Matroids and Complexity



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A thesis submitted for the degree of
Doctor of Philosophy
Trinity 2005

Acknowledgements

Most importantly, my thanks go to my supervisor, Professor Dominic Welsh, for his freely given time, encouragement, and hospitality, and for his generosity in sharing his considerable wealth of experience in matroids and complexity.

The following list of people includes some of those who have lent their assistance by providing useful references, helpful advice, or enjoyable discussion, and often a combination of all three. Magnus Bordewich, Henry Crapo, Graham Farr, Joseph Kung, Colin McDiarmid, Anna de Mier, James Oxley, John Talbot, Dirk Vertigan, Geoff Whittle, and Haidong Wu. I thank them all.

Abstract

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We consider different ways of describing a matroid to a Turing machine by listing the members of various families of subsets, and we construct an order on these different methods of description. We show that, under this scheme, several natural matroid problems are complete in classes thought not to be equal to P.

We list various results linking parameters of basis graphs to parameters of their associated matroids. For small values of k we determine which matroids have the clique number, chromatic number, or maximum degree of their basis graphs bounded above by k. If $\mathcal{P}$ is a class of graphs that is closed under isomorphism and induced subgraphs, then the set of matroids whose basis graphs belong to $\mathcal{P}$ is closed under minors. We characterise the minor-closed classes that arise in this way, and exhibit several examples.

One way of choosing a basis of a matroid at random is to select a total ordering of the ground set uniformly at random and use the greedy algorithm. We consider the class of matroids having the property that this procedure chooses a basis uniformly at random.

Finally we consider a problem mentioned by Oxley. He asked if, for every two elements and $n-2$ cocircuits in an n-connected matroid, there is a circuit that contains both elements and that meets every cocircuit. We show that a slightly stronger property holds for regular matroids.

Contents

1 Introduction **1**

2 Complexity issues in matroid theory **4**
2.1 Introduction 4
2.2 Various types of inputs 6
2.3 A comparison of inputs 10
2.4 Matroid intersection 21
2.5 Matroid isomorphism 24
2.6 Detecting minors 28

3 Basis graphs **35**
3.1 Introduction 35
3.2 Reconstructing the matroid 36
3.3 A key lemma 39
3.4 Graph and matroid parameters 41
3.5 Graph parameters define classes of matroids 46
3.6 Induced classes of matroids 50
3.7 A characterisation of near-graphic matroids 55
3.8 An induced class with infinitely many excluded minors 63
3.9 Basis graphs and complexity 67

4 Equitable matroids **71**
4.1 Introduction 71
4.2 Counting linear extensions 76
4.3 Super-equitable matroids 78
4.4 Basis-transitive matroids and PMDs 84
4.5 Some other classes 89
4.6 Summary 91

5 Circuits and cocircuits in regular matroids **94**
5.1 Introduction 94
5.2 Proof of the main result 96

6 Conclusions and questions **101**

A Excluded-minor characterisations **106**

B Computer calculations I **119**

C Computer calculations II **121**

References **123**

Chapter 1

Introduction

In a sense, matroids were present at the dawn of complexity theory. It was while working on the problems of graph matchings, matroid intersection, and matroid partitioning in the mid-1960's that Jack Edmonds elucidated the seminal ideas of "good algorithms", that is, polynomial-time algorithms, and "polynomial-time checkable proofs", which correspond to problems in NP ([14, 15], see also [19]).

Since that time, complexity theoretical approaches to matroid theory have diverged into two directions. In one direction the model of a deterministic Turing machine is abandoned; instead a modified machine is used, one that is equipped with an oracle which can, in unit time, give information about a subset of the ground set. The second direction abandons the ability to accept all matroids as input; instead the input is restricted to a small class of matroids that have "succinct" descriptions.

A third approach would not have to make either of these compromises. A matroid is essentially a finite set, along with a structured family, $\mathcal{A}$, of subsets. An obvious way to describe the matroid is to simply list the members of $\mathcal{A}$. This description would be acceptable to an ordinary deterministic Turing machine, and furthermore, any matroid could be described in this way, not merely those belonging to a restricted class.

The reason that this method of describing a matroid has not received as much attention as the other approaches is the concern that the large size of the input will artificially inflate the class of problems that can be solved in polynomial time, so that, in a trivial way, every algorithmic matroid problem will become "easy".

The situation appears to be more subtle than this. In Chapter 2 we show that there are several natural matroid problems that are unlikely to be solvable in polynomial time, even when the matroid is described by some "large" input.

There are numerous ways of describing a matroid by listing the members of a family of subsets. There is no reason to believe that all these methods of description are equivalent and, indeed, some seem to be intrinsically more compact than others. In order to formalise this idea, we describe an order on different input types in Section 2.3, and we explicitly display this ordering on a set of ten natural methods of description. Our ideas in this section operate by analogy to the comparison of matroid oracles performed by Hausmann and Korte [22], and Robinson and Welsh [45].

It seems to be an interesting quirk of matroid complexity that the difficulty of a

problem is highly dependent on the type of input that we use. A decision problem for matroids can be NP-complete if the input is, say, a list of circuits, but in P if the input is a list of bases.

Matroid basis graphs are the subject of an important problem in matroid complexity theory. If random walks on basis graphs were known to be rapidly mixing, then it would be possible to efficiently choose a basis of a matroid almost uniformly at random. This would in turn lead to an efficient method for approximating the number of bases of a matroid. This problem has received a considerable amount of attention. Basis graphs are attractive objects in their own right, and we dedicate Chapter 3 to their study.

We consider the connections between natural parameters of basis graphs, and parameters of their associated matroids. This leads to questions such as: what are the matroids whose basis graphs have their clique number, chromatic number, or maximum degree bounded above by some integer k? We explicitly answer this question for some small values of k. This idea can be generalised further: suppose that $\mathcal{P}$ is a class of graphs that is closed under isomorphism. We let $\mathcal{M}(\mathcal{P})$ be the family of matroids whose basis graphs belong to $\mathcal{P}$. If $\mathcal{P}$ has the additional property that it is closed under induced subgraphs, then $\mathcal{M}(\mathcal{P})$ has various nice qualities. For one thing, it is closed under taking minors. We give simple necessary and sufficient conditions for a minor-closed family of matroids to be induced by a class of graphs in this way. Several naturally defined families of matroids satisfy these conditions. They include: the binary matroids, the regular matroids, the polygon matroids of planar graphs, and those matroids which can be formed from matroids that are either graphic or cographic, using either direct sums, direct sums and 2-sums, or direct sums, 2-sums, and 3-sums. These last three classes had hitherto not been characterised by their excluded minors; we present such characterisations in Section 3.7. We also show that the class of matroids for which every connected component is either binary, or can be obtained from a binary matroid by relaxing a single circuit-hyperplane is an induced class, and we prove that this class has infinitely many excluded minors.

As mentioned before, being able to choose a basis of a matroid uniformly, or almost uniformly at random, would lead to an efficient method for approximately counting the bases of that matroid. An extremely simple way to choose a basis at random involves choosing a total ordering of the ground set uniformly at random, and then using this ordering and the greedy algorithm to select a basis. In general this procedure will not choose a basis uniformly at random. The class of matroids for which this procedure does select a basis uniformly at random is studied in Chapter 4. It contains some very well-known families of matroids, and some curious sporadic members.

In Chapter 5 we consider a problem mentioned by Oxley. Dirac [12] showed that in an n-connected graph there is a cycle that passes through any specified pair of edges, and any specified set of $n-2$ vertices. Oxley asked whether a certain natural analogue for matroids holds. By using Seymour's decomposition theorem [47] and results of Oxley [43] and Denley and Wu [10], we show that a slightly stronger property holds for the class of regular matroids.

We use Oxley's *Matroid Theory* [42] as a reference for the fundamental concepts

of matroid theory. Our terminology and notation will follow that source. Similarly, our reference for the basic material of complexity theory will be Garey and Johnson's *Computers and Intractability : a guide to the theory of* NP*-completeness* [19].

Chapter 2

Complexity issues in matroid theory

2.1 Introduction

The study of complexity issues in matroid theory has been coloured by one fundamental fact: the number of matroids is large. Let $f(n)$ be the number of non-isomorphic matroids on a ground set of size n. (Henceforth we will refer to such matroids as "matroids of size n" and, in general, we will refer to the cardinality of the ground set of a matroid as the "size" of that matroid.) There are a number of results on the behaviour of $f(n)$. One such result is due to Piff and Welsh [44]. They show that

$$f(n) > 2^{2^n n^{-5/2} \log_2 n}$$

for all sufficiently large n.

Suppose that Σ is an alphabet of size q, where $q \geq 2$, and that $g : \mathbb{Z}^+ \to \mathbb{Z}^+$ is a function on the positive integers. We will denote the set of finite words formed from characters in Σ by Σ^*. If n is a positive integer, then the number of words in Σ^* of length at most $g(n)$ is

$$\sum_{i=0}^{g(n)} q^i = \frac{q^{g(n)+1} - 1}{q - 1} < q^{g(n)+1} = 2^{g'(n)},$$

where $g'(n)$ is the function $(\log_2 q)(g(n) + 1)$. If $g(n)$ bounded above by a polynomial function of n, then so is $g'(n)$, and it is easy to see that $2^n n^{-5/2} \log_2 n$ will exceed $g'(n)$ for large enough values of n. Therefore, if $g(n)$ is polynomially-bounded, as n increases the number of non-isomorphic matroids of size n will eventually exceed the number of words in Σ^* of length at most $g(n)$.

It follows from these observations that there can exist no encoding function, $\sigma : \mathcal{M} \to \Sigma^*$, which takes the set, $\mathcal{M}$, of all matroids into words in Σ^* in such a way that there exists a polynomial function, h, so that the length of the word, $\sigma(M)$, is bounded above by $h(|E(M)|)$, for all matroids M. Thus, if σ is an encoding scheme for all matroids, $|E(M)|$ and $|\sigma(M)|$ cannot be polynomially-related parameters.

This stands in contrast to such mathematical objects as graphs, where there are established encoding schemes having the property that the "size" of the structure (the number of vertices in the case of a graph) is polynomially related to the length of the word that represents that structure.

This phenomenon has motivated two different approaches to matroid complexity. The first involves the idea of a Turing machine equipped with an oracle. In this case the input written on the machine's tape at the start of the computation does not include a description of the matroid. Instead the machine can, in unit time, query the oracle about any subset of the ground set of the matroid. Various models of oracles have been considered. The output of the oracle will depend on the subset about which it is queried, and on the type of oracle used. For example, an *independence oracle* will return the answer YES if the subset being queried is independent, and will return NO otherwise. A *rank oracle* will return the rank of the queried subset. Although there are many other conceivable oracles, the rank and independence oracles have become the standard models.

The focus of the oracle approach to matroid complexity has been to find efficient algorithms to decide whether a matroid has particular properties. Suppose that $\mathcal{N}$ is a class of matroids that is closed under isomorphism. If there exists a polynomial, p, and a Turing machine equipped with an independence or rank oracle, such that, for any matroid, M, of size n, the machine can decide whether $M \in \mathcal{N}$ in time bounded by $p(n)$, then we say that a *good algorithm* exists for deciding membership of $\mathcal{N}$. It is known that there are good algorithms for deciding whether a matroid is, for example, connected [2] or graphic [48].

The other approach to matroid complexity uses the standard model of the Turing machine, but restricts the input to a small class of matroids, so that the input size is bounded by some polynomial function of the size of the ground set. Generally this means restricting the input to matroids that are representable over a finite field. Any such matroid on a set of size n can be described by an $n \times n$ matrix. A classic result by Jaeger, Vertigan, and Welsh [25], which uses this approach, shows that evaluating the Tutte polynomial of a matroid given some such "succinct" representation is, in general, #P-hard.

Although these approaches have produced many attractive results, they have certain disadvantages. There is no clear notion of what a "polynomial transformation" would be for a Turing machine which receives its description of a matroid via an oracle. Therefore, if we use the first approach to matroid complexity, we cannot use powerful tools such as completeness.

Furthermore, the number of matroids for which there is a realistic oracle description seems likely to be small. It would be useful to have matroid algorithms that can accept any matroid as input, and this means algorithms that are not limited to input via an oracle or a succinct description.

A third approach to matroid complexity would avoid these drawbacks. In this approach we consider algorithms that accept as their input a description of a matroid via a list of its independent sets, flats, or some other family of subsets. Any matroid can be described in this way. Furthermore, since a matroid is essentially a finite set with a structured family of subsets, this approach has the advantage of being

consistent with the standard methods used in complexity theory to describe other objects of this sort.

However, if we adopt this approach, we will have to abandon the size of the ground set as a measure of the input size of a matroid. Historically, the concern with this approach has been that, because of the large size of the input, any interesting question that one might ask about a matroid would be solvable in polynomial time, and that therefore the orthodox distinction between algorithms that solve problems efficiently, and those that do not, would be eroded. However, this does not appear to be the case. Many natural matroid problems do not seem to have obvious polynomial time algorithms, even when the input is exponentially large in the size of the ground set. Moreover, we shall show that some of these natural problems are complete in complexity classes that are thought not to be equal to P.

2.2 Various types of inputs

In the next two sections we will look at different ways to describe a matroid, and, by analogy with the work of Hausmann and Korte [22], and Robinson and Welsh [45], we will construct an order on these different forms of input.

It is a well-known feature of matroids that they have many equivalent descriptions. We will concentrate on the following types of input.

Rank

Independent Sets

Spanning Sets

Bases

Flats

Circuits

Hyperplanes

Non-Spanning Circuits

Dependent Hyperplanes

Cyclic Flats

Obviously there are many more ways of describing a matroid, but to simplify the analysis we have eliminated certain redundancies. In particular, if one of the forms of input mentioned above involves describing the matroid, M, by listing the members of $\mathcal{A}$, a family of subsets of $E(M)$, we will not, in general, consider the forms of input that involve listing the members of

$$\{E(M) - A \mid A \in \mathcal{A}\},\ \{A' \subseteq E(M) \mid A' \notin \mathcal{A}\},\ \text{or}\ \{E(M) - A' \mid A' \notin \mathcal{A}\}.$$

Thus, since we will be discussing the description of a matroid via its spanning sets, we will ignore the descriptions via its coindependent sets (the complements of spanning sets), its non-spanning sets, and its codependent sets (the complements of non-spanning sets).

Note that we could choose to simplify our scheme further still by taking duality into account. We could then, for example, consider INDEPENDENT SETS and SPANNING SETS to be essentially the same form of input, as the independent sets of M^* are exactly the complements of the spanning sets of M.

Let us now formalise our notion of input. We will assume that our input always describes a labelled matroid. In particular, we will assume that, if we are describing a matroid, M, on a ground set of size n, then that ground set is $\{1, \ldots, n\}$. Our alphabet, Σ, will always consist of "0", "1", and ",". Any subset of $E(M)$ will be indexed by a characteristic vector: if A is a subset of $E(M)$, then A will be indexed by the vector $a_1 \ldots a_n$, where $a_i = 1$ if $i \in A$, and $a_i = 0$ otherwise, for $1 \leq i \leq n$.

The following types of input are especially simple: INDEPENDENT SETS, SPANNING SETS, BASES, FLATS, CIRCUITS, and HYPERPLANES. If we list the members of any of these families of subsets, then we have completely determined the matroid. Therefore, in these cases, our input will consist of the list of characteristic vectors, separated by commas. In order for the input string to be uniquely determined, let us impose an order on the set of characteristic vectors. We will order them first by increasing number of non-zero entries, and then order vectors of the same weight lexicographically. We will let (M, INPUT) denote the word in Σ^* that describes M via the form of description specified by INPUT, and let $|(M, \text{INPUT})|$ denote the length of this word. In particular, if INPUT is a method of description of M that involves listing the members of a family, $\mathcal{A}$, of subsets of $E(M)$, then $|(M, \text{INPUT})|$ is equal to $(|E(M)| + 1)|\mathcal{A}| - 1$.

As an illustration, let us give various descriptions of the rank-3 matroid, N, shown in Figure 2.1. Table 2.1 gives examples of strings in Σ^* that represent N according to different methods of input.

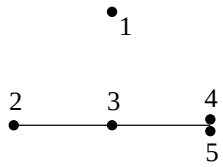


Figure 2.1: The rank-3 matroid N.

The families of independent sets, spanning sets, bases, and flats always contain at least one member, even if it is only the empty set. However, the family of circuits is empty exactly for those matroids isomorphic to $U_{n,n}$. Therefore we will define ($U_{n,n}$, CIRCUITS) to be equal to the string of n zeroes. This is not to be interpreted as stating that the empty set is a circuit, which, in any case, is impossible. Dually, the family of hyperplanes of $U_{0,n}$ is empty, so we will define ($U_{0,n}$, HYPERPLANES) to be the string of n ones.

The other forms of input may need a little more explanation. In all the following

$$
\begin{aligned}
(N, \textsc{Independent Sets}) &= 00000, 10000, 01000, 00100, 00010, 00001,\\
&\quad 11000, 10100, 10010, 10001, 01100, 01010,\\
&\quad 01001, 00110, 00101, 11100, 11010, 11001,\\
&\quad 10110, 10101\\
(N, \textsc{Spanning Sets}) &= 11100, 11010, 11001, 10110, 10101,\\
&\quad 11110, 11101, 11011, 10111, 11111\\
(N, \textsc{Bases}) &= 11100, 11010, 11001, 10110, 10101\\
(N, \textsc{Flats}) &= 00000, 10000, 01000, 00100, 11000,\\
&\quad 10100, 00011, 10011, 01111, 11111\\
(N, \textsc{Circuits}) &= 00011, 01110, 01101\\
(N, \textsc{Hyperplanes}) &= 11000, 10100, 10011, 01111
\end{aligned}
$$

Table 2.1: Different descriptions of N.

methods of description, sets of characteristic vectors will be ordered according to the method given above.

The RANK input will consist of the characteristic vector of each subset of $E(M)$ in turn, followed by a comma, followed by the binary representation of the rank of that subset. We will pad out each binary representation with zeroes where necessary, so that every such representation has length $\lceil \log_2 r(M) \rceil$. Therefore,

$$|(M, \textsc{Rank})| = 2^{|E(M)|}(|E(M)| + \lceil \log_2 r(M) \rceil + 2) - 1.$$

We illustrate the RANK input by giving $(N, \textsc{Rank})$ in Table 2.2, where N is the matroid shown in Figure 2.1.

$$
\begin{aligned}
(N, \textsc{Rank}) = {} & 00000, 00, 10000, 01, 01000, 01, 00100, 01, 00010, 01, 00001, 01,\\
& 11000, 10, 10100, 10, 10010, 10, 10001, 10, 01100, 10, 01010, 10,\\
& 01001, 10, 00110, 10, 00101, 10, 00011, 01, 11100, 11, 11010, 11,\\
& 11001, 11, 10110, 11, 10101, 11, 10011, 10, 01110, 10, 01101, 10,\\
& 01011, 10, 00111, 10, 11110, 11, 11101, 11, 11011, 11, 10111, 11,\\
& 01111, 10, 11111, 11
\end{aligned}
$$

Table 2.2: $(N, \textsc{Rank})$.

A *cyclic flat* is a flat that is also a (possibly empty) union of circuits. It is known (see, for example, [4] or [42, 2.1 Exercise 13(iii)]) that listing the cyclic flats

and specifying their ranks completely determines a matroid. Therefore, the Cyclic Flats input will follow the same template as Rank, except that, instead of listing every subset of $E(M)$, we will list only the cyclic flats. Therefore

$$(N, \text{Cyclic Flats}) = 00000, 00, 00011, 01, 01111, 10.$$

A matroid always contains at least one cyclic flat (the closure of the empty set). Note that

$$|(M, \text{Cyclic Flats})| = z(|E(M)| + \lceil \log_2 r(M) \rceil + 2) - 1,$$

where z is the number of cyclic flats of M.

Oxley and Whittle [40] observed that a list of non-spanning circuits uniquely specifies a matroid, as long as the rank of that matroid is also specified. Therefore, the Non-Spanning Circuits input will start with a binary representation of $r(M)$, followed by a comma, followed by a list of the characteristic vectors of the non-spanning circuits of M. If N is the matroid in Figure 2.1, then

$$(N, \text{Non-Spanning Circuits}) = 11, 00011, 01110, 01101.$$

As before, if there are no non-spanning circuits in M (which is the case exactly when M is uniform), then we define $(M, \text{Non-Spanning Circuits})$ to be the binary representation of $r(M)$, followed by a comma, followed by the string of n zeroes.

If M contains at least one non-spanning circuit, then

$$|(M, \text{Non-Spanning Circuits})| = \lceil \log_2 r(M) \rceil + c(|E(M)| + 1),$$

where c is the number of non-spanning circuits of M. If $c = 0$, then

$$|(M, \text{Non-Spanning Circuits})| = \lceil \log_2 r(M) \rceil + |E(M)| + 1.$$

It is easy to see that C is a non-spanning circuit of M if and only if $E(M) - C$ is a dependent hyperplane of M^*. Thus listing the dependent hyperplanes of a matroid and specifying its rank completely describes that matroid. The Dependent Hyperplanes input will therefore take exactly the same form as the Non-Spanning Circuits input. So, if M contains at least one dependent hyperplane, then

$$|(M, \text{Dependent Hyperplanes})| = \lceil \log_2 r(M) \rceil + h(|E(M)| + 1),$$

where h is the number of dependent hyperplanes of M. If M contains no dependent hyperplanes, then define $(M, \text{Dependent Hyperplanes})$ to be a binary representation of $r(M)$, followed by a comma, followed by a string of n ones.

We conclude by observing that

$$(N, \text{Dependent Hyperplanes}) = 11, 10011, 01111.$$

2.3 A comparison of inputs

It is apparent from the examples in the last section that the length of a description of a matroid can vary widely according to which type of input we employ. It is natural to ask whether one form of description is intrinsically more compact than another. In this section we attempt to answer that question.

Hausmann and Korte [22], and Robinson and Welsh [45] have compared various types of matroid oracle. Following their ideas, we say that ORACLE1 $\leq$ ORACLE2 if there exists a polynomial, p, such that the information which is obtained from ORACLE1 about a subset, A, of the ground set, can be obtained by at most $p(n)$ calls to ORACLE2 for any matroid of size n. Thus, for example, the oracle which returns YES if the queried subset is a basis and NO otherwise, is below the independence oracle in this order, for to check whether a subset, A, is a basis, we need make at most $n+1$ calls to an independence oracle for any n-element matroid: one call to check that A is independent, and then no more than n calls to check that $A \cup e$ is dependent for any $e \notin A$.

Our comparison of inputs will operate by analogy to this definition.

Definition 2.3.1. Suppose that INPUT1 and INPUT2 are two types of input. Then INPUT1 $\leq$ INPUT2 if there exist a polynomial, p, and a Turing machine which, given (M, INPUT1) as input, will produce (M, INPUT2) as output in time that is bounded by $p(|(M, \text{INPUT1})|)$, for any matroid M.

It is obvious that the relation, $\leq$, is both reflexive and transitive on forms of input. Observe that this definition is closely related to polynomial reducibility between languages: suppose that $\mathcal{N}$ is a class of matroids that is closed under isomorphism. Then

$$L_1 = \{(M, \text{INPUT1}) \mid M \in \mathcal{N}\} \text{ and } L_2 = \{(M, \text{INPUT2}) \mid M \in \mathcal{N}\}$$

are languages. It is clear that, if INPUT1 $\leq$ INPUT2, then L_1 is polynomially reducible to L_2. Thus, in this case, if there is a polynomial time algorithm for deciding membership of $\mathcal{N}$ given (M, INPUT2), then membership of $\mathcal{N}$ is also decidable in polynomial time given (M, INPUT1). Similarly, if a matroid problem is "difficult" when the description received takes the form specified by INPUT1, then it will also be difficult if the description is as specified by INPUT2. In particular, if L_1 is an NP-complete language, then L_2 is NP-hard.

Let **INPUT** be the set of inputs defined on page 6.

> **INPUT** = {RANK, INDEPENDENT SETS, SPANNING SETS, BASES, FLATS, CIRCUITS, HYPERPLANES, NON-SPANNING CIRCUITS, DEPENDENT HYPERPLANES, CYCLIC FLATS}.

The rest of this section will be devoted to proving Theorem 2.3.2.

Theorem 2.3.2. *The Hasse diagram in Figure 2.2 shows* **INPUT***, ordered according to Definition 2.3.1*

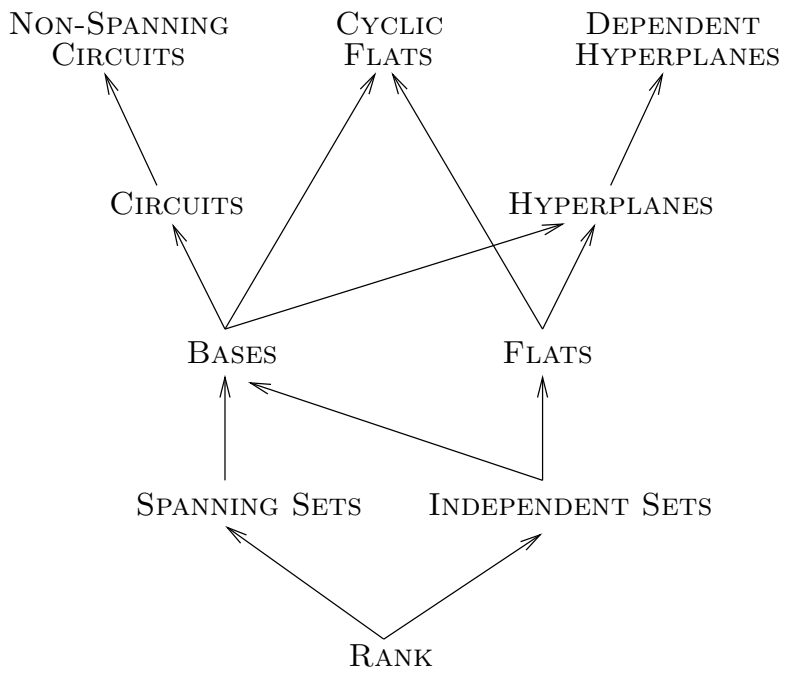


Figure 2.2: An ordering of input types.

It is interesting to note that the members of **INPUT** that are maximal under this order, Non-Spanning Circuits, Dependent Hyperplanes, and Cyclic Flats, are often used, both in the literature of matroid theory and informally, to describe matroids that have no succinct representation by a graph or a matrix or some other structure. The order given in Definition 2.3.1 seems to capture some intuitive notion of the "compactness" of a description.

Theorem 2.3.2 will follow from the subsequent lemmas. In the analysis that follows, we will generally take the number of comparisons of digits as the measure of the complexity of an algorithm.

Lemma 2.3.3. Rank $\leq$ Spanning Sets *and* Rank $\leq$ Independent Sets.

Proof. A Turing machine which produces (M, Spanning Sets) as its output, given (M, Rank) as input, need only check the rank of $E(M)$, and then work its way through the subsets of $E(M)$, writing onto its output the characteristic vectors of those subsets whose rank equals $r(M)$.

Similarly, to produce a list of the independent sets, the machine need only write onto its output the characteristic vectors of those subsets of $E(M)$ whose rank is equal to their size. Both of these procedures can be done in linear time in terms of $|(M, \text{Rank})|$, so Rank $\leq$ Spanning Sets and Rank $\leq$ Independent Sets. □

Lemma 2.3.4. Spanning Sets $\leq$ Bases *and* Independent Sets $\leq$ Bases.

Proof. A Turing machine which produces a list of bases given a list of spanning sets need only scan the list of spanning sets to determine the minimum size of such sets,

and then write onto its output the characteristic vectors of the spanning sets that have that size.

Similarly, a machine which produces a list of bases from a list of independent sets need only write onto its output the characteristic vectors of independent sets with maximum size. Both these procedures can clearly be done in polynomial time. □

Lemma 2.3.5. INDEPENDENT SETS $\leq$ FLATS.

Proof. Given the list of independent sets of M, we can create the list of flats by finding, in turn, the closure of each independent set, and then eliminating duplications.

Suppose that $|E(M)| = n$, and that the number of independent sets of M is i. Finding the closure of the independent set, I, can be accomplished by checking $I \cup e$, where $e \notin I$, against the list of independent sets. The element e is in the closure of I if and only if $I \cup e$ is dependent, so by repeating this procedure for each element $e \notin I$, we will find the closure of I. Comparing the characteristic vector of $I \cup e$ against the list of independent sets will take at most ni digit comparisons, and this procedure will be repeated at most n times for each of the i independent sets. Thus constructing the list of closed sets has a time complexity of $O((ni)^2)$. The list that we construct in this way contains exactly i closed sets, counting duplications. Eliminating these duplications will essentially involve comparing each pair of closed sets. Each such comparison will mean comparing n digits. Thus this task can be accomplished with $O(ni^2)$ digit comparisons, and the entire algorithm will have time complexity $O((ni)^2)$. Since the size of the input is $(n+1)i-1$, the algorithm runs in polynomial time. □

Lemma 2.3.6. BASES $\leq$ CIRCUITS.

Proof. If B is a basis of a matroid, M, and $e \in E(M) - B$, then $B \cup e$ contains a unique circuit, $C(e, B)$, which contains e, and which is known as the *fundamental circuit of e with respect to B*. Since every circuit is a fundamental circuit with respect to some basis, we can construct the list of circuits of M, given the list of bases, by creating the list of fundamental circuits with respect to each of the bases, and then eliminating duplications.

Suppose that $|E(M)| = n$, and that b is the number of bases of M. If B is a basis of M, and $e \in E(M) - B$, then we can find $C(e, B)$ by comparing $(B \cup e) - f$ against the list of bases for each $f \in B$. Since $f \in C(e, B)$ if and only if $(B \cup e) - f$ is a basis, this will enable us to find $C(e, B)$. Comparing $(B \cup e) - f$ with the list of bases will involve at most nb digit comparisons, and this will have to be done at most n^2 times for each of the b bases. Therefore, constructing the list of fundamental circuits will take time $O(n^3b^2)$. When this list is completed, it will contain no more than $O(nb)$ circuits. Eliminating duplications from the list will take n digit comparisons for each pair of circuits in the list, so this procedure will take no more than $O(n^3b^2)$ digit comparisons. Since the input has size $(n+1)b-1$, the algorithm runs in polynomial time. □

Lemma 2.3.7. BASES $\leq$ HYPERPLANES.

Proof. Since the bases of M^* are just the complements of the bases of M, we can certainly construct the list of cobases in polynomial time if we are given the list of bases. By Lemma 2.3.6, if we have the list of cobases, then we can construct the list of cocircuits in polynomial time. Then finding the list of hyperplanes is simply a matter of taking the complements of the cocircuits. □

Lemma 2.3.8. Bases $\leq$ Cyclic Flats.

Let the numbers of bases and cyclic flats in the matroid M be denoted by $b(M)$ and $z(M)$, respectively. The proof of Lemma 2.3.8 depends heavily on the following fact.

Proposition 2.3.9. *Let M be a matroid. Then $z(M) \leq b(M)$.*

Proof. The proof will be by induction of the size of the ground set of M. If $|E(M)| \leq 1$, then $b(M) = z(M) = 1$, so the proposition holds.

Let M be a matroid such that $|E(M)| = n > 1$, and assume that the proposition holds for all matroids of size $n - 1$. We may assume that there exists an element, $e \in E(M)$, such that e is neither a loop nor a coloop, for if every element is either a loop or a coloop, then $b(M) = z(M) = 1$.

Let $b_e(M)$ be the number of bases of M that contain e, and let $b_{\bar{e}}(M)$ be the number of bases of M that do not contain e. It is well known that $b_e(M) = b(M/e)$ and $b_{\bar{e}}(M) = b(M\backslash e)$. Similarly, let $z_e(M)$ be the number of cyclic flats of M that contain e, and let $z_{\bar{e}}(M)$ be the number of cyclic flats that do not contain e. It is easy to see that any cyclic flat that does not contain e is also a cyclic flat of $M\backslash e$, so $z_{\bar{e}}(M) \leq z(M\backslash e)$. Furthermore, if Z is a cyclic flat of M, and $e \in Z$, then $Z - e$ is a cyclic flat of M/e. Thus $z_e(M) \leq z(M/e)$, and hence

$$z(M) = z_e(M) + z_{\bar{e}}(M) \leq z(M/e) + z(M\backslash e).$$

By the inductive hypothesis, $z(M/e) \leq b(M/e)$ and $z(M\backslash e) \leq b(M\backslash e)$, so

$$z(M) \leq b_e(M) + b_{\bar{e}}(M) = b(M).$$ □

Proof of Lemma 2.3.8. We recursively define a parameter for cyclic flats. Suppose that Z is a cyclic flat of the matroid M. Then define $d(Z) = 1$ if $Z = \mathrm{cl}(C)$ for some circuit C, or if $Z = \varnothing$. If $d(Z) \neq 1$, then define $d(Z)$ to be equal to $s + 1$, where s is the smallest number such that there exist cyclic flats, Z_1 and Z_2, having the property that $Z = \mathrm{cl}(Z_1 \cup Z_2)$, and $d(Z_1)$, $d(Z_2) \leq s$. It is clear that $d(Z)$ is defined for any cyclic flat, Z, and that $d(Z) \leq r(M) + 1$.

Our algorithm for generating (M, Cyclic Flats), given the list of bases as input, will start by constructing the closures of all circuits. At each repetition of the loop, the algorithm will find the closure of the union of every pair of cyclic flats already on the list. Thus, after the loop has run i times, the algorithm will have found every cyclic flat, Z, such that $d(Z) \leq i + 1$.

Suppose that M is a matroid on a ground set of size n, and that the number of bases of M is b. The input size is therefore $(n + 1)b - 1$. First let us show that we can find the closure of any set in polynomial time if we have a list of bases.

We can check whether a subset of $E(M)$ is independent by comparing it against the family of bases. This will take at most nb digit comparisons. Since we can check whether a set is independent in polynomial time, we can find a maximal independent set in a subset, $A \subseteq E(M)$, in polynomial time by using the following implementation of the greedy algorithm. Recall that we are assuming the ground set of M to be $\{1, \ldots, n\}$. Initialise S to the empty set, and T to A. At each step of the algorithm, take the smallest element, i, in T. Update T to $T - i$. If $S \cup i$ is independent, update S to $S \cup i$, otherwise make no change to S. Continue until T is empty. It is easy to see that this is procedure will take at most $n^2 b$ digit comparisons.

If A is a subset of $E(M)$, then we can find the closure of A by first using the greedy algorithm to find a basis, $I \subseteq A$, of $M|A$. Finding $\mathrm{cl}(A) = \mathrm{cl}(I)$ involves checking $I \cup e$ against the set of bases to see whether it is dependent, for every element $e \notin A$. This will take at most $n^2 b$ digit comparisons.

From the proof of Lemma 2.3.6 we see that we can find the list of circuits in polynomial time, given the list of bases. Hence we can also construct the list of closures of circuits, and then eliminate duplications from the list, in polynomial time. If M has no loops, then the empty set is also a cyclic flat. In this case we also add the empty set to our list. Thus we have constructed the list containing every cyclic flat, Z, such that $d(Z) = 1$. This completes the preprocessing the algorithm will do before entering the loop.

Suppose that $Z_1, \ldots, Z_t$ is the list of cyclic flats that has been constructed after the loop has been repeated i times. In the next repetition of the loop, the algorithm will take each pair, $\{Z_j, Z_k\}$, of cyclic flats, and find $\mathrm{cl}(Z_j \cup Z_k)$. At the completion of the loop the algorithm will add these new cyclic flats to the list and then eliminate duplications. By Proposition 2.3.9, the length of the list at the beginning of the loop will never exceed b. Thus, in each repetition of the loop, the closure of at most b^2 unions of flats have to be found. Since $d(Z) \leq r(M) + 1$ for any cyclic flat Z, the algorithm can terminate after $r(M)$ repetitions of the loop. At this point the algorithm will have found every cyclic flat. Finding the rank of these flats can clearly be accomplished in polynomial time, using the greedy algorithm. Therefore the algorithm can write (M, Cyclic Flats) onto its output tape in polynomial time. □

Lemma 2.3.10. Flats $\leq$ Hyperplanes.

Proof. To find the hyperplanes of a matroid, we need only check, for each flat, $F \neq E(M)$, whether there is a flat, F', such that $F \subset F' \subset E(M)$. Such a flat, F', exists if and only if F is not a hyperplane. This procedure can clearly be done in polynomial time. □

Lemma 2.3.11. Flats $\leq$ Cyclic Flats.

We will use the following result to prove Lemma 2.3.11.

Proposition 2.3.12. *There exists a Turing machine that will find the rank of a specified subset, $A \subseteq E(M)$, in polynomial time, given a description of M via the* Flats *input.*

Proof. First we will prove that we can find the rank of any flat in polynomial time, given (M, Flats) as our input.

Suppose that $|E(M)| = n$, and that f is the number of flats of M. If F is a flat, then we wish to find a chain of flats,

$$F_0 \subset F_1 \subset \cdots \subset F_{t-1} \subset F_t,$$

where F_0 is the rank-zero flat, $F_t = F$, and, for $0 \leq i \leq t-1$, there is no flat, F', such that $F_i \subset F' \subset F_{i+1}$. In this case the rank of F is t [42, Lemma 1.7.6], so finding such a chain will give us $r(F)$.

We proceed by looking for a flat, G, that is maximal with respect to being properly contained in F. This can be accomplished by setting G equal to the rank-zero flat, and then working through the list of flats. Whenever a flat, G', such that $G \subset G' \subset F$ is encountered, update G to be equal to G'. When this procedure is complete, the output, G, will be the desired maximal flat. This procedure will take at most nf digit comparisons. By repeating this procedure recursively for G, we can find a maximal chain of flats joining the rank-zero flat to F. The entire algorithm will take time $O(n^2 f)$.

We will now suppose that A is an arbitrary subset of $E(M)$. We can find the closure of A by searching for the minimal flat that contains A, and then find the rank of A by finding the rank of $\mathrm{cl}(A)$ using the method described above. Finding the closure of A will take nf digit comparisons, so the entire algorithm for finding the rank of A will take time $O(nf)+O(n^2 f) = O(n^2 f)$. Since the input size is $(n+1)f-1$, this is a polynomial time algorithm. □

Proof of Lemma 2.3.11. By Proposition 2.3.12, a Turing machine which receives (M, Flats) as its input may be assumed to have a polynomial-time oracle for finding the rank of any subset of $E(M)$. Thus such a machine could construct the list of cyclic flats of M by writing onto its output the characteristic vector of each flat, F, that satisfies the condition that $r(F-e) = r(F)$ for every element $e \in F$. This condition holds if and only if F is cyclic. Once the cyclic flats of M have been found, the algorithm can find the rank of each such flat, and thus construct (M, Cyclic Flats). □

Lemma 2.3.13. Circuits $\leq$ Non-Spanning Circuits.

Proof. If a Turing machine receives (M, Circuits) as its input, then it can check in polynomial time whether a subset, $A \subseteq E(M)$, is independent by determining whether A contains a circuit. Thus such a machine could find a basis of M by using the greedy algorithm.

Once the rank of M is known, the algorithm may construct the list of non-spanning circuits by writing onto its output those circuits whose size does not exceed $r(M)$. This procedure can clearly be done in polynomial time. □

Lemma 2.3.14. Hyperplanes $\leq$ Dependent Hyperplanes.

Proof. Given a list of hyperplanes, a Turing machine can certainly find the list of cocircuits in polynomial time. By Lemma 2.3.13 it is possible to find the list of non-spanning circuits of M^* in polynomial time, given the list of circuits of M^*. The

dependent hyperplanes of M are exactly the complements of these sets. Furthermore, given the list of cocircuits, it is possible to find the corank of M by using the greedy algorithm. Thus it is possible to find the rank of M, and hence construct (M, Dependent Hyperplanes). □

The proof of Theorem 2.3.2 will be completed by that showing that there can be no further relations in the order we have imposed on the types of input. It might seem that we must consider every pair of inputs and check whether they are related, but our analysis will be greatly simplified by the following proposition, which follows immediately from the transitivity of the order.

Proposition 2.3.15. *Suppose that* Input1 $\not\leq$ Input2*. If* Input1 $\leq$ Input3*, and* Input4 $\leq$ Input2*, then* Input3 $\not\leq$ Input4*.*

Using Proposition 2.3.15, it is a relatively simple matter to check that proving the following facts will suffice to complete the proof of Theorem 2.3.2.

(i) Spanning Sets $\not\leq$ Flats

(ii) Independent Sets $\not\leq$ Spanning Sets

(iii) Flats $\not\leq$ Non-Spanning Circuits

(iv) Circuits $\not\leq$ Dependent Hyperplanes

(v) Hyperplanes $\not\leq$ Cyclic Flats

(vi) Circuits $\not\leq$ Cyclic Flats

(vii) Non-Spanning Circuits $\not\leq$ Circuits

(viii) Dependent Hyperplanes $\not\leq$ Hyperplanes

(ix) Cyclic Flats $\not\leq$ Dependent Hyperplanes

Suppose that $f = f_1, f_2, f_3, \ldots,$ and $g = g_1, g_2, g_3, \ldots,$ are sequences of numbers. We will say that f *grows super-polynomially with respect to* g, if, for every positive integer d, there exists an integer N, such that $n \geq N$ implies that $f_n > g_n^d$. Our method of proving that Input1 $\not\leq$ Input2 involves showing that a description of a matroid via Input2 will, in general, be too large. Observe that if Input1 $\leq$ Input2, then there must be some polynomial, p, such that

$$|(M, \text{Input2})| \leq p(|(M, \text{Input1})|)$$

for every matroid M. Thus, if we can find a sequence of matroids, $M_1, M_2, M_3, \ldots,$ such that $|(M_n, \text{Input2})|$ grows super-polynomially with respect to $|(M_n, \text{Input1})|$, then we have shown that Input1 $\not\leq$ Input2.

If f and g are sequences of non-zero numbers, and $\lim_{n\to\infty} f_n/g_n = 1$, then we shall say that f and g are *asymptotically equal*, and we shall denote this relation by $f \sim g$. We will be making implicit use of the following results, which follow in a straightforward fashion from the algebra of limits.

Proposition 2.3.16. *If f, g, and h are sequences of non-zero numbers, and the relations $f \sim g$ and $g \sim h$ hold, then $f \sim h$.*

If f and g are sequences of non-zero numbers, then we define fg and f/g to be the sequences f_1g_1, f_2g_2, $f_3g_3 \ldots$ and f_1/g_1, f_2/g_2, $f_3/g_3 \ldots$ respectively.

Proposition 2.3.17. *Suppose that f, f', g, and g' are sequences of non-zero numbers such that $f \sim f'$ and $g \sim g'$. Then $fg \sim f'g'$ and $f/g \sim f'/g'$.*

Proposition 2.3.18. *If f and g are sequences of non-zero numbers such that f grows super-polynomially with respect to g, and f' and g' are sequences such that $f \sim f'$ and $g \sim g'$, then f' grows super-polynomially with respect to g'.*

We will also make use of Stirling's well-known formula.

Proposition 2.3.19. [Stirling's Approximation] $n! \sim \sqrt{2\pi n}(n/e)^n$.

Lemma 2.3.20. Spanning Sets $\not\leq$ Flats.

Proof. For $n \geq 1$, let M_n be isomorphic to $U_{n-1,n}$. By the definitions in Section 2.2, it follows that

$$|(M_n, \text{Spanning Sets})| = (n+1)(n+1) - 1 \text{ and}$$
$$|(M_n, \text{Flats})| = (n+1)(2^n - n) - 1.$$

It follows immediately that the sequence $|(M_n, \text{Flats})|$ grows super-polynomially compared to $|(M_n, \text{Spanning Sets})|$. Thus Spanning Sets $\not\leq$ Flats. □

Lemma 2.3.21. Independent Sets $\not\leq$ Spanning Sets.

Proof. For $n \geq 1$, let M_n be isomorphic to $U_{1,n}$. Observe that

$$|(M_n, \text{Independent Sets})| = (n+1)(n+1) - 1 \text{ and}$$
$$|(M_n, \text{Spanning Sets})| = (n+1)(2^n - 1) - 1,$$

so $|(M_n, \text{Spanning Sets})|$ grows super-polynomially in comparison with $|(M_n, \text{Independent Sets})|$. □

It is easy to see that, if M is a matroid on the ground set E, then

$$\{I \subseteq E \mid I \text{ is independent in } M \text{ and } |I| \leq r(M) - 1\}$$

is the family of independent sets of a matroid, $T(M)$, known as the *truncation of M* (see [42, 1.3 Exercise 3]).

Lemma 2.3.22. Flats $\not\leq$ Non-Spanning Circuits.

Proof. If m is a positive integer, then we shall define $mU_{r,n}$ to be the matroid obtained by replacing each element in $U_{r,n}$ with a parallel class of size m.

For $n \geq 3$, define M_n to be $T(nU_{n-1,n} \oplus U_{2,2})$. That is, M_n is the matroid obtained by adding $n-1$ parallel elements to each element of a circuit of size n, then adding two coloops and truncating. Note that M_n contains n^2+2 elements, and $r(M_n) = n$.

There are $n+2$ parallel classes in M_n. Of these parallel classes, two are trivial, and n are non-trivial. Any set of these parallel classes is a flat, except for those that contain:

(i) exactly $n-2$ non-trivial parallel classes and both trivial parallel classes;

(ii) exactly $n-1$ non-trivial parallel classes; or,

(iii) n non-trivial parallel classes, and exactly one trivial parallel class.

The number of these sets is $\binom{n}{2}$, $4n$, and 2, respectively. Thus the number of flats of M_n is

$$2^{n+2} - \binom{n}{2} - 4n - 2 = 2^{n+2} - \frac{1}{2}(n^2 + 7n + 4),$$

so

$$|(M_n, \textsc{Flats})| = (n^2+2+1)(2^{n+2} - \frac{1}{2}(n^2+7n+4)) - 1.$$

It is easy to see that the number of non-spanning circuits of M_n is exactly $n^n + n\binom{n}{2}$. Thus

$$|(M_n, \textsc{Non-Spanning Circuits})| = \lceil \log_2 n \rceil + (n^2+2+1)(n^n + \frac{1}{2}n^2(n-1)).$$

Since $|(M_n, \textsc{Flats})| \sim n^2 2^{n+2}$, and $|(M_n, \textsc{Non-Spanning Circuits})| \sim n^{n+2}$, it follows that any polynomial function of $|(M_n, \textsc{Flats})|$ will eventually be exceeded by $|(M_n, \textsc{Non-Spanning Circuits})|$. Thus $\textsc{Flats} \not\leq \textsc{Non-Spanning Circuits}$. □

Lemma 2.3.23. $\textsc{Circuits} \not\leq \textsc{Dependent Hyperplanes}$.

Proof. Suppose that there exists a Turing machine which, given $(M, \textsc{Circuits})$ as input, produces $(M, \textsc{Dependent Hyperplanes})$ as output in polynomial time, for any matroid M. Since the hyperplanes of M are exactly the complements of circuits of M^*, and the dependent hyperplanes of M^* are exactly the complements of non-spanning circuits of M, it is not difficult to see that such a machine could be modified to produce $(M, \textsc{Non-Spanning Circuits})$ in polynomial time, given $(M, \textsc{Hyperplanes})$ as input. Thus $\textsc{Hyperplanes} \leq \textsc{Non-Spanning Circuits}$. This contradicts Proposition 2.3.15, as $\textsc{Flats} \leq \textsc{Hyperplanes}$ by Lemma 2.3.10, and $\textsc{Flats} \not\leq \textsc{Non-Spanning Circuits}$ by Lemma 2.3.22. □

Lemma 2.3.24. $\textsc{Hyperplanes} \not\leq \textsc{Cyclic Flats}$.

Proof. For $n \geq 3$, let M_n be isomorphic to $2U_{n-1,n}$. The hyperplanes of M_n are exactly the sets of $n-2$ parallel classes, while any set of parallel classes is a cyclic flat as long as it does not contain exactly $n-1$ such classes. Thus

$$|(M_n, \textsc{Hyperplanes})| = \frac{1}{2}n(n-1)(2n+1) - 1,$$

while

$$|(M_n, \textsc{Cyclic Flats})| = (2^n - n)(2n + \lceil \log_2(n-1) \rceil + 2) - 1,$$

and it follows immediately that $\textsc{Hyperplanes} \not\leq \textsc{Cyclic Flats}$. □

Lemma 2.3.25. $\textsc{Circuits} \not\leq \textsc{Cyclic Flats}$.

Proof. Suppose that there exists some Turing machine which, given $(M, \textsc{Circuits})$ as input, produces $(M, \textsc{Cyclic Flats})$ in polynomial time, for any matroid M.

Given $(M, \textsc{Hyperplanes})$ as input, we can certainly construct $(M^*, \textsc{Circuits})$ in polynomial time. Then, using our hypothetical Turing machine, we can construct $(M^*, \textsc{Cyclic Flats})$. Since the cyclic flats of M^* are exactly the complements of cyclic flats of M, we can then find the cyclic flats of M in polynomial time. Also, since we have a list of the circuits of M^*, it is easy to see that we can find the rank of any set in M^*, and hence in M, in polynomial time. Thus we can construct $(M, \textsc{Cyclic Flats})$, and this implies that $\textsc{Hyperplanes} \leq \textsc{Cyclic Flats}$, in contradiction to Lemma 2.3.24. □

Lemma 2.3.26. $\textsc{Non-Spanning Circuits} \not\leq \textsc{Circuits}$.

Proof. If M_n is isomorphic to $U_{n,2n}$, then M_n contains no non-spanning circuits, so, by definition,

$$|(M_n, \textsc{Non-Spanning Circuits})| = \lceil \log_2 n \rceil + 2n + 1.$$

On the other hand, the number of circuits is $\binom{2n}{n+1}$, so

$$|(M_n, \textsc{Circuits})| = (2n+1)\binom{2n}{n+1} - 1.$$

By applying Stirling's approximation it is easy to see that

$$\binom{2n}{n+1} \sim \frac{2^{2n}\sqrt{n}}{\sqrt{\pi(n^2-1)}},$$

so it is clear that $|(M_n, \textsc{Circuits})|$ increases super-polynomially in comparison to $|(M_n, \textsc{Non-Spanning Circuits})|$. □

Lemma 2.3.27. $\textsc{Dependent Hyperplanes} \not\leq \textsc{Hyperplanes}$.

Proof. Using duality, it is easy to see that if there were a Turing machine that could produce $(M, \textsc{Hyperplanes})$ in polynomial time, given $(M, \textsc{Dependent Hyperplanes})$, for any matroid M, then it would follow that $\textsc{Non-Spanning Circuits} \leq \textsc{Circuits}$. This contradiction to Lemma 2.3.26 completes the proof. □

Lemma 2.3.28. Cyclic Flats $\not\leq$ Dependent Hyperplanes.

Proof. For $n \geq 2$, let M_n be the matroid obtained by adding a single parallel element to a member of the ground set of $U_{n,2n}$. The only cyclic flats of M_n are the empty set, the non-trivial parallel class, and the entire ground set. Thus

$$|(M_n, \text{Cyclic Flats})| = 6n + 3\lceil \log_2 n \rceil + 8.$$

Any hyperplane that contains the non-trivial parallel class is dependent, so the number of such hyperplanes is $\binom{2n-1}{n-2}$. Therefore

$$|(M_n, \text{Dependent Hyperplanes})| = \lceil \log_2 n \rceil + (2n+2)\binom{2n-1}{n-2}.$$

Since

$$\binom{2n-1}{n-2} \sim \frac{2^{2n-1}\sqrt{2n-1}}{\sqrt{2\pi(n^2-n-2)}},$$

it follows that $|(M_n, \text{Dependent Hyperplanes})|$ grows super-polynomially in comparison to $|(M_n, \text{Cyclic Flats})|$. □

With this lemma we have completed the proof of Theorem 2.3.2.

We conclude this section by noting that, although the order defined in Definition 2.3.1 is anti-symmetric when applied to the set **INPUT**, this need not be true in general. Let the Cospanning Sets input consist of the list of cospanning sets. If we receive a list of the members of a family of subsets, then we can certainly construct the list of complements of those subsets in polynomial time. Thus, given the description, (M, Cospanning Sets), for some matroid M, we can construct (M, Independent Sets) in polynomial time. Thus Cospanning Sets $\leq$ Independent Sets. It is similarly easy to see that Independent Sets $\leq$ Cospanning Sets.

It is also possible for two types of input to be equivalent to each other under the order in Definition 2.3.1, even though they do not correspond to each other in the trivial way by which Independent Sets and Cospanning Sets are related. Suppose that M is a matroid on the ground set E. Hausmann and Korte [22] define the *girth* of a subset, $X \subseteq E$, to be equal to the minimum cardinality of a circuit contained in X, if such a circuit exists, and infinity otherwise. It is possible to determine M up to isomorphism by specifying the girth of every subset of E. Thus the Girth input will follow the same template as the Rank input, except the girth of each subset will be given, instead of its rank. We will use the string, "0", to represent the girth of a subset that contains no circuit. (Note that this will cause no confusion.)

We will show that Rank $\leq$ Girth and that Girth $\leq$ Rank. First note that a subset is independent if and only if its girth is infinite. Thus a Turing machine could, in polynomial time, decide which subsets of the ground set are independent, given the Girth description of a matroid. Finding the rank of each subset, X, is then merely a matter of comparing it against all the independent subsets and noting the maximum size of an independent set contained in X. Similarly, given a Rank

description of a matroid, a Turing machine could determine whether a subset, C, is a circuit by checking whether $r(C) = r(C-e) = |C|-1$ for every element $e \in C$. Once the subsets that are circuits have been identified, the Turing machine could find the girth of any subset by comparing it against the list of circuits. This procedure can also be done in polynomial time.

2.4 Matroid intersection

We now turn our attention to the problem of determining whether an algorithmic question may or may not be solved efficiently, given a description of a matroid via one of the inputs discussed in earlier sections.

We start with the straightforward observation that any matroid problem that can be efficiently solved by a Turing machine equipped with an oracle, as defined in Section 2.1, can also be solved efficiently by a Turing machine which receives its description of a matroid via one of the forms of input that we have discussed.

We make this assertion more formal in the next theorem.

Theorem 2.4.1. *Suppose that N is a Turing machine equipped with a matroid oracle. Let* Input *be a type of input that belongs to* **INPUT**. *There exists a Turing machine, N', such that, when acting on $(M,$ Input$)$, N' will produce the same output as N acting on an oracle description of the matroid M. Furthermore, if N runs in time bounded by $p(n)$ for any n-element matroid, where p is some polynomial, then there exists a polynomial, p', such that N' will run in time bounded by $p'(|(M,$ Input$)|)$ for any matroid M.*

Proof. We will prove this theorem by showing that a Turing machine which receives $(M,$ Input$)$ can efficiently simulate an independence oracle for M. That is, there exists a fixed polynomial, f, such that it can be decided, in time bounded by $f(|(M,$ Input$)|)$, whether or not a subset, $A \subseteq E(M)$, is independent in M. This will complete the proof, as $|(M,$ Input$)| \geq n$, where $|E(M)| = n$, and Robinson and Welsh [45] have shown that a call to any of the standard matroid oracles can be simulated using at most $g(n)$ calls to an independence oracle for any n-element matroid, where g is some fixed polynomial function.

Note that, by Theorem 2.3.2, we may assume that Input is one of Cyclic Flats, Non-Spanning Circuits, or Dependent Hyperplanes, for at least one of these types of description can be constructed in polynomial time from all of the methods of input that we have discussed.

Let us first assume that Input = Cyclic Flats. It is easy to see that a subset, $A \subseteq E(M)$, is independent in M if and only if $|A \cap F| \leq r(F)$ for every cyclic flat, F, of M. Thus checking the independence of A involves comparing the size of its intersection with each cyclic flat against the rank of that flat. This can clearly be done in time that is bounded by a linear function of $|(M,$ Cyclic Flats$)|$.

It is also easy to see that A is independent in M if and only if $|A| \leq r(M)$, and A contains no non-spanning circuits. Thus the independence of A can be checked in polynomial time given the input $(M,$ Non-Spanning Circuits$)$.

Let us now suppose that INPUT = DEPENDENT HYPERPLANES. We will determine, in time bounded by a polynomial function of $|(M, \text{DEPENDENT HYPERPLANES})|$, whether $E(M) - A$ is cospanning. Since $E(M) - A$ is cospanning if and only if A is independent, this will complete the proof. Note that a subset, $X \subseteq E(M)$, is coindependent if and only if $|X| \leq |E(M)| - r(M)$, and $E(M) - X$ is not contained in any dependent hyperplane of M. Thus we can determine whether X is coindependent in time that is bounded by a linear function of $|(M, \text{DEPENDENT HYPERPLANES})|$. Hence we can find the corank of $E(M) - A$ by applying the greedy algorithm. This can clearly be done in polynomial time. Obviously $E(M) - A$ is cospanning if and only if its corank is equal to $|E(M)| - r(M)$. □

Given the previous result, it is natural to ask whether there is some matroid problem that can be solved in polynomial time, given one of the matroid descriptions defined on page 6, but that cannot be solved efficiently by a Turing machine equipped with an oracle. We will show that the answer to this question is yes: a machine given a "large" description of a matroid is, in general, strictly more powerful than a machine equipped with an oracle.

In [45], Robinson and Welsh note that a Turing machine equipped with an oracle cannot decide whether a matroid is uniform in time bounded by a polynomial function of the size of the ground set. To justify this claim, we need only consider a rank-n matroid with a ground set of size $2n$. In order to determine whether such a matroid is uniform a Turing machine equipped with an independence oracle may need to check the independence of every set of size n; for otherwise it may be impossible to establish whether the matroid is isomorphic to $U_{n,2n}$ or to the paving matroid with exactly one non-spanning circuit. Thus in the worst case such a machine must make $\binom{2n}{n}$ calls to the oracle, and this quantity is not bounded by any polynomial function of $2n$. Thus there is no good algorithm for determining whether a matroid is uniform using the oracle approach to matroid complexity.

Similarly, it may be difficult to determine whether a matroid is uniform given a succinct description of that matroid. A *Hamiltonian cycle* of a graph is a cycle that contains every vertex. Suppose that G is a simple graph, not isomorphic to a cycle, with n vertices and at least n edges. Jerrum [26] has noted that we may define a rank-n matroid on the edge set of G, such that the only non-spanning circuits of this matroid are the edge sets of Hamiltonian cycles of G. Obviously this provides us with a succinct description of a matroid: deciding whether a set of edges is independent is easily achieved. But deciding whether the matroid is uniform is equivalent to deciding whether G contains a Hamiltonian cycle, and this is a notoriously difficult problem.

In contrast to this, a Turing machine which receives (M, INPUT) as input, where INPUT $\in$ **INPUT**, can decide whether M is uniform in polynomial time. By Theorem 2.3.2 we need demonstrate this only when INPUT is equal to one of NON-SPANNING CIRCUITS, CYCLIC FLATS, or DEPENDENT HYPERPLANES. A matroid is uniform if and only if it has no non-spanning circuits, or no dependent hyperplanes. Furthermore a matroid is uniform if and only if it has exactly one cyclic flat (the entire ground set), or exactly two cyclic flats (the empty set and the entire ground

set). Therefore determining from (M, INPUT) whether M is uniform is essentially a trivial problem.

Given this observation, we might wonder whether the concerns over this approach to matroid complexity are justified: perhaps every natural algorithmic question one can ask about matroids is solvable in polynomial time. However, the following material shows that this is unlikely to be the case.

Suppose that M_1 and M_2 are matroids on the same ground set. The problem of finding a set of maximum cardinality that is independent in both matroids is closely related to many problems in combinatorial optimisation, and has therefore received much attention. It is well known that this problem is solvable in polynomial time by a Turing machine equipped with an independence or rank oracle (see, for example, [29, Chapter 8]). The analogous problem for three matroids is not known to be solvable in polynomial time, and, indeed, Lawler [28] has remarked that the following problem is reducible to the three matroid intersection problem.

> 3-DIMENSIONAL MATCHING
>
> Instance: A set of triples, $M \subseteq X_1 \times X_2 \times X_3$, where X_1, X_2, and X_3 are pairwise disjoint sets having the same cardinality.
>
> Question: Does M contain a matching, that is, a subset, $M' \subseteq M$, such that every element in $X_1 \cup X_2 \cup X_3$ is contained in exactly one triple in M'?

This problem is known to be NP-complete (see [19]). Lawler observes that 3-DIMENSIONAL MATCHING is reducible to the special case of the three matroid intersection problem when the three matroids are *partition matroids*, that is, matroids that are the direct sum of rank-one uniform matroids. He does not specify a form of matroid input, but it is implicit in his work that a partition matroid is to be described by the partition of its ground set into connected components. Thus we can modify his proof to show that the following problem is, in general, unlikely to be solvable in polynomial time.

> 3-MATROID INTERSECTION
>
> Instance: An integer k, and three descriptions of the same type, (M_1, INPUT), (M_2, INPUT), and (M_3, INPUT), of three matroids on the same ground set, E.
>
> Question: Does there exist a set, $A \subseteq E$, such that $|A| = k$ and A is independent in M_1, M_2, and M_3?

Theorem 2.4.2. *If* CIRCUITS $\leq$ INPUT, *then* 3-MATROID INTERSECTION *is* NP*-complete.*

Proof (due to Lawler). It is clear that the problem is in NP. Obviously we need only prove the theorem in the case that INPUT = CIRCUITS.

Let $M \subseteq X_1 \times X_2 \times X_3$ be an instance of 3-DIMENSIONAL MATCHING, and suppose that for $1 \leq i \leq 3$, the set X_i is equal to $\{x_1^i, \ldots, x_s^i\}$. Suppose that M

contains t triples, $p_1, \ldots, p_t$. We will construct three partition matroids, M_1, M_2, and M_3, on the ground set $E = \{1, \ldots, t\}$. Each matroid contains s connected components. The connected components of M_i correspond to the elements of X_i. In particular, the connected component that corresponds to x_j^i is equal to the set $\{k \mid p_k \text{ contains } x_j^i,\ 1 \leq k \leq t\}$.

Suppose that I is a subset of $\{1, \ldots, t\}$. Every element in $X_1 \cup X_2 \cup X_3$ is contained exactly once in $M' = \{p_i \mid i \in I\}$ if and only if I contains exactly one element from each of the connected components of M_1, M_2, and M_3. Thus M contains a matching if and only if M_1, M_2, and M_3 contain a common independent set of size s.

It remains to show that (M_i, CIRCUITS) can be constructed in polynomial time. Note that every circuit of M_i is a pair of elements from E. Thus each matroid contains no more than $\binom{t}{2}$ circuits, and hence $|(M_i, \text{CIRCUITS})| = O(t^3)$. We may assume that $s \leq t$, so the input size of 3-DIMENSIONAL MATCHING is polynomially equivalent to t. Thus an algorithm that simply checks every pair of elements in E to see if it is a circuit of M_i can construct (M_i, CIRCUITS) in polynomial time. □

The proof of Theorem 2.4.2 shows that, if CIRCUITS $\leq$ INPUT, then 3-MATROID INTERSECTION remains NP-complete even when $k = r(M_1) = r(M_2) = r(M_3)$. Moreover, it is not difficult to see that if M_i is one of the matroids constructed in the proof of Theorem 2.4.2, then (M_i, HYPERPLANES) can also be constructed in polynomial time. Thus 3-MATROID INTERSECTION remains NP-complete when HYPERPLANES $\leq$ INPUT.

This problem provides a dramatic illustration of how the type of input used in a matroid problem can affect the complexity of that problem. If we have a list of the bases of M_1, M_2, and M_3, then we can find a common independent set of maximum size by considering the intersection of every triple of bases from the three matroids. This can clearly be done in polynomial time, so 3-MATROID INTERSECTION is in P if INPUT $\leq$ BASES, even though CIRCUITS covers BASES in the ordering of input types.

2.5 Matroid isomorphism

Although matroid intersection has been important in combinatorial optimisation, it does not frequently arise in classical matroid theory. To provide further evidence that this model of matroid complexity can be useful in distinguishing between easy and difficult computational problems we shall, in this section, consider a problem that arises more naturally.

The following problem has attracted much attention.

GRAPH ISOMORPHISM

Instance: Two graphs, G and G'.

Question: Is G isomorphic to G'?

Despite several decades of effort, GRAPH ISOMORPHISM is not known to belong either to P or to the class of NP-complete problems. If NP $\neq$ P, then there must be

problems in NP that are neither NP-complete nor in P, and GRAPH ISOMORPHISM is thought to be a good candidate for one of these problems (see [19]).

If a decision problem is polynomially reducible to GRAPH ISOMORPHISM, and it also the case that GRAPH ISOMORPHISM is polynomially reducible to that problem, then it is said to be *polynomially equivalent to* GRAPH ISOMORPHISM. Such problems are *isomorphism-complete*. Many problems are known to be isomorphism-complete. Most of them are the analogous isomorphism problem for some other class of mathematical objects (see, for example, [7]). We shall show that the analogous matroid problem is, in general, isomorphism-complete.

MATROID ISOMORPHISM

Instance: Descriptions of the same type, (M, INPUT) and (M', INPUT), of two matroids, M and M'.

Question: Is M isomorphic to M'?

Lemma 2.5.1. MATROID ISOMORPHISM *is polynomially reducible to* GRAPH ISOMORPHISM.

Proof. We shall develop a scheme for taking the description, (M, INPUT), to a graph, $G(M, \text{INPUT})$, that completely encodes the matroid M up to isomorphism.

A set of characteristic vectors shall be represented by a bipartite graph. Each element of the ground set of M and each vector shall be represented by a vertex, and two vertices shall be adjacent if and only if they correspond to a vector, and an element which is contained in the set represented by that vector. So as to distinguish the two sides of the bipartition, we shall adjoin a triangle to each vertex that represents a vector.

Let us now establish a way of graphically representing a non-negative integer in binary notation. If $b > 0$, then let $b_t \dots b_1$ be the binary representation of b, where $t = \lceil \log_2 b \rceil$. If $b = 0$, then define $t = 1$ and let the binary representation of b be the single digit b_1, where $b_1 = 0$. We shall represent the binary expansion of b by a graph, $n(b)$, that contains the vertices $\{v_1, \dots, v_t\} \cup \{v'_1, \dots, v'_t\} \cup \{x, y, z, w\}$. Make all of the vertices in $\{x, y, z, w\}$ adjacent to each other, and make x adjacent to v_1 and y adjacent to v'_1. For $i \in \{1, \dots, t-1\}$ make v_i adjacent to v_{i+1} and v'_i adjacent to v'_{i+1}. Furthermore, for $i \in \{1, \dots, t\}$, make v_i adjacent to v'_i if and only if $b_i = 1$. Finally, create two additional copies of K_4, the complete graph on four vertices, and adjoin them to the vertices x and y respectively.

If INPUT is such that the rank M needs to be specified, we shall construct $n(r(M))$ and let this graph be a connected component of $G(M, \text{INPUT})$.

If INPUT is such that the rank of each subset corresponding to a characteristic vector needs to be specified, we shall construct $n(r(F))$ for each set, F, that is represented by a characteristic vector, and identify z with the vertex in the bipartite graph that corresponds to that vector.

Recall that (N, NON-SPANNING CIRCUITS) and (N, CYCLIC FLATS) are $11, 00011, 01110, 01101$ and $00000, 00, 00011, 01, 01111, 10$ respectively, where N is

the rank-3 matroid that was displayed in Figure 2.1. We illustrate our construction of the graph, $G(M, \textsc{Input})$ by showing $G(N, \textsc{Non-Spanning Circuits})$ and $G(N, \textsc{Cyclic Flats})$ in Figure 2.3.

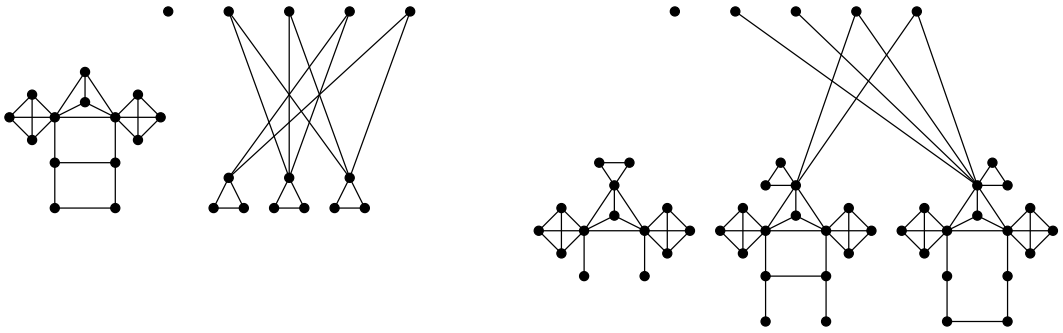


Figure 2.3: $G(N, \textsc{Non-Spanning Circuits})$ and $G(N, \textsc{Cyclic Flats})$.

It is clear that $G(M, \textsc{Input})$ can be constructed in time that is bounded by a polynomial function of $|(M, \textsc{Input})|$. Furthermore, if M is isomorphic to M', then $G(M, \textsc{Input})$ is isomorphic to $G(M', \textsc{Input})$.

It remains to show that, if $G(M, \textsc{Input})$ is isomorphic to $G(M', \textsc{Input})$, then M is isomorphic to M'. This is true since $G(M, \textsc{Input})$ has been constructed in such a way that, if we are given $G(M, \textsc{Input})$, then, regardless of how it is labelled, we can reconstruct a description, $(M', \textsc{Input})$, where $M' \cong M$. □

To show that GRAPH ISOMORPHISM is reducible to MATROID ISOMORPHISM, we develop a polynomial transformation from graphs to matroids. Suppose that G is a simple graph with n vertices, $\{v_1, \ldots, v_n\}$, and m edges, $\{e_1, \ldots, e_m\}$. Assume that $n \geq 3$. Let $X = \{x_1, \ldots, x_n\}$, $X' = \{x'_1, \ldots, x'_n\}$, and $Y = \{y_1, \ldots, y_m\}$ be disjoint sets. The matroid, $F(G)$, has rank 3, and the ground set of $F(G)$ is $X \cup X' \cup Y$. The non-spanning circuits of $F(G)$ are sets of the form $\{x_i, x'_i\}$, where $i \in \{1, \ldots, n\}$, and the sets

$$\{\{a, b, y_k\} \mid a \in \{x_i, x'_i\},\ b \in \{x_j, x'_j\},\ \text{and } e_k \text{ joins } v_i \text{ to } v_j\}.$$

Thus $F(G)$ can be formed by placing n parallel pairs in the plane in general position, and placing an element on the line between the parallel pairs that correspond to adjacent vertices in such a way that no additional dependencies are formed.

The *bicircular matroid of a graph, G*, denoted by $B(G)$, has the edge set of G as its ground set. The circuits of $B(G)$ are exactly the edge sets of *bicycles of* G, which are the subgraphs of G that are homeomorphic to one of the graphs in Figure 2.4. Thus a set, E, of edges is independent in $B(G)$ if and only if the subgraph of G which is induced by E contains at most one cycle in every connected component. Bicircular matroids have been studied in, for example, [33].

Using bicircular matroids, we may formulate an alternative description of the matroid $F(G)$. Suppose that G is simple and has at least three vertices. Let $G^{\circ\circ}$ be the graph which is obtained by adding two loops at each vertex of G. It is easy to see



Figure 2.4: Bicycles.

that $F(G)$ is isomorphic to the matroid obtained by repeatedly truncating $B(G^{\circ\circ})$ so that its rank is reduced to three.

Proposition 2.5.2. *There exists a polynomial-time Turing machine that computes* $(F(G), \textsc{Input})$ *for any simple graph* G *with at least three vertices, where* Input *is a method of description such that* Independent Sets $\leq$ Input.

Proof. Clearly it will be sufficient to prove the proposition when Input = Independent Sets. Suppose that G contains n vertices and m edges. Then the size of the ground set of $F(G)$ is $2n + m$. Since $r(F(G)) = 3$, there are no more than $\binom{2n+m}{3} + \binom{2n+m}{2} + 2n + m + 1$ independent sets in $F(G)$. This quantity is bounded above by $O(n^6)$ (since G is simple). We may assume that G is described via an adjacency matrix, so the size of the input is $O(n^2)$. Hence an algorithm that simply checked every subset of $E(F(G))$ of size at most three to see if it is independent in $F(G)$ would run in polynomial time. Thus $(F(G),$ Independent Sets$)$ can be constructed in polynomial time. □

Lemma 2.5.3. *If* Independent Sets $\leq$ Input*, then* GRAPH ISOMORPHISM *is polynomially reducible to* MATROID ISOMORPHISM.

Proof. Suppose that G and G' correspond to an instance of GRAPH ISOMORPHISM. It is not difficult to construct a transformation which demonstrates that the isomorphism problem for simple graphs is isomorphism-complete. Indeed, the graph isomorphism problem is often defined only for simple graphs (see, for example, [7]). Thus we will assume that G and G' are both simple. We may also assume that both G and G' contain at least three vertices. By Proposition 2.5.2 it will be sufficient to show that G and G' are isomorphic if and only if $F(G)$ and $F(G')$ are isomorphic.

It is clear that, if G and G' are isomorphic, then $F(G)$ and $F(G')$ are also isomorphic. Suppose that $F(G)$ is isomorphic to $F(G')$. The isomorphism must take parallel pairs of $F(G)$ to parallel pairs of $F(G')$, so it induces a bijection between vertices of G and G'. Two vertices of G are adjacent if and only if there is an element on the line between the two corresponding parallel pairs of $F(G)$; that is, if and only if the union of the two corresponding parallel pairs of $F(G)$ is not a closed set. This is the case if and only if there is an element on the line between the corresponding parallel pairs of $F(G')$, and this is equivalent to there being an edge between the corresponding vertices in G'. Thus G and G' are isomorphic. □

It follows from Lemmas 2.5.1 and 2.5.3 that, if Independent Sets $\leq$ Input, then MATROID ISOMORPHISM is polynomially equivalent to GRAPH ISOMORPHISM, even in the special case that $r(M) = r(M') = 3$. Note that a rank-2 matroid is determined up to isomorphism by the sizes of its parallel classes, so the isomorphism problem is certainly solvable in polynomial time in the case that $r(M) = r(M') = 2$.

It is similarly easy to see that the isomorphism problem for matroids of rank less than two is trivial.

Consider the transformation that takes a simple graph, G, with at least three vertices, to the matroid $(F(G))^*$. Certainly it is the case that G and G' are isomorphic graphs if and only if $(F(G))^*$ and $(F(G'))^*$ are isomorphic matroids. Furthermore, the spanning sets of $(F(G))^*$ are exactly the complements of independent sets of $F(G)$. It follows easily that there exists a Turing machine which, given a description of G, a simple graph with at least three vertices, can construct $((F(G))^*$, Spanning Sets) in polynomial time. Therefore GRAPH ISOMORPHISM is also polynomial reducible to MATROID ISOMORPHISM when Input = Spanning Sets.

We have shown that MATROID ISOMORPHISM can, in general, be solved in polynomial time if and only if GRAPH ISOMORPHISM is in P. The consensus is that GRAPH ISOMORPHISM is unlikely to be solvable in polynomial time, so MATROID ISOMORPHISM is probably a computationally intractable problem.

2.6 Detecting minors

It is a well-known observation of Seymour's [48] that there can be no "good" algorithm for deciding whether a matroid is binary, which is to say, whether or not it has a $U_{2,4}$-minor. That is, if p is a polynomial, then there exists no oracle Turing machine that decides whether a matroid has a $U_{2,4}$-minor in time bounded by $p(n)$ for any n-element matroid.

In contrast to this, we will show that deciding whether a matroid, M, contains a minor isomorphic to some fixed matroid can be done in polynomial time if M is described via a method of input that is either equal to Circuits, or that lies below Circuits in the order defined in Definition 2.3.1. Before we do so we note the following result.

Proposition 2.6.1. *Suppose that M is a matroid on the ground set E, and that* Input $\in$ **INPUT**. *There exists a polynomial-time Turing machine which computes* $(M/X\backslash Y,$ Input$)$, *given* $(M,$ Input$)$ *and disjoint subsets X, $Y \subseteq E$.*

Proof. Let e be an element of E. It is not difficult to show that $|(M\backslash e,$ Input$)| \leq |(M,$ Input$)|$ and $|(M/e,$ Input$)| \leq |(M,$ Input$)|$. From this it follows that, if it is possible to construct both $(M\backslash e,$ Input$)$ and $(M/e,$ Input$)$ in polynomial time, then the Turing machine can construct $(M/X\backslash Y,$ Input$)$ in polynomial time by simply contracting the elements of X and deleting the elements of Y in turn.

In most cases this result follows easily from elementary matroid theory, so we will merely illustrate the method of proof by considering the case when Input is equal to either Non-Spanning Circuits or Cyclic Flats.

Suppose first that Input = Non-Spanning Circuits. If e is not a coloop then the non-spanning circuits of $M\backslash e$ are exactly the non-spanning circuits of M that do not contain e. If e is not a loop, then the non-spanning circuits of M/e are exactly the sets of the form $C-e$, where C is a non-spanning circuit of M that contains e; and all sets C, where C is a non-spanning circuit of M not containing e, such that $r_M(C\cup e) <$

$r(M)$ and $(C-f)\cup e$ is independent for all elements $f \in C$. If e is a coloop or a loop then $M\backslash e = M/e$. It follows easily that both ($M\backslash e$, Non-Spanning Circuits) and (M/e, Non-Spanning Circuits) can be constructed in polynomial time.

Similarly, the cyclic flats of $M\backslash e$ are exactly the cyclic flats of M that do not contain e; and sets of the form $F - e$, where F is a cyclic flat of M that does contain e, such that $r_M(F - (e \cup f)) = r_M(F - e)$ for each element $f \in F - e$. Thus ($M\backslash e$, Cyclic Flats) can be efficiently constructed. Since (M^*, Cyclic Flats) can be found in polynomial time given (M, Cyclic Flats), it follows that a description of $M^*\backslash e$ can be constructed, and thus (M/e, Cyclic Flats) can also be found in polynomial time. □

We will need the following result.

Proposition 2.6.2. *Let N be a matroid and let t be the number of circuits of N, where $t > 0$. If M is a matroid on the ground set E, and M has a minor isomorphic to N on the set $A \subseteq E$, then there exist circuits, $C_1, \ldots, C_t$, of M, such that, if $X = (C_1 \cup \cdots \cup C_t) - A$, and $Y = E - (A \cup X)$, then $M/X\backslash Y \cong N$.*

Proof. Suppose that $M/X'\backslash Y'$ is isomorphic to N, where (X', Y') is a partition of $E - A$. We may assume that X' is independent. There are exactly t circuits, $C_1', \ldots, C_t'$, in $M/X'\backslash Y'$. For each circuit, C_i', there exists a circuit, C_i, of M, such that $C_i \subseteq C_i' \cup X'$ and $C_i' = C_i - X'$. Let X be the set $(C_1 \cup \cdots \cup C_t) - A$ and let Y be $E - (A \cup X)$. It remains to show that $M/X\backslash Y \cong N$. Note that $X \subseteq X'$. If $X = X'$, then we are done, so suppose that x is an element in $X' - X$. Let $M' = M/(X' - x)\backslash(Y' \cup x)$. To complete the proof, it will suffice to show that $M' = M/X'\backslash Y'$. Every set, $C_i' = C_i - X$, is certainly dependent in M'. If $C_i - X$ is not a circuit of M', then it must properly contain a circuit, C, of M'. But C contains a circuit of $M/X'\backslash Y'$, so $C_i - X$ properly contains a circuit, C_j', of $M/X'\backslash Y'$. But $C_i' = C_i - X$ is a circuit of $M/X'\backslash Y'$. This contradiction shows that every circuit, C_i', of $M/X'\backslash Y'$ is also a circuit of M'. Thus, if M' is not isomorphic to N, there must be a circuit, C, of M' that is independent in $M/X'\backslash Y'$. But every circuit of M' contains a circuit of $M/X'\backslash Y'$. Thus we have shown that $M' \cong N$. □

Let N be some fixed matroid. It is a consequence of Propositions 2.6.1 and 2.6.2 that the following problem can, in general, be efficiently solved.

DETECTING AN N-MINOR

Instance: A description, (M, Input), of the matroid M.

Question: Does M have a minor isomorphic to N?

Theorem 2.6.3. *If* Input $\leq$ Circuits, *then* DETECTING AN N-MINOR *is in* P *for any fixed matroid N.*

Proof. It will suffice to prove the result only when Input = Circuits. We will assume that we have the description, (N, Circuits), for if we have access only to a description of N via some other form of input, we may construct (N, Circuits) in some constant time, as N is fixed.

Suppose that the ground set of M is E, where $|E| = n$, and that M contains c circuits, so that $|(M, \textsc{Circuits})| = (n+1)c - 1$. Suppose also that $|E(N)| = s$, and that N has exactly t circuits. We may assume that $t > 0$, for otherwise the problem is trivially in P.

An algorithm to check whether M has an N-minor could simply work its way through every subset, A, of E, such that $|A| = s$, and every set, $\{C_1, \ldots, C_t\}$, of t circuits of M. By Proposition 2.6.1, it is possible to construct $(M/X\backslash Y, \textsc{Circuits})$, where $X = (C_1 \cup \cdots \cup C_t) - A$ and $Y = E - (A \cup X)$, and check whether $M/X\backslash Y$ is isomorphic to N in polynomial time. Proposition 2.6.2 guarantees that if M does have an minor isomorphic to N, then this procedure will find such a minor. Checking whether an isomorphism exists between N and $M/X\backslash Y$ will take some constant time, so the running time of the algorithm is determined by the number of subsets, $A \subseteq E$, and the number of families of circuits we must check. The first quantity is $\binom{n}{s}$, and the second is $\binom{c}{t}$, so the running time of the algorithm is $O(n^s c^t)$. Since s and t are fixed constants, the algorithm runs in polynomial time. □

Deciding whether a matroid has a fixed minor can, in general, be accomplished in polynomial time, much as it can be checked in polynomial time whether a graph has a fixed subgraph. However, the following material shows that deciding whether a matroid has a minor, when that minor is part of the input of the problem, is very unlikely to be solvable in polynomial time.

The following problem is well known to be NP-complete (see, for example, [19]).

> SUBGRAPH ISOMORPHISM
>
> Instance: Two graphs, G and H.
>
> Question: Does G have a subgraph that is isomorphic to H?

We shall show that the following generalisation to matroids is probably intractable.

> MINOR ISOMORPHISM
>
> Instance: Descriptions of the same type, $(M, \textsc{Input})$ and $(N, \textsc{Input})$, of two matroids, M and N.
>
> Question: Does M have a minor that is isomorphic to N?

Theorem 2.6.4. *If* $\textsc{Independent Sets} \leq \textsc{Input}$*, then* MINOR ISOMORPHISM *is* NP*-complete.*

Proof. It is easy to see that MINOR ISOMORPHISM is in NP, for, by Proposition 2.6.1, a certificate that consists of disjoint subsets, X and Y, of $E(M)$, and an isomorphism between $M/X\backslash Y$ and N, can be checked in polynomial time.

We will prove the theorem by showing that the transformation that takes a graph, G, to $F(G)$ provides us with a reduction from SUBGRAPH ISOMORPHISM to MINOR ISOMORPHISM. Suppose that the graphs G and H correspond to an instance of SUBGRAPH ISOMORPHISM. It is not difficult to show that SUBGRAPH ISOMORPHISM remains NP-complete if the input is restricted to simple graphs (indeed,

Garey and Johnson implicitly define SUBGRAPH ISOMORPHISM in terms of simple graphs [19]). Thus we will assume that G and H are simple, and that, moreover, both have at least three vertices. It follows from Proposition 2.5.2 that the proof of the theorem will be completed if we demonstrate that G contains a subgraph isomorphic to H if and only if $F(G)$ contains a minor isomorphic to $F(H)$.

Suppose that G has n vertices, $\{v_1, \ldots, v_n\}$, and m edges, $\{e_1, \ldots, e_m\}$. It is straightforward to see that if G' is a subgraph of G containing vertices $\{v_i \mid i \in I\}$ and edges $\{e_j \mid j \in J\}$, where I and J are subsets of $\{1, \ldots, n\}$ and $\{1, \ldots, m\}$ respectively, then $F(G')$ is isomorphic to the restriction of $F(G)$ to the set

$$\{x_i,\, x_i' \mid i \in I\} \cup \{y_j \mid j \in J\}.$$

Thus, if G has a subgraph isomorphic to H, then $F(G)$ certainly has a minor isomorphic to $F(H)$.

For the converse, let us suppose that N is a minor of $F(G)$, and that f is an isomorphism taking $F(H)$ to N. Note that $r(F(H)) = r(F(G))$, so N must be a restriction of $F(G)$, which is to say that N can be obtained from $F(G)$ without contracting elements. Therefore parallel pairs of N are also parallel pairs of $F(G)$. Hence f induces an injective map from parallel pairs of $F(H)$ to parallel pairs of $F(G)$. By taking the composition of this map with the bijection that takes vertices of H to the corresponding parallel pairs of $F(H)$, and the bijection that takes parallel pairs of $F(G)$ to the corresponding vertices of G, we obtain an injection from the vertices of H to the vertices of G. If two vertices are adjacent in H, then there is an element placed on the line between the corresponding parallel pairs in $F(H)$, and hence in N. This is the case only if there is an element placed on the line between the same parallel pairs in $F(G)$, and in this case, the corresponding vertices of G must be adjacent. This shows that the injective map from vertices of H into vertices of G has the property that the adjacency of two vertices in H implies the adjacency of the corresponding vertices in G. In other words, G has a subgraph that is isomorphic to H. □

Note that the proof of Theorem 2.6.4 implies that MINOR ISOMORPHISM remains NP-complete even when $r(M) = r(N) = 3$ and no line of M or N contains more than three parallel classes. Furthermore, by using duality we can show that G has a subgraph isomorphic to H if and only if $(F(G))^*$ has a minor isomorphic to $(F(H))^*$. We have already noted that we can construct $((F(G))^*$, Spanning Sets) is polynomial time, so MINOR ISOMORPHISM is NP-complete when Input = Spanning Sets.

A *clique* of a graph is a set of pairwise adjacent vertices. The following problem is well known to be NP-complete.

CLIQUE

Instance: An integer, k, and a graph G.

Question: Does G contain a clique of size k?

Often we want to know whether a matroid contains a minor of a specified size from a particular family of matroids. Such problems seem likely to be computationally intractable in general. In particular, using the same techniques as in the last proof, we can construct a reduction from CLIQUE, and thus show that the following problem is NP-complete when INDEPENDENT SETS $\leq$ INPUT.

$F(K_n)$-MINOR

Instance: An integer, n, and a description, (M, INPUT), of a matroid, M.

Question: Does M has a minor isomorphic to $F(K_n)$?

This result would be more interesting if the family $F(K_n)$ arose frequently in matroid theory. This is not the case, so we consider the more natural problem of finding a uniform minor of fixed rank.

$U_{r,n}$-MINOR

Instance: An integer, n, and a description, (M, INPUT), of a matroid, M.

Question: Does M has a minor isomorphic to $U_{r,n}$?

Theorem 2.6.5. *If r is a fixed integer greater than two, and* INDEPENDENT SETS $\leq$ INPUT*, then $U_{r,n}$-*MINOR *is* NP*-complete.*

An *independent set of vertices* in a graph is a set of pairwise non-adjacent vertices. We will prove Theorem 2.6.5 by constructing a reduction from the following NP-complete problem.

INDEPENDENT SET

Instance: An integer, k, and a graph, G.

Question: Does G contain an independent set of vertices of size k?

Proof of Theorem 2.6.5. In the discussion that followed Theorem 2.4.1, we noted that it is possible to decide in polynomial time whether a matroid, M, is uniform, given a word (M, INPUT), where INPUT $\in$ **INPUT**. Therefore it follows from Proposition 2.6.1 that $U_{r,n}$-MINOR is in NP.

Let r be a fixed integer that exceeds two. Suppose that the integer k and the simple graph G are an instance of INDEPENDENT SET. Let m be the number of edges in G. Let t be $\lceil (r-1)/2 \rceil$, and let tG° be the graph obtained by adding a loop to each vertex of G and then replacing each non-loop edge with a path of length t. The matroid, $F_r(G)$, is the bicircular matroid of tG°, repeatedly truncated so that its rank is equal to r.

Checking whether a set of edges of tG° is independent in $B(tG^\circ)$ can certainly be done in polynomial time. Since r is a fixed integer, and no independent set of $F_r(G)$ exceeds r in size, it follows, by analogy with the proof of Proposition 2.5.2, that ($F_r(G)$, INPUT) can be constructed in polynomial time, where INDEPENDENT SETS $\leq$ INPUT.

We complete the proof by showing that G has an independent set of k vertices if and only if $F_r(G)$ has a rank-r uniform minor of size $k + mt$.

First note that every non-loop cycle of tG° contains at least $3t$ edges, so if a bicycle of tG° contains such a cycle, then it has size at least $3t + 1$. This quantity is greater than r, as $r \geq 3$. If a bicycle contains two loops, then either it contains exactly $t + 2$ elements, or its size is at least $2t + 2$. It is straightforward to confirm that $t + 2 \leq r$ and that $2t + 2 > r$. This shows that the non-spanning circuits of $F_r(G)$ are exactly the sets containing the t edges in a path joining two vertices of G and the two loops incident with those vertices.

Suppose that G contains an independent set of k vertices. These vertices correspond to k loops of tG°. The set containing these loops and all the non-loop edges of tG° cannot contain a non-spanning circuit of $F_r(G)$, by the above discussion. Thus restricting $F_r(G)$ to this set of $k + mt$ elements gives a uniform minor. We may assume that $m \geq 3$, so $k + mt$ is certainly no less than r. Therefore $F_r(G)$ contains a rank-r uniform minor with $k + mt$ elements.

For the converse suppose that $F_r(G)$ has a rank-r uniform minor on the set A, where $|A| = k + mt$, and assume that G has no independent set of k vertices. Clearly the number of non-loop edges in A is at most mt. If it is exactly mt, then A contains no pair of loops that are attached to adjacent vertices in G, for otherwise A contains a non-spanning circuit of $F_r(G)$. Hence, in this case, the loops contained in A correspond to an independent set of k vertices in G. Therefore we must assume that the number of non-loop edges in A is less than mt. Suppose that A is chosen so that it contains as many non-loop edges as possible. Since A contains more than k loops, there must be a pair of loops, l and l', in A that are attached to adjacent vertices in G. Therefore one of the t edges that join l to l' in tG° does not belong to A. Let us call this edge e. Then $(A - l) \cup e$ contains no non-spanning circuits of $F_r(G)$, and our assumption on the maximality of A is contradicted. Therefore G has an independent set of k vertices. □

The proof of Theorem 2.6.5 shows that, if $r \geq 3$, and M is a rank-r matroid, then deciding whether M contains a rank-r uniform minor of a specified size is NP-complete. This statement is very unlikely to be true when $r \leq 2$, for finding a largest rank-2 uniform minor in a rank-2 matroid is merely a matter of deleting all loops and then deleting all but one element from each parallel class, and this can clearly be done in polynomial time. Similarly, finding a largest rank-one uniform minor in a rank-one matroid can be done by deleting all loops, and, of course, a rank-zero matroid is necessarily uniform. Thus if r is a fixed integer that does not exceed two, then deciding whether a rank-r matroid has a rank-r uniform minor of specified size is in P. In contrast, deciding whether a matroid (whose rank may exceed r) contains a rank-r uniform minor of specified size may not always be in P when $r \leq 2$. This issue is discussed more fully in Chapter 6.

We observe that $F_r(G)$ has a $U_{r,n}$-minor if and only if $(F_r(G))^*$ has a $U_{n-r,n}$-minor. The spanning sets of $(F_r(G))^*$ are the complements of the independent sets of $F_r(G)$. Thus we can construct $((F_r(G))^*$, Spanning Sets) in polynomial time from $(F_r(G)$, Independent Sets). Thus deciding whether a matroid contains a uniform

minor of specified size and corank is, in general, NP-complete. In particular, it follows that the following problem is NP-complete if SPANNING SETS $\leq$ INPUT, and r is a fixed integer that exceeds two.

$U_{n-r,n}$-MINOR

Instance: An integer, n, and a description, (M, INPUT), of a matroid, M.

Question: Does M have a minor isomorphic to $U_{n-r,n}$?

Chapter 3

Basis graphs

3.1 Introduction

Suppose that M is a matroid and that $\mathcal{B}$ is its set of bases. The *basis graph of* M, which we will denote by $\mathrm{BG}(M)$, has $\mathcal{B}$ as its set of vertices. Two bases, B and B', are adjacent in $\mathrm{BG}(M)$ if and only if their symmetric difference, $B\Delta B' = (B - B') \cup (B' - B)$, has size two. Equivalently B and B' are adjacent if and only if $|B - B'| = |B' - B| = 1$. It follows immediately from the axioms for bases that basis graphs are connected. We shall, when convenient, blur the distinction between a basis of M and a vertex of $\mathrm{BG}(M)$.

Basis graphs have many properties that are interesting in their own right, and they have been subject to a considerable amount of study. Maurer [34] has characterised those graphs that are isomorphic to the basis graph of a matroid. Holzmann, Norton, and Tobey [23], and, independently, Maurer, have studied the information about a matroid that can be derived from its basis graph. It is easy to see that adding a loop or a coloop to a matroid will not change its basis graph, so a basis graph certainly does not uniquely represent a matroid. However, Holzmann, Norton, and Tobey show that a basis graph uniquely defines a loopless and coloopless matroid up to a natural form of equivalence, which we now discuss.

Suppose that M and N are matroids, and that $M = M_1 \oplus \cdots \oplus M_m$ and $N = N_1 \oplus \cdots \oplus N_n$ are the decompositions of M and N into their connected components. We shall say that N is a *generalised dual of* M (equivalently, M is a generalised dual of N, or M and N are generalised duals) if $m = n$, and there exists a permutation, $\pi \in S_m$, such that either $M_i \cong N_{\pi(i)}$ or $M_i \cong N^*_{\pi(i)}$ for all $i \in \{1, \ldots, m\}$. In particular, if M is connected and N is a generalised dual of M, then either $N \cong M$ or $N \cong M^*$. It is clear that the relation of being generalised duals is an equivalence relation.

Suppose that $N = N_1 \oplus \cdots \oplus N_t$ is a generalised dual of $M = M_1 \oplus \cdots \oplus M_t$. There is a natural bijection, which we will denote by $\beta_{M,N}$, between the bases of M and the bases of N. By relabelling, we will assume that, for $i \in \{1, \ldots, t\}$, either $M_i \cong N_i$ or $M_i \cong N_i^*$. Therefore there is a bijection, ρ, between $E(M)$ and $E(N)$, such that ρ restricted to $E(M_i)$ is an isomorphism between M_i and one of N_i or N_i^*.

Define $\beta_{M,N}$ to be the bijection between $\mathcal{B}(M)$ and $\mathcal{B}(N)$, so that if B is a basis of M and $i \in \{1, \ldots, t\}$, then $\beta_{M,N}(B) \cap E(N_i) = \rho(B \cap E(M_i))$ when ρ takes M_i to N_i, and $\beta_{M,N}(B) \cap E(N_i) = E(N_i) - \rho(B \cap E(M_i))$ when ρ takes M_i to N_i^*.

It is not difficult to see that $\beta_{M,N}$ is an isomorphism between $\mathrm{BG}(M)$ and $\mathrm{BG}(N)$. Thus generalised duals have isomorphic basis graphs. Holzmann, Norton, and Tobey showed that a converse result also holds.

Theorem 3.1.1. [23, Theorem 5.3] *Two loopless and coloopless matroids have isomorphic basis graphs if and only if they are generalised duals.*

3.2 Reconstructing the matroid

Suppose that G is isomorphic to some basis graph, $\mathrm{BG}(M)$. A *proper labelling of* G is a bijection $\sigma : V(G) \to \mathcal{B}(N)$, where N is some matroid, such that two vertices are adjacent in G if and only if the symmetric difference of their labels has size two. Note that M and N may not be equal, or, indeed, isomorphic.

Holzmann, Norton, and Tobey [23] remark that, given a graph that is known to be isomorphic to the basis graph of a matroid, it is possible to reconstruct a proper labelling of that graph. This means that, given a description of the basis graph, $\mathrm{BG}(M)$, via, say, its adjacency matrix, we are able to construct the list of bases of M, or, at least, of a matroid that is equivalent to M under generalised duality and the addition or removal of loops and coloops. This method will be of use later in the chapter, and is of intrinsic interest, so we provide details of it now.

We start by stating some results on the structure of basis graphs. Let v be a vertex of a graph G. We will refer to the vertices that are adjacent to v as its *neighbours*. Let $\hat{N}_G(v)$ (or $\hat{N}(v)$, when the context is clear), denote the set containing v and all its neighbours. We will also define $N_G(v)$ (or $N(v)$) to be the set produced by removing v from $\hat{N}_G(v)$. Let v_1 and v_2 be vertices that are in the same connected component of G. The length of a shortest path of G between v_1 and v_2 is denoted by $d_G(v_1, v_2)$ (or by $d(v_1, v_2)$).

Suppose that n is an integer greater than one. The cycle of length n is denoted by C_n. If an extra vertex, adjacent to all other vertices, is added to C_n, the resulting graph is called the *wheel on* $n+1$ *vertices*, and is denoted by W_n.

Lemma 3.2.1. [23, Lemma 3.2] and [34, Lemma 1.4] *Suppose that u and v are two vertices of a basis graph. If $d(u, v) = 2$, then the subgraph induced by u, v, and $N(u) \cap N(v)$ is isomorphic to either C_4, W_4, or the octahedron (see Figure 3.1).*

Corollary 3.2.2. *If u and v are vertices in a basis graph and $d(u, v) = 2$, then there are two non-adjacent vertices in $N(u) \cap N(v)$.*

Proposition 3.2.3. [23, Corollary 3.2.1] *Let $\sigma : V(G) \to \mathcal{B}(N)$ be a proper labelling of a basis graph, G. Suppose that u and v are vertices of G and that $d(u, v) = 2$. Let x and y be two non-adjacent vertices in $N(u) \cap N(v)$. Then*

$$\sigma(v) = (\sigma(x) \cap \sigma(y)) \cup (\sigma(x) - \sigma(u)) \cup (\sigma(y) - \sigma(u)).$$

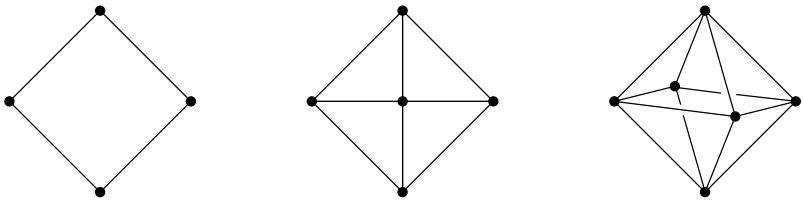


Figure 3.1: C_4, W_4, and the octahedron.

Lemma 3.2.4. [23, Lemma 4.1] *Let v be a vertex in the basis graph of a loopless and coloopless matroid, M. There exist partitions, π and π', of $N(v)$, into non-empty sets, such that:*

(i) If x, $y \in N(v)$, then x and y are adjacent in $\mathrm{BG}(M)$ *if and only if a member of π or π' contains both x and y.*

(ii) If $p \in \pi$ and $q \in \pi'$, then $|p \cap q| \leq 1$.

Proof. Suppose that v corresponds to the basis $B = \{x_1, \ldots, x_r\}$ of M, and that $E(M) - B = \{y_1, \ldots, y_{n-r}\}$, where $r = r(M)$ and $n = |E(M)|$. For $1 \leq i \leq r$, define p_i to be the set $\{B' \in \mathcal{B}(M) \mid B' \cap B = B - x_i\}$, and, for $1 \leq i \leq n - r$, let $q_i = \{B' \in \mathcal{B}(M) \mid B' - B = \{y_i\}\}$. Define $\pi(B, M)$ to be the collection $\{p_1, \ldots, p_r\}$ and $\pi'(B, M)$ to be $\{q_1, \ldots, q_{n-r}\}$. It is easy to verify that $\pi(B, M)$ and $\pi'(B, M)$ are partitions of $N(v)$ that satisfy the conditions of the lemma. □

The partitions discussed in the statement of Lemma 3.2.4 need not be unique. If π and π' are partitions of $N_{\mathrm{BG}(M)}(v)$ that satisfy the conditions of Lemma 3.2.4, then π and π' need not be the same as the natural partitions, $\pi(B, M)$ and $\pi'(B, M)$, defined in the proof of the lemma. However, π and π' must correspond to the natural partitions of some matroid, in fact a generalised dual of M. The next result follows from [23, Lemma 4.1].

Lemma 3.2.5. *Let v be a vertex in the basis graph of a loopless and coloopless matroid, M. Suppose that v corresponds to the basis B. If π and π' are partitions of $N(v)$ that satisfy the conditions of Lemma 3.2.4, then there is a generalised dual, N, of M, such that $\beta_{M,N}(\pi) = \pi(\beta_{M,N}(B), N)$ and $\beta_{M,N}(\pi') = \pi'(\beta_{M,N}(B), N)$, where $\beta_{M,N}$ is the natural bijection between the bases of M and N.*

In the statement of the last result we used the following, obvious, definitions: if $\mathcal{B}_1 \subseteq \mathcal{B}(M)$ is a set of bases of M, then $\beta_{M,N}(\mathcal{B}_1) = \{\beta_{M,N}(B) \mid B \in \mathcal{B}_1\}$; if $\tau = \{\mathcal{B}_1, \ldots, \mathcal{B}_n\}$ is a collection of sets of bases, then $\beta_{M,N}(\tau) = \{\beta_{M,N}(\mathcal{B}_1), \ldots, \beta_{M,N}(\mathcal{B}_n)\}$.

We are now able to describe the procedure that we discussed earlier for constructing a proper labelling of a basis graph. Let G be a graph, and suppose that $G \cong \mathrm{BG}(M)$ for some matroid M. Select a vertex v_0. We wish to find π and π', partitions of $N(v_0)$ that satisfy the conditions of Lemma 3.2.4. If v is any vertex in $N(v_0)$, then Lemma 3.2.4 implies that $N(v_0) \cap \hat{N}(v)$ is the union of two sets, X and

Y, such that $X \cap Y = \{v\}$, and the subgraph induced by X or Y is a clique, but no edge joins a vertex in $X - v$ to a vertex in $Y - v$. We can find the vertex sets of these cliques as follows: let v' be any vertex in $N(v_0) \cap \hat{N}(v)$ that is not equal to v. (If no such vertex exists, then $N(v_0) \cap \hat{N}(v) = \{v\}$, and the vertex sets of both cliques is $\{v\}$.) Then

$$A = N(v_0) \cap \hat{N}(v) \cap \hat{N}(v')$$

is the vertex set of one of the cliques, and

$$B = ((N(v_0) \cap \hat{N}(v)) - A) \cup v$$

is the vertex set of the other.

To find the desired partitions, π and π', we perform the following operation for each connected component, H, of the graph induced by $N(v_0)$: select a vertex, v, of H. Let A and B be the vertex sets of the cliques that make up the graph induced by $N(v_0) \cap \hat{N}(v)$. Add $\{A\}$ to π and $\{B\}$ to π'. We then recursively perform the same operation for every other vertex, v', in H. Suppose that A' and B' are the vertex sets of the cliques that contain v'. Without loss of generality we will suppose that $\{A'\}$ has already been added to either π or π', let us say π. Then we add $\{B'\}$ to π'. If $\{A'\}$ has been added to π', then we add $\{B'\}$ to π. We continue in this fashion until every vertex in H is contained in exactly one member of π and exactly one member of π'.

When this algorithmic procedure is complete, π and π' will satisfy the conditions of Lemma 3.2.4. By Lemma 3.2.5, there exists some generalised dual, N, of M, such that $\pi = \pi(B, N)$ and $\pi' = \pi'(B, N)$, where v_0 corresponds to the basis B of N. Suppose that $\pi = \{p_1, \ldots, p_s\}$ and $\pi' = \{q_1, \ldots, q_t\}$. Let $X = \{x_1, \ldots, x_s\}$ and $Y = \{y_1, \ldots, y_t\}$ be disjoint sets. We will construct a proper labelling that takes vertices of G to subsets of $X \cup Y$. Label v_0 with the set X. If v is a neighbour of v_0, then v is a member of exactly one set in $\{p_1, \ldots, p_s\}$ and exactly one set in $\{q_1, \ldots, q_t\}$. Suppose $v \in p_i \cap q_j$. Label v with the set $(X - x_i) \cup y_j$. Repeat this procedure with all the neighbours of v_0.

The rest of the proper labelling is constructed recursively. Suppose that the labelling has been applied to all the vertices in G that are joined to v_0 by a path of length at most i (where $i \geq 1$), and suppose that v is a vertex of G such that $d(v_0, v) = i + 1$. Let P be a path of length $i + 1$ that joins v_0 to v. Suppose that u is the vertex in P that is of distance two from v, and that x and y are two non-adjacent vertices in $N(u) \cap N(v)$. Since u, x, and y are joined to v_0 by paths of length at most i, they all have labels. The label of v can be found by applying Proposition 3.2.3. By repeating this procedure we arrive at a labelling of the entire graph. Proposition 3.2.3 and Lemma 3.2.5 guarantee that the labelling we derive is a proper labelling. In fact, if N' is the matroid on the set $X \cup Y$ whose bases are the labels of the vertices of G, then $N' \cong N$. It is not difficult to see that the algorithm we have just described can be completed in polynomial time.

3.3 A key lemma

If e is not a coloop of the matroid M, then the bases of $M\backslash e$ are $\{B \mid B \in \mathcal{B}(M),\ e \notin B\}$. Similarly, if e is not a loop, then the bases of M/e are $\{B - e \mid B \in \mathcal{B},\ e \in B\}$. It is clear that, if e is not a coloop of M, then two bases of $M\backslash e$ are adjacent in $\mathrm{BG}(M\backslash e)$ if and only if they are adjacent in $\mathrm{BG}(M)$. Furthermore, if B and B' are distinct bases of M/e, where e is not a loop of M, then B and B' are adjacent in $\mathrm{BG}(M/e)$ if and only if $B \cup e$ and $B' \cup e$ are adjacent bases of $\mathrm{BG}(M)$. Since $M\backslash e = M/e$ if e is a loop or a coloop, the next proposition follows easily from these remarks.

Proposition 3.3.1. *If M and N are matroids, and N is a minor of M, then* $\mathrm{BG}(N)$ *is isomorphic to an induced subgraph of* $\mathrm{BG}(M)$.

It is easy to see that the converse is not true. For example, the basis graph of $U_{2,4}$ is the octahedron, which has an induced subgraph isomorphic to W_4. The matroid obtained by adding a parallel element to $U_{2,3}$ has W_4 as its basis graph, and yet it is clearly not a minor of $U_{2,4}$. (See Figure 3.2.)

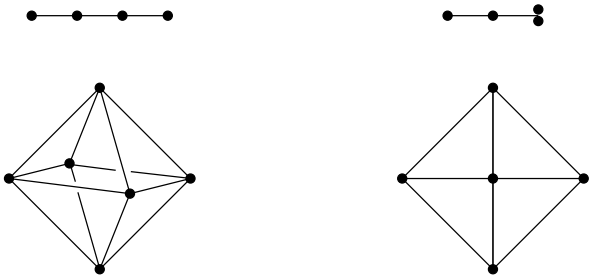


Figure 3.2: Two matroids and their basis graphs.

Although the two matroids in Figure 3.2 are unrelated in the sense that neither is a minor of the other, they are related in another sense, which we now describe.

Suppose that M and N are two matroids such that $r(M) = r(N)$. If N is isomorphic to a matroid N', such that $E(N') = E(M)$, and $\mathcal{B}(N') \subseteq \mathcal{B}(M)$, then we shall say that N is a *rank-preserving weak-map image of* M. This relation is denoted by $M \overset{\text{r.p.}}{\to} N$.

It is clear that the matroid on the right-hand side of Figure 3.2 is a rank-preserving weak-map image of $U_{2,4}$. This illustrates a more general phenomenon, which is true almost by definition.

Proposition 3.3.2. *Let M and N be two matroids. If $M \overset{\text{r.p.}}{\to} N$, then* $\mathrm{BG}(N)$ *is isomorphic to an induced subgraph of* $\mathrm{BG}(M)$.

Thus, if N is a rank-preserving weak-map image of a minor of M, then $\mathrm{BG}(N)$ is isomorphic to an induced subgraph of $\mathrm{BG}(M)$. The converse is also true.

Lemma 3.3.3. *Suppose that M is a matroid. If G is an induced subgraph of* $\mathrm{BG}(M)$, *and G is itself isomorphic to a basis graph, then there exist matroids, M_1 and M_2, such that M_1 is a minor of M, and $M_1 \overset{\text{r.p.}}{\to} M_2$, and, furthermore,* $\mathrm{BG}(M_2) \cong G$.

We remark here that, although this result seems not to appear in the literature, it is almost certainly known.

Proof of Lemma 3.3.3. Let us suppose that $\sigma : V(\mathrm{BG}(M)) \to \mathcal{B}(M)$ is the labelling that maps vertices of $\mathrm{BG}(M)$ to the corresponding bases of M. Suppose also that v_0 is a vertex of G, and that $B = \sigma(v_0) = \{x_1, \ldots, x_r\}$, while $E(M) - B = \{y_1, \ldots, y_{n-r}\}$, where $r = r(M)$ and $n = |E(M)|$.

The partitions, $\pi(B, M) = \{p_1, \ldots, p_r\}$ and $\pi'(B, M) = \{q_1, \ldots, q_{n-r}\}$, of $N_{\mathrm{BG}(M)}(v_0)$ were defined in the proof of Lemma 3.2.4. These partitions naturally induce two partitions on $N_G(v_0)$, although some of the blocks of these induced partitions may be empty. Let us therefore introduce two sets of indices: $I = \{i \mid p_i \cap N_G(v_0) \neq \varnothing,\ 1 \leq i \leq r\}$ and $J = \{j \mid q_j \cap N_G(v_0) \neq \varnothing,\ 1 \leq j \leq n-r,\ \}$. We may now define the partitions $\pi = \{p_i \cap N_G(v_0) \mid i \in I\}$ and $\pi' = \{q_j \cap N_G(v_0) \mid j \in J\}$. It is easy to see that π and π' satisfy the conditions of Lemma 3.2.4.

We now construct a proper labelling of G, exactly as in Section 3.2. Let X be equal to $\{x_i \mid i \in I\}$, and let $Y = \{y_j \mid j \in J\}$. The labelling, τ, will be from $V(G)$ to subsets of $X \cup Y$. Let $\tau(v_0) = X$. If $v \in N_G(v_0)$, then v is in exactly one member of π and exactly one member of π'. If $v \in p_i \cap q_j$, where $i \in I$ and $j \in J$, then label v with $(X - x_i) \cup y_j$. The rest of the labelling is constructed recursively using Proposition 3.2.3, as described earlier. Proposition 3.2.3 and Lemma 3.2.5 guarantee that τ is a proper labelling.

Next we show that the labelling, τ, of G essentially coincides with the labelling, σ, of $\mathrm{BG}(M)$.

3.3.3.1. *If $v \in V(G)$, then $\sigma(v) = \tau(v) \cup (B - X)$.*

Proof. The proof of this claim will be by induction on distance from v_0. The claim is true by construction for all vertices in $\hat{N}_G(v_0)$. Suppose that it is true for all vertices of distance at most i from v_0 (where $i \geq 1$), and suppose that $v \in V(G)$ is such that $d_G(v_0, v) = i + 1$. Let P be a path of length $i + 1$ from v_0 to v, and let u be the vertex in P such that $d_G(u, v) = 2$. Suppose that x and y are two non-adjacent vertices in $N_G(u) \cap N_G(v)$. By the inductive assumption, $\sigma(u) = \tau(u) \cup (B - X)$, $\sigma(x) = \tau(x) \cup (B - X)$, and $\sigma(y) = \tau(y) \cup (B - X)$. Proposition 3.2.3 shows that

$$\begin{aligned}\sigma(v) &= (\sigma(x) \cap \sigma(y)) \cup (\sigma(x) - \sigma(u)) \cup (\sigma(y) - \sigma(u)) \\ &= (\tau(x) \cap \tau(y)) \cup (\tau(x) - \tau(u)) \cup (\tau(y) - \tau(u)) \cup (B - X) \\ &= \tau(v) \cup (B - X).\end{aligned}$$

This proves the claim. □

We have constructed a proper labelling, τ, of G. Let M_2 be the matroid on the ground set $X \cup Y$ that has $\tau(V(G))$ as its set of bases. Let M_1 be the matroid $M/(B-X)\backslash(E(M)-(X\cup Y\cup B))$. It remains only to show that $M_1 \overset{\text{r.p.}}{\to} M_2$. Let B' be any basis of M_2, and let v' be the vertex of G such that $\tau(v') = B'$. Statement 3.3.3.1 implies that $\sigma(v') = B' \cup (B - X)$. It follows that B' is also a basis of M_1. This completes the proof of Lemma 3.3.3. □

We note that the proof of Lemma 3.3.3 can be modified to prove an apparently stronger result.

Corollary 3.3.4. *Suppose that M is a matroid. If G is an induced subgraph of $\mathrm{BG}(M)$, and G is itself isomorphic to a basis graph, then there exists a matroid, M_3, such that $\mathrm{BG}(M_3) \cong G$ and $M \overset{\text{r.p.}}{\to} M_3$.*

Proof. Using the notation of the proof of Lemma 3.3.3, let M_3 be the matroid obtained from M_2 by adding $B - X$ as a set of coloops, and adding $E(M) - (X \cup Y \cup B)$ as a set of loops. Since adding loops and coloops does not affect the basis graph of a matroid, it follows that $\mathrm{BG}(M_3) \cong \mathrm{BG}(M_2) \cong G$. Furthermore, 3.3.3.1 implies that every basis of M_3 is a basis of M. □

3.4 Graph and matroid parameters

This section contains a collection of results that link parameters of basis graphs with parameters of their associated matroids, mainly the size of their ground sets. We shall in particular be interested in numerical parameters of graphs having the property that, if H is an induced subgraph of G, then the parameter associated with H does not exceed that associated with G. The reason for this particular interest will become clear in subsequent sections. Thus we will largely concentrate on the parameters clique number, chromatic number, and maximum degree. First though we shall find an expression for the number of edges in a basis graph in terms of matroid parameters. We will do this by finding an expression for the degree of a vertex.

Suppose that B is a basis of a matroid, M. Recall that, if $e \notin B$, then the fundamental circuit of e with respect to B is denoted by $C(e, B)$.

Proposition 3.4.1. *Let B be a basis of the matroid M, and let $E(M) - B = \{x_1, \ldots, x_{n-r}\}$, where $r = r(M)$ and $n = |E(M)|$. The degree of B in $\mathrm{BG}(M)$ is*

$$\left(\sum_{i=1}^{n-r} |C(x_i, B)|\right) - (n - r).$$

Proof. Any basis that is adjacent to B can be obtained by adding an element, x_i, of $E(M) - B$ to B, and removing an element in $C(x_i, B) - x_i$. The result follows. □

By summing the degrees of all the vertices of $\mathrm{BG}(M)$ and dividing by two, we obtain the following expression for the number of edges in $\mathrm{BG}(M)$.

$$\frac{1}{2}\left(\sum_{\substack{B \in \mathcal{B}(M) \\ e \notin B}} |C(e, B)|\right) - \frac{1}{2}(n - r)|\mathcal{B}(M)|,$$

where $r = r(M)$, and $n = |E(M)|$.

The next result is easily demonstrated.

Proposition 3.4.2. [23, Lemma 3.1] *Let B and B' be bases of the matroid M. Then $d_{\mathrm{BG}(M)}(B,\, B') = |B \Delta B'|/2$.*

Let G be a connected graph. The *diameter of* G, denoted by $\mathrm{diam}(G)$, is $\max_{v,\, v' \in V(G)}\{d_G(v,\, v')\}$.

Corollary 3.4.3. *Let M be a matroid. The diameter of* $\mathrm{BG}(M)$ *is*

$$\max_{I \subseteq E(M)} \{|I| \mid I \text{ is independent in } M \text{ and } M^*\}.$$

Proof. Let B and B' be bases of M such that $d_{\mathrm{BG}(M)}(B,\, B') = \mathrm{diam}(\mathrm{BG}(M))$, and let $t = \max\{|I| \mid I \text{ is independent in } M \text{ and } M^*\}$. Since $B - B'$ is contained in B, and in the complement of B', it is both independent and coindependent. Therefore $\mathrm{diam}(\mathrm{BG}(M)) = |B - B'| \leq t$. Also, if I is a maximum-sized set that is both independent and coindependent, then there exist bases, B and B', such that $I \subseteq B - B'$. Therefore $\mathrm{diam}(\mathrm{BG}(M)) \geq d(B,\, B') = |B - B'| \geq |I| = t$. □

Corollary 3.4.3 illustrates the fact that basis graphs are "dense", in the sense that their diameters are small in comparison to the number of vertices. In fact, we may give a lower bound on the number of vertices in a basis graph that is exponential in terms of its diameter.

Proposition 3.4.4. *Let M be a matroid. Then $|V(\mathrm{BG}(M))| \geq 2^{\mathrm{diam}(\mathrm{BG}(M))}$.*

Proof. Let $t = \mathrm{diam}(\mathrm{BG}(M))$, and let B and B' be bases of M such that $|B - B'| = t$. It is easy to see that if X is any subset of $B - B'$, then there exists a basis, B_X, in $B' \cup X$ such that $B_X - B' = X$. If X and Y are distinct subsets of $B - B'$, then B_X and B_Y must be distinct bases. It immediately follows that M has at least 2^t bases. □

The bound in the last result is the best possible. To see this, let us define $U^t_{r,n}$ to be the matroid which has t connected components, each of which is isomorphic to $U_{r,n}$. Define Q_n, the *rank-n hypercube*, to be the graph that has the set of zero-one vectors of length n as its vertex set, so that two vertices are joined by an edge if and only if their corresponding vectors differ in exactly one place. It is easy to see that the basis graph of $U^n_{1,2}$ is isomorphic to Q_n. Therefore $\mathrm{diam}(\mathrm{BG}(U^n_{1,2})) = n$, and $|V(\mathrm{BG}(U^n_{1,2}))| = 2^n$.

We will now consider cliques in basis graphs. Let $\{B_1, \ldots, B_t\}$ be a collection of bases of the matroid M. Let $X = \cap_{i=1}^t B_i$. We shall say that $\{B_1, \ldots, B_t\}$ has *property I in M* if $|X| = r(M) - 1$, and there exists a set $Y = \{y_1, \ldots, y_t\}$ such that $B_i = X \cup y_i$ for all $i \in \{1, \ldots, t\}$. We shall say that $\{B_1, \ldots, B_t\}$ has *property II in M* if $|X| = r(M) - t + 1$, and there exists a set $Y = \{y_1, \ldots, y_t\}$ such that $B_i = (X \cup Y) - y_i$ for all $i \in \{1, \ldots, t\}$. It is easy to see that a set of bases, $\mathcal{B}$, has property I in M if and only if the corresponding set of cobases, $\mathcal{B}^* = \{E(M) - B \mid B \in \mathcal{B}\}$, has property II in M^*. Dually, $\mathcal{B}$ has property II in M if and only if $\mathcal{B}^*$ has property I in M^*.

It is obvious that a set of bases with property I or II forms a clique in the basis graph. The converse also holds.

Lemma 3.4.5. *Let $\{B_1, \ldots, B_t\}$ (where $t \geq 2$) be a set of distinct bases of M that are pairwise adjacent in* $\mathrm{BG}(M)$. *Then $\{B_1, \ldots, B_t\}$ has either property I or property II.*

Proof. The proof will be by induction on t. If $t = 2$, then $|X| = |B_1 \cap B_2| = r(M)-1$, since B_1 and B_2 are adjacent in $\mathrm{BG}(M)$. Let y_1 be the single element in $B_1 - B_2$ and y_2 the element in $B_2 - B_1$. It is now clear that $\{B_1, B_2\}$ has property I.

Let us suppose that $t \geq 3$, and that the lemma holds for all collections of $t-1$ pairwise adjacent bases. We shall consider the collection $\{B_1, \ldots, B_{t-1}\}$. Suppose that $\{B_1, \ldots, B_{t-1}\}$ has property I. Then $X' = \bigcap_{i=1}^{t-1} B_i$ has cardinality $r(M)-1$, and there exists a set $Y' = \{y'_1, \ldots, y'_{t-1}\}$ such that $B_i = X' \cup y'_i$ for all $i \in \{1, \ldots, t-1\}$. Since B_t is adjacent to B_1, there exist elements, $x \in B_1 - B_t$ and $y \in B_t - B_1$, such that $B_t = (B_1 - x) \cup y$. First assume that $x \in X'$. Then $y'_1 \in B_t$, but $y'_1 \notin B_2$. Since B_t is adjacent to B_2, it follows that $|B_t - B_2| = 1$, so $B_t - B_2 = \{y'_1\}$. Since $y \in B_t - B_1$ it follows that $y \neq y'_1$. Therefore $y \in B_2$, but $y \notin B_1$. Since $B_2 - B_1 = \{y'_2\}$, this implies that $y = y'_2$. It follows that $t = 3$, for, if $t > 3$, then $B_t \neq B_3$, and since $B_t - B_3$ contains both y'_1 and y'_2, the bases B_t and B_3 cannot be adjacent. Make the following definitions: $y_1 = y'_2$; $y_2 = y'_1$; and $y_3 = x$. Also, let $Y = \{y_1, y_2, y_3\}$, and let X be $X' - x = \bigcap_{i=1}^{3} B_i$. We may now observe that $\{B_1, B_2, B_3\}$ has property II.

We will now assume that $x \notin X'$. It follows that $x = y'_1$. Clearly, $y \notin \{y'_1, \ldots, y'_{t-1}\}$. Therefore we may set $y_i = y'_i$ for all $i \in \{1, \ldots, t-1\}$ and $y_t = y$. Then $\{B_1, \ldots, B_t\}$ has property I.

Let us assume that $\{B_1, \ldots, B_{t-1}\}$ has property II. Therefore $\{(E(M) - B_1), \ldots, (E(M) - B_{t-1})\}$ has property I in M^*. We may use the same arguments as above to show that $\{(E(M) - B_1), \ldots, (E(M) - B_t)\}$ has either property I or II in M^*, and hence $\{B_1, \ldots, B_t\}$ has property I or II in M. □

Suppose that B is a basis of a matroid, M. If $e \in B$, then $(E(M) - B) \cup e$ contains a unique cocircuit, the *fundamental cocircuit of e with respect to $E(M) - B$*, denoted by $C^*(e, E(M) - B)$.

Corollary 3.4.6. *Let M be a matroid on the ground set E, and let $\{B_1, \ldots, B_t\}$ be the vertex set of a maximal clique in* $\mathrm{BG}(M)$. *Either there exists a basis, B, and an element $e \in B$, such that $C^*(e, E - B) = \{e_1, \ldots, e_t\}$, and $B_i = (B - e) \cup e_i$ for all $i \in \{1, \ldots, t\}$; or, there exists a basis, B, and an element $e \notin B$, such that $C(e, B) = \{e_1, \ldots, e_t\}$, and $B_i = (B \cup e) - e_i$ for all $i \in \{1, \ldots, t\}$.*

Proof. We will first suppose that $\{B_1, \ldots, B_t\}$ has property I, so that $X = \bigcap_{i=1}^{t} B_i$ has cardinality $r(M)-1$, and there exists a set $Y = \{y_1, \ldots, y_t\}$ such that $B_i = X \cup y_i$ for all $i \in \{1, \ldots, t\}$. Clearly $\mathrm{cl}(X)$ is a hyperplane, and $Y \subseteq E - \mathrm{cl}(X)$. Assume that Y is not equal to $E - \mathrm{cl}(X)$ and let y be an element in $E - (\mathrm{cl}(X) \cup Y)$. Then $X \cup y$ is a basis, distinct from, and adjacent to, the bases $B_1, \ldots, B_t$. This contradicts our assumption on the maximality of the clique. Therefore $Y = E - \mathrm{cl}(X)$. If we take an arbitrary element $e \in Y$, then $B = X \cup e$ is the desired basis, and $C^*(e, E - B) = E - \mathrm{cl}(X) = Y$ is the desired cocircuit.

The case when $\{B_1, \ldots, B_t\}$ has property II is similar. □

Let G be a graph. The *clique number of* G, denoted by $\omega(G)$, is the size of a largest clique in G. If M is a matroid containing at least one circuit, then define $c(M)$ to be the size of a largest circuit. If M has no circuits then define $c(M)$ to be zero. Let $c^*(M) = c(M^*)$. Since every circuit or cocircuit of a matroid is fundamental with respect to some basis or cobasis, the next result follows easily.

Corollary 3.4.7. *Let M be a non-empty matroid. Then* $\omega(\mathrm{BG}(M)) = \max\{c(M), c^*(M)\}$.

The next result follows immediately from Corollary 3.4.7.

Corollary 3.4.8. *Let M be a non-empty matroid. Then* $\omega(\mathrm{BG}(M)) \leq |E(M)|$.

The bound in Corollary 3.4.8 cannot be improved. To see this we need only observe that $\mathrm{BG}(U_{n-1,n}) \cong K_n$ for all $n \geq 1$. Finding a lower bound for $\omega(\mathrm{BG}(M))$ in terms of $|E(M)|$ is less straightforward.

If G and H are two graphs, then the *Cartesian product of G and H*, denoted by $G\Box H$, has $V(G) \times V(H)$ as its vertex set. The vertices, (u, v) and (u', v'), are adjacent in $G\Box H$ if and only if either $u = u'$, and v is adjacent to v' in H; or $v = v'$, and u is adjacent to u' in G. The next proposition is easily proved.

Proposition 3.4.9. *If M and N are two matroids, then* $\mathrm{BG}(M \oplus N) \cong \mathrm{BG}(M)\Box\mathrm{BG}(N)$.

It is clear that for any two graphs, G and H, the clique number of $G\Box H$ is equal to $\max\{\omega(G), \omega(H)\}$. This gives us a way to construct arbitrarily large matroids whose basis graphs have small clique number. For instance, $|E(U^n_{1,2})| = 2n$, but, since $\mathrm{BG}(U^n_{1,2}) \cong Q_n$, it follows that $\omega(\mathrm{BG}(U^n_{1,2})) = 2$.

The situation is more interesting if we restrict ourselves to connected matroids. There has been a considerable amount of attention paid to the problem of how large a connected matroid may be if the sizes of its largest circuit and largest cocircuit are bounded. The best possible such result is due to Lemos and Oxley [30].

Proposition 3.4.10. [30, Theorem 1.4] *Let M be a connected matroid with at least two elements. Then* $|E(M)| \leq \lfloor c(M)c^*(M)/2 \rfloor$.

Corollary 3.4.11. *If M is a connected matroid with at least two elements, then* $\omega(\mathrm{BG}(M)) \geq \sqrt{2|E(M)|}$.

Proof. Corollary 3.4.7 implies that $\omega(\mathrm{BG}(M)) \geq c(M)$ and $\omega(\mathrm{BG}(M)) \geq c^*(M)$. Therefore

$$\frac{1}{2}(\omega(\mathrm{BG}(M)))^2 \geq \frac{1}{2}c(M)c^*(M) \geq \lfloor \frac{1}{2}c(M)c^*(M) \rfloor \geq |E(M)|,$$

and the result follows. □

The bound in Corollary 3.4.11 is sharp in the following sense.

Proposition 3.4.12. *If $n > 1$ is an integer, then there exists a connected matroid, M, such that* $\omega(\mathrm{BG}(M)) = n = \lceil \sqrt{2|E(M)|} \rceil$.

Proof. First suppose that n is even. Let G be the graph obtained from a cycle of length n by replacing each edge with a parallel class of size $n/2$. It is easy to check that the size of a largest circuit or cocircuit of $M(G)$ is n, so $\omega(\mathrm{BG}(M(G))) = n$. Also, $|E(M(G))| = n^2/2$, so $\lceil\sqrt{2|E(M(G))|}\rceil = n$.

In the case where n is odd, let G be obtained from the cycle of length n by replacing each edge with a parallel class of size $(n-1)/2$. Again, it is easy to check that $\omega(\mathrm{BG}(M(G))) = n$. Furthermore, $|E(M(G))| = n(n-1)/2$. Therefore $n^2 > 2|E(M(G))| > (n-1)^2$, so $\lceil\sqrt{2|E(M(G))|}\rceil = n$. □

Let G be a graph. The *chromatic number of* G, denoted by $\chi(G)$, is the least integer, n, such that G has a proper n-colouring. We now turn our attention to this parameter in the context of basis graphs.

Since $\chi(G) \geq \omega(G)$ for all graphs G, Corollary 3.4.11 immediately gives us the following result.

Proposition 3.4.13. *Let M be a connected matroid. Then $\chi(\mathrm{BG}(M)) \geq \sqrt{2|E(M)|}$.*

Estivill-Castro, Noy, and Urrutia [16] provide the following upper bound on $\chi(\mathrm{BG}(M))$ in terms of $|E(M)|$. Their result applies to graphic matroids, but, as observed by the authors, the proof extends to general matroids almost without modification.

Proposition 3.4.14. [16, Theorem 1] *If M is a matroid, then $\chi(\mathrm{BG}(M)) \leq |E(M)|$.*

To see that the bound in Proposition 3.4.14 is the best possible we need only observe that $\mathrm{BG}(U_{n-1,n}) \cong K_n$.

Let $\Delta(G)$ be the maximum degree of the graph G.

Lemma 3.4.15. *Let M be a matroid. If e is not a loop or coloop of M, then $\Delta(\mathrm{BG}(M\backslash e)) < \Delta(\mathrm{BG}(M))$ and $\Delta(\mathrm{BG}(M/e)) < \Delta(\mathrm{BG}(M))$.*

Proof. Proposition 3.3.1 implies that $\mathrm{BG}(M\backslash e)$ is an induced subgraph of $\mathrm{BG}(M)$. Since e is not a coloop, there exist bases that do not contain e, and $\mathrm{BG}(M\backslash e)$ is induced by exactly these bases. If no such basis has maximum degree in $\mathrm{BG}(M)$, then clearly $\Delta(\mathrm{BG}(M\backslash e)) < \Delta(\mathrm{BG}(M))$. Therefore we will assume that B is a basis of M which has maximum degree in $\mathrm{BG}(M)$, and that $e \notin B$. To prove that $\Delta(\mathrm{BG}(M\backslash e)) < \Delta(\mathrm{BG}(M))$, it will suffice to show that B is adjacent to a basis in M that is not a basis of $M\backslash e$. Since e is not a loop, there exists an element $f \in C(e, B) - e$. Then $(B - f) \cup e$ is the desired basis.

The fact that $\Delta(\mathrm{BG}(M/e)) < \Delta(\mathrm{BG}(M))$ follows by duality. □

Proposition 3.4.16. *If M is a connected matroid, then $\Delta(\mathrm{BG}(M)) \geq |E(M)| - 1$.*

Proof. The proof will be by induction on $\Delta(\mathrm{BG}(M))$. If M is a connected matroid, and $\Delta(\mathrm{BG}(M)) = 0$, then $\mathrm{BG}(M) \cong K_1$, and M must be isomorphic to $U_{0,0}$, $U_{0,1}$, or $U_{1,1}$. Note that these matroids are vacuously connected. This provides us with the base case for our induction.

Suppose that the claim is true for all connected matroids, M', such that $\Delta(\mathrm{BG}(M')) < k$ (where $k > 0$). Let M be a connected matroid with $\Delta(\mathrm{BG}(M)) = k$. Let e be an element of M. It is well known that either $M\backslash e$ or M/e must be connected [42, Theorem 4.3.1]. By duality we may assume the former. Since M is connected, e is neither a loop nor a coloop. Lemma 3.4.15 implies that $\Delta(\mathrm{BG}(M\backslash e)) < k$, so $|E(M\backslash e)| \leq k$, by the inductive assumption. Therefore $|E(M)| \leq k + 1$. □

Corollary 3.4.17. *If M is a matroid with t connected components, then $\Delta(\mathrm{BG}(M)) \geq |E(M)| - t$.*

Proof. Suppose $M = M_1 \oplus \cdots \oplus M_t$ is the decomposition of M into connected components. Then $\mathrm{BG}(M) \cong \mathrm{BG}(M_1)\Box \cdots \Box \mathrm{BG}(M_t)$. It is easily seen that $\Delta(G\Box H) = \Delta(G) + \Delta(H)$ for any two graphs, G and H. It follows from Proposition 3.4.16 that

$$\Delta(\mathrm{BG}(M)) = \sum_{i=1}^{t} \Delta(\mathrm{BG}(M_i)) \geq \sum_{i=1}^{t} (|E(M_i)| - 1) = |E(M)| - t. \qquad \Box$$

The bounds in Proposition 3.4.16 and Corollary 3.4.17 cannot be improved. Let $k \geq 0$ and $t \geq 1$ be integers. We can find a matroid, M, with t connected components, such that $\Delta(\mathrm{BG}(M)) = k$, and $|E(M)| = k + t$. Let $x_1, \ldots, x_t$ be any positive integers such that $x_1 + \cdots + x_t = k + t$. Let $M = U_{x_1-1,x_1} \oplus \cdots \oplus U_{x_t-1,x_t}$. Then $|E(M)| = x_1 + \cdots + x_t = k + t$, and $\Delta(\mathrm{BG}(M)) = (x_1 - 1) + \cdots (x_t - 1) = k$.

Proposition 3.4.18. *Let M be a matroid. Then $\Delta(\mathrm{BG}(M)) \leq |E(M)|^2/4$.*

Proof. Let B be any basis of M, and suppose that $E(M) - B = \{x_1, \ldots, x_{n-r}\}$, where $r = r(M)$, and $n = |E(M)|$. Since the size of any circuit is at most $r + 1$, Proposition 3.4.1 shows that the degree of B is at most $(n-r)(r+1)-(n-r) = r(n-r)$. Also, $0 \leq r \leq n$, so $r(n - r) \leq (n/2)^2$. □

Once again, the bound in Proposition 3.4.18 is the best possible. Let t be any non-negative integer. Then $\mathrm{BG}(U_{t,2t})$ is regular with degree t^2. Also, $|E(U_{t,2t})|^2/4 = (2t)^2/4 = t^2$.

3.5 Graph parameters define classes of matroids

The study of the graph parameters of basis graphs leads naturally to questions such as: what are the matroids with bipartite basis graphs? More generally, we shall investigate families consisting of matroids, M, such that some parameter of $\mathrm{BG}(M)$, clique number, chromatic number, or maximum degree, is bounded above by an integer, k. For small values of k, we will explicitly determine these classes.

Define $\mathcal{M}(\omega \leq k)$ to be the set of matroids that have no clique larger than k in their basis graphs. Similarly, define $\mathcal{M}(\chi \leq k)$ to be the set $\{M \mid M \text{ is a matroid and } \chi(\mathrm{BG}(M)) \leq k\}$, and define $\mathcal{M}(\Delta \leq k)$ to be $\{M \mid M \text{ is a matroid and } \Delta(\mathrm{BG}(M)) \leq k\}$.

We start by observing some obvious inclusions.

Proposition 3.5.1. *Let k be a positive integer. Then $\mathcal{M}(\chi \leq k) \subseteq \mathcal{M}(\omega \leq k)$, and $\mathcal{M}(\Delta \leq k-1) \subseteq \mathcal{M}(\omega \leq k)$.*

Finding the clique number or chromatic number of a graph is a computationally difficult task, so we would expect that, in general, determining whether a matroid belongs to one of the classes $\mathcal{M}(\omega \leq k)$ or $\mathcal{M}(\chi \leq k)$ will not be straightforward. We shall attempt to solve a special case of this problem, and determine when a uniform matroid belongs to one of the classes $\mathcal{M}(\omega \leq k)$, $\mathcal{M}(\chi \leq k)$, or $\mathcal{M}(\Delta \leq k)$. The next result follows from the observation that if $0 < r < n$, then the clique number of $\mathrm{BG}(U_{r,n})$ is $\max\{r+1, n-r+1\}$.

Proposition 3.5.2. *Let k be a positive integer. Then $U_{0,n}$ and $U_{n,n}$ belong to $\mathcal{M}(\omega \leq k)$ for all values of n. If $0 < r < n$, then $U_{r,n}$ belongs to $\mathcal{M}(\omega \leq k)$ if and only if $r \leq k-1$, and $n-r \leq k-1$.*

Proposition 3.5.3. *Let k be a positive integer. Then $U_{r,n} \in \mathcal{M}(\Delta \leq k)$ if and only if $r(n-r) \leq k$.*

Proof. If $r < n$, then every circuit in $U_{r,n}$ has size $r+1$, so it follows from Proposition 3.4.1 that $\mathrm{BG}(U_{r,n})$ is regular with degree $(n-r)(r+1)-(n-r) = r(n-r)$. If $r = n$, then $\mathrm{BG}(U_{r,n}) \cong K_1$, so $\mathrm{BG}(U_{r,n})$ is again regular with degree $r(n-r)$. The result follows. □

Analogous results for the chromatic number are much more difficult to obtain. The basis graphs of uniform matroids are also known as *Johnson graphs*, and have been well studied in non-matroidal contexts. Etzion and Bitan provide a survey of the work done on the chromatic number of Johnson graphs in [17]. We note here only the following.

Proposition 3.5.4. *Let k be a positive integer. If k is even, then $U_{2,n} \in \mathcal{M}(\chi \leq k)$ if and only if $n \leq k$. If k is odd, then $U_{2,n} \in \mathcal{M}(\chi \leq k)$ if and only if $n \leq k+1$.*

We now concentrate on characterising the classes $\mathcal{M}(\omega \leq k)$, $\mathcal{M}(\chi \leq k)$, and $\mathcal{M}(\Delta \leq k)$ for some small values of k. The proofs of these results are routine, but often lengthy, so we relegate most of them to Appendix A.

The classes of graphs that have clique number, chromatic number, or maximum degree at most k are all closed under induced subgraphs. Therefore Proposition 3.3.1 implies the following.

Theorem 3.5.5. *Let k be an integer. The classes, $\mathcal{M}(\omega \leq k)$, $\mathcal{M}(\chi \leq k)$, and $\mathcal{M}(\Delta \leq k)$ are all closed under minors.*

Therefore we can characterise these classes via their excluded minors. Our analysis will be simplified by the following proposition, which follows from the comments preceding Theorem 3.1.1.

Proposition 3.5.6. *Let k be an integer. The classes, $\mathcal{M}(\omega \leq k)$, $\mathcal{M}(\chi \leq k)$, and $\mathcal{M}(\Delta \leq k)$ are all closed under generalised duality.*

It has already been noted that $\omega(G\Box H) = \max\{\omega(G), \omega(H)\}$ for all graphs G and H. It is less obvious that $\chi(G\Box H) = \max\{\chi(G), \chi(H)\}$. The first proof of this fact is due to Sabidussi [46]. The next result follows from this discussion, and from Proposition 3.4.9.

Proposition 3.5.7. *Let k be an integer. The classes $\mathcal{M}(\omega \leq k)$ and $\mathcal{M}(\chi \leq k)$ are closed under direct sums, and therefore the excluded minors for these classes are connected.*

Clearly, $\mathcal{M}(\Delta \leq k)$ need not be closed under direct sums.

The class $\mathcal{M}(\omega \leq k)$ turns out to be especially easy to characterise in terms of its excluded minors.

Theorem 3.5.8. *Let k be a positive integer. A matroid belongs to $\mathcal{M}(\omega \leq k)$ if and only if it has no minor isomorphic to $U_{1,k+1}$ or $U_{k,k+1}$.*

Proof. Let M be a matroid. Note that $\mathrm{BG}(U_{1,k+1}) \cong \mathrm{BG}(U_{k,k+1}) \cong K_{k+1}$, so if M has a $U_{1,k+1}$- or $U_{k,k+1}$-minor, then Proposition 3.3.1 implies that the clique number of $\mathrm{BG}(M)$ is at least $k+1$, and hence $M \notin \mathcal{M}(\omega \leq k)$.

Suppose that M is not a member of $\mathcal{M}(\omega \leq k)$. Corollary 3.4.7 implies that M contains a circuit or cocircuit with at least $k+1$ elements, and thus M has a minor isomorphic to either $U_{1,k+1}$ or $U_{k,k+1}$. □

There seems to be no obvious way to explicitly characterise the uniform excluded minors for $\mathcal{M}(\Delta \leq k)$. However, we will make use of the following result.

Proposition 3.5.9. *Let k be a positive integer, and let $x_1, \ldots, x_t$ be integers such that $x_i > 1$ for all $i \in \{1, \ldots, t\}$, and $x_1 + \cdots + x_t = k + t + 1$. Then the matroid $M = \bigoplus_{i=1}^{t} U_{1,x_i}$ is an excluded minor for $\mathcal{M}(\Delta \leq k)$.*

Proof. The basis graph of M is isomorphic to $K_{x_1}\Box \cdots \Box K_{x_t}$ and therefore $\Delta(\mathrm{BG}(M)) = (x_1 - 1) + \cdots + (x_t - 1) = k + 1$. Hence $M \notin \mathcal{M}(\Delta \leq k)$. There are no loops or coloops in M, and Lemma 3.4.15 implies that a loopless and coloopless matroid with a basis graph that has maximum degree $k+1$ must be an excluded minor for $\mathcal{M}(\Delta \leq k)$. □

The class $\mathcal{M}(\Delta \leq k)$ has a property seemingly not common amongst minor-closed classes: every loopless and coloopless matroid in the class is a minor of an excluded minor for the class.

Proposition 3.5.10. *Suppose that M is a loopless and coloopless matroid and that $M \in \mathcal{M}(\Delta \leq k)$. Then there exists a matroid, M', such that M' is an excluded minor for $\mathcal{M}(\Delta \leq k)$ and M is a minor of M'.*

Proof. Let t be equal to $k - \Delta(\mathrm{BG}(M))$. Note that $t \geq 0$. Define M' to be $M \oplus U_{1,2}^{t+1}$. The maximum degree of $\mathrm{BG}(M')$ is $(t+1)\Delta(\mathrm{BG}(U_{1,2})) + \Delta(\mathrm{BG}(M)) = k + 1$, so $M' \notin \mathcal{M}(\Delta \leq k)$. Since M has no loops or coloops, M' is also loopless and coloopless. Therefore Lemma 3.4.15 implies that M' is an excluded minor for $\mathcal{M}(\Delta \leq k)$. □

We first note that the only connected matroids in $\mathcal{M}(\omega \leq 1)$ and $\mathcal{M}(\chi \leq 1)$ are $U_{0,1}$ and $U_{1,1}$. (For the sake of brevity in the statement of later results we will consistently ignore the empty matroid, $U_{0,0}$.) The single excluded minor for these classes is therefore $U_{1,2}$. The only loopless and coloopless matroid in $\mathcal{M}(\Delta \leq 1)$ is $U_{1,2}$. Therefore the excluded minors for $\mathcal{M}(\Delta \leq 1)$ are $U_{1,3}$, $U_{2,3}$, and $U_{1,2}^2$.

Proposition 3.5.11. *The class $\mathcal{M}(\chi \leq 2)$ is equal to the class $\mathcal{M}(\omega \leq 2)$, and therefore the only excluded minors for $\mathcal{M}(\omega \leq 2) = \mathcal{M}(\chi \leq 2)$ are $U_{1,3}$ and $U_{2,3}$. The only connected matroids in $\mathcal{M}(\omega \leq 2)$ and $\mathcal{M}(\chi \leq 2)$ are, up to duality, $U_{0,1}$ and $U_{1,2}$.*

Proof. Since $\mathrm{BG}(U_{1,3}) \cong K_3$, it is easy to see that $U_{1,3}$ and $U_{2,3}$ are excluded minors for $\mathcal{M}(\chi \leq 2)$. Therefore, if M is a connected member of $\mathcal{M}(\omega \leq 2)$ or $\mathcal{M}(\chi \leq 2)$, then $c(M)$, $c^*(M) \leq 2$. Hence $|E(M)| \leq 2$ by Proposition 3.4.10. The result follows. □

Proposition 3.5.11 answers the question posed at the beginning of this section. The only matroids with bipartite basis graphs can be formed by adding loops or coloops to an isomorphic copy of $U_{1,2}^n$, for some n. Therefore the only bipartite basis graphs are those formed from copies of K_2 using the Cartesian product, in other words, the hypercubes.

Proposition 3.5.12. [Proposition A.1.3] *The excluded minors for $\mathcal{M}(\Delta \leq 2)$ are $U_{1,4}$, $U_{2,4}$, $U_{1,2} \oplus U_{1,3}$, $U_{1,2}^3$, the matroid $M(W_2)$, shown in Figure 3.3, and their generalised duals.*

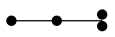


Figure 3.3: $M(W_2)$, an excluded minor for $\mathcal{M}(\Delta \leq 2)$.

Proposition 3.5.13. [Proposition A.1.4] *The class $\mathcal{M}(\chi \leq 3)$ is equal to $\mathcal{M}(\omega \leq 3)$, and the only excluded minors for this class are $U_{1,4}$ and $U_{3,4}$. The only connected matroids in $\mathcal{M}(\chi \leq 3) = \mathcal{M}(\omega \leq 3)$ are, up to duality, $U_{0,1}$, $U_{1,2}$, $U_{1,3}$, $U_{2,4}$, and $M(W_2)$.*

Proposition 3.5.14. [Proposition A.1.5] *The excluded minors for $\mathcal{M}(\Delta \leq 3)$ are $U_{1,5}$, $U_{2,4}$, $U_{1,2} \oplus U_{1,4}$, $U_{1,3}^2$, $U_{1,2}^2 \oplus U_{1,3}$, $U_{1,2}^4$, $M(W_2)$, and their generalised duals.*

Proposition 3.5.15. [Proposition A.1.6] *The excluded minors for $\mathcal{M}(\chi \leq 4)$ are, up to duality, $U_{1,5}$, $U_{2,5}$, $M(K_4)$, and the matroid N_1, shown in Figure 3.4. The only connected matroids in $\mathcal{M}(\chi \leq 4)$ are $U_{0,1}$, $U_{1,2}$, $U_{1,3}$, $U_{1,4}$, $U_{2,4}$, the matroids shown in Figure 3.5, and their duals.*

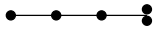


Figure 3.4: N_1, an excluded minor for $\mathcal{M}(\chi \leq 4)$.

Figure 3.5 shows eleven graphic matroids. Each is illustrated with a geometric representation and a graph directly beneath, which represents the same matroid.

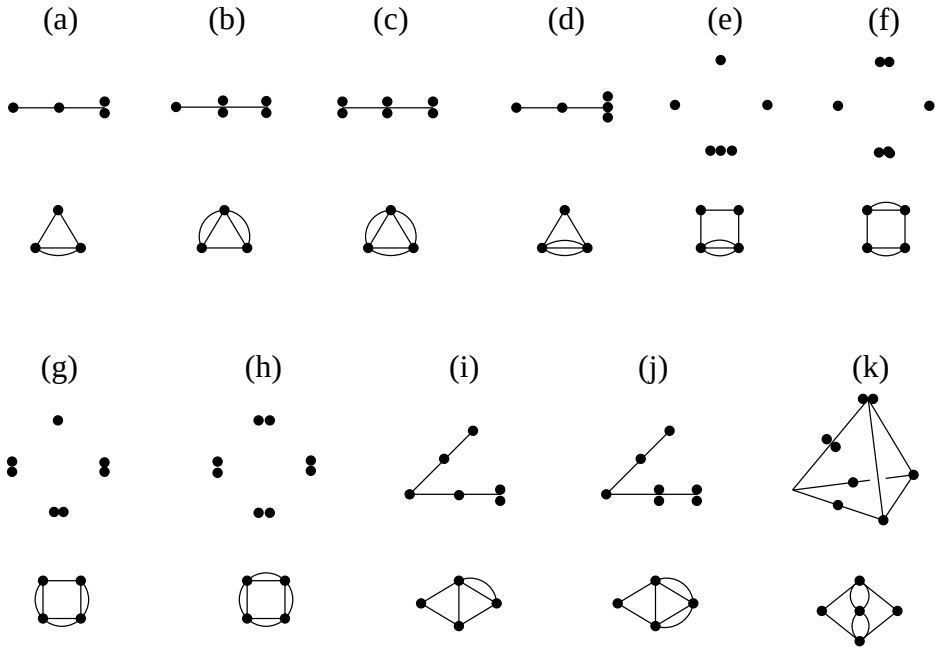


Figure 3.5: Matroids in $\mathcal{M}(\chi \leq 4)$.

Proposition 3.5.16. [Proposition A.1.7] *The excluded minors for $\mathcal{M}(\Delta \leq 4)$ are, up to generalised duality, $U_{1,6}$, $U_{2,5}$, $U_{1,2} \oplus U_{1,5}$, $U_{1,3} \oplus U_{1,4}$, $U_{1,2} \oplus U_{2,4}$, $U_{1,2} \oplus M(W_2)$, $U_{1,2}^2 \oplus U_{1,4}$, $U_{1,2} \oplus U_{1,3}^2$, $U_{1,2}^3 \oplus U_{1,3}$, $U_{1,2}^5$, N_1 (shown in Figure 3.4), and the matroids N_2 and N_3 shown in Figure 3.6.*



Figure 3.6: N_2 and N_3, excluded minors for $\mathcal{M}(\Delta \leq 4)$.

Proposition 3.5.10 tells us that the loopless and coloopless matroids in $\mathcal{M}(\Delta \leq k)$ are exactly the proper minors of the excluded minors for the class. Thus, by using the preceding propositions, we may explicitly characterise the classes $\mathcal{M}(\Delta \leq k)$ for $1 \leq k \leq 4$ by listing all their loopless and coloopless members; Table 3.1 does so up to generalised duality.

3.6 Induced classes of matroids

The ideas of the last section can be generalised much further. Suppose that $\mathcal{P}$ is a family of graphs that is closed under isomorphism. Such a family is known as a *graph property*. Define $\mathcal{M}(\mathcal{P})$ to be the set $\{M \mid M \text{ is a matroid and } \mathrm{BG}(M) \in \mathcal{P}\}$. For instance, $\mathcal{M}(\chi \leq k)$ is exactly the family $\mathcal{M}(\mathcal{P}_k)$, where $\mathcal{P}_k$ is the set of graphs that can be properly k-coloured.

k	Excluded minors of $\mathcal{M}(\Delta \leq k)$	Members of $\mathcal{M}(\Delta \leq k)$
1	$U_{1,3}$, $U_{1,2}^2$	$U_{1,2}$
2	$U_{1,4}$, $U_{2,4}$, $M(W_2)$ $U_{1,2} \oplus U_{1,3}$, $U_{1,2}^3$	$U_{1,2}$, $U_{1,3}$, $U_{1,2}^2$
3	$U_{1,5}$, $U_{2,4}$, $M(W_2)$ $U_{1,2} \oplus U_{1,4}$, $U_{1,3}^2$ $U_{1,2}^2 \oplus U_{1,3}$, $U_{1,2}^4$	$U_{1,2}$, $U_{1,3}$, $U_{1,4}$ $U_{1,2}^2$, $U_{1,2} \oplus U_{1,3}$ $U_{1,2}^3$
4	$U_{1,6}$, $U_{2,5}$, N_1, N_2, N_3 $U_{1,2} \oplus U_{1,5}$, $U_{1,3} \oplus U_{1,4}$ $U_{1,2} \oplus U_{2,4}$, $U_{1,2} \oplus M(W_2)$ $U_{1,2}^2 \oplus U_{1,4}$, $U_{1,2} \oplus U_{1,3}^2$ $U_{1,2}^3 \oplus U_{1,3}$, $U_{1,2}^5$	$U_{1,2}$, $U_{1,3}$, $U_{1,4}$, $U_{1,5}$, $U_{2,4}$, $M(W_2)$, $U_{1,2}^2$, $U_{1,2} \oplus U_{1,3}$ $U_{1,2} \oplus U_{1,4}$, $U_{1,3}^2$, $U_{1,2}^3$ $U_{1,2}^2 \oplus U_{1,3}$, $U_{1,2}^4$

Table 3.1: $\mathcal{M}(\Delta \leq k)$, for $1 \leq k \leq 4$.

Suppose that $\mathcal{P}$ is a graph property. It is clear that $\mathcal{M}(\mathcal{P})$ is closed under the addition of loops or coloops. The following result is a generalisation of Proposition 3.5.6 and is a consequence of the remarks preceding Theorem 3.1.1.

Proposition 3.6.1. *Let $\mathcal{P}$ be a graph property. The set $\mathcal{M}(\mathcal{P})$ is closed under generalised duality.*

The fact that $\mathcal{M}(\omega \leq k)$ and $\mathcal{M}(\chi \leq k)$ are both closed under direct sums is generalised in the next result.

Proposition 3.6.2. *Suppose that $\mathcal{P}$ is a graph property that is closed under Cartesian products. Then $\mathcal{M}(\mathcal{P})$ is closed under direct sums.*

We have already noted that $\mathcal{M}(\omega \leq k)$, $\mathcal{M}(\chi \leq k)$ and $\mathcal{M}(\Delta \leq k)$ are closed under minors. Proposition 3.3.2 implies that they are also closed under rank-preserving weak maps. This is an example of a more general phenomenon.

Proposition 3.6.3. *Suppose that $\mathcal{P}$ is a graph property that is closed under induced subgraphs. Then $\mathcal{M}(\mathcal{P})$ is closed under minors and rank-preserving weak maps.*

If $\mathcal{P}$ is a graph property that is also closed under induced subgraphs, then we shall say $\mathcal{P}$ is an *hereditary class*. In this case we will say that $\mathcal{M}(\mathcal{P})$ is *induced* by $\mathcal{P}$. If $\mathcal{M}$ is a family of matroids, and there exists some hereditary class, $\mathcal{P}$, such that $\mathcal{M} = \mathcal{M}(\mathcal{P})$, then we shall say that $\mathcal{M}$ is an *induced class*.

We will illustrate the idea of induced families of matroids by characterising $\mathcal{M}(\textsc{Planar})$, where Planar is the set of planar graphs.

Proposition 3.6.4. [Proposition A.1.8] *The excluded minors for $\mathcal{M}(\textsc{Planar})$ are $U_{1,5}$, $U_{2,5}$, N_1, N_2, N_3, $U_{1,2} \oplus U_{1,4}$, $U_{1,2} \oplus U_{2,4}$, $U_{1,3}^2$, $U_{1,2} \oplus M(W_2)$, $U_{1,2}^2 \oplus U_{1,3}$, $U_{1,2}^4$, and*

their generalised duals. Here the matroids $M(W_2)$, N_1, N_2, and N_3 are as illustrated in Figures 3.3, 3.4 and 3.6. The only loopless and coloopless matroids in $\mathcal{M}(\text{PLANAR})$ are, up to generalised duality, $U_{1,2}$, $U_{1,3}$, $U_{1,4}$, $U_{2,4}$, $M(W_2)$, $U^2_{1,2}$, $U_{1,2} \oplus U_{1,3}$, and $U^3_{1,2}$.

The proof of Proposition 3.6.4 can be found in Appendix A. Figure 3.7 shows the basis graphs of, respectively, $U_{0,1}$, $U_{1,2}$, $U_{1,3}$, $U^2_{1,2}$, $U_{1,4}$, $M(W_2)$, $U_{2,4}$, $U_{1,2} \oplus U_{1,3}$, and $U^3_{1,2}$.

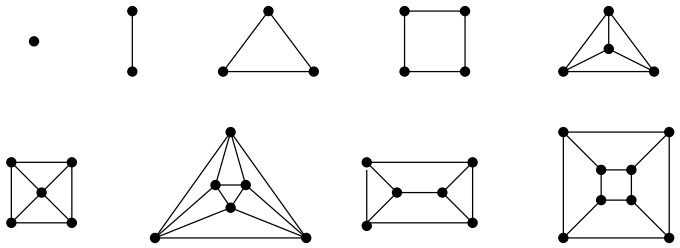


Figure 3.7: The only planar basis graphs.

We observe that none of the examples of induced classes of matroids that we have seen so far contains large connected matroids. This raises the question of exactly when an induced class of matroids does contain large connected members. The result of Lemos and Oxley supplies us with an answer. Informally speaking, an induced class, $\mathcal{M}(\mathcal{P})$, contains arbitrarily large connected matroids if and only if the hereditary class $\mathcal{P}$ contains arbitrarily large complete graphs.

Theorem 3.6.5. *Suppose that $\mathcal{P}$ is an hereditary class of graphs. The induced class, $\mathcal{M}(\mathcal{P})$, contains a connected matroid of size at least k, for every integer k, if and only if $\mathcal{P}$ contains K_n for every $n \geq 1$.*

Proof. First suppose that $\mathcal{P}$ contains K_n for every $n \geq 1$. Then $\mathcal{M}(\mathcal{P})$ contains $U_{1,n}$ for every $n \geq 1$, since $\text{BG}(U_{1,n}) \cong K_n$. Therefore $\mathcal{M}(\mathcal{P})$ contains a connected matroid of size at least k, for every integer k.

For the converse, suppose that $\mathcal{P}$ does not contain every complete graph. Then there must exist an integer, s, such that $K_s \in \mathcal{P}$, but K_t is not a member of $\mathcal{P}$ for all integers $t > s$. Suppose that $M \in \mathcal{M}(\mathcal{P})$, and that M has a circuit or a cocircuit of size t, where $t > s$. Then Corollary 3.4.7 implies that $\text{BG}(M)$ contains a clique of size at least t. Since $\text{BG}(M) \in \mathcal{P}$ by definition of $\mathcal{M}(\mathcal{P})$, and $\mathcal{P}$ is closed under induced subgraphs, this implies that $\mathcal{P}$ contains K_t. This contradiction implies that no matroid in $\mathcal{M}(\mathcal{P})$ contains a circuit or cocircuit of size more than s. Proposition 3.4.10 therefore implies that any connected matroid in $\mathcal{M}(\mathcal{P})$ has size at most $\lfloor s^2/2 \rfloor$. □

Any induced class of matroids is necessarily closed under the addition of loops or coloops, generalised duality, minors, and rank-preserving weak maps. It is an attractive fact that these simple necessary properties of induced classes are also sufficient.

Theorem 3.6.6. *Suppose that $\mathcal{M}$ is a family of matroids that is closed under isomorphism and taking minors. The following conditions are equivalent.*

(i) The family $\mathcal{M}$ is closed under the addition of loops or coloops, generalised duality, and rank-preserving weak maps.

(ii) All the excluded minors for $\mathcal{M}$ are loopless and coloopless. Whenever N is an excluded minor of $\mathcal{M}$, and N' is a generalised dual of N, then N' is also an excluded minor for $\mathcal{M}$. Furthermore, if $N'' \overset{\text{r.p.}}{\to} N$, then $N'' \notin \mathcal{M}$.

(iii) There exists an hereditary class, $\mathcal{P}$, such that $\mathcal{M} = \mathcal{M}(\mathcal{P})$.

Proof. The fact that induced classes of matroids are closed under generalised duality and rank-preserving weak maps is implied by Propositions 3.6.1 and 3.6.3. Furthermore the addition of loops or coloops does not affect the basis graph of a matroid. Therefore (iii) implies (i).

To show that (i) implies (ii) let us assume that $\mathcal{M}$ is closed under the addition of loops and coloops, generalised duality, and rank-preserving weak maps. Let $\mathrm{EX}(\mathcal{M})$ be the set of excluded minors for $\mathcal{M}$. Suppose that $N \in \mathrm{EX}(\mathcal{M})$. It is easy to see that N must be loopless and coloopless. If N' is a generalised dual of N, then N' cannot be in $\mathcal{M}$, for $\mathcal{M}$ is closed under generalised duality. Any proper minor of N' is a generalised dual of a proper minor of N, and must therefore be a member of $\mathcal{M}$. Thus $N' \in \mathrm{EX}(\mathcal{M})$.

Suppose that N'' is a matroid such that $N'' \overset{\text{r.p.}}{\to} N$. If $N'' \in \mathcal{M}$, then the fact that $\mathcal{M}$ is closed under rank-preserving weak maps implies that N too is a member of $\mathcal{M}$. This contradiction shows that $N'' \notin \mathcal{M}$.

To complete the proof we show that (ii) implies (iii). Suppose that (ii) holds. Let $\mathcal{BG}$ be the set $\{\mathrm{BG}(N) \mid N \in \mathrm{EX}(\mathcal{M})\}$. Define the hereditary class, $\mathcal{P}$, so that $G \in \mathcal{P}$ if and only if no induced subgraph of G is isomorphic to a member of $\mathcal{BG}$. We wish to show that $\mathcal{M} = \mathcal{M}(\mathcal{P})$. First suppose that $M \notin \mathcal{M}$. Then there must exist a matroid, $N \in \mathrm{EX}(\mathcal{M})$, such that M has an N-minor. Therefore $\mathrm{BG}(M)$ has an induced subgraph isomorphic to $\mathrm{BG}(N)$. Hence $\mathrm{BG}(M) \notin \mathcal{P}$ and $M \notin \mathcal{M}(\mathcal{P})$. From this we conclude that $\mathcal{M}(\mathcal{P}) \subseteq \mathcal{M}$.

Now suppose that $M \notin \mathcal{M}(\mathcal{P})$. Then there must exist a graph, $G \in \mathcal{BG}$, such that $\mathrm{BG}(M)$ contains an induced subgraph isomorphic to G. Let N be a member of $\mathrm{EX}(\mathcal{M})$ such that $\mathrm{BG}(N) \cong G$. Lemma 3.3.3 implies that there exist matroids, M_1 and M_2, such that M_1 is a minor of M and $M_1 \overset{\text{r.p.}}{\to} M_2$, while $\mathrm{BG}(M_2) \cong \mathrm{BG}(N) \cong G$. We may assume that $E(M_1) = E(M_2)$, and that $\mathcal{B}(M_2) \subseteq \mathcal{B}(M_1)$. Let L be the set of loops of M_2, and L^* the set of coloops. It is not difficult to see that $M_1/L^*\backslash L \overset{\text{r.p.}}{\to} M_2/L^*\backslash L$. Furthermore, $\mathrm{BG}(M_2/L^*\backslash L) \cong \mathrm{BG}(M_2) \cong \mathrm{BG}(N)$. Since both $M_2/L^*\backslash L$ and N are loopless and coloopless, Theorem 3.1.1 implies that $M_2/L^*\backslash L$ is a generalised dual of N, and must therefore be an excluded minor for $\mathcal{M}$. As $M_1/L^*\backslash L \overset{\text{r.p.}}{\to} M_2/L^*\backslash L$, it follows that $M_1/L^*\backslash L$, and therefore M, is not a member of $\mathcal{M}$. Hence $\mathcal{M} \subseteq \mathcal{M}(\mathcal{P})$, and the proof is complete. □

Suppose that $\mathcal{P}$ is an hereditary class of graphs. The graph G is a *forbidden subgraph for* $\mathcal{P}$ if $G \notin \mathcal{P}$, but all the proper induced subgraphs of G are in $\mathcal{P}$. The proof of Theorem 3.6.6 shows us that if a family of matroids, $\mathcal{M}$, is closed under

isomorphism, the addition of loops and coloops, generalised duality, minors, and rank-preserving weak maps, then $\mathcal{M} = \mathcal{M}(\mathcal{P})$, where the forbidden subgraphs of $\mathcal{P}$ are exactly the basis graphs of the excluded minors of $\mathcal{M}$. Thus, if we know the excluded minors of $\mathcal{M}$, we can describe an hereditary class, $\mathcal{P}$, such that $\mathcal{M} = \mathcal{M}(\mathcal{P})$, via its forbidden subgraphs. The converse is not quite so simple. If we know the forbidden subgraphs of $\mathcal{P}$, then we can do little more than repeat the definition, and say that the excluded minors of $\mathcal{M}(\mathcal{P})$ are those minor-minimal matroids that contain a forbidden subgraph in their basis graphs

To illustrate these ideas, observe that the forbidden subgraphs of the class of bipartite graphs are exactly the cycles of odd length. Therefore the excluded minors for $\mathcal{M}(\chi \leq 2)$ are exactly the minor-minimal matroids that contain an odd cycle in their basis graphs. We have shown that $U_{1,3}$ and $U_{2,3}$ are the only such matroids. This illustrates the fact that two different hereditary classes of graphs may induce the same family of matroids, as $\mathcal{M}(\chi \leq 2)$ is induced by both the family of bipartite graphs, and by the family of triangle-free graphs.

We will conclude this section by showing that some natural classes of matroids are induced classes.

Proposition 3.6.7. *The families of binary matroids, regular matroids, and the polygon matroids of planar graphs are induced classes.*

Proof. Maurer [35] has noted that the binary matroids are exactly those which have no induced subgraph isomorphic to the octahedron in their basis graphs. In any case, it is easy to see that these three classes are closed under the addition of loops or coloops, generalised duality, and minors. Lucas has proved that they are closed under rank-preserving weak maps [32, Theorem 6.5 and Proposition 6.13]. Hence they are induced classes by Theorem 3.6.6. □

Since binary matroids are an induced class, it is natural to ask whether this is true for the class of GF(q)-representable matroids, for other values of q. This question is easily answered.

Proposition 3.6.8. *Let q be a prime power. The set of GF(q)-representable matroids is an induced class if and only if $q = 2$.*

Proof. The previous result provides us with one direction of the implication. It remains to show that if $q > 2$, then the set of GF(q)-representable matroids is not an induced class. Figure 3.8 shows three matroids: $F_7^=$; F_7^-, also known as the non-Fano matroid; and F_7, the Fano plane. Suppose that q is not a power of two. Then F_7^- is GF(q)-representable, but F_7 is not [32]. Since $F_7^- \overset{\text{r.p.}}{\to} F_7$, this shows that GF($q$)-representability is not closed under rank-preserving weak maps, and therefore the set of GF(q)-representable matroids is not an induced class.

Similarly, if q is a power of two, and $q > 2$, then it is not difficult to see that $F_7^=$ is represented over GF(q) by the matrix shown below. (In this case $\alpha \in \text{GF}(q) - \{0, 1\}$.)

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & \alpha & 1 \end{bmatrix}$$

However, F_7^- is an excluded minor for GF(q)-representability [42, page 506], and $F_7^= \overset{\text{r.p.}}{\to} F_7^-$, so again, the set of GF(q)-representable matroids is not an induced class. □

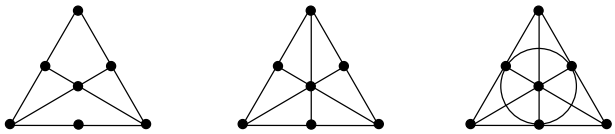


Figure 3.8: $F_7^=$, F_7^- and F_7.

Definition 3.6.9. A matroid is *near-graphic* if it is the generalised dual of a graphic matroid.

Equivalently, a matroid is near-graphic if and only if every connected component is either graphic or cographic. Before we prove that the class of near-graphic matroids is an induced class we note the following easy result.

Proposition 3.6.10. *If $M \overset{\text{r.p.}}{\to} N$, then $M^* \overset{\text{r.p.}}{\to} N^*$.*

Proposition 3.6.11. *The family of near-graphic matroids is an induced class.*

Proof. Obviously the class of near-graphic matroids is closed under minors and the addition of loops or coloops. It is closed under generalised duality by construction.

It remains to show that the class of near-graphic matroids is closed under rank-preserving weak maps. We first note that the class of graphic matroids is closed under rank-preserving weak maps [32, Proposition 6.13]. It follows from Proposition 3.6.10 that the class of cographic matroids is also closed under rank-preserving weak maps.

Suppose that M is a near-graphic matroid, and that $M \overset{\text{r.p.}}{\to} N$. We will assume that $E(M) = E(N)$, and that $\mathcal{B}(N) \subseteq \mathcal{B}(M)$. Let N' be a connected component of N. Then N' is contained in a connected component, M', of M [32, Proposition 5.2]. There exists a minor, M'', of M', such that $M'' \overset{\text{r.p.}}{\to} N'$ [32, Theorem 5.8]. By definition, M', and therefore M'', is either graphic or cographic. Hence N' is either graphic or cographic, and thus N is near-graphic. □

3.7 A characterisation of near-graphic matroids

The excluded-minor characterisations of binary matroids, regular matroids and the polygon matroids of planar graphs are classical results of Tutte's [51, 52]. However the near-graphic matroids have not been characterised via their excluded minors. We give such a characterisation in this section.

We first require some preliminary material. If M is a matroid, and (X, Y) is a partition of $E(M)$ such that $\min\{|X|, |Y|\} \geq k$ and

$$r(X) + r(Y) - r(M) \leq k - 1 \tag{3.1}$$

for some integer k, then (X, Y) is a *k-separation of M*. If equality holds in (3.1), then the separation is said to be *exact*. We say that M is *n-connected* if M has no k-separation where $k < n$.

Let M_1 and M_2 be two matroids such that $E(M_1) \cap E(M_2) = \{p\}$, where p is a loop or coloop of neither M_1 nor M_2. The *2-sum of M_1 and M_2 along the basepoint p*, denoted by $M_1 \oplus_2 M_2$, is a matroid on the ground set $(E(M_1) \cup E(M_2)) - p$. The collection of circuits of $M_1 \oplus_2 M_2$ is

$$\mathcal{C}(M_1\backslash p) \cup \mathcal{C}(M_2\backslash p) \cup \{(C \cup C') - p \mid C \in \mathcal{C}(M_1),\, C' \in \mathcal{C}(M_2), p \in C \cap C'\}.$$

It is well known that (X, Y) is a 1-separation of M if and only if $M = (M|X) \oplus (M|Y)$. Similarly, (X, Y) is an exact 2-separation of M if and only if there exist matroids, M_1 and M_2, on the sets $X \cup p$ and $Y \cup p$ respectively (where $p \notin E(M)$), such that M is equal to the 2-sum of M_1 and M_2 along p [47, (2.6)].

The next fact is well-known, and follows easily from [42, Proposition 7.1.15].

Proposition 3.7.1. *Suppose that M_1 and M_2 are two matroids, and that $E(M_1) \cap E(M_2) = \{p\}$, where p is a loop or coloop of neither M_1 nor M_2. If $e \in E(M_1) - p$, then $(M_1 \oplus_2 M_2)\backslash e = (M_1\backslash e) \oplus_2 M_2$, and $(M_1 \oplus_2 M_2)/e = (M_1/e) \oplus_2 M_2$.*

Suppose that M_1 and M_2 are two binary matroids and that $E(M_1) \cap E(M_2) = T$, where $M_1|T = M_2|T \cong U_{2,3}$. Suppose also that $|E(M_1)|, |E(M_2)| > 6$, and that T does not contain a cocircuit of M_1 or M_2. Then the *3-sum of M_1 and M_2*, denoted by $M_1 \oplus_3 M_2$, is the matroid on $E(M_1)\Delta E(M_2)$, the circuits of which are the minimal non-empty sets that can be expressed as the symmetric difference of a disjoint union of circuits of M_1, and a disjoint union of circuits of M_2.

The next result follows from [42, Proposition 12.4.14].

Proposition 3.7.2. *Suppose that M_1 and M_2 are binary matroids, and that $E(M_1) \cap E(M_2) = T$. Suppose also that $M_1|T = M_2|T \cong U_{2,3}$, while $|E(M_1)|, |E(M_2)| > 6$, and T contains no cocircuits of M_1 or M_2. If $e \in E(M_1) - T$, and $(M_1\backslash e) \oplus_3 M_2$ is defined, then $(M_1 \oplus_3 M_2)\backslash e = (M_1\backslash e) \oplus_3 M_2$. If $e \in E(M_1) - \mathrm{cl}_{M_1}(T)$, and $(M_1/e) \oplus_3 M_2$ is defined, then $(M_1 \oplus_3 M_2)/e = (M_1/e) \oplus_3 M_2$.*

The matroids R_{10} and R_{12} are binary self-dual matroids of rank five and six respectively. Matrices over GF(2) that represent R_{10} and R_{12} are shown in Figures 3.9 and 3.10.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Figure 3.9: A GF(2)-representation of R_{10}.

1	2	3	4	5	6	7	8	9	10	11	12
1	0	0	0	0	0	1	1	1	0	0	0
0	1	0	0	0	0	1	1	0	1	0	0
0	0	1	0	0	0	1	0	0	0	1	0
0	0	0	1	0	0	0	1	0	0	0	1
0	0	0	0	1	0	0	0	1	0	1	1
0	0	0	0	0	1	0	0	0	1	1	1

Figure 3.10: A GF(2)-representation of R_{12}.

The matroid R_{12} can also be expressed as the 3-sum of $M^*(K_{3,3})$ and $M(K_5)\backslash e$, where e is any element of $M(K_5)$. Figure 3.11 shows representations of $M^*(K_{3,3})$ and $M(K_5)\backslash e$. It is not possible to present an orthodox geometric representation of R_{12}, since it has rank 6. However Figure 3.12 illustrates how R_{12} can be decomposed as the 3-sum of $M^*(K_{3,3})$ and $M(K_5)\backslash e$. In this diagram the elements of R_{12} are labelled with the corresponding columns of the matrix given in Figure 3.10.

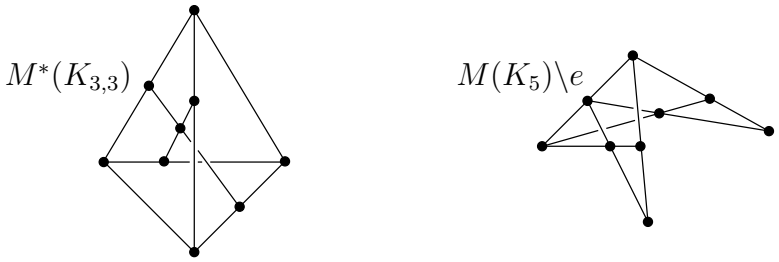


Figure 3.11: $M^*(K_{3,3})$ and $M(K_5)\backslash e$.

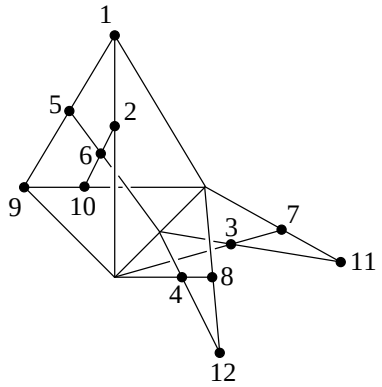


Figure 3.12: A representation of R_{12}.

The matroids R_{10} and R_{12} play a pivotal role in Seymour's decomposition theorem for regular matroids.

Theorem 3.7.3. [47, (14.3)] *Every regular matroid, M, can be constructed using direct sums, 2-sums, and 3-sums, starting with matroids each isomorphic to a minor of M, and each either graphic, cographic, or isomorphic to R_{10}.*

The next result is a central lemma in the proof of the decomposition theorem, and will be crucially important for our characterisation of near-graphic matroids.

Proposition 3.7.4. [47, (14.2)] *Let M be a 3-connected regular matroid. Then M is either graphic or cographic, or has a minor isomorphic to one of R_{10} or R_{12}.*

We now state and prove our excluded minor characterisation of near-graphic matroids.

Theorem 3.7.5. *The excluded minors for the class of near-graphic matroids are: $U_{2,4}$; the Fano plane, F_7, and its dual, F_7^*; R_{10}; R_{12}; and those matroids that are formed by taking the 2-sum of the polygon matroid of one of the graphs in Figure 3.13 with the bond matroid of one of the same graphs, using the element marked p as the basepoint.*

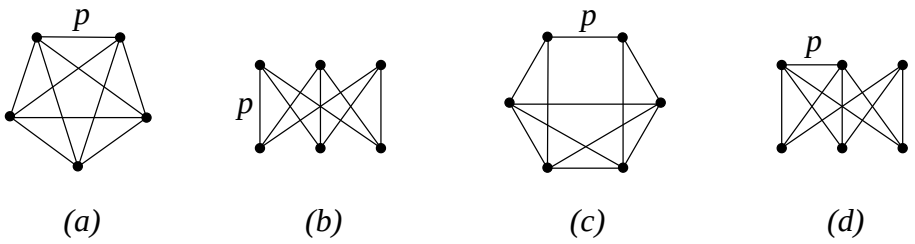


Figure 3.13:

Proof. Let us first note that the class of near-graphic matroids is closed under direct sums, so all the excluded minors for this class are connected. The excluded minors for graphic matroids are $U_{2,4}$, F_7, F_7^*, $M^*(K_5)$, and $M^*(K_{3,3})$, while the excluded minors for cographic matroids are $U_{2,4}$, F_7, F_7^*, $M(K_5)$, and $M(K_{3,3})$ [52]. Since $U_{2,4}$, F_7, and F_7^* are excluded minors for both graphic matroids and cographic matroids, and are connected, it follows that they are also excluded minors for near-graphic matroids.

It is known [47] that if e is any element of R_{10}, then $R_{10}\backslash e \cong M(K_{3,3})$, and that $R_{10}/e \cong M^*(K_{3,3})$. Therefore R_{10} is neither graphic nor cographic. However, if we remove (that is, either delete or contract) any element we clearly obtain a matroid that is either graphic or cographic. Since R_{10} is connected, it is therefore an excluded minor for near-graphic matroids.

The next result is slightly more difficult.

3.7.5.1. *R_{12} is an excluded minor for near-graphic matroids.*

Proof. It is known that R_{12} is neither graphic nor cographic [42, page 519] (in fact, it is easy to see that R_{12} has both an $M^*(K_{3,3})$-minor, and an $M(K_{3,3})$-minor). Let (A, B) be a partition of the ground set of R_{12} such that there are matroids,

$M_1 \cong M^*(K_{3,3})$ and $M_2 \cong M(K_5)\backslash e$, on the sets $A \cup T$ and $B \cup T$ respectively, and $R_{12} = M_1 \oplus_3 M_2$. (We are assuming here that T is a circuit of size three in both M_1 and M_2, and that T is disjoint from both A and B.) If R_{10} is labelled as in Figure 3.12, then $A = \{1, 2, 5, 6, 9, 10\}$ while $B = \{3, 4, 7, 8, 11, 12\}$. Since $M^*(K_{3,3})$ is an excluded minor for graphic matroids, it follows from Proposition 3.7.2 that if we delete or contract one of the elements in A from R_{12}, then the resulting matroid can be obtained by taking the 3-sum of two graphic matroids. It follows from [42, Proposition 12.4.19] that the class of graphic matroids is closed under 3-sums. Therefore removing an element of A from R_{12} produces a graphic matroid.

It is not difficult to confirm that R_{12} is self-dual, and that A and B are exchanged by the isomorphism that takes R_{12} to its dual [47]. Thus, if $e \in B$, then $R_{12}\backslash e$ is isomorphic to $(R_{12}^*/e)^*$ and, by the previous arguments, R_{12}^*/e is graphic. Thus $R_{12}\backslash e$ is cographic. We may similarly show that R_{12}/e is cographic. Since R_{12} is connected and neither graphic nor cographic, but all of its proper minors are graphic or cographic, it is an excluded minor for the class of near-graphic matroids. □

3.7.5.2. *Let G and H be two, not necessarily distinct, graphs from the collection shown in Figure 3.13. The 2-sum of $M(G)$ and $M^*(H)$ along the basepoint p is an excluded minor for near-graphic matroids.*

Proof. Both $M(G)$ and $M^*(H)$ are minors of $M(G) \oplus_2 M^*(H)$. Therefore $M(G) \oplus_2 M^*(H)$ is neither graphic nor cographic, as $M(G)$ has either an $M(K_5)$-minor, or an $M(K_{3,3})$-minor, and $M^*(H)$ has either an $M^*(K_5)$-minor, or an $M^*(K_{3,3})$-minor.

To complete the proof of the claim, we will show that removing any element (except p) from $M(G)$ produces a matroid that is both graphic and cographic. (It follows by duality that any matroid obtained from $M^*(H)$ by removing an element (except p) is also graphic and cographic.) Proposition 3.7.1 then implies that if $e \in E(M(G))-p$, both $(M(G) \oplus_2 M^*(H))\backslash e$ and $(M(G) \oplus_2 M^*(H))/e$ can be expressed as the 2-sum of two cographic matroids. The classes of graphic and cographic matroids are preserved under 2-sums [42, Corollary 7.1.23], so $(M(G) \oplus_2 M^*(H))\backslash e$ and $(M(G) \oplus_2 M^*(H))/e$ are both cographic. Similarly, if $e \in E(M^*(H)) - p$, then $(M(G) \oplus_2 M^*(H))\backslash e$ and $(M(G) \oplus_2 M^*(H))/e$ are both graphic.

It remains to prove our claim that removing an element (except p) from $M(G)$ produces a matroid that is both graphic and cographic; equivalently, deleting or contracting an edge other than p from any of the graphs in $\mathcal{G}$ produces a planar graph (since the matroids that are graphic and cographic are exactly the polygon matroids of planar graphs [42, Theorem 5.2.2]). If G is isomorphic to either K_5 or $K_{3,3}$, then this is well known. Therefore let us assume that G is one of the graphs (c) or (d) shown in Figure 3.13. Up to isomorphism, there are eight graphs that can be obtained by deleting or contracting an edge other than p from (c) or (d). These eight graphs are shown in Figures 3.14 and 3.15, and a planar drawing of each graph is given underneath it. □

We may now complete the proof of Theorem 3.7.5. Let M be an excluded minor for near-graphic matroids, and suppose that M has no minor isomorphic to $U_{2,4}$, F_7,

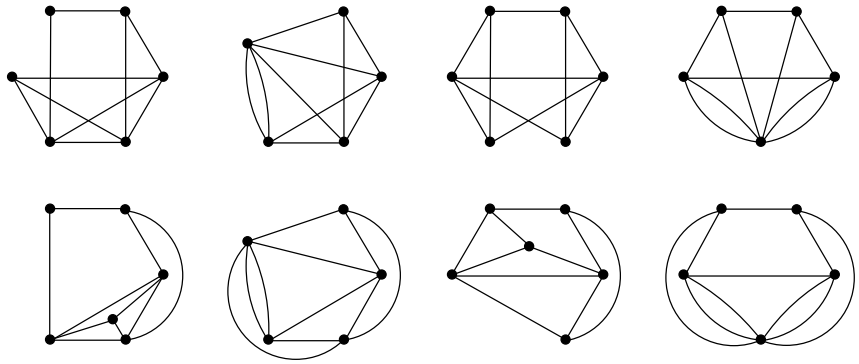


Figure 3.14:

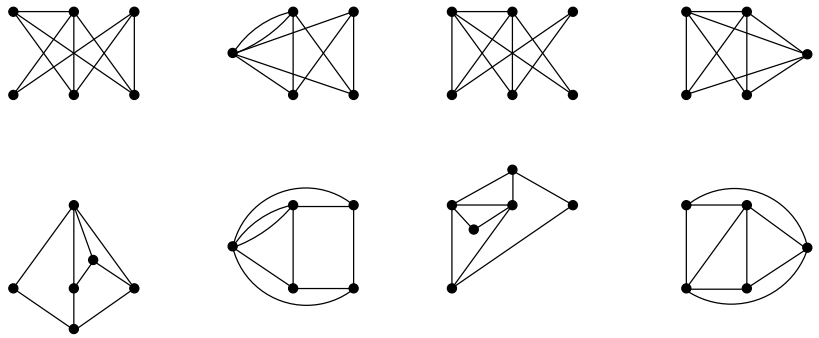


Figure 3.15:

F_7^*, R_{10}, or R_{12}. This implies that M is regular [51]. It follows immediately from Proposition 3.7.4 that M is not 3-connected.

Since M is not 3-connected, but is connected, there exist matroids, M_1 and M_2, such that $E(M_1) \cap E(M_2) = \{p\}$ and $M = M_1 \oplus_2 M_2$. Both M_1 and M_2 must be connected, for otherwise M is not connected [42, Proposition 7.1.20]. Since M_1 and M_2 are proper minors of M [42, Proposition 7.1.19], they must be either graphic or cographic. If both were graphic or cographic, then M would be graphic or cographic. Therefore, we will assume that M_1 is graphic, but not cographic, and that M_2 is cographic but not graphic.

It follows that M_1 must have an $M(K_5)$- or an $M(K_{3,3})$-minor. Let N be a minor of M_1 such that N is isomorphic to either $M(K_5)$ or $M(K_{3,3})$. Suppose that $e \in E(M_1) - p$. We wish to show that $M_1 \backslash e$ does not have an N-minor. Suppose that it does. Then $M \backslash e$ has an N-minor, and an M_2-minor, and is therefore neither graphic nor cographic. Since $M \backslash e$ is near-graphic, it follows that $M \backslash e$ is not connected. As $M \backslash e$ is the 2-sum of $M_1 \backslash e$ and M_2, it must be the case that $M_1 \backslash e$ is not connected. Let M' be a connected component of $M_1 \backslash e$ that has an N-minor. It cannot be the case that $e \in \mathrm{cl}_{M_1}(E(M'))$, for in that case M_1 is not connected. Therefore e is a coloop in $M_1|(E(M') \cup e)$, and hence, if we contract e from $M_1|(E(M') \cup e)$, we obtain a matroid that has an N-minor. Thus M_1/e has an N-minor. Since M is connected, either $M \backslash e$ or M/e is connected. Because $M \backslash e$ is not connected, M/e must be connected. However, M_1/e is a minor of M/e, and hence M/e has an N-minor. Since

M/e also has an M_2-minor, it follows that M/e is neither graphic nor cographic. This is a contradiction, as M/e is connected. We conclude that $M_1\backslash e$ does not have an N-minor.

Using duality, we may also show that, if $e \in E(M_1) - p$, then M_1/e does not have an N-minor. Hence, either M_1 is isomorphic to $M(K_5)$ or $M(K_{3,3})$, or M_1 is isomorphic to a matroid obtained by extending or coextending $M(K_5)$ or $M(K_{3,3})$ by a single element, p. Similarly, M_2 is either isomorphic to $M^*(K_5)$ or $M^*(K_{3,3})$, or can be obtained from one of these matroids by extending or coextending by p.

Let us suppose that M_1 is a single-element extension or coextension of $M(K_5)$ or $M(K_{3,3})$. Recall that M_1 is graphic. Since M_1 is connected, p cannot be a loop or a coloop of M_1. If p is a member of a parallel or series pair of M_1, then it is easy to see that $M_1 \oplus_2 M_2$ has a proper minor isomorphic to either $M(K_5) \oplus_2 M_2$ or $M(K_{3,3}) \oplus_2 M_2$. Neither of these matroids is near-graphic, so we have a contradiction. Given these restrictions, it follows that M_1 must be either the single-element coextension of $M(K_5)$ that is the polygon matroid of the graph (c) in Figure 3.13, or the single-element extension of $M(K_{3,3})$ that is the polygon matroid of graph (d). By similar reasoning, M_2 must be the bond matroid of one of the graphs in Figure 3.13. This completes the proof. □

We note that a matroid is near-graphic if and only if it can be constructed from matroids that are either graphic or cographic using direct sums. This suggests two obvious generalisations of the class of near-graphic matroids.

Definition 3.7.6. A matroid is 2-*graphic* if it can be constructed from matroids that are either graphic or cographic using direct sums and 2-sums. Similarly, a matroid is 3-*graphic* if it can be constructed from matroids that are either graphic or cographic using direct sums, 2-sums, and 3-sums.

The following characterisation of 3-graphic matroids follows without difficulty from Theorem 3.7.3 and the fact that any 3-connected regular matroid that has an R_{10}-minor is, in fact, isomorphic to R_{10} [47, (7.4)].

Proposition 3.7.7. *The class of* 3*-graphic matroids is equal to the class of regular matroids having no* R_{10}*-minor.*

We have noted that a matroid has a 1- or 2-separation if and only if it can be expressed as a direct sum or a 2-sum of two non-trivial matroids. The next result provides an analogue for 3-separations in binary matroids.

Proposition 3.7.8. [47, (2.9)] *Suppose that* (X, Y) *is an exact* 3*-separation of a binary matroid,* M*, such that* $\min\{|X|, |Y|\} \geq 4$*. Then there exist binary matroids,* M_1 *and* M_2*, on the ground sets* $X \cup T$ *and* $Y \cup T$ *respectively, where* T *is disjoint from both* X *and* Y*, such that* $M = M_1 \oplus_3 M_2$*. Conversely, if* M *can be expressed as the* 3*-sum of two matroids,* M_1 *and* M_2*, then* $(E(M_1) - E(M_2), E(M_2) - E(M_2))$ *is an exact* 3*-separation of* M*, and both blocks in this partition have size at least four.*

Proposition 3.7.9. *The families of 2-graphic and 3-graphic matroids are induced classes.* [1]

Proof. It is obvious that both classes are closed under the addition of loops or coloops. The fact that they are closed under taking minors follows from Proposition 3.7.1 and [42, Proposition 12.4.14].

Both the class of 2-graphic matroids and the class of 3-graphic matroids are closed under direct sums by construction, so to show that they are closed under generalised duality we need only show that they are closed under duality. The fact that the class of 3-graphic matroids is closed under duality follows from Proposition 3.7.7 and the fact that R_{10} is self-dual. Since $(M_1 \oplus_2 M_2)^* = M_1^* \oplus_2 M_2^*$ for any two matroids [42, Proposition 7.1.20], it is easy to see that the class of 2-graphic matroids is also closed under duality.

It remains to show that both classes are closed under rank-preserving weak maps. We shall prove this by induction on the size of the ground set. Every matroid with at most 3 elements is both 2-graphic and 3-graphic, so we shall use this as the base case of our induction. Suppose that M is an n-graphic matroid, where $n \in \{2, 3\}$. Suppose also that, if M' is a rank-preserving weak-map image of a n-graphic matroid, and $|E(M')| < |E(M)|$, then M' is also n-graphic. Let N be a rank-preserving weak-map image of M. We may assume that $E(M) = E(N)$, and that $\mathcal{B}(N) \subseteq \mathcal{B}(M)$.

If M has no k-separation, where $k \leq n$, then M must be either graphic or cographic. We have already noted that the classes of graphic and cographic matroids are closed under rank-preserving weak maps, so in this case there is nothing more to prove. Therefore we will assume that (X, Y) is a partition of $E(M)$ such that $\min\{|X|, |Y|\} \geq k$, and $r_M(X) + r_M(Y) - r(M) \leq k - 1$, where k is an integer no greater than n. Since $r_N(X) \leq r_M(X)$ and $r_N(Y) \leq r_M(Y)$, it follows that (X, Y) is an exact k'-separation of N, where $k' \leq k$. If $k' \leq 2$, then N can be obtained from two matroids, N_1 and N_2, using either a direct sum or a 2-sum. Both N_1 and N_2 are minors of N, and there exist minors, M_1 and M_2, of M, such that $M_1 \overset{\text{r.p.}}{\to} N_1$ and $M_2 \overset{\text{r.p.}}{\to} N_2$ [32, Theorem 5.8]. By the inductive hypothesis, both N_1 and N_2 are n-graphic, and therefore N itself is n-graphic.

Therefore we will suppose that $k' = 3$. Note that, since N is a rank-preserving weak-map image of a regular matroid, N is itself regular, by Proposition 3.6.7. We may assume that N is 3-connected, for otherwise N has a separation, (X', Y'), where $r_N(X') + r_N(Y') - r(N) \leq 2$, and we can therefore apply the arguments of the previous paragraph to show that N is n-graphic. If N has no 3-separation where both sides of the separation have size at least four, then N is either graphic or cographic [42, Corollary 13.2.6]. Therefore we will assume that (X, Y) is an exact 3-separation of N such that $\min\{|X|, |Y|\} \geq 4$. Thus, by Proposition 3.7.8, it follows that N is the 3-sum of two matroids, N_1 and N_2. Both N_1 and N_2 are minors of N [42, Proposition 12.4.16], and again, there are minors, M_1 and M_2, of M, such that $M_1 \overset{\text{r.p.}}{\to} N_1$ and $M_2 \overset{\text{r.p.}}{\to} N_2$. The inductive hypothesis implies that N_1 and N_2 are

[1] The observation that the families of 2-graphic and 3-graphic matroids are induced classes, and the excluded-minor characterisations of these classes, came from discussion with Professors Colin McDiarmid and James Oxley that took place during the examination of this thesis.

n-graphic, and since N can be obtained from N_1 and N_2 using a 3-sum, it follows that N is n-graphic. □

We conclude this section by noting the excluded-minor characterisations of 2-graphic and 3-graphic matroids.

Proposition 3.7.10. *The excluded minors for the class of 2-graphic matroids are $U_{2,4}$, F_7, F_7^*, R_{10}, and R_{12}. The excluded minors for the class of 3-graphic matroids are $U_{2,4}$, F_7, F_7^*, and R_{10}.*

Proof. The characterisation of 3-graphic matroids follows immediately from Proposition 3.7.7. Both R_{10} and R_{12} are 3-connected, and are neither graphic nor cographic, so cannot belong to the class of 2-graphic matroids. It is now easy to check that $U_{2,4}$, F_7, F_7^*, R_{10}, and R_{12} are excluded minors for 2-graphic matroids. Suppose that M is an excluded minor for 2-graphic matroids that is not listed in the statement of the proposition. Then M must be regular, and, since the class of 2-graphic matroids is closed under direct sums and 2-sums, M must be 3-connected. Proposition 3.7.4 shows that this is a contradiction. □

3.8 An induced class with infinitely many excluded minors

The classes of binary matroids, regular matroids, and the polygon matroids of planar graphs are known to have 1, 3, and 7 non-isomorphic excluded minors respectively. In the previous section we have shown that the classes of near-graphic, 2-graphic, and 3-graphic matroids have exactly 21, 5, and 4 non-isomorphic excluded minors respectively. It is natural to ask whether there exists an induced class of matroids that has infinitely many excluded minors. In this section we will show that such a class exists. The example we consider was suggested by a referee of [36].

If H is a circuit-hyperplane of M, then $\mathcal{B}(M) \cup \{H\}$ is the set of bases of a matroid, M', said to be produced by *relaxing the circuit-hyperplane, H, in M*. For every element $e \in E(M) - H$, the set $H \cup e$ is a circuit of M'. If B is any basis of a matroid, and, for every element, $e \notin B$, the set $B \cup e$ is a circuit, then we shall say that B is a *loose basis*. If B is a loose basis of M, then the set $\mathcal{B}(M) - \{B\}$ is the family of bases of a matroid on $E(M)$ [41, (1.5)]. This matroid will be said to be produced from M by *tightening the loose basis B*. Clearly this operation is the inverse of relaxing the circuit-hyperplane B.

Definition 3.8.1. Let $\mathcal{N}$ be the class of matroids defined so that $M \in \mathcal{N}$ if and only if every connected component of M is either binary, or can be obtained from a binary matroid by relaxing a single circuit-hyperplane.

Theorem 3.8.2. *The class $\mathcal{N}$ is an induced class.*

We defer the proof of Theorem 3.8.2. Proving that $\mathcal{N}$ is closed under generalised duality and taking minors is relatively simple, but the proof that it is closed under rank-preserving weak maps is more difficult.

We will note here that $\mathcal{N}$ has an infinite number of excluded minors: let J_n denote the $n \times n$ matrix of ones. For $n \geq 1$, let M_n be the binary matroid that is represented over GF(2) by the matrix $A_n = [I_n \mid J_n - I_n]$. Let the columns of A_n be labelled $a_1, \ldots, a_n, b_1, \ldots, b_n$. We will take the ground set of M_n to be the set of column labels. If n is even, then $H_1 = \{a_1, b_2, \ldots, b_n\}$ and $H_2 = \{a_2, \ldots, a_n, b_1\}$ are both circuit-hyperplanes of M_n. Let N_n be the matroid obtained by relaxing both of these circuit-hyperplanes.

Proposition 3.8.3. *If $n \geq 4$, and n is even, then N_n is an excluded minor for $\mathcal{N}$.*

Proof. If B is a loose basis of a binary matroid, M, and e and f are two elements of $E(M)-B$, then both $B \cup e$ and $B \cup f$ are circuits of M. Since the symmetric difference of two circuits in a binary matroid is itself a union of circuits [42, Theorem 9.1.2], it follows that $\{e, f\}$ must be a union of circuits of M. Note that H_1 is a loose basis of N_n. Since $n \geq 4$, it follows that $\{a_2, b_1\}$ is an independent pair of elements contained in $E(N_n) - H_1$. Therefore N_n cannot be binary. Furthermore, we can show that H_1 and H_2 are the only loose bases of N_n. Let N'_r be the matroid obtained by tightening H_1. Then H_2 is a loose basis in N'_n, and a_1 and b_2 are both contained in $E(N_n) - H_2$. Since $n \geq 4$, it is not the case that a_1 and b_2 are parallel, so N'_n is not binary. Using the same argument, we may show that tightening H_2 in N_n does not produce a binary matroid. Therefore $N_n \notin \mathcal{N}$.

However, it is easy to see that removing a single element from N_n produces a matroid that has exactly one loose basis, and tightening this basis produces a binary matroid. □

Proposition 3.8.4. *The class $\mathcal{N}$ is closed under generalised duality.*

Proof. Suppose that M is in $\mathcal{N}$, and that M_1 is a connected component of M. If M_1 is binary, then so is M_1^*, so we need only show that, if M_1 is obtained from a binary matroid by relaxing a circuit-hyperplane, then M_1^* can be obtained from a binary matroid in the same way. This follows immediately from [42, Proposition 2.1.7]. □

The next result follows without difficulty from [42, Proposition 3.3.9].

Proposition 3.8.5. *The class $\mathcal{N}$ is closed under taking minors.*

Let (M_1, M_2, M_3, M_4) be a sequence of matroids, such that, for $1 \leq i \leq 4$, the matroid M_i contains at least two elements, and is either uniform of rank one, or uniform of corank one. For $1 \leq i \leq 4$, let e_i be an element of $E(M_i)$. Let N be isomorphic to $U_{2,4}$, and suppose that the ground set of N is $\{e_1, \ldots, e_4\}$. We will use the notation $M(M_1, M_2, M_3, M_4)$ to denote the matroid

$$(((N \oplus_2 M_1) \oplus_2 M_2) \oplus_2 M_3) \oplus_2 M_4,$$

where the 2-sum that involves M_i uses the element e_i as its basepoint. Note that $M(M_1, M_2, M_3, M_4)$ is obtained from $U_{2,4}$ by a sequence of parallel or series extensions.

We will need the following result, which can easily be deduced from a theorem of Oxley's [41, Theorem 1.2].

Theorem 3.8.6. *Let M be a non-binary matroid, such that, for every element $e \in E(M)$, either $M\backslash e$ or M/e is binary. Then, either:*

(i) $M \in \mathcal{N}$*;*

(ii) the rank or the corank of M is equal to two; or,

(iii) there exists a sequence, (M_1, M_2, M_3, M_4), such that $M \cong M(M_1, M_2, M_3, M_4)$.

Lemma 3.8.7. *The class $\mathcal{N}$ is closed under rank-preserving weak maps.*

Proof. Let us suppose that the lemma does not hold. Then there exists a matroid, M, such that $M_1 \overset{\text{r.p.}}{\to} M$, where $M_1 \in \mathcal{N}$, but $M \notin \mathcal{N}$. We will assume that $E(M) = E(M_1)$ and that $\mathcal{B}(M) \subseteq \mathcal{B}(M_1)$. Among such counterexamples let M be chosen to be as small as possible, so that if M' is a rank-preserving weak-map image of a matroid in $\mathcal{N}$, and $|E(M')| < |E(M)|$, then M' itself is in $\mathcal{N}$.

3.8.7.1. *The matroid M is an excluded minor for the class $\mathcal{N}$.*

Proof. Suppose that M' is a proper minor of M. There exists a minor, M_1', of M_1 such that $M_1' \overset{\text{r.p.}}{\to} M'$ [32, Theorem 5.8]. Since $M_1' \in \mathcal{N}$, it follows that M' is in $\mathcal{N}$. □

Since $\mathcal{N}$ is, by construction, closed under direct sums, it follows that M is connected.

3.8.7.2. *The matroid M_1 is not binary.*

Proof. It has already been noted, and is easy to prove directly, that a rank-preserving weak-map image of a binary matroid is itself binary. Hence, if M_1 were binary, then M would be binary, and would therefore belong to $\mathcal{N}$. □

We conclude that M_1 can be obtained from a binary matroid, M_2, by relaxing a circuit-hyperplane, H.

3.8.7.3. $r(M) > 2$.

Proof. Suppose otherwise; since every matroid of rank at most one belongs to $\mathcal{N}$, the rank of M, and therefore M_1, must be two. Since M_1 is not binary, but can be obtained from a binary matroid by relaxing a circuit-hyperplane, it follows that M_1 must contain exactly four parallel classes. Any connected rank-preserving weak-map image of M_1 that is not isomorphic to M_1 contains at most three parallel classes, and is therefore binary. From this contradiction we conclude that $r(M) > 2$. □

Since $\mathcal{N}$ is closed under duality, M^* is also a minimal counterexample to Lemma 3.8.7. By applying the arguments above, we may conclude that $r(M^*) > 2$.

3.8.7.4. *If $e \in H$, then $M\backslash e$ is binary, and if $e \notin H$, then M/e is binary.*

Proof. Let e be an element of $E(M)$. It is not the case that e is a coloop of M_1 or M, for M is connected, and if e were a coloop of M_1 it would be a coloop of M. Therefore $M_1\backslash e \overset{\text{r.p.}}{\to} M\backslash e$. Suppose that $e \in H$. Then [42, Proposition 3.3.9] implies that $M_1\backslash e = M_2\backslash e$. Therefore $M_1\backslash e$ is binary. Since $M\backslash e$ is the rank-preserving weak-map image of a binary matroid, it is also binary.

Similarly, e is not a loop in M_1 or in M. Therefore $M_1/e \overset{\text{r.p.}}{\to} M/e$. Also, if $e \notin H$, then $M_1/e = M_2/e$. It follows that M/e is binary. □

From Theorem 3.8.6, and our assumption on the rank and corank of M, we conclude that there exists a sequence, (M_1, M_2, M_3, M_4), such that $M \cong M(M_1, M_2, M_3, M_4)$.

If, for some $i \in \{1, \ldots, 4\}$, the matroid M_i is isomorphic to $U_{1,2}$, we will say that M_i is *trivial*. Note that $M \oplus_2 U_{1,2} \cong M$, for any matroid M.

3.8.7.5. *No more than two of the non-trivial matroids in* (M_1, M_2, M_3, M_4) *are cocircuits, and no more than two are circuits.*

Proof. It is easy to see that

$$(M(M_1, M_2, M_3, M_4))^* = M(M_1^*, M_2^*, M_3^*, M_4^*).$$

Furthermore, the order of the matroids in (M_1, M_2, M_3, M_4) is insignificant. Therefore, if the claim in 3.8.7.5 is false, then, by duality and relabelling, we may assume that $M_i \cong U_{1,n_i}$, where $n_i > 2$, for all $i \in \{1, 2, 3\}$. It follows that M_4 is a circuit of size at least three, for otherwise $r(M) = 2$. By deleting all but two elements from $E(M_i) - e_i$, for $1 \leq i \leq 3$, and contracting all but one element from $E(M_4) - e_4$, we see that M has a minor isomorphic to $M' = M(U_{1,3}, U_{1,3}, U_{1,3}, U_{1,2})$. This is the rank-2 matroid that has three parallel classes of size two, and one parallel class of size one. It is not difficult to show that M' is an excluded minor for $\mathcal{N}$, so M must be isomorphic to M'. But this contradicts our assumption that $r(M) > 2$. □

3.8.7.6. *There are no trivial matroids in* (M_1, M_2, M_3, M_4).

Proof. If the claim is false, then we may assume that $M_1 \cong U_{1,2}$. It cannot be the case that every matroid in (M_1, M_2, M_3, M_4) is trivial, for then M would have rank two. By referring to 3.8.7.5, and using duality and relabelling, we may assume that M_2 is a circuit of size at least three, and that M_3 and M_4 are cocircuits. Then the rank of M is $|E(M_2)|$, and it is easy to see that $(E(M_1) \cup E(M_2)) - \{e_1, e_2\}$ is a loose basis of M.

The only non-spanning circuits in the matroid obtained by tightening this basis are: $(E(M_1) \cup E(M_2)) - \{e_1, e_2\}$; any pair of elements in $E(M_3) - e_3$; any pair of elements in $E(M_4) - e_4$; and, any triple of elements containing the single element of $E(M_1) - e_1$, an element from $E(M_3) - e_3$, and an element from $E(M_4) - e_4$. It is easy to see that this matroid is isomorphic to the 2-sum of two binary matroids, and is therefore binary. This implies that M is a member of $\mathcal{N}$. □

We may now assume that $|E(M_i)| > 2$ for all $i \in \{1, \ldots, 4\}$. Furthermore, from 3.8.7.5, and by relabelling if necessary, we may assume that M_1 and M_2 are cocircuits, while M_3 and M_4 are circuits. It is easily demonstrated that the rank of M is $|E(M_3)| + |E(M_4)| - 2$, and that $(E(M_3) \cup E(M_4)) - \{e_3, e_4\}$ is a loose basis of M. The only non-spanning circuits in the matroid obtained by tightening this basis are: $(E(M_3) \cup E(M_4)) - \{e_3, e_4\}$; any pair of elements in $E(M_1) - e_1$; any pair of elements in $E(M_2) - e_2$; the union of $E(M_3) - e_3$ with an element from $E(M_1) - e_1$ and an element from $E(M_2) - e_2$; and, the union of $E(M_4) - e_4$ with an element from $E(M_1) - e_1$ and an element from $E(M_2) - e_2$. It is not difficult to see that this matroid is binary, and that M is therefore in $\mathcal{N}$. This contradiction completes the proof of the lemma. □

The proof that $\mathcal{N}$ is an induced class follows from Theorem 3.6.6, Proposition 3.8.4, Proposition 3.8.5, Lemma 3.8.7, and the obvious observation that $\mathcal{N}$ is closed under the addition of loops and coloops.

3.9 Basis graphs and complexity

We conclude this chapter by briefly discussing basis graphs in reference to complexity theory.

An obvious question from a complexity perspective is: how difficult is it to decide whether an arbitrary graph is isomorphic to a basis graph? In other words, what is the complexity of the following problem?

> BASIS GRAPH
>
> Instance: A graph, G.
>
> Question: Does there exist a matroid, M, such that G is isomorphic to $\mathrm{BG}(M)$?

The problem of characterising basis graphs was solved by Maurer [34]. Although it is not explicitly mentioned in his work, it is not difficult to see that his characterisation is a good one; that is, his characterisation shows that BASIS GRAPH is in P.

Many of the classical NP-complete problems for graphs remains NP-complete even when the input is restricted to a smaller class of graphs. For instance, HAMILTONIAN CYCLE (defined below) remains NP-complete even when the input is restricted to the class of planar graphs [18].

> HAMILTONIAN CYCLE
>
> Instance: A graph, G.
>
> Question: Does G contain a Hamiltonian cycle?

Thus we might consider the following, non-specific, question.

Problem 3.9.1. *What is the complexity of graph-theoretical* NP*-complete problems, when the input graph is chosen from the set of basis graphs?*

Somewhat surprisingly, many of the classical NP-complete problems for graphs can be solved in polynomial time when the input is restricted in this way. Let us first note that HAMILTONIAN CYCLE is trivially in P when the input is restricted to basis graphs, for every basis graph has a Hamiltonian cycle. In fact, this seems to be one of the oldest known facts about basis graphs. Cummins [8] provided a proof for graphic matroids in 1966. Shank [49] gave a very short proof of the same result in 1968. His proof was also phrased in terms of graphic matroids but it can be made to apply to general matroids almost without alteration.

Proposition 3.9.2. CLIQUE *is in* P *when the input is restricted to basis graphs.*

Proof. Suppose that the graph, G, is isomorphic to $\mathrm{BG}(M)$, for some matroid M. If we are given a description of G via, say, its adjacency matrix, then, by the results in Section 3.2, we can construct the list of bases of M', where M' is some matroid that is equivalent to M up to generalised duality and the presence of loops and coloops. We observed in Section 3.2 that this procedure can be done in polynomial time.

By Lemma 2.3.6 we can, in polynomial time, construct the list of circuits of M' given the list of bases. Since it is easy to find the list of cobases from the list of bases, it also follows that we can construct the list of cocircuits in polynomial time. Therefore we can find the sizes of a largest circuit and a largest cocircuit of M' in polynomial time. But Corollary 3.4.7 then implies that we can find the clique number of $\mathrm{BG}(M')$ in polynomial time. This completes the proof, since $\mathrm{BG}(M') \cong G$. □

The following problem is NP-complete when k is a fixed integer that exceeds two.

GRAPH k-COLOURABILITY

Instance: A graph, G.

Question: Is G properly k-colourable?

Proposition 3.9.3. *Let k be a fixed integer that exceeds two. Then* GRAPH k-COLOURABILITY *is in* P *if the input is restricted to basis graphs.*

Proof. Suppose that G is isomorphic to $\mathrm{BG}(M)$, where M is some matroid. Deciding whether $\mathrm{BG}(M)$ is k-colourable is clearly equivalent to deciding whether $M \in \mathcal{M}(\chi \leq k)$. It follows immediately from Theorem 3.6.5 that there is a positive integer, t, such that no connected matroid in $\mathcal{M}(\chi \leq k)$ contains more than t elements. Therefore, for any fixed integer, k, it is possible (at least theoretically) to construct the list of all connected matroids in $\mathcal{M}(\chi \leq k)$ in some fixed time.

We have noted that it is possible, given G, to construct, in polynomial time, the list of bases of M', where M' is a loopless and coloopless matroid that is equivalent to M in the sense mentioned before.

It is well known that a Turing machine equipped with an independence oracle can find the partition of the ground set of a matroid into its connected components in polynomial time [2, Section 6.2]. (Essentially this amounts to finding the partition of a bipartite graph into its connected components.) It therefore follows from Theorem 2.4.1 that, given the list of bases of M', we can find the partition of $E(M')$ into

the connected components of M'. It follows from Proposition 3.6.2 that M' belongs to $\mathcal{M}(\chi \leq k)$ if and only if each of its connected components does. Checking whether a connected component of M belongs to $\mathcal{M}(\chi \leq k)$ will take some constant time, since the number of connected matroids in $\mathcal{M}(\chi \leq k)$ is fixed and finite. Therefore, checking each component of M' to see if it belongs to $\mathcal{M}(\chi \leq k)$ can be done in time that is bounded by the number of connected components multiplied by a constant. Since M' is loopless and coloopless, it is easy to see that the number of connected components does not exceed the number of bases of M', which is equal to the number of vertices in G. Thus deciding whether $M' \in \mathcal{M}(\chi \leq k)$ can be done in polynomial time. Since $\mathrm{BG}(M') \cong G$, this completes the proof. □

Of course, the last result may easily be generalised.

Proposition 3.9.4. *Suppose that $\mathcal{P}$ is a graph property that is closed under Cartesian products. If there exists some integer, k, such that $\omega(G) \leq k$ for every graph, $G \in \mathcal{P}$, then deciding membership of $\mathcal{P}$ is in* P *when the input is restricted to basis graphs.*

Proof. Suppose that M is a matroid and that $\mathrm{BG}(M) \in \mathcal{P}$. Corollary 3.4.7 implies that no circuit or cocircuit of M has size greater than k. Therefore, by Proposition 3.4.10, there is an integer, t, such that no connected matroid in $\mathcal{M}(\mathcal{P})$ contains more than t elements. Thus it is possible to construct the list of connected matroids in $\mathcal{M}(\mathcal{P})$ in some constant time.

To decide whether a graph, G, belongs to $\mathcal{P}$, where $G \cong \mathrm{BG}(M)$ for some matroid M, one could construct the list of bases of M', where $G \cong \mathrm{BG}(M')$, and then determine whether $M' \in \mathcal{M}(\mathcal{P})$ by finding the connected components of M', and checking in turn whether they belong to $\mathcal{M}(\mathcal{P})$. This entire procedure can be completed in polynomial time. □

Despite these results, it does seem that some NP-complete graph problems may still be hard for basis graphs: for example, there seems to be no obvious method for solving INDEPENDENT SET when the input is restricted to basis graphs. Moreover, the following problem might well be difficult.

> BASIS GRAPH SUBGRAPH
>
> Instance: An integer, k, and a graph, G.
>
> Question: Does there exist an induced subgraph, G', of G, such that the number of vertices in G' is at least k, and $G' \cong \mathrm{BG}(M)$ for some matroid M?

Problems of this sort are often computationally complex. Indeed, INDEPENDENT SET and CLIQUE are of the same type of problem as BASIS GRAPH SUBGRAPH.

However, as we discuss below, there are reasons why it may be very difficult to provide a certificate for the intractability of BASIS GRAPH SUBGRAPH by proving that it is NP-complete. For the same reasons it may be difficult to prove that an

NP-complete graph problem remains NP-complete when the input is restricted to basis graphs.

The method usually used to show that an NP-complete problem for graphs remains NP-complete for a restricted class of graphs is to construct a polynomial transformation that takes graphs into that restricted class. Such polynomial transformations often operate in a local way, for example by replacing each vertex with a particular graph (witness, for instance, the proof that GRAPH 3-COLOURABILITY remains NP-complete when the input is restricted to graphs of maximum degree four [19]).

In order to show that BASIS GRAPH SUBGRAPH is NP-complete, or to show that an NP-complete problem remains NP-complete for the class of basis graphs, we might wish to construct a polynomial transformation that operates on graphs, and that always produces a basis graph as output. A result by Donald, Holzmann, and Tobey [13] shows that such a transformation can probably not be obtained by acting locally on graphs. Apparently frustrated by the fact that all of the available characterisations of basis graphs relied on somewhat unattractive global properties, they proved that no set of local properties can ever characterise basis graphs. Therefore it seems as though providing a certificate of difficulty for these basis graph problems will require a different approach.

Chapter 4

Equitable matroids

4.1 Introduction

It is well known that counting the bases of a matroid is, in general, a computationally complex problem. For instance, Colbourn, Provan, and Vertigan [6] have shown that, given a transversal system, counting the bases of the associated transversal matroid is a #P-complete problem. The family of bicircular matroids is a subclass of the set of transversal matroids; recently Gimenez and Noy [20] have proved that counting the bases of a transversal matroid remains #P-complete even when the input is restricted to the special case of bicircular matroids.

As counting bases exactly seems in general to be intractable, much attention has been given to the problem of approximate counting of matroid bases. Work by Jerrum, Valiant, and Vazirani shows that the problem of producing an approximation of the number of bases of a matroid is intimately connected to the problem of choosing a basis uniformly at random. We will briefly discuss this connection, more details can be found in [27] or [50].

Suppose that M is a matroid, and that $X = \{e_1, \ldots, e_n\}$ is a coindependent subset of $E(M)$ with the property that $M\backslash X$ is small enough so that its bases can be counted exactly. For $i \in \{1, \ldots, n\}$, let $\mathcal{B}_i$ be the set of bases of M that do not intersect $\{e_1, \ldots, e_i\}$, and let $\mathcal{B}$ be the set of bases of M. If we are able to choose bases uniformly at random, then we may randomly sample the bases of M and, by counting how many do not contain e_1, we can gain an approximation to the proportion $|\mathcal{B}_1|/|\mathcal{B}|$. Since $\mathcal{B}_1$ is the set of bases of $M\backslash e_1$, we can, by repeating this process with $M\backslash e_1$, find an approximation to the proportion $|\mathcal{B}_2|/|\mathcal{B}_1|$. Proceeding in this way, we calculate an approximation to the product

$$\frac{|\mathcal{B}_1|}{|\mathcal{B}|}\frac{|\mathcal{B}_2|}{|\mathcal{B}_1|}\cdots\frac{|\mathcal{B}_{n-1}|}{|\mathcal{B}_{n-2}|}\frac{|\mathcal{B}_n|}{|\mathcal{B}_{n-1}|}.$$

Dividing this quantity by the number of bases in $M\backslash X$ and then taking the reciprocal gives us an approximation to $|\mathcal{B}|$.

Because of this connection between counting bases and choosing them uniformly at random, a great deal of effort has been spent in the search for efficient ways

to randomly select a basis of a matroid. In this chapter we will consider a class of matroids having the property that choosing a basis of one of these matroids uniformly at random is a trivial task, no more difficult than randomly ordering a set of integers. We will show that this class contains some very familiar families of matroids, although it also contains several sporadic members, meaning that there is probably little hope of a good characterisation.

An easy way to choose a basis of a matroid involves an application of the greedy algorithm. We will first define some terminology. Suppose that E is a set, and that τ is a subset of $E \times E$. If τ is reflexive, antisymmetric, and transitive, then τ is a *partial order*. In this case, we will write $x \leq_\tau y$ to indicate the fact that $(x, y) \in \tau$. If ρ is a partial order on E, and, for any pair of elements, $x, y \in E$, either $x \leq_\rho y$ or $y \leq_\rho x$, then ρ is a *total ordering of* E.

To implement the greedy algorithm on a matroid, M, we need a total ordering, ρ, of its ground set, E. Let the sets S and T take on the initial values $S = \varnothing$ and $T = E$. The algorithm proceeds as follows:

> **While** $T \neq \varnothing$ **do:**
>
> Choose $x \in T$ such that $x \leq_\rho y$ for all $y \in T$.
>
> $T \leftarrow T - x$
>
> **If** $S \cup x$ is independent, **then** $S \leftarrow S \cup x$, **otherwise** $S \leftarrow S$.

At the completion of this algorithm, the output, S, is a basis of M. Clearly S is dependent on the choice of total order. The output of the algorithm will be denoted by $B(\rho)$, and we shall say that ρ *selects* the basis $B(\rho)$.

Let M be a matroid on the ground set E. For every basis, B, of M, let p_B be the probability that B will be the output of the greedy algorithm if the total ordering of E is chosen uniformly at random. Let $O(B, M)$ be the number of total orderings of E that select B. Clearly $p_B = O(B, M)/n!$, where $n = |E(M)|$.

Definition 4.1.1. Let M be a matroid. If, for any two bases, B and B', of M, the equality $p_B = p_{B'}$ holds, then M is *equitable*.

Equivalently, we say that M is equitable if $O(B, M) = O(B', M)$ for any two bases, B and B', of M. A simple necessary condition for a matroid to be equitable follows almost immediately from the definition. We shall shortly see that this condition is not sufficient.

Proposition 4.1.2. *Suppose that M is an equitable matroid. If $|E(M)| = n$, then the number of bases of M divides $n!$.*

Proof. Let B be a basis of M, and let $b(M)$ be the number of bases of M. Because M is equitable, it follows that $p_B = 1/b(M)$. But p_B also equals $O(B, M)/n!$, so $O(B, M) = n!/b(M)$. □

The next result is hardly surprising.

Proposition 4.1.3. *All uniform matroids are equitable.*

Proof. Suppose that M is an isomorphic copy of $U_{r,n}$ on the ground set E, and that ρ is a total ordering of E. We may consider ρ as an ordered listing of the elements of E. It is easy to see that $B(\rho)$ consists exactly of the first r elements that appear in this listing. Thus, if B is a basis of M, the number of total orderings of E that select B is equal to the number of orderings that list the elements of B before any other elements. There are $r!(n-r)!$ such orderings. Since this is true for any basis, it follows that M is equitable. □

The class of equitable matroids also contains less trivial examples. Recall that the Fano plane is the rank-3 matroid represented in Figure 4.1.

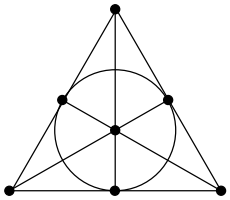


Figure 4.1: F_7, The Fano plane.

Proposition 4.1.4. *The Fano plane is an equitable matroid.*

Proof. Suppose that E is the ground set of F_7 and that B is a basis. It is not difficult to see that a total ordering of E that selects B must either: first list all the elements of B, in some order, followed by all the elements of $E-B$; or, list two elements, x and y, of B, followed by the single element in $\mathrm{cl}(\{x, y\}) - B$, followed by the remaining element in B and then the remaining elements in $E - B$.

In the first case, there are 3!4! ways of ordering the elements. In the second, we may choose x and y in 3 ways, and then there are 2!3! ways of ordering the elements. Therefore $O(B, F_7) = 3!4! + 3(2!3!) = 180$. Since this is true for any basis, it follows that F_7 is equitable. □

The number of bases of $U_{r,n}$ is $\binom{n}{r}$, which clearly divides $n!$. Similarly, the number of bases of F_7 is $\binom{7}{3} - 7 = 28$, which divides $7! = 5040$, so the necessary condition described in Proposition 4.1.2 holds in both cases. In fact, $7!/28 = 180 = O(B, F_7)$, as expected. We shall see later that all projective geometries, and their truncations, are equitable.

Recall that, if B is a basis of M and $e \in E - B$, then the fundamental circuit of e with respect to B is denoted by $C(e, B)$. Consider the relation $\tau(B, M) \subseteq E \times E$, so that $(x, y) \in \tau(B, M)$ if and only if either: $x = y$; or $x \in B$, $y \in E - B$, and $x \in C(y, B)$. It is easy to see that $\tau(B, M)$ is a partial order.

If τ and ρ are partial and total orders respectively on the set E, and $\tau \subseteq \rho$, then ρ is a *linear extension of* τ.

Proposition 4.1.5. *Let B be a basis of M, and let ρ be a total ordering of $E(M)$. Then $B(\rho) = B$ if and only if ρ is a linear extension of $\tau(B, M)$.*

Proof. Suppose that ρ is a linear extension of $\tau(B,\, M)$. Assume that, at some stage during the implementation of the greedy algorithm, $S \subseteq B$, and that y is the minimal element under ρ in T. Suppose also that $S \cup y$ is independent, but that $y \notin B$. It follows that $C(y,\, B)$ is not contained in $S \cup y$. Therefore there exists an element, x, in $C(y,\, B) - (S \cup y)$. Since y is the minimal element in T, and $x \leq_\rho y$, it must be the case that $x \notin T$. Therefore x has already been considered by the algorithm. As $x \notin S$, it follows that $S \cup x$ is dependent. But $S \cup x \subseteq B$, so this is a contradiction. Therefore $B(\rho) = B$.

We will now assume that $B(\rho) = B$. Suppose that $y \notin B$, and that $x \in C(y,\, B) - y$, while $y \leq_\rho x$, so that ρ is not a linear extension of $\tau(B,\, M)$. Suppose that the greedy algorithm proceeds until y is the minimal element in T under ρ. This implies that $x \in T$. Note that $(B - x) \cup y$ is a basis, and $S \cup y$ is a subset of this basis, so is therefore independent. Therefore $y \in B(\rho)$. This is a contradiction, so ρ must be a linear extension of $\tau(B,\, M)$. □

Corollary 4.1.6. *The number of total orderings that select a basis B in the matroid M is equal to the number of linear extensions of $\tau(B,\, M)$.*

Suppose that B is a basis of M, and that the ground set of M is E. Recall that $C^*(e,\, E - B)$ is the fundamental cocircuit of e with respect to $E - B$, where $e \in B$. It is elementary to demonstrate that, if $x \in B$ and $y \in E - B$, then $y \in C^*(x,\, E - B)$ if and only if $x \in C(y,\, B)$. It follows from this that $\tau(E - B,\, M^*) = (\tau(B,\, M))^{-1}$. Hence ρ is a linear extension of $\tau(B,\, M)$ if and only if ρ^{-1} is a linear extension of $\tau(E - B,\, M^*)$, and thus $O(B,\, M) = O(E - B,\, M^*)$. The next proposition follows immediately from this discussion.

Proposition 4.1.7. *If M is an equitable matroid, then M^* is also equitable.*

Recall that a loose basis of the matroid M is a basis, B, such that $B \cup e$ is a circuit for every element $e \in E(M) - B$. The next result provides a useful method of proving that a matroid is not equitable.

Proposition 4.1.8. *If M is an equitable matroid and has a loose basis, then M is uniform.*

Proof. Suppose that B is a loose basis of M. It follows that $\tau(B,\, M)$ comprises all pairs $(x,\, x)$, where $x \in E(M)$, and all pairs $(x,\, y)$, where $x \in B$ and $y \in E(M) - B$. It follows that if a total ordering of $E(M)$ selects B, then it must list the elements of B before any other elements. Hence $O(B,\, M) = r!(n - r)!$, where $r = |B| = r(M)$, and $n = |E(M)|$. We have already noted that if M is an equitable matroid, then $O(B,\, M) = n!/b(M)$. Therefore the number of bases of M is

$$\frac{n!}{O(B,\, M)} = \frac{n!}{r!(n - r)!} = \binom{n}{r}.$$

From this we deduce that every r-set of $E(M)$ is a basis of M, and, hence, that M is uniform. □

We now have enough material to show that the necessary condition in Proposition 4.1.2 for a matroid to be equitable is not sufficient. To see this, consider the rank-2 matroid on the set $\{1, \ldots, 6\}$, where the only non-trivial parallel class is the set $\{1, \ldots, 4\}$. It is easy to see that the number of bases in this matroid is 9, which divides 6!. However, $\{5, 6\}$ is a loose basis, and the matroid is certainly not uniform, so Proposition 4.1.8 implies that it is not equitable.

Proposition 4.1.9. *A matroid is equitable if and only if all of its connected components are.*

Proof. Suppose that M is a matroid on the ground set E, and that (X_1, X_2) is a separation of M. It is sufficient to show that M is equitable if and only if both $M|X_1$ and $M|X_2$ are equitable.

Suppose that ρ is a total ordering of $E(M)$, and, for $i \in \{1, 2\}$, let ρ_i be the total ordering of X_i that is induced by ρ. In other words, ρ_i is a total ordering of X_i, and $x \leq_{\rho_i} y$ if and only if x and y are both members of X_i, and $x \leq_\rho y$.

Let $B = B_1 \cup B_2$ be a basis of M, where B_i is a basis of $M|X_i$ for $i = 1, 2$. If ρ is a total ordering of E, then we claim that ρ selects B if and only if ρ_i selects B_i for all $i \in \{1, 2\}$.

To prove this claim, let us first assume that $B(\rho) = B$, but that ρ_1 does not select B_1. By Proposition 4.1.5 there must exist elements $x \in B_1$ and $y \in X_1 - B_1$ such that $x \in C_{M|X_1}(y, B_1)$, but $y \leq_{\rho_1} x$. However $C_{M|X_1}(y, B_1) = C_M(y, B)$, and $y \leq_\rho x$, so this implies that $B(\rho) \neq B$. The same argument shows that $B(\rho_2)$ must be equal to B_2.

Therefore let us assume that $B(\rho_i) = B_i$ for $i = 1, 2$. Suppose that ρ does not select B. This implies that there must be some element $y \in E - B$, and an element $x \in B$, such that $y \leq_\rho x$ and $x \in C_M(y, B)$. Hence, either $y \leq_{\rho_1} x$ or $y \leq_{\rho_2} x$, and this again leads us to a contradiction.

It follows from this claim that $O(B, M)$ is equal to the number of total orderings of E that induce orderings, ρ_1 and ρ_2, that select B_1 and B_2 respectively. If ρ_1 and ρ_2 are such orderings of X_1 and X_2, then there are $\binom{x_1+x_2}{x_1}$ total orderings of E that induce ρ_1 and ρ_2, where $x_i = |X_i|$, for if we think of a total ordering of E as a listing of the elements in E, then choosing an order that induces both ρ_1 and ρ_2 is equivalent to choosing the x_1 positions that we will fill with elements from X_1. Therefore it follows that $O(B, M) = \binom{x_1+x_2}{x_1} O(B_1, M|X_1) O(B_2, M|X_2)$.

Let us now suppose that $M|X_1$ and $M|X_2$ are both equitable, and let $B' = B'_1 \cup B'_2$ be another basis of M, where $B'_i = B' \cap X_i$ for $i = 1, 2$. Then, using the arguments above, we see that

$$\begin{aligned} O(B', M) &= \tbinom{x_1+x_2}{x_1} O(B'_1, M|X_1) O(B'_2, M|X_2) \\ &= \tbinom{x_1+x_2}{x_1} O(B_1, M|X_1) O(B_2, M|X_2) \\ &= O(B, M). \end{aligned}$$

It follows that M is equitable.

Conversely, suppose that M is equitable. Let B'_1 be a basis of $M|X_1$ that is distinct from B_1. Then, as $B = B_1 \cup B_2$ and $B' = B'_1 \cup B_2$ are both bases of M, it

follows that

$$\begin{aligned}\binom{x_1+x_2}{x_1}O(B_1,\, M|X_1)O(B_2,\, M|X_2) &= O(B,\, M) = O(B',\, M)\\ &= \binom{x_1+x_2}{x_1}O(B'_1,\, M|X_1)O(B_2,\, M|X_2),\end{aligned}$$

and therefore $O(B'_1,\, M|X_1) = O(B_1,\, M|X_1)$. Hence $M|X_1$ is equitable, and, by symmetrical arguments, so is $M|X_2$. □

Although the class of equitable matroids is closed under duality and direct sums, it would be too much to ask that it also closed under taking minors. We shall see later that, indeed, a minor of an equitable matroid need not be equitable.

4.2 Counting linear extensions

Proposition 4.1.5 indicates that counting the number of linear extensions of a partial order will be useful in the study of equitable matroids. In this section we will develop a systematic method for performing this task.

In general counting linear extensions is computationally difficult [3]. However, the partial orders that arise in this chapter are of a very restricted type. If τ is a partial order on the finite set E, and $x,\, y \in E$, then x and y are *comparable* if either $(x,\, y) \in \tau$, or $(y,\, x) \in \tau$. A *chain* is a set of pairwise comparable elements. The *height* of τ is the size of its largest chain. If B is a basis of a matroid, M, then it is easy to see that the partial order, $\tau(B,\, M)$, has height two.

The methods described in this section provide an organised way to count the linear extensions of a partial order of height two, but they do not lead to a polynomial-time algorithm for performing this task. Brightwell and Winkler [3] have proved that counting linear extensions is a #P-complete problem for partial orders of height five, and they claim that their methods can be extended to show that the problem remains #P-complete for partial orders of height three. The complexity of the problem for height two has not yet been established.

Let M be a matroid on the set E, and let B be a basis of M. We introduce two bipartite graphs, $G(B,\, M)$ and $\overline{G}(B,\, M)$. The vertex set of both graphs is E, and $(B,\, E - B)$ is a bipartition of both graphs. If $x \in B$ and $y \in E - B$, then x and y are adjacent in $G(B,\, M)$ if and only if $x \in C_M(y,\, B)$, while x and y are adjacent in $\overline{G}(B,\, M)$ if and only if $x \notin C_M(y,\, B)$.

Suppose that ρ is a total ordering of E. We shall associate a bipartite graph, which we will denote by $G_{B,\,M}(\rho)$, to ρ. Let X be the set $\{x \in B \mid \exists\, y \in E - B,\, y \leq_\rho x\}$. Also, let Y be equal to $\{y \in E - B \mid \exists\, x \in B,\, y \leq_\rho x\}$. The vertex set of $G_{B,\,M}(\rho)$ is $X \cup Y$, and $(X,\, Y)$ is a bipartition of $G_{B,\,M}(\rho)$. If $x \in X$ and $y \in Y$, then x and y are adjacent in $G_{B,\,M}(\rho)$ if and only if $y \leq_\rho x$. If we suppose that ρ is a linear extension of $\tau(B,\, M)$, then it follows that there cannot be an edge between x and y in $G_{B,\,M}(\rho)$ if $x \in C_M(y,\, B)$. Therefore $G_{B,\,M}(\rho)$ is a subgraph of $\overline{G}(B,\, M)$.

We now introduce a class of bipartite graphs, and show that $G_{B,\,M}(\rho)$ must belong to this class. The following material was developed by Yannakakis [56].

Suppose that e and e' are two edges in a graph, joining u to v, and u' to v' respectively. We will say that e and e' are *independent* if u, v, u', and v' are distinct

vertices, and there is no edge joining a vertex in $\{u, v\}$ to a vertex in $\{u', v'\}$. If G is a bipartite graph with no isolated vertices, and G contains no pair of independent edges, then G is a *chain graph*. Note that Yannakakis does not demand that a chain graph contains no isolated vertices.

Proposition 4.2.1. *Let B be a basis of the matroid M on the ground set E, and let ρ be a total ordering of E. Then $G_{B,M}(\rho)$ is a chain graph.*

Proof. By construction, $G_{B,M}(\rho)$ does not contain any isolated vertices. Let $\{x, x', y, y'\}$ be a set of distinct elements such that $x, x' \in B$, and $y, y' \in E - B$. Suppose that x is adjacent to y in $G_{B,M}(\rho)$ and x' is adjacent to y'. We wish to prove that either x is adjacent to y', or x' is adjacent to y. Assume that x is not adjacent to y'. Then $x \leq_\rho y'$. However, $y \leq_\rho x$, as x and y are adjacent in $G_{B,M}(\rho)$, and, similarly, $y' \leq_\rho x'$. It follows that $y \leq_\rho x'$, so x' and y are adjacent in $G_{B,M}(\rho)$. □

Our strategy for counting the linear extensions of $\tau(B, M)$ will be as follows: for each chain subgraph, H, of $\overline{G}(B, M)$, we determine how many linear extensions, ρ, have $G_{B,M}(\rho) = H$. By summing these quantities together, we arrive at the number of linear extensions. We need now only find a way to quantify the number of linear extensions that induce the chain subgraph H. To do this we will prove a characterisation of chain graphs.

Suppose that A is a set. An *ordered partition of A* is an ordered collection, $(A_1, \ldots, A_t)$, of non-empty, pairwise disjoint sets, such that $\cup_{i=1}^{t} A_i = A$.

Proposition 4.2.2. *Suppose that G is a bipartite graph, and that (X, Y) is an ordered partition that is also a bipartition of the vertex set of G. Then G is a chain graph if and only if, G is the empty graph, or there exists a pair of ordered partitions, $(X_1, \ldots, X_t)$ and $(Y_1, \ldots, Y_t)$, of X and Y respectively, such that, if $y \in Y_i$, then y is adjacent to the vertex x if and only if $x \in X_j$, where $i \leq j$. Moreover, if such a pair exists, then it is unique.*

Proof. First let us suppose that such partitions exist. Then there are no isolated vertices in G, for, if $y \in Y_i$, then y is adjacent to every vertex in X_i (which is non-empty). Similarly, if $x \in X_i$, then x is adjacent to every vertex in Y_i.

Let us suppose that e and e' are two edges of G, joining, respectively, x to y, and x' to y', where $x, x' \in X$, $y, y' \in Y$, and x, y, x', and y' are four distinct vertices. We wish to show that either x is adjacent to y', or x' is adjacent to y. If v is any vertex of G, then let $f(v)$ be the integer such that either $v \in X_{f(v)}$, or $v \in Y_{f(v)}$. Suppose that x is not adjacent to y'. Then $f(x) < f(y')$. However, $f(y) \leq f(x)$, and $f(y') \leq f(x')$, so $f(y) < f(x')$. Hence y is adjacent to x'. Thus G contains no pair of independent edges, and is therefore a chain graph.

Let us now suppose that G is a chain graph. If v is a vertex of G, then $N(v)$ denotes the neighbourhood set of v: the set of vertices that are adjacent to v. We will show that the neighbourhood sets of vertices in Y are totally ordered by set inclusion. That is, if y and y' are vertices in Y, then either $N(y) \subseteq N(y')$ or $N(y') \subseteq N(y)$. Suppose that this is not the case. Then there exist vertices $x \in N(y) - N(y')$, and

$x' \in N(y') - N(y)$. It is easy to see that the edges joining x to y, and x' to y' are independent, which contradicts our assumption that G is a chain graph.

Let $N_1, \ldots, N_t$ be the distinct members of the family $\{N(y) \mid y \in Y\}$, and assume that they are labelled so that N_j is properly contained in N_i whenever $i < j$; that is, the statement

$$N_1 \supset N_2 \supset \cdots \supset N_{t-1} \supset N_t$$

holds. Note that $N_1 = X$, for otherwise there exists a vertex in X that is adjacent to no vertex in Y, and we have assumed that G has no isolated vertices. For $i \in \{1, \ldots, t\}$, let Y_i be the set of vertices in Y that have N_i as their neighbourhood set. For $i \in \{1, \ldots, t-1\}$, let $X_i = N_i - N_{i+1}$, and let $X_t = N_t$. It is clear that none of the sets in $\{X_1, \ldots, X_t, Y_1, \ldots, Y_t\}$ is empty, and, furthermore, if $y \in Y_i$ and $x \in X_j$, then y and x are adjacent if and only if $i \leq j$.

It is easy to see that the partitions we have just described must be unique. □

Proposition 4.2.3. *Suppose that B is a basis of the matroid M, where the ground set of M is E, and that H is a chain subgraph of the graph $\overline{G}(B, M)$. Suppose that the vertex set of H is $X \cup Y$, where $X \subseteq B$, and $Y \subseteq E - B$. If H is the empty graph, then the number of linear extensions, ρ, of $\tau(B, M)$, such that $G_{B,M}(\rho) = H$ is $|B|!|E - B|!$. If G is not the empty graph, then let $(X_1, \ldots, X_t)$ and $(Y_1, \ldots, Y_t)$ be the ordered partitions of X and Y respectively that are described in the statement of Proposition 4.2.2. The number of linear extensions, ρ, of $\tau(B, M)$, such that $G_{B,M}(\rho) = H$, is*

$$|B - X|!|Y_1|!|X_1|! \cdots |Y_t|!|X_t|!|(E - B) - Y|!.$$

Proof. If we consider ρ as a listing of the elements of E, then it is clear that $G_{B,M}(\rho) = H$ if and only if the elements of $B - X$ come first, in some order, followed by the elements of Y_1, and then X_1, and so forth. The result follows immediately. □

We now have a method for counting the linear extensions of $\tau(B, M)$: for each chain subgraph, H, of $\overline{G}(B, M)$ we use Proposition 4.2.3 to calculate the number of linear extensions, ρ, of $\tau(B, M)$ that have $G_{B,M}(\rho) = H$. Summing these quantities gives us the desired number.

4.3 Super-equitable matroids

In this section we will describe a family of matroids that seems to dominate the set of equitable matroids.

Definition 4.3.1. A matroid, M, is *super-equitable* if $G(B, M)$ and $G(B', M)$ are isomorphic whenever B and B' are bases of M.

It is easy to see that every uniform matroid is super-equitable: if B is any basis of $U_{r,n}$, then $G(B, M)$ is isomorphic to $K_{r,n-r}$.

Proposition 4.3.2. *If M is super-equitable, then so is M^*.*

Proof. If the ground set of M is E, and B is a basis of M, then $G(B,\, M)$ is equal to $G(E - B,\, M^*)$, so the result follows immediately. □

We will use the next result to prove that super-equitable matroids are equitable. It follows from [42, Lemma 10.2.8, Corollary 10.2.9]. (See also [2, Theorem 6.3].)

Proposition 4.3.3. *Suppose that B is a basis of the matroid M, where $E = E(M)$. If $(X_1, \ldots, X_t)$ is the partition of E into the vertex sets of the connected components of $G(B,\, M)$, then $M = M|X_1 \oplus \cdots \oplus M|X_t$ is the decomposition of M into its connected components.*

Proposition 4.3.4. *A matroid is super-equitable if and only if all of its connected components are.*

Proof. Suppose that M_1 and M_2 are super-equitable matroids on the ground sets E_1 and E_2 respectively. We will show that $M_1 \oplus M_2$ is super-equitable. Let B and B' be bases of $M_1 \oplus M_2$, and, for $i \in \{1,\, 2\}$, let $B_i = B \cap E_i$ and $B'_i = B' \cap E_i$. Since M_1 and M_2 are super-equitable, there exist functions, f_1 and f_2, such that f_i is an isomorphism between $G(B_i,\, M_i)$ and $G(B'_i,\, M_i)$, where $i = 1,\, 2$. Proposition 4.3.3 implies that $G(B,\, M_1 \oplus M_2)$ is the disjoint union of $G(B_1,\, M_1)$ and $G(B_2,\, M_2)$, and, similarly, $G(B',\, M_1 \oplus M_2)$ is the disjoint union of $G(B'_1,\, M_1)$ and $G(B'_2,\, M_2)$. Define the function $f_1 \cup f_2$, so that, for every $e \in E_1 \cup E_2$, the image of e under $f_1 \cup f_2$ is $f_i(e)$, where $i \in \{1,\, 2\}$ and $e \in E_i$. Clearly $f_1 \cup f_2$ is an isomorphism between $G(B,\, M_1 \oplus M_2)$ and $G(B',\, M_1 \oplus M_2)$, and hence $M_1 \oplus M_2$ is super-equitable.

To prove the converse let us suppose that M is a super-equitable matroid, and that $M = M_1 \oplus \cdots \oplus M_t$ is the decomposition of M into its connected components. Let B_i and B'_i be two bases of M_i, where $i \in \{1, \ldots, t\}$, and, for $j \in \{1, \ldots, i - 1,\, i + 1, \ldots, t\}$, let B_j be a basis of M_j, so that

$$B = \left(\bigcup_{\substack{j=1 \\ j \neq i}}^{t} B_j \right) \cup B_i \quad \text{and} \quad B' = \left(\bigcup_{\substack{j=1 \\ j \neq i}}^{t} B_i \right) \cup B'_i$$

are two bases of M. Since M is super-equitable there exists an isomorphism, f, between $G(B,\, M)$ and $G(B',\, M)$.

Proposition 4.3.3 implies that the connected components of $G(B,\, M)$ are $G(B_j,\, M_j)$, where $1 \leq j \leq t$. The graph $G(B',\, M)$ can be produced from $G(B,\, M)$ by deleting $G(B_i,\, M_i)$ and replacing it with $G(B'_i,\, M_i)$. Obviously f takes connected components of $G(B,\, M)$ to connected components of $G(B',\, M)$, so consider the directed graph that has $\{v_1, \ldots, v_t\} \cup \{v'_i\}$ as its vertices. Each of the vertices, $v_j \in \{v_1, \ldots, v_t\}$, corresponds to the connected component, $G(B_j,\, M_j)$, of $G(B,\, M)$, while v'_i corresponds to the connected component, $G(B'_i,\, M_i)$, of $G(B',\, M)$. Place an arrow in the graph from vertex u to vertex v if f takes the connected component corresponding to u to the connected component corresponding to v. If $j \neq i$, then there is exactly one arrow going into v_j, and exactly one arrow coming out. On the other hand, v_i has outdegree one and indegree zero, while v'_i has indegree one and

outdegree zero. It follows immediately that there is a directed path from v_i to v_i', and that, therefore, $G(B_i, M_i)$ and $G(B_i', M_i)$ are isomorphic. This completes the demonstration that M_i is super-equitable. □

Proposition 4.3.5. *Every super-equitable matroid is equitable.*

Proof. It will suffice to prove the proposition for connected super-equitable matroids, for, if M is super-equitable, then each of its connected components is also super-equitable by Proposition 4.3.4. If each of these connected components is also equitable, then M is equitable by Proposition 4.1.9.

Let us therefore suppose that M is a connected super-equitable matroid on the ground set E. Let B and B' be two bases of M. We will show that $O(B, M) = O(B', M)$, and this will complete the proof. Both $G(B, M)$ and $G(B', M)$ are connected by Proposition 4.3.3. Therefore $(B, E - B)$ and $(B', E - B')$ are the only bipartitions of $G(B, M)$ and $G(B', M)$ respectively. Suppose that f is an isomorphism between $G(B, M)$ and $G(B', M)$. Then f must take B to either B' or $E - B'$.

First suppose that f takes B to B'. Since f is an isomorphism between $G(B, M)$ and $G(B', M)$, it follows that $x \in C(y, B)$ if and only if $f(x) \in C(f(y), B')$, for any elements $x \in B$ and $y \in E - B$. Therefore $(x, y) \in \tau(B, M)$ if and only if $(f(x), f(y)) \in \tau(B', M)$. Hence $\tau(B, M)$ and $\tau(B', M)$ are isomorphic partial orders. Suppose that ρ is a total ordering of E. It follows that ρ is a linear extension of $\tau(B, M)$ if and only if $f(\rho)$ is a linear extension of $\tau(B', M)$, where $f(\rho)$ is defined so that $x \leq_\rho y$ if and only $f(x) \leq_{f(\rho)} f(y)$ for any elements $x, y \in E$. Thus $O(B, M) = O(B', M)$.

Similarly, if f takes B to $E - B'$, then $x \in C(y, B)$ if and only if $f(y) \in C(f(x), B')$ for any elements $x \in B$ and $y \in E - B$. Thus $(x, y) \in \tau(B, M)$ if and only if $(f(y), f(x)) \in \tau(B', M)$. It follows that ρ is a linear extension of $\tau(B, M)$ if and only if $(f(\rho))^{-1}$ is a linear extension of $\tau(B', M)$. Therefore $O(B, M)$ is again equal to $O(B', M)$. □

The converse of Proposition 4.3.5 is not true. The class of super-equitable matroids is properly contained in the class of equitable matroids. However, examples of matroids that are equitable without being super-equitable seem to be very hard to find. The matroid in the following example was found with the aid of a computer search.

Example 1. Let A, B, and C be pairwise disjoint sets, where $|A| = |B| = 8$, and $|C| = 20$. Let M_{36} be the rank-3 matroid on $A \cup B \cup C$ that has as its non-spanning circuits any pair of elements from A, any pair of elements from B, and any triple of elements from C.

Thus M_{36} is isomorphic to the rank-3 matroid that contains a line of length twenty, and two parallel classes in general position, each of size eight.

Proposition 4.3.6. *M_{36} is equitable but not super-equitable.*

Proof. It is evident that there are only two types of bases in M_{36}. A basis of the first type contains one element from each of the sets A, B, and C. A basis of the second type contains two elements from C, and a single element from either A or B.

Let $B_1 = \{a_1, b_1, c_1\}$ be any basis of the first type, where $a_1 \in A$, $b_1 \in B$, and $c_1 \in C$. Then $\overline{G}(B_1, M_{36})$ has the following structure: there are nineteen isolated vertices (the members of $C - c_1$), seven vertices that are adjacent to both b_1 and c_1 (the members of $A - a_1$), and seven vertices adjacent to both a_1 and c_1 (the members of $B - b_1$).

If $B_2 = \{a_2, c_2, c_2'\}$ is a basis of the second type, where $a_2 \in A$, and $c_2, c_2' \in C$, then $\overline{G}(B_2, M_{36})$ has the following structure: there are eight isolated vertices (the members of B), eighteen vertices that are adjacent only to a_2 (the members of $C - \{c_2, c_2'\}$), and seven vertices that are adjacent to both c_2 and c_2' (the members of $A - a_2$). We will prove that $O(B_1, M_{36}) = O(B_2, M_{36})$, and it will follow easily from this that M_{36} is equitable.

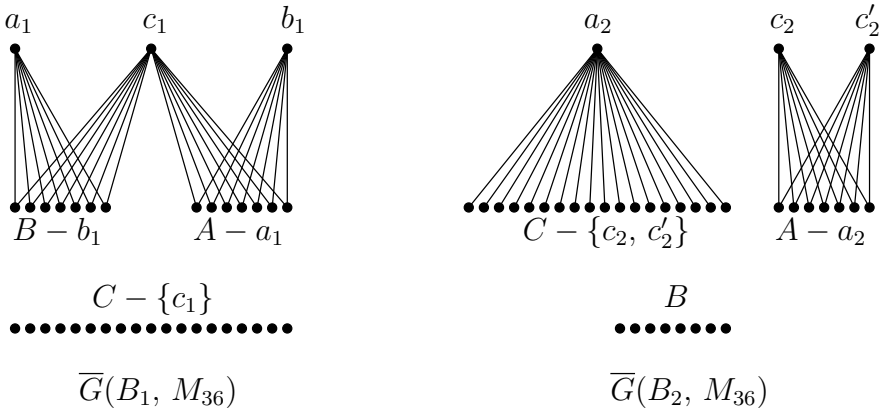


Figure 4.2: $\overline{G}(B_1, M_{36})$ and $\overline{G}(B_2, M_{36})$.

Let us consider the chain subgraphs of $\overline{G}(B_1, M_{36})$. First there is the empty graph. This subgraph is induced by the linear extensions of $\tau(B_1, M_{36})$ that have no element of $(A \cup B \cup C) - B_1$ preceding an element of B_1. There are 3!33! such linear extensions.

Next let us consider the chain subgraphs that contain only one element, x, from B_1. Such a graph is isomorphic to $K_{1,i}$, for some i. First suppose that x is equal to a_1 or b_1. Then $1 \leq i \leq 7$. There are $\binom{7}{i}$ ways to choose the vertices in $B - b_1$ or $A - a_1$ that are adjacent to a_1 or b_1 in this subgraph. It therefore follows from Proposition 4.2.3 that the number of linear extensions of $\tau(B_1, M_{36})$ that induce such a subgraph is

$$2\sum_{i=1}^{7} \binom{7}{i} 2!i!1!(33-i)!.$$

Let us define the function f_1, so that

$$f_1(x, r^*) = \sum_{i=1}^{x} \binom{x}{i} i!(r^* - i)!.$$

Then we may similarly show that the number of linear extensions that induce a chain subgraph that contains c_1 and no other vertex in B_1 is equal to $2f_1(14, 33)$.

No chain subgraph of $\overline{G}(B_1, M_{36})$ can contain more than two elements of B_1. Any such subgraph that does contain two elements of B_1 must contain c_1. Let us consider chain subgraphs that contain c_1 and a_1. Proposition 4.2.2 tells us that such a chain subgraph corresponds to an ordered partition of $\{a_1, c_1\}$. We will first suppose $\{a_1, c_1\}$ is the only block of this partition. Thus we are considering graphs that are isomorphic to $K_{2,i}$, where $1 \leq i \leq 7$. It follows from Proposition 4.2.3 that these graphs contribute a total of

$$\sum_{i=1}^{7} \binom{7}{i} 1!i!2!(33-i)! = 2f_1(7, 33)$$

to the number of linear extensions of $\tau(B_1, M_{36})$. Similarly, those chain subgraphs that are isomorphic to $K_{2,i}$ and that contain $\{b_1, c_1\}$ contribute $2f_1(7, 33)$ to the number of linear extensions of $\tau(B_1, M_{36})$.

We will now consider the chain subgraphs of $\overline{G}(B_1, M_{36})$ that contain a_1 and c_1, partitioned as $(\{a_1\}, \{c_1\})$. We have $\binom{7}{i}$ ways of choosing i vertices from $B - b_1$ to be adjacent to both a_1 and c_1, and then $\binom{7-i}{j}\binom{7}{k}$ ways of choosing j different vertices from $B - b_1$ and k vertices from $A - a_1$ to be adjacent to c_1 only. In this case i, j, and k are subject to the restrictions $1 \leq i$, $j + k$ and $i + j$, $k \leq 7$. Thus the total number of linear extensions of $\tau(B_1, M_{36})$ that induce a chain subgraph of this type is

$$\sum_{\substack{1 \leq i,\, j+k \\ i+j,\, k \leq 7}} \binom{7}{i}\binom{7-i}{j}\binom{7}{k} 1!i!1!(j+k)!1!(33-i-j-k)!.$$

Let us define the function, f_2, to be

$$f_2(x, y, r^*) = \sum_{\substack{1 \leq i,\, j+k \\ i+j \leq x \\ k \leq y}} \binom{x}{i}\binom{x-i}{j}\binom{y}{k} i!(j+k)!(r^*-i-j-k)!.$$

We may similarly show that the number of linear extensions of $\tau(B_1, M_{36})$ that induce a chain subgraph that contains b_1 and c_1, partitioned as $(\{b_1\}, \{c_1\})$, is equal to $f_2(7, 7, 33)$.

Next we will consider chain subgraphs of $\overline{G}(B_1, M_{36})$ that contain a_1 and c_1, partitioned as $(\{c_1\}, \{a_1\})$. There are $\binom{7}{i}$ ways of choosing i vertices from $B - b_1$ to be adjacent to both c_1 and a_1, and then $\binom{7-i}{j}$ of choosing j different vertices to be adjacent to a_1 only, where $1 \leq i$, j and $i + j \leq 7$. It follows that number of linear extensions that induce such subgraphs is equal to

$$\sum_{\substack{1 \leq i,\, j \\ i+j \leq 7}} \binom{7}{i}\binom{7-i}{j} 1!i!1!j!1!(33-i-j)! = f_2(7, 0, 33).$$

Similarly, the number of linear extensions that induce chain subgraphs that contain b_1 and c_1, partitioned as $(\{c_1\}, \{a_1\})$, is equal to $f_2(7, 0, 33)$.

We have shown that $O(B_1,\, M_{36})$ is equal to

$$3!33! + 8f_1(7,\, 33) + 2f_1(14,\, 33) + 2f_2(7,\, 7,\, 33) + 2f_2(7,\, 0,\, 33).$$

Mathematical software shows that this number is equal to

$$861095663865512077472220944793600000000$$

(see Appendix B).

Let us now count the linear extensions of $\tau(B_2,\, M_{36})$. The number of linear extensions that induce the empty chain subgraph of $\overline{G}(B_2,\, M_{36})$ is $3!33!$. The chain subgraphs that contain only one element, x, from B_2, are isomorphic to $K_{1,\,i}$. If x is equal to c_2 or c_2', then $1 \leq i \leq 7$, and there are $\binom{7}{i}$ ways in which we can choose the vertices to be adjacent to x. If $x = a_2$, then $1 \leq i \leq 18$, and there are $\binom{18}{i}$ ways in which we can choose i vertices from $C - \{c_2,\, c_2'\}$ to be adjacent to a_2. Thus the total number of linear extensions that induce a chain subgraph of this type is

$$2\sum_{i=1}^{7}\binom{7}{i}2!i!1!(33-i)! + \sum_{i=1}^{18}\binom{18}{i}2!i!1!(33-i)! = 4f_1(7,\, 33) + 2f_1(18,\, 33).$$

Let us now consider the chain subgraphs of $\overline{G}(B_2,\, M_{36})$ that contain two elements of B_2. Those two elements must be c_2 and c_2'. Consider those chain subgraphs of $\overline{G}(B_2,\, M_{36})$ that correspond to a partition of $\{c_2,\, c_2'\}$ that contains $\{c_2,\, c_2'\}$ as its only block. It is straightforward to see that the number of linear extensions of $\tau(B_2,\, M_{36})$ that induce such chain subgraphs is equal to $2f_1(7,\, 33)$.

Next we consider those chain subgraphs that contain c_2 and c_2', partitioned as either $(\{c_2\},\, \{c_2'\})$ or $(\{c_2\},\, \{c_2'\})$. The number of linear extensions that correspond to such chain subgraphs is $2f_2(7,\, 0,\, 33)$.

We have shown that the total number of linear extensions of $\tau(B_2,\, M_{36})$ is

$$3!33! + 6f_1(7,\, 33) + 2f_1(18,\, 33) + 2f_2(7,\, 0,\, 33)$$

This sum is again equal to

$$861095663865512077472220944793600000000.$$

By direct calculation we see that if B_1 is a basis of the first type, then $G(B_1,\, M_{36})$ contains $19.3 + 14 = 71$ edges, whereas if B_2 is a basis of the second type, then $G(B_2,\, M_{36})$ contains $8.3 + 18.2 + 7 = 67$ edges, so there is certainly no isomorphism between $G(B_1,\, M_{36})$ and $G(B_2,\, M_{36})$. Therefore M_{36} is not super-equitable. □

It is not difficult to see that M_{36} has $20.8^2 + 2.8.\binom{20}{2} = 4320$ bases. Dividing $36!$ by this number again gives us

$$861095663865512077472220944793600000000,$$

so the necessary condition for a matroid to be equitable, described in Proposition 4.1.2, holds.

The computer search that found M_{36} tested rank-3 matroids containing a line of length l and two freely placed parallel classes, each of size m, where every parallel class in the line has size n. A search of this class of matroids, where l ranges between 3 and 50, and m and n both range between 1 and 20, reveals only two equitable matroids out of the 19200 tested. One such matroid is M_{36} (obtained by setting (l, m, n) to (20, 8, 1)). The other is obtained by setting (l, m, n) to (20, 16, 2). (If we follow the pattern set by these two examples, and set (l, m, n) to (20, 32, 4) or (20, 64, 8), then we produce two more equitable matroids.)

We may deduce two things from this apparent scarcity of equitable matroids. Firstly, it follows immediately that the class of equitable matroids is not closed under taking minors. (If we set (l, m, n) to (19, 8, 1), then the resulting matroid is isomorphic to that obtained by deleting an element from M_{36}.) Secondly, it seems clear that the class of equitable matroids is highly sporadic. The fact that a matroid is equitable appears not to impose strong conditions on its structure; a matroid may be equitable merely as the result of an arithmetical coincidence, as seems to be the case with M_{36}. This indicates that a complete characterisation of equitable matroids may be unobtainable.

4.4 Basis-transitive matroids and PMDs

In this section we discuss two important classes of super-equitable matroids.

Observe that if B and B' are bases of a matroid, M, and σ is an automorphism of M that takes B to B', then σ is also an isomorphism between $G(B, M)$ and $G(B', M)$.

Definition 4.4.1. A matroid, M, is *basis-transitive* if, whenever B and B' are bases, there exists an automorphism, σ, of M, such that $\sigma(B) = B'$.

The discussion above implies the following:

Proposition 4.4.2. *Every basis-transitive matroid is super-equitable.*

Basis-transitive matroids have been studied in, for example, [9] and [31]. It is easy to see that uniform matroids are basis-transitive. Furthermore, it follows from [9] that the projective and affine geometries, PG(r, q) and AG(r, q), are basis-transitive, and so are their truncations.

We now describe another class of super-equitable matroids.

Definition 4.4.3. If M is a matroid, and, for $0 \leq i \leq r(M)$, all rank-i flats of M have the same cardinality, then M is a *perfect matroid design (PMD).*

A proof that every PMD is equitable can be found in [5, Proposition 3.2.2], although the terminology used is different.

Proposition 4.4.4. *Every PMD is super-equitable.*

Proof. Let M be a PMD on the set E. For $0 \leq i \leq r(M)$, let α_i be the size of the rank-i flats of M. If B is a basis of M, and X is a subset of B, then let $B(X)$ be the set

$$\{e \in E - B \mid e \text{ is adjacent to } x \text{ in } G(B, M) \text{ if and only if } x \in X\}.$$

Note that $B(X) = \{e \in E - B \mid C(e, B) = X \cup e\}$.

We claim that, for any integer $0 \leq i \leq r(M)$, there is an integer β_i, such that, if B is any basis of M, and X is any subset of B of size i, then $|B(X)| = \beta_i$. From this it follows easily that, if B and B' are any two bases of M, then $G(B, M)$ and $G(B', M)$ are isomorphic.

We prove the claim by induction on i. It is easy to see that β_0 is equal to the number of loops of M. Suppose that the claim is true if $i \leq k$, where $k \geq 0$, and let i be equal to $k + 1$. Suppose that X is a subset of size i of a basis, B. Note that

$$\mathrm{cl}_M(X) = X \cup \bigcup_{X' \subseteq X} B(X').$$

Since $B(Y)$ and $B(Y')$ are disjoint if $Y \neq Y'$, it follows from the inductive assumption that

$$\alpha_i = |\,\mathrm{cl}_M(X)| = |X| + |B(X)| + \sum_{j=0}^{i-1} \binom{i}{j} \beta_j.$$

Thus we may set β_i to be $\alpha_i - i - \sum_{j=0}^{i-1} \binom{i}{j} \beta_j$. This completes the proof. □

Perfect matroid designs have been investigated in [11] and [57]. The class of known PMDs includes the uniform matroids, the projective and affine geometries and their truncations. Although this indicates that the class of basis-transitive matroids has a large intersection with the class of PMDs, they are in fact unrelated classes, in the sense that neither is contained in the other, as shown by the next two examples.

A *Steiner system*, $S(t, k, v)$, is a set, S, of size v, along with a collection of subsets, called *blocks*, of size k, such that every t-subset of S is contained in exactly one block. It is known that every Steiner system defines a matroid, where the hyperplanes of the matroid are exactly the blocks of the Steiner system, and that, furthermore, any such matroid is a perfect matroid design [11]. In particular, any projective plane is an $S(2, n+1, n^2+n+1)$ Steiner system, where the blocks of the system are the lines of the plane, and every line has length $n + 1$.

Example 2. Consider a projective plane that is not a projective geometry; in other words, a plane in which Desargues' Theorem does not hold. The smallest such plane has 91 points (see, for example [24, IX.6]). By the above discussion, we know that such a plane is a Steiner system, and is therefore a PMD. However, Li's characterisation of rank-3 basis-transitive matroids [31] shows that a non-desarguesian plane cannot be basis-transitive. Therefore not every PMD is basis-transitive.

Next we will show that not every basis-transitive matroid is a PMD.

Example 3. Let r, n, and t be integers, such that $1 < r < n$ and $t \geq 2$. Let M be the matroid $T(M_1 \oplus \cdots \oplus M_t)$, where the ground set of M_i is X_i and $M_i \cong U_{r,n}$ for $1 \leq i \leq t$. Thus M can be obtained by truncating the direct sum of t isomorphic copies of $U_{r,n}$. If B is any basis of M, there exists an integer, $i \in \{1, \ldots, t\}$, such that B contains exactly $r-1$ elements from X_i, and exactly r elements from every X_j where $j \neq i$. It is now easy to see that M is basis-transitive. However, X_1 is a rank-r flat of M of size n, whereas a set containing $r-1$ elements from X_1 and a single element from X_2 is a rank-r flat of size r, which is strictly less than n. Hence M is not a PMD.

The next obvious question to ask is whether every super-equitable matroid is either basis-transitive or a PMD. It is not difficult to see that the answer is no. The direct sum of a coloop with a non-desarguesian plane is super-equitable, but is neither basis-transitive nor a PMD. Nor is it true that every connected super-equitable matroid is either basis-transitive or a PMD: one possible counter-example is the matroid obtained by truncating the direct sum of two isomorphic non-desarguesian planes. However, we will concentrate on another family of examples:

Example 4. Let n_1 and n_2 be two odd integers. For $i = 1, 2$ let $r_i = (n_i + 1)/2$, and let M_i be an isomorphic copy of U_{r_i, n_i} on the set X_i. Define M_{n_1, n_2} to be equal to $T(M_1 \oplus M_2)$, the matroid formed by truncating the direct sum of M_1 and M_2.

Note that the rank and the corank of M_{n_1, n_2} are both equal to $(n_1 + n_2)/2$. Figure 4.3 shows a representation of $M_{3,5}$, a rank-4 matroid with eight elements.

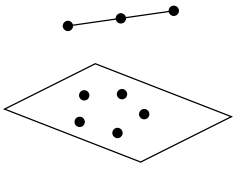


Figure 4.3: $M_{3,5}$

Proposition 4.4.5. *If n_1 and n_2 are odd integers such that $n_1, n_2 > 1$ and $n_1 \neq n_2$, then M_{n_1, n_2} is super-equitable, but neither basis-transitive nor a PMD.*

Proof. There are two types of bases of M_{n_1, n_2}. The first contains r_1 elements of X_1 and $r_2 - 1$ elements of X_2, while the second contains $r_1 - 1$ elements of X_1 and r_2 of X_2. Suppose that B_1 is any basis of the first type. Then $(B_1, (X_1 \cup X_2) - B_1)$ is a bipartition of $G(B_1, M_{n_1, n_2})$, and both sides of this bipartition have cardinality $(n_1 + n_2)/2$. Consider the partition, $(B_1 \cap X_1, B_1 \cap X_2)$, of B_1. The r_1 vertices in $B_1 \cap X_1$ are adjacent in $G(B_1, M_{n_1, n_2})$ to all the vertices in $(X_1 \cup X_2) - B_1$, while the $r_2 - 1$ vertices in $B_1 \cap X_2$ are adjacent only to the $n_2 - r_2 + 1$ vertices in $X_2 - B_1$.

Now consider B_2, a basis of the second type. Then $(B_2, (X_1 \cup X_2) - B_2)$ is a bipartition of $G(B_2, M_{n_1, n_2})$, and again, both sides of the partition have cardinality $(n_1 + n_2)/2$. Consider the partition, $(X_1 - B_2, X_2 - B_2)$, of $(X_1 \cup X_2) - B_2$. The

$n_1 - r_1 + 1$ vertices in $X_1 - B_2$ are adjacent to all the vertices in B_2, while the $n_2 - r_2$ vertices in $X_2 - B_2$ are adjacent only to the r_2 vertices in $B_2 \cap X_2$.

Observe that

$$r_1 = \frac{1}{2}(n_1 + 1) = n_1 - \frac{1}{2}(n_1 + 1) + 1 = n_1 - r_1 + 1,$$

and it is similarly easy to show that $n_2 - r_2 + 1 = r_2$, and hence $r_2 - 1 = n_2 - r_2$. Therefore

$$|B_1 \cap X_1| = |X_1 - B_2|,\ |X_2 - B_1| = |B_2 \cap X_2|,\ \text{and}\ |B_1 \cap X_2| = |X_2 - B_2|.$$

From this it follows that $G(B_1,\, M_{n_1,n_2})$ is isomorphic to $G(B_2,\, M_{n_1,n_2})$ (see Figure 4.4), and hence M_{n_1,n_2} is super-equitable.

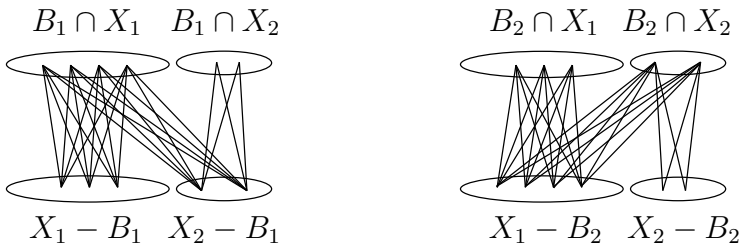


Figure 4.4: $G(B_1,\, M_{n_1,n_2})$ and $G(B_2,\, M_{n_1,n_2})$

Since $n_1,\, n_2 > 1$, it is also the case that $r_1,\, r_2 > 1$. It follows easily from this that $(X_1,\, X_2)$ is a partition of the ground set of M_{n_1,n_2} into two dependent flats, and, furthermore, that this is the only such partition. Since $|X_1| \neq |X_2|$, any automorphism of M_{n_1,n_2} must take X_1 to X_1, and X_2 to X_2. In particular no automorphism can take a basis of the first type to a basis of the second type, so M_{n_1,n_2} is not basis-transitive. Also, if we assume that $r_1 < r_2$, then X_1 is a rank-r_1 flat of size n_1, while any r_1-subset of X_2 is a rank-r_1 flat of size r_1. Since $r_1 < n_1$, this shows that M_{n_1,n_2} is not a PMD. □

We will conclude this section with some results on the properties of basis-transitive matroids and PMDs. The next result is obvious.

Proposition 4.4.6. *If M is a basis-transitive matroid, then M^* is also basis-transitive.*

In contrast to this, the dual of a PMD need not be a PMD. (See (96) and (101) of [58] or Section 12.6 of [53].)

The next proposition can be proved using an argument similar to that used to prove Proposition 4.3.4.

Proposition 4.4.7. *Suppose that M is a matroid. Then M is basis-transitive if and only if all its connected components are.*

On the other hand, the direct sum of two uniform matroids need not be a PMD, so the class of PMDs is not closed under direct sums.

Since the only effect that truncating a matroid has on its flats is to remove all its hyperplanes, it follows that the truncation of a PMD is also a PMD. We have already mentioned that projective and affine geometries are basis-transitive, and so are their truncations. In general, the truncation of a basis-transitive matroid need not be basis-transitive:

Example 5. Consider the matroid $M = T(U^3_{2,3})$. This is a connected rank-5 matroid, and we have already demonstrated, in Example 3, that M is basis-transitive. The truncation of M is a rank-4 matroid containing three pairwise-skew lines of length three. It is easy to see that $T(M)$ contains two types of bases. The first type avoids one of the non-trivial lines of $T(M)$, while the second type has a non-empty intersection with each non-trivial line. Since any automorphism of $T(M)$ must preserve its non-trivial lines, it follows that $T(M)$ is not basis-transitive.

Because $M = T(U^3_{2,3})$ is basis-transitive it is also super-equitable. However, if B is a basis of $T(M)$, then $G(B,\, M)$ may contain either 16 or 18 edges, according to whether B is a basis of the first or second type. Therefore $T(M)$ is not super-equitable. Furthermore, if B_1 is a basis of the first type, and B_2 a basis of the second type, then $\overline{G}(B_2,\, M)$ is isomorphic to a proper subgraph of $\overline{G}(B_1,\, M)$, and therefore $O(B_1,\, M)$ cannot possible be equal to $O(B_2,\, M)$, by the results of Section 4.2. It follows that $T(M)$ is not equitable.

We have proved the following:

Proposition 4.4.8. *The classes of basis-transitive, super-equitable, and equitable matroids are not closed under truncation.*

We have already noted that the class of equitable matroids is not closed under taking minors. The next example shows that the classes of super-equitable matroids, basis-transitive matroids, and PMDs are also not closed under taking minors.

Example 6. Consider the Fano plane, F_7. We have already noted that projective geometries belong to the intersection of basis-transitive matroids and PMDs, so F_7 is super-equitable, and hence equitable. However, by deleting two points from F_7 we obtain a matroid, M, which contains two lines of length three that intersect in a point (see Figure 4.5). Let B_1 be a basis of M that contains the point at the intersection of the two lines, and let B_2 be a basis that does not contain this point. Then $G(B_1,\, M)$ contains four edges, while $G(B_2,\, M)$ contains five. Thus M is neither basis-transitive, nor super-equitable. Obviously M is not a PMD.

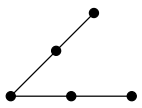


Figure 4.5: M.

We have demonstrated the following:

Proposition 4.4.9. *The classes of PMDs, basis-transitive matroids, super-equitable matroids, and equitable matroids, are not closed under taking minors.*

4.5 Some other classes

We will briefly consider some other classes of matroids that are related to those we have been studying.

An *equicardinal matroid* is one whose hyperplanes are equicardinal. The study of equicardinal matroids was initiated by Murty [39]. The next result is true by definition.

Proposition 4.5.1. *Every PMD is an equicardinal matroid.*

The converse is not true. The next example is due to Murty [39].

Example 7. Let F_7^+ be the matroid obtained by adding an element in parallel to each point of the basis that is obtained by relaxing a circuit-hyperplane of F_7, the Fano plane. It is not difficult to see that every hyperplane of F_7^+ has size four.

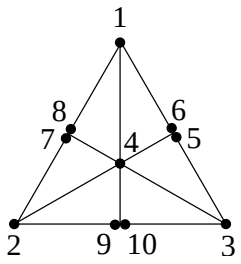


Figure 4.6: F_7^+.

Clearly F_7^+ is not a PMD.

The direct sum of two uniform matroids is basis-transitive and therefore super-equitable and equitable, but it clearly need not be equicardinal. Nor is it the case that an equicardinal matroid must be equitable, super-equitable, or basis-transitive.

Proposition 4.5.2. *F_7^+ is not equitable.*

Proof. Suppose that the ground set of F_7^+ is as labelled in Figure 4.6. Let B_1 be the basis $\{1, 2, 3\}$, and let B_2 be $\{1, 2, 5\}$. Then $\overline{G}(B_1, F_7^+)$ and $\overline{G}(B_2, F_7^+)$ are shown in Figures 4.7 and 4.8. Let f be the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 1 & 2 & 5 & 10 & 9 & 3 & 7 & 8 & 4 & 6 \end{pmatrix}.$$

The action of f on $\{1, \ldots, 10\}$ is represented by the alphabetic labels in Figures 4.7 and 4.8. If H is any chain subgraph of $\overline{G}(B_1, F_7^+)$, then $f(H)$ is also a chain subgraph of $\overline{G}(B_2, F_7^+)$. However, there are chain subgraphs, H, of $\overline{G}(B_2, F_7^+)$ such that $f^{-1}(H)$ is not a chain subgraph of $\overline{G}(B_1, F_7^+)$, for example, the chain subgraph containing the edges 16 and 26. This shows that $O(B_2, F_7^+)$ is strictly greater than $O(B_1, F_7^+)$, so F_7^+ is not equitable. □

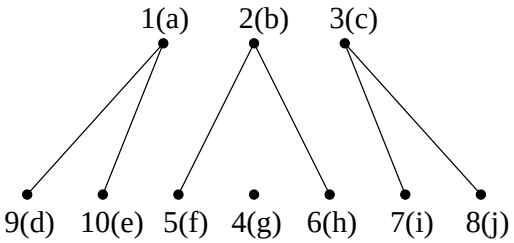


Figure 4.7: $\overline{G}(B_1,\, F_7^+)$

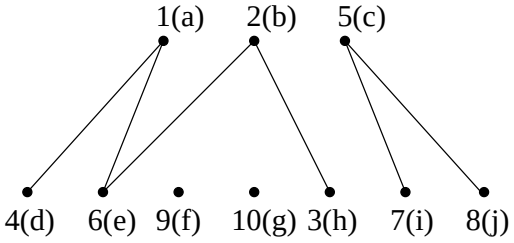


Figure 4.8: $\overline{G}(B_2,\, F_7^+)$

Since F_7^+ is not equitable, it is certainly not super-equitable or basis-transitive.

Proposition 3.4.1 shows that, if B is a basis of the matroid M, and $E(M) - B = \{x_1, \ldots, x_{n-r}\}$, where $r(M) = r$ and $|E(M)| = n$, then the degree of B in $\mathrm{BG}(M)$, the basis graph of M, is

$$\left(\sum_{i=1}^{n-r} |C(x_i,\, B)|\right) - (n - r), \tag{4.1}$$

It is not difficult to see that this quantity is equal to the number of edges in the bipartite graph $G(B,\, M)$. The next result follows immediately.

Proposition 4.5.3. *If M is a super-equitable matroid, then $\mathrm{BG}(M)$ is regular.*

The class of matroids that have regular basis graphs seems not to have been studied. There are a few easy facts we can prove about this class. The next result follows from Proposition 3.4.9.

Proposition 4.5.4. *Suppose that M is a matroid. Then M has a regular basis graph if and only if all of its connected components do.*

Similarly, the following proposition is a consequence of Proposition 3.6.1.

Proposition 4.5.5. *The class of matroids that have regular basis graphs is closed under generalised duality.*

Suppose that M is a matroid. From (4.1) it follows that, if all the circuits of M are equicardinal, then $\mathrm{BG}(M)$ is regular. Since $\mathrm{BG}(M) \cong \mathrm{BG}(M^*)$, this implies that if all the cocircuits of M are equicardinal, then $\mathrm{BG}(M)$ is regular. Thus we have proved the following.

Proposition 4.5.6. *If M is equicardinal, then* $\mathrm{BG}(M)$ *is regular.*

Therefore, the matroid F_7^+, presented in Example 7 has a regular basis graph. Thus Proposition 4.5.2 demonstrates that a matroid with a regular basis graph need not be equitable, super-equitable, or basis-transitive. Furthermore, the Fano plane, F_7, is equicardinal, so $\mathrm{BG}(F_7)$, and hence $\mathrm{BG}(F_7^*)$, is regular, but F_7^* contains hyperplanes of size 3 and 4. Thus the converse of Proposition 4.5.6 is not true.

Let M be a matroid on the ground set E. Suppose that $\{B_1, \ldots, B_n\}$ is the set of bases of M. If σ is an automorphism of M, then clearly $\sigma(B_i)$ is a basis of M, for all $i \in \{1, \ldots, n\}$. Therefore σ induces a permutation of $\{B_1, \ldots, B_n\}$ that takes B_i to $\sigma(B_i)$ for each i. It is clear that $|B_i \cap B_j|$ must be equal to $|\sigma(B_i) \cap \sigma(B_j)|$ for all pairs, B_i, $B_j \in \{B_1, \ldots, B_n\}$. Thus the permutation of $\{B_1, \ldots, B_n\}$ that is induced by σ is actually an automorphism of $\mathrm{BG}(M)$, so if the automorphism group of M acts transitively on the bases of M, then the automorphism group of $\mathrm{BG}(M)$ must act transitively on vertices. If the automorphism group of a graph is transitive upon vertices, then we shall say that the graph is *vertex-transitive*. Our discussion shows that the following result holds.

Proposition 4.5.7. *If M is a basis-transitive matroid, then* $\mathrm{BG}(M)$ *is vertex transitive.*

It is not known whether the converse of Proposition 4.5.7 holds. Astié-Vidal [1] has studied the relationship between the automorphism groups of matroids and the automorphism groups of their basis graphs. She shows that in most cases the automorphism group of a matroid will induce the entire automorphism group of the basis graph. In particular, it follows without difficulty from her work that any counterexample to the converse of Proposition 4.5.7 would have to be a self-dual matroid.

4.6 Summary

In this section we give a summary of the results of this chapter. The relationship between the classes of matroids we have discussed are shown by the Hasse diagram in Figure 4.9. The arrows show proper inclusion.

Table 4.1 provides a quick reference for some of the proofs and examples in the chapter. If Class A is contained in Class B, then the entry that is in the row labelled by A and the column labelled by B gives the reference for the result(s) that prove the containment. If Class A is not contained in B, then the reference is to the list below, and gives an example of a matroid that is contained in A, but not in B.

(i) The matroid M_{36}, described in Example 1.

(ii) A matroid M_{n_1,n_2}, where n_1 and n_2 are distinct odd integers that exceed one, as described in Example 4.

(iii) The matroid F_7^+, as described in Example 7.

(iv) The dual of the Fano plane.

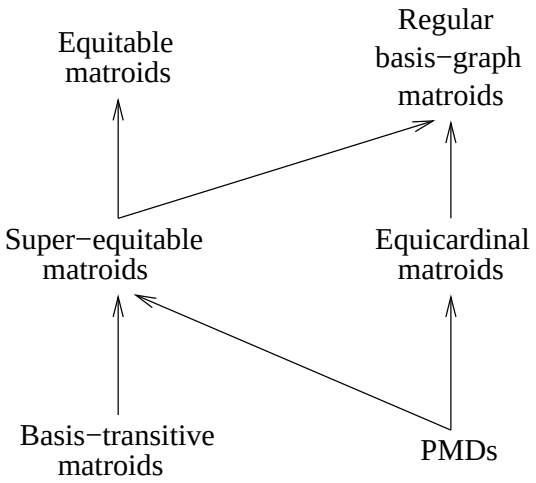


Figure 4.9:

(v) The matroid $T(U^t_{r,n})$, the truncation of the direct sum of t copies of $U_{r,n}$, where $1 < r < n$ and $t \geq 2$, as described in Example 3.

(vi) A non-desarguesian plane, as described in Example 2.

	Equitable	Super−equitable	Regular basis−graphs	Equicardinal	Basis−transitive	PMDs
Equitable		i	i	i	i	i
Super−equitable	4.3.5		4.5.3	ii	ii	ii
Regular basis−graphs	iii	iii		iv	iii	iii
Equicardinal	iii	iii	4.5.6		iii	iii
Basis−transitive	4.3.5 4.4.2	4.4.2	4.4.2 4.5.3	v		v
PMDs	4.3.5 4.4.4	4.4.4	4.4.4 4.5.3	4.5.1	vi	

Table 4.1:

Chapter 5

Circuits and cocircuits in regular matroids

5.1 Introduction

The cocircuits of the graphic matroid, $M(G)$, are exactly the minimal edge cut-sets of G. In particular, the set of edges incident with a non-isolated vertex always contains a cocircuit in the associated polygon matroid. For this reason it has become common, when seeking a matroid analogue of a graph-theoretical result, to replace the word "vertex" with the word "cocircuit" whenever it appears in the statement of the result, and to see whether the new statement holds for matroids in general.

As an example of this we consider a classical result of Dirac. Suppose that n is a positive integer. A graph is *n-connected* if it is connected, $n \leq |V(G)| - 1$, and deleting any set of $n-1$ vertices and the edges incident with them does not disconnect the graph. Dirac proved that for any two edges and any $n-2$ vertices in a simple n-connected graph, there is a cycle that contains both of the edges and that meets all $n-2$ of the vertices [12]. Oxley [43] asked whether the following matroid analogue of this result is true.

Problem 5.1.1. *Suppose that M is an n-connected matroid, where $n \geq 2$. Let e and f be distinct elements of M and let $C_1^*, \ldots, C_{n-2}^*$ be cocircuits. Does there always exist a circuit, C, of M, such that $e, f \in C$, and $C \cap C_i^* \neq \varnothing$ for all $i \in \{1, \ldots, n-2\}$?*

It is clear that the answer to this question is affirmative when $n = 2$. Oxley has proved that this is also the case when $n = 3$.

Theorem 5.1.2. [43, Theorem 1.1] *Suppose that M is a 3-connected matroid. For every pair, $\{e, f\}$, of distinct elements of $E(M)$ and every cocircuit, C^*, of M, there is a circuit that contains $\{e, f\}$ and meets C^*.*

Denley and Wu [10] proved that the matroid property described in Problem 5.1.1 holds for the class of graphic matroids. In fact they proved a stronger result. If S is a set of paths in a graph, and the members of S are pairwise vertex-disjoint, then we shall say that S is a set of *independent paths*.

Theorem 5.1.3. [10, Theorem 2.1] *Let G be an n-connected graph, where $n \geq 2$. Let S be a set of independent paths of total length s, and let T be a set of t cocircuits, where $s + t = n$ and $t \geq 1$. Then there is a cycle of G that contains each path of S and meets every cocircuit in T.*

In this chapter we will prove that the property described in Problem 5.1.1 holds for the class of regular matroids. In fact we will show that it holds even using a slightly stronger version of connectivity.

Recall that a *k-separation* of a matroid, M, is a partition, (X, Y), of $E(M)$, such that $\min\{|X|, |Y|\} \geq k$, and $r(X) + r(Y) - r(M) \leq k - 1$. The matroid M is said to be *n-connected* if M has no k-separation where $k < n$. The *Tutte connectivity* (often just the *connectivity*) of M, denoted by $\lambda(M)$, is the greatest integer, n, such that M is n-connected. If there exists no such greatest integer, then $\lambda(M)$ is taken to be infinite.

Tutte connectivity fails to completely capture the notion of graph connectivity, since an n-connected matroid with at least $2(n-1)$ elements has no circuit of size less than n. The next notion of connectivity that we consider is a direct analogue of graph connectivity. For an integer $k \geq 1$, a *vertical k-separation* of a matroid, M, is a partition, (X, Y), of $E(M)$ such that $\min\{r(X), r(Y)\} \geq k$, and $r(X) + r(Y) - r(M) \leq k - 1$. If $r(X) + r(Y) - r(M) = k - 1$, then we shall say that (X, Y) is an *exact vertical k-separation.*

We say that M is *vertically n-connected* if M has no vertical k-separations where $k < n$, and the *vertical connectivity of M*, denoted by $\kappa(M)$, is the greatest integer, n, for which M is vertically n-connected. If no such integer exists then we define $\kappa(M)$ to be equal to $r(M)$.

Proposition 5.1.4. [42, Theorem 8.2.5] *If G is a connected graph, then $\kappa(M(G))$ is equal to the connectivity of G.*

For the sake of brevity, we shall assign names to two properties of classes of matroids.

Definition 5.1.5. Suppose that $\mathcal{M}$ is a class of matroids and that $n \geq 2$ is an integer. We shall say that $\mathcal{M}$ has the *$\mathfrak{A}(n)$-property* (respectively, the *$\mathfrak{B}(n)$-property*) if, whenever e and f are distinct elements of M, an n-connected member (respectively, a simple, vertically n-connected member) of $\mathcal{M}$, and $C_1^*, \ldots, C_{n-2}^*$ are cocircuits of M, then there exists a circuit, C, of M, such that $e, f \in C$ and $C \cap C_i^* \neq \varnothing$ for all $i \in \{1, \ldots, n-2\}$.

If $\mathcal{M}$ has the $\mathfrak{A}(n)$-property (respectively, the $\mathfrak{B}(n)$-property) for all $n \geq 2$, we shall say that $\mathcal{M}$ has the *$\mathfrak{A}$-property* (respectively, the *$\mathfrak{B}$-property*). Thus it is known that the class of all matroids has the $\mathfrak{A}(2)$- and $\mathfrak{A}(3)$-properties, and, furthermore, the class of graphic matroids has the $\mathfrak{A}$-property. If $\mathcal{M}$ has the $\mathfrak{A}(n)$-property (respectively, the $\mathfrak{B}(n)$-property), and $M \in \mathcal{M}$, then we may abuse the terminology slightly and say that M has the respective property.

The next result shows the relation between the two properties.

Proposition 5.1.6. *Suppose that $\mathcal{M}$ is a class of matroids and that $n \geq 2$ is an integer. If $\mathcal{M}$ has the $\mathfrak{B}(n)$-property, then it also has the $\mathfrak{A}(n)$-property.*

Proof. We may assume that $n > 2$, for otherwise $\mathcal{M}$ certainly has the $\mathfrak{A}(n)$-property. Let M be an n-connected matroid in $\mathcal{M}$. Suppose that $e, f \in E(M)$, and $C_1^*, \ldots, C_{n-2}^*$ are cocircuits of M. If M is not uniform, then $\kappa(M) \geq \lambda(M)$ [42, Theorem 8.2.6]. Therefore $\kappa(M) \geq n$, so, in this case, M is vertically n-connected and simple (since $n > 2$). Since $\mathcal{M}$ has the $\mathfrak{B}(n)$-property, it follows that some circuit of M contains both e and f, and meets every one of the cocircuits.

If M is uniform, then every cocircuit meets every circuit of M, so the result follows. □

Thus, if a class of matroids has the $\mathfrak{B}$-property, it also has the $\mathfrak{A}$-property.

It is easy to demonstrate that, for $n \in \{2, 3\}$, a simple, vertically n-connected matroid is also n-connected. Thus the class of all matroids has the $\mathfrak{B}(2)$- and $\mathfrak{B}(3)$-properties. Furthermore, it follows from Theorem 5.1.3 and Proposition 5.1.4 that the class of graphic matroids has the $\mathfrak{B}$-property.

The main theorem of this chapter shows that the class of regular matroids has the $\mathfrak{B}$-property.

Theorem 5.1.7. *Suppose that M is a simple, vertically n-connected regular matroid, where $n \geq 2$. If e and f are elements of M, and $C_1^*, \ldots, C_{n-2}^*$ are cocircuits of M, then there exists a circuit, C, of M such that $e, f \in C$, and $C \cap C_i^* \neq \varnothing$ for all $i \in \{1, \ldots, n-2\}$.*

We prove Theorem 5.1.7 by showing that the class of cographic matroids has the $\mathfrak{B}$-property. Theorem 5.1.7 will then follow easily, as a consequence of Seymour's decomposition theorem for regular matroids [47], and the results by Oxley [43], and Denley and Wu [10].

5.2 Proof of the main result

Recall that R_{10} is the binary self-dual matroid represented by the matrix in Figure 5.1.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Figure 5.1: A GF(2)-representation of R_{10}.

The following proposition is a simple consequence of Seymour's decomposition theorem.

Proposition 5.2.1. *If M is a simple, vertically 4-connected regular matroid, then M is either graphic or cographic, or isomorphic to R_{10}.*

Proof. Assume that M is a simple, vertically 4-connected regular matroid. Then M is also vertically 3-connected. We have already noted that a simple, vertically 3-connected matroid must be 3-connected.

We will be done if we can show that, for every 3-separation, (X, Y), of M, either $|X| = 3$ or $|Y| = 3$, for in this case M is said to be *internally 4-connected*, and regular matroids that are 3-connected and internally 4-connected are either graphic, cographic, or isomorphic to R_{10} [42, Corollary 13.2.6].

Therefore let us suppose that $\min\{|X|, |Y|\} \geq 4$. It cannot be the case that (X, Y) is a vertical 3-separation of M, so either $r(X) < 3$, or $r(Y) < 3$. Without loss of generality we will assume the former. Since M is simple, $r(X) = 2$. But this implies that M contains a rank-2 flat containing at least four elements, and hence M has a $U_{2,4}$-minor. This contradicts our assumption that M is regular and completes the proof. □

Since we know that the class of all matroids has the $\mathfrak{B}(2)$- and $\mathfrak{B}(3)$-properties, and that graphic matroids have the $\mathfrak{B}$-property, to prove Theorem 5.1.7 it will suffice to prove that the class of cographic matroids has the $\mathfrak{B}$-property, and that R_{10} has the $\mathfrak{B}(n)$-property for all $4 \leq n \leq \kappa(R_{10})$. We will complete the latter task first.

It is easy to see that the first four columns of the matrix in Figure 5.1, along with the sixth and ninth columns, form a spanning circuit of R_{10}. Since the automorphism group of R_{10} acts transitively upon ordered pairs of elements [47], it follows that every pair of elements in R_{10} is contained in a spanning circuit. It is easy to see that every spanning circuit meets every cocircuit, so R_{10} has the $\mathfrak{B}(n)$-property whenever $4 \leq n \leq \kappa(R_{10})$.

Before we prove that the class of cographic matroids has the $\mathfrak{B}$-property, we will discuss some more preliminary material on vertical connectivity.

Unlike Tutte connectivity, vertical connectivity is not invariant under duality: the dual of a vertically n-connected matroid need not be vertically n-connected. It is not difficult to show that (X, Y) is an exact vertical k-separation of M if and only if $r(X) + r(Y) - r(M) = k - 1$, and X and Y both contain cocircuits of M. Thus, if (X, Y) is a partition of $E(M)$ such that

$$r^*(X) + r^*(Y) - r(M^*) = r(X) + r(Y) - r(M) \leq k - 1$$

and X and Y both contain circuits of M, then (X, Y) is a vertical k'-separation of M^* for some $k' \leq k$. We shall call such a separation a *cyclic k-separation of M*. We say that M is *cyclically n-connected* if M has no cyclic k-separations where $k < n$. Thus M is cyclically n-connected if and only if M^* is vertically n-connected.

We shall say that a graph, G, is cyclically n-connected if $M(G)$ is cyclically n-connected. Thus, if $G = (V, E)$ is a cyclically n-connected graph, and (X, Y) is a partition of E such that

$$r_{M(G)}(X) + r_{M(G)}(Y) - r(M(G)) < n - 1,$$

then it must be the case that no more than one of X and Y contains a cycle.

If X is a set of edges in G, let $V(X)$ denote the set of vertices incident with edges in X. It is well known that, if G is connected, then

$$r_{M(G)}(X) + r_{M(G)}(E - X) - r(M(G)) \leq |V(X) \cap V(E - X)| - 1.$$

We shall prove the following theorem.

Theorem 5.2.2. *Suppose that G is a cyclically n-connected graph (where $n \geq 3$), and that $M(G)$ is cosimple. For every pair of edges, e and f, and every set of cycles, $\{C_1, \ldots, C_{n-2}\}$, there exists a cocircuit of $M(G)$ that contains both e and f, and has edges in common with all the cycles in $\{C_1, \ldots, C_{n-2}\}$.*

Proving this theorem will clearly establish that cographic matroids have the $\mathfrak{B}$-property. The proof of Theorem 5.2.2 will follow almost immediately from the next lemma.

Lemma 5.2.3. *Suppose that $G = (V, E)$ is a cyclically n-connected graph (where $n \geq 3$), and that $M(G)$ is cosimple. Let e and f be edges of G, and let $C_1, \ldots, C_{n-2}$ be cycles of G. Suppose there is a cocircuit of $M(G)$ that contains e and f, and has edges in common with all the cycles in $\{C_1, \ldots, C_{n-3}\}$. Then there exists a cocircuit of $M(G)$ that contains e and f, and has edges in common with all the cycles in $\{C_1, \ldots, C_{n-2}\}$.*

Proof. We will assume that G has no isolated vertices. Then it is clear that G must be connected. Every cocircuit of $M(G)$ corresponds to a minimal edge cut-set of G, so there exists a partition, (A, B), of V, such that $G[A]$ and $G[B]$, the subgraphs induced by A and B, are connected, and $\Delta(A, B)$, the set of edges joining vertices in A to vertices in B, contains both e and f, and shares edges with all the cycles in $\{C_1, \ldots, C_{n-3}\}$. Suppose that the lemma is false, so there exists no partition, (A', B'), of V, such that $G[A']$ and $G[B']$ are connected and $\Delta(A', B')$ contains both e and f and edges from every cycle in $\{C_1, \ldots, C_{n-2}\}$. Then, in particular, C_{n-2} has no edges in common with $\Delta(A, B)$. We may assume that the vertices of C_{n-2} are contained in A. Among such partitions, let (A, B) have the property that $|A|$ is as small as possible.

Let $\delta(A)$ be the set of vertices in A that are adjacent to vertices in B. A vertex, $v \in \delta(A)$, is *bad* if v is incident with either e or f, or if there is some cycle $C_i \in \{C_1, \ldots, C_{n-3}\}$ such that $V(C_i) \cap A = \{v\}$. (Note that every cycle in $\{C_1, \ldots, C_{n-3}\}$ has vertices in both A and B.) We observe that there can be at most $n-1$ bad vertices in $\delta(A)$. Any vertex in $\delta(A)$ that is not bad is *good*.

If v is a vertex of the graph, H, and deleting v from H increases the number of connected components, then v is a *cut-vertex of H*.

5.2.3.1. *Every good vertex in $\delta(A)$ is a cut-vertex of $G[A]$.*

Proof. Suppose that $v \in \delta(A)$ is good, and is not a cut-vertex of $G[A]$. Consider the partition $(A - v, B \cup v)$ of V. Both $G[A - v]$ and $G[B \cup v]$ are connected, both the edges e and f are contained in the cocircuit $\Delta(A - v, B \cup v)$, and every circuit in

$\{C_1, \ldots, C_{n-3}\}$ has vertices in both $A - v$ and $B \cup v$. Therefore the minimality of A is contradicted. □

A *block* of a graph is a maximal connected subgraph that does not have a cut-vertex. Suppose that the graph, H, has blocks $B_1, \ldots, B_m$ and cut-vertices $c_1, \ldots, c_n$. The *block-cutpoint graph of* H, denoted by $\mathrm{bc}(H)$, has $\{B_1, \ldots, B_m\} \cup \{c_1, \ldots, c_n\}$ as its vertices. Two vertices are adjacent in $\mathrm{bc}(H)$ if and only if they correspond to a block and a cut-vertex contained in that block.

Let B_0 be the block of $G[A]$ that contains the cycle C_{n-2}, and let X be the set of edges of B_0. Suppose that $V(X) \cap V(E - X)$ contains exactly t vertices, $v_1, \ldots, v_t$. For $1 \leq i \leq t$, the vertex v_i is either a cut-vertex of $G[A]$, or a member of $\delta(A)$, perhaps both. By 5.2.3.1, it follows that any vertex in $\{v_1, \ldots, v_t\}$ that is not a cut-vertex of $G[A]$ must be a bad vertex in $\delta(A)$.

Next we will show that

$$t \leq n - 1. \tag{5.1}$$

If none of the vertices in $\{v_1, \ldots, v_t\}$ is a cut-vertex of $G[A]$, then t is no greater than the number of bad vertices, and, in particular, $t \leq n - 1$. Therefore we will assume that $\{v_1, \ldots, v_t\}$ contains at least one cut-vertex of $G[A]$. By relabelling, we will suppose that v_i is a cut-vertex of $G[A]$ if and only if $i \leq k$, for some integer $k \in \{1, \ldots, t\}$.

It is known that $\mathrm{bc}(G[A])$ is a tree [21, Theorem 4.4]. For each $i \in \{1, \ldots, k\}$, consider a maximal path in $\mathrm{bc}(G[A])$ that has B_0 as one of its end-vertices, and that passes through the cut-vertex v_i. Let the other end-vertex of this path be B_i (the degree-one vertices of $\mathrm{bc}(G[A])$ correspond to blocks). It is clear that the blocks $B_1, \ldots, B_k$ are pairwise distinct.

For $1 \leq i \leq k$, let c_i be the unique cut-vertex of $G[A]$ that is incident with the block B_i. Suppose that B_i contains no vertex in $\delta(A)$, other than perhaps c_i. Now B_i must contain a cycle, for otherwise B_i contains only a single edge, and in this case $M(G)$ is not cosimple. Let X' be the edge set of B_i. Then both X' and $E - X'$ contain cycles (since $E - X'$ contains C_{n-2}) and, since $|V(X') \cap V(E - X')| = 1$, it follows that $(X', E - X')$ is a cyclic 1-separation. This contradiction shows that there must exist a vertex, u_i, in B_i that is in $\delta(A)$ and is not a cut-vertex of $G[A]$.

The set $\{u_1, \ldots, u_k\} \cup \{v_{k+1}, \ldots, v_t\}$ contains t members of $\delta(A)$, none of which are cut-vertices of $G[A]$. Hence it again follows from 5.2.3.1 that each of these vertices is a bad vertex. We have shown that t does not exceed the number of bad vertices in $\delta(A)$, so (5.1) holds.

Since $t = |V(X) \cap V(E - X)| \leq n - 1$, and G is cyclically n-connected, $E - X$ can contain no cycle. Thus, in particular, every cycle in $\{C_1, \ldots, C_{n-3}\}$ contains at least one edge of X. Therefore there is no vertex, v, in $\delta(A)$, and no cycle, C_i, in $\{C_1, \ldots, C_{n-3}\}$, such that $V(C_i) \cap A = \{v\}$. Thus, if a vertex in $\delta(A)$ is bad, it must be incident with either e or f. Hence there are at most two bad vertices in $\delta(A)$. Since t does not exceed the number of bad vertices, we have shown that $t \leq 2$.

Now $E - X$ contains no cycle, and so is a forest. But the subgraph induced by $E - X$ can contain no degree-one vertices, other than those incident with some edge in X, for otherwise $M(G)$ would fail to be cosimple. Since $|V(X) \cap V(E - X)| \leq 2$, it

follows that the subgraph induced by $E-X$ contains at most two degree-one vertices, and is therefore a path. But G has minimum degree at least three, so $E - X$ must contain only one edge. However B contains at least one vertex, and all the edges incident with that vertex are in $E-X$. This contradiction completes the proof of the lemma. □

Proof of Theorem 5.2.2. We will prove the theorem by induction on n. We know that the class of all matroids has the $\mathfrak{B}(3)$-property, so, in particular, it follows that Theorem 5.2.2 holds when $n = 3$. Assume that $n > 3$, and that the theorem holds for $n - 1$. Let G be a graph such that $M(G)$ is cosimple and cyclically n-connected. Suppose that e and f are edges of G, and that $\{C_1, \ldots, C_{n-2}\}$ is a collection of cycles. By the inductive hypothesis, there is a cocircuit that contains both e and f and meets all the cycles in $\{C_1, \ldots, C_{n-3}\}$. The proof of the theorem now follows from an application of Lemma 5.2.3. □

Theorem 5.1.7 follows from Theorem 5.2.2, Proposition 5.2.1, and Theorem 5.1.3.

Chapter 6

Conclusions and questions

In this chapter we discuss some of the possible lines of investigation that arise from the material presented in this thesis.

Clearly there are many algorithmic questions that can be asked about a matroid, and it would be interesting to know the complexity of these questions, given various types of input. We saw in Chapter 2 that the complexity of a problem may vary greatly depending on the type of input used. In some cases we know that a problem is NP-complete for some forms of input, and in P for others, but for some problems the complexity may not be known for all the methods of description discussed in Section 2.2.

For instance, by Theorem 2.4.2, we know that 3-MATROID INTERSECTION is NP-complete when CIRCUITS $\leq$ INPUT in the order described in Definition 2.3.1, and we have observed that it is NP-complete if HYPERPLANES $\leq$ INPUT. We also noted that 3-MATROID INTERSECTION is in P when INPUT $\leq$ BASES. However, this does not cover every type of input.

Problem 6.1.4. *What is the complexity of* 3-MATROID INTERSECTION *if* INPUT *is equal to either* FLATS *or* CYCLIC FLATS*?*

There are several problems of this sort that arise from the material in Chapter 2. We will catalogue some of them here.

We know from Theorem 2.6.3 that testing whether a matroid contains a fixed minor can be done in polynomial time if INPUT $\leq$ CIRCUITS. It is not difficult to see by duality that this is also true if INPUT $\leq$ HYPERPLANES. This leaves the following open problem.

Problem 6.1.5. *What is the complexity of* DETECTING AN N-MINOR *when* INPUT *is equal to one of* NON-SPANNING CIRCUITS*,* CYCLIC FLATS*, or* DEPENDENT HYPERPLANES*?*

As a special case, we consider the complexity of deciding whether a matroid has a $U_{2,4}$-minor; which is to say, deciding whether or not it is binary.

BINARY MATROID

Instance: A description, (M, INPUT), of a matroid, M.

Question: Is M a binary matroid?

By the above discussion, BINARY MATROID is in P if INPUT $\leq$ CIRCUITS, or if INPUT $\leq$ HYPERPLANES. It would interesting to know the answer to the following problem.

Problem 6.1.6. *What is the complexity of* BINARY MATROID *when* INPUT *is equal to one of* NON-SPANNING CIRCUITS*,* CYCLIC FLATS*, or* DEPENDENT HYPERPLANES*?*

Note that BINARY MATROID is clearly in co-NP for any form of input, since if M has a $U_{2,4}$-minor, that can be easily certified. However it is not clear that BINARY MATROID is in NP for any of the forms of input mentioned in Problem 6.1.6.

Theorem 2.6.5 and the observations that follow its proof show that $U_{r,n}$-MINOR is NP-complete if INDEPENDENT SETS $\leq$ INPUT and $r \geq 3$. If $r < 3$, then the situation is varied. We first note that if $r = 0$, then $U_{r,n}$-MINOR is in P for any form of input, for a largest rank-zero minor is obtained by contracting a basis, and the size of such a minor is therefore equal to the corank.

It is not difficult to see that the size of a largest rank-one uniform minor of a matroid is equal to the size of a largest cocircuit. Therefore if we can efficiently find the size of a smallest hyperplane of a matroid, we can solve $U_{r,n}$-MINOR in polynomial time for the case when $r = 1$. This can obviously be done if INPUT $\leq$ HYPERPLANE. It is not so clear that it can be done when INPUT is equal to DEPENDENT HYPERPLANE. Finding the size of a smallest hyperplane of M, given the description (M, DEPENDENT HYPERPLANES), is equivalent to deciding whether M contains an independent hyperplane. If M has no independent hyperplane, then the size of a smallest hyperplane in M is simply the size of a smallest dependent hyperplane, whereas if M does have an independent hyperplane, then a minimum-sized hyperplane in M is as small as any hyperplane can be, which is to say that its cardinality is $r(M) - 1$.

Observe that M has an independent hyperplane if and only if M^* has a spanning circuit. Since we can obtain (M^*, NON-SPANNING CIRCUITS) in polynomial time, given (M, DEPENDENT HYPERPLANES), it follows that if the following problem is in P, then deciding whether M contains an independent hyperplane is also in P.

SPANNING CIRCUIT

Instance: A description, (M, NON-SPANNING CIRCUITS), of a matroid, M, via its non-spanning circuits.

Question: Does M have a spanning circuit?

Proposition 6.1.7. SPANNING CIRCUIT *is in* P.

Proof. Let $\mathcal{C}$ be the family of non-spanning circuits of M. Suppose that the rank of M is r. We may try to use the greedy algorithm to find a set of size $r + 1$ that contains no non-spanning circuit of M. If we succeed, then we have found a spanning circuit of M. If we do not succeed, then $\mathcal{C}$ is not the family of non-spanning circuits

of a matroid with rank $r+1$, for otherwise the greedy algorithm would have found an appropriate set of size $r+1$.

By the work of Oxley and Whittle [40], it follows that if $\mathcal{C}$ fails to be the family of non-spanning circuits of a matroid with rank $r+1$, then either:

(i) there exist two distinct circuits, C, $C' \in \mathcal{C}$, such that for some element, $e \in C \cap C'$, there is no member of $\mathcal{C}$ contained in $(C \cup C') - e$, and $|(C \cup C') - e| \leq r+1$; or

(ii) every subset of $E(M)$ of size $r+1$ contains a member of $\mathcal{C}$.

We can, in polynomial time, check each pair of circuits in $\mathcal{C}$ to decide whether (i) occurs. If it does, then $|(C \cup C') - e|$ must be equal to $r+1$, for otherwise $\mathcal{C}$ would fail to be the family of non-spanning circuits of a rank-r matroid, and we know that this is not the case. Thus we have found a subset of $E(M)$ of size $r+1$ that contains no member of $\mathcal{C}$; in other words, a non-spanning circuit of M.

If (i) does not occur, and the greedy algorithm fails to return a non-spanning circuit of M, then (ii) must occur, which implies that M has no spanning circuit.

It is clear that this procedure can be accomplished in polynomial time, so SPANNING CIRCUIT is in P. □

It follows that, given (M, Dependent Hyperplanes), we may, in polynomial time, find the size of a smallest hyperplane of M, and hence find the size of a largest cocircuit of M. Thus $U_{r,n}$-MINOR can be solved in polynomial time if $r = 1$ and Input $\leq$ Dependent Hyperplanes. This leave the following open question.

Problem 6.1.8. *What is the complexity of* $U_{r,n}$-MINOR *when* $r = 1$ *and* Input *is equal to either* Circuits, Non-Spanning Circuits, *or* Cyclic Flats*?*

If we are given the description (M, Bases), then we may, in polynomial time, generate a description of every rank-2 minor of M, by simply considering every pair of elements, e and f, and every basis, B, such that e, $f \in B$, and then considering the minor obtained by contracting $B - \{e, f\}$. The size of a largest rank-2 uniform minor of M is then equal to the maximum number of parallel classes in any of the rank-2 minors of M. Thus $U_{r,n}$-MINOR can be solved in polynomial time if $r = 2$ and Input $\leq$ Bases.

Problem 6.1.9. *What is the complexity of* $U_{r,n}$-MINOR *when* $r = 2$ *and* Input $\not\leq$ Bases*?*

Many of the algorithmic problems we have discussed are similar, in that they ask whether a matroid has a minor of a particular size that belongs to a specified family. Thus they belong to the following family of problems. Here $\mathcal{M}$ is a class of matroids that is closed under isomorphism.

MINOR IN $\mathcal{M}$

Instance: An integer, n, and a description, (M, Input), of a matroid, M.

Question: Does M have a minor, N, such that $|E(N)| \geq n$, and $N \in \mathcal{M}$?

This bears an obvious resemblance to the following graph-theoretical problem.

> INDUCED SUBGRAPH WITH PROPERTY Π
>
> Instance: An integer, n, and a graph, G.
>
> Question: Does G have an induced subgraph, H, such that H has at least n vertices, and H has property Π?

It has been shown that this problem is NP-hard for a wide choice of properties. In particular, Yannakakis [55] has proved that if the class of graphs that have property Π is an hereditary class that does not contain all graphs, but that does contain arbitrarily large graphs, then INDUCED SUBGRAPH WITH PROPERTY Π is NP-hard; moreover, if deciding whether a graph has property Π can be done in polynomial time, then INDUCED SUBGRAPH WITH PROPERTY Π is also NP-complete.

A similarly powerful result for MINOR IN $\mathcal{M}$ would be very attractive. It would be especially interesting to know whether the following matroid analogue of Yannakakis' result is true.

Problem 6.1.10. *Is it always true that, if $\mathcal{M}$ is a minor-closed family of matroids that does not contain every matroid, and there is no integer, k, such that the loopless and coloopless part of every matroid in $\mathcal{M}$ has size at most k, then* MINOR IN $\mathcal{M}$ *is* NP*-hard for at least one of the inputs discussed in Section 2.2?*

It seems as though these conditions on $\mathcal{M}$ may be necessary, in order to prevent MINOR IN $\mathcal{M}$ from being trivially in P. For example, if $\mathcal{M} = \{U_{n,n} \mid n \geq 1\}$, then MINOR IN $\mathcal{M}$ is certainly in P.

As we noted in Chapter 4, counting the bases of a matroid is a difficult problem. If a matroid, M, is described via the input (M, INPUT), where INPUT $\leq$ BASES, then the problem of counting the bases of M becomes simple. However, this raises to the following question.

Problem 6.1.11. *What is the complexity of counting the bases of the matroid M, given the description (M,* INPUT*), where* INPUT $\not\leq$ BASES*? In particular, is counting the bases of M a* #P*-complete problem for some type of input?*

The problem of counting bases is also the most obvious connection between basis graphs and matroid complexity. As we have seen in Chapter 4, this problem is linked to that of generating a basis of a matroid uniformly at random. One way of finding a basis at random is to perform a random walk on the basis graph. We could implement this walk as follows: suppose that after a certain number of steps, we have arrived at the basis B. With probability 1/2 let us stay at B, and, also with probability 1/2, let us choose elements $e \in B$ and $f \notin B$ uniformly at random. If $B' = (B - e) \cup f$ is a basis, let us shift to B'.

The stationary distribution of this Markov chain is the uniform distribution, but to be of assistance in efficiently approximating the number of bases of a matroid, we must be guaranteed that the Markov chain will approach the stationary distribution relatively quickly. This will be the case only if there are no "bottlenecks" in the basis

graph; that is, two large sets of vertices that are joined by only a small number of edges (see, for example, [38], [50], or [54]).

To make this concept more precise, we introduce the *cut-set expansion of a graph,* G, denoted by $\Omega(G)$. Suppose that A is a set of vertices of G. Let $C(A)$ denote the set of edges that join vertices in A to vertices not in A. We define $\Omega(G)$ to be $\min\{|C(A)|/|A|\}$, where the minimum is taken over all sets of vertices that satisfy $1 \leq |A| \leq |V(G)|/2$.

Problem 6.1.12. *Does there exist a polynomial, p, such that $\Omega(\mathrm{BG}(M)) \geq 1/p(n)$, for any matroid, M, on a ground set of size n?*

Showing that such a polynomial exists would immediately lead to an efficient way to approximate the number of bases of a matroid when that matroid is described via an independence or rank oracle. This is inarguably the most important open question about matroid basis graphs.

Mihail and Vazirani have made a much stronger conjecture, which states that a wide class of graphs (which includes basis graphs) has the property that every graph in the class has cut-set expansion at least one [38, 37].

Turning now to the subject of equitable matroids, we note that we still have an incomplete knowledge of how the six basic classes shown in Table 4.1 relate to each other. In particular, we know of no matroid with a regular basis graph that is equitable without being super-equitable. Thus the following problem is open.

Problem 6.1.13. *Is every equitable matroid with a regular basis graph also a super-equitable matroid?*

Appendix A

Excluded-minor characterisations

In this appendix we restate and prove some of the results of Sections 3.5 and 3.6. The next two results will help to simplify some of the subsequent proofs.

Proposition A.1.1. *Suppose that one of the connected components of the matroid M is isomorphic to $U_{1,2}$, so that M can be decomposed as $M = U_{1,2} \oplus M'$. If M is an excluded minor for $\mathcal{M}(\Delta \leq k+1)$, then M' is an excluded minor for $\mathcal{M}(\Delta \leq k)$.*

Proof. Suppose that M is an excluded minor for $\mathcal{M}(\Delta \leq k+1)$. Since $M \notin \mathcal{M}(\Delta \leq k+1)$, and $\Delta(\mathrm{BG}(M)) = \Delta(\mathrm{BG}(M')) + 1$, it follows that $M' \notin \mathcal{M}(\Delta \leq k)$. Let e be any element of $E(M')$. Note that $M\backslash e = U_{1,2} \oplus M'\backslash e$ is a member of $\mathcal{M}(\Delta \leq k+1)$. Therefore $\Delta(\mathrm{BG}(M'\backslash e)) \leq k$, and thus $M'\backslash e \in \mathcal{M}(\Delta \leq k)$. We may similarly show that $M'/e \in \mathcal{M}(\Delta \leq k)$. □

Proposition A.1.2. *If M is a connected matroid with more than one element, and M has no $U_{1,3}$- or $U_{2,3}$-minor, then M is isomorphic to $U_{1,2}$.*

Proof. Assume that $|E(M)| \geq 3$. If $e \in E(M)$, then either $M\backslash e$ or M/e is connected. It follows that M has a connected minor that contains exactly three elements. There are only two such matroids, $U_{1,3}$ and $U_{2,3}$. Therefore it must be the case that $|E(M)| = 2$, and hence $M \cong U_{1,2}$. □

Proposition A.1.3. [Proposition 3.5.12] *The excluded minors for $\mathcal{M}(\Delta \leq 2)$ are $U_{1,4}$, $U_{2,4}$, $U_{1,2} \oplus U_{1,3}$, $U^3_{1,2}$, the matroid $M(W_2)$, shown in Figure A.1, and their generalised duals.*

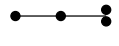


Figure A.1: $M(W_2)$, an excluded minor for $\mathcal{M}(\Delta \leq 2)$.

Proof. Let us first note that Proposition 3.5.3 enables us to check easily that $U_{1,4}$ and $U_{2,4}$ are excluded minors for $\mathcal{M}(\Delta \leq 2)$. Proposition 3.5.9 shows that $U_{1,2} \oplus U_{1,3}$ and $U^3_{1,2}$ are excluded minors for $\mathcal{M}(\Delta \leq 2)$ (use the values $x_1 = 2$, $x_2 = 3$, and $x_1 = x_2 = x_3 = 2$, respectively).

Since $\mathrm{BG}(M(W_2))$ is isomorphic to the wheel on five vertices, the maximum degree of $\mathrm{BG}(M(W_2))$ is four. If we ignore loops and coloops, the only matroids we can obtain from $M(W_2)$ by removing a single element are, up to duality, $U_{1,2}$, and $U_{1,3}$. It follows that $M(W_2)$ is indeed an excluded minor for $\mathcal{M}(\Delta \leq 2)$.

To complete the proof, let us assume that M is an excluded minor for $\mathcal{M}(\Delta \leq 2)$ that is not listed in the statement of the proposition. Therefore M has no minor isomorphic to $U_{1,4}$, $U_{2,4}$, $U_{1,2} \oplus U_{1,3}$, $U_{1,2}^3$, $M(W_2)$, or any of their generalised duals. Furthermore, M has no loops or coloops.

If M has at least three connected components, then Proposition A.1.2 implies that each component of M has a $U_{1,2}$-minor, so M has a $U_{1,2}^3$-minor. Therefore M has at most two connected components.

Suppose that M exactly two components. If any of these components has more than two elements, then Proposition A.1.2 implies that M has either a $U_{1,2} \oplus U_{1,3}$-minor or a $U_{1,2} \oplus U_{2,3}$-minor. Therefore M must be isomorphic to $U_{1,2} \oplus U_{1,2}$. But this matroid is a member of $\mathcal{M}(\Delta \leq 2)$, which contradicts our assumption that M is an excluded minor for this class.

We will now suppose that M is connected. By duality we may assume that there exists an element, e, of M, such that $M\backslash e$ is connected. Since $M\backslash e \in \mathcal{M}(\Delta \leq 2)$, Proposition 3.4.16 implies that $|E(M\backslash e)| \leq 3$. Therefore $|E(M)| \leq 4$. By duality we may assume that $r(M) \leq 2$. The rank of M cannot be zero, for otherwise $M \in \mathcal{M}(\Delta \leq 2)$. If $r(M) = 1$, then M is a proper minor of $U_{1,4}$, and is therefore in $\mathcal{M}(\Delta \leq 2)$. Let us assume that $r(M) = 2$. Since M is connected, it must have at least three parallel classes. Since it has no $U_{2,4}$-minor it must have exactly three such classes. No class can contain more than one element, for M does not have an $M(W_2)$-minor. Therefore $M \cong U_{2,3} \in \mathcal{M}(\Delta \leq 2)$. This contradiction completes the proof. □

Proposition A.1.4. [Proposition 3.5.13] *The class $\mathcal{M}(\chi \leq 3)$ is equal to $\mathcal{M}(\omega \leq 3)$, and the only excluded minors for this class are $U_{1,4}$ and $U_{3,4}$. The only connected matroids in $\mathcal{M}(\chi \leq 3) = \mathcal{M}(\omega \leq 3)$ are, up to duality, $U_{0,1}$, $U_{1,2}$, $U_{1,3}$, $U_{2,4}$, and $M(W_2)$.*

Proof. It is very easy to check that $U_{1,4}$ is an excluded minor for $\mathcal{M}(\chi \leq 3)$. Let M be a connected matroid that has no $U_{1,4}$- or $U_{3,4}$-minor. Since the circuits and cocircuits of M have size at most three, Proposition 3.4.10 shows that $|E(M)| \leq 4$.

The connected matroids of size at most four are listed in [42, Table 4.1]. Up to duality, they are $U_{0,1}$, $U_{1,2}$, $U_{1,3}$, $U_{1,4}$, $U_{2,4}$, and $M(W_2)$ (we again exclude the empty matroid). It remains only to show that $U_{0,1}$, $U_{1,2}$, $U_{1,3}$, $U_{2,4}$, and $M(W_2)$ belong to $\mathcal{M}(\chi \leq 3)$. This is easily done, since the basis graphs of these matroids are, respectively, K_1, K_2, K_3, the octahedron, and the wheel on five vertices. □

Proposition A.1.5. [Proposition 3.5.14] *The excluded minors for $\mathcal{M}(\Delta \leq 3)$ are $U_{1,5}$, $U_{2,4}$, $U_{1,2} \oplus U_{1,4}$, $U_{1,3}^2$, $U_{1,2}^2 \oplus U_{1,3}$, $U_{1,2}^4$, $M(W_2)$, and their generalised duals.*

Proof. It is easily checked that the matroids listed in the statement of the result are excluded minors for $\mathcal{M}(\Delta \leq 3)$.

Let us assume that M is an excluded minor for $\mathcal{M}(\Delta \leq 3)$ that is not listed in the statement of the proposition. Therefore M has no minor isomorphic to $U_{1,5}$, $U_{2,4}$, $U_{1,2} \oplus U_{1,4}$, $U^2_{1,3}$, $U^2_{1,2} \oplus U_{1,3}$, $U^4_{1,2}$, $M(W_2)$, or their generalised duals, and, furthermore, M has no loops or coloops.

The matroid M has at most three connected components, for otherwise it has a $U^4_{1,2}$-minor. If M has exactly three connected components, then none of them has more than two elements, for M has no $U^2_{1,2} \oplus U_{1,3}$- or $U^2_{1,2} \oplus U_{2,3}$-minors. Therefore M is isomorphic to $U^3_{1,2}$. This is a contradiction, as $U^3_{1,2} \in \mathcal{M}(\Delta \leq 3)$.

Suppose that M has two connected components. If both components have more than three elements, then M has a minor isomorphic to $U^2_{1,3}$, or one of its generalised duals, so at least one connected component of M is isomorphic to $U_{1,2}$. Proposition A.1.1 implies that the other component of M is an excluded minor for $\mathcal{M}(\Delta \leq 2)$, and is therefore isomorphic to $U_{1,4}$, $U_{3,4}$, $U_{2,4}$ or $M(W_2)$. This is a contradiction, as M is not isomorphic to $U_{1,2} \oplus U_{1,4}$ or $U_{1,2} \oplus U_{3,4}$, and M has no $U_{2,4}$- or $M(W_2)$-minors.

We will suppose that M is connected. Since M is an excluded minor for $\mathcal{M}(\Delta \leq 3)$, and we can remove a single element from M and stay connected, it follows from Proposition 3.4.16 that $|E(M)| \leq 5$. Therefore we may assume, by duality, that $r(M) \leq 2$. If M has rank one, then $M \cong U_{1,n}$, where $1 \leq n \leq 4$, and therefore M is in $\mathcal{M}(\Delta \leq 3)$. If M has rank two, then M has exactly three parallel classes. All of them are trivial, as M has no $M(W_2)$-minor. Therefore $M \cong U_{2,3} \in \mathcal{M}(\Delta \leq 3)$. □

Proposition A.1.6. [Proposition 3.5.15] *The excluded minors for $\mathcal{M}(\chi \leq 4)$ are, up to duality, $U_{1,5}$, $U_{2,5}$, $M(K_4)$, and the matroid N_1, shown in Figure A.2. The only connected matroids in $\mathcal{M}(\chi \leq 4)$ are $U_{0,1}$, $U_{1,2}$, $U_{1,3}$, $U_{1,4}$, $U_{2,4}$, the matroids shown in Figure A.3, and their duals.*

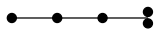


Figure A.2: N_1, an excluded minor for $\mathcal{M}(\chi \leq 4)$.

Proof. We will prove this result with a series of lemmas. It is easy to see that $U_{0,1}$, $U_{1,2}$, $U_{1,3}$, $U_{1,4}$ and $U_{2,4}$, are all members of $\mathcal{M}(\chi \leq 4)$.

A.1.6.1. *All of the matroids in Figure A.3 belong to $\mathcal{M}(\chi \leq 4)$.*

Proof. It will suffice to prove this claim for the matroids (e), (h), and (k), for all the other matroids in the diagram are minors of these.

It is easy to confirm that the basis graph of matroid (e) is isomorphic to the graph obtained by adding a new vertex, adjacent to all others, to $K_3 \square K_3$. The vertex that is adjacent to all others corresponds to the basis that contains no element of the non-trivial parallel class. Since the chromatic number of $K_3 \square K_3$ is three, it is clear that the basis graph of (e) can be 4-coloured.

Suppose that the ground set of matroid (h) is labelled with the integers between 1 and 8 in such a way that the parallel pairs are $\{1, 2\}$, $\{3, 4\}$, $\{5, 6\}$, and $\{7, 8\}$. Then it is possible to check that each of the 32 bases of (h) is contained in exactly

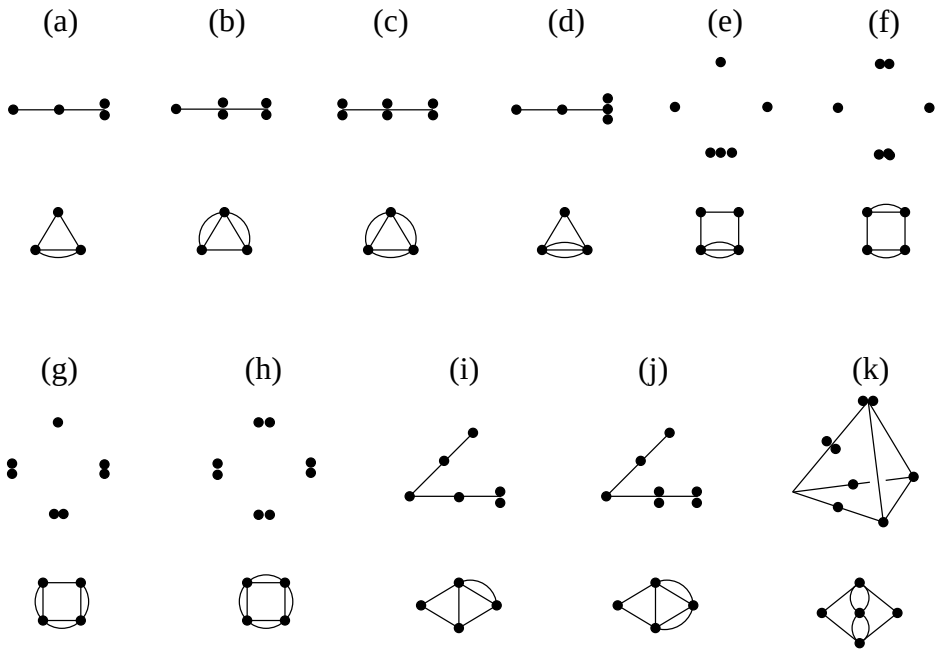


Figure A.3: Matroids in $\mathcal{M}(\chi \leq 4)$.

one of the following four sets, and, furthermore, none of the sets contains an adjacent pair. Thus (h) is contained in $\mathcal{M}(\chi \leq 4)$.

$$\{135, 148, 167, 237, 246, 258, 368, 457\}$$
$$\{136, 147, 158, 238, 245, 267, 357, 468\}$$
$$\{137, 145, 168, 236, 248, 257, 358, 467\}$$
$$\{138, 146, 157, 235, 247, 268, 367, 458\}$$

Similarly, suppose that the ground set of matroid (k) is labelled as in Figure A.4. Simple calculations show that (k) has 32 bases. The following four sets contain each of the bases of (k) exactly once, and, furthermore, no set contains a pair of bases that are adjacent in the basis graph of (k).

$$\{1235,\ 1246,\ 1347,\ 1368,\ 1458,\ 2348,\ 2367,\ 2457\}$$
$$\{1236,\ 1245,\ 1348,\ 1357,\ 1467,\ 2347,\ 2358,\ 2468\}$$
$$\{1237,\ 1248,\ 1346,\ 1358,\ 1457,\ 2345,\ 2368,\ 2467\}$$
$$\{1238,\ 1247,\ 1345,\ 1367,\ 1468,\ 2346,\ 2357,\ 2458\}$$ □

Checking these families of sets by hand is routine but tedious: a Mathematica program that performs the same task is shown in Appendix C.

It is easy to check that $U_{1,5}$ is an excluded minor for $\mathcal{M}(\chi \leq 4)$. That $U_{2,5}$ is also an excluded minor follows from Proposition 3.5.4. Showing that N_1 and $M(K_4)$ are excluded minors requires a little more work.

A.1.6.2. *The matroid N_1 is an excluded minor for $\mathcal{M}(\chi \leq 4)$.*

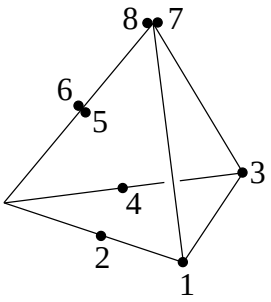


Figure A.4: Matroid (k).

Proof. Suppose that the ground set of N_1 is labelled with the integers between 1 and 5 in such a way that the only non-trivial parallel class of N_1 is $\{4, 5\}$. Then the basis graph for N_1 is shown in Figure A.5. We wish to show that this graph cannot be 4-coloured. We will assume the contrary, and suppose that there exists a proper 4-colouring of $\mathrm{BG}(N_1)$ with the colours R, B, G, and Y. Let us assume that 12 is coloured R. Because the vertices, 12, 13, 14, and 15 form a clique, they must be bijectively coloured with R, B, G, and Y. The same statement applies to 12, 23, 24, and 25. None of the vertices 13, 14, 23, or 24 can be coloured R, for they are all adjacent to 12. The same set of vertices cannot be coloured with exactly two colours, for that would imply that 15 and 25 have the same colour. Therefore 13, 14, 23, and 24 are coloured with all three of the colours B, G, and Y. Hence 34 is coloured R. Using the same reasoning, we can show that 13, 15, 23, and 25 must use all of the three colours B, G, and Y. It follows that 35 is also coloured R. This is a contradiction as 34 and 35 are adjacent.

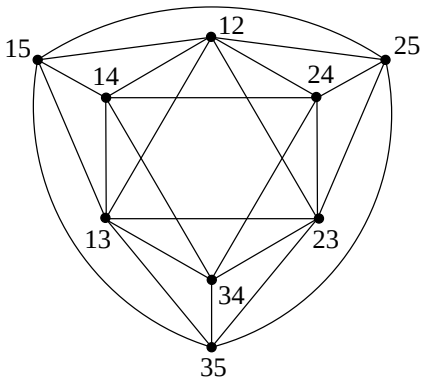


Figure A.5: The basis graph of N_1.

We have shown that $N_1 \notin \mathcal{M}(\chi \leq 4)$. The only matroids that can be produced by removing a single element from N_1 are (ignoring loops and coloops) $U_{1,3}$, $U_{1,4}$, $U_{2,4}$, and $M(W_2)$. Since all of these have 4-colourable basis graphs, it follows that N_1 is an excluded minor for $\mathcal{M}(\chi \leq 4)$. □

A.1.6.3. *The matroid $M(K_4)$ is an excluded minor for $\mathcal{M}(\chi \leq 4)$.*

Proof. Let the ground set of $M(K_4)$ be labelled as in Figure A.6. Then the basis graph of $M(K_4)$ is shown in Figure A.7. To avoid cluttering the diagram we have omitted all those edges that join a basis containing 6 to a basis avoiding 6.

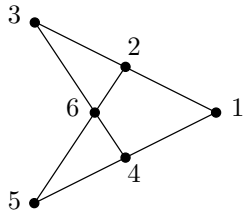


Figure A.6: $M(K_4)$

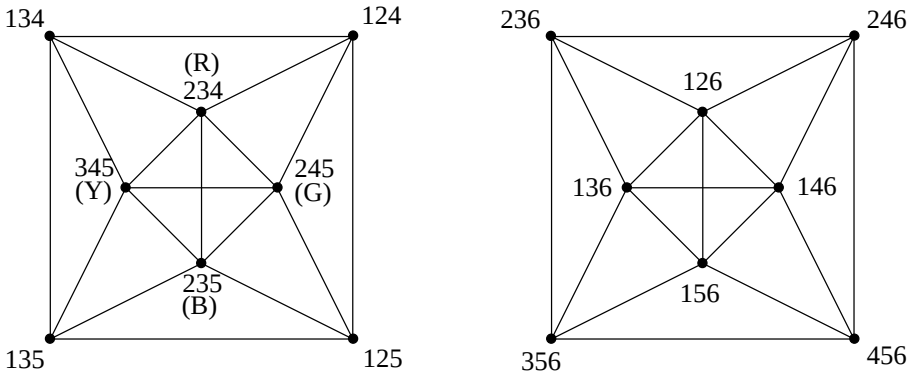


Figure A.7: The basis graph of $M(K_4)$ (with edges omitted).

Let us suppose that $\mathrm{BG}(M(K_4))$ can be properly 4-coloured with the colours R, B, G, and Y. Since 234, 235, 245, and 345 form a clique, we may assume that they are coloured R, B, G, and Y respectively. The vertices 126, 136, 146, and 156 also form a clique, so they must be bijectively coloured with R, B, G, and Y. Note that 124 and 246 are adjacent, and that both are adjacent to each of the vertices 126, 146, 234, and 245. This means that 126, 146, 234, and 245 must be coloured with exactly two colours, namely R and G. Similarly, 135 and 356 are adjacent, and are both adjacent to all the vertices 136, 156, 235, and 345. This implies that 136, 156, 235, and 345 are coloured using only B and Y.

Given these restrictions, there are only four ways in which the vertices 126, 136, 146, and 156 can be coloured.

	126	146	136	156
(i)	R	G	B	Y
(ii)	R	G	Y	B
(iii)	G	R	B	Y
(iv)	G	R	Y	B

Let us first consider case (i). In this case 126, 156, 235, and 245 use all four colours. This yields a contradiction, as 125 is adjacent to all four of these vertices.

Similarly, in case (iv), the vertices 146, 156, 245, and 345 use all four colours, but 456 is adjacent to all four vertices.

Now let us consider case (ii). The vertex 125 is adjacent to 126, 156, 235, and 245. These vertices are coloured R, B, B, and G respectively, so 125 must be coloured Y. By the same reasoning, 134, which is adjacent to 136, 146, 234, and 345 (coloured Y, G, R, and Y respectively) must be coloured B. This leads us to a contradiction, as 124 is adjacent to 125, 134, 234, and 245, and we have concluded that these vertices are coloured Y, B, R, and G.

Case (iii) is similar. The vertex 236 is adjacent to 126, 136, 234, and 235, and they are coloured G, B, R, and B, so 236 is coloured Y. Similarly, 456 is adjacent to 146, 156, 245, and 345 (R, Y, G, and Y), so 456 is coloured B. However, 246 is adjacent to 126, 146, 236, and 456 (G, R, Y, and B).

This contradiction shows that $M(K_4) \notin \mathcal{M}(\chi \leq 4)$. Up to duality, the only minor produced by removing a single element of $M(K_4)$ is the matroid (b) from Figure A.3. We have already shown, in A.1.6.1, that this matroid is in $\mathcal{M}(\chi \leq 4)$. This completes the proof that $M(K_4)$ is an excluded minor for $\mathcal{M}(\chi \leq 4)$. □

It remains to show that if a connected matroid has no minor isomorphic to $U_{1,5}$, $U_{2,5}$, $M(K_4)$, N_1, or their duals, then it must be one of the matroids listed as belonging to $\mathcal{M}(\chi \leq 4)$ in the statement of Proposition A.1.6. Let M be such a matroid. Since M has no $U_{1,5}$- or $U_{4,5}$-minors, the circuits and cocircuits of M have size at most four. Therefore Proposition 3.4.10 implies that $|E(M)| \leq 8$. By duality we may assume that $r(M) \leq 4$.

If M has rank zero, then clearly $M \cong U_{0,1}$. Similarly, if $r(M) = 1$, then $M \cong U_{1,n}$, where $1 \leq n \leq 4$. Suppose that M has rank two. Since M is connected it has at least three parallel classes, however, M has no $U_{2,5}$-minor, so it has no more than four parallel classes. If M has exactly four parallel classes, it must be isomorphic to $U_{2,4}$, for M has no N_1-minor. Any two parallel classes of M must contain no more than four elements in total, for otherwise M has a $U_{1,5}$-minor. Therefore, if M contains exactly three parallel classes, it must be isomorphic to one of the matroids (c) or (d) in Figure A.3, or to one of their restrictions.

We will now assume that $r(M) = 3$. By duality we may also assume that $|E(M)| \geq 6$. Suppose that M contains a non-trivial line, l. There must be at least two parallel classes that do not lie on l. We will initially assume that there are exactly two such classes, A and B. If $\mathrm{cl}(A \cup B)$ does not meet l, then M has a minor isomorphic to the matroid obtained by adding a point freely to the line between two elements of $U_{3,4}$. This matroid is the dual of N_1, so we have a contradiction. We must assume that $\mathrm{cl}(A \cup B)$ does meet l. If l contains four parallel classes, then M has an N_1-minor, so we will assume that l contains exactly three parallel classes. Therefore M must have at least one non-trivial parallel class, for otherwise $|E(M)| \leq 5$, in contradiction to our assumption. It is not difficult to see that if M contains more than one non-trivial parallel class, or if M contains a parallel class with more than two elements, then either M is isomorphic to matroid (j), or M has a minor isomorphic to $U_{1,5}$. Therefore we will assume that M contains exactly one non-trivial parallel class, and that this parallel class contains two elements. It follows that M is isomorphic to

either the dual of matroid (f), or matroid (i) in Figure A.3.

Let us now assume that there are three parallel classes, A, B, and C, which do not lie on l. If any two of these classes span a line which does not intersect l, then we again see that M has a minor isomorphic to the dual of N_1. Therefore the line spanned by any two of A, B, or C meets l. However, this leads to a contradiction, as it is now easy to see that M has a minor isomorphic to $M(K_4)$.

We will now assume that M contains no non-trivial lines. Since M has no $U_{3,5}$-minor, it contains at most four parallel classes, and since it is connected, it must contain exactly four such classes. No two parallel classes can contain more than four points in total, for otherwise M has a $U_{1,5}$-minor. It is now easy to see that M is isomorphic to one of the matroids (e) or (h), or one of their restrictions.

This completes the case that $r(M) = 3$. Let us assume that $r(M) = 4$. By duality we may assume $|E(M)| \geq 8$, and by an earlier comment, we conclude that $|E(M)|$ is exactly eight. Let e be any element of M. By duality we will assume that $M\backslash e$ is connected. Thus $(M\backslash e)^*$ is a connected, rank-3 matroid in $\mathcal{M}(\chi \leq 4)$ with exactly seven elements. By our earlier analysis we know there are exactly two such matroids, namely the matroids (g) and (j) in Figure A.3. Thus $M\backslash e$ is isomorphic to one of the matroids illustrated in Figure A.8. First suppose that $M\backslash e$ is isomorphic to the dual of matroid (g). Note that contracting the element x from $M\backslash e$ produces a connected matroid, so unless e is parallel to x in M, the matroid M/x is also connected. But in this case M/x is a connected, rank-3 member of $\mathcal{M}(\chi \leq 4)$ with exactly seven elements, and at least two non-trivial lines that intersect in a non-trivial parallel class. This is a contradiction, as our earlier analysis shows that no such matroid exists. Therefore e must be parallel with x, but this is a contradiction as M then has a minor isomorphic to $U_{1,5}$.

We must assume that $M\backslash e$ is isomorphic to the dual of (j). Contracting the element y from $M\backslash e$ produces a matroid isomorphic to matroid (i). It must be the case that M/y is connected, for if not, then e is parallel to y in M, and that implies that M has a $U_{1,5}$-minor. So M/y is isomorphic to a connected matroid in $\mathcal{M}(\chi \leq 4)$ that can be obtained from (i) by extending by the element e. Thus M/y is isomorphic to (j). It follows that e is placed on the line between y and z. However, by using the same arguments, we can show that e must be placed on the line between y' and z. Thus e must be parallel to z in M, and hence M is isomorphic to matroid (k). □

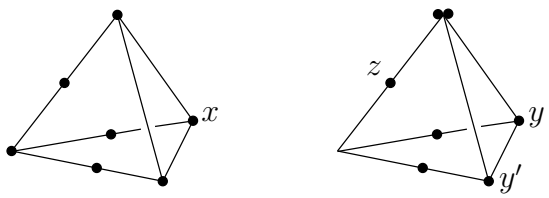


Figure A.8: The duals of matroids (g) and (j).

Proposition A.1.7. [Proposition 3.5.16] *The excluded minors for* $\mathcal{M}(\Delta \leq 4)$ *are, up to generalised duality,* $U_{1,6}$, $U_{2,5}$, $U_{1,2} \oplus U_{1,5}$, $U_{1,3} \oplus U_{1,4}$, $U_{1,2} \oplus U_{2,4}$, $U_{1,2} \oplus M(W_2)$,

$U_{1,2}^2 \oplus U_{1,4}$, $U_{1,2} \oplus U_{1,3}^2$, $U_{1,2}^3 \oplus U_{1,3}$, $U_{1,2}^5$, N_1 *(shown in Figure A.2), and the matroids* N_2 *and* N_3 *shown in Figure A.9.*



Figure A.9: N_2 and N_3, excluded minors for $\mathcal{M}(\Delta \leq 4)$.

Proof. Proposition 3.5.9 enables us easily to see that $U_{1,6}$, $U_{1,2} \oplus U_{1,5}$, $U_{1,3} \oplus U_{1,4}$, $U_{1,2}^2 \oplus U_{1,4}$, $U_{1,2} \oplus U_{1,3}^2$ $U_{1,2}^3 \oplus U_{1,3}$, and $U_{1,2}^5$ are all excluded minors for $\mathcal{M}(\Delta \leq 4)$.

Proposition 3.5.3 tells us that $\Delta(\mathrm{BG}(U_{2,5})) = 6$, but it is easy to see that all proper minors of $U_{2,5}$ do belong to $\mathcal{M}(\Delta \leq 4)$.

Because $\Delta(\mathrm{BG}(U_{2,4})) = 4$, it follows that $\Delta(\mathrm{BG}(U_{1,2} \oplus U_{2,4})) = 5$. Similarly, the maximum degree of $\mathrm{BG}(U_{1,2} \oplus M(W_2))$ is five. Lemma 3.4.15 therefore implies that $U_{1,2} \oplus U_{2,4}$ and $U_{1,2} \oplus M(W_2)$ must both be excluded minors for $\mathcal{M}(\Delta \leq 4)$.

Consider any basis of N_1 that does not contain an element of the parallel pair. Proposition 3.4.1 shows that the degree of this basis in $\mathrm{BG}(N_1)$ is six. However, the only matroids obtainable by removing a single element from N_1 are (ignoring loops and coloops), $U_{1,3}$, $U_{1,4}$, $U_{2,4}$, and $M(W_2)$. The basis graphs of these matroids are, respectively, K_3, K_4, the octahedron, and the wheel on five vertices. Since all of these have maximum degree at most four, N_1 is an excluded minor for $\mathcal{M}(\Delta \leq 4)$. Using a similar analysis, it is easy to show that the two matroids in Figure A.9 are also excluded minors for $\mathcal{M}(\Delta \leq 4)$.

Suppose that M is an excluded minor for $\mathcal{M}(\Delta \leq 4)$ that has not already been listed. Therefore M is a loopless and coloopless matroid that has no minor isomorphic to $U_{1,6}$, $U_{2,5}$, $U_{1,2} \oplus U_{1,5}$, $U_{1,3} \oplus U_{1,4}$, $U_{1,2} \oplus U_{2,4}$, $U_{1,2} \oplus M(W_2)$, $U_{1,2}^2 \oplus U_{1,4}$, $U_{1,2} \oplus U_{1,3}^2$, $U_{1,2}^3 \oplus U_{1,3}$, $U_{1,2}^5$, N_1, N_2, N_3, or one of their generalised duals.

Suppose that a connected component of M is isomorphic to $U_{1,2}$. Then Proposition A.1.1 and Proposition A.1.5 imply that M is isomorphic to $U_{1,2} \oplus U_{1,5}$, $U_{1,2} \oplus U_{2,4}$, $U_{1,2}^2 \oplus U_{1,4}$, $U_{1,2} \oplus U_{1,3}^2$, $U_{1,2}^3 \oplus U_{1,3}$, $U_{1,2}^5$, $U_{1,2} \oplus M(W_2)$, or one of their generalised duals. This is a contradiction, so no component of M is isomorphic to $U_{1,2}$. This means that M has at most two connected components, for if it has three, then it certainly has a minor isomorphic to $U_{1,2} \oplus U_{1,3}^2$ or one of its generalised duals.

Let us assume that M has exactly two connected components. Since neither component is isomorphic to $U_{1,2}$, Proposition A.1.2 implies that both components have a minor isomorphic to $U_{1,3}$ or $U_{2,3}$. Therefore neither component can have a minor isomorphic to $U_{1,4}$ or $U_{3,4}$, for M has no minor isomorphic to $U_{1,3} \oplus U_{1,4}$ or its generalised duals. Since M has no $U_{1,2} \oplus U_{2,4}$- or $U_{1,2} \oplus M(W_2)$-minors, it follows easily from Proposition A.1.4 that both components must be isomorphic to $U_{1,3}$ or $U_{2,3}$. This is a contradiction, as $\Delta(\mathrm{BG}(U_{1,3}^2)) = 4$, and therefore $M \in \mathcal{M}(\Delta \leq 4)$.

We may now suppose that M is connected. We can obtain a connected member of $\mathcal{M}(\Delta \leq 4)$ by removing a single element of M, and so it follows from Proposition 3.4.16 that $|E(M)| \leq 6$. Therefore we will assume that $r(M) \leq 3$. If $r(M) = 0$,

then $M \in \mathcal{M}(\Delta \leq 4)$, and similarly, if $r(M) = 1$, then M must be a proper minor of $U_{1,6}$.

Suppose that $r(M) = 2$. Then M must have either three or four parallel classes. If M has four such classes, then it must be isomorphic to $U_{2,4}$, for otherwise M has an N_1-minor. However, $\Delta(\mathrm{BG}(U_{2,4})) = 4$. Let us assume that M has exactly three parallel classes. Because M has no minor isomorphic to N_2 or N_3, it follows that no more than one parallel class can be non-trivial, and no parallel class can contain more than two elements. Therefore M is isomorphic to either $U_{2,3}$ or $M(W_2)$, and both of these belong to $\mathcal{M}(\Delta \leq 4)$.

To complete the proof we assume that $r(M) = 3$. By duality we may assume that $|E(M)| = 6$. First suppose that M contains a non-trivial line, l. There can be no more than four parallel classes in l. Suppose there are exactly four. Since M is connected there must be two non-parallel elements, a and b, that do not lie in l. Then M has a N_1- or $U_{2,5}$-minor, depending on whether or not $\mathrm{cl}(\{a, b\})$ meets l.

Let us assume that l contains exactly three parallel classes. Again, there must be two distinct parallel classes not contained in l, and if one of these parallel classes is non-trivial, then M has a minor isomorphic to either N_1 or N_2.

Similarly, if one of the parallel classes in l is non-trivial, then it is easy to see that M has a minor isomorphic to either N_1, N_2, or N_3. Therefore we will assume that M contains no non-trivial parallel classes. Since $|E(M)| = 6$, it is easy to see that there must be three pairwise non-parallel points, a, b, and c, that do not lie in l. Depending on whether or not l meets the lines $\mathrm{cl}(\{a, b\})$ and $\mathrm{cl}(\{a, c\})$, it must be the case that M has a minor isomorphic to $U_{2,5}$, N_1, or N_2.

Therefore we will assume that M contains no non-trivial lines. Since M has no $U_{3,5}$-minor, it must have exactly four parallel classes. Since M contains exactly six elements, M has either two non-trivial parallel classes, each containing exactly two elements, or one non-trivial parallel class, containing exactly three elements. In the first case M has a minor isomorphic to N_2, and in the second M has a minor isomorphic to N_3. This contradiction completes the proof. □

Proposition A.1.8. [Proposition 3.6.4] *The excluded minors for* $\mathcal{M}(\textsc{Planar})$ *are* $U_{1,5}$, $U_{2,5}$, N_1, N_2, N_3, $U_{1,2} \oplus U_{1,4}$, $U_{1,2} \oplus U_{2,4}$, $U^2_{1,3}$, $U_{1,2} \oplus M(W_2)$, $U^2_{1,2} \oplus U_{1,3}$, $U^4_{1,2}$, *and their generalised duals. Here the matroids* $M(W_2)$, N_1, N_2, *and* N_3 *are as illustrated in Figures A.1, A.2 and A.9. The only loopless and coloopless matroids in* $\mathcal{M}(\textsc{Planar})$ *are, up to generalised duality,* $U_{1,2}$, $U_{1,3}$, $U_{1,4}$, $U_{2,4}$, $M(W_2)$, $U^2_{1,2}$, $U_{1,2} \oplus U_{1,3}$, *and* $U^3_{1,2}$.

Proof. We first note that the basis graphs of $U_{1,2}$, $U_{1,3}$, $U_{1,4}$, $U_{2,4}$, $M(W_2)$, $U^2_{1,2}$, $U_{1,2} \oplus U_{1,3}$, and $U^3_{1,2}$ are, respectively, K_2, K_3, K_4, the octahedron, the wheel on five vertices, $K_2 \square K_2$, $K_2 \square K_3$, and $K_2 \square K_2 \square K_2$. These are all easily seen to be planar.

During the proof we will make frequent use of the well-known fact that a graph is planar if and only if it has no subgraph homeomorphic to $K_{3,3}$ or K_5.

It is clear that $U_{1,5}$ is an excluded minor for $\mathcal{M}(\textsc{Planar})$, for $\mathrm{BG}(U_{1,5}) \cong K_5$, and the only minors of $U_{1,5}$ produced by removing a single element are $U_{0,4}$ and $U_{1,4}$. The basis graphs of these matroids are, respectively, K_1 and K_4.

The easiest way of demonstrating that $\mathrm{BG}(U_{2,5})$ and $\mathrm{BG}(N_1)$ are not planar is to observe that, since $\mathrm{BG}(U_{2,5})$ and $\mathrm{BG}(N_1)$ are not 4-colourable (Proposition A.1.6), the Four Colour Theorem implies that they are not planar. It is now easy to show that $U_{2,5}$ and N_1 are excluded minors for $\mathcal{M}(\textsc{Planar})$.

Suppose that the ground sets of N_2 and N_3 are labelled with the integers between 1 and 5 so that the parallel classes of N_2 are $\{1\}$, $\{2, 3\}$, and $\{4, 5\}$, and the parallel classes of N_3 are $\{1\}$, $\{2\}$, and $\{3, 4, 5\}$. Figures A.10 and A.11 show $\mathrm{BG}(N_1)$ and $\mathrm{BG}(N_2)$, and subgraphs that are homeomorphic to K_5. It is easy to see that all the proper minors of N_2 and N_3 have planar basis graphs.

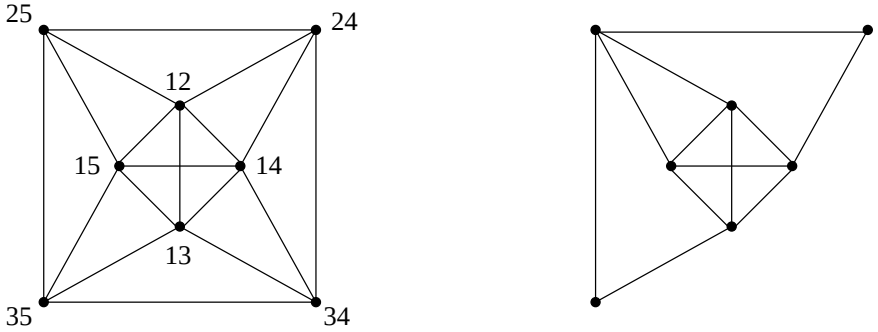


Figure A.10: The basis graph of N_2 is not planar.

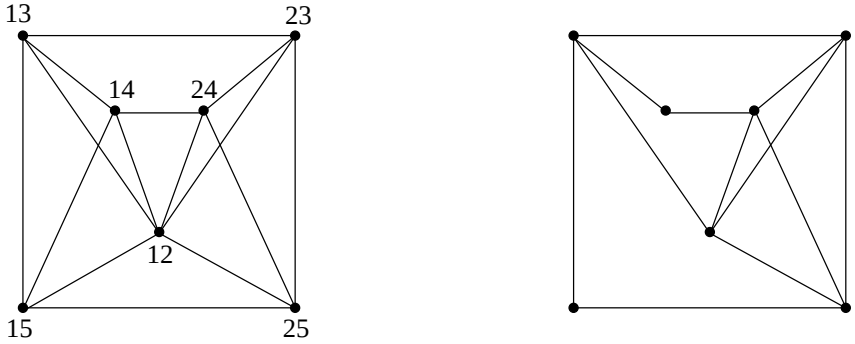


Figure A.11: The basis graph of N_3 is not planar.

Next we show that the basis graphs of $U_{1,2} \oplus U_{1,4}$, $U_{1,2} \oplus M(W_2)$, and $U_{1,2} \oplus U_{2,4}$ are not planar. It will follow easily that these matroids are excluded minors for $\mathcal{M}(\textsc{Planar})$. These three basis graphs are isomorphic to the Cartesian product of K_2 with, respectively, K_4, the wheel on five vertices, and the octahedron. The non-planarity of $K_2 \square K_4$ is demonstrated in Figure A.12. Figure A.13 shows that the basis graph of $U_{1,2} \oplus M(W_2)$ is not planar. This also shows that $\mathrm{BG}(U_{1,2} \oplus U_{2,4})$ is not planar, for $U_{1,2} \oplus U_{2,4} \overset{\text{r.p.}}{\to} U_{1,2} \oplus M(W_2)$, and Proposition 3.3.2 therefore implies that $\mathrm{BG}(U_{1,2} \oplus M(W_2))$ is an induced subgraph of the basis graph of $U_{1,2} \oplus U_{2,4}$.

The basis graphs of $U^2_{1,3}$ and $U^2_{1,2} \oplus U_{1,3}$ are isomorphic to $K_3 \square K_3$ and $K_2 \square K_2 \square K_3$ respectively. Figures A.14 and A.15 show that these graphs are not planar. It follows easily that $U^2_{1,3}$ and $U^2_{1,2} \oplus U_{1,3}$ are excluded minors for $\mathcal{M}(\textsc{Planar})$.

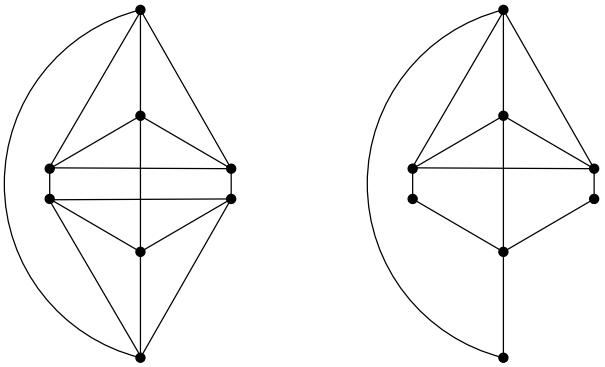


Figure A.12: The basis graph of $U_{1,2} \oplus U_{1,4}$ is not planar.

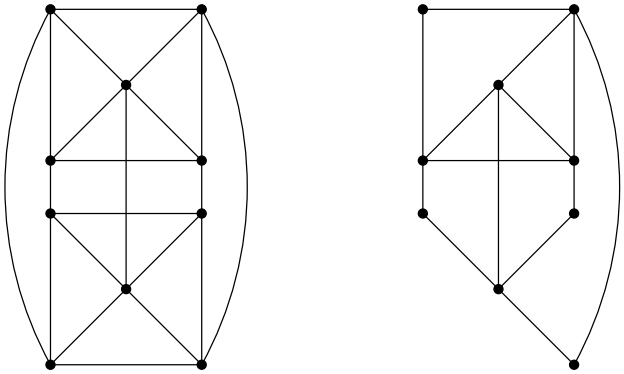


Figure A.13: The basis graph of $U_{1,2} \oplus M(W_2)$ is not planar.

To show that $U^4_{1,2}$ is not a member of $\mathcal{M}(\textsc{Planar})$ we note that $\mathrm{BG}(U^4_{1,2})$ is isomorphic to $K_2 \square K_2 \square K_2 \square K_2$, in other words, Q_4, the 4-dimensional hypercube. It follows from [21, Theorem 11.20] that this graph is not planar, whereas Q_3, the basis graph of $U^3_{1,2}$, is planar. It immediately follows that $U^4_{1,2}$ is an excluded minor for $\mathcal{M}(\textsc{Planar})$.

We have shown that the matroids listed in the statement of the proposition are indeed excluded minors for $\mathcal{M}(\textsc{Planar})$. To complete the proof we will let M be a loopless and coloopless matroid that has no minor isomorphic to $U_{1,5}$, $U_{2,5}$, N_1, N_2, N_3, $U_{1,2} \oplus U_{1,4}$, $U_{1,2} \oplus M(W_2)$, $U_{1,2} \oplus U_{2,4}$, $U^2_{1,3}$, $U^2_{1,2} \oplus U_{1,3}$, $U^4_{1,2}$, or one of their generalised duals. We wish to show that M is isomorphic to $U_{1,2}$, $U_{1,3}$, $U_{1,4}$, $U_{2,4}$,

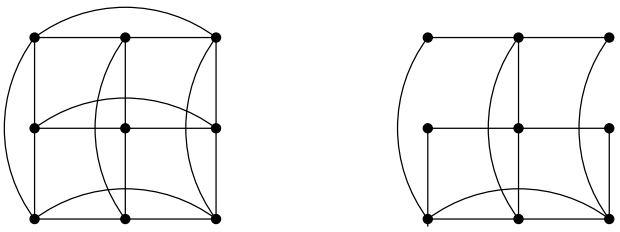


Figure A.14: The basis graph of $U^2_{1,3}$ is not planar.

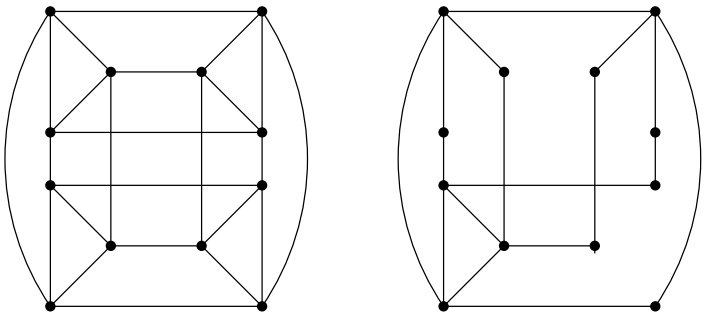


Figure A.15: The basis graph of $U_{1,2}^2 \oplus U_{1,3}$ is not planar.

$M(W_2)$, $U_{1,2}^2$, $U_{1,2} \oplus U_{1,3}$, $U_{1,2}^3$, or one of their generalised duals.

We first note that M has at most three connected components, and if it has exactly three, then it is isomorphic to $U_{1,2}^3$. If M has exactly two components, then one of these components must be isomorphic to $U_{1,2}$, for otherwise M has a minor isomorphic to a generalised dual of $U_{1,3}^2$. Let M' be the other component of M. No minor of M' can be isomorphic to $U_{1,4}$, $U_{2,4}$, $M(W_2)$, or one of their duals. Since the circuits and cocircuits of M' have size at most three, Proposition 3.4.10 shows that $|E(M')| \leq 4$, so we may assume that $r(M') \leq 2$. It is now easy to show that M' is isomorphic to $U_{1,2}$, $U_{1,3}$ or $U_{2,3}$.

We will now assume that M is connected. Then $|E(M)| \leq 8$, by Proposition 3.4.10, so we may assume that $r(M) \leq 4$. If $r(M) \leq 2$, then M is isomorphic to one of $U_{1,2}$, $U_{1,3}$, $U_{1,4}$, $U_{2,3}$, $M(W_2)$, or $U_{2,4}$. Suppose that $r(M) = 3$. We may assume that $|E(M)| \geq 6$. By modifying the proof of Proposition A.1.7 we may show that if a connected rank-3 matroid containing at least six elements has no $U_{2,5}$-, N_1-, N_2-, or N_3-minors, then that matroid can contain no non-trivial lines. Therefore we will assume that M contains no 3-circuits. There must be exactly four parallel classes in M, and none of these classes can be non-trivial, for otherwise M contains a minor isomorphic to the matroid obtained by adding an element in parallel to a point of $U_{3,4}$, and this is the dual of N_3. Therefore M must be isomorphic to $U_{3,4}$.

Finally, suppose that $r(M) = 4$, and that $|E(M)| = 8$. Since we can remove a single element of M and stay connected, it follows that there exists a connected rank-3 matroid in $\mathcal{M}(\textsc{Planar})$ with seven elements. However, our earlier analysis shows that no such matroid exists. This completes the characterisation of $\mathcal{M}(\textsc{Planar})$. $\square$

Appendix B

Computer calculations I

In this appendix we consider the matroid, M_{36}, introduced in Example 1 of Section 4.3. The Mathematica output shown here provides a verification that M_{36} is equitable.

We first define the functions, $f_1(x,\, r^*) = \sum_{i=1}^{x} \binom{x}{i} i!(r^* - i)!$ and

$$f_2(x,\, y,\, r^*) = \sum_{\substack{1 \le i,\, j+k \\ i+j \le x \\ k \le y}} \binom{x}{i}\binom{x-i}{j}\binom{y}{k} i!(j+k)!(r^* - i - j - k)!.$$

```
In[1] := f1[x_,r_] := Sum[Binomial[x,i]i!(r-i)!,{i,1,x}]
In[2] := f2[x_,y_,r_] := Sum[If[i+j <= x && 1 <= j+k,
            Binomial[x,i]Binomial[x-i,j]Binomial[y,k]
               i!(j+k)!(r-i-j-k)!,0],{i,1,x},{j,0,x},{k,0,y}]
```

In the proof of Proposition 4.3.6, we showed that $O(B_1,\, M_{36})$ is equal to $3!33! + 8f_1(7,\, 33) + 2f_1(14,\, 33) + 2f_2(7,\, 7,\, 33) + 2f_2(7,\, 0,\, 33)$.

```
In[3] := 3!33! + 8f1[7,33] + 2f1[14,33] + 2f2[7,7,33] + 2f2[7,0,33]
Out[3] = 861095663865512077472220944793600000000
```

Similarly, $O(B_2,\, M_{36})$ is equal to $3!33! + 6f_1(7,\, 33) + 2f_1(18,\, 33) + 2f_2(7,\, 0,\, 33)$.

```
In[4] := 3!33! + 6f1[7,33] + 2f1[18,33] + 2f2[7,0,33]
Out[4] = 861095663865512077472220944793600000000
```

These quantities are equal.

```
In[5] := %-%%
Out[5] = 0
```

The number of bases in M_{36} is $20.8^2 + 2.8.\binom{20}{2}$.

```
In[6] := 20*8^2 + 2*8*Binomial[20,2]
Out[6] = 4320
```

Dividing 36! by this quantity gives us $O(B_1,\, M_{36}) = O(B_2,\, M_{36})$, as expected.

```
In[7] := 36!/%
Out[7] = 861095663865512077472220944793600000000
```

A search of the rank-3 matroids that contain a line of length l, and two freely placed parallel classes, where the size of the parallel classes on the line is n, and the other two parallel classes have m elements, reveals two equitable matroids when l ranges between 3 and 50, and m and n both range between 1 and 20.

```
In[8] := Total1[l_,m_,n_] := f1[2m-2,2m+n*l-3] +
            f1[m-1,2m+n*l-3] + f2[m-1,m-1,2m+n*l-3]
In[9] := Total2[l_,m_,n_] := f1[n*l-2,2m+n*l-3] +
            f1[n-1,2m+n*l-3] + f2[n-1,n*l-n-1,2m+n*l-3]
In[10] := Do[If[Total1[l,m,n] == Total2[l,m,n],
             Print[{l,m,n}]],{l,3,50},{m,1,20},{n,1,20}]
          {20,8,1}
          {20,16,2}
```

Appendix C

Computer calculations II

During the proof of Proposition A.1.6 we stated that the basis graphs of matroids (h) and (k) (as shown in Figure A.3) can be 4-coloured. In this appendix we display the output from a Mathematica program that verifies this fact.

We start by defining the set of non-spanning circuits of (h) and (k).

```
In[1] := NSCircuits1 = {12,34,56,78};
In[2] := NSCircuits2 = {56,78,1234,1257,1258,1267,
            1268,3457,3458,3467,3468};
```

Next we list the families of bases that we claim are pairwise nonadjacent in the basis graphs.

```
In[3] := ColourClasses1 = {{135,148,167,237,246,258,368,457},
            {136,147,158,238,245,267,357,468},
            {137,145,168,236,248,257,358,467},
            {138,146,157,235,247,268,367,458}};
In[4] := ColourClasses2 ={{1235,1246,1347,1368,1458,2348,2367,2457},
            {1236,1245,1348,1357,1467,2347,2358,2468},
            {1237,1248,1346,1358,1457,2345,2368,2467},
            {1238,1247,1345,1367,1468,2346,2357,2458}};
```

The following commands check that there are no duplications in our lists of bases.

```
In[5] := Do[If[i != j &&
           IntegerDigits[Flatten[ColourClasses1][[i]]] ==
              IntegerDigits[Flatten[ColourClasses1][[j]]],
           Print[{i,j}]],{i,1,32},{j,1,32}]
In[6] := Do[If[i != j &&
           IntegerDigits[Flatten[ColourClasses2][[i]]] ==
              IntegerDigits[Flatten[ColourClasses2][[j]]],
           Print[{i,j}]],{i,1,32},{j,1,32}]
```

Next we check that our families of sets really are families of bases: that is, that none of the sets contains a non-spanning circuit.

```
In[7] := Do[If[MemberQ
            [Subsets[IntegerDigits[Flatten[ColourClasses1][[i]]]],
               IntegerDigits[NSCircuits1[[j]]]],
                  Print[{i,j}]],{i,1,32},{j,1,4}]
In[8] := Do[If[MemberQ
            [Subsets[IntegerDigits[Flatten[ColourClasses2][[i]]]],
               IntegerDigits[NSCircuits2[[j]]]],
            Print[{i,j}]],{i,1,32},{j,1,11}]
```

Finally we verify that, if two bases are contained in the same family of bases, then their intersection is no greater than one or two respectively; that is, we check that they are not adjacent in the basis graph.

```
In[9] := Do[Do[If[j != k &&
            Length[Intersection[IntegerDigits[ColourClasses1[[i,j]]],
               IntegerDigits[ColourClasses1[[i,k]]]]] > 1,
            Print[{i,j,k}]],{j,1,8},{k,1,8}],{i,1,4}]
In[10]:= Do[Do[If[j != k &&
            Length[Intersection[IntegerDigits[ColourClasses2[[i,j]]],
               IntegerDigits[ColourClasses2[[i,k]]]]] > 2,
            Print[{i,j,k}]],{j,1,8},{k,1,8}],{i,1,4}]
```

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