Developments in unitary quantum theory

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Abstract

This thesis describes advancements in unitary quantum theory. These can be divided into three main topics: Everett's relative-state formalism, the Page–Wootters construction, and unorthodox quantum theory.

Everett's relative-state formalism allows one to express the state of one quantum system relative to the state of another with which it is entangled. This formalism has only recently been translated to the Heisenberg picture by myself & Deutsch (2021). Our construction makes explicit the locality of Everettian multiplicity and demonstrates that the relative-states of a quantum system are fully fledged quantum systems in their own right. For instance, the observables of a relative state satisfy an appropriate form of a system's characteristic algebra, and the relative states can store locally inaccessible information, just like the absolute system.

The Heisenberg-picture relative-state formalism has been used by me (Kuypers (2022a)) to refine a construction due to Page & Wootters (1983). In their construction, it is shown that an external time parameter is unnecessary for explaining change and dynamics in quantum theory because one can instead rely on 'clock time' – i.e. the time as represented by a clock. This makes time manifestly physical: clock time is an observable attribute of a clock, whereas the time parameter is not an attribute of anything physical. I have shown (Kuypers (2022a)) that the Page–Wootters construction is a 'calculus for q-numbers'; that is to say that within that construction, one can define functions of q-numbers (which can be represented by matrices or operators on a Hilbert space) as well as derivatives of such functions.

Both the Page–Wootters construction and Everett's relative state formalism are here treated in the Heisenberg picture because that picture has the crucial advantage that it is a local description of quantum theory (see §2.5). Therefore, the Heisenberg picture has been used by me (Kuypers (2022b)) to describe quantum systems in chronology-violating regions of spacetime (i.e. regions of spacetime that contain closed timelike curves). The presence of closed timelike curves makes one of the axioms of 'orthodox' quantum theory problematic – namely, the axiom that the observables of spacelike-separated systems commute. In unorthodox quantum theory, that axiom is dropped. The foundations of unorthodox quantum theory are presented, and I demonstrate that this theory provides a resolution to the so-called grandfather paradox.

Throughout this work, $\hbar$ been set equal to 1.

Important terminology shall be introduced in italics.

You want to know the Secret – so did I,
Low in the dust I sought it, and on high
Sought it in awful flight from star to star,
The Sultan's watchman of the starry sky.

Up, up, where Parwin's hoofs stamp heaven's floor,
My soul went knocking at each starry door,
Till on the stilly top of heaven's stair,
Clear-eyed I looked – and laughed – and climbed no more.

Of all my seeking this is all my gain:
No agony of any mortal brain
Shall wrest the secret of the life of man;
The Search has taught me that the Search is vain.

Yet sometimes on a sudden all seems clear –
Hush! hush! my soul, the Secret draweth near;
Make silence ready for the speech divine –
If Heaven should speak, and there be none to hear!

Yea! sometimes on the instant all seems plain,
The simple sun could tell us, or the rain;
The world, caught dreaming with a look of heaven,
Seems on a sudden tip-toe to explain.

Like to a maid who exquisitely turns
A promising face to him who, waiting, burns
In hell to hear her answer – so the world
Tricks all, and hints what no man ever learns.

From *The Robiáyát of Omar Khayyám* (Le Gallienne (1900))

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1 Introduction

A guiding principle that connects my work (Kuypers & Deutsch (2021), Kuypers (2022a), and Kuypers (2022b)) is that the Heisenberg picture is the fundamental description of quantum theory. That is to say that the Heisenberg picture eliminates various misconceptions and errors that have been entrenched in the Schrödinger picture – in particular, the error of non-locality (see §2.5). For that reason, the material in this thesis is expressed predominantly in the Heisenberg picture.

Yet the foundations of quantum theory are conventionally expressed in the Schrödinger picture – e.g. there are scarce any introductory textbooks that provide a detailed account of the Heisenberg picture, making Heisenberg-picture formalism, terminology and notation less familiar to that of the Schrödinger picture. To remedy that problem, I elucidate the standard formalism of the Heisenberg picture in the following introduction, thereby providing context for the more technical discussions in later sections.

1.1 Descriptors of physical systems

Physical systems are objects that exist in the real world, which we attempt to describe via theory. In the mathematical sciences, we invoke mathematical objects (e.g. scalars, vectors, tensors) to represent the state of a physical system. A simple example of such a description is when an integer represents the number of objects in a collection. A collection of dogs, for instance, is a physical system that is described by an integer N, which represents how many dogs there are.

The dogs in such a collection are not integers; that is to say, they are not identical to integers, but they *are* equivalent to them, under some equivalence relation. To underscore this relation between physical objects that exist in the real world and the abstract objects used in their theoretical representation, I shall follow Deutsch & Hayden (2000) in referring to those mathematical entities that represent the state[1] of a system as the system's *descriptors*.

In classical physics, the descriptors of physical systems are c-numbers or fields over c-numbers – i.e. they are numbers that satisfy a commutative algebra, such as that of the real and complex numbers. An example of a classical descriptor is the aforementioned integer N, which can represent the number of physical objects in a collection. Other examples are that the classical electromagnetic field is a vector field over the reals, and that in general relativity, coordinates in spacetime are c-numbers.

With the invention of quantum theory came the realisation that c-numbers are not the only possible descriptors of physical systems. The descriptors of quantum systems are numbers that satisfy a non-commutative algebra, which are called q-numbers, following the terminology of Dirac (1967). In fact, it is a principle of quantum theory that all descriptors of physical systems are q-numbers, or vectors over q-numbers, etc.

Because of its importance, I shall call this the *fundamental principle of quantum theory* (Kuypers (2022a)), and 'quantum theory' is better thought of as a collection of theories that comply with this fundamental principle, as well as the other principles of quantum theory that I shall discuss in later sections.

The reason why quantum theory is better thought of as a set of principles is that it does not describe particular physical systems. Atoms and photons are distinct physical systems, but both are described by 'quantum theory'. Likewise, 'quantum theory' also describes large scale objects like superconductors (e.g. see van Wezel & van den Brink (2008)), measuring instruments (see §3.2), and quantum computers (see §2).[2] Quantum theory is not about any

[1] With the term 'state' or 'state of the system', I do not mean its 'state vector', like it conventionally does in quantum theory. Instead, I shall use the term 'state' to refer to 'the real factual situation' of the system, as Einstein (1970, p. 85) called it. Therefore, the term 'state' as I use it here does not exclusively apply to quantum systems but also to classical ones.

[2] Van Wezel & van den Brink (2008) explain superconductivity in terms of spontaneous symmetry breaking. Related to their work is Wallace's (2018) explanation of the relation between spontaneous symmetry breaking and decoherence, particularly in the case of ferromagnetism.

of those systems in particular, but about all of them – or rather, it is a theory about what kinds of theories can describe physical objects. A theory that constrains other theories of physics is what I shall call a *principle of physics*, using the terminology of Deutsch & Marletto (2015).

The fundamental principle has been enormously successful: it underpins our understanding of all elementary particles and fundamental interactions, with the sole but important exception of gravity. Quantum theory explains the phenomenon of superconductivity and the different phases of matter, such as the ferromagnetic phase, which fundamentally does not admit a classical description (Bohr (1911), van Vleck (1978)). That we understand the structure of the atom, the fusion reactions that fuel the sun, blackbody radiation, and much more besides, is all due to the success of the fundamental principle of quantum theory.

Despite its enormous success, modern physics only imperfectly adheres to quantum theory's fundamental principle because the principle conflicts with certain well-established theories. In particular, there is no quantum theory of spacetime, as the theory of general relativity is entirely classical. This does not imply that the fundamental principle is false, merely that our understanding of it and theories like general relativity is flawed. Better, deeper theories will resolve these issues.

On a related note, in conventional formulations of physics the time parameter is not a q-number. It is a c-number and cannot, therefore, be a descriptor of a physical system, as that is forbidden by the fundamental principle. Hence, either the time parameter is unphysical or the fundamental principle of quantum theory is false. This particular problem is part of *the problem of time*, to which I shall return in §4.

There is a further apparent shortcoming of the fundamental principle: the 'macroscopic world' does not seem to be described by q-numbers, since the world of our experience appears to be completely classical. But this apparent shortcoming is only a misconception: the 'macroscopic world' is exactly as it should appear if quantum theory is a universal physical theory (as explained in §3). That is to say that macroscopic objects are also represented by q-number descriptors, as required by quantum theory. The descriptors of macroscopic objects *appear* to be c-numbers to us because we cannot fully see everything those descriptors represent: macroscopic objects have counterparts in other parts of physical reality that go unseen by us. These other parts of reality are like our own 'world', and exist simultaneously with it, except that the histories of those other worlds at some point deviated from that of our own. Hence, our 'world' or 'universe'[3] is but a part of a much larger structure that quantum theory describes, which I will instead refer to as the *quantum multiverse*, or the *Everettian multiverse*, after its discoverer Hugh Everett III (1973), or simply the *multiverse* for short.[4]

1.2 The algebra of quantum descriptors

Physical systems possess attributes, and the totality of these attributes are what define a system's state at a particular time. Most of a system's attributes are different at different times – that is to say, those attributes change as the system evolves. Yet some attributes are time-invariant; they are *constitutive*, such as the charge of an electron, or the energy of a closed system. It is these constitutive attributes that characterise a system since it is by virtue of some attributes remaining unchanged while the system evolves that the system retains its identity over time.[5]

It is a principle of quantum theory that the algebra of a system's descriptors is a constitutive attribute. Consider some general quantum system $\mathfrak{S}$ with a set of time-dependent descriptors $\hat{X}_1(t), \hat{X}_2(t), \dots, \hat{X}_n(t)$. Though these descriptors change with time, their algebraic relations should be a constant of the motion, and to ensure that this is so, the descriptors must evolve unitarily, i.e.

$$\hat{X}_j(t) = U^\dagger(t)\hat{X}_j(0)U(t) \qquad (1 \le j \le n). \tag{1.1}$$

[3] I shall use the term 'universes' to match the terminology used by DeWitt & Graham (1973) and Everett (see Byrne 2010), but the terms 'worlds', 'branches' or 'histories' are also sometimes used to refer to what I have here called 'universes', and throughout this thesis I will use all those terms interchangeably.

[4] The multiverse I am referring to throughout this work should not be confused with that of other 'multiverse theories', such as the cosmological multiverse: whenever I use the term multiverse, I will be referring specifically to the quantum one.

[5] This is the *problem of change*, which Popper (1963, p. 192) phrased as: "how can a thing change, without losing its identity?" The answer that quantum theory provides us is that though some attributes of a system change, its constitutive attributes do not, and those attributes characterise the system.

Here $\hat{X}_j(0)$ is the j'th descriptor at the initial time $t = 0$; by definition, the unitary $U(t)$ must satisfy the constraint that

$$U^\dagger(t)U(t) = U(t)U^\dagger(t) = \hat{1}, \tag{1.2}$$

at any time t, where $\hat{1}$ is the unit observable, and where the dagger represents Hermitian conjugation. Another way of phrasing equation (1.2) is that $U^\dagger(t)$ is equal to the inverse of $U(t)$.

That unitary evolution preserves the descriptors' algebra can be proven as follows: if at time $t = 0$, the descriptors have the property that $\hat{X}_1(0)\hat{X}_2(0) = \hat{X}_3(0)$, then

$$\left.\begin{aligned} \hat{X}_1(t)\hat{X}_2(t) &= U^\dagger(t)\hat{X}_1U(t)U^\dagger(t)\hat{X}_2U(t) \\ &= U^\dagger(t)\hat{X}_1\hat{X}_2U(t) \\ &= U^\dagger(t)\hat{X}_3U(t) \\ &= \hat{X}_3(t) \end{aligned}\right\},$$

where the $t = 0$ labels have been suppressed for brevity's sake. The reverse is also true: if two sets of q-numbers have the same algebra, then there is a unitary transformation that transforms one set into the other. Hence, if a q-number's algebra is to be preserved, descriptors must evolve unitarily. I will prove this in §1.8.

Similarly, if at the initial time $t = 0$ the q-number $\hat{X}_j(0)$ is Hermitian, meaning that $\hat{X}_j(0)$ is equal to its Hermitian conjugate – i.e. $\left(\hat{X}_j(0)\right)^\dagger = \hat{X}_j(0)$, then it is also Hermitian at any other time because

$$\left.\begin{aligned} \left(\hat{X}_j(t)\right)^\dagger &= \left(U^\dagger(t)\hat{X}_jU(t)\right)^\dagger \\ &= U^\dagger(t)\hat{X}_j{}^\dagger U(t) \\ &= U^\dagger(t)\hat{X}_jU(t) \\ &= \hat{X}_j(t) \end{aligned}\right\},$$

where I again suppressed the $t = 0$ time labels. Evidently, when descriptors evolve unitarily, any initial algebraic relation between the descriptors will be preserved at a later time – as required by the principle that the algebra is a constitutive attribute.

1.3 Hamiltonian

Unitary evolution is generated by a Hamiltonian, defined as

$$\hat{H}(t) \stackrel{\text{def}}{=} iU^\dagger(t)\frac{dU(t)}{dt}. \tag{1.3}$$

The Hamiltonian is automatically Hermitian, as can be shown by differentiating (1.2) with respect to time. Moreover, using the Hamiltonian, one can differentiate (1.1) to obtain the *Heisenberg equation of motion*

$$\left.\begin{aligned}\frac{d\hat{X}_j(t)}{dt} &= \frac{dU^\dagger(t)}{dt}\hat{X}_j(0)U(t) + U^\dagger(t)\hat{X}_j(0)\frac{dU(t)}{dt}\\ &= -U^\dagger(t)\frac{dU(t)}{dt}\hat{X}_j(t) + \hat{X}_j(t)U^\dagger(t)\frac{dU(t)}{dt}\\ &= i\hat{H}(t)\hat{X}_j(t) - i\hat{X}_j(t)\hat{H}(t)\\ &= i\big[\hat{H}(t),\hat{X}_j(t)\big]\end{aligned}\right\} \qquad (1 \le j \le n). \tag{1.4}$$

Here I have used the fact that

$$\frac{dU^\dagger(t)}{dt} = -U^\dagger(t)\frac{dU(t)}{dt}U^\dagger(t),$$

as can be ascertained by differentiating (1.2) with respect to t. Thus, the time evolution of a system can be expressed as a set of differential equations, which depend on the expression for $\hat{H}(t)$.

Descriptors that commute with the Hamiltonian are constants of the motion. For example, the Hamiltonian is a constant because it commutes with itself:

$$\frac{d\hat{H}(t)}{dt} = i\big[\hat{H}(t),\hat{H}(t)\big] = 0.$$

So, I will drop the unnecessary time label in $\hat{H}(t)$ and denote it simply as $\hat{H}$. Sometimes an explicit time-dependence is included in the Hamiltonian to model an external influence on the system, such as in quantum computational networks, which require an outside influence to cause different gates to be enacted at different times. When the Hamiltonian has an explicit time dependence, the system will be called *open*. Otherwise, it is *isolated* or *closed*.

Typically, it will be sufficient to consider only isolated systems: an open system can be considered as a part of a larger, isolated system. And, on a more cosmic scale, the multiverse as a whole must be an isolated system since, by definition, there is nothing outside of it.

1.4 Observables

Descriptors represent the state of a physical system and do not necessarily correspond to anything measurable (though they could). I shall reserve the term *observable* to refer to measurable attributes of physical objects since an 'observable' invokes the idea that there is a recipe for how to observe that quantity, whereas the descriptors of a system need not necessarily come with such a recipe.

In quantum theory, an observable is a Hermitian q-number that can be expressed multiplicatively and additively in terms of a system's descriptors. The simplest example of an observable is a Hermitian descriptor $\hat{X}$. Conversely, descriptors that are not Hermitian are not observables, which is why the terms 'observables' and 'descriptors' have slightly different meanings.

1.5 Expectation values

Q-numbers are not *directly* observable: no measurement of a quantum system produces a measurement outcome that is at all like a q-number. Measurements of q-numbers produce c-number measurement results, and quantum theory allows us to make predictions about such results. In particular, we can compute the *expectation values* of observables. The expectation value $\langle\hat{X}\rangle$ of an arbitrary observable $\hat{X}$ is denoted as follows:

$$\langle\hat{X}\rangle = \langle\Psi|\hat{X}|\Psi\rangle, \tag{1.5}$$

where $|\Psi\rangle$ is the system's *Heisenberg state*.[6] The term 'expectation value' is somewhat misleading since it is only in specific circumstances that $\langle\hat{X}\rangle$ will correspond to the probabilistic or statistical expectation value of that q-number (see §1.7).

[6] The equality (1.5) is a version of the Born rule, which is typically regarded as axiomatic in quantum theory. However, it need not be: although it is outside of the scope of this thesis, the Born rule can be

I will assume that the Heisenberg state $|\Psi\rangle$ is a fixed constant – 'fixed' in the sense that the initial expectation values of a particular system will have to satisfy some predetermined restrictions which determine the expression for the Heisenberg state (see for example constraint (2.8)). Consequently, the Heisenberg 'state' is fixed by convention and cannot contain information about the state of the system. Information about a system's state is instead represented by its descriptors, making the name 'Heisenberg state' a misnomer.

From (1.5) directly follow three useful results: (i) the expectation value of the unit observable is equal to unity, i.e. $\langle\hat{1}\rangle = 1$; (ii) the expectation value of a sum of q-numbers is equivalent to the sum of their expectation values, i.e. $\langle\hat{X} + \hat{Y}\rangle = \langle\hat{X}\rangle + \langle\hat{Y}\rangle$; and (iii) for any real-valued constant a, the expectation value $\langle a\hat{X}\rangle$ is equivalent to the product $a\langle\hat{X}\rangle$.

A measurement result of $\hat{X}$ will on average have a value close to $\langle\hat{X}\rangle$. Yet, for the average measurement result to be equal to $\langle\hat{X}\rangle$, no particular result has to have that value. There could be situations in which there is a high probability of a measurement of $\hat{X}$ yielding a result that is significantly greater than $\langle\hat{X}\rangle$, or alternatively measurement results could be sharply peaked around $\langle\hat{X}\rangle$. The expectation value $\langle\hat{X}\rangle$ alone does not provide us with information about the spread in the results. Yet that spread is a significant physical quantity. So how is it quantified? An oft used measure of this spread is the *variance of* $\hat{X}$, defined as

$$V(\hat{X}) \stackrel{\text{def}}{=} \left\langle\left(\hat{X} - \langle\hat{X}\rangle\right)^2\right\rangle. \tag{1.6}$$

The term $\left(\hat{X} - \langle\hat{X}\rangle\right)^2$ is the squared deviation from the mean, where the square is included so that both positive and negative deviations from $\langle\hat{X}\rangle$ contribute equally towards the variance. Hence, the variance quantifies by how much a result of a measurement of $\hat{X}$ will on average deviate from $\langle\hat{X}\rangle$.

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deduced from the non-probabilistic axioms of quantum theory. For more about this topic, see the work of Wallace (2014).

As is evident from the definition (1.6), the variance of any Hermitian $\hat{X}$ is real- and positive-valued; i.e. $V(\hat{X}) \geq 0$ for all Hermitian $\hat{X}$. The interpretation of the values of $V(\hat{X})$ is that an $\hat{X}$ with a large variance will have a large spread in its measurement results, whereas a variance close to zero implies that measurements of $\hat{X}$ will yield results that deviate only slightly (or not at all) from $\langle\hat{X}\rangle$.

1.6 Eigenvalues of q-numbers

In the boundary case where $V(\hat{X}) = 0$, there is absolute certainty about the outcome of a perfect measurement of $\hat{X}$ since, in this case, there is no deviation between any measurement result and $\langle\hat{X}\rangle$. Put differently, when $V(\hat{X}) = 0$, a measurement of $\hat{X}$ will with certainty yield a result that is equal to $\langle\hat{X}\rangle$. When the variance of $\hat{X}$ is zero, the q-number has a well-defined value, and I shall then say that $\hat{X}$ is *sharp* with the value $\langle\hat{X}\rangle$. Otherwise, $\hat{X}$ is said to be *non-sharp*.

$\hat{X}$ is sharp when the Heisenberg state is an *eigenstate* of $\hat{X}$, and the sharp values of $\hat{X}$ are its *eigenvalues*.[7] The eigenvalues of $\hat{X}$ can be derived from the q-number's algebra. For instance, let $\hat{X}^2 = \hat{1}$; one then finds that $\hat{X}$ is sharp if and only if it has the expectation values $\langle\hat{X}\rangle = -1$ or $\langle\hat{X}\rangle = 1$ because

$$\left.\begin{aligned} \left\langle\left(\hat{X} - \langle\hat{X}\rangle\right)^2\right\rangle &= 0 \\ \leftrightarrow \langle\hat{X}\rangle^2 &= \langle\hat{X}^2\rangle \\ \leftrightarrow \langle\hat{X}\rangle^2 &= 1 \\ \leftrightarrow \langle\hat{X}\rangle &= \pm 1 \end{aligned}\right\}. \tag{1.7}$$

The complete set of eigenvalues of $\hat{X}$ is called the q-number's *spectrum of eigenvalues*, or *spectrum* for short, which I shall denote as $\mathrm{sp}(\hat{X})$. Using this notation, one can summarise the results obtained in (1.7) as follows:

[7] An eigenstate of $\hat{X}$ with eigenvalue x, denoted $|x\rangle$, has the property that $\hat{X}|x\rangle = x|x\rangle$.

$$\mathrm{sp}(\hat{X}) = \{1, -1\}.$$

Notably, $\mathrm{sp}(\hat{X})$ is a discrete set, as there are only two values for which $\hat{X}$ is sharp. Contrast this with classical physics, in which descriptors are invariably sharp and can assume a continuous infinity of such sharp values. Descriptors with discrete values are only approximative in classical physics, yet in quantum theory, such descriptors are fundamental – e.g. a quantum system's energy has a discrete spectrum – hence the name *quantum* theory.

In (1.7), the spectrum of $\hat{X}$ is derived from the q-number's algebra, which is a general recipe for obtaining a q-number's spectrum – i.e. the spectrum of any q-number follows from its algebra.[8] Since I have shown that a q-number's algebra is unaltered by an arbitrary unitary-transformation U, it follows that $\hat{X}$ and $U^\dagger X U$ must have identical spectra:

$$\mathrm{sp}(\hat{X}) = \mathrm{sp}(U^\dagger \hat{X} U).$$

Hence, when descriptors evolve unitarily, their spectra are constants of the motion.

1.7 Projectors

Roughly speaking, a q-number $\hat{X}$ 'keeps track' of all of its eigenvalues. That is to say that $\hat{X}$ and its algebra completely specify its spectrum of eigenvalues. All information about those eigenvalues, such as the probability of measuring a particular eigenvalue, can be deduced from $\hat{X}$ and the Heisenberg state.

Yet $\hat{X}$ is never equivalent to any of its eigenvalues – for example, though the eigenvalues of $\hat{X}$ may be 1 and -1 if $\hat{X}^2 = \hat{1}$, neither $(\hat{X} - 1)$ nor $(\hat{X} + 1)$ are ever equal to zero. Nevertheless, the individual eigenvalues and their probabilities can still be analysed. To study the individual eigenvalues of an arbitrary q-number $\hat{X}$, one can use an x-eigenvalue projector $\hat{\Pi}_x(\hat{X})$, whose effect on $\hat{X}$ is that

[8] One way to prove that the spectrum of any q-number depends only on its algebra is as follows: a network of qubits, as discussed in §2, can be used to simulate any other quantum system with arbitrary accuracy. This necessarily implies that the algebra of qubits can be used to approximate the algebra of any other quantum system, and since the spectrum of a qubit's observables is determined by their algebraic properties, so must the spectrum of any other q-number.

$$\hat{X}\hat{\Pi}_x(\hat{X}) = x\hat{\Pi}_x(\hat{X}). \tag{1.8}$$

By inspection of (1.8), one can see that the q-number $\hat{X}$ acts on $\hat{\Pi}_x(\hat{X})$ exactly as if $\hat{X}$ were a c-number, which provides a hand-waving argument for why q-numbers can approximate c-numbers in certain regions of the multiverse.

The x-eigenvalue projectors of an arbitrary q-number $\hat{X}$ satisfy the following characteristic algebra:

$$\left.\begin{array}{l} \hat{\Pi}_x(\hat{X})\hat{\Pi}_{x'}(\hat{X}) = \delta_{x,x'}\hat{\Pi}_x(\hat{X}) \\ \displaystyle\sum_{x\in \mathrm{sp}(\hat{X})} \hat{\Pi}_x(\hat{X}) = \hat{1} \end{array}\right\} \qquad \left(x, x' \in \mathrm{sp}(\hat{X})\right), \tag{1.9}$$

where $\delta_{x,x'}$ is the Kronecker delta function. The algebra (1.9) implies that a projector is sharp if and only if

$$\left.\begin{array}{r} \left\langle\left(\hat{\Pi}_x - \langle\hat{\Pi}_x\rangle\right)^2\right\rangle = 0 \\ \leftrightarrow \langle\hat{\Pi}_x\rangle^2 = \left\langle{\hat{\Pi}_x}^2\right\rangle \\ \leftrightarrow \langle\hat{\Pi}_x\rangle^2 = \langle\hat{\Pi}_x\rangle \\ \leftrightarrow \langle\hat{\Pi}_x\rangle = 0 \text{ or } \langle\hat{\Pi}_x\rangle = 1 \end{array}\right\}.$$

Here I have suppressed, for the sake of brevity, the projector's dependence on $\hat{X}$. Evidently, the projector has two eigenvalues, 1 and 0, and those eigenvalues have the following physical meaning: if $\hat{\Pi}_x(\hat{X})$ is sharp with value 1, then everywhere in the multiverse will $\hat{X}$ have the value x; and when $\hat{\Pi}_x(\hat{X})$ is sharp with value 0, then nowhere in the multiverse does $\hat{X}$ assume the value x (e.g. a measurement of $\hat{X}$ will with certainty *not* yield the eigenvalue x). We see repeated here that sharpness of a q-number corresponds to a definite property of a quantum system, such as a definite measurement result.

The set of projectors $\{\hat{\Pi}_x(\hat{X}) : x \in \mathrm{sp}(\hat{X})\}$ satisfy the algebraic constraints (1.9), making the set a so-called projection-valued measure (PVM), which is a set of projectors that sum to unity. The expectation values of the projector of a PVM represent the probabilities of measuring a particular eigenvalue of $\hat{X}$. This identification of the expectation value $\langle\hat{\Pi}_x(\hat{X})\rangle$ as a probability of measuring the value x is known as the Born rule, and although I shall basically treat the Born rule as if it were an axiom, it can be derived from the non-probabilistic axioms of quantum theory (Wallace (2012b, ch. 5)). As an example, if $\langle\hat{\Pi}_x(\hat{X})\rangle = \frac{1}{2}$, then according to the Born rule, the probability of measuring the eigenvalue x is equal to $\frac{1}{2}$.

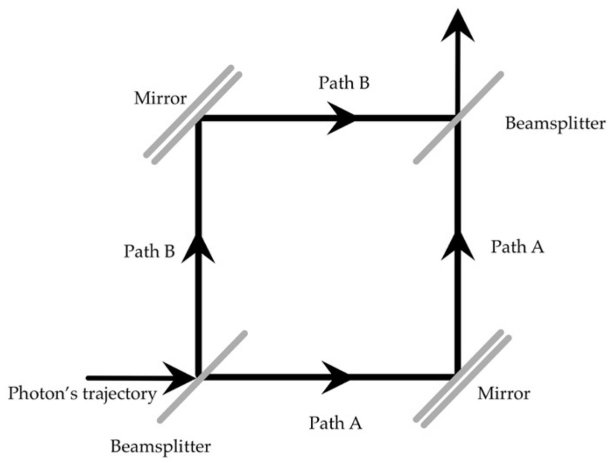

Figure 1. This is a pictorial representation of a Mach-Zehnder interferometer. In it, an incoming photon (shown at the bottom left of the diagram) interacts with a beamsplitter, causing the photon to branch into two instances. One of the instances follows path A while the other follows path B. On both paths, the photon encounters a mirror that ideally deflects it. The two instances of the photon interfere with one another through the second beamsplitter at the top right of the diagram. Crucially, the diagram represents the physical distance the photon's instances have to travel and not merely the topology of their trajectories. In particular, for there to be perfect interference at the second beamsplitter, the paths A and B must be equal in length.

The Born rule seems to imply that the eigenvalue x is a stochastic variable with an associated probability $\langle\hat{\Pi}_x(\hat{X})\rangle$. But the Born rule is not universally valid, as there are situations in which $\langle\hat{\Pi}_x(\hat{X})\rangle$ does not behave at all like the probability of a stochastic variable should – namely, when there is interference.[9] For instance, figure 1 shows a diagrammatic representation of a Mach-Zehnder interferometer, in which an incoming photon (which is initially moving to the right from the viewer's point of view) interacts with the beamsplitter located at the bottom left of the diagram. Immediately after the photon's interaction with the beamsplitter, there are two instances of the photon, one of which follows path A while the other follows path B. In the conventional setup of the Mach-Zehnder, the probability of observing the photon on path A is $\langle\hat{\Pi}_A\rangle = 0.5$, and the probability of observing it on path B is $\langle\hat{\Pi}_B\rangle = 0.5$.

After the photon encounters the second beamsplitter, at the top right of the diagram, the photon ends up having a single, definite trajectory in the upwards direction (again from the viewer's point of view). If the photon's interaction with the beamsplitter were a stochastic process in which an incoming photon is deflected with probability 0.5, the probability of it moving in the upwards direction after the second beamsplitter would be

$$0.5\langle\hat{\Pi}_A\rangle + 0.5\langle\hat{\Pi}_B\rangle = 0.5,$$

but that is not the case. The beamsplitter causes the different instances of the photon to interfere so that the photon has probability 1 of being found moving in the upwards direction. Hence, $\langle\hat{\Pi}_A\rangle$ and $\langle\hat{\Pi}_B\rangle$ do not add as probabilities should, implying that the Born rule does not hold when there is interference.

When the Born rule does hold, the expectation values of the projectors represent probabilities, implying that the projector $\hat{\Pi}_x(\hat{X})$ is sharp with value 1 if, and only if, $\hat{X}$ is sharp with value x, as that is the only situation in which measuring the value x has probability 1. Using this fact, as well as (1.9) and the algebra of $\hat{X}$, one can obtain an explicit expression for the projectors. For instance, when $\hat{X}$ has the algebraic property that $\hat{X}^2 = \hat{1}$, its projectors are

$$\hat{\Pi}_1(\hat{X}) \stackrel{\text{def}}{=} \frac{1}{2}(\hat{1} + \hat{X}), \qquad \hat{\Pi}_{-1}(\hat{X}) \stackrel{\text{def}}{=} \frac{1}{2}(\hat{1} - \hat{X}). \tag{1.10}$$

[9] Here I mean probability not merely in the sense of obeying the probability calculus but in particular the theory of probability as used in stochastic physical theories such as the theory of Brownian motion.

It can be readily verified that these expressions for the projectors satisfy (1.9) and that, furthermore, the projectors are sharp when $\hat{X}$ is sharp – e.g. when $\langle\hat{X}\rangle = -1$, it follows that $\langle\hat{\Pi}_{-1}(\hat{X})\rangle = 1$. It is also evident that the two projectors in (1.10) sum to unity.

By combining the results (1.8) and (1.9), I find that if $\hat{X}$ is a Hermitian q-number, it admits a decomposition in terms of its projectors:

$$\hat{X} = \hat{X}\hat{1} = \sum_{x\in\mathrm{sp}(\hat{X})} \hat{X}\hat{\Pi}_x(\hat{X}) = \sum_{x\in\mathrm{sp}(\hat{X})} x\hat{\Pi}_x(\hat{X}). \tag{1.11}$$

This is the *spectral decomposition* of $\hat{X}$. Notably, $\hat{X}$ can be decomposed in terms of projectors, yet projectors cannot themselves be decomposed in terms of anything more fundamental. For example, what would the spectral decomposition of the projector $\hat{\Pi}_x(\hat{X})$ be? Its eigenvalues are equal to 0 and 1, so let $\hat{\Pi}_0\left(\hat{\Pi}_x(\hat{X})\right)$ and $\hat{\Pi}_1\left(\hat{\Pi}_x(\hat{X})\right)$ be the projectors that project onto the eigenvalues 0 and 1 of $\hat{\Pi}_x(\hat{X})$. One can verify that $\hat{\Pi}_1\left(\hat{\Pi}_x(\hat{X})\right) = \hat{\Pi}_x(\hat{X})$ and that $\hat{\Pi}_0\left(\hat{\Pi}_x(\hat{X})\right) = \hat{1} - \hat{\Pi}_x(\hat{X})$, as these expression satisfy the equivalent of algebra (1.9), and they also fulfil the appropriate form of (1.8); namely

$$\left.\begin{array}{l}\hat{\Pi}_x(\hat{X})\hat{\Pi}_1\left(\hat{\Pi}_x(\hat{X})\right) = \hat{\Pi}_1\left(\hat{\Pi}_x(\hat{X})\right)\\ \hat{\Pi}_x(\hat{X})\hat{\Pi}_0\left(\hat{\Pi}_x(\hat{X})\right) = 0\end{array}\right\}.$$

The spectral decomposition of the projector $\hat{\Pi}_x(\hat{X})$ is therefore equivalent to … the projector $\hat{\Pi}_x(\hat{X})$.

From the spectral decomposition (1.13), it follows that the expectation value of $\hat{X}$ is

$$\langle\hat{X}\rangle = \sum_{x\in\mathrm{sp}(\hat{X})} x\langle\hat{\Pi}_x(\hat{X})\rangle,$$

where $\langle\hat{\Pi}_x(\hat{X})\rangle$ represents the probability of measuring the eigenvalue x. That the quantity $\langle\Psi|\hat{X}|\Psi\rangle$, which I have pre-emptively called 'the expectation value of $\hat{X}$', is equal to the expectation value of $\hat{X}$ is not at all clear from (1.5) alone. What I have proven above is that, whenever the Born rule holds, $\langle\Psi|\hat{X}|\Psi\rangle$ is the expectation value of $\hat{X}$ (i.e. I have proven that $\langle\Psi|\hat{X}|\Psi\rangle$ is equal to the weighted sum of $x \in \mathrm{sp}(\hat{X})$, weighted by the probabilities $\langle\hat{\Pi}_x(\hat{X})\rangle$). However, in other (that is, most) circumstances, the Born rule does not hold true, and the above result is then purely formal and does not refer to probabilities.

1.8 Diagonalisation

Having discussed the spectral decomposition, we are now in the position to prove a significant result that was alluded to in §1.2; namely, that two q-numbers with identical algebras must be related to one another via a unitary transformation.

As has been established in the previous section, any Hermitian q-number has an associated spectral decomposition, shown in (1.11). The spectral decomposition has an alternative formulation to that in (1.11), which is that a Hermitian q-number permits a *diagonalisation*. That is to say that if some arbitrary q-number $\hat{A}$ is Hermitian, it can be expressed follows:

$$\hat{A} = U^{\dagger}\hat{D}U, \tag{1.12}$$

where U is a unitary and $\hat{D}$ is a so-called 'diagonal matrix' (though one need not assume that either $\hat{D}$ or $\hat{A}$ are represented by matrices). The 'diagonal matrix' has as its diagonal entries the eigenvalues of $\hat{A}$. To link this formulation of the spectral decomposition to that shown in (1.11), consider that in (1.11) a Hermitian q-number $\hat{A}$ is diagonal in the basis of its eigenstates since $\hat{A}$ acts on its eigenstates exactly as if it is a c-number. By rotating to the basis of eigenstates, one diagonalises $\hat{A}$; this is what is stated in (1.12).

Now, consider two Hermitian q-numbers $\hat{X}$ and $\hat{Y}$, which have identical algebras. As a q-number's spectrum is determined solely by its algebra, their spectra must be identical, too, so that $\mathrm{sp}(\hat{X}) = \mathrm{sp}(\hat{Y})$. Both q-numbers admit of a diagonalisation,

$$\left.\begin{aligned} \hat{X} &= U_1^{\dagger}\hat{D}_1U_1 \\ \hat{Y} &= U_2^{\dagger}\hat{D}_2U_2 \end{aligned}\right\}.$$

Here U_1 and U_2 are unitaries, and $\widehat{D}_1$ and $\widehat{D}_2$ are 'diagonal matrices' with diagonal entries that are the eigenvalues of their respective q-numbers. Because $\mathrm{sp}(\hat{X}) = \mathrm{sp}(\hat{Y})$, I shall assume that $\widehat{D}_1$ and $\widehat{D}_2$ are identical, implying that

$$\hat{X} = U_1^\dagger \widehat{D}_1 U_1 = U_1^\dagger \widehat{D}_2 U_1 = U_1^\dagger U_2 \hat{Y} U_2^\dagger U_1.$$

This proves the claim that any two Hermitian q-numbers with the same algebra must be equivalent up to a unitary transformation. This is an important result since it implies that a system's algebra can *only* be a constitutive attribute if descriptors evolve unitarily: for if the descriptors' algebra is identical at different times, the descriptors at those different times must be related to one another by a unitary transformation.

1.9 Hilbert space and matrix representation

A q-number lives on a Hilbert space $\boldsymbol{H}$ of dimension N and can be represented as an N-dimensional matrix on $\boldsymbol{H}$. Such matrix representations often make calculations more transparent and intuitive. However, a matrix representation is not strictly necessary to perform any computations. One can opt to use the algebraic relations of the descriptors, combined with their initial expectation values, to derive the descriptors' expectation values at both earlier and later times.

The size of a system's Hilbert space is determined by the algebra of its descriptors. For instance, if two q-numbers $\hat{X}$ and $\hat{Y}$ commute, and $\hat{X} \neq \hat{Y}$, and if neither q-number is equal to $\hat{1}$, then $\hat{X}$ and $\hat{Y}$ live on separate Hilbert spaces, denoted $\boldsymbol{H}_X$ and $\boldsymbol{H}_Y$ respectively (this will be discussed in more detail in §2.3). The Hilbert space of the combined q-numbers is then a tensor product of their separate Hilbert spaces,

$$\boldsymbol{H} = \boldsymbol{H}_X \otimes \boldsymbol{H}_Y,$$

and so the dimension of the absolute Hilbert space is the product of those of the individual Hilbert spaces – that is to say, $\dim(\boldsymbol{H}) = \dim(\boldsymbol{H}_X)\dim(\boldsymbol{H}_Y)$.

1.10 Q-number functions

Having covered the foundations of quantum theory, I now proceed to more technical discussions. The projectors in (1.10) are examples of *q-number functions* – i.e. they are functions that take as their input a q-number and output a different q-number. More formally, I will define an *analytic q-number function* of $\hat{X}$ as a polynomial or convergent series in $\hat{X}$ with c-number coefficients

$$f(\hat{X}) = \sum_{n=0}^{\infty} a_n \hat{X}^n, \tag{1.13}$$

where, in the case that $f(\hat{X})$ is a polynomial, the c-number coefficients a_n will be zero for all values of n greater than the degree of the polynomial.

One might wonder where the c-number coefficients $\{a_n\}$ come from since they cannot be the descriptors of physical systems. What is their physical significance? It seems they do not have any, and as it turns out, we can do without those coefficients: in §4.6, a class of q-number functions that have q-number coefficients will be analysed.

Absolute convergence of a series has to be defined relative to a *norm* – and the relevant norm for a series of q-numbers is the *operator norm*. The operator norm of an arbitrary q-number $\hat{X}$ is defined as follows:

$$\|\hat{X}\| \stackrel{\text{def}}{=} \sqrt{\sup\left(\mathrm{sp}(\hat{X}^\dagger \hat{X})\right)}, \tag{1.14}$$

where 'sup' denotes the supremum, which takes as its input a set and outputs the largest element of that set. Hence, $\|\hat{X}\|$ is equal to the square root of the largest eigenvalue of the q-number $\hat{X}^\dagger \hat{X}$. The appearance of the product $\hat{X}^\dagger \hat{X}$ in the definition (1.14) has two primary functions: firstly, it ensure that $\hat{X}^\dagger \hat{X}$ will be Hermitian so that the operator norm is well-defined even for a non-Hermitian $\hat{X}$; and secondly, it ensures the operator norm is positive valued, as the eigenvalues of $\hat{X}^\dagger \hat{X}$ are always equal to or greater than zero.[10] Notably, the definition of the operator norm (1.14) is similar to the definition of the absolute value of a complex number: if $z \in \mathbb{C}$, then the absolute value of z is $|z| \stackrel{\text{def}}{=} \sqrt{z^\dagger z}$.

I will, for the moment, assume that $\hat{X}^\dagger \hat{X}$ has a largest eigenvalue so that $\hat{X}$ is bounded – unbounded q-numbers will be discussed in §4.2. With respect to bounded q-numbers, the 'operator norm' is a norm in the following sense: the operator norm is positive definite and satisfies both absolute homogeneity (i.e. $\|a\hat{X}\| = |a|\|\hat{X}\|$ for all $a \in \mathbb{R}$) and the triangle inequality (i.e. $\|\hat{X} + \hat{Y}\| \leq \|\hat{X}\| + \|\hat{Y}\|$).

Because of the operator norm (1.14), one can define absolute convergence of a series of q-numbers. I shall say that a series of q-numbers $f(\hat{X})$ converges absolutely if the sum $\sum_{n=0}^{N} \|a_n \hat{X}^n\|$ tends to some finite value in the limit where $N \to \infty$, i.e.

$$\lim_{N \to \infty} \sum_{n=0}^{N} \|a_n \hat{X}^n\| = L,$$

where L is a real-valued constant. If the q-number series $f(\hat{X})$ is absolutely convergent, then due to the triangle inequality we find that

$$\left\| \sum_{n=0}^{N} a_n \hat{X}^n \right\| \leq \sum_{n=0}^{N} \|a_n \hat{X}^n\|.$$

Therefore, by taking the limit of both sides of the inequality, one recovers that the norm $\|f(\hat{X})\|$ is automatically equal to or smaller than L, meaning that the function is conditionally convergent, too.

[10] Even for non-Hermitian $\hat{X}$, the product $\hat{X}^\dagger \hat{X}$ has only positive, real eigenvalues – i.e. $\hat{X}^\dagger \hat{X}$ is positive semi-definite. To prove this, consider the inner product $\langle \psi | \hat{X}^\dagger \hat{X} | \psi \rangle$, where $|\psi\rangle$ is a vector and $\langle \psi |$ its Hermitian conjugate. Defining the vector $|\phi\rangle \stackrel{\text{def}}{=} \hat{X} |\psi\rangle$, one finds that $\langle \psi | \hat{X}^\dagger \hat{X} | \psi \rangle = \langle \phi | \phi \rangle \geq 0$, where the inequality is the result of the inner product being positive semi-definite.

When the input of a q-number function is $U^\dagger \hat{X} U$, then the output of that function is $U^\dagger f(\hat{X}) U$. To prove this, consider a general q-number function $f(\hat{X})$, which is defined as a polynomial or convergent series in $\hat{X}$; it then follows that

$$\left.\begin{aligned} f(\hat{X}) &= \sum_{n=0}^{\infty} a_n \hat{X}^n \\ \rightarrow U^\dagger f(\hat{X}) U &= U^\dagger \left(\sum_{n=0}^{\infty} a_n \hat{X}^n \right) U \\ \rightarrow U^\dagger f(\hat{X}) U &= \sum_{n=0}^{\infty} a_n \left(U^\dagger \hat{X} U \right)^n \end{aligned}\right\},$$

where the unitary can be moved to the interior of the sum because it is a continuous function acting on a convergent series, and the series is over elements of a finite dimensional Hilbert space. Hence,

$$f\left(U^\dagger \hat{X} U\right) = U^\dagger f(\hat{X}) U. \tag{1.15}$$

As a consequence of (1.15), the algebraic relations of a q-number function $f(\hat{X})$ are a constant of the motion. For instance, if $\left(f(\hat{X})\right)^2 = \hat{1}$, then due to the result (1.15), the function must satisfy that same constraint at any other time; that is to say, it automatically follows that $\left(f\left(U^\dagger \hat{X} U\right)\right)^2 = \hat{1}$ for any unitary U.

If $\hat{X}$ is Hermitian, then $f(\hat{X})$ is Hermitian, too, as can be readily verified by inspection of (1.13). By virtue of it being Hermitian, the function $f(\hat{X})$ has a spectral decomposition in terms of the x-eigenvalue projectors of $\hat{X}$:

$$f(\hat{X}) = \sum_{x \in \mathrm{sp}(\hat{X})} f(x) \hat{\Pi}_x(\hat{X}). \tag{1.16}$$

where $f(x)$ is the function $f(\hat{X})$ with $\hat{X}$ replaced by its eigenvalue x. It is not self-evident that (1.16) is the spectral decomposition of $f(\hat{X})$. I will first prove (1.16) for polynomials in $\hat{X}$ by using the identity that the spectral decomposition of $\hat{X}^n$, for some integer n, is

$$\hat{X}^n = \hat{X}^n \hat{1} = \hat{X}^n \sum_{x \in \mathrm{sp}(\hat{X})} \hat{\Pi}_x(\hat{X}) = \sum_{x \in \mathrm{sp}(\hat{X})} x^n \hat{\Pi}_x(\hat{X}).$$

One therefore finds that the spectral decomposition of an arbitrary polynomial $P(\hat{X})$ of degree N is

$$\left.\begin{aligned} P(\hat{X}) &= \sum_{n=0}^{N} a_n \hat{X}^n \\ \xrightarrow{1} P(\hat{X}) &= \sum_{n=0}^{N} a_n \sum_{x \in \mathrm{sp}(\hat{X})} x^n \hat{\Pi}_x(\hat{X}) \\ \xrightarrow{2} P(\hat{X}) &= \sum_{x \in \mathrm{sp}(\hat{X})} \left(\sum_{n=0}^{N} a_n x^n \right) \hat{\Pi}_x(\hat{X}) \\ \xrightarrow{3} P(\hat{X}) &= \sum_{x \in \mathrm{sp}(\hat{X})} P(x) \hat{\Pi}_x(\hat{X}) \end{aligned}\right\}.$$

At (1), the spectral decomposition of $\hat{X}^n$ is inserted; at (2) the sums are exchanged, which is unproblematic because the sums are finite; and at (3) the definition for the polynomial is used. This concludes the proof of (1.16) for polynomials.

One can prove the same result for a general analytic function $f(\hat{X})$ as follows:

$$\left.\begin{aligned} f(\hat{X}) &= \sum_{n=0}^{\infty} a_n \hat{X}^n \\ \xrightarrow{1} f(\hat{X}) &= \sum_{n=0}^{\infty} a_n \sum_{x \in \mathrm{sp}(\hat{X})} x^n \hat{\Pi}_x(\hat{X}) \\ \xrightarrow{2} f(\hat{X}) &= \sum_{n=0}^{\infty} \left(\sum_{x \in \mathrm{sp}(\hat{X})} a_n x^n \hat{\Pi}_x(\hat{X}) \right) \\ \xrightarrow{3} f(\hat{X}) &= \sum_{x \in \mathrm{sp}(\hat{X})} \left(\sum_{n=0}^{\infty} a_n x^n \hat{\Pi}_x(\hat{X}) \right) \\ \xrightarrow{4} f(\hat{X}) &= \sum_{x \in \mathrm{sp}(\hat{X})} f(x) \hat{\Pi}_x(\hat{X}) \end{aligned}\right\}.$$

At (1), the spectral decomposition of $\hat{X}^n$ is inserted. At (2) the coefficient a_n is moved to the interior of the second sum; at (3) the sums are exchanged, which is permitted for the following reason: on any topological space where the sum is a continuous function (which is certainly the case in Hilbert space) if $\mathrm{sp}(\hat{X})$ is a finite set, and $\sum_{n=0}^{\infty} \hat{b}_{n,x}$ converges for all $x \in \mathrm{sp}(\hat{X})$, then

$$\sum_{x\in \mathrm{sp}(\hat{X})} \sum_{n=0}^{\infty} \hat{b}_{n,x} = \sum_{n=0}^{\infty} \sum_{x\in \mathrm{sp}(\hat{X})} \hat{b}_{n,x}.$$

Thus for (3), one need only show that $\sum_{x\in \mathrm{sp}(\hat{X})} a_n x^n \hat{\Pi}_x(\hat{X})$ converges, which it does since $\hat{\Pi}_x(\hat{X})$ is a bounded q-number and $\sum_{x\in \mathrm{sp}(\hat{X})} a_n x^n$ converges (see below). Since that term converges, one can use the definition of $f(x)$ at (4), which concludes the proof of (1.16) for analytic q-number functions. Because $f(\hat{X})$ has a spectral decomposition that is equal to (1.16), the spectrum of $f(\hat{X})$ is

$$\mathrm{sp}\left(f(\hat{X})\right) = \{f(x) : x \in \mathrm{sp}(\hat{X})\} = f\left(\mathrm{sp}(\hat{X})\right).$$

One can show that the c-number series $f(x)$ converges by using the fact that the series $f(\hat{X})$ converges absolutely. Note that for any integer n, the operator norm $\|\hat{X}^n\|$ is greater than or equal to the absolute value of any eigenvalue of $\hat{X}^n$, as follows from the definition of the operator norm for Hermitian q-numbers. Therefore, for some eigenvalue $x \in \mathrm{sp}(\hat{X})$, one finds that

$$\sum_{n=0}^{N} |a_n x^n| \le \sum_{n=0}^{N} \|a_n \hat{X}^n\|,$$

and because the q-number series is absolutely convergent, it follows that

$$\lim_{N\to\infty} \sum_{n=0}^{N} |a_n x^n| \le \lim_{N\to\infty} \sum_{n=0}^{N} \|a_n \hat{X}^n\| = L.$$

Evidently, the series of eigenvalues also converges absolutely to some finite value, so the series of eigenvalues is finite on its whole domain $\mathrm{sp}(\hat{X})$, as required.

1.11 The minimal polynomial and the trace

An important q-number function is the *minimal polynomial* of a bounded, Hermitian q-number $\hat{X}$, which is denoted $M_{\hat{X}}(\hat{X})$ and defined as the least polynomial in $\hat{X}$ of degree equal to or greater than one such that

$$M_{\hat{X}}(\hat{X}) = 0.$$

The subscript is included because $M_{\hat{X}}$ may have other inputs besides $\hat{X}$, but the polynomial is only equal to zero for $\hat{X}$. The characteristic polynomial of $\hat{X}$ can be expressed algebraically as

$$M_{\hat{X}}(\hat{X}) = \prod_{x \in \mathrm{sp}(\hat{X})} (x - \hat{X})^{m_x}. \tag{1.17}$$

where the integer m_x is the so-called degree of the algebraic multiplicity of x.[11] For example, if the algebra of a q-numbers $\hat{X}$ is such that $\hat{X}^2 = \hat{X}$, like the algebra of a projector, then its minimal polynomial $M_{\hat{X}}(\hat{X})$ is

$$M_{\hat{X}}(\hat{X}) = \hat{X}(\hat{1} - \hat{X}) = 0.$$

Alternatively, if a q-number $\hat{Y}$ has eigenvalues 3 and 2, where the latter has algebraic multiplicity equal to 2, then it must satisfy the following algebraic expression:

$$\left.\begin{aligned} M_{\hat{Y}}(\hat{Y}) &= 0 \\ \rightarrow \left(2 - \hat{Y}\right)^2\left(3 - \hat{Y}\right) &= 0 \\ \rightarrow \left(4 - 4\hat{Y} + \hat{Y}^2\right)\left(3 - \hat{Y}\right) &= 0 \\ \rightarrow 12 - 16\hat{Y} + 15\hat{Y}^2 - \hat{Y}^3 &= 0 \end{aligned}\right\}.$$

Q-number functions – such as the minimal polynomial – map q-numbers to q-numbers. There is also a second class of functions that depend on q-numbers, namely those functions that take as their input a q-number and output a c-number. An important example of this is the *trace* of a Hermitian q-number $\hat{X}$. Without referring to state vectors, the trace of $\hat{X}$ may be defined as follows:

[11] In the Schrödinger picture, the algebraic multiplicity of an eigenvalue x of $\hat{X}$ is the dimension of the subspace spanned by the eigenvectors of $\hat{X}$ with eigenvalue x.

$$\mathrm{tr}(\hat{X}) \stackrel{\text{def}}{=} \sum_{x \in \mathrm{sp}(\hat{X})} m_x x, \tag{1.18}$$

where m_x is again the algebraic multiplicity of the eigenvalue x and can be obtained from (1.17). For example, the unit observable $\hat{1}$ has a spectrum consisting of only one eigenvalue, i.e. $\mathrm{sp}(\hat{1}) = \{1\}$. But the algebraic multiplicity of this eigenvalue is equal to the dimension of the Hilbert space $\boldsymbol{H}$ on which that q-number lives:

$$\mathrm{tr}(\hat{1}) = \dim(\boldsymbol{H}).$$

(1.18) and (1.17) are similar in structure: the former is a sum of the eigenvalues while the latter is a product of them. Indeed, the trace of a q-number can be defined through its minimal polynomial. Let $M_{e^{\hat{X}}}$ be the minimal polynomial of $e^{\hat{X}}$, so that $M_{e^{\hat{X}}}(e^{\hat{X}}) = 0$; then $M_{e^{\hat{X}}}(0)$ is equal to

$$M_{e^{\hat{X}}}(0) = \prod_{x \in \mathrm{sp}(\hat{X})} (e^x)^{m_x},$$

and one finds that

$$\mathrm{tr}(\hat{X}) = \ln\left(M_{e^{\hat{X}}}(0)\right).$$

The trace is an inner product for Hermitian q-numbers; it can be straight-forwardly shown from the definition (1.18) that the trace satisfies the following three conditions:

- The trace is symmetric: $\mathrm{tr}(\hat{X}\hat{Y}) = \mathrm{tr}(\hat{Y}\hat{X})$ if $\hat{X}$ and $\hat{Y}$ are Hermitian and commute.
- The trace is linear: $\mathrm{tr}(\hat{X}a + \hat{Y}b) = \mathrm{tr}(\hat{X})a + \mathrm{tr}(\hat{Y})b$ for $a, b \in \mathbb{R}$.
- The trace is positive definite: $\mathrm{tr}(\hat{X}^2) = \sum_{x \in \mathrm{sp}(\hat{X})} x^2 > 0$ if $\hat{X}$ is Hermitian and non-zero.

The fundamental principle of quantum theory forbids c-numbers from being valid descriptors of physical systems. Yet c-numbers appear to accurately describe the real world – e.g. the number of dogs on Earth seems to be an integer whereas, according to quantum theory, this cannot be so. To borrow a phrase from Eugene Wigner (1960), how can the 'unreasonable effectiveness' of c-number descriptors be explained by quantum theory?

A partial solution to this is that one can define c-number functions in terms of q-number functions via the trace:

$$f(x) = \operatorname{tr}\left(f(\hat{X})\hat{\Pi}_x(\hat{X})\right) \qquad\qquad \left(x \in \operatorname{sp}(\hat{X})\right),$$

where I have used (1.16) and (1.9) to derive this result. That c-number functions can be defined in terms of q-number functions provides a hand-waving argument for why quantum theory can explain the unreasonable effectiveness of c-number descriptors, despite the world fundamentally being quantum. A more complete explanation of why c-number descriptors are so unreasonably effective will be provided in §3, where I will discuss Everett's solution to this problem.

1.12 The Schrödinger picture

In the Heisenberg picture of quantum theory, the time-dependent entities are the q-number descriptors. In the Schrödinger picture, all time-dependence is instead transferred to the state vector, so in that picture, the q-numbers become time-invariant.

The two pictures are related via a unitary transformation: given a Heisenberg state $|\Psi\rangle$ and a set of descriptors $\hat{X}_1(t), \hat{X}_2(t), \dots, \hat{X}_n(t)$ that evolve unitarily under the unitary $U(t)$, one can move to the Schrödinger picture by applying the unitary $U(t)$ to the state vector instead of the descriptors:

The Heisenberg picture		**The Schrödinger picture**
$\{\hat{X}_j(t)\}$	$\xleftrightarrow{\text{observables}}$	$\{\hat{X}_j(0)\}$
$\lvert\Psi\rangle$	$\xleftrightarrow{\text{state vector}}$	$\lvert\Psi(t)\rangle \stackrel{\text{def}}{=} U(t)\lvert\Psi\rangle$

The $\{\hat{X}_j(0)\}$ are the time-independent observables of the Schrödinger picture, which are equivalent to the Heisenberg observables $\{\hat{X}_j(t)\}$ at the initial time $t = 0$. Notably, this choice of observables makes both pictures completely identical at $t = 0$. That is to say, both the observables and state vectors of the Heisenberg and Schrödinger pictures are equivalent at this time so that the unitary that transforms between the pictures at $t = 0$ is the unit observable $\hat{1}$.

In the Schrödinger picture, $\langle\Psi(t)|\hat{X}_j(0)|\Psi(t)\rangle$ represents the expectation value of the observable $\hat{X}_j(0)$, whereas in the Heisenberg picture the expectation value of that same observable at time t is $\langle\Psi|\hat{X}_j(t)|\Psi\rangle$.[12] Due to the fact that these two quantities are equivalent, both pictures make identical predictions, i.e.

$$\langle\Psi(t)|\hat{X}_j(0)|\Psi(t)\rangle = \langle\Psi|\hat{X}_j(t)|\Psi\rangle.$$

A theory is, however, more than the sum of its predictions: it is first and foremost an explanation – a description of something in the world, how it behaves, and why (Deutsch (2016), Popper (2002)). The Schrödinger and Heisenberg pictures are significantly different *explanations* of quantum theory, despite the fact that they make identical predictions – as I shall stress repeatedly throughout this thesis. The most important explanatory difference between these pictures is that the Schrödinger picture is not a local description of quantum theory, whereas the Heisenberg picture is, as I shall discuss in detail in §2.5.[13]

[12] Incidentally, this result was first proven by Dirac; the history of that discovery is discussed by Farmelo (2009, p. 112).

[13] There are additional reasons to assume that the Heisenberg and Schrödinger picture are not identical. For instance, Dirac (1965) argued that there exist quantum field theories that admit of solutions in the Heisenberg picture but which result in pathological descriptions in the Schrödinger picture.

2 Quantum computational networks

2.1 Qubits

The theory of quantum computation revolves around *qubits* and the computational tasks they can perform. And although quantum computation is a theory of physics – and is therefore a theory about physical objects – qubits do not exist in the same way that, say, electrons, protons, and photons do. Unlike those physical systems, qubits are an abstraction: physical systems such as electrons, protons, and photons can instead be used to *instantiate* an abstract qubit. That is to say that the laws of physics do not put constraints, short of perfection, on how well physical systems can approximate a qubit.

A system instantiates a qubit when it has a *Boolean observable*, which is any observable that has exactly two eigenvalues. We have encountered a Boolean previously in §1: a Boolean is a q-number $\hat{q}_x$ that satisfies the algebra

$$(\hat{q}_x)^{\dagger} = \hat{q}_x \quad \text{and} \quad (\hat{q}_x)^{2} = \hat{1},$$

implying that it has exactly two eigenvalues, 1 and -1. Any other Boolean can be expressed as an affine transformation of a q-number with the algebra of $\hat{q}_x$; for instance, $\frac{1}{2}\left(\hat{q}_x + \hat{1}\right)$ is also a Boolean and has eigenvalues 0 and 1, as shown in (1.7).

Whenever a quantum system has a descriptor with the algebra of $\hat{q}_x$, it must also have other descriptors that fail to commute with $\hat{q}_x$. Otherwise, that system would have only commuting descriptors and violate the fundamental principle of quantum theory. In general, when the system instantiates a qubit, it has a triple of Hermitian descriptors, denoted $\hat{q}_x$, $\hat{q}_y$, and $\hat{q}_z$, that adhere to the *Pauli algebra*:

$$\left.\begin{aligned} \hat{q}_x\hat{q}_y &= i\hat{q}_z \\ (\hat{q}_x)^2 &= \hat{1} \end{aligned}\right\} \qquad \text{(and permutations over } x, y, z\text{).}$$

A physical example of a quantum Boolean is the spin of an electron since it is an observable whose sharp values are 1 and -1. Another Boolean is the path of a photon in an interference experiment, like shown in figure 1, during which a photon travels along one of two trajectories. Because of this restriction on the path of the photon, the observable that represents the trajectory that the photon has taken can only have two sharp values. This latter example is of interest because, unlike the spin of an electron – which is fundamentally a Boolean – the trajectory of a photon is typically an observable with a continuous spectrum. Yet by physically restricting and manipulating the photon, it too instantiates a qubit.

A quantum computational network is a system consisting of n qubits, denoted $\mathfrak{Q}_1,\ldots,\mathfrak{Q}_n$, that can interact with one another via gates. Following Deutsch & Hayden (2000) and Gottesman (1999), the state of a qubit $\mathfrak{Q}_a$ in the Heisenberg picture, at some general time t, is represented by a triple of time-dependent descriptors

$$\hat{\boldsymbol{q}}_a(t) = \left(\hat{q}_{ax}(t),\ \hat{q}_{ay}(t),\ \hat{q}_{az}(t)\right), \tag{2.1}$$

which invariably satisfy the Pauli algebra

$$\left.\begin{aligned} \hat{q}_{ai}(t)\hat{q}_{aj}(t) &= \delta_{ij}\hat{1} + i\epsilon_{ij}{}^{k}\hat{q}_{ak}(t) \\ \hat{q}_{ai}(t)^{\dagger} &= \hat{q}_{ai}(t) \end{aligned}\right\} \qquad \text{(for all } i,j,k \in \{x,y,z\}\text{).} \tag{2.2}$$

Here and throughout, I will use Einstein's summation convention: whenever an index appears twice in a product and is not otherwise defined, it will then implicitly be summed over all its values – for example, the index k in (2.2) is summed over all values in $\{x,y,z\}$.

When the descriptor $\hat{q}_{az}(t)$ is sharp with one of its eigenvalues (those are 1 and -1) and the qubit's dynamics are such that $\hat{q}_{az}(t)$ remains sharp for a certain duration, then during this time $\hat{q}_{az}(t)$ can function *as if* it is a classical bit. But even in such circumstances, the qubit as a whole is never literally equivalent to a classical bit: at least in principle, it is always possible for $\hat{q}_{az}(t)$ to become non-sharp, whereas a classical bit can *only* be sharp. And even when one of the descriptors of the qubit is sharp, there are also the other descriptors of the qubit, some of which must be non-sharp due to the non-commutative algebra of those descriptors.

This is true of the multiverse in general: the multiverse might sometimes be approximated as a collection of quasi-classical systems (i.e. Everettian 'universes'), but the multiverse as a whole is more than the sum of its quasi-classical parts. There are, for instance, always some descriptors of the multiverse that are non-sharp, and Everettian 'universes' have the counterfactual property that they can interfere with one another, even if they do not.[14] Hence why the Everettian 'universes' are called *quasi*-classical, which implies they are not simply classical.

The descriptors $\hat{q}_{ax}(t)$, $\hat{q}_{ay}(t)$, and $\hat{q}_{az}(t)$ of an individual qubit $\mathfrak{Q}_a$ are linearly independent, and therefore any observable of $\mathfrak{Q}_a$ can be expressed as a linear combination of these descriptors and the unit $\hat{1}$. This follows from the fact that the descriptors satisfy the identity that

$$\left.\begin{aligned} \operatorname{tr}\left(\hat{q}_{aj}(t)\right) &= 0 \\ \operatorname{tr}\left(\left(\hat{q}_{aj}(t)\right)^2\right) &= \operatorname{tr}(\hat{1}) \end{aligned}\right\} \qquad \text{(for all } j \in \{x, y, z\}\text{)}.$$

Since the trace is an inner product for q-numbers, the descriptors $\hat{q}_{ax}(t)$, $\hat{q}_{ay}(t)$, $\hat{q}_{az}(t)$ and $\hat{1}$ must be an orthogonal basis that spans a vector space of Hermitian q-numbers. This can be proven as follows: if the q-numbers $\hat{q}_{ax}(t)$, $\hat{q}_{ay}(t)$, $\hat{q}_{az}(t)$ and $\hat{1}$ are linearly independent then the expression

$$\alpha\hat{1} + \beta\hat{q}_{ax}(t) + \gamma\hat{q}_{ay}(t) + \delta\hat{q}_{az}(t) = 0 \tag{2.3}$$

should only be satisfied if the real-valued coefficients α, β, γ and δ are all equal to zero. And indeed, the only case in which (2.3) holds true is if they are all zero: by taking the trace of the left and right side of (2.3), one finds that $\alpha = 0$. By multiplying the left and right side of (2.3) by $\hat{q}_{ax}(t)$ and subsequently tracing over the resulting equation, one finds that $\beta = 0$. In like manner, one can show that γ and δ are equal to zero, too. Evidently, these descriptors are

[14] In the Heisenberg picture, there are always some descriptors that are non-sharp due to q-numbers having a non-commutative algebra. The equivalent statement in the Schrödinger picture is that the state vector can have a relative phase.

linearly independent and constitute an orthogonal basis that spans a vector space of q-numbers, denoted $\boldsymbol{O}_a(t)$, i.e.

$$\boldsymbol{O}_a(t) \stackrel{\text{def}}{=} \operatorname{span}\{\hat{q}_{ax}(t), \hat{q}_{ay}(t), \hat{q}_{az}(t)\}. \tag{2.4}$$

(To simplify calculations below, I have left out the unit observable from this vector space, but note that it too is linearly independent of the descriptors $\hat{q}_{ax}(t), \hat{q}_{ay}(t)$, and $\hat{q}_{az}(t)$.)

When considered in isolation, the descriptors in the vector space $\boldsymbol{O}_a(t)$ can be represented as 2×2 matrices. The Pauli matrices are conventionally used to represent these descriptors, so at the initial time $t = 0$, let the matrix representation of the descriptors, as defined through an isomorphism R that maps q-numbers to matrices, be as follows:

$$\left.\begin{array}{r} R(\hat{q}_{ax}(0)) = \hat{\sigma}_x \stackrel{\text{def}}{=} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ R\left(\hat{q}_{ay}(0)\right) = \hat{\sigma}_y \stackrel{\text{def}}{=} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\ R(\hat{q}_{az}(0)) = \hat{\sigma}_z \stackrel{\text{def}}{=} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{array}\right\}. \tag{2.5}$$

To abbreviate this matrix representation, it is convenient to use the Pauli vector, defined as $\hat{\boldsymbol{\sigma}} \stackrel{\text{def}}{=} (\hat{\sigma}_x,\ \hat{\sigma}_y,\ \hat{\sigma}_z)$, so that

$$R(\hat{\boldsymbol{q}}_a(0)) = \hat{\boldsymbol{\sigma}}.$$

The Pauli matrices live on a 2-dimensional Hilbert space, denoted $\boldsymbol{H}_a$, and on this Hilbert space, the unit observable's matrix representation is the 2×2 identity matrix

$$R(\hat{1}) = \hat{1}_1 \stackrel{\text{def}}{=} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

I have here explicitly included the isomorphism R, which maps q-numbers to matrices while preserving their algebra (as can be readily verified, the Pauli matrices satisfy the Pauli algebra (2.2), like their name suggests). As a slight abuse of notation, the isomorphism R shall be omitted in later sections because it is clear from context that the q-numbers are being represented by matrices. Note, however, that q-numbers are not identical to matrices; they are only equivalent to them under a suitable isomorphism such as R.[15]

An isolated qubit has descriptors that can be represented by Hermitian 2×2 matrices, but the algebra of the descriptors of larger systems (e.g. networks of qubits) is more complicated. The descriptors of those systems are therefore represented by larger matrices. For instance, as we shall see in §2.2, the matrix representation of a two-qubit system requires 4×4 matrices because the descriptors of spacelike-separated qubits must commute.

2.2 The commutation constraint

Although (2.2) specifies the algebra of individual qubits in the network, it leaves unspecified the algebraic relations between separate qubits. Therefore, it is conventional to impose the constraint that the descriptors of any pair of qubits mutually commute:

$$\left[\hat{q}_{ai}(t), \hat{q}_{bj}(t)\right] = 0 \qquad \text{(for all } i, j \in \{x, y, z\} \text{ and } a \neq b\text{).} \qquad (2.6)$$

This requirement that the descriptors of spacelike-separated systems must mutually commute is what I shall name the *commutation constraint*. The necessity of this constraint will be called into question in §5, but for the moment I will assume it to be a principle of quantum theory.[16]

Due to the commutation constraint, products such as $\hat{q}_{ai}(t)\hat{q}_{bj}(t)$ are Hermitian even when $a \neq b$ since

$$\left.\begin{aligned} \left(\hat{q}_{ai}(t)\hat{q}_{bj}(t)\right)^\dagger &= \hat{q}_{bj}(t)^\dagger \hat{q}_{ai}(t)^\dagger \\ &= \hat{q}_{bj}(t)\hat{q}_{ai}(t) \\ &= \hat{q}_{ai}(t)\hat{q}_{bj}(t) \end{aligned}\right\}.$$

[15] Notably, a matrix representation is not fundamental since it is merely a way to instantiate the algebra of q-numbers in terms of matrices through an isomorphism from the former to the latter, and there is a continuous infinity of such representations. It is because matrices are less abstract than q-numbers – and in the case of qubits, because the use of the Pauli matrices has become standardised – that a matrix representation often simplifies calculations.

[16] In this thesis, I will not explicitly treat fermions, which anti-commute at spacelike separations.

Here, to get from the penultimate line to the final line, one reorders the descriptors by using the commutation constraint. Therefore, $\hat{q}_{ai}(t)\hat{q}_{bj}(t)$ is an observable; its eigenvalues are 1 and -1 because

$$\left(\hat{q}_{ai}(t)\hat{q}_{bj}(t)\right)^2 = \hat{q}_{ai}(t)^2\hat{q}_{bj}(t)^2 = \hat{1},$$

so its minimal polynomial is

$$\left(\hat{1} - \hat{q}_{ai}(t)\hat{q}_{bj}(t)\right)\left(\hat{1} + \hat{q}_{ai}(t)\hat{q}_{bj}(t)\right) = 0.$$

It follows from the minimal polynomial that the trace of a product $\hat{q}_{ai}(t)\hat{q}_{bj}(t)$ and that of its square are

$$\left.\begin{aligned} \operatorname{tr}\left(\hat{q}_{ai}(t)\hat{q}_{bj}(t)\right) &= 0 \\ \operatorname{tr}\left(\left(\hat{q}_{ai}(t)\hat{q}_{bj}(t)\right)^2\right) &= \operatorname{tr}(\hat{1}) \end{aligned}\right\} \qquad \text{(for all } i,j \in \{x,y,z\} \text{ and } a \neq b\text{)}.$$

Therefore, the descriptors of separate qubits, and products of those descriptors, form an orthogonal basis of q-numbers, implying that an arbitrary observable $\hat{A}(t)$ of such a network can be expressed as

$$\hat{A}(t) = \alpha_0\hat{1} + \alpha^i\hat{q}_{ai}(t) + \beta^j\hat{q}_{aj}(t) + \gamma^{ij}\hat{q}_{ai}(t)\hat{q}_{aj}(t),$$

where α_0, α^i, β^j and γ^{ij} are real numbers for $i,j \in \{x,y,z\}$.

(2.5) is the conventional matrix representation of a singular qubit. That matrix representation can be generalised to a system of two qubits by using the tensor product. Let the matrix representation for a two-qubit system be such that, at the initial time $t = 0$, the qubits' descriptors are

$$\left.\begin{aligned} \hat{\boldsymbol{q}}_a(0) &= \hat{1}_1 \otimes \hat{\boldsymbol{\sigma}} \\ \hat{\boldsymbol{q}}_b(0) &= \hat{\boldsymbol{\sigma}} \otimes \hat{1}_1 \end{aligned}\right\}.$$

This ensures that in isolation, a qubits' descriptors satisfy the Pauli algebra, just as the representation (2.5) does. The tensor products with the 2×2 identity matrices ensure that all of the components $\hat{q}_{ai}(0)$ and $\hat{q}_{bj}(0)$ commute (where $i,j \in \{x,y,z\}$):

$$\hat{q}_{ai}(0)\hat{q}_{bj}(0) = (\hat{\sigma}_i \otimes \hat{1}_1)(\hat{1}_1 \otimes \hat{\sigma}_j) = \hat{\sigma}_i \otimes \hat{\sigma}_j = (\hat{1}_1 \otimes \hat{\sigma}_j)(\hat{\sigma}_i \otimes \hat{1}_1) = \hat{q}_{bj}(0)\hat{q}_{ai}(0).$$

Evidently, a system of two qubits has a matrix representation in terms of Hermitian 4×4 matrices, and since those matrices live on the tensor product of the Hilbert spaces of the individual qubits, the Hilbert space of the two-qubit system is $\boldsymbol{H}_a \otimes \boldsymbol{H}_b$.

The descriptors of the two qubits are linearly independent of one another, so no element in the set $\boldsymbol{\mathcal{O}}_a(t)$, as defined in (2.4), can be written in terms of an element in $\boldsymbol{\mathcal{O}}_b(t)$. In other words, the intersection of $\boldsymbol{\mathcal{O}}_a(t)$ and $\boldsymbol{\mathcal{O}}_b(t)$ is the empty set:

$$\boldsymbol{\mathcal{O}}_a(t) \cap \boldsymbol{\mathcal{O}}_b(t) = \emptyset.$$

As we have seen above, the descriptors in $\boldsymbol{\mathcal{O}}_a(t)$ and those in $\boldsymbol{\mathcal{O}}_b(t)$ commute with one another and live on separate Hilbert spaces. This result generalises: whenever the elements of one set of q-numbers commute with those of another set of q-numbers, and neither set contains the unit observable, and the intersection of the sets is the empty set – like is the case for $\boldsymbol{\mathcal{O}}_a(t)$ and $\boldsymbol{\mathcal{O}}_b(t)$ – then the elements of those sets live on separate Hilbert spaces. Additionally, the overall Hilbert space is then equal to the tensor product of their separate Hilbert spaces.

This straightforwardly generalises to products of descriptors of more than two qubits. Thus, such descriptors span a space of Hermitian q-numbers, and any observable of the network can be expressed algebraically in terms of those descriptors (i.e. they can be expressed as a combination of sums and products and scalar multiples of the q-numbers $\{\hat{q}_{aj}(t)\}$).

2.3 Matrix representation of a network of n qubits

For a network $\mathfrak{N}$ consisting of n qubits, each qubit's descriptors can be expressed as a Hermitian $2^n \times 2^n$ matrix. For the n-qubit network $\mathfrak{N}$, the initial representation (2.5) has the form

$$\hat{\boldsymbol{q}}_a(0) = \hat{1}_{a-1} \otimes \hat{\boldsymbol{\sigma}} \otimes \hat{1}_{n-a}, \tag{2.7}$$

where $\hat{1}_k$ is a tensor product of k instances of the 2×2 identity matrix $\hat{1}_1$, i.e.

$$\hat{1}_k \stackrel{\text{def}}{=} \underbrace{\hat{1}_1 \otimes \hat{1}_1 \otimes \ldots \otimes \hat{1}_1}_{k\ times}.$$

The initial matrix representation (2.7) of the descriptors will change as soon as the network starts to evolve. But since the descriptors of $\mathfrak{N}$ evolve unitarily, the algebra of the descriptors is guaranteed to be a constant of the motion, as discussed in §1.2.

To fully specify the network $\mathfrak{N}$, one requires not only the time-dependent descriptors but also $\mathfrak{N}$'s constant Heisenberg state $|\Psi\rangle$. I shall fix the Heisenberg state to be a standard constant by requiring that the initial ($t = 0$) expectation values of the network's z-observables are

$$\langle \hat{q}_{az}(0) \rangle = 1 \qquad\qquad (1 \leq a \leq n). \tag{2.8}$$

Thus, the Heisenberg state $|\Psi\rangle$ is an eigenstate of each of those z-observables, which specifies $|\Psi\rangle$ up to an irrelevant phase factor. Because of my convention (2.8), the Heisenberg 'state' contains no information about the physical state of $\mathfrak{N}$, and the state of the network is fully specified by the time-dependent descriptors of $\mathfrak{N}$.

2.4 Dynamics and gates

In the network model, the state of the qubits is assessed periodically, where the duration of the period is conventionally assumed to be 1 unit of time. During each such period, the network's qubits are made to enact a *gate*, which takes the qubits' input states and returns specific output states, determined by the particular gate that is enacted. Because the network's descriptors evolve unitarily, gates are represented by unitaries: a general gate **G** is implemented by the unitary $U_{\mathbf{G}}$, which is a function of the descriptors of $\mathfrak{N}$. If gate **G** acts between time t and $t+1$, its effect on $\mathfrak{Q}_a$ is

$$\hat{\boldsymbol{q}}_a(t+1) = U_{\mathbf{G}}^{\dagger}(\hat{\boldsymbol{q}}_1(t), \ldots, \hat{\boldsymbol{q}}_n(t))\, \hat{\boldsymbol{q}}_a(t) U_{\mathbf{G}}(\hat{\boldsymbol{q}}_1(t), \ldots, \hat{\boldsymbol{q}}_n(t)). \tag{2.9}$$

Perhaps the simplest quantum gate is the unit wire, denoted **I**. For an n-qubit network, the unitary $U_{\boldsymbol{I}}$ that implements the unit wire is represented by the unit observable:

$$U_{\boldsymbol{I}}(\hat{\boldsymbol{q}}_1(t), \dots, \hat{\boldsymbol{q}}_n(t)) = \hat{1}_n. \tag{2.10}$$

By using both (2.9) and (2.10), one finds that this gate leaves the qubits' descriptors completely unaffected during the period in which it acts – i.e. the descriptors of an arbitrary qubit in the network, denoted $\mathfrak{Q}_a$, evolve as follows under the unit wire:

$$\hat{\boldsymbol{q}}_a(t) \xrightarrow{\text{unit wire}} \hat{\boldsymbol{q}}_a(t+1) = \hat{\boldsymbol{q}}_a(t). \tag{2.11}$$

Hence, the effect of the unit wire is mathematically trivial since it merely delays the qubit by one unit of time. Implementing the unit wire on a physical qubit is, however, a non-trivial task. Qubits (and quantum systems generally) have a proclivity to interact with their environment, and these interactions change the state of the qubit, preventing the enactment of a unit wire. Counteracting those interactions with the environment is the main hurdle to overcome in building a quantum computer; that the hurdle can in principle be overcome is something we know from the theory of quantum error-correction (see for example the work of Bennett et al. (1996) and Ekert & Macchiavello (1996)).

In the Heisenberg picture, the unitary $U_{\mathbf{G}}$, which represents the general gate **G,** is time-dependent, so it does not have a time-invariant matrix representation, like it does in the Schrödinger picture. $U_{\mathbf{G}}$ does, however, have a characteristic expression in terms of the time-dependent descriptors. For example, the **not** gate is a single-qubit gate which, when it acts on qubit $\mathfrak{Q}_a$, toggles the value of that qubit's z-observable; the expression of the **not** gate in terms of the descriptors is

$$U_{\mathbf{not},a}\big(\hat{\boldsymbol{q}}_1(t), \dots, \hat{\boldsymbol{q}}_n(t)\big) = \hat{q}_{ax}(t). \tag{2.12}$$

Notably, the only unitary whose matrix representation is invariant is that of the unit wire, which is always $U_{\mathbf{I}} = \hat{1}_n$. That is so because the identity operator $\hat{1}_n$ commutes with all other matrices, leaving it entirely unaffected by any gate.

The effect of the unit wire has been expressed in two different ways: (2.10) depicts the gate's unitary, and (2.11) depicts the effect that this unitary has on the descriptors. This is a general feature of gates: a gate can be defined either in terms of its unitary or via its algebraic effect on the descriptors, and both those definitions are equivalent. That is so because a general gate **G** is implemented by a unitary U_G, which is a characteristic function of the descriptors. The effect that this unitary has on the descriptors is therefore determined solely by the algebraic relations of the network, which are invariant.

For instance, using the general formula (2.9) and the characteristic expression of the **not** gate (2.12), one finds that after the application of the **not** gate on $\mathfrak{Q}_a$, the descriptors of $\mathfrak{Q}_a$ at time $t+1$ have the following unique and fixed expression in terms of its descriptors at time t:

$$\textbf{not}: \qquad \hat{\boldsymbol{q}}_a(t+1) = \left(\hat{q}_{ax}(t),\ -\hat{q}_{ay}(t),\ -\hat{q}_{az}(t)\right), \tag{2.13}$$

while all other qubits in the network are unaffected. This way of defining a gate has the advantage that it makes clear how the gate affects the descriptors. It is for instance evident from the expression (2.13) that the eigenvalues of the z-observable of $\mathfrak{Q}_a$ are toggled by the **not** gate, since $\hat{q}_{az}(t+1) = -\hat{q}_{az}(t)$.

Another useful gate is the rotation $\boldsymbol{R}_x(\theta)$, which rotates the a'th qubit by an angle of θ radians about its x-axis; it is implemented by the unitary $e^{i\frac{\theta}{2}\hat{q}_{ax}(t)}$. The effect of $\boldsymbol{R}_x(\theta)$ on the descriptors of $\mathfrak{Q}_a$ is that[17]

$$\hat{\boldsymbol{q}}_a(t+1) = \left(\hat{q}_{ax}(t),\ \cos\theta\,\hat{q}_{ay}(t) + \sin\theta\,\hat{q}_{az}(t),\ \cos\theta\,\hat{q}_{az}(t) - \sin\theta\,\hat{q}_{ay}(t)\right), \tag{2.14}$$

while the descriptors of all other qubits remain unaffected. Hence it is exactly as if a rotation matrix has been applied to the descriptors $\hat{\boldsymbol{q}}_a(t+1)$.

2.5 Locality

In this thesis, I shall treat it as a principle of physics that all physical theories should be locally realistic. By this I mean that theories should satisfy *Einsteinian locality*, which requires that, as

[17] This can be verified by using the following identity: the exponential of a q-number $\hat{X}$ with the property that $\hat{X}^2 = \hat{1}$ is

$$e^{i\theta\hat{X}} = \cos\theta\,\hat{1} + i\sin\theta\,\hat{X}\,.$$

Einstein succinctly put it, 'the real factual situation of the system S_2 is independent of what is done with the system S_1, which is spatially separated from the former' (Einstein (1970, p. 85)).[18]

It is often assumed that quantum theory is non-local since the Schrödinger picture violates Einstein's criterion. In the Schrödinger picture, the real factual situation of a system is described by its state vector, which provides a 'complete' description of the system (i.e. it can be used to compute any expectation value of the system's observables), but it is not a local entity. That is to say that if the state vector represents the real factual situation of a system, then operations performed on one system appear to affect another system which is spacelike-separated from the former. Consider, for instance, a two-qubit system, consisting of the qubits $\mathfrak{Q}_1$ and $\mathfrak{Q}_2$, which are in the following entangled state at the initial time $t = 0$:

$$|\Psi(0)\rangle = |1\rangle_1|1\rangle_2 + |-1\rangle_1|-1\rangle_2, \tag{2.15}$$

where the subscripts indicate to which qubit the vectors belong, and the states $|1\rangle_a$ and $|-1\rangle_a$ are the eigenstates of $\hat{q}_{az}(0)$ with eigenvalues 1 and -1, respectively. Furthermore, let the Schrödinger-picture observables of each qubit be the time-independent triples $\hat{\boldsymbol{q}}_1(0)$ and $\hat{\boldsymbol{q}}_2(0)$.

When single-qubit gates are performed on the entangled state (2.15), the state vector does not retain any information about which gates were performed on which qubits. For instance, suppose the rotation $U(\hat{\boldsymbol{q}}_1(0)) = e^{-i\frac{\theta}{2}}e^{i\frac{\theta}{2}\hat{q}_{1z}(0)}$ is enacted, which rotates $\mathfrak{Q}_1$ about its z-axis by θ radians. (This unitary is similar to the single-qubit gate $\boldsymbol{R}_x(\theta)$ discussed earlier, except that $U(\hat{\boldsymbol{q}}_1(0))$ rotates the qubit about its z-axis and has an additional phase factor $e^{-i\frac{\theta}{2}}$ to simplify the results below.) The effect of $U(\hat{\boldsymbol{q}}_1(0))$ on the state vector is that

$$U(\hat{\boldsymbol{q}}_1(0))|\Psi(0)\rangle = |1\rangle_1|1\rangle_2 + e^{-i\theta}|-1\rangle_1|-1\rangle_2.$$

[18] Although it is a principle that physical theories are local in Einstein's sense, they can nonetheless be Bell non-local due to the different requirements that these two types of locality impose. For instance, quantum theory satisfies Einsteinian locality, but it does not admit a formulation in terms of locally hidden real-variables and is therefore Bell non-local.

Had $\mathfrak{Q}_2$ been rotated by the angle θ about its z-axis instead of $\mathfrak{Q}_1$, the state vector would not have revealed that fact. The unitary that performs this rotation on $\mathfrak{Q}_2$ is $U'(\hat{\boldsymbol{q}}_2(0)) = e^{-i\frac{\theta}{2}}e^{i\frac{\theta}{2}\hat{q}_{2z}(0)}$, and this rotation results in the follow state (Raymond-Robichaud (2021)):

$$U'(\hat{\boldsymbol{q}}_2(0))|\Psi(0)\rangle = |1\rangle_1|1\rangle_2 + e^{-i\theta}|-1\rangle_1|-1\rangle_2 = U(\hat{\boldsymbol{q}}_1(0))|\Psi(0)\rangle.$$

Therefore, if one assumes that the real factual situation of the two qubits is represented by the state vector, the factual situation of $\mathfrak{Q}_1$, after it is rotated by θ radians about its z-axis, is exactly what the state would have been had that same rotation been performed on $\mathfrak{Q}_2$ instead. That is so because both those rotations introduce the same relative phase $e^{-i\theta}$ to the state vector, and the relative phase is non-local – i.e. the phase $e^{-i\theta}$ does not belong exclusively to either qubit. We must therefore conclude that the Schrödinger picture violates Einstein's criterion.

Although the Schrödinger picture is a non-local description, quantum theory is nevertheless a locally realistic theory: when the situation described above is translated into the Heisenberg picture, one finds that $\mathfrak{Q}_1$ is unaffected by operations performed on $\mathfrak{Q}_2$, just as Einstein required be the case. And since quantum theory admits one such formulation in which the description of the two-qubit system is local, quantum theory satisfies Einstein's criterion. That the theory also admits a non-local formulation of the same situation is unimportant since it is a feature of local theories that they typically admit both local and non-local descriptions (Raymond-Robichaud (2021), Kuypers & Deutsch (2021)).[19]

To prove that the Heisenberg picture is a local formulation of quantum theory, consider the previously described qubits but this time in the Heisenberg picture. In that picture, the qubits' real factual situations are represented by the their triples of descriptors. If one performs the rotation $U(\hat{\boldsymbol{q}}_1(0))$ on $\mathfrak{Q}_1$ it will manifestly leave the descriptors of $\mathfrak{Q}_2$ unchanged, as only the descriptors of the qubit that participate in the rotation will be affected:

[19] For example, classical mechanics is a local theory which allows for a non-local formulation (Raymond-Robichaud (2021)).

$$\left.\begin{aligned} U^{\dagger}(\hat{\boldsymbol{q}}_1)\hat{\boldsymbol{q}}_1 U(\hat{\boldsymbol{q}}_1) &= \left(\cos\theta\,\hat{q}_{1x} + \sin\theta\,\hat{q}_{1z},\ \ \cos\theta\,\hat{q}_{1y} - \sin\theta\,\hat{q}_{1x},\ \ \hat{q}_{1z}\right) \\ U^{\dagger}(\hat{\boldsymbol{q}}_1)\hat{\boldsymbol{q}}_2 U(\hat{\boldsymbol{q}}_1) &= \left(\hat{q}_{2x},\ \ \hat{q}_{2y},\ \ \hat{q}_{2z}\right) \end{aligned}\right\},$$

where I have suppressed the $t = 0$ time labels for the sake of brevity. Likewise, the rotation $U'\big(\hat{\boldsymbol{q}}_2(0)\big)$ only affects $\mathfrak{Q}_2$ since

$$\left.\begin{aligned} U'^{\dagger}(\hat{\boldsymbol{q}}_2)\hat{\boldsymbol{q}}_1 U'(\hat{\boldsymbol{q}}_2) &= \left(\hat{q}_{1x},\ \ \hat{q}_{1y},\ \ \hat{q}_{1z}\right) \\ U'^{\dagger}(\hat{\boldsymbol{q}}_2)\hat{\boldsymbol{q}}_2 U'(\hat{\boldsymbol{q}}_2) &= \left(\cos\theta\,\hat{q}_{2x} + \sin\theta\,\hat{q}_{2y},\ \ \cos\theta\,\hat{q}_{2y} - \sin\theta\,\hat{q}_{2x},\ \ \hat{q}_{2z}\right) \end{aligned}\right\}.$$

So, unlike the Schrödinger-picture state vector, which does not contain information about which operations were performed on which qubits, the Heisenberg-picture descriptors manifestly do contain that information. In this sense the Heisenberg-picture descriptors are similar to the descriptors of classical physics: the time-dependent q-numbers are functions of q-numbers at earlier times and therefore represent the history of the qubits, just as the time-dependent c-number descriptors of classical systems do.

The above-described result about locality can be generalised to a network of n interacting qubits. For instance, let a gate $\mathbf{S}$ represent an operation on the first m qubits in an n-qubit network (where $m < n$) so that the unitary of gate $\mathbf{S}$ is a function of only those initial m qubits, i.e. $U_{\boldsymbol{S}}(\hat{\boldsymbol{q}}_1(t), \dots, \hat{\boldsymbol{q}}_m(t))$. Then due to the commutation constraint (2.6), the unitary $U_{\boldsymbol{S}}(\hat{\boldsymbol{q}}_1(t), \dots, \hat{\boldsymbol{q}}_m(t))$ will commute with all qubits that do not participate in the gate, and the effect of gate $\mathbf{S}$ on the k'th qubit, provided that $n \geq k > m$, is as follows:

$$\hat{\boldsymbol{q}}_k(t+1) = U_{\boldsymbol{S}}^{\dagger}\big(\hat{\boldsymbol{q}}_1(t), \dots, \hat{\boldsymbol{q}}_m(t)\big)\hat{\boldsymbol{q}}_k(t) U_{\boldsymbol{S}}\big(\hat{\boldsymbol{q}}_1(t), \dots, \hat{\boldsymbol{q}}_m(t)\big) = \hat{\boldsymbol{q}}_k(t). \qquad (2.16)$$

Evidently, the descriptors of the k'th qubit are left entirely unaltered between t and $t+1$, and only the descriptors of the qubits that participate in the gate (namely the first m qubits of the network) will change during this period, as required. This is true even when the qubits are entangled, for the following reason: (2.16) implies that Einsteinian locality is satisfied, yet I have made no assumptions about whether or not the qubits are in an entangled state, as that would be a feature of the descriptors *and* the stationary Heisenberg state. It is because the

Heisenberg state does not change that the result shown in (2.16) must be independent of the presence of entanglement between qubits in the network.

It follows from (2.16) that any quantum system which satisfies the commutation constraint will automatically satisfy Einstein's criterion of locality. The reverse is not true: the commutation constraint is a sufficient but not a necessary requirement for Einsteinian locality, as even without the commutation constraint, quantum systems can satisfy Einstein's criterion. For instance, fermions violate the commutation constraint because their descriptors anti-commute at spacelike separations, yet fermions do satisfy Einsteinian locality (Marletto et al. (2021), Tibau Vidal et al. (2022)). Similarly, in §5, I shall present a quantum theory of qubits that violate the commutation constraint, and it will be demonstrated that those qubits also satisfy Einsteinian locality.

2.6 Wallace–Timpson transformations

Some authors have argued that the Heisenberg picture has unnecessary degrees of freedom. For example, Timpson (2005) and Timpson & Wallace (2007) raised the objection that the Heisenberg picture admits of transformations that change the descriptors without altering their expectation values, in light of which the physical significance of the descriptors is unclear. In detail, their criticism is as follows: given a network of n qubits, whose state at some time t is represented by a set of descriptors $\{\hat{\boldsymbol{q}}_k(t)\}$ and a Heisenberg state $|\Psi\rangle$, one can perform a so-called Wallace-Timpson transformation, denoted $V(t)$, which has the defining property that

$$V(t)|\Psi\rangle = e^{i\phi(t)}|\Psi\rangle,$$

where $\phi(t)$ is some real-value function of time. Using the unitary $V(t)$, one can define a set of transformed descriptors, namely

$$\hat{\boldsymbol{q}}'_k(t) \stackrel{\text{def}}{=} V^\dagger(t)\hat{\boldsymbol{q}}_k(t)V(t) \qquad\qquad (1 \leq k \leq n).$$

The expectation values of each $\hat{\boldsymbol{q}}'_k(t)$ are equivalent to the expectation value of $\hat{\boldsymbol{q}}_k(t)$, which is why Wallace & Timpson (2007) write that, 'nothing whatever – no observable data, no theoretical considerations – can tell us that the physical state is given by [the $\hat{\boldsymbol{q}}_k(t)$] rather than [the $\hat{\boldsymbol{q}}'_k(t)$]'.

Although the Wallace–Timpson transformation leaves the descriptors' expectation values unaffected, it is not a dynamical symmetry of the system because $V(t)$ alters the dynamics of the descriptors, as Deutsch (2011) pointed out. So, to regard systems with descriptors $\hat{\boldsymbol{q}}_k(t)$ and $\hat{\boldsymbol{q}}'_k(t)$ as equivalent is tantamount to claiming there is no fact about what equations of motion those qubits obey – a questionable claim. For instance, that claim can be experimentally refuted: at least in principle, it is possible to reconstruct the unitary U that has acted on a system during a certain period, up to an irrelevant phase factor. One hypothetical procedure for doing so is the following: for some initial condition, evolve a network of qubits forward by some arbitrary unitary U (in practice, this amounts to preparing the physical qubits so that they will interact in some arbitrary way) for a duration of, say, 1 unit of time. By repeatedly preparing that same network and measuring it after that 1 unit of time has elapsed, one performs a tomographic procedure through which one recovers the network's final state. If that tomographic procedure is in turn enacted repeatedly, for many different initial conditions, one can recover from the experimental data about the initial and final states of the system what the unitary U must have been that acted on the system, showing that U is empirically significant.

In response to the criticisms of Wallace & Timpson, attempts have been made to determine the number of parameters needed to fully specify the descriptors of a network of qubits. For instance, Hewitt-Horsman & Vedral (2007a) worked on this problem, and later Bédard (2021), who was perhaps the first to successfully settle the issue. He solved the problem by noting that a unitary U, acting on a network between $t = 0$ and $t = 1$, is an encoding of all the descriptors of the network. That is to say: if one is told that the descriptors of the network at the initial and final times are $\{\hat{\boldsymbol{q}}_k(0)\}$ and $\left\{U^\dagger\hat{\boldsymbol{q}}_k(0)U\right\}$, then U can be reconstructed, up to an irrelevant phase factor, from those sets of descriptors (Bédard (2021a), Bédard (2021b)). Hence the descriptors of the network are a near exact encoding of the unitary U: everything that can be known about U can also be deduced from those descriptors, up to the irrelevant phase factor. Similarly, given $\{\hat{\boldsymbol{q}}_k(0)\}$ and U, one can deduce the expression for the descriptors at later times, so there is a bijection between the network's descriptors and its unitary.

Because of this bijection between the unitary U and the network's descriptors, Bédard (2021) found the parameters necessary to specify the descriptors by solving the simpler problem of determining the number of parameters needed to specify U. He proved that the number of parameters necessary to specify U for a network of n qubits is equal to $2^{2n} - 1$, whereas by comparison, the universal state vector of a network of n qubits has $2^{n+1} - 1$ parameters. Thus, a system's descriptors contain far more parameters than its state vector does.

Bédard (2021) has referred to this difference between the descriptors and the state vector as the 'cost of locality' because, in order to describe the real factual situation of a system in the Heisenberg picture, one must commit to many more real-valued parameters than seem necessary in the Schrödinger picture. Yet this 'cost' pales in comparison to what is gained from adopting the Heisenberg picture, namely a genuinely local description of quantum theory in which there is no 'spooky action at a distance'.

Moreover, Bédard's result about the descriptors can be interpreted another way: to specify a system's evolution in the Schrödinger picture, one requires a unitary U that satisfies the Schrödinger equation, which acts on the universal state-vector to evolve it. So, although the state vector has far fewer parameters than the descriptors, in both pictures of quantum theory one fundamentally has to rely on the unitary U, which has $2^{2n} - 1$ parameters. What is therefore presented as a cost of the Heisenberg picture is really no cost at all, as in both pictures a complete description of a system and its evolution require the unitary U. The only cost to consider when choosing between the Heisenberg and Schrödinger pictures is therefore between a local and a non-local description and not between different sets of parameters.

2.7 Locally inaccessible information

If quantum theory is locally realistic, what then accounts for the apparent 'spooky action at a distance' during Bell-type experiments? During these experiments, two spacelike-separated observers, commonly named Alice and Bob, each have access to one of a pair of Bell-state qubits, and both Alice and Bob have full knowledge of the initial state of the pair of qubits. Because of this, Alice can measure her qubit and use the results of her measurement and her knowledge of the qubits' state to predict, with complete certainty, what Bob will observe when he measures his qubit, despite Bob being entirely uncertain about the outcome of his measurement. So there appears to be 'action at a distance'.

The explanation for these phenomena is *locally inaccessible information* – i.e. information that is stored locally in at least two quantum systems, but which can only be retrieved via measurements on those combined systems (Deutsch & Hayden (2000)). In particular, locally inaccessible information cannot be retrieved via measurements on only one of those systems in isolation from the others. Consider, for instance, two entangled qubits $\mathfrak{Q}_1$ and $\mathfrak{Q}_2$, whose Schrödinger picture state-vector at time t is equal to $|1\rangle_1|1\rangle_2 + |-1\rangle_1|-1\rangle_2$, meaning the qubits are in a Bell state. This implies that, in the Heisenberg picture, the qubits' descriptors at time t have the following expectation values:

$$\left.\begin{array}{l}\langle\hat{\boldsymbol{q}}_1(t)\rangle = (0,0,0)\\ \langle\hat{\boldsymbol{q}}_2(t)\rangle = (0,0,0)\end{array}\right\},$$

and

$$\left.\begin{array}{l}\langle\hat{q}_{1x}(t)\hat{q}_{2x}(t)\rangle = 1\\ \langle\hat{q}_{1y}(t)\hat{q}_{2y}(t)\rangle = -1\\ \langle\hat{q}_{1z}(t)\hat{q}_{2z}(t)\rangle = 1\end{array}\right\}. \qquad (2.17)$$

Then, at time t, $\mathfrak{Q}_1$ effects a gate $\boldsymbol{R}_z(\theta)$, which rotates $\mathfrak{Q}_1$ by an angle of θ radians about its z-axis. That is to say, the descriptors of $\mathfrak{Q}_1$ evolve as follows:

$$\left.\begin{array}{l}\hat{\boldsymbol{q}}_1(t+1) = \left(\cos\theta\,\hat{q}_{1x}(t) - \sin\theta\,\hat{q}_{1y}(t),\;\; \cos\theta\,\hat{q}_{1y}(t) + \sin\theta\,\hat{q}_{1x}(t),\;\; \hat{q}_{1z}(t)\right)\\ \hat{\boldsymbol{q}}_2(t+1) = \left(\hat{q}_{2x}(t),\;\; \hat{q}_{2y}(t),\;\; \hat{q}_{2z}(t)\right)\end{array}\right\}. \qquad (2.18)$$

Manifestly, the information about the angle θ is located in the x- and y-observables of the triple $\hat{\boldsymbol{q}}_1(t+1)$, yet none of the descriptors' expectation values reflect the fact that it contains this information since

$$\langle \hat{\boldsymbol{q}}_1(t+1) \rangle = (0, \ 0, \ 0). \tag{2.19}$$

θ cannot, therefore, be detected through observing this specific qubit. Only by observing a descriptor of each qubit, and then bringing their resulting values to the same location, can that angle be retrieved – note for instance that

$$\langle \hat{q}_{1x}(t+1)\hat{q}_{2x}(t+1) \rangle = \cos\theta.$$

Thus, the angle θ, though measurable through measurements of both $\mathfrak{Q}_1$ and $\mathfrak{Q}_2$, is locally inaccessible: no measurement of either $\mathfrak{Q}_1$ and $\mathfrak{Q}_2$ in isolation, nor even repeated measurements or tomographic procedures performed separately on either of those qubits, will reveal that the descriptors of $\mathfrak{Q}_1$ depend on θ. It is only via comparing measurements performed on both qubits that θ can be retrieved. And such measurements are in the first instance performed locally on either qubit, after which their results must be carried to the same location before they can be compared.

Furthermore, during Bell-type experiments, observers can only make conditional predictions. Consider once more the observer Alice, who possesses the qubit $\mathfrak{Q}_1$, and who will make predictions about Bob and his qubit $\mathfrak{Q}_2$. At time t, those qubits are in a state for which $\hat{q}_{1z}(t)\hat{q}_{2z}(t)$ is sharp with value 1, as shown in (2.8), and both Alice and Bob have full knowledge of the state of $\mathfrak{Q}_1$ and $\mathfrak{Q}_2$ at that initial time. If Alice were to measure the z-observable of her qubit, having been told that Bob will likewise measure the z-observable of his qubit, she knows with certainty that the product of these measurement results will be 1, since Alice knows that

$$\langle \hat{q}_{1z}(t)\hat{q}_{2z}(t) \rangle = 1. \tag{2.20}$$

She can therefore deduce from her own measurement result and from (2.20) what Bob's result has to be – e.g. if her measurement of $\hat{q}_{1z}(t)$ yields the result 'it was a -1', then she knows that Bob's measurement of $\hat{q}_{2z}(t)$ will be 'it was a -1', too, because the product of the their results needs to be equal to 1, according to (2.20).

However, Alice's prediction depends entirely on Bob measuring the z-observable of $\mathfrak{Q}_2$, and it is possible that Bob instead decides to measure the x-observable of his qubit. In which case, Alice cannot predict with any certainty what Bob's result will be since $\hat{q}_{1x}(t)\hat{q}_{2z}(t)$ is non-sharp – its expectation values is $\langle\hat{q}_{1x}(t)\hat{q}_{2z}(t)\rangle = 0$.[20] Because quantum theory is local, Alice is unable to tell in which basis Bob performed his measurements, making her unable to predict Bob's measurement result in a general situation. To determine the result of Bob's measurement, she and Bob must meet to compare their results.

Moreover, it is misleading to say that Alice knows the result of a measurement that has yet to take place because, in quantum theory, measurements do not necessarily have a definite result. And information about Bob's measurement (such as information about which observable he measured) does not exist before that measurement is performed, after which it will have to flow to Alice (say, via a message) before she can definitively conclude anything about his result. Prior to her receiving this information about Bob's qubit and his measurement, she will be unaware of whether Bob has made a measurement at all. It is only when Alice and Bob agree beforehand what measurements they will perform that they can predict the other's result. Should either of them deviate from their prior agreements, whether by choice or through error, that will be unnoticeable by the other before they meet – because quantum theory is local.

[20] Q-numbers that do not commute cannot be sharp at the same time. $\hat{q}_{1x}(t)\hat{q}_{2z}(t)$ and $\hat{q}_{1z}(t)\hat{q}_{2z}(t)$ do not commute, and since the latter is sharp, the former must be non-sharp.

3 Everettian relative states in the Heisenberg picture

The following section concerns the work of myself & Deutsch (2021) on Everett's relative state construction. In §1, I discussed the 'unreasonable effectiveness' of c-number descriptors and the problem this presents to quantum theory. In fact, that problem is identical to the so-called measurement problem, which may be stated as follows: why do the observables of physical systems appear to have c-number values despite the underlying reality being represented by q-number descriptors? Many parochial and anthropomorphic solutions have been proposed to solve this problem, such as Wigner's (1967) theory that human consciousness plays a fundamental role in collapsing the state vector. Similarly, in Bohr's (1928) Copenhagen Interpretation, there is a fundamental separation between the small-scale 'quantum world' and the large-scale 'classical world', which raises the question of where the cut-off between those 'worlds' might be and whether it can ever be fundamental, as Everett (1973) pointed out.[21] Moreover, there is the *totalitarian principle* of quantum theory (see Marletto & Vedral (2022)), which states that any system that interacts with a quantum system must itself be a quantum system, casting further doubt upon such a separation.

The measurement problem was solved by Everett (1973). Instead of treating the quantum theory of measurements as a set of axioms of the theory, Everett created a physical model of measurement, which successfully reproduces all the results that are otherwise assumed to be axiomatic. Like all great ideas, there is a sense in which Everett's construction seems like it could not have been otherwise: measuring instruments (or measurers for short) are quantum systems (e.g. they consist of atoms) whose behaviour is therefore described by quantum

[21] Incidentally, the idea that there is a cut-off between the 'quantum world' and the 'classical world' does not make sense in light of Bohr's (1911) own discoveries about physics. In particular, the Bohr-van–Leeuwen theorem (van Vleck (1978)) states that no classical theory of physics can account for the existence of ferromagnetic materials. So, one must either believe that (macroscopic) ferromagnets are still part of the 'quantum world', raising the question of how large systems have to be before they are considered part of the 'classical world', or that there is no such cut-off.

theory. So, it makes sense to analyse the abilities and limitations of measurers via quantum theory.

In Everett's construction, a measurement performed by one system on another causes information to flow from the measured system to the measurer. Because of the primacy of information in Everett's construction, the most convenient – and the most fundamental – way of explaining his construction is in terms of quantum computation. Before describing his solution in terms of qubits, I shall first elaborate on some general features of the theory of measurement.

3.1 What are measurers and what is a measurement?

Let us deviate from quantum theory for a moment and consider what is meant by a measurement of a general physical system, whether it be classical or quantum. As an intuitive example of a measurement, one might consider how to measure the acidity of soil found in a typical garden. Certain flowers, such as the hydrangea, have petals that bloom either red or blue, depending on the acidity of the soil they are planted in. And although a hydrangea's petals are usually only a rough indicator of the acidity of soil, let us idealise the flower and assume that its petals being blue have the precise physical meaning 'the soil's pH is below 7', and likewise their being red means that 'the soil's pH is above 7'.[22] In this idealised case, measuring the colour of the petals will, with certainty, tell one whether the acidity of the soil has a pH above or below 7. Therefore, the flower's petals carry information about the acidity of the soil, and planting a hydrangea in soil would be one way of measuring the soil's acidity, albeit a somewhat time-consuming and unconventional method of doing so. Nonetheless, this implies that the flower and its petals are a measurer of the soil's acidity.

The details of this example can be abstracted away so that a measurement of some general system $\mathfrak{X}$ can be stated as follows: a measurement of a physical system $\mathfrak{X}$, which is assumed

[22] This is an idealisation: in reality a hydrangea is not a perfect measurer of the soil's acidity, so it would not, for instance, be able to distinguish sharply between soil with pH above and below 7. Instead, the flower is an imperfect measurer and error bars should be associated to its measurement results. Moreover, there will also be types of soil too acidic or basic for the flower to survive in, among other limitations of the flower.

to be in some state x, is a process in which $\mathfrak{X}$ interacts with a *measurer of* $\mathfrak{X}$ so that, through this interaction, the measurer should end up in a state M_x with the physical meaning '$\mathfrak{X}$ is in the state x'. To state this more succinctly, assume that the measurer is initially in some receptive state M_0 so that, before the measurement has taken place, the state of the composite system can be denoted by the Cartesian product (M_0, x). Then the complete measurement process can be summarised as follows:

$$(M_0, x) \xrightarrow{\text{measurement}} (M_x, x) \qquad\qquad (x \in \mathcal{D}_{\mathfrak{X}}). \tag{3.1}$$

Here $\mathcal{D}_{\mathfrak{X}}$ is some set of *distinguishable states* of the system $\mathfrak{X}$. Two states x and y are distinguishable from one another if there exists some measurer which can perform protocol (3.1) so that, after the measurement, the corresponding states M_x and M_y of the measurer represent distinct classical bit-strings of an appropriate length. Furthermore, the bit strings M_x and M_y should be able to participate in a classical reversible computation (i.e. a permutation of the set of such bit strings), and they must be copiable by other physical systems. The latter requirement ensures that the measurement result can be read from the measurer; that is to say that the measurer can itself be measured through a protocol of the form (3.1). For a more detailed explanation of these issues, see Deutsch & Marletto (2015).

The process in (3.1) can occur by mere happenstance, like the above-described hydrangea which just so happens to be a measurer for the acidity of garden-variety soil. Or the measurement can have been purposefully planned, such as when an experimenter carefully designs a measurer to determine, say, the energy spectrum of an atom in a lab. In either case, a measurement is an objective physical process: it is a matter of fact that information from the measured system flows to the measurer during their interaction.

That the state of one physical system can represent the state of another is a remarkable feature of nature, and one without which measurements – and science in general – would have been impossible. The world need not have been this way: neither logic nor mathematics require it to be a feature of nature that measurements can be performed or that information can flow from one system to another. It is the laws of physics that do: a system $\mathfrak{X}$ can be measured if the existence of measurer for $\mathfrak{X}$ is permitted by the laws of physics.

3.2 Measurers in quantum theory

Quantum theory allows for the measurement protocol (3.1) to be performed. This is perhaps best demonstrated in terms of a model consisting of two qubits $\mathfrak{Q}_M$ and $\mathfrak{Q}_S$, which are represented by the triples $\hat{\boldsymbol{q}}_M(t)$ and $\hat{\boldsymbol{q}}_S(t)$. During this model measurement, the qubit $\mathfrak{Q}_M$ will be an ideal measurer of the z-observable of $\mathfrak{Q}_S$, which represents some to-be-measured system. The ideal measurement interaction between these two qubits is modelled by a **cnot** gate. $\mathfrak{Q}_S$ will be the *control* and $\mathfrak{Q}_M$ will be the *target* of the **cnot**, and the unitary that represents this **cnot** gate is (Kuypers & Deutsch (2021))

$$U_{\mathbf{cnot}}\big(\hat{\boldsymbol{q}}_M(t), \hat{\boldsymbol{q}}_S(t)\big) = \hat{\Pi}_1\big(\hat{q}_{Sz}(t)\big) + \hat{\Pi}_{-1}\big(\hat{q}_{Sz}(t)\big)U_{\boldsymbol{not},M}\big(\hat{\boldsymbol{q}}_M(t)\big). \tag{3.2}$$

Here $\hat{\Pi}_a\big(\hat{q}_{Sz}(t)\big)$ denotes the a-eigenvalue projector of $\hat{q}_{Sz}(t)$. Consequently, the effect of the **cnot** is that it performs a **not** gate on the target $\mathfrak{Q}_M$ in those parts of the multiverse for which $\hat{q}_{Sz}(t)$ has value -1, whereas the **cnot** gate performs a unit wire on $\mathfrak{Q}_M$ in those parts of the multiverse for which $\hat{q}_{Sz}(t)$ has value 1.

Assuming the measurement starts at the initial time t, the effect of the **cnot** can be expressed algebraically as follows (Deutsch & Hayden (2000)):

$$\mathbf{cnot}: \qquad \begin{Bmatrix} \hat{\boldsymbol{q}}_M(t+1) \\ \hat{\boldsymbol{q}}_S(t+1) \end{Bmatrix} = \begin{Bmatrix} \big(\hat{q}_{Mx}(t),\ \hat{q}_{My}(t)\hat{q}_{Sz}(t),\ \hat{q}_{Mz}(t)\hat{q}_{Sz}(t)\big) \\ \big(\hat{q}_{Sx}(t)\hat{q}_{Mx}(t),\ \hat{q}_{Sy}(t)\hat{q}_{Mx}(t),\ \hat{q}_{Sz}(t)\big) \end{Bmatrix}. \tag{3.3}$$

In particular, the effect of the **cnot** on the z-observable of the target is that

$$\hat{q}_{Mz}(t) \xrightarrow{\mathbf{cnot}} \hat{q}_{Mz}(t+1) = \hat{q}_{Mz}(t)\hat{q}_{Sz}(t).$$

Due to the **cnot** gate, $\hat{q}_{Mz}(t+1)$ has come to depend on both $\hat{q}_{Mz}(t)$ and $\hat{q}_{Sz}(t)$, and it is because of this that information about $\hat{q}_{Sz}(t)$ has flowed to $\hat{q}_{Mz}(t+1)$. Conversely, the z-observable of the control is left entirely unchanged by the interaction:

$$\hat{q}_{Sz}(t) \xrightarrow{\mathbf{cnot}} \hat{q}_{Sz}(t+1) = \hat{q}_{Sz}(t). \tag{3.4}$$

The measurement interaction allows $\mathfrak{Q}_M$ to act as a measurer of $\mathfrak{Q}_S$'s z-observable: let $\mathfrak{Q}_M$ be in a receptive state with the property that $\langle \hat{q}_{Mz}(t) \rangle = 1$, and let the qubit $\mathfrak{Q}_S$ be sharp with the value $\langle \hat{q}_{Sz}(t) \rangle = \pm 1$. The totality of the qubit's expectation values at the initial time t are then

$$\left.\begin{aligned} \langle \hat{\boldsymbol{q}}_M(t) \rangle &= (0,\ \ 0,\ \ 1) \\ \langle \hat{\boldsymbol{q}}_S(t) \rangle &= (0,\ \ 0,\ \ \pm 1) \end{aligned}\right\}.$$

By using the algebraic relation (3.3) and the expectation values of the qubits, one can deduce that the expectation values at $t+1$ must be

$$\left.\begin{aligned} \langle \hat{\boldsymbol{q}}_M(t+1) \rangle &= (0,\ \ 0,\ \ \pm 1) \\ \langle \hat{\boldsymbol{q}}_S(t+1) \rangle &= (0,\ \ 0,\ \ \pm 1) \end{aligned}\right\}.$$

Accordingly, after the application of the **cnot**, the sharp values 1 and -1 of $\hat{q}_{Mz}(t)$ have the physical meaning 'the control was a 1' and 'the control was a -1'.

The z-observable of $\mathfrak{Q}_S$ initially stores a single bit of information, as it is either prepared in the state for which its z-observable is sharp with the value $+1$ or -1. This bit of information, stored in $\mathfrak{Q}_S$, is then copied into the z-observable $\mathfrak{Q}_M$ through the measurement interaction. And since that information has a definite location within the descriptors of the qubits (due to the Heisenberg picture being local), that information has unequivocally *flowed* from the system $\mathfrak{Q}_S$ to the measurer $\mathfrak{Q}_M$.

To show the connection between protocol (3.1) and the quantum theory of measurement as discussed here, consider the following *attributes* of $\mathfrak{Q}_S$ (Marletto & Vedral (2020))

$$\left.\begin{aligned} \mathbf{0}_S &= \{\hat{\boldsymbol{q}}_S : \langle \hat{q}_{Mz}(t) \rangle = 1\} \\ \mathbf{1}_S &= \{\hat{\boldsymbol{q}}_S : \langle \hat{q}_{Mz}(t) \rangle = -1\} \end{aligned}\right\}.$$

Here I have suppressed the time labels for the sake of brevity. The attributes $\mathbf{0}_S$ and $\mathbf{1}_S$ are sets of states for which $\mathfrak{Q}_S$ satisfies a particular condition. In the above-described attributes, the conditions are that the qubit's z-observable is sharp with the value 1 or -1. If the qubit is, for instance, said to have the attribute $\mathbf{0}_S$, this means the qubit's descriptors are contained in the set $\mathbf{0}_S$.

Likewise, $\mathfrak{Q}_M$ has attributes $\mathbf{0}_M$ and $\mathbf{1}_M$, in terms of which the measurement protocol discussed in this section can be summarised as follows: the measurer initially has the receptive attribute $\mathbf{0}_M$, so when $\mathfrak{Q}_S$ has the attributes $\mathbf{0}_S$ or $\mathbf{1}_S$, those qubits evolve as follows:

$$\left.\begin{matrix} (\mathbf{0}_M, \mathbf{0}_S) \xrightarrow{\mathbf{cnot}} (\mathbf{0}_M, \mathbf{0}_S) \\ (\mathbf{0}_M, \mathbf{1}_S) \xrightarrow{\mathbf{cnot}} (\mathbf{1}_M, \mathbf{1}_S) \end{matrix}\right\}.$$

Since after the measurement, $\mathbf{0}_M$ and $\mathbf{1}_M$ have the unequivocal physical meaning 'it was a 0' and 'it was a 1', this makes explicit the connection with (3.1). It also demonstrates that $\mathbf{0}_S$ and $\mathbf{1}_S$ are distinguishable attributes: the attributes of the measurer, after the measurement, represent bits that can participate in reversible classical computations, and they can be copied onto another system through the application of another **cnot** gate of which the measurer is the control.

Notably, the **cnot** gate also ensures that the x-observable of $\mathfrak{Q}_S$ is copied onto the x-observable of $\mathfrak{Q}_M$ since

$$\hat{q}_{Sx}(t) \xrightarrow{\mathbf{cnot}} \hat{q}_{Sx}(t+1) = \hat{q}_{Sx}(t)\hat{q}_{Mx}(t),$$

whereas the x-observable of $\mathfrak{Q}_M$ remains entirely unchanged,

$$\hat{q}_{Mx}(t) \xrightarrow{\mathbf{cnot}} \hat{q}_{Mx}(t+1) = \hat{q}_{Mx}(t).$$

The way the x-observables of the 'target' and 'control' evolve is analogous to how the qubits' z-observables evolve, except that in the x-basis the 'target' and 'control' qubits have opposite roles. That is to say that, in that basis $\mathfrak{Q}_S$ will be the target and $\mathfrak{Q}_M$ the control. Whether a system is being measured or is acting as a measurer is therefore not clear from the descriptors at time $t+1$ alone: it also depends on the Heisenberg state, which determines which of the qubits' descriptors contain copiable information, and on how the information ends up being used after the measurement interaction. Typically, one is not entirely free to choose how to use the information in a system, and this restricts which systems can be measurers – e.g. decoherence puts severe constraints on how the information in a system can be used (Zurek (1981), (1982)).

3.3 Measurement of non-sharp descriptors

A measurement of a sharp observable has a definite measurement outcome, but that is not the case when the observable being measured is non-sharp. During such measurements, there is no fact about what the measurer records, except that the measurer records all possible results. For instance, consider again the qubits $\mathfrak{Q}_M$ and $\mathfrak{Q}_S$ which interact via a **cnot** gate between some initial time $t = 0$ and a final time $t = 1$. As in the previous section, assume that $\mathfrak{Q}_M$ is in a receptive state with the property that $\langle \hat{q}_{Mz}(0) \rangle = 1$, but now let the z-observable of $\mathfrak{Q}_S$ be such that $\langle \hat{q}_{Sz}(0) \rangle = 0$, implying this z-observable is non-sharp. If $\mathfrak{Q}_M$ measures $\mathfrak{Q}_S$ via the **cnot** interaction, then the expectation values of those qubits evolve as follows:

$$\begin{Bmatrix} \langle \hat{\boldsymbol{q}}_M(0) \rangle \\ \langle \hat{\boldsymbol{q}}_S(0) \rangle \end{Bmatrix} = \begin{Bmatrix} (0, & 0, & 1) \\ (1, & 0, & 0) \end{Bmatrix} \xrightarrow{\mathbf{cnot}} \begin{Bmatrix} \langle \hat{\boldsymbol{q}}_M(1) \rangle \\ \langle \hat{\boldsymbol{q}}_S(1) \rangle \end{Bmatrix} = \begin{Bmatrix} (0, & 0, & 0) \\ (0, & 0, & 0) \end{Bmatrix}.$$

Thus, the **cnot** operation evolves the target and control qubits into a mixed state, implying that those qubits are entangled with one another after the application of the **cnot**.[23] And as the z-observable of the measurer is non-sharp, there is no fact about which exact result the measurer recorded.

Yet a measurement did take place, as is apparent from the effect that the **cnot** has on the target's descriptors, which is

$$\hat{\boldsymbol{q}}_M(1) = \hat{\boldsymbol{q}}_M(0)\, \hat{\Pi}_1\big(\hat{q}_{Sz}(0)\big) + \Big(U^\dagger_{\mathbf{not}}\big(\hat{\boldsymbol{q}}_M(0)\big)\hat{\boldsymbol{q}}_M(0) U_{\mathbf{not}}\big(\hat{\boldsymbol{q}}_M(0)\big)\Big)\, \hat{\Pi}_{-1}\big(\hat{q}_{Sz}(0)\big). \tag{3.5}$$

[23] That the qubits are in mixed states follows from their density operators and the expectation values of their descriptors. For instance, the density operator of $\mathfrak{Q}_S$ at some time t is

$$\hat{\rho}_S(t) = \frac{1}{\mathrm{tr}(\hat{1})}\Big(\hat{1} + \langle \hat{q}_{Sx}(t) \rangle \hat{q}_{Sx}(0) + \langle \hat{q}_{Sy}(t) \rangle \hat{q}_{Sy}(0) + \langle \hat{q}_{Sz}(t) \rangle \hat{q}_{Sz}(0)\Big).$$

So, using the expectation values of that qubit at time $t = 1$, one finds that $\hat{\rho}_S(t) = \frac{1}{\mathrm{tr}(\hat{1})}\hat{1}$, which is the characteristic expression for the density operator of a qubit in a maximally mixed state.

Described in words, the effect that the descriptors $\hat{\boldsymbol{q}}_M(1)$ experienced according to (3.5) is as follows: in that part of the multiverse in which $\hat{q}_{Sz}(0)$ has the value 1, the target was subjected to a unit wire, whereas in the part of the multiverse for which $\hat{q}_{Sz}(0)$ has the value -1, the target was subjected to a **not** gate. $\hat{q}_{Sz}(0)$ is non-sharp and assumes both the value 1 and -1 in different sectors of the multiverse, causing the measurer $\mathfrak{Q}_M$, upon measuring $\hat{q}_{Sz}(0)$, to *branch* into two instances, each of which records one of those eigenvalues of $\hat{q}_{Sz}(0)$. These branches are represented by triples of relative descriptors, defined as follows: the descriptors of $\mathfrak{Q}_M$ relative to $\hat{q}_{Sz}(0)$ having the value 1 or -1 are

$$\left.\begin{array}{ll} \mathfrak{Q}_{M,1}: & \hat{\boldsymbol{q}}_{M,1}(1) \stackrel{\text{def}}{=} \hat{\boldsymbol{q}}_M(1)\hat{\Pi}_1\big(\hat{q}_{Sz}(1)\big) \\ \mathfrak{Q}_{M,-1}: & \hat{\boldsymbol{q}}_{M,-1}(1) \stackrel{\text{def}}{=} \hat{\boldsymbol{q}}_M(1)\hat{\Pi}_{-1}\big(\hat{q}_{Sz}(t)\big) \end{array}\right\}. \tag{3.6}$$

Notably, because of the commutation constraint, the projectors $\hat{\Pi}_1\big(\hat{q}_{Sz}(1)\big)$ and $\hat{\Pi}_{-1}\big(\hat{q}_{Sz}(1)\big)$ commute with the descriptors of $\mathfrak{Q}_M$, which is why the products in (3.6) are well-defined. Additionally, because $\hat{q}_{Sz}(1) = \hat{q}_{Sz}(0)$, the projectors $\hat{\Pi}_1\big(\hat{q}_{Sz}(1)\big)$ and $\hat{\Pi}_{-1}\big(\hat{q}_{Sz}(1)\big)$ that appear in (3.6) project onto the eigenvalues of $\hat{q}_{Sz}(0)$, as required.

Although there is no definite measurement result for the absolute descriptors, the relative descriptors hold a definite record, as each of the instances $\mathfrak{Q}_{M,1}$ and $\mathfrak{Q}_{M,-1}$ recorded a distinct eigenvalue of $\hat{q}_{\mathrm{S}z}(1)$. For instance, the relative descriptor $\hat{q}_{(M,1)z}(1)$ is sharp with value 1 since

$$\frac{\langle \hat{q}_{(M,1)z}(1)\rangle}{\langle \hat{\Pi}_1\big(\hat{q}_{2z}(1)\big)\rangle} = \frac{\langle \hat{q}_{Mz}(1)\hat{\Pi}_1\big(\hat{q}_{Sz}(1)\big)\rangle}{\langle \hat{\Pi}_1\big(\hat{q}_{2z}(1)\big)\rangle} = 1,$$

where the term in the denominator ensures that probabilities remain normalised in each relative state. So, the instance of the measurer $\mathfrak{Q}_{M,1}$ has recorded the result 'it was a 1', and likewise the z-observable of $\mathfrak{Q}_{M,-1}$ is sharp with value -1, meaning it recorded 'it was a -1'.

$\mathfrak{Q}_{M,1}$ and $\mathfrak{Q}_{M,-1}$ represent distinct histories of the measurer because they each contain in their record a different eigenvalue of $\hat{q}_{\mathrm{S}z}(1)$. The measurer here considered can at most have two distinct histories due to only having a single-bit worth of memory, allowing it to, at most, measure a Boolean observable. This deficiency can be solved by enlarging the measurer so that it consists of more qubits. With an increased memory capacity, the measurer can perform repeated measurements on different qubits, and in so doing it can branch into multiple different histories.

It might seem like the relative descriptors no longer satisfy the Pauli algebra – for instance, note that the squares of the relative descriptors of $\mathfrak{Q}_{M,1}$ do not equal the unit observable $\hat{1}$ anymore since $\left(\hat{q}_{(M,1)i}\right)^2 = \hat{\Pi}_1(\hat{q}_{Sz})$ for $i \in \{x, y, z\}$, where I have dropped the time labels for clarity. Instead, within the relative states, the role of the unit observable is replaced by the projectors since

$$\hat{\Pi}_1(\hat{q}_{Sz})\,\hat{q}_{(M,1)i} = \hat{q}_{(M,1)i}\hat{\Pi}_1(\hat{q}_{Sz}) = \hat{q}_{(M,1)i} \qquad (i \in \{x, y, z\}),$$

and similar results can straightforwardly be obtained for the projectors $\hat{\Pi}_{-1}(\hat{q}_{Sz})$ and the relative descriptors $\hat{\boldsymbol{q}}_{M,-1}(0)$. Consequently, I shall define the *relative unit observables* as follows: $\hat{1}_{M,1} \stackrel{\text{def}}{=} \hat{\Pi}_1(\hat{q}_{Sz})$ and $\hat{1}_{M,-1} \stackrel{\text{def}}{=} \hat{\Pi}_{-1}(\hat{q}_{Sz})$. The physical significance of this result is that, with respect to $\hat{1}_{M,1}$ and $\hat{1}_{M,-1}$, the relative descriptors adhere to the Pauli algebra and that instances of the measurer $\mathfrak{Q}_{M,1}$ and $\mathfrak{Q}_{M,-1}$ are therefore qubits in their own right:

$$\left.\begin{aligned} \hat{q}_{(M,1)i}\hat{q}_{(M,1)j} &= \delta_{ij}\hat{1}_{M,1} + i\epsilon_{ij}{}^{k}\hat{q}_{(M,1)k} \\ \hat{q}_{(M,-1)i}\hat{q}_{(M,-1)j} &= \delta_{ij}\hat{1}_{M,-1} + i\epsilon_{ij}{}^{k}\hat{q}_{(M,-1)k} \end{aligned}\right\} \qquad (i, j, k \in \{x, y, z\}). \tag{3.7}$$

In a similar vein, the descriptors of $\mathfrak{Q}_S$ relative to the z-observable of the measurer having the value 1 and -1 are defined as[24]

$$\left.\begin{aligned} \mathfrak{Q}_{S,1}: &\quad \hat{\boldsymbol{q}}_{S,1}(1) \stackrel{\text{def}}{=} \hat{\boldsymbol{q}}_S(1)\hat{\Pi}_1\big(\hat{q}_{Mz}(1)\big) \\ \mathfrak{Q}_{S,-1}: &\quad \hat{\boldsymbol{q}}_{S,-1}(1) \stackrel{\text{def}}{=} \hat{\boldsymbol{q}}_S(1)\hat{\Pi}_{-1}\big(\hat{q}_{Mz}(1)\big) \end{aligned}\right\}. \tag{3.8}$$

And with respect to the relative unit observables $\hat{1}_{S,1} \stackrel{\text{def}}{=} \hat{\Pi}_1(\hat{q}_{Mz})$ and $\hat{1}_{S,-1} \stackrel{\text{def}}{=} \hat{\Pi}_{-1}(\hat{q}_{Mz})$, these descriptors also satisfy the Pauli algebra:

[24] A construction similar to the one presented here was created by Hewitt-Horsman & Vedral (2007b), but they do not stress that the relative descriptors satisfy an appropriate form of the Pauli algebra, nor that those descriptors admit a local foliation.

$$\left.\begin{aligned}\hat{q}_{(S,1)i}\hat{q}_{(S,1)j} &= \delta_{ij}\hat{1}_{S,1} + i\epsilon_{ij}{}^{k}\hat{q}_{(S,1)k}\\ \hat{q}_{(S,-1)i}\hat{q}_{(S,-1)j} &= \delta_{ij}\hat{1}_{S,-1} + i\epsilon_{ij}{}^{k}\hat{q}_{(S,-1)k}\end{aligned}\right\} \qquad (i,j,k \in \{x,y,z\}).$$

Borrowing a term from differential geometry, Kuypers & Deutsch (2021) say that the qubits can be *foliated* into relative states. An important feature of the Heisenberg-picture relative-state construction is that the foliation of $\mathfrak{Q}_M$ and $\mathfrak{Q}_S$ into relative states is entirely local. That is to say that the foliation of $\mathfrak{Q}_M$ into the relative states (3.6) is independent of that in (3.8), whereas in the Schrödinger picture, a foliation of the state vector into relative states simultaneously affects the description of both entangled qubits $\mathfrak{Q}_M$ and $\mathfrak{Q}_S$.

3.4 Everettian 'universes'

The relative-state construction allows us to express the separate instances of a quantum system by foliating it into relative states. Yet, a foliation into relative states merely consists of applying projectors to descriptors, and such a foliation is therefore not automatically physically meaningful: for instance, it does not necessarily represent a decomposition into instances of a system that will evolve autonomously for a certain period. Here I will investigate under what circumstances such a foliation is physically meaningful. Put differently, under what conditions do the Everettian universes evolve independently?

Let us once more consider the qubits $\mathfrak{Q}_M$ and $\mathfrak{Q}_S$ at the time $t = 1$, when the network is foliated into $\mathfrak{Q}_{S,1}$ with $\mathfrak{Q}_{M,1}$, and $\mathfrak{Q}_{S,-1}$ with $\mathfrak{Q}_{M,-1}$. Firstly, I shall discuss $\mathfrak{Q}_M$. In the ideal case, $\mathfrak{Q}_M$ can be validly foliated into relative states, relative to the eigenvalues 1 and -1 of the descriptor $\hat{q}_{Sz}(1)$, for the duration that the following two conditions are satisfied: (i) $\mathfrak{Q}_M$ and $\mathfrak{Q}_S$ are each in a maximally mixed state; and (ii) $\mathfrak{Q}_M$ contains a copy of $\hat{q}_{Sz}(1)$. I shall say that $\mathfrak{Q}_M$ contains a copy of $\hat{q}_{Sz}(1)$ at a time $t \geq 1$ if there is some observable of $\mathfrak{Q}_M$, denoted $\hat{A}(t)$, such that the product $\hat{A}(t)\hat{q}_{Sz}(1)$ is sharp. In this situation, measuring $\hat{A}(t)$ will provide information about $\hat{q}_{Sz}(1)$. For instance, in the above-described measurement protocol, the product $\hat{q}_{Mz}(1)\hat{q}_{Sz}(1)$ is sharp with value 1, which is why $\hat{q}_{Mz}(1)$ stores a copy of $\hat{q}_{Sz}(1)$.

When conditions (i) and (ii) are satisfied, $\mathfrak{Q}_M$ can be foliated in terms of $\mathfrak{Q}_{M,1}$ and $\mathfrak{Q}_{M,-1}$

$$\left.\begin{array}{rl}\mathfrak{Q}_{M,1}: & \hat{\boldsymbol{q}}_{M,1}(t) \stackrel{\text{def}}{=} \hat{\boldsymbol{q}}_M(t)\hat{\Pi}_1\big(\hat{q}_{Sz}(1)\big) \\ \mathfrak{Q}_{M,-1}: & \hat{\boldsymbol{q}}_{M,-1}(t) \stackrel{\text{def}}{=} \hat{\boldsymbol{q}}_M(t)\hat{\Pi}_{-1}\big(\hat{q}_{Sz}(1)\big)\end{array}\right\}. \tag{3.9}$$

Here $t \geq 1$. And similarly, for the duration that the qubit $\mathfrak{Q}_S$ is in a maximally mixed state and contains a copy of $\hat{q}_{Mz}(1)$, it can be foliated into relative states, too:

$$\left.\begin{array}{rl}\mathfrak{Q}_{S,1}: & \hat{\boldsymbol{q}}_{S,1}(t) \stackrel{\text{def}}{=} \hat{\boldsymbol{q}}_S(t)\hat{\Pi}_1\big(\hat{q}_{Mz}(1)\big) \\ \mathfrak{Q}_{S,-1}: & \hat{\boldsymbol{q}}_{S,-1}(t) \stackrel{\text{def}}{=} \hat{\boldsymbol{q}}_S(t)\hat{\Pi}_{-1}\big(\hat{q}_{Mz}(1)\big)\end{array}\right\}. \tag{3.10}$$

The foliation expressed above is valid not just at time $t = 1$, but also at later times, provided that conditions (i) and (ii) are still satisfied. For this to be the case, each of the instances of the qubits should evolve *autonomously* of all other instances. A qubit's instance, such as $\mathfrak{Q}_{S,1}$, evolves autonomously between t and $t + 1$ if its descriptors $\hat{\boldsymbol{q}}_{S,1}(t + 1)$ are functions only of $\hat{\boldsymbol{q}}_{S,1}(t)$.

What gates ensure that the instances of the qubits evolve autonomously? Clearly, a general gate does not preserve the foliation into relative states as, for example, a second application of the **cnot** at $t = 1$ would unentangle the qubits, evolving them into a pure state so that condition (i) would no longer be satisfied. Kuypers & Deutsch (2021) have shown that the most general gate that preserves the relative state structure is a gate **F**, whose unitary at time $t = 1$ is

$$U_{\mathbf{F}}\big(\hat{\boldsymbol{q}}_S(1), \hat{\boldsymbol{q}}_M(1)\big) = \alpha\hat{1} + \beta\hat{q}_{Mz}(1) + \gamma\hat{q}_{Sz}(1) + \delta\hat{q}_{Sz}(1)\hat{q}_{Mz}(1). \tag{3.11}$$

Here the coefficients α, β, γ, and δ are complex numbers such that $U_{\mathbf{F}}$ is unitary. As follows from (3.11) and (3.10), when gate **F** acts between $t = 1$ and $t = 2$, its effect on $\mathfrak{Q}_{S,1}$ and $\mathfrak{Q}_{S,-1}$ is

$$\left.\begin{array}{l}\hat{\boldsymbol{q}}_{S,1}(2) \stackrel{\text{def}}{=} \Big(U_{\mathbf{F}}^{\dagger}\big(\hat{\boldsymbol{q}}_S(1), \hat{\boldsymbol{q}}_M(1)\big)\hat{\boldsymbol{q}}_S(1)U_{\mathbf{F}}\big(\hat{\boldsymbol{q}}_S(1), \hat{\boldsymbol{q}}_M(1)\big)\Big)\hat{\Pi}_1\big(\hat{q}_{Mz}(1)\big) \\ \hat{\boldsymbol{q}}_{S,-1}(2) \stackrel{\text{def}}{=} \Big(U_{\mathbf{F}}^{\dagger}\big(\hat{\boldsymbol{q}}_S(1), \hat{\boldsymbol{q}}_M(1)\big)\hat{\boldsymbol{q}}_S(1)U_{\mathbf{F}}\big(\hat{\boldsymbol{q}}_S(1), \hat{\boldsymbol{q}}_M(1)\big)\Big)\hat{\Pi}_{-1}\big(\hat{q}_{Mz}(1)\big)\end{array}\right\}, \tag{3.12}$$

which can be rewritten as follows:

$$\left.\begin{aligned}\hat{\boldsymbol{q}}_{S,1}(2) = U^{\dagger}_{\mathbf{F}'}\left(\hat{\boldsymbol{q}}_{S,1}(1)\right)\hat{\boldsymbol{q}}_{S,1}(1)U_{\mathbf{F}'}\left(\hat{\boldsymbol{q}}_{S,1}(1)\right)\\ \hat{\boldsymbol{q}}_{S,-1}(2) = U^{\dagger}_{\mathbf{F}''}\left(\hat{\boldsymbol{q}}_{S,-1}(1)\right)\hat{\boldsymbol{q}}_{S,-1}(1)U_{\mathbf{F}''}\left(\hat{\boldsymbol{q}}_{S,-1}(1)\right)\end{aligned}\right\}.$$

Here the new unitaries are defined as $U_{\mathbf{F}'}\left(\hat{\boldsymbol{q}}_{S,1}(1)\right) \stackrel{\text{def}}{=} \left((\alpha+\beta)\hat{1}_{S,1} + (\gamma+\delta)\hat{q}_{(S,1)z}(1)\right)$ and $U_{\mathbf{F}''}\left(\hat{\boldsymbol{q}}_{S,-1}(1)\right) \stackrel{\text{def}}{=} \left((\alpha-\beta)\hat{1}_{S,-1} + (\gamma-\delta)\hat{q}_{(S,-1)z}(1)\right)$. As is evident from (3.12), $\mathfrak{Q}_{S,1}$ and $\mathfrak{Q}_{S,-1}$ evolve autonomously: for instance, $\mathfrak{Q}_{S,-1}$ is acted on by gate $\mathbf{F}''$, whose unitary is a function only of $\hat{\boldsymbol{q}}_{S,-1}(1)$, implying that the descriptors after that gate $\hat{\boldsymbol{q}}_{S,-1}(2)$ are a function only of $\hat{\boldsymbol{q}}_{S,-1}(1)$, as required. Likewise, $\mathfrak{Q}_{S,1}$ is acted on by gate $\mathbf{F}'$, which is a function only of $\hat{\boldsymbol{q}}_{S,1}(1)$. In the same way, one can deduce that the instances of $\mathfrak{Q}_M$ also evolve autonomously under the gate $\mathbf{F}$.

Besides gates of the form $U_{\mathbf{F}}$, the qubits can perform single qubit rotations, as those will invariably leave the qubits entangled, so the qubits' instances will evolve independently under the influence of such gates. The most general gate that maintains, between the times $t = 1$ and $t = 2$, the structure of the relatives states shown in (3.9) and (3.10) are therefore single qubit gates, a unitary of the form $U_{\mathbf{F}}$, or a product of $U_{\mathbf{F}}$ and a single qubit gate.

When the instances of a qubit evolve autonomously, they each represent non-interacting quantum systems and by another name are called 'Everettian universes'. These Everettian 'universes' are fully-fledged quantum systems: the instances of the qubits obey the Pauli algebra and, as we have seen above, they can perform single qubit gates without destroying the relative-state structure. There is a further sense in which instances of a qubit are quantum systems in their own rights: the instances of both $\mathfrak{Q}_M$ and $\mathfrak{Q}_S$ can store locally inaccessible information.

3.5 Locally inaccessible information in the relative states

Consider the case in which a gate $\boldsymbol{R}_z(\theta)$ acts on $\mathfrak{Q}_M$ between $t = 1$ and $t = 2$. The effect of $\boldsymbol{R}_z(\theta)$ is that it rotates $\mathfrak{Q}_M$ about its z-axis by an angle θ, and as it is a single qubit gate, it will leave the relative-state structure unchanged. After its application, at the final time $t = 2$, the relative descriptors of $\mathfrak{Q}_{M,1}$ and $\mathfrak{Q}_{M,-1}$ depend on the angle θ:

$$\left.\begin{aligned} \hat{\boldsymbol{q}}_{M,1}(2) &= \left(\cos\theta\,\hat{q}_{Mx} - \sin\theta\,\hat{q}_{My},\ \ \cos\theta\,\hat{q}_{My} + \sin\theta\,\hat{q}_{Mx},\ \ \hat{q}_{Mz}\right)\hat{\Pi}_1(\hat{q}_{Sz}) \\ \hat{\boldsymbol{q}}_{M,-1}(2) &= \left(\cos\theta\,\hat{q}_{Mx} - \sin\theta\,\hat{q}_{My}, -\cos\theta\,\hat{q}_{My} - \sin\theta\,\hat{q}_{Mx}, -\hat{q}_{Mz}\right)\hat{\Pi}_{-1}(\hat{q}_{Sz}) \end{aligned}\right\}. \tag{3.13}$$

Here I have suppressed all $t = 1$ time labels for brevity. At $t = 2$, expectation values of the descriptors are $\langle\hat{\boldsymbol{q}}_{M,1}(2)\rangle_{M,1} = (0,0,1)$ and $\langle\hat{\boldsymbol{q}}_{M,-1}(2)\rangle_{M,-1} = (0,0,-1)$, and evidently, neither of their expectation values depend on the angle θ, despite the descriptors $\mathfrak{Q}_{M,1}$ and $\mathfrak{Q}_{M,-1}$ depending on θ. Hence, there is information stored in those instances that cannot be retrieved through measurements performed on either of those instances in isolation.

The information about θ can be retrieved, but only through interreference experiments in which those instances are made to interact with one another. Therefore, the qubit instances store locally inaccessible information, further corroborating that they represent bona fide quantum systems.

3.6 Decoherence and Everettian 'universes'

When there is no interference between the separate branches of the multiverse, those branches will evolve independently of each other. That is to say, they will evolve exactly as if the other branches are not there at all.

Interference between branches is typically negligible because of *decoherence*, which is the phenomena in which a system is incapable of interfering because it is continually interacting with another system, such as its environment. Generally, 'large' quantum systems – meaning quantum systems that consist of many subsystems – have a proclivity to interact with other systems. So, we should generally expect them to be, to a high degree of accuracy, approximated as a collection of autonomously evolving branches. Moreover, in each of those branches, the world appears as if it is classical (Wallace (2012a), Wallace (2012b, ch. 3)).

To see how quasi-classical 'universes' emerge in the Heisenberg-picture relative-state construction, consider a quantum computational network, which I shall use to model a decoherent system. In a quantum computational network, the presence of decoherence can be modelled as a prohibition on the kinds of gates that the network can enact. In particular, I shall impose that the system can only enact **ccnot** gates (also known as **Toffoli** gates). This is a useful way to model decoherence because the **ccnot** and the **Hadamard** gate are a universal set: any gate can be expressed as products of **ccnot** and **Hadamard** gates. The **ccnot** models interactions between quantum systems and the **Hadamard** gate represents quantum interference. In a situation where there is ideal decoherence, there can be no interference, implying that the **Hadamard** gate cannot be implement in that situation. I will show how the foliation into relative states will then give rise to a collection of quasi-classical 'universes'.

Consider a quantum computational network $\mathfrak{C}$ consisting of $2n$ interacting qubits $\mathfrak{Q}_1, \mathfrak{Q}_2, \dots, \mathfrak{Q}_{2\mathrm{n}}$. I shall treat $\mathfrak{C}$ as two sub-networks, denoted $\mathcal{M}$ and $\mathcal{S}$, that consist of the qubits $\mathfrak{Q}_1, \mathfrak{Q}_2, \dots, \mathfrak{Q}_\mathrm{n}$ and $\mathfrak{Q}_{\mathrm{n}+1}, \mathfrak{Q}_{\mathrm{n}+2}, \dots, \mathfrak{Q}_{2\mathrm{n}}$, respectively. I shall also assume that these two sub-networks are entangled at some initial time t so that the network can be foliated into relative states (see below). Finally, $\mathcal{M}$ and $\mathcal{S}$ are ideally decoherent so that, after the initial time t, they are capable of enacting **ccnot** gates only.

The following two observables of the respective sub-networks will be crucial to their analysis:

$$\left.\begin{aligned} \hat{b}_M(t) &\overset{\text{def}}{=} 2^{n-1}\hat{\Pi}_{-1}\left(\hat{q}_{nz}(t)\right) + \cdots + 2\hat{\Pi}_{-1}\left(\hat{q}_{2z}(t)\right) + \hat{\Pi}_{-1}\left(\hat{q}_{1z}(t)\right) \\ \hat{b}_S(t) &\overset{\text{def}}{=} 2^{n-1}\hat{\Pi}_{-1}\left(\hat{q}_{(2n)z}(t)\right) + \cdots + 2\hat{\Pi}_{-1}\left(\hat{q}_{(n+2)z}(t)\right) + \hat{\Pi}_{-1}\left(\hat{q}_{(n+1)z}(t)\right) \end{aligned}\right\}. \tag{3.14}$$

The projectors in the definition of $\hat{b}_M(t)$ and $\hat{b}_S(t)$ have eigenvalues 0 and 1, and therefore the eigenvalues of $\hat{b}_M(t)$ and $\hat{b}_S(t)$ are binary numbers of length n. These binary-number eigenvalues are elements of Z_{2^n}, so physically, if $\hat{b}_M(t)$ and $\hat{b}_S(t)$ are sharp with those eigenvalues, they are bit strings that represent the states of a classical computer. When $\hat{b}_M(t)$ and $\hat{b}_S(t)$ are non-sharp, the sub-networks $\mathcal{M}$ and $\mathcal{S}$ will be in a superposition of those bit strings, which can evolve independently of one another and perform computations in parallel

– as shall be shown below. In such situations, $\hat{b}_M(t)$ and $\hat{b}_S(t)$ each represent an ensemble of classical computers.[25]

At the initial time t, the sub-network $\mathcal{M}$ is assumed to be entangled with $\mathcal{S}$; in particular, I shall assume that the product $\hat{b}_M(t)\hat{b}_S(t)$ is sharp while $\hat{b}_M(t)$ and $\hat{b}_S(t)$, separately, are non-sharp. This will allow me to foliate the sub-network $\mathcal{M}$ into relative states, relative to the eigenvalues s of $\hat{b}_S(t)$, as follows:

$$\hat{b}_{M,s}(t) \stackrel{\text{def}}{=} \hat{b}_M(t)\hat{\Pi}_s\Big(\hat{b}_S(t)\Big) \qquad\qquad (s \in Z_{2^n}), \qquad\qquad (3.15)$$

where $\hat{\Pi}_s\left(\hat{b}_S(t)\right)$ is the s-eigenvalue projector of $\hat{b}_S(t)$. The sub-networks are non-interacting, ensuring that (3.15) will remain a valid foliation throughout the analysis. And since the sub-networks are assumed to be decoherent, they can only enact **ccnot** gates, resulting in $\mathcal{M}$ being able to perform only reversible classical computations. That is to say that, during the $(t+1)$'th computational step, $\hat{b}_M(t)$ must evolve as follows (Kuypers & Deutsch (2021)):

$$\hat{b}_M(t+1) = f_t\left(\hat{b}_M(t)\right),$$

where f_t is a q-number function that permutes the set of eigenvalues of $\hat{b}_M(t)$. More precisely, f_t can be implemented as a sequence of **ccnot** gates, and using the spectral decomposition of $\hat{b}_M(t)$, which is

$$\hat{b}_M(t) = \sum_{m\in Z_2 n} m\hat{\Pi}_m\left(\hat{b}_M(t)\right),$$

one can express the effect of f_t as

$$\hat{b}_M(t+1) = \sum_{m\in Z_2 n} f_t(m)\hat{\Pi}_m\left(\hat{b}_M(t)\right),$$

[25] Note that neither $\hat{b}_M(t)$ nor $\hat{b}_S(t)$ is a full specification of the state of either sub-network: $\hat{b}_M(t)$ and $\hat{b}_S(t)$ commute with themselves and each other, so if they fully represent those systems, the fundamental principle of quantum theory would be violated.

with $f_t(m)$ itself an element of Z_{2^n} for every $m \in Z_{2^n}$.

If between times t and $t+1$, $\mathcal{M}$ effects the computation f_t, its relative descriptors (3.15) will evolve as follows:

$$\hat{b}_{M,s}(t+1) = f_t\left(\hat{b}_{M,s}(t)\right) \qquad (s \in Z_{2^n}). \tag{3.16}$$

Evidently, $\hat{b}_{M,s}(t+1)$ depends only on $\hat{b}_{M,s}(t)$, so each of those descriptors evolve autonomously of all others under f_t. Moreover, each element in $\{\hat{b}_{M,s}(t)\}$ are sharp with eigenvalues in Z_{2^n}, so they represent a collection of autonomously evolving Everettian 'universes'.

The evolution of these 'universes' is governed by the quasi-classical equation of motion (3.16), implying that such a 'universe' evolves *as if* it were a classical computer enacting the computation f_t. In isolation, one such Everettian 'universe' will appear to be entirely classical, behaving completely independently of anything that happens in the other 'universes', and appearing to be described by c-numbers (i.e. the binary numbers in Z_{2^n}).

Since an instance of $\mathcal{M}$ represents a quasi-classical computer, it can model the brain of a person who is, say, engaged in experimental research, trying to uncover the laws of physics. From his point of view, the world appears classical: he can describe himself and the world around him through the binary numbers in Z_{2^n} because these binary numbers represent his memories and thoughts, as well as the state of system $\mathcal{S}$ with which he is entangled. Evidently, he will be unaware of his counterparts in the other branches, whose existence he is unaffected by so long as the 'universes' remain autonomous.

This explains why the world will *appear to be* classical to observers in circumstances where there is significant decoherence. Yet it also explains why, despite the world appearing to be classical, it is not: the person described above has counterparts in other 'universes'. Likewise, the multiverse contains counterparts of our 'world' that exist simultaneously with our own, despite us being (mostly) unaware of them.

3.7 Locality and the Heisenberg-picture relative-state construction

In the Schrödinger picture, a foliation into relative states requires that one decomposes the entire universal state-vector into separate relative states. This makes it seem as if the whole multiverse is affected by a measurement – yet according to Einstein's criterion, only those systems involved in the measurement interaction should be affected and nothing besides.

To be more precise, a relative state in the Schrödinger picture is defined as a projector that acts on the state vector. For instance, consider the projector $\widehat{\Pi}_x(\hat{X})$ for some arbitrary q-number $\hat{X}$ that is both non-sharp and has been measured at some time t; then the relative state of the universal state vector $|\Psi(t)\rangle$ relative to $\hat{X}$ having the value x is $\widehat{\Pi}_x(\hat{X})|\Psi(\mathrm{t})\rangle$ up to a normalisation factor. All systems represented by $|\Psi(t)\rangle$ are simultaneously affected by this projection, and there is no way to foliate just the measured system or just the measurer into relative states while leaving other systems in the multiverse unchanged. So the measurement, performed locally in some bounded region of space, seems to have a superluminal effect on the entire multiverse. In that sense, the Everettian universe that $\widehat{\Pi}_x(\hat{X})|\Psi\rangle$ represents is really a universe: it represents the relative state of everything in existence relative to $\hat{X}$ having value x (Vaidman (2018)).

This is a shortcoming of the Schrödinger picture since, when Everett's construction is expressed in the Heisenberg picture, it is manifestly local. That is to say that only those systems involved in a measurement can be foliated into relative states. There will be no effect – no physical change whatsoever – on those systems not involved in the measurement.

So, in the Heisenberg picture, an Everettian 'universe' arises in the first instance locally – i.e. via local interactions between two or more systems. Some of those interactions go on to have cascading effects that result in ever more systems branching and becoming part of the same 'universe', but that 'universe' always has a finite spatial extent. It exists in a bounded region of space – where the boundary of that region cannot expand faster than the speed of light. Nothing outside of that region is part of this same quasi-classical structure – at least not until physical objects exterior to the region come into contact with its interior. 'Universe' is therefore a misnomer, as it is never the entire ‘universe’ (or multiverse) that splits. And only the Heisenberg picture of quantum theory makes this manifest.

4 The quantum theory of time

Throughout this thesis, I have assumed it to be a principle of physics that the descriptors of physical systems are q-numbers. What, then, is the c-number time parameter that is invoked in equations of motion? According to the fundamental principle, it cannot be a descriptor. So, either the principle is false, or c-number time is unphysical.

In the conventional treatment of quantum theory, the c-number time is like a backdrop against which quantum systems can change. It is, in fact, all but identical to Newton's (1687) absolute time, which

> *'of itself, and from its own nature flows equably without regard to anything external, and by another name is called duration: relative, apparent and common time, is some sensible and external (whether accurate or unequable) measure of duration by the means of motion…'*

Just like Newton's absolute time, quantum theory's c-number time 'flows' independently of the behaviour of physical systems. As Newton pointed out, this view of time should not be confused with what one might call *clock time* – i.e. time as represented by physical clocks, or what Newton called a *'measure of duration by the means of motion.'*

The idea of absolute time raises awkward conceptual issues even within classical physics: if time flows, where does it flow to? Yesterday did not 'flow' to today; yesterday will forever remain a fixed date on the calendar. Similarly, 'flow' implies change relative to time, so does the statement that 'time flows' mean only that 'time changes relative to time'?

Would we also have to conclude that time 'flows' in a completely stationary universe in which nothing ever changes? Or consider the following dramatisation: if time 'flows' independently of the behaviour of physical systems, does it also 'flow' in the hypothetical situation in which all physical systems in the multiverse suddenly 'freeze' for a certain duration? Nothing in the multiverse – no observer or measurer – could be in any way able to detect such a 'freeze' or its duration since the physical objects that could detect it (namely, clocks) are immobilised during the 'freeze'. This makes the c-number time just as undetectable as the duration of the 'freeze'.[26] This is the problem of duration, as for instance discussed by Poincaré (1913).

Clock time, on the other hand, is measurable, as it can be measured via the usual method, on the clock. Clock time is also manifestly physical, as clocks are physical objects. This has led some researchers (Barbour (2009), (2011), Page & Wootters (1983), DeWitt (1967), and Kuypers (2022a)) to develop *relational theories of time* in which dynamics are formulated in terms of clock time, not absolute time. Mach (2013) summarised the guiding principle of such relational theories of time as follows:

> *'It is utterly beyond our power to measure the changes of things by time. Quite the contrary, time is an abstraction at which we arrive through the changes of things...'*

An elegant example of a relational theory of time is that of Barbour (2009), who has constructed a model of a classical universe in which all dynamical laws can be expressed in terms of clock time. To be more specific, Barbour's model universe consists of celestial objects that interact solely via Newtonian gravity – there are no other forces at work in his model. Because of this, there exists a parameter whose expression depends only on the relative positions of the celestial bodies, and this parameter can entirely replace the role that Newton's absolute time plays in classical physics. Hence, the configuration of the celestial objects in Barbour's model universe are a natural clock relative to which the systems in that universe evolve in accordance with Newton's equations of motion, making Newton's absolute time entirely gratuitous.

Relational theories of time, such as Barbour's construction, are sometimes called *block universe theories* since all the states of such a universe's history exist 'simultaneously' in a timeless 'block'. For otherwise, different times would have to come into existence relative to some parameter that would effectively be Newton's absolute time by another name.

[26] This thought experiment of a universe that is momentarily 'frozen' is based on Shoemaker's (1969) discussion of absolute time.

Perhaps the earliest block universe construction was invented over two millennia ago, around 500 BC, by the pre-Socratic Greek philosopher Parmenides (Popper 2012). Parmenides expounded that the universe is fundamentally unchanging; that all movement is only apparent; and that through the use of reason one can discover that change is an illusion.[27] Parmenides' block universe is remarkably similar to the so-called Page–Wootters construction in quantum theory, as each describes a stationary world in which time and dynamics are emergent.

4.1 The Page–Wootters construction

Page & Wootters (1983) invented a construction, later refined by myself (Kuypers (2022a)), in which the c-number time is unimportant. They did this by proposing a model multiverse that is entirely stationary. Hence in this multiverse, the c-number time does not play any role – as required by the fundamental principle of quantum theory.

Page & Wootters modelled their multiverse as a non-relativistic system that evolves according to the Schrödinger equation. The Page–Wootters multiverse is typically described in the Schrödinger picture, in which that multiverse is stationary because its universal state-vector $|\Psi\rangle$ is an eigenstate of the multiverse's Hamiltonian and will therefore at most acquire an irrelevant phase factor as it evolves relative to c-number time. So, it is *as if* c-number time is not there at all. Page & Wootters then assume that the model consists of at least two systems – a clock and the 'rest of the universe'. Despite those combined systems being time invariant, they can evolve relative to one another; in particular the 'rest of the universe' is entangled with the clock and 'evolves' relative to the different states of the clock. In this way, evolution and dynamics can be formulated in terms of clock time. Moreover, in the Page–Wootters construction, clock time is represented by a bona-fide quantum observable of the clock $\mathfrak{C}$,

[27] In Popper's (2012) account of this episode in the history of science, Parmenides created the block-universe model by generalising a discovery he had made about the Moon. A contemporary of Parmenides, the philosopher Heraclitus, expounded that the lunar phases were caused by the Moon being a rotating bowl that contained fire. In Heraclitus' theory, the Moon waxes and wanes because the bowl is rotating. Parmenides realised that the Moon's waxing and waning are not due to the Moon changing (i.e. rotating), as it does in Heraclitus' theory, but there is instead only a change in the appearance of the Moon since in Parmenides' theory the Moon is a sphere lit by the Sun from different angles throughout a synodic month. Parmenides generalised his discovery by proposing that the universe is fundamentally unchanging, and that all movement is only apparent.

thereby resolving the conflict between the fundamental principle of quantum theory and c-number time.

It is not immediately obvious that the Page–Wootters construction eliminates the need for c-number time from the Heisenberg picture: although it is possible for a system's Schrödinger state-vector to be completely time-invariant, that system's descriptors in the Heisenberg picture will still depend on time (as will be shown in detail in §4.3). And if c-number time plays an explanatory role in the Heisenberg-picture Page–Wootters construction, the conflict with the fundamental principle of quantum theory would not have been resolved after all – at least not in that picture.

If the Page–Wootters construction cannot be adequately formulated in the Heisenberg picture, this might be indicative of a defect in that picture – e.g. perhaps the Heisenberg picture cannot validly describe the entire multiverse. In that case, the Heisenberg picture would be problematic, implying it cannot be fundamental. Yet that is not the case: in the following sections, I shall show that the Heisenberg picture admits a formulation of the Page–Wootters construction, which is based on my own work (Kuypers (2022a)). But first I will elaborate on the mathematical formulation of the Page–Wootters multiverse in the Schrödinger-picture so that the connections between it and my own construction can be made clear.

4.2 Page–Wootters in the Schrödinger picture

Page & Wootters (1983) expressed their construction in the Schrödinger picture by modelling their multiverse as a non-relativistic system that evolves according to the Schrödinger equation. Page & Wootters assumed their multiverse to be stationary, implying that its universal state-vector $|\Psi_S\rangle$ is an eigenstate of the multiverse's Hamiltonian $\mathcal{H}_S$,

$$\hat{\mathcal{H}}_S|\Psi_S\rangle = 0. \tag{4.1}$$

Here S indicates that a q-number or a vector is part of the Schrödinger picture. Due to (4.1), no measurable quantity of the Page–Wootters multiverse will depend on c-number time, yet subsystems of this multiverse can still evolve relative to one another.

In the Page–Wootters construction, systems evolve relative to a clock $\mathfrak{C}$, which is a quantum system that possesses a pair of canonically conjugate observables $\hat{h}_S$ and $\hat{t}_S$,

$$[\hat{t}, \hat{h}] = i\hat{1}. \tag{4.2}$$

$\hat{t}$ is the *time observable*, so-called because the eigenstate $|t\rangle$ of $\hat{t}_S$ with eigenvalue t has the physical meaning 'it is clock time t', just as the projectors $\hat{\Pi}_t(\hat{t})$ do in the Heisenberg picture.

The set of possible eigenvalues of $\hat{t}_S$ can be derived from its algebraic properties. For instance, one can readily verify that an eigenstate $|t\rangle$ of $\hat{t}_S$ is shifted by an arbitrary real-valued constant α when $|t\rangle$ is acted on by the operator $e^{-i\hat{h}_S\alpha}$. The proof of this is as follows: since $\hat{t}_S|t\rangle = t|t\rangle$, one finds that

$$\left.\begin{aligned} \hat{t}_S e^{-i\hat{h}_S\alpha}|t\rangle &= e^{-i\hat{h}_S\alpha}\left(e^{i\hat{h}_S\alpha}\hat{t}_S e^{-i\hat{h}_S\alpha}\right)|t\rangle \\ &= e^{-i\hat{h}_S\alpha}\left(\hat{t}_S + \alpha\hat{1}\right)|t\rangle \\ &= (t+\alpha)e^{-i\hat{h}_S\alpha}|t\rangle \end{aligned}\right\}. \tag{4.3}$$

Evidently, $e^{-i\hat{h}_S\alpha}|t\rangle$ is equal to the eigenstate of $\hat{t}_S$ with eigenvalue $t+\alpha$. Two further conclusions follow from that result: (i) $e^{-i\hat{h}_S\alpha}$ is a translation operator, as it translates the eigenstate $|t\rangle$ by the factor α; and (ii) if t is an eigenvalue of $\hat{t}_S$, then $t+\alpha$ is also an eigenvalue of $\hat{t}_S$. Since both these statements are true for any $\alpha \in \mathbb{R}$, the spectrum of $\hat{t}_S$ is equal to $\mathbb{R}$.

From the fact that $e^{-i\hat{h}_S\alpha}$ is a translation operator, one can also deduce the effect that $\hat{h}_S$ has on an eigenstate of $\hat{t}_S$. If one takes the limit in which α tends to zero, it follows from (4.3) that

$$\left.\begin{aligned} e^{-i\hat{h}_S\alpha}|t\rangle &= |t+\alpha\rangle \\ \rightarrow (\hat{1} + i\hat{h}_S\alpha + O(\alpha^2))|t\rangle &= |t+\alpha\rangle \\ \rightarrow \lim_{\alpha\to 0}\left(i\hat{h}_S + O(\alpha)\right)|t\rangle &= \lim_{\alpha\to 0}\frac{|t+\alpha\rangle - |t\rangle}{\alpha} \\ \rightarrow \hat{h}_S|t\rangle &= -i\frac{d}{dt}|t\rangle \end{aligned}\right\}, \tag{4.4}$$

where the terms of order one in α and higher are denoted $O(\alpha)$. Those will tend to zero in the limit $\alpha \rightarrow 0$. Consequently, one finds that $\hat{h}_S$ acts on the state $|t\rangle$ as a derivative with respect to the eigenvalue t.

For the system $\mathfrak{C}$ to function as an ideal clock, the 'rest of the universe', denoted $\mathfrak{R}$, must evolve according to the Schrödinger equation relative to $\mathfrak{C}$. That is to say that when $\mathfrak{C}$ is in the eigenstate $|t\rangle$, $\mathfrak{R}$ must be in the state $|\psi(t)\rangle$, where $|\psi(t)\rangle$ satisfies the Schrödinger equation. This is guaranteed to be the case when $\mathfrak{C}$ and $\mathfrak{R}$ are non-interacting. One might be concerned that if the clock and the 'rest of the universe' cannot interact, the clock cannot be measured. But the assumption that $\mathfrak{C}$ and $\mathfrak{R}$ do not interact is only a sufficient, and not a necessary, condition for $\mathfrak{C}$ to function as an ideal clock. There are other situations, including situations in which $\mathfrak{C}$ and $\mathfrak{R}$ can interact, that allow $\mathfrak{C}$ to be an ideal clock in the above-described sense (see for example the work of Smith & Ahmadi (2019)). I shall not analyse such general situations here and restrict attention to those cases in which $\mathfrak{C}$ and $\mathfrak{R}$ are non-interacting so that the Hamiltonian of the multiverse is

$$\hat{\mathcal{H}}_S = \hat{H}_S + \hat{h}_S. \tag{4.5}$$

Here the Hamiltonian of $\mathfrak{C}$ is $\hat{h}_S$, and $\hat{H}_S$ is the Hamiltonian of $\mathfrak{R}$. And because $\hat{\mathcal{H}}_S$ is the sum of the Hamiltonians of the subsystems (meaning there are no interaction terms), those systems evolve independently of one another.

The relative state of $\mathfrak{R}$ relative to the clock being in the state 'it is clock time t' is defined as

$$|\psi(t)\rangle \stackrel{\text{def}}{=} \langle t|\Psi_S\rangle\!\rangle. \tag{4.6}$$

I shall use the $\rangle\!\rangle$ to indicate, wherever clarification is needed, that $|\Psi_S\rangle$ resides on a larger Hilbert space than $|t\rangle$. By using (4.6), (4.5), (4.4), and (4.1), one can now deduce that $|\psi(t)\rangle$ obeys the Schrödinger equation (Wootters (1984)),

$$\left.\begin{aligned} i\frac{d}{dt}|\psi(t)\rangle &= i\frac{d}{dt}\langle t|\Psi_S\rangle\!\rangle \\ &= -\langle t|\hat{h}_S|\Psi_S\rangle\!\rangle \\ &= \langle t|\hat{H}_S|\Psi_S\rangle\!\rangle \\ &= \hat{H}_S|\psi(t)\rangle \end{aligned}\right\}.$$

Evidently, the eigenvalues of $\hat{t}_S$ correspond to the time-parameter of $\mathfrak{R}$, and thus dynamics have been formulated in terms of clock time, despite the universe being stationary with respect to the c-number time-parameter.

Due to these findings, the universal state vector is the 'sum total' of all the relative states $|\psi(t)\rangle|t\rangle$ since

$$|\Psi_S\rangle = \int_{-\infty}^{\infty} |t\rangle\langle t|\Psi_S\rangle\rangle dt = \int_{-\infty}^{\infty} |\psi(t)\rangle|t\rangle dt.$$

So, the universal state vector is like a stationary 'block' that contains all moments 'simultaneously'. This decomposition also shows that the Page–Wootters construction is fundamentally connected to Everett's relative state construction. The state $|\psi(t)\rangle|t\rangle$ is a relative state, representing the state of the multiverse relative to the clock being in the state $|t\rangle$, and hence different times are equivalent to different Everettian universes (Deutsch (1997)).[28]

4.3 Page-Wootters in the Heisenberg picture

I shall denote the c-number time parameter as λ. In the Schrödinger-picture Page–Wootters construction, the c-number time λ is irrelevant to the system's dynamics because the universal state-vector is invariant with respect to λ. Yet, in the Heisenberg picture, the descriptors can depend on c-number time even when their expectation values are independent of λ. For instance, in the Heisenberg picture, any isolated system has descriptors $\{\hat{X}_j(\lambda)\}$ that must satisfy the Heisenberg equation of motion

$$\frac{d\hat{X}_j(\lambda)}{d\lambda} = i[\hat{H}, \hat{X}_j(\lambda)],$$

[28] For $\mathfrak{C}$ to be an ideal clock, there can be no interactions between it and the rest of the universe. Hence, if one were to measure $\mathfrak{C}$, it would cease to be an ideal clock. So, time can only be measured imperfectly in the Page–Wootters construction. This does not affect other features of the construction – for instance, the universal state vector can nonetheless be decomposed into relative states (relative to clock time) because the clock and the rest of the universe need only be entangled for that.

where $\hat{H}$ is the system's Hamiltonian. The Heisenberg equation of motion is a differential equation in terms of λ, and only when $\hat{H}$ is equal to a multiple of the unit observable are all descriptors $\{\hat{X}_j(\lambda)\}$ invariant with respect to λ.

Hence, c-number time is apparently harder to remove from the foundations of the Heisenberg picture than from the Schrödinger picture. But it can be removed: in subsequent sections, I shall study functions of $\hat{t}$ and discuss how a derivative with respect to $\hat{t}$ can be defined, which makes the c-number time entirely unnecessary. To define a function of $\hat{t}$, one has to refer to the *possible states* that a clock can be in.

4.4 Counterfactual states

Prior to specifying a quantum system's dynamics, one has to define the system's set of possible states. Here the term 'possible' refers to a counterfactual property: these are the states that the system *could be in*, regardless of the dynamics of the system. The possible states of a quantum system are constrained by the principles of quantum theory discussed in earlier sections: for example, an isolated system has a constitutive algebra, which is invariant for all the system's possible states.

To illustrate this point, consider a qubit: its state is represented by a fixed Heisenberg 'state' $|\Psi\rangle$ and a triple of descriptors $\hat{\boldsymbol{q}}$ that satisfy the Pauli algebra. Given that $|\Psi\rangle$ is fixed and so contains no information about the actual state of the system, this qubit's set of possible states can be defined as follows:

$$\mathbb{P} \stackrel{\text{def}}{=} \{U^\dagger \hat{\boldsymbol{q}} U : U^\dagger U = U U^\dagger = \hat{1}\},$$

where the unitary U is a function solely of $\hat{\boldsymbol{q}}$.

The qubit's *history* is a one-parameter family of states, denoted $\mathbb{H}$. Each state in $\mathbb{H}$ is represented by a triple $\hat{\boldsymbol{q}}(\lambda) \in \mathbb{P}$ that depends on a c-number time λ. It must adhere to the qubit's equation of motion and satisfy an initial condition, making $\mathbb{H}$ a parameterised curve on $\mathbb{P}$.

There is a crucial difference between the time-dependent triple $\hat{\boldsymbol{q}}(\lambda)$ in $\mathbb{H}$ and those $\hat{\boldsymbol{q}}$ in $\mathbb{P}$: the triple $\hat{\boldsymbol{q}}$ represents a counterfactual state that the qubit *could be in*. In other words, regardless of the system's dynamical laws (as determined by its Hamiltonian and its initial conditions), the principles of quantum theory allow the qubit to be in the state represented by $\hat{\boldsymbol{q}}$, even if the qubit's history does not contain the state $\hat{\boldsymbol{q}}$. Hence, the triple $\hat{\boldsymbol{q}}$ represents a more fundamental, timeless notion of state, to which I shall refer throughout this section.

4.5 Quantum clocks

In the Heisenberg-picture Page–Wootters construction, clocks are physical systems whose descriptors are a pair of canonically conjugate q-numbers $\hat{h}$ and $\hat{t}$, so that

$$[\hat{t}, \hat{h}] = i\hat{1}. \tag{4.7}$$

The q-numbers $\hat{h}$ and $\hat{t}$ live on a Hilbert space $\boldsymbol{H}_C$, and $\hat{1}$ denotes the unit observable on that space. In the Heisenberg picture, the sharp values of $\hat{t}$ represent definite clock times. For instance, the clock for which $\hat{t}$ is sharp with value t has the unambiguous physical meaning 'it is clock time t'.

What are the eigenvalues of $\hat{t}$? As shown in previous chapters, one can deduce the eigenvalues of $\hat{t}$ from its algebra, which is shown in (4.7). Due to this algebra, $\hat{h}$ generates translations of $\hat{t}$: consider the unitary $e^{-i\alpha\hat{h}}$ for some real-value constant α, and let this unitary act on $\hat{t}$ so that one obtains

$$e^{i\alpha\hat{h}}\hat{t}e^{-i\alpha\hat{h}} = \hat{t} + \alpha\hat{1}. \tag{4.8}$$

Consequently, if the clock is in a state for which the variance of $\hat{t}$ is zero, it is also in a state for which the variance of $\hat{t} + \alpha\hat{1}$ is zero, since

$$\left.\begin{aligned} &\left\langle\left(\hat{t} + \alpha\hat{1} - \langle\hat{t} + \alpha\hat{1}\rangle\right)^2\right\rangle = 0 \\ \leftrightarrow\ &\langle\hat{t} + \alpha\hat{1}\rangle^2 = \left\langle\left(\hat{t} + \alpha\hat{1}\right)^2\right\rangle \\ \leftrightarrow\ &\langle\hat{t}\rangle^2 + 2\alpha\langle\hat{t}\rangle + \alpha^2 = \langle\hat{t}^2\rangle + 2\alpha\langle\hat{t}\rangle + \alpha^2 \\ \leftrightarrow\ &\langle\hat{t}\rangle^2 = \langle\hat{t}^2\rangle \end{aligned}\right\}.$$

Put differently, if $\hat{t}$ is sharp with value $\langle\hat{t}\rangle$, then $\hat{t}+\alpha\hat{1}$ is sharp with value $\langle\hat{t}\rangle+\alpha$, for any $\alpha\in\mathbb{R}$. And because $\hat{t}$ and $\hat{t}+\alpha\hat{1}$ are related by the unitary transformation in (4.8), those q-numbers must have identical spectra. So, we can conclude that the spectrum of $\hat{t}$ must be equal to $\mathbb{R}$. Consequently, $\hat{t}$ has the following spectral decomposition in terms of its t-eigenvalue projector $\widehat{\Pi}_t(\hat{t})$:

$$\hat{t}=\int_{-\infty}^{\infty}t\widehat{\Pi}_t(\hat{t})dt.$$

4.6 Q-number functions of clock time

Because of the fundamental principle of quantum theory, the c-number time cannot play a role in quantum theory and must instead be replaced by $\hat{t}$. For instance, equations of motion should be formulated as differential equations with respect to $\hat{t}$. Here I will explain what functions of $\hat{t}$, and derivatives with respect to it, are.

The q-number $\hat{t}$ is unbounded since $\|\hat{t}\|=\infty$. This brings with it technical problems that I shall obviate by instead considering the q-number $e^{i\omega\hat{t}}$, where ω has units of frequency, making the product $\omega\hat{t}$ dimensionless. $e^{i\omega\hat{t}}$ is well-behaved because the operator norm of $e^{i\omega\hat{t}}$ is equal to unity:

$$\left.\begin{aligned}\|e^{i\omega\hat{t}}\|&=\sqrt{\sup\left(\mathrm{sp}\left(e^{-i\omega\hat{t}}e^{i\omega\hat{t}}\right)\right)}\\&=\sqrt{\sup\left(\mathrm{sp}(\hat{1})\right)}\\&=1\end{aligned}\right\}.$$

Hence, I shall here study finite sums of $e^{i\omega\hat{t}}$:

$$f(\hat{t})=\sum_{n=-N}^{N}a_n\,e^{i\omega_n\hat{t}},\tag{4.9}$$

which are evidently bounded functions (using the results of §1.10, one could also study convergent series in $e^{i\omega\hat{t}}$, but I shall not do so here). Here, N is an integer; the coefficients $\{a_n\}$ are complex numbers; and the frequencies $\{\omega_n\}$ are real-valued functions of the integer n. To ensure that $f(\hat{t})$ is Hermitian, I also impose that $(a_n)^\dagger = a_{-n}$ and that $\omega_{-n} = -\omega_n$ for all integers n. The Hermiticity of $f(\hat{t})$ is useful since this q-number function is thereby guaranteed to have a spectrum of eigenvalues. As follows from the results of §1.10, the spectrum of $f(\hat{t})$ is

$$\mathrm{sp}\big(f(\hat{t})\big) = \left\{ f(t) = \sum_{n=-N}^{N} a_n\, e^{i\omega_n t} \,:\, t \in \mathrm{sp}(\hat{t}) \right\}. \tag{4.10}$$

and its spectral decomposition is

$$f(\hat{t}) = \int_{-\infty}^{\infty} f(t)\hat{\Pi}_t(\hat{t})dt. \tag{4.11}$$

If the fundamental principle of quantum theory is taken seriously, it implies that all functions of time are q-numbers functions such as $f(\hat{t})$.

4.7 Derivatives with respect to clock time

C-number time has thus far not entered my construction: I have yet to discuss dynamical laws and c-number time. Yet even without dynamics, an eigenvalue $f(t)$ of $f(\hat{t})$ has a well-defined derivative with respect to clock time, denoted $\frac{df(t)}{dt}$. This seems to indicate that c-number time and derivatives with respect to it can be replaced by $\hat{t}$ and its derivatives. In a simplified way, this problem may be summarised as follows:

$$\text{if} \quad \lambda \longrightarrow \hat{t}, \quad \text{then} \quad \frac{d}{d\lambda} \longrightarrow \; ? \tag{4.12}$$

Here λ represents c-number time, and the arrow '⟶' should be understood to mean 'is replaced by'.[29]

To solve the problem described in (4.12), I will define a q-number derivative as follows:

[29] Perhaps the first to define a derivative for q-numbers was Dirac (1926); however, he did not use the algebra of quantum systems to define his derivatives, as I shall do here.

$$\frac{d\bullet}{d\hat{t}} \stackrel{\text{def}}{=} i[\hat{h}, \bullet], \tag{4.13}$$

where $\bullet$ denotes the operator's input. This definition is chosen for two primary reasons. Firstly, just like the ordinary c-number derivative, the operator defined in (4.13) is linear: the effect of the derivative on a linear combination of two arbitrary q-numbers $\hat{A}$ and $\hat{B}$ is equal to the same linear combination of the effect on $\hat{A}$ and $\hat{B}$ separately. That is to say that

$$\left.\begin{aligned} \frac{d(\alpha\hat{A} + \beta\hat{B})}{d\hat{t}} &= i[\hat{h}, \alpha\hat{A} + \beta\hat{B}] \\ &= \alpha i[\hat{h}, \hat{A}] + \beta i[\hat{h}, \hat{B}] \\ &= \alpha\frac{d\hat{A}}{d\hat{t}} + \beta\frac{d\hat{B}}{d\hat{t}} \end{aligned}\right\},$$

where α and β are real numbers.

Secondly, according to definition (4.13), the eigenvalues of a q-number derivative are the derivatives of a c-number function. For instance, the q-number derivative of $e^{i\omega\hat{t}}$ is

$$\frac{d(e^{i\omega\hat{t}})}{d\hat{t}} = i[\hat{h}, e^{i\omega\hat{t}}] = i\omega e^{i\omega\hat{t}}, \tag{4.14}$$

where ω is some arbitrary frequency.[30] And from (4.14), it directly follows that the spectral decomposition of $\frac{d(e^{i\omega\hat{t}})}{d\hat{t}}$ is

$$\left.\begin{aligned} \frac{d(e^{i\omega\hat{t}})}{d\hat{t}} &= \int_{-\infty}^{\infty} i\omega e^{i\omega t}\widehat{\Pi}_t(\hat{t})dt \\ &= \int_{-\infty}^{\infty} \frac{d(e^{i\omega t})}{dt}\widehat{\Pi}_t(\hat{t})dt \end{aligned}\right\}.$$

Put differently, the eigenvalues of $\frac{d(e^{i\omega\hat{t}})}{d\hat{t}}$ are exactly the derivatives $\frac{d(e^{i\omega t})}{dt}$.

[30] The result (4.14) can be derived as follows: because of the commutation relation (2.9), the unitary $e^{i\omega\hat{t}}$ is a translation operator of $\hat{h}$ so that $e^{-i\omega\hat{t}}\hat{h}e^{i\omega_n\hat{t}} = \hat{h} + \omega\hat{1}$, which can be rearranged to give the desired identity.

Due to the definition of a q-number function, this result is true in general: under the action of the q-number derivative, the image of an arbitrary q-number function $f(\hat{t})$ is

$$\left.\begin{aligned} \frac{df(\hat{t})}{d\hat{t}} &= \frac{d}{d\hat{t}}\left(\sum_{n=-N}^{N} a_n e^{i\omega_n \hat{t}}\right) \\ &= \sum_{n=-N}^{N} a_n \frac{d\left(e^{i\omega_n \hat{t}}\right)}{d\hat{t}} \\ &= \sum_{n=-N}^{N} i a_n \omega_n e^{i\omega_n \hat{t}} \end{aligned}\right\}. \tag{4.15}$$

Notably, the coefficients $ia_n\omega_n$ in (4.15) satisfy the constraint that $(ia_n\omega_n)^\dagger = ia_{-n}\omega_{-n}$ for all n, so the derivative of a Hermitian q-number function is also Hermitian. And since $\frac{df(\hat{t})}{d\hat{t}}$ is Hermitian, it admits a spectral decomposition, the form of which is determined by (4.15):

$$\frac{df(\hat{t})}{d\hat{t}} = \int_{-\infty}^{\infty} \frac{df(t)}{dt} \hat{\Pi}_t(\hat{t}) dt, \tag{4.16}$$

where

$$\frac{df(t)}{dt} = \sum_{n=-\infty}^{\infty} i a_n \omega_n \, e^{i\omega_n t}.$$

It is due to the clock's unique algebra that the eigenvalues of q-number functions' derivatives are equal to the derivatives of c-number functions. I shall call the theory of q-number functions and their derivatives the *q-number calculus*.

4.8 The Heisenberg-picture Page–Wootters construction

In the previous section, I discussed how to define derivatives with respect to clock time in the absence of dynamics. This was done by studying functions of $\hat{t}$, where $\hat{t}$ represents a possible state of the clock. Here I shall show that the equation of motion of a qubit can be formulated in terms of $\hat{t}$. As such, consider a Page–Wootters universe that consists of a clock $\mathfrak{C}$ and a qubit $\mathfrak{Q}$, where the qubit is represented by a triple of descriptors that are functions of $\hat{t}$. That is to say:

$$\hat{\boldsymbol{q}}(\hat{t}) = \left(\hat{q}_x(\hat{t}),\ \ \hat{q}_y(\hat{t}),\ \ \hat{q}_z(\hat{t})\right), \tag{4.17}$$

which satisfy the Pauli algebra

$$\left.\begin{aligned} \hat{q}_i(\hat{t})\hat{q}_j(\hat{t}) &= \delta_{ij}\hat{1} + i\epsilon_{ij}{}^{k}\hat{q}_k(\hat{t}) \\ \left(\hat{q}_j(\hat{t})\right)^{\dagger} &= \hat{q}_j(\hat{t}) \end{aligned}\right\} \qquad (i, j, k \in \{x, y, z\}). \tag{4.18}$$

Here $\hat{1}$ is the unit observable of the composite system.

The descriptors $\hat{\boldsymbol{q}}(\hat{t})$ are a function of $\hat{t}$ in the following sense:

$$\hat{q}_j(\hat{t}) \stackrel{\text{def}}{=} \sum_{n=-N}^{N} \hat{A}_{j,n} e^{i\omega_n \hat{t}} \qquad (j \in \{x, y, z\}), \tag{4.19}$$

where the coefficients $\{\hat{A}_{j,n}\}$ are bounded q-numbers, and N is an integer so that the sum is automatically bounded, too. I shall impose two restrictions on the coefficients: (i) the $\{\hat{A}_{j,n}\}$ commute with $\hat{t}$ so that the products in (4.19) are well-defined; and (ii) the $\{\hat{A}_{j,n}\}$ commute with $\hat{h}$, ensuring that derivatives of $\hat{\boldsymbol{q}}(\hat{t})$ with respect to $\hat{t}$ can be defined exactly as in (4.13). Hence, the coefficients $\{\hat{A}_{j,n}\}$ are q-numbers (of which at least some are not equal to the unit observable) that commute with the descriptors of the clock; they therefore reside on a different Hilbert space from that of $\hat{t}$ and $\hat{h}$.

The Pauli algebra (4.18) requires each descriptor $\hat{q}_j(\hat{t})$ to be Hermitian. For that to be so, it is necessary that the following criteria be satisfied: $\hat{A}_{j,n}^{\dagger} = \hat{A}_{j,-n}$ and that $\omega_{-n} = -\omega_n$ for the values $-N \leq n \leq N$ and $j \in \{x, y, z\}$. It is noteworthy that, although the algebra of the coefficients is constrained in various ways, these constraints are not sufficient to determine their exact algebraic expression. The explicit algebraic expressions for the coefficients are instead determined by the qubit's equations of motion, as we shall see in later sections. Note also that some of the coefficients can be equal to zero for certain values of n and j.

The triple $\hat{\boldsymbol{q}}(\hat{t})$ commutes with $\hat{t}$, and because of this, $\hat{\boldsymbol{q}}(\hat{t})$ admits a decomposition in terms of the t-eigenvalue projectors $\hat{\Pi}_t(\hat{t})$, i.e.

$$\hat{\boldsymbol{q}}(\hat{t}) = \int_{-\infty}^{\infty} \hat{\boldsymbol{q}}(t)\hat{\Pi}_t(\hat{t})dt, \tag{4.20}$$

where $\hat{\boldsymbol{q}}(t)$ is equivalent to the expression (4.19), but with $\hat{t}$ replaced by t. In decomposition (4.20), the product $\hat{\boldsymbol{q}}(t)\hat{\Pi}_t(\hat{t})$ is a triple of *relative descriptors* (Kuypers (2022a), Kuypers & Deutsch (2021)), where $\hat{\boldsymbol{q}}(t)\hat{\Pi}_t(\hat{t})$ represents the state of the qubit relative to the clock being in the state 'it is clock time t'.

Using the definition (4.13), a derivative of $\hat{\boldsymbol{q}}(\hat{t})$ with respect to $\hat{t}$ is

$$\frac{d\hat{\boldsymbol{q}}(\hat{t})}{d\hat{t}} = i[\hat{h}, \hat{\boldsymbol{q}}(\hat{t})], \tag{4.21}$$

so that

$$\frac{d\hat{\boldsymbol{q}}(\hat{t})}{d\hat{t}} = \int_{-\infty}^{\infty} \frac{d\hat{\boldsymbol{q}}(t)}{dt}\hat{\Pi}_t(\hat{t})dt. \tag{4.22}$$

$\hat{\boldsymbol{q}}(\hat{t})$ does not necessarily commute with $\hat{h}$, otherwise its derivatives with respect to $\hat{t}$ would invariably be zero. This is a noteworthy feature of the construction because it now seems as if the descriptors of the qubit and those of the clock do not commute, yet they should according to the commutation constraint. Assuming that the triple $\hat{\boldsymbol{q}}(\hat{t})$ represents the state of the qubit, we must conclude that $\hat{h}$ is not a descriptor of the clock anymore. Although $\hat{h}$ still satisfies (4.2), the clock's Hamiltonian is represented by a different descriptor. I shall return to this issue in §4.11.

To formulate dynamics in terms of clock time, I will assume that the qubit obeys the Heisenberg equation of motion with respect to clock time, meaning the qubit's descriptors satisfy the following algebraic expression:

$$[\hat{h} - \hat{H}, \hat{\boldsymbol{q}}(\hat{t})] = 0 \;\Leftrightarrow\; \frac{d\hat{\boldsymbol{q}}(\hat{t})}{d\hat{t}} = i[\hat{H}, \hat{\boldsymbol{q}}(\hat{t})]. \tag{4.23}$$

Here $\hat{H}$ is the qubit's Hamiltonian relative to clock time. Once c-number time is introduced, the system will also have a Hamiltonian that generates translations with respect to that c-number time. That Hamiltonian need not be equal to $\hat{H}$; it shall be specified below. (4.23) demonstrates the pivotal result of my construction: equations of motion can be formulated algebraically, in terms of the q-number time $\hat{t}$.

The $\hat{\boldsymbol{q}}(\hat{t})$ is a triple of Hermitian q-numbers, so formally these descriptors are observables. Yet physically, they are not observables of the system: observers in a Page–Wootters multiverse experience time because they live within relative states, whereas $\hat{\boldsymbol{q}}(\hat{t})$ represents the entire history of the qubit with respect to clock time, making $\hat{\boldsymbol{q}}(\hat{t})$ unmeasurable by a measurer inside the Page–Wootters multiverse. Instead, the observables of the qubit are its relative descriptors $\hat{\boldsymbol{q}}(t)$. For the theory to be consistent with observation, the relative descriptors should therefore satisfy the Heisenberg equation of motion. Because of (4.23), the relative descriptors automatically satisfy that equation. This can be readily verified by using (4.23), (4.22), (4.21), and (4.20), from which it follows that

$$\int_{-\infty}^{\infty}\left(\frac{d\hat{\boldsymbol{q}}(t)}{dt}-\left[\hat{H},\hat{\boldsymbol{q}}(t)\right]\right)\hat{\Pi}_t(\hat{t})dt=0,$$

so

$$\frac{d\hat{\boldsymbol{q}}(t)}{dt}=i\left[\hat{H},\hat{\boldsymbol{q}}(t)\right] \qquad \left(\text{for all } t\in\mathrm{Sp}(\hat{t})\right). \tag{4.24}$$

Hence, the Page–Wootters multiverse, though devoid of time, is nonetheless in agreement with observation, meaning we cannot distinguish empirically the Page–Wootters multiverse from the one in which we live.

4.9 An example theory

To gain a more hands-on understanding of the Page–Wootters model, as it has so far been presented, one might consider that the qubit studied above has a Hamiltonian $\hat{H}=\frac{1}{2}\Omega\hat{\sigma}_x$, where Ω is a constant with units of frequency, and that the qubit's relative descriptors at clock time $t=0$ are equal to the Pauli matrices, i.e. $\hat{\boldsymbol{q}}(0)=\left(\hat{\sigma}_x,\ \hat{\sigma}_y,\ \hat{\sigma}_z\right)$. Then a solution to the Heisenberg equation of motion of (4.23) is that

$$\hat{\boldsymbol{q}}(\hat{t}) = \left(\hat{\sigma}_x,\ \ \hat{\sigma}_y \cos(\Omega\hat{t}) + \hat{\sigma}_z \sin(\Omega\hat{t}),\ \ \hat{\sigma}_z \cos(\Omega\hat{t}) - \hat{\sigma}_y \sin(\Omega\hat{t})\right).$$

From this equation one can reconstruct the coefficients $\{\hat{A}_{j,n}\}$ and the frequencies $\{\omega_n\}$ in (4.19); in particular, one can express the $\{\omega_n\}$ in terms of Ω, and the $\{\hat{A}_{j,n}\}$ can be expressed as linear combinations of the Pauli matrices and the unit observable. More generally, for an unspecified Hamiltonian, the solution to (4.23) can be expressed in terms of the qubit's descriptors at clock time $t = 0$, as follows:

$$\hat{\boldsymbol{q}}(\hat{t}) = e^{i\hat{H}\hat{t}}\hat{\boldsymbol{q}}(0)e^{-i\hat{H}\hat{t}}. \tag{4.25}$$

The physical system studied in this section has been a qubit, which is perhaps the simplest quantum system. But the construction is more general, as it can accommodate not just a model multiverse consisting of a qubit and a clock: the model can consist of any physical system, provided it is isolated and its Hilbert space is finite-dimensional. For instance, the role of the qubit could be replaced by a network of n interacting qubits that each evolve relative to clock time. Each descriptor of that network can therefore be expressed as a finite sum of the form (4.19), where the descriptors of different qubits have potentially different sets of frequencies $\{\omega_n\}$.

Q-number functions of the form (4.19) are sufficiently general to represent the descriptors of any finite physical system in the Page–Wootters construction. This can be proven as follows: consider an arbitrary descriptor $\hat{A}(\hat{t})$ of a system with a finite-dimensional Hilbert space. This descriptor is assumed to evolve relative to clock time, so $\hat{A}(\hat{t}) = e^{i\hat{H}\hat{t}}\hat{A}(0)e^{-i\hat{H}\hat{t}}$, where $\hat{A}(0)$ represents the relative descriptor at clock time $t = 0$. The spectral decomposition of the unitary $e^{-i\hat{H}\hat{t}}$ is

$$e^{-i\hat{H}\hat{t}} = \sum_{\omega \in \mathrm{sp}(\hat{H})} \hat{\Pi}_\omega(\hat{H})e^{-i\omega\hat{t}},$$

where the spectrum $\mathrm{sp}(\hat{H})$ has finitely many elements for systems with a finite-dimensional Hilbert space. Because of this, one finds that any $\hat{A}(\hat{t})$ that satisfies the Heisenberg equation of motion can be written as

$$\left.\begin{aligned}\hat{A}(\hat{t}) &= e^{i\hat{H}\hat{t}}\hat{A}(0)e^{-i\hat{H}\hat{t}} \\ &= \sum_{\omega,\omega'\in\mathrm{sp}(\hat{H})} \hat{\Pi}_{\omega'}(\hat{H})\hat{A}(0)\hat{\Pi}_{\omega}(\hat{H})e^{-i(\omega-\omega')\hat{t}}\end{aligned}\right\}.$$

If one defines a new variable ω_n that ranges over all the values in $\{\omega - \omega' : \omega, \omega' \in \mathrm{sp}(\hat{H})\}$, then one can express the sum as

$$\left.\begin{aligned}\hat{A}(\hat{t}) &= \sum_n \sum_{\omega\in\mathrm{sp}(\hat{H})} \left(\hat{\Pi}_{\omega}(\hat{H})\hat{A}(0)\hat{\Pi}_{\omega_n-\omega}(\hat{H})\right) e^{-i\omega_n\hat{t}} \\ &= \sum_n \hat{c}(\omega_n)e^{-i\omega_n\hat{t}}\end{aligned}\right\}.$$

Here I have defined the coefficients $\{\hat{c}(\omega_n)\}$ as follows:

$$\hat{c}(\omega_n) \stackrel{\text{def}}{=} \sum_{\omega\in\mathrm{sp}(\hat{H})} \left(\hat{\Pi}_{\omega}(\hat{H})\hat{A}(0)\hat{\Pi}_{\omega_n-\omega}(\hat{H})\right).$$

A crucial step in the derivation is that the Hamiltonian of the system has a finite spectrum; otherwise, there would have been infinitely many terms in the expansion. Hence the sums of the $\{e^{-i\omega_n\hat{t}}\}$ are sufficiently general to represent any system with a finite-dimensional Hilbert space.

4.10 Removing c-number time

The descriptors of a possible state of the Page–Wootters universe are $\hat{t}$, $\hat{h}$ and $\hat{\boldsymbol{q}}(\hat{t})$, where in particular $\hat{\boldsymbol{q}}(\hat{t})$ obeys the equation of motion (4.23). Because the notion of a 'possible state' is timeless (i.e. does not refer to c-number time), the descriptors do not yet depend on λ.

However, c-number time is still there because I must now introduce the dynamical laws of isolated systems, which require that the descriptors obey the Heisenberg equation of motion with respect to c-number time. Since the Page–Wootters multiverse is an isolated quantum system, an arbitrary descriptor of the model, denoted $\hat{A}(\lambda)$, should satisfy the Heisenberg equation

$$\frac{d\hat{A}(\lambda)}{d\lambda} = i\big[\hat{\mathcal{H}}, \hat{A}(\lambda)\big], \tag{4.26}$$

where λ represents c-number time, and $\hat{\mathcal{H}}$ is the Hamiltonian of the multiverse. Notably, this Hamiltonian is different from that of the qubit, denoted $\hat{H}$ in (4.23). In particular, the Hamiltonian $\hat{H}$ generates translations of clock time $\hat{t}$, whereas $\hat{\mathcal{H}}$ generates translations of c-number time λ.

The descriptor $\hat{A}(\lambda)$ is a solution to (4.26) provided that $\hat{A}(\lambda) = e^{i\lambda\hat{\mathcal{H}}}\hat{A}(0)e^{-i\lambda\hat{\mathcal{H}}}$, where $\hat{A}(0)$ is the descriptor at c-number time $\lambda = 0$. So, if the descriptors $\hat{t}$, $\hat{h}$ and $\hat{\boldsymbol{q}}(\hat{t})$ represent the state of the model multiverse at $\lambda = 0$ then, at a later time, the state of the model is represented by the descriptors

$$\left.\begin{array}{c} e^{i\lambda\hat{\mathcal{H}}}\hat{t}e^{-i\lambda\hat{\mathcal{H}}} \\ e^{i\lambda\hat{\mathcal{H}}}\hat{h}e^{-i\lambda\hat{\mathcal{H}}} \\ e^{i\lambda\hat{\mathcal{H}}}\hat{\boldsymbol{q}}(\hat{t})e^{-i\lambda\hat{\mathcal{H}}} \end{array}\right\}. \tag{4.27}$$

C-number time still appears to play a role in the construction. To solve this problem, I will assume that $\hat{\mathcal{H}} = \hat{h}$ because, for this choice of $\hat{\mathcal{H}}$, the equation of motion (4.26) becomes

$$\frac{d\hat{A}(\lambda)}{d\lambda} = i[\hat{h}, \hat{A}(\lambda)] \stackrel{\text{def}}{=} \frac{d\hat{A}(\lambda)}{d\hat{t}}. \tag{4.28}$$

In light of (4.28), any change relative to the unphysical λ can be reinterpreted as change relative to the physical $\hat{t}$. This solves the appearance of the unphysical c-number time in (4.28), since that equation now has an interpretation that is manifestly physical. For the descriptors in (4.27) to abide by requirement (4.28), they must be equal to

$$\left.\begin{array}{r} e^{i\lambda\hat{\mathcal{H}}}\hat{t}e^{-i\lambda\hat{\mathcal{H}}} = \hat{t} + \lambda\hat{1} \\ e^{i\lambda\hat{\mathcal{H}}}\hat{h}e^{-i\lambda\hat{\mathcal{H}}} = \hat{h} \\ e^{i\lambda\hat{\mathcal{H}}}\hat{\boldsymbol{q}}(\hat{t})e^{-i\lambda\hat{\mathcal{H}}} = \hat{\boldsymbol{q}}\big(\hat{t} + \lambda\hat{1}\big) \end{array}\right\}.$$

All other descriptors of the model can be deduced from these ones, and hence, I have provided a full specification of the model's evolution relative to λ.

There is a related issue: the expectation values of the model's descriptors might still depend on c-number time. That issue is solved if the Heisenberg state $|\Psi\rangle$ is an eigenstate of $\hat{\mathcal{H}}$ because, in that case, the expectation value of an arbitrary descriptor $\hat{A}(\lambda)$ will be

$$\langle\Psi|\hat{A}(\lambda)|\Psi\rangle = \langle\Psi|e^{i\lambda\hat{\mathcal{H}}}\hat{A}(0)e^{-i\lambda\hat{\mathcal{H}}}|\Psi\rangle = \langle\Psi|\hat{A}(0)|\Psi\rangle \qquad \text{(for all } \lambda\text{)}.$$

The totality of these constraints ensures that the c-number is entirely irrelevant to the Heisenberg-picture Page–Wootters construction.

4.11 Connections between Schrödinger- and Heisenberg-picture constructions

The chart below summarises the connections between the Heisenberg- and Schrödinger-picture variants of the Page–Wootters construction:

The Schrödinger picture	**correspondence**	**The Heisenberg picture**
$\|\Psi_S\rangle = \int_{-\infty}^{\infty} \|\psi(t)\rangle\|t\rangle dt$	$\xleftrightarrow{\text{absolute state}}$	$\hat{\boldsymbol{q}}(\hat{t}) = \int_{-\infty}^{\infty} \hat{\boldsymbol{q}}(t)\hat{\Pi}_t(\hat{t})dt$
$\|\psi(t)\rangle\|t\rangle$	$\xleftrightarrow{\text{relative state}}$	$\hat{\boldsymbol{q}}(t)\hat{\Pi}_t(\hat{t})$
$\hat{\mathcal{H}}_S = \hat{H}_S + \hat{h}_S$	$\xleftrightarrow{\text{Hamiltonian}}$	$\hat{\mathcal{H}} = \hat{h}$
$i\frac{d}{dt}\|\psi(t)\rangle = \hat{H}_S\|\psi(t)\rangle$	$\xleftrightarrow{\text{equation of motion}}$	$\frac{d\hat{\boldsymbol{q}}(\hat{t})}{d\hat{t}} = i[\hat{H}, \hat{\boldsymbol{q}}(\hat{t})]$

As was first demonstrated by Rijavec (2022), these variants of the Page–Wootters construction are mathematically equivalent: there exists a unitary transformation that maps the Schrödinger-picture construction to its Heisenberg-picture counterpart. For instance, a Schrödinger-picture Page–Wootters model that consists of a qubit and clock can be translated to the Heisenberg picture by applying the unitary $V \stackrel{\text{def}}{=} e^{-i\hat{H}_S\hat{t}_S}$ to the descriptors and state vector:

$$\left.\begin{array}{lll} V^\dagger \hat{h}_S V = \hat{h}_S - \hat{H}_S, & V^\dagger \hat{t}_S V = \hat{t}_S, & V^\dagger \hat{\boldsymbol{q}}(0) V = e^{i\hat{H}_S \hat{t}_S} \hat{\boldsymbol{q}}(0) e^{-i\hat{H}_S \hat{t}_S} \\ V^\dagger \hat{\mathcal{H}}_S V = \hat{h}_S, & V^\dagger \hat{H}_S V = \hat{H}_S, & V^\dagger |\Psi_S\rangle = |\psi(0)\rangle |TL\rangle \end{array}\right\}. \tag{4.29}$$

Here $|TL\rangle \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} |t\rangle dt$ is the so-called 'timeline' state (Giovannetti et al. (2015)).

Notably, V is similar to the conventional unitary-transformation that maps the Schrödinger picture to the Heisenberg picture except that (i) it depends not on the c-number time but on the q-number time $\hat{t}_S$, and (ii) it depends on the Hamiltonian of the qubit $\hat{H}_S$ and not on the Hamiltonian of the entire multiverse, $\hat{\mathcal{H}}_S$. Moreover, the unitary V is a function of a descriptor of the qubit (namely, the qubit's Hamiltonian $\hat{H}_S$) and a descriptor of the clock (namely, the clock's time observable $\hat{t}_S$). Consequently, the Heisenberg-picture descriptors that the transformation V gives rise to are a mix of the Schrödinger-picture ones. This is evident from (4.29): e.g. in the Heisenberg picture, the Hamiltonian of the clock is $V^\dagger \hat{h}_S V = \hat{h}_S - \hat{H}_S$, which is a linear combination of the clock's and the qubit's Schrödinger-picture Hamiltonians.

Incidentally, due to the commutation constraint, the descriptors of the clock and qubit must commute in both pictures. In the Schrödinger picture, the commutation constraint implies in particular that $[\hat{h}_S, \hat{\boldsymbol{q}}(0)] = 0$. It follows from this commutation relation that $\hat{h}_S - \hat{H}_S$ commutes with $\hat{\boldsymbol{q}}(\hat{t})$ since

$$\left.\begin{array}{r} [\hat{h}_S, \hat{\boldsymbol{q}}(0)] = 0 \\ \leftrightarrow V^\dagger [\hat{h}_S, \hat{\boldsymbol{q}}(0)] V = 0 \\ \leftrightarrow [\hat{h}_S - \hat{H}_S, \hat{\boldsymbol{q}}(\hat{t})] = 0 \end{array}\right\}.$$

Hence, the commutation constraint implies that the qubit must satisfy equation of motion (4.23).

4.12 Musings on determinism

Consider a relative descriptor $\hat{\boldsymbol{q}}(t)\hat{\Pi}_t(\hat{t})$. It seems natural to assume that each such relative descriptor 'occurs', by which I mean that the expectation value of that relative descriptor is non-zero

$$\langle \hat{\Pi}_t(\hat{t}) \rangle > 0 \qquad \text{(for all } t \in \mathrm{Sp}(\hat{t})\text{)}.$$

Without this condition, certain physical states that would conventionally be part of the system's history are not in that history.

But it is not clear to what extent this assumption is strictly necessary for the Page–Wootters construction to explain time and dynamics (see for instance Marletto & Vedral (2017)). If some set of clock times were 'missing' and so never 'occurred', no observer would be able detect that this is so. The reason for that is as follows: even if a range of clock times are 'missing' from the Page–Wootters multiverse, every system in this multiverse will nonetheless end up in exactly the state it would have been in had those 'missing' clock times been there. For instance, it is conceivable that I, the author of this thesis, have memories of writing the previous paragraph and of the coffee I drank while writing it, despite there being no state of the multiverse in which I actually wrote that paragraph or drank that coffee. Perhaps this set of events that 'would have happened' never did, despite the apparent evidence that indicates it did happen. This is reminiscent of a situation in the novel *Permutation City* (Greg Egan (1994)), in which a simulated person, Paul, is being rendered by a computer at half-second intervals. Paul is not rendered at intermediate moments, and in the paragraph below, he contemplates how he feels no different despite that:

> *'Paul counted – and the truth was, he felt no different. A little uneasy, yes – but factoring out any squeamishness, everything about his experience seemed to remain the same. And that made sense, at least in the long run – because nothing was being omitted, in the long run […] His model-of-a-brain was being fully described at half seconds (model time) intervals – but each description still included the results of everything that "would have happened" in between. Every half-second, his brain was ending up in exactly the state it would have been in if nothing had been left out.'*

Perhaps Paul's predicament is like our own in that, in our universe – without our being aware of it – certain moments are 'missing', despite us having memories of them.

On a related note, there is the issue of determinism in the Page–Wootters multiverse: the future, past and present state of any system always exist in the Page–Wootters construction. Does that mean the future is not open? This predicament is reminiscent of a stanza from Omar

Khayyam's Robiáyát (translated by Le Gallienne (1900)), whom I also quoted at the start of this thesis:

To me there is much comfort in the thought
That all our agonies can alter nought,
Our lives are written to their latest word,
We but repeat a lesson He hath taught.

And indeed, in a sense our lives are 'written to their latest word' since all that has happened and ever will happen to us already exists as part of the Page–Wootters multiverse.

I shall not attempt to reconcile the issue of free will and determinism here (though the interested reader may wish to consult the work of Dennett (1984)). I will note, however, that the Page–Wootters model is as deterministic as unitary quantum theory, so the construction neither complicates nor simplifies this particular problem.

4.13 Clocks and manifolds

In the Page–Wootters construction, there is a single ideal clock that can keep track of time arbitrarily well, relative to which all other systems in the multiverse evolve. Yet there are bounds on how accurately any particular system can keep track of time since there are limits on the amount of information that can be stored in any compact region of space (Bekenstein (1973)). We should therefore expect that nature has not one ideal clock, and that there instead exist many systems that function as *unideal clocks* – i.e. clocks that have a finite dimensional Hilbert space – which represent time in some compact region of space.[31]

Instead of a single clock, there must be various clocks that exist throughout space. Those clocks, in order to represent a global time coordinate, should be synchronised with one

[31] Notably, Wootters (1984) also argued that there are many systems that function as clocks in nature, though he did not refer to Bekenstein's bound as a motivation for that claim.

another, which can either be due the initial conditions of the multiverse, or due to the clocks having interacted with one another.[32]

The conventional Page–Wootters construction makes no reference to synchronisations, as there is only one clock. So, to study what it means for physical clocks to be synchronised, consider two quantum clocks $\mathfrak{C}_1$ and $\mathfrak{C}_2$, with respective time observables $\hat{t}_1$ and $\hat{t}_2$ that satisfy the commutation constraint. Notably, $\mathfrak{C}_1$ and $\mathfrak{C}_2$ can be (but need not be) idealised clocks.

The clocks are ideally synchronised if the clock time of $\mathfrak{C}_2$ invariably agrees with that of $\mathfrak{C}_1$. That is to say, if one measures the clock time of $\mathfrak{C}_2$ and $\mathfrak{C}_1$ separately and compares the results, then the clocks should invariably agree on what time it is. Any offset detected between the clocks would mean that they are not synchronised. The offset between them is represented by the observable $\hat{t}_1 - \hat{t}_2$, and therefore the clocks are synchronised if $\hat{t}_1 - \hat{t}_2$ is sharp with value zero. Notably, this definition of synchronicity does not require either $\hat{t}_1$ or $\hat{t}_2$ separately to be sharp.

It is here that a remarkable connection becomes clear. The synchronised clocks $\mathfrak{C}_1$ and $\mathfrak{C}_2$ bear striking similarities to a manifold, for two reasons: firstly, a manifold has an atlas, which consists of a set of charts (i.e. functions from the manifold to $\mathbb{R}^n$). These charts are similar to the spectrum of an ideal clock $\mathfrak{C}_1$, as it provides a function from $\hat{t}_1$ to the continuum through $\mathrm{sp}(\hat{t}_1) = \mathbb{R}$. Moreover, the eigenvalues in $\mathrm{sp}(\hat{t}_1)$ represent clock time. That is to say, they are time coordinates, just as the chart represents coordinates on a manifold.

Secondly, in regions where two charts overlap, one can transition from one chart to another through a transition map, which is a homeomorphism between those charts. Similarly, when two clocks are synchronised, their time observables must satisfy the constraint that $\hat{t}_1 - \hat{t}_2$ is

[32] It seems that clocks should be able to interact with one another, yet relative to what do the clocks interact? This is an unsolved problem that needs to be addressed within the Page–Wootters construction before interacting clocks can be studied.

sharp with value zero. This constraint tells us how the time coordinate of one clock is related to that of another and, therefore, plays the same role as the transition map does for manifolds.

This analogy between a collection of clocks and a manifold is not a coincidence: in the physical universe, one can only define coordinates relative to physical systems, so a physical description of a manifold must refer to those systems, like clocks, relative to which coordinates are defined.

It would be of great interest to recast the description of a manifold in terms of quantum systems, and in (Kuypers (2022a)), I conjectured that this reformulation should be possible. However, there are technical and conceptual issues that must be solved first. For instance, such a formulation will need to be preceded by a generalisation of the Page–Wootters construction – e.g. the Page–Wootters construction as I have explained here does not have any notion of space.[33] When space has been included into the model, it will be possible to consider clocks at spacelike separations, which would allow us to study the ways in which such clocks can be synchronised. I shall leave these issues to be addressed in future research.

[33] A theory of quantum space and time has been constructed by Singh (2022), but it has not yet been translated to the Heisenberg picture. So, it is not yet clear whether that model describes a q-number manifold.

5 Unorthodox quantum theory

5.1 Problems with the commutation constraint

Throughout this thesis, I have assumed it to be a principle of quantum theory that the descriptors of spacelike-separated systems should mutually commute – an assumption that I have named the *commutation constraint*. That is to say that if $\hat{A}(t)$ and $\hat{B}(t)$ are descriptors of the spacelike-separated systems $\mathfrak{A}$ and $\mathfrak{B}$ respectively, then for all times t

$$[\hat{A}(t), \hat{B}(t)] = 0. \tag{5.1}$$

There are some notable exceptions to this rule – e.g. the descriptors of fermions anti-commute at spacelike separations. Yet it is true that even for such systems, there is always assumed to be some fixed commutation relation for spacelike-separated systems, which is equally problematic as the commutation constraint (as shall become clear below). But for clarity, I will here focus my attention on the commutation constraint. Henceforth, I shall call any quantum theory that adheres to constraint (5.1) an *orthodox quantum theory.*

The commutation constraint ensures that orthodox quantum systems satisfy Einsteinian locality as proven in §2.5. However, Einsteinian locality does not require systems to commute at spacelike separations. Fermions, which also satisfy Einsteinian locality, do not adhere to the commutation constraint (see, for instance, the work of Marletto et al. (2021), and Nicetu et al. (2022)). Moreover, Deutsch (2001) has constructed a novel approach to quantum field theory, in which the commutation relations of the field's descriptors at spacelike separations are left completely unspecified; such fields are called *qubit fields.* Yet qubit fields are locally realistic, too.[34] And although the commutation constraint is useful in that it guarantees

[34] A qubit field is local because its equations of motion depend only on the field's descriptor at a particular position and derivatives at that position. Consequently, no influence – no physical effect whatsoever – from another spacelike-separated region can influence the field at that point, unless something physical travels between those regions.

locality, it is also a causes severe problems – two in particular, which I shall discuss in §5.1.1 and §5.1.2.

5.1.1 Closed timelike curves

In order for systems to commute at spacelike separations, the notion of spacelike-separated needs to be unambiguous. This is the case when spacetime can be foliated by spacelike hypersurfaces, as then events on the same spacelike hypersurface have a well-defined notion of spacelike separation. Yet not all spacetimes admit such foliations: there are also spacetimes that contain closed timelike curves – such as the Gödel (1949) metric, the Kerr metric (Hawking & Ellis (1973, ch. 5)), and spacetimes with traversable wormholes (Morris et al. (1988) and Lockwood (2007, ch. 6)).

In the presence of closed timelike curves, the notion of spacelike separation becomes muddled, implying that the commutation constraint cannot be satisfied (as observed by Hartle (1994) and Hawking (1995)). For instance, in the vicinity of a closed timelike curve, two spacelike-separated systems might also be timelike-separated. One of them might, say, be an older instance of the other because it travelled back in time on the closed timelike curve, implying that they are 'simultaneously' spacelike- and timelike-separated.

To demonstrate in more detail why quantum systems on closed timelike curves cannot satisfy the commutation constraint, I shall study a qubit $\mathfrak{Q}_1$ that travels on a closed timelike curve. The qubit at a general time t is represented by a triple of q-number descriptors

$$\hat{\boldsymbol{q}}_1(t) = \left(\hat{q}_{1x}(t),\ \ \hat{q}_{1y}(t),\ \ \hat{q}_{1z}(t)\right), \tag{5.2}$$

which at all times t satisfy the Pauli algebra

$$\left.\begin{aligned} \hat{q}_{1i}(t)\hat{q}_{1j}(t) &= \delta_{ij}\hat{1} + i\epsilon_{ij}{}^{k}\hat{q}_{1k}(t) \\ \hat{q}_{1i}(t)^{\dagger} &= \hat{q}_{1i}(t) \end{aligned}\right\} \qquad (i,j,k \in \{x,y,z\}). \tag{5.3}$$

The qubit is assumed to have a classical world line that traverses the closed timelike curve. The curve itself is modelled as two spacelike cuts in spacetime, as in figure 2.a, and I will abstract away all remaining spacetime geometry. This ensures that outside of the closed timelike curve (the closed timelike curve occupies a finite region of space), there is an unambiguous future and past, also depicted in figure 2.a.

Because $\mathfrak{Q}_1$ traverses the closed timelike curve, $\mathfrak{Q}_1$ travels back in time relative to the global time outside of the chronology-violating region. $\mathfrak{Q}_1$ then becomes a second qubit $\mathfrak{Q}_2$ at an earlier time and separate location. $\mathfrak{Q}_2$ is represented by a triple $\hat{\boldsymbol{q}}_2(t)$ which likewise confirm to the Pauli algebra.

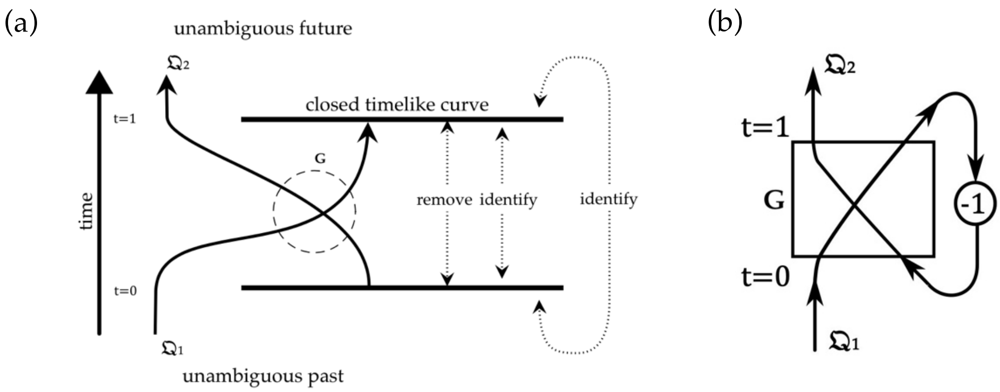

Figure 2. (a) A spacetime diagram depicting a closed timelike curve as a cut in spacetime, where its edges identified as shown. Furthermore, the diagram depicts the world lines of $\mathfrak{Q}_1$ and $\mathfrak{Q}_2$ and their interacting region G and the unambiguous future and past outside of the chronology-violating region. (b) A network diagram in which $\mathfrak{Q}_1$ enters a two-qubit gate **G**. After leaving the gate, $\mathfrak{Q}_1$ experiences a time-delay of -1 before participating in **G** again as the older qubit $\mathfrak{Q}_2$. These diagrams are from Kuypers (2022b).

Because $\mathfrak{Q}_1$ and $\mathfrak{Q}_2$ are qubits, their interactions in the interaction region G, as depicted in figure 2.a, can be modelled as a quantum computational gate **G** that acts on them for the duration of 1 unit of time. In such a model, the qubits enact the network shown in figure 2.b. After that interaction, $\mathfrak{Q}_1$ will travel back in time on the closed timelike curve, where it acquires a negative time-delay of -1 units (figure 2.b).

The qubits participate in gate **G** between time $t = 0$ and $t = 1$. In the Heisenberg picture, kinematic consistency requires the descriptors of the older qubit $\mathfrak{Q}_2$ at time $t = 0$ to be equal to the descriptors of the younger qubit $\mathfrak{Q}_1$ at time $t = 1$, i.e.

$$\hat{\boldsymbol{q}}_2(0) = \hat{\boldsymbol{q}}_1(1). \tag{5.4}$$

The kinematic consistency condition (5.4) is incompatible with the commutation constraint; that is to say that the descriptors of $\mathfrak{Q}_1$ cannot, in general, commute with those of $\mathfrak{Q}_2$. The clearest demonstration of this incompatibility is the case in which the gate $\mathbf{G}$ that acts on the qubits is the unit wire: due to this gate, the descriptors $\hat{\boldsymbol{q}}_1(1)$ are identical to what they are before the implementation of the gate so that $\hat{\boldsymbol{q}}_1(1) = \hat{\boldsymbol{q}}_1(0)$. Consequently, the descriptors of $\mathfrak{Q}_2$ at time $t = 0$ are identical to those of $\mathfrak{Q}_1$ since $\hat{\boldsymbol{q}}_2(0) = \hat{\boldsymbol{q}}_1(0)$, according to (5.4). Therefore, they cannot commute.

Previous models of qubits on closed timelike curves, such as those of Deutsch (1991) and Lloyd et al. (2011), have been formulated in terms of density matrices, which avoid the inconsistency caused by (5.4). Yet those models do not admit a Heisenberg-picture description of the same situation, so they violate the principle of locality. A local theory of qubits on a closed timelike curve requires those qubits to be described in the Heisenberg picture.

Even if one assumes that closed timelike curves do not exist in nature, they *could* exist according to general relativity. Therefore, the fact that the commutation constraint forbids the existence of closed timelike curves creates a conflict between the foundations of general relativity and those of orthodox quantum theory. This motivates the investigation of theories without the commutation constraint.

5.1.2 Quantum field theory

Another situation in which the commutation constraint results in conceptual issues is in quantum field theory. Orthodox field theory seems to imply that an arbitrary amount of information can be stored in a compact region of space, as was argued by Deutsch (2001). He argued as follows. Due to the commutation constraint, a scalar quantum field $\hat{\phi}(x)$ with spacetime coordinates x must satisfy the relation

$$\left[\hat{\phi}(x), \hat{\phi}(y)\right] = 0$$

when x and y are events on the same spacelike hypersurface. Consequently, there are infinitely many commuting observables in any region of space, each of which can be independently prepared and measured (Deutsch (2001)). So, one can store an arbitrary amount of information in a compact region of space according to quantum field theory. Yet it is widely believed that the amount of information that can be stored in any compact region of space has an upper bound, provided by Bekenstein's (1973) formula.

Another way to phrase the same argument is as follows: if there are infinitely many commuting observables in a compact region, then the Hilbert space of that region must be infinite-dimensional, due to the results of §1.9. But as Bao et al. (2017) have argued, the Hilbert space associated with such a region should be finite-dimensional; otherwise, there are infinitely many degrees of freedom within that region which could be entangled with those outside of it. In which case, the von Neumann entropy associated with $\mathcal{R}$ would be unbounded, which would again violate Bekenstein's bound.

This was one of the motivations of Deutsch (2001) for discarding that constraint and to instead create a quantum field theory in which the commutation relations at spacelike separations are not specified; the hypothetical fields he proposed are called *qubit fields*. A qubit field has a finite-dimensional Hilbert space because its descriptors do not necessarily commute at spacelike separation, which is why Deutsch conjectured that the fields can only store a finite amount of information.

However, to determine whether a given system can store information, and whether there are limits to its information storage capacity, one must have a theory of computation that applies to that system. That is to say that one must have a theory of how to use qubit fields to perform computational tasks. Such a theory currently does not exist in full, but in Kuypers (2022b), I demonstrated that a discretised version of a qubit field – i.e. a network of qubits that violate the commutation constraint – can perform computations tasks.

5.2 Unorthodox qubits

The commutation constraint is untenable in various circumstances – like in spacetimes containing closed timelike curves and situations in which a quantum field's information storage capacity is relevant. Because of this, I shall here present a theory of physical systems, first discussed by myself in Kuypers (2022b), that violate the commutation constraint, but which satisfies the other principles of quantum theory discussed in this thesis. Quantum theories that violate the commutation constraint are what I shall call *unorthodox quantum theories*. The simplest unorthodox system is an array $\mathfrak{A}$ of n interacting unorthodox qubits $\mathfrak{Q}_1, \mathfrak{Q}_2, \dots, \mathfrak{Q}_n$. I use the term *array* to mean that the qubits are continuously evolving; an array is different from a network, in which the state of the qubits need only be considered at integer times.

An unorthodox qubit $\mathfrak{Q}_a$ at time t is represented by a triple of q-number descriptors $\hat{\boldsymbol{q}}_a(t)$. These descriptors conform to the Pauli algebra, and additionally, I shall assume the descriptors to be analytic functions of time so that they have the following attributes: (i) they can be differentiated arbitrarily often, and (ii) they are equivalent to their Taylor series expansions.

If the qubits were orthodox quantum systems, one would now impose they obey the commutation constraint (2.6). But here, that constraint shall be dropped.

5.1 Dynamics

It is a principle of quantum theory that the algebra of a system's descriptors is a constitutive attribute. The descriptors of a qubit $\mathfrak{Q}_a$ must therefore conform to the Pauli algebra at all times, and, due to the results of §1.8, this implies that their evolution is unitary.

A problem now arises: local realism seems to require that the qubits' descriptors satisfy the commutation constraint because without it, the proof of §2.5 is no longer valid. This problem is caused by the fact that in orthodox quantum theory, a single unitary governs the evolution of all qubits. Consider, for instance, two unorthodox qubits $\mathfrak{Q}_a$ and $\mathfrak{Q}_b$ with descriptors $\hat{\boldsymbol{q}}_a(t)$ and $\hat{\boldsymbol{q}}_b(t)$ that violate the commutation constraint – not all of their descriptors mutually commute so that $[\hat{q}_{ai}(t), \hat{q}_{bj}(t)] \neq 0$ for some $i, j \in \{x, y, z\}$. If one then performs a non-trivial operation on $\mathfrak{Q}_a$, enacted by a unitary $U(\hat{\boldsymbol{q}}_a(t))$, which is a function only of $\hat{\boldsymbol{q}}_a(t)$, it is possible for it to affect $\mathfrak{Q}_b$ since without the commutation constraint, one finds that

$$U^\dagger(\hat{\boldsymbol{q}}_a(t))\hat{\boldsymbol{q}}_b(t)U(\hat{\boldsymbol{q}}_a(t)) \neq \hat{\boldsymbol{q}}_b(t).$$

How can the qubits' descriptors evolve unitarily and still be locally realistic without the commutation constraint?

The solution is to assign potentially different unitaries to the different qubits in the array so that individual operations can be performed on, say, a single qubit or a subset of qubits without this affecting the descriptors of other systems. This preserves both the algebra of the qubits, as each qubit evolves unitarily, and it simultaneously preserves local realism. Thus, an arbitrary qubit $\mathfrak{Q}_a$ is assigned a unitary $U_a(t)$, where the subscript a denotes that this unitary governs the evolution of that qubit specifically:

$$\hat{\boldsymbol{q}}_a(t) = U_a^\dagger(t)\hat{\boldsymbol{q}}_a(0)U_a(t) \qquad (1 \le a \le n), \tag{5.5}$$

where each $U_a(t)$ satisfies

$$U_a^\dagger(t)U_a(t) = \hat{1} \qquad (1 \le a \le n), \tag{5.6}$$

with $\hat{1}$ being the unit observable of system $\mathfrak{A}$.[35]

Hence, in unorthodox quantum theory, the evolution of the array of unorthodox qubits $\mathfrak{A}$ is specified by a set of initial conditions $\{\hat{\boldsymbol{q}}_a(0)\}$ and by the set of n unitaries $\{U_a(t)\}$. There are no constraints on the unitaries in the set $\{U_a(t)\}$, other than that they are unitaries, and in principle, each can be completely independent of the other.

This scheme preserves local realism because whenever some of the qubits should remain unaffected by an operation, this can be reflected by their independently chosen unitaries. For instance, one way of ensuring that a particular set of qubits remain unaffected for a certain duration is to require their unitaries to be equal to the unit observable. For example, consider the case in which $\mathfrak{Q}_a$ is subjected to the unitary $U_a(t) = e^{-\frac{1}{2}i\omega t\hat{q}_{ax}(0)}$, which rotates that qubit about its x-axis by ωt radians. If the remaining qubits in the array should be dynamically isolated from $\mathfrak{Q}_a$, then those remaining qubits can have unitaries that equal the unit

[35] Chao et al. (2017) treat a system of qubits that violate the commutation constraint. However, their construction is different from my own in that the evolution of their system is governed by a single unitary.

observable, ensuring that they will be unaffected. Using this set of unitaries and (5.5), I find that the descriptors of $\mathfrak{Q}_a$ are

$$\hat{\boldsymbol{q}}_a(t) = \left(\hat{q}_{ax},\ \ \cos(\omega t)\hat{q}_{ay} + \sin(\omega t)\hat{q}_{az}\,, \cos(\omega t)\hat{q}_{az} - \sin(\omega t)\hat{q}_{ay}\right), \tag{5.7}$$

while all other qubits in $\mathfrak{A}$ are left unchanged. Here I have suppressed the $t = 0$ labels for the sake of brevity. Evidently, Einstein's criterion is satisfied.

Notably, due to the potentially different unitaries $\{U_a(t)\}$, there is no Schrödinger picture for unorthodox qubits. For there to be a Schrödinger picture, there must be some unitary transformation which ensures that each of the $\{\hat{\boldsymbol{q}}_a(t)\}$ becomes time-independent and that all time-dependence is instead transferred to the state-vector. But such a transformation does not generally exist for the same reason that the qubits are not locally realistic under a single, global unitary: without the commutation constraint, there is in general no unitary transformation that can ensure that all the $\{\hat{\boldsymbol{q}}_a(t)\}$ become stationary. Therefore, if unorthodox qubits exist in nature, the Schrödinger picture must be fundamentally approximative, holding only in those circumstances in which unorthodox systems approximately have mutually commuting descriptors at spacelike separations.

5.2 Unorthodox Hamiltonians

Orthodox systems evolve unitarily, and that unitary evolution is generated by a Hamiltonian, as shown in §1.3. Likewise, because an unorthodox qubit $\mathfrak{Q}_a$ evolves unitarily, according to the unitary $U_a(t)$, one can define a Hamiltonian for it, too:

$$\hat{H}_a(t) \stackrel{\text{def}}{=} iU_a^\dagger(t)\frac{dU_a(t)}{dt}. \tag{5.8}$$

I shall call $\hat{H}_a(t)$ the *unorthodox Hamiltonian of* $\mathfrak{Q}_a$ or the *Hamiltonian of* $\mathfrak{Q}_a$ for short. $\hat{H}_a(t)$ is automatically Hermitian as one finds by differentiating (5.6).

Each qubit $\mathfrak{Q}_a$ has an equation of motion similar to the Heisenberg equation of motion (1.4) of orthodox quantum theory (1.4). $\mathfrak{Q}_a$'s equation of motion can be obtained by differentiating both sides of (5.5) with respect to t and using the definition (5.8); from that I obtain that

$$\left.\begin{aligned}\frac{d\hat{\boldsymbol{q}}_a(t)}{dt} &= \frac{dU_a^\dagger(t)}{dt}\hat{\boldsymbol{q}}_a(0)U_a(t) + U_a^\dagger(t)\hat{\boldsymbol{q}}_a(0)\frac{dU_a(t)}{dt} \\ &= -U_a^\dagger(t)\frac{dU_a(t)}{dt}\hat{\boldsymbol{q}}_a(t) + \hat{\boldsymbol{q}}_a(t)U_a^\dagger(t)\frac{dU_a(t)}{dt} \\ &= i\hat{H}_a(t)\hat{\boldsymbol{q}}_a(t) - i\hat{\boldsymbol{q}}_a(t)\hat{H}_a(t) \\ &= i\big[\hat{H}_a(t), \hat{\boldsymbol{q}}_a(t)\big]\end{aligned}\right\}. \tag{5.9}$$

Notably, the Hamiltonian $\hat{H}_a(t)$ only generates the motion of $\mathfrak{Q}_a$ and not that of any other qubit in the array. An arbitrary descriptor of the array that commutes with $\hat{H}_a(t)$ is therefore not automatically a constant of the motion; only if a descriptor $\hat{A}(t)$ commutes with every Hamiltonian of the array is it constant – i.e. only if $\big[\hat{H}_a(t), \hat{A}(t)\big] = 0$ for all $a \in \{1, \dots, n\}$ is $\hat{A}(t)$ time-independent.

Because $\hat{H}_a(t)$ generates the time evolution of the descriptors $\hat{\boldsymbol{q}}_a(t)$, it also generates the time evolution of any function of those descriptors $f\big(\hat{\boldsymbol{q}}_a(t)\big)$, provided that the function $f\big(\hat{\boldsymbol{q}}_a(t)\big)$ has an *algebraic expression* in terms of those descriptors. That is to say that $f\big(\hat{\boldsymbol{q}}_a(t)\big)$ should be expressed in terms of addition, multiplication, and scaling by real numbers of the components of $\hat{\boldsymbol{q}}_a(t)$. For such functions, it follows that

$$f\big(\hat{\boldsymbol{q}}_a(t)\big) = U_a^\dagger(t) f\big(\hat{\boldsymbol{q}}_a(0)\big) U_a(t),$$

as has been established in §1.10. So, one can reconstruct the proof (5.9) to find that

$$\frac{df\big(\hat{\boldsymbol{q}}_a(t)\big)}{dt} = i\big[\hat{H}_a(t), f\big(\hat{\boldsymbol{q}}_a(t)\big)\big]. \tag{5.10}$$

In general, $\hat{H}_a(t)$ does not commute with all other Hamiltonians of the array, so it changes with time and cannot represent the energy of a closed system. Currently, it is not known what the energy of an isolated system of unorthodox qubits is.

Not only does the Hamiltonian $\hat{H}_a(t)$ not necessarily commute with the other Hamiltonians of the array, but because it is not a constant of the motion, it will in general not commute with itself at different times either. What then is the most general expression of the unitary $U_a(t)$

in terms of the Hamiltonian? That expression can be obtained from the following differential equation, which the Hamiltonian and the unitary necessarily satisfy due to (5.8):

$$i\frac{dU_a(t)}{dt} = U_a(t)\hat{H}_a(t).$$

The unitary is assumed to satisfy the initial condition $U_a(0) = \hat{1}$ so that[36]

$$U_a^\dagger(0)\hat{\boldsymbol{q}}_a(0)U_a(0) = \hat{\boldsymbol{q}}_a(0). \tag{5.11}$$

From these considerations, it follows that $U_a(t)$ is a time-ordered product (Bauer et al. 2012)

$$U_a(t) = \mathcal{T}\exp\left(-i\int_0^t \hat{H}_a(t')\,dt'\right). \tag{5.12}$$

(5.12) establishes an important relation between the Hamiltonian $\hat{H}_a(t)$ and the unitary $U_a(t)$: we have already seen that from the unitaries $\{U_a(t)\}$ one can derive the expression of the Hamiltonian $\{\hat{H}_a(t)\}$ because of the definition (5.8). The opposite is shown in (5.12), as it provides a mapping from the Hamiltonians $\{\hat{H}_a(t)\}$ to the unitaries $\{U_a(t)\}$. Thus, no matter whether one starts with Hamiltonians or unitaries, (5.12) and (5.8) provide a way of translating back and forth between these two ways of expression the system's dynamics.

5.3 Isolated systems

Thus far, no restrictions have been placed on the Hamiltonians $\{\hat{H}_a(t)\}$: they can be functions of anything. However, for the array of unorthodox qubits to be an isolated system, its Hamiltonians can only be functions of the descriptors of the array, and not of anything outside of it. So, I now require that each Hamiltonian can be expressed algebraically in terms of the descriptors of $\mathfrak{A}$:

$$\hat{H}_a(t) = \hat{H}_a\big(\hat{\boldsymbol{q}}_1(t), \dots, \hat{\boldsymbol{q}}_n(t)\big) \qquad (1 \le a \le n). \tag{5.13}$$

[36] Prima facie, it seems possible that $U_a(0)$ is a function of descriptors that commute with $\hat{\boldsymbol{q}}_a(0)$, as then (5.11) would still be satisfied. It is currently unclear what – if anything – the physical relevance of such initial conditions would be.

This prevents the Hamiltonians from having an explicit time-dependence. (5.13) defines an isolated system, and when a system violates (5.13), it shall be called an open system.

(5.13) makes apparent that the equations of motion of the array $\mathfrak{A}$ can be unusually complicated compared to those of an isolated orthodox system. That is so because a Hamiltonian $\hat{H}_a(\hat{\boldsymbol{q}}_1(t), \ldots, \hat{\boldsymbol{q}}_n(t))$ can depend on any descriptor of $\mathfrak{A}$; it might, for instance, depend on the triple $\hat{\boldsymbol{q}}_b(t)$, for some b so that $a \neq b$. In that case, $\hat{H}_a(\hat{\boldsymbol{q}}_1(t), \ldots, \hat{\boldsymbol{q}}_n(t))$ may be denoted simply as $\hat{H}_a(\hat{\boldsymbol{q}}_b(t))$, and the derivative of $\hat{H}_a(\hat{\boldsymbol{q}}_b(t))$ will then depend on $\hat{H}_b(\hat{\boldsymbol{q}}_1(t), \ldots, \hat{\boldsymbol{q}}_n(t))$ because of (5.10):

$$\frac{d\hat{H}_a(\hat{\boldsymbol{q}}_b(t))}{dt} = i\left[\hat{H}_b(\hat{\boldsymbol{q}}_1(t), \ldots, \hat{\boldsymbol{q}}_n(t)), \hat{H}_a(\hat{\boldsymbol{q}}_b(t))\right].$$

Evidently, the evolution of $\hat{H}_a$ also depends on that of $\hat{H}_b$, which in turn might depend on that of another Hamiltonian. To solve the dynamics of an isolated system of unorthodox qubits, one must in general solve n coupled differential equations – not a straightforward task. I show how this can be done for a simple model theory in §5.8.

5.4 Constants of the motion

Orthodox quantum systems are characterised by the algebra of their descriptors – e.g. qubits are qubits because their descriptors obey the Pauli algebra. And in orthodox quantum theory, all algebraic relations of a system's descriptors are constitutive attributes, as there is a single unitary that acts on all descriptors in the same way. This is not true for unorthodox systems. An unorthodox qubit's descriptors invariably adhere to the Pauli algebra, but the commutation relations among qubits can vary with time. That is to say that the commutator $[\hat{q}_{ai}(t), \hat{q}_{bj}(t)]$ for $a \neq b$ and $i, j \in \{x, y, z\}$ is a function of time and not necessarily a constant.

If these algebraic relations of the descriptors can change with time, then in what sense does the array $\mathfrak{A}$ remain the same system? To identify $\mathfrak{A}$ as the same system, it must have some constitutive (i.e. time-invariant) attribute. This attribute is the algebraic closer of the array's descriptors, denoted $\boldsymbol{O}(t)$. That is to say that $\boldsymbol{O}(t)$ is the set of Hermitian q-numbers that can be expressed algebraically in terms of the descriptors in $\{\hat{q}_{aj}(t)\}$ (this set includes all values of $a \in \{1, .., n\}$ and $j \in \{x, y, z\}$).

$\boldsymbol{\mathcal{O}}(t)$ is a set of time-dependent descriptors, but it is itself a constant of the motion. That is so because any q-number in $\boldsymbol{\mathcal{O}}(t)$ is also present in $\boldsymbol{\mathcal{O}}(0)$ and visa versa, as I have proven in Kuypers (2022b). I shall here provide a rough outline of the proof in that paper. By definition, any element in $\boldsymbol{\mathcal{O}}(t)$ can be expressed algebraically in terms of the descriptors $\{\hat{q}_{aj}(t)\}$, and since those descriptors are analytic, any $\hat{q}_{aj}(t)$ in that set can be Taylor-expanded around, for instance, $t = 0$:

$$\hat{q}_{aj}(t) = \sum_{k=0}^{\infty} \left(\frac{t^k}{n!}\right) \frac{d^k \hat{q}_{aj}(0)}{dt^k}. \tag{5.14}$$

Evidently, the first term in the expansion is $\hat{q}_{aj}(0)$ and is an element of $\boldsymbol{\mathcal{O}}(0)$. Likewise, the second term $\frac{d\hat{q}_{aj}(0)}{dt}$ also belongs to $\boldsymbol{\mathcal{O}}(0)$, as can be shown by using (5.13) and (5.9) to obtain

$$\frac{d\hat{q}_{aj}(0)}{dt} = i\left[\hat{H}\left(\hat{\boldsymbol{q}}_1(0), \ldots, \hat{\boldsymbol{q}}_n(0)\right), \hat{q}_{aj}(0)\right]. \tag{5.15}$$

The right side of (5.15) is clearly an element of $\boldsymbol{\mathcal{O}}(0)$, so the left side of (5.15) must be, too. One can demonstrate, in like manner, that all higher order terms in the Taylor expansion are in $\boldsymbol{\mathcal{O}}(0)$, so any $\hat{q}_{aj}(t)$ must also be part of that set. Hence, any element of $\boldsymbol{\mathcal{O}}(t)$ can be expressed algebraically in terms of the $\{\hat{q}_{aj}(t)\}$, which in turn belong to $\boldsymbol{\mathcal{O}}(0)$ so that $\boldsymbol{\mathcal{O}}(t) \subseteq \boldsymbol{\mathcal{O}}(0)$.

This proof also works the other way around, as one can show that all elements of $\{\hat{q}_{aj}(0)\}$ are also part of $\boldsymbol{\mathcal{O}}(t)$ through a Taylor expansion. It therefore follows that $\boldsymbol{\mathcal{O}}(t) = \boldsymbol{\mathcal{O}}(0)$ for any time t, making $\boldsymbol{\mathcal{O}}(t)$ a constant of the motion, as promised. Since the time label of $\boldsymbol{\mathcal{O}}(t)$ is superfluous, I will henceforth denote it simply as $\boldsymbol{\mathcal{O}}$.

Unlike in the orthodox theory, the algebraic relations between descriptors of spacelike-separated systems are not conserved – i.e. they are not constitutive attributes of unorthodox systems. So those algebraic relations cannot characterise an array of unorthodox qubits.

Instead, unorthodox systems have a different invariant, namely the set $\boldsymbol{O}$ of q-numbers that can be generated algebraically from those qubits' descriptors, and it characterises an array.

5.5 Hilbert space

As discussed in §2.3, the algebraic relations of a system's descriptors determine the size of its Hilbert space. In particular, due to the commutation constraint the size of the Hilbert space of a system of n orthodox qubits scales exponentially with n since the size of its Hilbert space is 2^n. This is sometimes called the tensor product structure of Hilbert space.

To summarise the results of §2.3, consider, for instance, a system of two orthodox qubits, $\mathfrak{Q}_1$ and $\mathfrak{Q}_2$. The full set of Hermitian descriptors of $\mathfrak{Q}_1$ is spanned by the descriptors $\hat{q}_{1x}(t)$, $\hat{q}_{1y}(t)$, and $\hat{q}_{1z}(t)$, i.e.

$$\boldsymbol{O}_1(t) = \mathrm{span}\{\hat{q}_{1x}(t),\ \hat{q}_{1y}(t),\ \hat{q}_{1z}(t)\},$$

and these descriptors, in isolation, can be represented as 2×2 matrices that live on a 2-dimensional Hilbert space, denoted $\boldsymbol{H}_1$. Likewise, the vector space $\boldsymbol{O}_2(t)$ represents the set of all descriptors of $\mathfrak{Q}_2$, which live on the 2-dimensional Hilbert space $\boldsymbol{H}_2$. The commutation constraint ensures that the elements in $\boldsymbol{O}_1(t)$ are linearly independent of the elements in the set $\boldsymbol{O}_2(t)$ (for a proof, see §2.3) so that

$$\boldsymbol{O}_1(t) \cap \boldsymbol{O}_2(t) = \emptyset.$$

Put differently, the commutation constraint ensures that (i) all elements in $\boldsymbol{O}_1(t)$ commute with those of $\boldsymbol{O}_2(t)$; and (ii) that $\boldsymbol{O}_1(t)$ and $\boldsymbol{O}_2(t)$ are disjoint sets of q-numbers. Because the descriptors in $\boldsymbol{O}_1(t)$ are linearly independent of those in $\boldsymbol{O}_2(t)$, the descriptors in these two sets live on separate Hilbert spaces. And the Hilbert space of the two-qubit composite system that consists of $\mathfrak{Q}_1$ and $\mathfrak{Q}_2$ is equal to the tensor product of the qubits' separate Hilbert spaces $\boldsymbol{H}_1 \otimes \boldsymbol{H}_2$, which is 4-dimensional. This generalises to a system of n qubits, which by the same reasoning has a 2^n-dimensional Hilbert space.

What would the dimension of the Hilbert space of $\mathfrak{Q}_1$ and $\mathfrak{Q}_2$ be if those qubits are unorthodox? Unorthodox qubits do not have fixed algebraic relations, so the elements of $\boldsymbol{O}_1(t)$ do not necessarily commute with those in $\boldsymbol{O}_2(t)$, implying that the elements in $\boldsymbol{O}_1(t)$ are not necessarily linearly independent from those in $\boldsymbol{O}_2(t)$. The most extreme case of this is when two unorthodox qubits $\mathfrak{Q}_1$ and $\mathfrak{Q}_2$ have the following property:

$$\boldsymbol{O}_1(t) = \boldsymbol{O}_2(t). \tag{5.16}$$

In cases where (5.16) holds, any descriptor of $\mathfrak{Q}_1$ can be expressed as a linear combination of the descriptors of $\mathfrak{Q}_2$ and vice versa. The descriptors of both $\mathfrak{Q}_1$ and $\mathfrak{Q}_2$ must then be q-numbers on the same 2-dimensional Hilbert space so that the Hilbert space of the two-qubit composite system is 2-dimensional, in contrast to the 4-dimensional Hilbert space of two orthodox qubits.

When a pair of qubits has property (5.14), I shall say that they are *maximally non-commuting* ('maximal' because for these commutation relations, the dimension of the qubits' Hilbert space cannot be smaller). Importantly, if $\mathfrak{Q}_1$ maximally non-commutes with $\mathfrak{Q}_2$, and $\mathfrak{Q}_2$ maximally non-commutes with another qubits $\mathfrak{Q}_3$, this implies that

$$\boldsymbol{O}_1(t) = \boldsymbol{O}_2(t) = \boldsymbol{O}_3(t).$$

From which one can conclude that $\mathfrak{Q}_1$ and $\mathfrak{Q}_3$ maximally non-commute with one another, too.

Not all unorthodox qubits are maximally non-commuting. Unorthodox qubits might also be in states for which their descriptors mutually commute; in general, it is left unspecified what the qubits' commutation relations are.

When the descriptors of two qubits do not mutually commute, this can only make the dimension of their Hilbert space smaller. That is because it is only when the descriptors of qubits commute that those descriptors are linearly independent of one another, whereas in the non-commuting case that is not guaranteed. This implies that the dimension of Hilbert space of an array of n unorthodox qubits, denoted N, is largest when those qubits have mutually commuting descriptors, in which case $N = 2^n$. On the other hand, the dimension of the array's Hilbert space is smallest when all qubits in the array maximally non-commute, i.e. when

$$\boldsymbol{\mathcal{O}}_1(t) = \boldsymbol{\mathcal{O}}_2(t) = \cdots = \boldsymbol{\mathcal{O}}_n(t).$$

In this boundary case, the descriptors of any qubit can be represented as 2×2 matrices and live on the same 2-dimensional Hilbert space, which is the smallest dimension that the array's Hilbert space can have. Finally, the dimension of the array's Hilbert space must be a power of 2, as proven by Deutsch (2001). Put differently, $N = 2^m$ for some integer m, and since N cannot be larger than 2^n or smaller than 2, I find that $1 \leq m \leq n$.

5.6 Number of parameters needed for unorthodox qubits

Since the Hilbert space of unorthodox systems can be smaller than that of orthodox ones, might it similarly be the case that fewer parameters are necessary to specify the state of an unorthodox system? Consider for instance an array $\mathfrak{A}$ of n unorthodox qubits $\mathfrak{Q}_1,\ldots,\mathfrak{Q}_n$. When all qubits in this array maximally non-commute – i.e. when

$$\boldsymbol{\mathcal{O}}_1(t) = \boldsymbol{\mathcal{O}}_2(t) = \cdots = \boldsymbol{\mathcal{O}}_n(t),$$

the array's Hilbert space is 2-dimensional Hilbert space.

As the descriptor in the array lives on the same 2-dimensional Hilbert space, each such descriptor can be represented as a Hermitian 2×2 matrix. In particular, any descriptor can be represented as a linear combination of the Pauli matrices $\hat{\sigma}_x$, $\hat{\sigma}_y$, and $\hat{\sigma}_z$, with the restriction that a qubit's descriptors, in isolation, adhere to the Pauli algebra.

The most general linear combination of the triple $\hat{\boldsymbol{\sigma}} \stackrel{\text{def}}{=} (\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z)$ that preserves its Pauli algebra is a rigid rotation of those matrices. So, at some general time t, the descriptors of an arbitrary qubit in this particular array can be expressed as

$$\hat{\boldsymbol{q}}_a(t) = \hat{R}\big(\theta_a(t), \phi_a(t), \psi_a(t)\big)\hat{\boldsymbol{\sigma}} \qquad (1 \leq a \leq n). \tag{5.17}$$

Here $\hat{R}$ is a 3×3 rotation matrix which rotates the triple about its x-, y-, and z-axis by the respective angles $\theta_a(t)$, $\phi_a(t)$, and $\psi_a(t)$, which are real numbers in $[0,2\pi)$. Through inspection of (5.17), it is evident that the descriptors of each qubit contain at most three real-valued parameters, so the n-qubit array consists of no more than $3n$ parameters. An array of orthodox qubits requires exactly $2^{2n} - 1$ real-valued parameters, as shown by Bédard (2021) and discussed in §2.6.

The $3n$ parameters of all the maximally non-commuting qubits specify the state of the array. This implies that there are fewer states available to the array of maximally non-commuting qubits than for a network of orthodox ones and provides a handwaving argument in favour of the idea that less information can be stored in unorthodox qubits, as should be the case according to Deutsch's conjecture.

5.7 Heisenberg state

Besides an array's descriptors, which fully specify the dynamics of the system, there is also the array's stationary Heisenberg state $|\Psi\rangle$, which is necessary for predicting the expectation values of descriptors.[37] The expectation value of an arbitrary time-dependent descriptor $\hat{A}(t)$ of an array of unorthodox qubits $\mathfrak{A}$ shall, as usual, be denoted as

$$\langle \hat{A}(t) \rangle \overset{\text{def}}{=} \langle \Psi | \hat{A}(t) | \Psi \rangle.$$

Here I have assumed that $\langle \hat{A}(t) \rangle$ is a measurable quantity, but I have not yet proven this. That is to say that I have not used the theory of measurement presented in §3.1 to derive what the measurable attributes of unorthodox qubits are. That shall be done partly in §5.11.

I shall again fix the Heisenberg state so that it is a standard constant by imposing that the initial $(t = 0)$ z-observables of all qubits are sharp with value 1, i.e.

$$\langle \hat{q}_{az}(0) \rangle = 1 \qquad (1 \leq a \leq n). \tag{5.18}$$

[37] Note that the Heisenberg state $|\Psi\rangle$ is a ray in the Hilbert space $\boldsymbol{H}$ of the array $\mathfrak{A}$.

So, up to an irrelevant phase factor, the Heisenberg state $|\Psi\rangle$ is equivalent to the eigenstate of each of the descriptors in $\{\hat{q}_{az}(0)\}$ with eigenvalue 1. It is noteworthy that (5.18) can only be satisfied if all z-observables commute with one another. The reason being that two q-numbers can only be sharp simultaneously if they commute, and (5.18) requires each q-number in $\{\hat{q}_{az}(0)\}$ to be sharp.

No generality is lost due to (5.18) since no matter the commutation relations of the descriptors, and no matter the size of an array's Hilbert space, an array always has some descriptors that commute. The most notable example of this is that the z-observables of each descriptor can commute even if all qubits in an array maximally non-commute (see for example §5.8) because, in that case, each of those z-observables can be represented by the same q-number.

5.8 A model theory of two interacting unorthodox qubits

I have described the foundations of unorthodox qubits. To make the theory less abstract, I shall now provide a description of a toy model consisting of two unorthodox qubits, where some of the unique features of such qubits, discussed in previous sections, shall be highlighted.

Consider two interacting unorthodox qubits $\mathfrak{Q}_1$ and $\mathfrak{Q}_2$. At some general time t, they are represented by the following descriptors:

$$\left.\begin{aligned} \hat{\boldsymbol{q}}_1(t) &= \left(\hat{q}_{1x}(t),\ \ \hat{q}_{1y}(t),\ \ \hat{q}_{1z}(t)\right) \\ \hat{\boldsymbol{q}}_2(t) &= \left(\hat{q}_{2x}(t),\ \ \hat{q}_{2y}(t),\ \ \hat{q}_{2z}(t)\right) \end{aligned}\right\}.$$

As usual, each qubit in isolation is represented by a triple of descriptors that adhere to the Pauli algebra. Besides the algebra of the individual qubits, additional constraints are necessary to specify the algebraic relations of the descriptors of the separate qubits. Unorthodox quantum theory does not prescribe such constraints, as those are instead determined by the particular physical situation in question – e.g. the commutation relation of qubits on closed timelike curves is determined by their dynamics (see §5.12).

Here, I will consider a situation in which the qubits, at the initial time $t = 0$, have identical descriptors, i.e.

$$\hat{\boldsymbol{q}}_1(0) = \hat{\boldsymbol{q}}_2(0). \tag{5.19}$$

We have already seen a situation in which the descriptors of two qubits are identical – namely in §5.1.1, where a qubit was analysed while it travelled on a closed timelike curve. Due to that qubit enacting a unit wire while it traversed the closed timelike curve, it produced a second qubit (namely, an older version of itself) whose descriptors were identical to its own.

From requirement (5.18) follows that the qubits' initial commutation relation is

$$\hat{q}_{1i}(0)\hat{q}_{2j}(0) = \hat{1}\delta_{ij} + i\epsilon_{ij}{}^{k}\hat{q}_{1k}(0) \qquad\qquad (i, j, k \in \{x, y, z\}).$$

Because the descriptors of both qubits are initially identical, as shown in (5.18), and because a qubit in isolation can be represented by the 2×2 Pauli matrices, it follows that both qubits at the initial time $t = 0$ can be represented by those matrices:

$$\left.\begin{aligned} \hat{\boldsymbol{q}}_1(0) &= (\hat{\sigma}_x,\ \ \hat{\sigma}_y,\ \ \hat{\sigma}_z) \\ \hat{\boldsymbol{q}}_2(0) &= (\hat{\sigma}_x,\ \ \hat{\sigma}_y,\ \ \hat{\sigma}_z) \end{aligned}\right\}. \tag{5.20}$$

Notably, the z-observables of the qubits are both equal to $\hat{\sigma}_z$, implying that their z-observables commute and can be sharp simultaneously, as required by (5.18). This is despite the qubits maximally non-commuting: since the descriptors of both qubits are identical, it straightforwardly follows that

$$\boldsymbol{O}_1(t) = \boldsymbol{O}_2(t),$$

implying that $\mathfrak{Q}_1$ and $\mathfrak{Q}_2$ maximally non-commute.

Maximally non-commuting qubits such as $\mathfrak{Q}_1$ and $\mathfrak{Q}_2$ have unique properties. One notable feature of the qubits is that if, at time $t = 0$, $\mathfrak{Q}_1$ and $\mathfrak{Q}_2$ were to be swapped, this would be unnoticeable. Not only would there be no measurement that could show such an exchange had occurred, but at the level of the formalism, one could not detect this change either – unlike for orthodox qubits, whose descriptors would reveal that a swap happened. (One way this would be evident is from the matrix representation before and after the swap.) Again, this provides a handwaving argument that the unorthodox qubits store less information than their orthodox counterparts, though that question has yet to be settled.

Another noteworthy property of maximally non-commuting qubits, such as $\mathfrak{Q}_1$ and $\mathfrak{Q}_2$, is that the anticommutator of their descriptors is always a multiple of the unit observable. That is to say that at any time t, the descriptors of $\mathfrak{Q}_1$ and $\mathfrak{Q}_2$ satisfy the following identity:

$$\{\hat{q}_{1i}(t), \hat{q}_{2j}(t)\} = \mathrm{tr}\left(\hat{q}_{1i}(t)\hat{q}_{2j}(t)\right)\hat{1} \qquad \text{(for all } i, j \in \{x, y, z\}). \tag{5.21}$$

This can be proven as follows: the descriptors of $\mathfrak{Q}_1$ and $\mathfrak{Q}_2$ can invariably be expressed as linear combinations of the Pauli matrices because any descriptor of that system can be expressed in terms of those matrices (and the unit observable). So, let

$$\left.\begin{aligned} \hat{q}_{1i} &= \alpha_i{}^k\hat{\sigma}_k \\ \hat{q}_{2j} &= \beta_j{}^l\hat{\sigma}_l \end{aligned}\right\} \qquad \left(\alpha_i^k, \beta_j{}^l \in \mathbb{R}\right). \tag{5.22}$$

where I have suppressed the time labels for clarity and used the Einstein summation convention to sum the indices k and l over all their possible values in $\{x, y, z\}$. Notably, the sums do not include the unit observable because the qubits' descriptors are traceless. The equations (5.22) and (5.21) imply that

$$\{\hat{q}_{1i}, \hat{q}_{2j}\} = \alpha_i{}^k\beta_j{}^l\{\hat{\sigma}_k, \hat{\sigma}_l\}.$$

Furthermore, since the Pauli matrices anti-commute, this expression simplifies as follows:

$$\{\hat{q}_{1i}, \hat{q}_{2j}\} = \alpha_i{}^k\beta_j{}^l\{\hat{\sigma}_k, \hat{\sigma}_l\} = \alpha_i{}^k\beta_j{}^l\delta_{kl}\hat{1} = \alpha_i{}^k\beta_{j,k}\hat{1},$$

where $\alpha_i{}^k\beta_{j,k}$ is equal to the trace of the product $\hat{q}_{1i}\hat{q}_{2j}$ since

$$\mathrm{tr}\left(\hat{q}_{1i}\hat{q}_{2j}\right) = \mathrm{tr}\left(\alpha_i{}^k\hat{\sigma}_k\beta_j{}^l\hat{\sigma}_l\right) = \alpha_i{}^k\beta_j{}^l\mathrm{tr}(\hat{\sigma}_k\hat{\sigma}_l) = \alpha_i{}^k\beta_j{}^l\delta_{kl} = \alpha_i{}^k\beta_{j,k}.$$

Here I have used that $\mathrm{tr}\left(\hat{\sigma}_i\hat{\sigma}_j\right) = 2\delta_{ij}$ for $i, j \in \{x, y, z\}$. By combining these two results, one obtains the desired identity (5.21).

I shall now turn to the qubits' dynamics, which require that I specify their Hamiltonians. In this toy model, I shall treat the same dynamics as in Kuypers (2022b), where the qubits' Hamiltonians are

$$\left.\begin{aligned}\hat{H}_1(\hat{\boldsymbol{q}}_1(t),\hat{\boldsymbol{q}}_2(t)) &= \frac{1}{4}\{\{\hat{q}_{1z}(t),\hat{q}_{2z}(t)\},\hat{q}_{1x}(t)\}\\ \hat{H}_2(\hat{\boldsymbol{q}}_1(t),\hat{\boldsymbol{q}}_2(t)) &= 0\end{aligned}\right\}. \tag{5.23}$$

Evidently, the descriptors $\mathfrak{Q}_2$ are constants of the motion because the Hamiltonian of $\mathfrak{Q}_2$ is equal to zero; that is to say that for all times t

$$\hat{\boldsymbol{q}}_2(t) = \hat{\boldsymbol{q}}_2(0). \tag{5.24}$$

Using (5.24) and (5.21), one can now simplify the expression for $\hat{H}_1$ presented in (5.23) as follows:

$$\left.\begin{aligned}\hat{H}_1(\hat{\boldsymbol{q}}_1(t),\hat{\boldsymbol{q}}_2(t)) &= \frac{1}{4}\{\{\hat{q}_{1z}(t),\hat{q}_{2z}(t)\},\hat{q}_{1x}(t)\}\\ &= \frac{1}{4}\{\{\hat{q}_{1z}(t),\hat{q}_{2z}(0)\},\hat{q}_{1x}(t)\}\\ &= \frac{1}{4}\{\mathrm{tr}(\hat{q}_{1z}(t)\hat{q}_{2z}(0)),\hat{q}_{1x}(t)\}\\ &= \frac{1}{2}\mathrm{tr}(\hat{q}_{1z}(t)\hat{q}_{2z}(0))\hat{q}_{1x}(t)\end{aligned}\right\}.$$

Here, to get from the first to the second equality, I used (5.24), and to get from the second to the third I used the identity (5.21). A further fact follows from this simplification, namely that $\hat{q}_{1x}(t)$ commutes with $\hat{H}_1$ since

$$[\hat{q}_{1x}(t),\hat{H}_1(\hat{\boldsymbol{q}}_1(t),\hat{\boldsymbol{q}}_2(t))] = \frac{1}{2}\mathrm{tr}(\hat{q}_{1z}(t)\hat{q}_{2z}(0))[\hat{q}_{1x}(t),\hat{q}_{1x}(t)] = 0.$$

Thus, $\hat{q}_{1x}(t)$ is also a constant of the motion and the final expression one arrives at for $\hat{H}_1$ is that

$$\hat{H}_1(\hat{\boldsymbol{q}}_1(t),\hat{\boldsymbol{q}}_2(t)) = \frac{1}{2}\mathrm{tr}(\hat{q}_{1z}(t)\hat{q}_{2z}(0))\hat{q}_{1x}(0).$$

Through these simplifications, the unitaries become

$$\left.\begin{array}{l} U_1(t) = e^{-\frac{i}{2}\alpha(t)\hat{q}_{1x}(0)} \\ U_2(t) = \hat{1} \end{array}\right\},$$

where I introduced the variable

$$\alpha(t) \stackrel{\text{def}}{=} \int_0^t \operatorname{tr}\big(\hat{q}_{1z}(t')\hat{q}_{2z}(0)\big)dt'. \tag{5.25}$$

The unitary $U_1(t)$ represents a rotation about $\mathfrak{Q}_1$'s x-axis by $\alpha(t)$ radians, while the descriptors of $\mathfrak{Q}_2$ remain constant. So, using the initial matrix representation of those qubits, one obtains that the descriptors at a general time t are

$$\left.\begin{array}{l} \hat{\boldsymbol{q}}_1(t) = \big(\hat{\sigma}_x,\ \ \cos\big(\alpha(t)\big)\hat{\sigma}_y + \sin\big(\alpha(t)\big)\hat{\sigma}_z,\ \ \ \cos\big(\alpha(t)\big)\hat{\sigma}_z - \sin\big(\alpha(t)\big)\hat{\sigma}_y\big) \\ \hat{\boldsymbol{q}}_2(t) = \big(\hat{\sigma}_x,\ \ \hat{\sigma}_y,\ \ \hat{\sigma}_z\big) \end{array}\right\}. \tag{5.26}$$

Remarkably, by using the (5.26) and the definition (5.25), one can find an exact expression for $\alpha(t)$. This is due to the fact that the descriptors in (5.26) depend on $\alpha(t)$, but $\alpha(t)$ in turn depends on the expression for the descriptors, resulting in a self-consistency condition. I obtain that self-consistency condition by using (5.25) to deduce that

$$\frac{d\alpha(t)}{dt} = \operatorname{tr}\big(\hat{q}_{1z}(t)\hat{q}_{2z}(0)\big).$$

Then, by using (5.26), I find

$$\left.\begin{array}{l} \dfrac{d\alpha(t)}{dt} = \operatorname{tr}\big(\hat{q}_{1z}(t)\hat{q}_{2z}(0)\big) \\ \qquad\ = \operatorname{tr}\Big(\big(\hat{\sigma}_z\cos\big(\alpha(t)\big) - \hat{\sigma}_y\sin\big(\alpha(t)\big)\big)\hat{\sigma}_z\Big) \\ \qquad\ = \operatorname{tr}(\hat{\sigma}_z\hat{\sigma}_z)\cos\big(\alpha(t)\big) - \operatorname{tr}\big(\hat{\sigma}_z\hat{\sigma}_y\big)\sin\big(\alpha(t)\big) \\ \qquad\ = 2\cos\big(\alpha(t)\big) \end{array}\right\}.$$

The solution to this differential equation is that

$$\alpha(t) = 2\arctan(\tanh(c+t)),$$

where c is a real-valued constant. The value of that constant is $c = 0$ because of the boundary condition that $\alpha(0) = 0$, which follows from (5.25):

$$\alpha(0) \stackrel{\text{def}}{=} \int_0^0 \mathrm{tr}\big(\hat{q}_{1z}(t')\hat{q}_{2z}(0)\big)dt' = 0.$$

The Heisenberg state $|\Psi\rangle$ must the eigenstate of $\hat{\sigma}_z$ with eigenvalue 1 because, in my convention, the initial z-observables are both sharp with value 1. From this and (5.26), I find that

$$\left.\begin{array}{l}\langle\hat{\boldsymbol{q}}_1(t)\rangle = \big(0,\ \ \sin\big(\alpha(t)\big),\ \ \cos\big(\alpha(t)\big)\big)\\ \langle\hat{\boldsymbol{q}}_2(t)\rangle = (0,\ \ 0,\ \ 1)\end{array}\right\}.$$

Since any descriptor of the system can be expressed algebraically in terms of the components of $\hat{\boldsymbol{q}}_1(t)$ and $\hat{\boldsymbol{q}}_2(t)$, this concludes the analyses of the two qubits. Note, however, that it is not yet clear what the observables of the composite system are since products of descriptors of separate qubits are not necessarily Hermitian. Deriving the full set of observables would require a yet-to-be-completed theory of measurement for unorthodox qubits, based on the work of Deutsch & Marletto (2015) as described in §3.1. However, the foundations for that theory are described in §5.11.2.

5.9 Qubit field theory, Bekenstein's bound, and unorthodox qubits

In orthodox quantum theory, the descriptors of quantum fields obey the commutation constraint, and as discussed in §5.1, which causes various conceptual issues. Consider, for instance, that the commutation constraint immediately introduces infinities into that theory: a field has a descriptor $\hat{\phi}(x)$ and canonical momentum $\hat{\pi}(y)$ that satisfy the commutation relation

$$\big[\hat{\phi}(x), \hat{\pi}(y)\big] = i\delta(x-y),$$

where x and y are quadruples of coordinates on the same spacelike hypersurface (in a spacetime without closed timelike curves). Evidently, the commutator of $\hat{\phi}(x)$ and $\hat{\pi}(y)$ on that spacelike hypersurface is identically zero everywhere (as required by the commutation constraint) except when those descriptors are coincident, which is where the commutator blows up to infinity. The many subsequent infinities in quantum field theory seem to follow from these commutation relations.

Because of issues like this, such as those discussed in §5.1.2, Deutsch (2001) proposed an alternative quantum field theory in which a quantum field's descriptors violate the commutation constraint (that is to say, the field's descriptors do not necessarily mutually commute at spacelike separations). These fields are called *qubit fields* because their descriptors, at an individual spacetime event, are isomorphic to the descriptors of a qubit.

The theory of unorthodox qubits is ideally suited for studying qubit fields because the continuum limit of an array of unorthodox qubits is equivalent to a qubit field. Put differently, unorthodox qubits can be used as discretised models of qubit fields. The same is not true for orthodox qubits because those necessarily have commuting descriptors, which makes them insufficiently general for describing qubit fields.

The kinds of interactions and equations of motion of qubit fields, as well as the tasks these fields can perform, can therefore all be studied via unorthodox qubits. In particular, a theory of computation and measurement for unorthodox qubits would also be a theory of computation and measurement for qubit fields. Similarly, if unorthodox qubits can resolve the time-travel paradoxes that closed timelike curves give rise to (see §5.12), then so can qubit fields.

Deutsch (2001) conjectured that unorthodox systems with a finite Hilbert space can at most store a finite number of bits. If Deutsch's conjecture is true, it would follow that a qubit field must have bounded information storage capacity (because qubit fields have finite dimensional Hilbert spaces). In which case, qubit fields will have bounded entropy and satisfy Bekenstein's bound, resolving an important discrepancy between Bekenstein's bound and quantum field theory.

The theory of unorthodox qubits is currently not mature enough to settle these issues, but there are various hints which suggest that Deutsch's conjecture might be true. For example, as discussed in §5.6, the descriptors of an array of maximally non-commuting qubits have significantly fewer parameters than those of an orthodox array. So perhaps those qubits cannot store as much information as their orthodox counterparts can.

Before these issues can be settled, we need a theory of computation and measurement for unorthodox qubits. Such a theory is the only way to determine how much information can be stored in a qubit field. In the following sections I will provide the foundations of a theory of measurement and demonstrate that unorthodox qubits can instantiate classical information.

5.10 Computational networks of unorthodox qubits

So far I have studied arrays of unorthodox qubits, which are isolated systems with descriptors that evolve continuously. In order for the qubits to perform computational tasks, they will need to be manipulated by an outside influence – such as an experimenter – and hence the assumption that the qubits are isolated has to be dropped. To create a theory of computation for unorthodox qubits, I shall consider a *network* $\mathfrak{N}$ consisting of n interacting unorthodox qubits. In this network, the qubits perform gates that are implemented during periods of 1 unit of time, so the state of the qubits need only be assessed at integer times.

Between two integer times t and $t+1$, the qubits in the network are made to perform some general gate $\mathbf{G}$, which is implemented by a characteristic set of Hamiltonians $\{\hat{H}_{\mathrm{G},a}(\hat{\boldsymbol{q}}_1(t), \dots, \hat{\boldsymbol{q}}_n(t))\}$ that define the effect of $\mathbf{G}$ on $\mathfrak{N}$. At the final time $t+1$, the set of Hamiltonians changes instantaneously in the model (though in reality they are smoothly altered) so that $\mathfrak{N}$ performs a different gate during the subsequent time step. For the Hamiltonians of the qubits to change at $t+1$, there has to be something in the network's environment (e.g. an experimenter) that causes this change. However, because the way in which this change is brought about is unimportant for the dynamics of the qubits, the details of that environment can be abstracted away.

Between two integer times t and $t+1$, the network will evolve exactly *as if* it is an array (e.g. its descriptors will evolve continuously, and there is no outside influence acting on it during that time). Therefore, between t and $t+1$, the descriptors of the network are equivalent to their Taylor expansion on the interval $[t, t+1)$, implying the algebraic closure of the descriptors, denoted $\boldsymbol{\mathcal{O}}$, is a constant of the motion.

Since the network is equivalent to an array of unorthodox qubits on an interval $[t, t+1)$, one can use (5.12) and the characteristic set of Hamiltonians $\{\hat{H}_{\mathbf{G},a}(\hat{\boldsymbol{q}}_1(t), \dots, \hat{\boldsymbol{q}}_n(t))\}$ that define gate **G** to produce a set of unitaries for that gate on that interval. Those unitaries are defined as follows:

$$U_{\mathbf{G},a}(\hat{\boldsymbol{q}}_1(t), \dots, \hat{\boldsymbol{q}}_n(t)) = \mathcal{T} \exp\left(-i \int_t^{t+1} \hat{H}_a(t')\, dt'\right), \tag{5.27}$$

where the integral is from t to $t+1$, so the unitary $U_{\mathbf{G},a}$ evolves the descriptors of $\mathfrak{Q}_a$ forward by 1 unit of time. Doing this for each of the Hamiltonians in the set, one finds the set of unitaries $\{U_{\mathbf{G},a}(\hat{\boldsymbol{q}}_1(t), \dots, \hat{\boldsymbol{q}}_n(t))\}$ of gate **G**. The effect that gate **G** has on $\mathfrak{Q}_a$ between t and $t+1$, in terms of the unitaries, is as follows:

$$\hat{\boldsymbol{q}}_a(t+1) = U^\dagger_{\mathbf{G},a}(\hat{\boldsymbol{q}}_1(t), \dots, \hat{\boldsymbol{q}}_n(t))\hat{\boldsymbol{q}}_a(t)U_{\mathbf{G},a}(\hat{\boldsymbol{q}}_1(t), \dots, \hat{\boldsymbol{q}}_n(t)). \tag{5.28}$$

The unitaries studied here are not equivalent to the Hamiltonians because, whereas the Hamiltonians describe the qubit's evolution at any time in the interval $[t, t+1)$, the unitaries are already integrated over that interval, as is evident from (5.27). Consequently, there are certain unitaries which are not valid gates, whereas the Hamiltonians will invariably represent a valid gate.

Consider, for instance, the unorthodox qubits $\mathfrak{Q}_a$ and $\mathfrak{Q}_b$ with descriptors $\hat{\boldsymbol{q}}_a(t)$ and $\hat{\boldsymbol{q}}_b(t)$, which I shall assume to mutually commute at the initial time t. If the unitaries of those qubits could be chosen freely, one could enact the **swap** gate on $\mathfrak{Q}_a$ and simultaneously subject $\mathfrak{Q}_b$ to a unit wire, where these gates are enacted by the following two unitaries: $U_a(\hat{\boldsymbol{q}}_a(t), \hat{\boldsymbol{q}}_b(t)) = \frac{1}{4}\left(\hat{1} + \delta^{ij}\hat{q}_{ai}(t)\hat{q}_{bj}(t)\right)$, representing a **swap** gate, and $U_b(\hat{\boldsymbol{q}}_a(t), \hat{\boldsymbol{q}}_b(t)) = \hat{1}$, representing a unit wire. (Note that the products in $U_a(\hat{\boldsymbol{q}}_a(t), \hat{\boldsymbol{q}}_b(t))$ are well-defined because $\hat{\boldsymbol{q}}_a(t)$ and $\hat{\boldsymbol{q}}_b(t)$ commute at t.) The effect of these of unitaries would be to produce an exact copy of $\mathfrak{Q}_b$ since

$$\left.\begin{aligned}\hat{\boldsymbol{q}}_a(t+1) = \hat{\boldsymbol{q}}_b(t)\\ \hat{\boldsymbol{q}}_b(t+1) = \hat{\boldsymbol{q}}_b(t)\end{aligned}\right\}.$$

This would, however, imply that the qubits at time $t + 1$ are maximally non-commuting, whereas they initially had mutually commuting descriptors, so the dimension of must have shrunk the set of the system's q-number $\boldsymbol{O}$. This is impossible according to the results of §5.4, and therefore those unitaries do not represent a valid gate.

5.11 Classical computation

Having outlined how networks of unorthodox qubits are to be treated, we can now study the kinds of gates that those networks can enact. In particular, I shall address the following question: can the qubits be made to enact classical gates? The relevance of this question is: if measurement is to be possible on unorthodox qubits, it must be possible for such qubits to represent bit strings that can participate in classical computations (see §3.1). As I shall show below, unorthodox qubits can perform any classical gate and can therefore be used as measurers.

I shall first explain how an unorthodox qubit can represent a bit and how a classical gate is to be defined. Consider a qubit $\mathfrak{Q}_a$, which has attributes

$$\left.\begin{array}{l} \mathbf{0}_a = \{\hat{\boldsymbol{q}}_a : \langle \hat{q}_{az} \rangle = 1\} \\ \mathbf{1}_a = \{\hat{\boldsymbol{q}}_a : \langle \hat{q}_{az} \rangle = -1\} \end{array}\right\}. \tag{5.29}$$

For brevity's sake, the time labels have been dropped from the descriptors in (5.29). If the qubits in the network initially all have sharp z-observables, their descriptors are in a set such as (5.29), and then those z-observables represent a bit string. A classical gate is a gate which ensures that if all z-observables of the network are initially sharp, they are still sharp after the gate has had its effect. Hence, when the qubits' z-observables are sharp and a classical gate acts on those qubits, those z-observables can be treated *as if* they are bits in a classical computer.

5.11.1 The unorthodox not gate

The **not** gate is a classical gate whose characteristic effect on the network is that, when it acts on a qubit $\mathfrak{Q}_a$, causes the following transformation: $\left\{\mathbf{0}_a \xrightarrow{\mathbf{not}} \mathbf{1}_a,\ \mathbf{1}_a \xrightarrow{\mathbf{not}} \mathbf{0}_a\right\}$, while all other qubits in the array are left unaffected. The arrows show how the gate transforms input attributes into their corresponding output attributes.

The characteristic set of Hamiltonians that implements the **not** operation on $\mathfrak{Q}_a$, during the period starting at t and ending at $t+1$, is

$$\hat{H}_{\mathbf{not},a}\big(\hat{\boldsymbol{q}}_a(t)\big) = \frac{\pi}{2}\hat{q}_{ax}(t), \tag{5.30}$$

while all other Hamiltonians of $\mathfrak{N}$ are equal to zero, such that qubits other than $\mathfrak{Q}_a$ are not affected by this gate. When the Hamiltonian (5.30) is enacted for 1 unit of time, one finds that the descriptors of $\mathfrak{Q}_a$ change as follows:

$$\textbf{not}: \qquad \hat{\boldsymbol{q}}_a(t+1) = \Big(\hat{q}_{ax}(t),\ \ -\hat{q}_{ay}(t),\ \ -\hat{q}_{az}(t)\Big),$$

while all other qubits of $\mathfrak{N}$ remain unchanged. That the **not** gate toggles the z-observable of $\mathfrak{Q}_a$ is evident from the fact that $\hat{q}_{az}(t+1) = -\hat{q}_{az}(t)$, implying that the **not** gate performs the transformation $\left\{\mathbf{0}_a \xrightarrow{\mathbf{not}} \mathbf{1}_a,\ \mathbf{1}_a \xrightarrow{\mathbf{not}} \mathbf{0}_a\right\}$, as required.

5.11.2 Constructing an unorthodox cnot gate

For measurements to be possible, unorthodox qubits must be capable of enacting a **cnot**. This two-qubit gate acts on a control qubit $\mathfrak{Q}_a$ and a target qubit $\mathfrak{Q}_b$, whose descriptors are

$$\left.\begin{aligned} \hat{\boldsymbol{q}}_a(t) &= \Big(\hat{q}_{ax}(t),\ \hat{q}_{ay}(t),\ \hat{q}_{az}(t)\Big) \\ \hat{\boldsymbol{q}}_b(t) &= \Big(\hat{q}_{bx}(t),\ \hat{q}_{by}(t),\ \hat{q}_{bz}(t)\Big) \end{aligned}\right\}.$$

The characteristic effect of the **cnot** is that it performs a **not** gate on the target $\mathfrak{Q}_b$ if the control has attribute $\mathbf{1}_a$, whereas if the control has the attribute $\mathbf{0}_a$, the target is left unchanged. The **cnot** gate's effect can be summarised as follows:

$$\left.\begin{aligned} (\mathbf{1}_a, \mathbf{0}_b) &\xrightarrow{\mathbf{cnot}} (\mathbf{1}_a, \mathbf{1}_b), \quad (\mathbf{0}_a, \mathbf{1}_b) \xrightarrow{\mathbf{cnot}} (\mathbf{0}_a, \mathbf{1}_b) \\ (\mathbf{1}_a, \mathbf{1}_b) &\xrightarrow{\mathbf{cnot}} (\mathbf{1}_a, \mathbf{0}_b), \quad (\mathbf{0}_a, \mathbf{0}_b) \xrightarrow{\mathbf{cnot}} (\mathbf{0}_a, \mathbf{0}_b) \end{aligned}\right\}. \tag{5.31}$$

As we shall see below, unorthodox qubits can enact a **cnot** – i.e. there is a gate can that performs the task (5.31). However, the unorthodox **cnot** gate is notably different from its orthodox counterpart: for instance, the unorthodox **cnot** gate does not reduce to the orthodox one in cases where the qubits have commuting descriptors. This is not important for the current analysis: all that is required from a valid unorthodox **cnot** gate is that there be a set of Hamiltonians that comply with the principles established in §5.2 and that perform the transformation (5.31).

Because in unorthodox quantum theory, there are no predetermined commutation relations for separate qubits, it is crucial that the unorthodox **cnot** gate performs the task (5.31) regardless of the commutation relations of those qubits. It is for this reason that one cannot use the expression for the orthodox **cnot** as depicted in (3.2): that gate is defined only when qubits have mutually commuting descriptors. When they do not commute, the products in (3.2) are not Hermitian.

One can make the products in (3.2) Hermitian by, say, replacing such products with anti-commutators, but that would make the gate lose its functionality. For instance, when qubits maximally non-commute, the anti-commutator of their descriptors is proportional to the unit observable, which would make the gate act trivially.

To solve this issue, one needs a *reference qubit* $\mathfrak{Q}_c$ with descriptors

$$\hat{\boldsymbol{q}}_c(t) = \left(\hat{q}_{cx}(t),\ \hat{q}_{cy}(t),\ \hat{q}_{cz}(t)\right).$$

A defining feature of the reference qubit is that it maximally non-commutes with the control qubit and that its initial z-observable is sharp with value 1. Because it has this property, the reference qubits can be used to construct an operator similar to a projector, namely

$$P\left(\hat{q}_{az}(t), \hat{q}_{cz}(t)\right) \stackrel{\text{def}}{=} \frac{1}{4}\left(2 - \{\hat{q}_{az}(t), \hat{q}_{cz}(t)\}\right). \tag{5.32}$$

$P\left(\hat{q}_{az}(t), \hat{q}_{cz}(t)\right)$ is a q-number function and has two crucial properties:

(i) It commutes with all other q-numbers because it is a multiple of the unit observable.

(ii) When the control has attribute $\mathbf{0}_a$, P is equal to zero, whereas if the control has attribute $\mathbf{1}_a$, then P is equal to $\hat{1}$.

P has property (i) because the reference qubit must maximally non-commute with the target, so due to (5.21), expression (5.32) reduces to

$$P\big(\hat{q}_{az}(t), \hat{q}_{cz}(t)\big) = \frac{1}{4}\left(2 - \{\hat{q}_{az}(t), \hat{q}_{cz}(t)\}\right) = \frac{1}{4}\Big(2 - \mathrm{tr}\big(\hat{q}_{az}(t)\hat{q}_{cz}(t)\big)\Big)\,\hat{1},$$

which clearly commutes with all other descriptors of the network.

That P has property (ii) can be proven as follows. $\hat{q}_{cz}(t)$ is assumed to be sharp with value 1. Therefore, since that qubit maximally non-commutes with the target, $\hat{q}_{az}(t)$ can only be sharp with that same value if $\hat{q}_{az}(t) = \hat{q}_{cz}(t)$, in which case $\mathrm{tr}\big(\hat{q}_{az}(t)\hat{q}_{cz}(t)\big) = 2$, and then

$$P\big(\hat{q}_{az}(t), \hat{q}_{cz}(t)\big) = \frac{1}{4}\Big(2 - \mathrm{tr}\big(\hat{q}_{az}(t)\hat{q}_{cz}(t)\big)\Big)\,\hat{1} = 0.$$

Similarly, if $\hat{q}_{az}(t)$ is sharp with value -1, then $\hat{q}_{az}(t) = -\hat{q}_{cz}(t)$, implying that $\mathrm{tr}\big(\hat{q}_{az}(t)\hat{q}_{cz}(t)\big) = -2$ and

$$P\big(\hat{q}_{az}(t), \hat{q}_{cz}(t)\big) = \frac{1}{4}\Big(2 - \mathrm{tr}\big(\hat{q}_{az}(t)\hat{q}_{cz}(t)\big)\Big)\,\hat{1} = \hat{1}.$$

Because of these constraints, $P\big(\hat{q}_{az}(t), \hat{q}_{cz}(t)\big)$ is equal to 0 if the control has attribute $\mathbf{0}_a$ and is equal to $\hat{1}$ if the control has attribute $\mathbf{1}_a$.

The q-number function $P\big(\hat{q}_{az}(t), \hat{q}_{cz}(t)\big)$ is similar to a projector in that it assumes the values 0 and 1 for the eigenvalues 1 and -1 of $\hat{q}_{az}(t)$. Using this operator, one can construct an unorthodox **cnot** gate as follows: between time t and $t+1$, let the Hamiltonian of the target qubit $\mathfrak{Q}_b$ be

$$\hat{H}_{\mathbf{cnot},b}\big(\hat{\boldsymbol{q}}_a(t), \hat{\boldsymbol{q}}_b(t), \hat{\boldsymbol{q}}_c(t)\big) \stackrel{\text{def}}{=} \frac{\pi}{2}\hat{q}_{bx}(t) P\big(\hat{q}_{az}(t), \hat{q}_{cz}(t)\big), \tag{5.33}$$

while the Hamiltonians of the control and reference qubits are equal to zero. The latter requirement guarantees that $\mathfrak{Q}_a$ and $\mathfrak{Q}_c$ are stationary between t and $t+1$. The product in

(5.33) is well-defined irrespective of the commutation relations of $\mathfrak{Q}_a$ and $\mathfrak{Q}_b$ because of property (i). Due to property (ii), the Hamiltonian $\widehat{H}_{\mathbf{cnot},b}$ becomes that of a **not** gate if the control has attribute $\mathbf{1}_a$, whereas that Hamiltonian vanishes identically (implying it enacts a unit wire) when the control has the attribute $\mathbf{0}_a$. The control qubit is stationary in either case. Hence, this gate performs task (5.31) and is therefore a **cnot** gate for unorthodox qubits.

It appears that the **cnot** gate defined in (5.33) is a three-qubit gate rather than a two-qubit gate because it acts on $\mathfrak{Q}_a$, $\mathfrak{Q}_b$ and $\mathfrak{Q}_c$. However, $\mathfrak{Q}_c$ is similar to an ancilla qubit: it is left entirely unaffected by the gate, and its only role is enabling the **cnot** while being itself unaffected. It can be considered a resource for implementing the **cnot** (in fact, because $\mathfrak{Q}_c$ is left unchanged and enables the **cnot**, it is a *constructor*; see Deutsch & Marletto (2015)).

The unorthodox **cnot** defined in (5.33) provides the foundations for a theory of measurement. For instance, if the target qubit is prepared with the attribute $\mathbf{0}_b$, then attributes $\mathbf{0}_a$ and $\mathbf{1}_a$ of the control can be copied onto the target through the use of this **cnot**, as is evident from (5.31). Hence $\mathbf{0}_a$ and $\mathbf{1}_a$ are measurable attributes of unorthodox qubits. It is not yet clear what all observable attributes of the system are; this question will be left for future research.

5.11.3 Constructing an unorthodox ccnot gate

The **cnot** gate can copy attributes such as $\mathbf{0}_a$ and $\mathbf{1}_a$, but an ideal measurer also requires that measurement results (stored as bit strings) can be used in classical computation. And for this the **cnot** gate is not enough. That is to say that a general classical gate cannot be written in terms of only **cnot** gates. A gate which is universal for classical computations is the **ccnot** gate, a three-qubit gate whose effect on its two control qubits, $\mathfrak{Q}_a$ and $\mathfrak{Q}_b$, and its target $\mathfrak{Q}_c$ is that

$$\left.\begin{array}{ll} (\mathbf{0}_a,\mathbf{0}_b,\mathbf{0}_c) \xrightarrow{\mathbf{ccnot}} (\mathbf{0}_a,\mathbf{0}_b,\mathbf{0}_c), & (\mathbf{0}_a,\mathbf{0}_b,\mathbf{1}_c) \xrightarrow{\mathbf{ccnot}} (\mathbf{0}_a,\mathbf{0}_b,\mathbf{1}_c) \\ (\mathbf{1}_a,\mathbf{0}_b,\mathbf{0}_c) \xrightarrow{\mathbf{ccnot}} (\mathbf{1}_a,\mathbf{0}_b,\mathbf{0}_c), & (\mathbf{1}_a,\mathbf{0}_b,\mathbf{1}_c) \xrightarrow{\mathbf{ccnot}} (\mathbf{1}_a,\mathbf{0}_b,\mathbf{1}_c) \\ (\mathbf{0}_a,\mathbf{1}_b,\mathbf{0}_c) \xrightarrow{\mathbf{ccnot}} (\mathbf{0}_a,\mathbf{1}_b,\mathbf{0}_c), & (\mathbf{0}_a,\mathbf{1}_b,\mathbf{1}_c) \xrightarrow{\mathbf{ccnot}} (\mathbf{0}_a,\mathbf{1}_b,\mathbf{1}_c) \\ (\mathbf{1}_a,\mathbf{1}_b,\mathbf{0}_c) \xrightarrow{\mathbf{ccnot}} (\mathbf{1}_a,\mathbf{1}_b,\mathbf{1}_c), & (\mathbf{1}_a,\mathbf{1}_b,\mathbf{1}_c) \xrightarrow{\mathbf{ccnot}} (\mathbf{1}_a,\mathbf{1}_b,\mathbf{0}_c) \end{array}\right\}. \tag{5.34}$$

The expressions for the Hamiltonians for the **ccnot** gate are similar to those of the **cnot** gate. Like for the **cnot**, one requires that the control qubits $\mathfrak{Q}_a$ and $\mathfrak{Q}_b$ have reference-qubits, denoted $\mathfrak{Q}_d$ and $\mathfrak{Q}_e$ respectively, where each control qubit maximally non-commutes with its reference qubit. I shall assume that the z-observables of the reference qubits are sharp at the initial time t so that the **ccnot** is implemented between t and $t+1$ if $\mathfrak{Q}_c$'s Hamiltonian is

$$\begin{aligned} &\hat{H}_{\mathbf{ccnot},c}\big(\hat{\boldsymbol{q}}_a(t), \hat{\boldsymbol{q}}_b(t), \hat{\boldsymbol{q}}_c(t), \hat{\boldsymbol{q}}_d(t), \hat{\boldsymbol{q}}_e(t)\big) \\ &\qquad\qquad \stackrel{\text{def}}{=} \frac{\pi}{2}\hat{q}_{cx}(t)P\big(\hat{q}_{az}(t), \hat{q}_{dz}(t)\big)P\big(\hat{q}_{bz}(t), \hat{q}_{ez}(t)\big), \end{aligned} \tag{5.35}$$

while the Hamiltonian of the qubits $\mathfrak{Q}_a$, $\mathfrak{Q}_b$, $\mathfrak{Q}_d$, and $\mathfrak{Q}_e$ are each equal to zero.

The term $P\big(\hat{q}_{az}(t), \hat{q}_{dz}(t)\big)P\big(\hat{q}_{bz}(t), \hat{q}_{ez}(t)\big)$ in (5.35) is zero when the controls have any of the attributes $(\mathbf{0}_a, \mathbf{0}_b), (\mathbf{0}_a, \mathbf{1}_b)$, or $(\mathbf{1}_a, \mathbf{0}_b)$. However, when the control qubits have the value $(\mathbf{1}_a, \mathbf{1}_b)$, then expression (5.35) reduces to the Hamiltonian of a **not** gate, confirming that this is an unorthodox **ccnot** gate as defined in (5.31).

Since unorthodox qubits permit a **ccnot** gate to be enacted, those qubits can perform any classical reversible computation (i.e. permutations on the set of all bit strings). Consequently, unorthodox qubits permit the existence of perfect measurers: an unorthodox qubit can be an ideal measurer for another qubit, where their measurement interaction is modelled by a **cnot** gate. And because of the existence of the unorthodox **ccnot** gate, measurement results that can be stored as bit strings of some appropriate length and can also be used in any classical computations, as required (see §3.1).

Here I have proven that a network $\mathfrak{N}$ can enact any classical gate, but unorthodox qubits can perform more than just classical gates – e.g. they can enact any single-qubit gates, such as the **Hadamard** gate, which has no classical counterpart. It is not yet known what the full computational repertoire of a network of unorthodox qubits is; I shall leave this question to be addressed in future research.

5.12 Closed timelike curves revisited

When a qubit traverses a closed timelike curve, a second instance of the qubit will be produced at an earlier time and separate location so that there will be a pair of qubits (i.e. the original qubit and its older instance). That pair of qubits will have to be modelled as unorthodox qubits, as explained in §5.1.1.

Classically, problems arise when an observer (or a preprogramed machine) travels on closed timelike curves. A particularly well-known problem is the grandfather paradox, in which a physical system travels back in time only if it has observed that its future instance did not travel back in time. This paradox can be formulated in terms of a quantum computational network consisting of classical gates (Deutsch 1991, and Lloyd 2011). Because of the theory of classical computation established in the previous section, we can now analyse the grandfather paradox using unorthodox quantum theory.

5.12.1 The grandfather paradox in classical physics

When an observer (or a measurer) traverses a closed timelike curve, they can cause an inconsistent history known as the grandfather paradox. This paradoxical history is one in which an observer – named Alice – travels backwards in time on a closed timelike curve if she does not see herself emerge from it. If she does see herself emerge from the closed timelike curve, then she *will* traverse it. The inconsistency is thus that she only traverses the closed timelike curve if she does not do so.

The paradox can be embodied by a computational network consisting of three classical bits, following Deutsch (1991). The first two bits in this network are denoted $(x, -x)$, where $x \in \{-1,1\}$. These bits represent two possible paths that Alice can take. The first path with value x does not traverse the closed timelike curve, whereas the second path $-x$ does. If the bit has value -1, it means that Alice is occupying that path, and the value 1 means that she is not occupying it. Hence, if $x = 1$, then she enters the closed timelike curve, and if $x = -1$, she does not enter it.

If Alice enters the curve, the second bit $-x$ will become an older bit y. Alice enters the curve only if she is not occupying the path represented by y (i.e. only if $y = 1$ does she enter the curve). This is modelled by letting those bits interact via two classical **cnot** gates with the effect that

$$(x, -x,\ y) \rightarrow (xy, -xy,\ y).$$

Here $(x, -x,\ y)$ represents the states of the bits at time $t = 0$, and $(xy, -xy,\ y)$ are their states at a later time $t = 1$. Kinematic consistency now requires that the second bit at $t = 1$, when that bit is about the enter the closed timelike curve, becomes equal to the third bit at $t = 0$, i.e.

$$-xy = y.$$

So only when $x = -1$ does this self-consistency condition have a solution. Otherwise, if $x = 1$, one obtains that $-y = y$, in which case there is no value for $y \in \{-1,1\}$ for which it satisfies that equation, resulting in a paradox. So only when Alice avoids entering the closed timelike curve will she have a consistent history, at least according to classical physics.

However, classical physics is false, and a more accurate description of the physical situation described above requires that the bits instead be treated as qubits. This changes the analysis because the descriptors of qubits can be in superpositions of their eigenvalues, whereas the classical bits cannot. So, consider a qubit $\mathfrak{Q}_1$, with descriptors $\hat{\boldsymbol{q}}_1(t)$ that conform to the Pauli algebra; the sharp values of its z-observable correspond to that of the first classical bit, which at time $t = 0$ has value x. A second qubit $\mathfrak{Q}_2$, with descriptors $\hat{\boldsymbol{q}}_2(t)$, has a z-observable whose sharp values correspond to that of the second bit with the value $-x$.

When $\hat{q}_{2z}(t)$ is sharp with value -1, Alice occupies the path that traverses the closed timelike curve. Because the classical bits have the values $(x, -x)$, the qubits' z-observables must satisfy the constraint that

$$\langle \hat{q}_{1z}(0) \rangle = -\langle \hat{q}_{2z}(0) \rangle. \tag{5.36}$$

This implies that, counter to convention, not all z-observables are sharp with value 1 at $t = 0$.

This second qubit $\mathfrak{Q}_2$ travels back in time and produces a third qubit $\mathfrak{Q}_3$, whose z-observable corresponds to the classical bit y. Let $\mathfrak{Q}_1$, $\mathfrak{Q}_2$ and $\mathfrak{Q}_3$ interact for 2 units of time, during which they enact two **cnot** gates, of which $\mathfrak{Q}_1$ and $\mathfrak{Q}_2$ are the targets and $\mathfrak{Q}_3$ the control. In the

Heisenberg picture of orthodox quantum theory, the effect of these gates, which act between $t = 0$ and $t = 2$, is that

$$\left.\begin{array}{l} \begin{Bmatrix} \hat{\boldsymbol{q}}_1(0) \\ \hat{\boldsymbol{q}}_2(0) \\ \hat{\boldsymbol{q}}_3(0) \end{Bmatrix} = \begin{Bmatrix} (\hat{q}_{1x},\ \hat{q}_{1y},\ \hat{q}_{1z}) \\ (\hat{q}_{2x},\ \hat{q}_{2y},\ \hat{q}_{2z}) \\ (\hat{q}_{3x},\ \hat{q}_{3y},\ \hat{q}_{3z}) \end{Bmatrix} \\ \xrightarrow{\mathbf{cnot}(2,3)} \begin{Bmatrix} \hat{\boldsymbol{q}}_1(1) \\ \hat{\boldsymbol{q}}_2(1) \\ \hat{\boldsymbol{q}}_3(1) \end{Bmatrix} = \begin{Bmatrix} (\hat{q}_{1x},\ \hat{q}_{1y},\ \hat{q}_{1z}) \\ (\hat{q}_{2x},\ \hat{q}_{2y}\hat{q}_{3z},\ \hat{q}_{2z}\hat{q}_{3z}) \\ (\hat{q}_{3x}\hat{q}_{2x},\ \hat{q}_{3y}\hat{q}_{2x},\ \hat{q}_{3z}) \end{Bmatrix} \\ \xrightarrow{\mathbf{cnot}(1,3)} \begin{Bmatrix} \hat{\boldsymbol{q}}_1(2) \\ \hat{\boldsymbol{q}}_2(2) \\ \hat{\boldsymbol{q}}_3(2) \end{Bmatrix} = \begin{Bmatrix} (\hat{q}_{1x},\ \hat{q}_{1y}\hat{q}_{3z},\ \hat{q}_{1z}\hat{q}_{3z}) \\ (\hat{q}_{2x},\ \hat{q}_{2y}\hat{q}_{3z},\ \hat{q}_{2z}\hat{q}_{3z}) \\ (\hat{q}_{3x}\hat{q}_{2x}\hat{q}_{1x},\ \hat{q}_{3y}\hat{q}_{2x}\hat{q}_{1x},\ \hat{q}_{3z}) \end{Bmatrix} \end{array}\right\}. \tag{5.37}$$

I have suppressed the $t = 0$ labels for the sake of brevity, and I have added to the **cnot** gates labels that represent their target and control; for instance, **cnot**(2,3) means that the $\mathfrak{Q}_2$ is the target qubit and $\mathfrak{Q}_3$ is the control.

After those gates have been enacted, at time $t = 2$, qubit $\mathfrak{Q}_2$ traverses the closed timelike curve so that it experiences a negative time-delay of -2 units of time and becomes $\mathfrak{Q}_3$. To ensure kinematic consistency on the closed timelike curve, the descriptors of $\mathfrak{Q}_3$ at $t = 0$ must be

$$\hat{\boldsymbol{q}}_3(0) = \hat{\boldsymbol{q}}_2(2). \tag{5.38}$$

To satisfy the kinematic consistency, the qubits must violate the commutation constraint. For instance, using (5.38) and (5.37) and the Pauli algebra one finds that

$$\left.\begin{array}{l} \hat{q}_{3z}(0) = \hat{q}_{2z}(2) \\ \longrightarrow \hat{q}_{3z} = -\hat{q}_{3z}\hat{q}_{2z} \\ \longrightarrow \hat{q}_{2z} = -\hat{1} \end{array}\right\}.$$

If $\hat{q}_{2z} = -\hat{1}$, then this qubit does not adhere to the Pauli algebra since the descriptor $\hat{q}_{2z}$ then no longer has eigenvalues 1 and -1. We must therefore conclude that the Heisenberg picture of orthodox quantum theory does not allow of a resolution to the grandfather paradox.

However, the paradox can be resolved if the commutation constraint is dropped and $\mathfrak{Q}_1$, $\mathfrak{Q}_2$ and $\mathfrak{Q}_3$ are instead treated as unorthodox qubits.[38] I shall here treat the case in which the two qubits $\mathfrak{Q}_1$ and $\mathfrak{Q}_2$ have commute descriptors at $t = 0$, while $\mathfrak{Q}_3$ is assumed to maximally non-commute with $\mathfrak{Q}_2$ (this can be considered an ansatz about their commutation relations). To that end, let the qubits have the following initial matrix representation

$$\left.\begin{aligned} \hat{\boldsymbol{q}}_1(0) &= (\hat{\sigma}_x,\ \hat{\sigma}_y,\ \hat{\sigma}_z) \otimes \hat{1}_1 \\ \hat{\boldsymbol{q}}_2(0) &= \hat{1}_1 \otimes (\hat{\sigma}_x,\ -\hat{\sigma}_y,\ -\hat{\sigma}_z) \end{aligned}\right\}. \tag{5.39}$$

Let us also assume that the Heisenberg state is such that the operators $\hat{\sigma}_z \otimes \hat{1}_1$ and $\hat{1}_1 \otimes \hat{\sigma}_z$ are both sharp with value 1 so that

$$\left.\begin{aligned} \langle \hat{q}_{1z}(0) \rangle &= 1 \\ \langle \hat{q}_{2z}(0) \rangle &= -1 \end{aligned}\right\}.$$

This situation therefore corresponds to the case in which Alice occupies the trajectory that traverses the closed timelike curve – which would result in the impossible history in the classical version of the paradox. (5.39) specifies the matrix representation of $\mathfrak{Q}_1$ and $\mathfrak{Q}_2$, whereas that of $\mathfrak{Q}_3$ will be determined by the expression for $\hat{\boldsymbol{q}}_2(2)$, like in (5.38). The aim, now, is to show that the system's dynamics do not result in a paradoxical history.

$\mathfrak{Q}_1$, $\mathfrak{Q}_2$ and $\mathfrak{Q}_3$ will enact the same **cnot** gates as described in (5.37) except that the second **cnot**, which acts on the maximally non-commuting qubits $\mathfrak{Q}_2$ and $\mathfrak{Q}_3$, must be an unorthodox **cnot** gate and requires a reference qubit, denoted $\mathfrak{Q}_4$. As has been established in §5.11.2, the properties of the reference qubit $\mathfrak{Q}_4$ should be that it maximally non-commutes with the control $\mathfrak{Q}_3$, and that its z-observable should be sharp with value 1. $\mathfrak{Q}_4$ satisfies both of these constraints if its initial matrix representation is

[38] Marletto et al. (2019) have created a construction in which qubits on a closed timelike curve are represented by a so-called pseudo-density operator (see for instance Fitzsimons et al. (2016)). The pseudo-density operator is not only able to describe spatial correlations but also correlations in time. The relation between my own construction and that of Marletto et al. (2019) is currently unclear, but I conjecture that they are compatible descriptions of the same physical situation.

$$\hat{\boldsymbol{q}}_4(0) = \hat{1}_1 \otimes \left(\hat{\sigma}_x,\ \hat{\sigma}_y,\ \hat{\sigma}_z\right).$$

An unorthodox **cnot** gate with target $\mathfrak{Q}_2$ and control $\mathfrak{Q}_3$ is implemented between $t = 0$ and $t = 1$ if the Hamiltonian of $\mathfrak{Q}_2$ is

$$\hat{H}_{\mathbf{cnot},2}\left(\hat{\boldsymbol{q}}_2(t), \hat{\boldsymbol{q}}_3(t), \hat{\boldsymbol{q}}_4(t)\right) = \frac{\pi}{2}\hat{q}_{2x}(t)\vec{P}\left(\hat{q}_{3z}(t), \hat{q}_{4z}(t)\right), \tag{5.40}$$

while the Hamiltonians for $\mathfrak{Q}_1$, $\mathfrak{Q}_3$ and $\mathfrak{Q}_4$ are equal to zero during that period. Since the descriptors $\hat{q}_{2x}(t)$ and $\hat{q}_{3z}(t)$ do not evolve between $t = 0$ and $t = 1$, the expression (5.40) simplifies to

$$\hat{H}_{\mathbf{cnot},2}\left(\hat{\boldsymbol{q}}_2(t), \hat{\boldsymbol{q}}_3(t), \hat{\boldsymbol{q}}_4(t)\right) = \frac{\pi}{8}\hat{q}_{2x}(t)\left(2 - \mathrm{Tr}\left(\hat{q}_{3z}(0)\hat{q}_{4z}(0)\right)\right).$$

From this expression, it is clear that $\hat{H}_{\mathbf{cnot},2}$ commutes with $\hat{q}_{1x}(t)$, so that descriptor must be stationary too, resulting in a final simplification:

$$\hat{H}_{\mathbf{cnot},2}\left(\hat{\boldsymbol{q}}_2(t), \hat{\boldsymbol{q}}_3(t), \hat{\boldsymbol{q}}_4(t)\right) = \frac{\pi}{8}\hat{q}_{2x}(0)\left(2 - \mathrm{Tr}\left(\hat{q}_{3z}(0)\hat{q}_{4z}(0)\right)\right).$$

Therefore, the **cnot** rotates $\mathfrak{Q}_2$ about its x-axis by an angle

$$\phi \stackrel{\text{def}}{=} \frac{\pi}{4}\left(2 - \mathrm{Tr}\left(\hat{q}_{3z}(0)\hat{q}_{4z}(0)\right)\right),$$

so at $t = 1$, after the application of the unorthodox **cnot**, $\mathfrak{Q}_2$'s descriptors are

$$\hat{\boldsymbol{q}}_2(2) = \hat{1}_1 \otimes \left(\hat{\sigma}_x,\ -\hat{\sigma}_y\cos(\phi) - \hat{\sigma}_z\sin(\phi),\ -\hat{\sigma}_z\cos(\phi) + \hat{\sigma}_y\sin(\phi)\right). \tag{5.41}$$

To ensure kinematic consistency, the expression (5.41) must be equal to the descriptors of $\mathfrak{Q}_3$ at time $t = 0$. Using that expression and the definition of ϕ, one obtains that ϕ must satisfy the following self-consistency condition:

$$\phi = \frac{\pi}{2}(1 + \cos(\phi)).$$

The unique solution to this self-consistency is that $\phi = \frac{\pi}{2}$. For this value of ϕ, the evolution of $\mathfrak{Q}_1$, $\mathfrak{Q}_2$ and $\mathfrak{Q}_3$ between $t = 0$ and $t = 2$ is:

$$
\left.\begin{array}{rl}
& \begin{Bmatrix} \hat{\boldsymbol{q}}_1(0) \\ \hat{\boldsymbol{q}}_2(0) \\ \hat{\boldsymbol{q}}_3(0) \end{Bmatrix} = \begin{Bmatrix} \left(\hat{\sigma}_x, \otimes \hat{1}_1, \quad \hat{\sigma}_y \otimes \hat{1}_1, \quad \hat{\sigma}_z \otimes \hat{1}_1\right) \\ \left(\hat{1}_1 \otimes \hat{\sigma}_x, \ -\hat{1}_1 \otimes \hat{\sigma}_y, \ -\hat{1}_1 \otimes \hat{\sigma}_z\right) \\ \left(\hat{1}_1 \otimes \hat{\sigma}_x, \quad -\hat{1}_1 \otimes \hat{\sigma}_z, \ \hat{1}_1 \otimes \hat{\sigma}_y\right) \end{Bmatrix} \\
\xrightarrow{\text{(unorthodox) } \mathbf{cnot}(2,3)} & \begin{Bmatrix} \hat{\boldsymbol{q}}_1(1) \\ \hat{\boldsymbol{q}}_2(1) \\ \hat{\boldsymbol{q}}_3(1) \end{Bmatrix} = \begin{Bmatrix} \left(\hat{\sigma}_x \otimes \hat{1}_1, \ \hat{\sigma}_y \otimes \hat{1}_1, \ \hat{\sigma}_z \otimes \hat{1}_1\right) \\ \left(\hat{1}_1 \otimes \hat{\sigma}_x, -\hat{1}_1 \otimes \hat{\sigma}_z, \ \hat{1}_1 \otimes \hat{\sigma}_y\right) \\ \left(\hat{1}_1 \otimes \hat{\sigma}_x, -\hat{1}_1 \otimes \hat{\sigma}_z, \ \hat{1}_1 \otimes \hat{\sigma}_y\right) \end{Bmatrix} \\
\xrightarrow{\text{(orthodox) } \mathbf{cnot}(1,3)} & \begin{Bmatrix} \hat{\boldsymbol{q}}_1(2) \\ \hat{\boldsymbol{q}}_2(2) \\ \hat{\boldsymbol{q}}_3(2) \end{Bmatrix} = \begin{Bmatrix} \left(\hat{\sigma}_x \otimes \hat{1}_1, \ \hat{\sigma}_y \otimes \hat{\sigma}_y, \ \hat{\sigma}_z \otimes \hat{\sigma}_y\right) \\ \left(\hat{1}_1 \otimes \hat{\sigma}_x, -\hat{1}_1 \otimes \hat{\sigma}_z, \ \hat{1}_1 \otimes \hat{\sigma}_y\right) \\ \left(\hat{\sigma}_x \otimes \hat{\sigma}_x, -\hat{\sigma}_x \otimes \hat{\sigma}_z, \ \hat{1}_1 \otimes \hat{\sigma}_y\right) \end{Bmatrix}
\end{array}\right\}.
$$

During the entire procedure, which starts at $t = 0$ and ends at $t = 2$, the reference qubit's descriptors are constants of the motion, and are equal to $\hat{1}_1 \otimes \left(\hat{\sigma}_x, \ \hat{\sigma}_y, \ \hat{\sigma}_z\right)$. Note that an orthodox **cnot** is performed on $\mathfrak{Q}_2$ and $\mathfrak{Q}_3$ because those qubits have mutually commuting descriptors. These expressions for the descriptors completely specify the states of the qubits between $t = 0$ and $t = 2$, and manifestly there are no inconsistencies in the qubits' history, implying that the paradox has been resolved.

What does Alice experience according to this model of the grandfather paradox? Initially, at time $t = 0$, she intends to traverse the closed timelike curve (as indicated by the fact that $\langle \hat{q}_{2z}(0) \rangle = -1$). Before entering it, she performs a measurement to detect whether 'future Alice' has traversed the closed timelike curve; if 'future Alice' emerges, then Alice will decide against entering the closed timelike curve, and if 'future Alice' does not emerge, she enters it. The 'future Alice' is represented by $\mathfrak{Q}_3$, whose z-observable is non-sharp (its expectation value is $\langle \hat{q}_{3z}(0) \rangle = 0$), implying that 'future Alice' is in a superposition of having entered and not having entered the closed timelike curve. Therefore, during the measurements performed between $t = 0$ and $t = 2$, Alice evolves into a state in which her trajectory is no longer sharp either. Those two versions of Alice are entangled with one another since individually $\hat{q}_{1z}(2)$ and $\hat{q}_{3z}(2)$ are non-sharp while their product is sharp:

$$\langle \hat{q}_{1z}(2) \hat{q}_{3z}(2) \rangle = \left\langle \left(\hat{\sigma}_z \otimes \hat{\sigma}_y\right)\left(\hat{1}_1 \otimes \hat{\sigma}_y\right) \right\rangle = \left\langle \hat{\sigma}_z \otimes \hat{1}_1 \right\rangle = 1.$$

Therefore, in the unambiguous future, there will be two universes. In one, Alice decided against entering the closed timelike curve because she saw herself emerge from it. In the other universe, Alice never saw herself emerge from the closed timelike curve, which is why she entered it. Translated to the Schrödinger picture, Alice's state vector at $t = 2$ is

$$|1\rangle_1 |1\rangle_3 + |0\rangle_1 |0\rangle_3,$$

where the subscripts indicate to which qubit a vector belongs. I have not included the state of $\mathfrak{Q}_2$ because it does not exist in the unambiguous future, and if it had to be included, we could not use the Schrödinger picture because $\mathfrak{Q}_2$ maximally non-commutes with $\mathfrak{Q}_3$. Remarkably, the universe represented by $|0\rangle_1 |0\rangle_3$ is devoid of Alice because she has left that universe, whereas in the universe represented by $|1\rangle_1 |1\rangle_3$, there are two versions of her.[39]

[39] A notable difference between my construction and the one by Deutsch (1991) is that: in my construction, Alice is in a pure state in the unambiguous future, whereas in Deutsch's construction, she is in a mixed state.

6 Avenues for future research

Throughout this thesis, I have assumed that the Heisenberg picture is the most fundamental description of quantum theory and presented various constructions – such as Everett's relative-state construction and the Page–Wootters model – in that picture. The refinement of those constructions is far from complete and raises new and perhaps more interesting problems.

An open problem in the Page–Wootters construction, which I discussed in §4.13, may be phrased as follows: just as the c-number time was replaced by a q-number time, could the c-number coordinates of spacetime be replaced by q-number coordinates? There are various hints that this might indeed be possible. For instance, spacetime appears locally to be flat, meaning that curvature may be present but can only be detected by performing measurements on different regions of spacetime. This is remarkably similar to locally inaccessible information in quantum theory, which is present in a system but cannot be retrieved via that system alone. Refining the Page–Wootters formalism by addressing these issues could resolve important discrepancies between general relativity and quantum theory.

Furthermore, in the theory of unorthodox qubits as I presented it here, there are two primary open problems. Though I have shown that unorthodox qubits can model simple measurements and that the expectation values of such qubits' descriptors are, indeed, measurable quantities, it has yet to be shown what their *complete* set of observables is. For that, one may well have to refine the simple model of measurement that I presented; for this, the constructor theory of information (Deutsch & Marletto (2015)) should be used. Secondly, I have shown that unorthodox qubits allow for a resolution of the grandfather paradox, but unlike Deutsch (1991), who demonstrated that such solutions always exist in his Schrödinger-picture construction, it has not yet been proven that unorthodox qubits invariably have a kinematically consistent history. These issues may prove to be fruitful new research programs.

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