

Multiscale Turbulence in Magnetic Confinement Fusion Devices



Michael Hardman
Lady Margaret Hall
University of Oxford

A thesis submitted for the degree of
Doctor of Philosophy in Theoretical Physics

Trinity 2019

Multiscale Turbulence in Magnetic Confinement Fusion Devices

Michael Hardman

Lady Margaret Hall, University of Oxford

Submitted for the degree of Doctor of Philosophy in Theoretical Physics

Trinity 2019

Turbulence in magnetised plasmas is driven by microinstabilities which exist at the spatial scales of the ion and the electron thermal gyroradii, with time scales corresponding to the ion and the electron thermal speeds, respectively. Hence, plasma turbulence can have a well-separated long wavelength “ion scale” (IS) and short wavelength “electron scale” (ES), where the separation is controlled by the electron-to-deuterium mass ratio $(m_e/m_i)^{1/2} \simeq 1/60$. Resolving multiscale plasma turbulence in direct numerical simulations (DNS) is costly due to the small value of $(m_e/m_i)^{1/2}$; nonetheless, several DNS have been performed which show that interactions between the long wavelength and the short wavelength pieces of the turbulence can be important for determining the character of the turbulence. In this thesis we find scale-separated, coupled equations for the IS and the ES pieces of the turbulence, valid in the limit that $(m_e/m_i)^{1/2} \rightarrow 0$. These equations can be efficiently simulated in a system of coupled flux tubes. The equations provide insight into the mechanisms and the impact of the cross-scale interactions between the IS and the ES pieces of the turbulence, whilst avoiding the cost of DNS. We find that to leading order the IS turbulence evolves independently of the ES turbulence. The ES turbulence is sheared by parallel variation in the IS $\mathbf{E} \times \mathbf{B}$ drift, and the drive of instability at the ES is modified by the gradient of the IS distribution function. We study the effect of IS turbulence on the linear stability of the electron temperature gradient (ETG) mode. We find that a strongly driven ETG instability is stabilised by strongly driven IS turbulence, but is only weakly suppressed by IS turbulence driven near marginal stability. We identify the dominant suppression mechanism as parallel-to-the-field shear in the IS $\mathbf{E} \times \mathbf{B}$ drift: the simple nature of this suppression mechanism may explain why short wavelength turbulence saturates at a low level in the presence of strongly driven long wavelength turbulence in DNS.

Acknowledgements

I would like to thank my supervisors Michael Barnes and Colin Roach for their patience, time and support during the last 4 years. In addition, I have benefited from endless discussions with, and the support of, the other members of the Oxford plasma theory group and its like-minded collaborators: In particular, thanks to Felix Parra, Paul Dellar, George Wilkie, Jason Parisi, Valerian Hall-Chen, Plamen Ivanov, Yohei Kawazura; thanks to my examiners Bill Dorland and Alex Scheckochihin; and special thanks to Justin Ball, Alessandro Geraldini, Nicolas Christen, Mantas Abazorius, Ollie Beeke, Marek Strumik and Adnane Osmane for putting up with my dictation software and other bug-related mishaps!

This thesis would have been much more difficult to write without the support of my family, friends and housemates. I am not able to find the words to do them all justice here – thank you all!

This work has been carried out within the framework of the EUROfusion Consortium and has received funding from the Euratom research and training programme 2014-2018 and 2019-2020 under grant agreement No 633053 and from the RCUK Energy Programme [grant number EP/P012450/1]. The views and opinions expressed herein do not necessarily reflect those of the European Commission. The author acknowledges EUROfusion, the EUROfusion High Performance Computer (Marconi-Fusion), the use of ARCHER through the Plasma HEC Consortium EPSRC grant numbers EP/L000237/1 and EP/R029148/1 under the projects e281-gs2, software support from Joseph Parker through the Plasma-CCP Network under EPSRC grant number EP/M022463/1, and support from the Wolfgang Pauli Institute.

Contents

1	Introduction	12
1.1	Introduction to multiscale plasma turbulence	15
1.2	A physical picture of toroidal drift wave instabilities	16
1.2.1	Drift waves	16
1.2.2	Toroidal instabilities	18
1.2.3	The gyrokinetic ordering	20
1.3	Turbulence driven at the scales of the ion and the electron gyroradii . .	21
2	Multiscale Turbulence	26
2.1	The gyrokinetic equation	27
2.2	Separation of scales within the turbulence	34
2.3	Coupled gyrokinetic equations	38
2.4	Quasineutrality	44
2.5	Sizes of the IS and ES fluctuations and scale separation	44
2.5.1	Influence of ES fluctuations on ions at IS	47
2.5.2	Influence of ES fluctuations on electrons at IS	47
2.5.3	Influence of IS gradients on ES fluctuations	47
2.5.4	Ions at ES	48
2.5.5	Critical balance	49
2.6	Scale-separated, coupled equations	50
2.6.1	Fourier representation	52
2.7	The parallel boundary condition	53
2.7.1	The standard “twist-and-shift” parallel boundary condition . . .	54
2.7.2	The IS and ES “twist-and-shift” parallel boundary conditions .	56

2.7.3	The flux tube boundary conditions in Fourier space	57
2.7.4	The parallel coupling between ES flux tubes embedded in IS turbulence	58
2.7.5	Continuity of the cross-scale terms across the connections be- tween ES flux tubes	59
2.8	Boosting away the piece of $\overline{\mathbf{v}}^E$ that is constant in θ	61
2.9	IS and ES collision operators	62
2.9.1	The forms of the collision operators	64
2.9.2	Scale separation in the presence of collisions	66
2.9.3	Fourier representation for collisions	67
3	The Linear Stability of ES Microinstabilities in the Presence of IS Turbulence	68
3.1	Equations for IS and ES turbulence convenient for numerical implemen- tation	69
3.2	Separating the wave number spectrum into IS and ES components . . .	73
3.3	Simulating IS turbulence and computing a sample of IS gradients and drifts	77
3.4	Linear simulations of the ETG instability at ES	81
3.4.1	The effect of marginally unstable IS turbulence on the ETG in- stability	82
3.4.2	The effect of strongly driven IS turbulence on the ETG instability	83
3.4.3	Identifying the mechanism responsible for the suppression of the ETG instability	84
3.5	A simple model for the effect of the parallel-to-the-field shear in the IS $\mathbf{E} \times \mathbf{B}$ drift	87
4	Discussion	91
4.1	Scale-separated, coupled IS and ES turbulence	92
4.2	Relative sizes of the drives of instability at IS and ES	96
4.3	Suppression of the ETG instability by cross-scale interaction with IS turbulence	100

4.4	Limitations of the scale-separated model of multiscale turbulence and avenues for future research	102
-----	---	-----

A		107
----------	--	------------

A.1	The velocity space and gyroaverages	107
A.2	Useful properties of the ES average	108
A.3	Electrons at IS and orbital averaging	109
A.3.1	Proving the property (2.30) of the orbital average	109
A.3.2	Showing that $\bar{h}_e^{(0)} = 0$ for nonzonal passing electrons	110
A.3.3	Further properties of the orbital average	111
A.4	Obtaining the cross-scale terms	112
A.4.1	IS cross-scale terms	112
A.4.2	ES cross-scale terms	113
A.5	The sizes of the IS and ES turbulent fluctuations	114
A.5.1	ES turbulence enhanced by cross-scale interaction: $e\tilde{\phi}/T \gg \rho_{th,e}/a$	117
A.5.2	IS turbulence enhanced by cross-scale interaction: $e\bar{\phi}/T \gg \rho_{th,i}/a$	118
A.5.3	IS turbulence suppresses ES fluctuations: $e\bar{\phi}/T \sim \rho_{th,i}/a$ and $e\tilde{\phi}/T \ll \rho_{th,e}/a$	119
A.5.4	ES turbulence suppresses IS fluctuations: $e\bar{\phi}/T \ll \rho_{th,i}/a$ and $e\tilde{\phi}/T \sim \rho_{th,e}/a$	120
A.6	Obtaining the scale-separated ES quasineutrality relation	122
A.7	IS and ES heat fluxes	123
A.8	The “twist-and-shift” parallel boundary condition	125
A.9	The ES “twist-and-shift” parallel boundary condition	126
A.10	Satisfaction of relations (2.98) and (2.99)	127

B	128
B.1 Tests of the implementation of the cross-scale terms in GS2	128
B.2 Parallel-to-the-field and velocity space grids and resolutions	129
B.3 The model for electrons in IS simulations in GS2	130
B.4 Resolving IS turbulence	134
References	142

List of Figures

1.1	Diagram illustrating axisymmetric toroidal geometry.	13
1.2	Diagram illustrating the formation of drift waves.	17
1.3	Diagram illustrating the formation of toroidal instabilities.	18
1.4	Diagram illustrating how ES flux tubes (blue) may be nested within a larger IS flux tube (red).	24
2.1	Diagram illustrating magnetic-field-line-following coordinates.	28
2.2	A) shows a view, down the toroidal symmetry axis (labelled by Z) of a flux tube, in yellow, with parallel ends in magenta. The flux surface is indicated in grey. B) shows the same flux tube as viewed from the right of A) perpendicular to the toroidal symmetry axis.	55
2.3	The diagram shows the $\theta = \pi$ (left) and $\theta = -\pi$ (right) ends of the IS flux tube, represented by solid-lined parallelograms on the inboard mid-plane, which we parameterise with (ζ, ψ)	60
3.1	The linear growth rate γ calculated in GS2 in the absence of cross scale coupling, for equilibrium electron temperature gradient $a/L_{T_e} = 2.3$; and equilibrium ion temperature gradients of $a/L_{T_i} = 2.3$ and $a/L_{T_i} = 1.38$, and for the adiabatic ion (AI) model. Note that the modes with $\hat{\theta}_0 = 0$ are the most unstable modes. We observe that a/L_{T_i} has little effect on the ES ($k_y \rho_{th,i} \sim 60$) modes. (Left) shows γ for the value of the collision frequency used throughout this Chapter, $a\nu_{\nu\nu}/v_{th,\nu} = 10^{-2}$. (Right) shows γ for a collision frequency of $a\nu_{\nu\nu}/v_{th,\nu} = 10^{-4}$	75

3.2	(Left) The square of the IS electrostatic potential, averaged over the flux tube. The vertical dashed lines indicate the sampled times t_s for the $a/L_{Ti} = 2.3$ case. (Right) Contours of the IS electrostatic potential in the perpendicular-to-the-field plane. The sampled (x_s, y_s) positions are indicated by crosses. The electrostatic potential shown is for the last time sample from the left Figure, at the outboard midplane where $\theta = 0$.	79
3.3	(Left) The square of the IS electrostatic potential for the $a/L_{Ti} = 1.38$ case, averaged over the flux tube. The vertical dashed lines indicate the sampled times t_s . (Right) Contours of the IS electrostatic potential in the perpendicular-to-the-field plane. The sampled (x_s, y_s) positions are indicated by crosses. The electrostatic potential shown is for the last time sample from the left Figure, at the outboard midplane where $\theta = 0$.	80
3.4	The ES linear growth rate γ calculated using equations (3.11) and (3.14) with $\nabla_s \bar{g}_e = \bar{\mathbf{v}}_e^E = 0$.	81
3.5	ES linear growth rates γ calculated with $\nabla_s \bar{g}_e$ and $\bar{\mathbf{v}}_e^E$ sampled from the IS turbulence with $a/L_{Ti} = 1.38$, at distinct (x_s, y_s) points at a single t_s .	82
3.6	The average ES linear growth rate $\langle \gamma \rangle_{t_s, x_s, y_s}$ (Left) calculated with $\nabla_s \bar{g}_e$ and $\bar{\mathbf{v}}_e^E$ sampled from the IS turbulence with $a/L_{Ti} = 1.38$, and the standard error $\sigma_{\langle \gamma \rangle_{t_s, x_s, y_s}}$ on $\langle \gamma \rangle_{t_s, x_s, y_s}$ (Right).	83
3.7	The ES linear growth rates γ calculated with $\nabla_s \bar{g}_e$ and $\bar{\mathbf{v}}_e^E$ sampled from distinct (t_s, x_s, y_s) points within the IS turbulence driven at $a/L_{Ti} = 2.3$.	83
3.8	The average ES linear growth rate $\langle \gamma \rangle_{t_s, x_s, y_s}$ (Left) calculated with $\nabla_s \bar{g}_e$ and $\bar{\mathbf{v}}_e^E$ sampled from the IS turbulence with $a/L_{Ti} = 2.3$, and the standard error $\sigma_{\langle \gamma \rangle_{t_s, x_s, y_s}}$ on $\langle \gamma \rangle_{t_s, x_s, y_s}$ (Right).	84
3.9	The average ETG growth rate $\langle \gamma \rangle_{t_s, x_s, y_s}$ (Left) calculated with $\nabla_s \bar{g}_e$ sampled from the IS turbulence with $a/L_{Ti} = 2.3$ and $\bar{\mathbf{v}}_e^E = 0$; and the standard error $\sigma_{\langle \gamma \rangle_{t_s, x_s, y_s}}$ on $\langle \gamma \rangle_{t_s, x_s, y_s}$ (Right).	85
3.10	The average ETG growth rate $\langle \gamma \rangle_{t_s, x_s, y_s}$ (Left) calculated with $\bar{\mathbf{v}}_e^E$ sampled from the IS turbulence with $a/L_{Ti} = 2.3$ and $\nabla_s \bar{g}_e = 0$; and the standard error $\sigma_{\langle \gamma \rangle_{t_s, x_s, y_s}}$ on $\langle \gamma \rangle_{t_s, x_s, y_s}$ (Right).	85

3.11	ES linear growth rates γ calculated with $\nabla_s \bar{g}_e = 0$ and $\bar{\mathbf{v}}_e^E$ prescribed by (3.23), for $\hat{E} = 0.5 T/ea$ (Left) and $\hat{E} = T/ea$ (Right).	88
3.12	(Left) The ES linear growth rate γ calculated for $K_y \rho_{th,e} = 0.57$ and $\hat{\theta}_0 = 0$ using equations (3.11) and (3.14) with $\nabla_s \bar{g}_e = 0$ and $\bar{\mathbf{v}}_e^E \cdot \mathbf{k}_f$ as prescribed in equation (3.24). (Right) The maximum ES linear growth rate $\gamma^{\max}$ across all $K_y \rho_{th,e}$ and $\hat{\theta}_0$ as a function of $\hat{E}$	89
3.13	(Left) The ballooning eigenmode of the ETG instability, for $\hat{E}/(T/ea) = 0.0, 0.25$, and 0.5 , calculated using equations (3.11) and (3.14) with $\nabla_s \bar{g}_e = 0$ and $\bar{\mathbf{v}}_e^E \cdot \mathbf{k}_f$ as prescribed in equation (3.24). (Right) The coefficients $\omega_M = \mathbf{v}_e^M \cdot \mathbf{k}_f$ and $\bar{\omega}_E = \bar{\mathbf{v}}_e^E \cdot \mathbf{k}_f$ from equation (3.11) as a function of the extended ballooning coordinate for the calculation of the ETG ballooning eigenmodes of Figure 3.13. We do not show $\bar{\omega}_E$ in the case of $\hat{E}/(T/ea) = 0.0$ as $\bar{\omega}_E = 0$ in this case.	90
B.1	(Left) Comparing the ETG growth rate γ , and (Right) the ETG eigenmode for $K_y \rho_{th,e} = 0.57$, calculated in tests of the cross-scale terms. . .	128
B.2	The modulus of the ITG potential eigenmode $ \bar{\phi} $ calculated using (3.2)-(3.8) for $k_y \rho_{th,i} = 0.6$ and $\theta_0 = 0.0$, the fastest growing IS mode. The eigenmode is normalised to the peak value, which is at $\theta = 0$. We show the eigenmode for different electron models: for the model of the electron response at IS with $(m_e/m_i)^{1/2} = 1/60$ (ME); for the fully kinetic electron model with $(m_e/m_i)^{1/2} = 1/60$ (KE); and for the fully kinetic electron model with $(m_e/m_i)^{1/2} = (1/60) \times 10^{-3}$ (KE – RM). . .	132
B.3	The modulus of the ITG nonadiabatic density eigenmodes corresponding to the potential eigenmode $ \bar{\phi} $ in Figure B.2. We show the fluctuating nonadiabatic densities due to electrons $\bar{\delta n}'_e$ (Left) and ions $\bar{\delta n}'_i$ (Right), normalised to the corresponding peak in $e \bar{\phi} /T$, and the equilibrium density n_ν	133

B.4	The IS linear growth rate γ calculated using different models of the electron response at IS: firstly, the model used in this thesis, where the nonzonal passing electrons respond adiabatically and $(m_e/m_i)^{1/2} = 1/60$ (ME); the fully kinetic electron model with $(m_e/m_i)^{1/2} = 1/60$ (KE); and the fully kinetic electron model with $(m_e/m_i)^{1/2} = (1/60) \times 10^{-3}$ (KE – RM). (Left) The IS linear growth rate γ for CBC parameters. (Right) The IS linear growth rate for low electron collisionality, with all other parameters taking CBC values.	134
B.5	The modulus of the ITG, potential eigenmode calculated using (3.2)-(3.8) for $k_y \rho_{th,i} = 0.6$ and $\theta_0 = 0.0$, the fastest growing IS mode. The eigenmode is normalised to the peak value, which is at $\theta = 0$. Note that the eigenmode extends beyond $\theta = \pm\pi$, and so multiple k_x cells are required to resolve the mode.	136
B.6	The flux tube averaged turbulent IS heat flux calculated for CBC parameters in a simulation with $k_x^{\max} \rho_{th,i} = 3.17$ (Left) and $k_x^{\max} \rho_{th,i} = 6.3$ (Right). The simulation parameters are otherwise identical. Note the differing t_s axes between the left and right figures.	139
B.7	A snapshot of the modulus of the electrostatic potential $\bar{\phi}$, for $k_y \rho_{th,i} = 0.8$ and $\hat{\theta}_0 = 0.0$, from a nonlinear IS simulation with CBC parameters and $k_x^{\max} \rho_{th,i} = 3.17$ (Left) and $k_x^{\max} \rho_{th,i} = 6.38$ (Right). The simulation parameters are otherwise identical. Agreement between the left and right figure is not expected as in each instance $ \bar{\phi} $ is calculated from a different snapshot of a different simulation of turbulence.	140
B.8	Instances of the cross-scale drift frequency $\bar{\omega}_E = \mathbf{k}_f \cdot \bar{\mathbf{v}}_e^E$ as a function of the extended ballooning coordinate, for cases where $\bar{\mathbf{v}}_e^E$ is calculated from an IS simulation with $k_x^{\max} \rho_{th,i} = 3.17$ (Left) and $k_x^{\max} \rho_{th,i} = 6.38$ (Right). Note that $\omega_M = \mathbf{k}_f \cdot \mathbf{v}_e^M$ and note that we show the data in the electron normalised units. We show the $\bar{\omega}_E$ and ω_M coefficients for each K_x in the connected chain of ES modes separately to illustrate the numerical discontinuity which we resolve. We note that the mode with $K_y \rho_{th,e} = 0.57$ and $\hat{\theta}_0 = 0$ is the fastest growing ES mode.	141

Chapter 1

Introduction

A magnetic confinement fusion device uses a magnetic field to confine a hot hydrogenic plasma at densities and temperatures large enough so that a significant fraction of the nucleons undergo the process of nuclear fusion. For a pair of nucleons to fuse, they must be sufficiently close to one another so that the attractive strong nuclear force dominates their interaction. This requires kinetic energies large enough to overcome their mutual electrostatic Coulomb repulsion. The required kinetic energy is very large: A thermal hydrogenic gas undergoing fusion must have a very high temperature. A hydrogenic gas is ionised to form a plasma at temperatures much lower than the required temperature for fusion. Using the material walls of a vacuum vessel alone to confine a fusing, thermal hydrogenic plasma is not possible due to the cooling effect of the material walls on the plasma. Fortunately, a plasma may be insulated from the material wall of the vacuum vessel by a magnetic field: this provides one possible method for the confinement of a relatively pure, thermal hydrogenic population which undergoes fusion.

Magnetic confinement fusion devices operate on the following principle: As a result of the Lorentz force, charged particles execute helical orbits which follow magnetic field lines. This confines charged particles in the direction perpendicular to the magnetic field and drastically reduces the conduction of heat to the material wall in the directions perpendicular to the magnetic field. However, charged particles are still able to rapidly conduct heat along magnetic field lines. The leading concepts for magnetic confinement fusion devices, the tokamak and the stellarator [1], employ a confining magnetic field with toroidal geometry to reduce parallel-to-the-field transport to the

wall to acceptable levels. In a toroidal magnetic field geometry the majority of magnetic field lines are constrained to nested 2D “flux surfaces”: consequently, charged particles cannot rapidly transport heat to the wall by following the parallel-to-the-field direction. In Figure 1.1 we show a cutaway of an axisymmetric flux surface. We indicate the coordinate in the axial direction Z , the coordinate in the major radial direction R and the toroidal and poloidal angles, ζ and θ , respectively. The magnetic axis, where the magnetic field is purely toroidal, is indicated with dashes, and the solid lines indicate a segment of the magnetic field line which maps out the flux surface, with the assumed direction of $\mathbf{B}$ indicated with arrows. The flux surface label ψ , the poloidal flux, measures the minor radial distance from the flux surface to the magnetic axis.

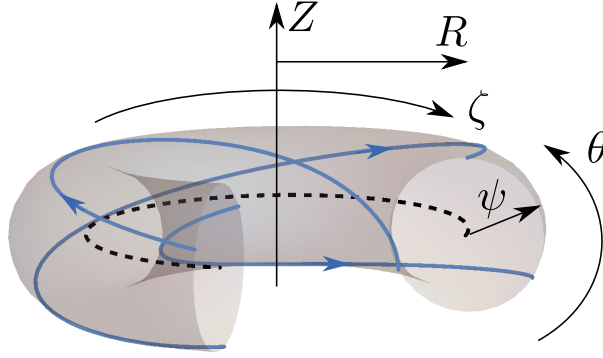


Figure 1.1: Diagram illustrating axisymmetric toroidal geometry.

Ultimately, magnetic confinement fusion devices will be designed to extract energy from the fusion reaction between the hydrogenic isotopes deuterium (D) and tritium (T). These devices aim to achieve steady-state “burning” of D-T with densities of order $2 \times 10^{20} \text{ m}^{-3}$, and temperatures of order 20 keV (in units of Kelvin (K), this means temperatures of order 10^8 K) [1–3]. Whilst terrestrial fusion has been achieved in magnetic confinement fusion devices, see e.g. [3–5], there are many physics problems that must be solved and engineering challenges that must be overcome before magnetic confinement fusion can produce energy for consumption [3, 6]. This thesis will focus on one of those physics problems: multiscale plasma turbulence in tokamaks.

Anomalous transport of heat and particles is a major limiting factor in the performance of tokamaks. In reactor conditions the dominant mechanism for anomalous transport in axisymmetric toroidal magnetic fields is plasma turbulence [6]. Plasma turbulence is characterised by chaotic collective motions within the plasma. These chaotic motions share many features in common with neutral fluid turbulence, but have access to more complicated dynamics due to the long-range electromagnetic forces present in the plasma. In addition, plasma dynamics can be strongly influenced by kinetic physics, which becomes important when the mean free path between particle collisions is large compared to the device size. As the mean plasma profiles are in a nonequilibrium state, the strong gradients of the mean profiles of the plasma density, mean velocity, and temperature act as sources of free energy to drive microinstabilities. Microinstabilities typically have time scales of evolution much shorter than the mean profiles, and have spatial scales much smaller than the scale of the device. In magnetic confinement fusion devices, plasma turbulence is driven by these microinstabilities. The form of the plasma profiles and the magnetic geometry determines which microinstabilities are driven unstable; the microinstabilities grow in amplitude until they saturate nonlinearly, and such that they carry significant particle, momentum, and heat flux. Particle, momentum, and heat fluxes then act to relax the underlying gradients in the plasma profiles. Therefore, a cornerstone to understanding the path to an operating regime for magnetic confinement fusion devices is a clear understanding of plasma turbulence.

The novel contribution of this thesis is an asymptotic study of multiscale plasma turbulence. We derive a novel set of equations which describes turbulence driven at asymptotically-separated space and time scales, and use these equations to investigate the cross-scale interactions between the scales. The remainder of this thesis is organised as follows. The following Sections in this Chapter introduce the field of multiscale turbulence. The introduction concludes with an executive summary of the major results of the thesis. In Chapter 2 we derive the model of multiscale plasma turbulence, and in Chapter 3 we use this model to investigate the effects of large ion-gyroradius-scale turbulent structures on microinstabilities driven at the much smaller scale of the electron gyroradius. Finally, Chapter 4 summarises and discusses the results of this thesis, and provides possible questions for future research.

The content of this thesis, including figures, is based on the contents of the papers Hardman et al. [7] and Hardman et al. [8].

1.1 Introduction to multiscale plasma turbulence

The mean plasma profiles typically have characteristic length scales of order the size of the device a in all directions. The characteristic turbulent wavenumbers perpendicular and parallel to the magnetic field $k_{\perp}$ and $k_{\parallel}$ typically satisfy $k_{\perp}\rho_{\text{th}} \sim 1$ and $k_{\parallel}a \sim 1$ respectively, where ρ_{th} is the thermal gyroradius of a particular particle species. In projected reactor conditions $\rho_{\text{th}}/a \ll 1$ due to the strong confining magnetic field. Consequently, one can assume a separation of spatial scales between the plasma equilibrium and the fluctuations in the plane perpendicular to the magnetic field line. In addition, the equilibrium profiles typically evolve much more slowly than the turbulence, which fluctuates at a characteristic frequency $\omega \sim v_{\text{th}}/a$, where v_{th} the thermal speed. Therefore, one can assume scale separation in time. Using the assumptions of scale separation in space and in time it is possible to derive separate but coupled evolution equations for the plasma profiles and the turbulent fluctuations [9–15].

Whilst the turbulence is often scale separated from the profiles, the turbulence itself contains subsidiary scales due to the presence of multiple species, labelled by ν , which introduces multiple thermal speeds $v_{\text{th},\nu} = \sqrt{2T_{\nu}/m_{\nu}}$ and multiple thermal gyroradii $\rho_{\text{th},\nu} = v_{\text{th},\nu}/\Omega_{\nu}$, where T_{ν} is the species temperature, m_{ν} is the species mass, and $\Omega_{\nu} = Z_{\nu}eB/m_{\nu}c$ is the species cyclotron frequency, where Z_{ν} is the species charge number, e is the proton charge, B is the magnetic field strength and c is the speed of light. Ions and electrons have vastly different masses, $m_{\text{i}} \gg m_{\text{e}}$. In the core of a magnetic confinement fusion device ions and electrons typically have temperatures of the same order $T_{\text{i}} \sim T_{\text{e}} \sim T$. As a consequence, distinct microinstabilities exist at the ion scale (IS), where $\omega \sim v_{\text{th},\text{i}}/a$ and $k_{\perp}\rho_{\text{th},\text{i}} \sim 1$, and at the electron scale (ES), where $\omega \sim v_{\text{th},\text{e}}/a$ and $k_{\perp}\rho_{\text{th},\text{e}} \sim 1$, due to the differing dynamics at each scale. The turbulence arising from the microinstabilities can thus be scale separated and have a multiscale character. This thesis will focus on the case where the dominant instability at the IS is the toroidal ion temperature gradient (ITG) instability [16, 17], and the dominant

instability at the ES is the toroidal electron temperature gradient (ETG) instability [18, 19]. To help develop intuition for the ITG and ETG microinstabilities which drive multiscale turbulence, in the following Section we give a basic physical picture for toroidally driven microinstabilities, and we motivate the gyrokinetic ordering used throughout this thesis. Following this, in Section 1.3 we continue the review of the physics of magnetised plasma turbulence driven at the scales of the ion gyroradius and the electron gyroradius.

1.2 A physical picture of toroidal drift wave instabilities

In this Section we give a physical picture of the dominant microinstabilities which drive transport in the core of tokamak plasmas. To simplify the discussion we consider the dominant microinstabilities in a simplified slab geometry. We use the coordinates x and y to represent the radial, and binormal directions, respectively. We take the magnetic field as $\mathbf{B} = B(x)\hat{\mathbf{y}} \times \hat{\mathbf{x}}$, with $\hat{\mathbf{y}}$ and $\hat{\mathbf{x}}$ the orthonormal unit vectors in the y and x directions, respectively. We assume that there are equilibrium, radial profiles of density $n(x)$ and temperature $T(x)$. This geometry represents the midplane of an idealised “cylindrical” tokamak with straight field lines. The relative size of the gradient scale lengths $L_n = -(\partial \ln n / \partial x)^{-1}$, $L_T = -(\partial \ln T / \partial x)^{-1}$, and $L_B = -(\partial \ln B / \partial x)^{-1}$ determine the behaviour and stability of linear perturbations to the equilibrium.

1.2.1 Drift waves

In Figure 1.2 we give a physical picture which illustrates the formation of stable drift waves. We neglect the gradients of the magnetic field and temperature. We consider the case where the electron species responds adiabatically by rapidly streaming down magnetic field lines faster than the evolution of the wave; i.e. where the fluctuating plasma density $\delta n = ne\phi/T$, with ϕ the fluctuating electrostatic potential. This means that we are implicitly looking at IS waves. We take the plasma profile to have a density gradient in the radial direction such that $\partial n / \partial x < 0$. In addition, note that a

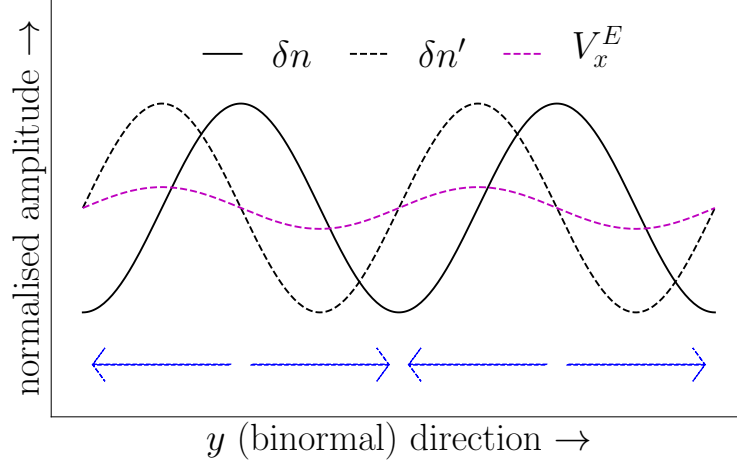


Figure 1.2: Diagram illustrating the formation of drift waves.

fluctuating electrostatic potential gives rise to an $\mathbf{E} \times \mathbf{B}$ drift in the radial direction

$$V_x^E = \frac{c}{B} \mathbf{b} \times \nabla \phi \cdot \hat{\mathbf{x}} = \frac{c}{B} \frac{\partial \phi}{\partial y} = \frac{cT}{eBn} \frac{\partial \delta n}{\partial y}. \quad (1.1)$$

An initial density perturbation δn with fluctuations of wavenumber k_y in the y direction, indicated by the black solid line, will generate an electric field, with directions indicated by the arrows in blue. This electric field creates a radial $\mathbf{E} \times \mathbf{B}$ drift V_x^E . Where V_x^E is positive (negative) the drift draws a larger (smaller) density from the core of the plasma as $\partial n / \partial x < 0$. This reinforces the fluctuation such that it moves to the left to become $\delta n'$, indicated by the black dashed line. This process continues indefinitely without instability. The result is a drift wave with a frequency, ω , which can be calculated using the following model equation for δn ($\propto \exp[ik_y y - i\omega t]$)

$$\frac{\partial \delta n}{\partial t} + V_x^E \frac{\partial \delta n}{\partial x} = 0. \quad (1.2)$$

Equation (1.2) can be obtained from the density moment of the linear gyrokinetic equation, in the collisional limit, neglecting parallel dynamics, in the ordering $L_n \ll L_B \sim L_T$. Using equations (1.1) and (1.2), we find that

$$\omega = \omega_*^n = -\frac{cT k_y}{eB L_n}. \quad (1.3)$$

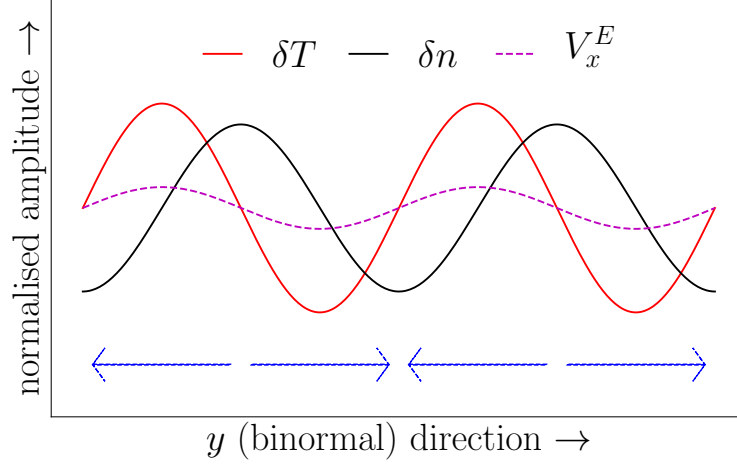


Figure 1.3: Diagram illustrating the formation of toroidal instabilities.

1.2.2 Toroidal instabilities

We now illustrate how the combination of equilibrium gradients of the magnetic field strength and temperature can lead to an instability. In Figure 1.3 we show the physical picture for the formation of the toroidal ion temperature gradient (ITG) instability. We again consider the case where electrons respond adiabatically. Consider the ion magnetic drift $\mathbf{v}_i^M$ in the y direction in the idealised slab geometry, where magnetic curvature is neglected,

$$V_y^M = \mathbf{v}_i^M \cdot \hat{\mathbf{y}} = \frac{v_\perp^2}{2\Omega_i B} \mathbf{b} \times \nabla B \cdot \hat{\mathbf{y}} = -\frac{v_\perp^2}{2\Omega_i B} \frac{\partial B}{\partial x}, \quad (1.4)$$

where $v_\perp$ is the magnitude of the particle velocity in the perpendicular-to-the-field direction. We consider the case where the magnetic field strength and equilibrium temperature decrease with increasing radius, i.e. $\partial B/\partial x < 0$ and $\partial T/\partial x < 0$. We neglect the equilibrium density gradient. Particles which are more energetic have a larger magnetic drift; note the velocity-dependence in (1.4). Hence, an initial ion temperature perturbation δT causes ions within regions of higher (lower) temperature to drift more rapidly (slowly) to the right in Figure 1.3. The result is a perturbation to the density δn . The density perturbation generates an electric field, indicated in blue in Figure 1.3. Finally, the electric field sets up a radial $\mathbf{E} \times \mathbf{B}$ drift $V_x^E > 0$. The

$\mathbf{E} \times \mathbf{B}$ drift reinforces the initial temperature perturbation by drawing hot plasma into the region where $\delta T > 0$ and cold plasma into the region where $\delta T < 0$. Hence, there exists an instability driven by the combination of the magnetic drift and temperature gradient. Note that had we considered the case where $V_y^M < 0$ then we would have found no instability. As a result, we can conclude that toroidally driven instabilities are unstable at the outboard midplane, where $(\partial B/\partial x)(\partial T/\partial x) > 0$; but are stable at the inboard, where $(\partial B/\partial x)(\partial T/\partial x) < 0$.

We can estimate the size of the instability growth rate γ with the following model fluid equations for δn and δT ($\propto \exp[ik_y y - i\omega t]$)

$$\frac{\partial}{\partial t} \frac{\delta T}{T} + V_x^E \frac{1}{T} \frac{\partial T}{\partial x} = 0, \quad (1.5)$$

and

$$\frac{\partial}{\partial t} \frac{\delta n}{n} + V_y^{M,\text{th}} \frac{\partial}{\partial y} \frac{\delta T}{T} = 0, \quad (1.6)$$

where $V_y^{M,\text{th}} = -(v_{\text{th},i}^2/2\Omega_i B)(\partial B/\partial x) = Tc/eBL_B$ is the magnetic drift of an ion gyrating with a speed $v_\perp = v_{\text{th},i}$. Equations (1.5) and (1.6) can be obtained from moments of the gyrokinetic equation, in the collisional limit, neglecting parallel dynamics and ordering $L_T \ll L_B \ll L_n$. Using equations (1.1), (1.5) and (1.6), we find the frequency ω of the instability

$$\omega^2 = -\omega_*^T \omega_*^B, \quad (1.7)$$

where $\omega_*^T = (L_n/L_T)\omega_*^n$, and $\omega_*^B = (L_n/L_B)\omega_*^n$. Equation (1.7) shows that the toroidal instability has an imaginary frequency and hence grows at a rate $\gamma = |\omega_*^n| \sqrt{L_n^2/L_T L_B}$.

We note that

$$\omega_*^n = -\frac{k_y T c}{e B L_n} \sim k_y v_{\text{th},i} \frac{\rho_{\text{th},i}}{a}. \quad (1.8)$$

Hence, in the fluid limit considered here, drift waves and instabilities have frequencies ω and growth rates γ which scale as

$$\omega \sim \gamma \sim k_y \rho_{\text{th},i} \frac{v_{\text{th},i}}{a}. \quad (1.9)$$

In fact we can argue that the maximum instability growth rate occurs where $k_y \rho_{\text{th},i} \sim 1$. In the physical picture presented here we have considered the long-wavelength limit, assuming that $k_y \rho_{\text{th},i} \ll 1$; i.e., the particle gyroradius is much smaller than the wavelength of the perturbation. The growth rate of the fluid instability indicates that modes with larger k_y are more unstable. In the case that $k_y \rho_{\text{th},i} \gg 1$ then the ion gyroradius is much larger than the wavelength of the perturbations. As the cyclotron frequency of the ions Ω_i is much larger than the mode frequency ω , the rapidly orbiting ions can only respond to an average of the density and temperature perturbations. Hence, when $k_y \rho_{\text{th},i} \gg 1$ the ions respond weakly and no drift wave or instability can form. This allows us to conclude that a general ordering for the perpendicular wave number $k_\perp$ of ion-driven drift wave instabilities is $k_\perp \rho_{\text{th},i} \sim 1$. We could have considered ES drift waves, where the ions respond adiabatically, and the electrons provide the drive of instability. By making analogous arguments to those in this section we would have found the electron temperature gradient (ETG) instability for which $k_\perp \rho_{\text{th},e} \sim 1$. Hence, in general it is possible to order the perpendicular wave number $k_\perp$ for drift waves as $k_\perp \rho_{\text{th}} \sim 1$, and the frequencies and growth rates as $\omega \sim \gamma \sim k_\perp \rho_{\text{th}} v_{\text{th}}/a \sim v_{\text{th}}/a$. Finally, we argue that the parallel length scale $l_\parallel$ of the turbulence is set by the connection length $qR \sim a$, where q is the safety factor. The connection length measures the distance along the field line for one poloidal transit of the tokamak. As toroidal instabilities are stable at the inboard of the tokamak, but unstable at the outboard, we expect regions of turbulence to develop centred around the outboard of the tokamak. The characteristic parallel-to-the-field scale of the turbulence is thus expected to be $l_\parallel \sim qR \sim a$.

1.2.3 The gyrokinetic ordering

In this Section we connect the physical pictures described in the last two Sections with the arguments which motivate the gyrokinetic ordering. By definition, in a magnetised plasma we must have $\rho^* = \rho_{\text{th}}/a \ll 1$. Consequently, the cyclotron frequency is much faster than the thermal streaming speed frequency $\Omega \sim v_{\text{th}}/\rho_{\text{th}} \sim \rho^{*-1} v_{\text{th}}/a \gg v_{\text{th}}/a$. Turbulence is driven by microinstabilities. In Sections 1.2.1 and 1.2.2 we give physical pictures which show that the frequencies and growth rates of these instabilities scale

as $\omega \sim \gamma \sim v_{\text{th}}/a$; and the perpendicular wave number scales as $k_{\perp}\rho_{\text{th}} \sim 1$, whilst the parallel scale length is $l_{\parallel} \sim a$. The mean radial profiles are assumed to have scale lengths of order a , with the evolution of these profiles in time slow compared to the turbulence: We estimate the time scale of the profile evolution τ_{P} with a random-walk argument. We assume that the effect of turbulent transport is to diffusively relax gradients in the profiles. For example, the temperature profile as a function of minor radius $T(r)$ heuristically would satisfy,

$$\frac{\partial T}{\partial t} \sim D_{\text{turb}} \frac{\partial^2 T}{\partial r^2}, \quad (1.10)$$

where D_{turb} is the turbulence diffusion coefficient, with Δr the typical step size made in the turbulent transport of T , and Δt the typical time scale over which the step Δr is taken. As turbulence is driven by microinstabilities, we have that $\Delta r \sim \rho_{\text{th}}$ and $\Delta t \sim v_{\text{th}}/a$. Hence, $D_{\text{turb}} \sim v_{\text{th}}\rho_{\text{th}}^2/a$ and $\tau_{\text{P}}^{-1} \sim \rho^{*2}v_{\text{th}}/a \ll \omega$. The relative amplitude of the turbulent fluctuations compared to the profiles is determined by the spatial size of the fluctuations. Given a background gradient ∇F_0 , a fluctuating eddy of size ρ_{th} can only stir particles to create an amplitude difference of at most $\delta f \sim \rho_{\text{th}}\nabla F_0 \sim (\rho_{\text{th}}/a)F_0$. This argument is often referred to as the ambient gradient argument.

1.3 Turbulence driven at the scales of the ion and the electron gyroradii

As discussed in the previous Sections, the existence of the ITG and the ETG microinstabilities means that plasma turbulence in magnetic confinement fusion devices can be driven with perpendicular-to-the-field scales of order the typical thermal ion and the electron gyroradii. Due to the small value of the mass ratio $(m_e/m_i)^{1/2}$, there can be a separation between the ion and electron gyroradius scales in the turbulence. It is useful to consider the turbulence as being composed of a long wavelength component – the ion scale (IS) – and a short wavelength component – the electron scale (ES).

Until relatively recently, the prevailing view in the magnetic confinement fusion community was that the turbulent transport was predominantly due to the larger wavelength modes in the turbulence (cf. [17, 20, 21]) $k_{\perp}\rho_{\text{th},i} \lesssim 1$; i.e., where ion physics

plays an important role. This has meant that investigations into turbulent transport have mostly been concerned with IS turbulence, studying it in isolation through single-scale simulation and theory. Implicitly, this approach assumes that there are no interactions between fluctuations at the disparate IS and ES in the turbulence. It was shown in [22] that ES instabilities can drive experimentally relevant levels of transport at short wavelengths $k_{\perp}\rho_{\text{th},e} \lesssim 1$. This challenged the prevailing view of transport in the community. More recently, ES transport has been observed in NSTX [23], and has been found to be a candidate for anomalous transport in MAST [24, 25]. However, the extreme expense of performing multiscale simulations of turbulence which resolve the widely separated IS and ES has meant that ES turbulence has, until recently, been studied independently of the IS turbulence under the same implicit assumption that there is no significant interaction between the IS and the ES. This assumption is likely to be valid when the IS turbulence is suppressed but is otherwise questionable. Examples may be found in [22, 24–30].

Without directly simulating or observing the full multiscale turbulence, it is difficult to assess the extent to which there are cross-scale interactions in the turbulence, and whether or not both scales will contribute significantly to the transport. Unfortunately, studying multiscale turbulence through direct simulation is made very challenging by the size of $(m_e/m_i)^{1/2}$ for a realistic deuterium plasma, $(m_e/m_i)^{1/2} \approx 1/60$, which determines the separation of $\rho_{\text{th},e}/\rho_{\text{th},i} \sim (m_e/m_i)^{1/2}$ and $v_{\text{th},i}/v_{\text{th},e} \sim (m_e/m_i)^{1/2}$. For example, if one wanted to extend the resolution of a well-resolved IS simulation to capture both the $a/v_{\text{th},i}$ and $a/v_{\text{th},e}$ time scales, then one must increase the resolution in time by approximately $v_{\text{th},e}/v_{\text{th},i} \sim (m_i/m_e)^{1/2}$. To resolve length scales perpendicular to the magnetic field line comparable to both $\rho_{\text{th},i}$ and $\rho_{\text{th},e}$ one must increase the resolution in both the perpendicular directions by $\rho_{\text{th},i}/\rho_{\text{th},e} \sim (m_i/m_e)^{1/2}$. Overall, the increased cost scales like $(m_i/m_e)^{3/2}$. For the deuterium mass, a well-resolved multiscale simulation could be expected to cost 60^3 more than the well-resolved IS simulation. This cost is currently prohibitive for routine investigation.

The earliest attempts to study multiscale turbulence via direct numerical simulation (DNS) were made in [31–33] using unphysically small ion to electron mass ratio. Recently, with improvements in computing power, it has been possible to perform small

numbers of multiscale DNS with the deuterium [34–38] and hydrogen [39–41] mass ratio. The multiscale simulations allow us to observe several features of multiscale turbulence. There can be a scale separation. The ES heat flux can be comparable to the IS heat flux, and even necessary to match experimental results [36], and there are nontrivial interactions between the IS and the ES. In [31, 32, 36–39, 41] it is shown that the IS fluctuations can affect the ES fluctuations. This is demonstrated in Figure 3 of [36], where we see that varying the ion temperature gradient drastically changes the ES fluctuations. As the ion response at the ES is negligible, the only mechanism through which the ion temperature gradient can affect the ES fluctuations is by cross-scale interactions. Further evidence for this cross-scale interaction appears in Figure 2 of [39], where high- $k_\perp$ modes in the multiscale simulation are suppressed compared to the same modes in the high- $k_\perp$ single-scale simulation. We also note that the ES turbulence can affect the IS turbulence [34, 36, 37, 39–41]. The presence of the ES turbulence can increase IS fluctuation amplitudes compared to an IS only simulation [34, 37, 39, 41]; see, e.g., Figure 2b of [39] and Figure 3 of [34]. In [40] the ES turbulence is able to effectively suppress the microtearing mode, which exists in the low- $k_\perp$ range.

We note that using unphysically large values of $(m_e/m_i)^{1/2}$ can lead to qualitatively unrealistic results in numerical experiments [35, 42]. For example, in Figure 3 of [35] we see that there are clearly-defined, separated, IS and ES peaks in the electron heat flux for the physical mass ratio $(m_e/m_i)^{1/2} = 1/60$. However, in the case with $(m_e/m_i)^{1/2} = 1/20$ we see only a single peak in the electron heat flux spectrum, indicating that for the unphysical value of $(m_e/m_i)^{1/2}$, for the parameters considered in [35], the IS and the ES can no longer be distinguished or separated.

Multiscale DNS demonstrate that there is a rich variety of physics to investigate. However, the high computational cost and the difficulty of diagnosing multiscale DNS means that there is a need for analytic theory to help provide a theoretical understanding of the mechanisms of the cross-scale interactions. In Chapter 2 of this thesis we will assume scale separation between the IS and the ES in the turbulence. We will take the gyrokinetic model as a starting point, and treat $(m_e/m_i)^{1/2}$ as an asymptotically-small parameter. Using the assumption of scale separation, we find scale-separated, coupled gyrokinetic equations for the IS and the ES pieces of the turbulence. These equations

may be solved using a system of coupled flux tubes, visualised in Figure 1.4, where the coordinates (ψ, α, θ) denote the usual magnetic-field-line-following radial, binormal, and parallel-to-the-field coordinates. This scale-separated approach is reminiscent of the approach taken in [9–15] to study the evolution of the turbulence and the profiles in a scale-separated way.

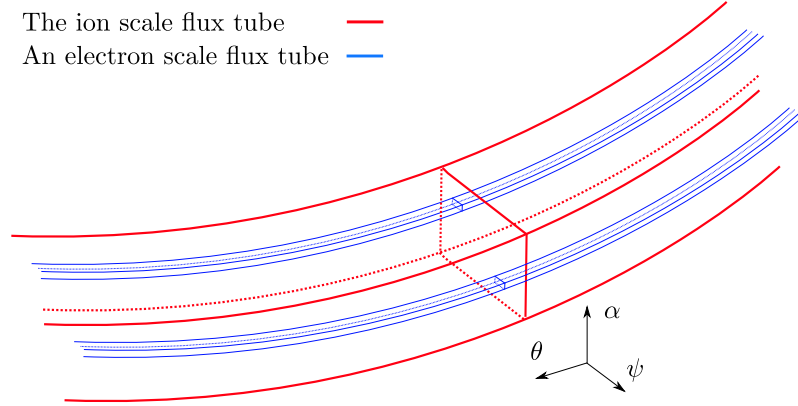


Figure 1.4: Diagram illustrating how ES flux tubes (blue) may be nested within a larger IS flux tube (red).

The major results of this thesis are as follows. In Chapter 2 we derive the scale-separated, coupled equations which describe IS and ES turbulence. At the IS the usual ion physics are retained. However, the equation for electrons at IS is averaged over the particle orbits in the direction parallel to the magnetic field. The IS fluctuations evolve independently of the ES turbulence, because there are no cross-scale terms appearing in the IS equations. At the ES the response of the ion species can be modelled as Maxwell-Boltzmann. The electron equation at the ES contains two new terms which depend on gradients of IS fluctuations. The first term is a drive term due to the gradient of the IS, electron distribution function. The gradient of the IS, electron distribution function has the effect of modifying the gradient of the equilibrium distribution function, the usual drive of instability. The second term represents an $\mathbf{E} \times \mathbf{B}$ advection due to the IS potential, which varies in the direction parallel to the magnetic field line. The effect of the $\mathbf{E} \times \mathbf{B}$ advection is to shear ES eddies in the parallel-to-the-field direction by differential flows. Note that perpendicular-to-the-field shearing of ES fluctuations due

to IS eddies does not appear in this model, as this effect is small by $(m_e/m_i)^{1/2}$ in the scale-separated limit.

In Chapter 3 we use numerical results from simulations of the coupled IS-ES system to investigate the effect of the two cross-scale interaction mechanisms appearing in the scale-separated ES gyrokinetic equation. We find that for cyclone base case (CBC) like parameters [43] a strongly driven ETG instability is suppressed in the presence of strongly driven IS turbulence, whereas IS turbulence driven near marginal stability has little effect on a strongly driven ETG instability. The most important cross-scale interaction mechanism in this case is mediated via the piece of the IS $\mathbf{E} \times \mathbf{B}$ drift which varies in the parallel-to-the-field direction. Finally, to aid understanding of the numerical results we consider a simplified model of the effect of parallel-to-the-field shear on a strongly driven ETG mode.

Chapter 4 discusses the results presented in this thesis. We discuss the analytical results presented in Chapter 2, describing the differences between the model of coupled IS-ES turbulence and the full gyrokinetic model, and the insights gained from its derivation. In Section 4.2 we anticipate the result of extending the scale-separated model by including the ratio of the IS to ES drives of instability as a parameter which scales with mass ratio, and we discuss the likely scenarios for steady state IS-ES turbulence. In Section 4.3 we discuss the simulation results of Chapter 3 and argue that the parallel-to-the-field shear in the IS $\mathbf{E} \times \mathbf{B}$ drift may be responsible for the suppression of ETG turbulence in DNS. Finally, in Section 4.4 we thoroughly discuss the limitations of the scale-separated model and suggest possible avenues of future work.

Chapter 2

Multiscale Turbulence

This chapter presents the theory for the derivation of scale-separated, coupled equations for the IS and the ES pieces of the turbulence. The structure of this chapter is as follows. Section 2.1 reviews the concepts used to derive the gyrokinetic equations that are necessary for the separation of the IS and the ES. In Section 2.2 we state the orderings for length and time scales and present the formalism which we will use to separate the IS and the ES. We introduce the method of multiple scales as a technique for deriving the coupled gyrokinetic equations. We define an average over fast (ES) time scales and short (ES) spatial scales in the plane perpendicular to the magnetic field line. With the assumption of statistical spatial periodicity of the ES turbulence, this ES average allows one to uniquely decompose the turbulence into IS and ES pieces. In Section 2.3 we apply the ES average to find the coupled IS and ES gyrokinetic equations, and retain the largest possible cross-scale interaction terms in the expansion in $(m_e/m_i)^{1/2}$. We do this without explicitly assuming the size of the amplitude of the fluctuations, to allow for the possibility of exotic orderings for the sizes of the fluctuations in the presence of cross-scale coupling. In Section 2.4 we use the ES average to find the quasineutrality relations which close the coupled equations. By using dominant balance arguments, we find in Section 2.5 the only self-consistent ordering for the size of the IS and the ES fluctuations that allows for statistically steady-state turbulence: the usual gyro-Bohm ordering. We show that it is possible to neglect the nonadiabatic response of the ion species at the ES, which is necessary for a local description of the ES. In Section 2.6 we use the results of the preceding Sections to obtain the maximally-ordered, scale-separated, coupled gyrokinetic equations. Section

2.7 provides and justifies a parallel boundary condition for the ES gyrokinetic equation, consistent with the “twist-and-shift” boundary condition [44] for the IS turbulence in a local flux tube domain. We show in Section 2.8 that part of the IS $\mathbf{E} \times \mathbf{B}$ advection may be removed from the ES equations by a rotation. Finally, in Section 2.9 we discuss the form taken by the collision operator within the assumptions of the model.

The appendices corresponding to this chapter are structured as follows. Appendices A.1-A.4 contain relations required to derive the scale-separated, coupled gyrokinetic equations. In Appendix A.5 we find the sizes of the fluctuation amplitudes which satisfy dominant balance in the coupled system of equations, and we discuss the possible impacts of cross-scale interaction. Appendix A.6 contains relations which are used to evaluate ES quasineutrality. In Appendix A.7 we find the scalings for the heat fluxes in the model. Appendices A.8-A.10 are devoted to the parallel boundary condition.

2.1 The gyrokinetic equation

The multiscale equations that describe the interaction between turbulent fluctuations at the IS and turbulent fluctuations at the ES are obtained from the δf gyrokinetic equations via an asymptotic expansion in the electron-ion mass ratio. This is a subsidiary expansion within the gyrokinetic ordering [9, 12, 45–50]. As such, we begin by briefly presenting the steps in the derivation of the δf gyrokinetic equation, which describes the evolution of turbulent “fluctuations” around a slowly changing “equilibrium” plasma profile in a plasma which is strongly magnetised. In this thesis we neglect the effects of equilibrium toroidal $\mathbf{E} \times \mathbf{B}$ rotation, and we only consider electrostatic fluctuations, where we can express the electric field $\mathbf{E}$ purely in terms of the electrostatic potential Φ , i.e.

$$\mathbf{E} = -\nabla\Phi. \tag{2.1}$$

We neglect electromagnetic fluctuations: In this thesis we consider cases where the dominant instabilities that drive the turbulence are the primarily electrostatic ion temperature gradient (ITG) [16, 17] and electron temperature gradient (ETG) [18, 19] modes. We consider the possible impacts of electromagnetic fluctuations in Section

4.4. We specialise to a toroidal magnetic geometry. The magnetic field $\mathbf{B}$ can always be written as

$$\mathbf{B} = \nabla\alpha \times \nabla\psi, \quad (2.2)$$

where the flux label ψ and field line label α will act as the coordinates perpendicular to the field line. We will take the poloidal angle θ as the coordinate which determines position along the field line. Figure 2.1 views a close-up of the magnetic geometry shown in Figure 1.1 to illustrate the magnetic-field-line-following coordinates ψ and α , with $\mathbf{b} = \mathbf{B}/B$ the direction of the magnetic field. The magnetic axis is indicated with the dashed line, and magnetic field lines are indicated in blue. The magnetic field direction $\mathbf{b}$, binormal direction $\nabla\alpha$ and radial direction $\nabla\psi$ are indicated with arrows, at a particular toroidal, poloidal and radial position in the geometry. Note that the coordinate set (ψ, α, θ) represents a left-handed coordinate system, where the product $\nabla\psi \times \nabla\alpha \cdot \nabla\theta < 0$.

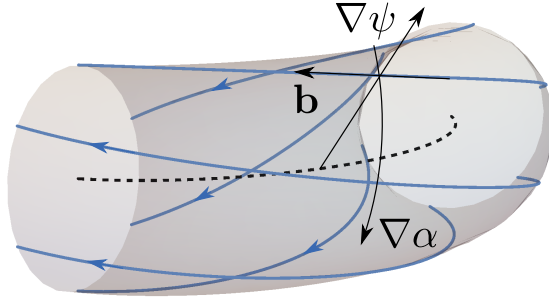


Figure 2.1: Diagram illustrating magnetic-field-line-following coordinates.

The δf gyrokinetic equation is derived by assuming a separation of space and time scales between the equilibrium profile and fluctuating parts of the particle distribution function. Explicitly, we take

$$\rho_{\text{th},\nu}/a = \rho_{\nu}^* \rightarrow 0 \quad (2.3)$$

and

$$\Omega_{\nu} \gg \omega \gg \tau_{\text{P}}^{-1}, \quad (2.4)$$

where ν is the species label, a is a characteristic equilibrium scale length, ω is a typical fluctuation frequency and τ_P is a characteristic time scale of the equilibrium profiles. It is conventional to order the collision frequency $\nu_C \sim \omega$ as a maximal ordering to allow for the possibility of both collisionless and collisional plasmas. As the electron-ion mass ratio is treated as an order unity parameter in the gyrokinetic ordering the gyrokinetic equation for each species has the same form. We thus simplify notation by suppressing the species index ν where it does not introduce ambiguity. We assume that the distribution function f and the electrostatic potential Φ have negligible amplitude at scales intermediate to $\rho_{th,\nu}$ and a . Moreover, we assume there is no direct cascade of energy between the scales. It is then possible to split the distribution function f ,

$$f = F + \delta f, \quad (2.5)$$

with F and δf the equilibrium and fluctuating parts of the distribution function respectively. We introduce a turbulent average $\langle \cdot \rangle^{\text{turb}}$, which averages over perpendicular-to-the-field spatial scales and time scales which are intermediate to the fluctuation and equilibrium scales, such that

$$F = \langle f \rangle^{\text{turb}}, \quad (2.6)$$

with the assumption of statistical periodicity

$$\langle \delta f \rangle^{\text{turb}} = 0. \quad (2.7)$$

Note that (2.7) can be satisfied by imposing periodicity on the turbulence in the plane perpendicular to the magnetic field. This is justified by noting that the turbulent fluctuations should have the same statistics everywhere within a domain that is asymptotically small compared to a . This can be ensured by deriving local evolution equations for the turbulence, choosing the perpendicular-to-the-field domain of simulations to be larger than the perpendicular-to-the-field correlation lengths of the turbulence, and imposing periodicity as the boundary condition in the perpendicular domain. We are able to make the assumption of statistical periodicity because of the assumptions of scale separation. This is in contrast to the study of neutral fluids, where the turbulence is universal and the inertial range spans the outer scale to the dissipation scale. As

there is no scale separation in neutral fluids, sub-grid models for neutral fluid turbulence are derived using large eddy simulation (LES) approaches (see e.g. [51–53]). Let time be t , ∇ be the gradient operator, and $\mathbf{I}$ be the identity matrix. Then $\nabla_{\parallel} = \mathbf{b} \cdot \nabla$ is the derivative parallel to the magnetic field, and $\nabla_{\perp} = (\mathbf{I} - \mathbf{b}\mathbf{b}) \cdot \nabla$ is the derivative perpendicular to the magnetic field. With these definitions the gyrokinetic orderings for F , δf and the fluctuating electrostatic potential ϕ are

$$\begin{aligned} \nabla_{\parallel} F &\sim \nabla_{\perp} F \sim \frac{F}{a}, & \frac{\partial F}{\partial t} &\sim \frac{F}{\tau_P}, \\ \nabla_{\parallel} \delta f &\sim \frac{\delta f}{a}, & \nabla_{\perp} \delta f &\sim \frac{\delta f}{\rho_{\text{th}}}, & \frac{\partial \delta f}{\partial t} &\sim \omega \delta f, \\ \frac{\delta f}{F} &\sim \frac{e\phi}{T} \sim \frac{\rho_{\text{th}}}{a}. \end{aligned} \tag{2.8}$$

The ordering (2.8) indicates that the fluctuations are highly anisotropic with respect to the magnetic field and evolve rapidly in time compared to the equilibrium. In Section 1.2 we motivate the ordering (2.8) with a physical picture of drift waves and the toroidal instabilities which drive plasma turbulence in tokamaks. In brief, the drift waves and instabilities which motivate the gyrokinetic ordering (2.8) arise due to the mean gradients of the plasma profiles and the nature of the particle motion in magnetic confinement fusion devices, which consists of rapid helical motion following magnetic field lines, and slow drifts across field lines.

Particles stream along the field at thermal speed time scales $a/v_{\text{th}} \sim \omega^{-1}$, which are much longer than the gyration time scale Ω^{-1} , i.e. $\Omega^{-1} \ll a/v_{\text{th}}$. In order to separate the rapid gyration from the particle streaming it is convenient to use gyrokinetic variables [45, 50] rather than the particle position $\mathbf{r}$ and particle velocity $\mathbf{v}$. We will use the guiding centre $\mathbf{R} = \mathbf{r} - \boldsymbol{\rho}_{\nu}$, where $\boldsymbol{\rho}_{\nu} = \mathbf{b} \times \mathbf{v}/\Omega_{\nu}$ is the vector gyroradius; the particle energy $\varepsilon = m_{\nu}v^2/2$, where $v = |\mathbf{v}|$; the sign of the parallel velocity $\sigma = v_{\parallel}/|v_{\parallel}|$, where $v_{\parallel} = \mathbf{b} \cdot \mathbf{v}$; the pitch angle $\lambda = v_{\perp}^2/(v^2 B)$, where $v_{\perp} = |\mathbf{v} - v_{\parallel}\mathbf{b}|$; and the gyrophase γ , which identifies the angular position of a particle in its gyromotion, and which is defined by

$$\tan \gamma = \frac{\mathbf{e}_2 \cdot \mathbf{v}}{\mathbf{e}_1 \cdot \mathbf{v}}, \tag{2.9}$$

where $\mathbf{e}_1$ and $\mathbf{e}_2$ are unit vectors which form an orthonormal basis with $\mathbf{b}$. We define $\mathbf{e}_1$ and $\mathbf{e}_2$ in terms of ψ and α in A.1. In these variables, variation on Ω^{-1} time scales occurs only through γ , over which one can conveniently average. We define the gyroaverage $\langle \cdot \rangle^\gamma$ as

$$\langle \cdot \rangle^\gamma = \frac{1}{2\pi} \int_0^{2\pi} \cdot d\gamma. \quad (2.10)$$

Note that the gyroaverage is taken at fixed ε , σ , and λ . In addition, either $\mathbf{r}$ or $\mathbf{R}$ is held fixed during the gyroaveraging. We will state explicitly whether $\mathbf{r}$ or $\mathbf{R}$ is held fixed in each gyroaverage. The gyroaverage introduces dependence on species by introducing ρ_ν dependence when averaging a function of $\mathbf{r}$ at fixed $\mathbf{R}$ and when averaging a function of $\mathbf{R}$ at fixed $\mathbf{r}$. When we wish to emphasise this, we denote the gyroaverage by $\langle \cdot \rangle_{\mathbf{R},\nu}^\gamma$ and $\langle \cdot \rangle_{\mathbf{r},\nu}^\gamma$ respectively.

To find equations which determine δf and F we use the turbulent average (2.6), the assumption of statistical periodicity (2.7), the orderings (2.8), the gyroaverage (2.10), and the Fokker-Planck equation for the plasma. We find that the equilibrium piece of the distribution function $F = F_0$, where F_0 is a Maxwellian distribution of velocities. The fluctuating piece of the distribution function δf is determined by the gyrokinetic equation, which is written in terms of the nonadiabatic response

$$h(\mathbf{R}, \varepsilon, \lambda, \sigma) = \delta f(\mathbf{r}, \mathbf{v}) + \frac{Ze\phi(\mathbf{r})}{T} F_0. \quad (2.11)$$

Note that h is independent of γ when h is regarded as a function of $\mathbf{R}$, and note that ϕ is a function of $\mathbf{r}$ only. The dependence on γ in δf arises due to ρ in the transformation between $\mathbf{r}$ and $\mathbf{R}$. In terms of h_ν , the gyrokinetic equation for each species is

$$\frac{\partial h_\nu}{\partial t} + v_\parallel \mathbf{b} \cdot \nabla \theta \frac{\partial h_\nu}{\partial \theta} + (\mathbf{v}_\nu^M + \mathbf{v}_\nu^E) \cdot \nabla h_\nu + \mathbf{v}_\nu^E \cdot \nabla F_{0\nu} = \frac{Z_\nu e F_{0\nu}}{T_\nu} \frac{\partial \varphi_\nu}{\partial t} + C_{1\nu}, \quad (2.12)$$

where

$$\varphi_\nu(\mathbf{R}) = \langle \phi(\mathbf{r}) \rangle_{\mathbf{R},\nu}^\gamma = \langle \phi(\mathbf{R} + \rho_\nu) \rangle_{\mathbf{R}}^\gamma \quad (2.13)$$

is the species dependent gyroaverage of the fluctuating potential at fixed $\mathbf{R}$. Physically φ_ν is the averaged potential experienced by a particle gyrating about its guiding centre.

We note that φ_ν gains species dependence due to the fact that $\mathbf{R}$, rather than $\mathbf{r}$, is held fixed in the gyroaverage in (2.13). $C_{l\nu}$ is the gyroaveraged, linearised Fokker-Planck collision operator for the species. $\mathbf{v}_\nu^M = (v_\perp^2/2\Omega_\nu B)\mathbf{b} \times \nabla B + (v_\parallel^2/\Omega_\nu)\mathbf{b} \times \mathbf{b} \cdot \nabla \mathbf{b}$ is the magnetic drift velocity. Finally,

$$\mathbf{v}_\nu^E = \frac{c}{B}\mathbf{b} \times \nabla \varphi_\nu \quad (2.14)$$

is the gyroaveraged $\mathbf{E} \times \mathbf{B}$ velocity for the species ν . To close the gyrokinetic equations for each species one requires an equation for the field; this is the quasineutrality relation,

$$\sum_\nu Z_\nu e \int d^3\mathbf{v} |_{\mathbf{r}} h_\nu(\mathbf{R}, \varepsilon, \lambda, \sigma) = \sum_\nu \frac{Z_\nu^2 e^2 n_\nu}{T_\nu} \phi(\mathbf{r}), \quad (2.15)$$

where $n_\nu = \int d^3\mathbf{v} F_{0\nu}$ is the equilibrium plasma density; and $\int d^3\mathbf{v} |_{\mathbf{r}}$ indicates a velocity integral taken at fixed $\mathbf{r}$, meaning that we take $\mathbf{R} = \mathbf{r} - \boldsymbol{\rho}_\nu$ when evaluating the velocity integral in (2.15). Where it does not introduce ambiguity, we will suppress $(\mathbf{R}, \varepsilon, \lambda, \sigma)$ dependences and species indices. We will take the collisionless limit $\nu_C \ll \omega$ in the greater part of this Chapter for simplicity. We demonstrate how the effects of collisions may be included within the scale-separated framework in Section 2.9. In Chapter 3 we employ a collision operator to regulate the the typical velocity scale δv_ν of the distribution function for a given species. Note that the collision operator $C_{l\nu}$ is a diffusive operator in velocity space, i.e. $C_{l\nu} \sim v_{\text{th},\nu}^2 \delta v_\nu^{-2} \nu_C$. In the presence of the collision operator structures with very small δv_ν are damped – this occurs when $\delta v_\nu/v_{\text{th},\nu} \ll (\nu_C/\omega)^{1/2}$. Hence, we can order $\delta v_\nu \sim v_{\text{th},\nu}$. The collision operator $C_{l\nu}$ provides the necessary, and physical, regularisation of δf and F [54, 55]. In a truly collisionless system δf and F could develop arbitrarily small velocity space structures through phase mixing [56–63], i.e. $\delta v \rightarrow 0$ as $\nu_C \rightarrow 0$.

In the δf gyrokinetic approach, equations (2.12) and (2.15) are solved in a field-following domain, termed a flux tube. We use the field-aligned coordinate system (ψ, α, θ) , which is theoretically convenient and allows for an efficient simulation domain which captures the structure of anisotropic, magnetised plasma turbulence with minimal resolution. The flux tube has a narrow extent around a central field line but extends for (typically) a 2π poloidal circuit along the field line. The assumptions of

scale separation in the plane perpendicular to $\mathbf{B}$, and statistical periodicity (2.7), permit the use of periodic boundary conditions in the (ψ, α) plane. However, we must also specify a boundary condition in the poloidal angle θ that gives the location along the magnetic field. The relevant physical boundary condition is periodicity in θ , but this must be applied at fixed toroidal angle ζ – not at fixed field line label α . The presence of shear in the pitch of the magnetic field makes enforcement of this boundary condition nontrivial. The standard flux tube parallel boundary condition is the so-called “twist-and-shift” boundary condition [44].

The gyrokinetic system (2.12) and (2.15) is the starting point for the model. Note that the gyrokinetic ordering allows for the possibility that the fluctuations have multiple distinct space and time scales arising from differences in the thermal speed $v_{\text{th},\nu}$ and thermal gyroradius $\rho_{\text{th},\nu}$ of each species. Indeed, equilibrium gradients in plasma density and temperature drive instabilities with distinct spatial scales of the ion and electron gyroradii, with corresponding time scales of $a/v_{\text{th},i}$ and $a/v_{\text{th},e}$, respectively. We suppose the existence of an IS and a separated ES within the turbulence. The separation between the scales is governed by the mass ratio $(m_e/m_i)^{1/2} \sim \rho_{\text{th},e}/\rho_{\text{th},i} \sim v_{\text{th},i}/v_{\text{th},e}$, which we treat as an asymptotic expansion parameter; i.e., $(m_e/m_i)^{1/2} \rightarrow 0$. We will assume that fluctuations in the scales intermediate to the IS and ES have negligible amplitude. Note that the expansion in $(m_e/m_i)^{1/2}$ is a subsidiary expansion within gyrokinetics, so it satisfies $\rho_{\text{th},\nu}/a \ll (m_e/m_i)^{1/2}$.

In the following Sections we proceed in analogy to the derivation of the coupled equilibrium-fluctuation equations to derive scale-separated, coupled IS-ES equations. These equations contain the physics of nonlocal (in wave number) cross-scale interaction. We propose an extension of the “twist-and-shift” flux tube parallel boundary condition to allow for efficient simulation of the IS-ES equations in a system of coupled flux tubes. We find the IS-ES equations are parallelisable in the sense that the ES equations may be integrated at multiple radial locations within the IS domain without reference to one another.

2.2 Separation of scales within the turbulence

We assume a separation of scales between IS fluctuations and ES fluctuations in the turbulence, i.e., $(m_e/m_i)^{1/2} \rightarrow 0$. We make the assumption that the turbulent fluctuations have negligible amplitude at scales intermediate to the IS and ES. This allows us to decompose the fluctuating distribution function δf into

$$\delta f = \overline{\delta f} + \widetilde{\delta f}, \quad (2.16)$$

where $\overline{\delta f}$ and $\widetilde{\delta f}$ are the fluctuating IS and ES pieces of the particle distribution function, respectively. In order to find evolution equations for $\overline{\delta f}$ and $\widetilde{\delta f}$ we introduce an ES average $\langle \cdot \rangle^{\text{ES}}$, which averages over spatial scales and time scales which are intermediate to the IS and ES, such that

$$\langle \delta f \rangle^{\text{ES}} = \overline{\delta f}, \quad (2.17)$$

with the assumption that the ES fluctuations are statistically periodic

$$\langle \widetilde{\delta f} \rangle^{\text{ES}} = 0. \quad (2.18)$$

This assumption allows us to asymptotically expand the gyrokinetic equation and find unique asymptotic series for $\overline{\delta f}$ and $\widetilde{\delta f}$ in the limit $(m_e/m_i)^{1/2} \rightarrow 0$, and is analogous to the assumption of statistical periodicity (2.7).

One could derive the coupled multiscale equations with an implicit assumption of statistical periodicity, as e.g., in [12]. However, here we explicitly show how (2.18) is satisfied by introducing fast and slow variables for space and time after the method of multiple scales (cf. [64]) and then homogenising the system. This approach clarifies how to deal with the nonlocal nature of gyrokinetics introduced by the gyroaverage.

In analogy to δf gyrokinetics, we adopt the following orderings for space and time scales,

$$\begin{aligned} \nabla_{\parallel} \overline{\delta f} &\sim \frac{\overline{\delta f}}{a}, & \nabla_{\perp} \overline{\delta f} &\sim \frac{\overline{\delta f}}{\rho_{\text{th},i}}, & \frac{\partial \overline{\delta f}}{\partial t} &\sim \frac{v_{\text{th},i}}{a} \overline{\delta f}, \\ \nabla_{\parallel} \widetilde{\delta f} &\sim \frac{\widetilde{\delta f}}{a}, & \nabla_{\perp} \widetilde{\delta f} &\sim \frac{\widetilde{\delta f}}{\rho_{\text{th},e}}, & \frac{\partial \widetilde{\delta f}}{\partial t} &\sim \frac{v_{\text{th},e}}{a} \widetilde{\delta f}. \end{aligned} \quad (2.19)$$

Note that we assume that the parallel scale is always set by the machine size, and hence we do not assume separation of scales in the parallel direction. We assume that the typical parallel-to-the-field wavenumber $k_{\parallel}$ of both the IS and ES pieces of the turbulence satisfies $(m_e/m_i)^{1/2} \ll k_{\parallel}a \sim 1 \ll (m_i/m_e)^{1/2}$. The ordering (2.19) assumes that the degree of spatial anisotropy in the turbulence in the plane perpendicular to the magnetic field line is an of order unity number in the $(m_e/m_i)^{1/2}$ expansion; this assumption excludes structures which have scales of size $\rho_{\text{th},i}$ in one direction perpendicular to the magnetic field, but scales of size $\rho_{\text{th},e}$ in another. In particular, this assumption excludes ETG streamers [22, 26, 27] if their radial extent scales like $\rho_{\text{th},i}$ rather than $\rho_{\text{th},e}$. We note that the orderings (2.19) exclude phenomena in IS turbulence due to passing electron dynamics, such as giant ballooning tails [65] and fine radial structures [66]. Long time scales associated with the saturation of $\rho_{\text{th},e}$ scale structures, such as those observed in [25], can be accommodated within the ordering (2.19) provided that the saturation time scale does not scale with $a/v_{\text{th},i}$. Finally, we note that the ordering (2.19) captures the electron precession resonance for $\rho_{\text{th},i}$ scale structures. The precession resonance is due to the slow toroidal drift of the parallel-to-the-field orbits of trapped electrons [67] and naturally appears in the IS equation for electrons.

Using the orderings (2.19) we proceed to find coupled IS-ES equations including all terms which might be relevant to leading order. We then use dominant balance arguments to find orderings for the size of the fluctuations consistent with (2.19), and neglect the terms in the coupled IS-ES equations which are small.

We introduce a fast spatial variable $\mathbf{r}_f$ and a slow spatial variable $\mathbf{r}_s$ in the 2-D plane perpendicular to the field line, such that all ES perpendicular-to-the-field variation in the solution appears through dependence on $\mathbf{r}_f$, and all IS perpendicular-to-the-field variation appears through dependence on $\mathbf{r}_s$. Functions of $\mathbf{r}$ will become functions of $(\theta, \mathbf{r}_s, \mathbf{r}_f)$, where the perpendicular and parallel spatial dependence is explicitly written out. We will also introduce fast and slow guiding centre coordinates, $\mathbf{R}_f = \mathbf{r}_f - \boldsymbol{\rho}_\nu$ and $\mathbf{R}_s = \mathbf{r}_s - \boldsymbol{\rho}_\nu$, respectively. We introduce the fast and slow times t_f and t_s , respectively. Functions of t will become functions of (t_s, t_f) . We will repeatedly use that the perpendicular-to-the-field fast and slow scales, $\rho_{\text{th},e}$ and $\rho_{\text{th},i}$, respectively, are

much shorter than the equilibrium scale a , i.e. $\rho_{\text{th,e}} \sim (m_e/m_i)^{1/2} \rho_{\text{th,i}} \ll \rho_{\text{th,i}} \ll a$. We will assume periodicity of the fluctuations in $\mathbf{r}_s$ to be consistent with the separation of scales between the fluctuations and the equilibrium assumed in δf gyrokinetics via condition (2.7). The gyrokinetic equation must be modified so that,

$$\delta f(t, \mathbf{r}) \rightarrow \delta f(t_s, t_f, \theta, \mathbf{r}_s, \mathbf{r}_f), \quad (2.20)$$

$$\nabla_\perp \rightarrow \nabla_s + \nabla_f, \quad \nabla_s \sim \rho_{\text{th,i}}^{-1}, \quad \nabla_f \sim \rho_{\text{th,e}}^{-1}, \quad (2.21)$$

$$\frac{\partial}{\partial t} \rightarrow \frac{\partial}{\partial t_s} + \frac{\partial}{\partial t_f}, \quad \frac{\partial}{\partial t_s} \sim \frac{v_{\text{th,i}}}{a}, \quad \frac{\partial}{\partial t_f} \sim \frac{v_{\text{th,e}}}{a}. \quad (2.22)$$

We then perform an asymptotic expansion on the resulting equations in the mass ratio $(m_e/m_i)^{1/2}$ to find the leading order, asymptotically valid equations. With the introduction of the new variables we may explicitly define the ES average,

$$\begin{aligned} \overline{\delta f}(t_s, \theta, \mathbf{r}_s) &= \langle \delta f(t_s, t_f, \theta, \mathbf{r}_s, \mathbf{r}_f) \rangle^{\text{ES}} = \\ &= \frac{1}{\tau_c A_e} \int_{t_s - \tau_c/2}^{t_s + \tau_c/2} dt_f \int_{\psi_s - \Delta_\psi/2}^{\psi_s + \Delta_\psi/2} d\psi_f \int_{\alpha_s - \Delta_\alpha/2}^{\alpha_s + \Delta_\alpha/2} d\alpha_f \mathcal{J} \delta f(t_s, t_f, \theta, \mathbf{r}_s, \mathbf{r}_f), \end{aligned} \quad (2.23)$$

where τ_c is a time intermediate to the ES and IS correlation times; Δ_ψ and Δ_α are intermediate to the ES and IS correlation lengths and angles in the ψ and α directions, respectively; ψ_f and ψ_s are the fast and slow flux labels, respectively; α_f and α_s are the fast and slow field line labels, respectively; $\mathcal{J}$ is the Jacobian of the transformation from $\mathbf{r}_f \rightarrow (\psi_f, \alpha_f)$, and

$$A_e = \int_{\psi_s - \Delta_\psi/2}^{\psi_s + \Delta_\psi/2} d\psi_f \int_{\alpha_s - \Delta_\alpha/2}^{\alpha_s + \Delta_\alpha/2} d\alpha_f \mathcal{J}. \quad (2.24)$$

Note that we regard $\mathbf{r}_f = \mathbf{r}_f(\psi_f, \alpha_f)$ and $\mathbf{r}_s = \mathbf{r}_s(\psi_s, \alpha_s)$ in the integration in (2.23) and (2.24). To satisfy (2.18) for every $(t_s, \theta, \mathbf{r}_s)$, we impose that the fluctuations are periodic in the fast variable $\mathbf{r}_f$. In particular we assume that

$$\delta f(t_s, t_f, \theta, \mathbf{r}_s, \mathbf{r}_f) = \delta f(t_s, t_f, \theta, \mathbf{r}_s, \mathbf{r}_f + j\Delta_\psi \mathbf{e}_\psi + l\Delta_\alpha \mathbf{e}_\alpha), \quad (2.25)$$

where j, l are any integers, $\mathbf{e}_\psi = (\mathbf{b} \times \nabla\alpha)/|\nabla\psi \cdot \mathbf{b} \times \nabla\alpha|$, and $\mathbf{e}_\alpha = (\nabla\psi \times \mathbf{b})/|\nabla\psi \cdot \mathbf{b} \times \nabla\alpha|$.

The use of periodic boundary conditions makes it advantageous to use a spectral representation of the gyrokinetic equation. We introduce fast and slow wave vectors, $\mathbf{k}_f$ and $\mathbf{k}_s$, respectively, which correspond to the fast and slow coordinates $\mathbf{r}_f$ and $\mathbf{r}_s$, respectively. It is necessary to project these wave vectors onto the field-line-following coordinate system. We define $\mathbf{k}_f = K_\alpha \nabla\alpha + K_\psi \nabla\psi$, where K_α is the fast binormal wave number, and K_ψ is the fast radial wave number. We define $\mathbf{k}_s = k_\alpha \nabla\alpha + k_\psi \nabla\psi$, where k_α is the slow binormal wave number, and k_ψ is the slow radial wave number. Note that the field-line-following coordinate system is nonorthogonal, and that $\nabla\psi$ and $\nabla\alpha$ in general depend on θ . As a result, there is a distinction between the radial wave numbers $K_\psi = \mathbf{k}_f \cdot \mathbf{e}_\psi$ and $k_\psi = \mathbf{k}_s \cdot \mathbf{e}_\psi$, and projection of the wave vectors in the radial $\nabla\psi$ direction, $\mathbf{k}_f \cdot \nabla\psi/|\nabla\psi|$ and $\mathbf{k}_s \cdot \nabla\psi/|\nabla\psi|$. Throughout this thesis we refer to K_ψ and k_ψ as the radial wave numbers. We define the fast and slow perpendicular-to-the-field wave numbers, $k_{f,\perp} = |\mathbf{k}_f|$ and $k_{s,\perp} = |\mathbf{k}_s|$, respectively. Note that $k_{f,\perp}$ and $k_{s,\perp}$ depend on θ because of the geometrical factors in $\nabla\alpha$ and $\nabla\psi$.

The ES average (2.23) is an average over areas perpendicular to the magnetic field line, and times intermediate to $a/v_{th,i}$ and $a/v_{th,e}$, with the average taken at fixed θ , ε , λ , and σ . To find the scale-separated, leading order equation for electrons at IS it is also necessary to average the IS electron gyrokinetic equation over electron orbits parallel to the magnetic field line, cf.[67, 68]. This necessity arises because electron parallel streaming, which introduces frequencies of order $v_{th,e}/a$, is faster than any other IS dynamics, which by assumption have frequencies of order $v_{th,i}/a \ll v_{th,e}/a$. Naively this would appear to break the ordering, as it would seem we would need to include ES time scales in the IS equation. Furthermore, it is known that rapid electron streaming can lead to very long tails in ballooning modes due to the passing electron response [65]. The long ballooning tails result in very fine radial structure in the turbulence near low order mode rational surfaces [66]. This problem is resolved by treating electron parallel streaming at IS as being asymptotically faster than IS frequencies, consistent with the expansion in $(m_e/m_i)^{1/2}$. To find the leading order equation for the electron distribution function at IS we introduce the orbital average $\langle \cdot \rangle^\circ$. After [2] we define

the orbital average $\langle \cdot \rangle^\circ$ in the passing and trapped parts of velocity space separately. In the passing region of velocity space

$$\langle \cdot \rangle^\circ = \frac{\int d\alpha_s d\theta (\cdot) / v_\parallel \mathbf{b} \cdot \nabla \theta}{\int d\alpha_s d\theta / v_\parallel \mathbf{b} \cdot \nabla \theta}, \quad (2.26)$$

where the integration $\int d\alpha_s d\theta$ is taken over the whole flux surface. In the trapped region of velocity space

$$\langle \cdot \rangle^\circ = \frac{\sum_\sigma \int_{\theta^-}^{\theta^+} d\theta (\cdot) / |v_\parallel| \mathbf{b} \cdot \nabla \theta}{2 \int_{\theta^-}^{\theta^+} d\theta / |v_\parallel| \mathbf{b} \cdot \nabla \theta}, \quad (2.27)$$

where the limits in the integration θ^+ and θ^- are the upper and lower bounce points of the trapped particle, respectively. Note that the orbital average commutes with fast and slow spatial derivatives,

$$\nabla_s \langle \cdot \rangle^\circ = \langle \nabla_s(\cdot) \rangle^\circ \quad (2.28)$$

and

$$\nabla_f \langle \cdot \rangle^\circ = \langle \nabla_f(\cdot) \rangle^\circ \quad (2.29)$$

as $v_\parallel \mathbf{b} \cdot \nabla \theta$ contains only equilibrium spatial variation. With the definitions (2.26) and (2.27) we show in A.3.1 that the orbital average $\langle \cdot \rangle^\circ$ has the very useful property

$$\left\langle v_\parallel \mathbf{b} \cdot \nabla \theta \frac{\partial H}{\partial \theta} \right\rangle^\circ = 0 \quad (2.30)$$

for any $H(\theta, \mathbf{R}_s, \varepsilon, \lambda, \sigma)$. The property (2.30) will allow us to use the orbital average $\langle \cdot \rangle^\circ$ to eliminate parallel streaming from the electron equation at IS.

2.3 Coupled gyrokinetic equations

In this Section we will use the ideas of Section 2.2 to find the scale-separated IS and ES gyrokinetic equations, where the largest possible cross-scale terms are retained. We take this approach, as it will allow us to consider whether or not the presence of cross-scale interaction can lead to novel mass ratio orderings for the sizes of the fluctuations.

We now apply the ES average (2.23) to the gyrokinetic equation (2.12) to find the IS gyrokinetic equation. The resulting gyrokinetic equation is a function of $\mathbf{R}_s$. We note that what is done here applies to both ion and electron species. Upon averaging, the gyrokinetic equation is,

$$\frac{\partial \bar{h}_\nu}{\partial t_s} + v_\parallel \mathbf{b} \cdot \nabla \theta \frac{\partial \bar{h}_\nu}{\partial \theta} + \mathbf{v}_\nu^M \cdot \nabla_s \bar{h}_\nu + \langle \mathbf{v}_\nu^E \cdot \nabla h_\nu \rangle^{\text{ES}} + \bar{\mathbf{v}}_\nu^E \cdot \nabla F_{0\nu} = \frac{Z_\nu e F_{0\nu}}{T_\nu} \frac{\partial \bar{\varphi}_\nu}{\partial t_s}, \quad (2.31)$$

where $\bar{h}_\nu = \langle h_\nu \rangle^{\text{ES}}$, the ES-averaged and gyroaveraged $\mathbf{E} \times \mathbf{B}$ drift $\bar{\mathbf{v}}_\nu^E$ is

$$\bar{\mathbf{v}}_\nu^E = \frac{c}{B} \mathbf{b} \times \nabla_s \bar{\varphi}_\nu, \quad (2.32)$$

and

$$\bar{\varphi}_\nu(\mathbf{R}_s) = \langle \bar{\phi}(\mathbf{r}_s) \rangle_{\mathbf{R}_s, \nu}^\gamma = \langle \bar{\phi}(\mathbf{R}_s + \boldsymbol{\rho}_\nu) \rangle_{\mathbf{R}_s}^\gamma, \quad (2.33)$$

with $\bar{\phi}(\mathbf{r}_s) = \langle \phi \rangle^{\text{ES}}$. The drift $\bar{\mathbf{v}}_\nu^E$ and the IS gyroaveraged potential $\bar{\varphi}_\nu$ depend on species because we hold $\mathbf{R}_s$ fixed, rather than $\mathbf{r}_s$, in the gyroaverage in (2.33) and so introduce dependence on the species gyroradius $\boldsymbol{\rho}_\nu$. Further, note that whilst the gyroaveraged potential φ_ν is a function of $\mathbf{R}_f$, the potential ϕ is a function of $\mathbf{r}_f$. To obtain (2.31)-(2.33), we used the fact that the ES average can be taken over either the real space variable $\mathbf{r}_f$ or the guiding centre $\mathbf{R}_f$, and that the ES average commutes with the gyroaverage,

$$\langle \langle \cdot \rangle^{\text{ES}} \rangle_{\mathbf{R}_f, \mathbf{R}_s}^\gamma = \langle \langle \cdot \rangle_{\mathbf{R}_f, \mathbf{R}_s}^\gamma \rangle^{\text{ES}}, \quad (2.34)$$

where the notation $\langle \cdot \rangle_{\mathbf{R}_f, \mathbf{R}_s}^\gamma$ indicates a gyroaverage with both $\mathbf{R}_f$ and $\mathbf{R}_s$ held fixed. Both of these properties are proven in A.2. The linear terms in (2.31) are simply filtered versions of the terms in the δf gyrokinetic equation (2.12). However, the nonlinear term yields a new, cross-scale coupling term. As shown in A.4.1, the form of this new term is

$$\langle \mathbf{v}_\nu^E \cdot \nabla h_\nu \rangle^{\text{ES}} = \bar{\mathbf{v}}_\nu^E \cdot \nabla_s \bar{h}_\nu + \nabla_s \cdot \left\langle \tilde{h}_\nu \frac{c}{B} \mathbf{b} \times (\nabla_s + \nabla_f) \tilde{\varphi}_\nu \right\rangle^{\text{ES}}, \quad (2.35)$$

where $\tilde{h}_\nu = h_\nu - \bar{h}_\nu$; the ES, gyroaveraged potential $\tilde{\varphi}_\nu$ is

$$\tilde{\varphi}_\nu(\mathbf{R}_s, \mathbf{R}_f) = \langle \tilde{\phi}(\mathbf{r}_s, \mathbf{r}_f) \rangle_{\mathbf{R}_s, \mathbf{R}_f, \nu}^\gamma = \langle \tilde{\phi}(\mathbf{R}_s + \boldsymbol{\rho}_\nu, \mathbf{R}_f + \boldsymbol{\rho}_\nu) \rangle_{\mathbf{R}_s, \mathbf{R}_f}^\gamma, \quad (2.36)$$

with $\tilde{\phi} = \phi - \bar{\phi}$; and $\bar{\mathbf{v}}_\nu^E \cdot \nabla_s \bar{h}_\nu$ is the usual $\mathbf{E} \times \mathbf{B}$ nonlinearity appearing in the IS gyrokinetic equation in the absence of cross-scale coupling. Note again that $\tilde{\varphi}_\nu$ depends on species because we hold $\mathbf{R}_s$ and $\mathbf{R}_f$ fixed, rather than $\mathbf{r}_s$ and $\mathbf{r}_f$, in the gyroaverage in (2.36). Dropping the $\nabla_s \tilde{\varphi}_\nu$ term in (2.35) since it is a factor $(m_e/m_i)^{1/2}$ smaller than the $\nabla_f \tilde{\varphi}_\nu$ term in (2.35), we find

$$\langle \mathbf{v}_\nu^E \cdot \nabla h_\nu \rangle^{\text{ES}} = \bar{\mathbf{v}}_\nu^E \cdot \nabla_s \bar{h}_\nu + \nabla_s \cdot \langle \tilde{\mathbf{v}}_\nu^E \tilde{h}_\nu \rangle^{\text{ES}}, \quad (2.37)$$

where

$$\tilde{\mathbf{v}}_\nu^E = \frac{c}{B} \mathbf{b} \times \nabla_f \tilde{\varphi}_\nu. \quad (2.38)$$

After filtering with the ES average, the IS equations for ions and electrons in the presence of cross-scale coupling are therefore

$$\begin{aligned} \frac{\partial \bar{h}_i}{\partial t_s} + v_\parallel \mathbf{b} \cdot \nabla \theta \frac{\partial \bar{h}_i}{\partial \theta} + \mathbf{v}_i^M \cdot \nabla_s \bar{h}_i + \bar{\mathbf{v}}_i^E \cdot \nabla_s \bar{h}_i + \nabla_s \cdot \langle \tilde{\mathbf{v}}_i^E \tilde{h}_i \rangle^{\text{ES}} \\ + \bar{\mathbf{v}}_i^E \cdot \nabla F_{0i} = \frac{Z_i e F_{0i}}{T} \frac{\partial \bar{\varphi}_i}{\partial t_s}, \end{aligned} \quad (2.39)$$

and

$$\begin{aligned} \frac{\partial \bar{h}_e}{\partial t_s} + v_\parallel \mathbf{b} \cdot \nabla \theta \frac{\partial \bar{h}_e}{\partial \theta} + \mathbf{v}_e^M \cdot \nabla_s \bar{h}_e + \bar{\mathbf{v}}_e^E \cdot \nabla_s \bar{h}_e + \nabla_s \cdot \langle \tilde{\mathbf{v}}_e^E \tilde{h}_e \rangle^{\text{ES}} \\ + \bar{\mathbf{v}}_e^E \cdot \nabla F_{0e} = -\frac{e F_{0e}}{T} \frac{\partial \bar{\varphi}_e}{\partial t_s}. \end{aligned} \quad (2.40)$$

The only modifications to the usual IS gyrokinetic equations are the inclusion of $\nabla_s \cdot \langle \tilde{\mathbf{v}}_i^E \tilde{h}_i \rangle^{\text{ES}}$ in (2.39), and $\nabla_s \cdot \langle \tilde{\mathbf{v}}_e^E \tilde{h}_e \rangle^{\text{ES}}$ in (2.40). The terms $\nabla_s \cdot \langle \tilde{\mathbf{v}}_i^E \tilde{h}_i \rangle^{\text{ES}}$ and $\nabla_s \cdot \langle \tilde{\mathbf{v}}_e^E \tilde{h}_e \rangle^{\text{ES}}$ physically represent the divergence of the spatial fluxes of $\bar{h}_i$ and $\bar{h}_e$ due to ES fluctuations. Note that (2.40) contains electron parallel streaming $v_\parallel \mathbf{b} \cdot \nabla \theta \partial \bar{h}_e / \partial \theta$. A naive estimate suggests that $v_\parallel \mathbf{b} \cdot \nabla \theta \partial \bar{h}_e / \partial \theta \sim (v_{\text{th},e}/a) \bar{h}_e$, whereas all other terms in (2.40) are at most of size $(v_{\text{th},i}/a) \bar{h}_e$. As a consequence, equation (2.40) is not properly scale separated, as the equation still contains $a/v_{\text{th},e}$ time scales and $\rho_{\text{th},e}$ spatial scales. To resolve this problem, we expand $\bar{h}_e$ in $(m_e/m_i)^{1/2}$. We take $\bar{h}_e =$

$\bar{h}_e^{(0)} + \bar{h}_e^{(1)} + O((m_e/m_i)\bar{h}_e)$, where $\bar{h}_e^{(0)}/F_{0e} \sim e\bar{\phi}/T$ and $\bar{h}_e^{(1)} \sim (m_e/m_i)^{1/2} \bar{h}_e^{(0)}$. Then, the leading order equation for $\bar{h}_e$ is

$$v_{\parallel} \mathbf{b} \cdot \nabla \theta \frac{\partial \bar{h}_e^{(0)}}{\partial \theta} = 0. \quad (2.41)$$

Equation (2.41) implies that $\bar{h}_e^{(0)}$ is constant in θ for passing particles, for which $v_{\parallel} > 0$. In addition, equation (2.41) implies that $\bar{h}_e$ is constant in θ for trapped particles in the region of θ where $v_{\parallel} > 0$. Defining the nonzonal piece of the distribution function to be the piece which contains variation in α_s , we show in A.3.2 that $\bar{h}_e^{(0)} = 0$ for the nonzonal passing piece of the electron distribution function. This is a result of the usual “twist-and-shift” boundary condition [44], combined with the observation that to be consistent with the use of a Fourier series for $\bar{h}_{\nu}$ we should impose $\bar{h}_{\nu \mathbf{k}_s} \rightarrow 0$ as $|\mathbf{k}_s \cdot \mathbf{e}_{\psi}| = |k_{\psi}| \rightarrow \infty$. To obtain the equation for the nonzero piece of $\bar{h}_e^{(0)}$, which is due to both zonal and nonzonal trapped electrons and only zonal passing electrons, we need to consider the next order equation for $\bar{h}_e$,

$$\begin{aligned} \frac{\partial \bar{h}_e^{(0)}}{\partial t_s} + v_{\parallel} \mathbf{b} \cdot \nabla \theta \frac{\partial \bar{h}_e^{(1)}}{\partial \theta} + \mathbf{v}_e^M \cdot \nabla_s \bar{h}_e^{(0)} + \bar{\mathbf{v}}_e^E \cdot \nabla_s \bar{h}_e^{(0)} + \nabla_s \cdot \left\langle \tilde{\mathbf{v}}_e^E \tilde{h}_e \right\rangle^{\text{ES}} \\ + \bar{\mathbf{v}}_e^E \cdot \nabla F_{0e} = -\frac{eF_{0e}}{T} \frac{\partial \bar{\varphi}_e}{\partial t_s}. \end{aligned} \quad (2.42)$$

We find an equation involving only leading-order piece of $\bar{h}_e$, $\bar{h}_e^{(0)}$, by using the orbital average $\langle \cdot \rangle^o$ to average over the rapid parallel orbits of the electrons in (2.42), and hence eliminate the term $v_{\parallel} \mathbf{b} \cdot \nabla \theta \partial \bar{h}_e^{(1)} / \partial \theta$ in (2.42). The orbital average, defined in (2.26) and (2.27), has the properties (2.30),

$$\left\langle \bar{h}_e^{(0)} \right\rangle^o = \bar{h}_e^{(0)} \quad (2.43)$$

and [69, 70]

$$\left\langle \mathbf{v}^M \cdot \nabla \psi \right\rangle^o = 0, \quad (2.44)$$

shown in A.3.3. The resulting scale-separated equation for the leading order piece of the electron distribution function at IS, $\bar{h}_e = \bar{h}_e^{(0)}$, is

$$\frac{\partial \bar{h}_e}{\partial t_s} + \left\langle \mathbf{v}_e^M \cdot \nabla \alpha \right\rangle^o \frac{\partial \bar{h}_e}{\partial \alpha_s} + \left\langle \bar{\mathbf{v}}_e^E \cdot \nabla_s \bar{h}_e \right\rangle^o + \left\langle \bar{\mathbf{v}}_e^E \cdot \nabla F_{0e} \right\rangle^o$$

$$+\nabla_s \cdot \left\langle \left\langle \tilde{\mathbf{v}}_e^E \tilde{h}_e \right\rangle^{\text{ES}} \right\rangle^o = -\frac{eF_{0e}}{T_e} \frac{\partial \langle \bar{\varphi}_e \rangle^o}{\partial t_s}, \quad (2.45)$$

where it is understood that $\bar{h}_e = 0$ for the nonzonal passing piece of the electron distribution function. Equation (2.45) captures the nonadiabatic response of zonal and nonzonal trapped electrons and zonal passing electrons at IS – all other components of the IS electron distribution function are small in the ordering. We note that equation (2.45) captures the magnetic drift of trapped particles in the α direction through the term $\langle \mathbf{v}_e^M \cdot \nabla \alpha \rangle^o \partial \bar{h}_e / \partial \alpha_s$. As a result, equation (2.45) describes the precession resonance due to the toroidal drift of the parallel-to-the-field orbits of trapped electrons, cf. [67]. To find the ES equation we subtract the IS equation (2.31) from the full equation (2.12). Again, the linear terms follow easily and the nonlinear term provides new cross-scale coupling terms:

$$\begin{aligned} \frac{\partial \tilde{h}_\nu}{\partial t_s} + \frac{\partial \tilde{h}_\nu}{\partial t_f} + v_{\parallel} \mathbf{b} \cdot \nabla \theta \frac{\partial \tilde{h}_\nu}{\partial \theta} + \mathbf{v}_\nu^M \cdot (\nabla_s + \nabla_f) \tilde{h}_\nu + \left[\mathbf{v}_\nu^E \cdot \nabla h_\nu - \langle \mathbf{v}_\nu^E \cdot \nabla h_\nu \rangle^{\text{ES}} \right] \\ + \left(\frac{c}{B} \mathbf{b} \times (\nabla_s + \nabla_f) \tilde{\varphi}_\nu \right) \cdot \nabla F_{0\nu} = \frac{Z_\nu e F_{0\nu}}{T_\nu} \left(\frac{\partial \tilde{\varphi}_\nu}{\partial t_s} + \frac{\partial \tilde{\varphi}_\nu}{\partial t_f} \right). \end{aligned} \quad (2.46)$$

We can further simplify (2.46) by neglecting sub-dominant terms in $(m_e/m_i)^{1/2}$; i.e., $\nabla_s + \nabla_f \approx \nabla_f$, $\partial/\partial t_s + \partial/\partial t_f \approx \partial/\partial t_f$, and $v_{\parallel} \mathbf{b} \cdot \nabla \theta \partial \tilde{h}_i / \partial \theta \ll \mathbf{v}^M \cdot \nabla_f \tilde{h}_i$ in the equation for ions. As shown in A.4.2, the nonlinear term reduces to

$$\mathbf{v}_\nu^E \cdot \nabla h_\nu - \langle \mathbf{v}_\nu^E \cdot \nabla h_\nu \rangle^{\text{ES}} = \tilde{\mathbf{v}}_\nu^E \cdot \nabla_f \tilde{h}_\nu + \bar{\mathbf{v}}_\nu^E \cdot \nabla_f \tilde{h}_\nu + \tilde{\mathbf{v}}_\nu^E \cdot \nabla_s \bar{h}_\nu, \quad (2.47)$$

where $\tilde{\mathbf{v}}_\nu^E \cdot \nabla_f \tilde{h}_\nu$ is the usual $\mathbf{E} \times \mathbf{B}$ nonlinearity appearing in the ES gyrokinetic equation in the absence of cross-scale coupling. Combining these results, the ES equation (2.46), for electrons, becomes

$$\begin{aligned} \frac{\partial \tilde{h}_e}{\partial t_f} + v_{\parallel} \mathbf{b} \cdot \nabla \theta \frac{\partial \tilde{h}_e}{\partial \theta} + (\mathbf{v}_e^M + \bar{\mathbf{v}}_e^E) \cdot \nabla_f \tilde{h}_e + \tilde{\mathbf{v}}_e^E \cdot \nabla_f \tilde{h}_e \\ + \tilde{\mathbf{v}}_e^E \cdot (\nabla F_{0e} + \nabla_s \bar{h}_e) = -\frac{eF_{0e}}{T} \frac{\partial \tilde{\varphi}_e}{\partial t_f}. \end{aligned} \quad (2.48)$$

For ions, equation (2.46) becomes

$$\frac{\partial \tilde{h}_i}{\partial t_f} + (\mathbf{v}_i^M + \bar{\mathbf{v}}_i^E) \cdot \nabla_f \tilde{h}_i + \tilde{\mathbf{v}}_i^E \cdot \nabla_f \tilde{h}_i + \tilde{\mathbf{v}}_i^E \cdot (\nabla F_{0i} + \nabla_s \bar{h}_i) = \frac{Z_i e F_{0i}}{T} \frac{\partial \tilde{\varphi}_i}{\partial t_f}. \quad (2.49)$$

Equations (2.48) and (2.49) contain the effect of IS gradients on ES fluctuations through two terms: $\bar{\mathbf{v}}^E \cdot \nabla_f \tilde{h}$, which represents advection of ES fluctuations by IS eddies; and $\tilde{\mathbf{v}}^E \cdot \nabla_s \bar{h}$, which represents the modification of the equilibrium gradient drive due to gradients in the IS fluctuations. Note that the term $\bar{\mathbf{v}}^E \cdot \nabla_f \tilde{h}$ cannot be eliminated by a change of reference frame because the advection velocity $\bar{\mathbf{v}}^E$ is a function of the parallel-to-the-field coordinate θ , as $\bar{\phi}$ varies along the magnetic field line. The ES equations (2.48) and (2.49) are scale separated in the sense that $\mathbf{R}_s$ appears only as a label of the IS gradients $\nabla_s \bar{\phi}(\mathbf{R}_s)$ and $\nabla_s \bar{h}(\mathbf{R}_s)$, and as a label on the fields $\tilde{\phi}(\mathbf{R}_s, \mathbf{R}_f)$ and $\tilde{h}(\mathbf{R}_s, \mathbf{R}_f)$. The equation for electrons at ES (2.48) can capture long electron temperature gradient (ETG) turbulence saturation times, such as observed in e.g. [25], provided that these saturation times does not scale with $a/v_{th,i}$.

As we will find in Section 2.5, we need not evolve equation (2.49) as a part of the scale-separated system. Physically, this is because the nonadiabatic response of ions at ES, $\tilde{h}_i$, is small, as the ions respond to the small scale ES potential averaged over scales of the large ion gyroradius. The ion cyclotron frequency Ω_i is much larger than the typical ES frequency $\omega \sim k_{\parallel} v_{th,e} \sim v_{th,e}/a$. We note that

$$\omega \sim \frac{v_{th,e}}{a} \sim \frac{\rho_{th,e}}{a} \Omega_e \sim \left(\frac{m_i}{m_e} \right)^{1/2} \frac{\rho_{th,i}}{a} \Omega_i \ll \Omega_i, \quad \text{if} \quad \frac{\rho_{th,i}}{a} \ll \left(\frac{m_e}{m_i} \right)^{1/2}. \quad (2.50)$$

Hence we find that the typical ES frequency $\omega \ll \Omega_i$ when $\rho_{th,i}/a \ll (m_e/m_i)^{1/2}$; this condition is naturally satisfied in the framework of this thesis where the asymptotic expansion in $(m_e/m_i)^{1/2}$ is subsidiary to the gyrokinetic expansion in $\rho_{th,i}/a$. The source terms which set the size of $\tilde{h}_i$ in equation (2.49), $\tilde{\mathbf{v}}_i^E \cdot (\nabla F_{0i} + \nabla_s \bar{h}_i)$ and $(Z_i e F_{0i}/T) \partial \tilde{\phi}_i / \partial t_f$, both depend on the ion gyroaveraged potential $\tilde{\phi}_i$. As we will discuss in Section 2.5, due to the large size of the ion gyroradius compared to the scale of ES fluctuations the ion gyroaveraged potential is small $e \tilde{\phi}_i / T \sim (m_e/m_i)^{1/4} e \tilde{\phi} / T$, and so $\tilde{h}_i$ becomes negligible as $(m_e/m_i)^{1/2} \rightarrow 0$.

2.4 Quasineutrality

Using (2.17) and (2.34), and noting that the integration variable for the ES average can be either $\mathbf{r}_f$, or $\mathbf{R}_f$ (see A.2), one can directly find the IS quasineutrality relation,

$$\sum_{\nu} Z_{\nu} e \int d^3 \mathbf{v} |_{\mathbf{r}_s} \bar{h}_{\nu}(\theta, \mathbf{R}_s, \varepsilon, \lambda, \sigma) = \sum_{\nu} \frac{Z_{\nu}^2 e^2 n_{\nu}}{T_{\nu}} \bar{\phi}(\theta, \mathbf{r}_s), \quad (2.51)$$

which is supplemented by the relation (2.33) for $\bar{\varphi}_{\nu}$. We find the quasineutrality relation for the ES by subtracting (2.51) from the full quasineutrality relation (2.15) to obtain

$$\sum_{\nu} Z_{\nu} e \int d^3 \mathbf{v} |_{\mathbf{r}_s, \mathbf{r}_f} \tilde{h}_{\nu}(\theta, \mathbf{R}_s, \mathbf{R}_f, \varepsilon, \lambda, \sigma) = \sum_{\nu} \frac{Z_{\nu}^2 e^2 n_{\nu}}{T_{\nu}} \tilde{\phi}(\theta, \mathbf{r}_s, \mathbf{r}_f), \quad (2.52)$$

which is supplemented by the relation (2.36) for $\tilde{\varphi}_{\nu}$. The system of equations (2.39), (2.45), and (2.48)-(2.52) constitutes a formally closed system of equations for the IS and ES quantities. To calculate the ion contribution to the ES quasineutrality relation (2.52) requires that we take a gyroaverage over a scale of the ion gyroradius, as does computing the ion gyroaveraged potential $\tilde{\varphi}_i$ according to equation (2.36). Naively one might think that the ion species would thus introduce ion gyroradius scale correlations in the ES turbulence. However, by considering the possible dominant balances in the IS-ES system we will find the contributions from ions at ES can always be ignored, and so we will arrive at a fully scale-separated system.

2.5 Sizes of the IS and ES fluctuations and scale separation

We now proceed to find all possible self-consistent orderings for $\bar{h}_i$, $\bar{h}_e$, $\tilde{h}_i$, $\tilde{h}_e$, $\bar{\phi}$, and $\tilde{\phi}$ that lead to statistically steady-state solutions to the system of equations (2.39), (2.45), and (2.48)-(2.52). For a statistically steady-state turbulence to be possible there must be a competition between the growth of linear instabilities and nonlinear interactions, which can be cross scale in nature. We look for dominant balances consistent with this observation.

Note that the presence of the gyroaverages can introduce mass ratio factors. Recalling that we have periodicity perpendicular to the field line in both the slow and

fast variables, we write a fluctuation $\phi(\mathbf{r}_s, \mathbf{r}_f)$ as,

$$\phi(\mathbf{r}_s, \mathbf{r}_f) = \sum_{\mathbf{k}_f} \phi_{\mathbf{k}_f}(\mathbf{r}_s) \exp[\mathbf{i}\mathbf{k}_f \cdot \mathbf{r}_f] = \sum_{\mathbf{k}_f, \mathbf{k}_s} \phi_{\mathbf{k}_f, \mathbf{k}_s} \exp[\mathbf{i}\mathbf{k}_f \cdot \mathbf{r}_f] \exp[\mathbf{i}\mathbf{k}_s \cdot \mathbf{r}_s], \quad (2.53)$$

where $\mathbf{k}_f \sim \rho_{\text{th},e}^{-1}$, $\mathbf{k}_s \sim \rho_{\text{th},i}^{-1}$, and we have used periodicity in $\mathbf{r}_s$ to write $\phi_{\mathbf{k}_f}(\mathbf{r}_s) = \sum_{\mathbf{k}_s} \phi_{\mathbf{k}_f, \mathbf{k}_s} \exp[\mathbf{i}\mathbf{k}_s \cdot \mathbf{r}_s]$. If we gyroaverage (2.53), recalling $\mathbf{R}_f = \mathbf{r}_f - \boldsymbol{\rho}$ and $\mathbf{R}_s = \mathbf{r}_s - \boldsymbol{\rho}$, we find

$$\begin{aligned} \langle \phi(\mathbf{r}_s, \mathbf{r}_f) \rangle_{\mathbf{R}_f, \mathbf{R}_s}^\gamma &= \sum_{\mathbf{k}_f, \mathbf{k}_s} \phi_{\mathbf{k}_f, \mathbf{k}_s} \exp[\mathbf{i}\mathbf{k}_f \cdot \mathbf{R}_f] \exp[\mathbf{i}\mathbf{k}_s \cdot \mathbf{R}_s] \langle \exp[\mathbf{i}(\mathbf{k}_f + \mathbf{k}_s) \cdot \boldsymbol{\rho}] \rangle^\gamma \\ &= \sum_{\mathbf{k}_f, \mathbf{k}_s} \phi_{\mathbf{k}_f, \mathbf{k}_s} \exp[\mathbf{i}\mathbf{k}_f \cdot \mathbf{R}_f] \exp[\mathbf{i}\mathbf{k}_s \cdot \mathbf{R}_s] J_0(|\mathbf{k}_f + \mathbf{k}_s||\boldsymbol{\rho}|), \end{aligned} \quad (2.54)$$

where $J_0(z)$ is the 0th Bessel function of the 1st kind, and we used the result

$$\langle \exp[\mathbf{i}\mathbf{k} \cdot \boldsymbol{\rho}] \rangle^\gamma = J_0(|\mathbf{k}||\boldsymbol{\rho}|), \quad (2.55)$$

shown in A.1. Note that $J_0(z)$ is an even function of z , and so the sign of Ω_ν does not enter in (2.55). In addition, note that $J_0(z) \sim 1$ for $z \lesssim 1$ and $J_0(z) \sim z^{-1/2}$ for $z \gg 1$. Because $|\mathbf{k}_f|\rho_{\text{th},e} \sim |\mathbf{k}_s|\rho_{\text{th},i} \sim 1$ and $|\mathbf{k}_s|\rho_{\text{th},e} \sim (m_e/m_i)^{1/2} \ll 1$, the gyroaverage introduces no additional mass ratio factors for IS quantities or for electrons at the ES. However, because $|\mathbf{k}_f|\rho_{\text{th},i} \sim (m_i/m_e)^{1/2} \gg 1$, the gyroaveraging operation does reduce the ion fluctuation amplitudes at the ES; i.e.,

$$J_0(|\mathbf{k}_f + \mathbf{k}_s||\boldsymbol{\rho}|) \sim J_0(|\mathbf{k}_f|\rho_{\text{th},i}) \sim \left(\frac{m_e}{m_i}\right)^{1/4}. \quad (2.56)$$

With these scalings for the Bessel functions, we show in A.5 that the only possibility for achieving a saturated dominant balance in the equations (2.39), (2.45), and (2.48)-(2.52) is for the fluctuations to obey the gyro-Bohm ordering

$$\begin{aligned} \frac{e\bar{\phi}}{T} &\sim \frac{\rho_{\text{th},i}}{a}, \quad \frac{e\tilde{\phi}}{T} \sim \frac{\rho_{\text{th},e}}{a}, \\ \frac{\bar{h}_i}{F_{0i}} &\sim \frac{\bar{h}_e}{F_{0e}} \sim \frac{e\bar{\varphi}_i}{T} \sim \frac{e\bar{\varphi}_e}{T} \sim \frac{e\bar{\phi}}{T}, \end{aligned} \quad (2.57)$$

$$\frac{\tilde{h}_e}{F_{0e}} \sim \frac{e\tilde{\varphi}_e}{T} \sim \frac{e\tilde{\phi}}{T},$$

$$\frac{\tilde{h}_i}{F_{0i}} \sim \frac{e\tilde{\varphi}_i}{T} \sim \left(\frac{m_e}{m_i}\right)^{1/4} \frac{e\tilde{\phi}}{T}.$$

Note that (2.57) excludes orderings in which either the IS or ES fluctuations have amplitudes which are greater or smaller than gyro-Bohm levels by a factor which scales with mass ratio. The ordering (2.57) is obtained with the assumption that numbers which control the saturation level of the IS and ES turbulence in detail are $O(1)$ numbers in the $(m_e/m_i)^{1/2}$ expansion. Such parameters include the degree of spatial anisotropy in the turbulence, the ratio of the nonzonal to zonal amplitudes and the ratio of the drives of instability at IS and ES. In A.5 we consider the possible ways in which cross-scale interaction might have allowed for novel mass ratio orderings which deviate from the gyro-Bohm scaling (2.57). We looked for balances where the ES turbulence is enhanced, with $e\tilde{\phi}/T \gg \rho_{th,e}/a$ (see A.5.1); the IS turbulence is enhanced, with $e\bar{\phi}/T \gg \rho_{th,i}/a$ (see A.5.2); the IS turbulence suppresses the ES turbulence, i.e. $e\tilde{\phi}/T \ll \rho_{th,e}/a$ and $e\bar{\phi}/T \sim \rho_{th,i}/a$ (see A.5.3); and the ES turbulence suppresses the IS turbulence, i.e. $e\tilde{\phi}/T \sim \rho_{th,e}/a$ and $e\bar{\phi}/T \ll \rho_{th,i}/a$ (see A.5.4). Of these possible impacts of cross-scale interactions, only the suppression of ES turbulence by IS turbulence is self-consistently allowed by dominant balance. Hence, the ordering (2.57) is the only ordering which gives a nonlinearly saturated steady-state turbulence. We discuss the physical meaning of ordering (2.57) in Section 2.6.

We now consider the relative sizes of the cross-scale terms appearing in (2.39), (2.45), (2.48), and (2.49) in the ordering (2.57), and demonstrate that the nonadiabatic response of ions at ES $\tilde{h}_i$ can be neglected. As a corollary, we need not evolve equation (2.49). Finally, we consider critical balance arguments to show that the orderings (2.57) for the sizes of fluctuations are consistent with the orderings for the parallel length scales (2.19). We use these considerations in Section 2.6, where we give the fully scale-separated IS and ES equations, with quasineutrality evaluated consistently with scale separation, and all small terms neglected.

2.5.1 Influence of ES fluctuations on ions at IS

Equation (2.39) contains a term which is a divergence of a turbulent flux driven at the ES. This cross-scale term is of size

$$\nabla_s \cdot \left\langle \tilde{\mathbf{v}}_i^E \tilde{h}_i \right\rangle^{\text{ES}} \sim \frac{c}{B} \frac{\tilde{\varphi}_i \tilde{h}_i}{\rho_{\text{th},i} \rho_{\text{th},e}} \sim \frac{c}{B} \frac{\tilde{\phi}}{\rho_{\text{th},i}^2} \frac{e\tilde{\phi}}{T} F_{0i}, \quad (2.58)$$

where we have used that $\tilde{h}_i/F_{0i} \sim e\tilde{\varphi}_i/T \sim (m_e/m_i)^{1/4} e\tilde{\phi}/T$ in the gyro-Bohm ordering (2.57). Hence, in the gyro-Bohm ordering (2.57),

$$\nabla_s \cdot \left\langle \tilde{\mathbf{v}}_i^E \tilde{h}_i \right\rangle^{\text{ES}} \sim \frac{m_e}{m_i} \frac{c}{B} \frac{\bar{\phi}}{\rho_{\text{th},i}^2} \frac{e\bar{\phi}}{T} F_{0i} \sim \frac{m_e}{m_i} \bar{\mathbf{v}}_i^E \cdot \nabla_s \bar{h}_i. \quad (2.59)$$

Therefore, the cross-scale term is small compared to the single scale nonlinear term $\bar{\mathbf{v}}_i^E \cdot \nabla_s \bar{h}_i$, which provides the IS saturation mechanism in the ordering (2.57). We conclude that we can neglect $\nabla_s \cdot \left\langle \tilde{\mathbf{v}}_i^E \tilde{h}_i \right\rangle^{\text{ES}}$ in the equation for ions at IS (2.39).

2.5.2 Influence of ES fluctuations on electrons at IS

The cross-scale term in (2.45) is of size

$$\nabla_s \cdot \left\langle \left\langle \tilde{\mathbf{v}}_e^E \tilde{h}_e \right\rangle^{\text{ES}} \right\rangle^o \sim \left(\frac{m_i}{m_e} \right)^{1/2} \frac{c}{B} \frac{\tilde{\phi}}{\rho_{\text{th},i}^2} \frac{e\tilde{\phi}}{T} F_{0e}, \quad (2.60)$$

and so in the gyro-Bohm ordering (2.57),

$$\nabla_s \cdot \left\langle \left\langle \tilde{\mathbf{v}}_e^E \tilde{h}_e \right\rangle^{\text{ES}} \right\rangle^o \sim \left(\frac{m_e}{m_i} \right)^{1/2} \frac{c}{B} \frac{\bar{\phi}}{\rho_{\text{th},i}^2} \frac{e\bar{\phi}}{T} F_{0e} \sim \left(\frac{m_e}{m_i} \right)^{1/2} \bar{\mathbf{v}}_e^E \cdot \nabla_s \bar{h}_e. \quad (2.61)$$

Hence in the gyro-Bohm ordering (2.57), we can neglect the electron cross-scale term $\nabla_s \cdot \left\langle \left\langle \tilde{\mathbf{v}}_e^E \tilde{h}_e \right\rangle^{\text{ES}} \right\rangle^o$ to leading order in the IS electron equation (2.45).

2.5.3 Influence of IS gradients on ES fluctuations

The cross-scale terms appearing in the ES gyrokinetic equation for electrons (2.48) have sizes

$$\bar{\mathbf{v}}_e^E \cdot \nabla_f \tilde{h}_e \sim \frac{c}{B} \frac{\bar{\phi} \tilde{h}_e}{\rho_{\text{th},i} \rho_{\text{th},e}}, \quad (2.62)$$

and

$$\tilde{\mathbf{v}}_e^E \cdot \nabla_s \bar{h}_e \sim \frac{c}{B} \frac{\tilde{\phi} \bar{h}_e}{\rho_{\text{th},i} \rho_{\text{th},e}}. \quad (2.63)$$

Hence, in the gyro-Bohm ordering (2.57), when $e\bar{\phi}/T \sim \rho_{\text{th},i}/a$ and $e\tilde{\phi}/T \sim \rho_{\text{th},e}/a$, we find that

$$\bar{\mathbf{v}}_e^E \cdot \nabla_f \tilde{h}_e \sim \tilde{\mathbf{v}}_e^E \cdot \nabla_s \bar{h}_e \sim \tilde{\mathbf{v}}_e^E \cdot \nabla F_{0e}. \quad (2.64)$$

The cross-scale terms modify the ES dynamics at leading order. Therefore, these terms can provide the mechanism for enhancement or suppression of the ES turbulence in the presence of IS fluctuations. The IS gradients can linearly stabilise the ES instability, but the enhancement of the ES fluctuation amplitude cannot scale with any power of the mass ratio.

2.5.4 Ions at ES

The contribution of ions to the ES electrostatic potential is small in $(m_e/m_i)^{1/2}$. This is seen by comparing the relative sizes of ion and electron contributions to the ES quasineutrality relation. Observe that

$$\begin{aligned} Z_i \int d^3\mathbf{v} |_{\mathbf{r}_s, \mathbf{r}_f} \tilde{h}_i(\theta, \mathbf{R}_s, \mathbf{R}_f, \varepsilon, \lambda, \sigma) &\sim \left(\frac{m_e}{m_i}\right)^{1/2} \frac{e\tilde{\phi}}{T} Z_i n_i \\ &\ll \int d^3\mathbf{v} |_{\mathbf{r}_s, \mathbf{r}_f} \tilde{h}_e(\theta, \mathbf{R}_s, \mathbf{R}_f, \varepsilon, \lambda, \sigma) \sim \frac{e\tilde{\phi}}{T} n_e. \end{aligned} \quad (2.65)$$

The factor of $(m_e/m_i)^{1/2}$ in the ion contribution to the ES electrostatic potential appears due to the combination of two effects. The velocity integration for ions introduces an ion gyroaverage, and hence a factor of $(m_e/m_i)^{1/4}$. The scaling of $\tilde{h}_i \sim (m_e/m_i)^{1/4} e\tilde{\phi}/T$ introduces a second factor of $(m_e/m_i)^{1/4}$. We conclude that the nonadiabatic response of ions at ES does not contribute to the ES potential to leading order. Physically we are able to neglect the ions at ES because the ion gyroradius is much larger than the ES domain. The ions rapidly gyrate at the ion cyclotron frequency Ω_i . As shown in the estimate (2.50), in the scale-separated model Ω_i is much larger than the typical ES turbulent frequencies $\omega \sim v_{\text{th},e}/a$, i.e. $\Omega_i \gg \omega$. Hence,

the ions rapidly sample many uncorrelated instances of ES turbulence because of their larger gyroradius. The ions effectively respond to a spatial average of ES turbulence at fixed time. In the asymptotic limit $(m_e/m_i)^{1/2} \rightarrow 0$, ions can only weakly respond to ES fluctuations because, as a consequence of statistical periodicity, spatial averages over ES turbulence should vanish. Taken with the conclusion of Section 2.5.1, we see that the nonadiabatic response of ions at the ES can be neglected entirely.

2.5.5 Critical balance

Observe that the gyro-Bohm ordering (2.57) is consistent with the critical balance argument [71] at the IS and ES separately. At both scales the $\mathbf{E} \times \mathbf{B}$ drift has the same magnitude

$$V^E \sim \frac{c}{B} \mathbf{b} \times \nabla_s \bar{\phi} \sim \frac{c}{B} \mathbf{b} \times \nabla_s \tilde{\phi} \sim \frac{v_{\text{th},i} \rho_{\text{th},i}}{a} \sim \frac{v_{\text{th},e} \rho_{\text{th},e}}{a}. \quad (2.66)$$

By assumption, the IS perpendicular correlation length $\bar{l}_\perp \sim \rho_{\text{th},i}$ and the ES perpendicular correlation length $\tilde{l}_\perp \sim \rho_{\text{th},e}$. The IS nonlinear turnover time $\bar{\tau}_{\text{nl}}$ obeys

$$\bar{\tau}_{\text{nl}} \sim \frac{\bar{l}_\perp}{V^E} \sim \frac{a}{v_{\text{th},i}}. \quad (2.67)$$

The ES nonlinear turnover time $\tilde{\tau}_{\text{nl}}$ obeys

$$\tilde{\tau}_{\text{nl}} \sim \frac{\tilde{l}_\perp}{V^E} \sim \frac{a}{v_{\text{th},e}}. \quad (2.68)$$

In critical balance, the parallel extent of the eddies is set by how far information can be transmitted by waves in one nonlinear turnover time. Structures much larger than this distance cannot remain correlated, and structures much shorter than this distance are Landau damped. This implies that the IS parallel correlation length $\bar{l}_\parallel$ is

$$\bar{l}_\parallel \sim v_{\text{th},i} \bar{\tau}_{\text{nl}} \sim a, \quad (2.69)$$

where we have used that at $\rho_{\text{th},i}$ scales the dominant waves which communicate information in the direction parallel to the field line will have group velocities of the order $v_{\text{th},i}$. Using similar arguments we find that the ES parallel correlation length $\tilde{l}_\parallel$ is

$$\tilde{l}_\parallel \sim v_{\text{th},e} \tilde{\tau}_{\text{nl}} \sim a, \quad (2.70)$$

where we have used that at $\rho_{\text{th},e}$ scales the dominant waves which communicate information in the direction parallel to the field line will have group velocities of the order $v_{\text{th},e}$. This result is consistent with the ordering (2.19).

2.6 Scale-separated, coupled equations

Following the discussion in the previous Section, we can now resolve how to take the gyroaverages in the quasineutrality relation (2.52) in a scale-separated way. We neglect the nonadiabatic response of ions at ES because the ion contribution to ES quasineutrality (2.52) is small by $(m_e/m_i)^{1/2}$ (see Section 2.5.4). At leading order, equation (2.52) becomes

$$-e \int d^3\mathbf{v} |_{\mathbf{r}_s, \mathbf{r}_f} \tilde{h}_e(\theta, \mathbf{R}_s, \mathbf{R}_f, \varepsilon, \lambda, \sigma) = \sum_{\nu} \frac{Z_{\nu}^2 e^2 n_{\nu}}{T_{\nu}} \tilde{\phi}(\theta, \mathbf{r}_s, \mathbf{r}_f). \quad (2.71)$$

When solving the ES gyrokinetic equation (2.48), the quantity that we need to close the equation is $\tilde{\varphi}_e(\mathbf{R}_s, \mathbf{R}_f)$. Noting that $|\mathbf{k}_f||\boldsymbol{\rho}_e| \sim |\mathbf{k}_f|\rho_{\text{th},e} \sim 1$ and $|\mathbf{k}_s||\boldsymbol{\rho}_e| \sim |\mathbf{k}_s|\rho_{\text{th},e} \sim (m_e/m_i)^{1/2}$, in A.6 we expand the Bessel function $J_0(|\mathbf{k}_f + \mathbf{k}_s||\boldsymbol{\rho}_e|)$, due to electron gyroaverages, in the expression for $\tilde{\varphi}_e(\mathbf{R}_s, \mathbf{R}_f)$ to find

$$\tilde{\varphi}_e(\mathbf{R}_s, \mathbf{R}_f) = - \left(\sum_{\nu} \frac{Z_{\nu}^2 e n_{\nu}}{T_{\nu}} \right)^{-1} \sum_{\mathbf{k}_f} \exp[i\mathbf{k}_f \cdot \mathbf{R}_f] J_0(|\mathbf{k}_f||\boldsymbol{\rho}_e|) \times \int d^3\mathbf{v} |_{\mathbf{R}_s} \tilde{h}_{e\mathbf{k}_f}(\theta, \mathbf{R}_s, \varepsilon, \lambda, \sigma) J_0(|\mathbf{k}_f||\boldsymbol{\rho}_e|) \left(1 + \mathcal{O} \left(\left(\frac{m_e}{m_i} \right)^{1/2} \right) \right), \quad (2.72)$$

where we can hold $\mathbf{R}_s$ fixed in the velocity integration because electron gyroradii are small compared to IS structures. In equation (2.72) $\mathbf{R}_s$ only appears as a label. Therefore, we have found a scale-separated scheme where ES fluctuations at different $\mathbf{R}_s$ can be determined independently, as long as there is no coupling introduced by the parallel boundary condition (considered in Section 2.7).

We now present the full system of scale-separated equations keeping only those terms that appear at leading order in the gyro-Bohm ordering (2.57). At IS we have

$$\frac{\partial \bar{h}_i}{\partial t_s} + v_{\parallel} \mathbf{b} \cdot \nabla \theta \frac{\partial \bar{h}_i}{\partial \theta} + \mathbf{v}_i^M \cdot \nabla_s \bar{h}_i + \bar{\mathbf{v}}_i^E \cdot \nabla_s \bar{h}_i + \bar{\mathbf{v}}_i^E \cdot \nabla F_{0i} = \frac{Z_i e F_{0i}}{T_i} \frac{\partial \bar{\varphi}_i}{\partial t_s}, \quad (2.73)$$

and

$$\frac{\partial \bar{h}_e}{\partial t_s} + \langle \mathbf{v}_e^M \cdot \nabla \alpha \rangle^\circ \frac{\partial \bar{h}_e}{\partial \alpha_s} + \langle \bar{\mathbf{v}}_e^E \cdot \nabla_s \bar{h}_e \rangle^\circ + \langle \bar{\mathbf{v}}_e^E \cdot \nabla F_{0e} \rangle^\circ = -\frac{eF_{0e}}{T_e} \frac{\partial \langle \bar{\varphi}_e \rangle^\circ}{\partial t_s}, \quad (2.74)$$

where $\bar{\varphi}_i = \langle \bar{\phi} \rangle_{\mathbf{R}_s, i}^\gamma$, $\bar{\varphi}_e = \langle \bar{\phi} \rangle_{\mathbf{R}_s, e}^\gamma$ as defined in equation (2.33), $\langle \cdot \rangle^\circ$ is the orbital average defined in (2.26) and (2.27), with the properties (2.28) and (2.30), and it is understood that $\bar{h}_e = 0$ for nonzonal passing electrons. These equations are closed by the quasineutrality relation

$$\sum_\nu Z_\nu e \int d^3 \mathbf{v} |_{\mathbf{r}_s} \bar{h}_\nu(\theta, \mathbf{R}_s, \varepsilon, \lambda, \sigma) = \sum_\nu \frac{Z_\nu^2 e^2 n_\nu}{T_\nu} \bar{\phi}(\theta, \mathbf{r}_s). \quad (2.75)$$

At ES we have,

$$\begin{aligned} \frac{\partial \tilde{h}_e}{\partial t_f} + v_\parallel \mathbf{b} \cdot \nabla \theta \frac{\partial \tilde{h}_e}{\partial \theta} + (\mathbf{v}_e^M + \bar{\mathbf{v}}_e^E) \cdot \nabla_f \tilde{h}_e + \tilde{\mathbf{v}}_e^E \cdot \nabla_f \tilde{h}_e \\ + \tilde{\mathbf{v}}_e^E \cdot (\nabla F_{0e} + \nabla_s \bar{h}_e) = -\frac{eF_{0e}}{T_e} \frac{\partial \tilde{\varphi}_e}{\partial t_f}, \end{aligned} \quad (2.76)$$

which is closed by the quasineutrality relation,

$$\tilde{\varphi}_e(\theta, \mathbf{R}_s, \mathbf{R}_f) = -\left(\sum_\nu \frac{Z_\nu^2 e n_\nu}{T_\nu} \right)^{-1} \left\langle \int d^3 \mathbf{v} |_{\mathbf{R}_s} \langle \tilde{h}_e(\theta, \mathbf{R}_s, \mathbf{R}_f, \varepsilon, \lambda, \sigma) \rangle_{\mathbf{R}_s, \mathbf{r}_f, e}^\gamma \right\rangle_{\mathbf{R}_s, \mathbf{R}_f, e}^\gamma \quad (2.77)$$

The notation in (2.77) indicates that the gyroaverage does not average over the slow variable $\mathbf{R}_s$, which is left fixed during the integrations.

Inspecting equations (2.73)-(2.77), we can see that the gyro-Bohm ordering (2.57) is an ordering in which the IS is dominant: the ES turbulence is modified at leading order by the cross scale terms $\bar{\mathbf{v}}_e^E \cdot \nabla_f \tilde{h}_e$ and $\tilde{\mathbf{v}}_e^E \cdot \nabla_s \bar{h}_e$ in (2.76) (see Section 2.5.3); the IS turbulence evolves independently of the ES fluctuations, as the largest possible cross-scale term in the IS equations, $\nabla_s \cdot \left\langle \left\langle \tilde{\mathbf{v}}_e^E \tilde{h}_e \right\rangle^{\text{ES}} \right\rangle^\circ$, is small in the $(m_e/m_i)^{1/2}$ ordering (see Sections 2.5.1 and 2.5.2); and the IS heat flux for ions $\bar{Q}_i$ and electrons $\bar{Q}_e$ dominates the contribution to the heat flux from electrons at ES $\tilde{Q}_e$, i.e.

$$\bar{Q}_i \sim \bar{Q}_e \sim nT v_{\text{th}, i} \left(\frac{\rho_{\text{th}, i}}{a} \right)^2 \gg \tilde{Q}_e \sim nT v_{\text{th}, e} \left(\frac{\rho_{\text{th}, e}}{a} \right)^2. \quad (2.78)$$

We use the model to recover the usual gyro-Bohm estimate for the heat fluxes (2.78) in A.7. We note that the heat flux contribution from nonzonal passing electrons at

IS, which we self-consistently neglect, is formally comparable to $\tilde{Q}_e$. This is not an inconsistency, but the result of the dominance of IS transport in the gyro-Bohm ordering. The estimates (2.78) do not capture the behaviour of the larger than electron gyro-Bohm heat flux observed in [22, 26, 27]. In [22, 26, 27] $\tilde{Q}_e$ was large due to the presence of spatially anisotropic ETG streamers. Recall that we have assumed that the degree of spatial anisotropy in the turbulence in the plane perpendicular to the magnetic field does not scale with $(m_e/m_i)^{1/2}$. We have regarded all physical parameters besides $\rho_{th,\nu}/a$ and $(m_e/m_i)^{1/2}$ as of order unity. Deviation from the gyro-Bohm ordering (2.57), and the consequent dominance of IS transport (2.78), may be possible in a model which breaks these assumptions. We do not consider such a model in this Chapter.

2.6.1 Fourier representation

For equations (2.73)-(2.77) to be implemented in a code it is convenient for them to be expressed spectrally. For clarity we give the full system of equations (2.73)-(2.77) in Fourier space. At IS we have

$$\begin{aligned} \frac{\partial \bar{h}_{i\mathbf{k}_s}}{\partial t_s} + v_{\parallel} \mathbf{b} \cdot \nabla \theta \frac{\partial \bar{h}_{i\mathbf{k}_s}}{\partial \theta} + [\mathbf{i} \mathbf{v}_i^M \cdot \mathbf{k}_s] \bar{h}_{i\mathbf{k}_s} - \sum_{\mathbf{k}'_s} \frac{c}{B} \mathbf{b} \times \mathbf{k}'_s \cdot \mathbf{k}_s \bar{\varphi}_{i\mathbf{k}'_s} \bar{h}_{i\mathbf{k}_s - \mathbf{k}'_s} \\ + \left[\mathbf{i} \frac{c}{B} \mathbf{b} \times \mathbf{k}_s \cdot \nabla F_{0i} \right] \bar{\varphi}_{i\mathbf{k}_s} = \frac{Z_i e F_{0i}}{T_i} \frac{\partial \bar{\varphi}_{i\mathbf{k}_s}}{\partial t_s}, \end{aligned} \quad (2.79)$$

and

$$\begin{aligned} \frac{\partial \bar{h}_{e\mathbf{k}_s}}{\partial t_s} + \left[\mathbf{i} \langle \mathbf{v}_e^M \cdot \nabla \alpha \rangle^o \mathbf{e}_\alpha \cdot \mathbf{k}_s \right] \bar{h}_{e\mathbf{k}_s} - \left\langle \sum_{\mathbf{k}'_s} \frac{c}{B} \mathbf{b} \times \mathbf{k}'_s \cdot \mathbf{k}_s \bar{\varphi}_{e\mathbf{k}'_s} \bar{h}_{e\mathbf{k}_s - \mathbf{k}'_s} \right\rangle^o \\ + \left\langle \left[\mathbf{i} \frac{c}{B} \mathbf{b} \times \mathbf{k}_s \cdot \nabla F_{0e} \right] \bar{\varphi}_{e\mathbf{k}_s} \right\rangle^o = - \frac{e F_{0e}}{T_e} \frac{\partial \langle \bar{\varphi}_{e\mathbf{k}_s} \rangle^o}{\partial t_s}, \end{aligned} \quad (2.80)$$

where $\bar{h}_{e\mathbf{k}_s} = 0$ for the nonzonal ($\mathbf{k}_s \cdot \mathbf{e}_\alpha \neq 0$) passing piece of phase space, $\bar{\varphi}_{i\mathbf{k}_s} = J_0(|\mathbf{k}_s||\boldsymbol{\rho}_i|) \bar{\phi}_{\mathbf{k}_s}$ and $\bar{\varphi}_{e\mathbf{k}_s} = J_0(|\mathbf{k}_s||\boldsymbol{\rho}_e|) \bar{\phi}_{\mathbf{k}_s} = \bar{\phi}_{\mathbf{k}_s}$ as $J_0(|\mathbf{k}_s||\boldsymbol{\rho}_e|) = 1$ in the mass ratio expansion. Equations (2.79) and (2.80) are closed by the quasineutrality relation

$$Z_i \int d^3 \mathbf{v} J_0(|\mathbf{k}_s||\boldsymbol{\rho}_i|) \bar{h}_{i\mathbf{k}_s}(\theta, \varepsilon, \lambda, \sigma) - \int d^3 \mathbf{v} \bar{h}_{e\mathbf{k}_s}(\theta, \varepsilon, \lambda, \sigma) = \sum_{\nu} \frac{Z_{\nu}^2 e n_{\nu}}{T_{\nu}} \bar{\phi}_{\mathbf{k}_s}(\theta). \quad (2.81)$$

At ES we have

$$\begin{aligned} \frac{\partial \tilde{h}_{e\mathbf{k}_f}}{\partial t_f} + v_{\parallel} \mathbf{b} \cdot \nabla \theta \frac{\partial \tilde{h}_{e\mathbf{k}_f}}{\partial \theta} + \left[i(\mathbf{v}_e^M + \bar{\mathbf{v}}_e^E) \cdot \mathbf{k}_f \right] \tilde{h}_{e\mathbf{k}_f} - \sum_{\mathbf{k}'_f} \frac{c}{B} \mathbf{b} \times \mathbf{k}'_f \cdot \mathbf{k}_f \tilde{\varphi}_{e\mathbf{k}'_f} \tilde{h}_{e\mathbf{k}_f - \mathbf{k}'_f} \\ + \left[i \frac{c}{B} \mathbf{b} \times \mathbf{k}_f \cdot (\nabla F_{0e} + \nabla_s \bar{h}_e) \right] \tilde{\varphi}_{e\mathbf{k}_f} = - \frac{eF_{0e}}{T_e} \frac{\partial \tilde{\varphi}_{e\mathbf{k}_f}}{\partial t_f}. \end{aligned} \quad (2.82)$$

Equation (2.82) is closed by the quasineutrality relation

$$\tilde{\varphi}_{e\mathbf{k}_f}(\mathbf{R}_s) = - \left(\sum_{\nu} \frac{Z_{\nu}^2 e n_{\nu}}{T_{\nu}} \right)^{-1} J_0(|\mathbf{k}_f||\boldsymbol{\rho}_e|) \int d^3\mathbf{v} |_{\mathbf{R}_s} \tilde{h}_{e\mathbf{k}_f}(\theta, \mathbf{R}_s, \varepsilon, \lambda, \sigma) J_0(|\mathbf{k}_f||\boldsymbol{\rho}_e|), \quad (2.83)$$

and the relations for $\bar{\mathbf{v}}_e^E$ and $\nabla_s \bar{h}_e$,

$$\bar{\mathbf{v}}_e^E = \bar{\mathbf{v}}_e^E(\mathbf{R}_s) = \frac{c}{B} \mathbf{b} \times \left[\sum_{\mathbf{k}_s} \exp[i\mathbf{k}_s \cdot \mathbf{R}_s] i\mathbf{k}_s \bar{\varphi}_{e\mathbf{k}_s} \right], \quad (2.84)$$

$$\nabla_s \bar{h}_e = \nabla_s \bar{h}_e(\mathbf{R}_s) = \sum_{\mathbf{k}_s} \exp[i\mathbf{k}_s \cdot \mathbf{R}_s] i\mathbf{k}_s \bar{h}_{e\mathbf{k}_s}. \quad (2.85)$$

Note that the dependence of $\tilde{h}_{e\mathbf{k}_f}$ and $\tilde{\varphi}_{e\mathbf{k}_f}$ on $\mathbf{R}_s$ is parametric in equation (2.82), as the dependence on $\mathbf{R}_s$ only appears through the quantities $\bar{\mathbf{v}}_e^E$ and $\nabla_s \bar{h}_e$. The evolution equations for $\tilde{h}_{e\mathbf{k}_f}(\mathbf{R}_s)$ and $\tilde{\varphi}_{e\mathbf{k}_f}(\mathbf{R}_s)$ may be solved in a system of flux tubes, with a single ES flux tube for each of the considered $\mathbf{R}_s$ locations within the IS flux tube. We discuss the parallel boundary conditions for the IS-ES system of flux tubes in the next section.

2.7 The parallel boundary condition

In the previous Section we found the scale-separated, coupled IS-ES equations (2.73)-(2.77). Due to the assumptions of statistical periodicity, (2.7) and (2.18), these equations are solved with periodic boundary conditions in the plane perpendicular to the magnetic field line. In this section we briefly review the standard “twist-and-shift” flux tube parallel boundary condition [44]. We note that a key assumption in deriving the “twist-and-shift” boundary condition is that the geometrical and equilibrium quantities in the equation for the turbulence are axisymmetric. As the IS quantities

$\nabla_s \bar{h}_e$ and $\bar{v}_e^E$ in the electron ES equation (2.76) are not axisymmetric, we require a generalised “twist-and-shift” boundary condition for the ES flux tubes embedded in the IS turbulence. Here, we propose parallel boundary conditions for the IS-ES equations which allow the ES turbulence to be simulated in a system of connected flux tubes nested within a single IS flux tube. We express the parallel boundary conditions spectrally, and then discuss the parallel coupling between ES flux tubes. Finally, we show that the cross-scale terms in the ES electron equation (2.76) are continuous across the parallel boundaries of the ES flux tubes when the IS turbulence satisfies the “twist-and-shift” boundary condition and the ES turbulence satisfies the proposed ES flux tube boundary condition.

2.7.1 The standard “twist-and-shift” parallel boundary condition

We visualise a proto-typical flux tube domain in Figure 2.2. Note that in general the parallel ends of the flux tube do not align and have differing orientations of their perpendicular-to-the-field coordinate axes, due to the presence of magnetic shear. To derive the usual parallel boundary condition of [44] we note that the assumption of statistical periodicity of the turbulence in the binormal (α) direction perpendicular to the field line is equivalent to assuming that only large toroidal mode numbers $k_\alpha \geq k_\alpha^{\min} \gg 1$, with k_α integers, are required to accurately describe the turbulence. In consequence, in combination with the assumptions that the background geometry and equilibrium plasma profiles are axisymmetric, and the assumption that the parallel length of the flux tube is longer than the parallel-to-the-field correlation length of the turbulence, in the flux tube model we can map out the turbulence on the whole flux surface using $k_\alpha^{\min}$ identical flux tubes. In this picture, the parallel boundary condition is the requirement that the turbulence mapped out by the $k_\alpha^{\min}$ copies of the flux tube is single valued in the poloidal angle θ , at fixed toroidal angle ζ and flux label ψ . It is important to impose the single-valued condition in the physical coordinates (ψ, ζ, θ) , rather than in the flux tube coordinates (ψ, α, θ) , where $\alpha(\psi, \theta, \zeta)$ is defined by

$$\alpha(\psi, \theta, \zeta) - \alpha_0 = \zeta - q(\psi)\theta = \zeta - q_0\theta - q'_0\theta(\psi - \psi_0), \quad (2.86)$$

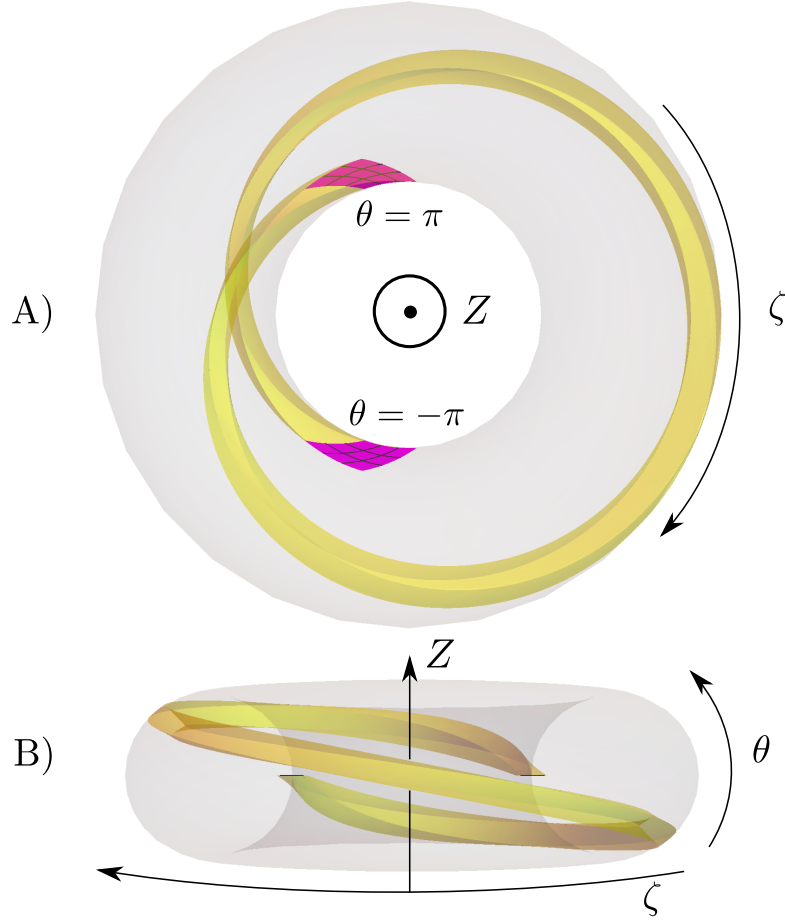


Figure 2.2: A) shows a view, down the toroidal symmetry axis (labelled by Z) of a flux tube, in yellow, with parallel ends in magenta. The flux surface is indicated in grey. B) shows the same flux tube as viewed from the right of A) perpendicular to the toroidal symmetry axis.

with $q(\psi)$ the safety factor, $q_0 = q(\psi_0)$, $q'_0 = dq/d\psi|_{\psi_0}$, and (ψ_0, α_0) the coordinates of the central field line in the flux tube. We cannot use the flux tube coordinates (ψ, α, θ) to apply the boundary condition because the basis vector defining the radial direction $\mathbf{e}_\psi = (\mathbf{b} \times \nabla \alpha) / |\nabla \psi \cdot \mathbf{b} \times \nabla \alpha|$ changes direction along the field line due to the presence of magnetic shear ($q'_0 \neq 0$), and thus the radial directions are different at the parallel ends of the flux tube, $\mathbf{e}_\psi(\theta) \neq \mathbf{e}_\psi(\theta + 2\pi)$. Finally, imposing the single-valued condition in the flux tube gives the parallel boundary condition [44]

$$h(\theta, \psi, \alpha(\psi, \theta, \zeta)) = h(\theta + 2\pi, \psi, \alpha(\psi, \theta + 2\pi, \zeta)), \quad (2.87)$$

where we have suppressed the velocity space dependences in h .

2.7.2 The IS and ES “twist-and-shift” parallel boundary conditions

In the case of ES turbulence embedded within IS turbulence, we must be careful when generalising the single-valued condition to find the ES flux tube boundary condition. The assumptions of statistical periodicity in the binormal direction for the ES turbulence, and the presence of the IS quantities $\nabla_s \bar{h}_e$ and $\bar{\mathbf{v}}_e^E$ in the ES equation for electrons (2.76), means that the ES turbulence is represented with two groups of toroidal mode numbers: the ES toroidal mode numbers $K_\alpha \geq K_\alpha^{\min} \gg k_\alpha^{\max}$, with K_α integers, and the IS toroidal mode numbers k_α . Equivalently, the turbulence within an ES flux tube depends on the IS guiding centre $\mathbf{R}_s$ at which it is located, through dependence on $\nabla_s \bar{h}_e$ and $\bar{\mathbf{v}}_e^E$. Therefore, it is not possible to map out the ES turbulence on the whole flux surface with copies of a single ES flux tube. Instead, the IS-ES turbulence must be mapped out by copies of the IS flux tube with the ES flux tubes embedded within it. The flux tube boundary conditions are found by imposing that the turbulence mapped out in this way is single valued in θ at fixed ζ . For the IS turbulence we find

$$\bar{h}(\theta, \mathbf{R}_s(\psi_s, \alpha(\psi_s, \theta, \zeta))) = \bar{h}(\theta + 2\pi, \mathbf{R}_s(\psi_s, \alpha(\psi_s, \theta + 2\pi, \zeta))), \quad (2.88)$$

where $\mathbf{R}_s(\psi_s, \alpha_s) = (\psi_s - \psi_0)\mathbf{e}_\psi + (\alpha_s - \alpha_0)\mathbf{e}_\alpha$, ψ_s is the slow flux label, and α_s the slow field line label. Equation (2.88) is equation (2.87) recast in the notation of this Chapter. For the ES turbulence we find

$$\begin{aligned} & \tilde{h}(\theta, \mathbf{R}_f(\psi_f, \alpha(\psi_f, \theta, \zeta)), \mathbf{R}_s(\psi_s, \alpha(\psi_s, \theta, \zeta))) \\ &= \tilde{h}(\theta + 2\pi, \mathbf{R}_f(\psi_f, \alpha(\psi_f, \theta + 2\pi, \zeta)), \mathbf{R}_s(\psi_s, \alpha(\psi_s, \theta + 2\pi, \zeta))), \end{aligned} \quad (2.89)$$

where $\mathbf{R}_f(\psi_f, \alpha_f) = (\psi_f - \psi_0)\mathbf{e}_\psi + (\alpha_f - \alpha_0)\mathbf{e}_\alpha$, with ψ_f the fast flux label and α_f the fast field line label. Note that although we measure the guiding center position $\mathbf{R}_f$ from the center of the IS flux tube (ψ_0, α_0) , each ES flux tube should be visualised as being centered at a different IS position, labelled by (ψ_s, α_s) , consistent with the ES dependence on $\mathbf{R}_s(\psi_s, \alpha_s)$. As we will discuss in Section 2.7.4, the ES boundary

condition (2.89) connects the ES flux tubes at distinct IS positions $\mathbf{R}_s(\psi_s, \alpha(\psi_s, \theta, \zeta))$ and $\mathbf{R}_s(\psi_s, \alpha(\psi_s, \theta + 2\pi, \zeta))$.

2.7.3 The flux tube boundary conditions in Fourier space

To implement the equations (2.79)-(2.83) requires a spectral parallel boundary condition for both the IS and ES pieces of the turbulence. We first focus on the boundary condition for IS turbulence. To write the IS distribution function in Fourier components we recall that $\mathbf{e}_\alpha = (\nabla\psi \times \mathbf{b})/|\nabla\psi \cdot \mathbf{b} \times \nabla\alpha|$, and note that $\mathbf{k}_s = k_\psi \nabla\psi + k_\alpha \nabla\alpha$. Then we have,

$$\begin{aligned} \bar{h}(\theta, \mathbf{R}_s(\psi_s, \alpha_s)) &= \sum_{\mathbf{k}_s} \exp[\mathbf{i}\mathbf{k}_s \cdot \mathbf{R}_s] \bar{h}_{\mathbf{k}_s}(\theta) = \\ &= \sum_{\mathbf{k}_s} \exp[\mathbf{i}k_\psi(\psi_s - \psi_0) + \mathbf{i}k_\alpha(\alpha_s - \alpha_0)] \bar{h}_{(k_\psi, k_\alpha)}(\theta). \end{aligned} \quad (2.90)$$

Using this, the IS boundary condition (2.87) in Fourier space is, after [44],

$$\bar{h}_{(k_\psi, k_\alpha)}(\theta) = \underbrace{\exp[-\mathbf{i}2\pi q_0 k_\alpha]}_{=1} \bar{h}_{(k_\psi + 2\pi q'_0 k_\alpha, k_\alpha)}(\theta + 2\pi). \quad (2.91)$$

The phase $\exp[-\mathbf{i}2\pi q_0 k_\alpha]$ can be set to 1 because as $\rho_{\text{th}}/a \rightarrow 0$ the integer $k_\alpha^{\text{min}} \rightarrow \infty$ and hence $q_0 k_\alpha$ can be made arbitrarily close to a very large integer for all k_α . Hence, we choose $q_0 = M/k_\alpha^{\text{min}}$, with M a large integer: the centre of the flux tube ψ_0 is a high order rational surface [44]. We will refer to the surface ψ_0 as an “IS rational surface” due to the particular rational value of q_0 . To aid the understanding of the arguments in this section, we reproduce the calculation from [44] in A.8 using the notation of this Chapter.

We now discuss the spectral boundary condition for ES turbulence. Note that we take q_0 to be the safety factor in every ES flux tube. Naively, we should take the safety factor in an ES flux tube on the surface ψ_s to be $q_s = q_0 + q'_0(\psi_s - \psi_0)$. However, q_s differs from q_0 only by $\mathcal{O}(\rho_{\text{th},i}/a)$. To be consistent with the scale separation between the equilibrium and the IS turbulence, we must take all equilibrium variables and their gradients to be the same in all ES flux tubes. By using that

$$\tilde{h}(\theta, \mathbf{R}_f(\psi_f, \alpha_f), \mathbf{R}_s(\psi_s, \alpha_s)) =$$

$$\sum_{K_\psi, K_\alpha} \exp [iK_\psi(\psi_f - \psi_0) + iK_\alpha(\alpha_f - \alpha_0)] \tilde{h}_{(K_\psi, K_\alpha)}(\theta, \mathbf{R}_s(\psi_s, \alpha_s)), \quad (2.92)$$

where K_ψ and K_α are the ES wave numbers corresponding to ψ_f and α_f , and $\mathbf{k}_f = K_\psi \nabla \psi + K_\alpha \nabla \alpha$, in A.9 we show that boundary condition (2.89) for the ES turbulence leads to the following spectral boundary condition,

$$\tilde{h}_{(K_\psi, K_\alpha)}(\theta, \mathbf{R}_s(\psi_s, \alpha(\psi_s, \theta, \zeta))) = \underbrace{\exp [-i2\pi q_0 K_\alpha]}_{=1} \tilde{h}_{(K_\psi + 2\pi q'_0 K_\alpha, K_\alpha)}(\theta + 2\pi, \mathbf{R}_s(\psi_s, \alpha(\psi_s, \theta + 2\pi, \zeta))). \quad (2.93)$$

The phase-factor $\exp [-i2\pi q_0 K_\alpha]$ in equation (2.93) is set to 1, because as $\rho_{\text{th,e}}/\rho_{\text{th,i}} \rightarrow 0$ we can again make the integer K_α^{min} increasingly large and hence $q_0 K_\alpha$ will be arbitrarily close to a very large integer for all K_α ; i.e. $q_0 = N/K_\alpha$ with N a large integer, and consequently ψ_0 is also an “ES rational surface”.

2.7.4 The parallel coupling between ES flux tubes embedded in IS turbulence

In the presence of IS turbulence, the ES flux tubes are in general coupled in the parallel direction by the ES boundary condition (2.89). This is because the IS gradients at the IS positions $\mathbf{R}_s(\psi_s, \alpha(\psi_s, \theta, \zeta))$ and $\mathbf{R}_s(\psi_s, \alpha(\psi_s, \theta + 2\pi, \zeta))$ appearing in (2.89) are in general distinct. Note that the expression for the change in α in a poloidal turn at fixed ζ is

$$\alpha(\psi, \theta + 2\pi, \zeta) - \alpha(\psi, \theta, \zeta) = -2\pi q_0 - 2\pi q'_0(\psi - \psi_0). \quad (2.94)$$

As the phase factor $\exp [-i2\pi q_0 k_\alpha]$ is set to 1 in the IS boundary condition (2.91), we may write the real space boundary condition corresponding to (2.91) as

$$\bar{h}(\theta, \mathbf{R}_s(\psi_s, \alpha(\psi_s, \theta, \zeta))) = \bar{h}(\theta + 2\pi, \mathbf{R}_s(\psi_s, \alpha(\psi_s, \theta, \zeta) - 2\pi q'_0(\psi_s - \psi_0))). \quad (2.95)$$

Note that on an IS rational surface, e.g. where $\psi_s = \psi_0$, the IS boundary condition (2.95) ensures that the IS fluctuations are periodic in θ at fixed α_s . On these surfaces an ES flux tube couples to a periodic copy of itself after a single 2π poloidal turn. For

radial locations in the IS flux tube which do not lie on an IS rational surface, i.e. where $\psi_s \neq \psi_0$, IS fluctuations are not periodic in θ at fixed α_s ; the IS boundary condition (2.95) encapsulates the effect of local magnetic shear by coupling field lines at different α_s at the parallel boundaries of the flux tubes in θ . An ES flux tube at the IS position (ψ_s, α_s) , where $\psi_s \neq \psi_0$, couples in the parallel direction to the ES flux tubes at the IS positions $(\psi_s, \alpha_s - 2\pi q'_0(\psi_s - \psi_0))$ and $(\psi_s, \alpha_s + 2\pi q'_0(\psi_s - \psi_0))$. By connecting multiple ES flux tubes in this way we arrive at a chain of ES flux tubes with a parallel boundary condition that is consistent with the parallel boundary condition of the IS turbulence. The chain of ES flux tubes here is reminiscent of the “flux tube train” of [72]. Figure 2.3 gives a visualisation of the coupling between ES flux tubes introduced by the boundary condition (2.89). The $\theta = \pm\pi$ ends of the flux tube are at ζ points different by $2\pi q_0$ (see Figure 2.2). We do not show the $2\pi q_0$ shift in ζ . The dotted-lined parallelograms are periodic continuations of the IS flux tube. The boundary condition (2.88) enforces that the turbulence on the (ζ, ψ) plane mapped out by the parallelograms at $\theta = \pm\pi$ must be the same. At $\psi = \psi_0$, the field line labelled by a_0 begins and ends at points in the (ζ, ψ) plane where the fluctuations have the same phase in ζ ; the field line a_0 is coupled to a periodic copy of itself after a poloidal turn. However, at $\psi = \psi_1$, a field line begins and ends at points in the (ζ, ψ) plane where the fluctuations do not have the same phase in ζ . Instead of coupling to a periodic copy of itself, the field line labelled by a_1 at $\theta = -\pi$ couples to a different field line, b_1 at $\theta = \pi$. The field line labelled by b_1 at $\theta = -\pi$ likewise couples to the periodic copy of a_1 at $\theta = \pi$. When the ES turbulence uses the boundary condition (2.89), the ES flux tubes within the IS flux tube (see Figure 1.4) couple in the same manner as the field lines that they follow.

2.7.5 Continuity of the cross-scale terms across the connections between ES flux tubes

To demonstrate that it is necessary to satisfy the proposed ES boundary condition (2.89), consider one of the new ES terms appearing in (2.76) in Fourier space, written in components,

$$\tilde{\mathbf{v}}_e^E \cdot \nabla_s \bar{h}_e = \left[i \frac{c}{B} \mathbf{b} \times \mathbf{k}_f \cdot \nabla_s \bar{h}_e \right] \tilde{\varphi}_{e\mathbf{k}_f}$$

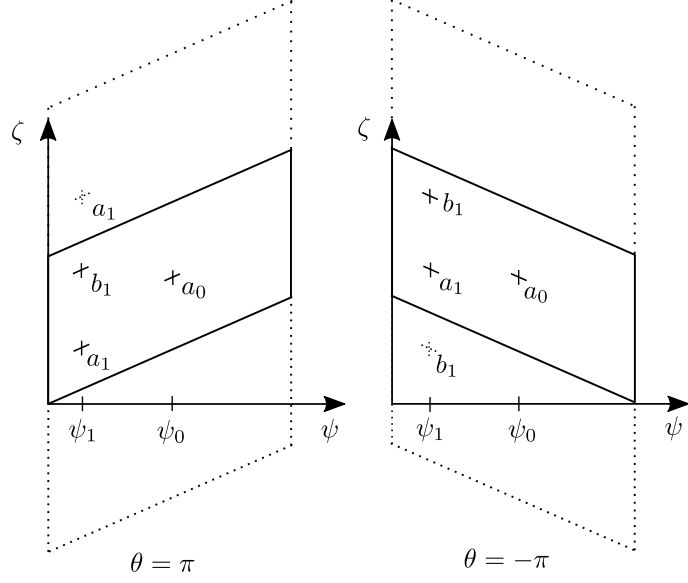


Figure 2.3: The diagram shows the $\theta = \pi$ (left) and $\theta = -\pi$ (right) ends of the IS flux tube, represented by solid-lined parallelograms on the inboard mid-plane, which we parameterise with (ζ, ψ) .

$$= c \left[K_\psi \frac{\partial}{\partial \alpha_s} \bar{h}_e(\theta, \mathbf{R}_s(\psi_s, \alpha_s)) - K_\alpha \frac{\partial}{\partial \psi_s} \bar{h}_e(\theta, \mathbf{R}_s(\psi_s, \alpha_s)) \right] \tilde{\varphi}_{e(K_\psi, K_\alpha)}, \quad (2.96)$$

where the ψ_s derivative is taken at fixed α_s , as is the convention unless otherwise stated. To ensure continuity of the ES coefficient multiplying $\tilde{\varphi}_{e(K_\psi, K_\alpha)}$ at the flux tube parallel boundaries $(\theta, \theta + 2\pi)$ we need to have,

$$\begin{aligned} & K_\psi \frac{\partial}{\partial \alpha_s} \bar{h}_e(\theta, \mathbf{R}_s(\psi, \alpha_s)) - K_\alpha \frac{\partial}{\partial \psi_s} \bar{h}_e(\theta, \mathbf{R}_s(\psi_s, \alpha_s)) = \\ & (K_\psi + 2\pi q'_0 K_\alpha) \frac{\partial}{\partial \alpha_s} \bar{h}_e(\theta + 2\pi, \mathbf{R}_s(\psi, \alpha_s + \delta \alpha_s)) \\ & - K_\alpha \frac{\partial}{\partial \psi_s} \bar{h}_e(\theta + 2\pi, \mathbf{R}_s(\psi_s, \alpha_s + \delta \alpha_s)), \end{aligned} \quad (2.97)$$

where $\delta \alpha_s = -2\pi q'_0(\psi_s - \psi_0)$. The K_α should be independent of each other, and independent of the IS, so,

$$\frac{\partial}{\partial \alpha_s} \bar{h}_e(\theta, \mathbf{R}_s(\psi_s, \alpha_s)) = \frac{\partial}{\partial \alpha_s} \bar{h}_e(\theta + 2\pi, \mathbf{R}_s(\psi_s, \alpha_s + \delta \alpha_s)), \quad (2.98)$$

and,

$$\begin{aligned} & \frac{\partial}{\partial \psi_s} \bar{h}_e(\theta + 2\pi, \mathbf{R}_s(\psi_s, \alpha_s + \delta\alpha_s)) - 2\pi q'_0 \frac{\partial}{\partial \alpha_s} \bar{h}_e(\theta + 2\pi, \mathbf{R}_s(\psi_s, \alpha_s + \delta\alpha_s)) \\ &= \frac{\partial}{\partial \psi_s} \bar{h}_e(\theta, \mathbf{R}_s(\psi_s, \alpha_s)), \end{aligned} \quad (2.99)$$

should be satisfied independently. As shown in A.10, these relations are satisfied independently when the IS turbulence satisfies the boundary condition (2.88).

2.8 Boosting away the piece of $\bar{\mathbf{v}}^E$ that is constant in θ

In this Section we show that if we write the equation for the ES fluctuations (2.76) in terms of the guiding centre distribution functions $\tilde{g}_e = \left\langle \tilde{\delta f}_e \right\rangle_{\mathbf{R}_f, \mathbf{R}_s}^\gamma = \tilde{h}_e + e\tilde{\varphi}_e F_{0e}/T$ and $\bar{g}_e = \left\langle \bar{\delta f}_e \right\rangle_{\mathbf{R}_s}^\gamma = \bar{h}_e + e\bar{\varphi}_e F_{0e}/T$ then we can remove the component of the IS advection $\bar{\mathbf{v}}_e^E$ due to the component $\bar{\phi}_c$ of the potential $\bar{\phi}$ which is constant in θ . By writing equation (2.76) in terms of $\tilde{g}_e$ and $\bar{g}_e$ we find that

$$\begin{aligned} & \frac{\partial \tilde{g}_e}{\partial t_f} + v_{\parallel} \mathbf{b} \cdot \nabla \theta \frac{\partial \tilde{g}_e}{\partial \theta} + (\mathbf{v}_e^M + \bar{\mathbf{v}}_e^E) \cdot \nabla_f \tilde{g}_e + \tilde{\mathbf{v}}_e^E \cdot \nabla_f \tilde{g}_e + \tilde{\mathbf{v}}_e^E \cdot (\nabla F_{0e} + \nabla_s \bar{g}_e) \\ &= -\frac{ZeF_{0e}}{T} v_{\parallel} \mathbf{b} \cdot \nabla \theta \frac{\partial \tilde{\varphi}_e}{\partial \theta} - \frac{ZeF_{0e}}{T} \mathbf{v}_e^M \cdot \nabla_f \tilde{\varphi}_e, \end{aligned} \quad (2.100)$$

where we have used that $\bar{\mathbf{v}}_e^E \cdot \nabla_f \tilde{\varphi}_e = -\tilde{\mathbf{v}}_e^E \cdot \nabla_s \bar{\varphi}_e$ and $\tilde{\mathbf{v}}_e^E \cdot \nabla_f \tilde{\varphi}_e = 0$. Note that if $\bar{\phi}_c$ is constant in θ this means that $\partial \bar{\phi}_c / \partial \theta|_\alpha = 0$. For irrational values of the safety factor q , this implies that $\bar{\phi}_c$ is also a constant in α . We will henceforth make this assumption. We explicitly separate out the $\mathbf{E} \times \mathbf{B}$ advection arising from $\bar{\phi}_c$ in (2.100) by writing (2.100) as

$$\frac{\partial \tilde{g}}{\partial t_f} + \bar{\mathbf{u}} \cdot \nabla \tilde{g} = S, \quad (2.101)$$

where

$$\bar{\mathbf{u}} = \frac{c}{B} \mathbf{b} \times \nabla \bar{\phi}_c, \quad \nabla = \nabla \psi \left(\frac{\partial}{\partial \psi_s} + \frac{\partial}{\partial \psi_f} \right) + \nabla \alpha \left(\frac{\partial}{\partial \alpha_s} + \frac{\partial}{\partial \alpha_f} \right), \quad (2.102)$$

and S contains the terms in (2.100) which do not explicitly appear in (2.101). Expanding the vector expression $\bar{\mathbf{u}} \cdot \nabla \tilde{g}$ in (2.101), we find that

$$\bar{\mathbf{u}} \cdot \nabla \tilde{g} = -\frac{c}{B} \mathbf{b} \cdot \underbrace{\nabla \alpha \times \nabla \psi}_B \frac{\partial \bar{\phi}_c}{\partial \psi_s} \frac{\partial \tilde{g}}{\partial \alpha_f} = -c \frac{\partial \bar{\phi}_c}{\partial \psi_s} \frac{\partial \tilde{g}}{\partial \alpha_f}. \quad (2.103)$$

Here $\partial \bar{\phi}_c / \partial \psi_s$ has no dependence on θ . This means that at a given IS location (ψ_s, α_s) the drift velocities are constant within the ES flux tube. The equation (2.101) is now

$$\frac{\partial \tilde{g}}{\partial t_f} - \underbrace{c \frac{\partial \bar{\phi}_c}{\partial \psi_s}}_{\text{constant}} \frac{\partial \tilde{g}}{\partial \alpha_f} = S. \quad (2.104)$$

Let

$$t_f' = t_f, \quad \psi_f' = \psi_f, \quad \alpha_f' = \alpha_f - c \frac{\partial \bar{\phi}_c}{\partial \psi_s} t_f. \quad (2.105)$$

Changing to the coordinates $(t_f', \psi_f', \alpha_f')$, we find

$$\frac{\partial \tilde{g}}{\partial t_f'} = S, \quad (2.106)$$

i.e. we have boosted to a frame rotating with the constant drift $\bar{\mathbf{u}}$.

2.9 IS and ES collision operators

In this Section we discuss how to include the effect of collisions in the scale-separated model of coupled IS-ES turbulence (2.73)-(2.77). We first discuss the orderings for collision frequencies. Then in Section 2.9.1 we find the forms of the IS and ES collision operators. Using the same techniques as used in A.6 to find the scale-separated quasineutrality relation (2.77) for the ES gyrokinetic equation (2.76), we find in Section 2.9.2 the scale-separated ES collision operator which should appear in (2.76). Finally in Section 2.9.3 we give the Fourier representation of the collision operators.

We have noted that collisions are required to regularise velocity space. In order to retain the regularising effect of collisions without obtaining a purely adiabatic electron response at IS, we order

$$\nu_{ee} \sim \nu_{ei} \sim \frac{v_{\text{th},i}}{a}, \quad (2.107)$$

where ν_{ee} is the electron-electron collision frequency and ν_{ei} is the electron-ion collision frequency. To be consistent, the ion-ion collision frequency

$$\nu_{ii} \sim \left(\frac{m_e}{m_i}\right)^{1/2} \frac{v_{th,i}}{a} \quad (2.108)$$

because ions collide with other ions at a rate $(m_e/m_i)^{1/2}$ times slower than the rate at which electrons collide with other electrons. The ion-electron collision frequency is negligible,

$$\nu_{ie} \sim \frac{m_e}{m_i} \nu_{ei} \sim \left(\frac{m_e}{m_i}\right)^{1/2} \nu_{ii}. \quad (2.109)$$

Due to the diffusive nature of the collision operator [70], we have that, for the species ν , at both spatial scales,

$$C_{l\nu} \sim \nu_{\nu\nu} v_{th,\nu}^2 (\delta v_\nu)^{-2} h_\nu. \quad (2.110)$$

In the orderings (2.107) and (2.108) the bulk of the ion distribution functions, where $\delta v_i \sim v_{th,i}$, and the bulk of the electron distribution function at ES, where $\delta v_e \sim v_{th,e}$, are unaffected by collisions at leading order. This can be observed by inspecting (2.110) and comparing the size of the collision terms with the linear timescale at each scale, i.e.

$$C_{li} \sim \left(\frac{m_e}{m_i}\right)^{1/2} \frac{v_{th,i}}{a} h_i \ll \frac{v_{th,i}}{a} h_i, \quad (2.111)$$

and

$$C_{le} \sim \left(\frac{m_e}{m_i}\right)^{1/2} \frac{v_{th,e}}{a} h_e \sim \frac{v_{th,i}}{a} h_e \ll \frac{v_{th,e}}{a} h_e. \quad (2.112)$$

For electrons at IS in the bulk of the distribution, where $\delta v_e \sim v_{th,e}$, the frequency of collisions is fast enough to modify the trapped piece of velocity space at leading order, but not fast enough to detrap the electrons entirely, which would require $C_{le} \gg (v_{th,i}/a) h_e$. This allows us to keep a fully kinetic description for the electrons. Nonetheless, pieces of the distribution functions inevitably develop small velocity space structures due to the parallel streaming terms, which introduce phase mixing. In the presence of collisions these small velocity space structures are damped, because the collision terms, which

introduce dissipation, become as large as the parallel streaming terms for each species. If $\delta v_i \lesssim (m_e/m_i)^{1/4} v_{th,i}$ for a piece of the ion distribution function then by (2.110) the effect of collisions will be significant: for this piece of the ion distribution function, $C_{li} \gtrsim (v_{th,i}/a) h_i \sim v_{\parallel} \mathbf{b} \cdot \nabla \theta \partial h_i / \partial \theta$. Similarly if $\delta v_e \lesssim (m_e/m_i)^{1/4} v_{th,e}$ for a piece of the electron distribution function, then by (2.110) the effect of collisions will be significant: for this piece of the electron distribution function, $C_{le} \gtrsim (v_{th,e}/a) h_e \sim v_{\parallel} \mathbf{b} \cdot \nabla \theta \partial h_e / \partial \theta$. Altogether, this allows us to assume that $\delta v_e \sim v_{th,e}$ and $\delta v_i \sim v_{th,i}$ at both spatial scales.

2.9.1 The forms of the collision operators

To order $O\left((m_e/m_i)^{1/2} (v_{th,i}/a) h_i\right)$, the ion collision operator has the form [70]

$$C_{li} = \langle C_{ii} [h_i] \rangle_{\mathbf{R}}^{\gamma}, \quad (2.113)$$

where C_{ii} is the linearised ion-ion self collision operator, and the velocity space derivatives acting on h_i are taken at fixed θ and $\mathbf{r}$. Note that because of ordering (2.109), the contribution of ion-electron collisions, of order $(m_e/m_i)(v_{th,i}/a) h_i$, can be neglected because it is $O(m_e/m_i)$ small compared to the linear terms. To find the IS collision operator which should appear in (2.73), we apply the ES average to (2.113). We find that

$$\overline{C_{li}} = \langle C_{ii} [\overline{h_i}(\theta, \mathbf{r}_s - \boldsymbol{\rho}_i, \varepsilon, \lambda, \sigma)] \rangle_{\mathbf{R}_s}^{\gamma}, \quad (2.114)$$

where the notation indicates velocity derivatives are held at fixed θ and $\mathbf{r}_s$. To derive (2.114) we have used the properties proved in A.2: the ES average commutes with the gyroaverage (2.10); and either $\mathbf{R}_f$ or $\mathbf{r}_f$ can be used as the ES average integration variable. The ions at ES are adiabatic and so we will not require $\widetilde{C_{li}}$. Keeping terms in the electron collision operator relevant in the ordering, $O\left((m_e/m_i)^{1/2} (v_{th,i}/a) h_e\right)$, we have [70]

$$C_{le} = \langle C_{ee} [h_e] \rangle_{\mathbf{R}}^{\gamma} + \left\langle L_{ei} \left[h_e - \frac{m_e \mathbf{u}_i \cdot \mathbf{v}}{T_e} F_{0e} \right] \right\rangle_{\mathbf{R}}^{\gamma}, \quad (2.115)$$

where C_{ee} is the electron-electron self collision operator, L_{ei} is the Lorentz pitch angle scattering operator, which appears due to electron-ion collisions, and $\mathbf{u}_i =$

$n_i^{-1} \int d^3 \mathbf{v} |_{\mathbf{r}} \mathbf{v} h_i$, is the the mean ion velocity. Again using the ES average (2.23) and the properties proved in A.2, we find

$$\overline{C_{le}} = \langle C_{ee} [\bar{h}_e] \rangle_{\mathbf{R}_s}^\gamma + \left\langle L_{ei} \left[\bar{h}_e - \frac{m_e \bar{\mathbf{u}}_i \cdot \mathbf{v}}{T_e} F_{0e} \right] \right\rangle_{\mathbf{R}_s}^\gamma, \quad (2.116)$$

where

$$\bar{\mathbf{u}}_i = \frac{1}{n_i} \int d^3 \mathbf{v} |_{\mathbf{r}_s} \mathbf{v} \bar{h}_i, \quad (2.117)$$

and the velocity derivatives in C_{ee} and L_{ei} are held at fixed θ and $\mathbf{r}_s$. In (2.116) we regard $\bar{h}_e = \bar{h}_e(\theta, \mathbf{r}_s - \boldsymbol{\rho}_e, \varepsilon, \lambda, \sigma)$. The collision operator which should appear in (2.74) is

$$\langle \overline{C_{le}} \rangle^o = \left\langle \left\langle C_{ee} [\bar{h}_e^{(0)}] \right\rangle_{\mathbf{R}_s}^\gamma \right\rangle^o + \left\langle \left\langle L_{ei} [\bar{h}_e^{(0)}] \right\rangle_{\mathbf{R}_s}^\gamma \right\rangle^o, \quad (2.118)$$

where $\bar{h}_e^{(0)}$ is the leading order piece of the IS electron distribution function, and we have neglected

$$\frac{m_e \bar{\mathbf{u}}_i \cdot \mathbf{v}}{T_e} F_{0e} \sim \left(\frac{m_e}{m_i} \right)^{1/2} \frac{\rho_{th,i}}{a} F_{0e} \ll \bar{h}_e^{(0)}. \quad (2.119)$$

By taking the difference between (2.115) and (2.116), we find the ES electron collision operator

$$\widetilde{C_{le}} = \left\langle C_{ee} [\tilde{h}_e] \right\rangle_{\mathbf{R}_s, \mathbf{R}_f}^\gamma + \left\langle L_{ei} \left[\tilde{h}_e - \frac{m_e \tilde{\mathbf{u}}_i \cdot \mathbf{v}}{T_e} F_{0e} \right] \right\rangle_{\mathbf{R}_s, \mathbf{R}_f}^\gamma, \quad (2.120)$$

where

$$\tilde{\mathbf{u}}_i = \frac{1}{n_i} \int d^3 \mathbf{v} |_{\mathbf{r}_f, \mathbf{r}_s} \mathbf{v} \tilde{h}_i. \quad (2.121)$$

In (2.120) the velocity derivatives in C_{ee} and L_{ei} are held at fixed θ , $\mathbf{r}_s$ and $\mathbf{r}_f$, and we regard $\tilde{h}_e = \tilde{h}_e(\theta, \mathbf{r}_s - \boldsymbol{\rho}_e, \mathbf{r}_f - \boldsymbol{\rho}_e, \varepsilon, \lambda, \sigma)$. Note that in the electron collision operator at ES, we only need to retain terms of $O(\nu_{ee} \tilde{h}_e) \sim O((v_{th,i}/a) \tilde{h}_e)$. As discussed in Section 2.2, in the ordering for the electron collision frequency (2.107), we imposed that collisions at ES appear as a sub-dominant term designed to regularise sharp gradients in velocity space. Note that $\tilde{\mathbf{u}}_i/v_{th,i}$ is small,

$$\frac{\tilde{\mathbf{u}}_i}{v_{th,i}} \sim \left(\frac{m_e}{m_i} \right)^{1/2} \frac{e\tilde{\phi}}{T}, \quad (2.122)$$

where in (2.122) one factor of $(m_e/m_i)^{1/4}$ is due to the velocity integration in the definition of $\tilde{\mathbf{u}}_i$ (2.121), which introduces an ion gyroaverage, and the second appears because $\tilde{h}_i \sim (m_e/m_i)^{1/4} e\tilde{\phi}/T$ by the scaling (2.57). Hence, we can neglect the contribution to the electron-ion collision operator from the ion response at ES $\tilde{h}_i$,

$$\frac{m_e \tilde{\mathbf{u}}_i \cdot \mathbf{v}}{T_e} F_{0e} \sim \frac{\tilde{\mathbf{u}}_i}{v_{th,e}} F_{0e} \ll \tilde{h}_e. \quad (2.123)$$

The ES electron collision operator neglecting ions at ES is therefore

$$\begin{aligned} \widetilde{C}_{le} = & \left\langle C_{ee} \left[\tilde{h}_e(\theta, \mathbf{r}_s - \boldsymbol{\rho}_e, \mathbf{r}_f - \boldsymbol{\rho}_e, \varepsilon, \lambda, \sigma) \right] \right\rangle_{\mathbf{R}_s, \mathbf{R}_f}^{\gamma} \\ & + \left\langle L_{ei} \left[\tilde{h}_e(\theta, \mathbf{r}_s - \boldsymbol{\rho}_e, \mathbf{r}_f - \boldsymbol{\rho}_e, \varepsilon, \lambda, \sigma) \right] \right\rangle_{\mathbf{R}_s, \mathbf{R}_f}^{\gamma}, \end{aligned} \quad (2.124)$$

where we note that (2.124) does not yet represent a scale-separated collision operator due to the gyroaverages and velocity derivatives held at fixed $\mathbf{r}_s$; holding $\mathbf{r}_s$ fixed in these operations introduces perpendicular-to-the-field coupling between ES flux tubes.

2.9.2 Scale separation in the presence of collisions

To obtain a scale-separated ES gyrokinetic equation the slow spatial coordinate $\mathbf{R}_s$ must appear only as a label. Gyroaverages and velocity derivatives held at fixed $\mathbf{r}_s$ introduce coupling between perpendicular locations in the IS flux tube which naively appear to break scale separation.

To deal with the gyroaverages and velocity derivatives held at fixed $\mathbf{r}_s$ in (2.120), we perform the same operation of approximating the Bessel functions as in the ES quasineutrality relation (2.72) to write the gyroaverage and velocity derivatives in $\widetilde{C}_{le}$ at fixed θ , $\mathbf{r}_f$ and $\mathbf{R}_s$ with only $O\left((m_e/m_i)^{1/2}\right)$ error (as with quasineutrality)

$$\begin{aligned} \widetilde{C}_{le} = & \left\langle C_{ee} \left[\tilde{h}_e(\theta, \mathbf{R}_s, \mathbf{r}_f - \boldsymbol{\rho}_e, \varepsilon, \lambda, \sigma) \right] \right\rangle_{\mathbf{R}_s, \mathbf{R}_f}^{\gamma} \left(1 + O\left(\left(\frac{m_e}{m_i}\right)^{1/2}\right) \right) \\ & + \left\langle L_{ei} \left[\tilde{h}_e(\theta, \mathbf{R}_s, \mathbf{r}_f - \boldsymbol{\rho}_e, \varepsilon, \lambda, \sigma) \right] \right\rangle_{\mathbf{R}_s, \mathbf{R}_f}^{\gamma} \left(1 + O\left(\left(\frac{m_e}{m_i}\right)^{1/2}\right) \right), \end{aligned} \quad (2.125)$$

and so obtain a scale-separated collision operator for (2.76).

2.9.3 Fourier representation for collisions

We now give the Fourier representation of the collision operators which appear in gyrokinetic equations (2.79), (2.80) and (2.82) when the effect of collisions is included. In the equation for ions at IS (2.79) the collision operator is

$$\overline{C}_{\text{li}\mathbf{k}_s} = \left\langle \exp[\mathbf{i}\mathbf{k}_s \cdot \boldsymbol{\rho}_i] C_{\text{ii}} \left[\exp[-\mathbf{i}\mathbf{k}_s \cdot \boldsymbol{\rho}_i] \overline{h}_{\text{i}\mathbf{k}_s} \right] \right\rangle^\gamma. \quad (2.126)$$

In the equation for electrons at IS (2.80) the collision operator is

$$\overline{C}_{\text{le}\mathbf{k}_s} = \left\langle \left\langle C_{\text{ee}} [\overline{h}_{\text{e}\mathbf{k}_s}] \right\rangle^\gamma \right\rangle^\circ + \left\langle \left\langle L_{\text{ei}} [\overline{h}_{\text{e}\mathbf{k}_s}] \right\rangle^\gamma \right\rangle^\circ, \quad (2.127)$$

where $\overline{h}_{\text{e}\mathbf{k}_s} = \overline{h}_{\text{e}\mathbf{k}_s}^{(0)}$, and where we have neglected sub-leading-order “finite Larmor radius” corrections due to nonzero $\mathbf{k}_s \cdot \boldsymbol{\rho}_e \sim (m_e/m_i)^{1/2} \ll 1$. We note the presence of the orbital average $\langle \cdot \rangle^\circ$. At ES in (2.82) we require the electron collision operator

$$\begin{aligned} \widetilde{C}_{\text{le}\mathbf{k}_f} &= \left\langle \exp[\mathbf{i}\mathbf{k}_f \cdot \boldsymbol{\rho}_e] C_{\text{ee}} \left[\exp[-\mathbf{i}\mathbf{k}_f \cdot \boldsymbol{\rho}_e] \overline{h}_{\text{e}\mathbf{k}_f} \right] \right\rangle_{\mathbf{R}_s}^\gamma \\ &+ \left\langle \exp[\mathbf{i}\mathbf{k}_f \cdot \boldsymbol{\rho}_e] L_{\text{ei}} \left[\exp[-\mathbf{i}\mathbf{k}_f \cdot \boldsymbol{\rho}_e] \overline{h}_{\text{e}\mathbf{k}_f} \right] \right\rangle_{\mathbf{R}_s}^\gamma, \end{aligned} \quad (2.128)$$

where the velocity derivatives appearing in C_{ee} and L_{ei} are held at fixed $\mathbf{R}_s$.

Chapter 3

The Linear Stability of ES Microinstabilities in the Presence of IS Turbulence

The theory presented in the previous Chapter assumes that multiscale plasma turbulence can be decomposed into well separated ion scale (IS) and electron scale (ES) pieces. This assumption is used to derive scale-separated, coupled equations for the IS and the ES pieces of plasma turbulence, equations (2.73)-(2.77). The parallel-to-the-field wavenumber of the turbulence $k_{\parallel}$ is assumed to scale as $(m_e/m_i)^{1/2} \ll k_{\parallel} a \sim 1 \ll (m_i/m_e)^{1/2}$ at both scales. The IS has a perpendicular-to-the-field scale of order the ion thermal gyroradius $\rho_{th,i}$, and a typical time scale of order $a/v_{th,i}$, with $v_{th,i}$ the ion thermal speed. The ES has a perpendicular-to-the-field scale of order the electron thermal gyroradius $\rho_{th,e}$, and a typical time scale of order $a/v_{th,e}$, with $v_{th,e}$ the electron thermal speed. The equations (2.73)-(2.77) assumed electrostatic fluctuations, and neglected the effects of equilibrium perpendicular flow shear on the turbulence.

The IS-ES equations (2.73)-(2.77) represent a reduced model of the full gyrokinetic equation (2.12). The IS-ES reduced model allows for theoretical investigation into the cross-scale interactions between the IS and the ES pieces of the turbulence. Due to the complexity of the equations, solving (2.73)-(2.77) requires numerical simulations; in particular, the gradient of the nonadiabatic piece of the IS distribution function $\nabla_s \bar{h}_e$, and the IS $\mathbf{E} \times \mathbf{B}$ drift $\bar{\mathbf{v}}_e^E$ appearing in the ES gyrokinetic equation (2.76) must be calculated from a nonlinear simulation of IS turbulence using (2.73)-(2.75). By design, these simulations are relatively inexpensive compared to multiscale DNS.

In this Chapter we present an investigation into the effect of IS turbulence on ES microinstabilities using the IS-ES equations (2.73)-(2.77). In Section 3.1 we briefly review the forms of the coupled, scale-separated gyrokinetic equations for IS and ES turbulence which are convenient for the numerical study. In Section 3.2 we motivate the separation of the binormal wave number range into IS and ES pieces. In Section 3.3 we present nonlinear flux tube simulations of the IS wave number range, and we discuss the method for sampling the IS gradients and IS drifts from the IS flux tube. In Section 3.4 we use the calculated ensemble of IS gradients and IS drifts to study the effect of cross-scale interaction on the electron temperature gradient (ETG) instability, which can drive ES turbulence. Finally, in Section 3.5 we present a model of the effect of parallel-to-the-field shear in $\overline{\mathbf{v}}_e^E$ on the ETG instability.

The appendices corresponding to this chapter are structured as follows. Appendix B.1 briefly outlines the numerical tests of the ES cross-scale terms, which are implemented in the δf gyrokinetic code **GS2** [73]. Appendix B.2 details the parallel-to-the-field and velocity space grids and resolutions used in the simulations. In Appendix B.3 we demonstrate that the model response of electrons at IS used in the IS simulations correctly captures the behaviour of electrons at IS in the limit that $(m_e/m_i)^{1/2} \rightarrow 0$. Appendix B.4 details the method used to obtain simulations of IS turbulence where the IS gradients and drifts obey the IS real space parallel boundary condition to sufficiently high accuracy to be used in studies of cross-scale coupling with ES simulations.

3.1 Equations for IS and ES turbulence convenient for numerical implementation

In order to make a numerical study of the effect of cross-scale interaction between the IS and ES pieces of the turbulence, the spectral form of the scale-separated, coupled equations (2.79)-(2.83) were implemented using the δf gyrokinetic code **GS2**. The implementation of the cross-scale terms is described briefly in Appendix B.1. In this Section we re-express the spectral equations for coupled IS-ES turbulence (2.79)-(2.83) in a form convenient for the discussion in this Chapter. As **GS2** evolves the Fourier components of the distribution of gyrocentres $g_\nu = \langle \delta f_\nu \rangle_{\mathbf{R}}^\gamma$, where $\langle \cdot \rangle_{\mathbf{R}}^\gamma$ is the gyrophase

average at fixed guiding centre $\mathbf{R}$, we express the IS equations (2.73)-(2.75) in terms of the Fourier components of the IS gyrocentre distribution function $\bar{g}_\nu = \langle \bar{\delta f}_\nu \rangle_{\mathbf{R}_s}^\gamma$, with $\mathbf{R}_s$ the slow, IS guiding centre coordinate introduced in Section 2.2, and we express the ES equations (2.76)-(2.77) in terms of the Fourier components of the ES distribution of gyrocentres $\tilde{g}_e = \langle \tilde{\delta f}_e \rangle_{\mathbf{R}_s, \mathbf{R}_f}^\gamma$, with $\mathbf{R}_f$ the fast, ES guiding centre coordinate. This has the added benefit of giving a straightforward physical interpretation of the cross-scale terms appearing in the ES equations: In Section 2.8 we demonstrate that when the ES gyrokinetic equation (2.76) is written in terms of $\bar{g}_e$ and $\tilde{g}_e$ then the component of $\bar{\mathbf{v}}_e^E$ which corresponds to toroidal rotation can be removed by changing to a rotating frame; as a corollary, in this form of the equation, the potential gradient $\nabla_s \bar{\phi}_e$ is responsible purely for the effect of parallel-to-the-field shear in the $\mathbf{E} \times \mathbf{B}$ drift $\bar{\mathbf{v}}_e^E$; and the gradient of the IS distribution function $\nabla_s \bar{g}_e$ contains both the nonadiabatic and the adiabatic response, and modifies the usual drive of instability, the gradient of the equilibrium distribution function ∇F_{0e} . Finally, we will use the results of Section 2.9 to include the effect of collisions explicitly in the equations below.

We write the IS gyrokinetic equations in terms of

$$\bar{g}_{i\mathbf{k}_s} = \bar{h}_{i\mathbf{k}_s} - \frac{Z_i e F_{0i}}{T_i} J_0(|\mathbf{k}_s| |\boldsymbol{\rho}_i|) \bar{\phi}_{\mathbf{k}_s} \text{ and } \bar{g}_{e\mathbf{k}_s} = \bar{h}_{e\mathbf{k}_s} + \frac{e F_{0e}}{T_e} \bar{\phi}_{\mathbf{k}_s}, \quad (3.1)$$

where $\mathbf{k}_s = k_\psi \nabla \psi + k_\alpha \nabla \alpha$ is the IS wave vector, with k_ψ , and k_α the IS radial and binormal wave numbers, respectively; $\boldsymbol{\rho}_\nu$ is the particle vector gyroradius of species ν ; $J_0(z)$ is the 0th Bessel function of the 1st kind; and we note that $\bar{g}_\nu(\mathbf{R}_s) = \langle \bar{\delta f}_\nu(\mathbf{r}_s) \rangle_{\mathbf{R}_s}^\gamma = \bar{h}_\nu(\mathbf{R}_s) - Z_\nu e F_{0\nu} \langle \bar{\phi}(\mathbf{r}_s) \rangle_{\mathbf{R}_s}^\gamma / T_\nu$, with $\bar{h}_\nu$ the nonadiabatic response of species ν at IS and $\mathbf{r}_s$ the IS particle position. As $|\mathbf{k}_s| |\boldsymbol{\rho}_e| \sim (m_e/m_i)^{1/2} \ll 1$ we take the Bessel functions due to the electron gyroaverage $J_0(|\mathbf{k}_s| |\boldsymbol{\rho}_e|) = 1$. Note that we refer to $k_\psi = \mathbf{k}_s \cdot \mathbf{e}_\psi$ as the IS radial wave number, where $\mathbf{e}_\psi = (\mathbf{b} \times \nabla \alpha) / |\nabla \psi \cdot \mathbf{b} \times \nabla \alpha|$, rather than the projection of $\mathbf{k}_s$ onto the $\nabla \psi$ direction, $\mathbf{k}_s \cdot \nabla \psi / |\nabla \psi|$. The equation for ions at IS is

$$\begin{aligned} \frac{\partial \bar{g}_{i\mathbf{k}_s}}{\partial t_s} + v_\parallel \mathbf{b} \cdot \nabla \theta \frac{\partial \bar{g}_{i\mathbf{k}_s}}{\partial \theta} + [\mathbf{i} \mathbf{v}_i^M \cdot \mathbf{k}_s] \bar{g}_{i\mathbf{k}_s} + \bar{\mathfrak{N}}_{i\mathbf{k}_s} + \left[\mathbf{i} \frac{c}{B} \mathbf{b} \times \mathbf{k}_s \cdot \nabla F_{0i} \right] J_0(|\mathbf{k}_s| |\boldsymbol{\rho}_i|) \bar{\phi}_{\mathbf{k}_s} \\ = - \frac{Z_i e F_{0i}}{T_i} J_0(|\mathbf{k}_s| |\boldsymbol{\rho}_i|) \left(v_\parallel \mathbf{b} \cdot \nabla \theta \frac{\partial \bar{\phi}_{\mathbf{k}_s}}{\partial \theta} + [\mathbf{i} \mathbf{v}_i^M \cdot \mathbf{k}_s] \bar{\phi}_{\mathbf{k}_s} \right) + \bar{C}_{li\mathbf{k}_s}, \end{aligned} \quad (3.2)$$

where the nonlinear term $\overline{\mathfrak{N}}_{i\mathbf{k}_s}$ is

$$\overline{\mathfrak{N}}_{i\mathbf{k}_s} = - \sum_{\mathbf{k}'_s} \left[\frac{c}{B} \mathbf{b} \times \mathbf{k}'_s \cdot \mathbf{k}_s \right] J_0(|\mathbf{k}'_s| |\boldsymbol{\rho}_i|) \overline{\phi}_{\mathbf{k}'_s} \overline{g}_{i\mathbf{k}_s - \mathbf{k}'_s}, \quad (3.3)$$

$\mathbf{v}_\nu^M = (v_\perp^2/2\Omega_\nu) \mathbf{b} \times \nabla B + (v_\parallel^2/\Omega_\nu) \mathbf{b} \times \mathbf{b} \cdot \nabla \mathbf{b}$ is the magnetic drift, t_s is the IS time coordinate, and the collision operator $\overline{C}_{li\mathbf{k}_s}$ is (see Section 2.9.3)

$$\overline{C}_{li\mathbf{k}_s} = \langle \exp[i\mathbf{k}_s \cdot \boldsymbol{\rho}_i] C_{ii} [\exp[-i\mathbf{k}_s \cdot \boldsymbol{\rho}_i] \overline{h}_{i\mathbf{k}_s}] \rangle^\gamma, \quad (3.4)$$

with $C_{ii}[\cdot]$ the ion-ion self-collision operator. The equation for electrons at IS is

$$\begin{aligned} \frac{\partial \overline{g}_{e\mathbf{k}_s}}{\partial t_s} + \left[i \langle \mathbf{v}_e^M \cdot \nabla \alpha \rangle^\circ \mathbf{e}_\alpha \cdot \mathbf{k}_s \right] \overline{g}_{e\mathbf{k}_s} + \overline{\mathfrak{N}}_{e\mathbf{k}_s} + \left\langle \left[i \frac{c}{B} \mathbf{b} \times \mathbf{k}_s \cdot \nabla F_{0e} \right] \overline{\phi}_{\mathbf{k}_s} \right\rangle^\circ \\ = \frac{eF_{0e}}{T_e} \frac{\partial}{\partial t_s} \left(\overline{\phi}_{\mathbf{k}_s} - \langle \overline{\phi}_{\mathbf{k}_s} \rangle^\circ \right) + \frac{eF_{0e}}{T_e} \left[i \langle \mathbf{v}_e^M \cdot \nabla \alpha \rangle^\circ \mathbf{e}_\alpha \cdot \mathbf{k}_s \right] \overline{\phi}_{\mathbf{k}_s} + \overline{C}_{le\mathbf{k}_s}, \end{aligned} \quad (3.5)$$

where the coordinate vector $\mathbf{e}_\alpha = \nabla \psi \times \mathbf{b}/B$, the nonlinear term $\overline{\mathfrak{N}}_{e\mathbf{k}_s}$ is

$$\overline{\mathfrak{N}}_{e\mathbf{k}_s} = - \left\langle \sum_{\mathbf{k}'_s} \frac{c}{B} \mathbf{b} \times \mathbf{k}'_s \cdot \mathbf{k}_s \overline{\phi}_{\mathbf{k}'_s} \overline{g}_{e\mathbf{k}_s - \mathbf{k}'_s} \right\rangle^\circ, \quad (3.6)$$

and the collision operator $\overline{C}_{le\mathbf{k}_s}$ is (see Section 2.9.3)

$$\overline{C}_{le\mathbf{k}_s} = \left\langle \langle C_{ee} [\overline{h}_{e\mathbf{k}_s}] \rangle^\gamma \right\rangle^\circ + \left\langle \langle L_{ei} [\overline{h}_{e\mathbf{k}_s}] \rangle^\gamma \right\rangle^\circ, \quad (3.7)$$

with $C_{ee}[\cdot]$ the electron-electron self collision operator, and $L_{ei}[\cdot]$ the electron-ion pitch angle scattering operator, and where we have neglected sub-leading-order “finite Larmor radius” corrections due to nonzero $\mathbf{k}_s \cdot \boldsymbol{\rho}_e \sim (m_e/m_i)^{1/2} \ll 1$. Finally, recall that $\overline{g}_{e\mathbf{k}_s} - e\overline{\phi}_{\mathbf{k}_s} F_{0e}/T_e = \overline{h}_{e\mathbf{k}_s} = 0$ for the nonzonal ($\mathbf{k}_s \cdot \mathbf{e}_\alpha \neq 0$) passing piece of phase space. Equations (3.2) and (3.5) are closed by the IS quasineutrality relation,

$$\frac{Z_i e \overline{\phi}_{\mathbf{k}_s}}{T_i} (1 - \Gamma_{0,i}) = \frac{1}{n_i} \int d^3 \mathbf{v} J_0(|\mathbf{k}_s| |\boldsymbol{\rho}_i|) \overline{g}_{i\mathbf{k}_s} - \frac{1}{n_e} \int d^3 \mathbf{v} \overline{g}_{e\mathbf{k}_s}, \quad (3.8)$$

where $\Gamma_{0,i} = n_i^{-1} \int d^3 \mathbf{v} F_{0i} J_0^2(|\mathbf{k}_s| |\boldsymbol{\rho}_i|)$, again we take $J_0(|\mathbf{k}_s| |\boldsymbol{\rho}_e|) = 1$, consistent with the $(m_e/m_i)^{1/2}$ expansion, and we use that $Z_i n_i = n_e$. Note that no cross-scale terms appear in the equations (3.2)-(3.8).

We note the definition of $\tilde{g}_e$,

$$\tilde{g}_e(\mathbf{R}_f, \mathbf{R}_s) = \left\langle \tilde{\delta f}_e(\mathbf{r}_f, \mathbf{r}_s) \right\rangle_{\mathbf{R}_f, \mathbf{R}_s}^\gamma = \tilde{h}_e(\mathbf{R}_f, \mathbf{R}_s) + \frac{eF_{0e}}{T_e} \left\langle \tilde{\phi}(\mathbf{r}_f, \mathbf{r}_s) \right\rangle_{\mathbf{R}_f, \mathbf{R}_s}^\gamma, \quad (3.9)$$

with $\tilde{h}_e$ the nonadiabatic response of electrons at ES, and $\mathbf{r}_f$ the ES particle position. We write the ES equations in terms of

$$\tilde{g}_{e\mathbf{k}_f}(\mathbf{R}_s) = \tilde{h}_{e\mathbf{k}_f}(\mathbf{R}_s) + \frac{eF_{0e}}{T_e} J_0(|\mathbf{k}_f||\boldsymbol{\rho}_e|) \tilde{\phi}_{\mathbf{k}_f}(\mathbf{R}_s), \quad (3.10)$$

where $\mathbf{k}_f = K_\psi \nabla \psi + K_\alpha \nabla \alpha$ is the ES wave vector, with K_ψ and K_α the ES radial and binormal wave numbers, respectively, and where we have used that $|\mathbf{k}_s| \rho_{\text{th},e} \sim (m_e/m_i)^{1/2} \ll 1$ to observe that $J_0(|\mathbf{k}_f + \mathbf{k}_s||\boldsymbol{\rho}_e|) \simeq J_0(|\mathbf{k}_f||\boldsymbol{\rho}_e|)$ and $\tilde{\phi}_{\mathbf{k}_f}(\mathbf{r}_s) \simeq \tilde{\phi}_{\mathbf{k}_f}(\mathbf{R}_s)$. Note that we refer to $K_\psi = \mathbf{k}_f \cdot \mathbf{e}_\psi$ as the ES radial wave number rather than the projection of $\mathbf{k}_f$ onto the $\nabla \psi$ direction, $\mathbf{k}_f \cdot \nabla \psi / |\nabla \psi|$. The equations for ES turbulence located at the IS perpendicular-to-the-field position (ψ_s, α_s) are

$$\begin{aligned} & \frac{\partial \tilde{g}_{e\mathbf{k}_f}}{\partial t_f} + v_\parallel \mathbf{b} \cdot \nabla \theta \frac{\partial \tilde{g}_{e\mathbf{k}_f}}{\partial \theta} + [\mathbf{i}(\mathbf{v}_e^M + \overline{\mathbf{v}}_e^E) \cdot \mathbf{k}_f] \tilde{g}_{e\mathbf{k}_f} \\ & + \left[\mathbf{i} \frac{c}{B} \mathbf{b} \times \mathbf{k}_f \cdot (\nabla F_{0e} + \nabla_s \bar{g}_e) \right] J_0(|\mathbf{k}_f||\boldsymbol{\rho}_e|) \tilde{\phi}_{\mathbf{k}_f} + \tilde{\mathfrak{N}}_{e\mathbf{k}_f} \\ & = \frac{eF_{0e}}{T_e} J_0(|\mathbf{k}_f||\boldsymbol{\rho}_e|) \left(v_\parallel \mathbf{b} \cdot \nabla \theta \frac{\partial \tilde{\phi}_{\mathbf{k}_f}}{\partial \theta} + [\mathbf{i} \mathbf{v}_e^M \cdot \mathbf{k}_f] \tilde{\phi}_{\mathbf{k}_f} \right) + \widetilde{C}_{le\mathbf{k}_f}. \end{aligned} \quad (3.11)$$

where t_f is the ES time coordinate, the nonlinear term $\tilde{\mathfrak{N}}_{e\mathbf{k}_f}$ is

$$\tilde{\mathfrak{N}}_{e\mathbf{k}_f} = - \sum_{\mathbf{k}'_f} \frac{c}{B} \mathbf{b} \times \mathbf{k}'_f \cdot \mathbf{k}_f J_0(|\mathbf{k}'_f||\boldsymbol{\rho}_e|) \tilde{\phi}_{\mathbf{k}'_f} \tilde{g}_{e\mathbf{k}_f - \mathbf{k}'_f}, \quad (3.12)$$

and the collision operator $\widetilde{C}_{le\mathbf{k}_f}$ is (see Section 2.9.3)

$$\begin{aligned} \widetilde{C}_{le\mathbf{k}_f} &= \left\langle \exp[\mathbf{i} \mathbf{k}_f \cdot \boldsymbol{\rho}_e] C_{ee} \left[\exp[-\mathbf{i} \mathbf{k}_f \cdot \boldsymbol{\rho}_e] \bar{h}_{e\mathbf{k}_f} \right] \right\rangle^\gamma \\ &+ \left\langle \exp[\mathbf{i} \mathbf{k}_f \cdot \boldsymbol{\rho}_e] L_{ei} \left[\exp[-\mathbf{i} \mathbf{k}_f \cdot \boldsymbol{\rho}_e] \bar{h}_{e\mathbf{k}_f} \right] \right\rangle^\gamma. \end{aligned} \quad (3.13)$$

Equation (3.11) is closed by the quasineutrality relation

$$\frac{e\tilde{\phi}_{\mathbf{k}_f}}{T_e} \left(1 - \Gamma_{0,e} + \frac{Z_i T_e}{T_i} \right) = - \frac{1}{n_e} \int d^3 \mathbf{v} \tilde{g}_{e\mathbf{k}_f} J_0(|\mathbf{k}_f||\boldsymbol{\rho}_e|), \quad (3.14)$$

where $\Gamma_{0,e} = n_e^{-1} \int d^3\mathbf{v} F_{0e} J_0^2(|\mathbf{k}_f||\boldsymbol{\rho}_e|)$, the ions have been taken to respond adiabatically; i.e., the nonadiabatic piece of the ion response at ES $\tilde{h}_i = 0$, and we use that $Z_i n_i = n_e$. Furthermore, the relations for $\bar{\mathbf{v}}_e^E$ and $\nabla_s \bar{g}_e$ appearing in the cross-scale terms in (3.11) are

$$\bar{\mathbf{v}}_e^E = \bar{\mathbf{v}}_e^E(\mathbf{R}_s) = \frac{c}{B} \mathbf{b} \times \left[\sum_{\mathbf{k}_s} \exp[i\mathbf{k}_s \cdot \mathbf{R}_s] i\mathbf{k}_s \bar{\phi}_{\mathbf{k}_s} \right], \quad (3.15)$$

$$\nabla_s \bar{g}_e = \nabla_s \bar{g}_e(\mathbf{R}_s) = \sum_{\mathbf{k}_s} \exp[i\mathbf{k}_s \cdot \mathbf{R}_s] i\mathbf{k}_s \bar{g}_{e\mathbf{k}_s}, \quad (3.16)$$

where we note that $\mathbf{R}_s = (\psi_s - \psi_0)\mathbf{e}_\psi + (\alpha_s - \alpha_0)\mathbf{e}_\alpha$ is the IS perpendicular-to-the-field guiding centre position as a function of (ψ_s, α_s) , with (ψ_0, α_0) the centre of the IS flux tube. As discussed in the previous Chapter, it is the gradient $\nabla_s \bar{g}_e$ and drift $\bar{\mathbf{v}}_e^E$ in (3.11) which give the ES flux tubes dependence on the IS coordinates (ψ_s, α_s) . The cross-scale terms which arise in the equation (3.11) for electrons at ES appear in analogy to the equilibrium drive term and equilibrium $\mathbf{E} \times \mathbf{B}$ flow shear in conventional gyrokinetics. These terms were implemented in the δf gyrokinetic flux tube code **GS2**, to allow for the study of scale-separated ES turbulence.

In contrast to the ES turbulence the IS turbulence evolves independently of the fluctuations at ES. We can study the effect of IS eddies on ES turbulence by calculating or prescribing the IS quantities $\bar{\mathbf{v}}_e^E$ and $\nabla_s \bar{g}_e$ and observing the consequences for ES physics. In this Chapter we focus on the effect of saturated IS turbulence on the ES linear instability, for cyclone base case (CBC) parameters [43], at different ion equilibrium temperature gradients.

3.2 Separating the wave number spectrum into IS and ES components

In the following Sections we describe an investigation into the effect of IS turbulence on ES microinstabilities. We assume that the IS and ES fluctuations are sufficiently well separated that we can describe the turbulence using the coupled system of gyrokinetic equations (3.2)-(3.14). In this section we describe how we choose to separate the wave number spectrum into IS and ES components.

We use radial and binormal wave numbers normalised to have dimensions of inverse length. We define the IS radial wave number $k_x = k_\psi(d\psi/dx)$ and IS binormal wave number $k_y = k_\alpha(d\alpha/dy)$, with $d\psi/dx = B_{\text{ref}}a\rho_0/q_0$ and $d\alpha/dy = B_{\text{ref}}a\partial\rho/\partial\psi|_{\rho_0}$: we choose a to be the half-diameter of the last closed flux surface at the height of the magnetic axis; q_0 is the safety factor on the flux surface ψ_0 at the centre of the IS flux tube; B_{ref} is a reference magnetic field, chosen here to be the toroidal magnetic field strength on the flux surface ψ_0 at the major radius of the magnetic axis; and $\rho = r/a$ is the normalised minor radial coordinate, with r the half-diameter of the flux surface at the height of the magnetic axis, i.e. $r = a$ at the last-closed-flux surface, and ρ_0 the normalised minor radial coordinate of the flux surface. To describe the locations of the ES flux tubes within the IS flux tube we use the IS perpendicular-to-the-field coordinates (x_s, y_s) that are defined to have units of length; i.e., $x_s = (\psi_s - \psi_0)(d\psi/dx)^{-1}$ and $y_s = (\alpha_s - \alpha_0)(d\alpha/dy)^{-1}$, respectively. Finally, we define the ES radial wave number $K_x = K_\psi(d\psi/dx)$ and the ES binormal wave number $K_y = K_\alpha(d\alpha/dy)$.

It is convenient to characterise the radial wave numbers in terms of the ballooning angle θ_0 , which defines the poloidal angle at which a nonzonal mode has no radial variation at fixed toroidal angle. For IS modes $\theta_0 = k_x/\hat{s}k_y$, and for ES modes $\theta_0 = K_x/\hat{s}K_y$. Both IS and ES modes have a $[-\pi, \pi]$ extent in poloidal angle θ and are connected at the $\pm\pi$ boundaries by the parallel boundary condition [44] (see Section 2.7). IS modes with $k_y \neq 0$ and $\theta_0 = \theta'_0$ are coupled at fixed k_y to the modes with $\theta_0 = \theta'_0 \pm 2\pi$. ES modes with $K_y \neq 0$ and $\theta_0 = \theta'_0$ are coupled at fixed K_y to the modes with $\theta_0 = \theta'_0 \pm 2\pi$. The coupled chains of modes constructed this way have an extended θ domain, and can be thought of as ballooning modes, cf. [74]. To uniquely label these ballooning modes we use $\hat{\theta}_0$, where $\hat{\theta}_0$ is defined to be the θ_0 of a mode within the coupled chain that satisfies $-\pi \leq \theta_0 < \pi$. We note that using $\hat{\theta}_0$ we can write θ_0 for every mode within the same coupled chain of modes as $\theta_0 = \hat{\theta}_0 + 2\pi N$, where N is an integer.

To satisfy the assumption of scale separation we choose the physical parameters such that the δf gyrokinetic equation predicts a gap in the linear growth rate spectrum between the IS and ES. This criterion makes it plausible that there is a gap in the nonlinear fluctuation spectrum. A gap in the fluctuation spectrum would allow us to

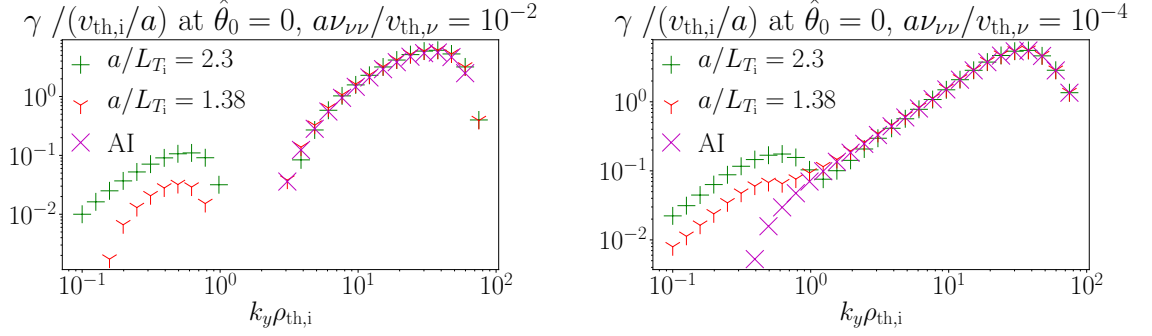


Figure 3.1: The linear growth rate γ calculated in **GS2** in the absence of cross scale coupling, for equilibrium electron temperature gradient $a/L_{T_e} = 2.3$; and equilibrium ion temperature gradients of $a/L_{T_i} = 2.3$ and $a/L_{T_i} = 1.38$, and for the adiabatic ion (AI) model. Note that the modes with $\hat{\theta}_0 = 0$ are the most unstable modes. We observe that a/L_{T_i} has little effect on the ES ($k_y \rho_{th,i} \sim 60$) modes. (Left) shows γ for the value of the collision frequency used throughout this Chapter, $a\nu_{\nu\nu}/v_{th,\nu} = 10^{-2}$. (Right) shows γ for a collision frequency of $a\nu_{\nu\nu}/v_{th,\nu} = 10^{-4}$.

divide the full toroidal mode number range into an IS part and an ES part. Figure 3.1 (Left) shows the linear growth rate γ calculated in **GS2** using the linear gyrokinetic equation in the absence of cross-scale coupling. We show γ for two values of equilibrium ion temperature gradient, $a/L_{T_i} = 2.3$ and $a/L_{T_i} = 1.38$, and for an adiabatic ion response (AI). We take $a/L_{T_e} = 2.3$ and use CBC parameters to define the magnetic geometry and the density gradient, for details see Table 3.1. For each k_y mode in Figure 3.1 we use the velocity space and parallel coordinate resolutions detailed in B.2. The simulations presented in Figure B.4 use time step sizes of $a\Delta t/v_{th,i} = 0.01/(k_y \rho_{th,i})$. We note that the gap in the growth rate spectrum depends strongly on the value of the species collision frequency $a\nu_{\nu\nu}/v_{th,\nu}$. For the moderate value of $a\nu_{\nu\nu}/v_{th,\nu} = 0.01$, Figure 3.1 (Left), there is a gap in the growth rate spectrum. However, in Figure 3.1 (Right) we see that for smaller values of $a\nu_{\nu\nu}/v_{th,\nu}$, there is no stable gap between the ITG and ETG instabilities. In such cases it is not clear how to define the minimum ES K_y mode. We note that the maximum ITG instability growth rate $\gamma_{ITG}^{\max}$ is larger than the ion collision frequency $a\nu_{ii}/v_{th,i} = 0.01 < a\gamma_{ITG}^{\max}/v_{th,i} \sim 0.1$, and the maximum ETG instability growth rate $\gamma_{ETG}^{\max}$ is larger than the electron collision frequency $a\nu_{ee}/v_{th,e} = 0.01 < a\gamma_{ETG}^{\max}/v_{th,e} \sim 0.1$. We consider an argument to show that there should be

Physical quantity	Description	Value
$r/a = \rho$	normalised mid-plane minor radius	0.54
R/a	normalised mid-plane major radius	3.0
dR/dr	Shrafranov shift	0.0
q	safety factor	1.4
$\hat{s} = rd \ln q / dr$	magnetic shear	0.8
κ	elongation	1.0
$d\kappa/d\rho$	elongation derivative	0.0
δ	triangularity	0.0
$d\delta/d\rho$	triangularity derivative	0.0
$a/L_{T_e} = -d \ln T_e / d\rho$	normalised electron temperature gradient	2.3
$a/L_{T_i} = -d \ln T_i / d\rho$	normalised ion temperature gradient	2.3 (and 1.38)
$-d \ln n / d\rho$	normalised density gradient	0.733
T_i/T_e	ion to electron temperature ratio	1.0
$a\nu_{ii}/v_{th,i}$	normalised ion collisionality	0.01
$a\nu_{ee}/v_{th,e}$	normalised electron collisionality	0.01
m_e/m_i	electron-ion mass ratio	1/3670

Table 3.1: The cyclone-base-case-like parameters used for this study

a dissipation range between the IS and ES fluctuations. According to the scalings inferred from critical balance [71], in the inertial range the spectrum of the fluctuation amplitudes goes like $k_{\perp}^{-2/3}$. Hence, the gradients of fluctuations have a spectrum which goes like $k_{\perp}^{1/3}$, i.e. the dominant contributions to gradients such as $\nabla_s \bar{\phi}$ and $\nabla_s \bar{h}_e$ come from the largest $k_{\perp}$ in the inertial range. If there is no gap between the IS and ES inertial ranges then resolving the the IS gradients $\nabla_s \bar{\phi}$ and $\nabla_s \bar{h}_e$ requires ES resolution. We discuss how to choose the level of collisional dissipation appropriate for scale-separated turbulence. Collisional dissipation occurs at a perpendicular scale $k_{\perp}$ when $\nu_{ii} k_{\perp}^2 \rho_{th,i}^2 \sim \omega$, where ω is the turbulent frequency corresponding to structures with the scale $k_{\perp}$. If we impose that $\omega \sim v_{th,i}/a$ and $k_{\perp} \rho_{th,e} \ll 1$, to ensure that the dissipation occurs at the IS, then we find that ν_{ii} should satisfy $(a\nu_{ii}/v_{th,i})^{-1/2} \ll (m_i/m_e)^{1/2}$. The choice of collision frequency used for IS simulations in this thesis satisfies this criterion, with $(a\nu_{ii}/v_{th,i})^{-1/2} = 10 \ll (m_i/m_e)^{1/2} \simeq 60$. As we discuss in the next Section, the IS simulations presented in this thesis use a hyperviscous model for high- $k_{\perp}$ dissipation in addition to the physical dissipation mechanism provided by collisions.

Motivated by the γ spectrum in Figure 3.1 (Left) for CBC parameters we define the

maximum IS toroidal mode number as $k_y^{\max} \rho_{\text{th},i} = 1.05$ and we define the minimum ES toroidal mode number as $K_y^{\min} \rho_{\text{th},e} = 0.041$ ($k_y \rho_{\text{th},i} = 2.5$). These definitions assume that the gap in the γ spectrum in Figure 3.1 (Left) corresponds to the gap in the fluctuation spectrum between the IS and ES fluctuations. Henceforth, we use ES units to normalise numerical results from the ES wave number range.

3.3 Simulating IS turbulence and computing a sample of IS gradients and drifts

In the IS equations (3.2)-(3.8) there are no cross-scale coupling terms. This means that the IS turbulence evolves independently of the ES turbulence in the limit that $(m_e/m_i)^{1/2} \rightarrow 0$. We use the δf gyrokinetic code **GS2** to integrate the equations (3.2)-(3.8). By simulating (3.2)-(3.8) for many $a/v_{\text{th},i}$ times we obtain a time history of fully developed IS turbulence. We sample the IS turbulence at specific times in this history to create an ensemble of IS gradients. We use these IS gradients to study the effects of cross-scale interaction on ES turbulence. We focus on two cases for which the effect of cross-scale interactions is vastly different: a case where the IS turbulence is driven far above marginal, and a case where the IS turbulence is near marginal stability. We realise IS turbulence driven far above marginal stability using CBC parameters (see Table 3.1), with an equilibrium ion temperature gradient $a/L_{T_i} = 2.3$. We find a near marginal IS turbulence by reducing a/L_{T_i} to 1.38 while holding all other parameters fixed.

In order to solve the equation for electrons at IS (3.5) using **GS2** we make use of the fact that **GS2** is an implicit code. Recall that (3.5) is derived in Chapter 2 by taking the $(m_e/m_i)^{1/2} \rightarrow 0$ limit in the electron, IS gyrokinetic equation. Making contact with the derivation of (3.5), we impose that $\bar{h}_e = 0$ for passing, nonzonal electrons; as indeed is the case in the asymptotic limit that $(m_e/m_i)^{1/2} \rightarrow 0$ (see Section 2.3). The remaining, nonzero piece of the electron distribution function is due to the trapped and the passing zonal electrons. These electrons follow periodic orbits, over which the implicit algorithm of **GS2** can average. For the averaging to be equivalent to the orbit averaging of Chapter 2, the time step size Δt must satisfy $\omega_b \Delta t \gg 1$, where ω_b is

the typical bounce frequency of the trapped electrons. In a small aspect ratio device, $\epsilon = r/R \ll 1$, we can estimate the size of $\omega_b \sim \epsilon^{1/2} v_{th,e}/(qR)$. Hence, we require that

$$\frac{a\Delta t}{v_{th,i}} \left(\frac{m_i}{m_e} \right)^{1/2} \frac{\epsilon^{1/2}}{qR} \gg 1. \quad (3.17)$$

In the nonlinear IS simulations presented in this thesis we take $(m_i/m_e)^{1/2} \simeq 60$, and $a\Delta t/v_{th,i} = 3.125 \times 10^{-3}$ as a result of the Courant-Friedrichs-Lewy (CFL) condition on the nonlinear term. With CBC parameters $\epsilon^{1/2}/(qR) \simeq 10^{-1}$. Hence, in the IS nonlinear simulations presented here the condition (3.17) is not satisfied, $(a\Delta t/v_{th,i})(m_i/m_e)^{1/2} \epsilon^{1/2}/qR \simeq 0.2$. As we discuss in Appendix B.3, for the CBC parameters used in this thesis, the imposition of that $\bar{h}_e = 0$ for passing, nonzonal electrons appears to be enough to correctly match the behaviour of the electron response in the limit $(m_e/m_i)^{1/2} \rightarrow 0$, even with the physical value of the mass ratio $(m_i/m_e)^{1/2} \simeq 60$. The end result is that we can use the **GS2** algorithm to take the asymptotic limit $(m_e/m_i)^{1/2} \rightarrow 0$ in the electron gyrokinetic equation that was the starting point in the derivation of (3.5), and hence solve (3.5) numerically as required.

To simulate IS turbulence we use a flux tube with a perpendicular cross-section of $40\pi\rho_{th,i} \times 40\pi\rho_{th,i}$. We use $n_y = 21$ toroidal modes with $k_y > 0$ and $k_y^{\max}\rho_{th,i} = 1.05$. We use $n_x = 255$ radial modes with $k_x^{\max}\rho_{th,i} = 6.38$. We use the poloidal angle and velocity space resolutions specified in B.2. We employ a hyperviscous model for high- k_x and high- k_y dissipation, which has the effect of adding a term

$$-D \left(\frac{k_{s,\perp}}{k_{c,\perp}} \right)^p \bar{h}_{\nu\mathbf{k}_s}, \quad (3.18)$$

to the RHS of equations (3.2) (with $\nu = i$) and (3.5) (with $\nu = e$), where $k_{s,\perp}(\theta) = |\mathbf{k}_s|$, $aD/v_{th,i} = 0.05$ and $k_{c,\perp}$ is a cut-off perpendicular wave number, specified to be the maximum $k_{s,\perp}$ (in θ) of the mode with $k_x = 0.50k_x^{\max}$ and $k_y = k_y^{\max}$. We specify the hyperviscous model in detail in Appendix B.4. We choose the maximum radial mode number $k_x^{\max}$ and model of hyperviscosity so that the amplitude of the IS potential $|\bar{\phi}_{k_x,k_y}|$ satisfies

$$|\bar{\phi}_{k_x,k_y}| \rightarrow 0, \text{ as } |k_x| \rightarrow k_x^{\max}. \quad (3.19)$$

This was necessary to ensure that the IS drift $\bar{\mathbf{v}}_e^E$ satisfied the parallel boundary condition [44] (see Section 2.7) to high accuracy, which is required for the ES simulations including the IS drift $\bar{\mathbf{v}}_e^E$ to be valid. We discuss this resolution issue in detail in Appendix B.4: It is difficult to quantify to what precision $\bar{\mathbf{v}}_e^E$ should satisfy the parallel boundary condition numerically; as $\bar{\mathbf{v}}_e^E$ is calculated from noisy turbulent fields, it is difficult to distinguish physical “noise” in $\bar{\mathbf{v}}_e^E$ from the piece of $\bar{\mathbf{v}}_e^E$ which fails to satisfy the parallel boundary condition if this latter piece is large. Ultimately, we find that $\bar{\mathbf{v}}_e^E$ satisfies the parallel boundary condition to within a few percent of relative error, and that this numerical error is sufficiently small to be confident in the results of this Chapter. We note that the IS gradient $\nabla_s \bar{g}_e$ exactly obeys the IS parallel boundary condition when calculated using the model of electrons at IS. Note that the passing electrons in the model of IS turbulence respond adiabatically and so therefore choosing $k_x^{\max} \rho_{\text{th},i} = 6.38$ does not introduce ES structures into the IS turbulence; the ballooning structure of the ITG mode is purely due to passing ions.

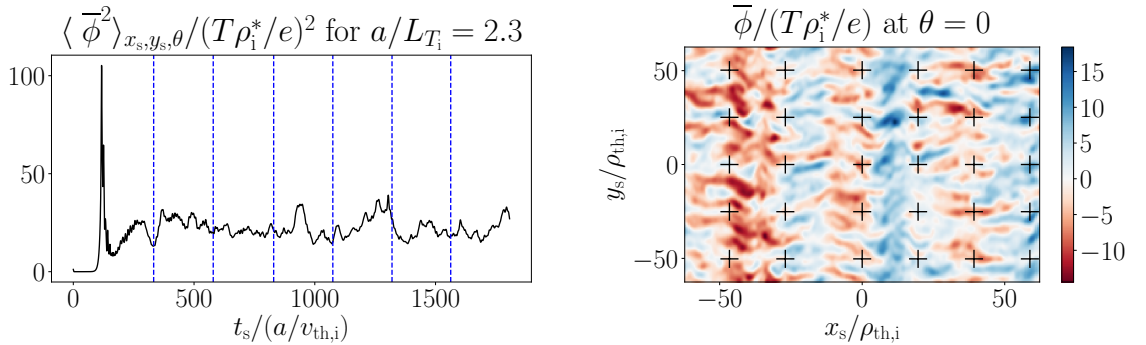


Figure 3.2: (Left) The square of the IS electrostatic potential, averaged over the flux tube. The vertical dashed lines indicate the sampled times t_s for the $a/L_{T_i} = 2.3$ case. (Right) Contours of the IS electrostatic potential in the perpendicular-to-the-field plane. The sampled (x_s, y_s) positions are indicated by crosses. The electrostatic potential shown is for the last time sample from the left Figure, at the outboard midplane where $\theta = 0$.

In Figure 3.2 (Left) we show the square of the IS electrostatic potential averaged over the IS flux tube $\langle \bar{\phi}^2 \rangle_{x_s, y_s, \theta}$ as a function of t_s for IS turbulence with $a/L_{T_i} = 2.3$. The vertical lines indicate the times at which we sample the instantaneous values of the IS gradient $\nabla_s \bar{g}_e$ and drift $\bar{\mathbf{v}}_e^E$. We sample $\nabla_s \bar{g}_e$ and $\bar{\mathbf{v}}_e^E$ at a set of (x_s, y_s) positions

in the plane perpendicular to the magnetic field within the IS flux tube: Figure 3.2 (Right) shows an instantaneous contour plot of $\bar{\phi}(x_s, y_s)$ and indicates the sampling positions with crosses. At a given (t_s, x_s, y_s) each IS gradient and drift is sampled as a function of $(\theta, \varepsilon, \lambda, \sigma)$. The ensemble of the gradients consists of a sample of size $6(t_s) \times 6(x_s) \times 5(y_s)$.

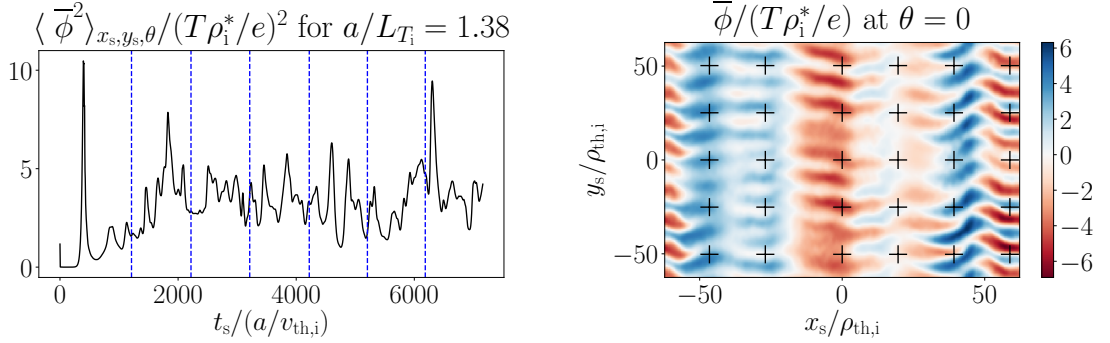


Figure 3.3: (Left) The square of the IS electrostatic potential for the $a/L_{Ti} = 1.38$ case, averaged over the flux tube. The vertical dashed lines indicate the sampled times t_s . (Right) Contours of the IS electrostatic potential in the perpendicular-to-the-field plane. The sampled (x_s, y_s) positions are indicated by crosses. The electrostatic potential shown is for the last time sample from the left Figure, at the outboard midplane where $\theta = 0$.

We take the same approach to sampling the IS turbulence that is driven by an ion temperature gradient of $a/L_{Ti} = 1.38$. In Figure 3.3 we show $\langle \bar{\phi}^2 \rangle_{x_s, y_s, \theta}$ as a function of time, and we indicate the sampling positions within the IS flux tube. The sample (x_s, y_s) positions are the same for both the $a/L_{Ti} = 1.38$ and the $a/L_{Ti} = 2.3$ cases. Figures 3.2 and 3.3 indicate that the instantaneous samples are taken from statistically converged states of the IS turbulence. This allows us to regard the sample of IS gradients as representative of the turbulence. We note that simulations of IS turbulence using the model for the electron response with artificially reduced $(m_e/m_i)^{1/2}$ (so that (3.17) is satisfied) have quantitatively similar statistics as the IS simulations presented in this thesis: The time averaged IS fluxes, and the average effect of the IS gradients and drifts on the ETG instability are quantitatively similar.

3.4 Linear simulations of the ETG instability at ES

In this section we present linear simulations which solve the ES equations (3.11) and (3.14), neglecting the nonlinear term $\tilde{\mathfrak{N}}_{\mathbf{e}\mathbf{k}_f}$ and using IS gradients $\nabla_s \bar{g}_e$ and drifts $\bar{\mathbf{v}}_e^E$ sampled from the simulations of IS turbulence, as described in Section 3.3. In the absence of IS turbulence – for example, in cases where the IS turbulence is suppressed – the ES equations (3.11) and (3.14) are the familiar single-scale gyrokinetic equations for ES turbulence with an adiabatic ion response. Figure 3.4 shows the linear growth rate γ as a function of K_y and $\hat{\theta}_0$, calculated using equations (3.11) and (3.14) with $\nabla_s \bar{g}_e = \bar{\mathbf{v}}_e^E = 0$. To simulate each $(K_y, \hat{\theta}_0)$ mode we use the velocity space and parallel coordinate resolutions detailed in B.2, with time step sizes of $a\Delta t/v_{\text{th},e} = 2.7 \times 10^{-6}/(K_y \rho_{\text{th},e})$. These resolutions are used in all the ES simulations presented in this thesis. The dashed line in Figure 3.4 is the linear stability boundary. We estimate the position of this contour by using a cubic spline fit of interpolated $\gamma(K_y, \hat{\theta}_0)$ data. The modes within the dashed line have $\gamma > 0$ and so are unstable.

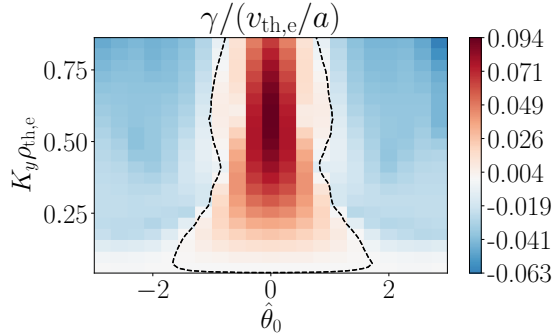


Figure 3.4: The ES linear growth rate γ calculated using equations (3.11) and (3.14) with $\nabla_s \bar{g}_e = \bar{\mathbf{v}}_e^E = 0$.

We now discuss the effects of nonzero $\nabla_s \bar{g}_e$ and $\bar{\mathbf{v}}_e^E$ on the ES linear growth rate. For each of the sampled $6(t_s) \times 6(x_s) \times 5(y_s)$ points within the IS flux tube we calculate the IS gradient $\nabla_s \bar{g}_e$ and drift $\bar{\mathbf{v}}_e^E$ as a function of $(\theta, \varepsilon, \lambda, \sigma)$. We then perform an initial value, linear calculation using equations (3.11) and (3.14) for each of the sampled (x_s, y_s) points within the IS flux tube at the distinct sample times t_s . We do this using gradients sampled from both the $a/L_{T_i} = 1.38$ and the $a/L_{T_i} = 2.3$ IS turbulences.

3.4.1 The effect of marginally unstable IS turbulence on the ETG instability

In this Section we discuss the effects of IS turbulence driven near marginal stability ($a/L_{Ti} = 1.38$) on the ES instability. Figure 3.5 shows the ETG growth rates calculated including the effects of IS gradients and drifts from two distinct (x_s, y_s) points within the IS flux tube, at the same time t_s . This illustrates that at a given (t_s, x_s, y_s) the effect of the IS gradients and drifts can be variable; the range of unstable ($K_y, \hat{\theta}_0$) modes can change, as can the value of the maximum growth rate $\gamma^{\max}$. However, if we compute the ETG growth rate $\gamma(t_s, x_s, y_s)$ including the effects of IS gradients and drifts at all of the points and times within the $6(t_s) \times 6(x_s) \times 5(y_s)$ sample from the IS flux tube, we find that the average effect of the IS turbulence is negligible. Figure 3.6 shows the average ETG growth rate $\langle \gamma \rangle_{t_s, x_s, y_s}$, computed by averaging over $\gamma(t_s, x_s, y_s)$, and the standard error on this average $\sigma_{\langle \gamma \rangle_{t_s, x_s, y_s}}$. Note the similarity between the averaged growth rate $\langle \gamma \rangle_{t_s, x_s, y_s}$ in Figure 3.6 and the growth rate calculated without IS gradients and drifts in Figure 3.4. The scale of the standard error $\sigma_{\langle \gamma \rangle_{t_s, x_s, y_s}}$ in Figure 3.6 indicates that the value of the average growth rate is well-converged. The result here is not surprising: sufficiently small amplitude IS turbulence should not affect the ES fluctuations.

IS turbulence driven near marginal stability can be realised when there is a strong equilibrium flow shear [75–77]. The effect of shear-suppressed IS turbulence on ETG linear stability should be examined. The near-marginal IS turbulence presented here may not be universally representative of near marginal states.

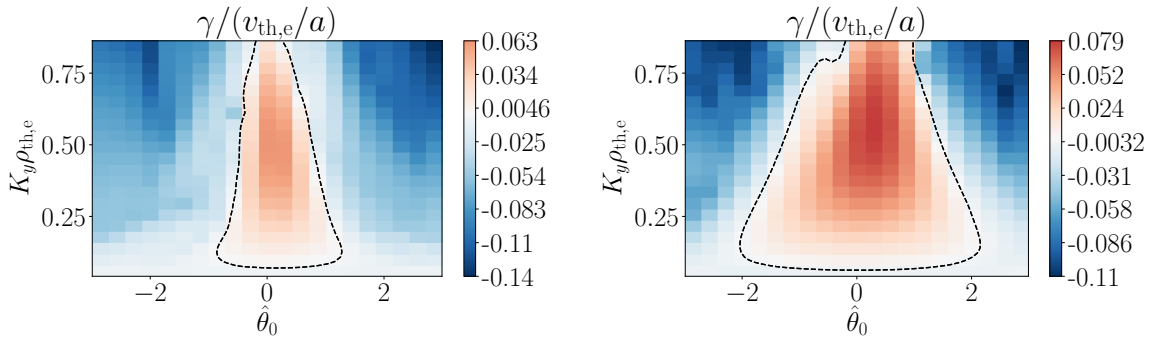


Figure 3.5: ES linear growth rates γ calculated with $\nabla_s \bar{g}_e$ and $\bar{\mathbf{v}}_e^E$ sampled from the IS turbulence with $a/L_{Ti} = 1.38$, at distinct (x_s, y_s) points at a single t_s .

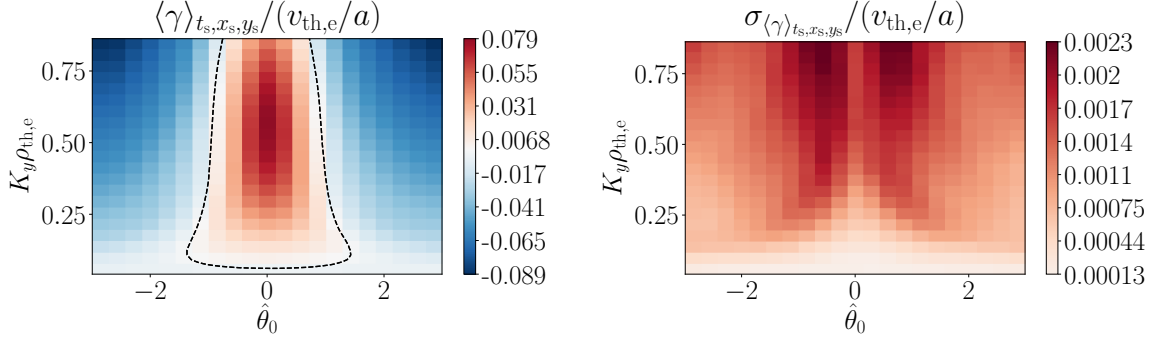


Figure 3.6: The average ES linear growth rate $\langle \gamma \rangle_{t_s, x_s, y_s}$ (Left) calculated with $\nabla_s \bar{g}_e$ and $\bar{\mathbf{v}}_e^E$ sampled from the IS turbulence with $a/L_{T_i} = 1.38$, and the standard error $\sigma \langle \gamma \rangle_{t_s, x_s, y_s}$ on $\langle \gamma \rangle_{t_s, x_s, y_s}$ (Right).

3.4.2 The effect of strongly driven IS turbulence on the ETG instability

We now discuss the effects of strongly driven IS turbulence ($a/L_{T_i} = 2.3$) on the ES instability. Figure 3.7 shows the ETG growth rate calculated using the IS gradient $\nabla_s \bar{g}_e$ and drift $\bar{\mathbf{v}}_e^E$ sampled from two distinct IS (x_s, y_s) within the IS flux tube, at two distinct t_s . In contrast to the near marginal case, these examples show that the ES fluctuations can be completely stabilised by IS turbulence. To demonstrate the average

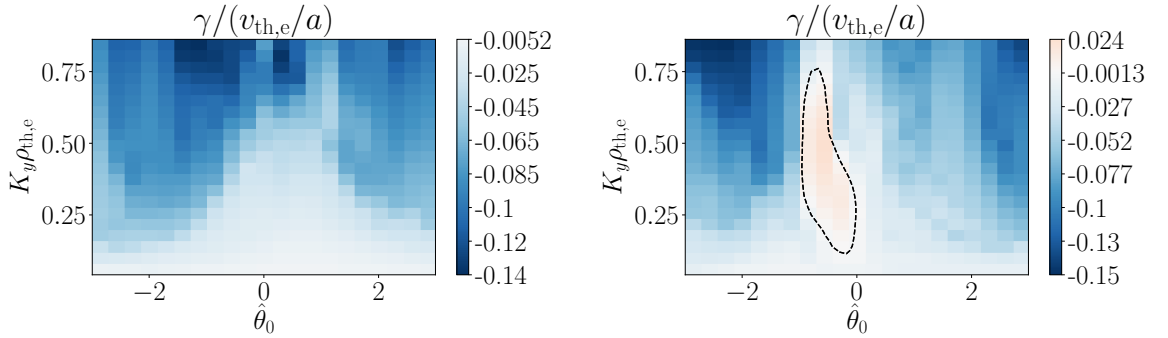


Figure 3.7: The ES linear growth rates γ calculated with $\nabla_s \bar{g}_e$ and $\bar{\mathbf{v}}_e^E$ sampled from distinct (t_s, x_s, y_s) points within the IS turbulence driven at $a/L_{T_i} = 2.3$.

effect of the IS turbulence when driven far from marginal stability, we again compute the ETG growth rate $\gamma(t_s, x_s, y_s)$ including the effects of IS gradients and drifts at all of the points and times within the $6(t_s) \times 6(x_s) \times 5(y_s)$ sample from the IS flux tube,

and compute the average ETG growth rate $\langle\gamma\rangle_{t_s, x_s, y_s}$. In this case we find that the average effect of IS turbulence is to suppress the ETG instability. In Figure 3.8 we show the average ETG growth rate $\langle\gamma\rangle_{t_s, x_s, y_s}$, and the standard error on this average $\sigma_{\langle\gamma\rangle_{t_s, x_s, y_s}}$. Note that in Figure 3.8 the ES instability has been completely suppressed by the presence of the IS gradients and drifts. The size of $\sigma_{\langle\gamma\rangle_{t_s, x_s, y_s}}$ relative to $\langle\gamma\rangle_{t_s, x_s, y_s}$ indicates that the sample is large enough to give converged statistics: the minimum value of $|\langle\gamma\rangle_{t_s, x_s, y_s}|$ is 2.3 times larger than the maximum value of $\sigma_{\langle\gamma\rangle_{t_s, x_s, y_s}}$; i.e. the effect of almost all the IS gradients and drifts within the $6(t_s) \times 6(x_s) \times 5(y_s)$ IS sample is to stabilise the ETG mode. We now discuss the cross-scale interaction mechanisms

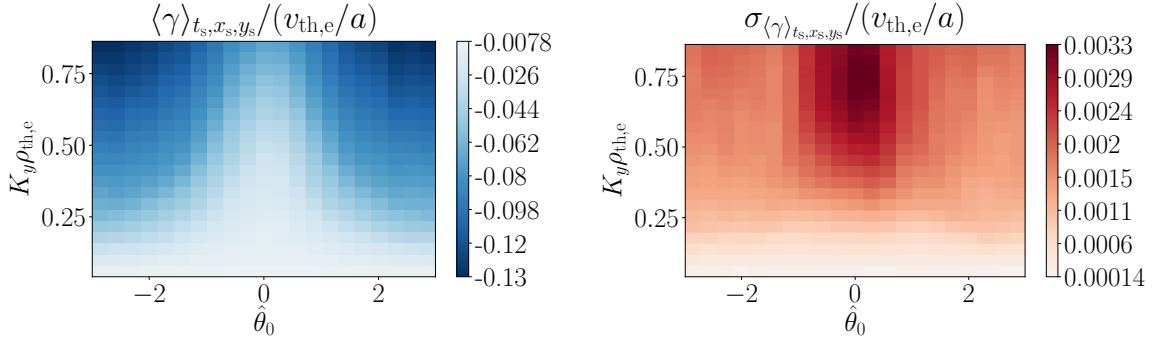


Figure 3.8: The average ES linear growth rate $\langle\gamma\rangle_{t_s, x_s, y_s}$ (Left) calculated with $\nabla_s \bar{g}_e$ and $\bar{v}_e^E$ sampled from the IS turbulence with $a/L_{Ti} = 2.3$, and the standard error $\sigma_{\langle\gamma\rangle_{t_s, x_s, y_s}}$ on $\langle\gamma\rangle_{t_s, x_s, y_s}$ (Right).

responsible for the suppression of the ES fluctuations in the $a/L_{Ti} = 2.3$ case.

3.4.3 Identifying the mechanism responsible for the suppression of the ETG instability

In Section 3.4.2 we demonstrated that a strongly-driven ETG instability is stabilised in the presence of strongly driven IS turbulence. In this section we consider which of the two cross-scale terms is responsible for this striking result. We again calculate the ETG growth rate $\gamma(t_s, x_s, y_s)$ but we include only the effect of either the IS gradient $\nabla_s \bar{g}_e$ or the IS drift $\bar{v}_e^E$; we do this for every sampled (x_s, y_s) point within the IS flux tube, and at every sampled time t_s . We only examine the case of $a/L_{Ti} = 2.3$ where the average effect of the IS turbulence on the ETG instability was significant. This approach allows

us to identify the effect of each cross-scale mechanism independently. We then calculate the average ETG growth rate $\langle \gamma \rangle_{t_s, x_s, y_s}$ including only the effect of the IS gradient or the IS drift. Figure 3.9 shows the average ETG growth rate as calculated using equations (3.11) and (3.14), with $\nabla_s \bar{g}_e$ taken from the IS $6(t_s) \times 6(x_s) \times 5(y_s)$ sample, and with $\bar{\mathbf{v}}_e^E = 0$. Figure 3.10 shows the average ETG growth rate as calculated with $\bar{\mathbf{v}}_e^E$ taken

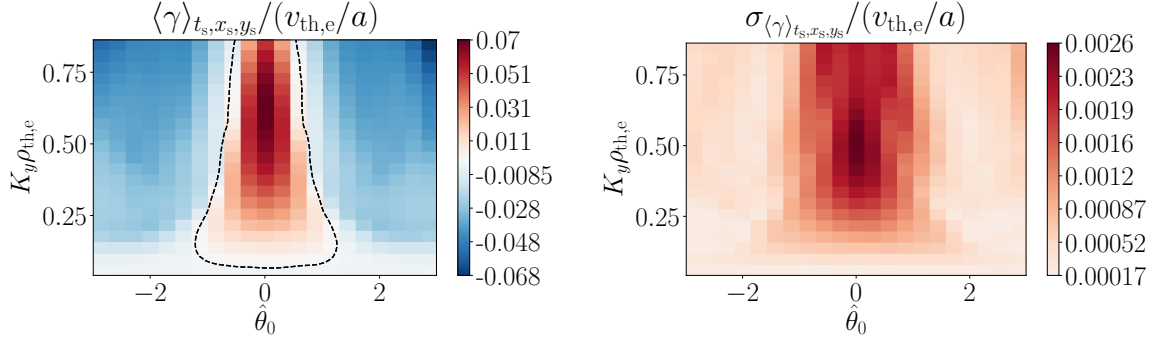


Figure 3.9: The average ETG growth rate $\langle \gamma \rangle_{t_s, x_s, y_s}$ (Left) calculated with $\nabla_s \bar{g}_e$ sampled from the IS turbulence with $a/L_{T_i} = 2.3$ and $\bar{\mathbf{v}}_e^E = 0$; and the standard error $\sigma \langle \gamma \rangle_{t_s, x_s, y_s}$ on $\langle \gamma \rangle_{t_s, x_s, y_s}$ (Right).

from the IS $6(t_s) \times 6(x_s) \times 5(y_s)$ sample, and with $\nabla_s \bar{g}_e = 0$.

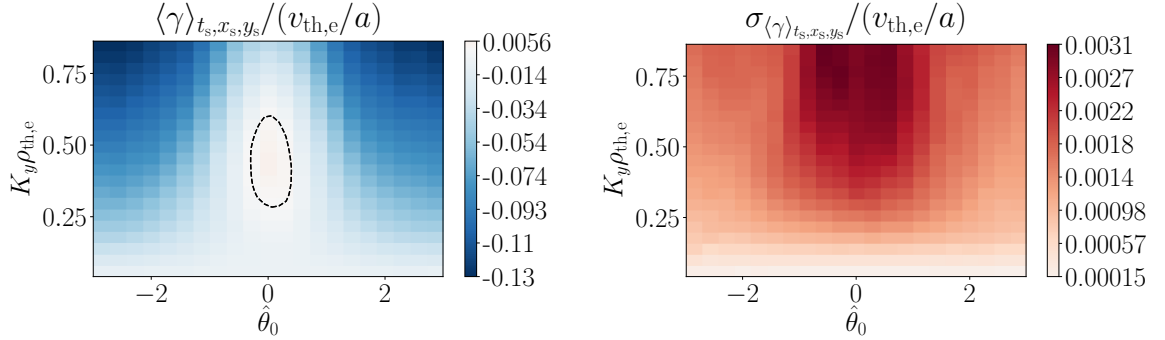


Figure 3.10: The average ETG growth rate $\langle \gamma \rangle_{t_s, x_s, y_s}$ (Left) calculated with $\bar{\mathbf{v}}_e^E$ sampled from the IS turbulence with $a/L_{T_i} = 2.3$ and $\nabla_s \bar{g}_e = 0$; and the standard error $\sigma \langle \gamma \rangle_{t_s, x_s, y_s}$ on $\langle \gamma \rangle_{t_s, x_s, y_s}$ (Right).

By comparing Figure 3.4, which shows the ETG growth rate in the absence of IS turbulence, to Figure 3.9 we see that for CBC parameters the gradient of the electron distribution function $\nabla_s \bar{g}_e$ has only a small suppressing effect on the ETG instability.

This result may be a consequence of the size of the collision frequency used in the simulations, and model for electrons at IS. As we discussed in Section 3.2, in order to produce a gap in the linear growth rate spectrum – and so have a scale-separated system – a collision frequency of $a\nu_{ee}/v_{th,e} = 0.01 \sim 0.1a\gamma_{ETG}^{\max}/v_{th,e}$ was required. For $a\nu_{ee}/v_{th,e} \gg 1$ and $(m_e/m_i)^{1/2} \rightarrow 0$ we can expect the electron distribution function $\bar{g}_e$ to tend towards a flux surface averaged adiabatic response [68]. This constrains the form of the electron distribution function at IS, which reduces the possible impacts of $\nabla_s \bar{g}_e$ to simply changing the effective background density gradient. We note that in the limit of small $\epsilon = r/R$ the effective collision frequency for trapped electrons at IS is $a\nu_{ee}^{\text{trapped}}/v_{th,e} \sim a\nu_{ee}/(\epsilon v_{th,e}) \simeq 0.05$. This is enhanced with respect to the collision frequency for passing particles due to the narrow range in velocity space which trapped particles occupy. This indicates that the response of electrons at IS is likely to be zonally dominated – in the model for electrons used in this thesis the nonzonal component of the nonadiabatic response of electrons vanishes in the collisional limit. However, the linear simulations shown in B.3 indicate that there is a significant nonzonal trapped electron response for CBC parameters, i.e. the simulations presented here do not represent the collisional limit. Understanding the effect of $\nabla_s \bar{g}_e$ on the ETG instability could be the subject of future work.

By comparing Figures 3.4, 3.8 and 3.10 we see that the inclusion of the cross-scale term involving the IS drift $\bar{\mathbf{v}}_e^E$ accounts for almost all of the suppression observed when both ES cross-scale terms are included. We conclude that for the parameters considered here the ES cross-scale term due to $\bar{\mathbf{v}}_e^E$ is the dominant mechanism by which the IS turbulence affects the ES fluctuations.

The piece of $\bar{\mathbf{v}}_e^E$ which is constant in the parallel coordinate θ may be removed from the ES equation (3.11) by a change to a rotating frame, see Section 2.8. It is thus the parallel variation in the IS drift $\bar{\mathbf{v}}_e^E$ that must act to suppress the ETG instability. In the following section we introduce a simplified model for this cross-scale suppression mechanism that relies upon the parallel-to-the-field shear introduced by $\bar{\mathbf{v}}_e^E$.

3.5 A simple model for the effect of the parallel-to-the-field shear in the IS $\mathbf{E} \times \mathbf{B}$ drift

In this Section we describe a simplified model for the effect of parallel-to-the-field shear in the IS $\mathbf{E} \times \mathbf{B}$ drift on a ES linear instability. This model gives a qualitative understanding for how the IS drift $\bar{\mathbf{v}}_e^E$ robustly suppresses ETG instability in the simulation results of Sections 3.4.2 and 3.4.3. To derive this model we prescribe the simplest possible IS drift $\bar{\mathbf{v}}_e^E$ that is consistent with the IS parallel boundary condition, and that has the effect of parallel-to-the-field shear. We then give simulation results where we examine the effect of local parallel-to-the-field shear in the IS $\mathbf{E} \times \mathbf{B}$ drift on the ETG instability.

We focus on the IS drift term $[\mathbf{i}\bar{\mathbf{v}}_e^E \cdot \mathbf{k}_f] \tilde{g}_{e\mathbf{k}_f}$ in equation (3.11). In real space notation, $[\mathbf{i}\bar{\mathbf{v}}_e^E \cdot \mathbf{k}_f] \tilde{g}_{e\mathbf{k}_f}$ corresponds to the drift term $\bar{\mathbf{v}}_e^E \cdot \nabla_f \tilde{g}_e$ appearing in (2.100). In this simplified model we neglect the effect of gradients of the IS distribution function, and so we restrict ourselves to cases with $\nabla_s \bar{g}_e = 0$. Writing $\bar{\mathbf{v}}_e^E \cdot \mathbf{k}_f$ in components we find that

$$\bar{\mathbf{v}}_e^E \cdot \mathbf{k}_f = \frac{c}{B_{\text{ref}}} \kappa_\rho \left(K_x \frac{\partial \bar{\phi}}{\partial y_s} \Big|_{x_s} - K_y \frac{\partial \bar{\phi}}{\partial x_s} \Big|_{y_s} \right), \quad (3.20)$$

where the geometrical factor κ_ρ is $\kappa_\rho = B_{\text{ref}}(dx/d\psi)(dy/d\alpha)$; $\kappa_\rho \simeq 1.0$ for CBC parameters. We must prescribe model values for $\partial \bar{\phi}/\partial y$ and $\partial \bar{\phi}/\partial x$ which are consistent with the IS parallel-to-the-field boundary condition [44], equation (2.87). Explicitly, $\partial \bar{\phi}/\partial y_s$ and $\partial \bar{\phi}/\partial x_s$ must satisfy

$$\frac{\partial \bar{\phi}}{\partial y_s} \Big|_{x_s} (\pi, x_s, y_s + \delta y_s) = \frac{\partial \bar{\phi}}{\partial y_s} \Big|_{x_s} (-\pi, x_s, y_s), \quad (3.21)$$

and

$$\frac{\partial \bar{\phi}}{\partial x_s} \Big|_{y_s} (\pi, x_s, y_s + \delta y_s) - 2\pi \hat{s} \frac{\partial \bar{\phi}}{\partial y_s} \Big|_{x_s} (\pi, x_s, y_s + \delta y_s) = \frac{\partial \bar{\phi}}{\partial x_s} \Big|_{y_s} (-\pi, x_s, y_s), \quad (3.22)$$

where $\delta y_s = -2\pi \hat{s} x_s$, and we choose $\pm\pi$ as the parallel coordinates of the parallel boundary of the flux tube. Note that equations (3.21) and (3.22) are simply equations (2.98) and (2.99) rewritten in terms of x_s and y_s . Whilst not necessary, for simplicity we will locate the model ES flux tube at $x_s = 0$ so that $\delta y_s = 0$. In this case, we are

only required to prescribe $\partial\bar{\phi}/\partial y_s$ and $\partial\bar{\phi}/\partial x_s$ at a single (x_s, y_s) point. The simplest possible pair of $\partial\bar{\phi}/\partial y_s$ and $\partial\bar{\phi}/\partial x_s$ which satisfy equations (3.21) and (3.22), at $x_s = 0$, is

$$\left.\frac{\partial\bar{\phi}}{\partial y_s}\right|_{x_s} = -\hat{E}, \quad \left.\frac{\partial\bar{\phi}}{\partial x_s}\right|_{y_s} = -\hat{D} - \hat{E}\hat{s}\theta, \quad (3.23)$$

where $\hat{D}$ and $\hat{E}$ are constants, with units of electric field T/ea . The potential gradients specified in equation (3.23) will result in an $\mathbf{E} \times \mathbf{B}$ drift in the y_s direction (the α direction) which varies linearly in the parallel-to-the-field direction. As such, the prescription for the potential gradients (3.23) gives a model of local parallel-to-the-field shear in the IS $\mathbf{E} \times \mathbf{B}$ drift. Using the prescriptions (3.23) we can find the resulting model form for $i\bar{\mathbf{v}}_e^E \cdot \mathbf{k}_f \tilde{g}_{e\mathbf{k}_f}$,

$$i\bar{\mathbf{v}}_e^E \cdot \mathbf{k}_f \tilde{g}_{e\mathbf{k}_f} = i\frac{c}{B_{\text{ref}}} \kappa_\rho \left(K_y \left(\hat{E}\hat{s}\theta + \hat{D} \right) - K_x \hat{E} \right) \tilde{g}_{e\mathbf{k}_f}. \quad (3.24)$$

As the effect of nonzero $\hat{D}$ enters only as a constant term multiplying K_y in (3.24), note that the effect of $\hat{D}$ is simply to Doppler shift the real frequency of the ES modes (see Section 2.8). Having specified the model of local parallel-to-the-field shear in the IS $\mathbf{E} \times \mathbf{B}$ drift, we now examine the effect of nonzero $\hat{E}$ on the ETG instability. We will take $\hat{D} = 0$ for the remainder of the discussion.

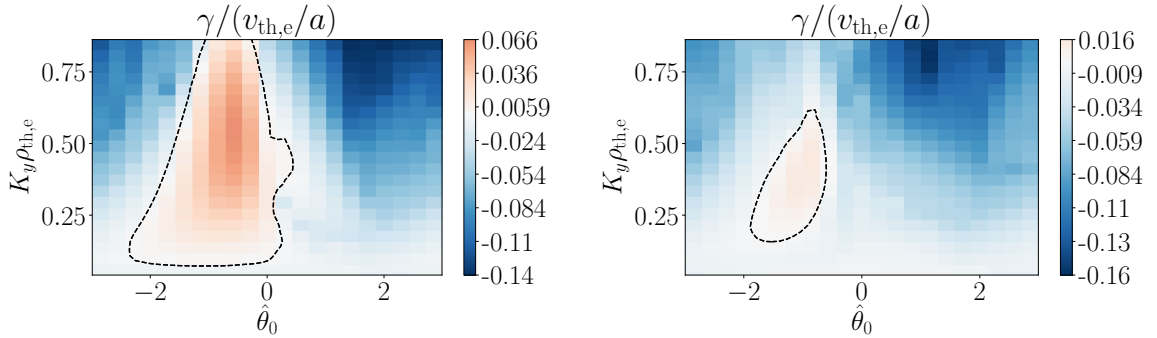


Figure 3.11: ES linear growth rates γ calculated with $\nabla_s \bar{g}_e = 0$ and $\bar{\mathbf{v}}_e^E$ prescribed by (3.23), for $\hat{E} = 0.5 T/ea$ (Left) and $\hat{E} = T/ea$ (Right).

We show the ETG growth rate γ for $\hat{E} = 0.5 T/ea$ and $\hat{E} = T/ea$ in Figure 3.11. By comparing Figure 3.4 with Figure 3.11 we observe that the effect of local parallel-to-the-field shear in the IS $\mathbf{E} \times \mathbf{B}$ drift is to suppress the value of the maximum mode growth rate and to shift the peak growth rate in $\hat{\theta}_0$.

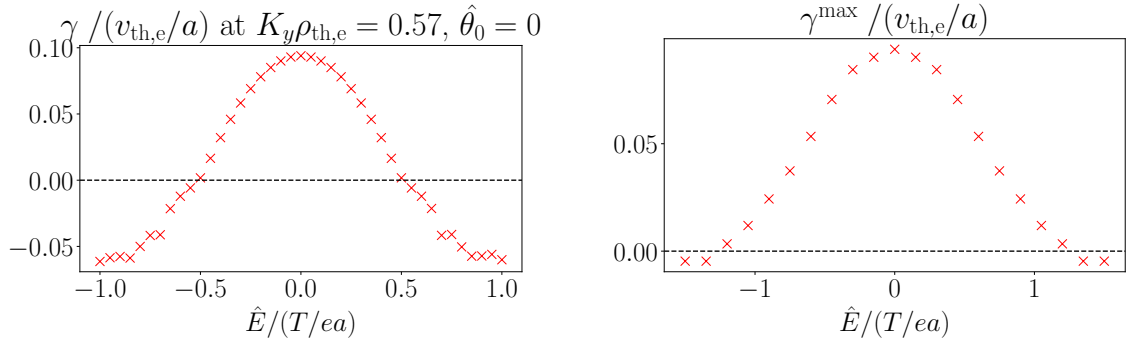


Figure 3.12: (Left) The ES linear growth rate γ calculated for $K_y \rho_{th,e} = 0.57$ and $\hat{\theta}_0 = 0$ using equations (3.11) and (3.14) with $\nabla_s \bar{g}_e = 0$ and $\bar{\mathbf{v}}_e^E \cdot \mathbf{k}_f$ as prescribed in equation (3.24). (Right) The maximum ES linear growth rate $\gamma^{\max}$ across all $K_y \rho_{th,e}$ and $\hat{\theta}_0$ as a function of $\hat{E}$.

We now focus on the effect of local parallel-to-the-field shear in the IS $\mathbf{E} \times \mathbf{B}$ drift on the modes with $K_y \rho_{th,e} = 0.57$. The mode with $K_y \rho_{th,e} = 0.57$ and $\hat{\theta}_0 = 0$ is the mode with the largest growth rate in the case with vanishing IS gradients. In Figure 3.12 (Left) we show the ES growth rate γ as a function of $\hat{E}$ for the mode with $K_y \rho_{th,e} = 0.57$ and $\hat{\theta}_0 = 0$. Note that γ is even in $\hat{E}$; parallel-to-the-field shear in $\bar{\mathbf{v}}_e^E$ always suppresses the ETG instability regardless of its sign. In Figure 3.12 (Right) we show the maximum ETG growth rate $\gamma^{\max}$ for all $\hat{\theta}_0$ modes with $K_y \rho_{th,e} = 0.57$; note that both positive and negative $\hat{E}$ also reduce $\gamma^{\max}$. This qualitatively explains why the presence of the IS drift $\bar{\mathbf{v}}_e^E$ generally results in suppression of the ETG instability even though $\langle \bar{\mathbf{v}}_e^E \rangle_{(x_s, y_s)} = 0$: In this simple model, the ES instability will be suppressed provided there is parallel-to-the-field linear shear of any sign in $\bar{\mathbf{v}}_e^E \cdot \mathbf{k}_f$ local to the peak of the ES ballooning mode. Inspecting equation (3.24), we can define a parallel shear rate

$$\hat{\omega}_E = \frac{c}{B_{\text{ref}}} \kappa_\rho \hat{s} \hat{E} K_y = \kappa_\rho \frac{\hat{s} e a \hat{E}}{2 T} K_y \rho_{th,e} \frac{v_{th,e}}{a}. \quad (3.25)$$

Note that the maximum ETG growth rate in the case of no parallel-to-the-field shear $\gamma^{\max}(\hat{E} = 0) \simeq 0.1 v_{th,e}/a$, and occurs at $K_y \rho_{th,e} = 0.57$. The value of $\hat{E}/(T/ea)$ for which the instability at $K_y \rho_{th,e} = 0.57$ is suppressed is $\hat{E}/(T/ea) \simeq 1$. Hence, the parallel shear rate $\hat{\omega}_E$ for $K_y \rho_{th,e} = 0.57$ is $\hat{\omega}_E = 0.29 v_{th,e}/a$, which is comparable in size to $\gamma^{\max}(\hat{E} = 0)$. This is consistent with the usual picture of flow shear; the shear

rate should be comparable to the growth rate of the unsheared instability for the shear to be relevant.

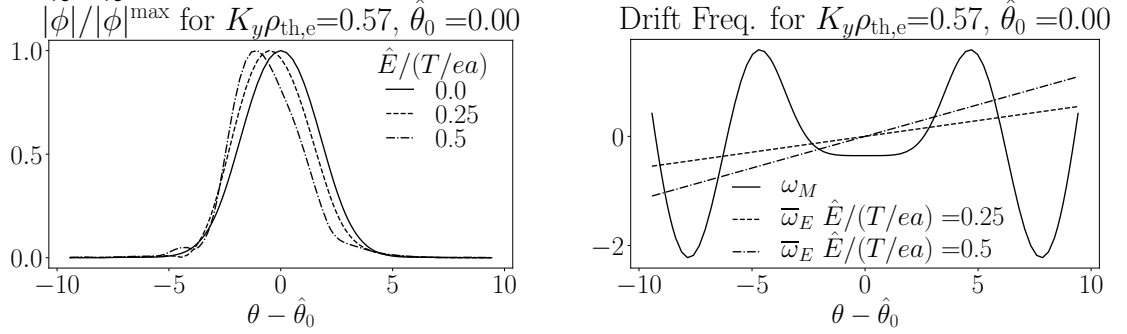


Figure 3.13: (Left) The ballooning eigenmode of the ETG instability, for $\hat{E}/(T/ea) = 0.0, 0.25$, and 0.5 , calculated using equations (3.11) and (3.14) with $\nabla_s \bar{g}_e = 0$ and $\bar{\mathbf{v}}_e^E \cdot \mathbf{k}_f$ as prescribed in equation (3.24). (Right) The coefficients $\omega_M = \mathbf{v}_e^M \cdot \mathbf{k}_f$ and $\bar{\omega}_E = \bar{\mathbf{v}}_e^E \cdot \mathbf{k}_f$ from equation (3.11) as a function of the extended ballooning coordinate for the calculation of the ETG ballooning eigenmodes of Figure 3.13. We do not show $\bar{\omega}_E$ in the case of $\hat{E}/(T/ea) = 0.0$ as $\bar{\omega}_E = 0$ in this case.

In Figure 3.13 we show the ballooning eigenmode and the drift frequencies of the ETG instability at $K_y \rho_{th,e} = 0.57$ and $\hat{\theta}_0 = 0$ for $\hat{E}/(T/ea) = 0.0, 0.25$, and 0.5 . We note that increasing $\hat{E}$ shifts the peak of the eigenmode away from $\theta - \hat{\theta}_0 = 0$. Note that $\omega_M = \mathbf{v}_e^M \cdot \mathbf{k}_f$ is the magnetic drift frequency, shown for $v_{\parallel}^2 + v_{\perp}^2/2 = v_{th,i}^2$; and $\bar{\omega}_E = \bar{\mathbf{v}}_e^E \cdot \mathbf{k}_f$, with $\bar{\mathbf{v}}_e^E \cdot \mathbf{k}_f$ as prescribed in equation (3.24). Comparing Figure 3.13 (Left) and Figure 3.13 (Right) we see that the peak of the ETG eigenmode shifts towards the $\theta - \hat{\theta}_0$ position where $\omega_M + \bar{\omega}_E$ has a local minimum; i.e., where the parallel-to-the-field shear in the total drift vanishes. In this simple picture the mode is stabilised when the peak of the ETG eigenmode is forced into the region where $\omega_M > 0$. In the region where $\omega_M > 0$ the magnetic drift is no longer in the electron diamagnetic direction, which removes the toroidal ETG mechanism for instability.

Chapter 4

Discussion

This thesis describes a study of cross-scale interactions in scale-separated turbulence in magnetic confinement fusion devices. The starting point of this investigation is the derivation of a system of coupled gyrokinetic equations, (2.73)-(2.77), which describe turbulent fluctuations driven at the scales of the ion and electron gyroradii, and the leading order cross-scale interactions between the two scales. We use these equations to study the properties of the electron temperature gradient (ETG) instability in the presence of ion temperature gradient (ITG) driven turbulence. We find that strongly driven ITG turbulence suppresses a strongly driven ETG instability, whereas a strongly driven ETG instability survives in the presence of weakly driven ITG turbulence. We demonstrate that the primary mechanism responsible for the suppression of the ETG instability is parallel-to-the-field shear in the IS $\mathbf{E} \times \mathbf{B}$ flow $\overline{\mathbf{v}}_e^E$.

In this Chapter we give a discussion of the results presented in this thesis. The structure of this discussion is as follows. In Section 4.1 we give an outline of the physics contained in the model equations (2.73)-(2.77), which are derived in the limit that $(m_e/m_i)^{1/2} \rightarrow 0$. We discuss the assumptions of the model; the implications of the model for the possible scenarios for cross-scale coupling between IS and ES turbulence; and the implications of the model for the possible levels of transport driven by IS and ES turbulence. As the fluctuations in the IS-ES model are restricted to have gyro-Bohm fluctuation amplitudes, the ES heat flux is always small compared to the IS heat flux within the assumptions of the model. In light of the evidence which suggests that ES transport can be experimentally relevant [22–25, 34–38], this feature is possibly problematic. In Section 4.2 we consider if it is possible to drive ES heat fluxes

that are comparable to IS heat fluxes within a scale-separated model, by considering possible $(m_e/m_i)^{1/2}$ orderings for a parameter which controls the ratio of the drives of instability at IS and ES. We specialise to the case of a hydrogenic plasma, with $Z_i = 1$, where $T_i = T_e = T$, and where the ion and electron temperature gradients are much larger than the critical values for instability. Under such conditions, the ratio of the equilibrium temperature length scales of electrons and ions, L_{T_e}/L_{T_i} , measures the relative sizes of the drives of instability at the IS and the ES. We find that the stiffness of turbulent transport indeed allows for the ES heat flux to be comparable to the IS heat flux with only a relatively small difference between L_{T_e} and L_{T_i} . This analysis allows us to note that there are likely only two distinct scenarios for scale-separated IS-ES turbulence: In the first scenario, the IS and ES fluctuations are driven equally; the ES turbulence is affected by cross-scale interactions with IS turbulent structures; the IS fluctuations evolve independently of the ES fluctuations; and the IS contribution to the heat flux is much larger than the ES contribution to the heat flux. In the second scenario, ES fluctuations are driven more strongly than IS fluctuations; the ES contribution to the heat flux is comparable to the IS contribution to the heat flux; and cross-scale interaction can be neglected at all scales. We briefly discuss mechanisms which may allow for other scenarios for the turbulence. In Section 4.3 we reinforce this picture of the possible scenarios for transport and cross-scale interactions with a discussion of the numerical results of Chapter 3. Finally, in Section 4.4 we discuss the limitations and advantages of the scale-separated approach and possible avenues of future research.

4.1 Scale-separated, coupled IS and ES turbulence

In Chapter 2 we obtain the equations (2.73)-(2.77) via an asymptotic expansion in the smallness of the electron-ion mass ratio $(m_e/m_i)^{1/2}$, subsidiary to the gyrokinetic expansion in ρ_{th}/a . The derivation relies on the existence of turbulence with a separated IS, where $k_\perp \rho_{th,i} \sim 1$ and $\omega \sim v_{th,i}/a$, and ES, where $k_\perp \rho_{th,e} \sim 1$ and $\omega \sim v_{th,e}/a$. We assume that the degree of spatial anisotropy in the turbulence in the plane perpendicular to the magnetic field is an $O(1)$ number in the $(m_e/m_i)^{1/2}$ expansion. We assume that

the parallel-to-the-field wavenumber $k_{\parallel}$ satisfies $(m_e/m_i)^{1/2} \ll k_{\parallel}a \sim 1 \ll (m_i/m_e)^{1/2}$ at both the spatial scales where $k_{\perp}\rho_{\text{th},i} \sim 1$ and the spatial scales where $k_{\perp}\rho_{\text{th},e} \sim 1$. Finally, we treat the ratio of the zonal to nonzonal fluctuation amplitudes as an $O(1)$ number, and we neglect electromagnetic fluctuations and the effects of equilibrium toroidal rotation and flow shear.

The model of coupled IS and ES turbulence, equations (2.73)-(2.77), differs from the full gyrokinetic system, the δf gyrokinetic equation (2.12) closed with quasineutrality (2.15), in the following key ways: the nonadiabatic ion response at ES is neglected; fast electron time scales due to electron parallel streaming and small radial structures due to passing electrons are ordered out of the IS equations; cross-scale interaction terms appear in the equation for the ES turbulence; and the IS decouples from the ES. Under the assumptions of the model, we have shown that the only possible ordering for the fluctuation amplitudes that results in saturated dominant balance is the gyro-Bohm ordering (2.57). We find that the presence of cross-scale interaction does not allow for steady-state, scale-separated IS-ES turbulence where the fluctuation amplitudes differ from the gyro-Bohm estimate by factors of mass ratio. In contrast to standard flux tube models, in this model the ES turbulence is simulated with equations (2.76)-(2.77) and boundary condition (2.89), using many flux tubes embedded within a single IS flux tube, which is evolved using equations (2.73)-(2.75) and boundary condition (2.88) [44]. The ES flux tubes are coupled in the binormal direction α by the parallel boundary condition (2.89), in a manner which is consistent with the IS parallel boundary condition (2.88). The model allows for the rigorous use of single-scale simulations, in a way that allows for efficient parallelisation, while still capturing cross-scale interactions.

We are able to neglect the nonadiabatic response of ions at ES because of the size of the large ion gyroradius compared to the scale of the correlation length perpendicular to the field for ES fluctuations $\tilde{l}_{\perp}$,

$$\tilde{l}_{\perp} \sim \rho_{\text{th},e} \ll \rho_{\text{th},i}. \quad (4.1)$$

The ions rapidly gyrate perpendicular to the field line at the ion cyclotron frequency Ω_i , which is much larger than any turbulent frequency ω , i.e., $\Omega_i \gg \omega$. Consequently the ion orbits rapidly sample many uncorrelated instances of ES turbulence. In the

asymptotic limit, because of the assumption of statistical periodicity (2.18), the averaged ES turbulence which the ions experience is vanishingly small and so the ions can only respond weakly to the turbulence at ES. Hence, they do not contribute to the ES potential (see Section 2.5.4). Because of the weakness of the ion response at ES, IS fluctuations are not influenced by the ion response at ES (see Section 2.5.1). It is thus not necessary to evolve ions at ES in the scale-separated system.

In order to retain only IS time and space scales in the IS equations, it is necessary to remove the fast time scales introduced by electron parallel streaming. It is also necessary to remove very fine radial structures that are introduced by passing electrons near rational surfaces in IS simulations, as a consequence of very long tails in the electron response in ballooning modes [65, 66]. We achieve this separation by taking parallel streaming for electrons to be asymptotically fast compared to IS frequencies, consistent with the assumption that $k_{\parallel}a \sim 1$ in the $(m_e/m_i)^{1/2}$ expansion. This allows us to neglect the $(m_e/m_i)^{1/2}$ small piece of the electron distribution function due to nonzonal passing electrons and use the orbital average to find the leading order equation for electrons at IS, cf. [67, 68].

There are two novel cross-scale interaction terms appearing in the coupled system of equations (2.73)-(2.77). These new terms appear in the gyrokinetic equation (2.76) for electrons at ES: $\tilde{\mathbf{v}}_e^E \cdot \nabla_s \bar{h}_e$, and $\bar{\mathbf{v}}_e^E \cdot \nabla_f \tilde{h}_e$. The term $\tilde{\mathbf{v}}_e^E \cdot \nabla_s \bar{h}_e$ arises due to gradients in the electron distribution function at IS and is analogous to the typical equilibrium drive term $\tilde{\mathbf{v}}_e^E \cdot \nabla F_{0e}$, with the caveat that the equilibrium distribution is a Maxwellian in velocities and has spatial variation only in ψ . It may seem surprising that gradients in the IS distribution function can drive (or suppress) ES instability on an equal footing with gradients of the equilibrium distribution function. However, while the equilibrium distribution function is large, it varies on spatial scales long compared to the size of the fluctuations. The net effect is that $\nabla_s \bar{h}_e \sim \nabla F_{0e}$. The second novel cross-scale coupling term in (2.76), $\bar{\mathbf{v}}_e^E \cdot \nabla_f \tilde{h}_e$, represents the advection of ES eddies perpendicular to the field line by IS drift wave motion $\bar{\mathbf{v}}_e^E$. The perpendicular correlation length of IS eddies $\bar{l}_{\perp}$ is much larger than the perpendicular correlation length of ES eddies $\tilde{l}_{\perp}$;

$$\frac{\tilde{l}_{\perp}}{\bar{l}_{\perp}} \sim \frac{\rho_{\text{th},e}}{\rho_{\text{th},i}} \sim \left(\frac{m_e}{m_i} \right)^{1/2}. \quad (4.2)$$

Furthermore, the nonlinear turnover time of IS eddies $\bar{\tau}_{\text{nl}}$ is much longer than the nonlinear turnover time of ES eddies $\tilde{\tau}_{\text{nl}}$;

$$\frac{\tilde{\tau}_{\text{nl}}}{\bar{\tau}_{\text{nl}}} \sim \frac{v_{\text{th},i}}{v_{\text{th},e}} \sim \left(\frac{m_e}{m_i}\right)^{1/2}. \quad (4.3)$$

Therefore, to leading order in the $(m_e/m_i)^{1/2}$ expansion, the ES turbulence sees an IS drift $\bar{\mathbf{v}}_e^E$ and an IS gradient $\nabla_s \bar{h}_e$ which are constant in the plane perpendicular to the field line and constant in time. Due to the parallel orbit averaging in (2.74), the gradient $\nabla_s \bar{h}_e$ is constant in the parallel-to-the-field coordinate θ . Like the magnetic drift $\mathbf{v}_e^M$, the IS drift $\bar{\mathbf{v}}_e^E$ varies in θ along the field line, and so $\bar{\mathbf{v}}_e^E$ causes nontrivial advection of the ES eddies. Because IS and ES eddies have the same parallel length scale (see (2.19) and Section 2.5.5), the IS drift $\bar{\mathbf{v}}_e^E$ has variation in θ on scales comparable to the ES turbulence. The IS drift is dynamically relevant, and has the effect of shearing ES eddies parallel to the field line. We emphasise that in the leading order equations the IS drift does not cause a shear perpendicular to the field line, which is commonly thought of as the relevant cross-scale mechanism for suppressing turbulence. The shear perpendicular to the field line arising from $\bar{\mathbf{v}}_e^E$ is small by $(m_e/m_i)^{1/2}$ compared to the drifts that are kept, and must be dropped from the scale-separated equations like the ion nonadiabatic response at ES and the other terms small by $(m_e/m_i)^{1/2}$. The piece of $\bar{\mathbf{v}}_e^E$ that is constant in θ may be removed from the equation for the ES fluctuations (2.76) by writing equation (2.76) in terms of the guiding centre distribution functions $\tilde{g}_e = \langle \tilde{\delta f}_e \rangle_{\mathbf{R}_f, \mathbf{R}_s}^\gamma$ and $\bar{g}_e = \langle \bar{\delta f}_e \rangle_{\mathbf{R}_s}^\gamma$, and changing coordinates to a rotating frame. It is the piece of $\bar{\mathbf{v}}_e^E$ which has variation parallel to the magnetic field line that is relevant for cross-scale interaction.

There are no cross-scale terms in the leading order IS equations. This means that the IS turbulence in the model is unaffected by ES eddies, and therefore evolves independently. This is in contrast to inferred interactions in some multiscale simulations [34, 36, 37, 39–41]; we note that in direct numerical simulations the ES turbulence can strongly affect IS turbulence when the IS instabilities are near marginal stability, as in [36, 39]. The largest cross-scale term at the IS, $\nabla_s \cdot \left\langle \left\langle \tilde{\mathbf{v}}_e^E \tilde{h}_e \right\rangle^{\text{ES}} \right\rangle_o$, appeared in the equation (2.45), and was small by a factor of $(m_e/m_i)^{1/2}$.

The model assumptions result in gyro-Bohm orderings for the fluxes; the IS heat flux for ions $\overline{Q}_i$ and electrons $\overline{Q}_e$ dominate the contribution to the heat flux from electrons at ES $\tilde{Q}_e$ by a factor of $(m_i/m_e)^{1/2}$. We note that the heat flux from ions at ES $\tilde{Q}_i$ is small compared to $\tilde{Q}_e$ by a factor of $(m_e/m_i)^{1/2}$ and so $\tilde{Q}_i$ can always be neglected.

In summary, the scale-separated model of IS-ES turbulence indicates that cross-scale interaction is important for determining the dynamics of ES fluctuations; cross-scale interaction has negligible influence on the evolution of IS fluctuations; and ES transport is always small compared to IS transport. Hence, under the strict assumptions of this model we should neglect ES turbulence and simply focus on the transport due to IS turbulence. This conclusion is not generally valid, because, for example, the IS-ES model presented in this thesis does not order the relative size of the IS and ES drives of instability with $(m_e/m_i)^{1/2}$. We expect that ES transport should be relevant when the ES turbulence is driven strongly compared to IS turbulence; this can occur when the IS turbulence is suppressed, for example by equilibrium flow shear, or when the electron temperature gradient is larger than the ion temperature gradient. Moreover, we might expect in such a scenario that the evolution of IS fluctuations should be influenced by cross-scale interaction with ES turbulence. To gain further insight into the scenarios available to scale-separated turbulence, in the following Section we consider the result of taking the ratio of the equilibrium electron and ion temperature gradient length scales L_{Te}/L_{Ti} to scale with $(m_e/m_i)^{1/2}$ to a fractional power. The parameter L_{Te}/L_{Ti} is a measure of the relative sizes of the drives of instability at IS and ES in hydrogenic, $Z_i = 1$, plasmas with $T_i = T_e$ and ion and electron temperature gradients much larger than the critical values for instability.

4.2 Relative sizes of the drives of instability at IS and ES

The discussion of the previous Section motivates an investigation into the scalings of IS and ES fluctuations with parameters which measure the relative sizes of the drives of instability at IS and ES. We consider temperature gradient driven instabilities, and we

specialise to the case of a plasma with $Z_i = 1$, $T_i = T_e = T$. We assume that the ion and electron temperature gradients are much larger than the critical values for instability. In this scenario, the relevant parameters controlling the drives of instability at IS and ES are the equilibrium ion and electron temperature gradients a/L_{T_i} and a/L_{T_e} , respectively. In this Section we briefly describe a preliminary investigation into the effect of taking $L_{T_e}/L_{T_i} \rightarrow 0$ alongside the mass ratio expansion $(m_e/m_i)^{1/2} \rightarrow 0$. We take as the starting point the gyro-Bohm scalings for the IS and ES fluctuations (2.57), and the estimates for the scaling of the outer scale ITG turbulence with L_{T_i} , derived from critical balance [71]. No such scalings have been tested for ETG turbulence; however, for the purpose of these estimates we will assume that ETG turbulence scales with L_{T_e} as ITG turbulence scales with L_{T_i} . This assumption may not be valid; ETG streamers may break the assumption of spatial isotropy implicit in the analysis in [71]. However, this estimate will qualitatively capture the stiffness of ETG transport.

Following [71], by balancing the linear drive with the parallel streaming term in the IS, ion gyrokinetic equation (2.39) and in the ES, electron gyrokinetic equation (2.48), we find that the perpendicular-to-the-field scale of IS fluctuations $\bar{l}_\perp$, and the perpendicular-to-the-field scale of ES fluctuations $\tilde{l}_\perp$, scale as

$$\bar{l}_\perp \sim \rho_{\text{th},i} \frac{qR}{L_{T_i}}, \quad \tilde{l}_\perp \sim \rho_{\text{th},e} \frac{qR}{L_{T_e}}, \quad (4.4)$$

where R is the major radius of the device, and q is the safety factor. Again following [71], by balancing the linear drive with the nonlinear term in the IS, ion gyrokinetic equation (2.39) and in the ES, electron gyrokinetic equation (2.48), we find that the IS and ES pieces of the fluctuations scale as

$$\frac{e\bar{\phi}}{T} \sim \frac{\bar{l}_\perp}{L_{T_i}} \sim \frac{qR\rho_{\text{th},i}}{L_{T_i}^2}, \quad \frac{e\tilde{\phi}}{T} \sim \frac{\tilde{l}_\perp}{L_{T_e}} \sim \frac{qR\rho_{\text{th},e}}{L_{T_e}^2}. \quad (4.5)$$

Finally, we write the scalings for the IS heat flux $\bar{Q}_i$, and the ES heat flux $\tilde{Q}_e$, under these assumptions

$$\bar{Q}_i \sim \frac{qR\rho_{\text{th},i}^2}{L_{T_i}^3} v_{\text{th},i} T n, \quad \tilde{Q}_e \sim \frac{qR\rho_{\text{th},e}^2}{L_{T_e}^3} v_{\text{th},e} T n. \quad (4.6)$$

In order for the scale-separated approach to remain valid, we require that $\tilde{l}_\perp$ and $\bar{l}_\perp$ are well separated

$$\frac{\tilde{l}_\perp}{\bar{l}_\perp} \sim \left(\frac{m_e}{m_i}\right)^{1/2} \frac{L_{T_i}}{L_{T_e}} \ll 1, \quad (4.7)$$

and we require that $\nabla_s \cdot \langle \tilde{\mathbf{v}}_i^E \tilde{h}_i \rangle^{\text{ES}} \ll \bar{\mathbf{v}}_i^E \cdot \nabla_s \bar{h}_i$ in (2.39), so that the nonadiabatic response of ions at ES can be neglected. Note that

$$\bar{\mathbf{v}}_i^E \cdot \nabla_s \bar{h}_i \sim v_{\text{th},i} \rho_{\text{th},i} \left(\frac{\rho_{\text{th},i}}{\bar{l}_\perp}\right)^2 \left(\frac{e\bar{\phi}}{T\rho_{\text{th},i}}\right)^2 F_{0i} \sim \frac{v_{\text{th},i} \rho_{\text{th},i}}{L_{T_i}^2} F_{0i}, \quad (4.8)$$

and

$$\nabla_s \cdot \langle \tilde{\mathbf{v}}_i^E \tilde{h}_i \rangle^{\text{ES}} \sim v_{\text{th},i} \rho_{\text{th},i} \frac{\rho_{\text{th},e}^2}{\bar{l}_\perp \tilde{l}_\perp} \tilde{J}^2 \left(\frac{e\tilde{\phi}}{T\rho_{\text{th},e}}\right)^2 F_{0i} \sim \frac{m_e}{m_i} v_{\text{th},i} \rho_{\text{th},i} \frac{L_{T_i}}{L_{T_e}^3} F_{0i}, \quad (4.9)$$

where $\tilde{J} \sim (m_e/m_i)^{1/4}$ appears due to the ion gyroaverage over ES fluctuations. Hence, we require that

$$\frac{\nabla_s \cdot \langle \tilde{\mathbf{v}}_i^E \tilde{h}_i \rangle^{\text{ES}}}{\bar{\mathbf{v}}_i^E \cdot \nabla_s \bar{h}_i} \sim \frac{m_e}{m_i} \left(\frac{L_{T_i}}{L_{T_e}}\right)^3 \ll 1, \quad (4.10)$$

meaning that for scale separation we must have $(m_e/m_i)^{1/3} \ll L_{T_e}/L_{T_i}$, which oversatisfies condition (4.7).

The IS and ES heat fluxes are comparable when

$$\frac{\tilde{Q}_e}{\bar{Q}_i} \sim \left(\frac{m_e}{m_i}\right)^{1/2} \left(\frac{L_{T_i}}{L_{T_e}}\right)^3 \sim 1. \quad (4.11)$$

The scaling (4.11) indicates that $\bar{Q}_i \sim \tilde{Q}_e$ when $L_{T_e}/L_{T_i} \sim (m_e/m_i)^{1/6}$, so a relatively small difference in the electron and ion drives can lead to comparable heat fluxes even without ETG streamers – $(m_e/m_i)^{1/6} \approx 1/4$ for a deuterium plasma. Note that the precise L_{T_e}/L_{T_i} necessary to make $\bar{Q}_i \simeq \tilde{Q}_e$ in an experiment or direct numerical simulation (DNS) depends on the ratio T_i/T_e , Z_i , and the distance to marginal stability. We took $T_i/T_e = 1$ and $Z_i = 1$ here, and assumed that the turbulence was driven far from marginal stability. The condition $L_{T_e}/L_{T_i} \sim (m_e/m_i)^{1/6}$ satisfies the requirements for scale separation to hold, relations (4.7) and (4.10). We now consider the possible

impact of cross-scale coupling with the scalings (4.4) and (4.5). For ES turbulence to influence the evolution of the IS fluctuations by cross-scale interaction we must have that $\nabla_s \cdot \left\langle \left\langle \tilde{\mathbf{v}}_e^E \tilde{h}_e \right\rangle^{\text{ES}} \right\rangle^o \sim \langle \bar{\mathbf{v}}_e^E \cdot \nabla_s \bar{h}_e \rangle^o$ in (2.45). We note that

$$\langle \bar{\mathbf{v}}_e^E \cdot \nabla_s \bar{h}_e \rangle^o \sim v_{\text{th},i} \rho_{\text{th},i} F_{0e} \left(\frac{\rho_{\text{th},i}}{\bar{l}_\perp} \right)^2 \left(\frac{e\bar{\phi}}{T\rho_{\text{th},i}} \right)^2 \sim \frac{v_{\text{th},i} \rho_{\text{th},i}}{L_{T_i}^2} F_{0e}, \quad (4.12)$$

and

$$\begin{aligned} \nabla_s \cdot \left\langle \left\langle \tilde{\mathbf{v}}_e^E \tilde{h}_e \right\rangle^{\text{ES}} \right\rangle^o &\sim \left(\frac{m_e}{m_i} \right)^{1/2} v_{\text{th},i} \rho_{\text{th},i} F_{0e} \left(\frac{\rho_{\text{th},e} \rho_{\text{th},i}}{\tilde{l}_\perp \bar{l}_\perp} \right) \left(\frac{e\bar{\phi}}{T\rho_{\text{th},i}} \right) \left(\frac{e\tilde{\phi}}{T\rho_{\text{th},e}} \right) \\ &\sim \left(\frac{m_e}{m_i} \right)^{1/2} \frac{v_{\text{th},i} \rho_{\text{th},i} L_{T_i}}{L_{T_e}^3} F_{0e}. \end{aligned} \quad (4.13)$$

Hence, in order for cross-scale interaction to be relevant at IS we must have that $L_{T_e}/L_{T_i} \sim (m_e/m_i)^{1/6}$ – the same scaling for which $\bar{Q}_i \sim \tilde{Q}_e$. For IS turbulence to influence the ES fluctuations via cross-scale interaction we must have that $\bar{\mathbf{v}}_e^E \cdot \nabla_f \tilde{h}_e \sim \tilde{\mathbf{v}}_e^E \cdot \nabla_f \tilde{h}_e$ in (2.48). We note that

$$\bar{\mathbf{v}}_e^E \cdot \nabla_f \tilde{h}_e \sim v_{\text{th},e} \rho_{\text{th},e} F_{0e} \left(\frac{\rho_{\text{th},e} \rho_{\text{th},i}}{\tilde{l}_\perp \bar{l}_\perp} \right) \left(\frac{e\bar{\phi}}{T\rho_{\text{th},i}} \right) \left(\frac{e\tilde{\phi}}{T\rho_{\text{th},e}} \right) \sim \frac{v_{\text{th},e} \rho_{\text{th},e}}{L_{T_e} L_{T_i}} F_{0e}, \quad (4.14)$$

and

$$\tilde{\mathbf{v}}_e^E \cdot \nabla_f \tilde{h}_e \sim v_{\text{th},e} \rho_{\text{th},e} F_{0e} \left(\frac{\rho_{\text{th},e}}{\tilde{l}_\perp} \right)^2 \left(\frac{e\tilde{\phi}}{T\rho_{\text{th},e}} \right)^2 \sim \frac{v_{\text{th},e} \rho_{\text{th},e}}{L_{T_e}^2} F_{0e}, \quad (4.15)$$

Hence, for $\bar{\mathbf{v}}_e^E \cdot \nabla_f \tilde{h}_e \sim \tilde{\mathbf{v}}_e^E \cdot \nabla_f \tilde{h}_e$ we require that $L_{T_e} \sim L_{T_i}$; i.e., under the assumptions made here if the ES turbulence is driven more strongly than the IS turbulence then the ES fluctuations evolve independently of the IS. As a corollary, under these assumptions strongly driven ES turbulence cannot influence the evolution of IS fluctuations through the cross-scale term $\nabla_s \cdot \left\langle \left\langle \tilde{\mathbf{v}}_e^E \tilde{h}_e \right\rangle^{\text{ES}} \right\rangle^o$ in (2.45): As we argue in Appendix (A.5.4), if no cross-scale terms appear in the equation for electrons at ES then the divergence term $\nabla_s \cdot \left\langle \left\langle \tilde{\mathbf{v}}_e^E \tilde{h}_e \right\rangle^{\text{ES}} \right\rangle^o$ in (2.45) is zero – ES fluctuations do not depend on the IS coordinate.

Finally, recall that we have assumed that the degree of spatial anisotropy in the turbulence in the plane perpendicular to the magnetic field does not scale with $(m_e/m_i)^{1/2}$.

Simple considerations show that $\bar{Q}_i \sim \tilde{Q}_e$ is incompatible with significant cross-scale interactions in the scale-separated model of turbulence. For cross-scale interaction to be possible the IS-ES turbulence must have fluctuation amplitudes which satisfy the ambient gradient scaling $\nabla_s \bar{\phi} \sim \nabla_f \tilde{\phi}$, i.e.,

$$\frac{\bar{\phi}}{\bar{l}_\perp} \sim \frac{\tilde{\phi}}{\tilde{l}_\perp}. \quad (4.16)$$

However, for the IS and ES heat fluxes to be comparable we require that

$$\frac{\bar{\phi}^2}{\bar{l}_\perp} \sim \frac{\tilde{\phi}^2}{\tilde{l}_\perp}. \quad (4.17)$$

It is impossible to satisfy both the ordering (4.16) and the ordering (4.17) when $\bar{l}_\perp \sim \rho_{\text{th},i} \gg \tilde{l}_\perp \sim \rho_{\text{th},e}$.

These observations indicate that there are two distinct scenarios for scale-separated IS-ES turbulence. In the first scenario, the IS turbulence and ES turbulence have comparable drives of instability; the IS turbulence affects the ES turbulence through cross-scale interaction; and the ES contribution to the heat flux is negligible. In the second scenario, the ES turbulence is driven more strongly than the IS turbulence; IS and ES contributions to the heat flux are comparable; and the effect of cross-scale interactions is negligible at all scales. For a scenario different from these to emerge, the IS and ES pieces of the turbulence must be mutually coupled by cross-scale interaction. Finding the physical mechanism which allows this to occur may involve considering a combination of parameters which have been treated as of order unity in this thesis, such as the degree of spatial anisotropy of the ES turbulence, and the ratio of the zonal to nonzonal fluctuation amplitudes. If it is possible to show that no such exotic regime exists, then we could make the interesting conclusion that the reported cross-scale interactions which affect the IS turbulence in DNS [34, 36, 37, 39–41] are a result of the imperfect separation between the IS and ES.

4.3 Suppression of the ETG instability by cross-scale interaction with IS turbulence

In Chapter 3 we use the analytical results of Chapter 2 to make a numerical study of the effect of IS turbulent structures on ES linear stability. These results reinforce

the picture of the available scenarios for IS-ES turbulence, discussed in the previous Section: when the ETG drive of instability and the ITG drive of instability are identical then not only is the ETG mode affected by the IS turbulence – it is stabilised; and IS turbulence driven near marginal stability has, on average, only a weak effect on the strongly driven ETG mode.

In order to make contact with the assumptions of the analytical theory we use simulation parameters based on the cyclone base case (CBC) [43]. In the absence of cross-scale coupling, linear simulations with these parameters show a stable gap between the ITG and ETG modes in the growth rate spectrum as a function of binormal wave number k_y . We define the IS and ES binormal wave number ranges based on this gap. This choice of parameters and wave numbers places the numerical investigation in a region of parameter space where the scale-separated, coupled model of plasma turbulence is likely to be valid.

We perform ES simulations including the effects of the IS gradient $\nabla_s \bar{g}_e$ and IS drift $\bar{\mathbf{v}}_e^E$ for two distinct cases: the case where both IS and ES turbulence are strongly driven $a/L_{T_i} = a/L_{T_e} = 2.3$, and the case where the ES turbulence is driven more strongly than the IS turbulence $a/L_{T_e} = 2.3 > a/L_{T_i} = 1.38$. The IS gradients $\nabla_s \bar{g}_e$ and IS drifts $\bar{\mathbf{v}}_e^E$ for these ES simulations are calculated with nonlinear IS flux tube simulations of equations (3.2)-(3.8). We find that, on average, the ETG instability is stabilised in the case that $a/L_{T_i} = a/L_{T_e} = 2.3$, and weakly suppressed in the case that $a/L_{T_e} = 2.3 > a/L_{T_i} = 1.38$. By only including the effect of the IS gradient $\nabla_s \bar{g}_e$ or only including the effect of the IS drift $\bar{\mathbf{v}}_e^E$ in the ES simulation we identify that the parallel-to-the-field shear in the IS $\mathbf{E} \times \mathbf{B}$ drift $\bar{\mathbf{v}}_e^E$ is responsible for the suppression of the ETG instability. By considering a simplified model of the parallel-to-the-field shear in the IS $\mathbf{E} \times \mathbf{B}$ drift we demonstrate that a linear parallel shear in $\bar{\mathbf{v}}_e^E$, of any sign, can suppress the ETG instability; this qualitatively explains how a turbulent IS drift $\bar{\mathbf{v}}_e^E$ can suppress the ETG instability, even though the mean value of $\bar{\mathbf{v}}_e^E$ across the ES flux tubes within the IS flux tube is zero.

The parallel-to-the-field shear suppression mechanism may explain why ES turbulence saturates at a low amplitude in DNS of multiscale turbulence, compared to ES

only simulations [38, 39, 41]. This suppression mechanism may also explain why increasing the drive of IS instability reduces the amplitude of the ES components of the turbulence [36, 37]. IS turbulence will generally have $\mathbf{E} \times \mathbf{B}$ drifts that have parallel-to-the-field variation, hence we can expect that the parallel-to-the-field shear suppression mechanism will generally be present. Moreover, this mechanism persists even in the limit of vanishing $(m_e/m_i)^{1/2}$ – the limit of infinite scale separation between the IS and ES pieces of the turbulence. A competing explanation for the suppression of ES modes in DNS is the perpendicular-to-the-field shear in the IS $\mathbf{E} \times \mathbf{B}$ drift [37, 39, 41]. In Chapter 2 we demonstrate that this mechanism does not appear in the ES equation (2.76) in the limit that $(m_e/m_i)^{1/2} \rightarrow 0$. Hence, IS perpendicular-to-the-field flow shear can only suppress ES turbulence if the IS and ES pieces of the turbulence are in fact not scale separated. The extent to which the turbulence in DNS is scale separated, and the extent to which either of shear mechanisms are responsible for the behaviour seen in DNS must be verified. It is important to note that [39, 41] demonstrated free energy transfer from “high-k” to “low-k” modes such that the “high-k” modes are suppressed; however, the mechanism of the coupling is not identified. The IS $\mathbf{E} \times \mathbf{B}$ perpendicular-to-the-field shearing rate was found to be small compared to the ES growth rates and correlation times in [37]. Intuitively we would expect that for a flow shear to be an effective suppression mechanism the shear rate should be comparable to the time scales of the unsheared turbulence. In agreement with this expectation, we find that the parallel-to-the-field shear rate $\hat{\omega}_E$, defined in Section 3.5 equation (3.25), is comparable to the unsheared ETG growth rate in the cases that the parallel-to-the-field shear stabilises the ETG mode.

4.4 Limitations of the scale-separated model of multiscale turbulence and avenues for future research

We stress that the key assumption in deriving the model equations (2.73)-(2.77) is the assumption of scale separation between the ion and the electron space and time scales

in the turbulence. We assume that the turbulent wave number and frequency spectrum has negligible amplitude in the intermediate range between the IS and the ES. We assume scale separation for both the radial (ψ) and binormal (α) directions perpendicular to the magnetic field line, and we assume that the degree of spatial anisotropy in the turbulence does not scale with $(m_e/m_i)^{1/2}$ at either the IS or ES. Altogether, this means that the model is unable to describe cases where there are significant fluctuations between the IS and the ES, which for example might be driven by trapped electron mode instability (TEM) or ITG and ETG when the macroscopic temperature gradients are very far above marginal stability. The assumption that the degree of spatial anisotropy of the turbulence in the plane perpendicular to the magnetic field is an of order unity number in the $(m_e/m_i)^{1/2}$ expansion means that the model cannot describe individual modes with both scales of order the electron gyroradius and scales of order the ion gyroradius. However, it is known that ETG turbulence is sometimes characterised by radially extended structures [22, 26, 27, 29, 30], often referred to as “streamers”. The radial size of ETG streamers can be many multiples of the electron thermal gyroradius $\rho_{\text{th,e}}$. In DNS, structures which are identified as ETG streamers appear in snapshots of the turbulence with a radial size comparable to the radial extent of the flux tube [34–36, 38]. This raises the question as to whether or not it is valid to use a scale-separated approach to decompose the IS and ES pieces of the turbulence. In this thesis we assumed that the radial correlation length of ES structures $\tilde{l}_x$ scaled like $\rho_{\text{th,e}}$, not $\rho_{\text{th,i}}$; this is certainly true in the limit that $(m_e/m_i)^{1/2} \rightarrow 0 - \rho_{\text{th,i}}$ does not appear in the equations for ES turbulence. However, from a modelling perspective, where we want to take $(m_e/m_i)^{1/2} = 1/60$, it is important to know the exact size of $\tilde{l}_x$: the scale-separated model of IS-ES turbulence can only be valid when the IS radial correlation length $\bar{l}_x$ satisfies $\bar{l}_x \gg \tilde{l}_x$. A careful study of the saturation mechanism for ETG turbulence in the absence of cross-scale coupling, perhaps inspired by critical balance arguments [71], would partially address these concerns if the study could give estimates for the radial and binormal correlation lengths, $\tilde{l}_x$ and $\tilde{l}_y$, respectively. This would enable comparison to the IS radial and binormal correlation lengths $\bar{l}_x$ and $\bar{l}_y$, respectively. There are other modes which are thought to be spatially anisotropic, for example the electromagnetic microtearing mode (MTM). The MTM is localised around

mode rational surfaces, and has scales of order $\rho_{\text{th},i}$ in the binormal direction, but much smaller scales in the radial direction [40, 78]. Future work could address if it is possible to develop a scale-separated model where the scale separation is in either the binormal or radial direction only.

In addition to the concern regarding separation of spatial scales, it is known that ETG turbulence can have a very long saturation timescale [25, 28]. In comparison to ITG turbulence, ETG turbulence is a weak generator of zonal flows. The lack of strong zonal flow generation is the explanation for the large ETG heat fluxes observed in simulation [22, 26, 27]. However in some circumstances, ETG turbulence generates a large amplitude zonal flow over many $a/v_{\text{th},e}$ times [25, 28], causing a very long transient before true saturation is reached. When $(m_e/m_i)^{1/2} = 1/60$, these saturation times can be so long as to be longer than the time it takes for ITG driven IS turbulence to saturate, making the scale-separated model of IS-ES turbulence problematic for quantitative modelling of multiscale plasmas with these parameters: the scale-separated model assumes that structures in the turbulence with perpendicular correlation lengths comparable to $\rho_{\text{th},e}$ have correlation times which scale as $a/v_{\text{th},e} \ll a/v_{\text{th},i}$. It is not clear if long-lived ES zonal structures survive in the presence of IS turbulence. Considering whether it is possible to develop a scale-separated model in the absence of a straightforward separation of time scales is important for future work.

The study of cross-scale interactions in this thesis has largely focused on the effect of cross-scale interaction on the ETG linear instability. We have developed understanding of the suppressing effect of the IS drift $\overline{\mathbf{v}}_e^E$, but further work is needed to understand the effect of the gradient of the IS electron distribution function $\nabla_s \overline{g}_e$. By changing which ES modes are linearly unstable, the cross-scale interactions which affect ES fluctuations may change the quantitative and perhaps even the qualitative behaviour of a saturated ES turbulent state. This observation motivates nonlinear simulations of ES turbulence that include the effects of cross-scale interaction. Intuitively, it would seem that the stabilisation of ES instabilities by the presence of IS turbulence would result in the quenching of ES turbulence. However, in analogy to the existence of sub-critical turbulence in the case where equilibrium flow shear suppresses the IS linear instability, e.g. as in [75–77], it is possible that there exists sub-critical ES turbulence

even when the ES linear instability is suppressed by cross-scale interaction. Sub-critical ES turbulence could perhaps be relevant for transport in a plasma that is transitioning from a regime where ES turbulence is driven more strongly than IS turbulence to a regime where the drives of IS and ES turbulence are comparable. A wider study could examine the effects of IS turbulence on ETG linear stability by considering other IS turbulent states. For example, in Chapter 3 we realised an IS turbulent state driven near marginal stability by reducing the ion temperature gradient from $a/L_{T_i} = 2.3$ to $a/L_{T_i} = 1.38$. Instead, we could have realised IS turbulence driven near marginal stability by imposing a strong equilibrium flow shear. The impact of these differently realised near-marginal IS turbulences on the ETG instability may be distinct.

Finally, in this thesis the equations for IS-ES turbulence were derived in the electrostatic approximation. This approximation is valid only for sufficiently small plasma $\beta = 8\pi nT/B^2$. Extending the multiscale treatment to the electromagnetic case is of interest for future work for two reasons. Firstly, the presence of electromagnetic fields may allow for other cross-scale interaction mechanisms. For example, we might expect that field line bending due to magnetic flutter at IS would be relevant to ES fluctuations. One of the most striking examples of cross-scale interaction appears in the fully electromagnetic simulations of [40], where turbulence with an IS k_y , driven by a MTM, is suppressed by the presence of ETG driven ES turbulence. Secondly, the underlying instabilities which drive the turbulence may change with β . Whilst the ITG instability can be stabilised by the effects of nonzero β [79–81], the kinetic ballooning mode (KBM) [82] can become unstable for $\beta \sim 0.01$ [79, 80]. There is evidence which indicates that the ETG instability is less sensitive to β , and remains unchanged even for β as large as 0.1 [79], though nonlinear ETG transport can be more strongly affected [79]. These observations are relevant for a multiscale model of plasma turbulence, as it is the underlying microinstabilities that define which scales exist in the turbulence: a complete analysis should include modes which are neglected in the electrostatic approximation, such as the MTM and KBM. We note that including electromagnetic fluctuations within the scale-separated model represents a significant challenge. The electromagnetic fluctuations have amplitudes which scale with a power of β . Hence, it seems that in any scale-separated model of turbulence β must be ordered with respect

to $(m_e/m_i)^{1/2}$. There may be many possible orderings to consider, depending on the underlying nature of the electromagnetic instabilities relevant to cross-scale coupling. A challenging question to resolve is how self-consistently include the fluctuating plasma current at IS in the model. This current can have an equal contribution from electrons and ions. A leading contribution to this current comes from the $(m_e/m_i)^{1/2}$ small correction to the electron distribution function $\bar{h}_e^{(1)}$ due to passing electrons, which was neglected in the derivation of the equation for electrons at IS (2.74).

The limitations of the IS-ES model of multiscale plasma turbulence are a result of the greatest strength of the model: the asymptotic expansion in $(m_e/m_i)^{1/2}$. This asymptotic expansion allows us to recover the dominant physics which governs the evolution of scale-separated IS and ES turbulence, including the nature of the leading-order cross-scale interactions and the allowed scenarios for steady-state, scale-separated turbulence. The ultimate purpose of this model is not to compete with DNS as a modelling tool, but rather to provide understanding of the rich physics observed in multiscale plasma turbulence.

Appendix A

A.1 The velocity space and gyroaverages

In this Section we show that in Fourier space taking the gyroaverage leads to the appearance of Bessel functions. Recalling that $\mathbf{B} = \nabla\alpha \times \nabla\psi$ is the representation for the magnetic field, where ψ is a flux label and α is the field line label, we can define an orthonormal field aligned coordinate system with basis vectors $\mathbf{b}$, $\mathbf{e}_1$, and $\mathbf{e}_2$, where

$$\mathbf{b} = \frac{\mathbf{B}}{B}, \quad \mathbf{e}_1 = \frac{\nabla\psi}{|\nabla\psi|}, \quad \text{and} \quad \mathbf{e}_2 = \mathbf{b} \times \frac{\nabla\psi}{|\nabla\psi|}. \quad (\text{A.1})$$

Using this coordinate system, and using the energy ε , pitch angle λ , sign σ , and gyrophase γ as coordinates, we can express the particle velocity $\mathbf{v}$ in the following manner,

$$\mathbf{v} = v_{\parallel}\mathbf{b} + v_{\perp}(\cos\gamma\mathbf{e}_1 + \sin\gamma\mathbf{e}_2), \quad (\text{A.2})$$

where $v_{\parallel} = \sigma(2\varepsilon/m_{\nu})^{1/2}(1 - \lambda B)^{1/2}$, and $v_{\perp} = 2\varepsilon\lambda B/m_{\nu}$. The form (A.2) is especially convenient for magnetised plasmas, where there is rapid gyromotion in the plane perpendicular to the magnetic field. Taking the dot product of the vector gyroradius $\boldsymbol{\rho} = \mathbf{b} \times \mathbf{v}/\Omega$ with the wave vector $\mathbf{k}$, a wave vector in the plane perpendicular to the direction of the magnetic field, i.e. $\mathbf{b} \cdot \mathbf{k} = 0$, and using (A.2), we find

$$\mathbf{k} \cdot \boldsymbol{\rho} = -|\mathbf{k}||\boldsymbol{\rho}|\sin(\gamma + \Psi), \quad (\text{A.3})$$

where $|\boldsymbol{\rho}| = v_{\perp}/|\Omega|$, $\sin\Psi = -(\mathbf{k} \cdot \mathbf{e}_2/|\mathbf{k}|)(|\Omega|/\Omega)$ and $\cos\Psi = (\mathbf{k} \cdot \mathbf{e}_1/|\mathbf{k}|)(|\Omega|/\Omega)$. Note that $v_{\perp} \geq 0$. Using the definition of the gyroaverage operator (2.10), we take the

gyroaverage over the phase of $\exp[\mathbf{i}\mathbf{k} \cdot \boldsymbol{\rho}]$ appearing in the gyrokinetic equations when the gyrokinetic equations are expressed in Fourier components. We find

$$\langle \exp[\mathbf{i}\mathbf{k} \cdot \boldsymbol{\rho}] \rangle^\gamma = \frac{1}{2\pi} \int_0^{2\pi} d\gamma \exp[-\mathbf{i}|\mathbf{k}||\boldsymbol{\rho}| \sin(\gamma + \Psi)] = J_0(|\mathbf{k}||\boldsymbol{\rho}|), \quad (\text{A.4})$$

where in the final equality we have recognised the definition of the 0th Bessel function of the 1st kind, and used that $J_0(z)$ is even in z . The sign of Ω in $\sin \Psi$ and $\cos \Psi$ becomes relevant for gyroaverages in the presence of electromagnetic fluctuations, due to the appearance of the 1st Bessel function of the 1st kind $J_1(z)$, which is odd in z .

A.2 Useful properties of the ES average

In this Section we prove the properties of the ES average necessary to derive the coupled ES and IS gyrokinetic equations. Periodicity perpendicular to the field line in both slow and fast variables allows us to write a fluctuation h as

$$h(\mathbf{r}_s, \mathbf{r}_f) = \sum_{\mathbf{k}_f} h_{\mathbf{k}_f}(\mathbf{r}_s) \exp[\mathbf{i}\mathbf{k}_f \cdot \mathbf{r}_f] = \sum_{\mathbf{k}_f, \mathbf{k}_s} h_{\mathbf{k}_f, \mathbf{k}_s} \exp[\mathbf{i}\mathbf{k}_f \cdot \mathbf{r}_f] \exp[\mathbf{i}\mathbf{k}_s \cdot \mathbf{r}_s]. \quad (\text{A.5})$$

This form allows us to show that one may take the ES average using real space $\mathbf{r}_f$ or guiding centre $\mathbf{R}_f$ as the integration variable. First using $\mathbf{r}_f$ as the integration variable,

$$\langle h(\mathbf{r}_s, \mathbf{r}_f) \rangle^{\text{ES}} = \sum_{\mathbf{k}_f} h_{\mathbf{k}_f}(\mathbf{r}_s) \langle \exp[\mathbf{i}\mathbf{k}_f \cdot \mathbf{r}_f] \rangle^{\text{ES}} = \sum_{\mathbf{k}_f} h_{\mathbf{k}_f}(\mathbf{r}_s) \delta_{0, \mathbf{k}_f} = h_0(\mathbf{r}_s). \quad (\text{A.6})$$

Then using $\mathbf{R}_f = \mathbf{r}_f - \boldsymbol{\rho}$ as the integration variable, where $\boldsymbol{\rho} = (\mathbf{b} \times \mathbf{v})/\Omega$ is the vector gyroradius, and noting that for fixed ε , λ , and γ , $\boldsymbol{\rho}$ is a constant vector shift which does not depend on either $\mathbf{r}_f$ or $\mathbf{r}_s$,

$$\begin{aligned} \langle h(\mathbf{r}_s, \mathbf{r}_f) \rangle^{\text{ES}} &= \sum_{\mathbf{k}_f} h_{\mathbf{k}_f}(\mathbf{r}_s) \langle \exp[\mathbf{i}\mathbf{k}_f \cdot \mathbf{R}_f] \rangle^{\text{ES}} \exp[\mathbf{i}\mathbf{k}_f \cdot \boldsymbol{\rho}] \\ &= \sum_{\mathbf{k}_f} h_{\mathbf{k}_f}(\mathbf{r}_s) \exp[\mathbf{i}\mathbf{k}_f \cdot \boldsymbol{\rho}] \delta_{0, \mathbf{k}_f} = h_0(\mathbf{r}_s). \end{aligned} \quad (\text{A.7})$$

Hence the choice of integration variable is unimportant. We are also able to show that the ES average commutes with the gyroaverage,

$$\left\langle \left\langle \cdot \right\rangle^{\text{ES}} \right\rangle_{\mathbf{R}_f, \mathbf{R}_s}^\gamma = \left\langle \left\langle \cdot \right\rangle_{\mathbf{R}_f, \mathbf{R}_s}^\gamma \right\rangle^{\text{ES}} \quad (\text{A.8})$$

To show this we apply the operations and find an identical result in the two cases:

$$\left\langle \langle h(\mathbf{r}_s, \mathbf{r}_f) \rangle_{\mathbf{R}_f, \mathbf{R}_s}^{\text{ES}} \right\rangle^\gamma = \langle h_0(\mathbf{r}_s) \rangle^\gamma = \sum_{\mathbf{k}_s} h_{0, \mathbf{k}_s} \langle \exp[\mathbf{i} \mathbf{k}_s \cdot \boldsymbol{\rho}] \rangle^\gamma \exp[\mathbf{i} \mathbf{k}_s \cdot \mathbf{R}_s], \quad (\text{A.9})$$

and

$$\begin{aligned} \left\langle \langle h(\mathbf{r}_s, \mathbf{r}_f) \rangle_{\mathbf{R}_f, \mathbf{R}_s}^\gamma \right\rangle^{\text{ES}} &= \left\langle \sum_{\mathbf{k}_f, \mathbf{k}_s} h_{\mathbf{k}_f, \mathbf{k}_s} \langle \exp[\mathbf{i}(\mathbf{k}_f + \mathbf{k}_s) \cdot \boldsymbol{\rho}] \rangle^\gamma \exp[\mathbf{i} \mathbf{k}_s \cdot \mathbf{R}_s] \exp[\mathbf{i} \mathbf{k}_s \cdot \mathbf{R}_f] \right\rangle^{\text{ES}} \\ &= \sum_{\mathbf{k}_f, \mathbf{k}_s} h_{\mathbf{k}_f, \mathbf{k}_s} \langle \exp[\mathbf{i}(\mathbf{k}_f + \mathbf{k}_s) \cdot \boldsymbol{\rho}] \rangle^\gamma \exp[\mathbf{i} \mathbf{k}_s \cdot \mathbf{R}_s] \langle \exp[\mathbf{i} \mathbf{k}_s \cdot \mathbf{R}_f] \rangle^{\text{ES}} \\ &= \sum_{\mathbf{k}_f, \mathbf{k}_s} h_{\mathbf{k}_f, \mathbf{k}_s} \langle \exp[\mathbf{i}(\mathbf{k}_f + \mathbf{k}_s) \cdot \boldsymbol{\rho}] \rangle^\gamma \exp[\mathbf{i} \mathbf{k}_s \cdot \mathbf{R}_s] \delta_{0, \mathbf{k}_f} \\ &= \sum_{\mathbf{k}_s} h_{0, \mathbf{k}_s} \langle \exp[\mathbf{i} \mathbf{k}_s \cdot \boldsymbol{\rho}] \rangle^\gamma \exp[\mathbf{i} \mathbf{k}_s \cdot \mathbf{R}_s]. \end{aligned} \quad (\text{A.10})$$

Hence, the ES average commutes with the gyroaverage.

A.3 Electrons at IS and orbital averaging

In this Section we prove the statements that we use to remove the $a/v_{\text{th},e}$ time scales and $\rho_{\text{th},e}$ spatial scales from the IS equation for electrons.

A.3.1 Proving the property (2.30) of the orbital average

To prove the property (2.30) in the passing region we first note that because the integration $\int d\alpha_s d\theta$ in the definition of the orbital average in the passing region (2.26) is taken over the whole flux surface we are free to write the integration in terms of the toroidal angle ζ in place of the field line label α_s . Hence, an equivalent definition of the orbital average in the passing region is

$$\langle \cdot \rangle^\circ = \frac{\int d\zeta d\theta (\cdot) / v_\parallel \mathbf{b} \cdot \nabla \theta}{\int d\zeta d\theta / v_\parallel \mathbf{b} \cdot \nabla \theta}. \quad (\text{A.11})$$

Using the definition (A.11), we see that

$$\left\langle v_\parallel \mathbf{b} \cdot \nabla \theta \frac{\partial H}{\partial \theta} \Big|_\alpha \right\rangle^\circ = \frac{\int d\zeta d\theta \partial H / \partial \theta|_\alpha}{\int d\zeta d\theta / v_\parallel \mathbf{b} \cdot \nabla \theta} = 0, \quad (\text{A.12})$$

where we have used the relation

$$\left. \frac{\partial H}{\partial \theta} \right|_{\alpha} = \left. \frac{\partial H}{\partial \theta} \right|_{\zeta} + \left. \frac{\partial \zeta}{\partial \theta} \right|_{\alpha} \left. \frac{\partial H}{\partial \zeta} \right|_{\theta}; \quad (\text{A.13})$$

physical periodicity in the poloidal and toroidal directions, $H(\theta, \zeta) = H(\theta + 2\pi, \zeta)$, and $H(\theta, \zeta) = H(\theta, \zeta + 2\pi)$; and that $\partial \zeta / \partial \theta|_{\alpha}$ is only a function of (ψ, θ) in axisymmetric devices. In the trapped region

$$\left\langle v_{\parallel} \mathbf{b} \cdot \nabla \theta \frac{\partial H}{\partial \theta} \right\rangle^{\circ} = \frac{\sum_{\sigma} \int_{\theta^{-}}^{\theta^{+}} d\theta \sigma \partial H / \partial \theta}{2 \int_{\theta^{-}}^{\theta^{+}} d\theta / |v_{\parallel}| \mathbf{b} \cdot \nabla \theta} = \frac{\sum_{\sigma} \sigma [H(\theta)]_{\theta^{-}}^{\theta^{+}}}{2 \int_{\theta^{-}}^{\theta^{+}} d\theta / |v_{\parallel}| \mathbf{b} \cdot \nabla \theta} = 0, \quad (\text{A.14})$$

where we have used that in the trapped region H satisfies the bounce condition $H(\theta^{\pm}, \sigma = 1) = H(\theta^{\pm}, \sigma = -1)$, where θ^{+} and θ^{-} are the poloidal coordinates for the upper and lower bounce points respectively.

A.3.2 Showing that $\bar{h}_{\mathbf{e}}^{(0)} = 0$ for nonzonal passing electrons

The leading order equation for electrons at IS is (2.41). Using the Fourier decomposition for IS fluctuations, we find that

$$v_{\parallel} \mathbf{b} \cdot \nabla \theta \frac{\partial \bar{h}_{\mathbf{e}\mathbf{k}_s}^{(0)}}{\partial \theta} = 0. \quad (\text{A.15})$$

Equation (A.15) implies that $\bar{h}_{\mathbf{e}\mathbf{k}_s}^{(0)}$ is constant in θ , for both passing and trapped pieces of the distribution function. To be consistent with the Fourier representation of $\bar{h}$, it is conventional to take $\bar{h}_{\mathbf{k}_s} \rightarrow 0$ as $|k_{\psi}| \rightarrow \infty$ as a boundary condition for the piece of the distribution function which represents passing particles. Note that the “twist-and-shift” parallel boundary condition (2.91) [44] (discussed in A.8) connects the lower parallel boundary ($\theta = -\pi$ in this thesis) of the mode labelled by the radial and binormal wave numbers (k_{ψ}, k_{α}) to the upper parallel boundary ($\theta = \pi$) of the mode labelled by $(k_{\psi} + 2\pi q_0 k_{\alpha}, k_{\alpha})$. Hence, taken with the boundary conditions (2.91) equation (A.15) gives the result that the leading piece of the IS electron distribution function $\bar{h}_{\mathbf{e}\mathbf{k}_s}^{(0)} = 0$ for nonzonal ($\mathbf{e}_{\alpha} \cdot \mathbf{k}_s = k_{\alpha} \neq 0$) passing particles.

A.3.3 Further properties of the orbital average

We now show that $\langle \bar{h}_e^{(0)} \rangle^\circ = \bar{h}_e^{(0)}$. In the passing region only the zonal component of the electron distribution function is nonzero. This means that in the passing region we can write

$$\bar{h}_e^{(0)} = \bar{h}_e^{(0)}(\psi_s, \varepsilon, \lambda, \sigma), \quad (\text{A.16})$$

i.e. $\bar{h}_e^{(0)}$ is constant in θ and α_s . Hence, applying the orbital average in the passing region (2.26) we find

$$\langle \bar{h}_e^{(0)} \rangle^\circ = \frac{\int d\alpha_s d\theta \bar{h}_e^{(0)} / v_{\parallel} \mathbf{b} \cdot \nabla \theta}{\int d\alpha_s d\theta / v_{\parallel} \mathbf{b} \cdot \nabla \theta} = \bar{h}_e^{(0)}, \quad (\text{A.17})$$

where the constancy of $\bar{h}_e^{(0)}$ in θ and α_s allows us to take $\bar{h}_e^{(0)}$ out of the integral in the numerator. In the trapped region $\bar{h}_e^{(0)}$ is constant in θ , and obeys the bounce condition $\bar{h}_e^{(0)}(\theta^\pm, \sigma = 1) = \bar{h}_e^{(0)}(\theta^\pm, \sigma = -1)$. This has the consequence that $\bar{h}_e^{(0)}$ is also a constant in σ . We can therefore write

$$\bar{h}_e^{(0)} = \bar{h}_e^{(0)}(\psi_s, \alpha_s, \varepsilon, \lambda). \quad (\text{A.18})$$

The orbital average in the trapped region (2.27) only averages over θ and σ ; hence,

$$\langle \bar{h}_e^{(0)} \rangle^\circ = \frac{\sum_\sigma \int_{\theta^-}^{\theta^+} d\theta \bar{h}_e^{(0)} / |v_{\parallel}| \mathbf{b} \cdot \nabla \theta}{2 \int_{\theta^-}^{\theta^+} d\theta / |v_{\parallel}| \mathbf{b} \cdot \nabla \theta} = \bar{h}_e^{(0)}, \quad (\text{A.19})$$

where we are again able to extract $\bar{h}_e^{(0)}$ from the integral in the numerator. Using identical arguments one can show that $\langle \partial \bar{h}_e^{(0)} / \partial \alpha_s \rangle^\circ = \partial \bar{h}_e^{(0)} / \partial \alpha_s$ and $\langle \partial \bar{h}_e^{(0)} / \partial \psi_s \rangle^\circ = \partial \bar{h}_e^{(0)} / \partial \psi_s$.

Finally, we show that $\langle \mathbf{v}^M \cdot \nabla \psi \rangle^\circ = 0$ in an axisymmetric magnetic field. In an axisymmetric device the magnetic field $\mathbf{B}$ may be expressed in a more restrictive form than (2.2) [2, 69, 70],

$$\mathbf{B} = I(\psi) \nabla \zeta + \nabla \zeta \times \nabla \psi, \quad (\text{A.20})$$

where I is a flux function. The magnetic drift in the radial direction $\mathbf{v}^M \cdot \nabla \psi$ may be written as [69, 70]

$$\mathbf{v}^M \cdot \nabla \psi = v_{\parallel} \mathbf{b} \cdot \nabla \theta \frac{\partial}{\partial \theta} \left(\frac{I v_{\parallel}}{\Omega} \right). \quad (\text{A.21})$$

Using the property of the orbital average (2.30) and (A.21), we see that

$$\langle \mathbf{v}^M \cdot \nabla \psi \rangle^o = 0. \quad (\text{A.22})$$

A.4 Obtaining the cross-scale terms

In this Section we derive the form of the cross-scale terms appearing in the coupled IS and ES gyrokinetic equations. As what is done here applies to both species, we suppress species indices, but note that φ is the species dependent, gyroaveraged potential.

A.4.1 IS cross-scale terms

Applying the ES average to the nonlinear term, and using the definitions $\varphi = \bar{\varphi} + \tilde{\varphi}$, $h = \bar{h} + \tilde{h}$, and $\nabla = \nabla_s + \nabla_f$, we find that

$$\begin{aligned} \langle \mathbf{v}^E \cdot \nabla h \rangle^{\text{ES}} &= \frac{c}{B} \langle \mathbf{b} \times \nabla \varphi \cdot \nabla h \rangle^{\text{ES}} \\ &= \frac{c}{B} \mathbf{b} \times \nabla_s \bar{\varphi} \cdot \nabla_s \bar{h} + \frac{c}{B} \left\langle \mathbf{b} \times (\nabla_s + \nabla_f) \tilde{\varphi} \cdot (\nabla_s + \nabla_f) \tilde{h} \right\rangle^{\text{ES}}. \end{aligned} \quad (\text{A.23})$$

Note that

$$\left\langle \mathbf{b} \times \nabla_f \tilde{\varphi} \cdot \nabla_f \tilde{h} \right\rangle^{\text{ES}} = \left\langle \nabla_f \cdot (\mathbf{b} \times \nabla_f \tilde{\varphi} \tilde{h}) \right\rangle^{\text{ES}} = 0, \quad (\text{A.24})$$

where we have used the fact that equilibrium quantities do not depend on the fast spatial variable $\mathbf{r}_s$. Furthermore note that

$$\left\langle \mathbf{b} \times \nabla_s \tilde{\varphi} \cdot \nabla_f \tilde{h} \right\rangle^{\text{ES}} = - \left\langle \tilde{h} \nabla_f \cdot (\mathbf{b} \times \nabla_s \tilde{\varphi}) \right\rangle^{\text{ES}} = \left\langle \tilde{h} \nabla_s \cdot (\mathbf{b} \times \nabla_f \tilde{\varphi}) \right\rangle^{\text{ES}}, \quad (\text{A.25})$$

where first we integrated by parts, and then performed an anti-cyclic permutation of the gradients acting on $\tilde{\varphi}$ recalling that equilibrium quantities do not depend on the slow spatial variable $\mathbf{r}_s$. Finally, note that the slow derivative ∇_s can pass through the average:

$$\left\langle \mathbf{b} \times \nabla_s \tilde{\varphi} \cdot \nabla_s \tilde{h} \right\rangle^{\text{ES}} = \nabla_s \cdot \left\langle (\mathbf{b} \times \nabla_s \tilde{\varphi}) \tilde{h} \right\rangle^{\text{ES}}, \quad (\text{A.26})$$

and

$$\left\langle \tilde{h} \nabla_s \cdot (\mathbf{b} \times \nabla_f \tilde{\varphi}) \right\rangle^{\text{ES}} + \left\langle (\mathbf{b} \times \nabla_f \tilde{\varphi}) \cdot \nabla_s \tilde{h} \right\rangle^{\text{ES}} = \nabla_s \cdot \left\langle (\mathbf{b} \times \nabla_f \tilde{\varphi}) \tilde{h} \right\rangle^{\text{ES}}. \quad (\text{A.27})$$

Thus, we find that

$$\left\langle \mathbf{b} \times (\nabla_s + \nabla_f) \tilde{\varphi} \cdot (\nabla_s + \nabla_f) \tilde{h} \right\rangle^{\text{ES}} = \nabla_s \cdot \left\langle (\mathbf{b} \times (\nabla_s + \nabla_f) \tilde{\varphi}) \tilde{h} \right\rangle^{\text{ES}}. \quad (\text{A.28})$$

Equation (A.23) now becomes

$$\left\langle \mathbf{v}^E \cdot \nabla h \right\rangle^{\text{ES}} = \frac{c}{B} \mathbf{b} \times \nabla_s \bar{\varphi} \cdot \nabla_s \bar{h} + \nabla_s \cdot \left\langle \tilde{h} \frac{c}{B} \mathbf{b} \times (\nabla_s + \nabla_f) \tilde{\varphi} \right\rangle^{\text{ES}}. \quad (\text{A.29})$$

Dropping the term $\nabla_s \cdot \left\langle \tilde{h} (c/B) \mathbf{b} \times \nabla_s \tilde{\varphi} \right\rangle^{\text{ES}}$, which is small by $(m_e/m_i)^{1/2}$, we find that

$$\left\langle \mathbf{v}^E \cdot \nabla h \right\rangle^{\text{ES}} = \bar{\mathbf{v}}^E \cdot \nabla_s \bar{h} + \nabla_s \cdot \left\langle \tilde{\mathbf{v}}^E \tilde{h} \right\rangle^{\text{ES}}, \quad (\text{A.30})$$

where we have used the definitions (2.32) and (2.38).

Physically the cross-scale term $\nabla_s \cdot \left\langle \tilde{\mathbf{v}}^E \tilde{h} \right\rangle^{\text{ES}}$ represents a divergence of a flux of particle density. One might have expected that the IS cross-scale term would have contained two fast derivatives, noting that

$$\tilde{\mathbf{v}}^E \cdot \nabla_f \tilde{h} \gg \nabla_s \cdot \left\langle \tilde{\mathbf{v}}^E \tilde{h} \right\rangle^{\text{ES}}, \quad (\text{A.31})$$

and so the cross-scale term would have been of the form $\left\langle \tilde{\mathbf{v}}^E \cdot \nabla_f \tilde{h} \right\rangle^{\text{ES}}$. However, because of the assumption of statistical periodicity (2.18), the term $\left\langle \tilde{\mathbf{v}}^E \cdot \nabla_f \tilde{h} \right\rangle^{\text{ES}}$ vanishes as shown in (A.24), and the leading order term in the IS cross-scale term is the one given in (A.30).

A.4.2 ES cross-scale terms

The ES nonlinear term is

$$\begin{aligned} \mathbf{v}^E \cdot \nabla h - \left\langle \mathbf{v}^E \cdot \nabla h \right\rangle^{\text{ES}} = & \frac{c}{B} \mathbf{b} \times \nabla_s \bar{\varphi} \cdot (\nabla_s + \nabla_f) \tilde{h} + \frac{c}{B} \mathbf{b} \times (\nabla_s + \nabla_f) \tilde{\varphi} \cdot \nabla_s \bar{h} \\ & + \frac{c}{B} \mathbf{b} \times (\nabla_s + \nabla_f) \tilde{\varphi} \cdot (\nabla_s + \nabla_f) \tilde{h} - \nabla_s \cdot \left\langle \left(\frac{c}{B} \mathbf{b} \times (\nabla_s + \nabla_f) \tilde{\varphi} \right) \tilde{h} \right\rangle^{\text{ES}}. \end{aligned} \quad (\text{A.32})$$

Keeping only leading order terms from each group of terms, and using definitions (2.32) and (2.38), we have that

$$\mathbf{v}^E \cdot \nabla h - \left\langle \mathbf{v}^E \cdot \nabla h \right\rangle^{\text{ES}} = \bar{\mathbf{v}}^E \cdot \nabla_f \tilde{h} + \tilde{\mathbf{v}}^E \cdot \nabla_s \bar{h} + \tilde{\mathbf{v}}^E \cdot \nabla_f \tilde{h}. \quad (\text{A.33})$$

A.5 The sizes of the IS and ES turbulent fluctuations

In this Section we determine the allowed scalings for the quantities $\bar{h}_i$, $\bar{h}_e$, $\tilde{h}_i$, $\tilde{h}_e$, $\bar{\phi}$ and $\tilde{\phi}$ in the coupled system of equations (2.39), (2.45), (2.48), (2.49), (2.51), and (2.52). For clarity, we first give estimates for the sizes of all the terms in the gyrokinetic equations (2.39), (2.45), (2.48), and (2.49). We then discuss the gyro-Bohm ordering and show that the scalings (2.57) allows for a self-consistent, saturated dominant balance. We observe that the only mechanism through which the fluctuation amplitudes can differ from the gyro-Bohm estimate (2.57) is cross-scale interaction. In A.5.1 through to A.5.4 we consider, and rule out, all possible deviations of the fluctuation amplitudes from the estimate (2.57) by powers of $(m_e/m_i)^{1/2}$. We conclude that the gyro-Bohm estimates for the fluctuation amplitudes (2.57) are the only scalings which allow for a self-consistent, saturated dominant balance, within the assumptions of scale separation. Note that the analysis in this Appendix will treat factors that may be relevant for determining the precise saturation levels of the IS and ES turbulence as of order unity in the $(m_e/m_i)^{1/2}$ expansion. Such factors might include the degree of spatial anisotropy of ES turbulence in the plane perpendicular to the magnetic field, or the ratio of the drive of IS instability to the drive of ES instability.

Writing the coupled gyrokinetic equations in real space, with the size of each term in terms of $\bar{h}_i$, $\bar{h}_e$, $\tilde{h}_i$, $\tilde{h}_e$, $\bar{\phi}$ and $\tilde{\phi}$, we find the equation for ions at IS,

$$\begin{aligned}
 & \underbrace{\frac{v_{\text{th},i}}{a} \bar{h}_i}_{\frac{\partial \bar{h}_i}{\partial t_s}} + v_{\parallel} \mathbf{b} \cdot \nabla \theta \underbrace{\frac{v_{\text{th},i}}{a} \bar{h}_i}_{\frac{\partial \bar{h}_i}{\partial \theta}} + \underbrace{\frac{v_{\text{th},i}}{a} \bar{h}_i}_{\mathbf{v}_i^M \cdot \nabla_s \bar{h}_i} + \underbrace{\frac{v_{\text{th},i}}{\rho_{\text{th},i}} \frac{e\bar{\phi}}{T} \bar{h}_i}_{\bar{\mathbf{v}}_i^E \cdot \nabla_s \bar{h}_i} \\
 & + \underbrace{\frac{v_{\text{th},e}}{\rho_{\text{th},i}} \frac{e\tilde{\phi}}{T} \tilde{h}_i}_{\nabla_s \cdot \langle \tilde{\mathbf{v}}_i^E \tilde{h}_i \rangle^{\text{ES}}} + \underbrace{\frac{v_{\text{th},i}}{a} \frac{e\bar{\phi}}{T} F_{0i}}_{\bar{\mathbf{v}}_i^E \cdot \nabla F_{0i}} = \underbrace{\frac{v_{\text{th},i}}{a} \frac{e\bar{\phi}}{T} F_{0i}}_{\frac{Z_i e}{T_i} F_{0i} \frac{\partial \bar{\varphi}_i}{\partial t_s}}, \tag{A.34}
 \end{aligned}$$

where $\tilde{J} \sim J_0(|\mathbf{k}_f| \rho_{\text{th},i}) \sim (m_e/m_i)^{1/4}$. We recall that an ion gyroaverage over an ES quantity introduces the additional mass ratio scaling factor $\tilde{J}$. See equations (2.53)-

(2.56) for a full discussion. The equation for electrons at IS is

$$\begin{aligned}
& \underbrace{\frac{v_{\text{th},i}}{a} \bar{h}_e^{(0)}}_{\frac{\partial \bar{h}_e}{\partial t_s}} + \underbrace{\frac{v_{\text{th},i}}{a} \bar{h}_e^{(0)}}_{\langle \mathbf{v}_e^M \cdot \nabla \alpha \rangle^o \frac{\partial \bar{h}_e}{\partial \alpha_s}} + \underbrace{\frac{v_{\text{th},i}}{\rho_{\text{th},i}} \frac{e\bar{\phi}}{T} \bar{h}_e^{(0)}}_{\langle \bar{\mathbf{v}}_e^E \cdot \nabla_s \bar{h}_e \rangle^o} \\
& + \underbrace{\frac{v_{\text{th},i}}{a} \frac{e\bar{\phi}}{T} F_{0e}}_{\langle \bar{\mathbf{v}}_e^E \cdot \nabla F_{0e} \rangle^o} + \underbrace{\frac{v_{\text{th},e}}{\rho_{\text{th},i}} \frac{e\tilde{\phi}}{T} \tilde{h}_e}_{\nabla_s \cdot \left\langle \left\langle \tilde{\mathbf{v}}_e^E \tilde{h}_e \right\rangle^{\text{ES}} \right\rangle^o} = \underbrace{\frac{v_{\text{th},i}}{a} \frac{e\bar{\phi}}{T} F_{0e}}_{-\frac{e}{T} F_{0e} \frac{\partial \langle \bar{\varphi}_e \rangle^o}{\partial t_s}}, \tag{A.35}
\end{aligned}$$

where $\bar{h}_e^{(0)}$ is the leading order piece of the electron distribution function at IS. The equation for electrons at ES is

$$\begin{aligned}
& \underbrace{\frac{v_{\text{th},e}}{a} \tilde{h}_e}_{\frac{\partial \tilde{h}_e}{\partial t_f}} + v_{\parallel} \mathbf{b} \cdot \nabla \theta \frac{\partial \tilde{h}_e}{\partial \theta} + \underbrace{\frac{v_{\text{th},e}}{a} \tilde{h}_e}_{\mathbf{v}_e^M \cdot \nabla_f \tilde{h}_e} + \underbrace{\frac{v_{\text{th},i}}{\rho_{\text{th},e}} \frac{e\bar{\phi}}{T} \tilde{h}_e}_{\bar{\mathbf{v}}_e^E \cdot \nabla_f \tilde{h}_e} + \underbrace{\frac{v_{\text{th},e}}{\rho_{\text{th},e}} \frac{e\tilde{\phi}}{T} \tilde{h}_e}_{\tilde{\mathbf{v}}_e^E \cdot \nabla_f \tilde{h}_e} \\
& + \underbrace{\frac{v_{\text{th},e}}{a} \frac{e\tilde{\phi}}{T} F_{0e}}_{\tilde{\mathbf{v}}_e^E \cdot \nabla F_{0e}} + \underbrace{\frac{v_{\text{th},e}}{\rho_{\text{th},i}} \frac{e\tilde{\phi}}{T} \bar{h}_e^{(0)}}_{\tilde{\mathbf{v}}_e^E \cdot \nabla_s \bar{h}_e} = -\frac{e F_{0e}}{T} \frac{\partial \tilde{\varphi}_e}{\partial t_f}. \tag{A.36}
\end{aligned}$$

Finally we find the equation for ions at ES,

$$\begin{aligned}
& \underbrace{\frac{v_{\text{th},e}}{a} \tilde{h}_i}_{\frac{\partial \tilde{h}_i}{\partial t_f}} + \underbrace{\frac{v_{\text{th},e}}{a} \tilde{h}_i}_{\mathbf{v}_i^M \cdot \nabla_f \tilde{h}_i} + \underbrace{\frac{v_{\text{th},i}}{\rho_{\text{th},e}} \frac{e\bar{\phi}}{T} \tilde{h}_i}_{\bar{\mathbf{v}}_i^E \cdot \nabla_f \tilde{h}_i} + \underbrace{\frac{v_{\text{th},e}}{\rho_{\text{th},e}} \tilde{J} \frac{e\tilde{\phi}}{T} \tilde{h}_i}_{\tilde{\mathbf{v}}_i^E \cdot \nabla_f \tilde{h}_i} \\
& + \underbrace{\frac{v_{\text{th},e}}{a} \tilde{J} \frac{e\tilde{\phi}}{T} F_{0i}}_{\tilde{\mathbf{v}}_i^E \cdot \nabla F_{0i}} + \underbrace{\frac{v_{\text{th},e}}{\rho_{\text{th},i}} \tilde{J} \frac{e\tilde{\phi}}{T} \bar{h}_i}_{\tilde{\mathbf{v}}_i^E \cdot \nabla_s \bar{h}_i} = \underbrace{\frac{v_{\text{th},e}}{a} F_{0i} \tilde{J} \frac{e\tilde{\phi}}{T}}_{\frac{Z_i e}{T} F_{0i} \frac{\partial \tilde{\varphi}_i}{\partial t_f}}, \tag{A.37}
\end{aligned}$$

where again the factor $\tilde{J}$ appears due to ion gyroaverages over the ES fluctuations. We will use equations (A.34)-(A.37) for the remainder of the discussion in this Section.

The gyrokinetic ordering (2.8) naturally suggests the gyro-Bohm ordering, where

$$\nabla_{\perp} \bar{\delta f} \sim \nabla_{\perp} \tilde{\delta f}. \quad (\text{A.38})$$

The ordering (A.38) allows for a separation of scales between electron and ion spatial scales, in analogy to the separation between turbulent and equilibrium scales in ordinary gyrokinetics. In the usual picture used to motivate the gyrokinetic orderings there are eddies of size ρ_{th} stirring up a background equilibrium gradient of scale a and size F_0/a , where F_0 is the typical equilibrium amplitude, which results in turbulent fluctuations of amplitude $(\rho_{\text{th}}/a)F_0$. In the gyro-Bohm ordering the picture is the same with $\rho_{\text{th}} \rightarrow \rho_{\text{th,e}}$, $a \rightarrow \rho_{\text{th,i}}$, $F_0 \rightarrow \bar{\delta f}$, i.e. the ES plays the role of the fluctuation and the IS plays the role of the equilibrium. With the orderings for length and time scales, ordering (2.19), we arrive at the gyro-Bohm ratio for the turbulent amplitudes,

$$\tilde{\delta f} \sim \frac{\rho_{\text{th,e}}}{\rho_{\text{th,i}}} \bar{\delta f} \sim \frac{v_{\text{th,i}}}{v_{\text{th,e}}} \bar{\delta f} \sim \left(\frac{m_e}{m_i} \right)^{1/2} \bar{\delta f}. \quad (\text{A.39})$$

We use the ordering (A.39) and look for a saturated dominant balance for ions at the IS in equation (A.34), where

$$\bar{\mathbf{v}}_i^E \cdot \nabla_s \bar{h}_i \sim \bar{\mathbf{v}}_i^E \cdot \nabla F_{0i} \Rightarrow \frac{v_{\text{th,i}}}{\rho_{\text{th,i}}} \frac{e\bar{\phi}}{T} \bar{h}_i \sim \frac{v_{\text{th,i}}}{a} \frac{e\bar{\phi}}{T} F_{0i}, \quad (\text{A.40})$$

and similarly for electrons at IS in equation (A.35), where

$$\bar{\mathbf{v}}_e^E \cdot \nabla_s \bar{h}_e \sim \bar{\mathbf{v}}_e^E \cdot \nabla F_{0e} \Rightarrow \frac{v_{\text{th,i}}}{\rho_{\text{th,i}}} \frac{e\bar{\phi}}{T} \bar{h}_e \sim \frac{v_{\text{th,i}}}{a} \frac{e\bar{\phi}}{T} F_{0e}. \quad (\text{A.41})$$

We do the same for electrons at ES in equation (A.36), where

$$\tilde{\mathbf{v}}_e^E \cdot \nabla_f \tilde{h}_e \sim \tilde{\mathbf{v}}_e^E \cdot \nabla F_{0e} \Rightarrow \frac{v_{\text{th,e}}}{\rho_{\text{th,e}}} \frac{e\tilde{\phi}}{T} \tilde{h}_e \sim \frac{v_{\text{th,e}}}{a} \frac{e\tilde{\phi}}{T} F_{0e}, \quad (\text{A.42})$$

and note that the scale of $\tilde{h}_i$ in (A.37) is set by the drive terms, rather than the advection terms,

$$\mathbf{v}_i^M \cdot \nabla_f \tilde{h}_i \sim \tilde{\mathbf{v}}_i^E \cdot \nabla F_{0i} \Rightarrow \frac{v_{\text{th,e}}}{a} \tilde{h}_i \sim \frac{v_{\text{th,e}}}{a} \tilde{J} \frac{e\tilde{\phi}}{T} F_{0i}. \quad (\text{A.43})$$

Then with quasineutrality equations (2.51) and (2.52), which indicate

$$\frac{e\bar{\phi}}{T} \sim \max \left\{ \frac{\bar{h}_i}{F_{0i}}, \frac{\bar{h}_e}{F_{0e}} \right\}, \quad \text{and} \quad \frac{e\tilde{\phi}}{T} \sim \max \left\{ \frac{\tilde{h}_i}{F_{0i}}, \frac{\tilde{h}_e}{F_{0e}} \right\}, \quad (\text{A.44})$$

we arrive at the self-consistent scalings (2.57).

Using the scalings (2.57) we see that the gyro-Bohm ordering captures ES turbulence where $\tilde{h}_e/F_{0e}$ is modified at leading order by IS gradients, where the nonadiabatic response of ions at ES can be ignored, and where the largest possible cross-scale terms can be neglected in the IS equations. The cross-scale term $\nabla_s \cdot \left\langle \tilde{\mathbf{v}}_i^E \tilde{h}_i \right\rangle^{\text{ES}}$ in (A.34) is neglected because it is small by $O(m_e/m_i)$. The cross-scale term $\nabla_s \cdot \left\langle \left\langle \tilde{\mathbf{v}}_e^E \tilde{h}_e \right\rangle^{\text{ES}} \right\rangle^o$ in (A.35) is small by $O\left((m_e/m_i)^{1/2}\right)$, and is neglected along with the small correction $\bar{h}_e^{(1)}$ to the electron distribution function at IS. We now consider if it is possible to modify the scalings with mass ratio, by considering the possible impacts of cross-scale interaction. Within the scale-separated model, cross-scale interactions are the only mechanisms which might alter the gyro-Bohm scaling (2.57). However, we conclude that (2.57) is the only self-consistent scaling for the fluctuation amplitudes, and that (2.73)-(2.77) is the most general set of leading order, scale-separated equations.

A.5.1 ES turbulence enhanced by cross-scale interaction:

$$e\tilde{\phi}/T \gg \rho_{\text{th},e}/a$$

First, consider the possibility that the presence of cross-scale interactions causes the ES turbulent amplitude to be enhanced by a factor of mass ratio compared to the gyro-Bohm estimate, i.e. $e\tilde{\phi}/T \gg \rho_{\text{th},e}/a$. The dominant terms in the ES equation for electrons (A.36) would be $\tilde{\mathbf{v}}_e^E \cdot \nabla_f \tilde{h}_e$, $\tilde{\mathbf{v}}_e^E \cdot \nabla F_{0e}$, and $\tilde{\mathbf{v}}_e^E \cdot \nabla_s \bar{h}_e$. The dominant terms in the ES equation for ions (A.37) would be $\tilde{\mathbf{v}}_i^E \cdot \nabla_f \tilde{h}_i$, $\tilde{\mathbf{v}}_i^E \cdot \nabla F_{0i}$, and $\tilde{\mathbf{v}}_i^E \cdot \nabla_s \bar{h}_i$. If the equilibrium drive terms are dominant then we would have a balance between $\tilde{\mathbf{v}}_e^E \cdot \nabla F_{0e}$ and the nonlinear term $\tilde{\mathbf{v}}_e^E \cdot \nabla_f \tilde{h}_e$ in (A.36) and a balance between $\tilde{\mathbf{v}}_i^E \cdot \nabla F_{0i}$ and the nonlinear term $\tilde{\mathbf{v}}_i^E \cdot \nabla_f \tilde{h}_i$ in (A.37). This results in

$$\tilde{h}_e \sim \frac{\rho_{\text{th},e}}{a} F_{0e}, \quad \tilde{h}_i \sim \frac{\rho_{\text{th},e}}{a} F_{0i}, \quad (\text{A.45})$$

and so with quasineutrality (A.44) we would find

$$\frac{e\tilde{\phi}}{T} \sim \frac{\rho_{\text{th},e}}{a}, \quad (\text{A.46})$$

inconsistent with the initial assumption.

This indicates that for $e\tilde{\phi}/T \gg \rho_{\text{th},e}/a$ to be possible we would need to have dominant cross-scale interaction terms. For electrons, this implies that

$$\tilde{\mathbf{v}}_e^E \cdot \nabla_{\mathbf{f}} \tilde{h}_e \sim \tilde{\mathbf{v}}_e^E \cdot \nabla_s \bar{h}_e \Rightarrow \frac{v_{\text{th},e}}{\rho_{\text{th},e}} \frac{e\tilde{\phi}}{T} \tilde{h}_e \sim \frac{v_{\text{th},e}}{\rho_{\text{th},i}} \frac{e\tilde{\phi}}{T} \bar{h}_e, \quad (\text{A.47})$$

and for ions that

$$\tilde{\mathbf{v}}_i^E \cdot \nabla_{\mathbf{f}} \tilde{h}_i \sim \tilde{\mathbf{v}}_i^E \cdot \nabla_s \bar{h}_i \Rightarrow \frac{v_{\text{th},e}}{\rho_{\text{th},e}} \tilde{J} \frac{e\tilde{\phi}}{T} \tilde{h}_i \sim \frac{v_{\text{th},e}}{\rho_{\text{th},i}} \tilde{J} \frac{e\tilde{\phi}}{T} \bar{h}_i. \quad (\text{A.48})$$

Therefore we need,

$$\tilde{h}_e \sim \frac{\rho_{\text{th},e}}{\rho_{\text{th},i}} \bar{h}_e, \quad \tilde{h}_i \sim \frac{\rho_{\text{th},e}}{\rho_{\text{th},i}} \bar{h}_i. \quad (\text{A.49})$$

In addition, we must have that the cross-scale interaction is much stronger than the equilibrium drive, and so we must have that

$$\frac{\bar{h}_e}{F_{0e}} \gg \frac{\rho_{\text{th},i}}{a} \quad \text{and} \quad \frac{\bar{h}_i}{F_{0i}} \gg \frac{\rho_{\text{th},i}}{a}. \quad (\text{A.50})$$

With (A.44) this requires

$$\frac{e\bar{\phi}}{T} \gg \frac{\rho_{\text{th},i}}{a}. \quad (\text{A.51})$$

We now determine if (A.51) is possible.

A.5.2 IS turbulence enhanced by cross-scale interaction:

$$e\bar{\phi}/T \gg \rho_{\text{th},i}/a$$

We consider if it is possible for the IS turbulent amplitude to be enhanced by a mass ratio factor compared to the gyro-Bohm estimate, i.e. $e\bar{\phi}/T \gg \rho_{\text{th},i}/a$. We consider the possibility that the IS cross-scale term $\nabla_s \cdot \left\langle \left\langle \tilde{\mathbf{v}}_e^E \tilde{h}_e \right\rangle^{\text{ES}} \right\rangle^0$ is larger than the equilibrium drive term $\langle \bar{\mathbf{v}}_e^E \cdot \nabla F_{0e} \rangle^0$, and is sufficiently large to balance the IS nonlinear term $\langle \bar{\mathbf{v}}_e^E \cdot \nabla_s \bar{h}_e \rangle^0$ in the electron equation (A.35). This is the only possible way to obtain $e\bar{\phi}/T \gg \rho_{\text{th},i}/a$, as all other balances lead to gyro-Bohm scaling. The dominant terms in (A.35) would now be $\langle \bar{\mathbf{v}}_e^E \cdot \nabla_s \bar{h}_e \rangle^0$ and $\left\langle \nabla_s \cdot \left\langle \tilde{\mathbf{v}}_e^E \tilde{h}_e \right\rangle^{\text{ES}} \right\rangle^0$, which would imply

$$\langle \bar{\mathbf{v}}_e^E \cdot \nabla_s \bar{h}_e \rangle^0 \sim \nabla_s \cdot \left\langle \left\langle \tilde{\mathbf{v}}_e^E \tilde{h}_e \right\rangle^{\text{ES}} \right\rangle^0 \Rightarrow \frac{v_{\text{th},i}}{\rho_{\text{th},i}} \frac{e\bar{\phi}}{T} \bar{h}_e \sim \frac{v_{\text{th},e}}{\rho_{\text{th},i}} \frac{e\tilde{\phi}}{T} \tilde{h}_e. \quad (\text{A.52})$$

With (A.49) this leads to

$$\frac{e\tilde{\phi}}{T} \frac{T}{e\bar{\phi}} \sim 1, \quad (\text{A.53})$$

which is inconsistent with the amplitude ratio which is implied by (A.49)

$$\frac{e\tilde{\phi}}{T} \frac{T}{e\bar{\phi}} \sim \frac{\rho_{\text{th,e}}}{\rho_{\text{th,i}}}. \quad (\text{A.54})$$

Therefore, neither $e\tilde{\phi}/T \gg \rho_{\text{th,e}}/a$ nor $e\bar{\phi}/T \gg \rho_{\text{th,i}}/a$ are possible.

We could repeat this argument by attempting to balance the IS nonlinear term for ions, $\bar{\mathbf{v}}_i^E \cdot \nabla_s \bar{h}_e$, with the IS cross-scale term for ions, $\nabla_s \cdot \langle \tilde{\mathbf{v}}_i^E \tilde{h}_i \rangle^{\text{ES}}$, and we would find that for the cross-scale term to compete we would need

$$\frac{e\tilde{\phi}}{T} \frac{T}{e\bar{\phi}} \sim \tilde{J}^{-1} \gg 1, \quad (\text{A.55})$$

which is again inconsistent with (A.54) and therefore not an allowed balance.

A.5.3 IS turbulence suppresses ES fluctuations: $e\bar{\phi}/T \sim \rho_{\text{th,i}}/a$ and $e\tilde{\phi}/T \ll \rho_{\text{th,e}}/a$

Now let us consider the possibility that as a result of cross-scale interaction, the ES fluctuation amplitude is suppressed by a mass ratio factor compared to the gyro-Bohm estimate, i.e. $e\bar{\phi}/T \sim \rho_{\text{th,i}}/a$ and $e\tilde{\phi}/T \ll \rho_{\text{th,e}}/a$. In this case, at the ES the usual nonlinearity $\tilde{\mathbf{v}}_\nu^E \cdot \nabla_f \tilde{h}_\nu$ is negligible for each species. The ES equations are still modified at leading order by the IS gradients, and so linear and cross-scale physics are now dominant in the ES equation (A.36), which gives us

$$\begin{aligned} v_\parallel \mathbf{b} \cdot \nabla \theta \frac{\partial \tilde{h}_e}{\partial \theta} &\sim \tilde{\mathbf{v}}_e^E \cdot \nabla F_{0e} \sim \tilde{\mathbf{v}}_e^E \cdot \nabla_s \bar{h}_e \\ \Rightarrow \frac{v_{\text{th,e}}}{a} \tilde{h}_e &\sim \frac{v_{\text{th,e}}}{a} \frac{e\tilde{\phi}}{T} F_{0e}. \end{aligned} \quad (\text{A.56})$$

The nonadiabatic response of ions at ES is still ignorable. Since linear and cross-scale physics are dominant, we still find

$$\tilde{h}_i \sim \tilde{J} \frac{e\tilde{\phi}}{T} F_{0i}, \quad (\text{A.57})$$

as in the gyro-Bohm regime (A.39). Since we retain scaling (A.57), the cross-scale term in the IS ion equation (A.34) may be ignored. We can ignore the IS cross-scale term in the IS electron equation (A.35), which now appears at an order smaller than $O((m_e/m_i)^{1/2})$ in the expansion,

$$\nabla_s \cdot \langle \tilde{\mathbf{v}}_e^E \tilde{h}_e \rangle^{\text{ES}} \sim \frac{v_{\text{th},e}}{\rho_{\text{th},i}} \frac{e\tilde{\phi}}{T} \tilde{h}_e \sim \frac{v_{\text{th},e}^2}{\rho_{\text{th},i} a} \left(\frac{e\tilde{\phi}}{T} \right)^2 F_{0e} \ll \left(\frac{m_e}{m_i} \right)^{1/2} \frac{v_{\text{th},i}}{a} \bar{h}_e. \quad (\text{A.58})$$

Hence, the ordering $e\bar{\phi}/T \sim \rho_{\text{th},i}/a$ and $e\tilde{\phi}/T \ll \rho_{\text{th},e}/a$ captures the modification of ES linear physics by IS profiles, where the IS is unaffected by the ES. To realise this ordering the ES fluctuations must be linearly stable, and have vanishing amplitude. Note that the IS gradients $\nabla_s \bar{\varphi}_e$ and $\nabla_s \bar{h}_e$ do not depend on t_f , the ES time coordinate. Hence, in this ordering the ES turbulence cannot be saturated. Either the presence of IS gradients suppresses the ES instability, in which case the ES turbulence vanishes, or the IS gradients enhance the ES instability, in which case the ES fluctuation amplitudes grow until they saturate at gyro-Bohm levels through the usual nonlinear term $\tilde{\mathbf{v}}_e^E \cdot \nabla_f \tilde{h}_e$.

A.5.4 ES turbulence suppresses IS fluctuations: $e\bar{\phi}/T \ll \rho_{\text{th},i}/a$ and $e\tilde{\phi}/T \sim \rho_{\text{th},e}/a$

Finally let us consider the possibility that as a result of cross-scale interaction, the IS fluctuation amplitude is suppressed by a mass ratio factor compared to the gyro-Bohm estimate, i.e. $e\bar{\phi}/T \ll \rho_{\text{th},i}/a$ and $e\tilde{\phi}/T \sim \rho_{\text{th},e}/a$. Here, the ES turbulence is nonlinearly saturated by the single scale nonlinearity $\tilde{\mathbf{v}}_e^E \cdot \nabla_f \tilde{h}_e$ in (A.36). The ES turbulence is unmodified by IS fluctuations as the gradient terms $\tilde{\mathbf{v}}_\nu^E \cdot \nabla_s \bar{h}_\nu$ and $\bar{\mathbf{v}}_\nu^E \cdot \nabla_f \tilde{h}_\nu$ are small for each species. The single scale nonlinear terms of the IS are negligible; both $\bar{\mathbf{v}}_i^E \cdot \nabla_s \bar{h}_i$ in equation (A.34) and $\langle \bar{\mathbf{v}}_e^E \cdot \nabla_s \bar{h}_e \rangle^o$ in equation (A.35) are small because $e\bar{\phi}/T \ll \rho_{\text{th},i}/a$. The only way for the IS to be saturated is through the cross-scale term $\nabla_s \cdot \langle \langle \tilde{\mathbf{v}}_e^E \tilde{h}_e \rangle^{\text{ES}} \rangle^o$ in (A.35), which is a time varying source. This would require

$$\langle \mathbf{v}_e^M \cdot \nabla \alpha \rangle^o \frac{\partial \bar{h}_e}{\partial \alpha_s} \sim \nabla_s \cdot \langle \langle \tilde{\mathbf{v}}_e^E \tilde{h}_e \rangle^{\text{ES}} \rangle^o \Rightarrow \frac{v_{\text{th},i}}{a} \bar{h}_e \sim \frac{v_{\text{th},e}}{\rho_{\text{th},i}} \frac{e\tilde{\phi}}{T} \tilde{h}_e. \quad (\text{A.59})$$

The balance (A.59) implies

$$\frac{\bar{h}_e}{F_{0e}} \sim \frac{\rho_{\text{th},e}}{a}. \quad (\text{A.60})$$

If we assume that the electrons set the scale of $e\bar{\phi}/T$ then the ion IS equation (A.34) has a dominant balance between only linear terms,

$$\mathbf{v}_i^M \cdot \nabla_s \bar{h}_i \sim \bar{\mathbf{v}}_i^E \cdot \nabla F_{0i} \Rightarrow \frac{v_{th,i}}{a} \bar{h}_i \sim \frac{v_{th,i}}{\rho_{th,i}} \frac{e\bar{\phi}}{T} F_{0i}, \quad (\text{A.61})$$

and so

$$\frac{\bar{h}_i}{F_{0i}} \sim \frac{\rho_{th,e}}{a}. \quad (\text{A.62})$$

Using (A.44), (A.60) and (A.62) we see that,

$$\frac{e\tilde{\phi}}{T} \sim \frac{\rho_{th,e}}{a}, \quad (\text{A.63})$$

is a consistent scaling. Therefore, naively $e\bar{\phi}/T \sim \rho_{th,e}/a \ll \rho_{th,i}/a$ and $e\tilde{\phi}/T \sim \rho_{th,e}/a$ appears to be a consistent ordering for the IS-ES system. However, note that in this regime the ES cross-scale terms, $\bar{\mathbf{v}}_e^E \cdot \nabla_f \tilde{h}_e$ and $\tilde{\mathbf{v}}_e^E \cdot \nabla_s \bar{h}_e$ in (A.36) and $\bar{\mathbf{v}}_i^E \cdot \nabla_f \tilde{h}_i$ and $\tilde{\mathbf{v}}_i^E \cdot \nabla_s \bar{h}_i$ in (A.37), are small. The ES turbulence only has dependence on the IS spatial coordinate $\mathbf{R}_s$ through the IS gradients appearing in the ES cross-scale terms, and the parallel boundary condition (2.89). As discussed in Section 2.7.4, the ES parallel boundary condition (2.89) couples the ES flux tubes in α_s at fixed ψ_s , through connections in the parallel coordinate θ at the flux tube parallel boundaries. The flux tubes coupled this way form a chain. In general, the length of the chain of flux tubes is a function of ψ_s . Hence, if the correlation length of the ES fluctuations is comparable to the length of a single flux tube then the ES fluctuations may have different statistics in chains of flux tubes with different lengths. If we assume that the length of a single flux tube is longer than the ES parallel correlation length then the parallel boundary condition (2.89) should not introduce significant spatial inhomogeneity. With this assumption, in the regime that $e\bar{\phi}/T \ll \rho_{th,i}/a$ and $e\tilde{\phi}/T \sim \rho_{th,e}/a$ the fluxes $\langle \tilde{\mathbf{v}}_e^E \tilde{h}_e \rangle^{\text{ES}}$ and $\langle \tilde{\mathbf{v}}_i^E \bar{h}_i \rangle^{\text{ES}}$ are not functions of $\mathbf{R}_s$. Hence,

$$\nabla_s \cdot \langle \tilde{\mathbf{v}}_i^E \bar{h}_i \rangle^{\text{ES}} = \nabla_s \cdot \left\langle \langle \tilde{\mathbf{v}}_i^E \bar{h}_i \rangle^{\text{ES}} \right\rangle^0 = 0 \quad (\text{A.64})$$

identically. Therefore, the regime where $e\bar{\phi}/T \ll \rho_{th,i}/a$ and $e\tilde{\phi}/T \sim \rho_{th,e}/a$ can only exist when the IS fluctuations are linearly stable, and therefore have no amplitude.

In this ordering there is no saturation mechanism for the IS turbulence. If the IS fluctuations are linearly unstable they will therefore grow linearly until the gyro-Bohm saturation level is reached. We conclude that only the gyro-Bohm scaling (2.57) gives saturated dominant balance in the equations (A.34)-(A.37). Note that the precise saturation level of coupled IS-ES turbulence with nonzero $(m_e/m_i)^{1/2}$ depends in detail on parameters that have been treated as of order unity in the $(m_e/m_i)^{1/2}$ expansion here. An example of such a parameter is the ratio of the drive of IS to ES instability. We give a brief discussion of the effect of this parameter in Section 4.2.

A.6 Obtaining the scale-separated ES quasineutrality relation

In this Section we show how to evaluate ES quasineutrality in a local, scale-separated way for the electron species. Recalling that for electrons $\mathbf{R}_f = \mathbf{r}_f - \boldsymbol{\rho}_e$ and $\mathbf{R}_s = \mathbf{r}_s - \boldsymbol{\rho}_e$, where $\boldsymbol{\rho}_e = (\mathbf{b} \times \mathbf{v})/\Omega_e$ is the vector gyroradius, and using the Fourier representation (2.53), then

$$\begin{aligned} \tilde{\varphi}_e(\mathbf{R}_s, \mathbf{R}_f) &= \left\langle \tilde{\phi}(\mathbf{r}_s, \mathbf{r}_f) \right\rangle_{\mathbf{R}_s, \mathbf{R}_f}^\gamma \\ &= - \left(\sum_\nu \frac{Z_\nu^2 e n_\nu}{T_\nu} \right)^{-1} \sum_{\mathbf{k}_f, \mathbf{k}_s} \exp[i\mathbf{k}_s \cdot \mathbf{R}_s] \exp[i\mathbf{k}_f \cdot \mathbf{R}_f] J_0(|\mathbf{k}_f + \mathbf{k}_s||\boldsymbol{\rho}_e|) \times \\ &\quad \int d^3\mathbf{v} \tilde{h}_{e\mathbf{k}_f, \mathbf{k}_s}(\theta, \varepsilon, \lambda, \sigma) J_0(|\mathbf{k}_f + \mathbf{k}_s||\boldsymbol{\rho}_e|), \end{aligned} \quad (\text{A.65})$$

where we have used (2.55). By noting that $|\mathbf{k}_f||\boldsymbol{\rho}_e| \sim |\mathbf{k}_f|\rho_{\text{th},e} \sim 1$ and $|\mathbf{k}_s||\boldsymbol{\rho}_e| \sim |\mathbf{k}_s|\rho_{\text{th},e} \sim (m_e/m_i)^{1/2}$, we can expand the Bessel function in its argument,

$$\begin{aligned} J_0(|\mathbf{k}_f + \mathbf{k}_s||\boldsymbol{\rho}_e|) &= J_0(|\mathbf{k}_f||\boldsymbol{\rho}_e|) + \text{O} \left(\underbrace{\frac{\mathbf{k}_f}{|\mathbf{k}_f|} \cdot \mathbf{k}_s |\boldsymbol{\rho}_e|}_{\sim (m_e/m_i)^{1/2}} \underbrace{\frac{dJ_0(z)}{dz}}_{\sim 1} \bigg|_{z=|\mathbf{k}_f||\boldsymbol{\rho}_e|} \right) \\ &= J_0(|\mathbf{k}_f||\boldsymbol{\rho}_e|) + \text{O} \left(\left(\frac{m_e}{m_i} \right)^{1/2} \right), \end{aligned} \quad (\text{A.66})$$

and so we are able to rewrite (A.65) in the form

$$\tilde{\varphi}_e(\mathbf{R}_s, \mathbf{R}_f) = - \left(\sum_{\nu} \frac{Z_{\nu}^2 e n_{\nu}}{T_{\nu}} \right)^{-1} \sum_{\mathbf{k}_f, \mathbf{k}_s} \exp[\mathbf{i} \mathbf{k}_s \cdot \mathbf{R}_s] \exp[\mathbf{i} \mathbf{k}_f \cdot \mathbf{R}_f] J_0(|\mathbf{k}_f| |\boldsymbol{\rho}_e|) \times \int d^3 \mathbf{v} \tilde{h}_{e\mathbf{k}_f, \mathbf{k}_s}(\theta, \varepsilon, \lambda, \sigma) J_0(|\mathbf{k}_f| |\boldsymbol{\rho}_e|) \left(1 + \mathcal{O} \left(\left(\frac{m_e}{m_i} \right)^{1/2} \right) \right). \quad (\text{A.67})$$

This allows us to resum the slow Fourier series, and thus return to a parametric representation where $\mathbf{R}_s$ only appears as a label,

$$\tilde{\varphi}_e(\mathbf{R}_s, \mathbf{R}_f) = - \left(\sum_{\nu} \frac{Z_{\nu}^2 e n_{\nu}}{T_{\nu}} \right)^{-1} \sum_{\mathbf{k}_f} \exp[\mathbf{i} \mathbf{k}_f \cdot \mathbf{R}_f] J_0(|\mathbf{k}_f| |\boldsymbol{\rho}_e|) \times \int d^3 \mathbf{v} |_{\mathbf{R}_s} \tilde{h}_{e\mathbf{k}_f}(\theta, \mathbf{R}_s, \varepsilon, \lambda, \sigma) J_0(|\mathbf{k}_f| |\boldsymbol{\rho}_e|) \left(1 + \mathcal{O} \left(\left(\frac{m_e}{m_i} \right)^{1/2} \right) \right). \quad (\text{A.68})$$

Therefore, we have found a scale-separated scheme for evaluating quasineutrality; at the ES we can parallelise over the label $\mathbf{R}_s$. Consequently, ES flux tubes which are labelled by different $\mathbf{R}_s$ may be integrated in isolation, up to coupling introduced by the parallel boundary condition. We derive the ES parallel boundary condition, equation (2.89), in Section 2.7.2. In Section 2.7.4 we discuss the coupling between ES flux tubes that results as a consequence of the ES parallel boundary condition.

A.7 IS and ES heat fluxes

In this Section we discuss the scalings of the heat fluxes predicted by the coupled system of equations in the ordering (2.57). In the electrostatic limit, the heat flux Q is defined as

$$Q = \left\langle \int d^3 \mathbf{v} |_{\mathbf{r}} \varepsilon \delta f \frac{c}{B} \mathbf{b} \times \nabla \phi \right\rangle^{\text{FT}} = \left\langle \int d^3 \mathbf{v} |_{\mathbf{r}} \varepsilon h \frac{c}{B} \mathbf{b} \times \nabla \phi \right\rangle^{\text{FT}}, \quad (\text{A.69})$$

where $\langle \cdot \rangle^{\text{FT}}$ is a spatial average over the entire flux tube. The average $\langle \cdot \rangle^{\text{FT}}$ is a flux surface average with an additional average in the ψ direction. In equation (A.69) we used that in electrostatic turbulence δf is related to h by (2.11), and integration by

parts with the periodic flux tube boundary conditions, to show that the heat flux is due to the nonadiabatic response h . For scale-separated IS-ES turbulence we can write

$$Q = \left\langle \int d^3\mathbf{v}|_{\mathbf{r}} \varepsilon(\bar{h} + \tilde{h}) \left(\frac{c\mathbf{b}}{B} \times (\nabla_s + \nabla_f)\phi \right) \right\rangle^{\text{FT}} = \frac{1}{V_i} \int d\mathbf{r}_{\parallel} \int d^2\mathbf{r}_s \int d^3\mathbf{v}|_{\mathbf{r}} \varepsilon\bar{h} \frac{c\mathbf{b}}{B} \times \nabla_s \bar{\phi} \\ + \frac{1}{N_{\mathbf{r}_s} V_e} \sum_{\mathbf{r}_s} \int d\mathbf{r}_{\parallel} \int d^2\mathbf{r}_f \int d^3\mathbf{v}|_{\mathbf{r}} \varepsilon\tilde{h}(\mathbf{r}_s) \frac{c\mathbf{b}}{B} \times \nabla_f \tilde{\phi}(\mathbf{r}_s), \quad (\text{A.70})$$

where the cross terms vanished because of the assumption of statistical periodicity (2.18), $V_i = \int d\mathbf{r}_{\parallel} \int d^2\mathbf{r}_s$ and $V_e = \int d\mathbf{r}_{\parallel} \int d^2\mathbf{r}_f$ are the volumes of the IS and ES flux tubes, respectively, and $N_{\mathbf{r}_s} = \sum_{\mathbf{r}_s} 1$ is the number of ES flux tubes. The sum $\sum_{\mathbf{r}_s}$ is over the ES flux tubes within the IS flux tube. We can identify the IS heat flux

$$\bar{Q} = \frac{1}{V_i} \int d\mathbf{r}_{\parallel} \int d^2\mathbf{r}_s \int d^3\mathbf{v}|_{\mathbf{r}} \varepsilon\bar{h} \frac{c\mathbf{b}}{B} \times \nabla_s \bar{\phi} = \frac{1}{V_i} \int d\mathbf{r}_{\parallel} A_i \sum_{\mathbf{k}_s} \int d^3\mathbf{v}|_{\mathbf{r}} \varepsilon\bar{h}_{-\mathbf{k}_s} \frac{c\mathbf{b}}{B} \times i\mathbf{k}_s \bar{\phi}_{\mathbf{k}_s} \quad (\text{A.71})$$

and the ES heat flux

$$\tilde{Q} = \frac{1}{N_{\mathbf{r}_s} V_e} \sum_{\mathbf{r}_s} \int d\mathbf{r}_{\parallel} \int d^2\mathbf{r}_f \int d^3\mathbf{v}|_{\mathbf{r}} \varepsilon\tilde{h}(\mathbf{r}_s) \frac{c\mathbf{b}}{B} \times \nabla_f \tilde{\phi}(\mathbf{r}_s) \\ = \frac{1}{N_{\mathbf{r}_s} V_e} \sum_{\mathbf{r}_s} \int d\mathbf{r}_{\parallel} A_e \sum_{\mathbf{k}_f} \int d^3\mathbf{v}|_{\mathbf{r}} \varepsilon\tilde{h}_{-\mathbf{k}_f}(\mathbf{r}_s) \frac{c\mathbf{b}}{B} \times i\mathbf{k}_f \tilde{\phi}_{\mathbf{k}_f}(\mathbf{r}_s), \quad (\text{A.72})$$

where $A_i = \int d^2\mathbf{r}_s$ and $A_e = \int d^2\mathbf{r}_f$ are the areas of the cross sections of the ion and ES flux tubes respectively. With these observations, we are able to write down the scaling of the fluxes of each species with the potentials $e\bar{\phi}/T$ and $e\tilde{\phi}/T$,

$$\bar{Q}_i \sim \bar{Q}_e \sim nT v_{\text{th},i} \left(\frac{e\bar{\phi}}{T} \right)^2, \quad (\text{A.73})$$

$$\tilde{Q}_i \sim nT v_{\text{th},e} \left(\frac{e\tilde{\phi}}{T} \right)^2 \left(\frac{m_e}{m_i} \right)^{1/2}, \quad \tilde{Q}_e \sim nT v_{\text{th},e} \left(\frac{e\tilde{\phi}}{T} \right)^2, \quad (\text{A.74})$$

Here the extra factor of $(m_e/m_i)^{1/2}$ in $\tilde{Q}_i$ appears due to the smallness of the ion response at ES and the Bessel functions in the ion gyroaverage at ES.

In the gyro-Bohm scaling (2.57),

$$\bar{Q}_i \sim \bar{Q}_e \sim nT v_{\text{th},i} \left(\frac{\rho_{\text{th},i}}{a} \right)^2, \quad (\text{A.75})$$

$$\tilde{Q}_i \sim nT v_{th,e} \left(\frac{\rho_{th,e}}{a} \right)^2 \left(\frac{m_e}{m_i} \right)^{1/2}, \quad \tilde{Q}_e \sim nT v_{th,e} \left(\frac{\rho_{th,e}}{a} \right)^2. \quad (\text{A.76})$$

Hence

$$\tilde{Q}_e \sim \left(\frac{m_e}{m_i} \right)^{1/2} \bar{Q}_i \sim \left(\frac{m_e}{m_i} \right)^{1/2} \bar{Q}_e. \quad (\text{A.77})$$

This means that the heat flux from the ES is small by $(m_e/m_i)^{1/2}$ compared to the heat flux driven at IS, within the assumptions of the scale-separated model. However, it is important to note that coupled IS-ES turbulence with nonzero $(m_e/m_i)^{1/2}$ may drive IS and ES heat fluxes which are comparable due to the effect of parameters which are considered of order unity in the $(m_e/m_i)^{1/2}$ expansion. For example, the IS heat flux may be small if the ion temperature gradient is close to the critical value for IS instability. The ES heat flux may be large when the ES instability is strongly driven due to the stiffness of turbulent transport. We expect that the heat flux from nonzonal passing electrons at IS, which we neglect, to always be negligible. The heat flux from nonzonal passing electrons at IS is a small correction to $\bar{Q}_e$. In scenarios where the IS heat flux dominates the ES heat flux neglecting a small piece of the IS heat flux is justified, providing we neglect $\tilde{Q}_e$. In cases where the ES heat flux $\tilde{Q}_e$ is comparable to or dominates the heat flux from IS, then again, neglecting a small piece of $\bar{Q}_e$ is justified. The ion ES contribution to the heat flux can always be neglected as $\tilde{Q}_i \sim (m_e/m_i)^{1/2} \tilde{Q}_e$.

A.8 The “twist-and-shift” parallel boundary condition

In this Section we reproduce the calculation of the IS spectral parallel boundary condition, first proposed in [44], in the notation of this thesis to aid the discussion in Section 2.7. The real space statement of the IS boundary condition is equation (2.88). Expanding the guiding centre variable $\mathbf{R}_s$ in equation (2.88) in Fourier modes after (2.90), we find that equation (2.88) implies

$$\sum_{k_\psi, k_\alpha} \bar{h}_{(k_\psi, k_\alpha)}(\theta) \exp [ik_\psi(\psi_s - \psi_0) + ik_\alpha(\alpha(\psi_s, \theta, \zeta) - \alpha_0)]$$

$$= \sum_{k'_\psi, k_\alpha} \bar{h}_{(k'_\psi, k_\alpha)}(\theta + 2\pi) \exp [ik'_\psi(\psi_s - \psi_0) + ik_\alpha(\alpha(\psi_s, \theta + 2\pi, \zeta) - \alpha_0)]. \quad (\text{A.78})$$

Using the definition of $\alpha(\psi, \theta, \zeta) - \alpha_0 = \zeta - q_0\theta - q'_0(\psi - \psi_0)$, equation (A.78) can be written as

$$\begin{aligned} & \sum_{k_\psi, k_\alpha} \bar{h}_{(k_\psi, k_\alpha)}(\theta) \exp [ik_\psi(\psi_s - \psi_0) + ik_\alpha(\alpha(\psi_s, \theta, \zeta) - \alpha_0)] \\ &= \sum_{k'_\psi, k_\alpha} \bar{h}_{(k'_\psi, k_\alpha)}(\theta + 2\pi) \exp [i(k'_\psi - 2\pi q'_0 k_\alpha)(\psi_s - \psi_0)] \\ & \quad \times \exp [+ik_\alpha(\alpha(\psi_s, \theta, \zeta) - \alpha_0)] \exp [-i2\pi q_0 k_\alpha] \end{aligned} \quad (\text{A.79})$$

Equating Fourier coefficients with the same exponent, we find that the IS Fourier space boundary condition is equation (2.91).

A.9 The ES “twist-and-shift” parallel boundary condition

In this Section we calculate the spectral boundary condition corresponding to the real space ES parallel boundary condition (2.89). Expanding the fast guiding centre variable $\mathbf{R}_f$ in equation (2.89) in Fourier modes as in (2.92), we find that equation (2.89) implies

$$\begin{aligned} & \sum_{K_\psi, K_\alpha} \tilde{h}_{(K_\psi, K_\alpha)}(\theta, \mathbf{R}_s(\psi_s, \alpha(\psi_s, \theta, \zeta))) \exp [iK_\psi(\psi_f - \psi_0) + iK_\alpha(\alpha(\psi_f, \theta, \zeta) - \alpha_0)] \\ &= \sum_{K'_\psi, K_\alpha} \tilde{h}_{(K'_\psi, K_\alpha)}(\theta + 2\pi, \mathbf{R}_s(\psi_s, \alpha(\psi_s, \theta + 2\pi, \zeta))) \times \\ & \quad \exp [iK'_\psi(\psi_f - \psi_0) + iK_\alpha(\alpha(\psi_f, \theta + 2\pi, \zeta) - \alpha_0)], \end{aligned} \quad (\text{A.80})$$

where K_ψ and K_α are the ES wave numbers corresponding to ψ_f and α_f . Equation (A.80) can be written as

$$\sum_{K_\psi, K_\alpha} \tilde{h}_{(K_\psi, K_\alpha)}(\theta, \mathbf{R}_s(\psi_s, \alpha_s)) \exp [iK_\psi(\psi_f - \psi_0) + iK_\alpha(\alpha(\psi_f, \theta, \zeta) - \alpha_0)]$$

$$\begin{aligned}
&= \sum_{K'_\psi, K_\alpha} \tilde{h}_{(K'_\psi, K_\alpha)}(\theta + 2\pi, \mathbf{R}_s(\psi_s, \alpha(\psi_s, \theta + 2\pi, \zeta))) \times \\
&\exp[i(K'_\psi - 2\pi q'_0 K_\alpha)(\psi_f - \psi_0)] \exp[iK_\alpha(\alpha(\psi_f, \theta, \zeta) - \alpha_0)] \exp[-i2\pi q_0 K_\alpha]. \quad (\text{A.81})
\end{aligned}$$

Equating Fourier coefficients with the same exponent, we find that the ES Fourier space boundary condition is equation (2.93).

A.10 Satisfaction of relations (2.98) and (2.99)

In this Section we show that the relations (2.98) and (2.99) are satisfied when the IS turbulence satisfies the boundary condition (2.88). Hence, we demonstrate that equation (2.89) is a sensible parallel boundary condition for the ES turbulence. First, we write the ψ_s derivatives at fixed ζ instead of fixed α_s ,

$$\left. \frac{\partial}{\partial \psi_s} \right|_{\alpha_s, \theta} = \left. \frac{\partial}{\partial \psi_s} \right|_{\zeta, \theta} + q'_0 \theta \left. \frac{\partial}{\partial \zeta} \right|_{\psi_s, \theta}, \quad \left. \frac{\partial}{\partial \alpha_s} \right|_{\psi_s, \theta} = \left. \frac{\partial}{\partial \zeta} \right|_{\psi_s, \theta}. \quad (\text{A.82})$$

Using the second relation displayed in (A.82), we note that if (2.88) is satisfied then (2.98) is satisfied. Next, we consider the LHS of equation (2.99), and demonstrate that it is equal to the RHS of equation (2.99). Note that

$$\begin{aligned}
&\left. \frac{\partial}{\partial \psi_s} \right|_{\alpha_s, \theta + 2\pi} \bar{h}(\theta + 2\pi, \mathbf{R}_s(\psi_s, \alpha(\psi_s, \theta + 2\pi, \zeta))) \\
&- 2\pi q'_0 \left. \frac{\partial}{\partial \alpha_s} \right|_{\psi_s, \theta + 2\pi} \bar{h}(\theta + 2\pi, \mathbf{R}_s(\psi_s, \alpha(\psi_s, \theta + 2\pi, \zeta))) \\
&= \left[\left. \frac{\partial}{\partial \psi_s} \right|_{\zeta, \theta + 2\pi} + q'_0 \theta \left. \frac{\partial}{\partial \zeta} \right|_{\psi_s, \theta + 2\pi} \right] \bar{h}(\theta + 2\pi, \mathbf{R}_s(\psi_s, \alpha(\psi_s, \theta + 2\pi, \zeta))) \\
&= \left[\left. \frac{\partial}{\partial \psi_s} \right|_{\zeta, \theta} + q'_0 \theta \left. \frac{\partial}{\partial \zeta} \right|_{\psi_s, \theta} \right] \bar{h}(\theta, \mathbf{R}_s(\psi_s, \alpha(\psi_s, \theta, \zeta))) \\
&= \frac{\partial}{\partial \psi_s} \bar{h}(\theta, \mathbf{R}_s(\psi_s, \alpha(\psi_s, \theta, \zeta))), \quad (\text{A.83})
\end{aligned}$$

where we have used the relations in (A.82) to rewrite the derivatives in a way so that we can use equation (2.88), and we have used that

$$\left. \frac{\partial}{\partial \psi_s} \right|_{\zeta, \theta + 2\pi} = \left. \frac{\partial}{\partial \psi_s} \right|_{\zeta, \theta}, \quad \text{and} \quad \left. \frac{\partial}{\partial \zeta} \right|_{\psi_s, \theta + 2\pi} = \left. \frac{\partial}{\partial \zeta} \right|_{\psi_s, \theta}. \quad (\text{A.84})$$

We can therefore see that relations (2.98) and (2.99) are satisfied if equation (2.88) is satisfied.

Appendix B

B.1 Tests of the implementation of the cross-scale terms in GS2

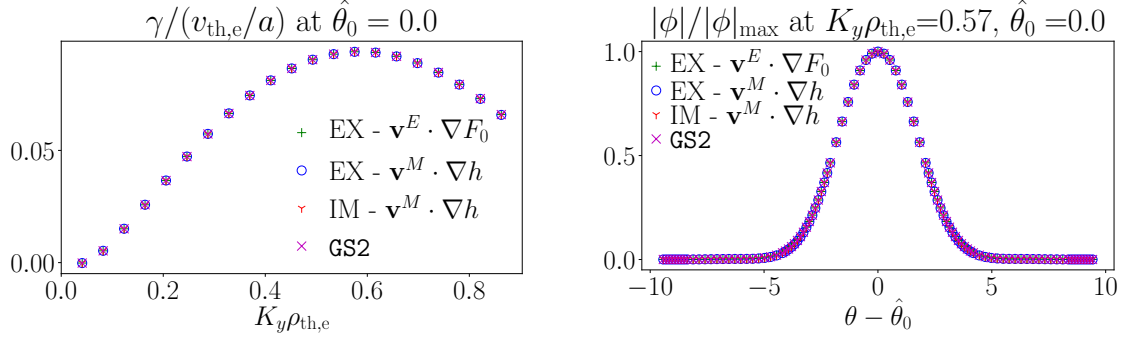


Figure B.1: (Left) Comparing the ETG growth rate γ , and (Right) the ETG eigenmode for $K_y \rho_{\text{th},e} = 0.57$, calculated in tests of the cross-scale terms.

In this Section we briefly describe the implementation and testing of the cross-scale terms $[i(c/B)\mathbf{b} \times \mathbf{k}_f \cdot \nabla_s \bar{h}_e] J_0(|\mathbf{k}_f| |\boldsymbol{\rho}_e|)$ and $[i\bar{\mathbf{v}}_e^E \cdot \mathbf{k}_f] \tilde{h}_{e\mathbf{k}_f}$ appearing in the ES, electron gyrokinetic equation (2.82). Both cross-scale terms are implemented using an explicit three-step Adams-Bashforth method [83], in terms of the nonadiabatic piece of the electron distribution function. In addition, the cross-scale term $[i\bar{\mathbf{v}}_e^E \cdot \mathbf{k}_f] \tilde{h}_{e\mathbf{k}_f}$ is implemented as a part of the GS2 implicit algorithm [73] for testing purposes. Both cross-scale terms are implemented so they can take as input a general array with the shape $(\theta, \sigma, \varepsilon, \lambda)$ in place of the IS gradients $\nabla_s \bar{h}_e$ or $\nabla_s \bar{\phi}$. We use this feature to test the cross-scale terms: In Figure B.1 we compare the growth rates γ for the ETG instability where the various implementations of the cross-scale terms are used to solve

the δf gyrokinetic equation (2.12). We use CBC parameters (see Table 3.1), adiabatic ions and the parallel-to-the-field and velocity space resolutions described in Appendix B.2. We compare four different cases: **GS2** — the standard implicit algorithm without cross-scale terms; EX - $\mathbf{v}^E \cdot \nabla F_0$ — the equilibrium drive term is supplied explicitly by replacing the IS gradient $\nabla_s \bar{h}_e$ with ∇F_{0e} ; EX - $\mathbf{v}^M \cdot \nabla h$ the magnetic drift term is supplied explicitly by replacing the IS drift $\bar{\mathbf{v}}_e^E$ with $\mathbf{v}_e^M$; and IM - $\mathbf{v}^M \cdot \nabla h$ the magnetic drift term is supplied implicitly by replacing the IS drift $\bar{\mathbf{v}}_e^E$ with $\mathbf{v}_e^M$. We find excellent agreement in all cases. Furthermore, we separately verify that supplying IS gradients which represent pure toroidal rotation leads to a Doppler shift in the frequency of the mode only; i.e., setting the gradients $\nabla_s \bar{g}_e = 0$ and $\bar{\mathbf{v}}_e^E$ constant in θ and directed purely in the α direction leave the growth rate γ unchanged.

B.2 Parallel-to-the-field and velocity space grids and resolutions

In this section we describe the parallel-to-the-field and velocity space grids and resolutions used in the IS and ES simulations. The gyrokinetic code **GS2** is implemented using field line following coordinates, with the poloidal angle θ as the coordinate measuring distance in the parallel-to-the-field direction. We specify that each (k_x, k_y) mode has a $[-\pi, \pi]$ extent in the parallel-to-the-field direction. For each (k_x, k_y) mode we choose the number of points in θ to be $n_\theta = 33$ with equal spacing in θ . The linear simulations use $n_c = 3$ connected k_x modes at fixed k_y to represent a ballooning mode. Equivalently the extent of the ballooning modes in extended poloidal angle is $[-3\pi, 3\pi]$. The nonlinear IS simulations use the spatial resolutions detailed in Section 3.3.

In the gyrokinetic code **GS2** velocity space is represented with energy ε and pitch angle λ coordinates for both signs $\sigma = \pm 1$ of the parallel velocity. To allow for spectrally accurate velocity space integration **GS2** employs an ε grid derived from a spectral speed $v = \sqrt{2\varepsilon/m}$ grid [55, 63, 73]. We employ an ε grid with $n_\varepsilon = 12$ points. Following [63], for the range of energies $0 < \varepsilon/T \leq v_c^2/2$, with $v_c = 2.5$, we use $n_{\varepsilon<} = 8$ points chosen according to Gauss-Legendre rules [84]; and for the energies $\varepsilon/T > v_c^2/2$ we use $n_{\varepsilon>} = 4$ points chosen according to Gauss-Laguerre rules [84].

The **GS2** λ grid is chosen to be spectral for the passing piece of the velocity space where $0 < \lambda \leq 1/B^{\max}$. We choose λ points in the range $0 < \lambda \leq 1/B^{\max}$ using Gauss-Radau rules [84]. Gauss-Radau rules are a variant of Gauss-Legendre rules which give nonzero weight to one endpoint. We chose to give nonzero weight in the passing particle integral to the $\lambda = 1/B^{\max}$ endpoint. This choice removed potential issues resulting from the fact that in the Gauss-Legendre grid of [55, 63] the phase space point $\lambda = 1/B^{\max}$ for $\theta = \pm\pi$ had zero weight in the velocity integral. For the trapped piece of the velocity space where $1/B^{\max} < \lambda \leq 1/B^{\min}$ we employ a nonspectral, unequally spaced grid in λ [55, 63, 73]. For both signs of the velocity, we choose to have $n_\lambda = 27$ points on the λ grid. We choose the number of passing particles $n_{\lambda,p}$, for which $0 < \lambda \leq 1/B^{\max}$, to be $n_{\lambda,p} = 11$. The number of trapped particles $n_{\lambda,t}$, for which $1/B^{\max} < \lambda \leq 1/B^{\min}$, is linked to the size of the θ grid n_θ and is $n_{\lambda,t} = (n_\theta - 1)/2 = 16$ in this case. The link between the λ and θ grids stems from the fact that we choose the λ points in the range $1/B^{\max} < \lambda \leq 1/B^{\min}$ according to the rule $\lambda_j = 1/B(\theta_j)$ for every θ_j on the θ grid [73]. This enforces that for every λ_j in the λ grid the θ grid contains the bounce point θ_j at which $v_{\parallel} = (2\varepsilon/m_\nu)^{1/2} (1 - \lambda_j B(\theta_j))^{1/2} = 0$; and that for every θ_j point in the θ grid there is a class of trapped particles which bounce at θ_j . In this discussion we have assumed that $B(\theta)$ is symmetric about $\theta = 0$.

B.3 The model for electrons in IS simulations in **GS2**

In the IS simulations presented in this thesis we used a reduced model for electrons in **GS2** [73] which has the effect that the the electron distribution function obeys equation (3.5). Equation (3.5) is derived by taking the asymptotic limit $(m_e/m_i)^{1/2} \rightarrow 0$ and noting that the electron thermal speed time scales $a/v_{\text{th},e}$ are asymptotically smaller than ion thermal speed time scales $a/v_{\text{th},i}$, and so should be averaged over. In consequence, nonzonal, passing electrons respond adiabatically; and the equation for the trapped and zonal, passing electrons, equation (3.5), is parallel orbit averaged. Because nonzonal, passing electrons respond adiabatically at the IS, the long ballooning tails observed in the passing electron distribution function [65] are ordered out of the IS

equations. Equivalently, the sharp radial structures due to passing electrons at mode rational surfaces [66] do not appear in the IS equations.

GS2 can simulate fully kinetic electrons. **GS2** uses a velocity space grid with energy ε , pitch angle λ and the sign of the parallel velocity σ as coordinates [55, 63, 73]. Note that **GS2** has a θ grid such that the bounce point of each class of trapped particles is represented. The θ grid contains the points such that the parallel velocity $v_{\parallel} = (2\varepsilon/m_{\nu})^{1/2} (1 - \lambda B(\theta))^{1/2} = 0$ for some θ for every trapped particle on the λ grid [73]. In consequence of this, and because (ε, λ) are conserved quantities of the single particle motion, **GS2** is able to accurately capture the motion of trapped particles, as well as passing particles with periodic orbits. **GS2** uses an implicit algorithm [73], and hence it is possible to capture the parallel orbit average behaviour of electrons simply by taking a time step Δt such that $\Delta t \sim a/v_{\text{th},i} \gg \omega_b^{-1}$ [73], where ω_b is the trapped electron bounce frequency. To implement equation (3.5) in **GS2** we simply enforce that the nonzonal, passing electrons respond adiabatically, whilst continuing to solve the usual gyrokinetic equation for trapped and zonal, passing electrons. We note that the electron distribution function found this way only corresponds precisely to $\bar{h}_e^{(0)}$ of Section 2.3 when $\omega_b \Delta t \gg 1$. If this constraint is not satisfied then the electron distribution function can contain components which are not constant in θ .

We now demonstrate that the implementation of (3.5) in **GS2** corresponds to taking the $(m_e/m_i)^{1/2} \rightarrow 0$ limit in the full electron gyrokinetic equation solved in standard **GS2**, whilst retaining the physical value for $(m_e/m_i)^{1/2}$. In Figures B.2 and B.3 we show the eigenmode of the fastest growing IS mode for CBC parameters, for the model of the electron response at IS with $(m_e/m_i)^{1/2} = 1/60$ (ME); for the fully kinetic electron model with $(m_e/m_i)^{1/2} = 1/60$ (KE); and for the fully kinetic electron model with $(m_e/m_i)^{1/2} = (1/60) \times 10^{-3}$ (KE – RM). The linear simulations presented in Figures B.2 and B.3 use time step sizes of $a\Delta t/v_{\text{th},i} = 0.05$. Note that the condition on the trapped electron bounce frequency ω_b for effective orbit averaging, relation (3.17), is only satisfied in the KE – RM case due to the artificially small value of $(m_e/m_i)^{1/2}$. In Figure B.2 we show the potential $\bar{\phi}$ eigenmode along the extended ballooning coordinate, normalised to the peak value. We see that the $|\bar{\phi}|$ is indistinguishable between the cases of the electron models ME and KE – RM, whereas for realistic mass fully kinetic

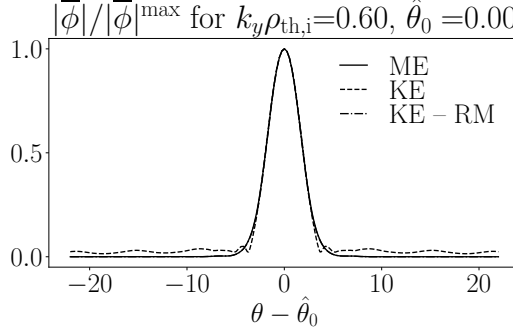


Figure B.2: The modulus of the ITG potential eigenmode $|\bar{\phi}|$ calculated using (3.2)-(3.8) for $k_y \rho_{th,i} = 0.6$ and $\theta_0 = 0.0$, the fastest growing IS mode. The eigenmode is normalised to the peak value, which is at $\theta = 0$. We show the eigenmode for different electron models: for the model of the electron response at IS with $(m_e/m_i)^{1/2} = 1/60$ (ME); for the fully kinetic electron model with $(m_e/m_i)^{1/2} = 1/60$ (KE); and for the fully kinetic electron model with $(m_e/m_i)^{1/2} = (1/60) \times 10^{-3}$ (KE - RM).

electrons (KE) we see an extended, but small, tail in $|\bar{\phi}|$. If sufficiently extended and large in amplitude, these tails could be responsible for breaking of the scale separation between IS and ES turbulence. In Figure B.3 (Left) we show the fluctuating nonadiabatic density of electrons $\bar{\delta n}'_e$ for each of the electron models, normalised to the corresponding peak value of $e|\bar{\phi}|/T$. We see that the extended tails in $|\bar{\phi}|$ are due to the electron nonadiabatic response [66]. In Figure B.3 (Right) we show the fluctuating nonadiabatic density of ions $\bar{\delta n}'_i$ for each of the electron models, normalised to the corresponding peak value of $e|\bar{\phi}|/T$.

In Figure B.4 (Left) we compare the IS linear growth rate γ calculated with the ME, KE, KE - RM and adiabatic electron (AE) treatment of electrons for CBC parameters. The linear simulations presented in Figure B.4 use time step sizes of $a\Delta t/v_{th,i} = 0.01/(k_y \rho_{th,i})$. Again note that the condition on the trapped electron bounce frequency ω_b for effective orbit averaging, relation (3.17), is only satisfied in the KE - RM case due to the artificially small value of $(m_e/m_i)^{1/2}$. There is qualitatively good agreement between the growth rates calculated using fully kinetic electrons (KE) and using the nonzonal, passing electron model (ME). We note that the growth rate calculated using the ME treatment of electrons is in excellent agreement

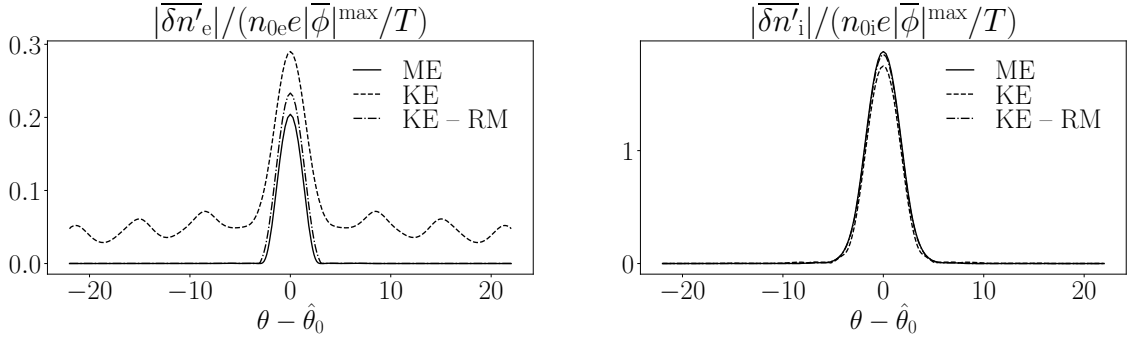


Figure B.3: The modulus of the ITG nonadiabatic density eigenmodes corresponding to the potential eigenmode $|\bar{\phi}|$ in Figure B.2. We show the fluctuating nonadiabatic densities due to electrons $\bar{\delta n}'_e$ (Left) and ions $\bar{\delta n}'_i$ (Right), normalised to the corresponding peak in $e|\bar{\phi}|/T$, and the equilibrium density n_ν .

with the growth rate calculated using fully kinetic electrons with a reduced mass of $(m_e/m_i)^{1/2} = 1/60 \times 10^{-3}$ (KE – RM). There is good agreement between the growth rate calculated using the AE and ME treatment of electrons for $k_y \rho_{th,i} < 0.7$. The discrepancy between the values of the growth rate for $k_y \rho_{th,i} > 0.7$ is due to the presence of the nonadiabatic trapped electron response retained in the ME case.

In Figure B.4 (Right) we compare the IS linear growth rate γ , as calculated using the ME, KE, KE – RM, and AE models for the electron response, for low electron collisionality $a\nu_{ee}/v_{th,e} = 0.01 \times 10^{-6}$, with all other parameters taken as CBC. We see that for low electron collisionality the AE model fails to capture the electron response; the growth rate calculated using the AE model for the electron response does not match the growth rate as calculated using the fully kinetic electron model (KE). In contrast, the growth rates calculated with both the KE – RM and ME models for the electron response agree well with the growth rate calculated using the full KE model. This is because it is predominantly the contribution of trapped electrons, whose nonadiabatic response is a leading order contribution to the IS potential, which contribute to the destabilisation of the ITG mode. For low electron collisionality, the trapped and passing electrons can respond differently. This is because transfer of particles between the trapped and passing pieces of velocity space is mediated by collisions. We recall that the model for the electron nonadiabatic response used for the IS simulations presented in this thesis (ME) takes the nonzonal passing electrons

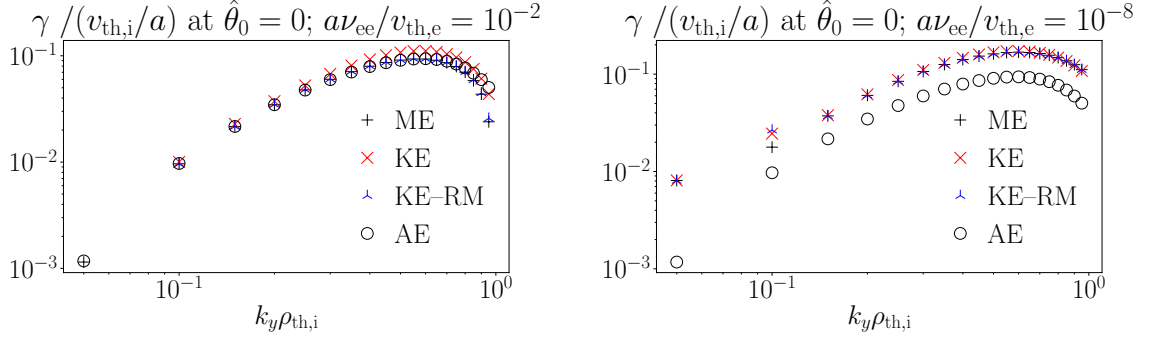


Figure B.4: The IS linear growth rate γ calculated using different models of the electron response at IS: firstly, the model used in this thesis, where the nonzonal passing electrons respond adiabatically and $(m_e/m_i)^{1/2} = 1/60$ (ME); the fully kinetic electron model with $(m_e/m_i)^{1/2} = 1/60$ (KE); and the fully kinetic electron model with $(m_e/m_i)^{1/2} = (1/60) \times 10^{-3}$ (KE – RM). (Left) The IS linear growth rate γ for CBC parameters. (Right) The IS linear growth rate for low electron collisionality, with all other parameters taking CBC values.

to respond adiabatically, whereas the trapped electrons respond nonadiabatically to the parallel orbit averaged potential. Therefore, we should expect that the ME model performs best for low electron collisionality.

In conclusion, we have shown that the model for the electron nonadiabatic response used in the IS simulations presented in this thesis correctly captures the $(m_e/m_i)^{1/2} \rightarrow 0$ limit for the behaviour of the electron nonadiabatic response at IS. By neglecting the nonadiabatic response due to nonzonal passing electrons, we avoid the formation of extended ballooning tails in the linear eigenmode structure of the ITG mode, and hence avoid the formation of sharp radial structures in the IS turbulence due to electron physics. The model for the electron nonadiabatic response accurately captures the effect of trapped electrons on ITG linear growth rates.

B.4 Resolving IS turbulence

The IS gradient $\nabla_s \bar{g}_e$ and drift $\bar{\mathbf{v}}_e^E = (c/B) \mathbf{b} \times \nabla_s \bar{\phi}$ appearing in the ES equation (3.11) must be calculated using turbulence simulated by a nonlinear gyrokinetic code. In order to be able to impose the correct ES parallel boundary condition on equation (3.11) then the IS distribution function for each species must exactly obey the IS real

space parallel boundary condition [44] (cf. Section 2.7)

$$\bar{g}_\nu(\theta, \psi, \alpha(\psi, \theta, \zeta)) = \bar{g}_\nu(\theta + 2\pi, \psi, \alpha(\psi, \theta + 2\pi, \zeta)), \quad (\text{B.1})$$

where we suppress velocity space dependence in $\bar{g}_\nu$ and we regard $\bar{g}_\nu$ as a function of the physical coordinates (θ, ψ, ζ) , with ζ is the toroidal angle. This requirement ensures that the IS gradients $\nabla_s \bar{g}_e$ and $\nabla_s \bar{\phi}$ have α and ψ components which satisfy the relations (2.98) and (2.99), as needed for the coefficients in the ES equation (3.11) to be continuous in the extended ballooning coordinate. The gyrokinetic equation is typically implemented using spectral methods, and hence the actual parallel boundary condition imposed is the spectral boundary condition [44]

$$\bar{g}_{\nu(k_\psi, k_\alpha)}(\theta) = \bar{g}_{\nu(k_\psi + 2\pi q'_0 k_\alpha, k_\alpha)}(\theta + 2\pi). \quad (\text{B.2})$$

Boundary conditions (B.1) and (B.2) are equivalent. In practice, only a finite set of k_ψ can be retained, and hence the boundary condition (B.1) is only satisfied if the fluctuation amplitude $\bar{g}_{\nu(k_\psi, k_\alpha)} \rightarrow 0$ for large $|k_\psi|$ on the k_ψ grid. Therefore, requiring that the IS turbulence exactly obeys the boundary condition (B.1) imposes a strict radial resolution requirement. This resolution requirement can be understood as the requirement that all extended ballooning chains of modes linked by the parallel boundary condition (B.2) must support modes whose amplitudes tend to zero at the ends of each ballooning chain.

The IS equations (3.2)-(3.8) capture the response of ions at IS and the response of trapped electrons and zonal, passing electrons. Note that the nonzonal, passing electrons can be assumed to respond adiabatically because of the asymptotically fast electron parallel streaming velocity in the limit that $(m_e/m_i)^{1/2} \rightarrow 0$, as discussed in Appendix B.3. As we take nonzonal, passing electrons to respond adiabatically, then the IS turbulence is free from fine radial structures generated by rapidly streaming electrons. However, radial structures can still develop in the turbulence due to passing ions. Compared to the bulk of the thermal ions, energetic ions from the tail of the distribution can stream rapidly along the field line, which by equation (B.2) transmits the ion response from the dominant instability at $k_\psi \approx 0$ to $k_\psi \neq 0$ modes. Sufficient resolution in k_ψ (or equivalently k_x) must be used to adequately capture the decay of

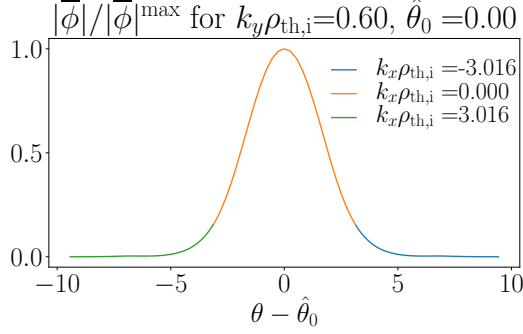


Figure B.5: The modulus of the ITG, potential eigenmode calculated using (3.2)-(3.8) for $k_y \rho_{th,i} = 0.6$ and $\theta_0 = 0.0$, the fastest growing IS mode. The eigenmode is normalised to the peak value, which is at $\theta = 0$. Note that the eigenmode extends beyond $\theta = \pm\pi$, and so multiple k_x cells are required to resolve the mode.

the amplitude of the linear ballooning modes. A typical ITG ballooning mode can be resolved with a range of $[-3\pi, 3\pi]$ in the extended ballooning coordinate; see Figure B.5.

The method used here for achieving an IS simulation where equation (B.1) is satisfied numerically to high precision is twofold: firstly, we require that the k_α (or equivalently k_y) modes with a growth rate close to the peak IS growth rate have at least 3 k_x connections, and so therefore cover a minimum range of $[-3\pi, 3\pi]$ in the extended ballooning coordinate; and secondly, we apply a hyperviscous model of grid scale dissipation to damp high k_y and high k_x modes, such that the amplitude of the ballooning modes resolved by the simulation smoothly tends to zero in the extended ballooning coordinate.

The set of modes retained within an IS simulation are

$$k_y = j_{ky} k_y^{\min}, \quad j_{ky} = [0, 1, \dots, n_y];$$

$$k_x = j_{kx} k_x^{\min}, \quad j_{kx} = \left[-\frac{n_x - 1}{2}, \dots, -1, 0, 1, \dots, \frac{n_x - 1}{2}\right], \quad (\text{B.3})$$

where we note that $j_{ky} < 0$ are eliminated by using that the fluctuations are real-valued fields. The parallel boundary condition requires that $J_t = 2\pi \hat{s} k_y^{\min} / k_x^{\min}$ is an integer [44], where $\hat{s} = r d \ln q / dr$ is the normalised magnetic shear. This is to ensure

that $k'_x = k_x + 2\pi\hat{s}k_y$ is always a member of the k_x grid. The integer $J_t j_{ky}$ is the spacing in j_{kx} between connected k_x modes for a given k_y [44], and hence $J_t j_{ky}$ is the number of ballooning chains at each k_y . At a given k_y , the number of k_x connections is bounded from below by $n_c = n_x // J_t j_{ky}$, where $//$ represents integer division. In order to perform simulations where $k_y^{\min} \simeq k_x^{\min}$, i.e. the simulation domain is square, for CBC parameters we take $J_t = 5$. We find that if we take $n_x = 255$, and hence $k_x^{\max} = 6.38$, then $n_c > 3$ for $k_y < 0.85$ ($j_{ky} < 17$). For this value of n_x almost all k_y modes within the IS simulations have more than 3 k_x connections, and hence for these k_y modes the amplitude of the ballooning mode can tend to zero within the grid of resolved k_x . We damp the remaining high k_y modes with a hyperviscous model which we now describe.

Hyperviscosity is a model of unresolved, sub-grid dissipation mechanisms [51–53, 85]. Hyperviscosity can be necessary to correctly resolve the turbulent cascade within a fluid or gyrokinetic simulation if the physical dissipation scale is barely resolved or not at all resolved. Here we use hyperviscosity to reduce the required k_x resolution necessary for the simulation to satisfy the condition (3.19). To illustrate the method of implementation of hyperviscosity used in this thesis, we write the gyrokinetic equation schematically as

$$\frac{\partial h}{\partial t} = \mathfrak{G}\mathfrak{K} * h + \mathfrak{D} * h, \quad (\text{B.4})$$

where $\mathfrak{G}\mathfrak{K}$ is an operator containing all the terms appearing in one of the IS gyrokinetic equations (3.2) or (3.5), and $\mathfrak{D}$ is an operator which represents the effect of hyperviscosity. Equation (B.4) admits a formal solution

$$h(t + \Delta t) = \exp[\Delta t(\mathfrak{G}\mathfrak{K} + \mathfrak{D})] * h(t). \quad (\text{B.5})$$

Using the technique of Gudonov operator splitting [55, 86] we approximate $\exp[\Delta t(\mathfrak{G}\mathfrak{K} + \mathfrak{D})]$ in (B.5) as $\exp[\Delta t\mathfrak{D}] \exp[\Delta t\mathfrak{G}\mathfrak{K}]$ with $\mathcal{O}([\mathfrak{G}\mathfrak{K}, \mathfrak{D}]\Delta t)$ error, where $[\mathfrak{G}\mathfrak{K}, \mathfrak{D}]$ is the commutator of $\mathfrak{G}\mathfrak{K}$ and $\mathfrak{D}$. This allows us to write the formal solution of equation (B.4) as

$$h(t + \Delta t) = \exp[\Delta t\mathfrak{D}] * h^*, \text{ with } h^* = \exp[\Delta t\mathfrak{G}\mathfrak{K}] * h(t). \quad (\text{B.6})$$

We take the hyperviscous operator to be

$$\mathfrak{D} = -D \left(\frac{k_{s,\perp}}{k_{c,\perp}} \right)^p, \quad (\text{B.7})$$

where $k_{s,\perp}(\theta) = |\mathbf{k}_s| = |k_\alpha \nabla \alpha + k_\psi \nabla \psi|$, D is the coefficient of hyperviscosity, p is the hyperviscous power and

$$k_{c,\perp} = \max \{ |k_{\alpha,c} \nabla \alpha(\theta) + k_{\psi,c} \nabla \psi(\theta)| \} \quad (\text{in } \theta), \quad (\text{B.8})$$

with $k_{\alpha,c}$ and $k_{\psi,c}$ the cut-off wavenumbers in the α and ψ directions, respectively. The hyperviscosity (B.7) damps modes of high perpendicular wave number $k_{s,\perp}$, keeping track of geometrical factors which cause $k_{s,\perp}$ to vary with θ . The cut-off wave number $k_{c,\perp}$ is the maximum $k_{s,\perp}(\theta)$ of the mode with $k_\alpha = k_{\alpha,c}$ and $k_\psi = k_{\psi,c}$. Finally, we can apply hyperviscosity as a filter: we first find h^* by integrating equation (B.4), with $\mathfrak{D} = 0$ and a time step of Δt . We then use the form of the hyperviscous operator (B.7) to find

$$h(t + \Delta t) = \exp[-D \Delta t (k_{s,\perp}/k_{c,\perp})^p] h^*. \quad (\text{B.9})$$

A discussion of the use of operator splitting in **GS2** is given in [55].

In the IS simulations we take $Da/v_{\text{th},i} = 0.05$, $p = 4$, and we take the normalised binormal and radial cut-off wave numbers to be $k_{y,c}\rho_{\text{th},i} = 1.05$ and $k_{x,c}\rho_{\text{th},i} = 3.17$, respectively. Note that $k_y\rho_{\text{th},i} = 1.05$ is the maximum k_y resolved. However, we take $k_{x,c}\rho_{\text{th},i} = 3.17 < k_x^{\text{max}}\rho_{\text{th},i} = 6.38$ such that any large k_x modes which might have non-negligible amplitudes are damped, even for low k_y . We do this to improve the numerical satisfaction of the parallel boundary condition.

We note in Figure B.6 that the heat fluxes $\overline{Q}_\nu$ of the turbulence are the same to 5% between simulations with $k_x^{\text{max}}\rho_{\text{th},i} = 3.17$ and $k_x^{\text{max}}\rho_{\text{th},i} = 6.38$, holding all other physical parameters, and modelling parameters such as hyperviscosity, fixed. Furthermore, in IS simulations with identical maximum radial wavenumber $k_x^{\text{max}}\rho_{\text{th},i} = 6.38$ but with differing values for the hyperviscous cut-off parameter $k_{x,c}$, $k_{x,c}\rho_{\text{th},i} = 3.17$ and $k_{x,c}\rho_{\text{th},i} = 6.3$, we find that the time averaged turbulent heat fluxes $\overline{Q}_\nu$ agree to within 3%. Therefore, we can be confident that the choice of hyperviscosity did not change the character of the turbulence.

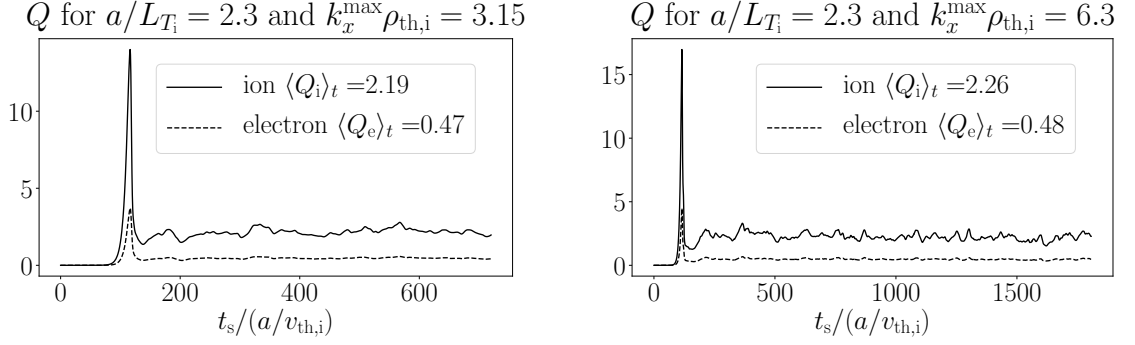


Figure B.6: The flux tube averaged turbulent IS heat flux calculated for CBC parameters in a simulation with $k_x^{\max} \rho_{\text{th},i} = 3.17$ (Left) and $k_x^{\max} \rho_{\text{th},i} = 6.3$ (Right). The simulation parameters are otherwise identical. Note the differing t_s axes between the left and right figures.

To compare the effect of using $k_x^{\max} \rho_{\text{th},i} = 6.38$ in the IS simulations, as opposed to $k_x^{\max} \rho_{\text{th},i} = 3.17$, in Figure B.7 we show the resolved part of the ballooning chain with $k_y \rho_{\text{th},i} = 0.8$ and $\hat{\theta}_0 = 0.0$ for both the $k_x^{\max} \rho_{\text{th},i} = 6.38$ case and the $k_x^{\max} \rho_{\text{th},i} = 3.17$ case. Note that $\hat{\theta}_0$ is defined to be the $\theta_0 = k_x/k_y \hat{s}$ in the connected chain of k_x modes that is closest to zero. Note that in the case with $k_x^{\max} \rho_{\text{th},i} = 3.17$, Figure B.7 (Left), the mode does not fully decay to zero amplitude at the parallel boundaries, whereas in the case with $k_x^{\max} \rho_{\text{th},i} = 6.3$, Figure B.7 (Right), the mode is able to decay to zero amplitude at the parallel boundaries. The fact that there are insufficient k_x connections in the case with $k_x^{\max} \rho_{\text{th},i} = 3.17$ means that the IS drift $\bar{\mathbf{v}}_e^E$ forms a coefficient in equation (3.11) which is not continuous in the parallel coordinate – we note that the model of the electron response at IS (Section B.3) ensures that IS gradient $\nabla_s \bar{g}_e$ forms a continuous ES coefficient. This observation results in two concerns for IS-ES calculations where the IS simulations have insufficient k_x resolution to exactly satisfy the boundary condition (B.1): the issue of whether or not IS structures are correctly resolved in the IS flux tube simulation; and the issue of the sensitivity of the numerical algorithm which solves the ES equation (3.11) to discontinuities in the term involving $\bar{\mathbf{v}}_e^E$ at $\theta = \pm\pi$.

To measure the average size of the discontinuity we introduce measures of the satisfaction of the parallel boundary conditions on the α and ψ components of $\nabla_s \bar{\phi}$,

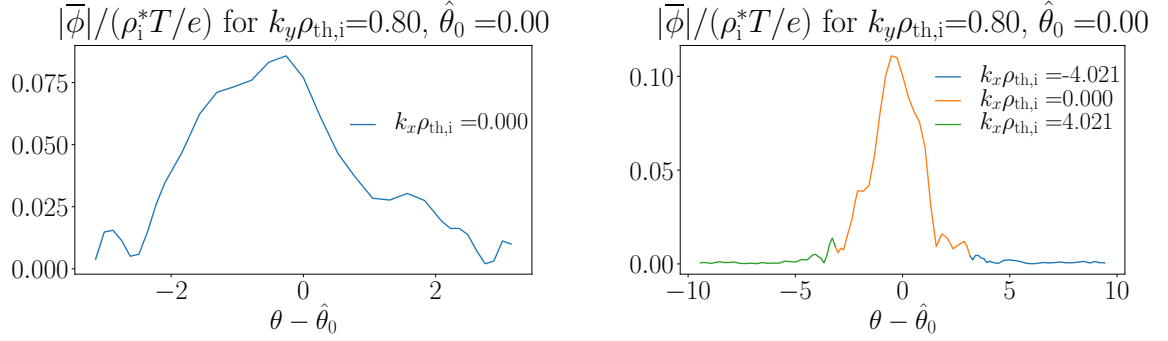


Figure B.7: A snapshot of the modulus of the electrostatic potential $\bar{\phi}$, for $k_y \rho_{th,i} = 0.8$ and $\hat{\theta}_0 = 0.0$, from a nonlinear IS simulation with CBC parameters and $k_x^{\max} \rho_{th,i} = 3.17$ (Left) and $k_x^{\max} \rho_{th,i} = 6.38$ (Right). The simulation parameters are otherwise identical. Agreement between the left and right figure is not expected as in each instance $|\bar{\phi}|$ is calculated from a different snapshot of a different simulation of turbulence.

equations (2.98) and (2.99). We write these measures in terms of y_s and x_s , cf. equations (3.21) and (3.22). To measure the continuity of $\partial \bar{\phi} / \partial y_s$ we define

$$\sigma_y = \frac{\sum_{x_s, y_s} |\partial \bar{\phi} / \partial y_s(-\pi, y_s) - \partial \bar{\phi} / \partial y_s(\pi, y_s + \delta y_s)|}{\sum_{x_s, y_s} |\partial \bar{\phi} / \partial y_s(-\pi, y_s)|}, \quad (\text{B.10})$$

with $\delta y_s = -2\pi \hat{s} x_s$ and x_s dependences suppressed. To measure the continuity of $\partial \bar{\phi} / \partial x_s$ we define

$$\sigma_x = \frac{\sum_{x_s, y_s} |\partial \bar{\phi} / \partial x_s(-\pi, y_s) - \partial \bar{\phi} / \partial x_s(\pi, y_s + \delta y_s) + 2\pi \hat{s} \partial \bar{\phi} / \partial y_s(\pi, y_s + \delta y_s)|}{\sum_{x_s, y_s} |\partial \bar{\phi} / \partial x_s(-\pi, y_s)|}. \quad (\text{B.11})$$

In Figure B.8 we show that $\mathbf{k}_f \cdot \bar{\mathbf{v}}_e^E$ is discontinuous at K_x cell boundaries when $\bar{\mathbf{v}}_e^E$ is calculated from an IS simulation with $k_x^{\max} \rho_{th,i} = 3.17$, and that these discontinuities become much smaller when we calculate $\bar{\mathbf{v}}_e^E$ from an IS simulation with $k_x^{\max} \rho_{th,i} = 6.38$. To confirm this, we use the error measures σ_y and σ_x . We find that for the case with $k_x^{\max} \rho_{th,i} = 3.17$ that $\sigma_y \simeq \sigma_x \simeq 0.4$, whereas, in the case with $k_x^{\max} \rho_{th,i} = 6.38$, we find that $2\sigma_y \simeq \sigma_x \simeq 0.06$. This indicates that increased resolution at fixed hyperviscosity did indeed improve the satisfaction of the real space parallel boundary conditions (2.98) and (2.99) (equivalently (3.21) and (3.22)).

To test the sensitivity of the ES simulation results to the errors in the satisfaction of (3.21) and (3.22), we recomputed the ES growth rate using IS drifts where the

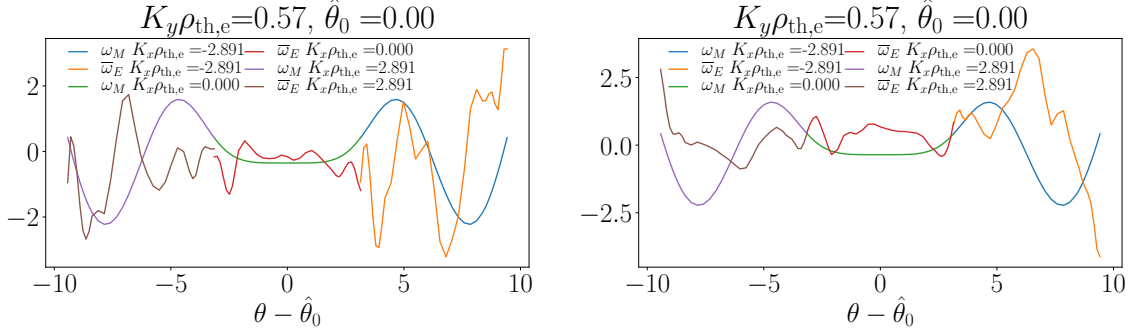


Figure B.8: Instances of the cross-scale drift frequency $\bar{\omega}_E = \mathbf{k}_f \cdot \bar{\mathbf{v}}_e^E$ as a function of the extended ballooning coordinate, for cases where $\bar{\mathbf{v}}_e^E$ is calculated from an IS simulation with $k_x^{\max} \rho_{\text{th},i} = 3.17$ (Left) and $k_x^{\max} \rho_{\text{th},i} = 6.38$ (Right). Note that $\omega_M = \mathbf{k}_f \cdot \mathbf{v}_e^M$ and note that we show the data in the electron normalised units. We show the $\bar{\omega}_E$ and ω_M coefficients for each K_x in the connected chain of ES modes separately to illustrate the numerical discontinuity which we resolve. We note that the mode with $K_y \rho_{\text{th},e} = 0.57$ and $\hat{\theta}_0 = 0$ is the fastest growing ES mode.

errors σ_y and σ_x had been artificially reduced using two separate methods. In the first method the values of $\partial \bar{\phi} / \partial y_s$ and $\partial \bar{\phi} / \partial x_s$ are averaged at $\theta = \pm \pi$ such that $\sigma_y = \sigma_x = 0$ identically. We find that the result of the ES growth rate calculation differs by less than 1% as a result of this change. In the second method, the gradients $\partial \bar{\phi} / \partial y_s$ and $\partial \bar{\phi} / \partial x_s$ are artificially smoothed with a spectral filter which smoothly reduces the errors σ_y and σ_x . As this filter is spectral, it changes the value of $\partial \bar{\phi} / \partial y_s$ and $\partial \bar{\phi} / \partial x_s$ everywhere. We find that we are able to reduce σ_y and σ_x by a factor of 20 in the $k_x^{\max} \rho_{\text{th},i} = 6.38$ case before the ES growth rates change by more than 4%. We conclude that the ES simulations use converged IS gradients.

We note that if fully kinetic electrons are retained in an IS simulation, rather than using the reduced model for electrons consistent with $(m_e/m_i)^{1/2} \rightarrow 0$ – equation (3.5), then the IS parallel boundary condition (B.1) is violated by the piece of the electron distribution function corresponding to energetic passing electrons. This violation of the parallel boundary condition led to $O(1)$ discontinuities in the coefficient $[i(c/B) \mathbf{b} \times \mathbf{k}_f \cdot \nabla_s \bar{g}_e] J_0(|\mathbf{k}_f| |\boldsymbol{\rho}_e|)$ appearing in (3.11) which could not be resolved with affordable and IS k_x resolution or hyperviscosity. This observation motivates careful treatment of electrons at IS.

References

- [1] A. H. Boozer. Physics of magnetically confined plasmas. *Rev. Mod. Phys.*, 76: 1071–1141, 2005.
- [2] R. D. Hazeltine and J. D. Meiss. *Plasma Confinement*. Dover, New York, 2003.
- [3] ITER Physics Basis Editors, ITER Physics Expert Group Chairs and Co-Chairs, ITER Joint Central Team, and Physics Unit. Chapter 1: Overview and summary. *Nuclear Fusion*, 39:2137–2174, 1999.
- [4] J. D. Strachan et al. Fusion power production from TFTR plasmas fueled with deuterium and tritium. *Phys. Rev. Lett.*, 72:3526–3529, 1994.
- [5] A. Gibson. Deuterium-tritium plasmas in the Joint European Torus (JET): Behavior and implications. *Physics of Plasmas*, 5:1839–1847, 1998.
- [6] M. Shimada et al. Chapter 1: Overview and summary. *Nuclear Fusion*, 47:S1–S17, 2007.
- [7] M. R. Hardman, M. Barnes, C. M. Roach, and F. I. Parra. A scale-separated approach for studying coupled ion and electron scale turbulence. *Plasma Physics and Controlled Fusion*, 61:065025, 2019.
- [8] M. R. Hardman, M. Barnes, C. M. Roach, and F. I. Parra. Suppression of the electron temperature gradient instability via cross-scale interactions with ion gyroradius scale turbulence. *In Preparation*, 2019.
- [9] H. Sugama and W. Horton. Transport processes and entropy production in toroidally rotating plasmas with electrostatic turbulence. *Phys. Plasmas*, 4:405, 1997.

- [10] H. Sugama and W. Horton. Nonlinear electromagnetic gyrokinetic equation for plasmas with large mean flows. *Physics of Plasmas*, 5:2560–2573, 1998.
- [11] P. J. Catto, A. N. Simakov, F. I. Parra, and G. Kagan. Electrostatic turbulence in tokamaks on transport time scales. *Plasma Physics and Controlled Fusion*, 50:115006, 2008.
- [12] I. G. Abel, G. G. Plunk, E. Wang, M. Barnes, S. C. Cowley, W. Dorland, and A. A. Schekochihin. Multiscale gyrokinetics for rotating tokamak plasmas: fluctuations, transport, and energy flows. *Reports on Progress in Physics*, 76:116201, 2013.
- [13] J. Candy, C. Holland, R. E. Waltz, M. R. Fahey, and E. Belli. Tokamak profile prediction using direct gyrokinetic and neoclassical simulation. *Phys. Plasmas*, 16:060704, 2009.
- [14] M. Barnes, I. G. Abel, W. Dorland, T. Goerler, G. W. Hammett, and F. Jenko. Direct multiscale coupling of a transport code to gyrokinetic turbulence codes. *Phys. Plasmas*, 17:056109, 2010. arxiv:0912.1974.
- [15] I. Calvo and F. I. Parra. Long-wavelength limit of gyrokinetics in a turbulent tokamak and its intrinsic ambipolarity. *Plasma Physics and Controlled Fusion*, 54:115007, 2012.
- [16] F. Romanelli. Ion temperature-gradient-driven modes and anomalous ion transport in tokamaks. *Physics of Fluids B: Plasma Physics*, 1:1018–1025, 1989.
- [17] S. C. Cowley, R. M. Kulsrud, and R. Sudan. Considerations of ion-temperature-gradient-driven turbulence. *Phys. Fluids B*, 3:2767, 1991.
- [18] Y. C. Lee, J. Q. Dong, P. N. Guzdar, and C. S. Liu. Collisionless electron temperature gradient instability. *The Physics of Fluids*, 30:1331–1339, 1987.
- [19] W. Horton, B. G. Hong, and W. M. Tang. Toroidal electron temperature gradient driven drift modes. *The Physics of Fluids*, 31:2971–2983, 1988.

- [20] C. M. Surko and R. E. Slusher. Waves and turbulence in a tokamak fusion plasma. *Science*, 221:817, 1983.
- [21] W. Horton. Drift waves and transport. *Rev. Mod. Phys.*, 71:735–778, 1999.
- [22] F. Jenko and W. Dorland. Prediction of significant tokamak turbulence at electron gyroradius scales. *Phys. Rev. Lett.*, 89:225001, 2002.
- [23] Y. Ren, E. Belova, N. Gorelenkov, W. Guttenfelder, S. M. Kaye, E. Mazzucato, J. L. Peterson, D. R. Smith, D. Stutman, K. Tritz, W. X. Wang, H. Yuh, R. E. Bell, C. W. Domier, and B. P. LeBlanc. Recent progress in understanding electron thermal transport in NSTX. *Nuclear Fusion*, 57:072002, 2017.
- [24] C. M. Roach, I. G. Abel, R. J. Akers, W. Arter, M. Barnes, Y. Camenen, F. J. Casson, G. Colyer, J. W. Connor, S. C. Cowley, D. Dickinson, W. Dorland, A. R. Field, W. Guttenfelder, G. W. Hammett, R. J. Hastie, E. Highcock, N. F. Loureiro, A. G. Peeters, M. Reshko, S. Saarelma, A. A. Schekochihin, M. Valovic, and H. R. Wilson. Gyrokinetic simulations of spherical tokamaks. *Plasma Physics and Controlled Fusion*, 51:124020, 2009.
- [25] G. J. Colyer, A. A. Schekochihin, F. I. Parra, C. M. Roach, M. A. Barnes, Y. c. Ghim, and W. Dorland. Collisionality scaling of the electron heat flux in ETG turbulence. *Plasma Physics and Controlled Fusion*, 59:055002, 2017.
- [26] W. Dorland, F. Jenko, M. Kotschenreuther, and B. N. Rogers. Electron temperature gradient turbulence. *Phys. Rev. Lett.*, 85:5579–5582, 2000.
- [27] F. Jenko, W. Dorland, M. Kotschenreuther, and B. N. Rogers. Electron temperature gradient driven turbulence. *Physics of Plasmas*, 7:1904–1910, 2000.
- [28] M. Nakata, T.-H. Watanabe, H. Sugama, and W. Horton. Formation of coherent vortex streets and transport reduction in electron temperature gradient driven turbulence. *Physics of Plasmas*, 17:042306, 2010.
- [29] W. Guttenfelder and J. Candy. Resolving electron scale turbulence in spherical tokamaks with flow shear. *Physics of Plasmas*, 18:022506, 2011.

- [30] W. Guttenfelder, J. L. Peterson, J. Candy, S. M. Kaye, Y. Ren, R. E. Bell, G. W. Hammett, B. P. LeBlanc, D. R. Mikkelsen, W. M. Nevins, and H. Yuh. Progress in simulating turbulent electron thermal transport in NSTX. *Nuclear Fusion*, 53:093022, 2013.
- [31] R. E. Waltz, J. Candy, and M. Fahey. Coupled ion temperature gradient and trapped electron mode to electron temperature gradient mode gyrokinetic simulations a. *Physics of plasmas*, 14:056116, 2007.
- [32] J. Candy, R. E. Waltz, M. R. Fahey, and C. Holland. The effect of ion-scale dynamics on electron-temperature-gradient turbulence. *Plasma Physics and Controlled Fusion*, 49:1209, 2007.
- [33] T. Görler and F. Jenko. Scale separation between electron and ion thermal transport. *Physical review letters*, 100:185002, 2008.
- [34] N. T. Howard, C. Holland, A. E. White, M. Greenwald, and J. Candy. Synergistic cross-scale coupling of turbulence in a tokamak plasma. *Physics of Plasmas*, 21:112510, 2014.
- [35] N. T. Howard, C. Holland, A. E. White, M. Greenwald, and J. Candy. Fidelity of reduced and realistic electron mass ratio multi-scale gyrokinetic simulations of tokamak discharges. *Plasma Physics and Controlled Fusion*, 57:065009, 2015.
- [36] N. T. Howard, C. Holland, A. E. White, M. Greenwald, and J. Candy. Multi-scale gyrokinetic simulation of tokamak plasmas: Enhanced heat loss due to cross-scale coupling of plasma turbulence. *Nuclear Fusion*, 56:014004, 2016.
- [37] N. T. Howard, C. Holland, A. E. White, M. Greenwald, J. Candy, and A. J. Creely. Multi-scale gyrokinetic simulations: Comparison with experiment and implications for predicting turbulence and transport. *Physics of Plasmas*, 23:056109, 2016.

- [38] N. Bonanomi, P. Mantica, J. Citrin, T. Goerler, and B. Teaca and. Impact of electron-scale turbulence and multi-scale interactions in the JET tokamak. *Nuclear Fusion*, 58:124003, 2018.
- [39] S. Maeyama, Y. Idomura, T.-H. Watanabe, M. Nakata, M. Yagi, N. Miyato, A. Ishizawa, and M. Nunami. Cross-scale interactions between electron and ion scale turbulence in a tokamak plasma. *Physical review letters*, 114:255002, 2015.
- [40] S. Maeyama, T.-H. Watanabe, and A. Ishizawa. Suppression of ion-scale microtearing modes by electron-scale turbulence via cross-scale nonlinear interactions in tokamak plasmas. *Phys. Rev. Lett.*, 119:195002, 2017.
- [41] S. Maeyama, T.-H. Watanabe, Y. Idomura, M. Nakata, A. Ishizawa, and M. Nunami. Cross-scale interactions between turbulence driven by electron and ion temperature gradients via sub-ion-scale structures. *Nuclear Fusion*, 57:066036, 2017.
- [42] N. T. Howard, A. E. White, M. Greenwald, C. Holland, and J. Candy. Multi-scale gyrokinetic simulation of Alcator C-Mod tokamak discharges. *Physics of Plasmas*, 21:032308, 2014.
- [43] A. M. Dimits, G. Bateman, M. A. Beer, B. I. Cohen, W. Dorland, G. W. Hammett, C. Kim, J. E. Kinsey, M. Kotschenreuther, A. H. Kritz, L. L. Lao, J. Mandrekas, W. M. Nevins, S. E. Parker, A. J. Redd, D. E. Shumaker, R. Sydora, and J. Weiland. Comparisons and physics basis of tokamak transport models and turbulence simulations. *Physics of Plasmas*, 7:969–983, 2000.
- [44] M. A. Beer, S. C. Cowley, and G. W. Hammett. Field-aligned coordinates for nonlinear simulations of tokamak turbulence. *Phys. Plasmas*, 2:7, 1995.
- [45] P. J. Catto. Linearized gyro-kinetics. *Plasma Phys.*, 20:719, 1978.
- [46] D. A. Hitchcock and R. D. Hazeltine. A gyrokinetic equation for low and high frequency problems. *Plasma Physics*, 20:1241–1252, 1978.

- [47] E. A. Frieman and L. Chen. Nonlinear gyrokinetic equations for low-frequency electromagnetic waves in general plasma equilibria. *Phys. Fluids*, 25:502, 1982.
- [48] W. W. Lee. Gyrokinetic approach in particle simulation. *The Physics of Fluids*, 26:556–562, 1983.
- [49] A. J. Brizard and T. S. Hahm. Foundations of nonlinear gyrokinetic theory. *Rev. Mod. Phys.*, 79:421, 2007.
- [50] F. I. Parra and P. J. Catto. Limitations of gyrokinetics on transport time scales. *Plasma Phys. Control. Fusion*, 50:065014, 2008.
- [51] M. Germano. Turbulence: the filtering approach. *Journal of Fluid Mechanics*, 238:325336, 1992.
- [52] L. C. Berselli, T. Iliescu, and W. J. Layton. *Mathematics of Large Eddy Simulation of Turbulent Flows, First Edition*. Springer, 2005.
- [53] E. Garnier, N. Adams, and P. Sagaut. *Large Eddy Simulation for Compressible Flows*. Springer, 2009.
- [54] I. G. Abel, M. Barnes, S. C. Cowley, W. Dorland, G. W. Hammett, and A. A. Schekochihin. Linearised model fokker-planck collision operators for gyrokinetic simulations. I. Theory. *Phys. Plasmas*, 15:122509, 2008. arXiv:0806.1069.
- [55] M. Barnes, I. G. Abel, W. Dorland, D. R. Ernst, G. W. Hammett, P. Ricci, B. N. Rogers, A. A. Schekochihin, and T. Tatsuno. Linearized model fokker-planck collision operators for gyrokinetic simulations. II. Numerical implementation and tests. *Phys. Plasmas*, 16:072107, 2009.
- [56] N. G. Van Kampen. *Physica (Amsterdam)*, 21:949, 1955.
- [57] K. M. Case. *Ann. Phys.*, 7:349, 1959.
- [58] J. A. Krommes and G. Hu. The role of dissipation in the theory and simulations of homogeneous plasma turbulence, and resolution of the entropy paradox. *Phys. Plasmas*, 1:3211, 1994.

- [59] J. A. Krommes. Thermostatted δf . *Phys. Plasmas*, 6:1477, 1999.
- [60] A. A. Schekochihin, S. C. Cowley, W. Dorland, G. W. Hammett, G. G. Howes, G. G. Plunk, E. Quataert, and T. Tatsuno. *Plasma Phys. Control. Fusion*, 50: 124024, 2008. arXiv: 0806.1069.
- [61] A. A. Schekochihin, S. C. Cowley, W. Dorland, G. W. Hammett, G. G. Howes, E. Quataert, and T. Tatsuno. Astrophysical gyrokinetics: Kinetic and fluid turbulent cascades in magnetized weakly collisional plasmas. *Astrophys. J. Suppl*, 182: 310, 2009. arXiv: 0704.0044.
- [62] T. Tatsuno, W. Dorland, A. A. Schekochihin, G. G. Plunk, M. Barnes, S. C. Cowley, and G. G. Howes. Nonlinear phase mixing and phase-space cascade of entropy in gyrokinetic plasma turbulence. *Phys. Rev. Lett.*, 103:015003, 2009.
- [63] M. Barnes, W. Dorland, and T. Tatsuno. Velocity space resolution in gyrokinetic simulations. *Phys. Plasmas*, 17:032106, 2010.
- [64] C. M. Bender and S. A. Orszag. *Advanced Mathematical Methods for Scientists and Engineers I: Asymptotic Methods and Perturbation Theory*. Springer, New York, 1999.
- [65] K. Hallatschek and W. Dorland. Giant electron tails and passing electron pinch effects in tokamak-core turbulence. *Phys. Rev. Lett.*, 95:055002, 2005.
- [66] J. Dominski, S. Brunner, T. Görler, F. Jenko, D. Told, and L. Villard. How non-adiabatic passing electron layers of linear microinstabilities affect turbulent transport. *Physics of Plasmas*, 22:062303, 2015.
- [67] M. A. Beer and G. W. Hammett. Bounce averaged trapped electron fluid equations for plasma turbulence. *Physics of Plasmas*, 3:4018–4022, 1996.
- [68] I. G. Abel and S. C. Cowley. Multiscale gyrokinetics for rotating tokamak plasmas: II. Reduced models for electron dynamics. *New Journal of Physics*, 15:023041, 2013.

- [69] F. L. Hinton and R. D. Hazeltine. Theory of plasma transport in toroidal confinement systems. *Rev. Mod. Phys.*, 25:239–308, 1976.
- [70] P. Helander and D. J. Sigmar. *Collisional transport in magnetized plasmas*. Cambridge University Press, Cambridge, 2002.
- [71] M. Barnes, F. I. Parra, and A. A. Schekochihin. Critically balanced ion temperature gradient turbulence in fusion plasmas. *Phys. Rev. Lett.*, 107:115003, 2011. arxiv:1104.4514.
- [72] T.-H. Watanabe, H. Sugama, A. Ishizawa, and M. Nunami. Flux tube train model for local turbulence simulation of toroidal plasmas. *Physics of Plasmas*, 22:022507, 2015.
- [73] M. Kotschenreuther, G. Rewoldt, and W.M. Tang. Comparison of initial value and eigenvalue codes for kinetic toroidal plasma instabilities. *Computer Physics Communications*, 88:128 – 140, 1995.
- [74] J. W. Connor, R. J. Hastie, and J. B. Taylor. High mode number stability of an axisymmetric toroidal plasma. *Proceedings of the Royal Society of London. A. Mathematical and Physical Sciences*, 365:1–17, 1979.
- [75] S. L. Newton, S. C. Cowley, and N. F. Loureiro. Understanding the effect of sheared flow on microinstabilities. *Plasma Physics and Controlled Fusion*, 52:125001, 2010.
- [76] M. Barnes, F. I. Parra, E. G. Highcock, A. A. Schekochihin, S. C. Cowley, and C. M. Roach. Turbulent transport in tokamak plasmas with rotational shear. *Phys. Rev. Lett.*, 106:175004, 2011.
- [77] F. van Wyk, E. G. Highcock, A. R. Field, C. M. Roach, A. A. Schekochihin, F. I. Parra, and W. Dorland. Ion-scale turbulence in MAST: Anomalous transport, subcritical transitions, and comparison to BES measurements. *Plasma Physics and Controlled Fusion*, 59:114003, 2017.

- [78] J. F. Drake, N. T. Gladd, C. S. Liu, and C. L. Chang. Microtearing modes and anomalous transport in tokamaks. *Phys. Rev. Lett.*, 44:994–997, 1980.
- [79] F. Jenko and W. Dorland. Nonlinear electromagnetic gyrokinetic simulations of tokamak plasmas. *Plasma Physics and Controlled Fusion*, 43:A141–A150, 2001.
- [80] P. B. Snyder and G. W. Hammett. Electromagnetic effects on plasma microturbulence and transport. *Physics of Plasmas*, 8:744–749, 2001.
- [81] C. Bourdelle, W. Dorland, X. Garbet, G. W. Hammett, M. Kotschenreuther, G. Rewoldt, and E. J. Synakowski. Stabilizing impact of high gradient of ω on microturbulence. *Physics of Plasmas*, 10:2881–2887, 2003.
- [82] W. M. Tang, J. W. Connor, and R. J. Hastie. Kinetic-ballooning-mode theory in general geometry. *Nuclear Fusion*, 20:1439–1453, 1980.
- [83] T. Tatsuno and W. Dorland. Magneto-flow instability in symmetric field profiles. *Physics of Plasmas*, 13:092107, 2006.
- [84] F. B. Hildebrand. *Introduction to Numerical Analysis, Second edition*. Dover, New York, 1987.
- [85] S. A. Smith and G. W. Hammett. Eddy viscosity and hyperviscosity in spectral simulations of 2D drift wave turbulence. *Physics of Plasmas*, 4:978–990, 1997.
- [86] S. K. Godunov. Finite difference methods for numerical computation of discontinuous solutions of the equations of fluid dynamics. *Mat. Sbornik*, 47:271–295, 1959.