

The Density of Zeros of Forms for which Weak Approximation Fails

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Abstract

The weak approximation principle fails for the forms $x^3 + y^3 + z^3 = kw^3$, when $k = 2$ or 3 . The question therefore arises as to what asymptotic density one should predict for the rational zeros of these forms. Evidence, both numerical and theoretical, is presented, which suggests that, for forms of the above type, the product of the local densities still gives the correct global density.

Let $f(x_1, \dots, x_n) \in \mathbb{Q}[x_1, \dots, x_n]$ be a rational form. We say that f satisfies the weak approximation principle if the following condition holds.

(WA): *Given an $\varepsilon > 0$ and a finite set S of places of \mathbb{Q} , and zeros*

$$(x_1^\nu, \dots, x_n^\nu) \in \mathbb{Q}_\nu^n$$

of the form f , we can find a rational zero (x_1, \dots, x_n) of f such that,

$$|x_i - x_i^\nu|_\nu < \varepsilon, \text{ for } 1 \leq i \leq n \text{ and } \nu \in S.$$

Alternatively, we may write $X(K)$ for the points on the hypersurface $f = 0$ whose coordinates lie in the field K , and consider the product

$$\Pi_S = \prod_{\nu \in S} X(\mathbb{Q}_\nu),$$

with the product topology. Then (WA) says that for any finite set S of places, the image of $X(\mathbb{Q})$ in Π_S , under the diagonal embedding, is dense. Thus, for example, non-singular quadratic forms satisfy (WA) when $n \geq 3$. However the cubic forms $x^3 + y^3 + z^3 - kw^3$ ($k = 2, 3$) do not satisfy (WA), as we shall show.

Conjectures about the density of zeros of f are often made by calculating the product of all the local densities. These conjectures can sometimes be proved

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using the Hardy-Littlewood circle method, and the product of densities one obtains is just the singular series multiplied by the singular integral. However such conjectures must seem highly dubious when (WA) fails. For then one knows that there are subsets of Π_S where there are a positive density of points, but no rational points. Two possible reformulations of the conjecture on the global density seem plausible. Either one may make no adjustment, in the expectation that the density of rational points in the admissible part of Π_S is correspondingly increased; or one may reduce the product of local densities by removing those parts of Π_S which the failure of (WA) excludes. When the Hasse Principle fails for f we must clearly follow the second procedure. This would be the case for the example $5x^3 + 12y^3 + 9z^3 + 10w^3$ of Cassels and Guy [2], for instance.

The purpose of this paper is to investigate further the above options, principally through an examination of the forms $x^3 + y^3 + z^3 - kw^3$. We first prove:-

Theorem 1 *If $(x, y, z, w) \in \mathbb{Z}^4$ is a primitive solution of $x^3 + y^3 + z^3 = 2w^3$ (so that $\text{hcf}(x, y, z, w) = 1$), then one of x, y, z is divisible by 6. If $(x, y, z, w) \in \mathbb{Z}^4$ is a primitive solution of $x^3 + y^3 + z^3 = 3w^3$, then either $x \equiv y \equiv z \pmod{9}$ or one of x, y, z is divisible by 9.*

This demonstrates the failure of (WA) for the above forms. The first form can have no rational zero close to both $(0, 1, 1, 1) \in \mathbb{Q}_2^4$ and $(1, 0, 1, 1) \in \mathbb{Q}_3^4$. Similarly, the second form has no rational zero close to $(-2, 1, 4, \sqrt[3]{19}) \in \mathbb{Q}_3^4$.

In a letter to the author, Professor Colliot-Thélène has shown that the above congruence restrictions are exactly those implied by the Brauer-Manin obstruction. Moreover, for the general equation $x^3 + y^3 + z^3 = kw^3$, with a non-cube integer k , there is always a non-trivial obstruction, eliminating two-thirds of the adèle points.

The proof of the theorem is based on a method of Cassels [1], and uses cubic reciprocity in $\mathbb{Q}(\omega)$, where $\omega = \frac{-1+\sqrt{-3}}{2}$. Let $k = 2$ or 3 as appropriate and suppose that $x + y\omega$ has a prime factor π in $\mathbb{Q}(\omega)$. We write p for the rational prime above π , and we suppose that $p \nmid 3k$. If p is inert then

$$\left(\frac{k}{\pi}\right)_3 = 1 \tag{1}$$

automatically, and otherwise the congruence $z^3 \equiv kw^3 \pmod{\pi}$ implies (1) unless $p \mid \text{hcf}(z, w)$. In the latter case, suppose that $\pi^e \parallel x + y\omega$. Since x, y, z and w are coprime we have $p \nmid \text{hcf}(x, y)$, so that $\pi \nmid x + y, x + y\omega^2$. Thus $\pi^e \parallel x^3 + y^3$, whence $\pi^e \parallel z^3 - kw^3$. Hence either (1) holds or $3 \mid e$. We may therefore conclude that

$$\left(\frac{k}{\pi^e}\right)_3 = 1 \tag{2}$$

whenever $\pi^e \parallel x + y\omega$ and $\pi \nmid 3k$.

If $k = 2$ then exactly one of x, y, z will be even, so that $2 \nmid x + y\omega$. Moreover a congruence (mod 9) shows that 3 divides exactly one of x, y, z (y say). Then $\text{hcf}(x + y\omega, 3) = 1$ and (2) holds for all prime factors of $x + y\omega$. It follows that

$$\left(\frac{2}{x + y\omega}\right)_3 = 1$$

on multiplying up the various relations (2). The law of cubic reciprocity now yields

$$\left(\frac{x + y\omega}{2}\right)_3 = 1,$$

since $3 \mid y$. This implies that y must be even and the theorem, for this case, follows.

If $k = 3$ then a congruence (mod 3) shows that either

$$x \equiv y \equiv z \equiv \pm 1 \pmod{3},$$

or 3 divides exactly one of x, y, z (y say). In the former case $x + y\omega$ has a primary associate $x\omega + \omega^2 y$, and the fact that

$$\left(\frac{3}{x\omega + y\omega^2}\right)_3 = 1$$

yields

$$x\omega + y\omega^2 \equiv \pm 1, \pm 2, \text{ or } \pm 4 \pmod{9},$$

by the supplement to the law of reciprocity. Hence $x \equiv y \pmod{9}$, and, by symmetry, $x \equiv y \equiv z \pmod{9}$. In case $3 \mid y$ and $3 \nmid xz$, then $x + y\omega$ is primary, and an analogous argument yields $9 \mid y$. This proves the theorem in the second case.

For a fixed $k \neq 0$, the local density of zeros at a finite place p is given by

$$\sigma_{p,k} = \lim_{n \rightarrow \infty} p^{-3n} N(p^n),$$

where

$$N(p^n) = N_k(p^n) =$$

$$\#\{(x, y, z, w) \pmod{p^n} : p \nmid \text{hcf}(x, y, z, w), x^3 + y^3 + z^3 \equiv kw^3 \pmod{p^n}\}.$$

This expression can be deduced, for example, by considering the major arc contribution to a suitable integral. Thus, if we write $e(\alpha) = \exp(2\pi i\alpha)$, then

$$R_k(N) =$$

$$\#\{(x, y, z, w) \in \mathbb{Z}^4 : |x|, |y|, |z| \leq N, \text{hcf}(x, y, z, w) = 1, x^3 + y^3 + z^3 = kw^3\}$$

is given by

$$\begin{aligned} \int_0^1 \sum_{\substack{|x|, |y|, |z| \leq N \\ |w| \leq 2N}}^{(1)} e(\alpha(x^3 + y^3 + z^3 - kw^3)) d\alpha \\ = \sum_{d=1}^{\infty} \mu(d) \int_0^1 \sum_{\substack{|x|, |y|, |z| \leq N \\ |w| \leq 2N}}^{(2)} e(\alpha(x^3 + y^3 + z^3 - kw^3)) d\alpha, \end{aligned}$$

where $\sum^{(1)}$ denotes the condition $\text{hcf}(x, y, z, w) = 1$, and $\sum^{(2)}$ the conditions $d|x, y, z, w$ and $(x, y, z, w) \neq (0, 0, 0, 0)$. For the above mentioned region, the real density of solutions will be $\sigma_{\infty, k} = k^{-1/3} I$, where

$$I = \frac{1}{3} \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 \frac{dx dy dz}{(x^3 + y^3 + z^3)^{2/3}}.$$

We shall not reproduce the calculation of these densities. The reader may refer to Vaughan [4; §§2.4 and 2.6] and Davenport [3; §6] for the necessary techniques.

We now see that the conjectured number of solutions, without any adjustment for the failure of (WA), would be

$${}_k N, \quad k = \sigma_{\infty, k} \prod_p \sigma_{p, k}.$$

To evaluate this more explicitly we observe that, if k is square-free, then

$$N(p^n) = \begin{cases} p^{3n-3} N(p), & n \geq 1, p \nmid 3k \\ p^{3n-6} N(p^2), & n \geq 2, p \mid 3k, \end{cases}$$

by Hensel's Lemma. We can calculate $N(p)$ and $N(p^2)$ with the aid of cubic Gauss sums. We find that, for $p \nmid 3k$, we have

$$N(p) = p^3 - 1, \quad \text{if } p \equiv 2 \pmod{3},$$

while for $p \equiv 1 \pmod{3}$ we have

$$N(p) = p^3 - 1 + 3p(p-1) \left\{ \left(\frac{k}{\pi} \right)_3 + \overline{\left(\frac{k}{\pi} \right)_3} \right\},$$

if p splits as $\pi \bar{\pi}$, with $\pi \equiv 1 \pmod{3}$, in $\mathbb{Q}(\omega)$. When $p \mid k$, $p \neq 3$, we have

$$N(p^2) = p^6 - p^4$$

and

$$N(p^2) = p^6 - p^4 - p^4(p-1)(\pi + \bar{\pi})$$

in the two cases. Finally

$$N(3^2) = \begin{cases} 2 \cdot 3^6, & k \equiv \pm 1 \pmod{9}, \\ 2^2 \cdot 3^5, & k \equiv \pm 2 \pmod{9}, \\ 2^3 \cdot 3^4, & k \equiv \pm 3 \pmod{9}, \\ 2 \cdot 3^5, & k \equiv \pm 4 \pmod{9}. \end{cases}$$

The product defining $\sigma_{\infty,k}$ is thus conditionally convergent only. However, by comparing it with the Euler product for $\zeta_{\mathbb{Q}(\sqrt[3]{k})}(s)$, we find that

$$\sigma_{\infty,k} = \sigma_{\infty,k} a_k b_k^3 c \Pi_{1,k} \Pi_{2,k} \quad (k = 2, 3),$$

where

$$\begin{aligned} a_2 &= \frac{2^9}{3^3 \cdot 7}, \quad a_3 = \frac{2^3}{3^2}; \\ b_k &= \lim_{s \rightarrow 1} (s-1) \zeta_{\mathbb{Q}(\sqrt[3]{k})}(s); \\ c &= \prod_{p \equiv 2 \pmod{3}} \left(1 - \frac{1}{p^2}\right)^3 \left(1 - \frac{1}{p^3}\right); \\ \Pi_{1,k} &= \prod \left(1 - \frac{1}{p}\right)^7 \left(1 + \frac{7}{p} + \frac{1}{p^2}\right), \end{aligned}$$

the product being over primes $p \equiv 1 \pmod{3}$ which split in $\mathbb{Q}(\sqrt[3]{k})$; and

$$\Pi_{2,k} = \prod \left(1 - \frac{1}{p^3}\right)^3,$$

the product being for primes $p \equiv 1 \pmod{3}$ which do not split in $\mathbb{Q}(\sqrt[3]{k})$. The class number formula gives

$$b_k = \begin{cases} \frac{\pi \log(1 + \sqrt[3]{2} + \sqrt[3]{4})}{3\sqrt{3}}, & k = 2, \\ \frac{2\pi \log(4 + 3\sqrt[3]{3} + 2\sqrt[3]{9})}{9\sqrt{3}}, & k = 3. \end{cases}$$

Moreover the products are now absolutely convergent and can be computed numerically to give

$$\begin{aligned} c &= 0.30663\dots \\ \Pi_{1,k} &= \begin{cases} 0.98938\dots, & k = 2, \\ 0.98927\dots, & k = 3, \end{cases} \\ \Pi_{2,k} &= \begin{cases} 0.95401\dots, & k = 2, \\ 0.97617\dots, & k = 3. \end{cases} \end{aligned}$$

We may also calculate

$$\sigma_{\infty,k} = \begin{cases} 9.81424\dots, & k = 2, \\ 8.57353\dots, & k = 3, \end{cases}$$

and

$$b_k = \begin{cases} 0.814624\dots, & k = 2, \\ 1.017615\dots, & k = 3, \end{cases}$$

so that

$$k = \begin{cases} 4.1598\dots, & k = 2, \\ 2.3780\dots, & k = 3. \end{cases}$$

For comparison one may find, by a naive computer search, all primitive solutions of $x^3 + y^3 + z^3 = kw^3$ for $k = 2, 3$, in the region $|x|, |y|, |z| \leq 1000$. Of course, the solutions generally occur in multiples of 12, because of the various symmetries involved. In searching for solutions one may use the restrictions found in Theorem 1, together with simple congruence conditions to moduli 2, 7, 9 etc. In this way we find

$$R_2(1000) = 3906, \quad R_3(1000) = 2462,$$

so that

$$\frac{R_2(1000)}{1000} \frac{1}{2} = 0.938\dots, \quad \frac{R_3(1000)}{1000} \frac{1}{3} = 1.035\dots$$

Figures 1 and 2, which show $R_k(N)$ for $N \leq 1000$, indicate a linear growth rate in each case. The evidence therefore suggests that

$$R_k(N) \sim kN,$$

despite the failure of (WA).

Although it seems hopeless to prove an asymptotic formula for $R_k(N)$, we can make a plausible heuristic investigation into the mean value

$$\sum_{K < k \leq 2K} R_k(N),$$

for suitably sized K . Allowing for solutions with $w = 0$, we can write

$$\sum_{K < k \leq 2K} R_k(N) = 6K + 2 \sum_{w=1}^{\infty} \#S(w), \quad (3)$$

where

$$S(w) = \{(x, y, z) \in \mathbb{Z}^3 : \text{hcf}(x, y, z, w) = 1, |x|, |y|, |z| \leq N, \\ w^3 |x^3 + y^3 + z^3|, K < \frac{x^3 + y^3 + z^3}{w^3} \leq 2K\}.$$

Of course $S(w)$ is empty unless $Kw^3 \leq 3N^3$. To estimate $\#S(w)$ in general, we observe that each admissible triple of residue classes for x, y and z modulo w^3 contains

$$V(w) + O(N^2 w^{-6}) + O(1)$$

elements, where

$$V(w) = \text{mes}\{(x, y, z) \in \mathbb{R}^3 : |x|, |y|, |z| \leq Nw^{-3}, Kw^{-6} < x^3 + y^3 + z^3 \leq 2Kw^{-6}\}.$$

Let

$$n(q) = \#\{x, y, z \pmod{q} : \text{hcf}(x, y, z, q) = 1, q|x^3 + y^3 + z^3\},$$

so that $n(w^3)$ is the number of available triples of residue classes. Then

$$n(w^3) = \prod_{p^e || w^3} n(p^e),$$

and

$$n(p^e) = p^{2e} \frac{n(p^2)}{p^4}, \quad \text{for } e \geq 2,$$

by Hensel's Lemma. We now define

$$\rho(w) = \prod_{p|w} \rho(p),$$

where

$$\rho(p) = p^{-4} n(p^2).$$

The Möbius inversion formula then yields

$$\rho(w) = \sum_{d|w} \nu(d),$$

where

$$\nu(d) = \mu(d) \prod_{p|d} (1 - \rho(p)).$$

Since $n(p^2) = p^4 + O(p^{7/2})$, we have $\nu(d) \ll d^{-1/2+\varepsilon}$, for any $\varepsilon > 0$, whence

$$\begin{aligned} \sum_{w \leq W} \rho(w) &= \sum_{d=1}^{\infty} \nu(d) \left[\frac{W}{d} \right] \\ &= W \sum_{d=1}^{\infty} \frac{\nu(d)}{d} + O(W^{1/2+\varepsilon}) \\ &= W \prod_p \left(1 + \frac{\rho(p) - 1}{p} \right) + O(W^{1/2+\varepsilon}). \end{aligned} \tag{4}$$

We now have to examine

$$2 \sum_w w^6 \rho(w) \{V(w) + O(N^2 w^{-6}) + O(1)\},$$

the sum being over values $w \ll NK^{-1/3}$. According to (4) the error terms are $O(N^3 K^{-1/3}) + O(N^7 K^{-7/3})$, whence (3) yields

$$\sum_{K < k \leq 2K} R_k(N) = 2 \sum_w w^6 \rho(w) V(w) + O(K) + O(N^3 K^{-1/3}) + O(N^7 K^{-7/3}).$$

We may write $V(w)$ as the difference $V_2(w) - V_1(w)$ of two monotonically decreasing functions given by

$$V_j(w) = \text{mes}\{(x, y, z) \in \mathbb{R}^3 : |x|, |y|, |z| \leq Nw^{-3}, 0 \leq x^3 + y^3 + z^3 \leq jKw^{-6}\}.$$

A partial summation then produces

$$\begin{aligned} \sum_w w^6 \rho(w) V(w) &= \prod_p \left(1 + \frac{\rho(p) - 1}{p}\right) \int w^6 V(w) dw \\ &\quad + O\left(\max_{w \ll NK^{-1/3}} w^{1/2+\varepsilon} \cdot w^6 \cdot V_2(w)\right) \\ &= \prod_p \left(1 + \frac{\rho(p) - 1}{p}\right) \int w^6 V(w) dw + O(N^{1/2+2\varepsilon} K^{5/6}). \end{aligned}$$

The integral above is

$$\begin{aligned} \frac{1}{2} \int_{-\infty}^{\infty} w^{-3} \text{mes}\{(x, y, z) \in \mathbb{R}^3 : |x|, |y|, |z| \leq N, K < \frac{x^3 + y^3 + z^3}{w^3} \leq 2K\} dw \\ = \frac{1}{2} \int_K^{2K} \int_{-N}^N \int_{-N}^N \int_{-N}^N \frac{dx dy dz}{(x^3 + y^3 + z^3)^{2/3}} \frac{dk}{3k^{1/3}}, \end{aligned}$$

Thus

$$\sum_{K < k \leq 2K} R_k(N) \sim \prod_p \left(1 + \frac{\rho(p) - 1}{p}\right) I \left\{ \int_K^{2K} \frac{dk}{k^{1/3}} \right\} N, \quad (5)$$

providing that $N^{2+\delta} \leq N \leq N^{3-\delta}$ for some constant $\delta > 0$. It seems likely that the lower bound for N could be improved somewhat. We shall want to exclude values of k which are cubes, $k = j^3$, say. However, if $x^3 + y^3 + z^3 = j^3 w^3$ then z, j and w determine $O(N^\varepsilon)$ values of x and y , except when $z = jw$, in which case there are $O(N)$ pairs x, y . It follows that

$$R_{j^3}(N) \ll j^{-1} N^{2+\varepsilon},$$

whence such terms contribute $O(N^{2+2\varepsilon})$ to (5). It therefore follows that

$$\sum_{K < k \leq 2K}^* R_k(N) \sim \prod_p \left(1 + \frac{\rho(p) - 1}{p}\right) I \left\{ \int_K^{2K} \frac{dk}{k^{1/3}} \right\} N,$$

for the same range of K as before, where \sum^* indicates that cube values of k are omitted.

We shall now attempt to compare this with

$$N \sum_{K < k \leq 2K}^* k.$$

When k is not a cube we write

$$\begin{aligned} k &= \sigma_{\infty, k} \lim_{X \rightarrow \infty} \prod_{p \leq X} \frac{N_k(p^{n(p, X)})}{p^{3n(p, X)}} \\ &= \lim_{X \rightarrow \infty} \frac{N_k(q(X))}{q(X)^3}, \end{aligned} \quad (6)$$

where

$$n(p, X) = \left[\frac{\log X}{\log p} \right]$$

and

$$q(X) = \prod_p p^{n(p, X)}.$$

If the convergence were uniform, we could write

$$\sum_{K < k \leq 2K}^* k \sim \sum_{K < k \leq 2K}^* \sigma_{\infty, k} \frac{N_k(Q)}{Q^3}, \quad (7)$$

with $Q = q(\log \log N)$, say. Unfortunately the convergence in (6) is not uniform. None the less the relation (7) seems plausible. Indeed it seems likely that one could prove it with a little effort, and we therefore continue our investigation on the assumption that it does indeed hold.

We now observe that

$$\begin{aligned} \sum_{k \leq X}^* N_k(Q) &= \sum_{k \leq X} N_k(Q) + O(X^{1/3} N^\varepsilon) \\ &= \left(\frac{X}{Q} + O(1) \right) N^*(Q) + O(X^{1/3} N^\varepsilon), \end{aligned}$$

for any $\varepsilon > 0$, where

$$N^*(s) = \#\{(x, y, z, w, k) \pmod{s} : \text{hcf}(x, y, z, w, s) = 1, x^3 + y^3 + z^3 \equiv kw^3 \pmod{s}\}.$$

Thus summation by parts yields

$$\sum_{K < k \leq 2K}^* \sigma_{\infty, k} N_k(Q) = \frac{N^*(Q)}{Q} I \int_K^{2K} \frac{dk}{k^{1/3}} + O(K^{-1/3} N^*(Q)) + O(N^\varepsilon). \quad (8)$$

Moreover

$$N^*(Q) = \prod_{p^e || Q} N^*(p^e)$$

and

$$N^*(p^e) = N_1(p^e) + N_2(p^e),$$

where $N_1(p^e)$ counts solutions with $p \nmid w$, and $N_2(p^e)$ counts those with $p | w$. Clearly

$$N_1 = p^{4e} \left(1 - \frac{1}{p}\right),$$

since x, y, z, w determine k uniquely. Moreover Hensel's Lemma yields

$$N_2(p^e) = p^{4e} \frac{N_2(p^2)}{p^8},$$

providing that $e \geq 2$ when $p = 3$. Since, in our earlier notation, we have $N_2(p^2) = p^3 n(p^2) = p^7 \rho(p)$, we deduce that

$$\frac{N^*(Q)}{Q} = Q^3 \prod_{p \leq \log \log N} \left(1 + \frac{\rho(p) - 1}{p}\right), \quad (9)$$

so long as $\log \log n \geq 9$. In view of (7), (8) and (9) we now have

$$\begin{aligned} \sum_{K < k \leq 2K}^* k &\sim \frac{N^*(Q)}{Q^4} I \int_K^{2K} \frac{dk}{k^{1/3}} + O(Q^{-3} K^{-1/3} N^*(Q)) + O(Q^{-3} N^\varepsilon) \\ &= \prod_{p \leq \log \log N} \left(1 + \frac{\rho(p) - 1}{p}\right) \int_K^{2K} \frac{dk}{k^{1/3}} + O(N^\varepsilon) \\ &\sim \prod_p \left(1 + \frac{\rho(p) - 1}{p}\right) \int_K^{2K} \frac{dk}{k^{1/3}}, \end{aligned}$$

if $K \geq N^\delta$, for some constant $\delta > 0$. We therefore have:-

Theorem 2 *If the conjecture (7) holds, then*

$$\sum_{K < k \leq 2K}^* R_k(N) \sim N \sum_{K < k \leq 2K}^* k.$$

Thus, for the equation $x^3 + y^3 + z^3 = kw^3$, the evidence all points to the conclusion that the Hardy-Littlewood formula is valid, despite the failure of (WA).

We conclude by investigating the Hardy-Littlewood formula for the equation $x^3 + y^3 + z^3 = k$. The local densities will be

$$\sigma_p = \lim_{n \rightarrow \infty} p^{-2n} N(p^n),$$

where

$$N(p^n) = \#\{(x, y, z) \pmod{p^n} : x^3 + y^3 + z^3 \equiv k \pmod{p^n}\},$$

while, for the range

$$N_1 \leq \max(|x|, |y|, |z|) \leq N_2,$$

the real density of solutions is

$$\begin{aligned} \sigma_\infty &= 6 \int_{N_1}^{N_2} \int_0^z \frac{dy}{3(z^3 - y^3)^{2/3}} dz \\ &= 2(\log \frac{N_2}{N_1}) \int_0^1 \frac{dt}{(1 - t^3)^{2/3}} \\ &= \frac{2}{3}(\log \frac{N_2}{N_1}) B(\frac{1}{3}, \frac{1}{3}) \\ &= 3.5332 \dots \log \frac{N_1}{N_2}. \end{aligned}$$

For cube-free k we have

$$\sigma_p = \frac{N(p)}{p^2} \quad (p \neq 3) \quad \text{and} \quad \sigma_3 = \frac{N(9)}{81}.$$

Then, as in our previous calculation, we obtain

$$\sigma_p = \begin{cases} 1, & p \nmid k, \\ 1 - p^{-2}, & p \mid k, \end{cases}$$

for $p \equiv 2 \pmod{3}$, and

$$\sigma_p = \begin{cases} 1 + \frac{3}{p} \{(\frac{k}{\pi})_3 + \overline{(\frac{k}{\pi})_3}\} - \frac{a}{p^2}, & p \nmid k, \\ 1 + \frac{(p-1)a}{p^2} - \frac{1}{p^2}, & p \mid k, \end{cases}$$

if $p \equiv 1 \pmod{3}$ splits as $\pi\bar{\pi}$ in $\mathbb{Q}(\sqrt{-3})$, and $4p = a^2 + 3b^2$ with $a \equiv 1 \pmod{3}$. As before we may compare σ_p with the Euler factor $E(p, k)$ in $\zeta_{\mathbb{Q}(\sqrt[3]{k})}(s)$. We then find that

$$\prod_p \sigma_p = A_k^3 \prod_p \sigma'_p,$$

where

$$A_k = \lim_{s \rightarrow 1} (s-1) \zeta_{\mathbb{Q}(\sqrt[3]{k})}(s),$$

and

$$\sigma'_p = \sigma_p \left(\frac{1 - 1/p}{E(p, k)} \right)^3.$$

We now have a convergent product, and we can calculate

$$\sigma_{\infty} \prod_p \sigma_p = \begin{cases} 0.366 \dots (\log N_2/N_1), & k = 3, \\ 0.539 \dots (\log N_2/N_1), & k = 30, \end{cases}$$

for example. In the light of our investigations above, we conjecture that these figures give, approximately, the number of solutions of $x^3 + y^3 + z^3 = k$; and that no adjustment for the failure of (WA), as implied by Theorem 1, is necessary. In particular we conjecture that these Diophantine equations have infinitely many solutions. However, since the solutions have 6 symmetries, the density of essentially different solutions would be expected to be one sixth of that indicated above. Thus one might predict, very roughly speaking, one new solution of $x^3 + y^3 + z^3 = 3$ in the range

$$N \leq \max\{|x|, |y|, |z|\} \leq \exp(6/0.366 \dots) N \doteq 1.3 \times 10^7 N,$$

and similarly, for $x^3 + y^3 + z^3 = 30$, one new solution in the range

$$N \leq \max\{|x|, |y|, |z|\} \leq 7 \times 10^4 N.$$

A search for solutions to these equations (Gardiner, Lazarus and Stein [4]) found only $(1, 1, 1)$ and $(4, 4, -5)$, corresponding to $k = 3$, and no solutions for $k = 30$. Indeed it was suggested that these equations have no more solutions. In as much as the search range was only to $2^{16} \doteq 6.5 \times 10^4$, the figures above indicate that any such conjecture is premature. In particular, in the case of $k = 3$, it would have been somewhat unexpected if a solution had been found.

References

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