Journal of Mathematical Analysis
ISSN: ..., URL: http://www.ilirias.com
Volume x Issue x (201x), Pages x-xx.

CLOSED-FORM EXACT SOLUTIONS TO THE DENSITY DEPENDENT FITZHUGH–NAGUMO TELEGRAPH EQUATION

M. HAJIKETABI, R. A. VAN GORDER, S. ABBASBANDY

Abstract. We use the first integral method to construct closed-from exact solutions of the density dependent Fitzhugh–Nagumo telegraph equation. While this method is very useful for obtaining formal solutions, many of the solutions are not physically relevant. Hence, we carefully determine which of these solutions is physically useful for the problem. For standard diffusion and linear density dependent diffusion, we obtain physically meaningful exact solutions. Interestingly, the case with linear density dependence shows strong qualitative agreement the standard diffusion case. For both cases, the wave solutions result in an initial concentration which either dissipates or tends to some finite positive value for large time. While one can obtain wave solutions with general wave speeds for the standard diffusion case, when there is density dependence in diffusion, the resulting wave speeds will take specific values depending on the values of the model parameters. For stronger density dependent diffusion involving quadratic and higher order functions of the concentration, we show that the equation is not integrable, and exact solutions should not be expected for such cases.

1. Introduction

The Nagumo equation is a nonlinear reaction–diffusion equation that models an active pulse transmission line simulating a nerve axon [1, 2] and it is used widely in biology, area of population genetics [3], circuit theory, and other fields [4, 5, 6, 7]. Telegraph equations, in particular, have been proposed to model diffusion in biological, social, and economic systems [8, 9]. In this paper, we seek exact solutions $u(x,t)$ to the density dependent Fitzhugh–Nagumo telegraph equation

$$\tau\frac{\partial^2 u}{\partial t^2}+(1+\tau g'(u))\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\left(u^m\frac{\partial u}{\partial x}\right)-g(u), \tag{1.1}$$

where $g(u)=u(1-u)(\alpha-u)$, $\alpha\in[0,1]$, $\tau\in\mathbb{R}$, and $m\geq 0$. In Eq. (1.1), $u(x,t)$ measures a concentration (such as that of a chemical species) at a point $x\in\mathbb{R}$ in space and time $t>0$. Regarding the model parameters, τ represents a measure of the memory (delay) effect [9, 10], α represents the value of the ambient concentration or density, and $g(u)=u(1-u)(\alpha-u)$ represents the reaction term [9, 11].

2010 *Mathematics Subject Classification.* 35C07, 35C05, 35M10.
Key words and phrases. exact solutions; first integral method; density dependent Fitzhugh–Nagumo telegraph equation.

Submitted . Published .

If parameter $\tau = 0$, we have the standard density dependent Fitzhugh–Nagumo equation. If $m = 0$, we have the standard Fitzhugh–Nagumo telegraph equation and if both $\tau = m = 0$ we have the standard Fitzhugh–Nagumo equation. Kawahara and Tanaka have found new exact solutions by using Hirota method [12], Nucci and Clarkson have obtained some new solutions with Jacobi elliptic function [13], Li and Guo have obtained new series of exact solutions by the first integral method [14], and Abbasbandy have obtained the soliton solution with the homotopy analysis method [15], of the standard Fitzhugh–Nagumo equation ($\tau = m = 0$). Solutions have been obtained in terms of series solutions and numerical solutions by the Runge–Kutta–Fehlberg 45 method [16] and by the δ-expansion method [17]. Masemola obtained reductions and then new exact solutions by applying the generalized double–reduction theorem via associated symmetries [18], Hashemi, Darvishi, and Baleanu used Lie symmetry group-preserving scheme (LSGPS) [19], of the standard density dependent Fitzhugh–Nagumo equation ($\tau = 0$). Van Gorder and Vajravelu have obtained numerical solutions for the standard Fitzhugh–Nagumo telegraph equation ($m = 0$) via a variational technique [20].

The inclusion of telegraph effects is an extension of works on related reaction-diffusion models, and may still be of physical relevance. Hadeler [21] suggested that the analysis of the Hodgkin-Huxley hyperbolic system, where a second order time term is added to the classical reaction-diffusion equation, by Fitzgibbon and Parrot [22, 23] was motivated by original theoretical investigations by Hodgkin and Huxley themselves, later dropped because of poorer analytical properties in comparison with the more common parabolic setting. Motivated by this, Hadeler [21] seems to be among the first to study the hyperbolic FitzHugh-Nagumo equation. Eq. (1.1) has been studied for certain parameter values in [24, 25], where approximate analytical and numerical solutions were obtained.

For reaction-diffusion models, a number of exact solutions have been reported previously, and some of this work involves using the integrability of the relevant reaction-diffusion equation to obtain closed-form exact solutions [26]-[40]. Having exact solutions to such equations is desirable for multiple reasons. First, as numerical solutions are approximations, we would obviously prefer exact solutions in closed-form in order to verify the true nature of the solutions. Second, these kinds of reaction-diffusion equations often have traveling wave solutions, and the wave speeds may not be arbitrary parameters that can be picked. Rather, the wave speed of a traveling wave solution may, in some cases, depend strongly on the model parameters. Finally, traveling wave solutions may not exist for all parameter regimes, and hence exact closed-form solutions allow us to demonstrate the existence of such traveling waves. These latter two points are linked: for some parameter values, the solutions may become degenerate, with zero or infinite wave speeds obtained. For such cases numerical approaches will give difficulties, yet with exact solutions we can better understand the qualitative features of the model, which will alert us to parameter regions for which traveling waves exist or do not exist. Therefore, in the present paper we shall explore the possible exact solutions to the density dependent Fitzhugh–Nagumo telegraph equation given by (1.1).

In Section 2, using the first integral method, we obtain exact closed-form solutions of Eq. (1.1) for different values of τ, α and $m = 0, 1, 2$. While a number of solutions are obtained, not all of these solutions are physically reasonable. Indeed, as has been pointed out in different contexts, many solution approaches such as this obtain some non-physical solutions, therefore one needs to take care that the solutions obtained actually are relevant to the physical problem [41]-[44]. As such, in Section 3 we focus on the physically meaningful solutions of (1.1). We find that bounded traveling wave solutions exist for the Fitzhugh–Nagumo telegraph equation ($m = 0$) or for linear density dependence ($m = 1$). For strong density dependence ($m = 2$), we show that (1.1) is not integrable, and therefore we expect no physically meaningful closed-form exact solution exists. We give concluding remarks in Section 4.

2. First integral method and exact solutions

The first integral method which is based on the ring theory of commutative algebra was first introduced in Feng [45] for finding the exact 1-soliton solutions of the Burger-KdV equation. This method was further developed by the same author in [46]-[50] and some other mathematicians [50]-[64]. The method is accurate, efficient, trustworthy and does not require complicated and tedious computations. The basic idea of this method is to construct a first integral with polynomial coefficients of an explicit form to an equivalent autonomous planar system by using the Division Theorem.The method provides exact and explicit solutions. This fact is important when looking for multiple solutions. Often times, numerical methods or analytical approaches will only find a single solution. However, since the first integral method gives us a way to find exact solutions, one will be able to find multiple solutions (if they do exist).

2.1. **Outline of the first integral method.**

Consider a general nonlinear partial differential equation (PDE) in the form

$$P(u, u_t, u_x, u_{tt}, u_{xx}, u_{tx}, u_{xxx}, ...) = 0. \tag{2.1}$$

Assume (2.1) has the traveling wave solution as

$$u(t, x) = u(z), \qquad z = x - ct. \tag{2.2}$$

Substituting (2.2) into Eq. (2.1) yields into the following ordinary differential equation (ODE):

$$Q(u, u', u'', u''', ...) = 0, \tag{2.3}$$

where prime denotes the derivative with respect to the same variable ξ. Next, we introduce a new independent variable

$$x(z) = u(z), \qquad y(z) = u'(z), \tag{2.4}$$

which change (2.3) to a system of ODEs

$$\begin{cases} x'(z) = y(z), \\ y'(z) = f\left(x(z), y(z)\right). \end{cases} \tag{2.5}$$

According to the qualitative theory of differential equations, if one can find two first integrals to system (2.5) under the same conditions, then analytic solutions to (2.5) can be solved directly. However, in general, it is difficult to realize this even for a single first integral, because for a given plane autonomous system, there is no general theory telling us how to find its first integrals in a systematic way. We shall apply the Division Theorem to obtain one first integral to (2.5) which reduces (2.3) to a first-order integrable ordinary differential equation. An exact solution to (2.3) is then obtained by solving this equation. Now, let us recall the Division Theorem for two variables in the complex domain $\mathbb{C}$ [45]:

Theorem 2.1 (Division Theorem)**.** *Suppose that $P(x,y)$ and $Q(x,y)$ are polynomials of two variables x and y in $\mathbb{C}(x,y)$ and $P(x,y)$ is irreducible in $\mathbb{C}(x,y)$. If $Q(x,y)$ vanishes at all zero points of $P(x,y)$, then there exists a polynomial $G(x,y)$ in $\mathbb{C}(x,y)$ such that $Q(x,y)=G(x,y)P(x,y)$.*

2.2. **Application of the first integral method.** We will be interested in solving the travelling wave problem for Eq. (1.1). To this end, let us assume a solution of the form $u(x,t)=f(z)$ where $z=x-ct$ is the wave variable and $c\in\mathbb{R}$ is the wave speed. Then Eq. (1.1) becomes

$$\begin{aligned}\tau c^2 f''(z)-cf'(z)\left(1+3\tau f^2(z)-2\tau(\alpha+1)f(z)+\tau\alpha\right)&\\ -\left(f^m(z)f'(z)\right)'+f(z)(1-f(z))(\alpha-f(z))=0.&\end{aligned} \tag{2.6}$$

The above equation can also be written as

$$\begin{aligned}&\left(\tau c^2-f^m(z)\right)f''(z)-mf^{m-1}(z)\left(f'(z)\right)^2\\ &-cf'(z)\left(1+3\tau f^2(z)-2\tau(\alpha+1)f(z)+\tau\alpha\right)\\ &\qquad+f^3(z)-(\alpha+1)f^2(z)+\alpha f(z)=0.\end{aligned} \tag{2.7}$$

If we rearrange Eq. (2.7) in terms of the second derivative, we find

$$\begin{aligned}f''(z)=\frac{1}{\tau c^2-f^m(z)}&\Big(mf^{m-1}(z)\left(f'(z)\right)^2\\ +cf'(z)&\left(1+3\tau f^2(z)-2\tau(\alpha+1)f(z)+\tau\alpha\right)\\ &\quad+f^3(z)-(\alpha+1)f^2(z)+\alpha f(z)\Big),\end{aligned} \tag{2.8}$$

which is a second order nonlinear equation. Next, we introduce new independent variables

$$x(z)=f(z),\qquad y(z)=f'(z), \tag{2.9}$$

which change Eq. (2.8) to a system of ODEs

$$\begin{aligned}x'(z)&=y(z),\\ y'(z)&=\frac{1}{\tau c^2-x^m(z)}\Big(mx^{m-1}(z)y^2(z)+cy(z)\left(1+3\tau x^2(z)-2\tau(\alpha+1)x(z)+\tau\alpha\right)\\ &\qquad+x^3(z)-(\alpha+1)x^2(z)+\alpha x(z)\Big).\end{aligned} \tag{2.10}$$

Now, we make the transformation $d\eta=dz/\left(-x^m(z)+\tau c^2\right)$ in Eq. (2.10) to avoid the singular line $-x^m(z)+\tau c^2=0$, temporarily. Thus, system (2.10) becomes

$$\begin{aligned}\frac{dx}{d\eta}&=\frac{dx}{dz}\frac{dz}{d\eta}=y(z)(-x^m(z)+\tau c^2),\\ \frac{dy}{d\eta}&=\frac{dy}{dz}\frac{dz}{d\eta}=mx^{m-1}(z)y^2(z)+cy(z)\left(1+3\tau x^2(z)-2\tau(\alpha+1)x(z)+\tau\alpha\right)\\ &\qquad+x^3(z)-(\alpha+1)x^2(z)+\alpha x(z).\end{aligned} \tag{2.11}$$

Now, we apply the Division Theorem to seek the first integral to system (2.11). Suppose that $x=x(\eta)$ and $y=y(\eta)$ are the nontrivial solutions to (2.11), and $q(x,y)=\sum_{i=0}^{n}a_i(x)y^i$ is an irreducible polynomial in $\mathbb{C}[x,y]$, such that

$$q\left(x\left(\eta\right),y\left(\eta\right)\right)=\sum_{i=0}^{n}a_i\left(x\left(\eta\right)\right)y^i(\eta)=0, \tag{2.12}$$

where $a_i(x)(i = 0, 1, ..., n)$, are polynomials of x and $a_n(x) \neq 0$ for all n. Eq. (2.12) is called the first integral to system (2.11). We put $n = 1$ in Eq. (2.12). According to the Division Theorem, there exists a polynomial $G(x, y) = g(x)+h(x)y$ in $\mathbb{C}[x, y]$ such that

$$\frac{dq}{d\eta}\bigg|_{(2.11)} = \left(\frac{dq}{dx}\frac{dx}{d\eta} + \frac{dq}{dy}\frac{dy}{d\eta}\right)\bigg|_{(2.11)} = \left(\sum_{i=0}^{1} a_i'(x)y^i\right)(-yx^m + y\tau c^2)$$
$$+\left(\sum_{i=0}^{1} ia_i(x)y^{i-1}\right)\Big(mx^{m-1}y^2 + cy\left(1 + 3\tau x^2 - 2\tau(\alpha+1)x + \tau\alpha\right)$$
$$+x^3 - (\alpha+1)x^2 + \alpha x\Big) = (g(x) + h(x)y)\left(\sum_{i=0}^{1} a_i(x)y^i\right), \qquad (2.13)$$

where prime denotes differentiation with respect to the variable x. By comparing with the coefficients of y^i (for $i = 0, 1, 2$) of both sides of (2.13), we have

$$a_1'(x)(-x^m + \tau c^2) = a_1(x)(h(x) - mx^{m-1}), \qquad (2.14)$$

$$a_0'(x)(-x^m + \tau c^2) + a_1(x)\left(3c\tau x^2 - 2c\tau(\alpha+1)x + c\tau\alpha + c\right)$$
$$= g(x)a_1(x) + h(x)a_0(x), \qquad (2.15)$$

$$a_1(x)(-x^3 + (\alpha+1)x^2 - \alpha x) = g(x)a_0(x). \qquad (2.16)$$

Now, we should obtain $a_0(x)$ and $a_1(x)$ from Eqs. (2.14)-(2.16) and putting in Eq. (2.12) we conclude the first integral for system (2.11).

2.3. **Exact traveling wave solutions.** We now apply the above results to Eq. (1.1) for various values of the model parameters. We shall organize our results based on the strength of the nonlinear diffusion parameter, m, which takes values 0, 1, or 2. We remark that all solutions obtained in this paper have been checked with Mathematica software by plugging each solution back into Eq. (1.1) and ensuring that the output is identically zero. We must point out that the Fitzhugh-Nagumo problem is for a density or concentration, so the solutions should have a minimal and maximal finite value. Hence, unbounded solutions or solutions which are complex-valued rather than real-valued, are non-physical, and hence are ruled out.

Theorem 2.2. *In the case of $m = 0$, applying the first integral method to Eq.* (1.1) *yields the following bounded, real-valued traveling wave solutions:*

$$u(x,t) = \frac{1}{1 + exp\left(\pm\frac{x}{\sqrt{2}} + k\right)}, \qquad (2.17)$$

where $\alpha = \frac{1}{2}$ and k is an arbitrary constant,

$$u(x,t) = \frac{2}{1 + exp\left(\pm\sqrt{2}x + 2k\right)}, \qquad (2.18)$$

where $\alpha = 2$ and k is an arbitrary constant,

$$u(x,t) = -tanh\left(\pm\frac{x}{\sqrt{2}} + k\right), \qquad (2.19)$$

where $\alpha = -1$ *and* k *is an arbitrary constant, and*

$$u(x,t) = \frac{1}{1 + exp\left(\pm\frac{x}{\sqrt{2}} + \left(-\frac{1}{2} + \alpha\right)t + k\right)}, \tag{2.20}$$

where $\tau = 0$, α *is not specified, and* k *is an arbitrary constant.*

Proof. First we put $m = 0$, in this case, Eqs. (2.14)-(2.16) are converted to:

$$a_1'(x)(-1 + \tau c^2) = a_1(x)h(x), \tag{2.21}$$

$$a_0'(x)(-1 + \tau c^2) + a_1(x)\left(3c\tau x^2 - 2c\tau(\alpha + 1)x + c\tau\alpha + c\right) = g(x)a_1(x) + h(x)a_0(x), \tag{2.22}$$

$$a_1(x)(-x^3 + (\alpha + 1)x^2 - \alpha x) = g(x)a_0(x). \tag{2.23}$$

Since $a_i(x)(i = 0, 1)$ and $h(x)$ are polynomials, then from Eq. (2.21) we deduce that $h(x) = 0$ only, and for simplicity $a_1(x)$ is constant and take $a_1(x) = 1$. Balancing the degrees of $g(x)$ and $a_0(x)$ in Eqs. (2.22) and (2.23), we conclude that $\deg(a_0(x)) = 1$ and $\deg(g(x)) = 2$ or $\deg(a_0(x)) = 2$ and $\deg(g(x)) = 1$.

In the case of $\deg(a_0(x)) = 1$ and $\deg(g(x)) = 2$, assuming $a_0(x) = A_1x + A_0$ ($A_1 \neq 0$) and $g(x) = B_2x^2 + B_1x + B_0$ ($B_2 \neq 0$), where A_0, A_1, B_0, B_1 and B_2 are all constants to be determined. Substituting $a_1(x), h(x), a_0(x)$ and $g(x)$ in Eqs. (2.22) and (2.23), and setting all the coefficients of powers x to be zero, we obtain a system of nonlinear algebraic equations and by solving it, we obtain the six following solutions:

$$A_1 = \mp\frac{\sqrt{-\tau(4+3\tau)}}{3\sqrt{2}\tau}, \quad A_0 = \pm\frac{\sqrt{-\tau(4+3\tau)}}{6\sqrt{2}\tau}, \quad B_2 = \pm\frac{3\sqrt{2}\tau}{\sqrt{-\tau(4+3\tau)}},$$
$$B_1 = \mp\frac{3\sqrt{2}\tau}{\sqrt{-\tau(4+3\tau)}}, \quad B_0 = 0, \quad c = \pm\frac{\sqrt{2}}{\sqrt{-\tau(4+3\tau)}}, \quad \alpha = \frac{1}{2}, \tag{2.24}$$

$$A_1 = \mp\frac{\sqrt{2+6\tau}}{3\sqrt{-\tau}}, \ A_0 = \pm\frac{\sqrt{2+6\tau}}{3\sqrt{-\tau}}, \ B_2 = \pm\frac{3\sqrt{-\tau}}{\sqrt{2+6\tau}},$$
$$B_1 = \pm\frac{3\sqrt{2}\tau}{\sqrt{-\tau(1+3\tau)}}, \ B_0 = 0, \ c = \mp\frac{1}{\sqrt{-\tau(2+6\tau)}}, \ \alpha = 2, \tag{2.25}$$

$$A_1 = \mp\frac{\sqrt{2+6\tau}}{3\sqrt{-\tau}}, \quad A_0 = 0, \quad B_2 = \pm\frac{3\sqrt{-\tau}}{\sqrt{2+6\tau}}, \quad B_1 = 0,$$
$$B_0 = \mp\frac{3\sqrt{-\tau}}{\sqrt{2+6\tau}}, \quad c = \mp\frac{1}{\sqrt{-\tau(2+6\tau)}}, \quad \alpha = -1. \tag{2.26}$$

Setting Eqs. (2.24)-(2.26) in Eq. (2.12) and using Eq. (2.9) we obtain, respectively,

$$\mp\frac{\sqrt{-\tau(4+3\tau)}}{3\sqrt{2}\tau}f(z) \pm \frac{\sqrt{-\tau(4+3\tau)}}{6\sqrt{2}\tau} + f'(z) = 0, \tag{2.27}$$

$$\mp\frac{\sqrt{2+6\tau}}{3\sqrt{-\tau}}f(z) \pm \frac{\sqrt{2+6\tau}}{3\sqrt{-\tau}} + f'(z) = 0, \tag{2.28}$$

$$\mp\frac{\sqrt{2+6\tau}}{3\sqrt{-\tau}}f(z) + f'(z) = 0. \tag{2.29}$$

Now, with solving Eqs. (2.27)-(2.29), we obtain, respectively

$$f(z)=\frac{1}{2}+k\exp\left(\pm\frac{\sqrt{-\tau(4+3\tau)}}{3\sqrt{2}\tau}z\right), \tag{2.30}$$

$$f(z)=1+k\exp\left(\pm\frac{\sqrt{2+6\tau}}{3\sqrt{-\tau}}z\right), \tag{2.31}$$

$$f(z)=k\exp\left(\pm\frac{\sqrt{2+6\tau}}{3\sqrt{-\tau}}z\right). \tag{2.32}$$

Now, from $u(x,t)=f(z)$ where $z=x-ct$ and using Eqs. (2.24)-(2.26) the exact solutions to Eq. (1.1) for $m=0$ can be written, respectively

$$u(x,t)=\frac{1}{2}+k\exp\left(\frac{-2t\pm\sqrt{-2\tau(4+3\tau)}x}{6\tau}\right), \tag{2.33}$$

where $\alpha=\frac{1}{2}$, $\tau\neq-\frac{4}{3},0$ and k is an arbitrary constant (if $-\frac{4}{3}<\tau<0$ this solutions is real),

$$u(x,t)=1+k\exp\left(\frac{-t\mp\sqrt{-\tau(2+6\tau)}x}{3\tau}\right), \tag{2.34}$$

where $\alpha=2$, $\tau\neq-\frac{1}{3},0$ and k is an arbitrary constant (if $-\frac{1}{3}<\tau<0$ this solutions is real), and

$$u(x,t)=k\exp\left(\frac{-t\mp\sqrt{-\tau(2+6\tau)}x}{3\tau}\right), \tag{2.35}$$

where $\alpha=-1$, $\tau\neq-\frac{1}{3},0$ and k is an arbitrary constant (if $-\frac{1}{3}<\tau<0$ this solutions is real). From the form of (2.33)-(2.35), it is clear that all three solutions are unfortunately unbounded.

For our next case, consider $\deg(a_0(x))=2$ and $\deg(g(x))=1$. We assume $a_0(x)=A_2x^2+A_1x+A_0$ ($A_2\neq0$) and $g(x)=B_1x+B_0$ ($B_1\neq0$), where A_0, A_1, A_2, B_0 and B_1 are all constants to be determined. Substituting $a_1(x),h(x),a_0(x)$ and $g(x)$ in Eqs. (2.22) and (2.23), and setting all the coefficients of powers x to be zero, we obtain a system of nonlinear algebraic equations and by solving it, we obtain the following solutions:

$$A_2=\mp\frac{1}{\sqrt{2}},\ \ A_1=\pm\frac{1}{\sqrt{2}},\ \ A_0=0,\ \ B_1=\pm\sqrt{2},\ \ B_0=\mp\frac{1}{\sqrt{2}},\ \ c=0,\ \ \alpha=\frac{1}{2}, \tag{2.36}$$

$$A_2=\mp\frac{1}{\sqrt{2}},\ \ A_1=\pm\sqrt{2},\ \ A_0=0,\ \ B_1=\pm\sqrt{2},\ \ B_0=\mp\sqrt{2},\ \ c=0,\ \ \alpha=2, \tag{2.37}$$

$$A_2=\mp\frac{1}{\sqrt{2}},\ \ A_1=0,\ \ A_0=\pm\frac{1}{\sqrt{2}},\ \ B_1=\pm\sqrt{2},\ \ B_0=0,\ \ c=0,\ \ \alpha=-1, \tag{2.38}$$

$$A_2=\mp\frac{1}{\sqrt{2}},\ \ A_1=\pm\frac{1}{\sqrt{2}},\ \ A_0=0,\ \ B_1=\pm\sqrt{2},\ \ B_0=\mp\frac{1}{\sqrt{2}}+c,\ \ \alpha=\frac{1}{2}\mp\frac{c}{\sqrt{2}},\ \ \tau=0, \tag{2.39}$$

$$A_2 = \mp\frac{1}{\sqrt{2}},\ \ A_1 = c,\ \ A_0 = \pm\frac{1}{\sqrt{2}} - c,\ \ B_1 = \pm\sqrt{2},\ \ B_0 = 0,\ \ \alpha = -1 \pm \sqrt{2}c,\ \ \tau = 0, \tag{2.40}$$

$$A_2 = \mp\frac{1}{\sqrt{2}},\ \ A_1 = \pm\sqrt{2} + c,\ \ A_0 = 0,\ \ B_1 = \pm\sqrt{2},\ \ B_0 = \mp\sqrt{2},\ \ \alpha = 2 \pm \sqrt{2}c,\ \ \tau = 0. \tag{2.41}$$

Setting Eqs. (2.36)-(2.41) in Eq. (2.12) and using Eq. (2.9) we have

$$\mp\frac{1}{\sqrt{2}}f^2(z) \pm \frac{1}{\sqrt{2}}f(z) + f'(z) = 0, \tag{2.42}$$

$$\mp\frac{1}{\sqrt{2}}f^2(z) \pm \sqrt{2}f(z) + f'(z) = 0, \tag{2.43}$$

$$\mp\frac{1}{\sqrt{2}}f^2(z) \pm \frac{1}{\sqrt{2}} + f'(z) = 0, \tag{2.44}$$

$$\mp\frac{1}{\sqrt{2}}f^2(z) \pm \frac{1}{\sqrt{2}}f(z) + f'(z) = 0, \tag{2.45}$$

$$\mp\frac{1}{\sqrt{2}}f^2(z) + cf(z) \pm \frac{1}{\sqrt{2}} - c + f'(z) = 0, \tag{2.46}$$

$$\mp\frac{1}{\sqrt{2}}f^2(z) + \left(\pm\sqrt{2} + c\right)f(z) + f'(z) = 0. \tag{2.47}$$

We solve Eqs. (2.42)-(2.47), respectively, and obtain the relevant solutions $f(z)$. Then using $u(x,t) = f(z)$ where $z = x - ct$ and the conditions outlined in Eqs. (2.36)-(2.41), the exact solutions can be written as

$$u(x,t) = \frac{1}{1 + \exp\left(\pm\frac{x}{\sqrt{2}} + k\right)}, \tag{2.48}$$

where $\alpha = \dfrac{1}{2}$ and k is an arbitrary constant,

$$u(x,t) = \frac{2}{1 + \exp\left(\pm\sqrt{2}x + 2k\right)}, \tag{2.49}$$

where $\alpha = 2$ and k is an arbitrary constant,

$$u(x,t) = -\tanh\left(\pm\frac{x}{\sqrt{2}} + k\right), \tag{2.50}$$

where $\alpha = -1$ and k is an arbitrary constant,

$$u(x,t) = \frac{1}{1 + \exp\left(\pm\frac{x}{\sqrt{2}} + \left(-\frac{1}{2} + \alpha\right)t + k\right)}, \tag{2.51}$$

where $\tau = 0$, α is not specified, and k is an arbitrary constant,

$$u(x,t) = 1 + \frac{2(\alpha - 1)}{2 - \sqrt{2}\exp\left(-\frac{1}{2}(\alpha^2 - 1)t \pm \frac{\sqrt{2}}{2}(\alpha - 1)(x \pm 2k)\right)}, \tag{2.52}$$

where $\tau = 0$, α is not specified, and k is an arbitrary constant, and finally

$$u(x,t) = \frac{\sqrt{2}\alpha}{\sqrt{2} - \exp\left(\frac{1}{2}\alpha(2 - \alpha)t \pm \frac{\sqrt{2}}{2}\alpha(x \pm 2k)\right)}, \tag{2.53}$$

where $\tau = 0$, α is not specified, and k is an arbitrary constant. We point out that the solutions (2.48)-(2.50) are not dependent on variable t because for them $c = 0$. The solutions (2.48)-(2.50) are therefore stationary solutions of Eq. (1.1) for each τ. Since we should restrict $0 < \alpha < 1$ for physical reasons, the solution (2.48) is the physically relevant stationary solution. Note that the solution (2.48) is actually of the same form of the solution (2.51), only the latter solution is valid for $\tau = 0$. □

Note that the latter three solutions (2.51)-(2.53) are valid only for $\tau = 0$, i.e. when there are no telegraph effects present. Such solutions are then really exact solutions of the Fitzhugh-Nagumo reaction diffusion equation. Note that while the solution (2.51) is bounded, the solutions (2.52) and (2.53) become unbounded and hence are not physically valid.

Since they are valid only for $\tau = 0$, the solutions given in Eqs. (2.51)-(2.53) are equivalent to those which have been reported in [14]. Again, we stress that only (2.51) is bounded and hence a reasonable solution to the physical problem. Meanwhile, the stationary solutions are only trivially solutions of the telegraph equation, since the time derivatives are all zero. Therefore, the first integral method gives no time dependent solutions of the telegraph equation (1.1).

Theorem 2.3. *In the case of $m = 1$, applying the first integral method to Eq.* (1.1) *yields the following bounded, real-valued traveling wave solution:*

$$u(x,t) = \frac{1}{1 + exp\Big(\pm x + \left(-\frac{1}{2} + \alpha\right) t + k\Big)}, , \tag{2.54}$$

$\alpha \neq \dfrac{1}{2}$ *and k is an arbitrary constant, provided that the telegraph constant satisfies* $\tau = \dfrac{2}{-1 + 2\alpha}$.

Proof. If we put $m = 1$, Eqs. (2.14)-(2.16) are converted to:

$$a_1'(x)\left(-x + \tau c^2\right) = a_1(x)\left(h(x) - 1\right), \tag{2.55}$$

$$\begin{aligned} a_0'(x)\left(-x + \tau c^2\right) + a_1(x)\left(3c\tau x^2 - 2c\tau(\alpha+1)x + c\tau\alpha + c\right) \\ = g(x)a_1(x) + h(x)a_0(x), \end{aligned} \tag{2.56}$$

$$a_1(x)\left(-x^3 + (\alpha+1)\, x^2 - \alpha x\right) = g(x)a_0(x). \tag{2.57}$$

Since $a_i(x)$ (for $i = 0, 1$) and $h(x)$ are polynomials, then from Eq. (2.55) we deduce that $h(x)$ must be a constant. For simplicity we put $h(x) = a_1(x) = 1$. Balancing the degrees of $g(x)$ and $a_0(x)$ in Eqs. (2.56) and (2.57), we conclude that $\deg(a_0(x)) = 1$ and $\deg(g(x)) = 2$ or $\deg(a_0(x)) = 2$ and $\deg(g(x)) = 1$.

Consider $\deg(a_0(x)) = 1$ and $\deg(g(x)) = 2$, assuming $a_0(x) = A_1 x + A_0$ ($A_1 \neq 0$), and $g(x) = B_2 x^2 + B_1 x + B_0$ ($B_2 \neq 0$). We put $a_1(x), h(x), a_0(x)$ and $g(x)$ in Eqs. (2.56) and (2.57), and set all the coefficients of powers x to be zero, we obtain a system of nonlinear algebraic equations and by solving it, we have

$$\begin{aligned} A_1 = \pm\frac{\sqrt{-1-\alpha}}{\sqrt{6}}, \quad A_0 = 0, \quad B_2 = \mp\frac{\sqrt{6}}{\sqrt{-1-\alpha}}, \quad B_1 = \mp\sqrt{6(-1-\alpha)}, \\ B_0 = \mp\frac{\sqrt{6}\alpha}{\sqrt{-1-\alpha}}, \quad c = \mp\frac{\sqrt{6}\alpha}{\sqrt{-1-\alpha}}, \quad \tau = \frac{1}{3\alpha}, \end{aligned} \tag{2.58}$$

$$A_1 = \pm\frac{\sqrt{-1+2\alpha}}{\sqrt{6}},\ \ A_0 = \mp\frac{\alpha\sqrt{-1+2\alpha}}{\sqrt{6}},\ \ B_2 = \mp\frac{\sqrt{6}}{\sqrt{-1+2\alpha}},$$
$$B_1 = \pm\frac{\sqrt{6}}{\sqrt{-1+2\alpha}},\ \ B_0 = 0,\ \ c = \mp\frac{\sqrt{3}\alpha(-3+2\alpha)}{\sqrt{-8+16\alpha}},\ \ \tau = -\frac{4}{9\alpha-6\alpha^2}, \tag{2.59}$$

$$A_1 = \frac{\sqrt{\pm(2-3\alpha)}\sqrt{\alpha-2}}{\sqrt{6}\sqrt{\mp(2-3\alpha)}},\ \ A_0 = \frac{\sqrt{\mp(2-3\alpha)}\sqrt{\alpha-2}}{\sqrt{6}\sqrt{\pm(2-3\alpha)}},\ \ B_2 = \frac{\sqrt{6}\sqrt{\pm(2-3\alpha)}}{\sqrt{\mp(2-3\alpha)}\sqrt{\alpha-2}},$$
$$B_1 = \frac{\alpha\sqrt{6}\sqrt{\mp(2-3\alpha)}}{\sqrt{\pm(2-3\alpha)}\sqrt{\alpha-2}},\ \ B_0 = 0,\ \ c = \mp\frac{\sqrt{-\frac{3}{2}(-2+3\alpha)^2}}{2\sqrt{-2+\alpha}},\ \ \tau = \frac{4}{6-9\alpha}. \tag{2.60}$$

Setting Eqs. (2.58)-(2.60) in Eq. (2.12) and using Eq. (2.9) we obtain the first integrals in terms of $f(z)$ that with solving them, we have

$$f(z) = k\exp\left(\mp\frac{\sqrt{-1-\alpha}}{\sqrt{6}}z\right), \tag{2.61}$$

$$f(z) = \alpha + k\exp\left(\mp\frac{\sqrt{-1+2\alpha}}{\sqrt{6}}z\right), \tag{2.62}$$

$$f(z) = 1 + k\exp\left(\mp\frac{\sqrt{2-3\alpha}\sqrt{\alpha-2}}{\sqrt{6}\sqrt{-2+3\alpha}}z\right). \tag{2.63}$$

Each of these solutions is unbounded in z (and hence in x), therefore the corresponding solutions to (1.1) will be unbounded, so we do not consider these solutions further.

In the case $\deg(a_0(x)) = 2$ and $\deg(g(x)) = 1$, we assume $a_0(x) = A_2x^2 + A_1x + A_0$ $(A_2 \neq 0)$ and $g(x) = B_1x + B_0$ $(B_1 \neq 0)$. Substituting $a_1(x), h(x), a_0(x)$ and $g(x)$ in Eqs. (2.56) and (2.57), and setting all the coefficients of powers x to be zero, we obtain a system of nonlinear algebraic equations and by solving it, we obtain the following solutions:

$$A_2 = \mp 1,\ \ A_1 = \pm 1,\ \ A_0 = 0,\ \ B_1 = \pm 1,\ \ B_0 = \mp\alpha,$$
$$\tau = \frac{2}{-1+2\alpha},\ \ c = \pm\frac{1}{2}\mp\alpha, \tag{2.64}$$

$$A_2 = \pm\frac{1}{\sqrt{\alpha}},\ \ A_1 = \mp\alpha,\ \ A_0 = 0,\ \ B_1 = \mp\sqrt{\alpha},\ \ B_0 = \pm\sqrt{\alpha},$$
$$\tau = \frac{-2}{\alpha(-2+\alpha)},\ \ c = \mp\frac{1}{2}\sqrt{\alpha}(\alpha-2), \tag{2.65}$$

$$A_2 = \pm\frac{1}{\sqrt{1+\alpha}},\ \ A_1 = \mp\sqrt{1+\alpha},\ \ A_0 = \frac{\pm\alpha}{\sqrt{1+\alpha}},\ \ B_1 = \mp\sqrt{1+\alpha},$$
$$B_0 = 0,\ \ \tau = \frac{-2}{(1+\alpha)^2},\ \ c = \mp\frac{1}{2}\sqrt{(1+\alpha)^3}. \tag{2.66}$$

We put this solutions in Eqs. (2.12) and obtain the first integrals. Then by combining first integrals with Eq. (2.10), we obtain the first order differential equations in terms of $f(z)$. Then with solving them, we have

$$f(z) = \frac{1}{1+\exp(\pm z + k)}, \tag{2.67}$$

$$f(z)=\frac{\alpha\exp(\alpha k)}{\exp(\alpha k)-\exp\left(\mp\sqrt{\alpha}z\right)}, \tag{2.68}$$

$$f(z)=\alpha+\frac{-1+\alpha}{-1+\exp\left(\dfrac{(-1+\alpha)\left(\pm z+k\sqrt{1+\alpha}\right)}{\sqrt{1+\alpha}}\right)}. \tag{2.69}$$

The functions given in (2.68) and (2.69) are unbounded in z, and therefore are neglected. The exact solution of (1.1) corresponding to (2.67) is given by

$$u(x,t)=\frac{1}{1+\exp\Big(\pm x+\left(-\frac{1}{2}+\alpha\right)t+k\Big)}, \tag{2.70}$$

where $\alpha\neq\dfrac{1}{2}$, $\tau=\dfrac{2}{-1+2\alpha}$, and k is an arbitrary constant. □

This solution gives physically desirable behaviors. The solution remains bounded like $0\leq u(x,t)\leq 1$ for all $x\in\mathbb{R}$ and all $t>0$. Therefore, the first integral method has furnished us with an exact solution to (1.1) in the presence of density-dependent diffusion ($m=1$). We explore these kinds of solutions further in Section 3.

Theorem 2.4. *In the case of $m=2$, applying the first integral method to Eq.* (1.1) *yields no bounded, real-valued traveling wave solutions.*

Proof. We now consider $m=2$. In this case, Eqs. (2.14)-(2.16) are converted to

$$a_1'(x)\left(-x^2+\tau c^2\right)=a_1(x)\left(h(x)-2x\right), \tag{2.71}$$

$$a_0'(x)\left(-x^2+\tau c^2\right)+a_1(x)\left(3c\tau x^2-2c\tau(\alpha+1)x+c\tau\alpha+c\right)$$
$$=g(x)a_1(x)+h(x)a_0(x), \tag{2.72}$$

$$a_1(x)\left(-x^3+(\alpha+1)x^2-\alpha x\right)=g(x)a_0(x). \tag{2.73}$$

Since $a_i(x)$ (for $i=0,1$) and $h(x)$ are polynomials, from Eq. (2.71) we deduce that $h(x)$ is a polynomial of first degree, only. For simplicity we put $h(x)=2x$ and $a_1(x)=1$. Balancing the degrees of $g(x)$ and $a_0(x)$ in Eqs. (2.72) and (2.73), we conclude that $\deg(a_0(x))=1$ and $\deg(g(x))=2$, only. We assume $a_0(x)=A_1x+A_0$ ($A_1\neq 0$) and $g(x)=B_2x^2+B_1x+B_0$ ($B_2\neq 0$). We put $a_1(x),h(x),a_0(x)$ and $g(x)$ into Eqs. (2.72) and (2.73), and set all of the coefficients of powers x to be zero. We obtain a system of nonlinear algebraic equations and by solving it, we obtain the solutions

$$A_1=\mp\frac{i}{\sqrt{6}},\quad A_0=0,\quad B_2=\mp\sqrt{6}i,\quad B_1=\pm\sqrt{6}(\alpha+1)i,\quad B_0=\mp\sqrt{6}\alpha i,$$
$$\tau=\frac{1}{2\alpha},\quad c=\mp\sqrt{6}\alpha i, \tag{2.74}$$

$$A_1=\pm\frac{\sqrt{(-1+2\alpha)(3+2\alpha)^2}}{\sqrt{6}(3+2\alpha)},\quad A_0=\mp\frac{\alpha\sqrt{(-1+2\alpha)(3+2\alpha)^2}}{\sqrt{6}(3+2\alpha)},$$
$$B_2=\mp\frac{\sqrt{6}(3+2\alpha)}{\sqrt{(-1+2\alpha)(3+2\alpha)^2}},\quad B_1=\pm\frac{\sqrt{6}(3+2\alpha)}{\sqrt{(-1+2\alpha)(3+2\alpha)^2}},$$
$$B_0=0,\quad \tau=\frac{-3-2\alpha}{6\alpha},\quad c=\pm\frac{\sqrt{6}\alpha(3-2\alpha)}{\sqrt{(-1+2\alpha)(3+2\alpha)^2}}, \tag{2.75}$$

$$A_1 = \pm \frac{i\sqrt{(-2+\alpha)(2+3\alpha)^2}}{\sqrt{6\alpha}(2+3\alpha)}, \quad A_0 = \mp \frac{i\sqrt{(-2+\alpha)(2+3\alpha)^2}}{\sqrt{6\alpha}(2+3\alpha)},$$
$$B_2 = \pm \frac{i\sqrt{6\alpha}(2+3\alpha)}{\sqrt{(-2+\alpha)(2+3\alpha)^2}}, \quad B_1 = \mp \frac{i\sqrt{6\alpha^3}(2+3\alpha)}{\sqrt{(-2+\alpha)(2+3\alpha)^2}},$$
$$B_0 = 0, \quad \tau = \frac{-2-3\alpha}{6\alpha^2}, \quad c = \pm \frac{i\sqrt{6\alpha^3}(2-3\alpha)}{\sqrt{(-2+\alpha)(2+3\alpha)^2}}. \tag{2.76}$$

Then we obtain

$$f(z) = k \exp\left(\pm \frac{iz}{\sqrt{6}}\right), \tag{2.77}$$
$$f(z) = \alpha + k \exp\left(\mp \frac{\sqrt{(-1+2\alpha)(3+2\alpha)^2}}{\sqrt{6}(3+2\alpha)} z\right), \tag{2.78}$$
$$f(z) = 1 + k \exp\left(\mp \frac{i\sqrt{(-2+\alpha)(2+3\alpha)^2}}{\sqrt{6\alpha}(2+3\alpha)} z\right). \tag{2.79}$$

The corresponding exact solutions of Eq. (1.1) are given by

$$u(x,t) = k \exp\left(\pm \frac{ix}{\sqrt{6}} - \alpha t\right), \tag{2.80}$$

where $\alpha \neq 0$, $\tau = \frac{1}{2\alpha}$, and k is an arbitrary constant,

$$u(x,t) = \alpha + k \exp\left(\frac{\sqrt{6}\alpha(3-2\alpha)t \mp \sqrt{(-1+2\alpha)(3+2\alpha)^2}x}{\sqrt{6}(3+2\alpha)}\right), \tag{2.81}$$

where $\alpha \neq 0, \frac{1}{2}, -\frac{3}{2}$, $\tau = \frac{-3-2\alpha}{6\alpha}$, and k is an arbitrary constant (if $\alpha > \frac{1}{2}$ this solution is real-valued),

$$u(x,t) = 1 + k \exp\left(\frac{-\sqrt{6\alpha}\alpha(2-3\alpha)t \mp \sqrt{(-2+\alpha)(2+3\alpha)^2}ix}{\sqrt{6\alpha}(2+3\alpha)}\right), \tag{2.82}$$

where $\alpha \neq 0, 2, -\frac{2}{3}$, $\tau = \frac{-2-3\alpha}{6\alpha^2}$ and k is an arbitrary constant (if $0 < \alpha < 2$ this solution is real-valued).

These solutions are either complex-valued or, when real-valued, they are unbounded. Therefore, the first integral method gives no bounded, real-valued traveling wave solutions for $m = 2$, the case of strong density dependence in the diffusion term. □

3. Properties of physically meaningful solutions

While the first integral method permitted us to obtain a number of exact solutions, many of these solutions were either unbounded or were not real-valued. Physically, one is interested in bounded, real-valued solutions of (1.1).

From the analysis of [65], we know that for a degree three polynomial reaction function such as the $g(u)$ we take in (1.1), equation (1.1) with $\tau = 0$ is conditionally integrable for $m = 0, 1$ and non-integrable for larger values of m. That analysis was restricted to the real case (which is the physically relevant case), which may explain why complex solutions were found for $m = 2$ in the previous section. When $\tau \neq 0$, the situation is more complicated.

3.1. **Exact solutions for $m = 0$.** When $m = 0$ there is no density dependent diffusion, and (1.1) reduces to the Fitzhugh-Nagumo telegraph equation. Some exact solutions for this case were recovered by the first integral method when $\tau = 0$ in Section 2, and these correspond to exact solutions of [14] when there are no telegraph effects. We now determine if solutions of this type may exist when $\tau \neq 0$.

Since we seek only bounded solutions with monotone increase or decrease in both space and time, it makes sense to consider.

$$f(z) = \frac{A}{C + \exp(Bz)} \quad \text{with} \quad z = x - ct\,. \tag{3.1}$$

Placing this solution representation into (2.6) we find an equation of the form

$$\eta_3 \exp(3Bz) + \eta_2 \exp(2Bz) + \eta_1 \exp(Bz) + \eta_0 = 0\,, \tag{3.2}$$

where

$$\eta_3 = (c^2\tau - 1)B^2 + c(\alpha\tau + 1)B + \alpha\,, \tag{3.3}$$

$$\eta_2 = 2c\,(C - \tau A + (C - A)\tau\alpha)\,B + (3C - A)\alpha - A\,, \tag{3.4}$$

$$\begin{aligned}\eta_1 = (1 - c^2\tau)C^2B^2 + \big(c(1 + \alpha\tau)C^2 - 2A(1 + \alpha)c\tau C + 3A^2c\tau\big)\,B \\ +3C^2\alpha + A^2 - 2A(1 + \alpha)C\,,\end{aligned} \tag{3.5}$$

$$\eta_0 = C(A - C)(A - \alpha C)\,. \tag{3.6}$$

Setting $\eta_0 = 0$, we have two relevant cases: $C = A$ or $C = \frac{A}{\alpha}$.

Let us first assume $C = A$. We obtain from the other three conditions that either $\tau = 0$ or $\tau = 1$ and $\alpha = \frac{1}{2}$. The first case is simply that where the telegraph effects are neglected, and the solution is simply that found in Section 2. For the latter case, we find that the final parameter, B, takes the form

$$B = J_\pm(c) = \begin{cases} \frac{1}{4}\left(\frac{3c \pm \sqrt{8 + c^2}}{1 - c^2}\right), & \text{if } c \neq \pm 1\,, \\ -\frac{1}{3}, & \text{if } c = 1\,, \\ \frac{1}{3}, & \text{if } c = -1\,. \end{cases} \tag{3.7}$$

Therefore, we obtain the physical solution

$$u_\pm(x,t) = \frac{A}{A + \exp\left(J_\pm(c)x - cJ_\pm(c)t\right)} \tag{3.8}$$

to (1.1) when $m = 0$, $\tau = 1$, and $\alpha = \frac{1}{2}$. Note that the wave speed, $c \in \mathbb{R}$, for these solutions is free, as well as the shape parameter $A > 0$. The drawback is that the solution is valid for only $\alpha = \frac{1}{2}$, which implies that this solution exists only due to a symmetry of the polynomial $g(u)$.

When we next consider $C = \frac{A}{\alpha}$, we find that either $\tau = 0$ or $\alpha = 2$ and $\tau = \frac{1}{4}$. The first case is again not desirable, as it would neglect telegraph effects. The second case is not desirable, as it makes sense to have $0 \leq \alpha \leq 1$. Therefore, the $C = \frac{A}{\alpha}$ case is not of physical interest.

Returning to the solution (3.8), we plot this solution for various values of the wave speed in Figure 1.

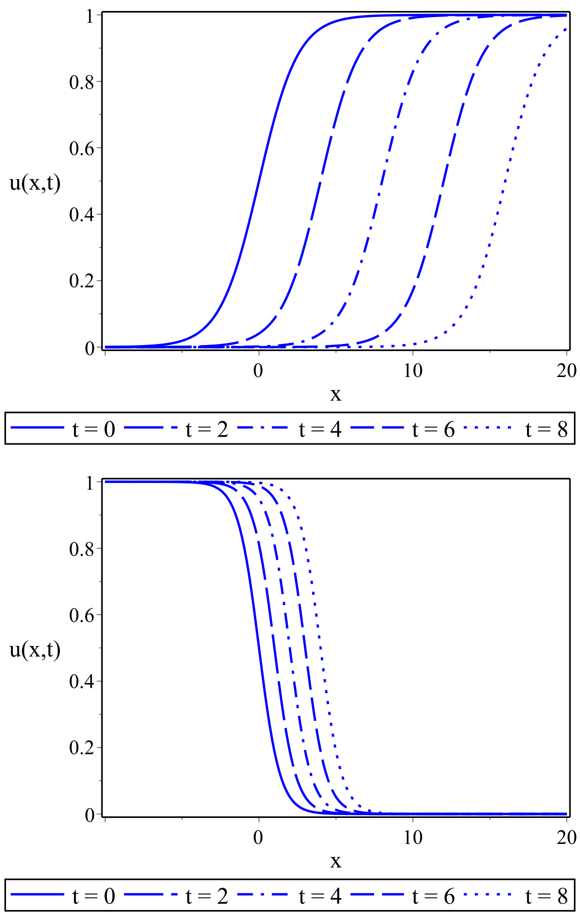

FIGURE 1. Traveling wave solutions corresponding to $u(x,t) = u_+(x,t)$ in (3.8) for $A = 1$ and either (top) $c = 2$ or (bottom) $c = 0.5$. It is interesting to note that the size of the wave speed will determine the features of the wave. For instance, if $c > 1$, then the wave will result in a decrease in concentration to zero as it passed a fixed location $x = x_0$. On the other hand, if $0 < c < 1$, then the wave results in an increase in concentration once it passes a fixed location.

3.2. **Exact solutions for $m = 1$.** We would like to calibrate our solutions so that they tend toward one of the three steady states, 0, α or 1, as either $|x| \to \infty$ or $t \to \infty$. Therefore, motivated by the bounded exact solutions in (2.70), we may again assume a solution

$$f(z) = \frac{A}{C + \exp(Bz)} \quad \text{with} \quad z = x - ct\,. \tag{3.9}$$

Placing this solution representation into (2.6) we find an equation of the form

$$\eta_3 \exp(3Bz) + \eta_2 \exp(2Bz) + \eta_1 \exp(Bz) + \eta_0 = 0\,, \tag{3.10}$$

where

$$\eta_3 = (Bc+\alpha)(Bc\tau - 1)\,, \tag{3.11}$$

$$\eta_2 = -A(2B^2 + 2c\tau(1+\alpha)B + \alpha + 1) + 2C(c(\alpha\tau+1)B + \frac{3}{2}\alpha)\,, \tag{3.12}$$

$$\eta_1 = (3\alpha + c(\alpha\tau+1)B - c^2B^2\tau)C^2 + AC(B^2 - 2\tau c(1+\alpha)B - 2(1+\alpha)) \\ +A^2(3Bc\tau+1)\,, \tag{3.13}$$

$$\eta_0 = C(A-C)(A-\alpha C)\,. \tag{3.14}$$

Setting $\eta_3 = 0$ gives either $B = -\frac{\alpha}{c}$ or $B = -\frac{1}{c\tau}$, while setting $\eta_0 = 0$ gives either $C = A$ or $A = \alpha C$ (yet $C \neq 0$). Then, setting $\eta_2 = 0$ and $\eta_1 = 0$, we obtain six possible cases. However, four of these cases result in complex solutions and must be discarded. The remaining two cases end up giving

$$C = \frac{A}{\alpha},\ B = -\frac{1}{c\tau},\ c = \pm\frac{\sqrt{\alpha}(2-\alpha)}{2},\ \tau = \pm\frac{2}{\alpha(2-\alpha)} \tag{3.15}$$

or

$$C = A,\ B = -\frac{\alpha}{c},\ c = \pm\frac{|1-2\alpha|}{2},\ \tau = \pm\frac{2}{2\alpha-1}\,. \tag{3.16}$$

While A, B, C and c are parameters that can be picked, note that τ is a model parameter. Therefore, there are two possible value of τ (as a function of α) which permit exact solutions.

For the exact solution corresponding to (3.15), we find

$$f(z) = \frac{\alpha A}{A + \alpha\exp(-\sqrt{\alpha}z)}\,. \tag{3.17}$$

Since there are two possible wave speeds, we find two exact solutions

$$u_\pm(x,t) = \frac{\alpha A}{A + \alpha\exp\left\{-\sqrt{\alpha}\left(x \mp \frac{\sqrt{\alpha}(2-\alpha)}{2}t\right)\right\}} \tag{3.18}$$

for $\alpha \in (0,1]$ and $A > 0$.

We note that both solutions in (3.18) are bounded like $0 \leq u_\pm(x,t) \leq \alpha$ for all $x \in \mathbb{R}$ and all $t > 0$. In particular, $u_\pm \to 0$ as $x \to -\infty$ while $u_\pm \to \alpha$ as $x \to \infty$. The wave propagates toward the left for u_+, therefore $u_+ \to \alpha$ as $t \to \infty$ and the density uniformly increases toward α over time. On the other hand, the wave propagates toward the right for u_- and hence the density uniformly decreases to zero over time. We demonstrate the behavior of the solutions given in (3.18), in Figure 2.

Regarding the exact solution corresponding to (3.16), we obtain

$$f(z) = \frac{A}{A + \exp(\mathrm{sgn}(1-2\alpha)z)}\,, \tag{3.19}$$

where $\mathrm{sgn}(j)$ takes the value -1 if $j < 0$, the value 0 if $j = 0$, and the value 1 if $j > 0$. Since there are two wave speeds, we again find a pair of solutions. However, note that we cannot have $\alpha = 1/2$, otherwise $\tau \to \infty$. When $\alpha \in (0, 1/2)$ and $A > 0$, we have the two solutions

$$u_\pm(x,t) = \frac{A}{A + \exp\left\{x \pm \left(\frac{1-2\alpha}{2}\right)t\right\}}\,. \tag{3.20}$$

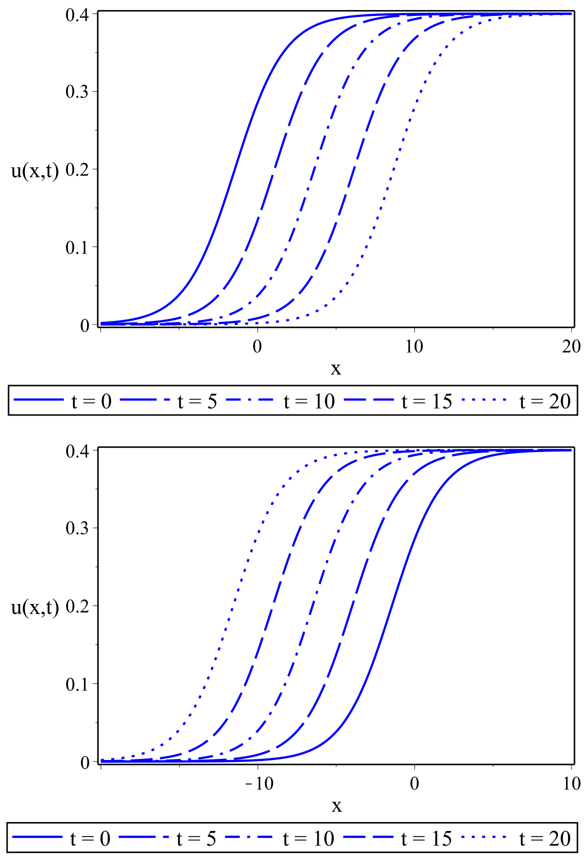

FIGURE 2. Traveling wave solutions corresponding to (top) $u(x,t) = u_+(x,t)$ or (bottom) $u(x,t) = u_-(x,t)$ in (3.18) for $A = 1$ and $\alpha = 0.4$. The solution $u_+(x,t)$ has a right moving wave which lowers the concentration to zero as it passes. Meanwhile, the solution $u_-(x,t)$ has a left moving wave which increases the concentration to a value of α as it passes a fixed location. Here the wave speed is given by $c_\pm(\alpha) = \mp\frac{\sqrt{\alpha}(2-\alpha)}{2}$ while solutions correspond to $\tau_\pm(\alpha) = \pm\frac{2}{\alpha(2-\alpha)}$.

Both solutions have the property that $u_\pm \to 1$ as $x \to -\infty$ and $u_\pm \to 0$ as $x \to \infty$. However, $u_+ \to 0$ as $t \to \infty$, while $u_- \to 1$ as $t \to \infty$. We give plots of the solutions in (3.20), in Figure 3.

Meanwhile, when $\alpha \in (1/2, 1]$ and $A > 0$, we have the two solutions

$$u_\pm(x,t) = \frac{A}{A + \exp\left\{-x \mp \left(\frac{2\alpha-1}{2}\right) t\right\}}\,. \tag{3.21}$$

Both solutions have the property that $u_\pm \to 0$ as $x \to -\infty$ and $u_\pm \to 1$ as $x \to \infty$. However, $u_+ \to 1$ as $t \to \infty$, while $u_- \to 0$ as $t \to \infty$. We give plots of the solutions in (3.21), in Figure 4.

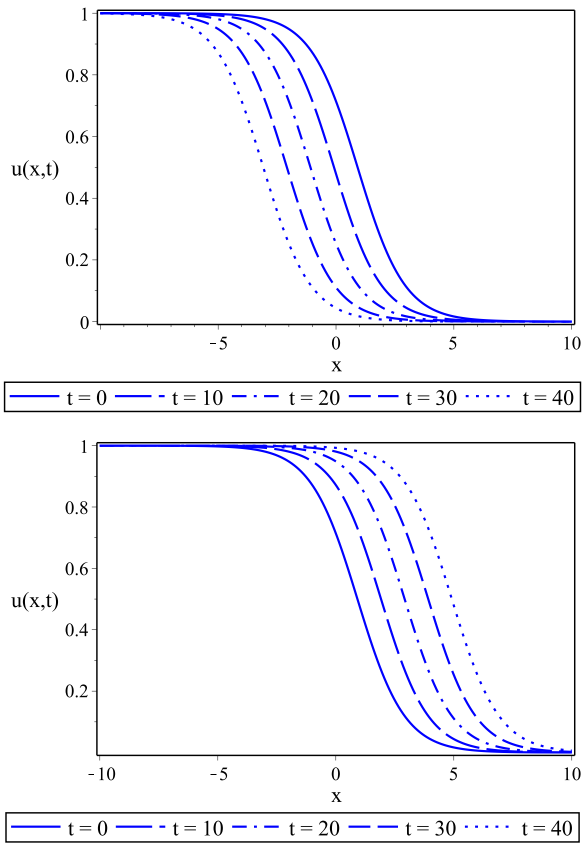

FIGURE 3. Traveling wave solutions corresponding to (top) $u(x,t) = u_+(x,t)$ or (bottom) $u(x,t) = u_-(x,t)$ in (3.20) for $A = 1$ and $\alpha = 0.4$. The solution $u_+(x,t)$ has a left moving wave which lowers the concentration to zero as it passes. Meanwhile, the solution $u_-(x,t)$ has a right moving wave which increases the concentration to unity as it passes a fixed location. Here the wave speed is given by $c_\pm(\alpha) = \pm\frac{1-2\alpha}{2}$ while solutions correspond to $\tau_\pm(\alpha) = \pm\frac{2}{2\alpha-1}$.

In order to see the influence of the shape parameter, $A > 0$, note that this shape parameter ends up shifting the center of the wave. To demonstrate this fact, observe from (3.18) that for $A > 0$ we have

$$\begin{aligned} u_\pm(x,t) &= \frac{\alpha}{1 + \alpha A^{-1}\exp\left\{-\sqrt{\alpha}\left(x \mp \frac{\sqrt{\alpha}(2-\alpha)}{2}t\right)\right\}} \\ &= \frac{\alpha}{1 + \alpha\exp\left\{-\sqrt{\alpha}\left(x - x_0(A,\alpha) \mp \frac{\sqrt{\alpha}(2-\alpha)}{2}t\right)\right\}}, \end{aligned} \tag{3.22}$$

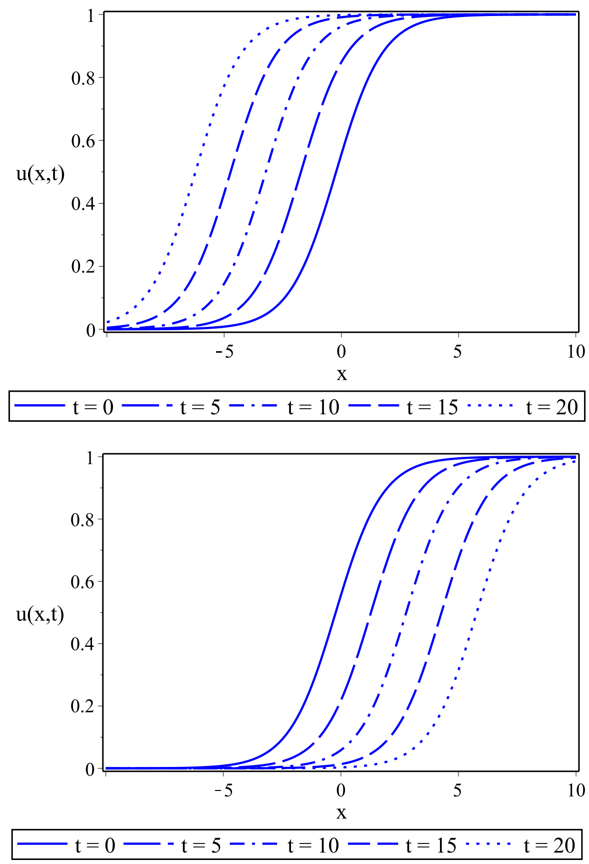

FIGURE 4. Traveling wave solutions corresponding to (top) $u(x,t) = u_+(x,t)$ or (bottom) $u(x,t) = u_-(x,t)$ in (3.18) for $A = 1$ and $\alpha = 0.8$. The solution $u_+(x,t)$ has a left moving wave which increases the concentration to unity as it passes. On the other hand, the solution $u_-(x,t)$ has a right moving wave which decreases the concentration to zero as it passes. Here the wave speed is given by $c_\pm(\alpha) = \mp\frac{2\alpha-1}{2}$ while solutions correspond to $\tau_\pm(\alpha) = \pm\frac{2}{2\alpha-1}$.

where

$$x_0(A,\alpha) = \frac{1}{\sqrt{\alpha}} \ln\left(\frac{1}{A}\right) . \tag{3.23}$$

Therefore, the parameter $A > 0$ shifts the wave location by $x_0(A,\alpha)$ from the reference location $x_0(1,\alpha) = 0$. One can obtain similar modifications of the solutions (3.20) and (3.21) when the shape parameter, $A > 0$, is modified.

Similarly, while we know that α will influence the wave speed of the solutions for the density dependent case ($m = 1$), the parameter α will also scale the solution envelope for the solutions (3.18). These solutions tend to concentrations of α for large x (rather than concentrations of 1). On the other hand, for the solutions

(3.20) and (3.21), the parameter α only enters into the wave speed, and hence the shape of those waves is determined simply from the parameter A.

These exact solutions hold the desired behaviors of solutions to equation (1.1). The solutions give monotone changes in density concentrations in space, and these concentration changes propagate as waves which eventually lead to complete saturation ($u \to 1$), uniform partial saturation ($u \to \alpha \in (0,1)$), or dissipation of the concentration ($u \to 0$) as $t \to \infty$.

These results are interesting, as they show that by including a density dependence in the telegraph equation (1.1) we still obtain physically relevant solutions which demonstrate qualitative behaviors consistent with the standard Fitzhugh-Nagumo equation.

3.3. **Non-integrability of the $m = 2$ case.** To better understand why the $m = 2$ case is difficult, let us better determine the dependence of the model (1.1) on the strength of density dependence, m. We shall consider a Pinalevé analysis in order to determine if the $m = 2$ case can be integrable. Note that this was done in [65], for general reaction-diffusion models of the type (1.1), but there were no telegraph terms included in that study.

Theorem 3.1. *When $m \geq 2$, Eq.* (1.1) *is not integrable.*

Proof. Consider the equation

$$\tau \frac{\partial^2 u}{\partial t^2} + (1 + \tau g'(u)) \frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(G(u) \frac{\partial u}{\partial x} \right) - g(u), \tag{3.24}$$

where we now consider a general polynomial function G in the diffusion term. Let $\deg(G) \geq 0$ denote the degree of G as a polynomial in u. We assume a leading order solution of the form

$$u(x,t) = u_0(\phi(x,t))^{-\ell} \quad \text{for} \quad \ell > 0\,, \tag{3.25}$$

where $\phi(x,t)$ denotes the singular manifold. Then, this expression denotes the lowest order term in the Laurent expansion of $u(x,t)$ about the singular manifold $\phi(x,t)$. The possible dominant terms are then

$$\frac{\partial^2 u}{\partial t^2} \sim \frac{1}{\phi^{\ell+2}}\,, \tag{3.26}$$

$$g(u) \sim \frac{1}{\phi^{3\ell}}\,, \tag{3.27}$$

$$\frac{\partial}{\partial x}\left(G(u) \frac{\partial u}{\partial x} \right) \sim \frac{1}{\phi^{(\deg(G)+1)\ell+2}}\,. \tag{3.28}$$

Since $\deg(G) \geq 0$, we have $\ell + 2 \leq (\deg(G)+1)\ell + 2$ and hence the term (3.26) will not be dominant. Therefore, in order to balance the dominant terms, we scale the terms in (3.27) and (3.28) so that

$$\frac{1}{\phi^{3\ell}} \sim \frac{1}{\phi^{(\deg(G)+1)\ell+2}}\,. \tag{3.29}$$

In order to have such a scaling, we need $3\ell = (\deg(G)+1)\ell + 2$. Rearranging, this means that $(2 - \deg(G))\ell = 2$.

If there is no density dependence (this is the case of $m = 0$), $\deg(G) = 0$ and we have $\ell = 1$. If we have linear density dependence (this is the case of $m = 1$), $\deg(G) = 1$ and we have $\ell = 2$. Therefore, both cases are candidates for passing the

Painlevé test (for details see [65]). On the other hand, consider $\deg(G) = 2$, which corresponds to the case of $m = 2$. For any ℓ, we obtain $0 = 2$, a contradiction. Therefore, Eq. (3.24) fails the Painlevé test for $\deg(G) = 2$. This means that the more specific telegraph equation (1.1) fails the Painlevé test when $m = 2$, and hence it is not integrable.

A similar derivation shows that Eq. (1.1) is not integrable for $m > 2$ since ℓ would not be a positive integer for such a case. □

This suggests that closed-form exact bounded solutions are not likely to be found for (1.1) in the case when $m \geq 2$. Hence, one would instead need to consider approximate or numerical solutions, such as what was done in [20, 24], in cases where the density dependence term is of high order.

4. Discussion and Conclusions

We obtained closed-form exact solutions for the density dependent Fitzhugh–Nagumo telegraph equation (1.1) for different value α and τ and $m = 0, 1, 2$, by using the first integral method. Importantly, some of these solutions are physically meaningful and generalize what was shown in the literature for the non-telegraph, non-density dependent special case [14].

When we have density dependence ($m = 1$), we find that solutions take the form of a right or left propagating wave, with the wave speed a function of the parameter α. Note also that these solutions exist conditionally, for very specific values of the telegraph parameter, τ, which are determined by the value of α. This suggests that telegraph effects preserve the general structure of the density-dependent Nagumo equation in specific cases, and destroy these structures otherwise. As shown in other work [24], other values of τ can result in oscillations of the solutions around the wave solutions given here, meaning that these values of τ result in a type of instability in the solutions. In contrast, the "preferred" values $\tau = \tau(\alpha)$ result in smooth, monotone wave structures. This is interesting, in that it tells us when one can find reaction-diffusion like dynamics for certain specific telegraph parameters, while for other parameters the dynamics of the solutions will be more oscillatory and less diffusive in nature.

As one of the waves passes a fixed location, with the concentration will decrease to zero or it will increase to some fixed positive value (either α or unity, depending on the form of the solution). There are examples of right and left moving waves with both types of behaviors. One can view these wave solutions are reaction fronts. When the wave results in a decrease in concentration, the wave has used up the ambient concentration of the chemical species, and behind this wave the concentration tends asymptotically to zero. On the other hand, when a wave results in an increase in concentration, one can view the process as a reaction front which creates a chemical species and deposits a fixed concentration of the chemical species (either α or 1 depending on the relevant scalings and physical properties) after the reaction occurs. In our exact solutions this process is smooth and monotone. If the telegraph parameters were not picked appropriately (as functions of α), then as shown in [24] the reactions were exhibit some oscillations, and the reactions would not progress as effectively. Therefore, our analysis has actually revealed 'optimal' telegraph parameters which allow for rapid, monotone reactions as the traveling wave front passes. The solutions therefore give monotone changes in density concentrations in space, and eventually lead to complete saturation ($u \to 1$), uniform partial saturation ($u \to \alpha \in (0,1)$), or dissipation of the concentration ($u \to 0$) as time becomes large.

Regarding the actual structure of the density dependent Fitzhugh–Nagumo telegraph equation, we show that (1.1) may be integrable for only $m = 0$ or $m = 1$, which is likely why we find the closed-form exact solutions for those cases. This may suggest that the proper form of the model is either $m = 0$ (the standard telegraph Fitzhugh-Nagumo equation) or $m = 1$. Therefore, the kind of density dependence like

$$\frac{\partial}{\partial x}\left(u\frac{\partial u}{\partial x}\right) \tag{4.1}$$

is physically interesting, since we still obtain the traveling wave type solutions known for the standard telegraph Fitzhugh-Nagumo equation. Stronger density dependence, like

$$\frac{\partial}{\partial x}\left(u^2\frac{\partial u}{\partial x}\right), \tag{4.2}$$

appears to destroy this nice structure. Note, however, that one may still obtain numerical solutions for the latter case, such as those of [20, 24]. Yet, one would not necessarily be able to obtain a 'natural' wave speed, and would instead treat the wave speed of the numerical solutions as an additional parameter in the problem.

References

[1] R. Fitzhugh, *Impulse and physiological states in models of nerve membrane*, Biophysics Journal **1** (1961) 445–466.

[2] J.S. Nagumo. S. Arimoto, S. Yoshizawa, *An active pulse transmission line simulating nerve axon*, Proceedings of the IRE **50** (1962) 2061–2070.

[3] D.G. Aronson, H.F. Weinberger, *Multidimensional nonlinear diffusion arising in population genetics*, Advances in Mathematics **30** (1978) 33–76.

[4] A.C. Scott, *Neunstor propagation on a tunnel diode loaded transmission line*, Proceedings of IEEE **51** (1963) 240–249.

[5] J. Nagumo, S. Yoshizawa, S. Arimoto, *Bistable transmission lines*, Transactions on IEEE Circuit Theory **12** (1965) 400–412.

[6] W.F. Pickard, *On the propagation of the nervous impulse down medullated and unmedullated fibers*, Journal of Theoretical Biology **11** (1966) 30–45.

[7] R.J. Buratti, A.G. Lindgren, *Neuristor wave forms and stability by the linear approximation*, Proceedings of IEEE **56** (1968) 1392–1393.

[8] E. Ahmed, S.Z. Hassan, *On diffusion in some biological and economic systems*, Z. Naturforsch. **55a** (2000) 669–672.

[9] E. Ahmed, H.A. Abdusalam, E.S. Fahmy, *On telegraph reaction diffusion and coupled map lattice in some biological systems*, International Journal of Modern Physics C **12** (2001) 717–726.

[10] V. Afraimovich, B. Fernandez, *Topological properties of linearly coupled expanding map lattices*, Nonlinearity **13** (2000) 973.

[11] H.A. Abdusalam, *Analytic and approximate solutions for Nagumo telegraph reaction diffusion equation*, Applied Mathematics and Computation **157** (2004) 515–522.

[12] T. Kawahara, M. Tanaka, *Interaction of travelling fronts: an exact solution of a nonlinear diffusion equation*, Physics Letters A **97** (1983) 311–314.

[13] M.C. Nucci, P.A. Clarkson, *The nonclassical method is more general than the direct method for symmetry reductions: an example of the Fitzhugh–Nagumo equation*, Physics Letters A **164** (1992) 49–56.

[14] H. Li, Y. Guo, *New exact solutions to the Fitzhugh–Nagumo equation*, Applied Mathematics and Computation **180** (2006) 524–528.

[15] S. Abbasbandy, *Soliton solutions for the Fitzhugh–Nagumo equation with the homotopy analysis method*, Applied Mathematical Modelling **32** (2008) 2706–2714.

[16] R. A. Van Gorder, K. Vajravelu, *Analytical and numerical solutions of the density dependent diffusion Nagumo equation*, Physics Letters A **372** (2008) 5152–5158.
[17] R. A. Van Gorder, *Travelling waves for a density dependent diffusion nagumo equation over the real line*, Communications in Theoretical Physics **58(1)** (2012) 5-11.
[18] P. Masemola, A. H. Kara, K. Fakhar, *Reductions and new exact solutions of the density-dependent Nagumo and Fisher equations*, Journal of Engineering Mathematics **82** (2013) 77–83.
[19] M. S. Hashemi, E. Darvishi, D. Baleanu, *A geometric approach for solving the density-dependent diffusion Nagumo equation*, Advances in Difference Equations **2016:89** (2016) 1–13.
[20] R. A. Van Gorder, K. Vajravelu, *A variational formulation of the Nagumo reaction–diffusion equation and the Nagumo telegraph equation*, Nonlinear Analysis: Real World Applications **11** (2010) 2957–2962.
[21] K.P. Hadeler, *Reaction transport equations in biological modeling*, Mathematical and Computer Modelling **31** (2000) 75-81.
[22] W.E. Fitzgibbon, *A hyperbolic system from a neural transmission model*, Computers & Mathematics with Applications **13** (1987) 767-769.
[23] W.E. Fitzgibbon, M.E. Parrott, *Convergence of singularly perturbed hodgkin-huxley systems*, Nonlinear Analysis: Theory, Methods, and Applications **22** (1994) 363-379.
[24] R. A. Van Gorder, K. Vajravelu, *Analytical and numerical solutions of the density dependent Nagumo telegraph equation*, Nonlinear Analysis: Real World Applications **11** (2010) 3923–3929.
[25] V. S. Erturk, Z. M. Odibat, S. Momani, *Application of multi–step differential transform method for the analytical and numerical solutions of the density dependent Nagumo telegraph equation*, Romanian Journal of Physics **57** (2012) 1065–1078.
[26] S. R. Choudhury, *Painlevé analysis and partial integrability of a class of reaction–diffusion equations*, Nonlinear Analysis: Theory, Methods & Applications **18** (1992) 445–459.
[27] N.A. Kudryashov, *Exact soliton solutions of the generalized evolution equation of wave dynamics*, PMM Journal of Applied Mathematics and Mechanics **52** (1988) 361–365.
[28] N.A. Kudryashov, *Exact solutions of the generalized Kuramoto–Sivashinsky equation*, Physics Letters A **147** (1990) 287–291.
[29] N.A. Kudryashov, *Partial differential equations with solutions having movable first order singularities*, Physics Letters A **169** (1992) 237–242.
[30] N.A. Kudryashov, *Exact solutions of the Burgers–Huxley equation*, PMM Journal of Applied Mathematics and Mechanics **68** (2004) 413–420.
[31] N.A. Kudryashov, *Exact solitary waves of the Fisher equation*, Physics Letters A **342** (2005) 99–106.
[32] A.M. Wazwaz, *New solitary wave solutions to the Kuramoto–Sivashinsky and the Kawahara equations*, Applied Mathematics and Computation **182** (2006) 1642–1650.
[33] X.N. Li, G.M. Wei, Y.Q. Liang, *Painlevé analysis and new analytic solutions for variable-coefficient Kadomtsev–Petviashvili equation with symbolic computation*, Applied Mathematics and Computation **216** (2010) 3568–3577.
[34] N.A. Kudryashov, M.V. Demina, *Traveling wave solutions of the generalized nonlinear evolution equations*, Applied Mathematics and Computation **210** (2009) 551–557.
[35] D.S. Wang, *Integrability of a coupled KdV system: Painlevé property, Lax pair and Bäcklund transformation*, Applied Mathematics and Computation **216** (2010) 1349–1354.
[36] F.J. Romeiras, *Exact travelling wave solutions of the generalized Bretherton equation*, Applied Mathematics and Computation **215** (2009) 1791–1805.
[37] K. Singh, R.K. Gupta, S. Kumar, *Benjamin-Bona-Mahony (BBM) equation with Variable coefficients: similarity reductions and Painlevé analysis*, Applied Mathematics and Computation **217** (2011) 7021-7027.
[38] M.V. Demina, N.A. Kudryashov, *On elliptic solutions of nonlinear ordinary differential equations*, Applied Mathematics and Computation **217** (2011) 9849–9853.
[39] T. Geng, X.H. Meng, W.R. Shan, B. Tian, *Bäcklund transformation and multi-soliton solutions for a $(2+1)$–dimensional Korteweg–de Vries system via symbolic computation*, Applied Mathematics and Computation **217** (2010) 1470–1475.
[40] R.J. Wiltshire, C. Sophocleous, *An application of the Weiss algorithm of Painleve analysis to create solutions of the reaction diffusion equation*. The 10th International Conference in

Modern Group Analysis, in: N.H. Ibragimov, C. Sophocleous, P.A. Damianou (Eds.), Group Analysis of Differential Equations, 2005.

[41] N.A. Kudryashov, N.B. Loguinova, *Be careful with the Exp-function method*, Communications in Nonlinear Science and Numerical Simulation **14** (2009) 1881–1890.

[42] Z. Navickas, M. Ragulskis, L. Bikulciene, *Be careful with the Exp-function method–Additional remarks*, Communications in Nonlinear Science and Numerical Simulation **15** (2010) 3874–3886.

[43] N.A. Kudryashov, *Seven common errors in finding exact solutions of nonlinear differential equations*, Communications in Nonlinear Science and Numerical Simulation **14** (2009) 3507–3529.

[44] N.A. Kudryashov, *On "new travelling wave solutions" of the KdV and the KdV-Burgers equations*, Communications in Nonlinear Science and Numerical Simulation **14** (2009) 1891–1900.

[45] Z. Feng, *On explicit exact solutions to the compound Burgers–KdV equation*, Physics Letters A **293** (2002) 57–66.

[46] Z. Feng, *Traveling wave behavior for a generalized Fisher equation*, Chaos, Solitons and Fractals **38** (2008) 481–488.

[47] Z. Feng, *Exact solution to an approximate sine-Gordon equation in (n+ 1)-dimensional space*, Physics Letters A **302** (2002) 64–76.

[48] Z. Feng, X. Wang, *The first integral method to the two-dimensional Burgers-Korteweg-de Vries equation*, Physics Letters A **308** (2003) 173–178 .

[49] Z. Feng, G. Chenb, *Solitary wave solutions of the compound Burgers-Korteweg-de Vries equation*, Physica A **352** (2005) 419–435.

[50] Z. Feng, R. Knobel, *Traveling waves to a Burgers–Kortewegde Vries type equation with higher order nonlinearities*, Journal of Mathematical Analysis and Applications **328** (2007) 1435–1450.

[51] H. Li, Y. Guo, *New exact solutions to the Fitzhugh-Nagumo equation*, Applied Mathematics and Computation **180** (2006) 524–528.

[52] X. Deng, *Travelling wave solutions for the generalized Burgers-Huxley equation*, Applied Mathematics and Computation **204(2)** (2008) 733–737.

[53] K. R. Raslan, *The first integral method for solving some important nonlinear partial differential equations*, Nonlinear Dynamics **53** (2008) 281–286.

[54] F. Tascan, A. Bekir, M. Koparan, *Travelling wave solutions of nonlinear evolution equations by using the first integral method*, Communications in Nonlinear Science and Numerical Simulation **14** (2009) 1810–1815.

[55] F. Tascan, A. Bekir, *Travelling wave solutions of the Cahn–Allen equation by using first integral method*, Applied Mathematics and Computation **207** (2009) 279–282.

[56] S. Abbasbandy, A. Shirzadi, *The first integral method for modified Benjamin–Bona–Mahony equation*, Communications in Nonlinear Science and Numerical Simulation **15** (2010) 1759–1764.

[57] N. Taghizadeh, M. Mirzazadeh, *Exact solutions of the nonlinear Schrodinger equation by the first integral method*, Journal of Mathematical Analysis and Applications **374** (2011) 549–553.

[58] N. Taghizadeh, M. Mirzazadeh, *The first integral method to some complex nonlinear partial differential equations*, Journal of Computational and Applied Mathematics **235** (2011) 4871–4877.

[59] H. Jafari, A. Sooraki, Y. Talebi, A. Biswas, *The first integral method and traveling wave solutions to Davey-Stewartson equation*, Nonlinear Analysis: Modelling and Control **17(2)** (2012) 182–193.

[60] B. Lu, *The first integral method for some time fractional differential equations*, Journal of Mathematical Analysis and Applications **395(2)** (2012) 684–693.

[61] M. Mirzazadeh, M. Eslami, *Exact solutions of the Kudryashov-Sinelshchikov equation and nonlinear telegraph equation via the first integral method*, Nonlinear Analysis: Modelling and Control **17(4)** (2012) 481–488.

[62] N. Taghizadeh, M. Mirzazadeh, A. Samiei Paghaleh, *Exact solutions of some nonlinear evolution equations via the first integral method*, Ain Shams Engineering Journal **4** (2013) 493–499.

[63] M. Hajiketabi, R. A. Van Gorder, E. Shivanian, *Complex–valued travelling wave solutions to the Casimir equation for the Ito system via the first integral method*, Communications in Numerical Analysis **2014** (2014) 1–11.

[64] M. Alquran, Q. Katatbeh and B. Al–Shrida, *Applications of first integral method to some complex nonlinear evolution systems*, Applied Mathematics & Information Sciences **9** (2015) 825–831.

[65] J. Hearns, R. A. Van Gorder, S. R. Choudhury, *Painlevé test, integrability, and exact solutions for density-dependent reaction-diffusion equations with polynomial reaction functions*, Applied Mathematics and Computation **219** (2012) 3055–3064.

M. Hajiketabi, S. Abbasbandy
Department of Mathematics, Imam Khomeini International University, Qazvin, 34149-16818, Iran
E-mail address: abbasbandy@yahoo.com

R. A. Van Gorder
Mathematical Institute, University of Oxford, Andrew Wiles Building, Radcliffe Observatory Quarter, Woodstock Road, Oxford, OX2 6GG, United Kingdom
E-mail address: Robert.VanGorder@maths.ox.ac.uk