K_\zeta \otimes_A M) = V_H(\zeta) \cap V_H(M) = \ell_\mu^\perp \cap V_H(M)$, induction gives $V_H(K_\zeta \otimes_A M) \subseteq F(V_A^r(K_\zeta \otimes_A M))$. Therefore $\ell_\mu^\perp \cap V_H(M) \subseteq F(V_A^r(M))$, and since this inclusion holds for every nonzero point $\mu \in k^c$ such that $\dim(\ell_\mu^\perp \cap V_H(M))$ is strictly less than $\dim V_H(M)$, we see that $V_H(M) \subseteq F(V_A^r(M))$. \square

Consequently, the dimension of the rank variety of an A -module is the complexity of the module. In particular, an indecomposable module is periodic if and only if its rank variety is of dimension one.

Corollary 3.7. For every A -module M , the dimension of $V_A^r(M)$ is the complexity of M .

Corollary 3.8. An indecomposable A -module M is periodic if and only if the dimension of $V_A^r(M)$ is one.

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