

THE SPREAD OF FIRE ON A RANDOM MULTIGRAPH

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Abstract

We study a model for the destruction of a random network by fire. Suppose that we are given a multigraph of minimum degree at least 2 having real-valued edge-lengths. We pick a uniform point from along the length and set it alight; the edges of the multigraph burn at speed 1. If the fire reaches a vertex of degree 2, the fire gets directly passed on to the neighbouring edge; a vertex of degree at least 3, however, passes the fire either to all of its neighbours or none, each with probability $1/2$. If the fire goes out before the whole network is burnt, we again set fire to a uniform point. We are interested in the number of fires which must be set in order to burn the whole network, and the number of points which are burnt from two different directions. We analyse these quantities for a random multigraph having n vertices of degree 3 and $\alpha(n)$ vertices of degree 4, where $\alpha(n)/n \rightarrow 0$ as $n \rightarrow \infty$, with i.i.d. standard exponential edge-lengths. Depending on whether $\alpha(n) \gg \sqrt{n}$ or $\alpha(n) = O(\sqrt{n})$, we prove that as $n \rightarrow \infty$ these quantities converge jointly in distribution when suitably rescaled to either a pair of constants or to (complicated) functionals of Brownian motion.

We use our analysis of this model to make progress towards a conjecture of Aronson, Frieze and Pittel concerning the number of vertices which remain unmatched when we use the Karp-Sipser algorithm to find a matching on the Erdős-Rényi random graph.

Keywords: Random multigraph; Karp-Sipser algorithm; differential equations method; reflected stochastic differential equation

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1. Introduction

1.1. The model

Suppose that we are given a finite connected multigraph with strictly positive real-valued edge-lengths. We introduce a model for the destruction of such a network by fire. The edges are flammable. First, a point is picked uniformly (i.e. according to the normalised Lebesgue measure) and set alight. (With probability 1, this point will lie in the interior of an edge.) The fire passes at speed 1 along the edge (in both directions) until it reaches a node. A node of degree $d \geq 3$ will pass the fire onto all of its other neighbouring edges with probability $1/2$ or stop the fire with probability $1/2$. If it stops the fire, it becomes a vertex of degree $d - 1$. A vertex of degree 2 necessarily passes a fire arriving along from one of its neighbouring edges onto the other one. The fire spreads until it either goes out or has burnt the whole network. If it goes out

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before the whole network is burnt, a new uniform point is picked and set alight, and the process continues as before. We are interested in two aspects of this process:

1. How many new fires must be set in order to burn the whole network?
2. How many times does it happen that a point is burnt from two different directions?

We refer to the second phenomenon as a *clash*.

Let $\alpha : \mathbb{N} \rightarrow \mathbb{N}$ be a function such that $\alpha(n)/n \rightarrow 0$ as $n \rightarrow \infty$. We will study these questions in the setting where the base network is a random multigraph with n vertices of degree 3 and $\alpha(n)$ vertices of degree 4, sampled according to the configuration model (see below for a description). Throughout this paper, we will implicitly assume n to be even. We take the edge-lengths to be independent and identically distributed standard exponential random variables.

Let F^n be the number of fires we must set in order to burn the whole network. Let C^n be the number of clashes. We will study the limiting behaviour of these quantities as $n \rightarrow \infty$. It turns out that both scale as \sqrt{n} as long as $\alpha(n) = O(\sqrt{n})$ and as $\alpha(n)$ if $\alpha(n) \gg \sqrt{n}$. In order to state our results more precisely, we introduce an auxiliary stochastic process.

Let $(B_t)_{t \geq 0}$ be a standard Brownian motion and for $a \geq 0$ let $(X_t^a, L_t^a)_{0 \leq t \leq 1}$ be the unique solution to the stochastic differential equation with reflection determined by

$$dX_t^a = -\frac{2}{3} \frac{X_t^a}{(1-t)} dt + \frac{2a}{3} (1-t)^{1/3} dt + dB_t + dL_t^a, \quad 0 \leq t < 1, \quad (1)$$

where $(L_t^a)_{0 \leq t < 1}$ is the local time process of X^a at level 0. We construct the solution explicitly below, and show that both of these quantities have finite almost sure limits as $t \rightarrow 1$, which we call L_1^a and X_1^a .

Theorem 1.1. (i) Suppose that $\alpha(n)/\sqrt{n} \rightarrow a$ as $n \rightarrow \infty$, where $a \geq 0$. Then, as $n \rightarrow \infty$,

$$\frac{1}{\sqrt{n}}(F^n, C^n) \xrightarrow{d} \left(\frac{1}{2} L_1^a, \int_0^1 \frac{X_s^a}{3(1-s)} ds \right).$$

where the limiting random variables are almost surely finite.

(ii) Suppose that $\alpha(n) \gg \sqrt{n}$. Then, as $n \rightarrow \infty$,

$$\frac{1}{\alpha(n)} F^n \xrightarrow{p} 0, \quad \frac{1}{\alpha(n)} C^n \xrightarrow{p} \frac{1}{4}.$$

1.2. Motivation: the Karp-Sipser algorithm on a random graph

The Karp-Sipser algorithm, introduced in [12], is a greedy algorithm for finding a matching in a fixed graph. The algorithm works as follows. Call a vertex of degree 1 a *pendant* vertex. If there is at least one pendant vertex in the graph, choose one uniformly at random and include the edge incident to it in the matching. Remove this edge, the two vertices that form it and any other edges incident to them. If, on the other hand, there are no pendant vertices in the graph, choose one of the existing edges uniformly at random, and include it in the matching. Remove the chosen edge together with the two vertices that form it, as well as any other edges incident to those

vertices. Now repeat the procedure on the resulting graph, until there are no edges remaining.

A key observation is that whenever there exists a pendant vertex in a graph, that vertex and its neighbour are included in *some* maximum matching. So the Karp-Sipser algorithm never makes a “mistake” in including such an edge in its matching. On the other hand, in the other type of move (picking a uniform edge and including it in the matching) it is possible that it includes an edge which would *not* be in any maximum matching.

The Karp-Sipser algorithm turns out to be very successful at finding a near-maximum matching in certain classes of (sparse) random graphs [12, 1, 6]. Suppose that we take the graph to be the Erdős-Rényi random graph $G(N, M)$, with N vertices and M edges, where $M = \lfloor cN/2 \rfloor$ and $c > 0$ is a constant. Let D_N be the difference between the size of a maximum matching on $G(N, M)$ and the matching produced by the Karp-Sipser algorithm. Let A_N be the number of vertices remaining in the graph at the point of the first uniform random choice which are left unmatched by the Karp-Sipser algorithm. Then $D_N \leq A_N/2$. Aronson, Frieze and Pittel [1] proved the following theorem.

Theorem 1.2. (i) If $c < e$, then $\mathbb{P}(D_N = 0) \rightarrow 1$ as $N \rightarrow \infty$.

(ii) If $c > e$, there exist constants $C_1, C_2 > 0$ such that

$$C_1 N^{1/5} / (\log N)^{75/2} \leq \mathbb{E}[A_N] \leq C_2 N^{1/5} (\log N)^{12}.$$

Aronson, Frieze and Pittel conjecture that, in fact, $N^{-1/5} \mathbb{E}[A_N]$ converges as $N \rightarrow \infty$; indeed, one might also reasonably conjecture that $N^{-1/5} A_N$ possesses a limit in distribution as $N \rightarrow \infty$. One of our aims in this paper is to make progress towards understanding how such a limit in distribution might arise.

The Karp-Sipser algorithm proceeds in two phases. In the first phase (Phase I), the algorithm recursively attacks the pendant subtrees in the graph, matching as it goes. Phase II starts the first time that the algorithm is forced to pick a uniform edge. At the start of Phase II, the graph necessarily contains only vertices of degree 2 or more. It will, in general, have several components, of which some may consist of isolated cycles. In these cycle components, Karp-Sipser necessarily yields a maximum matching. So the source of the “mistakes” is the complex components of the graph present at the start of Phase II. Our original motivation for introducing the model studied in this paper is to understand the behaviour of the Karp-Sipser algorithm on the type of complex components appearing in Phase II.

For $c < e$, Phase I is essentially the whole story, apart from a few isolated cycles (which cannot cause “mistakes”). On the other hand, for $c > e$, a non-trivial graph remains at the end of Phase I. The work of Aronson, Frieze and Pittel [1] suggests that, for $c > e$, the part of the Karp-Sipser process which gives the dominant contribution to A_N arises close to the end of Phase II. Their analysis indicates the following heuristic picture for the structure of the graph at this point: ignoring log-corrections, it is approximately a uniform random graph with $\Theta_p(N^{3/5})$ vertices of degree 2, $\Theta_p(N^{2/5})$ vertices of degree 3 and $\Theta_p(N^{1/5})$ vertices of degree 4. In a moment, we will describe the structure of this graph.

Before going any further, it will be useful to introduce the configuration model [3, 7, 18, 19, 20]. Fix n and a sequence $\mathbf{d} = (d_1, d_2, \dots, d_n)$, where $d_i \geq 0$ is the degree of vertex i and $\sum_{i=1}^n d_i$ is even. First assign vertex i a number d_i of half-edges. Then

choose a uniform random pairing of the half-edges. This generates a random multigraph $CM(\mathbf{d})$. In particular, assuming that there exists at least one simple graph with degrees \mathbf{d} , conditionally on the event that $CM(\mathbf{d})$ is simple, it is uniformly distributed on the set of simple graphs with those degrees. (See the account in Chapter 7 of van der Hofstad [16] for proofs of these results.)

For a fixed $k \geq 2$, we will write $\text{Dirichlet}_k(1, 1, \dots, 1)$ for the uniform distribution on the simplex $\{\mathbf{x} = (x_1, x_2, \dots, x_k) : x_1, x_2, \dots, x_k \geq 0, \sum_{i=1}^k x_i = 1\}$ (which is the simplest of the family of Dirichlet distributions). We will omit the subscript k when the number of co-ordinates is clear from context.

Now fix $t, u > 0$ and $v \geq 0$, and suppose we start from a degree sequence with $\lfloor tN^{3/5} \rfloor$ vertices of degree 2, $\lfloor uN^{2/5} \rfloor$ of degree 3 and $\lfloor vN^{1/5} \rfloor$ of degree 4. Let G_N be a uniform random graph with these degrees. Let K_N be the *kernel* of G_N , that is the multigraph obtained by contracting paths of vertices of degree 2. It is straightforward to see that K_N is distributed according to the configuration model with $\lfloor uN^{2/5} \rfloor$ vertices of degree 3 and $\lfloor vN^{1/5} \rfloor$ vertices of degree 4. With probability tending to 1 as $N \rightarrow \infty$, G_N possesses a giant complex component C_N , containing all of the vertices of degrees 3 and 4 (i.e. the kernel), as well as some random number t'_N of the vertices of degree 2, where $t'_N/N^{3/5} \xrightarrow{P} t$ as $n \rightarrow \infty$. On this event of high probability, C_N is a uniform connected graph with its degree sequence. Outside the giant, there is a collection of $O_p(\log N)$ disjoint cycles, containing the remaining vertices of degree 2.

Now consider the following way of constructing a complex component which we claim has approximately the same distribution as C_N . First generate the kernel K_N according to the configuration model with $\lfloor uN^{2/5} \rfloor$ vertices of degree 3 and $\lfloor vN^{1/5} \rfloor$ vertices of degree 4. Then, one-by-one, allocate t'_N vertices of degree 2 to the edges of K_N : at each step, an edge of the current structure is chosen uniformly at random, split into two edges in series, and the vertex is inserted into the middle. Thus the lengths of the paths of degree-2 vertices we insert between neighbouring vertices in K_N evolve according to a multicolour Pólya's urn with $(3\lfloor uN^{2/5} \rfloor + 4\lfloor vN^{1/5} \rfloor)/2$ colours and a single ball of each colour to start. Then for large N , the proportions of the t'_N vertices of degree 2 which get allocated to each of the edges of K_N look approximately like a $\text{Dirichlet}(1, 1, \dots, 1)$ vector, with $(3\lfloor uN^{2/5} \rfloor + 4\lfloor vN^{1/5} \rfloor)/2$ co-ordinates.

Let us now consider how the Karp-Sipser algorithm behaves on G_N . We may neglect the cycle components, as they can give rise to at most $O(\log N)$ unmatched vertices. The algorithm first picks a uniform edge, matches its end-points, and then removes the neighbouring edges. Typically we matched two vertices somewhere inside a long path of degree 2 vertices, and so we are now left with two pendant vertices, each at the end of a path of degree-2 vertices with a higher-degree vertex at its end. If such

We have not found a good reference for these statements, which are folklore in the random graphs literature; statements in a similar spirit may be found, for example, in the recent paper of Joos, Perarnau, Rautenbach and Reed [10]. Their Theorem 2 implies that G_N contains a giant component with high probability. By Theorem 3.15 of van der Hofstad [17], K_N is connected with high probability. That the giant component of G_N contains K_N and a proportion 1 of all vertices of degree 2 comes down to the fact that the number of ways of generating a random 2-regular graph is negligible compared to the number of ways of generating a graph with kernel K_N . Finally, a random 2-regular graph on n vertices has $O_p(\log n)$ components (see Arratia, Barbour and Tavaré [2]).

By Proposition 7.13 of [16], K_N possesses constant-order numbers of self-loops and multiple edges; in the urn process, there is negligible probability that we fail to allocate any vertices of degree 2 to the self-loops or to at least two of a set of edges between the same two vertices.

a path is of odd length, Karp-Sipser will “consume” the degree-2 vertices but leave the higher-degree vertex untouched (except to reduce its degree by 1). If, on the other hand, the path is of even length, the higher-degree vertex gets matched and removed, causing its neighbours to become pendant vertices. Thus, in this case, the algorithm eats further away into the graph. If ever a particular path gets eaten away at from both ends (which can happen since the graph has cycles), there is a chance that some vertex in the path will remain unmatched. Again, whether this in fact happens or not depends on the parity of the path of degree-2 vertices. For large enough N , we expect that such paths will be of odd and even lengths with approximately equal probability. So we expect that paths which get burnt from both ends will give rise to an unmatched vertex with probability $1/2$.

As we have already argued, since the number of edges in the multigraph is much smaller than the number of degree-2 vertices, the proportions of degree-2 vertices assigned to each edge of the multigraph will be, for large N , close to $\text{Dirichlet}(1, 1, \dots, 1)$. For fixed $k \geq 2$, a vector (D_1, D_2, \dots, D_k) with $\text{Dirichlet}_k(1, 1, \dots, 1)$ distribution may be obtained by sampling E_1, E_2, \dots, E_k , independent and identically distributed standard exponential random variables and setting

$$(D_1, D_2, \dots, D_k) = \frac{1}{\sum_{i=1}^k E_i} (E_1, E_2, \dots, E_k). \quad (2)$$

The right-hand side is independent of the random variable $\sum_{i=1}^k E_i$. Once we have accounted for parity, the lengths of the paths of degree-2 vertices play a role only when we pick a new uniform edge to match, which we do with probability proportional to length. So only *relative* lengths matter, and we can equivalently think of E_1, \dots, E_k as the “lengths” of the paths of degree 2 vertices in our approximate model. (In what follows, we will primarily use $\text{Dirichlet}(1, 1, \dots, 1)$ edge-lengths, but it is convenient to be able to move back and forth between these two points of view.)

In summary, letting n be the number of vertices of degree 3 and $\alpha(n)$ be the number of vertices of degree 4, we obtain the model described in Section 1.1 as an approximation. We do not attempt here to assess the quality of this approximation (and we only make rigorous statements about the model described in Section 1.1). Rather our interest is in the mechanism by which the distribution of the number of vertices which remain unmatched at the end of the Karp-Sipser algorithm arises. With the scaling suggested by Aronson, Frieze and Pittel, we would have $n = \Theta(N^{2/5})$ and $\alpha(n) = \Theta(N^{1/5})$ i.e. $\alpha(n) = \Theta(\sqrt{n})$. So we should be in regime (i) of Theorem 1.1, which supports the conjecture that $N^{-1/5}A_N$ possesses a limit in distribution.

1.3. Our analysis

Our model has two convenient features which make it amenable to analysis: the distributional properties of the edge-lengths and the fact that we may sample the multigraph edge by edge at the same time as we burn it. Let us first address the edge-lengths. We will make use of the following result (Proposition 1 of [4]), which follows from standard properties of exponential random variables via the relationship (2).

Proposition 1.1. (i) Suppose that $(D_1, D_2, \dots, D_k) \sim \text{Dirichlet}_k(1, 1, \dots, 1)$ and let I be a random index from $\{1, 2, \dots, k\}$ with conditional distribution $\mathbb{P}(I = i | D_1, \dots, D_k) =$

D_i , $1 \leq i \leq k$. Let U be independent of everything else with uniform distribution on $[0, 1]$. Then defining

$$D'_i = D_i \mathbb{1}_{\{1 \leq i < I\}} + U D_i \mathbb{1}_{\{i=I\}} + (1-U) D_i \mathbb{1}_{\{i=I+1\}} + D_{i-1} \mathbb{1}_{\{I+1 < i \leq k+1\}},$$

we have that $(D'_1, D'_2, \dots, D'_{k+1})$ has $\text{Dirichlet}_{k+1}(1, 1, \dots, 1)$ distribution.

(ii) Suppose that $(D'_1, D'_2, \dots, D'_{k+1}) \sim \text{Dirichlet}_{k+1}(1, 1, \dots, 1)$ and that J is chosen independently and uniformly at random from $\{1, 2, \dots, k\}$. Then defining

$$D_i = D'_i \mathbb{1}_{\{1 \leq i < J\}} + (D'_i + D'_{i+1}) \mathbb{1}_{\{i=J\}} + D'_{i+1} \mathbb{1}_{\{J+1 \leq i \leq k\}},$$

we have that $(D_1, D_2, \dots, D_k) \sim \text{Dirichlet}_k(1, 1, \dots, 1)$.

Using part (i) of this proposition, we see that if we start from $\text{Dirichlet}(1, 1, \dots, 1)$ edge-lengths and pick a point uniformly from the length measure, then splitting at it yields distances from the sampled point to the adjacent vertices of degree at least 3 either side of it which are again part of a $\text{Dirichlet}(1, 1, \dots, 1)$ vector (with one more co-ordinate). We will see in what follows that the property that (given the number of edges in the multigraph) the relative edge-lengths are $\text{Dirichlet}(1, 1, \dots, 1)$ is preserved. Consider the process at an instant when the fire has just reached a vertex, or when a point has just been set alight. In general there are several edges burning, and using the memoryless property of the exponential distribution, their relative distances to the adjacent vertices are still standard exponential divided by the sum of those exponentials. In particular, whenever there are multiple edges burning, the next to reach its vertex is chosen uniformly from among all those present. Since we are not interested in how long the whole process takes but rather in the number of times we must set light to the network and how many clashes we observe, we perform our analysis in discrete time: that is, we consider the multigraph *without* edge-lengths, always set fire to a uniformly-chosen edge (which has the effect of splitting it into two edges, both alight), and we choose which edge next finishes burning uniformly from among those currently alight.

Recall the description of the configuration model from the previous section. We may generate the pairing of the half-edges in any order we like, which makes the configuration model particularly amenable to an exploration-process-type analysis. In particular, given that we have revealed the pairings of a particular collection of half-edges, assuming we keep track of the degree sequence of the rest of the graph, the rest of the graph is again a configuration model with that degree sequence. We exploit this property below. Observe that we have n vertices of degree 3 and $\alpha(n)$ of degree 4 in our configuration model. Our process starts by picking a uniform edge, splitting it in two and setting each resulting edge alight. (Let us refer to this as the beginning of a *wave*, with a new wave beginning every time we set light to a point in the multigraph.) A uniform one of these two edges reaches its vertex next. It samples its vertex from among those of degree 3 with probability $3n/(3n + 4\alpha(n))$ and from those of degree 4 with probability $4\alpha(n)/(3n + 4\alpha(n))$. Thereafter, we may think of edges which are alight as the first half-edge of a pair, whose second half-edge we have yet to sample.

Suppose that we currently have x edges burning, and that there are u vertices with 3 unattached half-edges and v vertices with 4 unattached half-edges. We pick the next half-edge to process uniformly from those currently burning, and pick its pair uniformly at random from among those available, including any which are themselves burning.

The pair half-edge is already burning with probability $(x-1)/(3u+4v+x-1)$, in which case we form an edge which is burning from both ends and generate a clash. Otherwise, if we connect to a vertex of degree 3, which occurs with probability $3u/(3u+4v+x-1)$, the fire is either passed to the two other half-edges (with probability $1/2$) or stopped. If it is stopped, the vertex of degree 3 becomes a vertex of degree 2. There are two different things that might happen to this vertex of degree 2. With probability $1/(3u+4v+x-3)$, its two half-edges are in fact connected to each other to form an isolated cycle. (As observed above, this is a rare event: there are only $O(1)$ self-loops in the whole multigraph.) Any such isolated cycle necessarily yields a clash. With the complementary probability, the two remaining half-edges are not connected to each other but rather to other half-edges. Since vertices of degree 2 cannot stop fires, in this case we may simply remove the vertex and contract the path of length 2 in which it sat to a single edge. (Using part (ii) of Proposition 1.1, this results the relative lengths of the edges in the unseen parts of the multigraph still being $\text{Dirichlet}(1, 1, \dots, 1)$ distributed). In either case, the remaining unseen part of the multigraph (after deletion of the loop, or contraction of a path of length 2) is still distributed according to the configuration model with the updated degree distribution. Finally, if we connect to a vertex of degree 4, which occurs with probability $4v/(3u+4v+x-1)$, then either the fire is passed to all three other neighbours (with probability $1/2$) or to none of them, in which case the result is that we get another vertex of degree 3.

Note that we must treat the very first edge of a wave differently: although we start with two burning half-edges, they cannot be paired to each other (since otherwise there would be an edge in the original graph with no vertex, which is impossible). So let us treat the first step of a wave as consisting of picking a uniform edge, splitting it in two, sampling the vertex to which one of the resulting burning edges is attached and seeing whether it passes the fire on or not. So if at some step, there are no burning half-edges, on the next step there will be either 1, 3 or 4 burning half-edges, corresponding to the events that the first of the two fires was stopped, that it was passed on through a vertex of degree 3, or that it was passed on through a vertex of degree 4.

In this way, we see that at each step of the procedure, we either process one vertex or generate a clash. A vertex of degree 3 is processed precisely once; a vertex of degree 4 is processed once or twice, depending on whether it stops the first fire it encounters or not. If we track the numbers of vertices of degree 3 and 4 and the numbers of currently burning half-edges, we have a Markovian evolution, which we may hope to analyse using the tools of stochastic process theory. In particular, in what follows we make extensive use of martingales.

The rest of the paper is organized as follows. In Section 2, we write down explicitly the transition probabilities of our Markov chain, and give some first estimates relevant for the forthcoming analysis. In particular, we identify a coupling which facilitates our analysis of the end of the process. In Section 3, we prove fluid limits for the suitably rescaled number of nodes of degree 3 and 4; that is, we show that these processes remain close to deterministic functions on a time-interval which is bounded away from the end of the process (this is an application of the so-called differential equations method [8, 21]). In Section 4, we analyse the case $\alpha(n) \gg \sqrt{n}$, and prove a fluid limit result for the (suitably rescaled) numbers of fires and clashes we observe, as long as we are bounded away from the end of the process. Section 5 deals with the limiting properties of the numbers of fires and the number of clashes we observe, again as long

as we bounded away from the end of the process, for $\alpha(n) = O(\sqrt{n})$. In this case, the limiting process for the number of fires is a reflected diffusion, and the proof is based on an invariance principle for reflecting Markov chains (in the spirit of [9, 11, 15]). Finally, in Section 6, we prove that the end of the process does not contribute significantly to any of these quantities, and so the convergence results can be extended into that range also.

2. The Markov chain

For $i \geq 0$, let $U^n(i)$ and $V^n(i)$ represent the numbers of nodes of degree 3 and 4 respectively after i steps of the burning procedure. Let $X^n(i)$ be the number of burning half-edges we have after i steps. Let $N^n(i)$ be the counting process of the number of clashes observed up to step i . Set $U^n(0) = n$, $V^n(0) = \alpha(n)$, $X^n(0) = 0$, and $N^n(0) = 0$. We have already argued that the process $(U^n(i), V^n(i), X^n(i), N^n(i))_{i \geq 0}$ evolves in a Markovian manner until time $\zeta_n = \inf\{i \geq 0 : U^n(i) + V^n(i) + X^n(i) = 0\}$, when it stops. Write $L^n(k) = 2 \sum_{i=0}^{k-1} \mathbb{1}_{\{X^n(i)=0\}}$ (we will think of this quantity as a local time). Then $F^n = \frac{1}{2}L^n(\zeta_n)$ and $C^n = N^n(\zeta_n)$. We will find it convenient to rescale time by n (essentially because we start with $n + \alpha(n)$ vertices and a typical step involves the removal of a vertex of degree 3).

Recall the definition of the process X^a from (1). We also let $x : [0, 1] \rightarrow \mathbb{R}_+$ be the (deterministic) function $x(t) = (1 - t)^{2/3} - (1 - t)^{4/3}$. For $0 \leq t \leq 1$, let

$$N_t^a = \int_0^t \frac{X_s^a}{3(1-s)} ds, \quad m(t) = \frac{1}{4} - \frac{1}{2}(1-t)^{2/3} + \frac{1}{4}(1-t)^{4/3}.$$

Let $\mathbb{D}(\mathbb{R}_+, \mathbb{R})$ denote the space of càdlàg functions from \mathbb{R}_+ to \mathbb{R} , equipped with the Skorokhod topology. (In fact, since our limit processes will always be continuous, we will rather obtain convergence with respect to the uniform norm.) The crux of our argument is the following scaling limit theorem which straightforwardly implies Theorem 1.1.

Theorem 2.1. (i) Suppose $\alpha(n)/\sqrt{n} \rightarrow a$ as $n \rightarrow \infty$, for $a \geq 0$. Then, uniformly as $n \rightarrow \infty$

$$\frac{1}{\sqrt{n}} (X^n(\lfloor nt \rfloor), L^n(\lfloor nt \rfloor), N^n(\lfloor nt \rfloor), 0 \leq t \leq \zeta_n/n) \xrightarrow{d} (X_t^a, L_t^a, N_t^a, 0 \leq t \leq 1).$$

(ii) Suppose $\alpha(n) \gg \sqrt{n}$. Then, uniformly as $n \rightarrow \infty$

$$\frac{1}{\alpha(n)} (X^n(\lfloor nt \rfloor), L^n(\lfloor nt \rfloor), N^n(\lfloor nt \rfloor), 0 \leq t \leq \zeta_n/n) \xrightarrow{d} (x(t), 0, m(t), 0 \leq t \leq 1).$$

2.1. Transition probabilities

We have already described the possible transitions of our four-dimensional Markov chain in the Introduction. Let us be a little more explicit about the transition probabilities.

Suppose that $(U^n(i), V^n(i), X^n(i), N^n(i)) = (u, v, x, m) \in \mathbb{Z}_+^4$. If $x = 0$ but $u + v >$

0, then

$$(U^n(i+1), V^n(i+1), X^n(i+1), N^n(i+1)) \\ = \begin{cases} (u-1, v, 1, m+1) & \text{with probability } \frac{3u}{2(3u+4v)(3u+4v-2)}, \\ (u-1, v, 1, m) & \text{with probability } \frac{3u(3u+4v-3)}{2(3u+4v)(3u+4v-2)}, \\ (u-1, v, 3, m) & \text{with probability } \frac{3u}{2(3u+4v)}, \\ (u+1, v-1, 1, m) & \text{with probability } \frac{4v}{2(3u+4v)}, \\ (u, v-1, 4, m) & \text{with probability } \frac{4v}{2(3u+4v)}. \end{cases}$$

If $x > 0$ then

$$(U^n(i+1), V^n(i+1), X^n(i+1), N^n(i+1)) \\ = \begin{cases} (u-1, v, x-1, m+1) & \text{with probability } \frac{3u}{2(3u+4v+x-1)(3u+4v+x-3)}, \\ (u-1, v, x-1, m) & \text{with probability } \frac{3u(3u+4v+x-4)}{2(3u+4v+x-1)(3u+4v+x-3)}, \\ (u-1, v, x+1, m) & \text{with probability } \frac{3u}{2(3u+4v+x-1)}, \\ (u+1, v-1, x-1, m) & \text{with probability } \frac{4v}{2(3u+4v+x-1)}, \\ (u, v-1, x+2, m) & \text{with probability } \frac{4v}{2(3u+4v+x-1)}, \\ (u, v, x-2, m+1) & \text{with probability } \frac{x-1}{(3u+4v+x-1)}. \end{cases}$$

Writing $(\mathcal{F}_i^n)_{i \geq 0}$ for the natural filtration of the four-dimensional process, we have

$$\begin{aligned} \mathbb{E}[U^n(i+1) - U^n(i) | \mathcal{F}_i^n] &= \frac{2V^n(i) - 3U^n(i)}{3U^n(i) + 4V^n(i) + X^n(i) - \mathbb{1}_{\{X^n(i) > 0\}}}, \\ \mathbb{E}[V^n(i+1) - V^n(i) | \mathcal{F}_i^n] &= \frac{-4V^n(i)}{3U^n(i) + 4V^n(i) + X^n(i) - \mathbb{1}_{\{X^n(i) > 0\}}}, \\ \mathbb{E}[X^n(i+1) - X^n(i) | \mathcal{F}_i^n] &= 2\mathbb{1}_{\{X^n(i)=0\}} + \frac{2V^n(i) - 2X^n(i) + 2\mathbb{1}_{\{X^n(i) > 0\}}}{3U^n(i) + 4V^n(i) + X^n(i) - \mathbb{1}_{\{X^n(i) > 0\}}}, \\ \mathbb{E}[N^n(i+1) - N^n(i) | \mathcal{F}_i^n] &= \end{aligned} \quad (3)$$

$$= \frac{3U^n(i) + 2(X^n(i) - \mathbb{1}_{\{X^n(i) > 0\}})(3U^n(i) + 4V^n(i) + X^n(i) - 3)}{2(3U^n(i) + 4V^n(i) + X^n(i) - \mathbb{1}_{\{X^n(i) > 0\}})(3U^n(i) + 4V^n(i) + X^n(i) - \mathbb{1}_{\{X^n(i) > 0\}} - 2)}. \quad (4)$$

We will also need the following conditional second moments:

$$\begin{aligned} \mathbb{E}[(U^n(i+1) - U^n(i))^2 | \mathcal{F}_i^n] &= \frac{3U^n(i) + 2V^n(i)}{3U^n(i) + 4V^n(i) + X^n(i) - \mathbb{1}_{\{X^n(i) > 0\}}}, \\ \mathbb{E}[(V^n(i+1) - V^n(i))^2 | \mathcal{F}_i^n] &= \frac{4V^n(i)}{3U^n(i) + 4V^n(i) + X^n(i) - \mathbb{1}_{\{X^n(i) > 0\}}}, \\ \mathbb{E}[(X^n(i+1) - X^n(i))^2 | \mathcal{F}_i^n] &= \begin{cases} 5 + \frac{14V^n(i)}{3U^n(i) + 4V^n(i)} & \text{if } X^n(i) = 0, \\ 1 + \frac{6V^n(i) + 3X^n(i) - 3}{3U^n(i) + 4V^n(i) + X^n(i) - 1} & \text{if } X^n(i) > 0, \end{cases} \\ \mathbb{E}[(N^n(i+1) - N^n(i))^2 | \mathcal{F}_i^n] &= \mathbb{E}[N^n(i+1) - N^n(i) | \mathcal{F}_i^n]. \end{aligned} \quad (5)$$

2.2. A coupling and some first estimates

The process runs for ζ_n steps. At each step, we process a whole edge of the multigraph, except at the start of a wave, when we possibly need two steps to process an edge. There are $(3n + 4\alpha(n))/2$ edges in total and so $\zeta_n \leq 3n + 4\alpha(n)$.

In the sequel, and particularly in Section 6, we will make extensive use of a coupling of a modified version of X^n and a reflecting simple symmetric random walk, which we now introduce. First, we divide the burning half-edges into two stacks, of sizes $X_1^n(i)$ and $X_2^n(i)$, such that $X^n(i) = X_1^n(i) + X_2^n(i)$ for all $0 \leq i \leq \zeta_n$. We may give these sub-processes whatever dynamics we choose, as long as their sum behaves as $(X^n(i))_{i \geq 0}$. We proceed as follows.

Whenever $X_1^n(i) > 0$, we select the next half-edge to process from the first stack. If the fire is absorbed, X_1^n is simply reduced by 1. If we connect to a vertex of degree 3 and pass the fire on, X_1^n increases by 1. If we connect to a vertex of degree 4 and pass the fire on, we let each of X_1^n and X_2^n increase by 1. If we create a clash with a half-edge from the first stack, X_1^n is reduced by 2. We may also create a clash with a vertex from the second stack, in which case X_1^n and X_2^n are both reduced by 1.

If $X_1^n(i) = 0$ but $X_2^n(i) > 0$, then we select the half-edge to process from the second stack, add any new half-edges arising from passing the fire on to a vertex of degree 3 to X_1^n and add 2 of the three burning half-edges arising from passing the fire on to a vertex of degree 4 to X_1^n and the last one to X_2^n . Finally, if $X_1^n(i) = X_2^n(i) = 0$, then we allocate all new half-edges to the first stack, except if we connect to a vertex of degree 4 and pass on the fire, in which case X_1^n jumps to 3 and X_2^n jumps to 1.

We will also track the clashes and split them according to whether they involve a half-edge from the second stack or not, yielding $N^n(i) = N_1^n(i) + N_2^n(i)$ for $i \geq 0$.

We will describe the transition probabilities of this process in detail below. Our aim is to couple X_1^n with a process Y^n in such a way that $X_1^n(i) \leq Y^n(i)$ for all $i \geq 0$ and Y^n is a simple symmetric random walk (SSRW), reflected at 2. In order to keep the notation to a reasonable level, we will describe the transitions of $(X_1^n(i), X_2^n(i), Y^n(i), N_1^n(i), N_2^n(i))_{i \geq 0}$, leaving those of $(U^n(i), V^n(i))_{i \geq 0}$ implicit.

Conditional on $X_1^n(i) = x_1 > 0, X_2^n(i) = x_2 \geq 0, Y^n(i) = y \geq 3, U^n(i) = u, V^n(i) = v$, with $x = x_1 + x_2$,

$$\begin{aligned}
 & (X_1^n(i+1) - X_1^n(i), X_2^n(i+1) - X_2^n(i), Y^n(i+1) - Y^n(i), N_1^n(i+1) - N_1^n(i), \\
 & \hspace{15em} N_2^n(i+1) - N_2^n(i)) \\
 = & \begin{cases} (-2, 0, -1, +1, 0) & \text{with probability } \frac{x_1-1}{2(3u+4v+x-1)} \\ (-2, 0, +1, +1, 0) & \text{with probability } \frac{x_1-1}{2(3u+4v+x-1)} \\ (-1, -1, -1, 0, +1) & \text{with probability } \frac{x_2}{2(3u+4v+x-1)} \\ (-1, -1, +1, 0, +1) & \text{with probability } \frac{x_2}{2(3u+4v+x-1)} \\ (-1, 0, -1, +1, 0) & \text{with probability } \frac{3u}{2(3u+4v+x-1)(3u+4v+x-3)} \\ (-1, 0, -1, 0, 0) & \text{with probability } \frac{4v}{2(3u+4v+x-1)} + \frac{3u(3u+4v+x-4)}{2(3u+4v+x-1)(3u+4v+x-3)} \\ (+1, 0, +1, 0, 0) & \text{with probability } \frac{3u}{2(3u+4v+x-1)} \\ (+1, +1, +1, 0, 0) & \text{with probability } \frac{4v}{2(3u+4v+x-1)}. \end{cases}
 \end{aligned}$$

Conditional on $X_1^n(i) = x_1 > 0, X_2^n(i) = x_2 \geq 0, Y^n(i) = y = 2, U^n(i) = u, V^n(i) = v$, with $x = x_1 + x_2$,

$$\begin{aligned}
& (X_1^n(i+1) - X_1^n(i), X_2^n(i+1) - X_2(i), Y^n(i+1) - Y^n(i), N_1^n(i+1) - N_1^n(i), \\
& \qquad \qquad \qquad N_2^n(i+1) - N_2^n(i)) \\
& = \begin{cases} (-2, 0, +1, +1, 0) & \text{with probability } \frac{x_1-1}{(3u+4v+x-1)} \\ (-1, -1, +1, 0, +1) & \text{with probability } \frac{x_2}{(3u+4v+x-1)} \\ (-1, 0, +1, +1, 0) & \text{with probability } \frac{3u}{2(3u+4v+x-1)(3u+4v+x-3)} \\ (-1, 0, +1, 0, 0) & \text{with probability } \frac{4v}{2(3u+4v+x-1)} + \frac{3u(3u+4v+x-4)}{2(3u+4v+x-1)(3u+4v+x-3)} \\ (+1, 0, +1, 0, 0) & \text{with probability } \frac{3u}{2(3u+4v+x-1)} \\ (+1, +1, +1, 0, 0) & \text{with probability } \frac{4v}{2(3u+4v+x-1)}. \end{cases}
\end{aligned}$$

Conditional on $X_1^n(i) = 0, X_2^n(i) = x_2 > 0, Y^n(i) = y \geq 3, U^n(i) = u, V^n(i) = v,$

$$\begin{aligned}
& (X_1^n(i+1) - X_1^n(i), X_2^n(i+1) - X_2(i), Y^n(i+1) - Y^n(i), N_1^n(i+1) - N_1^n(i), \\
& \qquad \qquad \qquad N_2^n(i+1) - N_2^n(i)) \\
& = \begin{cases} (0, -2, -1, 0, +1) & \text{with probability } \frac{x_2-1}{2(3u+4v+x_2-1)} \\ (0, -2, +1, 0, +1) & \text{with probability } \frac{x_2-1}{2(3u+4v+x_2-1)} \\ (0, -1, -1, +1, 0) & \text{with probability } \frac{3u}{2(3u+4v+x_2-1)(3u+4v+x_2-3)} \\ (0, -1, -1, 0, 0) & \text{with probability } \frac{4v}{2(3u+4v+x_2-1)} + \frac{3u(3u+4v+x_2-4)}{2(3u+4v+x_2-1)(3u+4v+x_2-3)} \\ (+2, -1, +1, 0, 0) & \text{with probability } \frac{3u}{2(3u+4v+x_2-1)} \\ (+2, 0, +1, 0, 0) & \text{with probability } \frac{4v}{2(3u+4v+x_2-1)}. \end{cases}
\end{aligned}$$

Conditional on $X_1^n(i) = 0, X_2^n(i) = x_2 > 0, Y^n(i) = y = 2, U^n(i) = u, V^n(i) = v,$

$$\begin{aligned}
& (X_1^n(i+1) - X_1^n(i), X_2^n(i+1) - X_2(i), Y^n(i+1) - Y^n(i), N_1^n(i+1) - N_1^n(i), \\
& \qquad \qquad \qquad N_2^n(i+1) - N_2^n(i)) \\
& = \begin{cases} (0, -2, +1, 0, +1) & \text{with probability } \frac{x_2-1}{(3u+4v+x_2-1)} \\ (0, -1, +1, +1, 0) & \text{with probability } \frac{3u}{2(3u+4v+x_2-1)(3u+4v+x_2-3)} \\ (0, -1, +1, 0, 0) & \text{with probability } \frac{4v}{2(3u+4v+x_2-1)} + \frac{3u(3u+4v+x_2-4)}{2(3u+4v+x_2-1)(3u+4v+x_2-3)} \\ (+2, -1, +1, 0, 0) & \text{with probability } \frac{3u}{2(3u+4v+x_2-1)} \\ (+2, 0, +1, 0, 0) & \text{with probability } \frac{4v}{2(3u+4v+x_2-1)}. \end{cases}
\end{aligned}$$

Conditional on $X_1^n(i) = 0, X_2^n(i) = 0, Y^n(i) = y \geq 3, U^n(i) = u, V^n(i) = v,$

$$\begin{aligned}
& (X_1^n(i+1) - X_1^n(i), X_2^n(i+1) - X_2(i), Y^n(i+1) - Y^n(i), N_1^n(i+1) - N_1^n(i), \\
& \qquad \qquad \qquad N_2^n(i+1) - N_2^n(i)) \\
& = \begin{cases} (+1, 0, -1, 0, 0) & \text{with probability } \frac{1}{2} \\ (+3, 0, +1, 0, 0) & \text{with probability } \frac{3u}{2(3u+4v)} \\ (+3, +1, +1, 0, 0) & \text{with probability } \frac{4v}{2(3u+4v)}. \end{cases}
\end{aligned}$$

Finally, conditional on $X_1^n(i) = 0, X_2^n(i) = 0, Y^n(i) = y = 2, U^n(i) = u, V^n(i) = v,$

$$(X_1^n(i+1) - X_1^n(i), X_2^n(i+1) - X_2^n(i), Y^n(i+1) - Y^n(i), N_1^n(i+1) - N_1^n(i), N_2^n(i+1) - N_2^n(i))$$

$$= \begin{cases} (+1, 0, +1, 0, 0) & \text{with probability } \frac{1}{2} \\ (+3, 0, +1, 0, 0) & \text{with probability } \frac{3u}{2(3u+4v)} \\ (+3, +1, +1, 0, 0) & \text{with probability } \frac{4v}{2(3u+4v)}. \end{cases}$$

By construction, Y^n performs a SSRW with upward reflection at 2. For fixed $i \geq 0$, as long as $Y^n(i) \geq 2$ and $X_1^n(i) \leq Y^n(i)$, we obtain $X_1^n(j) \leq Y^n(j)$ for all $j \geq i$.

Finally, we observe that since a vertex of degree 4 contributes to the size of the second stack at most once, $X_2^n(i) \leq V^n(0) - V^n(i)$ for all $i \geq 0$. Similarly, the number of clashes involving at least one half-edge from the second stack is bounded above by the number of vertices of degree 4 processed so far, $N_2^n(i) \leq V^n(0) - V^n(i)$.

Henceforth, we will use the notation $(\mathcal{F}_i^n)_{i \geq 0}$ for the natural filtration of the process

$$(X_1^n(i), X_2^n(i), Y^n(i), U^n(i), V^n(i), N_1^n(i), N_2^n(i))_{i \geq 0}.$$

For future reference, we note that

$$\mathbb{E}[X_1^n(i+1) - X_1^n(i) | \mathcal{F}_i^n] = \mathbb{1}_{\{X_1^n(i)=0, X_2^n(i)>0\}} + 2\mathbb{1}_{\{X_1^n(i)=0, X_2^n(i)=0\}} \\ - \frac{2(X_1^n(i) - \mathbb{1}_{\{X_1^n(i)>0\}}) + (X_2^n(i) - \mathbb{1}_{\{X_1^n(i)=0, X_2^n(i)>0\}})}{3U^n(i) + 4V^n(i) + X^n(i) - \mathbb{1}_{\{X^n(i)>0\}}}$$
(6)

and

$$\mathbb{E}[N_1^n(i+1) - N_1^n(i) | \mathcal{F}_i^n] \\ = \frac{3U^n(i) + 2(X_1^n(i) - \mathbb{1}_{\{X_1^n(i)>0\}})(3U^n(i) + 4V^n(i) + X^n(i) - 3)}{2(3U^n(i) + 4V^n(i) + X^n(i) - \mathbb{1}_{\{X^n(i)>0\}})(3U^n(i) + 4V^n(i) + X^n(i) - \mathbb{1}_{\{X^n(i)>0\}} - 2)}.$$
(7)

As a first consequence of our coupling, we show that X^n varies on a smaller scale than n .

Lemma 2.1. *We have that $\frac{1}{\alpha(n) \vee \sqrt{n}} \sup_{0 \leq i \leq \zeta_n} X^n(i)$ is bounded in L^4 . In particular, $\frac{1}{n} \sup_{0 \leq i \leq \zeta_n} X^n(i)$ converges in probability to 0 as $n \rightarrow \infty$.*

Proof. We have $X^n(0) = 0$; let $Y^n(0) = 2$. Then by the coupling, we have

$$\sup_{0 \leq i \leq \zeta_n} X^n(i) \leq \alpha(n) + \sup_{0 \leq i \leq \zeta_n} Y^n(i) \leq \alpha(n) + \sup_{0 \leq i \leq 3n+4\alpha(n)} Y^n(i).$$

Now let Z be a standard SSRW, and note that Z is a martingale. Then Y^n has the same law as $(2 + |Z(i)|)_{i \geq 0}$. Bearing in mind that for $i \geq 1$ we have $\mathbb{E}[(Z(i))^4] = i(3i-2) \leq 3i^2$, by Doob's L^4 inequality,

$$\mathbb{E} \left[\left(\sup_{0 \leq i \leq 3n+4\alpha(n)} |Z(i)| \right)^4 \right] \leq \left(\frac{4}{3} \right)^4 \mathbb{E} [Z(3n+4\alpha(n))^4] \leq \frac{256}{27} (3n+4\alpha(n))^2.$$

Hence,

$$\begin{aligned} \mathbb{E} \left[\left(\sup_{0 \leq i \leq \zeta_n} X^n(i) \right)^4 \right] &\leq \mathbb{E} \left[\left(\alpha(n) + 2 + \sup_{0 \leq i \leq 3n+4\alpha(n)} |Z(i)| \right)^4 \right] \\ &\leq 16 \left[(\alpha(n) + 2)^4 + \frac{256}{27} (3n + 4\alpha(n))^2 \right]. \end{aligned}$$

Thus, there exists a constant $C > 0$ such that for all $n \geq 1$,

$$\mathbb{E} \left[\left(\frac{1}{\alpha(n) \vee \sqrt{n}} \sup_{0 \leq i \leq \zeta_n} X^n(i) \right)^4 \right] \leq C$$

and since $\alpha(n)/n \rightarrow 0$, we may deduce the claimed convergence in probability. \square

3. Fluid limit approximations for the auxiliary processes

In this section we prove that U^n and V^n , after appropriate rescaling, remain concentrated around their expected trajectories. In order to do so, we employ the differential equations method [8, 21]. Here we briefly recall the main idea, closely following Darling and Norris [8], although our presentation is for discrete-time rather than continuous-time processes and is in a somewhat simplified setting. Suppose that we are given a stochastic process P^n evolving in discrete time with finite state-space $\mathcal{S}^n \subseteq \mathbb{Z}$. We assume that P^n is either itself Markov or is a co-ordinate of some higher-dimensional Markov process \mathbf{P}^n . Let the natural filtration of \mathbf{P}^n be denoted by $(\mathcal{F}_{\mathbf{P}^n}^n(i))_{i \geq 0}$. For each $i \geq 0$, the process $(M_P^n(i))_{i \geq 0}$ defined by

$$M_P^n(i) = P^n(i) - P^n(0) - \sum_{j=0}^{i-1} \mathbb{E} [P^n(j+1) - P^n(j) | \mathcal{F}_{\mathbf{P}^n}^n(j)]$$

is a martingale. We will use this notation throughout the paper, and will refer to M_P^n as the *standard martingale* associated with the process P^n . Then, for each fixed $t > 0$, we have

$$\frac{P^n(\lfloor nt \rfloor)}{n} = \frac{P^n(0)}{n} + \frac{M_P^n(\lfloor nt \rfloor)}{n} + \sum_{j=0}^{\lfloor nt \rfloor - 1} \frac{\mathbb{E} [P^n(j+1) - P^n(j) | \mathcal{F}_{\mathbf{P}^n}^n(j)]}{n}.$$

Fix $t_0 > 0$ and let $\mathcal{U} \subseteq \mathbb{R}$. We will make four assumptions.

- (1) For some constant $p(0) \in \mathcal{U}$ we have $|P^n(0)/n - p(0)| \xrightarrow{p} 0$.
- (2) Suppose that $\nu : [0, t_0] \times \mathcal{U} \rightarrow \mathbb{R}$ is a bi-Lipschitz function with Lipschitz constant K . Suppose that p , the unique solution to the differential equation $dp(t)/dt = \nu(t, p(t))$ with initial condition $p(0)$, is such that, for some $\epsilon > 0$, $p(t)$ lies at distance greater than ϵ from \mathcal{U}^c for all $0 \leq t \leq t_0$.
- (3) Let $T_n = \inf\{t \geq 0 : P^n(\lfloor nt \rfloor)/n \notin \mathcal{U}\}$. We assume that

$$\sup_{0 \leq t \leq t_0 \wedge T_n} \left| \sum_{j=0}^{\lfloor nt \rfloor - 1} \frac{\mathbb{E} [P^n(j+1) - P^n(j) | \mathcal{F}_{\mathbf{P}^n}^n(j)]}{n} - \int_0^t \nu \left(s, \frac{P^n(\lfloor ns \rfloor)}{n} \right) ds \right| \xrightarrow{p} 0.$$

(4) We have $\frac{\mathbb{E}[M_P^n(\lfloor n(t_0 \wedge T_n) \rfloor)^2]}{n^2} \rightarrow 0$ as $n \rightarrow \infty$.

Assumption (1) tells us that the initial condition is well-concentrated on the scale of n . Assumption (3) gives that, conditionally on P^n being in state $\lfloor nx \rfloor$ at time $\lfloor ns \rfloor$, the size of the expected increment of P^n is close to $\nu(s, x)$. It is then natural to compare P^n/n to the solution to the differential equation in (2), as long as P^n/n remains within the set \mathcal{U} .

Proposition 3.1. *Under assumptions (1), (2), (3) and (4), we have*

$$\sup_{0 \leq t \leq T} \left| \frac{P^n(\lfloor nt \rfloor)}{n} - p(t) \right| \xrightarrow{p} 0$$

as $n \rightarrow \infty$.

Proof. Since p solves the given differential equation, we may write it in integral form as $p(t) = p(0) + \int_0^t \nu(s, p(s)) ds$. Hence,

$$\begin{aligned} & \frac{P^n(\lfloor nt \rfloor)}{n} - p(t) \\ &= \frac{P^n(0)}{n} - p(0) + \frac{M_P^n(\lfloor nt \rfloor)}{n} + \sum_{j=0}^{\lfloor nt \rfloor - 1} \frac{\mathbb{E}[P^n(j+1) - P^n(j) | \mathcal{F}_P^n(j)]}{n} - \int_0^t \nu(s, p(s)) ds. \end{aligned}$$

By Doob's L^2 inequality, assumption (4) implies that $\sup_{0 \leq t \leq t_0 \wedge T_n} (|M_P^n(i)|/n) \xrightarrow{p} 0$ as $n \rightarrow \infty$. Fix $\delta > 0$ and let

$$\begin{aligned} \Omega_{n,\delta} = & \left\{ \left| \frac{P^n(0)}{n} - p(0) \right| \leq \frac{\delta e^{-Kt_0}}{3} \right\} \cap \left\{ \sup_{0 \leq t \leq t_0 \wedge T_n} \frac{|M_P^n(\lfloor nt \rfloor)|}{n} \leq \frac{\delta e^{-Kt_0}}{3} \right\} \cap \\ & \left\{ \sup_{0 \leq t \leq t_0 \wedge T_n} \left| \sum_{j=0}^{\lfloor nt \rfloor - 1} \frac{\mathbb{E}[P^n(j+1) - P^n(j) | \mathcal{F}_P^n(j)]}{n} - \int_0^t \nu\left(s, \frac{P^n(\lfloor ns \rfloor)}{n}\right) ds \right| \leq \frac{\delta e^{-Kt_0}}{3} \right\}. \end{aligned}$$

By the assumptions, we have $\mathbb{P}(\Omega_{n,\delta}^c) \rightarrow 0$ as $n \rightarrow \infty$. On the event $\Omega_{n,\delta}$, we have that for $0 \leq t \leq t_0 \wedge T_n$,

$$\begin{aligned} \left| \frac{P^n(\lfloor nt \rfloor)}{n} - p(t) \right| &\leq \delta e^{-Kt_0} + \int_0^t \left| \nu\left(s, \frac{P^n(\lfloor ns \rfloor)}{n}\right) ds - \nu(s, p(s)) \right| ds \\ &\leq \delta e^{-Kt_0} + K \int_0^t \left| \frac{P^n(\lfloor ns \rfloor)}{n} - p(s) \right| ds, \end{aligned}$$

by the Lipschitz property of ν on \mathcal{U} . Hence, by Gronwall's lemma, on the event $\Omega_{n,\delta}$, $\sup_{0 \leq t \leq t_0 \wedge T_n} \left| \frac{P^n(\lfloor nt \rfloor)}{n} - p(t) \right| \leq \delta$. The result follows. \square

We begin with a preparatory lemma.

Lemma 3.1. For $t < 1$ and $0 \leq i \leq \lfloor nt \rfloor$ we have

$$\begin{aligned}\mathbb{E}[U^n(i+1) - U^n(i) | \mathcal{F}_i^n] &= -1 + E_{U,1}^n(i) \\ \mathbb{E}[(U^n(i+1) - U^n(i))^2 | \mathcal{F}_i^n] &= 1 + E_{U,2}^n(i) \\ \mathbb{E}[V^n(i+1) - V^n(i) | \mathcal{F}_i^n] &= \frac{-4V^n(i)}{3U^n(i)} + E_V^n(i) \\ \mathbb{E}[(V^n(i+1) - V^n(i))^2 | \mathcal{F}_i^n] &= \frac{4V^n(i)}{3U^n(i)} - E_V^n(i),\end{aligned}$$

where $\sup_{0 \leq i \leq \lfloor nt \rfloor} |E_{U,1}^n(i)|$, $\sup_{0 \leq i \leq \lfloor nt \rfloor} |E_{U,2}^n(i)|$, and $\frac{n}{\alpha(n)} \sup_{0 \leq i \leq \lfloor nt \rfloor} |E_V^n(i)|$ all converge to 0 in L^2 as $n \rightarrow \infty$.

Proof. Let us note that

$$\mathbb{E}[U^n(i+1) - U^n(i) | \mathcal{F}_i^n] = -1 + \frac{6V^n(i) + X^n(i) - \mathbb{1}_{\{X^n(i) > 0\}}}{3U^n(i) + 4V^n(i) + X^n(i) - \mathbb{1}_{\{X^n(i) > 0\}}}.$$

Having in mind that in each step we remove at most one vertex, for $0 \leq i \leq \lfloor nt \rfloor$ we have $U^n(i) \geq n - i$, and so

$$\sup_{0 \leq i \leq \lfloor nt \rfloor} \frac{6V^n(i) + X^n(i) - \mathbb{1}_{\{X^n(i) > 0\}}}{3U^n(i) + 4V^n(i) + X^n(i) - \mathbb{1}_{\{X^n(i) > 0\}}} \leq \sup_{0 \leq i \leq \lfloor nt \rfloor} \frac{6\alpha(n) + X^n(i)}{3(n-i)},$$

which converges to 0 in L^2 as $n \rightarrow \infty$ by Lemma 2.1. A very similar argument shows that $\sup_{0 \leq i \leq \lfloor nt \rfloor} |E_{U,2}^n(i)|$ converges to 0 in L^2 as $n \rightarrow \infty$.

Turning now to V^n , we see that

$$\begin{aligned}\mathbb{E}[V^n(i+1) - V^n(i) | \mathcal{F}_i^n] &= -\frac{4V^n(i)}{3U^n(i)} + \frac{4V^n(i)}{3U^n(i)} \cdot \frac{4V^n(i) + X^n(i) - \mathbb{1}_{\{X^n(i) > 0\}}}{3U^n(i) + 4V^n(i) + X^n(i) - \mathbb{1}_{\{X^n(i) > 0\}}}, \\ \mathbb{E}[(V^n(i+1) - V^n(i))^2 | \mathcal{F}_i^n] &= \frac{4V^n(i)}{3U^n(i)} - \frac{4V^n(i)}{3U^n(i)} \cdot \frac{4V^n(i) + X^n(i) - \mathbb{1}_{\{X^n(i) > 0\}}}{3U^n(i) + 4V^n(i) + X^n(i) - \mathbb{1}_{\{X^n(i) > 0\}}}.\end{aligned}$$

Letting $E_V^n(i) = \frac{4V^n(i)}{3U^n(i)} \cdot \frac{4V^n(i) + X^n(i) - \mathbb{1}_{\{X^n(i) > 0\}}}{3U^n(i) + 4V^n(i) + X^n(i) - \mathbb{1}_{\{X^n(i) > 0\}}}$, we have

$$\sup_{0 \leq i \leq \lfloor nt \rfloor} |E_V^n(i)| \leq \frac{4\alpha(n)}{3(n-i)} \frac{4 \sup_{0 \leq i \leq \lfloor nt \rfloor} (X^n(i) + V^n(i))}{3(n-i)}$$

and so Lemma 2.1 implies that $\sup_{0 \leq i \leq \lfloor nt \rfloor} \frac{n}{\alpha(n)} |E_V^n(i)| \rightarrow 0$ in L^2 as $n \rightarrow \infty$. \square

Lemma 3.2. Let $t < 1$. Then as $n \rightarrow \infty$, we have

$$\sup_{0 \leq i \leq \lfloor nt \rfloor} \left| \frac{U^n(i)}{n} - \left(1 - \frac{i}{n}\right) \right| \xrightarrow{p} 0.$$

If, in addition, we assume that $\alpha(n) \rightarrow \infty$ as $n \rightarrow \infty$, then

$$\sup_{0 \leq i \leq \lfloor nt \rfloor} \left| \frac{V^n(i)}{\alpha(n)} - \left(1 - \frac{i}{n}\right)^{4/3} \right| \xrightarrow{p} 0.$$

Proof. Having in mind the size of expected increments given by Lemma 3.1, and $U^n(0) = n$ and $V^n(0) = \alpha(n)$, the candidate for the fluid limit is the solution to the system of differential equations given by

$$\frac{du(s)}{ds} = -1, \quad \frac{dv(s)}{ds} = -\frac{4v(s)}{3u(s)}, \quad \text{for } 0 \leq s \leq t \quad (8)$$

and the initial condition $u(0) = 1$, $v(0) = 1$. The solution is given by $u(s) = 1 - s$, $v(s) = (1 - s)^{4/3}$. We observe that $u(s)$ and $v(s)$ remain bounded away from 0 for all $0 \leq s \leq t$.

Let $T_U^n = \inf\{s \geq 0 : U^n(\lfloor ns \rfloor) = 0\}$. Since $U^n(i) \geq n - i$, it is clear that $T_U^n \geq 1$ for all n and so, in particular, $T_U^n > t$. For $i \geq 0$, let $(M_U^n(i))_{i \geq 0}$ be the standard martingale associated with U^n . By Lemma 3.1, we have that for $0 \leq s \leq t$,

$$\left| \sum_{j=0}^{\lfloor ns \rfloor - 1} \frac{\mathbb{E}[U^n(j+1) - U^n(j) | \mathcal{F}_j^n]}{n} + \int_0^s du \right| \leq \sup_{0 \leq s \leq t} \left| \frac{\lfloor ns \rfloor}{n} - s \right| + t \sup_{0 \leq i \leq \lfloor nt \rfloor} |E_{U,1}^n(i)|,$$

which converges in probability to 0 as $n \rightarrow \infty$. Moreover,

$$\mathbb{E}[M_U^n(i)^2] = \mathbb{E} \left[\sum_{j=0}^{i-1} \text{var}(U^n(j+1) - U^n(j) | \mathcal{F}_j^n) \right] \leq i \left(1 + \mathbb{E} \left[\sup_{0 \leq j \leq i} |E_{U,2}^n(j)| \right] \right)$$

and so by Lemma 3.1 we have

$$\frac{\mathbb{E}[M_U^n(\lfloor nt \rfloor)^2]}{n^2} \leq \frac{\lfloor nt \rfloor (1 + \mathbb{E}[\sup_{0 \leq i \leq \lfloor nt \rfloor} |E_{U,2}^n(i)|])}{n^2} \rightarrow 0. \quad (9)$$

Hence, by Proposition 3.1, $\sup_{0 \leq s \leq t} \left| \frac{U^n(\lfloor ns \rfloor)}{n} - u(s) \right| \xrightarrow{p} 0$. We now turn to V^n . Let us deal first with the second moment of the standard martingale M_V^n . We have

$$\begin{aligned} \mathbb{E}[M_V^n(i)^2] &= \mathbb{E} \left[\sum_{j=0}^{i-1} \text{var}(V^n(j+1) - V^n(j) | \mathcal{F}_j^n) \right] \\ &\leq i \cdot \mathbb{E} \left[\sup_{0 \leq j \leq i} \left(\left| \frac{4V^n(j)}{3U^n(j)} \right| + |E_V^n(j)| \right) \right]. \end{aligned}$$

Since $\frac{4V^n(j)}{3U^n(j)} \leq \frac{4\alpha(n)}{3(n-j)}$, Lemma 3.1 gives us $\mathbb{E}[M_V^n(\lfloor nt \rfloor)^2] / \alpha(n)^2 \rightarrow 0$. Turning now to the drift, we have

$$\begin{aligned} &\sup_{0 \leq s \leq t} \left| \sum_{j=0}^{\lfloor ns \rfloor - 1} \frac{\mathbb{E}[V^n(j+1) - V^n(j) | \mathcal{F}_j^n]}{\alpha(n)} + \int_0^s \frac{4v(r)}{3u(r)} dr \right| \\ &\leq \sup_{0 \leq s \leq t} \left| \int_0^{\lfloor ns \rfloor / n} \left(\frac{4V^n(\lfloor nr \rfloor) / \alpha(n)}{3U^n(\lfloor nr \rfloor) / n} - \frac{4v(r)}{3u(r)} \right) dr \right| + \sup_{0 \leq s \leq t} \left| \int_{\lfloor ns \rfloor / n}^s \frac{4v(r)}{3u(r)} dr \right| \\ &\quad + \frac{nt}{\alpha(n)} \sup_{0 \leq i \leq \lfloor nt \rfloor} |E_V^n(i)|. \end{aligned}$$

The penultimate term on the right-hand side clearly tends to 0, and the last term converges to 0 in probability by Lemma 3.1. We have

$$\begin{aligned}
& \sup_{0 \leq s \leq t} \left| \int_0^{\lfloor ns \rfloor / n} \left(\frac{4V^n(\lfloor nr \rfloor) / \alpha(n)}{3U^n(\lfloor nr \rfloor) / n} - \frac{4v(r)}{3u(r)} \right) dr \right| \\
& \leq \int_0^{\lfloor nt \rfloor / n} \frac{4V^n(\lfloor ns \rfloor) / \alpha(n)}{3u(s)} \left| \frac{u(s)}{U^n(\lfloor ns \rfloor) / n} - 1 \right| ds + \int_0^{\lfloor nt \rfloor / n} \frac{4}{3u(s)} \left| \frac{V^n(\lfloor ns \rfloor)}{\alpha(n)} - v(s) \right| ds \\
& \leq \frac{4}{3(1-t)} \sup_{0 \leq s \leq t} \left| \frac{u(s)}{U^n(\lfloor ns \rfloor) / n} - 1 \right| + \frac{4}{3(1-t)} \int_0^t \left| \frac{V^n(\lfloor ns \rfloor)}{\alpha(n)} - v(s) \right| ds.
\end{aligned}$$

The first term on the right-hand side converges to 0 in probability. So following the argument in the proof of Proposition 3.1, we easily obtain $\sup_{0 \leq s \leq t} \left| \frac{V^n(\lfloor ns \rfloor)}{\alpha(n)} - v(s) \right| \xrightarrow{p} 0$. \square

4. Fluid limit for $\alpha(n) \gg \sqrt{n}$

In the case where $\alpha(n) \gg \sqrt{n}$, we will also prove fluid limits for X^n and N^n .

Theorem 4.1. Fix $t \in (0, 1)$. For $\alpha(n) \gg \sqrt{n}$, as $n \rightarrow \infty$ we have

$$\begin{aligned}
& \sup_{0 \leq s \leq t} \left| \frac{X^n(\lfloor ns \rfloor)}{\alpha(n)} - \left((1-s)^{2/3} - (1-s)^{4/3} \right) \right| \xrightarrow{p} 0, \quad \frac{L^n(\lfloor nt \rfloor)}{\alpha(n)} \xrightarrow{p} 0, \\
& \sup_{0 \leq s \leq t} \left| \frac{N^n(\lfloor ns \rfloor)}{\alpha(n)} - \left(\frac{1}{4} - \frac{1}{2}(1-s)^{2/3} + \frac{1}{4}(1-s)^{4/3} \right) \right| \xrightarrow{p} 0.
\end{aligned}$$

To this end, we again study the expected increments.

Lemma 4.1. For $t < 1$ and $0 \leq i \leq \lfloor nt \rfloor$ we have

$$\begin{aligned}
\mathbb{E}[X^n(i+1) - X^n(i) | \mathcal{F}_i^n] &= 2\mathbb{1}_{\{X^n(i)=0\}} + \frac{2\alpha(n)(1-i/n)^{4/3} - 2X^n(i)}{3(n-i)} + E_{X,1}^n(i), \\
\mathbb{E}[(X^n(i+1) - X^n(i))^2 | \mathcal{F}_i^n] &= 4\mathbb{1}_{\{X^n(i)=0\}} + 1 + E_{X,2}^n(i), \\
\mathbb{E}[N^n(i+1) - N^n(i) | \mathcal{F}_i^n] &= \mathbb{E}[(N^n(i+1) - N^n(i))^2 | \mathcal{F}_i^n] = \frac{X^n(i)}{3(n-i)} + E_N^n(i),
\end{aligned}$$

where $\frac{n}{\alpha(n)\sqrt{\alpha(n)}} \sup_{0 \leq i \leq \lfloor nt \rfloor} |E_{X,1}^n(i)| \xrightarrow{p} 0$, $\sup_{0 \leq i \leq \lfloor nt \rfloor} |E_{X,2}^n(i)| \rightarrow 0$ in L^2 , and $\frac{n}{\alpha(n)\sqrt{\alpha(n)}} \sup_{0 \leq i \leq \lfloor nt \rfloor} |E_N^n(i)| \rightarrow 0$ in L^2 as $n \rightarrow \infty$.

Proof. By (3), we have

$$\begin{aligned}
& \mathbb{E}[X^n(i+1) - X^n(i) | \mathcal{F}_i^n] \\
&= 2\mathbb{1}_{\{X^n(i)=0\}} + \frac{2\alpha(n)(1-i/n)^{4/3} - 2X^n(i)}{3(n-i)} \\
&+ \frac{2V^n(i) - 2\alpha(n)(1-i/n)^{4/3} + 2\mathbb{1}_{\{X^n(i)>0\}}}{3(n-i)} \\
&+ \frac{2V^n(i) - 2X^n(i) + 2\mathbb{1}_{\{X^n(i)>0\}}}{3(n-i)} \left(\left(1 + \frac{3U^n(i) - 3(n-i) + 4V^n(i) + X^n(i) - \mathbb{1}_{\{X^n(i)>0\}}}{3(n-i)} \right)^{-1} - 1 \right) \\
&=: 2\mathbb{1}_{\{X^n(i)=0\}} + \frac{2\alpha(n)(1-i/n)^{4/3} - 2X^n(i)}{3(n-i)} + E_{X,1}^n(i).
\end{aligned}$$

By the fluid limit for V^n in Lemma 3.2, we have

$$\frac{n}{\alpha(n)} \sup_{0 \leq i \leq \lfloor nt \rfloor} \left| \frac{2V^n(i) - 2\alpha(n)(1-i/n)^{4/3} + 2\mathbb{1}_{\{X^n(i)>0\}}}{3(n-i)} \right| \xrightarrow{P} 0.$$

Since $(1+x)^{-1} \geq 1-x$ for $x > 0$, and $U^n(i) \geq n-i$, we have

$$\begin{aligned}
\Delta^n &:= \sup_{0 \leq i \leq \lfloor nt \rfloor} \left| \left(1 + \frac{3U^n(i) - 3(n-i) + 4V^n(i) + X^n(i) - \mathbb{1}_{\{X^n(i)>0\}}}{3(n-i)} \right)^{-1} - 1 \right| \\
&\leq \frac{1}{3(1-t)} \left(3 \sup_{0 \leq i \leq \lfloor nt \rfloor} \left| \frac{U^n(i)}{n} - \left(1 - \frac{i}{n} \right) \right| + \frac{4\alpha(n)}{n} + \frac{1 + \sup_{0 \leq i \leq t} X^n(i)}{n} \right).
\end{aligned} \tag{10}$$

The quantity on the right-hand side is bounded above by $4/(1-t)$. Using Lemma 2.1 (which says that X^n is of smaller order than n) and the fluid limit for U^n from Lemma 3.2, we see that Δ^n converges to 0 in probability and, since it is bounded, also in L^4 .

By Lemma 2.1, we have that $\frac{1}{\alpha(n)\sqrt{\sqrt{n}}} \sup_{0 \leq i \leq \lfloor nt \rfloor} (X^n(i) + V^n(i))$ is tight in n . It follows that $\frac{n}{\alpha(n)\sqrt{\sqrt{n}}} \sup_{0 \leq i \leq \lfloor nt \rfloor} \frac{2V^n(i) - 2X^n(i) + 2\mathbb{1}_{\{X^n(i)>0\}}}{3(n-i)}$ is also tight, and so $\frac{n}{\alpha(n)\sqrt{\sqrt{n}}} \sup_{0 \leq i \leq \lfloor nt \rfloor} |E_{X,1}^n(i)| \xrightarrow{P} 0$.

For the second moment, rewriting the expression (5) we have

$$\begin{aligned}
& \mathbb{E}[(X^n(i+1) - X^n(i))^2 | \mathcal{F}_i^n] \\
&= 4\mathbb{1}_{\{X^n(i)=0\}} + 1 + \frac{6V^n(i) + 8V^n(i)\mathbb{1}_{\{X^n(i)=0\}} + 3X^n(i) - 3\mathbb{1}_{\{X^n(i)>0\}}}{3U^n(i) + 4V^n(i) + X^n(i) - \mathbb{1}_{\{X^n(i)>0\}}} \\
&=: 4\mathbb{1}_{\{X^n(i)=0\}} + 1 + E_{X,2}^n(i).
\end{aligned}$$

But then $\sup_{0 \leq i \leq \lfloor nt \rfloor} |E_{X,2}^n(i)| \leq \frac{14\alpha(n) + 3 \sup_{0 \leq i \leq \lfloor nt \rfloor} X^n(i)}{3n(1-t)}$ which converges to 0 in probability and in L^4 , by Lemma 2.1. Turning now to the increments of N^n , we

have

$$\begin{aligned}
& \mathbb{E}[N^n(i+1) - N^n(i) | \mathcal{F}_i^n] \\
&= \frac{X^n(i)}{3(n-i)} + \frac{X^n(i)}{3(n-i)} \left(\left(1 + \frac{3U^n(i) - 3(n-i) + 4V^n(i) + X^n(i) - \mathbb{1}_{\{X^n(i) > 0\}}}{3(n-i)} \right)^{-1} - 1 \right) \\
&+ \frac{3U^n(i) - 2\mathbb{1}_{\{X^n(i) > 0\}}(3U^n(i) + 4V^n(i) + X^n(i) - 3)}{2(3U^n(i) + 4V^n(i) + X^n(i) - \mathbb{1}_{\{X^n(i) > 0\}})(3U^n(i) + 4V^n(i) + X^n(i) - \mathbb{1}_{\{X^n(i) > 0\}} - 2)} \\
&=: \frac{X^n(i)}{3(n-i)} + E_N^n(i).
\end{aligned}$$

For $0 \leq i \leq \lfloor nt \rfloor$ we have

$$\left| \frac{3U^n(i) - 2\mathbb{1}_{\{X^n(i) > 0\}}(3U^n(i) + 4V^n(i) + X^n(i) - 3)}{2(3U^n(i) + 4V^n(i) + X^n(i) - \mathbb{1}_{\{X^n(i) > 0\}})(3U^n(i) + 4V^n(i) + X^n(i) - \mathbb{1}_{\{X^n(i) > 0\}} - 2)} \right| \leq \frac{3}{2n(1-t)}.$$

Hence,

$$\sup_{0 \leq i \leq \lfloor nt \rfloor} |E_N^n(i)| \leq \frac{3}{2(1-t)} + \frac{\sup_{0 \leq i \leq \lfloor nt \rfloor} X^n(i)}{3(1-t)} \cdot \Delta^n.$$

We have $\Delta^n \rightarrow 0$ in L^4 . Moreover, $\frac{1}{\alpha(n)\sqrt{\sqrt{n}}} \sup_{0 \leq i \leq \lfloor nt \rfloor} X^n(i)$ is bounded in L^4 , and so by the Cauchy-Schwarz inequality, we get $\frac{n}{\alpha(n)\sqrt{\sqrt{n}}} \sup_{0 \leq i \leq \lfloor nt \rfloor} |E_N^n(i)| \rightarrow 0$ in L^2 , as desired. \square

Before we can proceed to the proof of Theorem 4.1, we need some technical lemmas to deal with the fact that X^n reflects off 0. Let $\gamma^n = \inf\{i \geq 0 : X^n(i) \geq V^n(i)\}$.

Lemma 4.2. *Fix $t \in (1/2, 1)$. Then there exists $t_0 \in (0, t)$ sufficiently small that $\mathbb{P}(\gamma^n < t_0 n) \rightarrow 0$ as $n \rightarrow \infty$.*

Proof. Let $\tilde{\gamma}^n = \inf\{i \geq 0 : X^n(i) \geq 3\alpha(n)/8\}$. Observe that V^n is decreasing and that if $s_0 = 1 - 2^{-3/4} \approx 0.405$ then $(1 - s_0)^{4/3} = 1/2 > 3/8$. Then by Lemma 3.2, we have $\mathbb{P}(V^n(\lfloor ns_0 \rfloor) > 3\alpha(n)/8) \rightarrow 1$ as $n \rightarrow \infty$. So if we can show that there exists $s_1 \in (0, t)$ such that $\mathbb{P}(\tilde{\gamma}^n < ns_1) \rightarrow 0$ as $n \rightarrow \infty$ then, setting $t_0 = s_0 \wedge s_1$, we obtain $\mathbb{P}(\gamma^n < nt_0) \rightarrow 0$. So it remains to prove that there exists such an s_1 .

The time-inhomogeneity of the process X^n makes explicit calculations awkward. We instead couple X^n with a simpler process A^n . If $A^n(i) = 0$ then $A^n(i+1) = 1$. If $A^n(i) > 0$ then

$$A^n(i+1) - A^n(i) = \begin{cases} -1 & \text{with probability } \frac{1}{2} \\ +1 & \text{with probability } \frac{1}{2} - \frac{4\alpha(n)}{3n(1-t)} \\ +2 & \text{with probability } \frac{4\alpha(n)}{3n(1-t)}. \end{cases}$$

(We implicitly take n sufficiently large that $4\alpha(n)/(3n(1-t)) < 1/2$.) We set $A^n(0) = 0$. It is straightforward to see that we may couple X^n and A^n in such a way that $X^n(i) \leq 3 + A^n(i)$ for all $0 \leq i \leq \lfloor nt \rfloor$. Let $\lambda_n = \inf\{i \geq 0 : A^n(i) \geq 3\alpha(n)/8 - 3\}$. Then we certainly have $\lambda_n \leq \tilde{\gamma}^n$. So we will find $s_1 \in (0, t)$ such that $\mathbb{P}(\lambda_n < ns_1) \rightarrow 0$. For simplicity, write $b = \lfloor 3\alpha(n)/8 \rfloor - 3$ for the upper barrier and $d = 4\alpha(n)/(3n(1-t))$ for the drift. The quantity

$$h_1 = \mathbb{P}(A^n \text{ hits } 0 \text{ before } \{b, b+1\} | A^n(0) = 1)$$

is calculated via a standard calculation in Lemma 7.1 in the Appendix. Substituting the given values of b and d in the (lengthy) expression there, we obtain that $1 - h_1 \sim 8\alpha(n)/(3n(1-t))$, as $n \rightarrow \infty$. By the strong Markov property, the random walk A^n visits 0 a Geometric($1 - h_1$) number of times before going above $3\alpha(n)/8 - 3$, and so

$$\frac{1}{\alpha(n)} \mathbb{E} \left[\sum_{i=0}^{\lambda_n-1} \mathbb{1}_{\{A^n(i)=0\}} \right] \sim \frac{3n(1-t)}{8\alpha(n)^2} \rightarrow 0,$$

since $\alpha(n) \gg \sqrt{n}$.

Now note that the standard martingale for A^n is

$$M_A^n(i) = A^n(i) - \frac{4\alpha(n)i}{3n(1-t)} - \sum_{j=0}^{i-1} \left(1 - \frac{4\alpha(n)}{3n(1-t)} \right) \mathbb{1}_{\{A^n(j)=0\}}.$$

Let $\mathcal{F}_A^n(j)$ be the natural filtration of A^n . We have

$$\text{var}(A^n(j+1) - A^n(j) | \mathcal{F}_A^n(j)) \leq \left(1 + \frac{4\alpha(n)}{n(1-t)} \right) \mathbb{1}_{\{A^n(j)>0\}} \leq 1 + \frac{4\alpha(n)}{n(1-t)},$$

and so

$$\mathbb{E}[(M_A^n(i))^2] \leq i \left(1 + \frac{4\alpha(n)}{n(1-t)} \right) \leq 2i$$

for n sufficiently large. For any $s > 0$, then,

$$\sup_{0 \leq i \leq \lambda_n \wedge \lfloor ns \rfloor} \left| \frac{A^n(i)}{\alpha(n)} - \frac{4i}{3n(1-t)} \right| \leq \frac{\sup_{0 \leq i \leq \lfloor ns \rfloor} |M_A^n(i)|}{\alpha(n)} + \frac{1}{\alpha(n)} \sum_{j=0}^{\lambda_n-1} \mathbb{1}_{\{A^n(i)=0\}}.$$

For any $\delta > 0$, it then follows by using Doob's L^2 inequality that

$$\begin{aligned} & \mathbb{P} \left(\sup_{0 \leq i \leq \lambda_n \wedge \lfloor ns \rfloor} \left| \frac{A^n(i)}{\alpha(n)} - \frac{4i}{3n(1-t)} \right| > \delta \right) \\ & \leq 4 \frac{\mathbb{E}[M_A^n(\lfloor ns \rfloor)^2]}{\delta^2 \alpha(n)^2} + \frac{1}{\delta \alpha(n)} \mathbb{E} \left[\sum_{j=0}^{\lambda_n-1} \mathbb{1}_{\{A^n(i)=0\}} \right] \\ & \leq \frac{8\lfloor ns \rfloor}{\delta^2 \alpha(n)^2} + \frac{1}{\delta \alpha(n)} \mathbb{E} \left[\sum_{j=0}^{\lambda_n-1} \mathbb{1}_{\{A^n(i)=0\}} \right] \rightarrow 0, \end{aligned}$$

as $n \rightarrow \infty$.

Finally, if we take $s_1 = 3(1-t)/16$ then $4\lfloor ns_1 \rfloor / (3n(1-t)) \leq 1/4 < 3/8$. Then for $\delta \in (0, 1/8)$,

$$\mathbb{P}(\lambda_n < ns_1) \leq \mathbb{P} \left(\sup_{0 \leq i \leq \lambda_n \wedge \lfloor ns_1 \rfloor} \left| \frac{A^n(i)}{\alpha(n)} - \frac{4i}{3n(1-t)} \right| > \delta \right) \rightarrow 0$$

as $n \rightarrow \infty$. The result follows. \square

Proposition 4.1. For the $t_0 \in (0, t)$ from Lemma 4.2, $\frac{L^n(\lfloor nt_0 \rfloor)}{\alpha(n)} \xrightarrow{P} 0$, as $n \rightarrow \infty$.

Proof. We again make use of a coupling, this time to produce a lower bound: we define a process D^n such that $D^n(i) \leq X^n(i)$ for $i \leq \gamma^n - 1$.

Conditionally on $(U^n(i), V^n(i), X^n(i), D^n(i)) = (u, v, x, d)$ with $D^n(i) = d > 1$, we let

$$D^n(i+1) - D^n(i) = \begin{cases} +2 & \text{with probability } \frac{4(x-1)\mathbb{1}_{\{x>0\}}}{2(3u+4v+(x-1)\mathbb{1}_{\{x>0\}})} \\ +1 & \text{with probability } \frac{3u+4v-4(x-1)\mathbb{1}_{\{x>0\}}}{2(3u+4v+(x-1)\mathbb{1}_{\{x>0\}})} \\ -1 & \text{with probability } \frac{3u+4v}{2(3u+4v+(x-1)\mathbb{1}_{\{x>0\}})} \\ -2 & \text{with probability } \frac{2(x-1)\mathbb{1}_{\{x>0\}}}{2(3u+4v+(x-1)\mathbb{1}_{\{x>0\}})}. \end{cases}$$

Conditionally on $(U^n(i), V^n(i), X^n(i), D^n(i)) = (u, v, x, d)$ with $D^n(i) = d = 1$, we let

$$D^n(i+1) - D^n(i) = \begin{cases} +2 & \text{with probability } \frac{2(x-1)\mathbb{1}_{\{x>0\}}}{2(3u+4v+(x-1)\mathbb{1}_{\{x>0\}})} \\ +1 & \text{with probability } \frac{3u+4v-2(x-1)\mathbb{1}_{\{x>0\}}}{2(3u+4v+(x-1)\mathbb{1}_{\{x>0\}})} \\ -1 & \text{with probability } \frac{3u+4v+2(x-1)\mathbb{1}_{\{x>0\}}}{2(3u+4v+(x-1)\mathbb{1}_{\{x>0\}})}. \end{cases}$$

Conditionally on $(U^n(i), V^n(i), X^n(i), D^n(i)) = (u, v, x, d)$ with $D^n(i) = d = 0$, we let $D^n(i+1) = 1$. Since we assume $i \leq \gamma^n - 1$, it is straightforward to see that we may produce a coupling such that $D^n(i) \leq X^n(i)$.

Let $\tilde{L}^n(i) = 2 \sum_{j=0}^{i-1} \mathbb{1}_{\{D^n(j)=0\}}$, and observe that we have $\tilde{L}^n(\gamma^n \wedge \lfloor nt_0 \rfloor) \geq L^n(\gamma^n \wedge \lfloor nt_0 \rfloor)$. But then for any $\delta > 0$,

$$\mathbb{P}\left(\frac{L^n(\lfloor nt_0 \rfloor)}{\alpha(n)} > \delta\right) \leq \mathbb{P}\left(\frac{\tilde{L}^n(\gamma^n \wedge \lfloor nt_0 \rfloor)}{\alpha(n)} > \delta, \gamma^n \geq \lfloor nt_0 \rfloor\right) + \mathbb{P}(\gamma^n < \lfloor nt_0 \rfloor).$$

By Lemma 4.2, we have that $\mathbb{P}(\gamma^n < \lfloor nt_0 \rfloor) \rightarrow 0$ and so it suffices to prove that the first term tends to 0.

We have

$$\mathbb{E}[D^n(i+1) - D^n(i) | \mathcal{F}_D^n(i)] = \mathbb{1}_{\{D^n(i)=0\}},$$

where $(\mathcal{F}_D^n(i))_{i \geq 0}$ denotes the natural filtration of (U^n, V^n, X^n, D^n) . Also,

$$\begin{aligned} \text{var}(D^n(i+1) - D^n(i) | \mathcal{F}_D^n(i)) &\leq 1 + \frac{9(X^n(i) - \mathbb{1}_{\{X^n(i)>0\}}) \mathbb{1}_{\{D^n(i) \geq 1\}}}{3U^n(i) + 4V^n(i) + X^n(i) - \mathbb{1}_{\{X^n(i)>0\}}} \\ &\leq 1 + \frac{3\alpha(n)}{n(1-s)}, \end{aligned}$$

for $i \leq \gamma^n \wedge \lfloor ns \rfloor$. Hence, $\left(D^n(i) - \sum_{j=0}^{i-1} \mathbb{1}_{\{D^n(j)=0\}}\right)_{i \geq 0}$ is a mean 0 martingale, with bounded steps and

$$\sum_{j=0}^{i-1} \mathbb{E}\left[\left(D^n(j+1) - D^n(j) - \mathbb{1}_{\{D^n(j)=0\}}\right)^2 \middle| \mathcal{F}_D^n(j)\right] = i + E_D^n(i)$$

for $i \leq \gamma^n \wedge \lfloor nt_0 \rfloor$, where $\sup_{i \leq \gamma^n \wedge \lfloor nt_0 \rfloor} \frac{n}{\alpha(n)} |E_D^n(i)|$ is bounded with high probability. By the martingale functional central limit theorem (Theorem 1.4, Section 7.1 of Ethier

and Kurtz [9]), we then have that, on the event $\{\gamma^n \geq \lfloor nt_0 \rfloor\}$,

$$\frac{1}{\sqrt{n}} \left(D^n(\lfloor nu \rfloor) - \sum_{j=0}^{\lfloor nu \rfloor - 1} \mathbb{1}_{\{D^n(j)=0\}} \right)_{0 \leq u \leq t_0} \xrightarrow{d} (W_u)_{0 \leq u \leq t_0},$$

where W is a standard Brownian motion. But we also have that

$$\frac{1}{\sqrt{n}} \sum_{j=0}^{\lfloor nt_0 \rfloor - 1} \mathbb{1}_{\{D^n(j)=0\}} = - \inf_{0 \leq i \leq \lfloor nt_0 \rfloor} \frac{1}{\sqrt{n}} \left(D^n(i) - \sum_{j=0}^{i-1} \mathbb{1}_{\{D^n(j)=0\}} \right),$$

and so by the continuous mapping theorem we deduce that on $\{\gamma^n \geq \lfloor nt_0 \rfloor\}$,

$$\frac{\tilde{L}^n(\lfloor nt_0 \rfloor)}{\sqrt{n}} = \frac{2}{\sqrt{n}} \sum_{j=0}^{\lfloor nt_0 \rfloor - 1} \mathbb{1}_{\{D^n(j)=0\}} \xrightarrow{d} -2 \inf_{0 \leq u \leq t_0} W_u.$$

The result then follows since $\alpha(n) \gg \sqrt{n}$. \square

We now turn to the proof of the main theorem in this section.

Proof of Theorem 4.1. Lemma 4.1 gives us that the expected increments of X^n and N^n are

$$\begin{aligned} \mathbb{E}[X^n(i+1) - X^n(i) | \mathcal{F}_i^n] &= 2 \mathbb{1}_{\{X^n(i)=0\}} + \frac{\alpha(n)}{n} \left(\frac{2(1-i/n)^{1/3}}{3} - \frac{2X^n(i)/\alpha(n)}{3(1-i/n)} \right) + E_{X,1}^n(i), \\ \mathbb{E}[N^n(i+1) - N^n(i) | \mathcal{F}_i^n] &= \frac{\alpha(n)}{n} \frac{X^n(i)/\alpha(n)}{3(1-i/n)} + E_N^n(i). \end{aligned}$$

This suggests that, as long as X^n does not hit 0 too often, the candidate for the fluid limit should be the solution to

$$\frac{dx(s)}{ds} = \frac{2}{3}(1-s)^{1/3} - \frac{2x(s)}{3(1-s)}, \quad \frac{dm(s)}{ds} = \frac{x(s)}{3(1-s)}, \quad (11)$$

with initial conditions $x(0) = 0$, $m(0) = 0$. The unique solution to this system of differential equations is given by $x(s) = (1-s)^{2/3} - (1-s)^{4/3}$ and $m(s) = \frac{1}{4} - \frac{1}{2}(1-s)^{2/3} + \frac{1}{4}(1-s)^{4/3}$ for $s \in [0, 1]$. We will use Proposition 4.1 to help control the local time at the start, and then prove the fluid limit result in the standard manner.

Let $T^n = \inf\{s \geq t_0 : X^n(\lfloor ns \rfloor) = 0\}$. Then for $t \geq t_0$ we have $L^n(\lfloor n(t \wedge T^n) \rfloor) = L^n(\lfloor nt_0 \rfloor)$, since we may only accumulate local time at 0. We have

$$\mathbb{E}[M_X^n(i)^2] = \mathbb{E} \left[\sum_{j=0}^{i-1} \text{var} (X^n(j+1) - X^n(j) | \mathcal{F}_j^n) \right] \leq i \left(5 + \mathbb{E} \left[\sup_{0 \leq j \leq i} |E_{X,2}^n(j)| \right] \right),$$

and so $\frac{\mathbb{E}[M_X^n(\lfloor nt \rfloor)^2]}{\alpha(n)^2} \rightarrow 0$ by Lemma 4.1 and the fact that $\alpha(n) \gg \sqrt{n}$. It follows as usual that

$$\frac{\sup_{0 \leq s \leq t \wedge T_n} |M_X^n(\lfloor nt \rfloor)|}{\alpha(n)} \xrightarrow{p} 0. \quad (12)$$

Then

$$\begin{aligned}
& \sup_{0 \leq s \leq t \wedge T^n} \left| \sum_{j=0}^{\lfloor ns \rfloor - 1} \frac{\mathbb{E}[X^n(j+1) - X^n(j) | \mathcal{F}_j^n]}{\alpha(n)} - \int_0^s \left(\frac{2}{3}(1-u)^{1/3} - \frac{2x(s)}{3(1-s)} \right) du \right| \\
& \leq \sup_{0 \leq s \leq t} \left| \frac{1}{n} \sum_{j=0}^{\lfloor ns \rfloor - 1} \frac{2}{3}(1-j/n)^{1/3} - \int_0^s \frac{2}{3}(1-u)^{1/3} du \right| + \sup_{0 \leq s \leq t \wedge T^n} \sum_{j=0}^{\lfloor ns \rfloor - 1} \frac{2\mathbb{1}_{\{X^n(j)=0\}}}{\alpha(n)} \\
& \quad + \sup_{0 \leq s \leq t \wedge T^n} \left| \frac{1}{n} \sum_{j=0}^{\lfloor ns \rfloor - 1} \frac{2}{3(1-j/n)} \frac{X^n(j)}{\alpha(n)} - \int_0^s \frac{2x(u)}{3(1-u)} du \right| \\
& \quad + \frac{\lfloor nt \rfloor}{\alpha(n)} \sup_{0 \leq i \leq \lfloor n(t \wedge T^n) \rfloor} |E_{X,1}^n(i)|.
\end{aligned}$$

The first term is clearly the difference between a Riemann approximation to an integral and that integral, and tends to 0. As we have $L^n(\lfloor n(t \wedge T^n) \rfloor) = L^n(\lfloor nt_0 \rfloor)$, the second term converges in probability to 0 by Proposition 4.1. The fourth term converges in probability to 0 by Lemma 4.1. The third term is bounded above by

$$\begin{aligned}
& \frac{2}{3(1-t)} \int_0^{\lfloor n(t \wedge T^n) \rfloor / n} \left| \frac{X^n(\lfloor nu \rfloor)}{\alpha(n)} - x(u) \right| du + \int_0^t \left| \frac{2}{3(1-u)} - \frac{2}{3(1-\lfloor nu \rfloor / u)} \right| x(u) du \\
& \quad + \sup_{0 \leq s \leq t} \int_{\lfloor ns \rfloor / n}^s \frac{2x(u)}{3(1-u)} du,
\end{aligned}$$

where the second and third terms clearly tend to 0 as $n \rightarrow \infty$. The rest of the argument then goes through as in the proof of Proposition 3.1 to show that

$\sup_{0 \leq s \leq t \wedge T^n} \left| \frac{X^n(\lfloor ns \rfloor)}{\alpha(n)} - x(s) \right| \xrightarrow{p} 0$. Now observe that for fixed $t > t_0$, $\inf_{t_0 \leq s \leq t} x(s) > 0$. So for sufficiently small $\delta > 0$, $\sup_{0 \leq s \leq t \wedge T^n} \left| \frac{X^n(\lfloor ns \rfloor)}{\alpha(n)} - x(s) \right| \leq \delta$ implies that $T^n > t$ and $L^n(\lfloor nt \rfloor) = L^n(\lfloor nt_0 \rfloor)$. Hence,

$$\sup_{0 \leq s \leq t} \left| \frac{X^n(\lfloor ns \rfloor)}{\alpha(n)} - x(s) \right| \xrightarrow{p} 0 \quad \text{and} \quad \frac{L^n(\lfloor nt \rfloor)}{\alpha(n)} \xrightarrow{p} 0.$$

Finally, to get the result for N^n , note that

$$\mathbb{E}[M_N^n(i)^2] \leq \mathbb{E} \left[\sum_{j=0}^{i-1} \mathbb{E}[N^n(j+1) - N^n(j) | \mathcal{F}_j^n] \right]$$

and so

$$\begin{aligned}
\frac{\mathbb{E}[M_N^n(\lfloor nt \rfloor)^2]}{\alpha(n)^2} & \leq \frac{1}{n\alpha(n)} \mathbb{E} \left[\sum_{i=0}^{\lfloor nt \rfloor - 1} \frac{X^n(i)/\alpha(n)}{3(1-i/n)} \right] + \frac{nt}{\alpha(n)^2} \mathbb{E} \left[\sup_{0 \leq i \leq \lfloor nt \rfloor} E_N^n(i) \right] \\
& \leq \frac{t}{3(1-t)\alpha(n)^2} \mathbb{E} \left[\sup_{0 \leq i \leq \lfloor nt \rfloor} X^n(i) \right] + \frac{nt}{\alpha(n)^2} \mathbb{E} \left[\sup_{0 \leq i \leq \lfloor nt \rfloor} E_N^n(i) \right] \rightarrow 0,
\end{aligned}$$

by Lemmas 2.1 and 4.1. Then

$$\begin{aligned} \sup_{0 \leq s \leq t} \left| \frac{N^n(\lfloor ns \rfloor)}{\alpha(n)} - m(s) \right| &\leq \frac{\sup_{0 \leq s \leq t} |M_N^n(\lfloor ns \rfloor)|}{\alpha(n)} + \frac{\sup_{0 \leq i \leq \lfloor nt \rfloor} |E_N^n(i)|}{\alpha(n)} \\ &\quad + \sup_{0 \leq s \leq t} \left| \frac{1}{n} \sum_{j=0}^{\lfloor ns \rfloor - 1} \frac{1}{3(1-j/n)} \frac{X^n(j)}{\alpha(n)} - \int_0^s \frac{x(s)}{3(1-s)} ds \right|, \end{aligned}$$

and the usual argument allows us to complete the proof. \square

5. Diffusion limit for $\alpha(n) = O(\sqrt{n})$

5.1. The limiting process

Proposition 5.1. *Let $a \geq 0$ and let $(B_t, 0 \leq t \leq 1)$ be a Brownian motion. There is a unique solution (X^a, L^a) to the SDE with reflection at 0 given by*

$$dX_t^a = -\frac{2}{3} \frac{X_t^a}{(1-t)} dt + \frac{2a}{3} (1-t)^{1/3} dt + dB_t + dL_t^a, \quad (13)$$

for $0 \leq t < 1$ with initial condition $X_0^a = 0$, $L_0^a = 0$, satisfying the conditions (a) $X_t^a \geq 0$ for $0 \leq t < 1$, (b) L_t^a is non-decreasing, and (c) $\int_0^t \mathbb{1}_{\{X_s^a > 0\}} dL_s^a = 0$ for $0 \leq t < 1$. Indeed, let

$$K_t^a = - \inf_{0 \leq s \leq t} \left[a - a(1-s)^{2/3} + \int_0^s \frac{dB_u}{(1-u)^{2/3}} \right].$$

Then the solution to (13) is given explicitly for $0 \leq t < 1$ by

$$\begin{aligned} L_t^a &= \int_0^t (1-s)^{2/3} dK_s^a, \\ X_t^a &= a(1-t)^{2/3} - a(1-t)^{4/3} + (1-t)^{2/3} \int_0^t \frac{dB_u}{(1-u)^{2/3}} + (1-t)^{2/3} K_t^a. \end{aligned}$$

Moreover, the following statements are true almost surely:

$$X_1^a := \lim_{t \rightarrow 1^-} X_t^a = 0, \quad \int_0^1 \frac{X_s^a}{3(1-s)} ds < \infty, \quad L_1^a := \lim_{t \rightarrow 1^-} L_t^a < \infty.$$

Finally, for $a \geq 0$ and $r \geq a+1$ we have that

$$\sup_{0 \leq t < 1} \mathbb{P} \left(\frac{X_t^a}{\sqrt{1-t}} > r \right) \leq e^{-(r-a)^2/6}.$$

Proof. Theorem 2.1.1 from [13] guarantees the existence of a unique solution to the reflected SDE on the time-interval $[0, t]$ for any fixed $t < 1$, since the diffusion and drift coefficients satisfy the appropriate Lipschitz and linear growth conditions.

For $0 \leq t < 1$, define

$$Y_t^a = a - a(1-t)^{2/3} + \int_0^t \frac{dB_u}{(1-u)^{2/3}} + K_t^a.$$

Then we straightforwardly have that $dY_t^a = \frac{2a}{3}(1-t)^{-1/3}dt + (1-t)^{-2/3}dB_t + dK_t^a$ and, by Skorokhod's lemma (see, for example, Lemma 2.1 of Chapter VI of [14]), K^a is the local time at 0 of Y^a .

Now let $X_t^a = (1-t)^{2/3}Y_t^a$ and $L_t^a = \int_0^t (1-s)^{2/3}dK_s^a$. Then, by construction, $Y_t^a \geq 0$ for $0 \leq t < 1$ and so $X_t^a \geq 0$ for $0 \leq t < 1$. Furthermore,

$$dX_t^a = -\frac{2}{3}(1-t)^{-1/3}Y_t^a dt + (1-t)^{2/3}dY_t^a = -\frac{2}{3}\frac{X_t^a}{(1-t)}dt + \frac{2a}{3}(1-t)^{1/3}dt + dB_t + dL_t^a.$$

Clearly $L_0^a = 0$ and L_t^a is non-decreasing. Let us show that the local time at 0 of X^a is L^a . By Tanaka's formula, the local time is given by

$$\begin{aligned} & |X_t^a| - \int_0^t \operatorname{sgn}(X_s^a)dX_s^a \\ &= (1-t)^{2/3}|Y_t^a| + \frac{2}{3} \int_0^t \operatorname{sgn}(Y_s^a)Y_s^a(1-s)^{-1/3}ds - \int_0^t (1-s)^{2/3}\operatorname{sgn}(Y_s^a)dY_s^a \\ &= \int_0^t (1-s)^{2/3}d|Y_s^a| - \int_0^t (1-s)^{2/3}\operatorname{sgn}(Y_s^a)dY_s^a, \end{aligned}$$

by integration by parts. But we also have $K_t^a = |Y_t^a| - \int_0^t \operatorname{sgn}(Y_s^a)dY_s^a$ and so the last line is equal to $\int_0^t (1-s)^{2/3}dK_s^a$, which is L_t^a . It follows that $\int_0^t \mathbb{1}_{\{X_s^a > 0\}}dL_s^a = 0$. Hence, (X^a, L^a) solves the reflected SDE.

For $a = 0$, we have

$$X_t^0 = (1-t)^{2/3} \int_0^t \frac{dB_u}{(1-u)^{2/3}} - (1-t)^{2/3} \inf_{0 \leq s \leq t} \int_0^s \frac{dB_u}{(1-u)^{2/3}}$$

and so (by Exercise 1.3.1 of [13]) we have

$$(X_t^0, 0 \leq t < 1) \stackrel{d}{=} \left((1-t)^{2/3} \left| \int_0^t \frac{dB_u}{(1-u)^{2/3}} \right|, 0 \leq t < 1 \right).$$

Let first show that $\lim_{t \rightarrow 1} X_t^0 = 0$. By the Dubins-Schwarz theorem we have

$$\left(\int_0^t \frac{dB_s}{(1-s)^{2/3}}, 0 \leq t < 1 \right) \stackrel{d}{=} (B_{f(t)}, 0 \leq t < 1), \quad (14)$$

where $f(t) = 3(1-t)^{-1/3} - 3$ is the quadratic variation of the process on the left-hand side of (14). Let us recall that $(B_t, t \geq 0) \stackrel{d}{=} (tB_{1/t}, t \geq 0)$. Thus, we have

$$(X_t^0, 0 \leq t < 1) \stackrel{d}{=} \left((1-t)^{2/3}f(t) \cdot |B_{1/f(t)}|, 0 \leq t < 1 \right).$$

Then we have

$$(1-t)^{2/3}f(t) \cdot |B_{1/f(t)}| = 3 \left((1-t)^{1/3} - (1-t)^{2/3} \right) \left| B_{\frac{(1-t)^{1/3}}{3(1-(1-t)^{1/3})}} \right| \rightarrow 0$$

almost surely as $t \rightarrow 1$. It follows that $\lim_{t \rightarrow 1} X_t^0 = 0$ almost surely. Moreover,

$$\int_0^1 \frac{X_t^0}{3(1-t)}dt \stackrel{d}{=} \frac{1}{3} \int_0^1 (1-t)^{-1/3} |B_{f(t)}| dt.$$

The change of variables $u = f(t)$ yields that this is in turn equal to $9 \int_0^\infty |B_u|(u+3)^{-3} du$. Since $B_u \stackrel{d}{=} \sqrt{u}B_1$ and $\mathbb{E}[|B_u|] = \sqrt{2u/\pi}$ we have

$$\mathbb{E} \left[\int_0^1 \frac{X_t^0}{3(1-t)} dt \right] = 9 \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{u}(u+3)^{-3} du < \infty$$

and so $\int_0^1 \frac{X_t^0}{3(1-t)} dt < \infty$ a.s. Since we have

$$X_t^0 = -2 \int_0^t \frac{X_s^0}{3(1-s)} ds + B_t + L_t^0,$$

and X_t^0 , $\int_0^t \frac{X_s^0}{3(1-s)} ds$ and B_t all possess finite almost sure limits as $t \rightarrow 1$, the same must also be true of L_t^0 .

To obtain the almost sure finiteness statements for a general $a \geq 0$, note that if we build X^0 and X^a from the same Brownian motion then

$$0 \leq X_t^a \leq a(1-t)^{2/3} + X_t^0. \quad (15)$$

It follows that $\lim_{t \rightarrow 1} X_t^a = 0$ almost surely. We also get

$$\int_0^1 \frac{X_s^a}{3(1-s)} ds \leq \frac{a}{2} + \int_0^1 \frac{X_s^0}{3(1-s)} ds < \infty$$

almost surely. Finally, by the same argument as in the $a = 0$ case, we must then also have that $L_1^a := \lim_{t \rightarrow 1} L_t^a < \infty$ almost surely.

We now turn to the final statement. By (15), it is sufficient to show the result for $a = 0$. For any $0 \leq t < 1$, we have

$$\mathbb{P} \left(\frac{X_t^0}{\sqrt{1-t}} > r \right) = \mathbb{P} \left((1-t)^{1/6} |B_{f(t)}| > r \right) \leq \mathbb{P} \left(|N(0,1)| > r/\sqrt{3} \right) \leq e^{-r^2/6},$$

by standard Gaussian tail bounds. \square

5.2. The invariance principle

Theorem 4.1 gives that for $\alpha(n) \gg \sqrt{n}$ the number of fires rescaled by $\alpha(n)$ possesses a fluid limit. This arises because the dominant contribution to the number of fires comes from connecting to nodes of degree 4. If we had no vertices of degree 4, we would connect only to vertices of degree 3, and the number of fires would make jumps of $+1$ or -1 only, each with probability $1/2$. This suggests that X^n should behave like a simple symmetric random walk. If $\alpha(n) \sim a\sqrt{n}$, $a > 0$ this random walk acquires a positive drift.

Recall that for $0 \leq i \leq \zeta^n$, we have $L^n(i) = 2 \sum_{j=0}^{i-1} \mathbb{1}_{\{X^n(j)=0\}}$. We will prove the following scaling limit.

Theorem 5.1. *Fix $t < 1$ and suppose $\alpha(n)/\sqrt{n} \rightarrow a$ for $a \geq 0$. Then*

$$\left(\frac{X^n(\lfloor ns \rfloor)}{\sqrt{n}}, \frac{L^n(\lfloor ns \rfloor)}{\sqrt{n}}, 0 \leq s \leq t \right) \xrightarrow{d} (X_s^a, L_s^a, 0 \leq s \leq t),$$

uniformly, where $(X_s^a, L_s^a, 0 \leq s \leq 1)$ is the process from Proposition 5.1.

We will deduce Theorem 5.1 from the following general invariance principle for reflected diffusions. Such invariance principles go back to Stroock and Varadhan [15], and we do not believe that this result is novel. But since we could not find a version in the existing literature adapted to our particular setting, we give a proof in the appendix, using ideas from Ethier and Kurtz [9] and Kang and Williams [11].

Theorem 5.2. *Let $q : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$, $b : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}$ satisfy the following conditions:*

(1) *(Lipschitz condition)*

$$\exists K > 0 \forall t \geq 0 \forall y_1, y_2 \in \mathbb{R}_+ : |q(t, y_1) - q(t, y_2)| + |b(t, y_1) - b(t, y_2)| \leq K|y_1 - y_2|;$$

(2) *(linear growth condition)*

$$\exists C > 0 \forall t \geq 0 \forall x \in \mathbb{R}_+ : |q(t, y)| + |b(t, y)| \leq C(1 + |y|).$$

Let Y^n and B^n be $\mathbb{D}(\mathbb{R}_+, \mathbb{R})$ -valued stochastic processes and let Q^n and L^n be increasing $\mathbb{D}(\mathbb{R}_+, \mathbb{R})$ -valued stochastic processes. Let $\mathcal{F}_t^n = \sigma(Y^n(s), L^n(s), Q^n(s), B^n(s) : s \leq t)$ and let $\tau^n(r) = \inf\{t \geq 0 : |Y^n(t)| \geq r \text{ or } |Y^n(t-)| \geq r\}$. Suppose that

(a) $M^n = Y^n - B^n$ is an (\mathcal{F}_t^n) -local martingale,

(b) $(M^n)^2 - Q^n$ is an (\mathcal{F}_t^n) -local martingale,

and that for all $r > 0$ and $T > 0$,

$$(c) \lim_{n \rightarrow \infty} \mathbb{E} \left[\sup_{t \leq T \wedge \tau^n(r)} |Y^n(t) - Y^n(t-)|^2 \right] = 0,$$

$$(d) \lim_{n \rightarrow \infty} \mathbb{E} \left[\sup_{t \leq T \wedge \tau^n(r)} |B^n(t) - B^n(t-)|^2 \right] = 0,$$

$$(e) \lim_{n \rightarrow \infty} \mathbb{E} \left[\sup_{t \leq T \wedge \tau^n(r)} |Q^n(t) - Q^n(t-)| \right] = 0,$$

$$(f) \sup_{t \leq T \wedge \tau^n(r)} \left| B^n(t) - L^n(t) - \int_0^t b(s, Y^n(s)) ds \right| \xrightarrow{P} 0 \text{ as } n \rightarrow \infty,$$

$$(g) \sup_{t \leq T \wedge \tau^n(r)} \left| Q^n(t) - \int_0^t q(s, Y^n(s)) ds \right| \xrightarrow{P} 0 \text{ as } n \rightarrow \infty,$$

$$(h) Y^n(0) \xrightarrow{P} 0 \text{ as } n \rightarrow \infty,$$

(i) *there exists a sequence $(\delta_n)_{n \geq 1}$ of constants with $\delta_n \rightarrow 0$ such that $L^n(0) = 0$, $L^n(t) - L^n(t-) \leq \delta_n$ and $L^n(t) = \int_0^t \mathbb{1}_{\{Y^n(s) \leq \delta_n\}} dL^n(s)$.*

Then $(Y^n, L^n) \xrightarrow{d} (Y, L)$ in the Skorokhod topology as $n \rightarrow \infty$, where (Y, L) is the unique solution to the reflected SDE specified by $Y_0 = L_0 = 0$, L is non-decreasing, and

$$dY_t = b(t, Y_t)dt + \sigma(t, Y_t)dW_t + dL_t,$$

where $\int_0^t \mathbb{1}_{\{Y_s > 0\}} dL_s = 0$, W is a standard Brownian motion and $\sigma(t, y) = \sqrt{q(t, y)}$.

Proof of Theorem 5.1. Define $\tilde{X}^n(s) = \frac{X^n(\lfloor ns \rfloor)}{\sqrt{n}}$ and $\tilde{L}^n(s) = \frac{L^n(\lfloor ns \rfloor)}{\sqrt{n}}$. Recall from Proposition 5.1 that $(X_s^a, L_s^a, 0 \leq s \leq t)$ is the solution to the reflected SDE

$$dX_s^a = -\frac{2}{3} \frac{X_s^a}{(1-s)} ds + \frac{2a}{3} (1-s)^{1/3} ds + dB_s + dL_s^a.$$

For $0 \leq s \leq t$ and $x \geq 0$, let $q(s, x) = 1$ and $b(s, x) = -\frac{2}{3} \frac{x}{(1-s)} + \frac{2a}{3} (1-s)^{1/3}$. It follows straightforwardly that for $0 \leq s \leq t$, the functions q and b satisfy the Lipschitz and linear growth conditions (1) and (2) in Theorem 5.2.

For $i \leq \lfloor nt \rfloor$, let

$$B^n(i) = \sum_{j=0}^{i-1} \mathbb{E} [X^n(j+1) - X^n(j) | \mathcal{F}_j^n],$$

so that $M_X^n(i) = X^n(i) - B^n(i)$, as well as

$$\begin{aligned} Q^n(i) &= \sum_{j=0}^{i-1} \mathbb{E} [M_X^n(j+1)^2 - M_X^n(j)^2 | \mathcal{F}_j^n] = \sum_{j=0}^{i-1} \mathbb{E} [(M_X^n(j+1) - M_X^n(j))^2 | \mathcal{F}_j^n] \\ &= \sum_{j=0}^{i-1} \left(\mathbb{E} [(X^n(j+1) - X^n(j))^2 | \mathcal{F}_j^n] - \mathbb{E} [X^n(j+1) - X^n(j) | \mathcal{F}_j^n]^2 \right). \end{aligned}$$

Since everything is bounded, M_X^n and $(M_X^n)^2 - Q^n$ are both (\mathcal{F}_i^n) -martingales. Let

$$\tilde{M}(s) = \frac{M_X^n(\lfloor ns \rfloor)}{\sqrt{n}}, \quad \tilde{B}^n(s) = \frac{B^n(\lfloor ns \rfloor)}{\sqrt{n}} \quad \text{and} \quad \tilde{Q}^n(s) = \frac{Q^n(\lfloor ns \rfloor)}{n}$$

be the appropriately rescaled versions of these quantities. Then (a) and (b) hold by construction and the main task facing us is to check that conditions (f) and (g) of the Theorem are fulfilled for \tilde{M}^n and $(\tilde{M}^n)^2 - \tilde{Q}^n$.

By Lemma 4.1,

$$\begin{aligned} B^n(i) &= 2 \sum_{j=0}^{i-1} \mathbb{1}_{\{X^n(j)=0\}} + \sum_{j=0}^{i-1} \frac{2\alpha(n)(1-j/n)^{4/3} - 2X^n(j)}{3(n-j)} + \sum_{j=0}^{i-1} E_{X,1}^n(j) \\ &= L^n(i) + \sum_{j=0}^{i-1} \frac{2\alpha(n)(1-j/n)^{4/3} - 2X^n(j)}{3(n-j)} + \sum_{j=0}^{i-1} E_{X,1}^n(j). \end{aligned}$$

Then

$$\begin{aligned} &\sup_{0 \leq s \leq t} \left| \tilde{B}^n(s) - \tilde{L}^n(s) - \int_0^s \left(\frac{2a}{3} (1-u)^{1/3} - \frac{2\tilde{X}^n(u)}{(1-u)} \right) du \right| \\ &\leq \sup_{0 \leq s \leq t} \left| \sum_{j=0}^{\lfloor ns \rfloor - 1} \frac{1}{n} \left(\frac{2\alpha(n)}{3\sqrt{n}} (1-j/n)^{1/3} - \frac{2\tilde{X}^n(j/n)}{3(1-j/n)} \right) \right. \\ &\quad \left. - \int_0^s \left(\frac{2a}{3} (1-u)^{1/3} - \frac{2\tilde{X}^n(u)}{3(1-u)} \right) du \right| + \sup_{0 \leq i \leq \lfloor nt \rfloor} \sqrt{n} |E_{X,1}^n(i)|, \end{aligned}$$

which converges to 0 in probability as $n \rightarrow \infty$ by Lemma 4.1. Hence, condition (f) holds.

By Lemma 4.1,

$$Q^n(i) = \sum_{j=0}^{i-1} \left(1 + E_{X,2}^n(j) - 4\mathbb{1}_{\{X^n(j)=0\}} \left(\frac{2\alpha(n)(1-j/n)^{4/3} - 2X^n(j)}{3(n-j)} + E_{X,1}^n(j) \right) - \left(\frac{2\alpha(n)(1-j/n)^{4/3} - 2X^n(j)}{3(n-j)} + E_{X,1}^n(j) \right)^2 \right).$$

Then

$$\begin{aligned} & \sup_{0 \leq s \leq t} |\tilde{Q}^n(s) - s| \\ &= \sup_{0 \leq s \leq t} \left| \frac{\lfloor ns \rfloor}{n} - s \right. \\ & \quad + \sum_{j=0}^{\lfloor ns \rfloor - 1} \left(\frac{E_{X,2}^n(j)}{n} - \left(\frac{2(\alpha(n)/\sqrt{n})(1-j/n)^{4/3} - 2\tilde{X}(j/n)}{3(n-j)} + \frac{E_{X,1}^n(j)}{\sqrt{n}} \right)^2 \right. \\ & \quad \left. \left. - \frac{4}{\sqrt{n}} \mathbb{1}_{\{\tilde{X}^n(j/n)=0\}} \left(\frac{2(\alpha(n)/\sqrt{n})(1-j/n)^{4/3} - 2\tilde{X}(j/n)}{3(n-j)} + \frac{E_{X,1}^n(j)}{\sqrt{n}} \right) \right) \right|. \end{aligned}$$

Now, for large enough n we have

$$\begin{aligned} & \left| \frac{2(\alpha(n)/\sqrt{n})(1-j/n)^{4/3} - 2\tilde{X}(j/n)}{3(n-j)} + \frac{E_{X,1}^n(j)}{\sqrt{n}} \right| \\ & \leq \frac{1}{n} \frac{4a}{3(1-t)^{1/3}} + \frac{1}{n} \frac{2}{3(1-t)} \sup_{0 \leq s \leq t} \tilde{X}^n(s) + \frac{\sup_{0 \leq i \leq \lfloor nt \rfloor} |E_{X,1}^n(i)|}{\sqrt{n}}. \end{aligned}$$

Since $\sup_{0 \leq i \leq \lfloor nt \rfloor} |E_{X,2}^n(i)| \xrightarrow{P} 0$, $\sup_{0 \leq i \leq \lfloor nt \rfloor} |E_{X,1}^n(i)| \xrightarrow{P} 0$ and $\sup_{0 \leq s \leq t} \tilde{X}^n(s)$ is tight, it follows that $\sup_{0 \leq s \leq t} |\tilde{Q}^n(s) - s| \xrightarrow{P} 0$ as $n \rightarrow \infty$, and so we have that condition (g) of Theorem 5.2 is fulfilled.

Since the increments of X^n are bounded, conditions (c), (d) and (e) of Theorem 5.2 are trivially satisfied. As $\tilde{X}^n(0) = 0$, condition (h) is satisfied. Finally, note that condition (i) is fulfilled if we take $\delta_n = 2/\sqrt{n}$, since $\tilde{L}^n(s) - \tilde{L}^n(s-) \leq 2/\sqrt{n}$ and

$$\int_0^s \mathbb{1}_{\{\tilde{X}^n(u) \leq \frac{2}{\sqrt{n}}\}} dL^n(u) = 2 \sum_{i=0}^{\lfloor ns \rfloor - 1} \mathbb{1}_{\{\tilde{X}^n(i)=0\}} = L^n(s).$$

Applying Theorem 5.2 completes the proof. \square

Finally, we are ready to prove the convergence of the suitably rescaled number of clashes we encounter up to a fixed time $t < 1$.

Lemma 5.1. *For fixed $0 < t < 1$ we have $\left(\frac{N^n(\lfloor ns \rfloor)}{\sqrt{n}}, 0 \leq s \leq t \right) \xrightarrow{d} (N_s^a, 0 \leq s \leq t)$ uniformly as $n \rightarrow \infty$.*

Proof. The argument is standard, since N^n is a counting process which is bounded by $(3n + 4\alpha(n))/2$. We have that

$$\mathbb{E} [M_N^n(k)^2] \leq \mathbb{E} \left[\sum_{i=0}^{k-1} \mathbb{E} [N^n(i+1) - N^n(i) | \mathcal{F}_i^n] \right] = \mathbb{E} [N^n(k)],$$

where the inequality holds since $N^n(i+1) - N^n(i) \in \{0, 1\}$. Now,

$$\begin{aligned} \frac{1}{\sqrt{n}} \mathbb{E} [N^n(\lfloor nt \rfloor)] &\leq \frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor nt \rfloor - 1} \mathbb{E} \left[\frac{3n + 2X^n(i)(3n + 4\alpha(n) + 2X^n(i))}{2(3n(1-t) - 1)(3n(1-t) - 3)} \right] \\ &\leq \frac{\sqrt{nt} (3n + 2\mathbb{E} [\sup_{0 \leq i \leq \zeta_n} X^n(i)] (3n + 4\alpha(n) + 2n + 2\alpha(n)))}{2(3n(1-t) - 1)(3n(1-t) - 3)} \leq C \end{aligned} \quad (16)$$

for some constant C , uniformly in n . In particular, $\mathbb{E} [N^n(\lfloor nt \rfloor)] / n \rightarrow 0$. Fix $\delta > 0$. Using Markov's inequality, we obtain that $M_N^n(\lfloor nt \rfloor) / \sqrt{n} \xrightarrow{P} 0$, as $n \rightarrow \infty$.

Now, by Lemma 4.1 we have $\sqrt{n} \sup_{0 \leq i \leq \lfloor nt \rfloor} \left| \mathbb{E} [N^n(i+1) - N^n(i) | \mathcal{F}_i^n] - \frac{X^n(i)}{3(n-i)} \right| \xrightarrow{P} 0$ and so

$$\sup_{0 \leq s \leq t} \frac{1}{\sqrt{n}} \left| \sum_{i=1}^{\lfloor ns \rfloor - 1} \mathbb{E} [N^n(i+1) - N^n(i) | \mathcal{F}_i^n] - \sum_{i=0}^{\lfloor ns \rfloor - 1} \frac{X^n(i)}{3(n-i)} \right| \xrightarrow{P} 0.$$

Finally, we have

$$\frac{1}{\sqrt{n}} \sum_{i=0}^{\lfloor ns \rfloor - 1} \frac{X^n(i)}{3(n-i)} = \frac{1}{\sqrt{n}} \int_0^{\lfloor ns \rfloor / n} \frac{X^n(\lfloor nu \rfloor)}{3 \left(1 - \frac{\lfloor nu \rfloor}{n}\right)} du.$$

But then using Theorem 5.1 and the continuous mapping theorem, we get

$$\frac{1}{\sqrt{n}} \int_0^{\lfloor ns \rfloor / n} \frac{X^n(\lfloor nu \rfloor)}{3 \left(1 - \frac{\lfloor nu \rfloor}{n}\right)} du \xrightarrow{d} \int_0^s \frac{X_u^a}{3(1-u)} du = N_s^a,$$

uniformly for $0 \leq s \leq t$. Hence, $N^n(\lfloor ns \rfloor) / \sqrt{n} \xrightarrow{d} N_s^a$, as $n \rightarrow \infty$, uniformly for $0 \leq s \leq t$, as desired. \square

6. The end of the process

We now understand the scaling behaviour of the process $(X^n(i), L^n(i), N^n(i))_{i \geq 0}$ on the time-interval $[0, \lfloor n(1-\epsilon) \rfloor]$ for any $\epsilon > 0$. It remains to get from $\lfloor n(1-\epsilon) \rfloor$ to ζ_n .

We first show that $V^n(n)$ is small and that $X^n(\lfloor n(1-\epsilon) \rfloor), L^n(\lfloor n(1-\epsilon) \rfloor), N^n(\lfloor n(1-\epsilon) \rfloor)$ are, for small ϵ , good approximations of $X^n(n), L^n(n), N^n(n)$, by careful use of the coupling from Section 2.2.

Proposition 6.1. *For any $\delta > 0$,*

(a) if $\alpha(n) \rightarrow \infty$ then $V^n(n) / \alpha(n) \xrightarrow{P} 0$ as $n \rightarrow \infty$,

- (b) $\lim_{\epsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \mathbb{P} \left(\frac{1}{\alpha(n)\sqrt{\sqrt{n}}} (N^n(n) - N^n(\lfloor n(1-\epsilon) \rfloor)) > \delta \right) = 0,$
- (c) $\lim_{\epsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \mathbb{P} \left(\frac{1}{\alpha(n)\sqrt{\sqrt{n}}} \sup_{\lfloor n(1-\epsilon) \rfloor \leq i \leq n} X^n(i) > \delta \right) = 0,$
- (d) $\lim_{\epsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \mathbb{P} \left(\frac{1}{\alpha(n)\sqrt{\sqrt{n}}} (L^n(n) - L^n(\lfloor n(1-\epsilon) \rfloor)) > \delta \right) = 0.$

Proof. (a) We have

$$\mathbb{E}[V^n(\lfloor n(1-\epsilon) \rfloor)] \leq 2\epsilon^{4/3}\alpha(n) + \alpha(n)\mathbb{P}(V^n(\lfloor (1-\epsilon)n \rfloor) > 2\epsilon^{4/3}\alpha(n))$$

and, by Lemma 3.2, $\mathbb{P}(V^n(\lfloor (1-\epsilon)n \rfloor) > 2\epsilon^{4/3}\alpha(n)) \rightarrow 0$ as $n \rightarrow \infty$. But V^n is decreasing and so

$$\limsup_{n \rightarrow \infty} \frac{\mathbb{E}[V^n(n)]}{\alpha(n)} \leq \limsup_{n \rightarrow \infty} \frac{\mathbb{E}[V^n(\lfloor n(1-\epsilon) \rfloor)]}{\alpha(n)} \leq 2\epsilon^{4/3}. \quad (17)$$

Since $V^n(n) \geq 0$ and this inequality holds for any $\epsilon > 0$, we must have that the left-hand side is, in fact, equal to 0. Markov's inequality then gives that $V^n(n)/\alpha(n) \xrightarrow{P} 0$.

(b) Recall the coupling and notation from Section 2.2. First note that

$$\begin{aligned} \mathbb{E}[N^n(n) - N^n(\lfloor n(1-\epsilon) \rfloor)] &= \mathbb{E}[N_1^n(n) - N_1^n(\lfloor n(1-\epsilon) \rfloor)] + \mathbb{E}[N_2^n(n) - N_2^n(\lfloor n(1-\epsilon) \rfloor)] \\ &\leq \mathbb{E}[N_1^n(n) - N_1^n(\lfloor n(1-\epsilon) \rfloor)] + \mathbb{E}[V^n(\lfloor n(1-\epsilon) \rfloor)]. \end{aligned} \quad (18)$$

From (7), the process

$$\left(N_1^n(k) - \sum_{i=0}^{k-1} \frac{3U^n(i) + 2(X_1^n(i) - \mathbb{1}_{\{X_1^n(i) > 0\}})(3U^n(i) + 4V^n(i) + X^n(i) - 3)}{2(3U^n(i) + 4V^n(i) + X^n(i) - \mathbb{1}_{\{X^n(i) > 0\}})(3U^n(i) + 4V^n(i) + X^n(i) - \mathbb{1}_{\{X^n(i) > 0\}} - 2)} \right)_{k \geq 0}$$

is the standard martingale. Hence,

$$\begin{aligned} &\mathbb{E}[N_1^n(n) - N_1^n(\lfloor n(1-\epsilon) \rfloor)] \\ &= \mathbb{E} \left[\sum_{i=\lfloor n(1-\epsilon) \rfloor}^{n-1} \frac{3U^n(i) + 2(X_1^n(i) - \mathbb{1}_{\{X_1^n(i) > 0\}})(3U^n(i) + 4V^n(i) + X^n(i) - 3)}{2(3U^n(i) + 4V^n(i) + X^n(i) - \mathbb{1}_{\{X^n(i) > 0\}})(3U^n(i) + 4V^n(i) + X^n(i) - \mathbb{1}_{\{X^n(i) > 0\}} - 2)} \right] \\ &\leq \frac{1}{6} \sum_{i=\lfloor n(1-\epsilon) \rfloor}^{n-1} \frac{1}{(n-i)} + \mathbb{E} \left[\sum_{i=\lfloor n(1-\epsilon) \rfloor}^{n-1} \frac{X_1^n(i) - \mathbb{1}_{\{X_1^n(i) > 0\}}}{3U^n(i) + 4V^n(i) + X^n(i) - \mathbb{1}_{\{X^n(i) > 0\}}} \right] \\ &\leq \frac{1}{6} (\log(\lceil n\epsilon \rceil) + 1) + \mathbb{E} \left[\sum_{i=\lfloor n(1-\epsilon) \rfloor}^{n-1} \frac{X_1^n(i) - \mathbb{1}_{\{X_1^n(i) > 0\}}}{3U^n(i) + 4V^n(i) + X^n(i) - \mathbb{1}_{\{X^n(i) > 0\}}} \right]. \end{aligned} \quad (19)$$

Let us fix $C > 6$ and define a sequence of stopping times in the following way. Let $T_0 = \inf\{k \geq \lfloor n(1-\epsilon) \rfloor : X_1^n(k) \geq \lfloor C\sqrt{n-k} \rfloor\}$ and, inductively, for $i \geq 1$, let

$$S_i = \inf\{k \geq T_{i-1} : X_1^n(k) = 0\} \wedge n \quad \text{and} \quad T_i = \inf\{k \geq S_i : X_1^n(k) = \lfloor C\sqrt{n-k} \rfloor\} \wedge n.$$

Then since $U^n(i) \geq n - i$,

$$\begin{aligned} & \mathbb{E} \left[\sum_{i=\lfloor n(1-\epsilon) \rfloor}^{n-1} \frac{X_1^n(i) - \mathbb{1}_{\{X_1^n(i) > 0\}}}{3U^n(i) + 4V^n(i) + X^n(i) - \mathbb{1}_{\{X^n(i) > 0\}}} \right] \\ & \leq \mathbb{E} \left[\sum_{i=\lfloor n(1-\epsilon) \rfloor}^{T_0-1} \frac{X_1^n(i) - \mathbb{1}_{\{X_1^n(i) > 0\}}}{3(n-i)} \right] + \mathbb{E} \left[\sum_{k=1}^n \sum_{i=S_k}^{T_k-1} \frac{X_1^n(i) - \mathbb{1}_{\{X_1^n(i) > 0\}}}{3(n-i)} \right] \\ & \quad + \mathbb{E} \left[\sum_{k=1}^n \sum_{i=T_{k-1}}^{S_k-1} \frac{X_1^n(i) - \mathbb{1}_{\{X_1^n(i) > 0\}}}{3U^n(i) + 4V^n(i) + X^n(i) - \mathbb{1}_{\{X^n(i) > 0\}}} \right]. \end{aligned}$$

Either we have $T_0 = \lfloor n(1-\epsilon) \rfloor$ (in which case the first sum is empty) or on the time interval $[\lfloor n(1-\epsilon) \rfloor, T_0 - 1]$ we have $X_1^n(i) - 1 \leq C\sqrt{n-i}$. Likewise, on the interval $[S_k, T_k - 1]$, we have $X_1^n(i) - 1 \leq C\sqrt{n-i}$. So

$$\begin{aligned} & \mathbb{E} \left[\sum_{i=\lfloor n(1-\epsilon) \rfloor}^{T_0-1} \frac{X_1^n(i) - \mathbb{1}_{\{X_1^n(i) > 0\}}}{3(n-i)} \right] + \mathbb{E} \left[\sum_{k=1}^n \sum_{i=S_k}^{T_k-1} \frac{X_1^n(i) - \mathbb{1}_{\{X_1^n(i) > 0\}}}{3(n-i)} \right] \\ & \leq \frac{C}{3} \sum_{i=\lfloor n(1-\epsilon) \rfloor}^{n-1} \frac{1}{\sqrt{n-i}}. \end{aligned}$$

Since $\sum_{i=1}^{\lceil n\epsilon \rceil} i^{-1/2} \leq \int_0^{\lceil n\epsilon \rceil} x^{-1/2} dx \leq 2\sqrt{n\epsilon + 1}$, we get

$$\mathbb{E} \left[\sum_{i=\lfloor n(1-\epsilon) \rfloor}^{T_0-1} \frac{X_1^n(i) - \mathbb{1}_{\{X_1^n(i) > 0\}}}{3(n-i)} \right] + \mathbb{E} \left[\sum_{k=1}^n \sum_{i=S_k}^{T_k-1} \frac{X_1^n(i) - \mathbb{1}_{\{X_1^n(i) > 0\}}}{3(n-i)} \right] \leq \frac{2C}{3} \sqrt{n\epsilon + 1}. \quad (20)$$

It therefore remains to deal with the term

$$\mathbb{E} \left[\sum_{k=1}^n \sum_{i=T_{k-1}}^{S_k-1} \frac{X_1^n(i) - \mathbb{1}_{\{X_1^n(i) > 0\}}}{3U^n(i) + 4V^n(i) + X^n(i) - \mathbb{1}_{\{X^n(i) > 0\}}} \right].$$

Now note from (6) that

$$\begin{aligned} & \left(X_1^n(k) + \sum_{i=0}^{k-1} \frac{2X_1^n(i) - 2\mathbb{1}_{\{X_1^n(i) > 0\}} + X_2^n(i) - \mathbb{1}_{\{X_1^n(i)=0, X_2^n(i) > 0\}}}{3U^n(i) + 4V^n(i) + X^n(i) - \mathbb{1}_{\{X^n(i) > 0\}}} \right. \\ & \quad \left. - \sum_{i=0}^{k-1} \mathbb{1}_{\{X_1^n(i)=0, X_2^n(i) > 0\}} - 2 \sum_{i=0}^{k-1} \mathbb{1}_{\{X_1^n(i)=X_2^n(i)=0\}}, k \geq 0 \right) \end{aligned}$$

is a martingale. Since X_1^n does not touch 0 on the time interval $[T_{k-1}, S_k - 1]$, we have

by the optional stopping theorem that

$$\begin{aligned}
& \mathbb{E} \left[\sum_{i=T_{k-1}}^{S_k-1} \frac{X_1^n(i) - \mathbb{1}_{\{X_1^n(i)>0\}}}{3U^n(i) + 4V^n(i) + X^n(i) - \mathbb{1}_{\{X^n(i)>0\}}} \right] \\
&= \frac{1}{2} \mathbb{E} [X_1^n(T_{k-1}) - X_1^n(S_k)] - \frac{1}{2} \mathbb{E} \left[\sum_{i=T_{k-1}}^{S_k-1} \frac{X_2^n(i) - \mathbb{1}_{\{X_1^n(i)=0, X_2^n(i)>0\}}}{3U^n(i) + 4V^n(i) + X^n(i) - \mathbb{1}_{\{X^n(i)>0\}}} \right] \\
&\leq \frac{1}{2} \mathbb{E} [X_1^n(T_{k-1})] \leq \frac{C \mathbb{E} [\sqrt{n - T_{k-1}}]}{2} \leq \frac{C \sqrt{n - \mathbb{E} [T_{k-1}]}}{2},
\end{aligned}$$

where the last inequality follows using Jensen's inequality. So we have

$$\mathbb{E} \left[\sum_{k=1}^n \sum_{i=T_{k-1}}^{S_k-1} \frac{X_1^n(i) - \mathbb{1}_{\{X_1^n(i)>0\}}}{3U^n(i) + 4V^n(i) + X^n(i) - \mathbb{1}_{\{X^n(i)>0\}}} \right] \leq \frac{C}{2} \sum_{k=0}^n \sqrt{n - \mathbb{E} [T_k]}. \quad (21)$$

We thus need a lower bound on $\mathbb{E} [T_k]$ for any k . Let us assume that at some time $\lfloor n(1 - \epsilon) \rfloor \leq \ell < n$, we have $X_1^n(\ell) = 0$. In order to find such a lower bound, we use the coupling from Section 2.2, which yields a SSRW reflected above 2, Y^n , such that $Y^n(\ell) = 2$ and $X_1^n(i) \leq Y^n(i)$ for all $i \geq \ell$. Recall that $(Y^n(i))_{i \geq \ell} \stackrel{d}{=} (2 + |Z(i)|)_{i \geq \ell}$ where Z is a SSRW with $Z(\ell) = 0$. Then if $\sigma = \inf\{i \geq \ell : 2 + |Z(i)| = \lfloor C\sqrt{n - i} \rfloor\}$, we have that σ is stochastically smaller than T_k conditioned on $S_k = \ell$. Now, $(Z(i)^2 - i)_{i \geq \ell}$ is a martingale, and so $\mathbb{E} [Z(\sigma)^2 - \sigma] = -\ell$. But

$$\begin{aligned}
\mathbb{E} [Z(\sigma)^2] &= \mathbb{E} [(\lfloor C\sqrt{n - \sigma} \rfloor - 2)^2] \geq \mathbb{E} [(C\sqrt{n - \sigma} - 3)^2] \\
&\geq C^2(n - \mathbb{E} [\sigma]) - 6C\mathbb{E} [\sqrt{n - \sigma}].
\end{aligned}$$

Since $n - \sigma \in \mathbb{Z}$, we have the very crude bound $\sqrt{n - \sigma} \leq n - \sigma$. Hence,

$$\mathbb{E} [\sigma] - \ell = \mathbb{E} [Z(\sigma)^2] \geq (C^2 - 6C)(n - \mathbb{E} [\sigma]).$$

Then $\mathbb{E} [\sigma] \geq \frac{(C^2 - 6C)n + \ell}{C^2 - 6C + 1}$. By the stochastic domination, for $k \geq 1$ we obtain $\mathbb{E} [T_k] \geq \frac{(C^2 - 6C)n + \mathbb{E} [S_k]}{C^2 - 6C + 1}$. Since $S_k \geq T_{k-1}$, we get $\mathbb{E} [T_k] \geq \frac{(C^2 - 6C)n + \mathbb{E} [T_{k-1}]}{C^2 - 6C + 1}$, and so $n - \mathbb{E} [T_k] \leq \frac{n - \mathbb{E} [T_{k-1}]}{C^2 - 6C + 1}$. By induction,

$$n - \mathbb{E} [T_k] \leq \frac{n - \mathbb{E} [T_0]}{(C^2 - 6C + 1)^k} \leq \frac{\lceil n\epsilon \rceil}{(C^2 - 6C + 1)^k},$$

since $T_0 \geq \lfloor n(1 - \epsilon) \rfloor$. It follows from (21) that

$$\mathbb{E} \left[\sum_{k=1}^n \sum_{i=T_{k-1}}^{S_k-1} \frac{X_1^n(i) - \mathbb{1}_{\{X_1^n(i)>0\}}}{3U^n(i) + 4V^n(i) + X^n(i) - \mathbb{1}_{\{X^n(i)>0\}}} \right] \leq \frac{C\sqrt{C^2 - 6C + 1}\sqrt{n\epsilon + 1}}{2(\sqrt{C^2 - 6C + 1} - 1)}.$$

Putting this together with (18), (19) and (20) we get

$$\begin{aligned}
& \mathbb{E} [N^n(n) - N^n(\lfloor n(1 - \epsilon) \rfloor)] \\
&\leq \frac{1}{6}(\log(\lceil n\epsilon \rceil) + 1) + C\sqrt{n\epsilon + 1} + \frac{C\sqrt{C^2 - 6C + 1}\sqrt{n\epsilon + 1}}{2(\sqrt{C^2 - 6C + 1} - 1)} + \mathbb{E} [V^n(\lfloor (1 - \epsilon)n \rfloor)].
\end{aligned}$$

Using (17), we have $\limsup_{n \rightarrow \infty} \mathbb{E}[V^n(\lfloor n(1-\epsilon) \rfloor)]/\alpha(n) \leq 2\epsilon^{4/3}$. Hence,

$$\limsup_{n \rightarrow \infty} \frac{1}{\alpha(n) \vee \sqrt{n}} \mathbb{E}[N^n(n) - N^n(\lfloor n(1-\epsilon) \rfloor)] \leq \left(C + \frac{C\sqrt{C^2-6C+1}}{2(\sqrt{C^2-6C+1}-1)} \right) \sqrt{\epsilon} + 2\epsilon^{4/3}. \quad (22)$$

Applying Markov's inequality and taking $\epsilon \rightarrow 0$ then yields that for $\delta > 0$,

$$\lim_{\epsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \mathbb{P} \left(\frac{1}{\alpha(n) \vee \sqrt{n}} (N^n(n) - N^n(\lfloor n(1-\epsilon) \rfloor)) > \delta \right) = 0.$$

(c) We will show that $\frac{1}{\alpha(n) \vee \sqrt{n}} \mathbb{E} \left[\sup_{\lfloor n(1-\epsilon) \rfloor \leq i \leq n} X^n(i) \right]$ is small. We again use the coupling of X_1^n with Y^n , but started now with $Y^n(\lfloor n(1-\epsilon) \rfloor) = X_1^n(\lfloor n(1-\epsilon) \rfloor) + 2$. This gives

$$(Y^n(\lfloor n(1-\epsilon) \rfloor + k))_{k \geq 0} \stackrel{d}{=} (2 + |X_1^n(\lfloor (1-\epsilon)n \rfloor) + Z(k)|)_{k \geq 0}.$$

Since $X^n(i) = X_1^n(i) + X_2^n(i)$ with $X_2^n(i) \leq V^n(\lfloor n(1-\epsilon) \rfloor)$ for $i \geq \lfloor n(1-\epsilon) \rfloor$, we have

$$\sup_{\lfloor n(1-\epsilon) \rfloor \leq i \leq n} X^n(i) \leq V^n(\lfloor n(1-\epsilon) \rfloor) + \sup_{\lfloor n(1-\epsilon) \rfloor \leq i \leq n} Y^n(i)$$

and so we get

$$\begin{aligned} & \frac{1}{\alpha(n) \vee \sqrt{n}} \mathbb{E} \left[\sup_{\lfloor n(1-\epsilon) \rfloor \leq i \leq n} X^n(i) \right] \\ & \leq \frac{2 + \mathbb{E}[V^n(\lfloor n(1-\epsilon) \rfloor)] + \mathbb{E}[X_1^n(\lfloor n(1-\epsilon) \rfloor)]}{\alpha(n) \vee \sqrt{n}} + \frac{1}{\sqrt{n}} \mathbb{E} \left[\sup_{0 \leq i \leq \lceil n\epsilon \rceil} |Z(i)| \right] \\ & \leq \frac{2 + \mathbb{E}[V^n(\lfloor n(1-\epsilon) \rfloor)] + \mathbb{E}[X_1^n(\lfloor n(1-\epsilon) \rfloor)]}{\alpha(n) \vee \sqrt{n}} + 2\sqrt{\epsilon}, \end{aligned}$$

by Doob's L^2 inequality. We have already shown that $\lim_{\epsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{\mathbb{E}[V^n(\lfloor n(1-\epsilon) \rfloor)]}{\alpha(n) \vee \sqrt{n}} =$

0. For $\alpha(n)/\sqrt{n} \rightarrow a$, we have $X^n(\lfloor n(1-\epsilon) \rfloor)/\sqrt{n} \xrightarrow{d} X_{1-\epsilon}^a$ as $n \rightarrow \infty$. By Lemma 2.1, $(X^n(\lfloor n(1-\epsilon) \rfloor)/\sqrt{n})$ is a uniformly integrable sequence of random variables, and so we obtain $\mathbb{E}[X^n(\lfloor n(1-\epsilon) \rfloor)/\sqrt{n}] \rightarrow \mathbb{E}[X_{1-\epsilon}^a]$ as $n \rightarrow \infty$. Hence,

$$\limsup_{n \rightarrow \infty} \frac{\mathbb{E} \left[\sup_{\lfloor n(1-\epsilon) \rfloor \leq i \leq n} X^n(i) \right]}{\sqrt{n}} \leq 2a\epsilon^{4/3} + \mathbb{E}[X_{1-\epsilon}^a] + 2\sqrt{\epsilon}.$$

By the final statement of Proposition 5.1, we have $\mathbb{P}(X_{1-\epsilon}^a > r\sqrt{\epsilon}) \leq e^{-(r-a)^2/6}$ for $r > a$ and it follows, in particular, that $\mathbb{E}[X_{1-\epsilon}^a] \leq C\sqrt{\epsilon}$ for some constant $C > 0$.

For $\alpha(n) \gg \sqrt{n}$, we have $X^n(\lfloor n(1-\epsilon) \rfloor)/\alpha(n) \xrightarrow{P} \epsilon^{2/3} - \epsilon^{4/3}$. Again, using the uniform integrability from Lemma 2.1, we get $\mathbb{E}[X^n(\lfloor n(1-\epsilon) \rfloor)]/\alpha(n) \rightarrow \epsilon^{2/3} - \epsilon^{4/3}$ and so we have $\limsup_{n \rightarrow \infty} \left(\mathbb{E} \left[\sup_{\lfloor n(1-\epsilon) \rfloor \leq i \leq n} X^n(i) \right] / \alpha(n) \right) \leq 2\epsilon^{4/3} + \epsilon^{2/3} - \epsilon^{4/3}$. Hence, for any $\alpha(n)$,

$$\lim_{\epsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{\mathbb{E} \left[\sup_{\lfloor n(1-\epsilon) \rfloor \leq i \leq n} X^n(i) \right]}{\alpha(n) \vee \sqrt{n}} = 0 \quad (23)$$

and another application of Markov's inequality gives

$$\lim_{\epsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \mathbb{P} \left(\frac{1}{\alpha(n) \vee \sqrt{n}} \sup_{\lfloor n(1-\epsilon) \rfloor \leq i \leq n} X^n(i) > \delta \right) = 0.$$

(d) Recall that the standard martingale associated with X^n is

$$M_X^n(k) = X^n(k) + \frac{1}{2}V^n(k) - L^n(k) + \sum_{i=0}^{k-1} \frac{2X^n(i) - 2\mathbb{1}_{\{X^n(i)>0\}}}{3U^n(i) + 4V^n(i) + X^n(i) - \mathbb{1}_{\{X^n(i)>0\}}} \quad (24)$$

It follows that

$$\begin{aligned} & \mathbb{E}[X^n(n) - X^n(\lfloor n(1-\epsilon) \rfloor)] \\ &= \mathbb{E}[L^n(n) - L^n(\lfloor n(1-\epsilon) \rfloor)] + \frac{1}{2}\mathbb{E}[V^n(\lfloor n(1-\epsilon) \rfloor) - V^n(n)] \\ & \quad - \mathbb{E} \left[\sum_{i=\lfloor n(1-\epsilon) \rfloor}^{n-1} \frac{2X^n(i) - 2\mathbb{1}_{\{X^n(i)>0\}}}{3U^n(i) + 4V^n(i) + X^n(i) - \mathbb{1}_{\{X^n(i)>0\}}} \right]. \end{aligned}$$

Now, from (4) we have

$$\mathbb{E}[N^n(n) - N^n(\lfloor n(1-\epsilon) \rfloor)] \geq \mathbb{E} \left[\sum_{i=\lfloor n(1-\epsilon) \rfloor}^{n-1} \frac{X^n(i) - \mathbb{1}_{\{X^n(i)>0\}}}{3U^n(i) + 4V^n(i) + X^n(i) - \mathbb{1}_{\{X^n(i)>0\}}} \right].$$

Hence,

$$\mathbb{E}[L^n(n) - L^n(\lfloor n(1-\epsilon) \rfloor)] \leq \mathbb{E}[X^n(n)] + 2\mathbb{E}[N^n(n) - N^n(\lfloor n(1-\epsilon) \rfloor)].$$

Applying Markov's inequality and using (22) and (23), we get that for any $\delta > 0$,

$$\lim_{\epsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \mathbb{P} \left(\frac{1}{\alpha(n) \vee \sqrt{n}} (L^n(n) - L^n(\lfloor n(1-\epsilon) \rfloor)) > \delta \right) = 0.$$

□

Lemma 6.1. *We have $U^n(n) \leq N^n(n) + 3V^n(0)$.*

Proof. At each step we either connect to a node of degree 3 or 4, or there is a clash. Each node of degree 3 is removed after we connect to it, and each node of degree 4 can be visited at most twice before being removed. Thus, during the first n steps, a node of degree 3 is removed at least $n - N^n(n) - 2V^n(0)$ times. Moreover, at most $V^n(0)$ nodes of degree 4 become nodes of degree 3. Hence, $U^n(n) \leq U^n(0) - (n - N^n(n) - 2V^n(0)) + V^n(0) = N^n(n) + 3V^n(0)$. □

We finally turn to the behaviour of our processes on the time-interval $[n, \zeta_n]$.

Proposition 6.2. *As $n \rightarrow \infty$,*

$$(a) \quad \zeta_n/n \xrightarrow{P} 1,$$

$$(b) \quad \frac{1}{\alpha(n) \vee \sqrt{n}} \sup_{n \leq i \leq \zeta_n} X^n(i) \xrightarrow{P} 0,$$

$$(c) \frac{1}{\alpha(n) \vee \sqrt{n}} (N^n(\zeta_n) - N^n(n)) \xrightarrow{P} 0,$$

$$(d) \frac{1}{\alpha(n) \vee \sqrt{n}} (L^n(\zeta_n) - L^n(n)) \xrightarrow{P} 0.$$

Proof. Write $\bar{\alpha}(n) = \alpha(n) \vee \sqrt{n}$.

(a) As in each step we remove at least two half-edges, by the same argument as gave us the bound $\zeta_n \leq 3n + 4\alpha(n)$ in Section 2.2, we have $\zeta_n - n \leq 3U^n(n) + 4V^n(n) \leq 3U^n(n) + 4\alpha(n)$. So Lemma 6.1 implies that

$$\zeta_n - n \leq 3N^n(n) + 13\alpha(n). \quad (25)$$

If $\alpha(n)/\sqrt{n} \rightarrow a$ then $N^n(n)/\sqrt{n} \xrightarrow{d} N_1^a$, and it follows that $\zeta_n/n \xrightarrow{P} 1$ and, indeed, that $(\zeta_n - n)/\sqrt{n}$ is a tight sequence of random variables. If, on the other hand, $\alpha(n) \gg \sqrt{n}$, we have $N^n(n)/\alpha(n) \xrightarrow{P} 1/4$, and again we get $\zeta_n/n \xrightarrow{P} 1$ with $(\zeta_n - n)/\alpha(n)$ a tight sequence of random variables.

(b) By the usual coupling with Y^n , we have

$$\sup_{n \leq i \leq \zeta_n} X^n(i) \leq V^n(n) + \sup_{n \leq i \leq \zeta_n} Y^n(i),$$

where $(Y^n(n+i))_{0 \leq i \leq \zeta_n - n} \stackrel{d}{=} (2 + |X_1^n(n) + Z(i)|)_{0 \leq i \leq \zeta_n - n}$. Fix $\delta > 0$. Then for any $C > 13$, since $\sup_{0 \leq i \leq C\bar{\alpha}(n)} |Z(i)|/\sqrt{\bar{\alpha}(n)}$ is bounded in L^2 , we have

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \mathbb{P} \left(\frac{\sup_{0 \leq i \leq \zeta_n - n} |Z(i)|}{\bar{\alpha}(n)} > \delta/2 \right) \\ & \leq \limsup_{n \rightarrow \infty} \mathbb{P} \left(\frac{\sup_{0 \leq i \leq C\bar{\alpha}(n)} |Z(i)|}{\bar{\alpha}(n)} > \delta/2 \right) + \limsup_{n \rightarrow \infty} \mathbb{P}(\zeta_n - n > C\bar{\alpha}(n)) \\ & \leq \mathbb{P}(N_1 > (C - 13)/3), \end{aligned}$$

where for the last inequality we recall (25). We have $V^n(n)/\bar{\alpha}(n) \xrightarrow{P} 0$ by Proposition 6.1 (a) and $X^n(n)/\bar{\alpha}(n) \xrightarrow{P} 0$, so since $C > 13$ was arbitrary, we obtain

$$\frac{1}{\bar{\alpha}(n)} \sup_{n \leq i \leq \zeta_n} X^n(i) \xrightarrow{P} 0.$$

(c) We have

$$\begin{aligned} & \mathbb{P} \left(\frac{1}{\bar{\alpha}(n)} (N^n(\zeta_n) - N^n(n)) > \delta \right) \\ & \leq \mathbb{P} \left(\frac{1}{\bar{\alpha}(n)} (N^n(\zeta_n) - N^n(n)) > \delta, \frac{N^n(n)}{\bar{\alpha}(n)} \leq \frac{1}{\epsilon} \right) + \mathbb{P} \left(\frac{N^n(n)}{\bar{\alpha}(n)} > \frac{1}{\epsilon} \right) \\ & \leq \frac{1}{\delta \bar{\alpha}(n)} \mathbb{E} \left[(N^n(\zeta_n) - N^n(n)) \mathbb{1}_{\{\frac{N^n(n)}{\bar{\alpha}(n)} \leq \frac{1}{\epsilon}\}} \right] + \epsilon \frac{\mathbb{E}[N^n(n)]}{\bar{\alpha}(n)}. \end{aligned} \quad (26)$$

Let $\eta_n = \inf\{i \geq n : U^n(i) \leq \epsilon \bar{\alpha}(n)\}$. We have that $\zeta_n - \eta_n \leq 3U^n(\eta_n) + 4V^n(n) \leq$

$3\epsilon\bar{\alpha}(n)$ and $\zeta_n - n \leq 3N^n(n)$. Now

$$\begin{aligned}
& \mathbb{E} \left[(N^n(\zeta_n) - N^n(n)) \mathbb{1}_{\{\frac{N^n(n)}{\bar{\alpha}(n)} \leq \frac{1}{\epsilon}\}} \right] \\
& \leq \mathbb{E} \left[\sum_{i=n}^{\zeta_n-1} \frac{3U^n(i) + 2(X^n(i) - \mathbb{1}_{\{X^n(i)>0\}})(3U^n(i) + 4V^n(i) + X^n(i) - 3)}{2(3U^n(i) + 4V^n(i) + X^n(i) - \mathbb{1}_{\{X^n(i)>0\}})(3U^n(i) + X^n(i) - \mathbb{1}_{\{X^n(i)>0\}} - 2)} \right. \\
& \quad \left. \times \mathbb{1}_{\{\frac{N^n(n)}{\bar{\alpha}(n)} \leq \frac{1}{\epsilon}\}} \right] \\
& \leq \mathbb{E} \left[\sum_{i=n}^{\zeta_n-1} \left(\frac{X^n(i) + 1}{3U^n(i)} \wedge 1 \right) \mathbb{1}_{\{\frac{N^n(n)}{\bar{\alpha}(n)} \leq \frac{1}{\epsilon}\}} \right] \\
& \leq \mathbb{E} \left[\sum_{i=n}^{\eta_n} \left(\frac{X^n(i) + 1}{3\epsilon\bar{\alpha}(n)} \wedge 1 \right) \mathbb{1}_{\{\frac{N^n(n)}{\bar{\alpha}(n)} \leq \frac{1}{\epsilon}\}} \right] + 3\epsilon\bar{\alpha}(n) \\
& \leq \frac{1}{\epsilon^2} \mathbb{E} \left[\left(\left(1 + \sup_{n \leq i \leq \zeta_n} X^n(i) \right) \wedge 3\epsilon\bar{\alpha}(n) \right) \mathbb{1}_{\{\frac{N^n(n)}{\bar{\alpha}(n)} \leq \frac{1}{\epsilon}\}} \right] + 3\epsilon\bar{\alpha}(n).
\end{aligned}$$

By the coupling, the expectation is bounded by

$$\mathbb{E} \left[\left(3 + V^n(n) + X^n(n) + \sup_{0 \leq i \leq 3\bar{\alpha}(n)/\epsilon} |Z(i)| \right) \wedge 3\epsilon\bar{\alpha}(n) \right].$$

Now

$$\frac{1}{\bar{\alpha}(n)} \left(3 + V^n(n) + X^n(n) + \sup_{0 \leq i \leq 3\bar{\alpha}(n)/\epsilon} |Z(i)| \right) \wedge 3\epsilon$$

is a bounded random variable which converges to 0 in probability, and so we get

$$\limsup_{n \rightarrow \infty} \frac{1}{\bar{\alpha}(n)} \mathbb{E} \left[(N^n(\zeta_n) - N^n(n)) \mathbb{1}_{\{\frac{N^n(n)}{\bar{\alpha}(n)} \leq \frac{1}{\epsilon}\}} \right] \leq 3\epsilon.$$

Now note that combining Theorem 4.1, (16) and (22), we have that there exists a constant $C > 0$ such that $\sup_{n \geq 1} (\mathbb{E}[N^n(n)] / \bar{\alpha}(n)) < C$ and so using (26) we obtain

$$\limsup_{n \rightarrow \infty} \mathbb{P} \left(\frac{1}{\bar{\alpha}(n)} (N^n(\zeta_n) - N^n(n)) > \delta \right) \leq \frac{3\epsilon}{\delta} + C\epsilon.$$

Since $\epsilon > 0$ was arbitrary, we get $(N^n(\zeta_n) - N^n(n)) / \bar{\alpha}(n) \xrightarrow{P} 0$.

Finally, we observe that using the martingale M_X^n defined in (24) and the optional stopping theorem, we have $0 \leq \mathbb{E} \left[(L^n(\zeta_n) - L^n(n)) \mathbb{1}_{\{\frac{N^n(n)}{\bar{\alpha}(n)} \leq \frac{1}{\epsilon}\}} \right]$ and

$$\begin{aligned}
& \mathbb{E} \left[(L^n(\zeta_n) - L^n(n)) \mathbb{1}_{\{\frac{N^n(n)}{\bar{\alpha}(n)} \leq \frac{1}{\epsilon}\}} \right] \\
& \leq \mathbb{E} \left[(X^n(\zeta_n) - X^n(n)) \mathbb{1}_{\{\frac{N^n(n)}{\bar{\alpha}(n)} \leq \frac{1}{\epsilon}\}} \right] + 2\mathbb{E} \left[(N^n(\zeta_n) - N^n(n)) \mathbb{1}_{\{\frac{N^n(n)}{\bar{\alpha}(n)} \leq \frac{1}{\epsilon}\}} \right] \\
& \leq 2\mathbb{E} \left[(N^n(\zeta_n) - N^n(n)) \mathbb{1}_{\{\frac{N^n(n)}{\bar{\alpha}(n)} \leq \frac{1}{\epsilon}\}} \right],
\end{aligned}$$

since $X^n(\zeta_n) = 0$ and $X^n(n) \geq 0$. So then

$$\begin{aligned} & \mathbb{P} \left(\frac{1}{\bar{\alpha}(n)} (L^n(\zeta_n) - L^n(n)) > \delta \right) \\ & \leq \mathbb{P} \left(\frac{1}{\bar{\alpha}(n)} (L^n(\zeta_n) - L^n(n)) > \delta, \frac{N^n(n)}{\bar{\alpha}(n)} \leq \frac{1}{\epsilon} \right) + \mathbb{P} \left(\frac{N^n(n)}{\bar{\alpha}(n)} > \frac{1}{\epsilon} \right) \\ & \leq \frac{2}{\delta \bar{\alpha}(n)} \mathbb{E} \left[(N^n(\zeta_n) - N^n(n)) \mathbb{1}_{\{\frac{N^n(n)}{\bar{\alpha}(n)} \leq \frac{1}{\epsilon}\}} \right] + \epsilon \frac{\mathbb{E}[N^n(n)]}{\bar{\alpha}(n)}, \end{aligned}$$

and the rest of the argument goes through as before. \square

We now have all the elements needed to prove Theorem 2.1.

Proof of Theorem 2.1. (i) From Theorem 5.1 and Lemma 5.1, for any $\epsilon > 0$ we have

$$\left(\frac{X^n(\lfloor ns \rfloor)}{\sqrt{n}}, \frac{L^n(\lfloor ns \rfloor)}{\sqrt{n}}, \frac{N^n(\lfloor ns \rfloor)}{\sqrt{n}}, 0 \leq s \leq 1 - \epsilon \right) \xrightarrow{d} (X_s, L_s, N_s, 0 \leq s \leq 1 - \epsilon).$$

By Proposition 5.1, we have $(X_{1-\epsilon}, L_{1-\epsilon}, N_{1-\epsilon}) \rightarrow (0, L_1, N_1)$ almost surely as $\epsilon \rightarrow 0$. The principle of accompanying laws (Theorem 3.2 of Billingsley [5]) gives that the statement of Proposition 6.1 is exactly what we need in order to deduce that

$$\left(\frac{X^n(\lfloor ns \rfloor)}{\sqrt{n}}, \frac{L^n(\lfloor ns \rfloor)}{\sqrt{n}}, \frac{N^n(\lfloor ns \rfloor)}{\sqrt{n}}, 0 \leq s \leq 1 \right) \xrightarrow{d} (X_s, L_s, N_s, 0 \leq s \leq 1).$$

Proposition 6.2 then allows us to conclude that

$$\left(\frac{X^n(\lfloor ns \rfloor)}{\sqrt{n}}, \frac{L^n(\lfloor ns \rfloor)}{\sqrt{n}}, \frac{N^n(\lfloor ns \rfloor)}{\sqrt{n}}, 0 \leq s \leq \zeta_n/n \right) \xrightarrow{d} (X_s, L_s, N_s, 0 \leq s \leq 1),$$

as desired.

(ii) From Theorem 4.1, for any $\epsilon > 0$, we have

$$\left(\frac{X^n(\lfloor ns \rfloor)}{\alpha(n)}, \frac{L^n(\lfloor ns \rfloor)}{\alpha(n)}, \frac{N^n(\lfloor ns \rfloor)}{\alpha(n)}, 0 \leq s \leq 1 - \epsilon \right) \xrightarrow{d} (x(s), 0, m(s), 0 \leq s \leq 1 - \epsilon).$$

It is straightforward that $x(1 - \epsilon) \rightarrow 0$ and that $m(1 - \epsilon) \rightarrow 1/4$ as $\epsilon \rightarrow 0$. By another application of the principle of accompanying laws, and Propositions 6.1 and 6.2, we may then conclude that

$$\left(\frac{X^n(\lfloor ns \rfloor)}{\alpha(n)}, \frac{L^n(\lfloor ns \rfloor)}{\alpha(n)}, \frac{N^n(\lfloor ns \rfloor)}{\alpha(n)}, 0 \leq s \leq \zeta_n/n \right) \xrightarrow{d} (x(s), 0, m(s), 0 \leq s \leq 1),$$

as desired. \square

7. Appendix

7.1. Proof of the invariance principle

Proof of Theorem 5.2. Let $\theta^n(r) = \inf\{t \geq 0 : Q^n(t) > \int_0^t \sup_{|y| \leq r} q(s, y) ds + 1\}$. Set $\tilde{M}_r^n := M^n(\cdot \wedge \theta^n(r) \wedge \tau^n(r))$. Relative compactness of $\{\tilde{M}_r^n\}_{n \geq 1}$ follows as in Theorem 4.1 of Chapter 7 of Ethier and Kurtz. Assumptions (c) and (d) imply that

any subsequential weak limit has sample paths in $\mathbb{C}(\mathbb{R}_+, \mathbb{R})$ almost surely. Then all of the conditions in Assumption 4.1 of Kang and Williams [11] hold for the processes stopped at $\tau^n(r)$. It follows by their Theorem 4.2 that the sequence of processes $\{(Y^n, M^n, L^n)\}_{n \geq 1}$ is tight and such that any subsequential weak limit almost surely has continuous sample paths.

Now fix $r_0 > 0$ and let $\{Y^{n_k}(\cdot \wedge \tau^{n_k}(r_0)), M^{n_k}(\cdot \wedge \tau^{n_k}(r_0)), L^{n_k}(\cdot \wedge \tau^{n_k}(r_0))\}_{k \geq 1}$ be a convergent subsequence with limit $Y_{r_0}, M_{r_0}, L_{r_0}$. Let $\tau_{r_0}(r) = \inf\{t \geq 0 : |Y_{r_0}(t)| \geq r\}$. Then for all but countably many $r < r_0$ (i.e. those such that $\mathbb{P}(\lim_{s \rightarrow r} \tau_{r_0}(s) = \tau_{r_0}(r)) = 1$), we have

$$\begin{aligned} & (Y^{n_k}(\cdot \wedge \tau^{n_k}(r)), M^{n_k}(\cdot \wedge \tau^{n_k}(r)), L^{n_k}(\cdot \wedge \tau^{n_k}(r)), \tau^{n_k}(r)) \\ & \xrightarrow{d} (Y_{r_0}(\cdot \wedge \tau_{r_0}(r)), M_{r_0}(\cdot \wedge \tau_{r_0}(r)), L_{r_0}(\cdot \wedge \tau_{r_0}(r)), \tau_{r_0}(r)). \end{aligned}$$

Conditions (f), (c) and (d) guarantee that $M^{n_k}(\cdot \wedge \tau^{n_k}(r_0))$ is a uniformly integrable martingale. Conditions (b), (g) and (e) guarantee that

$$(M^{n_k}(\cdot \wedge \tau^{n_k}(r_0)))^2 - Q^{n_k}(\cdot \wedge \tau^{n_k}(r_0))$$

is a uniformly integrable martingale. Thus, by Problem 7 from Chapter 7 of Ethier and Kurtz, as in the proof of Theorem 1.4(b) there, we must then have that

$$M_{r_0}(t \wedge \tau_{r_0}(r)) = Y_{r_0}(t \wedge \tau_{r_0}(r)) - L_{r_0}(t \wedge \tau_{r_0}(r)) - \int_0^{t \wedge \tau_{r_0}(r)} b(s, Y_{r_0}(s)) ds$$

and $M_{r_0}(t \wedge \tau_{r_0}(r))^2 - \int_0^{t \wedge \tau_{r_0}(r)} q(s, Y_{r_0}(s)) ds$ are continuous martingales. Then, it follows that $M_{r_0}(t \wedge \tau_{r_0}(r)) = \int_0^{t \wedge \tau_{r_0}(r)} \sigma(s, Y_{r_0}(s)) dW(s)$ for some Brownian motion W . But then we have

$$Y_{r_0}(t \wedge \tau_{r_0}(r)) = \int_0^{t \wedge \tau_{r_0}(r)} b(s, Y_{r_0}(s)) ds + \int_0^{t \wedge \tau_{r_0}(r)} \sigma(s, Y_{r_0}(s)) dW(s) + L_{r_0}(t \wedge \tau_{r_0}(r)).$$

Moreover, by (i), $L_{r_0}(0) = 0$, L_{r_0} is non-decreasing, and $\int_0^{t \wedge \tau_{r_0}(r)} \mathbb{1}_{\{Y_{r_0}(s) > 0\}} dL_{r_0}(s) = 0$. So (Y_{r_0}, L_{r_0}) solves a stopped version of the reflected SDE.

But if (Y, L) is the unique solution of the reflected SDE with Brownian motion W then $(Y(\cdot \wedge \tau(r)), L(\cdot \wedge \tau(r)))$ is the unique solution of the stopped version, where $\tau(r) = \inf\{t \geq 0 : |Y(s)| \geq r\}$. In consequence, $(Y^n(\cdot \wedge \tau^n(r)), L^n(\cdot \wedge \tau^n(r))) \xrightarrow{d} (Y(\cdot \wedge \tau(r)), L(\cdot \wedge \tau(r)))$ for any r such that $\mathbb{P}(\lim_{s \rightarrow r} \tau(s) = \tau(r)) = 1$. But $\tau(r) \rightarrow \infty$ as $r \rightarrow \infty$, since Y has continuous sample paths. Hence, $(Y^n, L^n) \xrightarrow{d} (Y, L)$. \square

7.2. A hitting probability calculation

Fix $d \in (0, 1/2)$ and an integer $b \geq 1$. Suppose that A is a random walk which makes steps of size -1 with probability $1/2$, $+1$ with probability $1/2 - d$ and $+2$ with probability d .

Lemma 7.1. *The probability, started from 1, that A hits 0 before $\{b, b+1\}$ is given by*

$$1 - \frac{4d(-1 - \sqrt{1+8d})^b + 4d(-1 + \sqrt{1+8d})^b}{(-2)^{b+1}\sqrt{1+8d} + 4d(-1 - \sqrt{1+8d})^b + 4d(-1 + \sqrt{1+8d})^b - (-1 - \sqrt{1+8d})^{b+1} - (-1 + \sqrt{1+8d})^{b+1}}.$$

Proof. For $0 \leq k \leq b+1$, let $h_k = \mathbb{P}(A \text{ hits } 0 \text{ before } \{b, b+1\} | A(0) = k)$. Then for $1 \leq k \leq b-1$, we have $h_k = \frac{1}{2}h_{k-1} + (\frac{1}{2} - d)h_{k+1} + dh_{k+2}$ and elementary calculations yield that $h_k = \varphi + \beta\lambda_-^k + \gamma\lambda_+^k$ for $0 \leq k \leq b+1$, where $\lambda_{\pm} = \frac{-1 \pm \sqrt{1+8d}}{4d}$. The boundary conditions are $h_0 = 1$, $h_b = h_{b+1} = 0$. Solving for the constants gives

$$\varphi = \frac{\lambda_-^{b+1}\lambda_+^b - \lambda_-^b\lambda_+^{b+1}}{\lambda_-^{b+1}\lambda_+^b - \lambda_-^b\lambda_+^{b+1} - \lambda_-^{b+1} + \lambda_-^b - \lambda_+^{b+1} + \lambda_+^b}, \quad \beta = \frac{\lambda_+^b - \lambda_+^{b+1}}{\lambda_-^{b+1}\lambda_+^b - \lambda_-^b\lambda_+^{b+1} - \lambda_-^{b+1} + \lambda_-^b - \lambda_+^{b+1} + \lambda_+^b},$$

$$\gamma = \frac{\lambda_-^b - \lambda_-^{b+1}}{\lambda_-^{b+1}\lambda_+^b - \lambda_-^b\lambda_+^{b+1} - \lambda_-^{b+1} + \lambda_-^b - \lambda_+^{b+1} + \lambda_+^b},$$

and simplifying the expression for h_1 gives the claimed result. \square

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