

Equivalent Blocks of Finite General Linear Groups in Non-describing Characteristic

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J. Chuang, R. Kessar, and J. Rickard have proved Broué's Abelian defect group conjecture for many symmetric groups. We adapt the ideas of Kessar and Chuang towards finite general linear groups (represented over non-describing characteristic). We then describe Morita equivalences between certain p -blocks of $GL_n(q)$ with defect group $C_{p^\alpha} \times C_{p^\alpha}$, as q varies (see Theorem 2). Here p and q are coprime. This generalizes work of S. Koshitani and M. Hyoe, who proved the same result for principal blocks of $GL_n(q)$ when $p = 3$, $\alpha = 1$, in a different way. © 2002 Elsevier Science

INTRODUCTION AND PRELIMINARIES

We prove a pair of results concerning the representation theory of finite general linear groups over non-describing characteristic.

Let p be a prime, and let q be a prime power, prime to p . Let e be the multiplicative order of $q \pmod p$. Let $GL_n(q)$ be the general linear group over a field of q elements. Let \mathcal{O} be a complete discrete valuation ring with maximal ideal \mathcal{J} , residue field $k = \mathcal{O}/\mathcal{J}$ of characteristic p , and fraction field K of characteristic zero, big enough for all the groups considered in this paper. Throughout this paper, for any \mathcal{O} -free \mathcal{O} -algebra A , and for any \mathcal{O} -free A -module M , we will write kB for $k \otimes_{\mathcal{O}} A$, we will write KB for $K \otimes_{\mathcal{O}} B$, and we will write \bar{M} for $k \otimes_{\mathcal{O}} M$. We do p -modular representation theory, as described in [1, 19].

Our results compare blocks of finite general linear groups. Here, a p -block of a finite group is a primitive central idempotent of a group algebra (over \mathcal{O} or k), or the corresponding indecomposable algebra (over \mathcal{O} or k), or the corresponding set of characters.

Our first result (Theorem 1) is proved in Section 2 and is analagous to a theorem of Chuang and Kessar [17] for symmetric groups. It describes a module category equivalence between certain blocks of general linear groups and certain of their Brauer correspondents. So we make use of the Brauer homomorphism (see [3] for more details):

For a finite group G , with a p -subgroup P , and an $\mathcal{O}G$ -module M , the Brauer homomorphism is defined to be the quotient map

$$Br: M^P \rightarrow M(P) = M^P / \left(\sum_{Q < P} Tr_Q^P(M^Q) + \mathcal{J}M^P \right).$$

The Brauer quotient $M(P)$ is the quotient of P -fixedpoints of M by relative traces from proper subgroups, reduced modulo p ; $M(P)$ is a $kN_G(P)$ -module. When M is a permutation module, $M(P) \neq 0$ if and only if M has a direct summand with a vertex containing P . In case that G acts on $M = \mathcal{O}G$ by conjugation, without doubt $\mathcal{O}G(P) = kC_G(P)$, and the quotient map

$$Br_P^G: (\mathcal{O}G)^P \rightarrow kC_G(P)$$

is an algebra homomorphism, the classical Brauer homomorphism with respect to P , given by the rule

$$Br_P^G \left(\sum_{g \in G} a_g g \right) = \sum_{g \in C_G(P)} \bar{a}_g g.$$

We are concerned with Br in the following situation: Let H be a subgroup of G containing P , and let i be a primitive idempotent in $(\mathcal{O}G)^H$. Then $\mathcal{O}Gi$ is an indecomposable summand of $\mathcal{O}G$ as an $\mathcal{O}G \times H$ -module, and $(\mathcal{O}Gi)(\Delta P) = kC_G(P)Br_P^G(i)$. In particular, if $Br_P^G(i)$ is non-zero, $\mathcal{O}Gi$ has a vertex containing ΔP . Here ΔP is the diagonal subgroup $\{(x, x) \mid x \in P\}$ of $G \times H$.

Our second result draws Morita equivalences between certain p -blocks of $GL_n(q)$, as q varies. Precisely, we prove that any two weight two unipotent p -blocks of $GL_n(q)$ ($2 < p, n, e$ fixed) with the the same e -core and the same defect group are Morita equivalent (Theorem 2). The defect groups of these blocks are $C_{p^\alpha} \times C_{p^\alpha}$. We assemble these equivalences by piecing together Morita equivalences of Theorem 1 with Morita equivalences of Jost [16] and derived equivalences like those of Chuang [6]. Koshitani and Hyoe [18] proved the same result for the principal blocks of $GL_4(q)$ and $GL_5(q)$ with defect group $C_3 \times C_3$.

In order to construct the derived equivalences mentioned above, we prove results analagous to those of Scopes concerning weight two blocks of symmetric groups. These are written (Theorem 3) in an appendix, in case they are of any independent interest.

The proof of Theorem 2 is made possible by similarities in the combinatorics of unipotent p -blocks of $GL_n(q)$ and $GL_n(q')$, where e is the multiplicative order of both q and q' . Indeed, e -combinatorics plays in the study of these blocks like p -combinatorics plays in the study of p -blocks of symmetric groups. So it is often convenient to picture partitions on an abacus with e runners (see [14, p. 78]).

We label the runners on an abacus $0, \dots, e-1$, from left to right, and label its rows $0, 1, \dots$, from the top downwards. If $\lambda = (\lambda_1, \lambda_2, \dots)$ is a partition with m parts or fewer then we may represent λ on the abacus with m beads: for $i = 1, \dots, m$, write $\lambda_i + m - i = s + et$, with $0 \leq s \leq e-1$, and place a bead on the s th runner in the t th row. Sliding a bead up one row on its runner into a vacant position corresponds to removing a rim e -hook from λ . Thus, given an abacus representation of a partition, sliding all the beads up as far as possible produces an abacus representation of the e -core of that partition, a partition from which no further hooks can be removed. The e -core is independent of the way in which hooks have been removed. The e -weight of a partition is the number of e -hooks removed to obtain the e -core.

Fix an abacus representation of a partition λ , and for $i = 0, \dots, e-1$, let λ_1^i be the number of unoccupied positions on the i th runner which occur above the lowest bead on that runner. Let λ_2^i be the number of unoccupied positions on the i th runner which occur above the second lowest bead on that runner, etc., etc. Then $\lambda^i = (\lambda_1^i, \lambda_2^i, \dots)$ is a partition, and the e -tuple $[\lambda^0, \dots, \lambda^{e-1}]$ is named the e -quotient of λ . The e -quotient depends on the number of beads in the abacus representation of λ . The weight of λ is the sum $|\lambda^0| + \dots + |\lambda^{e-1}|$.

The partitions with a given e -core κ and weight w can be parametrized by e -quotients:

Fix m so that any such partition has m parts or fewer. Representing the partitions on an abacus with m beads, there is an e -quotient for each one. We thus introduce a bijection between the set of partitions with e -core κ and weight w , and the set of e -tuples $[\sigma^0, \dots, \sigma^{e-1}]$ such that $|\sigma^0| + \dots + |\sigma^{e-1}| = w$.

Whenever we represent partitions with a given e -core on an abacus, we assume that m is fixed as above.

1. BLOCKS OF GENERAL LINEAR GROUPS

We describe the ordinary irreducible characters, the simple modules, and the p -blocks of $GL_n(q)$, following Section 7 of [11]. To compare notations, identify a polynomial with one of its roots. Then in [11], our

$\chi(\lambda_{1,s_1}, \dots, \lambda_{l,s_l})$ is the character of $S_K(s_1, \lambda_1) \circ \dots \circ S_K(s_l, \lambda_l) \uparrow^G$. And our $D(\lambda_{1,s_1}, \dots, \lambda_{l,s_l})$ is Dipper and James' $D_F(s_1, \lambda_1) \circ \dots \circ D_F(s_l, \lambda_l) \uparrow^G$.

1.1. Characters

Let \mathcal{F} be the set of monic irreducible polynomials of \mathbb{F}_q with non-zero roots, i.e., those not equal to the degree one polynomial X .

The characters of $GL_n(q)$, $\chi(\lambda_{1,s_1}, \dots, \lambda_{l,s_l})$ may be parametrised by multipartitions $\{\lambda_{1,s_1}, \dots, \lambda_{l,s_l}\}$, which are partitions $\{\lambda_i\}$, indexed by distinct elements s_i of \mathcal{F} , such that $\sum \deg(s_i) \cdot |\lambda_i| = n$.

Those characters $\chi(\lambda_{X-1})$, identified with one partition, indexed by the polynomial $X - 1$ are the *unipotent characters*.

1.2. Simple Modules

We call an irreducible polynomial over \mathbb{F}_q *p-regular* if one of its roots is *p-regular*. This is equivalent to all of its roots being *p-regular*. The set of *p-regular* elements of \mathcal{F} we name \mathcal{F}_p .

The simple modules of $GL_n(q)$, $D(\lambda_{1,s_1}, \dots, \lambda_{l,s_l})$ can be parametrized by multipartitions $\{\lambda_{1,s_1}, \dots, \lambda_{l,s_l}\}$, indexed by distinct elements s_i of \mathcal{F}_p , such that $\sum \deg(s_i) \cdot |\lambda_i| = n$.

The simple module $D(\lambda_{1,s_1}, \dots, \lambda_{l,s_l})$ is the unique simple quotient of a certain *p*-modular reduction of the character $\chi(\lambda_{1,s_1}, \dots, \lambda_{l,s_l})$.

1.3. Blocks

For a multipartition $I = \{\lambda_{1,s_1}, \dots, \lambda_{r,s_r}\}$, define functions F_I and G_I as

$$F_I: \mathcal{F}_p \rightarrow \mathbb{Z}$$

$$u \mapsto \sum \deg(s_i) \cdot |\lambda_i|,$$

where the summation is over those polynomials s_i whose roots, on taking their *p*-regular parts, become the roots of u :

$$G_I: \mathcal{F} \rightarrow \{\text{partitions}\}$$

$$v \mapsto \begin{cases} \text{the } \tilde{e}(\deg(s_i))\text{-core of } \lambda_i (\text{if } v = s_i) \\ \text{the empty partition (otherwise).} \end{cases}$$

Here $\tilde{e}(d)$ is the multiplicative order of $q^d \pmod{p}$.

Two characters $\chi(I)$ and $\chi(J)$ of $GL_n(q)$ are in the same *p*-block if and only if $F_I = F_J$ and $G_I = G_J$. This is a theorem of Fong and Srinivasan [12, 7A]. A block is *unipotent* if it contains a unipotent character. We name the unipotent block which contains the characters $\{\chi(\lambda_{X-1}) \mid \lambda \text{ has } e\text{-core } \tau\}$: it is *the block with e-core } \tau*. The *weight* of a

unipotent block is defined to be the e -weight of λ , for any $\chi(\lambda_{X-1})$ lying in that block.

We consider blocks of $GL_n(q)$ of abelian defect. Unless another is stated, a reference for the following is Section 3 of [12]. Let w be an integer smaller than p .

1.4. Subgroups

Consider a Levi subgroup of $GL_{we}(q)$ which is a (block diagonal) product of w copies of $GL_e(q)$. We write $GL_e(q)^i$ for the i th factor in this direct product, so the Levi subgroup is $GL_e(q)^1 \times \cdots \times GL_e(q)^w$. A Sylow p -subgroup, D , of this Levi subgroup, is also a Sylow p -subgroup of $GL_{we}(q)$. The group D is a direct product of w copies of a cyclic group C_{p^a} , where a is the greatest integer such that p^a divides $q^e - 1$. So $D = D^1 \times \cdots \times D^w$, where $D^i < GL_e(q)^i$. The normalizer of D in $GL_{we}(q)$ is contained in the subgroup $GL_e(q)^1 \times \cdots \times GL_e(q)^w \rtimes S_w \cong GL_e(q) \wr S_w$, where S_w is the group of block permutation matrices, whose conjugation action on $GL_e(q)^1 \times \cdots \times GL_e(q)^w$ permutes the $GL_e(q)^i$'s.

Let $v \geq we$. There is a (block diagonal) Levi subgroup $GL_{we}(q) \times GL_{v-we}(q) < GL_v(q)$, where $GL_{we}(q)$ contains D , such that $C_{GL_v}(D) = C_{GL_{we}}(D) \times GL_{v-we}(q) < GL_e(q)^1 \times \cdots \times GL_e(q)^w \times GL_{v-we}(q)$, and $N_{GL_v}(D) = N_{GL_{we}}(D) \times GL_{v-we}(q) < GL_e(q) \wr S_w \times GL_{v-we}(q)$.

1.5. Brauer Correspondence

Let κ be an e -core. Broué and Puig ([4, (3.5)]) have proved that $Br_D^{GL_v}(f_w) = 1_{kD} \otimes f_0$.

Here f_w (resp. f_0) is the block of $GL_v(q)$ (resp. $GL_{v-we}(q)$) containing those characters $\chi(\lambda_s)$ for which λ has e -core κ , and Br is the Brauer morphism. These blocks have D as a defect group.

1.6. Example

The principal block of $GL_e(q)$ has a cyclic defect group D . Let \mathcal{E} be the set of degree e polynomials in \mathcal{F} , all of whose roots have p' part 1. The characters in the principal block are

$$\begin{aligned} \chi(\lambda_{X-1}), \quad & \text{where } \lambda \text{ is an } e\text{-hook}, \quad \text{and} \\ \chi((1)_s), \quad & \text{where } s \in \mathcal{E}. \end{aligned}$$

Its Brauer tree is a line, with $e + 1$ vertices and end multiplicity $(|D| - 1)/e$:

$$\begin{array}{ccccccc} \circ & \text{---} & \circ & \text{---} & \cdots & \text{---} & \circ & \text{---} & \bullet \\ \chi((e)_{X-1}) & & \chi((e-1, 1)_{X-1}) & & & & \chi((1^e)_{X-1}) & & \chi((1)_s) \end{array}$$

2. AN ANALOGUE OF A THEOREM OF CHUANG AND KESSAR

We prove the Morita equivalence of certain blocks of certain general linear groups of abelian defect and their local blocks. This is analogous to a theorem for symmetric groups, proved in [17].

Let ρ be an e -core satisfying the following property: ρ has an abacus representation in which each runner has at least $w - 1$ more beads than the runner to its immediate left. The smallest such ρ has an abacus display with $i(w - 1)$ beads on the i th runner, for $i = 0, \dots, e - 1$. Let $|\rho| = r$. Let $v = we + r$. The following result imitates theorem 3 of [17].

THEOREM 1. *Let $w < p$, and let e be the multiplicative order of $q \pmod{p}$. The unipotent block of $GL_v(q)$ containing the characters $\chi(\lambda_{X-1})$, where λ has e -core ρ , is Morita equivalent (over \mathbb{C}) to the principal block of $GL_e(q) \wr S_w$.*

The proof is the length of this section.

The principal block of $GL_e(q) \wr S_w$ is Morita equivalent to that block tensored with the defect 0 block of $GL_r(q)$ containing $\chi(\rho_{X-1})$, a block of $GL_e(q) \wr S_w \times GL_r(q)$. We shall prove the theorem by showing that Green correspondence induces a Morita equivalence between this block of $GL_e(q) \wr S_w \times GL_r(q)$ and the block of $GL_v(q)$ with e -core ρ .

Let $GL_v(q) = G_0 > G_1 > \dots > G_w = L$ be a sequence of Levi subgroups of $GL_v(q)$, where

$$G_i = GL_e(q)^1 \times \dots \times GL_e(q)^i \times GL_{v-ie}(q).$$

Let $P_1 > \dots > P_w$ be a sequence of parabolic subgroups of $G_0 > \dots > G_{w-1}$ with Levi subgroups $G_1 > \dots > G_w$, and unipotent complements $U_1 > \dots > U_w$ such that $P_i = G_i \cdot U_i < G_{i-1}$.

Let U_i^+ be the sum $1/|U_i| \sum_{u \in U_i} u$, a central idempotent in $\mathbb{C}P_i$. Note that $|U_i|$ is a power of q^e , so equal to 1 (mod p), and that G_i commutes with U_i^+ .

Let a_i be the principal block idempotent of $GL_e(q)^i$, and let f_{w-i} be the block idempotent for the block of $GL_{v-ie}(q)$ with e -core ρ . Then let

$$b_i = a_1 \otimes \dots \otimes a_i \otimes f_{w-i} \quad (1 \leq i \leq w),$$

a block idempotent of G_i .

We set $G = G_0$, $b = b_0$, and $f = b_w$. Let $D = D^1 \times \dots \times D^w$ be a Sylow p -subgroup of $GL_e(q)^1 \times \dots \times GL_e(q)^w$.

Let S_w be the subgroup of permutation matrices of $GL_v(q)$ whose conjugation action on L permutes the factors of $GL_e(q)^1 \times \dots \times GL_e(q)^w$.

We define N to be the semi-direct product of L and S_w , a subgroup of $GL_n(q)$ isomorphic to $GL_e(q) \wr S_w \times GL_r(q)$.

To prove Theorem 1, we show that $\mathbb{C}Nf$ and $\mathbb{C}Gb$ are Morita equivalent.

LEMMA 1. (1) D is a defect group of $\mathcal{O}G_i b_i$, for $i = 0, \dots, p-1$.

(2) $Br_D^G(b_i) = 1_{kD} \otimes f_0$, and $Br_D^G(U_i^+) = 1$

(3) N stabilizes f , and as an $\mathcal{O}(N \times L)$ -module, $\mathcal{O}Nf$ is indecomposable with vertex ΔD . In particular, $\mathcal{O}Nf$ is a block of N .

(4) $\mathcal{O}Gb$ and $\mathcal{O}Nf$ both have defect group D and are Brauer correspondents.

Proof. (1) $D^{i+1} \times \dots \times D^w$ is a defect group of $\mathcal{O}GL_{v-wi}(q)f_{w-i}$, and D^j is a defect group of $\mathcal{O}GL_e(q)^j a_j$.

(2) $G_i > C_G(D)$, so $Br_D^G(b_i) = Br_D^{G_i}(b_i)$. Hence,

$$\begin{aligned} Br_D^G(b_i) &= Br_D^{G_i}(b_i) = Br_{D^1}^{GL_e^1}(a_1) \otimes \dots \otimes Br_{D^i}^{GL_e^i}(a_i) \otimes Br_{D^{i+1} \times \dots \times D^w}^{GL_{v-ie}}(f_{w-i}) \\ &= 1_{kD} \otimes f_0, \end{aligned}$$

where the last equality is by Puig and Broué's result of Section 1.5.

In addition, $P_i > C_G(D)$, and $Br_D^G(U_i^+) = Br_D^{P_i}(U_i^+) = 1$, because $U_i \cap C_G(D) = \{1\}$.

(3) There can be no doubt that N stabilises f . By part (1), $\mathcal{O}Lf$ has vertex ΔD . Since $C_G(D) < L$, the conjugate of ΔD by an element of $N \times L$ outside $L \times L$ is never conjugate to ΔD in $L \times L$. Consequently, the stabilizer of $\mathcal{O}Lf$ in $N \times L$ is exactly $L \times L$. So $\mathcal{O}Nf = \text{Ind}_{L \times L}^{N \times L}(\mathcal{O}Lf)$ is indecomposable and has vertex ΔD .

(4) Follows from (1) and (2) by Brauer's first main theorem.

By Alperin's description of Brauer correspondence [2, 6.2.7], the $\mathcal{O}(G \times G)$ -module $\mathcal{O}Gb$ and the $\mathcal{O}(N \times N)$ -module $\mathcal{O}Nf$ both have vertex ΔD and are Green correspondents. Let X be the Green correspondent of $\mathcal{O}Gb$ in $G \times N$, an indecomposable summand of $\text{Res}_{N \times N}^{G \times G}(\mathcal{O}Gb)$ with vertex ΔD . Because $\mathcal{O}Nf$ is a direct summand of $\text{Res}_{N \times N}^{G \times N}(X)$, we have $Xf \neq 0$, so $Xf = X$ and X is an $(\mathcal{O}Gb, \mathcal{O}Nf)$ -bimodule.

Now let Y be ${}_G Y_L = \mathcal{O}Gb_0 U_1^+ b_1 \dots U_w^+ b_w$, an $(\mathcal{O}Gb, \mathcal{O}Lf)$ -bimodule. So the functor $Y \otimes_L -$ from $L\text{-mod}$ to $G\text{-mod}$ is

$$HCInd_{G_1 b_1}^{G_0 b_0} \dots HCInd_{G_w b_w}^{G_{w-1} b_{w-1}},$$

where $HCInd$ is Harish-Chandra induction (see [8]).

We prove the main theorem by verifying the following:

PROPOSITION 1. *There is a sequence of \mathcal{O} -split monomorphisms of algebras*

$$\mathcal{O}Nf \hookrightarrow \text{End}_G(X) \hookrightarrow \text{End}_G(Y).$$

The algebras $\mathcal{O}Nf$ and $\text{End}_G(Y)$ have the same \mathcal{O} -rank, so the above maps are isomorphisms.

The left $\mathcal{O}G$ -module ${}_G X$ is a progenerator for $\mathcal{O}Gb$. Hence $X \otimes_N -$ induces a Morita equivalence between $\mathcal{O}Nf$ and $\mathcal{O}Gb$.

Proof. Truly, ${}_G\mathcal{G}Gb_G$ is isomorphic to a direct summand of $Ind_{G \times N}^{G \times G}(X)$, since X and $\mathcal{G}Gb$ are Green correspondents. Thus ${}_G\mathcal{G}Gb$ is a direct summand of $|G : N|$ copies of ${}_GX$, and ${}_GX$ is a progenerator for $\mathcal{G}Gb$.

As well, there is an \mathcal{G} -split homomorphism of algebras $\mathcal{G}Nf \rightarrow End_G(X)$, born of the $G \times N$ -module structure of X . Since ${}_N\mathcal{G}Nf_N$ is a direct summand of $Res_{N \times N}^{G \times N}(X)$, this homomorphism is a monomorphism.

But $Res_{G \times L}^{G \times N}(X)$ is indecomposable with vertex ΔD . First, it is at least a direct sum of indecomposable modules whose vertices are the conjugates of ΔD , by Mackey's formula— X has vertex ΔD , and $G \times L$ is a normal subgroup of $G \times N$ containing ΔD . Secondly, $Res_{G \times L}^{G \times N}(X) = {}_GX \otimes_N \mathcal{G}Nf_L$ is a summand of $Ind_{N \times L}^{G \times L}(\mathcal{G}Nf)$, which by Green correspondence has exactly one summand with vertex ΔD . (${}_N\mathcal{G}Nf_N$ has vertex ΔD and $N_{G \times L}(\Delta D) < N \times L$).

So ${}_GX_L$ is the only summand of ${}_G\mathcal{G}Gb_L$ with vertex ΔD , by Mackey's formula—for ${}_GX_N$ is the only summand of ${}_G\mathcal{G}Gb_N$ with a vertex not strictly contained in ΔD .

Let ${}_GY_L = \mathcal{G}Gb_0 U_1^+ b_1 \cdots U_w^+ b_w$. Each b_i and each U_i^+ is an idempotent in $(\mathcal{G}G)^L$; these idempotents commute with each other, and their product is an idempotent contained in $(\mathcal{G}Gb)^L$. Hence, ${}_GY_L$ is also a direct summand of ${}_G\mathcal{G}Gb_L$. We show that it has as direct summands all summands of ${}_G\mathcal{G}Gb_L$ with vertex ΔD , using the Brauer morphism. So no doubt ${}_GY_L = {}_GX_L \oplus *$, revealing an \mathcal{G} -split monomorphism $End_G(X) \hookrightarrow End_G(Y)$. The calculation goes

$$\begin{aligned} Y(\Delta D) &= \mathcal{G}Gb_0 U_1^+ b_1 \cdots U_w^+ b_w(\Delta D) \\ &= \mathcal{G}C_G(D) Br_D^G(b_0 U_1^+ b_1 \cdots U_w^+ b_w) \\ &= \mathcal{G}C_G(D) Br_D^G(b_0) Br_D^G(U_1^+) Br_D^G(b_1) \cdots Br_D^G(U_w^+) Br_D^G(b_w) \\ &= \mathcal{G}C_G(D) Br_D^G(b_0) Br_D^G(b_1) \cdots Br_D^G(b_w) \\ &= \mathcal{G}C_G(D) Br_D^G(b), \quad \text{by part (2) of Lemma 1} \\ &= \mathcal{G}Gb(\Delta D). \end{aligned}$$

So Y has indeed as summands all summands of ${}_G\mathcal{G}Gb_L$ with vertex ΔD (${}_G\mathcal{G}Gb_L$ is a permutation module).

Finally, we show that $\mathcal{G}Nf$ and $End_G(Y)$ have the same \mathcal{G} -rank, actually equal to $w! \dim_K(KLf)$. The algebra $End_G(Y)$ is \mathcal{G} -free. Hence it is enough to calculate the dimensions over K of $End_G(K \otimes_G Y)$ and KNf . One of these is straightforward— ${}_NKNf$ is the induced module $Ind_L^N(KLf)$, so certainly has dimension $w! \dim(KLf)$. The proposition will be proved when we have shown that $End_G(K \otimes Y)$ has the same dimension.

We calculate ${}_GY = {}_GY \otimes_{KL} KLf = HCInd_{G_1, b_1}^{G_0, b_0} \cdots HCInd_{G_w, b_w}^{G_{w-1}, b_{w-1}}({}_LKLf)$. This is to be done by first computing $HCInd_{G_1, b_1}^{G_0, b_0} \cdots HCInd_{G_w, b_w}^{G_{w-1}, b_{w-1}}(\psi)$,

when ψ is an irreducible character of KLf , using the Littlewood–Richardson rule (see [9, Lemma 4.8]). The relevant combinatorics are described by Lemma 4 of [17], which we record below as Lemma 2. Here, if $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots)$ and $\mu = (\mu_1 \geq \mu_2 \geq \dots)$ are partitions, we write $\mu \subset \lambda$ exactly when $\mu_i \leq \lambda_i$ for $i = 1, 2, \dots$. And an abacus is fixed so that all relevant e -quotients are well defined (cf. Introduction).

LEMMA 2 (Chuang, Kessar). *Let λ be a partition with e -core ρ and weight $v \leq w$. Let $\mu \subset \lambda$ be a partition with e -core ρ and weight $v - 1$. Then there exists α with $0 \leq \alpha \leq e - 1$ such that $\mu^i = \lambda^i$ for $i \neq \alpha$ and $\mu^\alpha \subset \lambda^\alpha$ with $|\mu^\alpha| = |\lambda^\alpha| - 1$. Moreover the complement of the Young diagram of μ in that of λ is the Young diagram of the hook partition $(\alpha + 1, 1^{e-\alpha-1})$.*

In terms of character theory, by the Littlewood–Richardson rule, this means that Harish–Chandra induction from (a block with core ρ of weight $v - 1$) \otimes (the principal block of GL_e) to (a block with core ρ of weight $v \leq w$) takes $\chi(\mu_{X-1}) \otimes \chi((\alpha + 1, 1^{e-\alpha-1})_{X-1})$ to the sum of $\chi(\lambda_{X-1})$'s, such that λ is obtained from μ by moving a bead up the α th runner.

Let us count the number of ways of sliding single beads up the l th runner of a core j times, so that on the resulting runner the top bead has been raised σ_1^l times, the second top bead has been raised σ_2^l times, etc., so that $\sigma_1^l \geq \sigma_2^l \geq \dots$ and $\sum_i \sigma_i^l = j$. It is equal to the number of ways of writing the numbers $1, \dots, j$ in the Young diagram of $(\sigma_1^l, \sigma_2^l, \dots)$ so that numbers increase across rows and down columns—that is, the degree of the character χ^{σ^l} of the symmetric group S_j [14, 7.2.7].

The characters in the block KLf are of the form $\chi(\lambda_{1,s_1}) \otimes \dots \otimes \chi(\lambda_{w,s_w}) \otimes \chi(\rho_{X-1})$, where the indexed partitions λ_i are either e -cores indexed by the polynomial $X - 1$, or the partition (1) indexed by an element of \mathcal{E} .

Let χ_Y be the character of ${}_G Y_L$. A combinatorial description of the multiplicity of a given irreducible character of KGb in $\chi_Y \otimes_{KL} \chi(\lambda_{1,s_1}) \otimes \dots \otimes \chi(\lambda_{w,s_w}) \otimes \chi(\rho_{X-1})$ is now visible:

Suppose that s_1, \dots, s_{r_0} are all equal to $X - 1$, that $\lambda_i = (1)$ for $i \geq r_0 + 1$, that $s_{r_0+1} = \dots = s_{r_0+r_1} =: \theta_1$ are elements of \mathcal{E} , that $s_{r_0+r_1+1} = \dots = s_{r_0+r_1+r_2} =: \theta_2$ are elements of \mathcal{E} not equal to θ_1 , etc., etc. Also suppose that λ_i , for $i = 1, \dots, r_0$ is an e -hook, that $\lambda_1 = \dots = \lambda_{l_0} = (1^e)$, that $\lambda_{l_0+1} = \dots = \lambda_{l_0+l_1} = (2, 1^{e-1}), \dots$, and that $\lambda_{l_0+\dots+l_{e-2}+1} = \dots = \lambda_{l_0+\dots+l_{e-1}} = (e)$, where $l_0 + \dots + l_{e-1} = r_0$. Then $\chi_Y \otimes_{KL} \chi(\lambda_{1,s_1}) \otimes \dots \otimes \chi(\lambda_{w,s_w}) \otimes \chi(\rho_{X-1})$ is equal to the character sum

$$\sum (\dim \chi^{\sigma^0} \dots \dim \chi^{\sigma^{e-1}} \cdot \dim \chi^{\nu^1} \cdot \dim \chi^{\nu^2} \dots) \chi(\mu_{X-1}, \nu_{\theta_1}^1, \nu_{\theta_2}^2, \dots).$$

Here, the summation is over partitions $\mu = [\sigma^0, \dots, \sigma^{e-1}]$ of $|\rho| + r_0 e$ with core ρ , such that $(|\sigma^0|, \dots, |\sigma^{e-1}|) = (l_0, l_1, \dots, l_{e-1})$, over partitions ν^1 of

r_1 , over partitions ν^2 of r_2 , etc. And χ^α is the irreducible character of the symmetric group corresponding to α .

If, when we selected a character of L , we had permuted some of the λ_{i, s_i} 's (there are $(w!/l_0! \cdots l_{e-1}!r_1!r_2! \cdots)$ different arrangements), we would have seen the same character when we applied Y .

So the character of ${}_G Y$ is the sum of characters

$$\begin{aligned} \sum_{|\sigma^i|=l_i, |\nu^i|=r_i} & [(w!/l_0! \cdots l_{e-1}!r_1!r_2! \cdots) \\ & \times \dim \chi^{\sigma^0} \cdots \dim \chi^{\sigma^{e-1}} \cdot \dim \chi^{\nu^1} \cdot \dim \chi^{\nu^2} \cdots \\ & \times \dim \chi((1^e)_{X-1})^{l_0} \cdot \dim \chi((2, 1^{e-2})_{X-1})^{l_1} \cdots \\ & \times \dim \chi((e)_{X-1})^{l_{e-1}} \dim \chi(\nu_{\theta_1}^1)^{r_1} \cdot \dim \chi(\nu_{\theta_2}^2)^{r_2} \cdots \\ & \times \dim \chi(\rho_{X-1}) \chi([\sigma^0, \dots, \sigma^{e-1}]_{X-1}, \nu_{\theta_1}^1, \nu_{\theta_2}^2, \dots)]. \end{aligned}$$

What, then, is the dimension of the semisimple algebra $\text{End}_G(Y) = \text{End}_G(Y \otimes_L KL)$? It is (remembering that $\sum_{|\sigma|=m} |\chi^\sigma|^2 = m!$)

$$\begin{aligned} \sum_{l_0+\cdots+l_{e-1}+r_1+r_2+\cdots=w} & \left[(w!/l_0! \cdots l_{e-1}!r_1!r_2! \cdots)^2 l_0! \cdots l_{e-1}!r_1!r_2! \cdots \right. \\ & \times \dim \chi((1^e)_{X-1})^{2l_1} \cdot \dim \chi((2, 1^{e-2})_{X-1})^{2l_2} \cdots \\ & \times \dim \chi((e)_{X-1})^{2l_e} \dim \chi((1)_{\theta_1})^{2r_1} \cdot \dim \chi((1)_{\theta_2})^{2r_2} \cdots \dim \chi(\rho_{X-1})^2 \left. \right] \\ & = w! \sum_{l_0+\cdots+l_{e-1}+r_1+r_2+\cdots=w} \left[(w!/l_0! \cdots l_{e-1}!r_1!r_2! \cdots) \right. \\ & \quad \times \dim \chi((1^e)_{X-1})^{2l_0} \cdot \dim \chi((2, 1^{e-2})_{X-1})^{2l_1} \cdots \dim \chi((e)_{X-1})^{2l_{e-1}} \\ & \quad \times \dim \chi((1)_{\theta_1})^{2r_1} \cdot \dim \chi((1)_{\theta_2})^{2r_2} \cdots \dim \chi(\rho_{X-1})^2 \left. \right] \\ & = w! \cdot \dim(KLf). \end{aligned}$$

Remark. The correspondence between indecomposable modules of kNf and indecomposable modules of kGb given by Theorem 1 above is exactly Green correspondence between G and N . For if M is an indecomposable of kNf with vertex Q , the kG -module $X \otimes_{kN} M$ cannot have a smaller vertex than Q , as then $M = X^* \otimes_{kG} X \otimes_{kN} M$ would have a smaller vertex than Q .

3. MATCHING OF CHARACTERS UNDER MORITA EQUIVALENCE

From now on, assume $w = 2$. We seek knowledge of the character correspondence induced by the equivalences of Theorem 1. Our ideas and notation are stolen from Chuang—see [6, Sect. 4].

3.1. Representations of $GL_e(q) \wr S_2$

Let $GL_e(q) \wr S_2 = (GL_e \times GL_e) \rtimes S_2$, where S_2 is generated by an involution σ , whose action swaps the two GL_e 's. Given a GL_e representation V , the tensor product $V \otimes V$ is a $GL_e \times GL_e$ representation and can be extended in two ways to a $GL_e \wr S_2$ representation: either let σ act by $(v_1 \otimes v_2 \rightarrow v_2 \otimes v_1)$ (and call the resulting module $V \otimes V^+$), or let σ act by $(v_1 \otimes v_2 \rightarrow -v_2 \otimes v_1)$ (and call the resulting module $V \otimes V^-$).

3.2. Characters of $GL_e(q) \wr S_2$

The characters of $GL_e(q) \wr S_2$ in the principal block are

$$\chi^{(i,j)} = \text{Ind}_{GL_e \times GL_e}^{GL_e \wr S_2} (\chi((i+1, 1^{e-i-1})_{X-1}) \otimes \chi((j+1, 1^{e-j-1})_{X-1})),$$

for $0 \leq i < j \leq e-1$.

$$\chi^{(i)+} = \chi((i+1, 1^{e-i-1})_{X-1}) \otimes \chi((i+1, 1^{e-i-1})_{X-1})^+,$$

for $0 \leq i \leq e-1$.

$$\chi^{(i)-} = \chi((i+1, 1^{e-i-1})_{X-1}) \otimes \chi((i+1, 1^{e-i-1})_{X-1})^-,$$

for $0 \leq i \leq e-1$.

$$\chi^{(i,(1)_s)} = \text{Ind}_{GL_e \times GL_e}^{GL_e \wr S_2} (\chi((i+1, 1^{e-i-1})_{X-1}) \otimes \chi((1)_s)),$$

for $0 \leq i \leq e-1$, and for $s \in \mathcal{C}$.

$$\chi^{((1)_s, (1)_t)} = \text{Ind}_{GL_e \times GL_e}^{GL_e \wr S_2} (\chi((1)_s) \otimes \chi((1)_t)),$$

for $s \neq t$ elements of \mathcal{C} .

$$\chi^{((1)_s)+} = \chi((1)_s) \otimes \chi((1)_s)^+,$$

for $s \in \mathcal{C}$.

$$\chi^{((1)_s)-} = \chi((1)_s) \otimes \chi((1)_s)^-,$$

for $s \in \mathcal{C}$.

3.3. Simple Modules of $GL_e(q) \wr S_2$

The simple modules in the principal block are

$$D^{(i,j)} = \text{Ind}_{GL_e \times GL_e}^{GL_e \wr S_2} (D((i+1, 1^{e-i-1})_{X-1}) \otimes D((j+1, 1^{e-j-1})_{X-1})),$$

$$\text{for } 0 \leq i < j \leq e-1.$$

$$D^{(i)+} = D((i+1, 1^{e-i-1})_{X-1}) \otimes D((i+1, 1^{e-i-1})_{X-1})^+,$$

$$\text{for } 0 \leq i \leq e-1.$$

$$D^{(i)-} = D((i+1, 1^{e-i-1})_{X-1}) \otimes D((i+1, 1^{e-i-1})_{X-1})^-,$$

$$\text{for } 0 \leq i \leq e-1.$$

3.4. Decomposition Numbers for $GL_e(q) \wr S_2$

Some of the decomposition numbers for the principal block of $GL_e(q) \wr S_2$ are as follows (read $D(i, j)$ as zero if $i < 0$ or $j > e-1$ and read $D(i)^+$ and $D(i)^-$ as zero if $i < 0$ or $i > e-1$):

$$[\bar{\chi}^{(i,j)} : D] = \begin{cases} 1 & \text{if } D \in \{D^{(i,j)}, D^{(i+1,j)}, D^{(i,j+1)}, D^{(i+1,j+1)}\} \\ 0 & \text{otherwise,} \end{cases}$$

$$\text{for } 0 \leq i < j \leq e-1 \text{ and } j-i > 1.$$

$$[\bar{\chi}^{(i-1,i)} : D] = \begin{cases} 1 & \text{if } D \in \{D^{(i-1,i)}, D^{(i)+}, D^{(i)-}, D^{(i-1,i+1)}, D^{(i,i+1)}\} \\ 0 & \text{otherwise,} \end{cases}$$

$$\text{for } 0 \leq i \leq e-1.$$

$$[\bar{\chi}^{(i)+} : D] = \begin{cases} 1 & \text{if } D \in \{D^{(i)+}, D^{(i+1)+}, D^{(i,i+1)}\} \\ 0 & \text{otherwise,} \end{cases}$$

$$\text{for } 0 \leq i \leq e-1.$$

$$[\bar{\chi}^{(i)-} : D] = \begin{cases} 1 & \text{if } D \in \{D^{(i)-}, D^{(i+1)-}, D^{(i,i+1)}\} \\ 0 & \text{otherwise,} \end{cases}$$

$$\text{for } 0 \leq i \leq e-1.$$

3.5. Chuang's Notation

To describe the character correspondence, we introduce some notation. First, in the principal block of $GL_e(q) \wr S_2$, are defined

$$\chi^{(i)\pm} = \begin{cases} \chi^{(i)+}, & \text{if } e-1-i \text{ is even} \\ \chi^{(i)-}, & \text{if } e-1-i \text{ is odd,} \end{cases}$$

$$\chi^{(i)\mp} = \begin{cases} \chi^{(i)+}, & \text{if } e-1-i \text{ is odd} \\ \chi^{(i)-}, & \text{if } e-1-i \text{ is even,} \end{cases}$$

$$D^{(i)\pm} = \begin{cases} D^{(i)+}, & \text{if } e-1-i \text{ is even} \\ D^{(i)-}, & \text{if } e-1-i \text{ is odd,} \end{cases}$$

$$D^{(i)\mp} = \begin{cases} D^{(i)+}, & \text{if } e-1-i \text{ is odd} \\ D^{(i)-}, & \text{if } e-1-i \text{ is even.} \end{cases}$$

Second, in the unipotent block of $GL_{2e+r}(q)$ with core ρ , we define (in the abacus notation from the end of the Introduction)

$$\chi^{[i,j]} = \chi([\emptyset, \dots, \emptyset, (1), \emptyset, \dots, \emptyset, (1), \emptyset, \dots, \emptyset]_{X-1}),$$

where (1)'s appear in the i and j positions ($0 \leq i < j \leq e-1$),

$$\chi^{[i]} = \chi([\emptyset, \dots, \emptyset, (2), \phi, \dots, \phi]_{X-1}),$$

where (2) appears in the i th position ($0 \leq i \leq e-1$).

$$\chi^{[i,i]} = \chi([\phi, \dots, \phi, (1^2), \phi, \dots, \phi]_{X-1}),$$

where (1^2) appears in the i th position ($0 \leq i \leq e-1$).

The simple modules $D^{[i,j]}$, $D^{[i]}$, and $D^{[i,i]}$ are defined similarly.

3.6. Decomposition Numbers for $GL_{2e+r}(q)$

Given an e -core τ , there is a notion of a block of a Hecke algebra with e -core τ [10]. It follows from the combinatorial observations of Chuang [6, Sect. 4] that Richards conjectured the numbers given below as decomposition numbers of the blocks of Hecke algebras $\mathcal{H}_q(S_{2e+r})$ with e -core ρ in [23, 4.7]. All that was missing from a complete proof of Richards' conjecture was a Schaper formula for Hecke algebras. Such a formula has since been proved by James and Mathas [15, 4.7]. The decomposition numbers are also decomposition numbers of the unipotent block of $GL_{2e+r}(q)$ with e -core ρ because the decomposition matrix of the Hecke algebra $\mathcal{H}_q(S_n)$ may be seen as a submatrix of the unipotent part of the decomposition

matrix of the general linear group $GL_n(q)$, by results of Dipper ([7, 5.14], in case $s = 1$).

In this way, the decomposition numbers for the unipotent block of $GL_{2e+r}(q)$ with core ρ are known,

$$[\bar{\chi}^{[i,j]} : D] = \begin{cases} 1 & \text{if } D \in \{D^{[i,j]}, D^{[i+1,j]}, D^{[i,j+1]}, D^{[i+1,j+1]}\} \\ 0 & \text{if } D \in \{D(\lambda_{X-1}) \text{ s.t. } \lambda \text{ is } e\text{-regular}\}, \end{cases}$$

for $0 \leq i < j \leq e-1$ and $j-i > 1$.

$$[\bar{\chi}^{[i-1,i]} : D] = \begin{cases} 1 & \text{if } D \in \{D^{[i-1,i]}, D^{[i]}, D^{[i,i]}, D^{[i-1,i+1]}, D^{[i,i+1]}\} \\ 0 & \text{if } D \in \{D(\lambda_{X-1}) \text{ s.t. } \lambda \text{ is } e\text{-regular}\}, \end{cases}$$

for $0 \leq i \leq e-1$.

$$[\bar{\chi}^{[i]} : D] = \begin{cases} 1 & \text{if } D \in \{D^{[i]}, D^{[i+1,i+1]}, D^{[i,i+1]}\} \\ 0 & \text{if } D \in \{D(\lambda_{X-1}) \text{ s.t. } \lambda \text{ is } e\text{-regular}\}, \end{cases}$$

for $0 \leq i \leq e-1$.

$$[\bar{\chi}^{[i,i]} : D] = \begin{cases} 1 & \text{if } D \in \{D^{[i,i]}, D^{[i+1]}, D^{[i,i+1]}\} \\ 0 & \text{if } D \in \{D(\lambda_{X-1}) \text{ s.t. } \lambda \text{ is } e\text{-regular}\}, \end{cases}$$

for $0 \leq i \leq e-1$.

Look at the character correspondence!

PROPOSITION 2. *Under the Morita equivalence between the principal block of $GL_e(q) \wr S_2$ and $GL_{2e+r}(q)$, the following characters match up:*

$$\begin{aligned} \chi^{(i,j)} &\leftrightarrow \chi^{([i,j])}, \\ \chi^{(i)\pm} &\leftrightarrow \chi^{([i])}, \\ \chi^{(i)\mp} &\leftrightarrow \chi^{([i,i])}. \end{aligned}$$

Proof. Take $w = 2$ in Theorem 1, and assume the notation used there.

The proof of Theorem 1 shows that the following characters correspond (for ${}_G Y \otimes_L \psi = {}_G X \otimes_N KN_L \otimes_L \psi = {}_G X \otimes_N \text{Ind}_L^N \psi$, whenever ψ is a character of L):

$$\begin{aligned} \chi^{(i,j)} &\leftrightarrow \chi^{[i,j]} \\ \chi^{(i)\pm} + \chi^{(i)\mp} &\leftrightarrow \chi^{[i]} + \chi^{[i,i]}. \end{aligned}$$

So we analyse the correspondence of the characters appearing in the second of these equations. By the Littlewood–Richardson rule,

$$HCR_{GL_{2e} \times GL_r f_2}^{GL_{2e+r} b}(\chi^{[e-1]}) = \chi((2e)_{X-1}) \otimes \chi(\rho_{X-1}).$$

$\chi((2e)_{X-1})$ is the trivial character of $GL_{2e}(q)$, so on restriction to $GL_e(q) \wr S_2$, it remains the trivial character. The Green correspondent of the kN -module $k \otimes D(\rho_{X-1})$ is thus any p -modular reduction of $\chi^{[e-1]}$. Hence under the Morita equivalence $\chi^{[e-1]}$ matches up with $\chi^{(e-1)+}$. An examination of the decomposition numbers of the two blocks reveals the only possibility for the matching of unipotent characters is that stated, $D^{(e-1)+} \leftrightarrow D^{[e-1]}$, and from (3.4) and (3.6), $[\bar{\chi}^{(e-2)+} : D^{(e-1)+}] = 1 = [\bar{\chi}^{[e-2, e-2]} : D^{[e-1]}]$, whereas $[\bar{\chi}^{(e-2)-} : D^{(e-1)+}] = 0 = [\bar{\chi}^{[e-2]} : D^{[e-1]}]$. So $\chi^{(e-2)+} \leftrightarrow \chi^{[e-2, e-2]}$. Similarly, $D^{(e-2)+} \leftrightarrow D^{[e-2, e-2]}$, etc.

4. BLOCKS OF GENERAL LINEAR GROUPS OF WEIGHT TWO

J. Rickard has discovered complexes of bimodules which induce derived equivalences between any two blocks of symmetric groups of a given weight $w \leq 5$.

We specialise to the case $w = 2$, where there is a published version of this (see [6, Sect. 2]), and use these complexes (adapted for the general linear group), together with the Chuang–Kessar type theorem above, to describe Morita equivalences between unipotent blocks of weight two of $GL_n(q_1)$ and $GL_n(q_2)$ which have isomorphic defect groups, and the same core. The idea is as follows: such Morita equivalences exist for blocks $B(q)$ of $GL_v(q)$ featured in Theorem 1, as there are equivalences between local blocks. Take a Rickard complex between $GL_n(q_1)$ and a block $B(q_1)$ of $GL_v(q_1)$, and between $GL_n(q_2)$ and a block $B(q_2)$ of $GL_v(q_2)$. Show the composition $GL_n(q_1) \rightarrow B(q_1) \rightarrow B(q_2) \rightarrow GL_n(q_2)$ is concentrated in degree zero, and this gives a Morita equivalence.

What is the theorem?

THEOREM 2. *Suppose that q_1 and q_2 are powers of primes r_1 and r_2 , both prime to $p > 2$, and that q_1 and q_2 are both primitive e th roots of unity (mod p). Suppose also that $v_p(q_1^e - 1)$, the greatest power of p dividing $q_1^e - 1$, is equal to $v_p(q_2^e - 1)$. Let τ be an e -core, and let $n = 2e + |\tau|$. Then the weight two block of $\mathcal{O}GL_n(q_1)$ with e -core τ is Morita equivalent to the weight two block of $\mathcal{O}GL_n(q_2)$ with e -core τ .*

Koshitani and Hyoue [18] proved this result when $p = 3$, $v_p = 1$, and $n = 4, 5$. And in case $e = 1$, this result was proved by Puig in [22].

Proof. Take two blocks of weight two of $\mathcal{O}GL_n(q_1)$ and $\mathcal{O}GL_n(q_2)$ as above. Assume first that $\tau = \rho$. Then a Morita equivalence exists by Theorem 1. First, the principal blocks of $GL_e(q_1)$ and $GL_e(q_2)$ are both “Brauer lines” with end multiplicity $(p^\alpha - 1)/e$, and so by a theorem of

Linckelmann [20, Theorem 2.7], they are Morita equivalent over \mathcal{O} . Second, this implies the principal blocks of $GL_e(q_1) \wr S_2$ and $GL_e(q_2) \wr S_2$ are Morita equivalent over \mathcal{O} by a theorem of Marcus [21, 4.3(a)].

The equivalence in this case is then

$$GL_{2e+r}(q_1) \rightarrow GL_e(q_1) \wr S_2 \rightarrow GL_e(q_2) \wr S_2 \rightarrow GL_{2e+r}(q_2).$$

Note that for this equivalence (by Proposition 2), we have the property

Υ : the characters $\chi(\lambda_{1,X-1})$ correspond under the Morita equivalence.

We now work by induction. Suppose that two weight 2 blocks $B(q_1)$ and $\tilde{B}(q_1)$ form a $[2:k]$ pair (for the definition and relevance of a $[2:k]$ pair, look in the Appendix). We show that if $\tilde{B}(q_1)$ and $\tilde{B}(q_2)$ are Morita equivalent, then there is a Morita equivalence between $B(q_1)$ and $B(q_2)$. Likewise, if there is a Morita equivalence between $B(q_1)$ and $B(q_2)$, the blocks $\tilde{B}(q_1)$ and $\tilde{B}(q_2)$ will be Morita equivalent. This will establish Theorem 2. If $k \geq 2$ then all the above blocks are Morita equivalent over \mathcal{O} , by the work of Jost [16]. Furthermore Υ holds for the resulting equivalence between $\tilde{B}(q_1)$ and $\tilde{B}(q_2)$, since Jost's equivalences for q_1 and q_2 involve identical combinatorics. To complete the proof then, suppose that $B(q_i)$ and $\tilde{B}(q_i)$ form $[2:1]$ pairs corresponding to the same pair of e -cores, τ and $\tilde{\tau}$. Assume the notation of Theorem 3. Let $B_i = B(q_i)$ and $\tilde{B}_i = \tilde{B}(q_i)$ for $i = 1, 2$.

Let M_i be a \tilde{B}_i - B_i bimodule which induces Harish-Chandra restriction from B_i to \tilde{B}_i over \mathcal{O} . It is projective as a right B_i -module and as a left \tilde{B}_i -module. Let $\delta'_i: P'_i \rightarrow M_i$ be a projective cover of M_i . Lemma 2 of [25] together with part (e) of Theorem 3 of our Appendix implies that

$$P'_i \cong \bigoplus_{\lambda} P_i(D_i(\Phi(\lambda)_{X-1})) \otimes_{\mathcal{O}} P_i(D_i(\lambda)_{X-1})^*,$$

where the sum runs over all partitions of $t + 2e$ with e -core τ , and $P_i(D_i)$ is the projective cover of a simple module D_i in the block B_i .

Let P_i be a direct summand of P'_i isomorphic to $P_i(D_i(\Psi(\alpha)_{X-1})) \otimes_{\mathcal{O}} P_i(D_i(\alpha_{X-1}))^*$, and let δ_i be the restriction of δ'_i to P_i . Put

$$X_i = (0 \rightarrow P_i \xrightarrow{\delta_i} M_i \rightarrow 0),$$

a complex of \tilde{B}_i - B_i bimodules with M_i in degree 0. The bimodules X_i induce derived equivalences at the level of characters,

$$\begin{aligned} \text{ch}(K \otimes_{\mathcal{O}} M_i) &= \sum_{\lambda \neq \alpha, \beta, \gamma \text{ wt. } 2} \chi(\Phi(\lambda)_{X-1}) \otimes \chi(\lambda_{X-1})^* \\ &\quad + (\chi(\Phi(\alpha)_{X-1}) + \chi(\Phi(\beta)_{X-1})) \otimes \chi(\alpha_{X-1})^* \\ &\quad + (\chi(\Phi(\alpha)_{X-1}) + \chi(\Phi(\gamma)_{X-1})) \otimes \chi(\beta_{X-1})^* \end{aligned}$$

$$\begin{aligned}
& + (\chi(\Phi(\beta)_{X-1}) + \chi(\Phi(\gamma)_{X-1})) \otimes \chi(\gamma_{X-1})^* \\
& + \sum_{\lambda \text{ wt. 1, } s \in \mathcal{E}} \chi(\Phi(\lambda_{X-1}), (1)_s) \otimes \chi(\lambda_{X-1}), (1)_s)^* \\
& + \sum_{s \neq t \in \mathcal{E}} \chi(\tilde{\tau}, (1)_s, (1)_t) \otimes \chi(\tau, (1)_s, (1)_t)^* \\
& + \sum_{\mu \text{ wt. 2, } s \in \mathcal{E}} \chi(\tilde{\tau}, \mu_s) \otimes \chi(\tau, \mu_s)^*
\end{aligned}$$

and

$$\begin{aligned}
\text{ch}(K \otimes_{\mathcal{O}} P_i) &= [\chi(\Phi(\alpha)_{X-1}) + \chi(\Phi(\beta)_{X-1}) + \chi(\Phi(\gamma)_{X-1})] \\
&\otimes [\chi(\alpha_{X-1}) + \chi(\beta_{X-1}) + \chi(\gamma_{X-1})]^*
\end{aligned}$$

by Theorem 3(a)–(d), so that the Lefschetz character of $K \otimes_{\mathcal{O}} X$ is

$$\begin{aligned}
& \sum_{\lambda \neq \alpha, \beta, \gamma \text{ wt. 2}} \chi(\Phi(\lambda)_{X-1}) \otimes \chi(\lambda_{X-1})^* \\
& - (\chi(\Phi(\gamma)_{X-1}) \otimes \chi(\alpha_{X-1})^* - \chi(\Phi(\beta)_{X-1}) \otimes \chi(\beta_{X-1})^* \\
& - \chi(\Phi(\alpha)_{X-1}) \otimes \chi(\gamma_{X-1})^* \\
& + \sum_{\lambda \text{ wt. 1, } s \in \mathcal{E}} \chi(\Phi(\lambda_{X-1}), (1)_s) \otimes \chi(\lambda_{X-1}), (1)_s)^* \\
& + \sum_{s \neq t \in \mathcal{E}} \chi(\tilde{\tau}, (1)_s, (1)_t) \otimes \chi(\tau, (1)_s, (1)_t)^* \\
& + \sum_{\mu \text{ wt. 2, } s \in \mathcal{E}} \chi(\tilde{\tau}, \mu_s) \otimes \chi(\tau, \mu_s)^*.
\end{aligned}$$

Let T be a bimodule describing a Morita equivalence over \mathcal{O} between \tilde{B}_1 and \tilde{B}_2 , with the property Y.

Consider the complex $X_1^* \otimes_{\tilde{B}_1} T \otimes_{\tilde{B}_2} X_2$. This induces an equivalence at the level of characters where all signs are +’s (the negative signs in the characters of X_1 and X_2 cancel since T has the property Y). We show that this complex is split and has homology concentrated in degree 0:

Consider $\bar{P}_1^* \otimes_{k\tilde{B}_1} \bar{T} \otimes_{k\tilde{B}_2} \bar{M}_2$. It is a projective kB_1 - kB_2 -bimodule because \bar{P}_1^* is a projective kB_1 - $k\tilde{B}_2$ -bimodule, and $-\otimes_{k\tilde{B}_2} \bar{M}_2$ is a direct summand of induction. Also, by theorem 3(e) of the appendix, for all $\lambda \neq \alpha$,

$$\begin{aligned}
\bar{P}_1^* \otimes_{k\tilde{B}_1} \bar{T} \otimes_{k\tilde{B}_2} \bar{M}_2 \otimes_{kB_2} D_2(\lambda_{X-1}) &\cong \bar{P}_1 \otimes_{k\tilde{B}_1} D_1(\Phi(\lambda)_{X-1}) \\
&\cong \text{Hom}_{k\tilde{B}_1}(\bar{P}_1, D_1(\Phi(\lambda)_{X-1})) = 0,
\end{aligned}$$

so as a right kB_2 -module, $\bar{P}_1^* \otimes_{k\tilde{B}_1} \bar{T} \otimes_{k\tilde{B}_2} \bar{M}_2$ is a direct sum of copies of $\bar{P}(D_2(\alpha_{X-1}))^*$. On the other hand, the quotient module

$$(\bar{P}_1^* \otimes_{k\tilde{B}_1} \bar{T} \otimes_{k\tilde{B}_2} \bar{P}_2') / (\bar{P}_1^* \otimes_{k\tilde{B}_1} \bar{T} \otimes_{k\tilde{B}_2} \bar{P}_2)$$

as a right kB_2 -module has no summand isomorphic to $\bar{P}(D_2(\alpha_{X-1}))^*$, so applying [25, Lemma 1] to the surjective map

$$\text{id} \otimes \bar{\delta}_2': (\bar{P}_1^* \otimes_{k\tilde{B}_1} \bar{T} \otimes_{k\tilde{B}_2} \bar{P}_2') \rightarrow (\bar{P}_1^* \otimes_{k\tilde{B}_1} \bar{T} \otimes_{k\tilde{B}_2} \bar{M}_2)$$

we conclude the map

$$\text{id} \otimes \bar{\delta}_2: (\bar{P}_1^* \otimes_{k\tilde{B}_1} \bar{T} \otimes_{k\tilde{B}_2} \bar{P}_2) \rightarrow (\bar{P}_1^* \otimes_{k\tilde{B}_1} \bar{T} \otimes_{k\tilde{B}_2} \bar{M}_2)$$

is also surjective. Tensoring complexes,

$$X_1^* \otimes_{\tilde{B}_1} \otimes T \otimes_{\tilde{B}_2} X_2 = \begin{pmatrix} 0 \rightarrow M_1^* \otimes_{\tilde{B}_1} T \otimes_{\tilde{B}_2} P_2 \\ \rightarrow P_1^* \otimes_{\tilde{B}_1} T \otimes_{\tilde{B}_2} P_2 \oplus M_1^* \otimes_{\tilde{B}_1} T \otimes_{\tilde{B}_2} M_2 \\ \rightarrow P_1^* \otimes_{\tilde{B}_1} T \otimes_{\tilde{B}_2} M_2 \rightarrow 0, \end{pmatrix}$$

where the first map is $(\delta_1^* \otimes \text{id}, \text{id} \otimes \delta_2)$, and the second map is $(\text{id} \otimes \delta_2 - \delta_1^* \otimes \text{id})$. The map $\text{id} \otimes \bar{\delta}_2$ is surjective. Hence, so is $(\text{id} \otimes \bar{\delta}_2 - \bar{\delta}_1^* \otimes \text{id})$. Therefore by Nakayama's lemma $(\text{id} \otimes \delta_2 - \delta_1^* \otimes \text{id})$ is surjective, and this map splits because $P_1^* \otimes_{\tilde{B}_1} T \otimes_{\tilde{B}_2} \otimes M_2$ is projective. By a dual argument $(\delta_1^* \otimes \text{id}, \text{id} \otimes \delta_2)$ is injective and splits. Hence $X_1^* \otimes_{\tilde{B}_1} \otimes T \otimes_{\tilde{B}_2} X_2$ is a split complex with homology concentrated in degree 0, thus isomorphic to a complex consisting of a bimodule in degree 0. This bimodule is defined over \mathcal{O} , is projective on each side, and induces a 1–1 correspondence on characters. By a theorem of Broué [5, (0.2)] this bimodule induces a Morita equivalence with property Y.

A similar argument shows that if B_1 and B_2 are Morita equivalent blocks with property Y and are in $[2 : 1]$ pairs with \tilde{B}_1 and \tilde{B}_2 , respectively, then \tilde{B}_1 and \tilde{B}_2 are Morita equivalent blocks with the property Y. This completes the proof of Theorem 2.

APPENDIX: WEIGHT TWO UNIPOTENT BLOCKS OF $GL_n(q)$

We study weight two unipotent blocks of $GL_n(q)$. Our methods are strongly based on those of Scopes [27].

Notation for Weight Two Blocks

Here is some notation, devised by Scopes. Let us fix an e -core τ , and let us fix an abacus representation of τ , with runners $0, 1, \dots, e-1$. In the unipotent block of $GL_{2e+r}(q)$ with core τ , we write the partitions of $2e+r$ with e -core τ as (in the abacus notation from the end of the Introduction)

$$[i, j] := [\emptyset, \dots, \emptyset, (1), \emptyset, \dots, \emptyset, (1), \emptyset, \dots, \emptyset],$$

where (1)'s appear in the i and j positions ($0 \leq i < j \leq e-1$).

$$[i] := [\emptyset, \dots, \emptyset, (2), \phi, \dots, \phi],$$

where (2) appears in the i th position ($0 \leq i \leq e-1$).

$$[i, i] := [\phi, \dots, \phi, (1^2), \phi, \dots, \phi]$$

where (1^2) appears in the i th position ($0 \leq i \leq e-1$).

Comparing Weight Two Blocks

We use $[2 : k]$ pairs to compare weight two unipotent blocks of $GL_n(q)$.

Let τ and $\tilde{\tau}$ be e -cores. Let $B^{\tau,2}(q)$ be the weight two unipotent block of $GL_{we+|\tau|}(q)$ with e -core τ . Let $B^{\tilde{\tau},2}(q)$ be the weight two unipotent block of $GL_{2e+|\tilde{\tau}|}(q)$ with e -core $\tilde{\tau}$. The blocks $B^{\tau,2}(q)$ and $B^{\tilde{\tau},2}(q)$ are said to form a $[2 : k]$ -pair if (in an abacus representation) $\tilde{\tau}$ can be obtained from τ by moving k beads from the i th column to the $i-1$ th column, for some i .

There is for any e -core τ a sequence of e -cores $\tau = \tau_1, \tau_2, \dots, \tau_l = \emptyset$ (where \emptyset denotes the empty e -core), such that for each $i = 1, 2, \dots, l-1$ one of the following possibilities occurs:

(1) $B^{\tau_i,2}(q)$ and $B^{\tau_{i+1},2}(q)$ form a $[2 : k]$ pair, where $k \geq 2$, and are Morita equivalent by Jost's work in [16].

(2) $B^{\tau_i,2}(q)$ and $B^{\tau_{i+1},2}(q)$ form a $[2 : 1]$ pair; that is to say, τ_{i+1} can be obtained from τ_i by removing a single node.

We investigate case (2) in more detail. Suppose that τ and $\tilde{\tau}$ are partitions of t and $t-1$ and that $B = B^{\tau,2}(q)$ and $\tilde{B} = B^{\tilde{\tau},2}(q)$ form a $[2 : 1]$ pair. Suppose that $\tilde{\tau}$ is obtained from τ by moving the top bead from runner i to runner $i-1$.

DEFINITION. Let α be the partition $[i, i]$ with core τ , let β be the partition $[i-1, i]$ with core τ , and let γ be the partition $[i-1]$ with core τ .

Let $\tilde{\alpha}$ be the partition $[i, i]$ with core $\tilde{\tau}$, let $\tilde{\beta}$ be the partition $[i-1, i]$ with core $\tilde{\tau}$, and let $\tilde{\gamma}$ be the partition $[i-1]$ with core $\tilde{\tau}$.

A study of the abacus representations of these partitions reveals that $\alpha > \beta > \gamma$ and that $\tilde{\alpha} > \tilde{\beta} > \tilde{\gamma}$. A generalization of some of the work in [27] is:

THEOREM 3. *There exists a bijection Φ from the set of partitions of $t + 2e$ with e -core τ to the set of partitions of $t - 1 + 2e$ with e -core $\tilde{\tau}$, such that the following hold:*

(a)

$$HCR_{\tilde{K}\tilde{B}}^{KB}(\chi(\alpha_{X-1})) = \chi(\Phi(\alpha)_{X-1}) + \chi(\Phi(\beta)_{X-1}),$$

$$HCR_{\tilde{K}\tilde{B}}^{KB}(\chi(\beta_{X-1})) = \chi(\Phi(\alpha)_{X-1}) + \chi(\Phi(\gamma)_{X-1}),$$

$$HCR_{\tilde{K}\tilde{B}}^{KB}(\chi(\gamma_{X-1})) = \chi(\Phi(\beta)_{X-1}) + \chi(\Phi(\gamma)_{X-1}),$$

and for all λ other than α, β, γ ,

$$HCR_{\tilde{K}\tilde{B}}^{KB}(\chi(\lambda_{X-1})) = \chi(\Phi(\lambda)_{X-1}).$$

(b) Φ can be extended to give a bijection between partitions of $t + e$ with e -core τ and partitions of $t - 1 + e$ with e -core $\tilde{\tau}$ in such a way that

$$HCR_{\tilde{K}\tilde{B}}^{KB}(\chi(\lambda_{X-1}, (1)_s)) = \chi(\Phi(\lambda)_{X-1}, (1)_s),$$

for $s \in \mathcal{E}$.

(c)

$$HCR_{\tilde{K}\tilde{B}}^{KB}(\chi(\tau_{X-1}, \mu_s)) = \chi(\tilde{\tau}_{X-1}, \mu_s),$$

for $s \in \mathcal{E}$, and for μ a partition of 2.

$$HCR_{\tilde{K}\tilde{B}}^{KB}(\chi(\tau_{X-1}, (1)_{s_1}, (1)_{s_2})) = \chi(\tilde{\tau}_{X-1}, (1)_{s_1}, (1)_{s_2}),$$

if s_1 and s_2 are distinct elements of \mathcal{E} .

(d) Let $P(D(\alpha_{X-1}))$ be the projective cover of $D(\alpha_{X-1})$ as a B -module and let $P(D(\Phi(\alpha)_{X-1}))$ be the projective cover of $D(\Phi(\alpha)_{X-1})$ as a \tilde{B} -module. Then

$$\text{ch}(K \otimes_{\mathcal{O}} P(D(\alpha_{X-1}))) = \chi(\alpha_{X-1}) + \chi(\beta_{X-1}) + \chi(\gamma_{X-1}),$$

and

$$\text{ch}(K \otimes_{\mathcal{O}} P(D(\Phi(\alpha)_{X-1}))) = \chi(\Phi(\alpha)_{X-1}) + \chi(\Phi(\beta)_{X-1}) + \chi(\Phi(\gamma)_{X-1}).$$

(e) There is a bijection between the sets $\{D(\lambda_{X-1}) \in k\tilde{B} \mid \lambda \neq \alpha\}$ and $\{D(\mu_{X-1}) \in k\tilde{B} \mid \mu \neq \Phi(\alpha)\}$ induced by $HCR_{k\tilde{B}}^{k\tilde{B}}$.

The head of $HCR_{k\tilde{B}}^{k\tilde{B}}(D(\alpha_{X-1}))$ is isomorphic to $D(\Phi(\alpha)_{X-1})$.

Proof. Essentially, most of this is proved in [27]. More details are given there, if the reader requires them. Parts (a), (b), and (c) are straightforward consequences of the branching rule for $GL_n(q)$. We set $\Phi(\alpha) = \tilde{\alpha}$, $\Phi(\beta) = \tilde{\beta}$, and $\Phi(\gamma) = \tilde{\gamma}$. For all other partitions λ of weight ≤ 2 with e -core τ , we let $\Phi(\lambda)$ be the unique partition obtained by moving a bead from row i into row $i - 1$. row $i - 1$.

A further consequence of the branching rule is that for all λ other than α, β, γ , one has

$$HCInd_{KB}^{KB}(\chi(\Phi(\lambda)_{X-1})) = \chi(\lambda_{X-1}).$$

(d) Scopes defines a partition ν of $t + 2e + 1$ to be that e -core with abacus representation obtained from the abacus representation of τ by moving the topmost bead from column $i - 1$ and placing it on top of the topmost bead in column i . We may Harish-Chandra induce to the block weight zero block $B^{\nu,0}$ of $GL_{t+2e+1}(q)$. By the branching theorem,

$$HCInd_{KB^{\nu,0}}^{KB}(\chi(\lambda_{X-1})) = \chi(\nu_{X-1}) \quad \text{if and only if } \lambda \in \{\alpha, \beta, \gamma\}.$$

Since ν is a core, $\chi(\nu_{X-1})$ has a unique simple p -modular reduction $D(\nu_{X-1})$. And some simple kB -module D , which is a composition factor of only $\chi(\alpha_{X-1}), \chi(\beta_{X-1}), \chi(\gamma_{X-1})$, is sent by $HCInd_{kB}^{kB^{\nu,0}}$ to $D(\nu_{X-1})$. But

$$HCInd_{kB}^{kB^{\nu,0}}(\chi(\nu_{X-1})) = \chi(\alpha_{X-1}) + \chi(\beta_{X-1}) + \chi(\gamma_{X-1})$$

is the character of a projective module, which must have a summand isomorphic to $P(D(\alpha)_{X-1})$ by the unitriangularity of the decomposition matrix. Further, since $HCRes$ and $HCInd$ are adjoint functors, we have $HCInd_{kB}^{kB^{\nu,0}}(D(\alpha_{X-1})) = D(\nu_{X-1})$. We have thus proved that $D(\alpha_{X-1})$ is a composition factor of $\chi(\alpha_{X-1}), \chi(\beta_{X-1}), \chi(\gamma_{X-1})$, implying that in fact

$$\text{ch}(K \otimes_{\mathcal{O}} P(D(\alpha_{X-1}))) = \chi(\alpha_{X-1}) + \chi(\beta_{X-1}) + \chi(\gamma_{X-1}).$$

That $\text{ch}(K \otimes_{\mathcal{O}} P(D(\Psi(\alpha)_{X-1}))) = \chi(\Phi(\alpha)_{X-1}) + \chi(\Phi(\beta)_{X-1}) + \chi(\Phi(\gamma)_{X-1})$ may be proved similarly, using the weight zero block of $GL_{2e+t-2}(q)$ with e -core obtained from that of $\tilde{\tau}$ by placing the top bead from column i on top of column $i - 1$.

(e) To prove this, we first note that $HCRes_{kB}^{kB}(D)$ is non-zero for all simple modules D in B . For the K -space spanned by the set of Brauer characters of the simple modules of kB is equal to the K -space spanned by the set of Brauer characters of the unipotent characters of KB (by unitriangularity of the decomposition matrix). This space has dimension equal to the number of unipotent characters of KB . And restricting this space, we get a space of the same dimension, by (a). So certainly, $HCRes_{kB}^{kB}(D) = 0$ is impossible for simple D in B . Likewise, $HCInd_{kB}^{kB}(D) = 0$ is impossible for simple D in \bar{B} .

Since no simples are killed by these functors, and $HCRes$, $HCInd$ are exact functors, we note that if $HCRes(\chi) = \psi$ and $HCInd(\psi) = \chi$, then any simple composition factor of χ must be sent to a simple composition factor of ψ and vice versa. Thus, by (a), for all λ different from α, β, γ , it is true that $HCRes_{k\tilde{B}}^{kB}(D(\lambda_{X-1})) = D(\Phi(\lambda)_{X-1})$ is a simple module. This simple $k\tilde{B}$ -module is a composition factor of $\chi(\Phi(\lambda)_{X-1})$ and so cannot be $D(\Phi(\alpha)_{X-1})$, by part (d).

The projective cover of $D(\gamma_{X-1})$ must contain at least two copies of $D(\gamma_{X-1})$, so $D(\gamma_{X-1})$ is a composition factor of some $\chi(\mu_{X-1})$, $\mu < \gamma$. Hence, $D(\gamma_{X-1})$ is sent under $HCRes_{k\tilde{B}}^{kB}$ to a simple $k\tilde{B}$ -module. This simple $k\tilde{B}$ -module is a composition factor of $\chi(\Phi(\mu)_{X-1})$ and so cannot be $D(\Phi(\alpha)_{X-1})$, by part (d).

To complete the proof of (e), we look at $HCRes_{k\tilde{B}}^{kB}(D(\lambda_{X-1}))$, for $\lambda = \beta, \alpha$.

We first suppose that β is e -singular. Then (by [9, 5.1]) $D(\beta)$ is a composition factor of $\chi(\lambda_{X-1}\mu_s)$, for some partition μ of size ≤ 2 , some partition λ of size $2e + t - e|\mu|$, and some degree e polynomial s with p' -part 1. But $HCRes_{\tilde{B}}^B/HCInd_{\tilde{B}}^B$ induce one-one correspondences between these kinds of characters by (b),(c). So $HCRes_{k\tilde{B}}^{kB}(D(\beta_{X-1}))$ is simple and not equal to $D(\phi(\alpha)_{X-1})$ by part (d).

Now suppose that β is e -regular, and suppose that $D(\beta_{X-1})$ is only a composition factor of $\chi(\lambda_{X-1})$ when λ is β or γ . The decomposition matrix of the Hecke algebra $\mathcal{H}_q(S_n)$ is a submatrix of the decomposition matrix of $GL_n(q)$ [7, 5.14], where the submatrix has rows corresponding to unipotent characters and columns corresponding to e -regular partitions. Thus, there is a simple module for the Hecke algebra which is only a composition factor of the characters $\chi(\gamma), \chi(\beta)$ of $\mathcal{H}_q(S_n)$. Twisting by the automorphism $\#$ of $\mathcal{H}_q(S_n)$ (as defined in [11, Sect. 2]), which sends characters $\chi(\lambda)$ to their conjugates $\chi(\lambda')$, we find that there is a simple module for $kGL_n(q)$ which is a composition factor of $\chi(\alpha_{c,X-1})$ and $\chi(\beta_{c,X-1})$, plus possibly some unipotent characters. Here α_c and β_c are the partitions $[k, k]$ and $[k, k-1]$ in the conjugate $[2 : 1]$ pair (following [27, 4.4]). This is a contradiction by (d). Hence, $D(\beta_{X-1})$ is a composition factor of $\chi(\lambda_{X-1})$, for some λ different from α, β, γ , and by our (now standard) argument, $HCRes_{k\tilde{B}}^{kB}(D(\beta_{X-1}))$ is simple and not equal to $D(\phi(\alpha)_{X-1})$ by part (d).

The map we have discovered from $\{D(\lambda_{X-1}) \in kB \mid \lambda \neq \alpha\}$ to $\{D(\mu_{X-1}) \in kB \mid \mu \neq \Phi(\alpha)\}$ induced by $HCRes_{k\tilde{B}}^{kB}$ is an isomorphism, since the dimensions of the spaces of Brauer characters spanned by these two sets are equal.

Finally, we show that $HCRes_{k\tilde{B}}^{kB}(D(\alpha_{X-1}))$ has a simple top isomorphic to $D(\Phi(\alpha)_{X-1})$. Note that $HCRes_{k\tilde{B}}^{kB}(D(\alpha_{X-1}))$ is a self-dual module, and that by the adjointness of $HCInd/HCRes$ the only irreducible

modules in the head or socle are isomorphic to $D(\Phi(\alpha)_{X-1})$. Likewise, $HCInd_{k\tilde{B}}^{kB}(D(\Phi(\alpha)_{X-1}))$ is self-dual and has irreducible modules in the head and socle isomorphic to $D(\alpha_{X-1})$. We deduce that if either $HCRes_{k\tilde{B}}^{kB}(D(\alpha_{X-1}))$ or $HCInd_{k\tilde{B}}^{kB}(D(\Phi(\alpha)_{X-1}))$ have composition length greater than two then both modules have irreducible head and socle.

Suppose for a contradiction that both of these do have composition length two. Then

$$\begin{aligned} & \text{the composition length of } \chi(\alpha_{X-1}) \\ &= (\text{the composition length of } \chi(\Phi(\alpha)_{X-1}) + \chi(\Phi(\beta)_{X-1})) - 1 \\ &= (\text{the composition length of } 2\chi(\alpha_{X-1}) + \chi(\beta_{X-1}) + \chi(\gamma_{X-1})) - 3. \end{aligned}$$

Hence the composition length of the projective cover of $D(\alpha_{X-1})$ is 3 by (d). This is impossible.

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