

Homogeneous Sets in Graphs and Hypergraphs



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This thesis is dedicated to the memory of my grandmother Tanka
Chakalova, an inspiring woman whom I still miss every day.

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Abstract

A set of vertices in a graph or a hypergraph is called *homogeneous* if it is *independent*, that is it does not contain any edge, or if it is *complete*, that is it contains all possible pairs or subsets of it as edges. We investigate the properties of graphs and hypergraphs in two cases of imposed restrictions on the structure of their homogeneous sets.

First we study the asymptotic structure of random perfect graphs. We give a generation model which yields such graphs almost uniformly, with an additive error of $e^{-\Omega(n)}$ in the total variation distance. We use this model to determine a number of properties of random perfect graphs, including the distribution of the stability and the clique number, the probability of containing a fixed induced subgraph, Hamiltonicity, clique-colourability, connectivity, edge colouring, and the limit of a uniformly drawn sequence of perfect graphs.

In the second part, we give a hypergraph parameter $\mu(H)$, called *minor-matching number*, with the property that hypergraphs H with bounded rank and minor-matching number contain a polynomially-bounded number of maximal independent sets. In the other direction, every hypergraph H contains at least $2^{\mu(H)}$ maximal independent sets. A number of hard hypergraph problems, including maximum-sized independent set, k -colouring and hypergraph homomorphism can be solved in polynomial time if a list with all maximal independent sets of the hypergraph is given as part of the input, and hence a family of instances with bounded minor-matching number of the input hypergraph form a new polynomial class for the problems above. The class can further be generalised by considering the maximum minor-matching number of a bag in a tree decomposition as a new treewidth measure. We explain how to use this measure, defined as *minor-matching treewidth*, to solve hard problems and how to algorithmically construct a tree decomposition with approximate minimal width.

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Chapter 1

Introduction

The results in this thesis are inspired by the more general question of how imposing restrictions on the structure of the homogeneous sets in graphs and hypergraphs affects the computational complexity of instances with such restrictions.

We address two main questions: what is the structure of a uniformly sampled perfect graph and how to use this structure to design algorithms that are fast on average, and second, finding a parameter that controls the number of maximal independent sets of a hypergraph, and using this parameter to find new tractable cases for different natural problems on hypergraphs.

Perfect graphs are a good example of a large class with a condition on the cliques and the stable sets that leads to polynomial-time algorithms for the clique, independence and chromatic numbers [GLS84]. Unfortunately, these algorithms involve interior-point methods, and could be quite slow in practice. We investigate the asymptotic structure of a random perfect graph P_n sampled uniformly from the set of perfect graphs on vertex set $\{1, \dots, n\}$. Our approach is based on the result of Prömel and Steger that almost all perfect graphs are generalised split graphs, together with a method to generate such graphs almost uniformly. We show that the distribution of the maximum of the stability number $\alpha(P_n)$ and the clique number $\omega(P_n)$ is close to a concentrated distribution $L(n)$ which plays an important role in our generation method. We also prove that the probability that P_n contains any given graph H as an induced subgraph is asymptotically 0 or $\frac{1}{2}$ or 1. Further we show that almost all perfect graphs are 2-clique-colourable, improving a result of Bacsó et al from 2004; they are almost all Hamiltonian; they almost all have connectivity $\kappa(P_n)$ equal to their minimum degree; they are almost all in class one (edge-colourable using Δ colours, where Δ is the maximum degree); and a sequence of independently and uniformly sampled perfect graphs of increasing size converges almost surely to the graphon $W_P(x, y) = \frac{1}{2}(\mathbb{1}[x \leq 1/2] + \mathbb{1}[y \leq 1/2])$.

In the second part, we study the number of maximal independent sets in hypergraphs and how to control it. Finding the maximum number of maximal independent

sets in an n -vertex graph G , $i(G)$, from a restricted class is an extensively studied problem. Let kK_2 denote the matching of size k , that is a graph with $2k$ vertices and k disjoint edges. A graph with an induced copy of kK_2 contains at least 2^k maximal independent sets. The other direction was established in a series of papers [Far89, BY89, FHT93] finally yielding $i(G) \leq (n/k)^{2k}$ for a graph G without an induced $(k+1)K_2$. Alekseev proved in [Ale07] that $i(G)$ is at most the number of induced matchings of G .

In this thesis we generalise the aforementioned results to clutters. The right substructures in this setting are minors rather than induced subgraphs. Maximal independent sets of a clutter $\mathcal{H}$ are in one-to-one correspondence to the sets of its blocker, $b(\mathcal{H})$, hence $i(\mathcal{H}) = |b(\mathcal{H})|$. We show that

$$|b(\mathcal{H})| \leq \sum_{m=0}^{k \cdot f(r)} \binom{|\mathcal{H}|}{m} \binom{r}{2}^m$$

for a $(k+1)K_2$ -minor-free clutter $\mathcal{H}$, where $f(r) = (2r-3)2^{r-2}$ and r is the maximum size of a set in $\mathcal{H}$. A key step in the proofs is, similarly to Alekseev's result, showing that $i(\mathcal{H})$ is bounded by the number of appearances of a substructure called semi-matching, and then proving a dependence between the number of semi-matchings and the number of minor matchings. Note that similarly to graphs, a clutter containing a kK_2 minor has at least 2^k maximal independent sets.

From a computational perspective, a polynomial number of maximal independent sets is particularly interesting. Our results lead to polynomial-time algorithms for restricted instances of many problems including **Set Cover**, **Maximal Independent Set**, **k-Colouring** and **Hypergraph Homomorphism**.

To further extend the class of tractable instances, we present a new width measure for a tree decomposition, *minor-matching hypertree width*, $\mu\text{-tw}$, for graphs and hypergraphs, such that bounding the width guarantees that the set of maximal independent sets has a polynomially-sized restriction to each decomposition bag. The relaxed conditions of the decomposition allow a much wider class of graphs and hypergraphs to have bounded width compared to other tree decompositions. We show that, for fixed k , there are $2^{(1-\frac{1}{k}+o(1))\binom{n}{2}}$ n -vertex graphs of minor-matching hypertree width at most k . The aforementioned problems remain tractable for hypergraphs with bounded minor-matching hypertree width and bounded rank. We show that, given a fixed k , for any n -vertex graph G , it is possible to construct a decomposition of minor-matching hypertree width at most $O(k^3)$, or to prove that $\mu\text{-tw}(G) > k$ in time $n^{O(k^3)}$. This is done by presenting a general algorithm for approximating the hypertree width of well-behaved measures, and reducing $\mu\text{-tw}$ to such a measure. The result relating the restriction of the maximal independent sets to a given set S to the set

of induced matchings intersecting S in graphs, and minor matchings intersecting S in hypergraphs, may be of independent interest.

All work in the thesis is original, unless explicitly stated and accompanied with a reference. The chapter on perfect graphs is based on joint work with Colin McDiarmid on random perfect graphs [MY17b] and Hamilton cycles [MY17a]; and the chapter on clutters and blockers is based on a paper of estimating blocker size for clutters of bounded rank [Yol18a] and a paper using this result for creating a new tree decomposition [Yol18b].

Chapter 2

Preliminaries and overview of results

2.1 Perfect graphs

2.1.1 Perfect graph theory

Given a graph G and a set of vertices S of G , we use $G[S]$ to denote the subgraph of G induced by S . The *clique number*, $\omega(G)$, is the maximum size of a complete subgraph, and the *chromatic number*, $\chi(G)$, is the minimum size of a cover of the vertices of G with independent sets. Clearly $\omega(G) \leq \chi(G)$, since a clique can intersect an independent set at most once. We may ask when does $\omega(G)$ equal $\chi(G)$, but this restriction does not reveal the structure of G , as for every graph G we may add a disjoint complete graph of size $\chi(G)$ to obtain a new graph G' with $\omega(G') = \chi(G')$. To avoid such cases, one may, in addition, impose this restriction on all induced subgraphs.

Definition 2.1.1 (Perfect Graph). *A graph G is called perfect if $\omega(H) = \chi(H)$ for every induced subgraph H of G .*

Important classes of perfect graphs include bipartite graphs, chordal graphs, comparability graphs, interval graphs and line graphs of bipartite graphs. Any complement of a graph in either class above is also perfect. This is no coincidence, as we have the following theorem:

Theorem 2.1.2 (Weak perfect graph theorem [[Lov72a](#), [Lov72b](#)]). *A graph is perfect if and only if its complement is perfect.*

Many theorems in combinatorics can be reformulated as stating that a certain family of graphs is perfect. For instance, König's Theorem (closely related to Hall's marriage theorem) is equivalent to saying that co-bipartite graphs are perfect. Dilworth's theorem is equivalent to saying that comparability graphs are perfect. Therefore, determining whether or not a graph is perfect is a critical problem in combinatorics.

Minimal examples of non-perfect graphs are odd cycles, C_{2k+1} , since $\omega(C_{2k+1}) = 2 < 3 = \chi(C_{2k+1})$, and complements of such cycles, $\overline{C_{2k+1}}$, since $\omega(\overline{C_{2k+1}}) = k < k + 1 = \chi(\overline{C_{2k+1}})$ (also by the weak perfect graph theorem) for $k \geq 2$.

Definition 2.1.3 (Berge Graph). *A graph is called Berge if it does not contain induced C_{2k+1} and $\overline{C_{2k+1}}$ for any $k \geq 2$.*

Somehow surprisingly, these are all the minimal non-perfect graphs.

Theorem 2.1.4 (Strong perfect graph theorem [CRST06]). *A graph is perfect if and only if it is Berge.*

Hence it suffices to see if a hereditary family of graphs contains C_{2k+1} or $\overline{C_{2k+1}}$ for some k in order to determine whether the entire family is perfect or not. Here a class of graphs is called *hereditary* if it is closed under taking induced subgraphs.

Perfect graphs are so fascinating because they play a central role not only in structural combinatorics, but also in polyhedral combinatorics, information theory and combinatorial optimisation.

Given an n -vertex graph G , define $\text{STAB}(G) \subseteq \mathbb{R}^n$ to be the convex hull of the incidence vectors of stable sets of G . A related polytope is $\text{QSTAB}(G) \subseteq \mathbb{R}^n$, defined as the set of points $x = (x_v, v \in V(G))$ such that $x_v \geq 0$ for $v \in V(G)$ and $\sum_{v \in C} x_v \leq 1$ for every clique C of G . Clearly $\text{STAB}(G) \subseteq \text{QSTAB}(G)$ for every graph G , and note that the former polytope is properly contained in the latter for $G = C_{2k+1}$ and $G = \overline{C_{2k+1}}$. The Weak Perfect Graph Theorem easily follows as a corollary from the theorem below.

Theorem 2.1.5 ([Chv75]). *A graph G is perfect if and only if $\text{STAB}(G) \equiv \text{QSTAB}(G)$.*

Given a matrix $A = \{a_{i,j}\}_{i,j \in [n]}$, we say that a row $a_i = (a_{i,1}, \dots, a_{i,n})$ is *undominated* if there is no other row $a_j = (a_{j,1}, \dots, a_{j,n})$, such that $a_{i,\ell} \leq a_{j,\ell}$ for each ℓ between 1 and n . The maximal-clique-versus-vertex matrix of a graph G , $\text{MC}(G)$, has rows corresponding to the vertices of G and columns corresponding to the maximal cliques of G , such that $\text{MC}(G)_{i,j}$ is 1 if the i -th vertex of G is included in the j -th clique of G and 0 otherwise.

Theorem 2.1.6 ([Chv75]). *For a 0,1-matrix A , the following two conditions are equivalent:*

1. *The linear program*

$$\max \langle c, x \rangle \text{ subject to } Ax \leq 1, \text{ and } x \geq 0.$$

has an optimal integral solution for every objective c ,

2. *the undominated rows of A form a maximal-clique-versus-vertex matrix of a perfect graph.*

The result is useful, since solving linear programs can be done efficiently in both the practical and the theoretical sense, while solving integer linear programs is NP-hard. In other words, the theorem above allows us to solve integer linear programs given the promise that the input matrix is of the proper form. Naturally, an algorithm that either solves an instance of the problem or rejects it if it does not meet the perfect graph criteria is more useful. To this end we need a recognition algorithm for perfect graphs.

Theorem 2.1.7. *Deciding whether an n -vertex graph is perfect or not can be decided in $O(n^{10})$ time [CLV03], or even in $O(n^9)$ time [CCL⁺05].*

The two results above were obtained almost simultaneously and are mostly independent of each other.

A major and relatively early result in perfect graph theory is due to Grötschel, Lovász and Schrijver.

Theorem 2.1.8 ([GLS84]). *Given an n -vertex perfect graph G , the parameters $\alpha(G)$, $\omega(G)$, $\chi(G)$, and their weighted versions can be found in polynomial time.*

The result is significant in at least two more ways. First, it demonstrated the importance of a graph parameter $\theta(G)$, called *Lovász theta function*, originally used to bound the Shannon capacity of a graph. New applications of $\theta(G)$ are still found in different fields. Second, it led to increased interest in semidefinite programming and interior point methods. The algorithm in [GLS84] is still the only known algorithm for computing the parameters $\alpha(G)$, $\omega(G)$ and $\chi(G)$ of perfect graphs, and finding a *combinatorial* counterpart is a major open problem.

Theorems 2.1.6 and 2.1.8 are much in the spirit of our central discussion, namely how imposing conditions on the homogeneous sets of a graph can lead to tractability. There are many other beautiful alternative definitions and results for perfect graphs arising from information theory, including Berge's original motivation to study perfect graphs, but unfortunately they require too many preliminaries, and are unrelated to the original results in this thesis.

2.1.2 Probability and measure theory

We use only the most basic definitions from probability and measure theory, first and second moment methods and Chernoff's inequality. Assume all measures below are defined for some fixed measurable space $(\Omega, \mathcal{F})$. Recall that $\|X\|_p^\mu = (\int |X|^p d\mu)^{1/p}$

is the L_p norm w.r.t. μ , and that $\|X\|_p^\mu \leq \|X\|_q^\mu$ for $1 \leq p \leq q \leq \infty$ when μ is a probability measure (that is, $\mu(\Omega) = 1$). The total variation distance of two bounded measures μ and λ is $d_{TV}(\mu, \lambda) = \sup_{S \in \mathcal{F}} |\mu(S) - \lambda(S)|$, and it forms a metric for the space of bounded measures over $(\Omega, \mathcal{F})$. In particular, $d_{TV}(\mu, \lambda) \leq d_{TV}(\mu, \rho) + d_{TV}(\rho, \lambda)$. A measure μ is *absolutely continuous* with respect to measure λ , denoted $\mu \ll \lambda$, if $\mu(S) = 0$ whenever $\lambda(S) = 0$. A measurable function f is called *Radon–Nikodym derivative* of μ with respect to λ and denoted by $d\mu/d\lambda$ if $\mu(S) = \int_S f d\lambda$ for each $S \in \mathcal{F}$. The Radon–Nikodym Theorem states that a Radon–Nikodym derivative exists when μ and λ are σ -finite and $\mu \ll \lambda$. We use the following result:

$$2d_{TV}(\mu, \lambda) = \left\| \frac{d\mu}{d(\mu + \lambda)} - \frac{d\lambda}{d(\mu + \lambda)} \right\|_1^{\mu + \lambda},$$

for finite μ and λ . We use measure theory very rarely, only in the proofs of few theorems, but whenever we do, the result and notation above simplify the presentation significantly.

2.1.3 Random Perfect Graphs

In the previous section we discussed how input structures restricted to be perfect lead to tractable algorithms for otherwise hard problems. Unfortunately, these algorithms are only theoretically efficient. Despite serious efforts, neither the recognition of perfect graphs, nor the algorithm for finding the stability and chromatic numbers of perfect graphs have been improved significantly. On the other hand, very efficient and elegant algorithms are known for the fundamental classes of perfect graphs, including chordal graphs, interval graphs, comparability graphs, etc. It is a natural question to ask if it is possible to design an algorithm that is fast on the majority of inputs, and use the less-efficient, general algorithms as a fallback. Equivalently, we look for a worst-case polynomial and deterministic algorithm for perfect graphs with low expected running time given a uniformly sampled n -vertex perfect graph.

These algorithms will in fact be very simple, but the real challenge is to build a framework for efficiently analysing random perfect graphs. One possible first step might be to use the machinery developed for the proof of the strong perfect graph theorem, but the problem there is with one of the decompositions, called *skew partition*. Skew partitions are non-constructive in the sense that gluing two perfect graphs with a skew partition can yield a non-perfect graph, and therefore this decomposition cannot be used to approximately count the number of n -vertex perfect graphs, or equivalently, to build a probabilistic framework. Finding a family of basic classes and operations whose closure is exactly the set of perfect graphs is an exciting open problem, and answering it will be a major breakthrough in the theory of perfect graphs, but it appears to be quite challenging.

Definition 2.1.9 (Generalised split graph). *A graph G is unipolar if for some $k \geq 0$ it permits a partition of its vertices into $k + 1$ cliques $C_0, \dots, C_k$, such that there are no edges between C_i and C_j for $1 \leq i < j \leq k$. A graph G is said to be a generalised split graph if it or its complement is unipolar.*

We denote the set of n -vertex generalised split graphs by $\mathcal{GS}_n$ and the set of n -vertex perfect graphs by $\mathcal{P}_n$. The main result of [PS92] states that $|\mathcal{GS}_n|/|C_5\text{-free}_n| = 1 - e^{-\Theta(n)}$, where $C_5\text{-free}_n$ is the set of n -vertex graphs without C_5 as an induced subgraph, and this will be our starting point. Since $\mathcal{GS}_n \subseteq \mathcal{P}_n \subseteq C_5\text{-free}_n$, we can deduce that $|\mathcal{GS}_n|/|\mathcal{P}_n| = 1 - e^{-\Theta(n)}$ (to be precise, it has to be shown that the term in the exponent is $-\Theta(n)$, as opposed to $-\Omega(n)$, but this is easy to see). Given a family of graphs $\mathcal{F}$, we use the notation $G_n \in_u \mathcal{F}_n$ to denote the discrete probability space over all n -vertex graphs with a uniform measure over the n -vertex members of $\mathcal{F}$. The result of Prömel and Steger [PS92] can be reformulated as $d_{TV}(GS_n, P_n) = e^{-\Theta(n)}$, where $GS_n \in_u \mathcal{GS}_n$ and $P_n \in_u \mathcal{P}_n$. Thus, accepting an additive error of $e^{-\Theta(n)}$, we can entirely focus on generalised split graphs. The advantage here is that this class is constructive, and its members are easy to describe.

We present a probability space $Gen(n)$ that is easy to generate. In essence,

1. first pick an integer $k \sim L(n)$ for the size of the central clique, C_0 , where $L(n)$ is a very concentrated distribution around $\mu_n = (n - \log_2 n + \log_2 \ln n)/2 + c_n$ with $|c_n| \leq 1$, and set $C_0 = [k] = \{1, \dots, k\}$,
2. construct the side cliques, $C_1, \dots, C_\ell$, according to a partition $\pi = (C_1, \dots, C_\ell)$ of $[n] \setminus [k] = \{k + 1, \dots, n\}$, sampled uniformly at random from the set of all partitions of this $(n - k)$ -set;
3. put every edge between C_0 and $\bigcup_{i=1}^\ell C_i$ uniformly at random and independently;
4. finally, flip an unbiased coin and depending on the outcome either set $Gen(n)$ to be the generated graph so far or its complement.

We will show that $L(n)$ is very close to $\max(\alpha(P_n), \omega(P_n))$, namely

$$d_{TV}\{L(n), \max(\alpha(P_n), \omega(P_n))\} = e^{-\Omega(n)}.$$

Note that the weak perfect graph theorem implies that $\alpha(P_n) \sim \omega(P_n)$, but the two random variables are far from independent.

Theorem 2.1.10. *For $P_n \in_u \mathcal{P}_n$ and $G_n \sim Gen(n)$ we have $d_{TV}(P_n, G_n) = e^{-\Theta(n)}$.*

In addition, we provide tools to work with $Gen(n)$, namely concentration bounds for $L(n)$ and the different parameters of a random partition like the number of parts, the size of the maximum part and others. This framework is used to describe a number of properties of random perfect graphs, namely the distribution of $\omega(P_n) \sim \chi(P_n) \sim \alpha(P_n)$, the subgraph densities, the probability that P_n contains a Hamilton cycle, the distribution of the connectivity and the chromatic index of P_n , and the graphon limit of a sequence of independent samples of P_n with increasing size.

2.2 Transversal enumeration

2.2.1 Problem statement

A hypergraph transversal is a generalisation of a hitting set in a graph. More precisely, a subset T of the vertices is called *transversal*, if it has a non-empty intersection with all hyperedges, or equivalently, the complement of T is independent, that is, it does not contain any edge. Transversal enumeration usually refers to the closely related problems of efficiently enumerating all inclusion-wise minimal transversals of a hypergraph $\mathcal{H}$; and determining whether a hypergraph $\mathcal{F}$ consists of all minimal transversals of $\mathcal{H}$ as hyperedges. Formally, given a hypergraph $\mathcal{H}$, we use $b(\mathcal{H})$ to denote the hypergraph with minimal transversals of $\mathcal{H}$ as hyperedges, called the *blocker* of $\mathcal{H}$.

TRANS-HYP

Instance: Two hypergraphs $\mathcal{H}, \mathcal{F}$.

Question: Is it true that $b(\mathcal{H}) \equiv \mathcal{F}$?

TRANS-ENUM

Instance: A hypergraph $\mathcal{H}$.

Output: The edges of $b(\mathcal{H})$.

Before we go any further, note that the output of **TRANS-ENUM** can be much larger than the input. An algorithm is said to have *polynomial total time* if its running time is polynomial with respect to its input *and* output. A stronger notion is called *incremental polynomial time*, meaning that the time between every two output bits is polynomial w.r.t. the input and the number of output bits produced so far. Even stronger notion is *polynomial delay*, where the time between two output bits is polynomial w.r.t. the size of the input.

Theorem 2.2.1. *The following three are equivalent [EG95]:*

1. **TRANS-HYP** is in P ,

2. there exists a polynomial total time algorithm for **TRANS-ENUM**, and
3. there exists an incremental polynomial time algorithm for **TRANS-ENUM**.

Proof sketch. The implications $2 \Rightarrow 1$ and $3 \Rightarrow 2$ are obvious. Suppose 1. holds. We present an algorithm that incrementally builds a sequence of hypergraphs $\mathcal{F}_0 = \emptyset, \dots, \mathcal{F}_\ell = b(\mathcal{H})$, such that $\mathcal{F}_i \subseteq \mathcal{F}_{i+1}$ and the two differ on one hyperedge. We can test at step j if $b(\mathcal{H}) = \mathcal{F}_j$, given the initial assumption, and if this is the case the algorithm terminates. For a hypergraph $\mathcal{G}$ and a vertex x we define $\mathcal{G} \setminus x = \{h \in \mathcal{G} : x \notin h\}$ and $\mathcal{G}/x = \{h \setminus \{x\} : h \in \mathcal{G}\}$, where all non-inclusion-wise minimal edges of the latter are removed. These operations are called *deletion* and *contraction*. They have the nice property that $b(\mathcal{H}) = \mathcal{F}$ iff $b(\mathcal{H} \setminus x) = \mathcal{F}/x$ and $b(\mathcal{H}/x) = \mathcal{F} \setminus x$. Since $b(\mathcal{H}) \neq \mathcal{F}_j$, the equality fails for at least one pair of the form above, i.e. either $b(\mathcal{H} \setminus x) \neq \mathcal{F}_j/x$ or $b(\mathcal{H}/x) \neq \mathcal{F}_j \setminus x$. We may write that pair as $(\widehat{\mathcal{H}}, \widehat{\mathcal{F}}_j)$, then find a pair $(\widehat{\mathcal{H}} \setminus x', \widehat{\mathcal{F}}_j/x')$ or $(\widehat{\mathcal{H}}/x', \widehat{\mathcal{F}}_j \setminus x')$, such that $b(\widehat{\mathcal{H}} \setminus x') \neq \widehat{\mathcal{F}}_j/x'$ or $b(\widehat{\mathcal{H}}/x') \neq \widehat{\mathcal{F}}_j \setminus x'$ respectively, and continue like this until the two hypergraphs have non-empty vertex sets. Finally, we can backtrack the sequence of deletions and contractions to create a set $T \in b(\mathcal{H}) \setminus \mathcal{F}_j$, and write $\mathcal{F}_{j+1} = \mathcal{F}_j \cup \{T\}$. $\square$

In general, an algorithm for **TRANS-HYP** with running time $f(|\mathcal{F}| + |\mathcal{H}|)$ can be converted to an algorithm for **TRANS-ENUM** with incremental time $n^{O(1)}f(n)$, where $n = |\mathcal{F}| + |\mathcal{H}'|$ and $\mathcal{H}'$ is the set of transversals output so far. Therefore, it often suffices to focus on the complexity of algorithms for the decision problem **TRANS-HYP**.

2.2.2 Algorithms and Complexity

There is no known polynomial algorithm for **TRANS-HYP**, but the existence of such an algorithm would not directly contradict any widely accepted complexity conjecture like ETH or $\text{FTP} \neq \text{W}[1]$. In their seminal paper [FK96], Fredman and Khachiyan gave the first quasi-polynomial time algorithm for **TRANS-HYP** with running time $n^{O(\frac{\log n}{\log \log n})}$, so it is unlikely that **TRANS-HYP** is NP-complete. Most likely **TRANS-HYP** is NP-intermediate, but there is little concrete evidence for that.

Tractable cases include:

1. The rank of a hypergraph H , $rk(H)$, is the maximum size of a hyperedge. In [EGM03] it is shown that **TRANS-HYP** can be solved in time

$$O\{(n|\mathcal{H}|)^{rk(\mathcal{H})-1}||\mathcal{F}||\},$$

and therefore **TRANS-HYP** can be solved in polynomial time whenever $rk(\mathcal{H})$ is bounded.

2. The *degree* of a vertex in a hypergraph is the number of hyperedges that contain it; a hypergraph is said to be *k-degenerate* if every induced subhypergraph, or equivalently, every minor obtained with deletion only, contains a vertex of degree at most k . If $\mathcal{H}$ or $\mathcal{F}$ is $O\{\log(|\mathcal{H}| + |\mathcal{F}|)\}$ -degenerate, then **TRANS-HYP** can be solved in polynomial time [EGM03].

Other conditions, for instance acyclicity of one of the input hypergraphs or logarithmic treewidth of the underlying graphs, are implied by the two cases above.

2.2.3 Equivalent formulations

Transversal enumeration is a fundamental problem in computer science and it has appeared in various forms since the 1950/1960s [Mag59, Ben66, Gro74]. Suppose f is a monotone Boolean formula, that is $f(\bar{x}) \leq f(\bar{y})$ whenever $\bar{x} \preceq \bar{y}$. It can be shown that f has a unique minimal CNF representation with only positive literals. This representation is called *canonical CNF* and is denoted by $\varphi(f)$. The dual of a monotone function $f(\bar{x})$ is the function $\neg f(\neg \bar{x})$, and the dual of a canonical CNF $\varphi = \varphi(f)$ is defined as $\varphi^d = \varphi(\neg f(\neg \bar{x}))$. Every canonical CNF φ corresponds uniquely to a *clutter*, that is a hypergraph whose edges form a Sperner family, denoted by $H(\varphi)$, and vice versa. Under this bijection, the blocker relation over clutters corresponds exactly to the dual relation over monotone formulae, more precisely $H(\varphi^d) = b(H(\varphi))$. Since obviously $(\varphi^d)^d = \varphi$, this representation makes it easy to see that $b(b(H)) = H$ for every clutter H . The problem of determining if two canonical CNFs are dual is called **Monotone Dualisation**, and it is equivalent to **TRANS-HYP**.

Another formulation of the problem is the following. An instance of **SAT** is called **MSAT** if every clause is composed of entirely positive or negative literals, and further called **IMSAT** if every pair of a positive clause c^+ and negative clause c^- resolve, that is c^+ and c^- contain a shared variable x that appears unnegated in c^+ and negated in c^- . It is easy to give a polynomial, even linear, reduction from **SAT** to **MSAT**, and hence **MSAT** is NP-c. For instance, for each variable x_i add two new variables y_i, y_i^c ; and clauses $y_i \vee y_i^c, \neg y_i \vee \neg y_i^c$. Now we can rewrite every old clause c as a new clause c' , by replacing each variable x_j by y_j or y_j^c depending on whether x_j appears negated in c or not. Note that every new clause is either entirely composed of positive or negative literals, and hence the new formula is an instance of **MSAT**. The complexity of **IMSAT** is more interesting. The key observation here is that an instance $\varphi(\bar{x}) = \varphi^+(\bar{x}) \wedge \varphi^-(\neg \bar{x})$ of **MSAT** is an instance of **IMSAT** if and only if for every Boolean n -vector $\bar{x}$ at least one of $\varphi^+(\bar{x})$ and $\varphi^-(\neg \bar{x})$ evaluates to true, or equivalently, $\varphi^+(\bar{x}) \vee \varphi^-(\neg \bar{x}) \equiv 1$. Therefore, an instance $\varphi^+(\bar{x}) \wedge \varphi^-(\neg \bar{x})$ is unsatisfiable if and

only if $\varphi^+(\bar{x}) = \neg\varphi^-(\neg\bar{x})$, or equivalently, $\varphi^+ = (\varphi^-)^d$ and hence **IMSAT** is exactly the same problem as **TRANS-HYP**.

Transversal enumeration can be phrased in terms of colouring. Suppose we are given two hypergraphs $\mathcal{F}$ and $\mathcal{G}$ over a shared vertex set V and want to determine if there is a red/blue colouring $c : V \rightarrow \{r, b\}$ such that $\mathcal{F}$ does not contain a red monochromatic edge and $\mathcal{G}$ does not contain a blue monochromatic edge. The problem is easily seen to be NP-c; for instance, consider the reduction forming $\mathcal{F}$ and $\mathcal{G}$ from the positive and negative clauses of an **MSAT** instance. We call the problem above **IH 2-COL** (intersecting hypergraph 2-colouring) if it has the additional restriction that every edge of $\mathcal{F}$ intersects every edge of $\mathcal{G}$. The problem **IH 2-COL** is essentially a reformulation of **MISAT**, and hence it is equivalent to **TRANS-HYP**.

All problems above can be stated in a stronger form where the input hypergraphs/CNFs are required to be the same! Namely, **SELFDUALITY** is the problem of determining if $\varphi = \varphi^d$ for a monotone CNF formula φ ; in **MISAT** we may impose an additional restriction that $\varphi^+ = \varphi^-$ to define a new problem **DMISAT**, and in **IH 2-COL** we may ask for a proper 2-colouring (in the traditional sense) of a hypergraph where every pair of edges has a non-empty intersection (this is how the problem is normally stated), without changing the complexity. All problems above are **TRANS-HYP**-complete. Indeed, on instance $(\mathcal{F}, \mathcal{G})$ of **TRANS-HYP**, create a new hypergraph $\mathcal{H}$, with two new vertices x, y and where every edge of $\mathcal{F}$ is extended to contain x and every edge of $\mathcal{G}$ is extended to contain y . In addition, add the set $\{x, y\}$ as an edge of $\mathcal{H}$. It is easy to see that $\mathcal{F} = b(\mathcal{G})$ if and only if $\mathcal{H} = b(\mathcal{H})$. To see this, observe that there are three types of minimal transversals of $\mathcal{H}$: the edge $\{x, y\}$; transversals of the type $\{x\} \cup T_G$, where $T_G \in b(\mathcal{G})$, and transversals of the type $\{y\} \cup T_F$, where $T_F \in b(\mathcal{F})$.

2.2.4 Insight into Fredman-Khachiyan's algorithm

Even though Fredman-Khachiyan's result [FK96] has little relevance to our main focus, we have developed sufficient machinery to present their main idea, which is too beautiful to be overlooked.

Lemma 2.2.2. *Suppose $\mathcal{F}$ is a clutter and $\mathcal{G} = b(\mathcal{F})$. There is an edge $h \in \mathcal{F} \cup \mathcal{G}$ such that $|h| \leq \log_2(|\mathcal{F}| + |\mathcal{G}|)$.*

Proof. Assume there is no such edge. Consider a red/blue colouring c , where every vertex of $\mathcal{G}$ is coloured uniformly and independently at random. We call c a *proper colouring* if there is no red monochromatic edge of $\mathcal{F}$ and no blue monochromatic edge of $\mathcal{G}$.

$$\mathbb{P}(c \text{ is not proper}) \leq \sum_{h \in \mathcal{F}} 2^{-|h|} + \sum_{h \in \mathcal{G}} 2^{-|h|} < 1,$$

and therefore a proper colouring exists. However, as we saw during the discussion about **IH 2-COL**, a proper colouring exists if and only if $\mathcal{G} \neq b(\mathcal{F})$, a contradiction. $\square$

Now consider a recursive branching algorithm, that given a pair of clutters, $(\mathcal{G}, \mathcal{F})$, performs the usual cheap rejection checks, $\bigcup \mathcal{G} = \bigcup \mathcal{F}$, $\mathcal{G} \subseteq b(\mathcal{F})$ and $\mathcal{F} \subseteq b(\mathcal{G})$; and if any of them fails it rejects, otherwise it picks a vertex $v \in V(\mathcal{G})$ and branches to $(\mathcal{G} \setminus v, \mathcal{F}/v)$ and $(\mathcal{G}/v, \mathcal{F} \setminus v)$. The leaves of the recursion are clutters with no vertices, but since there is a single such duality pair, $b(\emptyset) = \{\emptyset\}$, it is easy to solve the problem in this case. The algorithm is correct by the property

$$\mathcal{G} = b(\mathcal{F}) \quad \text{iff} \quad \mathcal{G} \setminus v = b(\mathcal{F}/v) \text{ and } \mathcal{G}/v = b(\mathcal{F} \setminus v).$$

This approach is very standard and it can be traced back to Claude Berge [Ber84]. The innovation comes from the choice for v . Suppose w.l.o.g. that the short edge h from Lemma 2.2.2 is contained in $\mathcal{G}$. Since h is a transversal of $\mathcal{F}$, we can pick $v \in h$ to intersect at least $|\mathcal{F}|/|h|$ edges of $|\mathcal{F}|$. Therefore, if we set $U = |\mathcal{F}||\mathcal{G}|$ as the *volume* of the instance $(\mathcal{F}, \mathcal{G})$, the volume of at least one child of the node $(\mathcal{F}, \mathcal{G})$ decreases by a factor of $1 - 1/\log_2(|\mathcal{F}| + |\mathcal{G}|)$. Let us call an edge *red* if the volume of the subproblem is reduced by a factor of $1 - 1/\log_2(|\mathcal{F}| + |\mathcal{G}|)$, and *blue* otherwise. In case the volume of both subproblems is reduced by the factor above, resolve the tie arbitrarily. Every leaf of the recursion tree is identified by the colouring of the path leading to it, and there are at most $m = 2 \log_2(|\mathcal{F}| + |\mathcal{G}|)^2$ red edges along such path, since

$$|\mathcal{F}||\mathcal{G}| \left(1 - \frac{1}{\log_2(|\mathcal{F}| + |\mathcal{G}|)}\right)^m \leq |\mathcal{F}||\mathcal{G}| \exp\{-2 \log_2(|\mathcal{F}| + |\mathcal{G}|)\} < 1.$$

By setting $n = |\bigcup \mathcal{F}| = |\bigcup \mathcal{G}|$, we see that there are at most $\binom{n}{m} \leq n^m$ leaves in the recursion tree. A more sophisticated decomposition improves the running time to $m^{O(\frac{\log m}{\log \log m})}$

2.2.5 Estimates on the size of the blocker

2.2.5.1 Personal Motivation

I started working on the problem by trying to find a hardness result based on the (Strong) Exponential Time hypothesis, (S)ETH. Let s_k be the infimum of those δ for which there exists an algorithm solving k-SAT in time $O(2^{\delta n})$. Clearly, $s_2 \leq s_3 \leq \dots$, and $s_2 = 0$, because there is a polynomial-time algorithm for 2-SAT. ETH is a conjecture stating that $s_3 > 0$, and if true, it would imply $P \neq NP$. The sequence $\{s_k\}_{k \geq 2}$ is monotonic and bounded by 1, so it must converge to a limit s_∞ . SETH states that $s_\infty = 1$. The initial plan was to give a $O(2^{\epsilon n})$ reduction from k-SAT to

MISAT, which would imply that there is no polynomial algorithm for **TRANS-HYP** with degree s_k/ϵ . If $\epsilon = \epsilon(n)$ can be chosen to converge to 0, this would imply that there is no P-time algorithm for **TRANS-HYP**.

I tried different approaches, but a recurring problem was that reducing an instance I of **k-SAT** to a clutter $\mathcal{H}(I)$ such that the solution of I can be read from $b(\mathcal{H}(I))$ would contradict ETH, assuming a P-time algorithm for **TRANS-HYP**, as long as $|b(\mathcal{H}(I))| = 2^{o(n)}$. And hence I felt it is natural to find a sufficient, and hopefully necessary, condition for a clutter $\mathcal{H}$ to have a blocker of size $2^{o(n)}$. Such conditions are not rare in combinatorics; for instance, a set system $\mathcal{F}$ has size $2^{o(n)}$ if and only if $\text{VC-dim}(\mathcal{F}) = o(n)$. Unfortunately, it is difficult to understand the structure of a clutter with high VC-dimension of its blocker, or equivalently, to explain how the blocker of a clutter with high VC-dimension looks like. Finding a natural parameter, $\lambda(\mathcal{H})$, such that, say $2^{\lambda(\mathcal{H})} \leq |b(\mathcal{H})| \leq \binom{|\mathcal{H}|}{\leq \lambda(\mathcal{H})}$, can be useful for at least three reasons:

1. as mentioned before, it would help establish a computational lower bound for **TRANS-HYP**;
2. as long as the parameter is natural, most likely the class of clutters $\{\mathcal{H} : \lambda(\mathcal{H}) \leq k\}$ would form a new tractable family for transversal enumeration, generalising previous algorithms;
3. the class above will correspond to new tractable classes for fundamental problems like maximum independent set, k-colouring, and graph/hypergraph homomorphism, and more generally, problems with solutions that can be read from the blocker.

We give full details in the next section.

2.2.5.2 Previous work: Results for graphs

The problem discussed above has been completely solved for the case of graphs. Recall that $T \in b(G)$ if and only if $V(G) \setminus T$ is a maximal independent set. It is easy to see that a graph G with an induced matching of size k contains at least 2^k maximal independent sets. Farber proved in [Far89] that an n -vertex graph G without an induced $2K_2$ has $O(n^2)$ maximal independent sets. Let $\rho = \rho(G) = |\{uv \in E(G) : N(u) \cup N(v) \neq V(G)\}|$, and let $\mu = \mu(G)$ be the maximal k such that G contains an induced copy of kK_2 . Balas and Yu generalised the result of Farber in [BY89] showing that $|b(G)| \leq \rho^\mu + 1$. Farber, Hujter and Tuza proved in [FHT93] that $|b(G)| < \sqrt[3]{\mu} e^{1-\mu} \binom{n}{2\mu}$ for a graph on $n \geq 4\mu$ vertices. All these results are much improved by an elegant theorem by Alekseev [Ale07] stating that $|b(G)|$ is at most the number of induced matchings.

Theorem 2.2.3 (Alekseev, [Ale07]). *The number of maximal stable sets in a graph is at most the number of induced matchings.*

Proof. Let us denote the set of maximal stable sets of a graph G by $i(G)$, i.e. $I \in i(G)$ iff $I^c \in b(G)$, and the set of induced matchings of G by $m(G)$. The proof is by induction on the number of vertices of G . The claim is trivial when G has no vertices, since $|i(E_0)| = 1 = |m(E_0)|$ (we count the empty graph as an induced matching). Now suppose we are given a graph G and that the statement holds for smaller graphs. Similarly to clutters, we use $G \setminus S$ to denote the graph induced by the vertices outside S . Pick an arbitrary vertex $v \in V(G)$, and let $I \in i(G)$ be arbitrary. There are two cases for the choice for I : (i) $I \setminus \{v\} \in i(G \setminus \{v\})$ and (ii) $I \setminus \{v\} \notin i(G \setminus \{v\})$. We can pick I to fit in the first case in precisely $|i(G \setminus \{v\})|$ ways. Now suppose I falls in the second case, in particular $v \in I$. Since $I \setminus \{v\}$ is stable in $G \setminus \{v\}$, we see that $I \setminus \{v\}$ is not maximal, i.e. there is $J \in i(G \setminus \{v\})$ and $I \setminus \{v\} \subsetneq J$. Every vertex in $J \setminus (I \setminus \{v\})$ must be a neighbour of v in G , because I is maximal in G . Let $u \in J \setminus (I \setminus \{v\})$ be arbitrary. Every neighbour of u is outside $I \setminus \{v\}$, because J is stable, and therefore we see that

1. $I \setminus \{v\}$ is disjoint from $N(u) \cup N(v)$,
2. $I \setminus \{v\}$ is stable in $G \setminus (N(u) \cup N(v))$ (obviously), and
3. $(I \setminus \{v\}) \cup \{w\}$ is not stable in G for every non-neighbour w of v , because I is maximal in G ,

and therefore $I \setminus \{v\} \in i(G \setminus (N(u) \cup N(v)))$. We conclude that

$$|i(G)| \leq |i(G \setminus \{v\})| + \sum_{u \in N(v)} |i(G \setminus (N(u) \cup N(v)))|.$$

Now let us turn our attention to induced matchings. Given a vertex v and a set of edges M inducing a matching, we have two cases: (i) v is not an endpoint of an edge of M , hence $M \in m(G \setminus v)$; and (ii) v is an endpoint of an edge of M , say $uv \in M$, and no other vertex in M can be a neighbour of u or v , hence $M - uv \in m(G \setminus (N(u) \cup N(v)))$. Picking a different edge results in a different matching, hence

$$|m(G)| = |m(G \setminus \{v\})| + \sum_{u \in N(v)} |i(G \setminus (N(u) \cup N(v)))|.$$

By putting everything together, and using the induction hypothesis we get:

$$\begin{aligned} |b(G)| = |i(G)| &\leq |i(G \setminus \{v\})| + \sum_{u \in N(v)} |i(G \setminus (N(u) \cup N(v)))| \\ &\leq |m(G \setminus \{v\})| + \sum_{u \in N(v)} |i(G \setminus (N(u) \cup N(v)))| = |m(G)|. \quad \square \end{aligned}$$

Now let us compare $\mu(G)$ with the Vapnik-Chervonenkis dimension [CV71]. A set S is said to be *shattered* by a set system $\mathcal{F}$ if the restriction of $\mathcal{F}$ over S is the powerset of S , i.e. $tr_S(\mathcal{F}) = 2^S$; and let $sh(\mathcal{F})$ denote the set system composed of sets shattered by $\mathcal{F}$.

Theorem 2.2.4 (Sauer–Shelah lemma; this form is due to Pajor [Paj85]). *A set system $\mathcal{F}$ shatters at least $|\mathcal{F}|$ sets, i.e. $|\mathcal{F}| \leq |sh(\mathcal{F})|$.*

Proof. The proof is by induction on $|\bigcup \mathcal{F}|$. We have $\mathcal{F} \equiv sh(\mathcal{F})$ when $|\bigcup \mathcal{F}| = 0$. Now suppose we are given a set system $\mathcal{F}$ and that the statement holds for systems $\mathcal{G}$ with $|\bigcup \mathcal{G}| < |\bigcup \mathcal{F}|$. Pick an arbitrary $x \in \bigcup \mathcal{F}$, and let $\mathcal{F}_0 = \{F \in \mathcal{F} : x \notin F\}$, $\mathcal{F}_1 = \{F \setminus \{x\} : x \in F \in \mathcal{F}\}$. For every $S \in sh(\mathcal{F}_0) \cap sh(\mathcal{F}_1)$ we have $S \cup \{x\} \in sh(\mathcal{F})$, and hence $|sh(\mathcal{F}_0)| + |sh(\mathcal{F}_1)| \leq |sh(\mathcal{F})|$. By the IH we get

$$|\mathcal{F}| = |\mathcal{F}_0| + |\mathcal{F}_1| \leq |sh(\mathcal{F}_0)| + |sh(\mathcal{F}_1)| \leq |sh(\mathcal{F})|. \quad \square$$

The VC-dimension of a set system $\mathcal{F}$, $\text{VC-dim}(\mathcal{F})$, is the maximum size of a shattered set, i.e. $\text{VC-dim}(\mathcal{F}) = \max\{|S| : S \in sh(\mathcal{F})\}$. The usual statement of Sauer–Shelah lemma is now immediate:

$$|\mathcal{F}| \leq \binom{|\bigcup \mathcal{F}|}{\leq \text{VC-dim}(\mathcal{F})}.$$

Given a graph G , let $\nu(G) := \text{VC-dim}(b(G))$. The discussion above gives us the following bounds on $|b(G)|$ for an n -vertex graph G with m edges:

$$2^{\nu(G)} \leq |b(G)| \leq \binom{n}{\leq \nu(G)} = \sum_{k=0}^{\nu(G)} \binom{n}{k}, \quad (2.1)$$

$$2^{\mu(G)} \leq |b(G)| \leq \binom{n}{\leq 2\mu(G), \text{ even}} = \sum_{k=0}^{\mu(G)} \binom{n}{2k}, \quad (2.2)$$

$$2^{\mu(G)} \leq |b(G)| \leq \binom{m}{\leq \mu(G)} = \sum_{k=0}^{\mu(G)} \binom{m}{k}. \quad (2.3)$$

When $\mu(G)$ is small, equations (2.2) and (2.3) give a polynomial bound on $b(G)$. When $\mu(G)$ is large, we have the following observation:

Note 2.2.5. *Given a sequence $\{G_n\}_{n \in \mathbb{N}}$, where G_n is an n -vertex graph, the following three statements are equivalent:*

1. $|b(G_n)| = 2^{\Theta(n)}$,
2. $\nu(G_n) = \Theta(n)$, and
3. $\mu(G_n) = \Theta(n)$.

This can be seen from the approximation $\binom{n}{pn} \approx (2\pi p(1-p)n)^{-\frac{1}{2}} 2^{\mathcal{H}(p)n}$, where $p \in (0, 1)$ and $\mathcal{H}(p) = -p \log_2 p - (1-p) \log_2 (1-p)$, which can easily be derived from Stirling's formula. The important term on the RHS is $2^{\mathcal{H}(p)n}$, and $\mathcal{H}(p) \rightarrow 0$ as $p \rightarrow 0$.

It is easy to see that $\mu(G) \leq \nu(G)$: for every induced matching M , form a set S by picking an arbitrary endpoint from every edge of M , and observe that $b(G)$ shatters S . Therefore, the two lower bounds are aligned: $2^{\mu(G)} \leq 2^{\nu(G)} \leq |b(G)|$. The upper bounds however, are incomparable. For instance, take $G = K_n$ and $G = nK_2$ (the complete graph with n vertices and the graph with n disjoint edges). Since $\nu(K_n) = \mu(K_n) = 1$, we have:

$$|b(K_n)| = n < \binom{v(K_n)}{\leq \nu(K_n)} = n + 1 < \binom{e(K_n)}{\leq \mu(K_n)} = \binom{v(K_n)}{\leq 2\mu(K_n), \text{even}} = \binom{n}{2} + 1;$$

and since $\nu(nK_2) = \mu(nK_2) = n$, we have:

$$|b(nK_2)| = 2^n = \binom{e(nK_2)}{\leq \mu(nK_2)} = \binom{n}{\leq n} < \binom{v(nK_2)}{\leq \nu(nK_2)} = \binom{2n}{\leq n}.$$

In general, the μ upper bounds, (2.2) and (2.3), are bounded from above by the square of the ν upper bound, (2.1), since $e(G) \leq v(G)^2/2$ and $\mu(G) \leq \nu(G)$. There is no polynomial bound in the other direction. Indeed, let A be a k -set, and let $B = \{v_S : S \subseteq A\}$ be a 2^k -set, disjoint from A . Now suppose G is a graph over $A \cup B$, where each v_S is joined by an edge to every vertex of $S \cup (B \setminus \{v_S\})$. Since G is co-bipartite (A is stable and B is complete), we have that $\mu(G) = 1$ (as long as $k \geq 1$). On the other hand, the maximal independent sets of G are of the form $S \cup \{v_{A \setminus S}\}$ for every $S \subseteq A$, and therefore A is shattered and $\nu(G) = k$. Setting $n := 2^k + k$ to be the number of vertices in G , we see that $\nu(G) \approx \mu(G) \log_2 n$. The example for large gap above is close to the best possible since,

$$2^{\mu(G)} \leq 2^{\nu(G)} \leq |b(G)| \leq \binom{n}{\leq 2\mu(G), \text{even}} \leq n^{2\mu(G)},$$

and hence $\mu(G) \leq \nu(G) \leq 2\mu(G) \log_2 n$.

We end the section with two more notes. First, all upper bounds in (2.1 - 2.3) can be crude at the same time. For instance, take the graph composed of ℓ disjoint copies of complete $k \times k$ bipartite graphs, $\ell K_{k,k}$. We have $\mu(G) = \nu(G) = \ell$ and $|b(\ell K_{k,k})| = 2^\ell$. Now, if we set $k = \ell = \sqrt{n}$,

$$(2.1) \quad \text{yields} \quad 2^{\sqrt{n}} = |b(\ell K_{k,k})| \leq \binom{2n}{\sqrt{n}} \geq (2\sqrt{n})^{\sqrt{n}},$$

$$(2.2) \quad \text{yields} \quad 2^{\sqrt{n}} = |b(\ell K_{k,k})| \leq \binom{2n}{\leq 2\sqrt{n}, \text{even}} \geq (\sqrt{n})^{2\sqrt{n}}, \text{ and}$$

$$(2.3) \quad \text{yields} \quad 2^{\sqrt{n}} = |b(\ell K_{k,k})| \leq \binom{n^{\frac{3}{2}}}{\sqrt{n}} \geq n^{\sqrt{n}}.$$

Here we used the inequality $\binom{a}{b} \geq (\frac{a}{b})^b$. Second, it is not clear whether or not the ν -bound is ever tight. In fact, Péter Frankl conjectured in [Fra89] that the following inequality holds for every clutter $\mathcal{F}$:

$$|\mathcal{F}| \leq \binom{|\bigcup \mathcal{F}|}{\text{VC-dim}(\mathcal{F})}.$$

Note the lack of “ $\leq$ ” in front of $\text{VC-dim}(\mathcal{F})$. The bound is easily seen to be tight by picking $\mathcal{F}$ to contain all k -subsets of a ground set U .

To summarise this section, we defined a new measure, $\mu(G)$, for graphs, inspired by Alekseev’s Theorem, such that:

1. $\mu(G)$ behaves similarly to the VC-dimension of the blocker of G , capturing the key properties, and sometimes giving much better bounds, while having good worst-case guarantees;
2. unlike VC-dim, $\mu(G)$ is defined explicitly over the structure of G , and a reduction R could potentially use this structure to bound the value of $\mu(R(G))$.

2.2.5.3 New results

The main idea is to generalise $\mu(G)$ to hypergraphs (or more precisely, clutters), derive similar bounds as for graphs, and prove the usefulness of the parameter by giving interesting applications and connections to other fields. This will prove to be quite difficult, mainly because bounding $|b(\mathcal{H})|$ from below and above is equivalent to finding estimates on the size of every clutter (antichain, Sperner family), since every clutter is the blocker of another clutter, namely its own blocker (recall that $b(b(\mathcal{H})) = \mathcal{H}$).

We start by carefully defining clutter, blocker and minor. An *independent set* of a hypergraph $\mathcal{H}$ is a subset of $V(\mathcal{H})$ containing no $S \in \mathcal{H}$. Note that if a hypergraph $\mathcal{H}$ contains two hyperedges g and h , such that $g \subsetneq h$, in which case we say that h is *subsumed* by g , the set of maximal independent sets of $\mathcal{H}$ does not change if h is removed. Therefore, subsumed edges play no role here, and hence we lose nothing by assuming there are no such edges. This leads us to the first definition.

Definition 2.2.6 (Clutter). *A set system is a finite set of finite subsets of the natural numbers $\mathbb{N}$. A set system is called clutter, Sperner family or antichain if its sets are incomparable. Every set system $\mathcal{H}$ we associate with a vertex set $V(\mathcal{H}) := \bigcup \mathcal{H}$ and rank $rk(\mathcal{H}) := \max_{S \in \mathcal{H}} |S|$. Given a set system $\mathcal{H}$ we define $cl(\mathcal{H})$ to be the clutter obtained from $\mathcal{H}$ by removing all non-minimal (subsumed) edges.*

The *edge clutter* of a graph G is the clutter composed of the edges of G .

In the context of clutters, the complements of independent sets, called *transversals*, are more natural to work with.

Definition 2.2.7 (Blocker). A transversal of a clutter $\mathcal{H}$ is a set intersecting every $S \in \mathcal{H}$. A minimal transversal is a transversal not containing another transversal as a subset. The blocker of $\mathcal{H}$, denoted $b(\mathcal{H})$, is the clutter consisting of all minimal transversals of $\mathcal{H}$.

One reason why transversals are preferred is because $b(b(\mathcal{H})) = \mathcal{H}$ for every clutter $\mathcal{H}$, and this property is not shared with maximal independent sets. For example, take $C_6 = \{12, 23, 34, 45, 56, 61\}$ and note that $i(C_6) = \{14, 25, 36, 135, 246\}$ and $i(i(C_6)) = \{123, 234, 345, 456, 561, 612\} \neq C_6$.

Definition 2.2.8 (Deletion and Contraction). Suppose $\mathcal{H}$ is a clutter and $v \in \mathbb{N}$. Define

$$\begin{aligned}\mathcal{H} \setminus v &:= \{S \in \mathcal{H} : v \notin S\} \text{ to be } \mathcal{H} \text{ with } v \text{ deleted, and} \\ \mathcal{H}/v &:= cl(\{S - v : S \in \mathcal{H}\}) \text{ to be } \mathcal{H} \text{ with } v \text{ contracted.}\end{aligned}$$

Definition 2.2.9 (Minor). We say that $\mathcal{F}$ is a minor of $\mathcal{H}$ if $\mathcal{F}$ can be obtained from $\mathcal{H}$ through a series of deletions and contractions. We write $\mathcal{F} \subseteq_{\mathcal{M}} \mathcal{H}$ to denote that $\mathcal{F}$ is a minor of $\mathcal{H}$ and say that $\mathcal{H}$ is $\mathcal{F}$ -minor-free if $\mathcal{H}$ contains no minor isomorphic to $\mathcal{F}$.

Here we refer to the standard definition of clutter minor. Note that this notion is different from the more widely used notion of graph minor. By “minor”, in this thesis, we always mean clutter minor and never graph minor. The names coincide because a graph minor renders a clutter minor in the graphic matroid.

A critical property of minors is that deletion and contraction commute, that is $\mathcal{H} \setminus v/u = \mathcal{H}/u \setminus v$, $\mathcal{H} \setminus v/u = \mathcal{H}/u \setminus v$ and $\mathcal{H}/v/u = \mathcal{H}/u/v$ (see Lemma 4.1.6), and therefore the order of the operations in the definition of minor does not matter. Even though the objects above have a rich algebraic structure, we only use some very basic observations: $b(b(\mathcal{H})) = \mathcal{H}$, $b(\mathcal{H} \setminus v) = b(\mathcal{H})/v$, $b(\mathcal{H}/v) = b(\mathcal{H}) \setminus v$ and $\mathcal{F} \subseteq_{\mathcal{M}} \mathcal{H}$ if and only if $b(\mathcal{F}) \subseteq_{\mathcal{M}} b(\mathcal{H})$ (follows from Lemma 4.1.6).

A minor $\mathcal{F}$ of a graph G contains sets of size 0, 1 and 2. In case $\mathcal{F}$ contains the empty set, then $\mathcal{F}$ contains no other sets and has no transversals. If $\mathcal{F}$ contains a set of size 1, say $\{v\}$, no other set contains the vertex v , since $\mathcal{F}$ is a clutter, so these vertices play no significant role in the structure of $\mathcal{F}$, and we may ignore them. The rest of $\mathcal{F}$ is an *induced* subgraph of G . In particular, we see that if $\mathcal{F}$ is a graph, then $\mathcal{F}$ is an induced subgraph of G . We conclude that the minor and induced subgraph relations coincide over the set of graphs.

Another way to view clutter minors is via partial assignments. Recall that every clutter $\mathcal{H}$ corresponds uniquely to a monotone Boolean formula $\chi = \varphi(\mathcal{H})$. Here a

minor $\mathcal{F}$ of $\mathcal{H}$ corresponds to a partial assignment $\varphi(\mathcal{F})$ of χ , namely $\varphi(\mathcal{F})$ can be obtained from χ by substituting some of its variables with *true* or *false* and removing the consequently redundant clauses.

We are in position to state the first new theorem.

Theorem 2.2.10. *Suppose $\mathcal{H}$ is a $(k+1)K_2$ -minor-free rank r clutter. Then*

$$|b(\mathcal{H})| \leq \sum_{i=0}^{k \cdot (2r-3)2^{r-2}} \binom{|\mathcal{H}|}{i} \binom{r}{2}^i.$$

The theorem holds with equality for $\mathcal{H} \cong kK_2$. Similarly to the case of graphs, we see that a minor-closed class of clutters has polynomially bounded $|b(\mathcal{H})|$ if and only if it does not contain kK_2 for some k . Theorem 2.2.10 is actually a corollary of a more general result, the conjunction of theorems 4.2.3 and 4.2.5, generalising Theorem 2.2.3 when r is set to 2.

Let $\mu(\mathcal{H})$ denote the maximum size of a minor matching of a clutter $\mathcal{H}$. Theorem 2.2.10 can be used in the following way: suppose P is a hard problem, taking clutter $\mathcal{H}$ as input. If P can be solved in polynomial time with respect to its input when we imagine that both $\mathcal{H}$ and $b(\mathcal{H})$ are included in the input, then P can be solved in polynomial time when both $rk(\mathcal{H})$ and $\mu(\mathcal{H})$ are bounded. In this case we say that *the solution of P can be read from the blocker*. This framework yields a completely new and unexplored approach to tractability. The archetypal example of such problems is the maximum stable set problem: if we are given the list of all maximal stable sets, we can simply iterate through it to find a maximum-sized. There are, of course, more elaborate problems that fit into this framework, for instance k -colouring and (hyper)graph homomorphism. In general, $b(\mathcal{H})$ contains important structural information about $\mathcal{H}$, that is difficult to access otherwise.

It is somehow unsatisfactory that very simple clutters, like the matching nK_2 , do not fit into the framework above. On the other hand, natural problems can be solved on each edge individually, and the global solution can be glued together in the end. A very successful approach to tackle tractability is the use of separators. Suppose a hypergraph $\mathcal{H}$ can be separated by a set S into a pair of hypergraphs, $\mathcal{H}_1$ and $\mathcal{H}_2$. An optimal solution of a natural problem P for $\mathcal{H}$ can usually be recovered from a list mapping each restriction over S of a feasible solution of P to the optimal value of a feasible solution extending this restriction. The delicate part is to find a separator S guaranteeing that the restriction of the set of feasible solutions of P to S is not too large, say polynomially-sized. If $\mathcal{H}$ permits a hierarchical structure of separators with this property, then the otherwise hard problem P can be solved in polynomial time for $\mathcal{H}$, provided that we can efficiently find these separators. Let us come back to the

example of maximum independent set. For example, if $\mathcal{H}$ is a chordal graph, we can find a list of clique separators to decompose $\mathcal{H}$, and since the size of the intersection of every clique and stable set is at most 1, we only need to remember the size of the maximal independent set containing at most one specified vertex from the separator, i.e. we need to store $|S| + 1$ numbers. If $\mathcal{H}$ is a graph with bounded treewidth, say $k - 1$, each separator can be chosen to have size at most k , and hence there are at most 2^k restrictions. The two examples above can be unified in the following way: we require to choose the separators $S_1, \dots, S_t$, such that $\alpha(S_i) \leq k$. Here there are at most $\binom{|S_i|}{\leq k} = O(n^k)$ restrictions of an independent set over the separator S_i . This is in a sense the best possible, but since an independent set of maximum size is a *maximal* independent set, we can do better, and only require the restriction of the set of *maximal independent sets* with the separator S , i.e. $tr_S(b(\mathcal{H}))$, to be polynomially-sized. Directly requiring a list of separators to satisfy $|tr_S(b(\mathcal{H}))| \leq n^c$ will indeed be sufficient to solve the maximum independent set problem on $\mathcal{H}$ in polynomial time, but such a decomposition can be both difficult to find and verify to be correct. Instead we give a combinatorial condition to restrict the size of $tr_S(b(\mathcal{H}))$.

Definition 2.2.11. *Given a clutter $\mathcal{H}$ and a set $S \subseteq V(\mathcal{H})$, we say that a $kK_2 \cong \mathcal{F} \subseteq_{\mathcal{M}} \mathcal{H}$ minor is S -intersecting if every edge of $\mathcal{F}$ has a non-empty intersection with S . We define $\mu_S(\mathcal{H})$ to be the maximum size of an S -intersecting minor-matching.*

Theorem 2.2.12. *Suppose $\mathcal{H}$ is a rank r clutter, $S \subseteq \mathbb{N}$ and $k = \mu_S(\mathcal{H})$. Then*

$$|tr_S(b(\mathcal{H}))| \leq \sum_{i=0}^{k(2r-3)2^{r-2}} \binom{|E_S(\mathcal{H})|}{i} \binom{r}{2}^i,$$

where $E_S(\mathcal{H})$ is the set of edges of $\mathcal{H}$ intersecting S . In particular, when $\mu_S(\mathcal{H})$ and $rk(\mathcal{H})$ are bounded, $tr_S(b(\mathcal{H}))$ is polynomially-sized, and it can be computed in polynomial time.

Note that Theorem 2.2.12 reduces to Theorem 2.2.10 when $S = V(\mathcal{H})$ (or $S = \mathbb{N}$).

The theorem essentially shows that we can solve problems with solutions that can be read from the blocker given a tree decomposition of a hypergraph with bounded rank where every bag has bounded size of an intersecting minor-matching. There is of course the legitimate question whether such a decomposition can be found efficiently. Let $\mu\text{-}tw(\mathcal{H})$ denote the minimum over all tree decompositions of the maximum value of $\mu_B(\mathcal{H})$, taken over all bags B of the decomposition. We will prove the following theorem:

Theorem 2.2.13. *There exists an algorithm with running time $v(G)^{O(k^3)}$ that given a graph G and an integer k as input, either*

- *finds a tree decomposition $\mathcal{T}$ of G with $\mu\text{-tw}(\mathcal{T}) = O(k^3)$, or*
- *correctly concludes that $\mu\text{-tw}(G) > k$.*

Note that given a graph with bounded μ -width, the algorithm in the theorem above will construct a decomposition with bounded μ -width, so Theorem [2.2.13](#) suffices to render the class of graph with bounded μ -width tractable.

Chapter 3

Random Perfect Graphs

3.1 Introduction

A graph is *perfect* if the chromatic number equals the clique number in each of its induced subgraphs. Perfect graphs have formed a central field in graph theory for some decades, partly because of the challenging open problems and partly because of the connections to polyhedral combinatorics, linear optimisation and computational complexity, see for example [RAR01]. Fundamental classes of graphs that are all perfect include bipartite graphs, chordal graphs, comparability graphs and interval graphs.

Generalised split graphs form another important class of perfect graphs. A graph G is *unipolar* if for some $k \geq 0$ its vertex set $V(G)$ can be partitioned into $k+1$ cliques $C_0, C_1, \dots, C_k$, so that there are no edges between C_i and C_j for $1 \leq i < j \leq k$. We call C_0 the *central clique*, and the C_i for $i \geq 1$ the *side cliques*; and we call the pair (G, C_0) a *unipolar arrangement of order $v(G)$* . A graph G is *co-unipolar* if its complement $\overline{G}$ is unipolar; and it is a *generalised split graph* if it is unipolar or co-unipolar. We denote the classes of perfect, unipolar, co-unipolar and generalised split graphs by $\mathcal{P}$, $\mathcal{GS}^+$, $\mathcal{GS}^-$ and $\mathcal{GS}$ respectively. Given a class $\mathcal{G}$ of graphs, we let $\mathcal{G}_n$ denote the set of graphs in $\mathcal{G}$ on vertex set $[n] := \{1, \dots, n\}$. We say that a sequence of events $(A_n)_{n \geq 1}$ holds *with high probability (or whp)* if $\mathbb{P}(A_n) = 1 - o(1)$, and that it holds *with very high probability (or wvhp)* if $\mathbb{P}(A_n) = 1 - e^{-\Omega(n)}$.

Our goal is to describe the asymptotic properties of perfect graphs. We rely on the key theorem of Prömel and Steger [PS92] that almost all perfect graphs are generalised split graphs. We present and analyse an almost uniform generation process for the graphs in $\mathcal{GS}_n$. This generation process, together with the Prömel-Steger theorem, yields a powerful method for working with uniformly sampled perfect graphs. The usefulness of the method will be seen when we prove a range of results about the asymptotic behaviour of perfect graphs, described in the next subsection.

3.1.1 Plan of the chapter

In § 3.2 we present our method for analysing random perfect graphs. The main result is that the total variation distance between a naturally generated random generalised split graph in $\mathcal{GS}_n$ and the uniformly sampled perfect graph P_n is $e^{-\Theta(n)}$. We introduce a family of concentrated distributions $L(n)$ which are used in the generation process, and give a number of results essential for our analysis, including surveying relevant results about random partitions, but we defer the detailed proofs concerning the generation process to § 3.9 near the end of the chapter.

In § 3.3 we study the stability number $\alpha(P_n)$ and clique number $\omega(P_n)$. We show that the minimum of the two numbers is asymptotically normally distributed (with specified mean and variance), and the distribution of the maximum is very close to the distribution $L(n)$ mentioned above.

In § 3.4 we discuss which induced subgraphs are likely to be found in a random perfect graph, and we find a trichotomy. Depending on whether (i) $H \in \mathcal{P} \setminus \mathcal{GS}$, (ii) $H \in \mathcal{GS} \setminus \mathcal{GS}^-$ or $H \in \mathcal{GS} \setminus \mathcal{GS}^+$ and (iii) $H \in \mathcal{GS}^+ \cap \mathcal{GS}^-$, the probability that P_n contains an induced copy of H is respectively (i) $e^{-\Theta(n)}$, (ii) $1/2 \pm e^{-\Omega(n)}$ and (iii) $1 - e^{-\Omega(n)}$. For example, the complete bipartite graph $K_{2,3} \in \mathcal{GS}^- \setminus \mathcal{GS}^+$, so the probability that P_n contains an induced $K_{2,3}$ is well-estimated by $\frac{1}{2}$.

In § 3.5 we show that almost all perfect graphs are 2-clique colourable, improving the result of Bacsó et al [BGG⁺04] from 2004 that almost all perfect graphs are 3-clique colourable.

In § 3.6 we discuss Hamilton cycles, and find that the distribution $L(n)$ appears again. We show that, if $X \sim L(n)$, the probability that P_n is Hamiltonian is

$$1 - \frac{1}{2}\mathbb{P}(X > n/2) \pm e^{-\Omega(n)} = 1 - 2^{-\frac{1}{4}(1+o(1))\log^2 n} = 1 - o(1). \quad (3.1)$$

In § 3.7 we discuss connectivity and the chromatic index of perfect graphs. We show that for a random perfect graph P_n the connectivity $\kappa(P_n)$ equals the minimum degree wvhp, and the chromatic index equals the maximum degree whp.

In § 3.8 we study the limit of a sequence of independently sampled perfect graphs P_n . In particular, we show that

$$\begin{aligned} \mathbb{P}(\delta_{\square}(P_n, W_P) \leq n^{-1/2}) &= 1 - e^{-\Omega(\sqrt{n} \log n)} \text{ and} \\ \mathbb{P}(\delta_{\square}(P_n, W_P) \leq (\log n)^{-2}) &= 1 - e^{-\Theta(n)}, \end{aligned}$$

where $\delta_{\square}$ is the cut distance, and $W_P(x, y)$ is the graphon $\frac{1}{2}(\mathbb{1}[x \leq 1/2] + \mathbb{1}[y \leq 1/2])$. This implies that a sequence of uniformly and independently sampled perfect graphs of increasing size converges almost surely to W_P and provides an analytic tool for estimating subgraph densities of P_n .

In § 3.9 we give the proofs concerning the generation process which were deferred from § 3.2.

3.2 Generating perfect graphs

Before we go any further, let us check that generalised split graphs are perfect. The weak perfect graph theorem states that the complement of a perfect graph is perfect [Lov72a, Lov72b], so it suffices to consider a unipolar graph G , and show it is perfect. Also, each induced subgraph of a unipolar graph is unipolar, so it suffices to show that $\chi(G) = \omega(G)$.

Consider a unipolar arrangement (G, C_0) , where C_0 is the central clique, and let $C_1, \dots, C_k$ be the side cliques. Let G_j denote the induced subgraph $G[C_0 \cup C_j]$. Bipartite graphs are perfect, so co-bipartite graphs are perfect, and thus $\chi(G_j) = \omega(G_j)$ for each j . Therefore we can colour each G_j with $\max_j \omega(G_j)$ colours, and then make sure that the different colourings agree on the central clique C_0 . It is easy to see that $\omega(G) = \max_j \omega(G_j)$ and it follows that $\chi(G) = \omega(G)$, as required.

We next discuss some probability distributions on the set $\mathcal{G}_n$ of all graphs on $[n]$. Probably the most well-known is that of the binomial random graph $G(n, p)$, which assigns probability $p^{e(G)}(1-p)^{e(\overline{G})}$ to each graph G in $\mathcal{G}_n$. We are predominantly interested in the following probability distribution on $\mathcal{G}_n$:

$$\mathbb{P}_{\mathcal{P}_n}(G) = \begin{cases} 1/|\mathcal{P}_n| & \text{if } G \in \mathcal{P}_n \\ 0 & \text{otherwise.} \end{cases}$$

Here is another distribution on $\mathcal{G}_n$:

$$\mathbb{P}_{\mathcal{GS}_n}(G) = \begin{cases} 1/|\mathcal{GS}_n| & \text{if } G \in \mathcal{GS}_n \\ 0 & \text{otherwise.} \end{cases}$$

We rely heavily on Theorem 2.4 from [PS92], which for our purposes can be reformulated as follows:

$$|\mathcal{GS}_n| = (1 - e^{-\Omega(n)})|\mathcal{P}_n|. \quad (3.2)$$

In other words, the class of $\mathcal{GS}$ -graphs is a very good asymptotic approximation to the class of perfect graphs, as all $\mathcal{GS}$ -graphs are perfect and almost all perfect graphs are $\mathcal{GS}$ -graphs. The bound in (3.2) is sharp. Fix any graph $U \in \mathcal{P} \setminus \mathcal{GS}$, for example take the disjoint union of $K_{2,3}$ and K_3 , and consider the family $\mathcal{F}^U$ of the graphs that are a disjoint union of U and a perfect graph. We have $\mathcal{F}^U \subseteq \mathcal{P} \setminus \mathcal{GS}$, $|\mathcal{F}_n^U| \geq |\mathcal{P}_{n-v(U)}|$, and hence

$$\frac{|\mathcal{GS}_n|}{|\mathcal{P}_n|} = 1 - \frac{|\mathcal{P}_n \setminus \mathcal{GS}_n|}{|\mathcal{P}_n|} \geq 1 - \frac{|\mathcal{F}_n^U|}{|\mathcal{P}_n|} \geq 1 - \frac{|\mathcal{P}_{n-v(U)}|}{|\mathcal{P}_n|}.$$

To bound the ratio we need estimates on $|\mathcal{P}_n|$, for instance

$$|\mathcal{P}_n| = 2^{\frac{n^2}{4} + \frac{n}{2} \log n - \frac{n}{2} \log \ln n - \frac{n}{2} \log \frac{e}{2} + O\left(\frac{n}{\ln \ln(n+3)}\right)} \quad [\text{PS92}],$$

and we see that

$$|\mathcal{GS}_n|/|\mathcal{P}_n| \geq 1 - 2^{-\frac{nv(U)}{2} + o(n)}. \quad (3.3)$$

The combination of equations (3.2) and (3.3) yields

$$|\mathcal{GS}_n| = (1 - e^{-\Theta(n)})|\mathcal{P}_n|. \quad (3.4)$$

We can also express (3.4) using total variation distance:

$$\begin{aligned} d_{TV}(\mathbb{P}_{\mathcal{GS}_n}, \mathbb{P}_{\mathcal{P}_n}) &= \sup_{A \subseteq \mathcal{G}_n} |\mathbb{P}_{\mathcal{GS}_n}(A) - \mathbb{P}_{\mathcal{P}_n}(A)| = \mathbb{P}_{\mathcal{P}_n}(\mathcal{P}_n \setminus \mathcal{GS}_n) \\ &= \frac{|\mathcal{P}_n \setminus \mathcal{GS}_n|}{|\mathcal{P}_n|} = e^{-\Theta(n)}. \end{aligned} \quad (3.5)$$

How many n -vertex perfect graphs are there? The number is very close to the number of n -vertex $\mathcal{GS}$ -graphs, which is approximately twice the number of n -vertex unipolar graphs. The last class is easy to estimate: for each k between 0 and n there are $\binom{n}{k}$ ways to choose which k vertices to include in the central clique, B_{n-k} ways to specify the partition of the remaining vertices and $2^{k(n-k)}$ ways to specify the edges between the central clique and the remaining vertices. Here B_n stands for the n -th Bell number, which is the number of ways to partition an n -element set. Note that the last calculation is not precise, since unipolar graphs may be part of more than one unipolar arrangement. If we denote $\binom{n}{k} 2^{k(n-k)} B_{n-k}$ by $\ell_{n,k}$, we would expect that the number of unipolar graphs is close to $\mathcal{L}_n := \sum_{k=0}^n \ell_{n,k}$. This turns out to be a very precise estimate and we show in § 3.9 that

$$|\mathcal{GS}_n| = 2\mathcal{L}_n(1 - e^{-\Omega(n)}). \quad (3.6)$$

(See Lemma 2.2 of [PS92] for an asymptotic estimate of $|\mathcal{GS}_n|$.)

Definition 3.2.1. *For each n let $L(n)$ be the discrete integer-valued distribution with probability mass function*

$$p_{L(n)}(x) = \ell_{n,x} / \mathcal{L}_n \quad \text{for } x = 0, \dots, n.$$

3.2.1 The generation process

Now we describe our method to sample almost uniformly from the set of generalised split graphs on $[n]$.

Definition 3.2.2. For each n let $Gen(n)$ be a random quadruple $(B, E, (k, \sigma), \pi)$, such that

1. B is a $\{-1, 1\}$ -valued random variable, taking each value with probability half,
2. E is the set of edges of an Erdős-Rényi random graph $G(n, \frac{1}{2})$,
3. (k, σ) is the result of the following random experiment: choose a number $k \in [0, n]$ with distribution $L(n)$, set $W = \{k+1, \dots, n\}$, and then sample σ uniformly at random from $\Pi(W)$, the set of all partitions of W ,
4. π is a uniformly sampled permutation from $Sym([n])$,
5. the components of the quadruple are independent.

Let ρ be the (deterministic) map from such quadruples to $\mathcal{G}_n$ defined as follows. Let $U = (B, E, (k, \sigma), \pi)$ be an outcome of $Gen(n)$. Start with an n -vertex empty graph $G_0 = ([n], \emptyset)$. It is convenient to write $C = [k]$ and $S = V(G) \setminus C$. Create G_1 by adding edges to G_0 , so that C induces a clique, and S induces a disjoint union of cliques corresponding to σ , i.e. $ij \in E(G_1)$ for $1 \leq i < j \leq k$; and $ij \in E(G_1)$ whenever i and j are contained in the same part of σ for $k < i < j \leq n$. Let G_2 be obtained from G_1 by adding the edges $E \cap (C \times S)$, and let G_3 be G_2 if $B = 1$ and $\overline{G_2}$ otherwise. Finally let G_4 be obtained by permuting the vertices of G_3 according to π , i.e. $\pi(i)\pi(j) \in E(G_4)$ iff $ij \in E(G_3)$. We define $\rho(U) = G_4$ for each such vector U .

Consider $U = (B, E, (k, \sigma), \pi) \sim Gen(n)$ and the corresponding random graph $G = \rho(U)$. It is easy to see that G is unipolar if $B = 1$ and co-unipolar if $B = -1$. Indeed, $(G, \pi(C))$ is a unipolar arrangement if $B = 1$ and $(\overline{G}, \pi(C))$ is a unipolar arrangement otherwise. We often refer to the appropriate pair as the *induced arrangement* of $\rho(U)$ by the generation. Whenever we want to test if G has some isomorphism-closed graph property, π does not affect the membership, so we can assume that π is the identity permutation. All graph properties in which we are interested are isomorphism-closed, so π will largely be ignored.

We define our last probability distribution, $\mathbb{P}_{Gen(n)}$ on $\mathcal{G}_n$, as follows:

$$\mathbb{P}_{Gen(n)}(G) = \mathbb{P}(\rho(Gen(n)) = G).$$

By the above, $\mathbb{P}_{Gen(n)}(G) > 0$ if and only if $G \in \mathcal{GS}_n$. We shall show that

$$d_{TV}(\mathbb{P}_{\mathcal{GS}_n}, \mathbb{P}_{Gen(n)}) = e^{-\Omega(n)}. \quad (3.7)$$

It is not hard to prove (3.7) from (3.6). However, both equations have technical proofs, so they are postponed to § 3.9.

Theorem 3.2.3. *The uniform measure $\mathbb{P}_{\mathcal{P}_n}$ and the generated measure $\mathbb{P}_{Gen(n)}$ satisfy*

$$d_{TV}(\mathbb{P}_{\mathcal{P}_n}, \mathbb{P}_{Gen(n)}) = e^{-\Theta(n)}.$$

Proof. From the triangle inequality for total variation distance, (3.5) and (3.7), we have

$$d_{TV}(\mathbb{P}_{\mathcal{P}_n}, \mathbb{P}_{Gen(n)}) \leq d_{TV}(\mathbb{P}_{\mathcal{P}_n}, \mathbb{P}_{\mathcal{GS}_n}) + d_{TV}(\mathbb{P}_{\mathcal{GS}_n}, \mathbb{P}_{Gen(n)}) = e^{-\Theta(n)}.$$

A lower bound follows from arguments similar to these for (3.4). $\square$

Theorem 3.2.3 describes the precision of our almost-uniform sampling. Note that any approach relying on (3.4) cannot hope to achieve better precision. The proof is rather technical, and is postponed to Section 3.9. The properties of the generated graph $\rho((B, E, (k, \sigma), \pi))$ depend a lot on the value of B . It is often the case that we break the analysis into different cases depending on the value of B . We introduce two new random 4-tuples - $Gen^+(n)$ and $Gen^-(n)$, which are the same as $Gen(n)$, but with overwritten values for B :

Definition 3.2.4. *For each n define $Gen^+(n)$ to be $(1, E, (k, \sigma), \pi)$ and $Gen^-(n)$ to be $(-1, E, (k, \sigma), \pi)$, where E , (k, σ) and π are the same as in the definition of $Gen(n)$.*

The following observation is easily verified: Let Q be an arbitrary graph property. If $G \sim \rho(Gen(n))$, $G^+ \sim \rho(Gen^+(n))$ and $G^- \sim \rho(Gen^-(n))$, then

$$\mathbb{P}(G \in Q) = \frac{1}{2}\mathbb{P}(G^+ \in Q) + \frac{1}{2}\mathbb{P}(G^- \in Q). \quad (3.8)$$

Still with $G \sim \rho(Gen(n))$, we shall see in Section 3.9 that wvhp G is not both unipolar and co-unipolar, that is

$$\mathbb{P}(G \in \mathcal{GS}^+ \cap \mathcal{GS}^-) = e^{-\Omega(n)}. \quad (3.9)$$

3.2.2 On the parts of the generation process

Theorem 3.2.3 will prove to be very useful. Let $P_n \in_u \mathcal{P}_n$. Whenever we want to estimate $q_n = \mathbb{P}_{\mathcal{P}_n}(Q) = \mathbb{P}(P_n \in Q)$ for some graph property Q , we can instead estimate $q'_n = \mathbb{P}_{Gen(n)}(Q) = \mathbb{P}(\rho(Gen(n)) \in Q)$, and use the fact that $|q_n - q'_n| = e^{-\Omega(n)}$. The approximate sampling, $Gen(n)$, is a convenient method for estimating probability, because it decomposes the resulting graph into three independent parts – a central clique, a partition of the remaining vertices and a bipartite part in between.

3.2.2.1 Central clique

An essential part of the generation process involves determining the size of the central clique. For this purpose we introduced the distribution $L(n)$. Recall that if $X \sim L(n)$ and x is an integer between 0 and n , then $\mathbb{P}(X = x) = \ell_{n,x}/\mathcal{L}_n$, where $\ell_{n,x} = \binom{n}{x} 2^{x(n-x)} B_{n-x}$ and $\mathcal{L}_n = \sum_{k=0}^n \ell_{n,k}$. The term $2^{x(n-x)}$ dwarfs the other two terms and heavily concentrates the distribution. The term B_{n-x} breaks the symmetry around $n/2$ and tilts the mean towards zero by about $\frac{1}{2} \log n$, where throughout the chapter $\log$ means $\log_2$.

Theorem 3.2.5. *Suppose $X \sim L(n)$. For sufficiently large n we have*

$$2^{-(x+1)^2-1} \leq \mathbb{P}(|X - \mu| \geq x) \leq 2^{-(x-2)^2+2} + n^{-n},$$

for each $x > 1$, where

$$\mu = \mu(n) = \frac{n - \log n + \log \ln n}{2}.$$

The upper bound in the theorem above is essential for handling the generation process. Indeed, Theorem 3.2.3 will always be used in combination with Theorem 3.2.5. Most often we take $x = \Omega(\sqrt{n})$ in order to have an exponential decay. However, sometimes there is a phase transition when the central clique becomes bigger than the remaining graph, most notably when we look at Hamiltonian cycles in co-unipolar graphs, and then we take x to be $n/2 - \mu \sim \frac{1}{2} \log n$. The proof of Theorem 3.2.5 is rather technical, and is postponed to § 3.9.3. In the proof we also show that if $X \sim L(n)$, then $|\mathbb{E}X - \mu(n)| < 1$.

3.2.2.2 Bipartite part

Let $U \sim \text{Gen}(n)$ and let (G, C) be the arrangement induced by the generation process. The edges between C and $\overline{C}$ are best seen as a random bipartite graph with colour classes C and $\overline{C}$ and edges in between, each present with probability $1/2$ and independently of the others. The sizes of C and $\overline{C}$ are not necessarily identical, but Theorem 3.2.5 shows that their difference is $O(\sqrt{n})$ wvhp. Random bipartite graphs share many properties with binomial random graphs $G(n, p)$, which have been exhaustively studied.

3.2.2.3 Partition part

Let Π_n be the set of all partitions of $[n]$. The elements of each partition will be referred to as *parts*. We turn Π_n into a probability space by taking each partition with equal probability, namely $1/|\Pi_n|$. Let $\sigma = \sigma(n)$ be an outcome from this probability space, i.e. $\sigma \in_u \Pi_n$. The structure of σ can vary from having a single part with n elements

to having n parts with one element each. However, σ has a well-defined shape with high probability.

Let $r = r(n)$ be the unique root of the equation $re^r = n$ and let $B_n = |\Pi_n|$ be the n th Bell number. It can be found in [DB70] that

$$r(n) = \ln n - (1 + o(1)) \ln \ln n, \quad \text{and} \quad (3.10)$$

$$B_n = \frac{1 + o(1)}{\sqrt{r}} e^{n(r-1+1/r)-1}. \quad (3.11)$$

There is a sizeable literature for random partitions, see in particular [Pit97]. Let $|\pi|$ be the number of parts in a partition π . If X is a random variable we will use $\widehat{X}$ to denote $(X - \mathbb{E}X)/\sqrt{\text{Var}X}$. One of the earliest results on random partitions is by Harper:

Theorem 3.2.6. (Harper [Har66]) For $\sigma \in_u \Pi_n$

1. $|\widehat{\sigma}| \xrightarrow{d} N(0, 1)$,
2. $\mathbb{E}|\sigma| = \frac{B_{n+1}}{B_n} - 1 = (1 + o(1)) \frac{n}{r} = (1 + o(1)) \frac{n}{\ln n}$,
3. $\text{Var}|\sigma| = \frac{B_{n+2}}{B_n} - \left(\frac{B_{n+1}}{B_n}\right)^2 - 1 = \frac{n}{r(r+1)} - 1 + o(1) = (1 + o(1)) \frac{n}{(\ln n)^2}$.

Let $Y_t(\pi)$ be the number of parts of size t of an arbitrary partition $\pi \in \Pi_n$, so that $|\pi| = \sum_{t=1}^n Y_t(\pi)$ and $n = \sum_{t=1}^n tY_t(\pi)$. Suppose that $\sigma \in_u \Pi_n$ from now on.

Theorem 3.2.7. (Pittel [Pit97]) Given a partition π and an interval $I = [k_1, k_2]$ where $1 \leq k_1 \leq k_2 \leq n$, let

$$Y_I(\pi) = \sum_{j \in I} Y_j(\pi), \quad \lambda_I = \sum_{j \in I} \frac{r^j}{j!}.$$

Then, uniformly for $I = [k_1, k_2]$ and $\xi \geq 0$,

$$\mathbb{P}(|Y_I(\sigma) - \lambda_I| \geq \xi) = O(e^{-\min(\frac{\xi^2}{4\lambda_I}, \frac{\xi}{2})}).$$

The last theorem is useful for small deviations, but rather crude for deviations proportional to the mean. A counting argument yields a better bound:

Lemma 3.2.8. Let $\epsilon > 0$ be fixed and let $\lambda = n/r = e^r$. Then for large n

$$\mathbb{P}(|\sigma| - \lambda \geq \epsilon\lambda) < ne^{-n(\epsilon - \ln(\epsilon+1))}.$$

In particular, $||\sigma| - \lambda| < \epsilon\lambda$ wvhp.

Proof. Any surjective function $f : [n] \rightarrow [k]$ induces a partition $\{f^{-1}[i]\}_{i=1}^k$ with exactly k parts. Every partition with k parts is induced by exactly $k!$ functions of the form above, as permuting the values of the image does not affect the partition. We conclude:

$$\begin{aligned}\mathbb{P}(|\sigma| = k)B_n &= |\{\pi \in \Pi_n \text{ s.t. } |\pi| = k\}| \\ &= |\{f : [n] \rightarrow [k] \text{ s.t. } f \text{ is surjective}\}|/k! \\ &\leq |\{f : [n] \rightarrow [k]\}|/k! = k^n/k! \\ &\leq k^n \left(\frac{k}{e}\right)^{-k} = \exp(n \ln k - k \ln k + k),\end{aligned}$$

for each $0 \leq k \leq n$.

Suppose that $k = (1 + c)\lambda$ for some $c = c(n) > -1$. Substituting in the inequality above we get:

$$\begin{aligned}\mathbb{P}(|\sigma| = (1 + c)\lambda) &\leq B_n^{-1} \exp(n \ln(1 + c)\lambda - (1 + c)\lambda \ln(1 + c)\lambda + (1 + c)\lambda) \\ &= B_n^{-1} \exp\{ n \ln \lambda - \lambda \ln \lambda + \lambda \\ &\quad + n \ln(1 + c) - c\lambda \ln \lambda \\ &\quad + \lambda(c - \ln(1 + c) - c \ln(1 + c)) \} \\ &\leq \exp(n\{\ln(1 + c) - c\} + \lambda\{c - \ln(1 + c) - c \ln(1 + c)\})\end{aligned}$$

for sufficiently large n . Here we have used $\lambda \ln \lambda = n$; and that (3.11) implies that we have

$$B_n^{-1} \exp(n(r - 1 + 1/r)) < 1 \quad \text{for sufficiently large } n.$$

Note that $c - \ln(1 + c) - c \ln(1 + c) \leq 0$ for all $c > -1$, so we can write

$$\mathbb{P}(|\sigma| = (1 + c)\lambda) \leq \exp(n\{\ln(1 + c) - c\})$$

for large n . Let $f(x) : (-1, +\infty) \rightarrow \mathbb{R}$ be defined as $f(x) = \ln(1 + x) - x$, so that

$$\mathbb{P}(|\sigma| = (1 + c)\lambda) \leq \exp(nf(c)).$$

The following properties are easily verified by taking derivatives: $f(x) \leq f(|x|)$; $f(0) = 0$; $f(x) < 0$ for $x \neq 0$; and f is strictly increasing in $(-1, 0]$ and strictly decreasing in $[0, \infty)$. We deduce that for every $I \subseteq (-1, \infty]$ we have $\sup_{x \in I} f(x) \leq f(\inf_{x \in I} |x|)$. Let $I_\delta = [0, n] \setminus [\lfloor (1 - \delta)\lambda \rfloor, \lceil (1 + \delta)\lambda \rceil]$, for $\delta > 0$. We deduce

$$\mathbb{P}(|\sigma| - \lambda \geq \epsilon\lambda) = \mathbb{P}(|\sigma| \in I_\epsilon) \leq \max_{k \in I_\epsilon} \mathbb{P}(|\sigma| = k) < n \exp\{f(\epsilon)n\}. \quad \square$$

For $\pi \in \Pi_n$ let $L(\pi)$ be the maximum size of a part.

Lemma 3.2.9. *Suppose $0 \leq x \leq n/2$ and let $\sigma \in_u \Pi_n$. We have*

$$\mathbb{P}(L(\sigma) \geq x) < (1 + o(1))e^{-x(\ln x - \ln r - 2) + \ln n}.$$

Proof. We may assume that $x \geq e^2 r$ (since otherwise the RHS is ≥ 1). Let E_x be the event that σ contains a part of size $\lfloor x \rfloor$. Then by (3.11) and noting that $r_{n-x} - 1 + \frac{1}{r_{n-x}} \leq r_n - 1 + \frac{1}{r_n}$ (this holds because $f(t) = t - 1 + t^{-1}$ and r_t are increasing in $t \in [1, \infty)$) we get

$$\begin{aligned} \mathbb{P}(E_x) &\leq \binom{n}{\lfloor x \rfloor} \frac{B_{n-\lfloor x \rfloor}}{B_n} \leq \left(\frac{en}{x}\right)^x (1 + o(1)) \sqrt{\frac{r_n}{r_{n-x}}} e^{-x(r_n - 1 + 1/r_n)} \\ &\leq (1 + o(1)) \left(\frac{en}{x}\right)^x \left(\frac{n}{er_n}\right)^{-x} \\ &= (1 + o(1)) e^{-x(\ln x - \ln r - 2)}. \end{aligned}$$

Thus

$$\begin{aligned} \mathbb{P}(L(\sigma) \geq x) &= \mathbb{P}\left(\bigcup_{z \geq x} E_z\right) \leq \sum_{z \geq x} \mathbb{P}(E_z) < n \max_{z \geq x} \mathbb{P}(E_z) \\ &\leq (1 + o(1)) e^{-x(\ln x - \ln r - 2) + \ln n}, \end{aligned}$$

since $\ln x - \ln r - 2 \geq 0$. □

Lemma 3.2.10. *Suppose $a = a(n) = o(n)$ is integral for all n . Then*

$$\frac{B_{n-a}}{B_n} = \exp(-ar_{n-a} + o(n)).$$

Proof. We have

$$\begin{aligned} \frac{B_{n-a}}{B_n} &= (1 + o(1)) \exp\left((n-a) \left(r_{n-a} - 1 + \frac{1}{r_{n-a}}\right) - n \left(r_n - 1 + \frac{1}{r_n}\right)\right) \\ &= \exp((1 + o(1))n \ln(1 - a/n) - a(r_{n-a} - 1) + o(n)) \\ &= \exp((1 + o(1))(-a) - a(r_{n-a} - 1) + o(n)) \\ &= \exp(-ar_{n-a} + o(n)). \end{aligned} \quad \square$$

Corollary 3.2.11. *We have*

$$\mathbb{P}(L(\sigma) \geq n/\log n) = e^{-\Theta(n)}. \quad (3.12)$$

and further, if $\epsilon(n) = o(1)$, then

$$\mathbb{P}(L(\sigma) \geq \epsilon(n)n/\log n) = e^{-o(n)}. \quad (3.13)$$

Proof. Let $a = \lceil n/\log n \rceil$. Then $\mathbb{P}(L(\sigma) \geq n/\log n)$ is at least $\frac{B_{n-a}}{B_n}$; and is at most $\binom{n}{a} n \frac{B_{n-a}}{B_n}$ (where the factor n is to account for choosing a part (or the empty set) to add to the chosen part of size a). Now we can use the last lemma. □

To close this section on generating random perfect graphs, let us briefly consider the graph G output by the generation process, and how to recover a unipolar arrangement for G or its complement (or how to seek such arrangements for the random perfect graph P_n). On input an arbitrary n -vertex graph H , in $O(n^2)$ -time we can test if H is unipolar or co-unipolar, and if it is either then output a corresponding vertex partition. This result is a joint work with Colin McDiarmid [MY15] from my master's thesis, improving on earlier $O(n^3)$ -time approaches. The method is not very complicated, but in the present case there is a trivial $O(n^2)$ -time heuristic.

Consider G . Let A be the set of vertices with degree at most $n/2$, and let B consist of the remaining vertices. Then wvhp exactly one of the following holds: (a) B is a clique and A is a disjoint union of cliques (forming a unipolar arrangement for G), or (b) A is stable and B is a disjoint union of stable sets (forming a unipolar arrangement for $\overline{G}$).

To check this, let $0 < \epsilon < 1/4$. We have seen that, in the generation process, wvhp $|k - n/2| < \epsilon n/2$ (by Theorem 3.2.5) and each part in σ has size $< \epsilon n/2$ (by Lemma 3.2.9). Hence, in the unipolar case ($B = 1$), by standard Chernoff bounds, wvhp each vertex in the central clique has degree in $(3n/4 - \epsilon n, 3n/4 + \epsilon n)$ and each other vertex has degree in $(n/4 - \epsilon n, n/4 + \epsilon n)$, so (a) holds. Similarly, in the co-unipolar case, wvhp each vertex in the central stable set has degree in $(n/4 - \epsilon n, n/4 + \epsilon n)$ and each other vertex has degree in $(3n/4 - \epsilon n, 3n/4 + \epsilon n)$, so (b) holds. Also, by (3.9) wvhp (a) and (b) cannot both hold.

3.3 Stability and Clique number

In this section we discuss the stability number $\alpha(P_n)$ and clique number $\omega(P_n)$ of the random perfect graph P_n . Our main result is Theorem 3.3.7. Of course everything we say also applies to the clique covering number and chromatic number, as they equal the former two parameters for perfect graphs.

Lemma 3.3.1. *Let $U = (1, E, (k, \sigma), \pi)$ be an arbitrary outcome of $\text{Gen}^+(n)$ and let $G^+ = \rho(U)$. Then $0 \leq \alpha(G^+) - |\sigma| \leq 1$.*

Proof. Let (G^+, C) be the induced unipolar arrangement from the generation. We know that σ is a partition of $\overline{C}$ into disjoint cliques, and therefore by picking a representative member from each part we obtain an independent set of size $|\sigma|$, hence $\alpha(G^+) \geq |\sigma|$. For the other direction, observe C together with σ gives a partition of $V(G^+)$ into $|\sigma| + 1$ cliques. Every stable set can intersect each of these cliques at most once, hence $\alpha(G^+) \leq |\sigma| + 1$. $\square$

The last lemma pins down $\alpha(G^+)$ to only two values, namely $|\sigma|$ and $|\sigma| + 1$. We next show that the second outcome is very unlikely.

Lemma 3.3.2. *Let $U = (1, E, (k, \sigma), \pi) \sim \text{Gen}^+(n)$, let $G^+ = \rho(U)$, and let $0 < \delta < 1 - \ln 2 \approx 0.3$. Then $\mathbb{P}(\alpha(G^+) = |\sigma| + 1) = O(e^{-n^\delta})$.*

We give an informal proof first. Condition on the induced unipolar arrangement (G^+, C) and on the partition σ . We have $\alpha(G^+) = |\sigma| + 1$ if and only if there is a vertex from C whose neighbourhood does not contain any side clique in $\overline{C}$. Let A_v be the event that v is such a vertex, and hence $\{\alpha(G^+) = |\sigma| + 1\} \equiv \bigcup_v A_v$. For a fixed vertex $u \in C$ and a fixed side clique R we have $\mathbb{P}(R \not\subseteq N(u)) = 1 - 2^{-|R|}$. Therefore

$$\mathbb{P}(A_v) = \prod_R (1 - 2^{-|R|}),$$

where that product is taken over the side cliques in $\overline{C}$. Up until now everything is rigorous. Observe from Theorem 3.2.7 that whp the number of side cliques of size i is close to $r^i/i!$, where r is the unique root of the equation $n - k = re^r$. For a moment assume that the number of side cliques of size i equals this value, for each i . Then

$$\begin{aligned} \mathbb{P}(A_v) &= \prod_{i=1}^n \prod_{R: |R|=i} (1 - 2^{-i}) = \prod_{i=1}^n (1 - 2^{-i})^{r^i/i!} \\ &\leq \prod_{i=1}^n \left(e^{-2^{-i}} \right)^{r^i/i!} \quad \text{since } 1 + x \leq e^x \\ &= \exp \left(- \sum_{i=1}^n (r/2)^i / i! \right) \leq \exp(-e^{r/2} + 2) \\ &= \exp \left(-\sqrt{(n-k)/r} + 2 \right) = \exp \left(-(1 + o(1))\sqrt{n/(2 \ln n)} \right). \end{aligned}$$

For the last step we assumed that $k \sim n/2$, which is easily justifiable from the concentration theorem, Theorem 3.2.5. We use the union bound to finalise our arguments:

$$\begin{aligned} \mathbb{P}(\alpha(G^+) = |\sigma| + 1) &\leq \sum_v \mathbb{P}(A_v) \leq \exp \left(-(1 + o(1))\sqrt{n/(2 \ln n)} \right) \\ &= O(\exp(-n^{1/2-\epsilon})). \end{aligned}$$

We observe that δ from the statement is $1/2 - \epsilon$ in this case, which may be the best possible.

Proof of Lemma 3.3.2. Let $\mathcal{J} = [n/2 - 2\sqrt{n}, n/2 + 1 + 2\sqrt{n}]$. We have $\mathbb{P}(k \in \mathcal{J}) \geq 1 - 2^{-n}$ from Theorem 3.2.5. Condition σ on $k = k_0$ for some $k_0 \in \mathcal{J}$. We use Theorem 3.2.7 for σ with $I = [0, r + r^{3/4}]$, $\lambda_I = e^r \mathbb{P}[Po(r) \leq r + r^{3/4}] \leq e^r$ and $\xi = 2(e^r)^{3/4}$, where r is the unique root of $n - k_0 = re^r$. The theorem says that

$$\mathbb{P}(Y_I(\sigma) \geq \lambda_I - \xi \mid k = k_0) \geq 1 - c \exp(-\sqrt{e^r}),$$

for some constant c . In order to give a lower bound on λ_I , we use a standard Chernoff bound for a Poisson random variable: $\mathbb{P}(Po(r) \geq (1 + \eta)r) < \exp(-r\eta^2/4)$, for $0 < \eta \leq 1$. Taking $\eta = r^{-1/4}$ we obtain:

$$\lambda_I - \xi > e^r(1 - \exp(-\sqrt{r}/4) - 2(e^r)^{-1/4}) = e^r(1 - o(1)).$$

Define the events A_v as in the preceding informal discussion. The event $\{Y_I(\sigma) \geq \lambda_I - \xi\}$ implies that in σ there are at least $e^r(1 + o(1))$ parts of size at most $r(1 + o(1))$, hence

$$\begin{aligned} \mathbb{P}(A_v | (Y_I(\sigma) \geq \lambda_I - \xi) \cap k = k_0) &\leq (1 - 2^{-r(1+o(1))})^{e^r(1+o(1))} \\ &\leq \exp\{-2^{-r(1+o(1))}e^r(1+o(1))\} \\ &= \exp\{-(e/2)^{r(1+o(1))}\} \\ &= \exp\{-\exp[r \cdot (1 - \ln 2)(1 + o(1))]\} \\ &= \exp\{((n - k_0)/r)^{(1 - \ln 2)(1+o(1))}\} \quad \text{since } e^r = (n - k_0)/r \\ &= O(\exp(-n^\delta)). \end{aligned}$$

From the union bound we get

$$\begin{aligned} \mathbb{P}(\alpha(G^+) = |\sigma| + 1 | k = k_0) &\leq \mathbb{P}(Y_I(\sigma) < \lambda_I - \xi | k = k_0) \\ &\quad + \sum_v \mathbb{P}(A_v | (Y_I(\sigma) \geq \lambda_I - \xi) \cap (k = k_0)) \\ &= O(\exp(-n^\delta)). \end{aligned}$$

Hence

$$\begin{aligned} \mathbb{P}(\alpha(G^+) = |\sigma| + 1) &\leq \mathbb{P}(k \notin \mathcal{J}) + \max_{k_0 \in \mathcal{J}} \mathbb{P}(\alpha(G^+) = |\sigma| | k = k_0) \\ &= O(\exp(-n^\delta)). \end{aligned} \quad \square$$

Note that the bound for δ in the last proof is $1 - \ln 2 \approx 0.3$, which is slightly worse than the constant in the informal discussion.

We shorten $G^+ \sim \rho(\text{Gen}^+(n))$ to $G^+ \sim \text{Gen}^+(n)$.

Proposition 3.3.3. *Let $G^+ \sim \text{Gen}^+(n)$, and denote $\alpha(G^+)$ by X . Then, as $n \rightarrow \infty$, $\widehat{X} \xrightarrow{d} N(0, 1)$, $\mathbb{E}[X] \sim \frac{n}{2 \ln n}$, and $\text{Var}[X] \sim \frac{n}{2 \ln^2 n}$.*

We write Y for $|\sigma|$ and Y_m for $|\sigma_m|$ when $\sigma_m \in_u \Pi_m$. To prove the last result we will use the following claim.

Claim 3.3.4. $\mathbb{E}Y_{n+1} = \mathbb{E}Y_n + \frac{1+o(1)}{\ln n}$.

Proof. From Theorem 3.2.6, $\mathbb{E}Y_n = \frac{B_{n+1}}{B_n} - 1$ and $\text{Var}Y_n = \frac{B_{n+2}}{B_n} - \left(\frac{B_{n+1}}{B_n}\right)^2 - 1$. Hence

$$\text{Var}Y_n = \frac{B_{n+2}}{B_{n+1}} \frac{B_{n+1}}{B_n} - \left(\frac{B_{n+1}}{B_n}\right)^2 - 1 = (\mathbb{E}Y_{n+1} + 1)(\mathbb{E}Y_n + 1) - (\mathbb{E}Y_n + 1)^2 - 1.$$

Therefore

$$(\mathbb{E}Y_{n+1} + 1) - (\mathbb{E}Y_n + 1) = \frac{\text{Var}Y_n + 1}{\mathbb{E}Y_n + 1} = \frac{1 + o(1)}{\ln n}. \quad \square$$

Proof of Proposition 3.3.3. Recall that σ is a random partition created by first choosing a value for $k \sim L(n)$, and then taking $\sigma \in_u \Pi_{n-k}$. It suffices to show that $\hat{Y} \xrightarrow{d} N(0, 1)$. Indeed, if this were the case we get $Y \leq X \leq Y + 1$, $\text{Var}(Y) \rightarrow \infty$, $\hat{Y} \xrightarrow{d} N(0, 1)$, and hence $\hat{X} \xrightarrow{d} N(0, 1)$.

Let $\Phi(x)$ be the CDF of $N(0, 1)$. For an arbitrary fixed x and $\epsilon > 0$ we show that for large n

$$|\mathbb{P}(\hat{Y} < x) - \Phi(x)| < \epsilon.$$

Find $\ell = \ell(\epsilon)$ such that $\mathbb{P}(k \notin \mathcal{I}) < \epsilon/2 + n^{-n}$ for $\mathcal{I} = \mathcal{I}(n) = [\lfloor \mathbb{E}k - \ell \rfloor, \lceil \mathbb{E}k + \ell \rceil]$. Such ℓ exists by Theorem 3.2.5. We emphasise that the length of $\mathcal{I}$ is at most $2\ell + 1$, which is independent of n .

$$\begin{aligned} |\mathbb{P}(\hat{Y} < x) - \Phi(x)| &= \sum_{c=0}^n \left| (\mathbb{P}(\hat{Y} < x | k = c) - \Phi(x)) \mathbb{P}(k = c) \right| \\ &< \sum_{c \in \mathcal{I}} \left| (\mathbb{P}(\hat{Y} < x | k = c) - \Phi(x)) \right| \mathbb{P}(k = c) + \epsilon/2 + n^{-n} \\ &\leq \max_{c \in \mathcal{I}} \left| \mathbb{P}(\hat{Y} < x | k = c) - \Phi(x) \right| + \epsilon/2 + n^{-n}. \end{aligned}$$

Fix any integer valued function $q(n)$ with $q(n) \in \mathcal{I}(n)$.

$$\begin{aligned} |\mathbb{E}Y - \mathbb{E}Y_{n-q}| &= |\mathbb{E} \{ \mathbb{E}[Y | k] \} - \mathbb{E}Y_{n-q}| \\ &\leq \sum_{i=0}^n |\mathbb{E}Y_{n-i} - \mathbb{E}Y_{n-q}| \mathbb{P}(k = i) \\ &\leq \sum_{i \in \mathcal{I}} |\mathbb{E}Y_{n-i} - \mathbb{E}Y_{n-q}| \mathbb{P}(k = i) + \epsilon/2 + n^{-n} \\ &\leq \max_{i \in \mathcal{I}} |\mathbb{E}Y_{n-i} - \mathbb{E}Y_{n-q}| + \epsilon/2 + n^{-n} \\ &\leq (1 + o(1)) \frac{2\ell + 1}{2 \ln n} + \epsilon/2 + n^{-n} \\ &= \epsilon/2 + o(1). \end{aligned} \tag{3.14}$$

We proceed with the variance in a similar fashion.

$$|\text{Var}Y - \text{Var}Y_{n-q}| \leq \max_{i \in \mathcal{I}} |\text{Var}Y_{n-i} - \text{Var}Y_{n-q}| + \epsilon/2 + n^{-n} = o(\text{Var}Y_{n-q}).$$

For the last equality we have used part 3 of Theorem 3.2.6. We conclude

$$\text{Var}Y = (1 + o(1))\text{Var}Y_{n-q}. \quad (3.15)$$

Write $y = (x(\text{Var}Y)^{1/2} + \mathbb{E}Y - \mathbb{E}Y_{n-q})(\text{Var}Y_{n-q})^{-1/2}$, so that $\mathbb{P}(\widehat{Y} < x | k = q) = \mathbb{P}(\widehat{Y}_{n-q} < y)$. From (3.14), (3.15) and Theorem 3.2.6 we have $\lim_{n \rightarrow \infty} y = x$. Find $\delta > 0$, so that $\Phi(x) \leq \Phi(x - \delta) + \epsilon/6$. From Theorem 3.2.6 we have $\mathbb{P}(\widehat{Y}_{n-q} < x - \delta) \rightarrow \Phi(x - \delta)$ if $n - q \rightarrow \infty$. For large n

$$\mathbb{P}(\widehat{Y}_{n-q} < y) \geq \mathbb{P}(\widehat{Y}_{n-q} < x - \delta) \geq \Phi(x - \delta) - \epsilon/6 \geq \Phi(x) - \epsilon/3.$$

We can do the same in the other direction and deduce that for large n

$$|\mathbb{P}(\widehat{Y} < x | k = q) - \Phi(x)| < \epsilon/3.$$

But q was an arbitrary function, hence for large n

$$\max_{c \in \mathcal{I}} |\mathbb{P}(\widehat{Y} < x | k = c) - \Phi(x)| < \epsilon/3,$$

and the proposition is proven. $\square$

We continue with $\omega(G^+)$. The overview is that usually there is a unique clique of maximum size in G^+ – the central clique. The vertices in the central clique have degree $(1 + o(1))3n/4$ whp, while the vertices outside have degree $(1 + o(1))n/4$ whp, so the central clique C from the generation is easily recognisable in the resulting graph (and in the 4-tuple U , as well as k and B , we know σ up to relabelling of $\overline{C}$, we know a little about E and we know $\pi([k]) = C$).

Lemma 3.3.5. *Suppose $G^+ = \rho(U)$, where $U = (B, E, (k, \sigma), \pi) \sim \text{Gen}^+(n)$. Then both the probability that C is not the unique clique of maximum size and $\mathbb{P}(\omega(G^+) \neq k)$ are $2^{-(\frac{1}{2} + o(1))n}$.*

Proof. As usual, let $\mathcal{I} = [n/2 - \sqrt{n}, n/2 + \sqrt{n}]$, and recall that $k \in \mathcal{I}$ wvhp by Theorem 3.2.5. Let $k_0 \in \mathcal{I}$, and condition on $k = k_0$.

We first give a lower bound for $\mathbb{P}(\omega(G^+) \neq k_0)$. With probability at least 2^{-k_0} there is a vertex outside the central clique which is adjacent to all vertices from the central clique, and hence $\mathbb{P}(\omega(G^+) > k_0) \geq 2^{-(\frac{1}{2} + o(1))n}$.

We now prove an upper bound. Of course, $\mathbb{P}(\omega(G^+) \neq k)$ is at most the probability that C is not the unique clique of maximum size: we upper bound the latter probability. Let (G^+, C) be the unipolar arrangement induced by U . There are two cases in which G^+ contains a clique $M \neq C$ of size at least k_0 . The first case is when $|\overline{C} \cap M| = 1$, when also $|C \setminus M| \leq 1$. The probability of that happening is at most $(n - k_0)k_0 2^{-(k_0 - 1)} = 2^{-(\frac{1}{2} + o(1))n}$.

Let us focus on the second case; $|\overline{C} \cap M| > 1$. This implies that there are at least two vertices from $\overline{C}$ with degree at least k_0 . If we substitute $n/\sqrt{\log n}$ for x in Lemma 3.2.9, we get that the probability that there is a part in σ of size more than $n/\sqrt{\log n}$ is less than e^{-n} for large n . Assume there is no such part in σ .

For every $v \in \overline{C}$ write $d(v) = d_C(v) + d_{\overline{C}}(v)$. We have $d_C(v) \sim \text{Bin}(k_0, 1/2)$ and $d_{\overline{C}}(v) \leq n/\sqrt{\log n} - 1$. Let E_v be the event $\{d_C(v) \geq k_0 - n/\sqrt{\log n}\}$. Next we use the Chernoff bound in the form: $\mathbb{P}(X \geq mp + t) \leq e^{-2t^2/m}$, for $X \sim \text{Bin}(m, p)$ and $t \geq 0$. We obtain

$$\mathbb{P}(E_v) \leq e^{-(1+o(1))k_0/2}.$$

The events E_v and E_u are independent for $u \neq v$, hence $\mathbb{P}(E_v \cap E_u) \leq e^{-(1+o(1))k_0}$ for every pair $u \neq v$. We deduce that the probability that there are two vertices from $\overline{C}$ with degree at least k_0 is at most $n^2 e^{-(1+o(1))k_0} + e^{-n} = e^{-(\frac{1}{2}+o(1))n}$.

Combining the two cases completes the proof of the upper bound. $\square$

Lemma 3.3.6. *For $G^+ \sim \text{Gen}^+(n)$ and a fixed $\epsilon > 0$ we have wvhp*

$$(1 - \epsilon) \frac{n}{2 \ln n} < \alpha(G^+) < (1 + \epsilon) \frac{n}{2 \ln n}, \text{ and} \\ \frac{n}{2} - \epsilon \sqrt{n} < \omega(G^+) < \frac{n}{2} + \epsilon \sqrt{n}.$$

In particular, $\alpha(G^+) < \omega(G^+)$ wvhp.

Proof. The first line follows from Lemma 3.3.1, Lemma 3.2.8 and Theorem 3.2.5; the second line follows from Theorem 3.2.5 and Lemma 3.3.5. $\square$

Observe that the proof of Theorem 3.2.5 will complete the proof of Lemma 3.3.6. For every graph G we have $\omega(G) = \alpha(\overline{G})$ and therefore everything we showed for $\alpha(G^+)$ and $\omega(G^+)$ directly translates to $\omega(G^-)$ and $\alpha(G^-)$. Let $h(G) = \min\{\alpha(G), \omega(G)\}$ and $H(G) = \max\{\alpha(G), \omega(G)\}$ for every graph G . (Here h and H come from homogeneous.)

Theorem 3.3.7. *For $P_n \in_u \mathcal{P}_n$,*

1. $h(P_n)$ is asymptotically normal with mean $\sim \frac{n}{2 \ln n}$ and variance $\sim \frac{n}{2 \ln^2 n}$,
2. $d_{TV}(H(P_n), L(n)) = e^{-\Omega(n)}$.

Proof. Let $U = (B, E, (k, \sigma), \pi) \sim \text{Gen}(n)$. We couple G_n , G_n^+ and G_n^- by writing $U^+ = (1, E, (k, \sigma), \pi)$, $U^- = (-1, E, (k, \sigma), \pi)$, $G_n = \rho(U)$, $G_n^+ = \rho(U^+)$ and $G_n^- = \rho(U^-)$. We have

$$\begin{aligned} d_{TV}(h(P_n), \alpha(G_n^+)) &\leq d_{TV}(h(P_n), h(G_n)) + d_{TV}(h(G_n), \alpha(G_n^+)) \\ &\leq d_{TV}(\mathbb{P}_{\mathcal{P}_n}, \mathbb{P}_{\text{Gen}(n)}) + \mathbb{P}(h(G_n) \neq \alpha(G_n^+)). \end{aligned}$$

But $d_{TV}(\mathbb{P}_{P_n}, \mathbb{P}_{Gen(n)}) = e^{-\Omega(n)}$ from Theorem 3.2.3; and, since $h(G_n) = h(G_n^+)$,

$$\mathbb{P}(h(G_n) \neq \alpha(G_n^+)) = \mathbb{P}(\omega(G_n^+) < \alpha(G_n^+)) = e^{-\Omega(n)}$$

from Lemma 3.3.6. Thus $d_{TV}(h(P_n), \alpha(G^+)) = e^{-\Omega(n)}$. The first part in the statement now follows from Proposition 3.3.3.

Similar arguments establish the second part of the theorem. Much as above we may write

$$d_{TV}(H(P_n), L(n)) \leq d_{TV}(H(P_n), H(G_n)) + d_{TV}(H(G_n), \omega(G_n^+)) + d_{TV}(\omega(G_n^+), L(n)).$$

The first and third terms in the upper bound here are $e^{-\Omega(n)}$ by Theorem 3.2.3 and Lemma 3.3.5. Since $H(G_n) = H(G_n^+)$, the second term is at most $\mathbb{P}(\alpha(G_n^+) > \omega(G_n^+))$, which is $e^{-\Omega(n)}$ by Lemma 3.3.6. Thus $d_{TV}(H(P_n), L(n)) = e^{-\Omega(n)}$, as required. $\square$

The last result shows that the distribution $L(n)$ is not only needed for the approximate generation process, but also corresponds to an important property of perfect graphs.

Finally here in this section on stability and clique numbers, let us briefly consider algorithmic aspects, following the comments at the end of the last section, and using Lemma 3.3.6.

Given a random perfect graph P_n , let A be the set of vertices with degree at most $n/2$, and let B consist of the remaining vertices. Then wvhp exactly one of the following two cases holds.

- (a) B is a clique and indeed is the unique maximum clique (by Lemma 3.3.5), so $\omega = |B|$; and A is a disjoint union of cliques, so α is either the number of parts (cliques) in A or this number plus 1 (by Lemma 3.3.1), and we can easily tell which in $O(n^2)$ -time.
- (b) A is the unique maximum stable set, so $\alpha = |A|$; and B induces a complete multipartite graph, so ω is either the number of parts or this number plus 1, and we can easily tell which in $O(n^2)$ -time.

In the first case, to tell if α is the number of parts plus 1 we just test if some vertex in B is non-adjacent to at least one vertex in each side clique; and similarly for the second case. In particular, wvhp in $O(n^2)$ time we can determine ω and α (and know they are correct).

3.4 Induced Subgraphs

Suppose $P_n \in_u \mathcal{P}_n$. From Lemma 3.3.6 and Theorem 3.2.3 we have $\omega(P_n) \geq \frac{(1-\epsilon)n}{2 \ln n}$ wvhp, so we can of course find any fixed graph H as a subgraph of P_n wvhp.

We use $H \subseteq_i G$ to denote that H is an induced subgraph of G . We observed already that the generated graphs, $\rho(\text{Gen}(n))$, are symmetric with respect to taking complements. The same is true for perfect graphs, that is $P_n \sim \overline{P_n}$, by the weak perfect graph theorem [Lov72a, Lov72b]. Thus we see that, if H is fixed and $P_n \in_u \mathcal{P}_n$, then

$$\mathbb{P}(H \subseteq_i P_n) = \mathbb{P}(\overline{H} \subseteq_i P_n). \quad (3.16)$$

Lemma 3.4.1. *Suppose that $G_n^+ \sim \text{Gen}^+(n)$ and that H is fixed. Then*

$$\mathbb{P}(H \subseteq_i G_n^+) = \begin{cases} 0 & \text{if } H \notin \mathcal{GS}^+, \\ 1 - e^{-\Omega(n \ln n)} & \text{if } H \in \mathcal{GS}^+. \end{cases}$$

To prove this lemma, we first establish two preliminary claims.

Claim 3.4.2. *Suppose $\sigma \in_u \Pi_n$, and let $a = a(n) \leq n^\delta$ for some $0 < \delta < 1$. The probability that σ contains at most a parts is at most $\exp\{-(1 - \delta + o(1))n \ln n\}$.*

Proof. The probability of the event in the statement is at most a^n/B_n , and the bound follows, since $\ln B_n = (1 + o(1))n \ln n$. $\square$

Claim 3.4.3. *Suppose $\sigma \in_u \Pi_n$, and let $l \geq 1$ be a fixed integer. The probability that all parts of σ are of size at most l is at most $\exp\{-(\frac{1}{l} + o(1))n \ln n\}$.*

Proof. We call a finite sequence $(a_i)_{i=1}^k$ *type* if $\sum_{i=1}^k a_i = n$. We say that σ has type (a_i) if (a_i) corresponds to the sizes of the parts of σ taken in any order. The number of partitions with type (a_i) with $a_i \leq l$ for all i is

$$\frac{n!}{k! \prod_{i=1}^k a_i!} \leq \frac{n!}{k!} \leq \frac{e\sqrt{n}}{\sqrt{2\pi k}} \cdot \frac{n^n e^{-n}}{k^k e^{-k}} = \frac{n^n}{\left(\frac{n}{l}\right)^{\frac{n}{l}}} 2^{O(n)} = \exp\left\{\left(1 - \frac{1}{l}\right)n \ln n + O(n)\right\}.$$

The number of types (a_i) with $a_i \leq l$ for all i is at most $l^n = \exp(O(n))$, and hence the number of partitions with parts at most l is $\exp\left\{\left(1 - \frac{1}{l}\right)n \ln n + O(n)\right\}$. The statement in the lemma follows after dividing by B_n . $\square$

Corollary 3.4.4. *For fixed $\delta > 0$ and $l \geq 1$, the probability that a random partition $\sigma \in_u \Pi_n$ contains at most n^δ parts of size more than l is $\exp(-\Theta(n \ln n))$.*

Proof. Let Π'_Q and Π''_Q be the classes of partitions of Q with at most n^δ parts and with parts of size at most l respectively. The number of partitions described in the statement can be bounded by $\sum_{S \subseteq [n]} |\Pi'_S| |\Pi''_{\overline{S}}| = \exp(-O(n \ln n))$ by the previous two claims. For a lower bound consider the partition π with n parts; $\mathbb{P}(\sigma = \pi) = B_n^{-1} = \exp(-\Theta(n \ln n))$. $\square$

Proof of Lemma 3.4.1. Since $\mathcal{GS}^+$ is a hereditary class of graphs and G_n^+ takes as values only $\mathcal{GS}^+$ -graphs with non-zero probability, we get that if $H \notin \mathcal{GS}^+$, then $\mathbb{P}(H \subseteq_i G_n^+) = 0$ for any n .

Now suppose that $H \in \mathcal{GS}^+$. Fix a unipolar arrangement (H, A) . Suppose that $\pi \in \Pi_{\overline{A}}$ corresponds to $H[\overline{A}]$, and let $l = L(\pi)$ be the maximum size of a part in π . Let I_n be the interval $[n/2 - n^{2/3}, n/2 + n^{2/3}]$. By Theorem 3.2.5, $\mathbb{P}(k \notin I_n) \leq 2n^{-n}$ for n sufficiently large; so we may condition on $k \in I_n$. Let (G_n^+, C) be the unipolar arrangement induced by the generation process.

By Corollary 3.4.4 (with $1/2 < \delta < 1$) there are at least $|\pi|n^{1/2}$ side cliques in $\overline{C}$ of size at least l with probability $1 - \exp(-\Theta(n \ln n))$, so we may condition on this event. Hence we can find a collection of at least $t_1 = \lceil n^{1/2} \rceil$ disjoint sets $T_j \subseteq \overline{C}$ such that $G_n^+[T_j] \cong H[\overline{A}]$. Pick $t_0 = \lceil n/(3|A|) \rceil$ disjoint subsets S_i of C , each of size $|A|$. Then $G_n^+[S_i] \cong H[A]$ for every i , and hence for $1 \leq i \leq t_0$ and $1 \leq j \leq t_1$ we have $p_0 = \mathbb{P}(G_n^+[S_i \cup T_j] \cong H) \geq 2^{-|A||\overline{A}|}$.

The probability that no such pair induces H is $(1 - p_0)^{t_0 t_1} = \exp(-\Theta(n^{3/2}))$. The failure probability is dominated by the term $\exp(-\Theta(n \ln n))$. $\square$

The bound in Lemma 3.4.1 cannot be improved. For suppose H is the 4-edge path: then

$$\mathbb{P}(H \not\subseteq_i G_n^+) \geq \mathbb{P}(\sigma \text{ has only parts of size } 1) \geq 1/B_n = e^{-O(n \ln n)}.$$

It is now immediate that for a co-unipolar graph H we have

$$\mathbb{P}(H \subseteq_i G_n^-) = \mathbb{P}(\overline{H} \subseteq_i G_n^+) = 1 - e^{-\Omega(n \lg n)},$$

and hence we arrive at the following theorem:

Theorem 3.4.5. *Let H be any (fixed) graph, and let $P_n \in_u \mathcal{P}_n$. Then*

$$\mathbb{P}(H \subseteq_i P_n) = \begin{cases} e^{-\Omega(n)} & \text{if } H \notin \mathcal{GS}, \\ 1/2 \pm e^{-\Omega(n)} & \text{if } H \in \mathcal{GS} \setminus (\mathcal{GS}^+ \cap \mathcal{GS}^-), \\ 1 - e^{-\Omega(n)} & \text{if } H \in (\mathcal{GS}^+ \cap \mathcal{GS}^-). \end{cases}$$

Proof. Consider $Q_n = \{G \in \mathcal{G}_n : H \subseteq_i G\}$ and $\mathbb{P}_{\mathcal{P}_n}(Q_n)$. Now the result follows from Lemma 3.4.1, Equation (3.8) and Theorem 3.2.3. $\square$

Note that Theorem 3.4.5 accords with (3.16). It also follows from Theorem 3.4.5 that for every graph H :

$$\mathbb{P}\left((H \subseteq_i P_n) \cup (\overline{H} \subseteq_i P_n)\right) = \begin{cases} e^{-\Omega(n)} & \text{if } H \notin \mathcal{GS}, \\ 1 - e^{-\Omega(n)} & \text{if } H \in \mathcal{GS}. \end{cases}$$

and

$$\mathbb{P}\left((H \subseteq_i P_n) \cap (\overline{H} \subseteq_i P_n)\right) = \begin{cases} e^{-\Omega(n)} & \text{if } H \notin \mathcal{GS}^+ \cap \mathcal{GS}^-, \\ 1 - e^{-\Omega(n)} & \text{if } H \in \mathcal{GS}^+ \cap \mathcal{GS}^-. \end{cases}$$

3.5 Clique Colouring

A j -clique colouring of a graph is a colouring of the vertices with j colours so that no maximal clique is monochromatic (ignoring any isolated vertices). We improve Corollary 6 of [BGG⁺04], which states that almost all perfect graphs are 3-clique-colourable.

Theorem 3.5.1. *Almost all perfect graphs are 2-clique-colourable.*

Corollary 6 of [BGG⁺04] is proved by showing that all generalised split graphs are 3-clique-colourable, and then using the theorem of Prömel and Steger discussed earlier. In the same article it is shown that there are generalised split graphs which are not 2-clique-colourable. There are several other subclasses of perfect graphs for which the clique-chromatic number is known to be at most 3, and it was conjectured in [DSSW91] that perfect graphs have bounded clique-chromatic number. This was recently disproved in [CPTT16]. See [MMP16] for recent work on clique-colourings of binomial random graphs and of geometric graphs. We prove Theorem 3.5.1 using Theorem 3.2.3. We consider unipolar and co-unipolar graphs separately.

3.5.1 Clique colouring of unipolar graphs

We start with a deterministic lemma. We say that a vertex *sees* a set of vertices if it has a neighbour from the set.

Lemma 3.5.2. *Let $G \in \mathcal{GS}_n^+$, with a central clique C . Suppose that some vertex $x \in C$ sees each side clique Q which is a maximal clique in G . Then G is 2-clique-colourable.*

Proof. We may suppose that G contains at least two vertices. Choose a vertex $y_Q \in N(x) \cap Q$ for each side clique Q with $N(x) \cap Q \neq \emptyset$. Define a colouring $c : V(G) \rightarrow \{1, 2\}$ in the following way:

$$c(v) = \begin{cases} 1 & \text{if } v = x, \\ 2 & \text{if } v \in C - x, \\ 2 & \text{if } v = y_Q \text{ for some side clique } Q, \\ 1 & \text{otherwise.} \end{cases}$$

We claim that c is a proper clique colouring. Note first that $N(x) \neq \emptyset$: this is trivial if $|C| \geq 2$; and if $C = \{x\}$ then there is a side clique Q , and Q must contain a neighbour of x (whether it is a maximal clique or not).

Suppose for a contradiction that M is a monochromatic maximal clique. Since M is a clique, it must lie in the co-bipartite graph induced by C and a side clique Q . Further $M \not\subseteq C$: for if so then $M = C$, and C is not monochromatic if $|C| \geq 2$ and C is not a maximal clique if $C = \{x\}$.

If $N(x) \cap Q$ is empty, then M contains a vertex of colour 1 in Q , does not contain x , and since M is monochromatic, M is contained in Q . By the definition of x , Q is not maximal; hence M is not maximal, but this is a contradiction.

Thus we may assume that $N(x) \cap Q$ is nonempty, and so contains y_Q . Then either (1) $M \subseteq \{x\} \cup (Q \setminus \{y_Q\})$ or (2) $M \subseteq (C \setminus \{x\}) \cup \{y_Q\}$. In case (1), we could add y_Q to M , and in case (2) we could add x . Thus again M is not maximal, and we have our final contradiction. $\square$

Let $t = t(n) = \log n - 2 \log \log n$ (t is for threshold). Given a unipolar arrangement (G, C) of order n , we use t to split the side cliques of (G, C) into two categories - the *big* side cliques with at least t vertices, and the *small* side cliques with less than t vertices. We shall see in the next two lemmas that, for $G^+ \sim \text{Gen}^+(n)$, whp the big side cliques can all be seen by some $x \in C$ and the small side cliques are not maximal, so we can apply the above deterministic lemma to deduce that whp G^+ is 2-clique-colourable.

Lemma 3.5.3. *Let $G^+ \sim \text{Gen}^+(n)$, with induced unipolar arrangement (G, C) . Then with probability $1 - 2^{-\Omega(\log^2 n)}$ there is a vertex $x \in C$ which sees each big side clique.*

Proof. Condition on $k = k_0 \in \mathcal{I} = [n/2 - \sqrt{n}, n/2 + \sqrt{n}]$, as usual. Condition also on the partition σ of $W = [n] \setminus [k]$; and assume wlog that the permutation π is the identity, so $C = [k]$. Let $\mathcal{B}$ be the set of all big side cliques, so $|\mathcal{B}| \leq (n - k_0)/t$.

For $v \in C$ and $Q \in \mathcal{B}$, the probability that v does not see Q is

$$2^{-|Q|} \leq 2^{-t} = \frac{\log^2 n}{n}.$$

Thus the probability that v sees each $Q \in \mathcal{B}$ is at least

$$\begin{aligned} \left(1 - \frac{\log^2 n}{n}\right)^{\frac{n-k_0}{t}} &= \exp\left(-\frac{\log^2 n}{n} \frac{n-k_0}{t} + O\left(\frac{\log^3 n}{n}\right)\right) \\ &= \exp\left(-\left(\frac{1}{2} + o(1)\right) \log n\right) = n^{-\alpha+o(1)} \end{aligned}$$

where $\alpha = 1/(2 \log e) \approx 0.35$. Then, the probability that for each vertex $v \in C$ there is some $Q \in \mathcal{B}$ which v fails to see is at most

$$(1 - n^{-\alpha+o(1)})^{k_0} \leq e^{-n^{\frac{1}{2}-\alpha+o(1)}},$$

which is easily at most $2^{-\Omega(\log^2 n)}$. $\square$

Lemma 3.5.4. *Let $G^+ \sim \text{Gen}^+(n)$, with induced unipolar arrangement (G, C) . Then with probability $1 - 2^{-\Omega(\log^2 n)}$ no small side clique is a maximal clique.*

Proof. Condition on $k = k_0 \in \mathcal{I} = [n/2 - \sqrt{n}, n/2 + \sqrt{n}]$, as usual. Let us say that a vertex $v \in C$ *extends* a side clique Q if $Q \subseteq N(v)$. For $v \in C$ and $Q \in \mathcal{B}$, the probability that v does not extend Q is

$$1 - 2^{-|Q|} \leq 1 - 2^{-t} = 1 - \frac{\log^2 n}{n} \leq e^{-\frac{\log^2 n}{n}}.$$

Thus the probability that no $v \in C$ extends Q is at most

$$\left(e^{-\frac{\log^2 n}{n}} \right)^{k_0} = e^{-(\frac{1}{2} + o(1)) \log^2 n}.$$

But there are at most n small side cliques, so the same upper bound holds for the probability that some small side clique is a maximal clique. $\square$

Lemma 3.5.5. *The graph $G^+ \sim \text{Gen}^+(n)$ is 2-clique-colourable with probability $1 - 2^{-\Omega(\log^2 n)}$.*

Proof. This follows from the last three lemmas. $\square$

3.5.2 Clique colouring of co-unipolar and perfect graphs

Co-unipolar graphs are easier to clique-colour than unipolar graphs.

Lemma 3.5.6. *The random graph $G^- \sim \text{Gen}^-(n)$ is 2-clique-colourable wvhp.*

Proof. We first prove an auxiliary claim.

Claim 3.5.7. *Let G be a co-unipolar graph, with a unipolar arrangement $(\overline{G}, C)$ for $\overline{G}$. Suppose that C is non-empty, and every vertex of C has neighbours in at least two side independent sets of $\overline{C}$. Then G is 2-clique-colourable.*

Proof. Since C is non-empty, there are at least two side independent sets in $\overline{C}$. Let Q be a side independent set. Colour the vertices in $C \cup Q$ with 1 and the remaining vertices with 2. A maximal clique contained completely in $\overline{C}$ contains a vertex from each side independent set and therefore is not monochromatic. There are no maximal cliques contained completely in C . A monochromatic clique which intersects both C and $\overline{C}$ must contain precisely two vertices – $c \in C$ and $q \in Q$. However, such a clique can always be extended with a neighbour of c from a different side independent set. $\square$

Let us check that G^- satisfies the condition of the claim wvhp. By Theorem 3.2.5 we may assume $k \in \mathcal{I} = [n/2 - \sqrt{n}, n/2 + \sqrt{n}]$. Let (G^-, C) be the co-unipolar arrangement induced by the generation. By Lemma 3.2.9 wvhp the maximum size of an independent set of $\overline{C}$ is at most $n/\sqrt{\ln n} = o(n)$. On the other hand every vertex of C has degree at least $n/5$ wvhp from Chernoff's inequality and the union bound.

If the minimum degree of the vertices in C is bigger than the maximum size of a side independent set $\overline{C}$, then every vertex of C must see at least two side independent sets. $\square$

Theorem 3.5.1 now follows from Equation (3.8), Lemma 3.5.5, Lemma 3.5.6 and Theorem 3.2.3. Indeed we see that the probability that P_n fails to be 2-clique-colourable is $2^{-\Omega(\log^2 n)}$.

Further, we may see from the proofs that there is an $O(n^2)$ -time algorithm that on input P_n first generates a unipolar representation for the graph or its complement, and then finds a 2-clique-colouring with failure probability $2^{-\Omega(\log^2 n)}$.

3.6 Hamiltonicity

A *Hamilton* cycle in a graph is a cycle visiting every vertex exactly once. Recall the distribution $L(n)$ in Definition 3.2.1. In this section we prove the following theorem.

Theorem 3.6.1. *Almost all perfect graphs are Hamiltonian. Indeed, for $P_n \in_u \mathcal{P}_n$ and $X \sim L(n)$,*

$$\mathbb{P}(P_n \text{ is Hamiltonian}) = 1 - \frac{1}{2}\mathbb{P}(X > n/2) \pm e^{-\Omega(n)}.$$

It follows from this result and Theorem 3.2.5 that

$$\mathbb{P}(P_n \text{ is Hamiltonian}) = 1 - 2^{-(\frac{1}{4}+o(1))\log^2 n} = 1 - o(1),$$

which will complete the proof of (3.1). Indeed, we may see from the proofs that there is an $O(n^3)$ -time algorithm that succeeds wvhp on input P_n , and either outputs a Hamilton cycle, or a stable set of size $> n/2$ (proving there is no Hamilton cycle), or fails. To prove Theorem 3.6.1 we consider unipolar and co-unipolar graphs separately, and in order to handle co-unipolar graphs we give a deterministic condition for hamiltonicity of bipartite graphs and show that a random such graph satisfies the requirements with very high probability. At the end of the section we generalise this condition to a tight extremal threshold of existence of Hamilton cycles in general graphs.

3.6.1 Hamilton cycles in random unipolar graphs

Lemma 3.6.2. *Suppose that (G, C) is a unipolar arrangement. Let $T_1, T_2 \subseteq \overline{C}$ each contain exactly one vertex from each side clique and be such that $T_1 \cap T_2$ consists of the vertices of $\overline{C}$ forming side cliques of size one. Let A_1 and A_2 be disjoint subsets of C . Suppose that in G there are complete matchings M_1 from T_1 into A_1 and M_2 from T_2 into A_2 . Then G is Hamiltonian.*

Proof. Fix some ordering of the side cliques of $\overline{C}$. There is a Hamilton cycle which enters each clique from an edge of M_1 , visits all vertices inside and then leaves the clique from an edge of M_2 . After all side cliques are visited, the Hamilton cycle visits any remaining vertices in C and finishes at the initial vertex. $\square$

Definition 3.6.3. Let $G_{n,m,\frac{1}{2}}$ denote a random $n \times m$ bipartite graph with the edges present independently with probability $1/2$.

Lemma 3.6.4. Suppose that $1 \leq n \leq m$. Then $G = (V_1, V_2, E) \sim G_{n,m,\frac{1}{2}}$ contains a complete matching from V_1 to V_2 with probability $1 - 2^{-m(1+o_m(1))}$.

Proof. We can find an isolated vertex in V_1 with probability at least 2^{-m} , hence 2^{-m} is a lower bound on the probability of failing to have a complete matching (from V_1 to V_2). Hall's marriage theorem states that G contains a complete matching if and only if $|N(S)| \geq |S|$ for every $S \subseteq V_1$. Let E_S be the event that $|N(S)| < |S|$. We have $\mathbb{P}(E_S) \leq \binom{m}{m-s+1} 2^{-s(m-s+1)}$, where $s = |S|$. Thus

$$\mathbb{P}(\cup_S E_S) \leq \sum_S \mathbb{P}(E_S) = \sum_{|S| \leq 2} \mathbb{P}(E_S) + \sum_{3 \leq |S| \leq n-2} \mathbb{P}(E_S) + \sum_{|S| \geq n-1} \mathbb{P}(E_S).$$

Clearly

$$\sum_{|S| \leq 2} \mathbb{P}(E_S) \leq n2^{-m} + \binom{n}{2} m 2^{-2(m-1)} \leq n 2^{-m+O(1)}.$$

For $3 \leq s \leq \min\{n, m-2\}$ we have $2^{-s(m-s+1)} \leq 2^{-3(m-2)}$, so

$$\sum_{3 \leq |S| \leq n-2} \mathbb{P}(E_S) \leq \sum_{3 \leq |S| \leq \min\{n, m-2\}} \mathbb{P}(E_S) \leq 2^{n+m} 2^{-3(m-2)} \leq 2^{-m+O(1)}.$$

If $m \geq n+2$ we are done, so assume that $m = n + \delta$ where δ is 0 or 1, and consider $s = n-1$ and $s = n$. We have $(n-1)(m-n+2) = (m-\delta-1)(\delta+2) \geq m$ for $m \geq 3$, and $n(m-n+1) = (m-\delta)(\delta+1) \geq m$ for $m \geq 2$. Hence

$$\sum_{|S| \geq n-1} \mathbb{P}(E_S) \leq n^2 2^{-m}$$

for $m \geq 3$, which completes the proof. $\square$

Lemma 3.6.5. The random graph $G^+ \sim \text{Gen}^+(n)$ is Hamiltonian wvhp.

Proof. Condition on $|k - n/2| < \sqrt{n}$, which happens with wvhp by Theorem 3.2.5. By Lemma 3.2.8, the number of side cliques is at most $n/16$ wvhp. Condition on this event and on the arrangement (G^+, C) induced by the generation process. Partition C into two near equal parts $A, B \subseteq C$ with $||A| - |B|| \leq 1$, so that $|A| \sim |B| \sim n/4$. Find $T_A, T_B \subset \overline{C}$ as in Lemma 3.6.2. By Lemma 3.6.4, we can find complete matchings between T_A and A and between T_B and B wvhp. We now see that G^+ satisfies the conditions of Lemma 3.6.2 wvhp and therefore G^+ is Hamiltonian wvhp. $\square$

3.6.2 Hamilton cycles in random bipartite graphs

Hamilton cycles in random bipartite graphs have been studied in Frieze [Fri85] and in Bollobás and Kohayakawa [BK91]. However, we are interested in the dense case and seek an exponentially small failure probability, which does not appear to have been done before. We shall show:

Theorem 3.6.6. *If $G \sim G_{n,n,\frac{1}{2}}$ then G is Hamiltonian with probability $1 - 2^{-(1+o(1))n}$.*

This theorem will follow from the deterministic Theorem 3.6.9 below, based on ‘Posa flips’, but first we need some preparation.

Definition 3.6.7. *Suppose G is a bipartite graph. We define $\tilde{\alpha}(G)$ to be the maximum integer k such that we can find k -sets $S_1 \subseteq V_1$ and $S_2 \subseteq V_2$ with $E(S_1, S_2) = \emptyset$ for a bipartition (V_1, V_2) of G . We call the pair (S_1, S_2) a bipartite hole of G .*

We denote the minimum degree of a graph G by $\delta(G)$. The (vertex) connectivity, $\kappa(G)$, of a non-complete graph G is the minimum size of a set S such that $G \setminus S$ is disconnected. The connectivity of an n -vertex complete graph is $n - 1$. We say that G is k -connected if $\kappa(G) \geq k$.

Lemma 3.6.8. *Suppose G is a bipartite graph such that $\delta(G) > \tilde{\alpha}(G)$. Then G is $(\delta(G) - \tilde{\alpha}(G))$ -connected.*

Proof. The statement easily follows if G is complete, as $\tilde{\alpha}(G) = 0$ and $\kappa(G) = \delta(G) = \delta(G) - \tilde{\alpha}(G)$. Suppose G is not complete. Let $S \subseteq V(G)$ be a set of less than $\delta(G) - \tilde{\alpha}(G)$ vertices. We must show that $G' = G[V(G) \setminus S]$ is connected. Let (V_1, V_2) be a bipartition of G .

Let $v_1 \in V_1 \setminus S$ and $v_2 \in V_2 \setminus S$. Then $|N(v_1) \setminus S| \geq \tilde{\alpha}(G) + 1$ and $|N(v_2) \setminus S| \geq \tilde{\alpha}(G) + 1$; and hence either v_1 and v_2 are adjacent, or a neighbour of v_1 is adjacent to a neighbour of v_2 in G' . Thus v_1 and v_2 are connected in G' by a path of length at most 3. If $v_1, v_2 \in V_1 \setminus S$, then v_1 has a neighbour $u \in V_2 \setminus S$, and we have already seen that v_2 and u must be connected in G' , so v_1 and v_2 are connected in G' . The same conclusion holds if $v_1, v_2 \in V_2 \setminus S$. $\square$

Theorem 3.6.9. *An $n \times n$ bipartite graph G with $n \geq 2$ is Hamiltonian if $\delta(G) \geq 2\tilde{\alpha}(G) + 1$.*

Proof. All paths in this proof have their terminal vertices labelled as *start* and *end*. Suppose $P = \{v_1, \dots, v_l\}$ is a path with start v_1 and end v_l , and suppose v_l is adjacent to v_i . A *flip of P around v_i* is the path $\{v_1, \dots, v_i, v_l, \dots, v_{i+1}\}$ over $V(P)$, starting at v_1 and ending at v_{i+1} . (If $i = l - 1$ the flip does nothing.)

Let $P = \{v_1, \dots, v_l\}$ be a path starting at v_1 , ending at v_l , and having maximum length of any path in G . For $k = 0, 1, 2$ let W_k be the set of ends of paths obtained from P by at most k flips. Clearly $W_0 = \{v_l\}$. Since P is maximal, v_l is adjacent to vertices of P only, hence $W_1 = \{v_{i+1} : v_i v_l \in E(G)\}$ satisfies $|W_1| \geq \delta(G)$.

Assume that $\delta(G) \geq 2\tilde{\alpha}(G) + 1$, and let (V_1, V_2) be a bipartition of G . Without loss of generality suppose that $v_l \in V_1$. Thus each set $W_k \subseteq V_1$. To complete the proof it is sufficient to show that every vertex of V_2 is adjacent to a vertex of W_2 . Indeed, suppose this is the case. If $v_1 \in V_2$, then P can be closed to a cycle, which must be Hamiltonian since G is connected by Lemma 3.6.8 and P has maximum length. If $v_1 \in V_1$, then there is a vertex $u \in V_2 \setminus V(P)$. But this contradicts the maximality of P , because there is a path Q with $V(Q) = V(P)$ ending at a neighbour of u obtained from P by two flips, and Q can be extended to contain u , yielding a longer path.

We claim that

$$|W_2| \geq n - 2\tilde{\alpha}(G). \quad (3.17)$$

Assuming this, and recalling that $\delta(G) \geq 2\tilde{\alpha}(G) + 1$, we see that every vertex of V_2 must be adjacent to a vertex of W_2 , and we are done. So it remains to establish the claim (3.17).

Suppose $v_{i+1} \in W_1$ is adjacent to $v_j \neq v_i$. Then

$$v_{j+1} \in W_2 \quad \text{if } j < i; \quad (3.18)$$

$$v_{j-1} \in W_2 \quad \text{if } j > i. \quad (3.19)$$

More precisely, the path obtained from P by first flipping around v_i and then v_j ends with v_{j+1} or v_{j-1} depending on whether v_j comes before v_i in P .

Since $|W_1| \geq 2\tilde{\alpha}(G) + 1$, we can find an integer t such that $v_{t+1} \in W_1$ and if we write

$$F_1 = \{v_i \in V(G) : i > t + 1\} \text{ and}$$

$$F_2 = \{v_i \in V(G) : i < t + 1\},$$

then $|F_1 \cap W_1| \geq \tilde{\alpha}(G)$ and $|F_2 \cap W_1| \geq \tilde{\alpha}(G)$. Let $M_1 = (F_1 \cap W_1) \cup \{v_{t+1}\}$ and $M_2 = (F_2 \cap W_1) \cup \{v_{t+1}\}$. It follows from (3.19) that if $v_j \in F_1$ has a neighbour in M_2 , then its predecessor, v_{j-1} , is contained in W_2 ; and from (3.18) it follows that if $v_j \in F_2, v_j \neq v_t$ has a neighbour in M_1 , then its successor, v_{j+1} , is contained in W_2 . But $v_{t+1} \in W_1$, and hence $v_{t+1} \in W_2$, so no special care is required for the case $v_j = v_t$. For $k = 1, 2$ let $N_k = N[M_{3-k}] \cap F_k$ and $N_k^c = (F_k \cap V_2) \setminus N_k$ (and hence $F_k \cap V_2 = N_k \cup N_k^c$); and let N_1^- be the set of predecessors in P of vertices in N_1 , and let N_2^+ be the set of successors of N_2 . We see that

1. $N_1^- \cup N_2^+ \subseteq W_2 \setminus \{v_l\}$;

$$2. |N_1^-| = |N_i|, |N_2^+| = |N_2|;$$

$$3. N_1^- \cap N_2^+ = \{v_{t+1}\};$$

$$4. v_l \in W_2.$$

Since $F_k \cap V_2 = N_k \cup N_k^c$ and $v_{t+1} \in V_1$, we observe that $n = |V_2| = |N_1| + |N_1^c| + |N_2| + |N_2^c|$, and therefore

$$\begin{aligned} |W_2| &\geq |N_1^- \cup N_2^+| + 1 = (|N_1^-| + |N_2^+| - 1) + 1 \\ &= |N_1| + |N_2| = n - (|N_1^c| + |N_2^c|). \end{aligned}$$

However, $E(M_{3-k}, N_k^c) = \emptyset$ (by the definition of N_k) and $|M_k| > \tilde{\alpha}(G)$, so $|N_k^c| \leq \tilde{\alpha}(G)$, and hence $|W_2| \geq n - 2\tilde{\alpha}(G)$. Thus we have proved (3.17), and the proof is complete. $\square$

Theorem 3.6.6 now follows because if $G \sim G_{n, n, \frac{1}{2}}$ and $t = \sqrt{n}$, then the events $\{\delta(G) < 2t + 1\}$ and $\{\tilde{\alpha}(G) > t\}$ both have probability $2^{-(1+o(1))n}$. As a lower bound consider the event that G contains an isolated vertex.

3.6.3 Hamilton cycles in random co-unipolar graphs

Lemma 3.6.10. *Suppose that $G^- \sim \text{Gen}^-(n)$ and $X \sim L(n)$. Then G^- is Hamiltonian with probability $\mathbb{P}(L(n) \leq n/2) - e^{-\Omega(n)}$.*

Proof. Recall that $\text{Gen}^-(n)$ is a random quadruple $(-1, E, (k, \sigma), \pi)$. If $k > n/2$, then $\alpha(G^-) > n/2$, which makes it impossible to contain a Hamilton cycle. We show that conditional on $k \leq n/2$, G^- is Hamiltonian wvhp, which will complete the proof.

From Theorem 3.2.5 and Lemma 3.2.8 we may condition on the event that $k \geq n/2 - \sqrt{n}$ and the event that the number of parts in σ is at least $n/(3 \ln n)$ (since both hold wvhp). Condition also on (G^-, C) being the co-unipolar arrangement induced by the generation. Partition $\overline{C}$ as $Q \cup T$, so that $|C| = |Q| = k$, $|T| = n - 2k \leq 2\sqrt{n}$, and T contains vertices only from different parts of σ . By Corollary 3.6.6, there is a Hamilton cycle in the induced subgraph $G^-[C \cup Q]$ wvhp. Condition on this event and on H being a Hamiltonian cycle. If T is empty we are done, so assume not.

Now we have a Hamilton cycle H in $G^-[C \cup Q]$, and a non-empty set T of vertices inducing a clique. Let u, v be vertices in T , and note that there is a u - v Hamilton path in $G^-[T]$ (where $u = v$ if $|T| = 1$). We may assume that the maximum size of a side stable set is at most $n/\log n$ by Lemma 3.2.9. Let S_u, S_v be the side stable sets containing u, v respectively, and let R be the set of vertices in $Q \setminus S_u$. Then u is adjacent to each vertex in R , and $|R| \geq k - |S_u| \geq n/3 + |S_v|$ for n sufficiently large.

The vertices in R have at least $|R| - |S_v| \geq n/3$ neighbours along H which are not in S_v , and if v is adjacent to any of these neighbours then we have a Hamilton cycle in the whole graph. But the probability that this fails is at most $2^{-n/3}$, since any neighbour in $\overline{C}$ must be adjacent to v and so we may assume there are none. $\square$

3.6.4 Hamilton cycles in general graphs

Recall that $\delta(G)$ denotes the minimum degree $d(v)$ of a vertex v in G . An early result by Dirac [Dir52] states:

Theorem 3.6.11 (Dirac's Theorem). *A graph G with $n \geq 3$ vertices is Hamiltonian if $\delta(G) \geq n/2$.*

The theorem is sharp, since the disjoint union of two complete n -vertex graphs has minimum degree $n-1$ and it does not contain a Hamilton cycle. This example contains a large bipartite hole, that is two disjoint sets of vertices with no edge between them. It is natural to ask if such a hole is necessary to construct a non-Hamiltonian graph with large minimum degree. We show that indeed this is the case.

Given disjoint sets S and T of vertices in a graph, we let $E(S, T)$ denote the set of edges with one end in S and one in T .

Definition 3.6.12. *An (s, t) -bipartite-hole in a graph G consists of two disjoint sets of vertices S and T with $|S| = s$ and $|T| = t$ such that $E(S, T) = \emptyset$. We define the bipartite-hole-number $\alpha^*(G)$ to be the least integer r which may be written as $r = s + t - 1$ for some positive integers s and t such that G does not contain an (s, t) -bipartite-hole.*

An equivalent definition of $\alpha^*(G)$ is the maximum integer r such that G contains an (s, t) -bipartite-hole for every pair of non-negative integers s and t with $s + t = r$. Observe that $\alpha^*(G) = 1$ if and only if G is complete, and $\alpha^*(G) \geq \alpha(G)$, where $\alpha(G)$ is the stability number of G . Also note that for $1 \leq a \leq b$, we have $\alpha^*(K_{a,b}) = b$ and $\alpha^*(\overline{K_{a,b}}) = \min\{b+1, 2a+1\}$. (Here $K_{a,b}$ denotes the complete bipartite graph with parts of sizes a and b , and $\overline{G}$ denotes the complement of G .)

Recall that the connectivity of G is denoted by $\kappa(G)$.

Lemma 3.6.13. *The following holds for every non-connected graph G :*

$$\kappa(G) \geq \delta(G) + 2 - \alpha^*(G).$$

Proof. Suppose for contradiction that the vertices v_1 and v_2 are separated by a set S of size less than $\delta(G) + 2 - \alpha^*(G)$. Let s and t be positive integers such that $\alpha^*(G) + 1 = s + t$ and G has no (s, t) -bipartite-hole. Then $\frac{\alpha^*(G)+1}{2} \leq \max(s, t) \leq \alpha^*(G)$. Now the

closed neighbourhoods $N[v_i]$ satisfy $|N[v_i] \setminus S| \geq \delta(G) + 1 - |S| \geq \alpha^*(G) \geq \max(s, t)$. The sets $N[v_1] \setminus S$ and $N[v_2] \setminus S$ are disjoint because S is a separator, but $|N[v_1] \setminus S| \geq s$ and $|N[v_2] \setminus S| \geq t$, so there is an edge between them and S does not separate v_1 from v_2 , a contradiction. $\square$

The inequality above does not hold for complete graphs, as $\kappa(K_n) = n - 1 < (n - 1) + 2 - 1 = \delta(K_n) + 2 - \alpha^*(K_n)$. It can be seen that the Lemma is sharp for every value of $\kappa(G)$ from the following example. Take three disjoint sets A_1 , A_2 and X , such that $|A_1| = |A_2| = a \geq 1$ and $|X| = x$. Let G be a graph over $A_1 \cup A_2 \cup X$ where $A_i \cup X$ is a clique and there are no other edges. We have $\alpha^*(G) = a + 1$, $\kappa(G) = x$ and $\delta(G) = a + x - 1$, hence the statement of Lemma 3.6.13 is met with equality. Hamiltonian graphs are 2-connected, so the least we can require for hamiltonicity is $\delta(G) \geq \alpha^*(G)$.

Theorem 3.6.14. *A graph G with at least 3 vertices is Hamiltonian if $\delta(G) \geq \alpha^*(G)$.*

Proof. If $\alpha^*(G) = 1$, then G is complete, and so G is Hamiltonian. Thus we may suppose that $\alpha^*(G) \geq 2$. We will show that if P is a maximal length path in G , then $G[V(P)]$ is Hamiltonian. This, together with the connectedness of G following from Lemma 3.6.13, is enough to complete the proof.

Indeed, suppose P is a maximal length path in G , $n = v(P)$, and label the vertices in $V(P)$ with $[n] := \{1, \dots, n\}$ in the order they appear in the path, after choosing an arbitrary orientation. We may assume that vertices 1 and n are not adjacent. For a set $S \subseteq V(P)$, define S^+ to be the set of successors x^+ of elements x in S , and define S^- to be the set of predecessors x^- . We leave S^+ undefined if $n \in S$ and S^- is undefined if $1 \in S$.

We now describe three situations when P can be closed to form a cycle. The first yields a standard proof of Dirac's and Ore's theorems, the second involves 'non-crossing' edges from the end vertices, and the third involves 'crossing edges'.

- (a) If for some $j \in (1, n)$ we have $j \in N(1)$ and $j^- \in N(n)$, then $1j - nj^- - 1$ is a spanning cycle of $V(P)$ (where we follow the path P from j to n and from j^- to 1). See Figure 1.

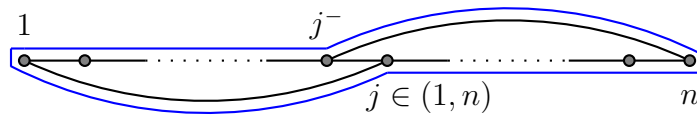


Figure 3.1: Single flip

- (b) If for some $k \in (1, n)$ there exist $i \in N(1) \cap (1, k]$ and $j \in N(n) \cap [k, n)$ such that i^- is adjacent to j^+ , then $1 - i^- j^+ - n j - i 1$ is a spanning cycle of $V(P)$. Here we may have $i = j$; see Figure 2.

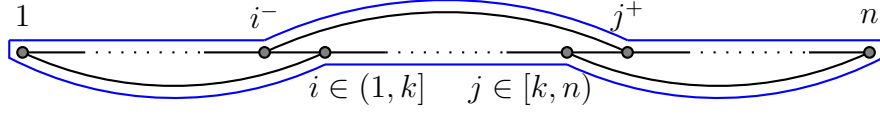


Figure 3.2: Double nested flip

- (c) If for some $k \in (1, n)$ there exist $i \in N(1) \cap [k, n)$ and $j \in N(n) \cap [1, k)$ such that i^+ is adjacent to j^+ , then $1 - j n - i^+ j^+ - i 1$ is a spanning cycle of $V(P)$. Here we may have $j^+ = i$; see Figure 3.

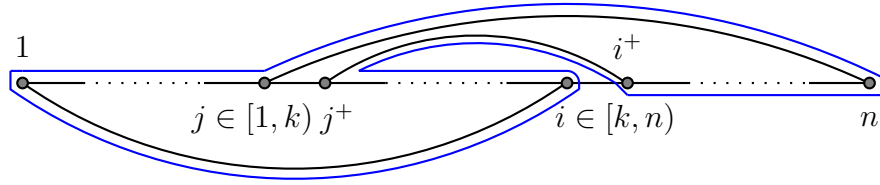


Figure 3.3: Double cross flip

We shall show that at least one of these situations must hold. Suppose for a contradiction that this is not the case. Then for every $k \in (1, n)$

$$E[(N(1) \cap (1, k])^-, (N(n) \cap [k, n))^+] = \emptyset \quad (3.20)$$

since (b) does not hold; and

$$E[\{1\} \cup (N(1) \cap [k, n))^+, (N(n) \cap [1, k))^+] = \emptyset \quad (3.21)$$

since (a) and (c) do not hold.

Let $1 \leq s \leq t$ be such that $\alpha^*(G) + 1 = s + t$ and G has no (s, t) -bipartite-hole. Since $\alpha^*(G) \geq 2$, we have $s \leq \frac{\alpha^*(G)+1}{2} < \alpha^*(G)$, and hence

$$|N(1) \cap (1, 2]| = 1 \leq s \leq \delta(G) - 1 < |N(1) \cap (1, n]| = d(1).$$

Therefore we can choose $k \in (1, n)$ such that $|N(1) \cap (1, k]| = s$. Equation (3.20) implies that $|N(n) \cap [k, n)| < t$. Since $|N(n) \cap [1, k)| + |N(n) \cap [k, n)| \geq \delta(G)$, we have $|N(n) \cap [1, k)| > \delta(G) - t \geq \alpha^*(G) - t = s - 1$, and so $|N(n) \cap [1, k)| \geq s$. Now from (3.21) we deduce $|N(1) \cap [k, n)| < t - 1$, hence $|N(1) \cap [k, n)| \leq t - 2$. Finally, since 1 is not adjacent to n , we have

$$\delta(G) \leq |N(1) \cap (1, k]| + |N(1) \cap [k, n)| \leq s + t - 2 \leq \delta(G) - 1,$$

and this contradiction completes the proof. $\square$

Another example where the theorem is sharp is the complete bipartite graph, $K_{r,r+1}$, with $\delta(K_{r,r+1}) = r$ and $\alpha^*(K_{r,r+1}) = r + 1$. Theorem 3.6.14 generalises Theorem 3.6.11 of Dirac. Indeed, a graph G with $\delta(G) \geq n/2$ has no $(1, \lfloor n/2 \rfloor)$ -bipartite-hole, and hence $\delta(G) \geq n/2 \geq \alpha^*(G)$.

A related theorem by Chvátal and Erdős [CE72], has a sharp condition close to Theorem 3.6.14.

Theorem 3.6.15 (Chvátal-Erdős Theorem). *A graph G with at least 3 vertices is Hamiltonian if $\kappa(G) \geq \alpha(G)$.*

There are interesting connections between Theorems 3.6.14 and 3.6.15, and between the parameters κ , δ , α and α^* . For example, $\kappa(G) \leq \delta(G) \leq v(G) - \alpha(G)$ and $\alpha(G) \leq \alpha^*(G) \leq v(G) - \kappa(G)$. Furthermore, as we saw in Lemma 3.6.13, $\kappa(G) \geq \delta(G) - \alpha^*(G) + 2$.

Comparing Theorems 3.6.14 and 3.6.15, neither condition implies the other. Here is an example of a graph G that meets the conditions of Theorem 3.6.14 but not Theorem 3.6.15. It has vertex set $V(G) = \{a\} \cup B \cup C \cup D$, such that $|B| = k + \ell$, $|C| = k$, $|D| = \ell + 1$, and all these sets are disjoint. All edges between $\{a\}$ and B , between B and C , and between C and D are present, B and D are complete, and C is independent. It is easy to see that $k = \kappa(G) < \alpha(G) = k + 1$, and $\delta(G) = k + \ell \geq \max\{k + 1, 2\ell + 3\} = \alpha^*(G)$ for $\ell \geq 1$ and $k \geq \ell + 3$. In the other direction, C_5 satisfies $\kappa = 2 = \alpha$ but $\delta = 2 < 3 = \alpha^*$.

Theorem 3.6.14 can be extended to provide a sufficient condition for the existence of many edge-disjoint Hamilton cycles; and in fact the next result will be deduced quickly from Theorem 3.6.14. We need a preliminary lemma. For graphs F and G with the same vertex set V , we define $F \cup G = (V, E(F) \cup E(G))$ and $F - G = (V, E(F) \setminus E(G))$.

Lemma 3.6.16. *Suppose $H_1, \dots, H_r$ are $r \geq 1$ Hamilton cycles in a graph G and let $H = H_1 \cup \dots \cup H_r$. Then $\alpha^*(G - H) + 1 \leq (r + 1)(\alpha^*(G) + 1)$.*

Proof. Let $1 \leq s \leq t$ be such that $\alpha^*(G) + 1 = s + t$ and G has no (s, t) -bipartite-hole. Let $U, W \subseteq V(G)$ be disjoint sets of size s and $2rs + t$ respectively. Now

$$|W \setminus N_H(U)| \geq |W| - \sum_{i=1}^r |N_{H_i}(U)| \geq 2rs + t - 2rs = t.$$

But G has no (s, t) -bipartite-hole, so $G - H$ has no $(s, 2rs + t)$ -bipartite-hole. Finally, we see that $\alpha^*(G - H) + 1 \leq s + 2rs + t \leq (r + 1)(\alpha^*(G) + 1)$, since $s \leq \frac{\alpha^*(G) + 1}{2}$. $\square$

Theorem 3.6.17. *Let $r \geq 0$ be an integer, and let G be a graph with at least 3 vertices such that $\delta(G) \geq (r + 1)\alpha^*(G) + 3r$. Then G contains $r + 1$ edge-disjoint Hamilton cycles.*

Proof. We sequentially find edge-disjoint Hamilton cycles $H_1, H_2, \dots$. Let $0 \leq i \leq r$ and suppose we have found $H_1, \dots, H_i$. Let $G_i = G - \cup_{j \leq i} H_j$. Then by Lemma 3.6.16

$$\alpha^*(G_i) \leq (i+1)(\alpha^*(G) + 1) - 1 \leq (r+1)\alpha^*(G) + r \leq \delta(G) - 2r \leq \delta(G_i).$$

Hence by Theorem 3.6.14 we can find H_{i+1} edge-disjoint from $H_1, \dots, H_i$. $\square$

Note that by setting $r = 0$ in Theorem 3.6.17 we regain Theorem 3.6.14.

We conclude the section by applying Theorem 3.6.17 to show quickly that for a sufficiently dense random graph G , the probability of G failing to contain many edge-disjoint Hamilton cycles is well-estimated by the probability that G contains a vertex with too small degree ($< 2r$), or indeed contains an isolated vertex.

Theorem 3.6.18. *Let $0 < \epsilon < 1$, let $0 \leq p = p(n) \leq 1 - \epsilon$, and let $r = r(n)$ be a positive integer. If $\frac{p(n)\sqrt{n}}{r(n)\log n} \rightarrow \infty$ as $n \rightarrow \infty$, then the probability that $G(n, p)$ fails to contain at least r edge-disjoint Hamilton cycles is $(1 - p)^{(1+o(1))n}$.*

Setting $r = 1$ we obtain:

Corollary 3.6.19. *If $p(n)\sqrt{n}/\log n \rightarrow \infty$ as $n \rightarrow \infty$, then the probability that $G(n, p)$ fails to be Hamiltonian is $(1 - p)^{(1+o(1))n}$.*

The following result is phrased to cover the existence of one Hamilton cycle, and of many.

Lemma 3.6.20. *Fix $0 < \epsilon < 1$ and let $0 \leq p = p(n) \leq 1 - \epsilon$ for all n . Given $r = r(n) \geq 1$, let A_r be the event that $G(n, p)$ contains at least r edge-disjoint Hamilton cycles, and let A_r^c be the complementary event. Then*

$$n \log(1 - p) \leq \log \mathbb{P}(A_r^c) \leq n \log(1 - p) + (2 + o(1)) r \sqrt{n} \log n.$$

Proof. Let $G \sim G(n, p)$, $t = \lceil \sqrt{n} \rceil$ and $d = r(2t) + 3r - 3$. From Theorem 3.6.17 we have $\{\alpha^*(G) \leq 2t\} \cap \{\delta(G) \geq d\} \subseteq A_r$, so

$$\{\delta(G) = 0\} \subseteq A_r^c \subseteq \{\alpha^*(G) > 2t\} \cup \{\delta(G) < d\}.$$

Clearly $\mathbb{P}(\delta(G) = 0) \geq (1 - p)^n = \exp(n \log(1 - p))$. Also, the probability that vertex n has degree at most $d - 1$ is at most the expected number of $(d - 1)$ -subsets of $[n - 1]$ such that each other vertex is not adjacent to n . Thus

$$\begin{aligned} \mathbb{P}(\delta(G) < d) &\leq n \binom{n-1}{d-1} (1-p)^{(n-1)-(d-1)} \leq n^d (1-p)^n \epsilon^{-d} \\ &= \exp(n \log(1 - p) + d(\log n + \log(1/\epsilon))). \end{aligned}$$

Further

$$\begin{aligned}
\mathbb{P}(\alpha^*(G) > 2t) &\leq \mathbb{P}(G \text{ has a } (t, t)\text{-bipartite-hole}) \\
&\leq \binom{n}{t}^2 (1-p)^{t^2} \leq \left(\frac{en}{t}\right)^{2t} (1-p)^{t^2} \leq e^{2t} n^t (1-p)^n \\
&= \exp(n \log(1-p) + \sqrt{n}(\log n + O(1)));
\end{aligned}$$

and the required upper bound on $\log \mathbb{P}(A_r^c)$ follows since $d = (2 + o(1)) r \sqrt{n}$. $\square$

Theorem 3.6.18 and Corollary 3.6.19 follow directly from Lemma 3.6.20.

3.7 Connectivity and Chromatic index

3.7.1 Connectivity

A set of vertices in a graph G is a *cutset* if removing these vertices leaves a disconnected graph. The (*vertex*) *connectivity* of G is the minimum size of a cutset, except that, if G is complete, then by convention $\kappa(G) = |V(G)| - 1$.

We use $\delta(G)$ to denote the minimum vertex degree. Observe that $\delta(G)$ is natural upper bound for $\kappa(G)$. We prove the following theorem:

Theorem 3.7.1. *The connectivity of almost all perfect graphs equals their minimum degree. Indeed, wvhp $\kappa(P_n) = \delta(P_n)$, where $P_n \in_u \mathcal{P}_n$.*

Proof. We first consider unipolar graphs. Let (G^+, C) be the unipolar arrangement induced by the generation of $Gen^+(n)$. Then, for each fixed ϵ with $0 < \epsilon < \frac{1}{20}$, G^+ satisfies each of the following properties wvhp:

- (a1) $\forall u \in \overline{C}, d_C(u) \leq (1 + \epsilon) \frac{1}{4}n$.
- (a2) $\forall u \in \overline{C}, d_{\overline{C}}(u) \leq \epsilon n$.
- (a3) $\forall u, v \in \overline{C}, |(N(v) \cup N(u)) \cap C| \geq (1 - \epsilon) \frac{3}{8}n$.
- (a4) $|C| > (1 - \epsilon) \frac{n}{2}$.

Chernoff's inequality is sufficient to prove (a1) and (a3); for (a2) we can use Lemma 3.2.9, and (a4) follows from the concentration theorem, Theorem 3.2.5.

These conditions are sufficient to ensure $\kappa(G^+) = \delta(G^+)$. Note first that if $\overline{C} = \emptyset$ then G^+ is a clique and the result holds; so we may assume that $\overline{C} \neq \emptyset$. Suppose for contradiction that $Q \subset V = V(G^+)$ is a separator of G^+ with $|Q| < \delta(G^+)$. Note that $\delta(G^+) \leq (1 + 5\epsilon) \frac{n}{4} \leq |C|$, and so $C \not\subseteq Q$. We claim that every vertex of $V \setminus Q$ is connected to a vertex in $C \setminus Q$, which will yield our contradiction. This is obvious

for vertices $u \in C \setminus Q$. If $u \in \overline{C}$, then since $|Q| < \delta(G^+)$, u must have a neighbour v outside Q . If $v \in C \setminus Q$ we are done, so suppose that $v \in N(u) \cap \overline{C}$. From (a3), u and v see together at least $(1 - \epsilon)\frac{3}{8}n$ vertices of C , while $|Q|$ is at most $(1 + 5\epsilon)\frac{n}{4}$, so there must be a vertex from $C \setminus Q$ connected to either u or v .

The situation is similar for co-unipolar graphs. Let (G^-, C) be the co-unipolar arrangement induced by the generation of $Gen^-(n)$. Then, for each fixed ϵ with $0 < \epsilon < \frac{1}{20}$, G^- satisfies the following conditions wvhp.

$$(b1) \forall u \in C, d(u) \leq (1 + \epsilon)\frac{1}{4}n.$$

$$(b2) \text{ Each side independent set in } \overline{C} \text{ has size at most } \epsilon n.$$

$$(b3) |\overline{C}| > (1 - \epsilon)\frac{n}{2}.$$

To prove these, we can use a Chernoff bound for (b1), Lemma 3.2.9 for (b2), and Theorem 3.2.5 for (b3).

These conditions are sufficient to ensure $\kappa(G^-) = \delta(G^-)$. Note first that if $C = \emptyset$ then G^- is a complete multipartite graph and the result holds; so we may assume that $C \neq \emptyset$. Thus $\delta(G^-) \leq (1 + \epsilon)\frac{1}{4}n$. Suppose that $Q \subset V = V(G^-)$ is a cutset of G^- and $|Q| < \delta(G^-)$. Fix some vertex $w \in V \setminus Q$. We claim that every vertex of $V \setminus Q$ is connected to w .

First consider $u \in \overline{C}$. Then w and u are not neighbours (if and) only if they are contained in the same side independent set I . But then there is a vertex $x \in \overline{C} \setminus Q$ in a different side independent set, since $|\overline{C} \setminus Q| > (1 - 3\epsilon)\frac{n}{4} \geq |I|$; and the path u, x, w connects u to w . Now consider $u \in C$. Since $|Q| < \delta(G)$, u must have a neighbour in $\overline{C}$, and we just seen that all vertices in $\overline{C}$ are connected to w .

We have now shown that $\mathbb{P}_{Gen(n)}(\kappa(G) = \delta(G)) = 1 - e^{-\Omega(n)}$, and Theorem 3.2.3 completes the proof. $\square$

3.7.2 Chromatic Index

The *chromatic index* of a graph G , $\chi'(G)$, is the minimum number of colours required to colour the edges, so that no edges with the same colour share a vertex. Vizing proved in [Viz64] that $\chi'(G) = \Delta(G)$ or $\chi'(G) = \Delta(G) + 1$. If $\chi'(G) = \Delta(G)$, then G is said to be class one, otherwise it is said to be class two.

In this subsection we prove

Theorem 3.7.2. *Almost all perfect graphs are class one graphs.*

Class one graphs have a natural connection with perfect graphs. Denote the line graph of a graph G by $L(G)$. Then, for graphs G with $\Delta(G) \geq 3$, $\Delta(G) = \omega(L(G))$ and

$\chi'(G) = \chi(L(G))$; and hence G is a class one graph iff $\omega(L(G)) = \chi(L(G))$. Vizing showed in [Viz65] that each graph of class two has at least 3 vertices of maximum degree (and indeed the vertices of maximum degree induce a subgraph with a cycle). Erdős and Wilson proved in [EW77] that almost all graphs have a unique vertex of maximum degree, and thus are class one, see also [FJMR88]. We shall show that almost all perfect graphs have a unique vertex of maximum degree, and thus are class one and can be Δ -edge-coloured in $O(n^3)$ time. (If G has a unique vertex v_0 of maximum degree then we can Δ -edge-colour G in $O(nm)$ -time as follows. Pick an edge e incident with v_0 , use for example the algorithm of [MG92] to $\Delta(G)$ -edge-colour $G \setminus e$, and then use a standard Vizing fan iteration to colour e .)

Let $d_j = d_j(G)$ be the j th largest degree in the list of all $v(G)$ degrees. Theorem 15 in Chapter III (The degree sequence) of the book [Bol01] concerns the first few gaps $d_i - d_{i+1}$ for a random graph $G_{n,p}$. The next lemma is a much easier result that may be proved along similar lines; we do not spell a proof here.

Lemma 3.7.3. *Whp $B := G_{n,m,1/2}$ with $m/n \rightarrow 1$ is such that*

$$d_1(B) - d_2(B) \geq \frac{n^{1/2}}{\log n}.$$

where $d_1(B)$ and $d_2(B)$ are the largest and second largest degrees of vertices in the first colour class.

Let (G^+, C) be the unipolar arrangement induced by the generation of $Gen^+(n)$, and let B denote the corresponding bipartite graph with parts C and $\overline{C}$. With high probability, $d(u) \geq (1 - \epsilon)\frac{3}{4}n$ for all u in C , and $d(v) \leq (1 + \epsilon)\frac{1}{4}n$ for all v in $\overline{C}$. Hence, by Lemma 3.7.3, whp

$$d_1(G^+) - d_2(G^+) = d_1(B) - d_2(B) \geq \frac{n^{1/2}}{2 \log n},$$

and so whp G^+ has a unique vertex of maximum degree.

Let (G^-, C) be the co-unipolar arrangement induced by the generation of $Gen^-(n)$, and let B denote the corresponding bipartite graph with parts $\overline{C}$ and C (in this order). With high probability, $d(u) \geq (1 - \epsilon)\frac{3}{4}n$ for all u in $\overline{C}$, and $d(v) \leq (1 + \epsilon)\frac{1}{4}n$ for all v in C . Also, by Lemma 3.2.9, with high probability $|d_{\overline{C}}(u) - d_{\overline{C}}(v)| < 10 \ln n$ for each $u, v \in \overline{C}$, and therefore

$$\begin{aligned} d_1(G^-) - d_2(G^-) &\geq d_1(B) - d_2(B) - 10 \ln n \\ &\geq (1 - \epsilon) \frac{n^{1/2}}{2 \log n} - 10 \ln n, \end{aligned}$$

and so whp G^- has a unique vertex of maximum degree.

We have now seen that whp both G^+ and G^- have a unique vertex of maximum degree, which implies they are class one; Theorem 3.7.2 now follows from Theorem 3.2.3.

3.8 The limit of a sequence of random perfect graphs

We show that a sequence of uniformly and independently sampled n -vertex perfect graphs converges with probability one to the graphon

$$W_P(x, y) = \frac{\mathbb{1}[x \leq 1/2] + \mathbb{1}[y \leq 1/2]}{2}$$

and show how to use this for estimates on the subgraph densities.

3.8.1 Notation

We use the notation for left limits from [Lov12]. Suppose throughout (as usual) that P_n is a uniformly sampled perfect graph on vertex set $[n]$. Let λ be the Lebesgue measure on $\mathbb{R}^2$. A *kernel* $W : [0, 1]^2 \rightarrow \mathbb{R}$ is an a.e. bounded symmetric measurable function. A *graphon* W is a kernel with $0 \leq W \leq 1$ a.s. The *cut norm*, $\|\cdot\|_\square$, is a norm on the vector space of kernels defined by

$$\|W\|_\square = \sup_{S, T \subseteq [0, 1]} \left| \int_{S \times T} W d\lambda \right|.$$

The supremum is taken over all measurable $S, T \subseteq [0, 1]$. Let $S_{[0, 1]}$ denote the set of invertible measure preserving maps on $[0, 1]$. Define $W^\varphi(x, y) := W(\varphi(x), \varphi(y))$ for $\varphi \in S_{[0, 1]}$. The *cut distance*, $\delta_\square$, is defined by

$$\delta_\square(U, W) = \inf_{\varphi \in S_{[0, 1]}} \|U - W^\varphi\|_\square$$

for kernels U, W . For a graph G we define the graphon W_G by partitioning $[0, 1] = S_1 \sqcup \dots \sqcup S_{v(G)}$, $\lambda(S_i) = 1/v(G)$, and writing $W(x, y) = \mathbb{1}_{ij \in E(G)}$ if $(x, y) \in S_i \times S_j$. Let $\text{hom}(F, G)$ and $\text{hom}_{\text{inj}}(F, G)$ be the number of homomorphisms (edge preserving maps), injective homomorphisms respectively, from graph F to graph G . Let $t(F, G) = \frac{\text{hom}(F, G)}{n^k}$ and let $t_{\text{inj}}(F, G) = \frac{\text{hom}_{\text{inj}}(F, G)}{n(n-1)\dots(n-k+1)}$, where $k = v(F)$ and $n = v(G)$. For a graph F and kernel W define

$$t(F, W) = \int_{[0, 1]^{v(F)}} \prod_{ij \in E(G)} W(x_i, x_j) \prod_{i \in V(G)} dx_i.$$

We have $t(F, G) = t(F, W_G^\varphi)$ for each pair of graphs F and G and each $\varphi \in S_{[0, 1]}$. A sequence of graphs, $G_1, G_2, \dots$, is said to converge to a graphon W , denoted by $G_n \rightarrow W$, if W_{G_n} converges to W with respect to $\delta_\square$.

3.8.2 Convergence results

Theorem 3.8.1. *We have*

$$\delta_{\square}(W_{P_n}, W_P) \leq n^{-1/2} \text{ with probability } 1 - e^{-\Omega(\sqrt{n} \log n)}, \quad (3.22)$$

$$\delta_{\square}(W_{P_n}, W_P) \leq (\log n)^{-2} \text{ with probability } 1 - e^{-\Omega(n)}. \quad (3.23)$$

Before we prove Theorem 3.8.1 we discuss a few corollaries. Suppose each P_n is sampled independently.

Corollary 3.8.2. *With probability one we have $P_n \rightarrow W_P$.*

Proof. This follows from Theorem 3.8.1 (either part), a Borel-Cantelli lemma, and Theorem 11.22 of [Lov12]. $\square$

Corollary 3.8.3. *For a fixed graph F*

$$|t_{\text{inj}}(F, P_n) - t(F, W_P)| \leq \binom{v(F)}{2} n^{-1} + e(F) n^{-1/2}$$

with probability at least $1 - e^{-\Omega(\sqrt{n} \log n)}$.

Proof. It is easy to see that for every graph G we have

$$|t_{\text{inj}}(F, G) - t(F, G)| \leq \binom{v(F)}{2} \frac{1}{v(G)}.$$

Theorem 10.23 in [Lov12] states that

$$|t(F, W_1) - t(F, W_2)| \leq e(F) \cdot \delta_{\square}(W_1, W_2)$$

for every two graphons W_1 and W_2 . The rest follows from part 1 of Theorem 3.8.1. $\square$

The notation in the second corollary may be misleading, in the sense that the statement has nothing to do with graph limits. Indeed, the value $t(F, W_P)$ can be calculated explicitly and then used as an estimate on the number of subgraphs F of P_n . For example, if F is the single edge K_2 then $t(F, W_P) = \frac{1}{2}$ so the random perfect graph P_n contains $(1 + O(n^{-1/2})) \frac{n^2}{4}$ edges with probability at least $1 - e^{-\Omega(\sqrt{n} \log n)}$; and similarly P_n contains $(1 + O(n^{-1/2})) \frac{5n^3}{96}$ triangles with probability at least $1 - e^{-\Omega(\sqrt{n} \log n)}$. Using an inclusion-exclusion argument we can estimate the densities of induced graphs and in particular Corollary 3.8.3 implies that the split graphs are the only graphs with induced subgraph density bounded away from zero.

We can be more precise for example concerning the number $e(P_n)$ of edges of P_n . Note that the expected value is exactly $\frac{1}{2} \binom{n}{2}$. Also, whp $e(G_n^+) = \frac{1}{4} n^2 + (\frac{1}{4} + o(1)) n \ln n$ and so whp $e(G_n^-) = \frac{1}{4} n^2 - (\frac{1}{4} + o(1)) n \ln n$; and hence whp $e(P_n) = \frac{1}{4} n^2 + O(n \log n)$, with a bimodal distribution.

3.8.3 Proof of Theorem 3.8.1

We consider the unipolar case $G^+ \sim \text{Gen}^+(n)$ and the co-unipolar case $G^- \sim \text{Gen}^-(n)$ separately, and start with the former. Suppose $\text{Gen}^+(n) = (1, E, (k, \sigma), \pi)$. Let $\varphi \in S_{[0,1]}$ be a measure preserving map, mapping each vertex of G^+ to an interval of the form $V_i = [\frac{i-1}{n}, \frac{i}{n})$ where $i \in [n]$, and in addition let φ map the central clique of W_{G^+} to $[0, k/n)$ and the side cliques to $[k/n, 1)$. Pick arbitrary measurable $S, T \subseteq [0, 1]$. To prove that $\delta_{\square}(G^+, W_P) \leq \epsilon$ it is sufficient to show that

$$\left| \int_{S \times T} (W_{G^+}^{\varphi} - W_P) \right| \leq \epsilon.$$

Claim 3.8.4. *We can find $S', T' \subseteq [0, 1]$ of the form $S' = \cup_{i \in I_s} V_i$ and $T' = \cup_{i \in I_t} V_i$, where $I_s, I_t \subseteq [n]$, such that*

$$\left| \int_{S \times T} (W_{G^+}^{\varphi} - W_P) \right| \leq \left| \int_{S' \times T'} (W_{G^+}^{\varphi} - W_P) \right|.$$

Proof. Initially set $S' = S$ and $T' = T$. For $i = 1, \dots, n$ in turn, check if $\int_{T'} (W_{G^+}^{\varphi} - W_P)(\frac{i-1}{n}, y) dy$ has the same sign as $\int_{S' \times T'} (W_{G^+}^{\varphi} - W_P) d\lambda$, and if so, add the entire interval V_i to S' , or remove V_i from S' otherwise. These operations can only increase the absolute value of the integral. Now repeat the same procedure for T' . It is clear that in the end we obtain S' and T' with the desired properties. $\square$

Every such pair (S', T') is measurable. We may assume that initially we are given S and T of this form.

Let $A_1 = [0, 1/2)$, $B_1 = [1/2, 1)$, $A_2 = [0, k/n)$, $B_2 = [k/n, 1)$, $A = A_1 \cap A_2$, $B = B_1 \cap B_2$ and $I = A \cup B$. Recall that we use k to denote the size of the central clique in the generation process. By Theorem 3.2.5, wvhp both $\lambda(A)$ and $\lambda(B)$ are contained in $[\frac{1}{2} - \frac{n^{-1/2}}{24}, \frac{1}{2}]$. Condition on this event, and now we have

$$\left| \int_{S \times T} (W_{G^+}^{\varphi} - W_P) \right| \leq \left| \int_{(S \times T) \cap (I \times I)} (W_{G^+}^{\varphi} - W_P) \right| + \frac{n^{-1/2}}{12}.$$

Let $O_1 = (S \times T) \cap (A \times A)$, $O_2 = (S \times T) \cap (A \times B)$, $O_3 = (S \times T) \cap (B \times A)$, $O_4 = (S \times T) \cap (B \times B)$, and $w_i = \left| \int_{O_i} (W_{G^+}^{\varphi} - W_P) \right|$.

We have $W_{G^+}^{\varphi} = W_P = 1$ over O_1 , so clearly $w_1 = 0$. The restriction of $W_{G^+}^{\varphi}$ over O_2 is a step function corresponding to a random bipartite graph with density $1/2$, while W_P is uniformly $1/2$. This classical example motivated the study of the cut distance, but we give full details here.

Claim 3.8.5. *Suppose M is a random $\{-1, 1\}$ -valued $m \times m$ matrix, where each entry has expected value 0 and is independent from the others. Then the maximum absolute value of a rectangular sum in M is at most $\frac{5}{3}m^{3/2}$ wvhp.*

Proof. Let $R \subseteq [m]$ be a subset of the rows of M , and let $Q \subseteq [m]$ be a subset of the columns of M . Clearly $M_{Q,R} := \sum_{r \in R, q \in Q} M_{r,q} \sim 2X - |Q||R|$, where $X \sim \text{Bin}(|Q||R|, 1/2)$; so by Chernoff's inequality

$$\mathbb{P}(|M_{Q,R}| \geq x) = \mathbb{P}\left(\left|X - \frac{|Q||R|}{2}\right| \geq \frac{x}{2}\right) \leq 2e^{-\frac{x^2}{2m^2}}.$$

Taking $x = \frac{5}{3}m^{3/2}$, and the union bound over all 2^{2m} possible choices for R, Q completes the proof. $\square$

Setting $m = \lceil n/2 \rceil$ and rescaling in Claim 3.8.5 shows that w_2 and w_3 are bounded by $\frac{5}{12}n^{-1/2}$ wvhp.

Finally, w_4 is bounded by twice the number of edges in the partition part of G^+ divided by n^2 . Lemma 3.2.9 states that the maximum size of a part in a random partition is at most x with probability at least $1 - (1 + o(1))e^{-x(\ln x - \ln \ln n - 2) + \ln n}$. The maximum size of a part bounds the maximum degree. Here we have to make a compromise between low cut distance and high probability. To favour the former, use Lemma 3.2.9 with $x = \frac{1}{12}\sqrt{n}$ and deduce that

$$\left| \int_{S \times T} (W_{G^+}^\varphi - W_P) \right| \leq \frac{1}{\sqrt{n}}$$

with probability $1 - e^{-\Omega(\sqrt{n} \log n)}$.

Now let us aim for a bound that holds wvhp. Let $\log n \leq x \leq n/2$. Observe that if x vertices are contained in parts of σ each of size at least s , then x vertices are contained in at most $\lceil x/s \rceil$ parts. Thus, assuming $s \leq x$ and so $\lceil x/s \rceil \leq 2x/s$, the probability that some set of x points is contained in parts of σ each of size at least s is at most

$$\begin{aligned} & \binom{n}{x} \left(\left\lceil \frac{x}{s} \right\rceil\right)^x n \frac{B_{n-x}}{B_n} \\ & \leq \left(\frac{en}{x} \cdot \frac{2x}{s}\right)^x (1 + o(1)) \left(\frac{r_n}{r_{n-x}}\right)^{\frac{1}{2}} n \exp\{-x(r_n - 1 + 1/r_n)\} \\ & = \left(\frac{2en}{s}\right)^x \exp\{-x(\ln n - (1 + o(1)) \log \log n)\} \\ & = \exp\{-x(\ln s - (1 + o(1)) \log \log n)\}. \end{aligned}$$

The factor n in the first bound above arises since the ‘last part’ for the x -set may need to be amalgamated with some part for the $(n-x)$ -set. Set $x = \lceil n/(3 \log n) \rceil$ and $s = n/\log^2 n$, to see that wvhp at most $n/(3 \log n)$ points are contained in parts of σ each of size at least $n/\log^2 n$. Since also wvhp no part has size greater than $n/\log n$ by (3.12), we see that wvhp the number of edges in $G^+[\bar{S}]$ is at most

$$\left(\frac{1}{2} + o(1)\right)n \cdot \frac{n}{\log^2 n} + \frac{n}{3 \log n} \frac{n}{\log n}$$

so wvhp $w_4 \leq \frac{5}{6}(\log n)^{-2}$. Hence wvhp $\delta_{\square}(G^+, W_P) \leq (\log n)^{-2}$.

Let J be the graphon with $J(x, y) = 1$ for each $x, y \in [0, 1]$; and note that $J - W_P$ may be written as W_P^{ψ} for some $\psi \in S_{[0,1]}$. Then

$$\int_{S \times T} (W_{G^-}^{\varphi} - W_{P^{\psi}}) = - \int_{S \times T} (W_{G^+}^{\varphi} - W_P).$$

Hence, the bounds for G^+ transfer to G^- . Theorem 3.2.3 completes the proof.

Using (3.13) we may see that the bound in (3.23) is best possible, in the sense that, if $\epsilon(n) = o(1)$, then $\delta_{\square}(W_{P_n}, W_P) \geq \epsilon(n)(\log n)^{-2}$ holds with probability $e^{-o(n)}$.

3.9 Proofs for the generation model

We conclude this chapter with the proofs of Theorems 3.2.3 and 3.2.5, which were deferred to here. Of course, the proofs of these theorems do not depend on any of the earlier work which used them.

3.9.1 Proof of Theorem 3.2.3

We shall prove (3.6) and (3.7) and complete the proof of Theorem 3.2.3. For a real number $p \geq 1$ we denote the $\mathcal{L}_p$ norm of X in the probability space $(\mathcal{G}_n, 2^{\mathcal{G}_n}, \mu)$ by $\|X\|_p^{\mu}$. Recall that $\|X\|_p^{\mu}$ is non-decreasing in p . Let ν and μ be discrete probability measures on (all subsets of) $\mathcal{G}_n$ such that $\nu(G) > 0$ implies $\mu(G) > 0$. In this case ν is said to be *absolutely continuous* with respect to μ , written $\nu \ll \mu$. The Radon–Nikodym derivative $\frac{d\nu}{d\mu}$ is given by $\frac{d\nu}{d\mu}(G) = \frac{\nu(G)}{\mu(G)} \mathbf{1}_{\mu(G)>0}$. We may express the total variation distance between ν and μ in terms of $\frac{d\nu}{d\mu}$ and the $\mathcal{L}_1$ norm $\|\cdot\|_1^{\mu}$: we have

$$2d_{TV}(\nu, \mu) = \sum_G |\nu(G) - \mu(G)| = \sum_G \left| \frac{d\nu}{d\mu}(G) - 1 \right| \mu(G) = \left\| \frac{d\nu}{d\mu} - 1 \right\|_1^{\mu}.$$

(The same hold for general, not necessarily discrete, probability measures, provided that $\nu \ll \mu$.) Also note that for each G

$$\frac{d\mathbb{P}_{Gen(n)}}{d\mathbb{P}_{\mathcal{P}_n}}(G) = \mathbb{P}(\rho(Gen(n)) = G) |\mathcal{P}_n|.$$

The main result in this section is stronger than Theorem 3.2.3.

Theorem 3.9.1. *For each $p \geq 1$ we have*

$$2d_{TV}(\mathbb{P}_{Gen(n)}, \mathbb{P}_{\mathcal{P}_n}) = \left\| \frac{d\mathbb{P}_{Gen(n)}}{d\mathbb{P}_{\mathcal{P}_n}} - 1 \right\|_1^{\mathbb{P}_{\mathcal{P}_n}} \leq \left\| \frac{d\mathbb{P}_{Gen(n)}}{d\mathbb{P}_{\mathcal{P}_n}} - 1 \right\|_p^{\mathbb{P}_{\mathcal{P}_n}} = e^{-\Theta(n)}.$$

We have just noted the first two (in)equalities in the statement of the theorem: the non-trivial part is the final equality. To prove it we start from the triangle inequality

$$\left\| \frac{d\mathbb{P}_{Gen(n)}}{d\mathbb{P}_{\mathcal{P}_n}} - 1 \right\|_p^{\mathbb{P}_{\mathcal{P}_n}} \leq \left\| \frac{d\mathbb{P}_{Gen(n)}}{d\mathbb{P}_{\mathcal{P}_n}} - \frac{d\mathbb{P}_{\mathcal{GS}_n}}{d\mathbb{P}_{\mathcal{P}_n}} \right\|_p^{\mathbb{P}_{\mathcal{P}_n}} + \left\| \frac{d\mathbb{P}_{\mathcal{GS}_n}}{d\mathbb{P}_{\mathcal{P}_n}} - 1 \right\|_p^{\mathbb{P}_{\mathcal{P}_n}}.$$

It is straightforward to show from (3.4) that the second term on the right is $e^{-\Omega(n)}$.

For the first term on the right, routine manipulations yield

$$\left\| \frac{d\mathbb{P}_{Gen(n)}}{d\mathbb{P}_{\mathcal{P}_n}} - \frac{d\mathbb{P}_{\mathcal{GS}_n}}{d\mathbb{P}_{\mathcal{P}_n}} \right\|_p^{\mathbb{P}_{\mathcal{P}_n}} = \left\| \frac{d\mathbb{P}_{Gen(n)}}{d\mathbb{P}_{\mathcal{GS}_n}} - 1 \right\|_p^{\mathbb{P}_{\mathcal{GS}_n}} \left(\frac{|\mathcal{P}_n|}{|\mathcal{GS}_n|} \right)^{1-1/p}.$$

Since $\frac{|\mathcal{P}_n|}{|\mathcal{GS}_n|} = 1 + e^{-\Omega(n)}$ by (3.4), it follows that it is sufficient to show that $\left\| \frac{d\mathbb{P}_{Gen(n)}}{d\mathbb{P}_{\mathcal{GS}_n}} - 1 \right\|_p^{\mathbb{P}_{\mathcal{GS}_n}} = e^{-\Omega(n)}$ to complete the proof of Theorem 3.9.1.

We need a few definitions to continue.

Definition 3.9.2. Let $\mathcal{CGS}_n^+$ (from coloured generalised split graphs) be the set of all unipolar arrangements of order n and let $\mathcal{CGS}_n = \mathcal{CGS}_n^+ \times \{-1, 1\}$.

Recall that $\mathcal{L}_n$ was introduced just before Definition 3.2.1. We see that $|\mathcal{CGS}_n^+| = \mathcal{L}_n$ and $|\mathcal{CGS}_n| = 2\mathcal{L}_n$. A closer look at the definitions reveals that if $((G, C), B) \in_u \mathcal{CGS}_n$ is uniformly selected, if we set $H_1 = G$ if $B = 1$ and $H_1 = \overline{G}$ otherwise, and set $H_2 \sim \rho(Gen(n))$, then H_1 and H_2 are equal in distribution. Therefore

$$\mathbb{P}_{Gen(n)}(G) = \frac{1}{|\mathcal{CGS}_n|} \sum_{C \subseteq [n]} \{ \mathbb{1}[(G, C) \in \mathcal{CGS}_n^+] + \mathbb{1}[(\overline{G}, C) \in \mathcal{CGS}_n^+] \}.$$

To simplify the notation we define $R_+(G) = \sum_{C \subseteq [n]} \mathbb{1}[(G, C) \in \mathcal{CGS}_n^+]$, $R_-(G) = R_+(\overline{G})$ and $R(G) = R_+(G) + R_-(G)$, so that $\sum_G R(G) = |\mathcal{CGS}_n|$ and

$$\mathbb{P}_{Gen(n)}(G) = \frac{R(G)}{|\mathcal{CGS}_n|}.$$

Note that $R(G) > 0$ iff $G \in \mathcal{GS}_n$. Now we see that (for each G)

$$\frac{d\mathbb{P}_{Gen(n)}}{d\mathbb{P}_{\mathcal{GS}_n}} = \frac{\frac{R}{|\mathcal{CGS}_n|}}{\frac{1}{|\mathcal{GS}_n|}} = R \frac{|\mathcal{GS}_n|}{|\mathcal{CGS}_n|}.$$

It is clear from the definitions that $\|R - 1\|_1^{\mathbb{P}_{\mathcal{GS}_n}} = \frac{|\mathcal{CGS}_n|}{|\mathcal{GS}_n|} - 1$. We have

$$\begin{aligned} \left\| \frac{d\mathbb{P}_{Gen(n)}}{d\mathbb{P}_{\mathcal{GS}_n}} - 1 \right\|_p^{\mathbb{P}_{\mathcal{GS}_n}} &= \left\| R \frac{|\mathcal{GS}_n|}{|\mathcal{CGS}_n|} - 1 \right\|_p^{\mathbb{P}_{\mathcal{GS}_n}} = \left\| R - \frac{|\mathcal{CGS}_n|}{|\mathcal{GS}_n|} \right\|_p^{\mathbb{P}_{\mathcal{GS}_n}} \frac{|\mathcal{GS}_n|}{|\mathcal{CGS}_n|} \\ &\leq \left\| R - 1 + 1 - \frac{|\mathcal{CGS}_n|}{|\mathcal{GS}_n|} \right\|_p^{\mathbb{P}_{\mathcal{GS}_n}} \leq \|R - 1\|_p^{\mathbb{P}_{\mathcal{GS}_n}} + \left| 1 - \frac{|\mathcal{CGS}_n|}{|\mathcal{GS}_n|} \right| \\ &= \|R - 1\|_p^{\mathbb{P}_{\mathcal{GS}_n}} + \|R - 1\|_1^{\mathbb{P}_{\mathcal{GS}_n}} \leq 2 \|R - 1\|_p^{\mathbb{P}_{\mathcal{GS}_n}}. \end{aligned}$$

Thus

$$\left\| \frac{d\mathbb{P}_{Gen(n)}}{d\mathbb{P}_{\mathcal{GS}_n}} - 1 \right\|_p^{\mathbb{P}_{\mathcal{GS}_n}} \leq 2 \|R - 1\|_p^{\mathbb{P}_{\mathcal{GS}_n}}. \quad (3.24)$$

Further simplifications lead us to

$$\begin{aligned} \|R - 1\|_p^{\mathbb{P}_{\mathcal{GS}_n}} &= \|(R_+ - \mathbb{1}_{\mathcal{GS}_n^+}) + (R_- - \mathbb{1}_{\mathcal{GS}_n^-}) + (\mathbb{1}_{\mathcal{GS}_n^+} + \mathbb{1}_{\mathcal{GS}_n^-} - 1)\|_p^{\mathbb{P}_{\mathcal{GS}_n}} \\ &\leq 2\|R_+ - \mathbb{1}_{\mathcal{GS}_n^+}\|_p^{\mathbb{P}_{\mathcal{GS}_n}} + \|\mathbb{1}_{\mathcal{GS}_n^+} \mathbb{1}_{\mathcal{GS}_n^-}\|_p^{\mathbb{P}_{\mathcal{GS}_n}} \\ &= 2\|R_+ - 1\|_p^{\mathbb{P}_{\mathcal{GS}_n^+}} \left(\frac{|\mathcal{GS}_n^+|}{|\mathcal{GS}_n|} \right)^{1/p} + \|\mathbb{1}_{\mathcal{GS}_n^+} \mathbb{1}_{\mathcal{GS}_n^-}\|_p^{\mathbb{P}_{\mathcal{GS}_n}} \\ &\leq 2\|R_+ - 1\|_p^{\mathbb{P}_{\mathcal{GS}_n^+}} + \|\mathbb{1}_{\mathcal{GS}_n^+} \mathbb{1}_{\mathcal{GS}_n^-}\|_1^{\mathbb{P}_{\mathcal{GS}_n}}. \end{aligned} \quad (3.25)$$

The only combinatorial argument we need to complete this proof is phrased as a lemma with a proof in the next section:

Lemma 3.9.3. *For every integral $j \geq 1$ we have*

$$\|R_+^j\|_1^{\mathbb{P}_{\mathcal{GS}_n^+}} \leq \|R_+\|_1^{\mathbb{P}_{\mathcal{GS}_n^+}} (1 + 2^{-n/2+o(n)}).$$

From now on until equation (3.26), all norms are for $\mu = \mathbb{P}_{\mathcal{GS}_n^+}$, and we drop the superscript $\mathbb{P}_{\mathcal{GS}_n^+}$ for legibility. By combining the lemma for $j = 2$ with Jensen's inequality we get

$$\|R_+\|_1^2 \leq \|R_+^2\|_1 \leq \|R_+\|_1 (1 + e^{-\Omega(n)}),$$

and so $\|R_+\|_1 \leq 1 + e^{-\Omega(n)}$. Hence, by the last lemma, we have $\|R_+^j\|_1 \leq 1 + e^{-\Omega(n)}$. Now $1 \leq R_+^j \leq R_+^{j+1}$ a.s. (that is, for all $G \in \mathcal{GS}_n^+$), and therefore

$$\|R_+^{j+1} - R_+^j\|_1 = \|R_+^{j+1}\|_1 - \|R_+^j\|_1 = e^{-\Omega(n)}.$$

Finally, we see that for each $j \geq 1$

$$\begin{aligned} \|(R_+ - 1)^{j+1}\|_1 &= \left\| (R_+ - 1) \sum_{i=0}^j \binom{j}{i} R_+^i (-1)^{j-i} \right\|_1 \leq \sum_{i=0}^j \binom{j}{i} \|(R_+ - 1) R_+^i\|_1 \\ &\leq 2^j \|(R_+ - 1) R_+^j\|_1 \leq e^{-\Omega(n)}. \end{aligned}$$

Therefore

$$\|R_+ - 1\|_p^{\mathbb{P}_{\mathcal{GS}_n^+}} = e^{-\Omega(n)} \quad (3.26)$$

for each real $p \geq 1$.

By Lemma 3.3.6 we have $\mathbb{P}_{\mathcal{GS}_n^+}(\alpha(G) \geq \omega(G)) = e^{-\Omega(n)}$ and $\mathbb{P}_{\mathcal{GS}_n^-}(\alpha(G) \leq \omega(G)) = e^{-\Omega(n)}$. Partition the set of graphs which are both unipolar and co-unipolar into $U_1 = \{G \in \mathcal{GS}_n^+ \cap \mathcal{GS}_n^- : \alpha(G) > \omega(G)\}$ and $U_2 = \{G \in \mathcal{GS}_n^+ \cap \mathcal{GS}_n^- : \alpha(G) \leq \omega(G)\}$. We see from Lemma 3.3.6 that U_1 consists of rare unipolar graphs and U_2 consists of

rare co-unipolar graphs; or more formally, $|U_1| = e^{-\Omega(n)}|\mathcal{GS}_n^+|$ and $|U_2| = e^{-\Omega(n)}|\mathcal{GS}_n^-|$. Hence

$$\|\mathbb{1}_{\mathcal{GS}_n^+} \mathbb{1}_{\mathcal{GS}_n^-}\|_1^{\mathbb{P}_{\mathcal{GS}_n}} = |\mathcal{GS}_n^+ \cap \mathcal{GS}_n^-|/|\mathcal{GS}_n| = e^{-\Omega(n)}. \quad (3.27)$$

Now (3.25), (3.26) and (3.27) imply that

$$\|R - 1\|_p^{\mathbb{P}_{\mathcal{GS}_n}} = e^{-\Omega(n)}. \quad (3.28)$$

We now have all the pieces to complete the story, assuming Lemma 4.2.3. By (3.26), $\mathcal{L}_n/|\mathcal{GS}_n^+| - 1 = \|R^+ - 1\|_1^{\mathbb{P}_{\mathcal{GS}_n^+}} = e^{-\Omega(n)}$, so using also (3.27)

$$|\mathcal{GS}_n| = 2|\mathcal{GS}_n^+| - |\mathcal{GS}_n^+ \cap \mathcal{GS}_n^-| = 2\mathcal{L}_n(1 - e^{-\Omega(n)}),$$

and we have proved (3.6). Also, by (3.24) and (3.28),

$$\left\| \frac{d\mathbb{P}_{Gen(n)}}{d\mathbb{P}_{\mathcal{GS}_n}} - 1 \right\|_p^{\mathbb{P}_{\mathcal{GS}_n}} = e^{-\Omega(n)},$$

which yields (3.7), and completes the proof of Theorem 3.9.1. Note finally that by the last equation and (3.27), for $G \sim Gen(n)$

$$\mathbb{P}(G \in \mathcal{GS}_n^+ \cap \mathcal{GS}_n^-) \leq (1 + e^{-\Omega(n)}) |\mathcal{GS}_n^+ \cap \mathcal{GS}_n^-|/|\mathcal{GS}_n| = e^{-\Omega(n)},$$

proving (3.9).

3.9.2 Proving Lemma 3.9.3

Fix $j \geq 1$. Suppose throughout this section that $G \in \mathcal{GS}_n^+$, $C_1, \dots, C_j \subseteq V(G)$ and (G, C_i) is a unipolar arrangement of order n for each $i \in [j]$. (For this given G , the number of choices for $C_1, \dots, C_j$ is $R^+(G)^j$, which we want to upper bound.) Further, let $C = \cap_{i=1}^j C_i$, $\tilde{C} = \cap_{i=1}^j \overline{C_i}$, and let $l = n - |C| - |\tilde{C}|$. The following claim is easy to verify.

Claim 3.9.4. *The pair $(G[C \cup \tilde{C}], C)$ is a unipolar arrangement of order $|C \cup \tilde{C}|$.*

Now let $C'_i = C_i \setminus C$ and $Q = V(G) \setminus (C \cup \tilde{C})$, so that $Q = \cup_{i=1}^j C'_i$.

Lemma 3.9.5. *Let $i \in [j]$. Then the pair $(G[Q], C'_i)$ is a unipolar arrangement of order l with at most $j - 1$ side cliques.*

Proof. Without loss of generality suppose that $i = 1$. It is easy to see that $G[Q]$ is a generalised split graph with a central clique C'_1 , since $C'_1 \subseteq C_1$, $Q \setminus C'_1 \subseteq \overline{C_1}$ and (G, C_1) is a unipolar arrangement. Further, $Q = \cup_{i=1}^j C'_i$, so every vertex in $Q \setminus C'_1$ is covered by a clique C'_i , where $2 \leq i \leq j$. Hence $G[Q \setminus C'_1]$ is a disjoint union of cliques, covered by $j - 1$ cliques, so the unipolar arrangement $(G[Q], C'_1)$ contains at most $j - 1$ side cliques. $\square$

Since each C'_i is complete to C and $Q = \cup_{i=1}^j C'_i$, it follows that Q is complete to C .

We now focus on C'_1 . Observe that, for each vertex $v \in C'_1$, since $v \notin C$ we can find C'_i such that $v \notin C'_i$. Thus C'_1 can be expressed as a disjoint union $C''_2 \cup \dots \cup C''_j$ where $C''_i \cap C'_i = \emptyset$ for each $i = 2, \dots, j$. Let $\mathcal{C}$ denote the family of sets $\{C'_i \setminus C'_1\}_{i=2}^j \cup \{C''_i\}_{i=2}^j$. Note that the union of these sets is Q .

Lemma 3.9.6. *For each $S \in \mathcal{C}$, either S is complete to a side clique of $(G[C \cup \tilde{C}], C)$ and non-adjacent to all other side cliques, or S is non-adjacent to $\tilde{C}$.*

Proof. In the unipolar arrangement (G, C_1) , each $C'_i \setminus C'_1$ is contained in a side clique, so it is either complete to a side clique of $(G[C \cup \tilde{C}], C)$, and non-adjacent to the others, or non-adjacent to any vertex of $\tilde{C}$. Similarly, in the unipolar arrangement (G, C_i) , C''_i is contained in a side clique, hence it has the same property. $\square$

Let f be the function mapping each set S in $\mathcal{C}$ to the side clique S' of $(G[C \cup \tilde{C}], C)$ such that S is complete to S' , or to $\emptyset$ if no vertex in S is adjacent to $\tilde{C}$.

Lemma 3.9.7. *The edge set $E(Q, \tilde{C})$ can be reconstructed from f .*

Proof. For each $v \in Q$ either $v \in C'_1$ and then v lies in some set $S \in \{C''_i\}_{i=2}^j$, or v lies in some set $S \in \{C'_i \setminus C'_1\}_{i=2}^j$. In either case, S is in $\mathcal{C}$ and the set of neighbours of v in $\tilde{C}$ is precisely the set $f(S)$. $\square$

The edges of G can be reconstructed from $G[C \cup \tilde{C}]$ – a unipolar graph, $G[Q]$ – a unipolar graph with at most $k - 1$ side cliques and $E(Q, C \cup \tilde{C})$, which in turn can be reconstructed from f , since Q is complete to C . It will turn out that the number of choices for $G[C \cup \tilde{C}]$ dominates the rest, and this number is maximised whenever $Q = \emptyset$. However, the vertex labels are a nuisance, and additional information is required to correctly recover G . The precise statement is given below.

Lemma 3.9.8. *The $j + 1$ tuple $(G, C_1, \dots, C_j)$ can be uniquely reconstructed from:*

1. *the pair $(G[C \cup \tilde{C}], C)$ seen as a unipolar $(n - l)$ -arrangement over the vertex set $[n - l]$ so that the order of the labels of the vertices is preserved;*
2. *the pair $(G[Q], C'_1)$ seen as a unipolar l -arrangement over the vertex set $[l]$ with at most $j - 1$ side cliques so that the order of the labels of the vertices is preserved;*
3. *the sets $C'_2, \dots, C'_j \subseteq [l]$;*
4. *the l -subset of $V(G)$ specifying the original labels of Q ;*
5. *the function f .*

Observe that there are at most $j^l 2^{\frac{l^2}{4}}$ choices for the pair $(G[Q], C'_1)$ above.

Recall that $\mathcal{L}_n = |\mathcal{CGS}_n^+|$. Let $T_n = 2^{\frac{n^2}{4} + \frac{n}{2} \log n - \frac{n}{2} \log \ln n - \frac{n}{2} \log \frac{e}{2}}$ for $n \geq 3$, and let $T_n = 1$ for $n = 0, 1, 2$. Prömel and Steger show in [PS92] that $\mathcal{L}_n \leq T_n \times 2^{O(\frac{n}{\ln \ln(n+3)})}$, hence for some c we have $|\mathcal{CGS}_n^+| \leq T_n \times 2^{\frac{cn}{\ln \ln(n+3)}}$. It follows that

$$U_{n,l} := \left(T_{n-l} \times 2^{\frac{cn}{\ln \ln(n+3)}} \right) \times \left(j^l 2^{\frac{l^2}{4}} \right) \times (2^{(j-1)l}) \times \binom{n}{l} \times ((n+1)^{2(j-1)})$$

is an upper bound on the number of choices for $(G, C_1, \dots, C_j)$ when $|Q| = l$, and $U_n = \sum_{l=1}^n U_{n,l}$ is an upper bound on the number of choices for $(G, C_1, \dots, C_j)$ when $Q \neq \emptyset$.

Lemma 3.9.9. *The number $U_{n,l}$ is maximised subject to $1 \leq l \leq n$ when $l = 1$.*

Proof. We may prove this by thinking of l as a continuous variable and taking the derivative of $U_{n,l}$ with respect to l . The analysis is straightforward but unpleasant: we leave the details to the reader. $\square$

Corollary 3.9.10. *We have*

$$U_n = |\mathcal{CGS}_n^+| 2^{-n/2+o(n)}.$$

Proof. Since $T_{n-1} = T_n 2^{-n/2+o(n)}$, we see that $\frac{U_{n,1}}{|\mathcal{CGS}_n^+|} = 2^{-n/2+o(n)}$, and hence

$$|\mathcal{CGS}_n^+| 2^{-n/2+o(n)} = U_{n,1} \leq U_n \leq n U_{n,1} = |\mathcal{CGS}_n^+| 2^{-n/2+o(n)}. \quad \square$$

Finally, the number of choices for the $j+1$ tuple $(G, C_1, \dots, C_j)$ when $Q = \emptyset$ is clearly $|\mathcal{CGS}_n^+|$, so we have

$$\begin{aligned} \|R_+^j\|_1^{\mathbb{P}_{\mathcal{GS}_n^+}} &= \sum_{G \in \mathcal{GS}_n^+} R_+^j(G) \frac{1}{|\mathcal{GS}_n^+|} \leq (U_n + |\mathcal{CGS}_n^+|) \frac{1}{|\mathcal{GS}_n^+|} \\ &= |\mathcal{CGS}_n^+| (1 + 2^{-n/2+o(n)}) \frac{1}{|\mathcal{GS}_n^+|} = \|R_+\|_1^{\mathbb{P}_{\mathcal{GS}_n^+}} (1 + 2^{-n/2+o(n)}). \end{aligned}$$

This completes the proof of Lemma 3.9.3, Theorem 3.9.1 and Theorem 3.2.3.

3.9.3 Concentration: Proof of Theorem 3.2.5

Let $\mu = (n - \log n + \log \ln n)/2$ as in the statement of the theorem (for $n \geq 3$), let $\hat{\mu} = \lceil \mu \rceil$ and let $\check{\mu} = \hat{\mu} - 1$. As before, let $\ell_{n,k} = \binom{n}{k} 2^{k(n-k)} B_{n-k}$. We fix n and drop it in the subscript for legibility, i.e. write ℓ_k instead of $\ell_{n,k}$.

Lemma 3.9.11. *Using the notation above, there is an n_0 such that $\ell_k/\ell_{\hat{\mu}} \leq n^{-n-1}$ as long as $|k - \hat{\mu}| \geq n^{2/3}$ for $n \geq n_0$.*

Proof. Let $x = k - \hat{\mu}$, we have

$$\frac{\ell_k}{\ell_{\hat{\mu}}} = \frac{\binom{n}{k} 2^{k(n-k)} B_{n-k}}{\binom{n}{\hat{\mu}} 2^{\hat{\mu}(n-\hat{\mu})} B_{n-\hat{\mu}}} = \frac{(n - \hat{\mu})!}{(n - \hat{\mu} - x)!} \left(\frac{(\hat{\mu} + x)!}{\hat{\mu}!} \right)^{-1} 2^{x(n-2\hat{\mu})} 2^{-x^2} \frac{B_{n-\hat{\mu}-x}}{B_{n-\hat{\mu}}}.$$

We can bound each of the terms $\frac{(n-\hat{\mu})!}{(n-\hat{\mu}-x)!}$, $\left(\frac{(\hat{\mu}+x)!}{\hat{\mu}!} \right)^{-1}$, $2^{x(n-2\hat{\mu})}$ and $\frac{B_{n-\hat{\mu}-x}}{B_{n-\hat{\mu}}}$ with $n^{|x|}$, so we get

$$\frac{\ell_k}{\ell_{\hat{\mu}}} < n^{|x|} n^{|x|} 2^{-|x|^2} n^{|x|} = 2^{3|x| \log n - |x|^2} \leq 2^{3n^{2/3} \log n - n^{4/3}} \leq n^{-n-1}$$

for large n . □

The bound is rather crude, and it can be improved, but we do not need a better bound for our purposes. From now we focus on $k = \hat{\mu} + x$ with $|x| \leq n^{2/3}$. This implies $k \sim n/2$, which allows us to use asymptotics for k as n grows, for instance $(n - k)/(k + 1) \rightarrow 1$ and more importantly

$$\frac{B_{n-k-1}}{B_{n-k}} = (1 + o_n(1)) \frac{\ln(n - k)}{n - k} = (1 + o_n(1)) \frac{2 \ln n}{n},$$

where the error term is uniform over k , provided that $|x| \leq n^{2/3}$.

Lemma 3.9.12. *There is an n_0 such that the following holds for each $n \geq n_0$. Suppose $0 \leq x \leq n^{2/3}$ is integral and $r \in \{\frac{\ell_{\hat{\mu}-x}}{\ell_{\hat{\mu}}}, \frac{\ell_{\hat{\mu}+x}}{\ell_{\hat{\mu}}}\}$. Then $2^{-x^2-2x} \leq r \leq 2^{-x^2+2x}$.*

Proof. Define the continuous, strictly decreasing function $d(y) = 2^{n-2y \frac{\ln n}{n}}$, so that for $k \sim n/2$

$$\frac{\ell_{k+1}}{\ell_k} = \frac{n - k}{k + 1} 2^{n-2k-1} (1 + o_n(1)) \frac{2 \ln n}{n} = (1 + o_n(1)) d(k).$$

Observe that $d(k) = 1 \iff 2k = n + \log \ln n - \log n \iff k = \mu$. We have $d(\mu + x) = d(\mu) \cdot 2^{-2x} = 2^{-2x}$, and

$$\ell_{\hat{\mu}+x} = \ell_{\hat{\mu}} \prod_{i=0}^{x-1} \frac{\ell_{\hat{\mu}+i+1}}{\ell_{\hat{\mu}+i}} = \ell_{\hat{\mu}} \prod_{i=0}^{x-1} (1 + o_n(1)) d(\hat{\mu} + i).$$

For each i we use the bounds

$$2^{-2i-2} = d(\mu + 1 + i) \leq d(\hat{\mu} + i) \leq d(\mu + i) = 2^{-2i},$$

while the error terms, $1 + o_n(1)$, we bound uniformly by $\frac{1}{2}$ and 2, provided $n \geq n_0$. By multiplying the terms we get $-2\binom{x}{2} - 3x$ and $-2\binom{x}{2} + x$ in the exponent for the lower and upper bounds respectively, which matches the statement of the lemma for the case $r = \ell_{\hat{\mu}+x}/\ell_{\hat{\mu}}$. The second case is analogous. □

Claim 3.9.13. For $a > 1$ we have $\sum_{x=a}^{\infty} 2^{-x^2} < \frac{2^{-(a-1)^2}}{2(a-1)\ln 2}$.

Proof. Since $-x^2$ is decreasing, we have

$$\begin{aligned} \sum_{x=a+1}^{\infty} 2^{-x^2} &< \int_a^{\infty} 2^{-x^2} dx \leq \frac{1}{a} \int_a^{\infty} x 2^{-x^2} dx \\ &= \frac{1}{-2a \ln 2} \left(2^{-x^2} \Big|_{x=a}^{x=\infty} \right) = \frac{2^{-a^2}}{2a \ln 2}. \end{aligned} \quad \square$$

In the claim below and its proof, for simplicity we have written $n^{2/3}$ rather than $\lfloor n^{2/3} \rfloor$.

Claim 3.9.14. For each integer $k \geq 2$ we have

$$\sum_{i=k}^{n^{2/3}} \ell_{\check{\mu}-i} \leq \ell_{\check{\mu}} 2^{-(k-1)^2} \left(2 + \frac{1}{(k-1)\ln 2} \right) \quad \text{and} \quad \sum_{i=k}^{n^{2/3}} \ell_{\hat{\mu}+i} \leq \ell_{\hat{\mu}} 2^{-(k-1)^2} \left(2 + \frac{1}{(k-1)\ln 2} \right).$$

Proof. Using Lemma 3.9.12 we see that

$$\begin{aligned} \sum_{i=k}^{n^{2/3}} \ell_{\check{\mu}-i} &\leq \ell_{\check{\mu}} \sum_{i=k}^{n^{2/3}} 2^{-i^2+2i} = 2\ell_{\check{\mu}} \sum_{i=k}^{n^{2/3}} 2^{-(i-1)^2} \leq 2\ell_{\check{\mu}} \left(2^{-(k-1)^2} + \sum_{i=k}^{\infty} 2^{-i^2} \right) \\ &= 2\ell_{\check{\mu}} \left(2^{-(k-1)^2} + \frac{2^{-(k-1)^2}}{2(k-1)\ln 2} \right). \end{aligned}$$

The last inequality follows from Claim 3.9.13. The second part of the statement is similar. $\square$

Now we put everything together to complete the proof of Theorem 3.2.5. Suppose $n \geq 3$, $X \sim L(n)$ and $x > 2$, then (noting the trivial bound $\mathcal{L}_n \geq \ell_{\hat{\mu}} + \ell_{\check{\mu}}$)

$$\begin{aligned} \mathbb{P}(|X - \mu| \geq x) &\leq \mathbb{P}(\check{\mu} - n^{2/3} \leq X \leq \check{\mu} - x + 1) \\ &\quad + \mathbb{P}(\hat{\mu} + x - 1 \leq X \leq \hat{\mu} + n^{2/3}) \\ &\quad + \mathbb{P}(|X - \mu| \geq n^{2/3}) \\ &\leq \frac{\ell_{\hat{\mu}}}{\ell_{\hat{\mu}} + \ell_{\check{\mu}}} \cdot 2^{-(x-2)^2} \cdot 4 + \frac{\ell_{\check{\mu}}}{\ell_{\hat{\mu}} + \ell_{\check{\mu}}} \cdot 2^{-(x-2)^2} \cdot 4 + n^{-n} \\ &= 2^{-(x-2)^2+2} + n^{-n}, \text{ and} \\ \mathbb{P}(|X - \mu| \geq x) &\geq \mathbb{P}(X = \hat{\mu} + x) + \mathbb{P}(X = \check{\mu} - x) \\ &\geq \frac{2^{-x^2-2x}(\ell_{\hat{\mu}} + \ell_{\check{\mu}})}{(3 + \frac{1}{2\ln 2})(\ell_{\hat{\mu}} + \ell_{\check{\mu}}) + n^{-n}\hat{\mu}} \\ &\geq 2^{-x^2-2x-2}. \end{aligned}$$

Chapter 4

Tractability via bounded Minor Matchings

4.1 Notation and preliminaries

We recall the basic definitions and also introduce a few more.

Definition 4.1.1 (Clutter). *A set system is a finite set of finite subsets of the natural numbers $\mathbb{N}$. A set system $\mathcal{H}$ is called clutter, Sperner family or antichain if its sets are inclusion-wise incomparable. Every set system $\mathcal{H}$ we associate with a vertex set $V(\mathcal{H}) := \bigcup \mathcal{H}$ and rank $rk(\mathcal{H}) := \max_{S \in \mathcal{H}} |S|$. Given a set system $\mathcal{H}$ we define $cl(\mathcal{H})$ to be the clutter obtained from $\mathcal{H}$ by removing all non-minimal (subsumed) edges.*

Definition 4.1.2 (Blocker). *A transversal of a clutter $\mathcal{H}$ is a set intersecting every $S \in \mathcal{H}$. A minimal transversal is a transversal not containing another transversal as a subset. The blocker of $\mathcal{H}$, denoted $b(\mathcal{H})$, is the clutter consisting of all minimal transversals of $\mathcal{H}$.*

Definition 4.1.3 (Deletion and Contraction). *Suppose $\mathcal{H}$ is a clutter and $v \in \mathbb{N}$. Define*

$$\begin{aligned}\mathcal{H} \setminus v &:= \{S \in \mathcal{H} : v \notin S\} \text{ to be } \mathcal{H} \text{ with } v \text{ deleted, and} \\ \mathcal{H} / v &:= cl(\{S - v : S \in \mathcal{H}\}) \text{ to be } \mathcal{H} \text{ with } v \text{ contracted.}\end{aligned}$$

Definition 4.1.4 (Minor). *We say that $\mathcal{F}$ is a minor of $\mathcal{H}$ if $\mathcal{F}$ can be obtained from $\mathcal{H}$ through a series of deletions and contractions. We write $\mathcal{F} \subseteq_{\mathcal{M}} \mathcal{H}$ to denote that $\mathcal{F}$ is a minor of $\mathcal{H}$ and say that $\mathcal{H}$ is $\mathcal{F}$ -minor-free if $\mathcal{H}$ contains no minor isomorphic to $\mathcal{F}$.*

Denote the set $\{1, \dots, n\}$ by $[n]$. It is often convenient to write $S \cup x$ and $S - x$ instead of $S \cup \{x\}$ and $S \setminus \{x\}$.

We quickly revise some well-known properties of clutters and blockers used throughout the proofs.

Definition 4.1.5 (Join and Meet). Suppose $\mathcal{H}$ and $\mathcal{F}$ are clutters. We define

$\mathcal{H} \vee \mathcal{F} := cl(\mathcal{H} \cup \mathcal{F})$ to be the join of $\mathcal{H}$ and $\mathcal{F}$, and

$\mathcal{H} \wedge \mathcal{F} := cl(\{S \cup T : S \in \mathcal{H}, T \in \mathcal{F}\})$ to be the meet of $\mathcal{H}$ and $\mathcal{F}$.

Lemma 4.1.6 (Algebraic properties [Cor01]). Let $\mathcal{H}$, $\mathcal{G}$ and $\mathcal{F}$ be clutters and let v and u be distinct elements of $\mathbb{N}$. The operations defined above have the following properties:

1. deletion and contraction commute, that is $\mathcal{H} \setminus v \setminus u = \mathcal{H} \setminus u \setminus v$, $\mathcal{H} \setminus v / u = \mathcal{H} / u \setminus v$ and $\mathcal{H} / v / u = \mathcal{H} / u / v$;
2. if we denote the set of all clutters by $\mathcal{C}(\mathbb{N})$, the clutters $\emptyset$ and $\{\emptyset\}$ by $\widehat{0}$ and $\widehat{1}$ respectively, then $(\mathcal{C}(\mathbb{N}), \vee, \wedge, \widehat{0}, \widehat{1})$ forms a bounded distributive lattice, that is

Commutative laws

$$\mathcal{F} \vee \mathcal{G} = \mathcal{G} \vee \mathcal{F}$$

$$\mathcal{F} \wedge \mathcal{G} = \mathcal{G} \wedge \mathcal{F}$$

Associative laws

$$\mathcal{F} \vee (\mathcal{G} \vee \mathcal{H}) = (\mathcal{F} \vee \mathcal{G}) \vee \mathcal{H}$$

$$\mathcal{F} \wedge (\mathcal{G} \wedge \mathcal{H}) = (\mathcal{F} \wedge \mathcal{G}) \wedge \mathcal{H}$$

Absorption laws

$$\mathcal{F} \vee (\mathcal{F} \wedge \mathcal{G}) = \mathcal{F}$$

$$\mathcal{F} \wedge (\mathcal{F} \vee \mathcal{G}) = \mathcal{F}$$

Identity laws

$$\mathcal{F} \vee \widehat{0} = \mathcal{F}$$

$$\mathcal{F} \wedge \widehat{1} = \mathcal{F}$$

Distributive laws

$$\mathcal{F} \wedge (\mathcal{G} \vee \mathcal{H}) = (\mathcal{F} \wedge \mathcal{G}) \vee (\mathcal{F} \wedge \mathcal{H})$$

$$\mathcal{F} \vee (\mathcal{G} \wedge \mathcal{H}) = (\mathcal{F} \vee \mathcal{G}) \wedge (\mathcal{F} \vee \mathcal{H}).$$

The lattice has the additional properties that $\mathcal{F} \vee \mathcal{H} = \widehat{1}$ iff $\mathcal{F} = \widehat{1}$ or $\mathcal{H} = \widehat{1}$, $\mathcal{F} \wedge \mathcal{H} = \widehat{0}$ iff $\mathcal{F} = \widehat{0}$ or $\mathcal{H} = \widehat{0}$, and $b(\widehat{0}) = \widehat{1}$;

3. the map $\mathcal{H} \mapsto b(\mathcal{H})$ is a duality operation: $b(b(\mathcal{H})) = \mathcal{H}$;
4. deletion is the dual operation of contraction and vice versa: $b(\mathcal{H} \setminus v) = b(\mathcal{H}) / v$ and $b(\mathcal{H} / v) = b(\mathcal{H}) \setminus v$;
5. join is the dual operation of meet and vice versa: $b(\mathcal{H} \vee \mathcal{F}) = b(\mathcal{H}) \wedge b(\mathcal{F})$ and $b(\mathcal{H} \wedge \mathcal{F}) = b(\mathcal{H}) \vee b(\mathcal{F})$;
6. the property of being a minor is preserved by duality: $\mathcal{F} \subseteq_{\mathcal{M}} \mathcal{H}$ if and only if $b(\mathcal{F}) \subseteq_{\mathcal{M}} b(\mathcal{H})$;
7. minors commute with join and meet:

$$(\mathcal{H} \vee \mathcal{F}) \setminus v = \mathcal{H} \setminus v \vee \mathcal{F} \setminus v$$

$$(\mathcal{H} \vee \mathcal{F}) / v = \mathcal{H} / v \vee \mathcal{F} / v$$

$$(\mathcal{H} \wedge \mathcal{F}) \setminus v = \mathcal{H} \setminus v \wedge \mathcal{F} \setminus v$$

$$(\mathcal{H} \wedge \mathcal{F}) / v = \mathcal{H} / v \wedge \mathcal{F} / v.$$

Definition 4.1.7 (Upset). We say that a possibly infinite set system $\mathcal{U}$ is an upset over $\mathbb{N}$ if $B \in \mathcal{U}$ whenever $A \subseteq B \subseteq \mathbb{N}$ and $A \in \mathcal{U}$.

Proof of Lemma 4.1.6. Let $\mathcal{U}(\mathbb{N})$ be the set of all upsets over $\mathbb{N}$. Since (i) $\mathcal{U}(\mathbb{N})$ is a collection of subsets of $2^{\mathbb{N}}$, (ii) $\mathcal{U}(\mathbb{N})$ is closed under union and intersection, and (iii) $\{\emptyset, 2^{\mathbb{N}}\} \subset \mathcal{U}(\mathbb{N})$, we have that $(\mathcal{U}(\mathbb{N}), \cup, \cap, \emptyset, 2^{\mathbb{N}})$ forms a bounded distributive lattice.

Recall that clutters have finite vertex sets. Given a clutter $\mathcal{H} \in \mathcal{C}(\mathbb{N})$, let $\lambda(\mathcal{H})$ be $\bigcap \{\mathcal{U} \in \mathcal{U}(\mathbb{N}) : \mathcal{H} \subseteq \mathcal{U}\}$. An arbitrary intersection of upsets is an upset, so $\lambda(\mathcal{H}) \in \mathcal{U}(\mathbb{N})$ is the smallest upset containing $\mathcal{H}$. Alternatively, $\lambda(\mathcal{H})$ is the smallest upset with inclusion-minimal sets $\mathcal{H}$. Now let $\mathcal{U}_0 \subseteq \mathcal{U}(\mathbb{N})$ be the range of λ . Since λ is injective, λ is a bijection between $\mathcal{C}(\mathbb{N})$ and $\mathcal{U}_0$. Note that $\lambda(\emptyset) = \emptyset$ and $\lambda(\{\emptyset\}) = 2^{\mathbb{N}}$, so $\{\emptyset, 2^{\mathbb{N}}\} \subset \mathcal{U}_0$.

It is easy to see that $\lambda(\mathcal{H} \vee \mathcal{F}) = \lambda(\mathcal{H}) \cup \lambda(\mathcal{F})$ and $\lambda(\mathcal{H} \wedge \mathcal{F}) = \lambda(\mathcal{H}) \cap \lambda(\mathcal{F})$, so $\mathcal{U}_0$ is closed under union and intersection, and therefore $(\mathcal{U}_0, \cup, \cap, \emptyset, 2^{\mathbb{N}})$ is a bounded distributive lattice. But λ is an isomorphism between $(\mathcal{U}_0, \cup, \cap, \emptyset, 2^{\mathbb{N}})$ and $(\mathcal{C}(\mathbb{N}), \vee, \wedge, \emptyset, \{\emptyset\})$, and hence the same holds for $(\mathcal{C}(\mathbb{N}), \vee, \wedge, \emptyset, \{\emptyset\})$. Part (2.) now is proved.

Suppose $\mathcal{U} \in \mathcal{U}(\mathbb{N})$ and $v \in \mathbb{N}$. We define:

- $\mathcal{U} \setminus v := \{F : F \setminus \{v\} \in \mathcal{U}\},$
- $\mathcal{U} / v := \{F \setminus \{v\} : F \in \mathcal{U}\} \cup \mathcal{U},$
- $b(\mathcal{U}) = \{\mathbb{N} \setminus F : F \notin \mathcal{U}\}.$

It is a routine to verify that $\lambda(\mathcal{H} \setminus v) = \lambda(\mathcal{H}) \setminus v$, $\lambda(\mathcal{H} / v) = \lambda(\mathcal{H}) / v$ and $\lambda(b(\mathcal{H})) = b(\lambda(\mathcal{H}))$. Now it is trivial to see that deletion and contraction commute: suppose $\mathcal{U}$ is an upset, and $\mathcal{M}$ is obtained from $\mathcal{U}$ by deleting and contracting the disjoint sets $D = \{d_1, \dots, d_k\}$ and $C = \{c_1, \dots, c_\ell\}$. Regardless of the order of the operations we have $\mathcal{M} = \{F \setminus S : S \subseteq C \text{ and } F \setminus D \in \mathcal{U}\}$, and hence part (1.) follows since λ is an isomorphism.

Proving the remaining points of Lemma 4.1.6 using the isomorphism λ is straight forward. Part (3.) directly follows from the definition of $b(\mathcal{U})$. Part (4.) for upsets is a definition chase. Part (5.) follows from the De Morgan's laws. Part (6.) follows directly from part (4.); if $\mathcal{F}$ is obtained from $\mathcal{M}$ by deleting D and contracting C , then $b(\mathcal{F})$ can be obtained from $b(\mathcal{M})$ by contracting D and deleting C . Part (7.) is a definition chase for upsets. It is worth noting that

$$\begin{aligned} (\mathcal{H} \wedge \mathcal{F}) \setminus v &= b(b((\mathcal{H} \wedge \mathcal{F}) \setminus v)) = b((b(\mathcal{H}) \vee b(\mathcal{F})) / v) = b(b(\mathcal{H}) / v \vee b(\mathcal{F}) / v) \\ &= \mathcal{H} \setminus v \wedge \mathcal{F} \setminus v, \end{aligned}$$

and similarly $(\mathcal{H} \wedge \mathcal{F}) / v = \mathcal{H} / v \wedge \mathcal{F} / v$. □

A more visual way to understand deletion and contraction is to associate a clutter $\mathcal{H}$ with an upset over $V(\mathcal{H})$. Here deletion and contraction correspond to projecting the upset over a hypercube of dimension less one. This gives a more intuitive justification of why deletion and contraction distribute with join and meet, and why they are dual with respect to the blocker. The isomorphism presented in the proof of Lemma 4.1.6 has its own merits, namely it puts all clutters into a lattice with shared minimum and maximum. This eliminates the need to additionally carry a vertex set along with the edges of a clutter, and differentiate between clutters (V_1, H) and (V_2, H) for distinct V_1 and V_2 .

Another isomorphic structure is the set of monotone boolean functions. A boolean function f is monotone if $f(\bar{x}) \leq f(\bar{y})$ whenever $\bar{x} \preceq \bar{y}$. Deletion and contraction of clutters correspond to substituting a variable with false and true respectively, join and meet correspond to disjunction and conjunction respectively, and the dual function of $f(\bar{x})$ is $\neg f(\neg \bar{x})$.

The Dedekind number: 2, 3, 6, 20, 168, 7581, 7828354, 2414682040998, 56130437228687557907788 (sequence A000372 in the OEIS), counts the number of clutters with at most n vertices, the number of upsets over $[n]$, the number of elements in a free distributive lattice with n generators, and the number of monotone boolean functions with n variables.

Given a k -set S , the notation $\mathcal{H} \setminus S$ stands for $\mathcal{H} \setminus s_1 \dots \setminus s_k$, where $\{s_1, \dots, s_k\}$ is an arbitrary ordering of S . The choice of the ordering does not matter by part (1.) of Lemma 4.1.6. We define $\mathcal{H}/S$ to be $\mathcal{H}/s_1 \dots /s_k$ likewise. Finally, given disjoint sets S and T we define $\mathcal{H}[S; T]$ to be $\mathcal{H} \setminus S/T$. This operation has the nice property that $\mathcal{H}[S_1 \cup S_2; T_1 \cup T_2] = \mathcal{H}[S_1; T_1][S_2; T_2]$ for any four sets S_1, S_2, T_1 and T_2 , such that $(S_1 \cup S_2) \cap (T_1 \cup T_2) = \emptyset$, which follows from part (1.) of Lemma 4.1.6.

Suppose $kK_2 \subseteq_{\mathcal{M}} \mathcal{H}$. Then $b(kK_2) \subseteq_{\mathcal{M}} b(\mathcal{H})$, and since deleting and contracting vertices can only decrease the number of edges, we have $|b(\mathcal{H})| \geq |b(kK_2)| = 2^k$. This observation is generalised by the next lemma.

Definition 4.1.8 (Trace). *Given a set system $\mathcal{F}$ and a set S , we define the trace of S over $\mathcal{F}$ to be*

$$tr_S(\mathcal{F}) = \{F \cap S : F \in \mathcal{F}\}.$$

Lemma 4.1.9. *Suppose $\mathcal{F} \subseteq_{\mathcal{M}} \mathcal{H}$ and $S \subseteq V(\mathcal{F})$. Then $tr_S(\mathcal{F}) \subseteq tr_S(\mathcal{H})$.*

Proof. Let $\mathcal{F} = \mathcal{H}[D; C]$. The edges of $\mathcal{F}$ are contractions of edges $\mathcal{H}$; for each $f \in \mathcal{F}$ pick $h_f \in \mathcal{H}$ such that $f \subseteq h_f \subseteq f \cup C$ to form $\mathcal{H}' = \{h_f : f \in \mathcal{F}\}$. Now we have $tr_S(\mathcal{F}) = tr_S(\mathcal{H}')$, because C and S are disjoint; and $tr_S(\mathcal{H}') \subseteq tr_S(\mathcal{H})$, because $\mathcal{H}' \subseteq \mathcal{H}$. $\square$

Recall that a set system $\mathcal{F}$ shatters a set S if $tr_S(\mathcal{F}) = 2^S$, and that $sh(\mathcal{F})$ denotes the set system of sets shattered by $\mathcal{F}$. The lemma above shows that if $\mathcal{F} \subseteq_{\mathcal{M}} \mathcal{H}$, then $sh(\mathcal{F}) \subseteq sh(\mathcal{H})$. In particular, $VC\text{-dim}(\mathcal{F}) \leq VC\text{-dim}(\mathcal{H})$.

Definition 4.1.10 (*S-intersecting minor-matching*). Suppose $S \subseteq \mathbb{N}$. A clutter $\mathcal{F}$ is called an *S-intersecting minor-matching* of a clutter $\mathcal{H}$ if $\mathcal{F} \cong kK_2$ for some k and every edge of $\mathcal{F}$ intersects S .

Lemma 4.1.11. Suppose $\mathcal{F}$ is an *S-intersecting minor-matching* of $\mathcal{H}$. Then

$$|tr_S(b(\mathcal{H}))| \geq 2^{|\mathcal{F}|}.$$

Proof. Construct $T \subseteq S$ by picking a vertex from each edge of $\mathcal{F}$. We have $tr_T(b(\mathcal{F})) = 2^T$, and since $b(\mathcal{F}) \subseteq_{\mathcal{M}} b(\mathcal{H})$, we have

$$2^{|\mathcal{F}|} = |tr_T(b(\mathcal{F}))| \leq |tr_T(b(\mathcal{H}))| \leq |tr_S(b(\mathcal{H}))|. \quad \square$$

Definition 4.1.12. Given a clutter $\mathcal{H} \neq \{\emptyset\}$ and a set $S \subseteq \mathbb{N}$ we denote the maximum size of an *S-intersecting minor-matching* of $\mathcal{H}$ by $\mu_S(\mathcal{H})$. Let $\mu_S(\emptyset)$ be -1 .

The lemma above can be rephrased as $|tr_S(b(\mathcal{H}))| \geq 2^{\mu_S(\mathcal{H})}$. The lower bound may appear very crude, but in fact it is in a sense the best possible, for clutters of bounded rank, as discussed in the next section.

The condition $\mathcal{H} \neq \{\emptyset\}$ in Definition 4.1.12 is a subtle one. There are two clutters with empty vertex set – $\widehat{0} = \emptyset$ and $\widehat{1} = \{\emptyset\}$. Recall that $b(\widehat{0}) = \widehat{1}$ and $b(\widehat{1}) = \widehat{0}$. From these two clutters, $\widehat{0}$ is a matching (with 0 edges), and $\widehat{1}$ is not. Every minor of $\widehat{1}$ is $\widehat{1}$, so there is no minor of $\widehat{1}$ that is a matching. On the contrary, if $\mathcal{H} \neq \widehat{1}$, then every edge of $\mathcal{H}$ contains a vertex, so we may delete all vertices to get as a minor $\widehat{0}$ – a matching. The empty matching is *S-intersecting* for every S , so $\mu_S(\mathcal{H})$ is well defined.

4.2 Minor-Matching Theorem

The main purpose of this section is to outline the proof of the following theorem, stated earlier as Theorem 2.2.12.

Theorem 4.2.1. Suppose $\mathcal{H}$ is a clutter and $S \subseteq \mathbb{N}$. We have

$$|tr_S(b(\mathcal{H}))| \leq \sum_{k=0}^{\mu_S(\mathcal{H})f(rk(\mathcal{H}))} \binom{|E_S(\mathcal{H})|}{k} \binom{rk(\mathcal{H})}{2}^k,$$

where $f(r) = (2r - 3)2^{r-2}$ and $E_S(\mathcal{H})$ is the set of the edges of $\mathcal{H}$ intersecting S . In particular, when $\mu_S(\mathcal{H})$ and $rk(\mathcal{H})$ are bounded, $tr_S(b(\mathcal{H}))$ has a polynomially-bounded size, and it can be computed in polynomial time.

Definition 4.2.2 (Semi-matching and expanded minor matching). *Consider a set $\mathcal{S}$ of the form $\{(E_i, H_i)\}_{i=1}^k$, a clutter $\mathcal{H}$ and the following conditions:*

1. $|E_i| = 2$, $E_i \subseteq H_i$ and $H_i \in \mathcal{H}$ for each $1 \leq i \leq k$,
2. $\{E_i\}_{i=1}^k$ are pairwise disjoint,
- 3a. $E_i \not\subseteq H_j$ for $i \neq j$,
- 3b. $E_i \cap H_j = \emptyset$ for $i \neq j$,
4. all sets of $\mathcal{H}$ contained in $\bigcup_{i=1}^k H_i$ contain as a subset at least one set E_i .

We call $\mathcal{S}$ a semi-matching if it satisfies properties 1, 2, 3a and 4; and an expanded minor matching if it additionally satisfies 3b. A semi-matching or an expanded minor matching is called S -intersecting if each E_i intersects a set $S \subseteq \mathbb{N}$.

Semi-matchings and expanded minor matchings coincide with induced matchings in edge clutters.

Theorem 4.2.3 (Decomposition theorem). *For every clutter $\mathcal{H}$ and set $S \subseteq \mathbb{N}$, $|tr_S(b(\mathcal{H}))|$ is at most the number of S -intersecting semi-matchings of $\mathcal{H}$.*

Note the lack of restrictions on the rank of $\mathcal{H}$. In particular, when $S = \mathbb{N}$, the Decomposition Theorem is a generalisation of Alekseev's theorem from [Ale07] to clutters.

The following lemma establishes that expanded minor matchings and minor matchings are essentially equivalent.

Lemma 4.2.4. *If $\{(E_i, H_i)\}_{i=1}^k$ is an expanded minor matching of a clutter $\mathcal{H}$, then $\{E_i\}_{i=1}^k \cong kK_2$ is a minor of $\mathcal{H}$. Conversely, if $\{E_i\}_{i=1}^k$ is a kK_2 minor of $\mathcal{H}$, we can find sets $\{H_i\}_{i=1}^k$ such that $\{(E_i, H_i)\}_{i=1}^k$ is an expanded minor matching of $\mathcal{H}$.*

Proof. To prove the first part, contract $(\bigcup_{i=1}^k H_i) \setminus (\bigcup_{i=1}^k E_i)$ and delete $V(\mathcal{H}) \setminus (\bigcup_{i=1}^k H_i)$. Any non-deleted edge is subsumed by some E_i .

For the second part, each E_i must be a contraction of a set $H_i \in \mathcal{H}$, such that E_i intersects H_j if and only if $i = j$. It is a routine to verify that $\{(E_i, H_i)\}_{i=1}^k$ is an expanded minor matching of $\mathcal{H}$. $\square$

Clearly, every expanded minor matching is a semi-matching. In the opposite direction, we see that, in clutters of bounded rank, every semi-matching contains an expanded minor matching containing at least a constant proportion of its pairs.

Theorem 4.2.5 (Matching theorem). *Suppose $\mathcal{H}$ is a clutter of rank at most r and $\mathcal{S}$ is a semi-matching of $\mathcal{H}$. There is an expanded minor matching $\mathcal{S}' \subseteq \mathcal{S}$ of size at least $|\mathcal{S}|2^{-(r-2)}/(2r-3)$.*

Theorem 4.2.5 is sharp in the sense that if we put $r = 2$, we have $\mathcal{S} \equiv \mathcal{S}'$. In this case the entire $\mathcal{S}$ corresponds to an induced matching (in graph theoretical sense).

For clutters of unbounded rank, a semi-matching need not contain an expanded minor matching with at least a constant proportion of its pairs. Consider $\mathcal{H}$ over the vertex set $\{a_1, \dots, a_n\} \cup \{b_1, \dots, b_n\}$ defined by $\mathcal{H} = \{S_i\}_{i=1}^n$, where $S_i = \{a_i\} \cup \{b_j : 1 \leq j \leq i\}$. We see that $\mathcal{H}$ contains a semi-matching $\mathcal{S} = \{(\{a_i b_i\}, S_i)\}_{i=1}^n$ of size n , but $\mathcal{H}$ is $2K_2$ -minor-free.

4.3 Decomposition and Matching theorems

We now present the main technical tool to handle semi-matchings.

Definition 4.3.1 (Expansion). *Suppose $R_1, \dots, R_k, C$ are subsets of $\mathbb{N}$ such that $\{R_i\}$ are pairwise disjoint and $\bigcup_{i=1}^k R_i \subseteq C$. We denote the set of functions $f : \{R_i\}_{i=1}^k \rightarrow C$, such that $f(R_i) \in R_i$ for each i , by $\mathcal{F}(\{R_i\}_{i=1}^k)$. For each such function $f \in \mathcal{F}(\{R_i\}_{i=1}^k)$ define $im_f := \{f(R_i) : 1 \leq i \leq k\}$ to be the image of f . For every clutter $\mathcal{H}$ we define*

$$\mathcal{H} \circ (\{R_i\}_{i=1}^k, C) := \bigvee_{f \in \mathcal{F}(\{R_i\}_{i=1}^k)} \mathcal{H}[im_f; C \setminus im_f].$$

Recall that for disjoint S and T , $\mathcal{H}[S; T]$ denotes $\mathcal{H} \setminus S/T$. Note that $\mathcal{H} \circ (\{R_i\}_{i=1}^k, C) \neq \hat{1}$ iff $C \setminus im_f$ is independent (does not contain any $S \in \mathcal{H}$) for each $f \in \mathcal{F}(\{R_i\}_{i=1}^k)$, and hence condition 4. of the definition of semi-matching is equivalent to $\mathcal{H} \circ (\{E_i\}_{i=1}^k, \bigcup_{i=1}^k H_i) \neq \hat{1}$. We normally shorten $\mathcal{H} \circ (\{E_i\}_{i=1}^k, \bigcup_{i=1}^k H_i)$ to $\mathcal{H} \circ \mathcal{S}$.

Lemma 4.3.2. *Suppose we are given a clutter $\mathcal{H}$, a minimal transversal T of $\mathcal{H}$, a set $S \subseteq \mathbb{N}$ and $x \in \mathbb{N}$. Exactly one of the following two holds:*

- (i) *there exists a minimal transversal $T' \subseteq T \setminus \{x\}$ of $\mathcal{H} \setminus x$ such that $T' \cap S = (T \setminus \{x\}) \cap S$; or*
- (ii) *there exists $z \in S \setminus \{x\}$ and a set $H \in \mathcal{H}$ such that $x \in H$, $H \cap T = \{z\}$ and $T \setminus \{z\} \in b(\mathcal{H} \circ (\{\{x, z\}\}, H))$.*

Proof. We consider two cases, corresponding to the two possible conclusions. It turns out that conclusion (i) must hold if $x \in T$.

Case 1: For each $z \in (T \setminus \{x\}) \cap S$ there exists $H \in \mathcal{H} \setminus x$ such that $H \cap T = \{z\}$.

Let $T' \subseteq T \setminus \{x\}$ and $T' \in b(\mathcal{H} \setminus x)$. Clearly $T' \cap S \subseteq (T \setminus \{x\}) \cap S$. To prove the opposite direction, suppose $t \in (T \setminus \{x\}) \cap S$. We can find $H \in \mathcal{H} \setminus x$ such that $H \cap T = \{t\}$. But since T' is a transversal of $\mathcal{H} \setminus x$ – hence $H \cap T'$ is non-empty – and $T' \subseteq T$, it follows that $H \cap T' = \{t\}$, and therefore $t \in T' \cap S$.

Case 2: There exists $z \in (T \setminus \{x\}) \cap S$ such that for each $H \in \mathcal{H}$ if $H \cap T = \{z\}$, then $x \in H$.

Since $z \in T$ and T is minimal, there is $H \in \mathcal{H}$ such that $H \cap T = \{z\}$. (Thus in this case $x \notin T$.) It is easy to see that $T \setminus \{z\} \in b(\mathcal{H}[z, H \setminus \{z\}])$. Let $F \in \mathcal{H}[x, H \setminus \{x\}]$, and let $\widehat{F} \in \mathcal{H}$ be such that $F \subseteq \widehat{F} \subseteq F \cup (H \setminus \{x\})$. Since $x \notin \widehat{F}$ we have $\widehat{F} \cap T \neq \{z\}$. Let $y \in \widehat{F} \cap (T \setminus \{z\})$, and hence $y \notin H$, so $y \in F$. But then $(T \setminus \{z\}) \cap F$ is non-empty, and F is arbitrary, so we conclude that $T \setminus \{z\}$ is (possibly non-minimal) transversal of $\mathcal{H}[x, H \setminus \{x\}]$. The last statement in conjunction with $T \setminus \{z\} \in b(\mathcal{H}[z, H \setminus \{z\}])$ implies that $T \setminus \{z\} \in b(\mathcal{H} \circ (\{x, z\}, H))$. $\square$

Lemma 4.3.3. *Suppose $\mathcal{H}$ is a clutter, $C \in \mathcal{H}$, $R \subseteq C$, $|R| = 2$ and $\mathcal{S}' = \{(L_i, S'_i)\}_{i=1}^k$ is a semi-matching in $\mathcal{H}' = \mathcal{H} \circ (\{R\}, C)$. Then there is a semi-matching $\mathcal{S} = \{(L_i, S_i)\}_{i=1}^k \cup (\{R\}, C)$ in $\mathcal{H}$, such that $\mathcal{H} \circ \mathcal{S} = \mathcal{H}' \circ \mathcal{S}'$ and $S'_i \subseteq S_i \subseteq S'_i \cup C$ for each i . Pick one such semi-matching, say the lex-first one, and call it $\text{ext}(\mathcal{S}', \mathcal{H}, R, C)$.*

Proof. It follows from the definition of $\mathcal{H} \circ (\{R\}, C)$ that since $S'_i \in \mathcal{H}'$ there is at least one set $S_i \in \mathcal{H}$ such that $S'_i \subseteq S_i \subseteq S'_i \cup C$ and $R \not\subseteq S_i$ for each $i \in [k]$. Fix an arbitrary such S_i for each i and consider $\mathcal{S} = \{(L_i, S_i)\}_{i=1}^k \cup (\{R\}, C)$. From the definitions it immediately follows that $\mathcal{S}$ satisfies conditions 1, 2 and 3a with respect to $\mathcal{H}$. To see that $\mathcal{S}$ satisfies 4 expand

$$\begin{aligned} \mathcal{H} \circ \mathcal{S} &= \bigvee_{f \in \mathcal{F}(\{L_1, \dots, L_k\} \cup \{R\})} \mathcal{H}[im_f; (\cup_i S'_i \cup C) \setminus im_f] \\ &= \bigvee_{f' \in \mathcal{F}(\{L_1, \dots, L_k\})} \bigvee_{r \in R} (\mathcal{H}[r; C - r][im_{f'}; \cup_i S'_i \setminus im_{f'}]) \\ &= \bigvee_{f' \in \mathcal{F}(\{L_1, \dots, L_k\})} \left(\bigvee_{r \in R} \mathcal{H}[r; C - r] \right) [im_{f'}; \cup_i S'_i \setminus im_{f'}] = \mathcal{H}' \circ \mathcal{S}' \neq \widehat{1}. \quad \square \end{aligned}$$

Proof of the Decomposition Theorem 4.2.3. We denote the set of all S -intersecting semi-matchings of a clutter $\mathcal{H}$ by $sm(S, \mathcal{H})$. We prove by induction on $|V(\mathcal{H})|$ that $|tr_S(b(\mathcal{H}))| \leq |sm(S, \mathcal{H})|$. If $|V(\mathcal{H})| = 0$, then $\mathcal{H}$ is either $\emptyset$ or $\{\emptyset\}$. Not that in either case, irrespective of the choice for S , $tr_S(b(\mathcal{H})) \equiv b(\mathcal{H}) \equiv sm(S, \mathcal{H})!$

Now suppose $V(\mathcal{H}) \neq \emptyset$. If $V(\mathcal{H}) \subseteq S$ pick some $x \in V(\mathcal{H})$, otherwise pick arbitrary $x \in V(\mathcal{H}) \setminus S$. Let $\mathcal{T}_1 \subseteq b(\mathcal{H})$ be the set of minimal transversals of the first type described in Lemma 4.3.2, and let $\mathcal{T}_2 = b(\mathcal{H}) \setminus \mathcal{T}_1$.

If $x \notin S$, then clearly $tr_S(\mathcal{T}_1) \subseteq tr_S(b(\mathcal{H} \setminus x))$. To prove that $tr_S(b(\mathcal{H} \setminus x)) \subseteq tr_S(\mathcal{T}_1)$, let $A \in tr_S(b(\mathcal{H} \setminus x))$ and let $T' \in b(\mathcal{H} \setminus x)$, such that $A = T' \cap S$. In particular, we can find $T \in \{T', T' \cup \{x\}\} \cap b(\mathcal{H})$. It follows that $T \in \mathcal{T}_1$, $A \in tr_S(\mathcal{T}_1)$ and finally $tr_S(b(\mathcal{H} \setminus x)) \subseteq tr_S(\mathcal{T}_1)$. We conclude that $tr_S(\mathcal{T}_1) \equiv tr_S(b(\mathcal{H} \setminus x))$, and therefore $|tr_S(\mathcal{T}_1)| = |tr_S(b(\mathcal{H} \setminus x))|$.

If $V(\mathcal{H}) \subseteq S$, then the mapping $T \mapsto T \setminus \{x\}$ is a bijection between $\mathcal{T}_1$ and $b(\mathcal{H} \setminus x)$, and hence

$$|tr_S(\mathcal{T}_1)| = |\mathcal{T}_1| = |b(\mathcal{H} \setminus x)| = |tr_S(b(\mathcal{H} \setminus x))|.$$

We see that $|tr_S(\mathcal{T}_1)| = |tr_S(b(\mathcal{H} \setminus x))|$ in either case, and hence by induction $|tr_S(\mathcal{T}_1)| \leq |sm(S, \mathcal{H} \setminus x)|$.

The transversals of $\mathcal{T}_2$ fall in the second case of Lemma 4.3.2. Let $A \in tr_S(\mathcal{T}_2)$. Then there is an $H \in \mathcal{H}$ and $z \in H \setminus \{x\}$ such that $z \in A$ and $A \setminus \{z\} \in tr_S(b(\mathcal{H} \circ (\{\{x, z\}\}, H)))$.

Let $\mathcal{R}$ be the set of pairs (R, C) such that $C \in \mathcal{H}$, $R \subseteq C$, and $R = \{x, z\}$ where $z \in S \setminus \{x\}$. We define $E_{R,C}$ to be the set

$$E_{R,C} := \{ext(\mathcal{S}', \mathcal{H}, R, C) : \mathcal{S}' \in sm(S, \mathcal{H} \circ (\{R\}, C))\}.$$

We see that $E_{R,C} \subseteq sm(S, \mathcal{H})$ and $E_{R,C} \cap E_{R',C'} = \emptyset$ for $(R, C) \neq (R', C')$ as the sets disagree on the pair of sets containing x . Furthermore, $sm(S, \mathcal{H} \setminus x) \subseteq sm(S, \mathcal{H})$ and $sm(S, \mathcal{H} \setminus x)$ is disjoint from each $E_{R,C}$ because $x \notin L$ for each $(L, S) \in \mathcal{S} \in sm(S, \mathcal{H} \setminus x)$ and every $\mathcal{S} \in E_{R,C}$ contains a pair, (R, C) , including $x \in R$. Now we see

$$\begin{aligned} |tr_S(b(\mathcal{H}))| &\leq |tr_S(\mathcal{T}_1)| + |tr_S(\mathcal{T}_2)| \\ &\leq |tr_S(b(\mathcal{H} \setminus x))| + \sum_{(R,C) \in \mathcal{R}} |tr_S(b(\mathcal{H} \circ (\{R\}, C)))| \\ &\leq |sm(S, \mathcal{H} \setminus x)| + \sum_{(R,C) \in \mathcal{R}} |sm(S, \mathcal{H} \circ (\{R\}, C))| \\ &= |sm(S, \mathcal{H} \setminus x)| + \sum_{(R,C) \in \mathcal{R}} |E_{R,C}| \leq |sm(S, \mathcal{H})|. \quad \square \end{aligned}$$

Proof of the Matching Theorem 4.2.5. Suppose $\mathcal{S} = \{(L_i, S_i)\}_{i=1}^\ell$ is a semi-matching in a rank r clutter $\mathcal{H}$ and let G be a graph over $[\ell]$ where i is connected to j if $|S_i \cap L_j| = 1$ or $|S_j \cap L_i| = 1$. It follows that G contains at most $(r-2)\ell$ edges. A classic result in graph theory states that $\alpha(G) \geq \frac{v(G)^2}{2e(G)+v(G)}$, and hence G contains an independent set I of size at least $\ell/(2r-3)$.

For $f \in \mathcal{F}(\{L_i\}_{i \in [\ell] \setminus I})$ define

$$I'(f) := \{i \in I : f(L_j) \notin S_i \text{ for each } j \in [\ell] \setminus I\}.$$

Sample g uniformly at random from $\mathcal{F}(\{L_i\}_{i \in [\ell] \setminus I})$. We have $\mathbb{E}|I'| \geq |I|2^{-(r-2)} \geq \ell 2^{-(r-2)}/(2\ell - 3)$, so there must be some $J = I'(g)$ of at least this size, where $g \in \mathcal{F}(\{L_i\}_{i \in [\ell] \setminus I})$.

We claim that $\mathcal{S}_J := \{(L_i, S_i)\}_{i \in J}$ is an expanded minor matching. Properties 1 and 2 are inherited from $\mathcal{S}$. Property 3b holds because $J \subseteq I$ is stable in G . Let $h \in \mathcal{F}(\{L_i\}_{i \in [\ell] \setminus J})$ be an arbitrary extension of g and, let $C := \bigcup_{i=1}^k S_i$ and $C_J := \bigcup_{i \in J} S_i$. We see that C_J is disjoint from im_h . Since every $S \in \mathcal{H}$ with $S \subseteq C$ contains L_i as a subset for some $i \in [l]$, it follows that if $S \subseteq C_J \subseteq C \setminus im_h$, then $L_i \subseteq S$ for some $i \in J$. $\square$

Proof of the Minor-Matching Theorem, Theorem 4.2.1. By the Decomposition Theorem, Theorem 4.2.3, we have $|tr_S(b(\mathcal{H}))| \leq |sm(S, \mathcal{H})|$. However, for every semi-matching $\mathcal{S} \in sm(S, \mathcal{H})$, we must have $|\mathcal{S}| \leq \mu_S(\mathcal{H})f(rk(\mathcal{H}))$, as otherwise by the Matching Theorem, Theorem 4.2.5, there is a subset $\mathcal{S}' \subseteq \mathcal{S}$ of size more than $\mu_S(\mathcal{H})$ that is an expanded minor matching. This is a contradiction, because $\mathcal{S}'$ is S -intersecting. We conclude that the size of $tr_S(b(\mathcal{H}))$ is bounded from above by the number of subsets of $E_S(\mathcal{H})$ of size at most $\mu_S(\mathcal{H})f(rk(\mathcal{H}))$, and the first part of the statement follows.

To make the theorem algorithmic, one may consider a branching algorithm, exploring all cases of Lemma 4.3.2. The size of the recursion tree is bounded by the number of quasimatchings, hence the total running time is polynomial. $\square$

4.4 Complexity of finding minor matchings

It is natural to ask whether a clutter $\mathcal{H}$ contains a given clutter $\mathcal{F}$ as a minor or not.

CLUTTER MINOR
Instance: Two clutters, $\mathcal{H}$ and $\mathcal{F}$.
Question: Is $\mathcal{F}$ isomorphic to a minor of $\mathcal{H}$?

A naive approach would be to guess an injective function $f : V(\mathcal{F}) \rightarrow V(\mathcal{H})$, sets H_F for each $F \in \mathcal{F}$ such that $f[F] \subseteq H_F \in \mathcal{H}$, and check if f is an isomorphism between

$$\mathcal{H} \left[V(\mathcal{H}) \setminus \bigcup_{F \in \mathcal{F}} H_F; \bigcup_{F \in \mathcal{F}} H_F \setminus E_F \right] \quad \text{and} \quad \mathcal{F}.$$

It is interesting to see if there is a clever algorithm for the last step, but since the overall complexity is dominated by the guessing steps, we can be crude and state that the total running time is $||\mathcal{H}||^{O(||\mathcal{F}||)}$. In particular, if $\mathcal{F}$ is fixed, the algorithm is P-time.

Clutter Minor is in general NP-complete. For instance, restricting $\mathcal{H}$ to be a graph, and $\mathcal{F}$ to K_k , makes the problem equivalent to the **Clique** problem. This also shows that **Clutter Minor** is W[1]-hard w.r.t. $||\mathcal{F}||$. We are primarily interested in the case when $\mathcal{F} \cong kK_2$. In the case when $\mathcal{H}$ is a graph and $\mathcal{F} \cong kK_2$, **Clutter Minor** is reduced to the **Induced Matching** problem. **Induced Matching** has similar properties to the **Clique Problem**; namely it is W[1]-hard [MT09], and there is no $(n/2)^{1-\epsilon}$ approximation algorithm unless P=NP [OFGZ08]. We now put this formally.

MINOR MATCHING
Instance: A clutter $\mathcal{H}$ and an integer k .
Question: Is kK_2 isomorphic to a minor of $\mathcal{H}$?

MAXIMUM MINOR MATCHING
Instance: A clutter $\mathcal{H}$.
Question: What is the maximum k such that $kK_2 \subseteq_{\mathcal{M}} \mathcal{H}$?

Lemma 4.4.1. *Minor Matching is W[1]-hard, and there is no $n^{1-\epsilon}$ approximation algorithm for the Maximum Minor Matching problem unless P=NP.*

The proof is very easy, so we present it here for completeness.

Proof. Consider the **Independent Set** problem and its optimisation variant, **Maximum Independent Set**. As noted before, **Independent Set** is W[1]-complete [MT09], and that there is no $n^{1-\epsilon}$ approximation algorithm for **Maximum Independent Set** unless P = NP [OFGZ08]. Reduce an instance $I = (G, k)$ of the **Independent Set**, where the vertices of G are $\{v_1, \dots, v_n\}$, to an instance $(M(G), k)$, where $M(G)$ is an edge clutter containing an isomorphic copy of G and new vertices $\{u_1, \dots, u_n\}$, with each vertex v_i joined by a binary edge to u_i . It is easy to see that $\alpha(G) = \mu(M(G))$. The same reduction suffices to prove the second part of the statement. $\square$

The reduction is trivial, but also very robust, and we will come back to it in § 4.7.1. As a side note, the authors of [OFGZ08] prove the $(n/2)^{1-\epsilon}$ -inapproximability of **Induced Matching** in their paper, but advertise a much weaker $n^{\frac{1}{2}-\epsilon}$ -inapproximability in the abstract and the introduction.

For completeness, we present one more problem.

INTERSECTING MINOR MATCHING
Instance: A clutter $\mathcal{H}$, a set $S \subseteq V(\mathcal{H})$, and an integer k .
Question: Is $\mu_S(\mathcal{H}) \leq k$?

The problem is a generalisation of **Minor Matching**, and hence all hardness results above apply here. The algorithm for clutter minors can be adapted to this problem as well.

4.5 Algorithms for clutters with bounded minor matchings

Definition 4.5.1. Suppose $\mathcal{J}$ is a countable set, and that for every clutter $\mathcal{H}$ and $J \in \mathcal{J}$ we are given a parameter $f_{\mathcal{H},J} \in \mathbb{Q}$. We say that $f_{\mathcal{H},J}$ can be read from the blocker if $f_{\mathcal{H},J}$ can be computed from $(\mathcal{H}, b(\mathcal{H}), J)$ in polynomial time with respect to $\|(\mathcal{H}, b(\mathcal{H}), J)\|$.

Lemma 4.5.2. Fix integers r and k . Any parameter $f_{\mathcal{H},J}$ that can be read from the blocker can be computed from $(\mathcal{H}, J)$ in polynomial time with respect to $\|(\mathcal{H}, J)\|$ for clutters $\mathcal{H}$ with $rk(\mathcal{H}) \leq r$ and $\mu(\mathcal{H}) \leq k$.

Many natural parameters can be read from the blocker, but are NP-hard to compute if only $\mathcal{H}$ is given as input. We give few examples below.

4.5.1 Weighted Independent Set

As a warm-up, our first example is **Weighted Independent Set**. Recall that a set of vertices S of a clutter $\mathcal{H}$ is called *independent* if it contains no set $H \in \mathcal{H}$ as a subset.

WEIGHTED INDEPENDENT SET
Instance: A clutter $\mathcal{H}$, and a weight function $w : V(\mathcal{H}) \rightarrow \mathbb{Q}$.
Question: Find the maximum value $w(J) = \sum_{j \in J} w(j)$ of an independent set J of $\mathcal{H}$.

Lemma 4.5.3. The maximum weight of an independent set of a clutter $\mathcal{H}$ w.r.t. a weight function w , $i_{\mathcal{H},w}$, can be read from the blocker.

Proof. Suppose J is an independent set of $\mathcal{H}$ of maximum weight, and with respect to that J is inclusion-minimal. For each $v \in J$ we have $w(v) > 0$, so we may delete the vertices of $\mathcal{H}$ with non-positive weight to obtain a new clutter $\mathcal{H}'$ where J is a maximal independent set. But now $V(\mathcal{H}') \setminus J \in b(\mathcal{H}')$, so it suffices to return $\max_{T \in b(\mathcal{H}')} (w(V(\mathcal{H}') \setminus T))$. $\square$

We will come back to this example later, but note that the problem can be generalised significantly. Indeed, suppose Q is a $\mathbb{Q}$ -valued monotone oracle over the independent sets of $\mathcal{H}$, that is $Q(S) \in \mathbb{Q}$ for each stable set S of $\mathcal{H}$, and $Q(S) \leq Q(T)$ whenever $S \subseteq T$. Then the parameter $\max_S Q(S)$, where the maximum is taken over all stable sets S of $\mathcal{H}$, can be read from the blocker.

4.5.2 k -Colouring

A more interesting example is **k -Colouring**. Colouring of a clutter $\mathcal{H}$ with k colours is a function $c : V(\mathcal{H}) \rightarrow [k]$. A colouring c is called *proper* if no edge is *monochromatic*, that is all of its vertices are coloured the same. The chromatic number of $\mathcal{H}$, $\chi(\mathcal{H})$, is the minimum number of colours in a proper colouring.

CLUTTER k -COLOURING
Instance: A clutter $\mathcal{H}$.
Question: Is $\chi(\mathcal{H}) \leq k$?

We need to phrase the problem in terms of maximal independent sets in order to tackle it. Suppose $c : V(\mathcal{H}) \rightarrow [k]$ is a proper colouring. Observe that $c^{-1}(a)$ is an independent set for each colour a . Therefore $\chi(\mathcal{H})$ is the minimum size of a cover of $\mathcal{H}$ with independent sets. We may assume that each independent set in the cover is maximal. Given $b(\mathcal{H})$, we can easily see if $\chi(\mathcal{H}) \leq k$: simply check if any k -tuple $(\overline{T}_1, \overline{T}_2, \dots, \overline{T}_k)$ of maximal independent sets covers $\mathcal{H}$.

We now consider the more general problem of determining the chromatic number of a clutter.

CHROMATIC NUMBER OF A CLUTTER
Instance: A clutter $\mathcal{H}$.
Question: What is $\chi(\mathcal{H})$?

It is not clear whether this parameter can be read from the blocker, but at least we can give an $O(\ln n)$ -approximation. This is a direct consequence of the celebrated $O(\ln n)$ -approximation of **Set Cover**.

4.5.3 Clutter Homomorphism

Suppose $\mathcal{H}$ and $\mathcal{F}$ are clutters. A (not-necessarily injective) mapping $f : V(\mathcal{H}) \rightarrow V(\mathcal{F})$ is called *homomorphism* if the image of each set of $\mathcal{H}$ is contained in $\mathcal{F}$. The existence of a homomorphism between $\mathcal{H}$ and $\mathcal{F}$ is denoted by $\mathcal{H} \rightarrow \mathcal{F}$. The framework of homomorphisms has a rich expressive power to describe graph-theoretic properties. For instance, $\chi(G) \leq k$ if and only if $G \rightarrow K_k$.

As usual, we start by formulating the problem in terms of maximal independent sets. A clutter is called *uniform* if all edges have the same size. Let us denote the set of all maximal independent sets of a clutter $\mathcal{H}$ by $i(\mathcal{H})$, i.e. $I \in i(\mathcal{H})$ if and only if $V(\mathcal{H}) \setminus I \in b(\mathcal{H})$.

Lemma 4.5.4. *Suppose $\mathcal{H}$ and $\mathcal{F}$ are r -uniform clutters. A mapping $f : V(\mathcal{H}) \rightarrow V(\mathcal{F})$ is a homomorphism if and only if $f^{-1}(I)$ is independent in $\mathcal{H}$ for every $I \in i(\mathcal{F})$.*

Proof. If f is a homomorphism, then clearly $f^{-1}(I)$ is stable for each $I \in i(\mathcal{F})$, as otherwise the image of any edge contained in $f^{-1}(I)$ is not an edge.

Suppose $f : V(\mathcal{H}) \rightarrow V(\mathcal{F})$ is a function such that $f^{-1}(I)$ is stable in $\mathcal{H}$ for every $I \in i(\mathcal{F})$. Let h be an edge of $\mathcal{H}$ and let $f(h)$ be its image. Since $\mathcal{H}$ and $\mathcal{F}$ are r -uniform, $f(h)$ is either an edge of $\mathcal{F}$ or stable. If $f(h)$ is stable, let $I \in i(\mathcal{F})$ be a maximal independent set of $\mathcal{F}$ containing $f(h)$. But now we see that $h \subseteq f^{-1}(I)$, which is a contradiction, and hence $f(h)$ is an edge for every $h \in E(\mathcal{H})$. $\square$

Lemma 4.5.5. *Suppose $\mathcal{H}$ and $\mathcal{F}$ are r -uniform clutters. The following two statements are equivalent:*

1. $\mathcal{H} \rightarrow \mathcal{F}$,
2. *there exists a function $g : i(\mathcal{F}) \rightarrow i(\mathcal{H})$ such that*

$$\left\{ A_x = \bigcap_{I \in i(\mathcal{F}), x \in I} g(I) : x \in V(\mathcal{F}) \right\}$$

covers $V(\mathcal{H})$.

Proof. First suppose $f : \mathcal{H} \rightarrow \mathcal{F}$ is a homomorphism. Each $f^{-1}(I)$ is stable in $\mathcal{H}$ for $I \in i(\mathcal{F})$, so we may set $g(I) = J$, where $J \in i(\mathcal{H})$ is an arbitrary maximal independent set containing $f^{-1}(I)$. Let v be an arbitrary vertex of $\mathcal{H}$. We have $v \in A_{f(v)}$, and therefore the sets A_x cover $V(\mathcal{H})$.

Now suppose g is a function satisfying condition 2. in the statement. For each vertex v of $\mathcal{H}$ pick some arbitrary vertex x of $\mathcal{F}$ such that $v \in A_x$ and set $f(v) = x$. Such a vertex exists, because $\{A_x\}_{x \in V(\mathcal{F})}$ covers $V(\mathcal{H})$. Let $I \in i(\mathcal{F})$ be arbitrary. We see that $f^{-1}(I) \subseteq g(I)$ is stable in $\mathcal{H}$, and therefore f is a homomorphism. $\square$

For each r -uniform clutter $\mathcal{F}$ we can consider the following problem:

$\mathcal{F}$-HOMOMORPHISM
Instance: An r -uniform clutter $\mathcal{H}$.
Question: Is there a homomorphism $f : \mathcal{H} \rightarrow \mathcal{F}$?

The corresponding boolean parameter $f_{\mathcal{F}}$ can clearly be read from the blocker: just test if any of the $|b(\mathcal{F})|$ -tuples $(\overline{T_1}, \overline{T_2}, \dots)$, $T_i \in b(\mathcal{H})$, satisfies the second condition of Lemma 4.5.5.

4.5.4 Other Problems

The problems discussed so far, **Independent Set**, k -**Colouring**, and $\mathcal{F}$ -**Homomorphism** are fundamental to theoretical computer science, and many other problems have a natural reduction to them. For instance, the **Set-Cover** problem can easily be reduced to **Hitting Set** by swapping the roles of vertices and sets. On the other hand, **Hitting Set** is identical to **Independent Set**. **MSAT** can be seen in the following way: given a Boolean formula where every clause is either entirely positive or entirely negative, $\varphi = \varphi^+ \wedge \varphi^-$, is there a transversal of the clauses of φ^+ (φ^-) that does not contain any clause of φ^- (φ^+). Therefore the solution of the instance could be read from the blocker of φ^+ or φ^- . As we discussed before, **SAT** has a very natural reduction to **MSAT**.

4.6 Minor-Matching Hypertree width

As pointed out in the introduction, some very simple clutters have exponentially-sized blockers, for example the matching nK_2 , and therefore the solutions of problems like **Independent Set** and **Homomorphism** cannot be read from their blocker. We address this issue in this section and the following one by generalising the methods in this chapter so far from clutters with excluded minor matching to clutters permitting a tree decomposition with no bag in the decomposition containing a large minor matching. Concretely, for each hypergraph H we introduce a parameter $\mu\text{-tw}(H)$, called *minor-matching hypertree width*, as the minimum over all tree decompositions $(T, \{B_t : t \in T\})$ of the maximum $\mu_{B_t}(H)$ over all bags B_t . For example, the matching nK_2 can be decomposed into n bags containing one edge each, joined arbitrarily to form a tree. No bag contains a minor matching of size two, and hence $\mu\text{-tw}(nK_2) = 1$.

When comparing different tree decompositions, usually the following four questions are considered. The first one is *on what structures does the decomposition work*. The original treewidth decomposition is defined on graphs, but later work introduces tree decompositions for hypergraphs, for example Hypertreewidth [GLS99] and Fractional hypertree width decompositions [GM06]. The decomposition in this chapter yields efficient algorithms for *hypergraphs with bounded rank*. It is not clear whether or not the restriction on the size of the hyperedges is necessary, but so far we can only guarantee polynomial-time solutions when the condition holds.

The second question is *what is the number of structures with bounded width*. Here minor-matching hypertree decompositions really stand out, as most graphs and hypergraphs have much lower minor-matching hypertree width compared to other width measures. We show that, for fixed k , there are $2^{(1-\frac{1}{k}+o(1))\binom{n}{2}}$ n -vertex graphs with minor-matching hypertree width at most k (the number of all n -vertex graphs is $2^{\binom{n}{2}}$).

In comparison, most tree decompositions permit $2^{O(n \ln n)}$ n -vertex graphs of width at most k . More detail is given in § 4.6.4.

The next question is *what family of problems can be solved in polynomial time provided a decomposition of the instance with bounded width is given as input*. For the moment there is no meta-theorem similar to Courcelle's theorem [Cou90], describing classes of computable problems, and instead we give concrete examples of how to solve the problems in § 4.5, namely **Independent Set**, **k -Colouring** and **Homomorphism**, via dynamic programming in § 4.6.3. Other problems, like **Set Cover** and **k -SAT**, also have natural solutions using the developed methods.

The final question is *is it possible to efficiently construct a tree decomposition of width k , or more generally $f(k)$, given a structure of width at most k* . This problem is essential, as it allows us to drop the requirement to provide a tree decomposition in the input of the problem, and find truly polynomial cases of hard problems. A tree decomposition without this property has limited use, since it may be as hard to find a decomposition as to solve the problem. Given a fixed k and a problem P that permits a polynomial solution of instances (I, T) , where T is a decomposition of width at most k , it suffices to construct decomposition of width $f(k)$ to solve $P(I)$, as $f(k)$ is also bounded from above. In § 4.7 we present an $n^{O(k^3)}$ -time algorithm that given a graph G either finds a tree decomposition of G with μ -width $O(k^3)$, or proves that $\mu\text{-tw}(G) > k$.

The rest of this chapter is dedicated to address these questions. We need to introduce a number of auxiliary concepts, and some of the proofs, especially when it comes to the approximation algorithm, can be very technical.

4.6.1 Tree decompositions

Definition 4.6.1 (Hypergraph). A hypergraph $H = (V, E)$ is a pair of a finite set V called vertex set and a set system E over V with elements called hyperedges or just edges. We use $\|H\|$ to denote $|V(H)| + \sum_{h \in E} |h|$.

Definition 4.6.2 (Tree decomposition [BB72, Hal76, RS84, GLS99]). We say that the pair $\mathcal{T} = (T, \{B_t : t \in V(T)\})$ is a (hyper)tree decomposition of a hypergraph H if T is a tree, B_t , called the bag of t , is a subset of $V(H)$ for each node t of T , and

1. for each vertex x of H , the nodes t of T such that $x \in B_t$, denoted T_x , induce a connected subtree of T , and
2. every hyperedge of H is contained in some bag B_t , $t \in V(T)$.

We sometimes shorten $(T, \{B_t : t \in V(T)\})$ to $(T, \{B_t\})$ or even (T, B_t) . Instead of tree/hypertree decomposition we sometimes just say decomposition.

Definition 4.6.3 (λ -treewidth). Suppose that λ_H is a real-valued function on the subsets of $V(H)$ for each hypergraph H . Given a hypergraph H and a tree decomposition $\mathcal{T} = (T, B_t)$ of H , we define

$$\lambda_H(\mathcal{T}) = \max_{t \in V(T)} \lambda_H(B_t) \quad \text{and} \quad \lambda\text{-}tw(H) = \min_{\mathcal{T}'} \lambda_H(\mathcal{T}'),$$

to be the λ -width of $\mathcal{T}$, and the λ -treewidth of H respectively, where the minimum is taken over all tree decompositions $\mathcal{T}'$ of H .

For instance, if we define $\alpha_H(S)$ to be the maximum number of independent vertices of S in a hypergraph H , we obtain a new treewidth measure, $\alpha\text{-}tw(\mathcal{T})$, corresponding to the maximum size of an independent set of a bag of $\mathcal{T}$.

If H is clear from the context, we normally abbreviate $\lambda_H(\mathcal{T})$ to $\lambda(\mathcal{T})$.

Definition 4.6.4 (Treewidth). The treewidth [BB72, Hal76, RS84] of a graph G , $tw(G)$, is defined as $\kappa\text{-}tw(G)$ for the function $\kappa_G(S) = |S| - 1$.

Treewidth plays a central role in the study of graph minors, but we are mainly interested in applications to algorithms. It is NP-complete to determine the treewidth of a graph G , but for fixed k there are linear-time algorithms, with an exponential dependency on k , to construct a tree decomposition of width at most k or to decide that $tw(G) > k$ [Bod96].

Definition 4.6.5 (Clique-width). The clique-width of a graph G , $cw(G)$, is the minimum number of labels needed to construct G from the following operations:

1. creating of a new vertex with label i ,
2. disjoint union of two labelled graphs,
3. joining by an edge every vertex with label i to every vertex of label j , for given $j \neq i$, and
4. renaming every label i to j .

A sequence of these operations creating a graph G using at most k labels is called k -expression of G .

In contrast to treewidth, dense graphs may have low clique-width. For example, the complete graph K_n has treewidth $n-1$, while $cw(K_n) = 2$ for $n > 1$. Moreover, a family of graphs with bounded treewidth also has bounded clique-width; more precisely, as shown in [CR05],

$$cw(G) \leq 3 \cdot 2^{tw(G)-1}.$$

The example with the complete graph shows that no general bound exists for the other direction, but nevertheless, interesting special bounds do exist:

1. $tw(G) \leq f(\Delta(G), cw(G))$ [CO00],
2. $tw(G) \leq g(k, cw(G))$ for K_k -minor-free graphs G [CO00],
3. $tw(G) \leq 3 \cdot cw(G)(t-1) - 1$ for $K_{t,t}$ -free graphs G [GW00],

where $\Delta(G)$ denotes the maximum degree of a graph G , and f and g are functions.

Similarly to treewidth, finding the clique-width of a graph is NP-complete, [FRRS06]. There is an $O(f(k)v(G)^3)$ -time algorithm finding a $(2^{k+1} - 1)$ -expression for a given input graph G with clique width at most k [HO08].

Rank-width is a closely related width measure to clique-width.

Definition 4.6.6 (Rank-width). *Suppose G is a graph and A is a subset of $V(G)$. The cut-rank of A , $cutrk_G(A)$, is the linear rank of the matrix $\{1_{ij \in E(G)} : i \in A, j \notin A\}$. A ternary tree is a tree where all non-leaves have degree 3. A pair (T, L) is called rank-decomposition of G if T is a ternary tree and L is a bijection from the leaves of T to the vertices of G . For every edge e of T the connected components of $T - e$ induce a partition (A, B) of the leaves of T . Define the width of e to be $cutrk_G(L(A)) = cutrk_G(L(B))$, the width of (T, L) to be the maximum width of an edge, and the rank-width of G , $rw(G)$, to be the minimum width of a rank-decomposition of G .*

A key link between rank-width and clique-width has been established in [OS06]:

$$rw(G) \leq cw(G) \leq 2^{rw(G)+1} - 1.$$

The inequalities reveal that a family of graphs has bounded rank-width if and only if it has bounded clique-width.

The importance of the width parameters above is best illustrated by Courcelle's theorem [Cou90].

Definition 4.6.7 (MSO_1 , MSO_2). *A graph can be seen as a finite model with its vertices as universe, and a single symmetric relation indicating adjacency. The monadic second-order graph logic, MSO_1 , consists of formulae for this model with quantifiers over vertices and sets of vertices. A graph can also be seen as a finite model with its vertices and edges as universe, and two symmetric relations indicating adjacency between vertices and edges. The extended monadic second-order graph logic, MSO_2 , consists of formulae for this model with quantifiers over vertices, edges, sets of vertices and sets of edges.*

Note that in the definition of MSO_2 we do not allow quantifiers over sets of arbitrary pairs of vertices.

Theorem 4.6.8 (Courcelle’s theorem [Cou90, BPT92, CMR00]). *For every MSO_1 formula ψ_1 and every MSO_2 formula ψ_2 , there are functions f_1 and f_2 , such that there are $O(f_1(cw(G))n)$ -time and $O(f_2(tw(G))n)$ -time algorithms to test if $G \models \phi_1$ and $G \models \phi_2$ respectively for every n -vertex graph G .*

We now turn our attention to tree decompositions of hypergraphs.

Definition 4.6.9 (Primal graph). *The primal graph $\underline{H}$ of a hypergraph H is a graph over the same vertex set as H with two vertices joined by an edge whenever they appear in the same hyperedge of H .*

The Helly property of trees implies that $\mathcal{T}$ is a decomposition of a hypergraph H if and only if it is a decomposition of $\underline{H}$.

The first hypergraph treewidth measure for which it is possible to construct a tree decomposition efficiently and it is not bounded in terms of the treewidth of the primal graph is *generalised hypertree width*.

Definition 4.6.10 (Generalised hypertree width [GLS99]). *The generalised hypertree width of a hypergraph H is defined as $\varrho\text{-}tw(H)$, where $\varrho_H(S)$ is the minimum size of a cover of S by edges of H .*

As the name suggests, there is a notion called *hypertree width*. Hypertree width, $\varrho'\text{-}tw(H)$, involves an additional technical restriction on the tree decomposition, but as shown in [AGG07], $\varrho\text{-}tw(H) \leq \varrho'\text{-}tw(H) \leq 3\varrho\text{-}tw(H)$, and hence the two notions are essentially equivalent. There is a polynomial-time algorithm to determine if $\varrho'\text{-}tw(H) \leq k$ for fixed k , therefore it is possible to approximate $\varrho\text{-}tw$ within a factor of 3 for small k .

Definition 4.6.11 (Fractional cover number). *A fractional cover of $S \subseteq V(H)$ by H , is a function $\gamma : E(H) \rightarrow [0, 1]$, such that for each $s \in S$ we have $\sum_{h \in E(H): s \in h} \gamma(h) \geq 1$. The weight of a fractional cover γ is $\sum_{h \in E(H)} \gamma(h)$. The fractional cover number, $\varrho_H^*(S)$, is defined as the minimum weight of a fraction cover.*

Definition 4.6.12 (Fractional hypertree width [GM14]). *We define $\varrho^*\text{-}tw$ to be fractional hypertree width.*

Clearly, $\varrho_H^*(S) \leq \varrho_H(S)$, and hence $\varrho^*\text{-}tw(H) \leq \varrho\text{-}tw(H) \leq \varrho'\text{-}tw(H)$. A recent result [FGP16] states that checking $\varrho^*\text{-}tw(H) \leq k$ is NP-c for any fixed $k \geq 2$. However, given a hypergraph H and a constant k , it is possible to construct a tree decomposition of H of ϱ^* -width $O(k^3)$, or to prove that $\varrho^*\text{-}tw(H) > k$ in $n^{O(k^3)}$ [Mar10] time.

A critical property of generalised hypertree width and fractional hypertree width is that constraint satisfaction problems can be solved efficiently on instances of bounded width.

Definition 4.6.13. A constraint satisfaction problem (CSP) is a triple of variables V , a domain D , and a set C of constraints, where each constraint is a relation on a subset of the variables. The task is to assign a value from D to each variable so that each constraint is satisfied.

For instance, graph colouring can be expressed as a CSP. The domain D is the set of available colours and there is a constraint requiring each edge to be coloured properly. We can also interpret 3-SAT as a CSP: the domain is $D = \{0, 1\}$ and each constraint corresponds to a clause.

Definition 4.6.14 (Constraint hypergraph). The hypergraph with the variables of an instance I of CSP as vertices and a hyperedge of each constraint is called constraint hypergraph of I , $C(I)$.

Theorem 4.6.15. Given a tree decomposition $\mathcal{T}$ of $C(I)$, it is possible to solve an instance I of a CSP in time $\|I\|^{q(\mathcal{T})+O(1)}$ [GLS99] and $\|I\|^{q^*(\mathcal{T})+O(1)}$ [GM14]. Here $\|I\|$ is the sum of the number of variables, the number of tuples in each constraint relation, and the size of the domain.

Recall that we defined the rank of a hypergraph H , $rk(H)$, to be the maximum size of an edge of H . It is easy to see that

$$q\text{-}tw(H) + 1 \leq tw(H) \leq rk(H)(q^*\text{-}tw(H) + 1) \leq rk(H)(q\text{-}tw(H) + 1),$$

and hence the classes of hypergraphs with restricted rank and width in either notion are essentially equivalent.

Theorem 4.6.16. Let $CSP(\mathcal{H})$ denote all CSP instances with constraint graph in a recursively enumerable class $\mathcal{H}$ of bounded rank. Assuming $FTW \neq W[1]$, the following two conditions are equivalent:

1. $CSP(\mathcal{H})$ is polynomially solvable,
2. $\mathcal{H}$ has bounded treewidth.

The condition $FTW \neq W[1]$ is a widely assumed hypothesis in parameterised complexity. On the contrary, minor-matching hypertree width cannot be bounded from below in terms of $rk(H)$ and $tw(H)$, and therefore CSPs are computationally unfeasible.

4.6.2 Definition of μ -tw

Recall that $\mu_S(H)$ is defined as the maximum size of an S -intersecting minor-matching of a clutter H and a set $S \subseteq \mathbb{N}$.

Definition 4.6.17 (Minor-matching hypertree width). *Let $\mu_H(S) = \mu_S(\text{cl}(H))$. We define $\mu\text{-tw}(H)$ to be the minor-matching hypertree width of H .*

Suppose H, H_1 and H_2 are hypergraphs. We write $H = H_1 \cup H_2$ if $V(H_1) \cup V(H_2) = V(H)$ and $E(H_1) \cup E(H_2) = E(H)$. If $H = H_1 \cup H_2$, we say that S is an (H_1, H_2) -separator if $V(H_1) \cap V(H_2) \subseteq S$.

Minor-matching treewidth can be used to compute hypergraph parameters. Indeed, suppose we want to compute a hypergraph parameter f_H . If it is possible to generalise f_H to a function f_H^S over $\text{tr}_S(b(H))$, say $f_H = f_H^\emptyset(\emptyset)$, for any $S \subseteq V(H)$, and it is possible to compute f_H^S from $f_{H_1}^S$ and $f_{H_2}^S$, where S is an (H_1, H_2) -separator, then, roughly speaking, we can compute f_H given a tree decomposition $\mathcal{T}$ with $\mu(\mathcal{T}) \leq k$. The next lemma formalises the discussion above.

Definition 4.6.18. *We say that a function $f_H^S : \text{tr}_S(b(H)) \rightarrow \mathbb{Q}$, or more generally, $f_H^S : \text{tr}_S(b(H))^p \rightarrow \mathbb{Q}$, can be read from a blocker decomposition if it has the following properties:*

1. *it is possible to compute $f_H^{V(H)}$ from $b(H)$ efficiently,*
2. *it is possible to compute f_H^S from $f_{H'}^{S'}$ efficiently if $S \subseteq S'$,*
3. *it is possible to compute $f_{H+v}^{S \cup \{v\}}$ from f_H^S , where $H + v$ is a hypergraph obtained from H by adding a new isolated vertex, v ,*
4. *it is possible to compute f_H^S from $\text{tr}_S(b(H))$, $f_{H_1}^S$ and $f_{H_2}^S$ efficiently if $V(H_1) \cap V(H_2) \subseteq S$ and $H = H_1 \cup H_2$,*

for every hypergraph H and set $S \subseteq V(H)$. Here efficiently means in polynomial time with respect to the size of the input, including $\text{tr}_S(b(H))$, and the size of the output, where all functions are represented in table form.

Lemma 4.6.19. *Suppose r and k are fixed constants. Let H be a hypergraph of rank at most r , $\mathcal{T} = (T, \{B_t\})$ be a tree decomposition of H with $\mu(\mathcal{T}) \leq k$, and f_H^S be a function that can be read from a blocker decomposition. It is possible to compute $f_H^{B_t}$ from $(H, \mathcal{T})$ in polynomial time w.r.t. $\|H\|$ for every bag B_t , $t \in T$. In particular, it is possible to compute $f_H^\emptyset$ in polynomial time.*

Proof. Given a bag B_r , pick r as the root of T and orient every edge from the root towards the leaves. Let T_t be the subtree of T composed of descendants of t , and let H_t be the subhypergraph of H induced by the vertices contained in bags of T_t . We compute $f_{H_t}^{B_t}$ recursively, from the leaves towards the root. We can compute $f_{H_l}^{B_l}$ for each leaf l of T using property 1. of f . Given a node t with direct descendants $t_1 \dots t_m$, first compute $f_{H_{t_i}}^{B_t \cap B_{t_i}}$ using property 2., then add each vertex in $B_t \setminus B_{t_i}$ as an isolated vertex to compute $f_{H_{t_i} + (B_t \setminus B_{t_i})}^{B_t}$ using property 3., and finally merge sequentially $f_{B_t}^{B_t}$ with $f_{H_{t_i} + (B_t \setminus B_{t_i})}^{B_t}$ to construct $f_{H_t}^{B_t}$. Theorem 4.2.1 guarantees that $tr_{B_t}(b(H))$ is polynomially sized and that it can be computed in polynomial time, and hence the input and the output of each step is polynomially sized. $\square$

4.6.3 Examples of functions that can be read from a blocker decomposition

In the previous section we defined when a function can be read from a blocker decomposition. These functions play a central role in tractability via minor-matching tree decompositions, as they can be computed in polynomial time via such decompositions of bounded width. In this section we revise the three functions that we saw can be read from the blocker in § 4.5, namely **Independent Set**, **k -Colouring** and **Homomorphism**, and show that in addition, they can be read from a blocker decomposition.

4.6.3.1 Weighted Independent Set

Recall that we use $i(H)$ to denote the set of all maximal independent sets of a hypergraph H , and note that $i(H)$ is easy to compute from $b(H)$: $i(H) = \{V(H) \setminus T : T \in b(H)\}$. We come back to **Weighted Independent Set**. Define $f_H^S(A) := \max\{w(J) : J \in i(H), A = S \cap J\}$. It suffices to show that f_H^S can be read from a blocker decomposition in order to prove that the problem can be solved in polynomial time using a tree decomposition of bounded μ -width. Properties 1.-3. are easy to verify. Suppose $H = H_1 \cup H_2$ and $V(H_1) \cap V(H_2) \subseteq S$ and that $f_H^S(A)$ is witnessed by $J \in i(H)$. Since J is stable in H_1 and H_2 , we may find $J_i \in i(H_i)$ such that $J \cap V(H_i) \subseteq J_i$. In particular, $A \subseteq J_i \cap S$. We have

$$w(J) = w(J_1) + w(J_2) - w(J_1 \cap S) - w(J_2 \cap S) + w(A).$$

It follows that $w(J_i) = f_{H_i}^S(J_i \cap S)$, and hence

$$f_H^S(A) = \max\{f_{H_1}^S(A_1) + f_{H_2}^S(A_2) - w(A_1) - w(A_2) + w(A) : A_1 \in tr_S(i(H_1)), A_2 \in tr_S(i(H_2)), A = A_1 \cap A_2\}.$$

4.6.3.2 k-Colouring

Consider the function

$$f_H^S(A_1, A_2, \dots, A_k) = \begin{cases} 1 & \text{if } (A_1, A_2, \dots, A_k) \text{ covers } S \text{ and} \\ & \text{there exists a cover } (I_1, I_2, \dots, I_k) \text{ of } H, \\ & \text{s.t. } I_i \in i(H) \text{ and } A_i \subseteq I_i \text{ for } 1 \leq i \leq k; \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that f_H^S satisfies properties 1.-3. of Definition 4.6.18.

Lemma 4.6.20. *Suppose $H = H_1 \cup H_2$, $V(H_1) \cap V(H_2) \subseteq S$, and $\{A_i \in tr_S(i(H))\}_{i \in [k]}$ covers S . The following two conditions are equivalent:*

1. $f_H^S(A_1, A_2, \dots, A_k) = 1$,
2. *we can find $A_i^j \in tr_S(i(H_j))$, $i \in [k]$, $j \in [2]$, such that $A_i \subseteq A_i^1 \cap A_i^2$, and $f_{H_j}^S(A_1^j, A_2^j, \dots, A_k^j) = 1$ for $j \in [2]$.*

Proof. First suppose that $f_H^S(A_1, A_2, \dots, A_k) = 1$ and that it is witnessed by $(I_1, I_2, \dots, I_k)$. We see that $J_i^j = V(H_j) \cap I_i$ is stable in H_j , hence we may find $I_i^j \in i(H_j)$ such that $J_i^j \subseteq I_i^j$. Observe that $(I_1^j, I_2^j, \dots, I_k^j)$ covers H_j . Therefore $f_{H_j}^S(I_1^j \cap S, I_2^j \cap S, \dots, I_k^j \cap S) = 1$.

Conversely, suppose that the second condition holds and that it is witnessed by $(I_1^j, I_2^j, \dots, I_k^j)$. We may write $J_i = (I_i^1 \cup I_i^2) \setminus S \cup A_i$ to cover H with k (not necessarily maximal) independent sets. Extend each J_i to a maximal independent set $I_i \in i(H)$ to complete the proof. $\square$

The lemma proves that f_H^S satisfies condition 4. of Definition 4.6.18. Clearly $f_H^\emptyset(\emptyset^k) = 1$ if and only if $\chi(H) \leq k$.

4.6.3.3 Hypergraph Homomorphism

Suppose H and F are r -uniform hypergraphs (hence clutters), and let $i(F) = \{I_1, \dots, I_k\}$. Let $f_H^S : tr_S(i(H))^k \rightarrow \{0, 1\}$ be defined as follows:

$$f_H^S(A_1, \dots, A_k) = 1 \text{ iff}$$

1. there exists a k -tuple $(J_1, \dots, J_k) \in i(H)^k$ such that $\{\cap_{i: x \in I_i} J_i\}_{x \in V(F)}$ covers H ;
2. $\{\cap_{i: x \in I_i} A_i\}_{x \in V(F)}$ covers S ; and
3. $A_i \subseteq J_i$.

From Lemma 4.5.5 we have that $f_H^\emptyset(\cdot) = 1$ iff $H \rightarrow F$. It is easy to see that f_H^S satisfies properties 1. - 3. of Definition 4.6.18. Property 4. follows from the following lemma.

Lemma 4.6.21. *Suppose $H = H_1 \cup H_2$, $V(H_1) \cap V(H_2) \subseteq S$, $A_i \in \text{tr}_S(i(H))$ for $1 \leq i \leq k$ and $\{\cap_{i:x \in I_i} A_i\}_{x \in V(F)}$ covers S . The following two conditions are equivalent:*

1. $f_H^S(A_1, \dots, A_k) = 1$,
2. *we can find $A_i^j \in \text{tr}_S(i(H_j))$, $i \in [k]$, $j \in [2]$, such that $A_i \subseteq A_i^1 \cap A_i^2$, and $f_{H_j}^S(A_1^j, \dots, A_k^j) = 1$ for $j \in [2]$.*

Proof. First suppose $(J_1, \dots, J_k)$ witnesses $f_H^S(A_1, \dots, A_k) = 1$. We can find $(J_1^j, \dots, J_k^j) \in i(H_j)^k$ such that $J_i \cap V(H_j) \subseteq J_i^j$. It can be checked that $f_{H_j}^S(J_1^j \cap S, \dots, J_k^j \cap S) = 1$.

Now suppose the second condition in the statement holds and that it is witnessed by $(J_1^j, \dots, J_k^j) \in i(H_j)^k$ and $(A_1^j, \dots, A_k^j) \in \text{tr}_S(i(H_j))^k$. Find a k -tuple $(J_1, \dots, J_k) \in i(H)^k$ such that $(J_i^1 \cup J_i^2) \setminus S \cup A_i \subseteq J_i$. It is easy to check that $(J_1, \dots, J_k)$ witnesses $f_H^S(A_1, \dots, A_k) = 1$. $\square$

4.6.4 Comparison with other tree decompositions

As discussed before, treewidth, hypertree width and fractional hypertree width are functionally bounded to each other for hypergraphs of bounded rank, while minor-matching hypertree width cannot be bounded from below by treewidth even for graphs. For instance, consider the complete graph, K_n . Since K_n is $2K_2$ -free, we have $\mu\text{-tw}(K_n) = 1$, while $\text{tw}(K_n) = n - 1$.

As for clique-width (or equivalently rank-width), it cannot be bounded functionally by $\mu\text{-tw}$. For instance, consider the graph G over Z_{4k+1} where $i \sim j$ if $i > k + j \pmod{4k+1}$. We have that G is $2K_2$ -free, hence $\mu\text{-tw}(G) = 1$, while $\text{cr}(G) \geq k$. In the other direction, let $G_0 = K_1$, let G_{2n+1} be a complete union of two copies of G_{2n} and let G_{2n+2} be a disjoint union of two copies of G_{2n+1} . Clearly $\text{rk}_w(G_n) = 1$, while using the tools developed in § 4.7 we can see that $\mu\text{-tw}(G_{2n+1}) = n$.

Nevertheless, minor-matching hypertree width permits many more graphs of bounded width and additionally it applies to hypergraphs as well as graphs. Indeed, every co-bipartite graph (graph G with $\chi(\overline{G}) \leq 2$) is $3K_2$ -free, and there are $e^{\Theta(n^2)}$ n -vertex co-bipartite graphs, while the number of graphs with bounded clique-width by a fixed constant is $e^{O(n \ln n)}$.

We can give a more precise estimate on the size of the class of n -vertex graphs G with $\mu\text{-tw}(G) \leq k$, call it $M_{n,k}$. Note that $M_{n,k}$ contains all graphs G with $\chi(\overline{G}) \leq k$, since $\mu(G) \leq \alpha(G) \leq \chi(\overline{G}) \leq k$; does not contain all bipartite graphs, not all split graphs, and not all graphs with $\chi(\overline{G}) \leq k+1$ (for large n); and therefore the colouring number of $M_{n,k}$ is exactly k (for large n) and $|M_{n,k}| = 2^{(1-\frac{1}{k}+o(1))\binom{n}{2}}$ by the colouring

number theorem [Ale93, BT97, BBS09, ABBM11b]. Similar argument holds for graphs G with $\alpha\text{-tw}(G) \leq k$.

Lemma 4.6.22. *The number of n -vertex graphs G with $\mu\text{-tw}(G) \leq k$ ($\alpha\text{-tw}(G) \leq k$) is $2^{(1-\frac{1}{k}+o(1))\binom{n}{2}}$.*

4.7 Approximation of $\mu\text{-tw}$ for graphs and $\alpha\text{-tw}$ for hypergraphs

This section is organised as follows. We start by arguing that finding decompositions of low α and μ width for graphs is essentially equivalent. In § 4.7.1 we explain how to reduce a graph G to a graph $M(G)$ such that $\alpha\text{-tw}(G) = \mu\text{-tw}(M(G))$. Moreover, for every decomposition $\mathcal{T}$ of $M(G)$ we can find a decomposition $M^{-1}(\mathcal{T})$ of G such that $\alpha(M^{-1}(\mathcal{T})) \leq \mu(\mathcal{T})$. The main result of § 4.7.2 states that a reduction from $\mu\text{-tw}$ to $\alpha\text{-tw}$ exists with the same properties.

Recall that $\underline{H}$ is the primal graph of H .

Definition 4.7.1. *We say that a hypergraph width measure λ_H is well-behaved if*

1. $\lambda_H(\{x\}) \geq 1$ for each vertex $x \in V(H)$,
2. $\lambda_H(S \cup T) \leq \lambda_H(S) + \lambda_H(T)$,
3. $\lambda_H(S \cup T) = \lambda_H(S) + \lambda_H(T)$ for disjoint sets S and T such that $E_{\underline{H}}(S, T) = \emptyset$,
4. $\lambda_F(S) \leq \lambda_H(T)$ for each pair of hypergraphs $F \supseteq H$ and sets $S \subseteq T$, and
5. $\lambda_H(S) \leq k$ can be decided in $||H||^{O(k)}$ time.

Note that the independence number α_H , the edge covering number ϱ_H (Definition 4.6.10), and the fractional edge covering number ϱ_H^* (Definition 4.6.12), are all well-behaved, while μ_H is not. The following theorem is proved in § 4.7.3.

Theorem 4.7.2. *Suppose λ_H is a well-behaved width measure for hypergraphs. There exists an algorithm with running time $||H||^{O(k^3)}$ that given a hypergraph H and an integer k as input, either*

- *finds a tree decomposition $\mathcal{T}$ of H with $\lambda(\mathcal{T}) = O(k^3)$, or*
- *correctly concludes that $\lambda\text{-tw}(H) > k$.*

Note the lack of restrictions on the rank.

The overall approximation for $\mu\text{-}tw$ is as follows: starting from a graph G and a number k , build a graph $L = L(G)$ as in § 4.7.2. We have $\mu\text{-}tw(G) = \alpha\text{-}tw(L)$, so the algorithm in Theorem 4.7.2 either finds a decomposition $\mathcal{T}$ of L such that $\alpha(\mathcal{T}) = O(k^3)$, or declares $\alpha\text{-}tw(L) > k$. In former case we get a decomposition $L^{-1}(\mathcal{T})$ of G such that $\mu(L^{-1}(\mathcal{T})) = O(k^3)$, in the latter we deduce that $\mu\text{-}tw(G) > k$.

4.7.1 Reducing $\alpha\text{-}tw$ to $\mu\text{-}tw$

In the proof of Theorem 4.4.1 we gave the following reduction: given a graph G let $M = M(G)$ be obtained from G by adding a new vertex v' and an edge vv' for each $v \in V(G)$. The key observation here is that given a set $S \subseteq V(G) \subseteq V(M)$, we have $\alpha_G(S) = \mu_M(S)$.

Lemma 4.7.3. *We have $\alpha\text{-}tw(G) = \mu\text{-}tw(M)$.*

Proof. If $\mu\text{-}tw(M) = 0$, then M contains no vertices, but in this case $\alpha\text{-}tw(G) = \mu\text{-}tw(M) = 0$. Suppose $\mathcal{T} = (T, \{B_t\})$ is a tree decomposition of G of α -width at least 1. Extend $\mathcal{T}$ to $\mathcal{M}(\mathcal{T})$ by adding new nodes with bags vv' for each vertex v and attached to an arbitrary node containing v , or an arbitrary node if no such node exists. It is easy to see that $\mathcal{M}(\mathcal{T})$ is a tree decomposition of M . The μ -width of each new bag is 1, hence a bag of maximum width can be found among the nodes of $\mathcal{T}$. As noted before, the α -width of a node of $\mathcal{T}$ equals its μ -width in M , hence $\mu(\mathcal{M}(\mathcal{T})) = \alpha(\mathcal{T})$, and therefore, since $\mathcal{T}$ was arbitrary, we have $\alpha\text{-}tw(G) \geq \mu\text{-}tw(M)$.

Conversely, let $\widehat{\mathcal{T}}$ be a tree decomposition of M . For each vertex v of G delete v' from each bag of $\widehat{\mathcal{T}}$ and add a new node with bag vv' attached to an arbitrary node of $\widehat{\mathcal{T}}$ containing v , or an arbitrary node if no such node exists. We see that the modified tree decomposition is of the form $\mathcal{M}(\mathcal{T})$ for a tree decomposition $\mathcal{T}$ of G . It is easy to see that $\mu(\mathcal{M}(\mathcal{T})) \leq \mu(\widehat{\mathcal{T}})$, and in the previous paragraph we saw that $\alpha(\mathcal{T}) = \mu(\mathcal{M}(\mathcal{T}))$, hence $\mu\text{-}tw(M) \geq \alpha\text{-}tw(G)$. $\square$

Combining $\mu\text{-}tw(G) \leq \alpha\text{-}tw(G)$ with the lemma above we get

$$\mu\text{-}tw(G) \leq \alpha\text{-}tw(G) = \mu\text{-}tw(M(G)) \leq \alpha\text{-}tw(M(G)).$$

For completeness, we note

$$\mu(G) \leq \alpha(G) = \mu(M(G)) \leq \alpha(M(G)).$$

4.7.2 Reducing μ -tw to α -tw

Recall that the *line graph* of G , denoted $L(G)$, has the edges of G as vertices, and two vertices of $L(G)$ are adjacent whenever the corresponding edges of G intersect.

Let $L^k(G)$ be the graph with vertices $E(G)$ and edges

$$\left\{ ef \in \binom{E(G)}{2} : 1 \leq d_{L(G)}(e, f) \leq k \right\}.$$

Here $d_G(u, v)$ is used to denote the *distance* between u and v in G , that is the length of the shortest u - v path in G measured in number of edges. If there is no u - v path, we set $d_G(u, v) = \infty$. Note that $L(G) = L^1(G)$. We focus on $L = L^2(G)$. Two vertices, e and f , of L are adjacent if $G[e \cup f]$ is connected, or equivalently $G[e \cup f] \not\cong 2K_2$. Let $\mathcal{L} : 2^{V(G)} \rightarrow 2^{V(L)}$ map sets $S \subseteq V(G)$ to the edges of G incident with S . We see that $\mu_G(S) = \alpha_L(\mathcal{L}(S))$.

Theorem 4.7.4. *We have $\mu\text{-tw}(G) = \alpha\text{-tw}(L)$.*

Proof. Suppose $\mathcal{T} = (T, B_t)$ is a tree decomposition of G . Consider $\mathcal{L}(\mathcal{T}) = (T, \{\mathcal{L}(B_t)\}_{t \in V(T)})$. If ef is an edge of L , then either e and f share a vertex, or there is an edge g intersecting both. Let h be e in the prior case and g in the latter. The edge h must be contained in some bag B_{t_0} , and note that $\mathcal{L}(B_{t_0})$ contains ef . Denote by T_e the nodes of $\mathcal{L}(\mathcal{T})$ with bags containing $e = uv \in V(L)$. Observe that $T_e = T_v \cup T_u$; T_v and T_u are trees that touch, hence T_e is connected. We conclude that $\mathcal{L}(\mathcal{T})$ is a tree decomposition of L . For each bag B_t we have $\mu_G(B_t) = \alpha_L(\mathcal{L}(B_t))$, so $\mu(\mathcal{T}) = \alpha(\mathcal{L}(\mathcal{T}))$.

Now suppose (T, B_t) is a tree decomposition of L . Let S_t be the vertices of G adjacent only to edges of B_t . Equivalently, $\mathcal{L}(S_t) \subseteq B_t$ and S_t is maximal with the property. Let $e = uv$ be an edge of G . The edges of G adjacent to u or v form a clique in L , hence there must be a bag B_{t_0} containing $\mathcal{L}(\{u, v\})$ (by Helly property for trees), and we conclude that uv is contained in S_{t_0} . Suppose a vertex u of G is contained in two bags S_r and S_s . The bags B_r and B_s contain all edges of G incident with u , E_u , hence E_u is contained in all bags B_t on the unique r - s path in T , and therefore u is contained in the bags S_t along the same path. We conclude that (T, S_t) is a tree decomposition of G with μ -width at most $\alpha(T, B_t)$. $\square$

Similarly as before, we have that $\mu\text{-tw}(G) \leq \alpha\text{-tw}(G)$, and combining this with the theorem above we get

$$\alpha\text{-tw}(G) \geq \mu\text{-tw}(G) = \alpha\text{-tw}(L^2(G)) \geq \mu\text{-tw}(L^2(G)).$$

For completeness, we note

$$\alpha(G) \geq \mu(G) = \alpha(L^2(G)) \geq \mu(L^2(G)).$$

4.7.3 Approximating well-behaved width measures

The overall strategy is similar to other treewidth approximation algorithms, in particular to the approximation of fractional hypertree width [Mar10]. Suppose λ_H is a well-behaved measure throughout.

Theorem 4.7.5. *There exists an $||H||^{O(k)}$ -time algorithm that on input hypergraph H ; sets of vertices A, B ; and an integer k , either*

- *finds an (A, B) -separator S with $\lambda_H(S) \leq \binom{k+1}{2}k$, or*
- *correctly concludes that A and B cannot be separated by a set S with $\lambda_H(S) \leq k$ or that $\lambda\text{-tw}(H) > k$.*

We need few preliminary lemmas first. Given a hypergraph H and a decomposition (T, B_t) of H , denote the subtree of T induced by the nodes containing $v \in V(H)$ by T_v . A set S is an (A, B) -separator in a hypergraph H if it is an (A, B) -separator in $\underline{H}$, that is there are no edges between $A \setminus S$ and $B \setminus S$ and $A \cap B \subseteq S$.

Lemma 4.7.6. *For every hypergraph H and set S we have*

$$\alpha_{\underline{H}}(S) \leq \lambda_H(S).$$

Proof. Let $I \subseteq S$ be an independent set of $\underline{H}$. We have $\lambda_H(\{i\}) \geq 1$ for each $i \in I$ by Property 1. of λ_H . There are no edges in $\underline{H}$ between any pair of vertices $i, j \in I$, hence by Property 2. of λ_H we get $\lambda_H(I) = \sum_{i \in I} \lambda_H(\{i\}) \geq |I|$. By Property 3. of λ_H we see that $\lambda_H(I) \leq \lambda_H(S)$. Now suppose $I \subseteq S$ is maximum-sized. We get

$$\alpha_{\underline{H}}(S) = |I| \leq \lambda_H(I) \leq \lambda_H(S). \quad \square$$

Lemma 4.7.7. *Suppose H is a hypergraph with $\lambda\text{-tw}(H) \leq k$. Let $H_0, H_1, \dots, H_\ell$ be a sequence with $H_0 := H$ and H_{i+1} be obtained from H_i by adding an arbitrary edge uv such that*

$$\lambda_{H_0}(N_{\underline{H}_i}(u) \cap N_{\underline{H}_i}(v)) > k.$$

Then every tree-decomposition of H of λ -width at most k is a tree-decomposition of H_ℓ . Furthermore, if S is an (A, B) -separator of H with $\lambda_H(S) \leq k$, then S is an (A, B) -separator of each H_i .

Proof. The proof is by induction on ℓ . The claim is trivial for $\ell = 0$. Suppose $\ell > 0$ and that the lemma holds for smaller values. Let $\mathcal{T} = (T, B_t)$ be an arbitrary tree decomposition of H such that $\lambda_{H_0}(\mathcal{T}) \leq k$. By the induction hypothesis, $\mathcal{T}$ is a decomposition of $H_{\ell-1}$ as well. If $\mathcal{T}$ is not a decomposition of H_ℓ , then the edge uv added on the ℓ -th step is not covered in $\mathcal{T}$, hence T_u and T_v are disjoint. Let

$Z = N_{\underline{H}_{\ell-1}}(u) \cap N_{\underline{H}_{\ell-1}}(v)$ and let x be an arbitrary vertex of Z . We see that T_x intersects both T_u and T_v . It follows that x is contained in every bag along the unique shortest T_u - T_v path in T , P_{uv} . Note that P_{uv} is non-empty; it contains at least two nodes of T . But x was arbitrary, so the entire Z is contained in every bag B_t , where $t \in P_{uv}$. This is a contradiction, since $\lambda_{H_0}(Z) > k$ and $\lambda_{H_0}(\mathcal{T}) \leq k$.

For the second part, suppose S is not an (A, B) -separator of H_i , and let i be minimal with this property. Clearly $i > 0$. Suppose the edge uv is added on the i -th step. Since S is not a separator of H_i , w.l.o.g. we have $u \in A \setminus S$ and $v \in B \setminus S$. But then every vertex of $Z = N_{\underline{H}_{i-1}}(u) \cap N_{\underline{H}_{i-1}}(v)$ is contained in S in order to prevent an u - v path of length 2 in $\underline{H}_{i-1}$, hence $Z \subseteq S$ and $k < \lambda_H(Z) \leq \lambda_H(S)$. We arrive at a contradiction, since

$$\lambda_H(S) \leq k < \lambda_H(Z) \leq \lambda_H(S). \quad \square$$

Definition 4.7.8. Let H be a hypergraph and let $\underline{\underline{H}}$ be obtained from $\underline{H}$ by adding edges uv such that $\lambda_H(N_{\underline{H}_i}(u) \cap N_{\underline{H}_i}(v)) > k$ as in the previous lemma until no longer possible and in any order, say Lex-first.

Note that $\underline{\underline{H}}$ can be built from H in $||H||^{O(k)}$ time. Indeed, at most $O(n^2)$ new edges are added, each step inspects at most $O(n^2)$ non-adjacent pairs, and for each pair (u, v) it can be checked if $\lambda_H(N_{\underline{H}_i}(u) \cap N_{\underline{H}_i}(v)) > k$ in $||H||^{O(k)}$ time, hence the total running time is $||H||^{O(k)}$.

Lemma 4.7.9. If H is a hypergraph with $\lambda\text{-tw}(H) \leq k$, then for every clique C of $\underline{\underline{H}}$ we have $\lambda_H(C) \leq k$. Furthermore, every (A, B) -separator S of H with $\lambda_H(S) \leq k$ is an (A, B) -separator in $\underline{\underline{H}}$.

Proof. Suppose C is a clique in $\underline{\underline{H}}$ and let $\mathcal{T}$ be a minimum λ -width tree decomposition of H . Then $\mathcal{T}$ is a decomposition of $\underline{\underline{H}}$ by Lemma 4.7.7, and since C is a clique, there must be a bag B of $\mathcal{T}$ containing C by the Helly property for trees. We conclude that

$$\lambda_H(C) \leq \lambda_H(B) \leq \lambda_H(\mathcal{T}) = \lambda\text{-tw}(H) \leq k.$$

The second part of the statement follows directly from Lemma 4.7.7. $\square$

Lemma 4.7.10. [Whi81, Tar85, Mar10] Given a graph G it is possible to construct in polynomial time a set $\mathcal{K}$ of at most $|V(G)|$ subsets of $V(G)$ such that

- each clique C of G is contained in a set $K \in \mathcal{K}$, and
- $K \setminus C$ is contained in a connected component of $G \setminus C$ for each set $K \in \mathcal{K}$ and every clique C of G .

We give a short proof for convenience.

Proof of Lemma 4.7.10. Create a sequence of graphs $G_0, G_1, \dots, G_\ell$, starting from $G_0 = G$, such that G_{i+1} is obtained from G_i by adding a new edge uv with both ends u, v contained in an induced cycle H of G_i of length at least four, until no such cycle exists, or equivalently, until G_i is chordal.

It is well-known that a chordal graph F has at most $|V(F)|$ maximal cliques and that they can be found in linear time [RT75]. Let $\mathcal{K}$ be the set of maximal cliques of G_ℓ .

Every clique C of G remains a clique in G_ℓ , hence it is contained in a maximal clique $K \in \mathcal{K}$ of G_ℓ . Suppose for contradiction that we can find a clique C of G and a set $K \in \mathcal{K}$, such that $K \setminus C$ is not contained in a connected component of $G_0 \setminus C$. Clearly $K \setminus C$ is connected in G_ℓ , so there must be some index j such that $K \setminus C$ is not contained in a component of G_j , but it is contained in a component of $G_{j+1} \setminus C$. Suppose G_{j+1} is obtained from G_j by adding the edge uv . If u or v is contained in C or u and v are in the same connected component of $G_j \setminus C$, then the components $G_j \setminus C$ are the same as the components of $G_{j+1} \setminus C$, which is a contradiction from the disagreement on how many components have a non-empty intersection with $K \setminus C$.

Now suppose u and v are contained in different connected components of $G_j \setminus C$. There is an induced cycle H in G_j of length at least four that contains both u and v . Since H is an induced cycle, and C is a clique, $H \setminus C$ is connected, and in particular it contains a u - v path. But this is a contradiction, since u and v are in different connected components of $G_j \setminus C$. $\square$

Proof of Theorem 4.7.5. Suppose there is an (A, B) -separator S with $\lambda_H(S) \leq k$ and that $\lambda_{tw}(H) \leq k$. Let I be a maximum-sized independent subset of S with respect to $\underline{H}$. Note that $|I| = \alpha_{\underline{H}}(S) \leq \alpha_{\underline{H}}(S) \leq \lambda_H(S) \leq k$, hence $|I| \leq k$. Let

$$\begin{aligned} X &= \bigcup_{u,v \in I; u \neq v} \left(N_{\underline{H}}(u) \cap N_{\underline{H}}(v) \right) \\ N_v &= N_{\underline{H}}(v) \setminus X \cup \{v\} \quad \text{for } v \in I, \text{ and} \\ Y &= \bigcup_{v \in I} N_v. \end{aligned}$$

It follows that $S \subseteq X \cup Y$ since I is maximal, and that $N_v \cap S$ is a clique of $\underline{H}$ since I is maximum-sized. For $v \in I$, let $\mathcal{K}_v$ be the sets of $\underline{H}[N_v]$ guaranteed by Lemma 4.7.10, and pick a $K_v \in \mathcal{K}_v$ to contain $N_v \cap S$ for each $v \in I$. Since $K_v \setminus (N_v \cap S) = K_v \setminus S$ is contained in a component of $\underline{H}[N_v] \setminus S$, it follows that $K_v \setminus S$ is contained in a component of $\underline{H} \setminus S$. By Lemma 4.7.9 we have that S is an (A, B) -separator of $\underline{H}$, so every vertex of $K_v \setminus S$ is either reachable from A in $\underline{H} \setminus S$, or none are. Let $J_1 \subseteq I$ be the vertices corresponding to the prior case, and $J_2 = I \setminus J_1$ – the latter. In case $K_v \setminus S$ is empty, put v in J_1 or J_2 arbitrarily. Let $Z = X \cup (\cup_{v \in I} K_v)$. Note that $S \subseteq Z$. For

$v \in J_2$, we call a vertex $u \in K_v$ *bad* if there is a path P_{Au} from A to u in $\underline{H}$ such that $P_{Au} \cap Z = \{u\}$, and for $u \in J_1$ we call a vertex $u \in K_v$ *bad* if there is a path P_{Bu} from B to u in $\underline{H}$ such that $P_{Bu} \cap Z = \{u\}$. Observe that by the definition of J_1 and J_2 , every bad vertex is contained in S . If $x \in J_1$, $y \in J_2$, $u \in K_x$ and $v \in K_y$, we call the pair $\{u, v\}$ *bad* if there is a path P_{uv} from u to v in $\underline{H}$, such that $P_{uv} \cap Z = \{u, v\}$. If $\{u, v\}$ is a bad pair, then S must contain at least one of the two vertices, as otherwise u and v are in the same connected component of $\underline{H} \setminus S$, contradicting the choices for J_1 and J_2 . To summarise:

- $S \subseteq Z$,
- $S \cap K_v$ is a clique in $\underline{H}$ for each $v \in I$,
- S contains all bad vertices, and
- S intersects all bad pairs.

Let S' be another set with these properties. We claim that $S' \cup X$ is an (A, B) -separator in $\underline{H}$ with $\lambda_H(S' \cup X) \leq \binom{k+1}{2}k$. Since X is a union of at most $\binom{k}{2}$ coneighbourhoods of non-adjacent vertices in $\underline{H}$, we have $\lambda_H(X) \leq \binom{k}{2}k$ by the definition of $\underline{H}$ and Property 2. of λ_H . Furthermore, $\lambda_H(S' \setminus X) \leq \sum_{v \in I} \lambda_H(S' \cap K_v) \leq k^2$, because $S' \cap K_v$ is a clique in $\underline{H}$, and hence $\lambda_H(S' \cap K_v) \leq k$ by Lemma 4.7.9. By putting everything together we conclude that $\lambda_H(S' \cup X) \leq k^2 + \binom{k}{2}k = \binom{k+1}{2}k$.

Now suppose $S' \cup X$ is not an (A, B) -separator in $\underline{H}$, and that $P_{a,b}$ is an a - b path in $\underline{H}$, disjoint from $S' \cup X$, where $a \in A$ and $b \in B$. Clearly, $P_{a,b}$ is disjoint from X , so $P_{a,b} \cap Z = P_{a,b} \cap (Z \setminus X)$. Let $p_1, \dots, p_m$ be the vertices of $P_{a,b} \cap Z$, ordered as the path is traversed from a to b . Since $S \subseteq Z$ is an (A, B) -separator in $\underline{H}$, $P_{a,b} \cap Z \supseteq P_{a,b} \cap S$ is non-empty. We claim that if $p_1 \in K_v$ and $p_m \in K_u$, then $v \in J_1$ and $u \in J_2$. Indeed, if $v \in J_2$, then stripping $P_{a,b}$ from a to p_1 yields a path P such that $P \cap Z = \{p_1\}$, hence p_1 is bad and it must be included in S' , a contradiction. Similarly, if $u \in J_1$, then by denoting the subpath of $P_{a,b}$ from p_m to b by P , we see that $P \cap Z = \{p_m\}$, hence p_m is bad, a contradiction. Therefore, there must be a pair $\{p_j, p_{j+1}\}$ of two consecutive vertices p_j, p_{j+1} of $P_{a,b} \cap Z$, such that $p_j \in K_u$ with $u \in J_1$ and $p_{j+1} \in K_v$ with $v \in J_2$. But then for the p_j - p_{j+1} subpath, P , of $P_{a,b}$, we have $P \cap Z = \{p_j, p_{j+1}\}$, and hence the pair $\{p_j, p_{j+1}\}$ is bad, and therefore it cannot be disjoint from S' , a contradiction.

Consider a 2-CNF formula ψ

- with variables x_v for each vertex $v \in Z \setminus X$,
- clauses $\neg x_v \vee \neg x_u$ for each pair of non-adjacent in $\underline{H}$ vertices v, u contained in the same set K_w ,

- with clauses $x_v \vee x_u$ for each bad pair $\{v, u\}$, and
- set all variables x_v corresponding to bad vertices v to *true*.

Observe that ψ has a satisfying assignment – setting the variables corresponding to vertices of $S \setminus X$ to *true* and the rest to *false* yields one. Conversely, if S' is the set of variables set to *true* in a satisfying assignment, then by the discussion above we have that $S' \cup X$ is an (A, B) -separator in $\underline{H}$, and hence also of H , and $\lambda_H(S' \cup X) \leq \binom{k+1}{2}k$.

The algorithm to find an (A, B) -separator of bounded λ -width is as follows:

1. Construct $\underline{H}$,
2. guess I ,
3. build X and N_v for each $v \in I$,
4. build $\mathcal{K}_v$, guess $K_v \in \mathcal{K}_v$ for each $v \in I$, and build Z ,
5. guess a partition (J_1, J_2) of I (J_1 or J_2 may be empty),
6. find which vertices are bad, which pairs are bad, and build ψ ,
7. if ψ does not have a satisfying assignment, reject the branch; otherwise find a set S' corresponding to a satisfying assignment,
8. if for any $v \in I$ we have $\lambda_H(S' \cap K_v) > k$, declare $\lambda\text{-tw}(H) > k$ and terminate; if $S' \cup X$ is not an (A, B) -separator in H , reject the branch; otherwise return $S' \cup X$.

As noted before, building $\underline{H}$ takes $||H||^{O(k)}$ time. Next, since $|I| \leq k$, we may try all $O(n^k)$ independent sets of size at most k one by one, and thus add a factor of $O(n^k)$ to the total running time. Constructing each $\mathcal{K}_v$ is cheap, but guessing each $K_v \in \mathcal{K}_v$ adds one more $O(n^k)$ factor. There are at most 2^k choices for (J_1, J_2) , so we get an additional factor of $O(2^k)$. Step 6. can be done in $n^{O(1)}$ time. Since finding a satisfying assignment or correctly concluding that a 2-CNF formula is unsatisfiable can be done in linear time with respect its length [APT79], Step 7. takes $O(n^2)$ time. The last step takes $||H||^{O(k)}$ time. The total running time is $||H||^{O(k)}$. $\square$

Lemma 4.7.11. *Suppose H is a hypergraph with $\lambda\text{-tw}(H) \leq k$ and $W \subseteq V(H)$. There exists a set $S \subseteq V(H)$ with $\lambda_H(S) \leq k$ such that for every connected component C_i of $\underline{H} \setminus S$ we have $\lambda_H(W \cap C_i) \leq \lambda_H(W)/2$.*

Proof. Let (T, B_t) be a tree decomposition of H of λ -width at most k . If $t_1 t_2$ is an edge of T , $T - t_1 t_2$ has two connected components, T_1 and T_2 respectively, and let us denote $\cup\{B_t : t \in V(T_i)\}$ by V_i . We have that $Z := B_{t_1} \cap B_{t_2} = V_1 \cap V_2$ is a (V_1, V_2) -separator. If $\lambda_H(W \cap (V_i \setminus Z)) \leq \lambda_H(W)/2$ for $i \in \{1, 2\}$, then we may choose Z as S and we are done. Otherwise, orient the edge $t_1 t_2$ towards t_i , where $\lambda_H(W \cap (V_i \setminus Z)) > \lambda_H(W)/2$.

Suppose S cannot be chosen as $B_{t_1} \cap B_{t_2}$, and all edges are oriented. Let t be a node of T with outdegree zero. Let $t_1, \dots, t_m$ be the neighbours of t in T , so that each edge $t_i t$ is oriented from t_i to t , and let $T_1, \dots, T_m$ be the corresponding components of $T \setminus t$. Define V_i as before. Every connected component of $\underline{H} \setminus B_t$ is contained in some set $V_i \setminus B_t$, but from the orientation of $t_i t$ we get that

$$\lambda_H(W \cap (V_i \setminus B_t)) \leq \lambda_H(W \cap (V_i \setminus (B_t \cap B_{t_i}))) < \lambda_H(W)/2,$$

so we may choose B_t as S . □

Theorem 4.7.12. *There is an $\|H\|^{O(r+k)}$ -time algorithm that given a hypergraph H and a set $W \subseteq V(H)$ with $\lambda_H(W) \leq r$ as input either*

- *finds a partition (A, B) of W and an (A, B) -separator S of H with $\lambda_H(S) \leq \binom{k+1}{2}k$ such that $\lambda_H(A \setminus S) \leq \frac{2}{3}r + k$ and $\lambda_H(B \setminus S) \leq \frac{2}{3}r + k$, or*
- *correctly concludes that $\lambda\text{-}tw(H) > k$.*

Proof. Suppose that $\lambda\text{-}tw(H) \leq k$, let S be the set guaranteed by Lemma 4.7.11, and let $C_1, \dots, C_t$ be the connected components of $\underline{H} \setminus S$, such that $\lambda_H(W \cap C_i) \leq r/2$, and further suppose that $\lambda_H(C_i \cap W) \leq \lambda_H(C_{i+1} \cap W)$ for each $i < t$. If $V(H) = S$ there is nothing to do, so suppose $t \geq 1$. We seek to find a partition $(\hat{A}, \hat{B})$ of $W \setminus S$ such that S is an $(\hat{A}, \hat{B})$ -separator and $\lambda_H(\hat{A}), \lambda_H(\hat{B}) \leq \frac{2}{3}r$. Let $R_j = \cup_{i=1}^j C_i \cap W$ and suppose ℓ is the last index such that $\lambda_H(R_\ell) \leq r/2$. If $\ell = t$, set $\hat{A} = R_\ell$ and $\hat{B} = \emptyset$. If $\ell + 1 = t$, set $\hat{A} = R_\ell$ and $\hat{B} = C_t \cap W$. Otherwise let $Q = \cup_{i=\ell+2}^t C_i \cap W$ and note that $\lambda_H(Q) \geq \lambda_H(C_{\ell+1} \cap W)$ from the ordering. We claim that at least one pair of $\{(R_\ell \cup (C_{\ell+1} \cap W), Q), (R_\ell, (C_{\ell+1} \cap W) \cup Q)\}$ is a possible choice for $(\hat{A}, \hat{B})$. Indeed, if $\lambda_H(R_\ell \cup (C_{\ell+1} \cap W)) > \frac{2}{3}r$, then $\frac{1}{3}r > \lambda_H(Q) \geq \lambda_H(C_{\ell+1} \cap W)$, hence $\lambda_H(Q \cup (C_{\ell+1} \cap W)) < \frac{2}{3}r$.

Let (A, B) be an arbitrary partition of W such that $\hat{A} \subseteq A$ and $\hat{B} \subseteq B$. The desired separator in the first condition of the statement can be found with the algorithm in Theorem 4.7.5 with arguments A and B since $\lambda_H(A) \leq \frac{2}{3}r + k$ and $\lambda_H(B) \leq \frac{2}{3}r + k$. It remains to show that a partition (A, B) can be found efficiently. Let I be a maximal independent subset of $\hat{A}$ in $\underline{H}$, and let $\Gamma(I)$ be $I \cup \{v \in V(H) : u \in I, uv \in E(\underline{H})\}$. We see that $\hat{A} \subseteq \Gamma(I) \cap W \subseteq \hat{A} \cup S$, and $\hat{B} \subseteq W \setminus \Gamma(I) \subseteq \hat{B} \cup S$, so $(\Gamma(I) \cap W, W \setminus \Gamma(I))$ is a possible choice for (A, B) . Furthermore, $|I| \leq \lambda_H(\hat{A}) \leq \frac{2}{3}r$, so I can be found by trying all stable sets of size at most $\frac{2}{3}r$ in time $O(n^{\frac{2}{3}r})$. The final algorithm is:

1. Guess I , and independent set of $\underline{H}$ with $|I| \leq \frac{2}{3}r$,
2. construct $A = \Gamma(I) \cap W$ and $B = W \setminus \Gamma(I)$,
3. reject the branch if $\lambda_H(A) > \frac{2}{3}r + k$ or $\lambda_H(B) > \frac{2}{3}r + k$,
4. run the algorithm in Theorem 4.7.5 with arguments H , A and B .

The total running time is $\|H\|^{O(r+k)}$. □

Theorem 4.7.13. *There exists a $\|H\|^{O(k^3)}$ -time algorithm that given a hypergraph H , an integer k and a set $W \subseteq V(H)$ with $\lambda_H(W) \leq \frac{3}{2}(k^3 + k^2) + 3k + 3$, either*

- *finds a tree decomposition, $\mathcal{T} = (T, B_t)$, of H , such that $\lambda(\mathcal{T}) \leq 2k^3 + 2k^2 + 3k + 3$ and $W \subseteq B_t$ for some $t \in V(T)$, or*
- *correctly concludes that $\lambda\text{-}tw(H) > k$.*

Proof. Let $K = \frac{3}{2}(k^3 + k^2) + 3k + 3$ and $s = \frac{1}{2}(k^3 + k^2)$. The algorithm starts by creating a set W^* such that $W \subseteq W^*$ and $\lambda_H(W^*) = \min\{K, \lambda_H(V(H))\}$. Let S and (A_1, A_2) be the separator and the partition of W^* from Theorem 4.7.12; let V_1 be the vertices of $\underline{H} \setminus S$ contained in connected components with a non-empty intersection with A_1 and let V_2 be $V(H) \setminus (V_1 \cup S)$. For $i \in \{1, 2\}$, the algorithm checks if $V_i \not\subseteq A_i$, in which case it makes a recursive call with arguments $H_i = H[V_i \cup S]$ and $W_i = A_i \cup S$ to create a tree decomposition $(T_i, B_{t,i})$ or conclude that $\lambda\text{-}tw(H) > k$ if $\lambda\text{-}tw(H_i) > k$. In case $V_i \subseteq A_i$, let T_i be the empty tree. After T_1 and T_2 have been created, the corresponding node containing W_i in each tree is joined to a new node with bag $W^* \cup S$. If either T_1 or T_2 is empty, ignore it.

If $\lambda\text{-}tw(H) \leq k$, then $\lambda\text{-}tw(H_i) \leq k$ and

$$\lambda(W_i) \leq \lambda(A_i \setminus S) + \lambda(S) \leq \frac{2}{3}K + k + s = K - 1,$$

so we can prove by induction that the algorithm creates tree decompositions $(T_1, B_{t,1})$ of H_1 and $(T_2, B_{t,2})$ of H_2 of width at most $K + s$. If $\lambda\text{-}tw(H) > k$, the algorithm will either create a decomposition of width at most $K + s$, or find an induced subhypergraph H' with $\lambda\text{-}tw(H') > k$, implying $\lambda\text{-}tw(H) > k$.

The total number of steps can be bounded by the number of recursive calls times the maximum number of steps for each call. The number of steps in each call is clearly $\|H\|^{O(k^3)}$. We claim that the number of calls is bounded by $|V(H) \setminus W^*| + 1$. The proof is by induction on $|V(H) \setminus W^*|$. If $|V(H) \setminus W^*| = 0$, then $\lambda_H(V(H)) \leq K$, hence the algorithm creates one bag and does not make recursive calls.

Now suppose $V(H) \not\subseteq W^*$. We claim that the number of recursive calls to create T_i is bounded by $|V_i \setminus W^*|$. Indeed, if $|V_i \setminus W^*| = |V_i \setminus A_i| = 0$, then T_i is empty, and there

are no recursive calls. If $|V_i \setminus W^*| > 0$, then by the induction hypothesis, there are at most $|V(H_i) \setminus W_i^*| + 1$ calls. Since $\lambda(W_i) \leq K - 1$, either $W_i^* \equiv V(H_i)$ or $\lambda(W_i^*) = K$, and hence W_i^* contains a vertex of $V_i \setminus W^*$, so in either case $|V(H_i) \setminus W_i^*| + 1 \leq |V_i \setminus W^*|$. Therefore the total number of calls is bounded by $|V_1 \setminus W^*| + |V_2 \setminus W^*| + 1 \leq |V \setminus W^*| + 1$.

Overall, the total number of recursive calls, and hence the number of bags created, is bounded by $|V(H)|$, and therefore the total running time is $\|H\|^{O(k^3)}$. $\square$

Theorem 4.7.2 is the special case of Theorem 4.7.13 when $W = \emptyset$.

Chapter 5

Conclusions and Future Research

In the perfect graphs section we presented a generation model for perfect graphs which yields such graphs almost uniformly, with error $e^{-\Theta(n)}$ (in terms of total variation distance); and we used this approach to investigate several questions about such graphs.

We aimed to investigate the most natural questions about random perfect graphs, but of course there are further natural open problems. For example, concerning induced subgraphs H , we found the limiting probability that P_n has an induced subgraph H , but we did not discuss the distribution of the number of such induced subgraphs.

Our methods do not let us approach the following question about automorphisms. It is well known that almost all graphs G_n on vertex set $[n]$ have no non-trivial automorphisms, and easy to see that in fact the probability that G_n has a non-trivial automorphism is $2^{-(1+o(1))n}$. It is also not hard to see that, if $R_n \sim \text{Gen}(n)$, then the probability that R_n has a non-trivial automorphism is $2^{-(\frac{1}{2}+o(1))n}$. It is natural to expect that the probability for P_n is similar to that for R_n , but we do not know: we would need a finer result than (3.4). Similarly, if H is a fixed graph in $\mathcal{GS}^+ \cap \mathcal{GS}^-$ and $R_n \sim \text{Gen}(n)$, then by Lemma 3.4.1 the probability that R_n fails to have an induced copy of H is $e^{-\Omega(n \log n)}$. Does such a result hold for P_n ?

A long standing open problem is to describe the structure of a random perfect graph G with a prescribed number of edges, say $e(G) \sim v(G) \log v(G)$. This problem has been discussed before [PS92], and even directly approached [BTW12], but to the best of our knowledge a complete answer is a distant possibility.

Let C_5 -free denote the class of graphs with no induced subgraph C_5 . Then $\mathcal{GS} \subset \mathcal{P} \subset C_5$ -free, and the key result (3.2) from Theorem 2.4 of Prömel and Steger [PS92] in fact extends as follows:

$$|\mathcal{GS}_n| = (1 - e^{-\Omega(n)})|(C_5\text{-free})_n|. \quad (5.1)$$

Thus our results refer also to random C_5 -free graphs. For related general results see [ABBM11a, BB11] and references there.

In the clutter chapter we presented a way to control the size of a blocker via the maximum size of a minor matching, used this condition to find new polynomial-time cases for hard problems, and finally we generalised these conditions to a width measure for tree decompositions.

It is not clear whether the restriction on the rank is necessary. We formulate the following conjectures:

Conjecture 5.1. *There exists a function f such that for every clutter $\mathcal{H}$ with m edges we have*

$$|b(\mathcal{H})| \leq m^{f(\mu(\mathcal{H}))}.$$

Conjecture 5.2. *There exists a function f such that for every clutter $\mathcal{H}$ with m edges and any subset S of its vertices we have*

$$|tr_S(b(\mathcal{H}))| \leq m^{f(\mu_{\mathcal{H}}(S))}.$$

It is also interesting to see if the approximation of μ -tw can be extended to clutters of any rank, and if there are efficient reductions of α -tw to μ -tw for clutters and vice-versa.

Another question is to find more examples of problems with solutions that can be read from the blocker. Indeed, $b(\mathcal{H})$ contains a lot of important structural information that is difficult to access otherwise. Finding an explicit family of problems of this type can lead to further developments of μ -tw.

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