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Density and Trace for Graph Spaces of First-Order Linear Operators

Max Jensen

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Abstract

We define and analyse graph spaces of first-order linear differential operators. In particular we consider the density of the set of smooth functions and the construction of a trace operator.

Oxford University Computing Laboratory
Wolfson Building
Parks Road
Oxford, England OX1 3QD

Introduction

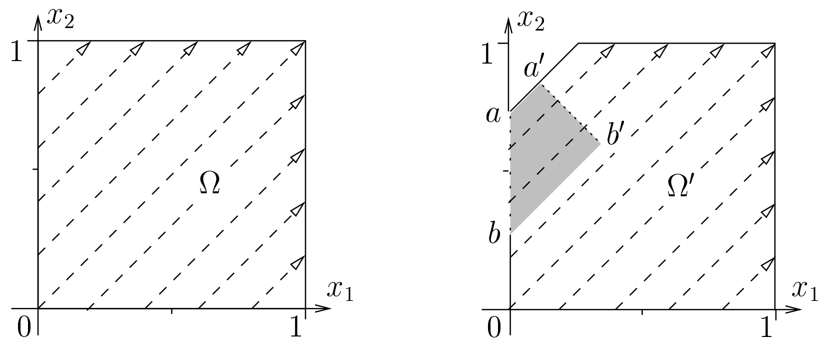
Let us begin with an example. Suppose that Ω is the open square $(0,1)^2 \subset \mathbb{R}^2$ on which we define the differential operator

$$\mathcal{L} : L^1(\Omega) \to \mathscr{D}'(\Omega), v \mapsto \partial_1 v + \partial_2 v,$$

$\mathscr{D}'(\Omega)$ denoting the space of distributions. Then the characteristics of $\mathcal{L}$ pass at an angle of $45°$ or $\pi/4$ through the domain, as indicated on the left figure below. We wish to consider the boundary value problem for data prescribed on the set

$$\partial_{-}\Omega := \partial\Omega \cap \{(x_1,x_2) : x_1 = 0\} \cap \{(x_1,x_2) : x_2 = 0\}.$$

The method of characteristics allows one to solve this problem for smooth and also non-smooth data on $\partial_{-}\Omega$. Clearly, discontinuities in the boundary data lead to discontinuous solutions which do not lie in $W^{1,q}(\Omega)$. Therefore we focus our attention on the graph space $\mathbf{W}^q_{\mathcal{L}}(\Omega)$ of $\mathcal{L}$, which is defined as the vector space of all functions $\mathbf{v} \in L^q(\Omega)$ for which the value $\mathcal{L}(\mathbf{v})$ lies in $L^q(\Omega)$ as well. In the context of this example it means that we require the existence of a weak derivative in the characteristic direction only, while admitting discontinuities in other directions.



In the main part of this report we address the issue of density of $\mathscr{C}^\infty$-functions for general first-order linear differential operators on Lipschitz domains. We are also interested in defining a trace for members of $\mathbf{W}^q_{\mathcal{L}}(\Omega)$. However we notice that certain restrictions emerge through the weaker requirements on differentiability, compared with $W^{1,q}(\Omega)$; considering, for instance, the characteristic boundary at the upper left corner of the domain depicted in the right plot; on the restriction to the slice $\overline{a'b'}$ in normal direction, a solution of the boundary value problem is identical to the boundary data on $\overline{ab} \subset \partial_{-}\Omega$, neglecting the compression factor $\cos(\pi/4)$. Therefore when the boundary data lie merely in $L^q(\partial\Omega)$ it appears unreasonable to expect existence of a trace of the solution in the sense of $W^{1,q}$-functions.

Our aim in this report is to formalise these observations and to introduce the graph space and its associated trace operator in a rigorous manner. We shall conclude the discussion with a few comments about related publications.

Definition and Density

Consider a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$ with boundary $\partial\Omega$ and unit outward normal $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_n)$. Choose a conjugate pair q, q', i.e., $q, q' \in \mathbb{R}$ such that

$$1 < q < \infty, \qquad q' = \frac{q}{q-1}.$$

Given a tensor $\mathbf{B} \in [W^{1,\infty}(\Omega)]^{m\times m\times n}$ and a matrix $\mathbf{C} \in [W^{1,\infty}(\Omega)]^{m\times m}$, we are interested in the graph space of the differential operator

$$\mathcal{L} : [L^q(\Omega)]^m \to [\mathscr{D}'(\Omega)]^m, \mathbf{v} \mapsto \partial_k(B_{ijk}\, v_j) + C_{ij}\, v_j.$$

Here and throughout the text we employ the Einstein summation convention. We also follow the notional convention to typeset entries of tensors, including vectors and matrices, in italic letters; tensors will be typeset in bold letters. Further, we set, for every manifold M,

$$\mathbf{L}^q(M) := [L^q(M)]^m, \quad \mathbf{W}^{r,q}(M) = [W^{r,q}(M)]^m.$$

Then the graph space of $\mathcal{L}$ is the set

$$\mathbf{W}^q_{\mathcal{L}}(\Omega) := \{\mathbf{v} \in \mathbf{L}^q(\Omega) : \mathcal{L}(\mathbf{v}) \in \mathbf{L}^q(\Omega)\},$$

which is normed by the mapping

$$\|\mathbf{v}\|_{\mathcal{L},q} = \|\mathbf{v}\|_{\mathbf{L}^q(\Omega)} + \|\mathcal{L}(\mathbf{v})\|_{\mathbf{L}^q(\Omega)}.$$

Our first investigations concern the density of smooth functions in $\mathbf{W}^q_{\mathcal{L}}(\Omega)$. Consider the approximate identity $\delta \mapsto \psi_\delta$, $\delta > 0$, with non-negative functions $\psi_\delta(\mathbf{x}) = \delta^{-n}\, \psi_1(\delta^{-1}\mathbf{x})$, satisfying the condition that the support of ψ_1 lies within the unit ball centred at the origin. We write, for $\mathbf{v} \in \mathbf{W}^q_{\mathcal{L}}(\Omega)$,

$$\mathbf{v}^\delta = (v_1^\delta, \ldots, v_n^\delta) = (v_1 * \psi_\delta, \ldots, v_n * \psi_\delta).$$

Theorem 1: *Let $\Omega' \subset\subset \Omega$ and suppose that, for $\mathbf{v} \in \mathbf{W}^q_{\mathcal{L}}(\Omega)$, $1 < q < \infty$, the support* $\operatorname{supp}(\mathbf{v})$ *is a subset of Ω'. Then, for every $\varepsilon > 0$ there exists a function $\mathbf{v}_\varepsilon \in [\mathscr{C}_0^\infty(\Omega')]^m$ with*

$$\|\mathbf{v} - \mathbf{v}_\varepsilon\|_{\mathcal{L},q} < \varepsilon.$$

Proof. Step I: Transformation of $\partial(\mathbf{B}\mathbf{v}) * \psi$
Let $\mathbf{u} \in \mathbf{L}^{q'}(\Omega')$ and assume $\delta < \operatorname{dist}(\operatorname{supp}(\mathbf{v}), \partial\Omega')$ and $\delta' < \operatorname{dist}(\partial\Omega', \partial\Omega)$. We denote differentiation with respect to $\bar{\mathbf{x}}$ by $\bar{\partial}_k$, this implies

$$\bar{\partial}_k \psi_\delta(\mathbf{x} - \bar{\mathbf{x}}) = -\partial_k \psi_\delta(\mathbf{x} - \bar{\mathbf{x}}).$$

Hence, if we extend $\mathbf{u}$ to Ω by setting $\mathbf{u}(\mathbf{x}) = 0$ outside Ω',

$$\begin{aligned}\int_\Omega u_i\,(\partial_k(B_{ijk}\,v_j) * \psi_\delta)\,\mathrm{d}\mathbf{x} &= \int_\Omega u_i(\mathbf{x}) \int_\Omega \bar{\partial}_k(B_{ijk}(\bar{\mathbf{x}})\,v_j(\bar{\mathbf{x}}))\,\psi_\delta(\mathbf{x}-\bar{\mathbf{x}})\,\mathrm{d}\bar{\mathbf{x}}\,\mathrm{d}\mathbf{x} \\ &= \int_\Omega \lim_{\delta'\to 0} u_i^{\delta'}(\mathbf{x}) \int_\Omega \bar{\partial}_k(B_{ijk}(\bar{\mathbf{x}})\,v_j(\bar{\mathbf{x}}))\,\psi_\delta(\mathbf{x}-\bar{\mathbf{x}})\,\mathrm{d}\bar{\mathbf{x}}\,\mathrm{d}\mathbf{x} \\ &= \lim_{\delta'\to 0} \int_\Omega \bar{\partial}_k(B_{ijk}(\bar{\mathbf{x}})\,v_j(\bar{\mathbf{x}})) \int_\Omega u_i^{\delta'}(\mathbf{x})\,\psi_\delta(\mathbf{x}-\bar{\mathbf{x}})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\bar{\mathbf{x}} \\ &= \lim_{\delta'\to 0} \int_\Omega -B_{ijk}(\bar{\mathbf{x}})\,v_j(\bar{\mathbf{x}})\,\bar{\partial}_k \int_\Omega u_i^{\delta'}(\mathbf{x})\,\psi_\delta(\mathbf{x}-\bar{\mathbf{x}})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\bar{\mathbf{x}} \\ &= \lim_{\delta'\to 0} \int_\Omega u_i^{\delta'}(\mathbf{x}) \int_\Omega -B_{ijk}(\bar{\mathbf{x}})\,v_j(\bar{\mathbf{x}})\,\bar{\partial}_k\psi_\delta(\mathbf{x}-\bar{\mathbf{x}})\,\mathrm{d}\bar{\mathbf{x}}\,\mathrm{d}\mathbf{x} \\ &= \int_\Omega u_i(\mathbf{x}) \int_\Omega \partial_k(B_{ijk}(\bar{\mathbf{x}})\,\psi_\delta(\mathbf{x}-\bar{\mathbf{x}}))\,v_j(\bar{\mathbf{x}})\,\mathrm{d}\bar{\mathbf{x}}\,\mathrm{d}\mathbf{x}.\end{aligned}$$

In the course of integration by parts we used that

$$\Big(\bar{\mathbf{x}} \mapsto \int_\Omega u_i^{\delta'}(\mathbf{x})\,\psi_\delta(\mathbf{x}-\bar{\mathbf{x}})\,\mathrm{d}\mathbf{x}\Big) \in \mathscr{D}(\Omega).$$

As $\mathbf{L}^{q'}(\Omega')$ is the dual space of $\mathbf{L}^q(\Omega')$ and $\mathrm{supp}(\psi_\delta * \partial_k(B_{ijk}\,v_j)) \subset \Omega'$, we obtain the identity of $\mathbf{L}^q(\Omega')$ functions

$$\Big(\mathbf{x} \mapsto (\partial_k(B_{ijk}\,v_j) * \psi_\delta)(\mathbf{x})\Big) = \Big(\mathbf{x} \mapsto \int_\Omega \partial_k(B_{ijk}(\bar{\mathbf{x}})\,\psi_\delta(\mathbf{x}-\bar{\mathbf{x}}))\,v_j(\bar{\mathbf{x}})\,\mathrm{d}\bar{\mathbf{x}}\Big).$$

Step II: Transformation of $\partial(\mathbf{B}\mathbf{v}^\delta)$
As above, we have

$$\begin{aligned}\int_\Omega u_i\,\partial_k(B_{ijk}\,(\psi_\delta * v_j))\,\mathrm{d}\mathbf{x} &= \int_\Omega u_i(\mathbf{x})\,\partial_k\big(B_{ijk}(\mathbf{x}) \int_\Omega v_j(\bar{\mathbf{x}})\,\psi_\delta(\mathbf{x}-\bar{\mathbf{x}})\,\mathrm{d}\bar{\mathbf{x}}\big)\,\mathrm{d}\mathbf{x} \\ &= \int_\Omega u_i(\mathbf{x}) \int_\Omega \partial_k(B_{ijk}(\mathbf{x})\,\psi_\delta(\mathbf{x}-\bar{\mathbf{x}}))\,v_j(\bar{\mathbf{x}})\,\mathrm{d}\bar{\mathbf{x}}\,\mathrm{d}\mathbf{x}.\end{aligned}$$

Therefore, in $\mathbf{L}^q(\Omega')$,

$$\Big(\mathbf{x} \mapsto \partial_k(B_{ijk}\,(\psi_\delta * v_j))(\mathbf{x})\Big) = \Big(\mathbf{x} \mapsto \int_\Omega \partial_k(B_{ijk}(\mathbf{x})\,\psi_\delta(\mathbf{x}-\bar{\mathbf{x}}))\,v_j(\bar{\mathbf{x}})\,\mathrm{d}\bar{\mathbf{x}}\Big).$$

Step III: Boundedness of $\mathbf{B}(\partial\psi_\delta)\mathbf{v}$
Suppose that $\delta < \mathrm{dist}(\mathrm{supp}(\mathbf{v}), \partial\Omega')$. We consider the operator

$$T : \mathbf{L}^1(\Omega) \to \mathbf{L}^1(\Omega),\ (T\mathbf{v})_i(\mathbf{x}) = \int_\Omega (B_{ijk}(\bar{\mathbf{x}}) - B_{ijk}(\mathbf{x}))\,\partial_k\psi_\delta(\mathbf{x}-\bar{\mathbf{x}})\,v_j(\bar{\mathbf{x}})\,\mathrm{d}\bar{\mathbf{x}}.$$

Since B_{ijk} lies in $W^{1,\infty}(\Omega)$, it is a Lipschitz continuous function for which

$$\|\mathbf{B}\| := \|\mathbf{B}\|_{[W^{1,\infty}(\Omega)]^{m\times m\times n}}$$

is a Lipschitz constant. Therefore we have continuity of T in the L^1-vector norm:

$$
\begin{aligned}
\|T\mathbf{v}\|_{\mathbf{L}^1(\Omega)} &= \sum_{i=1}^{m} \int_\Omega \Big| \int_\Omega (B_{ijk}(\bar{\mathbf{x}}) - B_{ijk}(\mathbf{x}))\, \partial_k \psi_\delta(\mathbf{x} - \bar{\mathbf{x}})\, v_j(\bar{\mathbf{x}})\, \mathrm{d}\bar{\mathbf{x}} \Big| \, \mathrm{d}\mathbf{x} \\
&\le \sum_{i=1}^{m} \int_\Omega \int_\Omega \Big| \frac{B_{ijk}(\bar{\mathbf{x}}) - B_{ijk}(\mathbf{x})}{\delta} \Big| \, |\delta\, \partial_k \psi_\delta(\mathbf{x} - \bar{\mathbf{x}})| \, |v_j(\bar{\mathbf{x}})| \, \mathrm{d}\bar{\mathbf{x}} \, \mathrm{d}\mathbf{x} \\
&\le \sum_{j=1}^{m} \|\mathbf{B}\| \cdot \int_\Omega \int_\Omega |\delta\, \partial_k \psi_\delta(\mathbf{x} - \bar{\mathbf{x}})| \, \mathrm{d}\mathbf{x} \, |v_j(\bar{\mathbf{x}})| \, \mathrm{d}\bar{\mathbf{x}} \\
&= \|\mathbf{B}\| \cdot \|\partial_k \psi_1\|_{L^1(\Omega)} \cdot \|\mathbf{v}\|_{\mathbf{L}^1(\Omega)},
\end{aligned}
$$

where we used that $\delta\, \partial_k \psi_\delta(\mathbf{x}) = \delta^{-n} (\partial_k \psi_1)(\delta^{-1} \mathbf{x})$ and the transformation of variables $\mathbf{x} \mapsto \delta \mathbf{x}$. In the case of $\mathbf{v} \in \mathbf{L}^\infty(\Omega)$, the same bound holds for the L^∞-norm, since

$$
\begin{aligned}
\|T\mathbf{v}\|_{\mathbf{L}^\infty(\Omega)} &= \max_i \operatorname*{ess-sup}_{\bar{\mathbf{x}}} \Big| \int_\Omega (B_{ijk}(\bar{\mathbf{x}}) - B_{ijk}(\mathbf{x}))\, \partial_k \psi_\delta(\mathbf{x} - \bar{\mathbf{x}})\, v_j(\bar{\mathbf{x}})\, \mathrm{d}\bar{\mathbf{x}} \Big| \\
&\le \max_i \operatorname*{ess-sup}_{\bar{\mathbf{x}}} \int_\Omega \Big| \frac{B_{ijk}(\bar{\mathbf{x}}) - B_{ijk}(\mathbf{x})}{\delta} \Big| \, |\delta\, \partial_k \psi_\delta(\mathbf{x} - \bar{\mathbf{x}})| \, |v_j(\bar{\mathbf{x}})| \, \mathrm{d}\bar{\mathbf{x}} \\
&\le \|\mathbf{B}\| \cdot \|\mathbf{v}\|_{\mathbf{L}^\infty(\Omega)} \cdot \int_\Omega |\delta\, \partial_k \psi_\delta(\mathbf{x} - \bar{\mathbf{x}})| \, \mathrm{d}\bar{\mathbf{x}} \\
&= \|\mathbf{B}\| \cdot \|\partial_k \psi_1\|_{L^1(\Omega)} \cdot \|\mathbf{v}\|_{\mathbf{L}^\infty(\Omega)}.
\end{aligned}
$$

Next we apply the Riesz-Thorin Interpolation Theorem to derive a bound for $\mathbf{v} \in \mathbf{L}^q(\Omega)$. For the reader's convenience we have stated the theorem in the Appendix; we apply it with $\theta = 1 - 1/q = 1/q'$ and $p = q$, so that, for $\mathbf{v} \in \mathbf{L}^q(\Omega)$,

$$
\begin{aligned}
&\Big\| \int_\Omega (B_{ijk}(\bar{\mathbf{x}}) - B_{ijk}(\cdot))\, \partial_k \psi_\delta((\cdot) - \bar{\mathbf{x}})\, v_j(\bar{\mathbf{x}})\, \mathrm{d}\bar{\mathbf{x}} \Big\|_{\mathbf{L}^q(\Omega)} \\
&\qquad\qquad = \|T\mathbf{v}\|_{\mathbf{L}^q(\Omega)} \le 2 \cdot \|\mathbf{B}\| \cdot \|\partial_k \psi_1\|_{L^1(\Omega)} \cdot \|\mathbf{v}\|_{\mathbf{L}^q(\Omega)}.
\end{aligned}
$$

Step IV: Boundedness of $(\mathfrak{L}\mathbf{v})^\delta - \mathfrak{L}\mathbf{v}^\delta$

Using that $\psi_\delta(\mathbf{x}) \le 0$ for all $\mathbf{x} \in \mathbb{R}^n$, we deduce that

$$
\begin{aligned}
&\Big\| \int_\Omega \partial_k (B_{ijk}(\bar{\mathbf{x}}) - B_{ijk}(\cdot))\, \psi_\delta((\cdot) - \bar{\mathbf{x}})\, v_j(\bar{\mathbf{x}})\, \mathrm{d}\bar{\mathbf{x}} \Big\|_{\mathbf{L}^q(\Omega)} \qquad (1) \\
&\quad \le \Big(\sum_{i=1}^{m} \int_\Omega \Big(\int_\Omega |\partial_k ((B_{ijk}(\bar{\mathbf{x}}) - B_{ijk}(\mathbf{x}))| \, |\psi_\delta(\mathbf{x} - \bar{\mathbf{x}}))\, v_j(\bar{\mathbf{x}})| \, \mathrm{d}\bar{\mathbf{x}} \Big)^q \mathrm{d}\mathbf{x} \Big)^{1/q} \\
&\quad \le 2 \cdot \|\mathbf{B}\| \cdot \|\, \psi_\delta * |\mathbf{v}|\, \|_{\mathbf{L}^q(\Omega)} \le 2 \cdot \|\mathbf{B}\| \cdot \|\psi_1\|_{L^1(\Omega)} \cdot \|\mathbf{v}\|_{\mathbf{L}^q(\Omega)}.
\end{aligned}
$$

Hence using the product rule

$$
\begin{aligned}
&\|\partial_k (B_{ijk}\, v_j) * \psi_\delta - \partial_k (B_{ijk}\, v_j^\delta)\|_{\mathbf{L}^q(\Omega)} \\
&\qquad = \Big\| \int_\Omega \partial_k \big((B_{ijk}(\bar{\mathbf{x}}) - B_{ijk}(\cdot))\, \psi_\delta((\cdot) - \bar{\mathbf{x}}) \big) v_j(\bar{\mathbf{x}})\, \mathrm{d}\bar{\mathbf{x}} \Big\|_{\mathbf{L}^q(\Omega)} \\
&\qquad \le 2 \cdot \|\mathbf{B}\| \cdot \|\psi_1\|_{W^{1,1}(\Omega)} \cdot \|\mathbf{v}\|_{\mathbf{L}^q(\Omega)}.
\end{aligned}
$$

Finally as in (1)

$$\|(C_{ij}v_j) * \psi_\delta - C_{ij}\, v_j^\delta\|_{\mathbf{L}^q(\Omega)} = \Big\| \int_\Omega ((C_{ij}(\bar{\mathbf{x}}) - C_{ij}(\cdot))\, \psi_\delta((\cdot) - \bar{\mathbf{x}}))\, v_j(\bar{\mathbf{x}})\, \mathrm{d}\bar{\mathbf{x}} \Big\|_{\mathbf{L}^q(\Omega)}$$
$$\leq 2 \cdot \|\mathbf{C}\|_{[L^\infty(\Omega)]^{m\times m}} \cdot \|\psi_1\|_{L^1(\Omega)} \cdot \|\mathbf{v}\|_{\mathbf{L}^q(\Omega)}.$$

Step V: Weakly converging sequence
The last step shows that the sequence

$$s_1(\ell) = \mathcal{L}(\mathbf{v}) * \psi_{1/\ell} - \mathcal{L}(\mathbf{v} * \psi_{1/\ell})$$

is bounded in $\mathbf{L}^q(\Omega)$. Hence by the Banach-Alaoglu Theorem there exists a sequence $t_1 : \mathbb{N} \to \mathbb{N}$ such that $s_1 \circ t_1$ is weakly converging to an element $\bar{\mathbf{v}} \in \mathbf{L}^q(\Omega)$. But, for $\mathbf{w} \in \mathbf{L}^{q'}(\Omega)$ and $\bar{\ell} = 1/t_1(\ell)$,

$$\begin{aligned}
\int_\Omega \bar{v}_i w_i \,\mathrm{d}\mathbf{x} &= \lim_{\delta\to 0} \int_\Omega \bar{v}_i\, w_i^{\delta'} \,\mathrm{d}\mathbf{x} \\
&= \lim_{\delta\to 0} \lim_{\ell\to\infty} \int_\Omega \big(\mathcal{L}(\mathbf{v}) * \psi_{\bar{\ell}} - \mathcal{L}\big(\mathbf{v} * \psi_{\bar{\ell}}\big)\big)\, w_i^\delta \,\mathrm{d}\mathbf{x} \\
&= \lim_{\delta\to 0} \Big(\int_\Omega \mathcal{L}(\mathbf{v})\, w_i^\delta \,\mathrm{d}\mathbf{x} - \lim_{\ell\to\infty} \int_\Omega \mathcal{L}\big(\mathbf{v} * \psi_{\bar{\ell}}\big)\, w_i^\delta \,\mathrm{d}\mathbf{x} \Big) \\
&= \lim_{\delta\to 0} \lim_{\ell\to\infty} \int_\Omega \big(v_j - v_j * \psi_{\bar{\ell}}\big) \big(C_{ij}\, w_i^\delta - B_{ijk}\, \partial_k w_i^\delta\big) \,\mathrm{d}\mathbf{x} \\
&= \lim_{\delta\to 0} \int_\Omega \lim_{\ell\to\infty} \big(v_j - v_j * \psi_{\bar{\ell}}\big) \big(C_{ij}\, w_i^\delta - B_{ijk}\, \partial_k w_i^\delta\big) \,\mathrm{d}\mathbf{x} = 0,
\end{aligned}$$

and so $\bar{\mathbf{v}} = 0$.
Step VI: Strongly converging sequence
Given $\varepsilon > 0$, we can select a sequence $t_2 : \mathbb{N} \to \mathbb{N}$, such that for $t_3 = t_1 \circ t_2$ and $\bar{\bar{\ell}} = 1/t_3(\ell)$

$$\|\mathbf{v} - \mathbf{v} * \psi_{\bar{\bar{\ell}}}\|_{\mathbf{L}^q(\Omega)} < \frac{\varepsilon}{3\cdot 2^\ell}, \qquad \|\mathcal{L}(\mathbf{v}) - \mathcal{L}(\mathbf{v}) * \psi_{\bar{\bar{\ell}}}\|_{\mathbf{L}^q(\Omega)} < \frac{\varepsilon}{3\cdot 2^\ell}.$$

Using Mazur's Theorem, cf. [4, Theorem 3.13], there exists a finite convex combination

$$\mathbf{v}_\varepsilon = \sum_{\ell=1}^s \lambda_\ell\, \mathbf{v} * \psi_{\bar{\bar{\ell}}}, \qquad \sum_{\ell=1}^s \lambda_\ell = 1, \lambda_\ell \in [0,1], s \in \mathbb{N},$$

such that

$$\Big\| \Big(\sum_{\ell=1}^s \lambda_\ell\, \mathcal{L}(\mathbf{v}) * \psi_{\bar{\bar{\ell}}} \Big) - \mathcal{L}(\mathbf{v}_\varepsilon) \Big\|_{\mathbf{L}^q(\Omega)} = \Big\| \sum_{\ell=1}^s \lambda_\ell \left(\mathcal{L}(\mathbf{v}) * \psi_{\bar{\bar{\ell}}} - \mathcal{L}(\mathbf{v} * \psi_{\bar{\bar{\ell}}}) \right) \Big\|_{\mathbf{L}^q(\Omega)} < \frac{\varepsilon}{3}.$$

Hence

$$\|\mathbf{v} - \mathbf{v}_\varepsilon\|_{\mathbf{L}^q(\Omega)} \leq \sum_{\ell=1}^s \lambda_\ell \|\mathbf{v} - \mathbf{v} * \psi_{\bar{\bar{\ell}}}\|_{\mathbf{L}^q(\Omega)} < \sum_{\ell=1}^\infty \frac{\varepsilon}{3\cdot 2^\ell} < \frac{\varepsilon}{3}.$$

Similarly, but by using the triangle inequality

$$\|\mathcal{L}(\mathbf{v}) - \mathcal{L}(\mathbf{v}_\varepsilon)\|_{\mathbf{L}^q(\Omega)} < \frac{\varepsilon}{3} + \frac{\varepsilon}{3}.$$

But then $\|\mathbf{v} - \mathbf{v}_\varepsilon\|_{\mathcal{L},q} < \varepsilon$ and $\mathbf{v}_\varepsilon \in [\mathscr{C}_0^\infty(\Omega')]^m$. ////

The transformations in Steps I, II and the first bound in Step III are based on ideas in [1]. We can extend the theorem to all functions in $\mathbf{W}^q_{\mathcal{L}}(\Omega)$ using the techniques introduced by Meyers and Serrin in [2], see also [7, p. 54]. However, due to the existence of functions $\mathbf{v} \in \mathbf{W}^q_{\mathcal{L}}(\Omega)$ whose support does not lie compactly in Ω, it becomes necessary to admit $\mathscr{C}^\infty(\Omega)$ as class of approximating functions instead of $\mathscr{C}_0^\infty(\Omega')$, $\Omega' \subset\subset \Omega$.

Theorem 2: *The space $[\mathscr{C}^\infty(\Omega)]^m \cap \mathbf{W}^q_{\mathcal{L}}(\Omega)$ is dense in $\mathbf{W}^q_{\mathcal{L}}(\Omega)$, $1 < q < \infty$.*

Proof. Let Ω_i be open subsets in Ω such that $\Omega_i \subset\subset \Omega_{i+1}$ and

$$\bigcup_{i=1}^{\infty} \Omega_i = \Omega.$$

Let $\mathfrak{F}$ be a partition of unity of Ω subordinate to the covering $(\Omega_{i+1} \setminus \overline{\Omega_{i-1}})_{i\in\mathbb{N}}$, where Ω_{-1} is taken as the empty set. This means that $\mathfrak{F}$ is a family of functions f_j, $j \in \mathbb{N}$, such that:

1. for each f_j there exists an i such that $\mathrm{supp}(f_j) \subset \Omega_{i+1} \setminus \overline{\Omega_{i-1}}$;
2. each compact set $K \subset \Omega$ intersects the support of finitely many f_j only;
3. for all $\mathbf{x} \in \Omega$ holds $\sum_j f_j(\mathbf{x}) = 1$.

Let $\bar{f}_i$ be the sum of all $f_j \in \mathfrak{F}$ for which i is the smallest index such that $\mathrm{supp}(f_j) \subset \Omega_{i+1} \setminus \overline{\Omega_{i-1}}$. Then the $\bar{f}_i$ sum to one, too. Choose $\varepsilon > 0$. For $i \in \mathbb{N}$ and $\mathbf{v} \in \mathbf{W}^q_{\mathcal{L}}(\Omega)$ there exists, according to the last theorem, a function $\mathbf{v}_{\varepsilon,i}$ in $[\mathscr{C}_0^\infty(\Omega_{i+1} \setminus \overline{\Omega_{i-1}})]^m$ such that

$$\|f_i \cdot \mathbf{v} - \mathbf{v}_{\varepsilon,i}\|_{\mathcal{L},q} < \frac{\varepsilon}{2^i}.$$

Since $\overline{\Omega_j}$ is compact, the supports of only finitely many $\mathbf{v}_{\varepsilon,i}$ intersect Ω_j. Hence the sum

$$\mathbf{v}_\varepsilon = \sum_{i=1}^{\infty} \mathbf{v}_{\varepsilon,i}$$

is defined and is a member of $[\mathscr{C}^\infty(\Omega)]^m$. Notice that because of the layout of the supports of the $\mathbf{v}_{\varepsilon,i}$, the sequence

$$j \mapsto \left(\mathbf{v} - \sum_{i=1}^{j} \mathbf{v}_{\varepsilon,i}\right)_+\Big|_{\Omega_j} = (\mathbf{v} - \mathbf{v}_\varepsilon)_+|_{\Omega_j}$$

of $\mathbf{L}^q(\Omega)$ functions exhibits monotonic and pointwise convergence to $(\mathbf{v} - \mathbf{v}_\varepsilon)_+$ as $j \to \infty$. Hence by the Monotonic Convergence Theorem $(\mathbf{v} - \mathbf{v}_\varepsilon)_+$ is L^q-integrable, as well. The same argument employed on $(\mathbf{v}-\mathbf{v}_\varepsilon)_-$, $(\partial_k B_{ijk}(v_j-(v_\varepsilon)_j))_+$ and $(\partial_k B_{ijk}(v_j-(v_\varepsilon)_j))_-$ asserts that $\mathbf{v}_\varepsilon \in \mathbf{W}^q_{\mathcal{L}}(\Omega)$. We conclude

$$\|\mathbf{v}-\mathbf{v}_\varepsilon\|_{\mathcal{L},q} \leq \sum_{i=1}^{\infty} \left\| f_i \cdot \mathbf{v} - \mathbf{v}_{\varepsilon,i} \right\|_{\mathcal{L},q} < \varepsilon.$$

This proves the density of smooth functions in $\mathbf{W}^q_{\mathcal{L}}(\Omega)$. ////

Trace

To define the trace of functions in $\mathbf{W}^q_{\mathcal{L}}(\Omega)$ we follow the construction for $W^q(\mathrm{div},\Omega)$; compare, for example, with [5]. Nečas [3] shows existence of a bounded extension operator

$$\Gamma_e : \mathbf{W}^{1-1/q',q'}(\partial\Omega) \to \mathbf{W}^{1,q'}(\Omega), \mathbf{g} = (\Gamma_e\, \mathbf{g})|_{\partial\Omega}, \quad 1<q<\infty.$$

Since, for $\mathbf{v} \in \mathbf{W}^{1,q}(\Omega)$ and $\mathbf{w} \in \mathbf{W}^{1,q'}(\Omega)$,

$$\int_{\partial\Omega} w_i\, B_{ijk}\, \nu_k\, v_j \,\mathrm{d}S = \int_\Omega (\partial_k w_i)\, B_{ijk}\, v_j \,\mathrm{d}V + \int_\Omega w_i\, \partial_k(B_{ijk}\, v_j) \,\mathrm{d}V, \tag{2}$$

we obtain, for all $\mathbf{w}$ in the image of Γ_e with $\mathbf{w} = \Gamma_e\, \mathbf{g}$,

$$\begin{aligned} |\langle \mathbf{g}, \mathbf{B}(\boldsymbol{\nu})\cdot \mathbf{v}\rangle_{\partial\Omega}| &\leq \|\partial_k w_i\|_{[L^{q'}(\Omega)]^{m\times m}} \cdot \|B_{ijk}\, v_j\|_{[L^q(\Omega)]^{m\times m}} + \|\mathbf{w}\|_{\mathbf{L}^{q'}(\Omega)} \cdot \|\partial_k(B_{ijk}\, v_j)\|_{\mathbf{L}^q(\Omega)} \\ &\leq C\cdot \|\mathbf{g}\|_{\mathbf{W}^{1-1/q',q'}(\partial\Omega)} \cdot (\|B_{ijk}\, v_j\|_{[L^q(\Omega)]^{m\times m}} + \|\partial_k(B_{ijk}\, v_j)\|_{\mathbf{L}^q(\Omega)}), \end{aligned}$$

setting $C = \|\Gamma_e\|$ and $\mathbf{B}(\boldsymbol{\nu}) = (B_{ijk}\, \nu_k)_{ij}$. This shows that, for all $\mathbf{v} \in \mathbf{W}^{1,q}(\Omega)$, the functional

$$\langle \cdot, \mathbf{B}(\boldsymbol{\nu})\cdot \mathbf{v}\rangle : \mathbf{W}^{1-1/q',q'}(\partial\Omega) \to \mathbb{R}, \mathbf{g} \mapsto \langle \mathbf{g}, \mathbf{B}(\boldsymbol{\nu})\cdot \mathbf{v}\rangle_{\partial\Omega}$$

is a continuous mapping. Therefore, $\langle \cdot, \mathbf{B}(\boldsymbol{\nu})\cdot \mathbf{v}\rangle$ belongs to the dual of $\mathbf{W}^{1-1/q',q'}(\partial\Omega)$, which is the space

$$(\mathbf{W}^{1-1/q',q'}(\partial\Omega))' = [(W^{1-1/q',q'}(\partial\Omega))']^m = \mathbf{W}^{-1/q,q}(\partial\Omega).$$

Also

$$\|\langle \cdot, \mathbf{B}(\boldsymbol{\nu})\cdot \mathbf{v}\rangle\|_{\mathbf{W}^{-1/q,q}(\partial\Omega)} \leq C\cdot (\|B_{ijk}\, v_j\|_{[L^q(\Omega)]^{m\times m}} + \|\partial_k(B_{ijk}\, v_j)\|_{\mathbf{L}^q(\Omega)}). \tag{3}$$

Because

$$\begin{aligned} \|B_{ijk} v_j\|_{[L^q(\Omega)]^{m\times m}} &\leq \|\mathbf{B}\|_{[W^\infty(\Omega)]^{m\times m\times n}}\, \|\mathbf{v}\|_{\mathbf{L}^q(\Omega)}, \\ \|\partial_k(B_{ijk}\, v_j)\|_{\mathbf{L}^q(\Omega)} &\leq \|\mathcal{L}(\mathbf{v})\|_{\mathbf{L}^q(\Omega)} + \|\mathbf{C}\|_{[L^\infty(\Omega)]^{m\times m}}\, \|\mathbf{v}\|_{\mathbf{L}^q(\Omega)}, \end{aligned}$$

we obtain from (3) the continuity of the linear operator

$$\Gamma_{\mathcal{L}} : \mathbf{W}^{1,q}(\Omega) \to \mathbf{W}^{-1/q,q}(\partial\Omega), \mathbf{v} \mapsto \langle \cdot, \mathbf{B}(\boldsymbol{\nu})\cdot \mathbf{v}\rangle_{\partial\Omega}.$$

in $\|\cdot\|_{\mathfrak{L},q}$. As $\mathbf{W}^{1,q}(\Omega)$ is dense in $\mathbf{W}^q_{\mathfrak{L}}(\Omega)$, $\Gamma_{\mathbf{B}}$ extends to a continuous linear operator on the graph space.

Definition: *The operator*

$$\Gamma_{\mathfrak{L}} : \mathbf{W}^q_{\mathfrak{L}}(\Omega) \to \mathbf{W}^{-1/q,q}(\partial\Omega), \mathbf{v} \mapsto \lim_{\varepsilon\to 0} \langle\cdot, \mathbf{B}(\boldsymbol{\nu}) \cdot \mathbf{v}_\varepsilon\rangle_{\partial\Omega} =: \langle\cdot, \mathbf{B}(\boldsymbol{\nu}) \cdot \mathbf{v}\rangle_{\partial\Omega}$$

is called trace operator for $\mathbf{W}^q_{\mathfrak{L}}(\Omega)$*; here* $\{\mathbf{v}_\varepsilon\}_{\varepsilon>0}$ *is smooth and converging to* $\mathbf{v}$ *in* $\mathbf{W}^q_{\mathfrak{L}}(\Omega)$*.*

Observe that the definition of the trace is independent of the extension operator Γ_e as the choice of Γ_e only influences the constant C in (3). As $\partial_k(B_{ijk}\, v_j) \in \mathbf{L}^q(\Omega)$, the identity (2) is meaningful for all elements of $\mathbf{W}^q_{\mathfrak{L}}(\Omega)$: Choose $\mathbf{v}_\varepsilon$ as above and let $\mathbf{w} \in \mathbf{W}^{1,q'}(\Omega)$, then

$$\begin{aligned}
\langle \mathbf{w}, \mathbf{B}(\boldsymbol{\nu}) \cdot \mathbf{v}\rangle_{\partial\Omega} &= \lim_{\varepsilon\to 0} \langle \mathbf{w}, \mathbf{B}(\boldsymbol{\nu}) \cdot \mathbf{v}_\varepsilon\rangle_{\partial\Omega} \\
&= \lim_{\varepsilon\to 0} \int_\Omega (\partial_k w_i)\, B_{ijk}\, v_{\varepsilon,j}\, \mathrm{d}V + \lim_{\varepsilon\to 0} \int_\Omega w_i\, \partial_k(B_{ijk}\, v_{\varepsilon,j})\, \mathrm{d}V \\
&= \int_\Omega (\partial_k w_i)\, B_{ijk} \lim_{\varepsilon\to 0} v_{\varepsilon,j}\, \mathrm{d}V + \int_\Omega w_i \lim_{\varepsilon\to 0} \partial_k(B_{ijk}\, v_{\varepsilon,j})\, \mathrm{d}V \\
&= \int_\Omega (\partial_k w_i)\, B_{ijk}\, v_j\, \mathrm{d}V + \int_\Omega w_i\, \partial_k(B_{ijk}\, v_j)\, \mathrm{d}V.
\end{aligned}$$

Comments

In [1] Friedrichs analyses the existence and uniqueness of linear symmetric hyperbolic systems with associated differential operator $\mathfrak{L}$. Initially he considers the weak and strong extension of these operators, but then shows that these extensions coincide. This allows him to apply a duality argument by which he proves the existence of a strong solution by uniqueness of weak solutions. To ensure coincidence of the weak and strong extension he introduces the integral operators

$$K_\varepsilon u(\mathbf{x}) = \int_\Omega k_\varepsilon(\mathbf{x}, \bar{\mathbf{x}}) u(\bar{\mathbf{x}})\, \mathrm{d}\bar{\mathbf{x}},$$

which generalize the idea of mollifiers. Indeed k_ε is used in the calculation as a combination of an approximate identity with terms of $\mathfrak{L}$ and its dual $\mathfrak{L}'$. Friedrichs presents three properties of k_ε which are sufficient to prove convergence of certain smooth functions $\mathbf{v}_\varepsilon$ to $\mathbf{v}$. We only used one of the properties, namely boundedness. The other two properties concern a suitable choice of the support of k_ε and the mean of k_ε. The analysis in [1] is given for lens-shaped domains since then symmetries in boundary terms can be exploited.

In 1964, ten years after the publication of [1], Meyers and Serrin proved that for $W^{r,q}(\Omega)$ the weak and strong extension are equivalent for general open sets Ω. The proof of Theorem 2 in this report uses very similar arguments to those in [2]. The only difference is that Meyers and Serrin were able to use mollified functions directly while for our problem the strongly convergent sequence obtained by Theorem 1 is required.

Appendix

We cite the Riesz-Thorin Interpolation Theorem in abbreviated form from [6].

Theorem 3: *Let* $1 \leq p_0, p_1, q_0, q_1 \leq \infty$*. Further let* $0 < \theta < 1$*, and let* p *and* q *be defined by*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

Let μ *and* ν *be* σ*-finite measures. If* T *is a linear mapping with*

$$\begin{aligned} T : L^{p_0}(\mu) \to L^{q_0}(\nu) \quad & \textit{continuous with norm } M_0, \\ T : L^{p_1}(\mu) \to L^{q_1}(\nu) \quad & \textit{continuous with norm } M_1, \end{aligned}$$

then

$$\|Tf\|_{L^q} \leq 2M_0^{1-\theta} M_1^{\theta} \|f\|_{L^p} \qquad \forall\, f \in L^{p_0}(\mu) \cap L^{p_1}(\mu).$$

Hence the operator is extendable to a continuous linear mapping

$$T : L^p(\mu) \to L^q(\nu)$$

with norm $2M_0^{1-\theta} M_1^{\theta}$*.*

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