ISSN 1471-0498



DEPARTMENT OF ECONOMICS

DISCUSSION PAPER SERIES

TESTING FOR PRODUCTION WITH COMPLEMENTARITIES

Pawel Dziewulski and John K.-H. Quah

Number 722
September 2014

Manor Road Building, Manor Road, Oxford OX1 3UQ

Testing for production with complementarities

Paweł Dziewulski John K.-H. Quah*

September 2014

Abstract

Suppose we observe a finite number of input decisions made by a firm, as well as the prices at which those inputs were acquired. What conditions on the set of observations are necessary and sufficient for it to be consistent with a firm choosing inputs to maximize profit, subject to a production function exhibiting production complementarities? In this paper, we develop an axiomatic characterisation of this hypothesis and also develop a test that can be easily applied to finite data sets.

Keywords: profit maximisation, production complementarities, supermodular production, modern manufacturing, cyclical monotonicity, quasilinear preferences
JEL Classification: D21, D24

1 Introduction

The importance of complementarities for modern manufacturing was highlighted by Milgrom and Roberts (1990, 1994, 1995), who observed a fundamental shift from mass production, which benefits from economies of scale, to a new pattern of manufacturing based on flexibility and economies of scope. To capture the intuitive notion of complementarity, they assume that the firm has a supermodular production function; in essence, this assumption means that the marginal productivity of any factor rises when the input of other factors is increased. Lattice programming techniques that do not rely on conventional (and often inappropriate) assumptions such as concavity (see, e.g., Milgrom and Roberts, 1990, Milgrom and Shannon, 1994, or Topkis, 1995) can be used to analyse

*Department of Economics, Manor Road Building, Manor Road, Oxford OX1 3UQ, United Kingdom. E-mails: pawel.dziewulski@economics.ox.ac.uk and john.quah@economics.ox.ac.uk.

a firm's behavior in this context. A basic conclusion of this analysis is that, when a profit-maximizing firm has a supermodular production function, an increase in the price of an input raises the demand for *all* inputs. But this result only serves to highlight a mathematically natural and economically important question: what restrictions on a profit-maximizing firm's behaviour will *exhaust* the implications of supermodularity? Put another way, given the ubiquity of complementarity in modern manufacturing, economists should be able to demarcate precisely the observable implications of complementarity on firm behavior. The purpose of this paper is to do just that.

The setting we have in mind is the following. Suppose we observe a finite number of input decisions made by a firm, as well as the prices at which those inputs were acquired; formally, each observation is an input-price pair $(x, p) \in \mathbb{R}^\ell_+ \times \mathbb{R}^\ell_+$. What conditions on this set of observations are necessary and sufficient for it to be consistent with a firm choosing inputs to maximize profit, subject to a production function exhibiting production complementarities? In other words, we would like to characterize a data set for which there exists a supermodular function f, defined over the set of all possible input vectors $X \subseteq \mathbb{R}^\ell_+$, such that, for any observation (x, p) in the data set,

$$f(x) - p \cdot x \geq f(y) - p \cdot y, \tag{1}$$

for any other input vector $y \in X$. We can interpret $f(x)$ as the revenue the firm receives when it uses the input vector x; alternatively, if we consider a price-taking firm producing a single good, then it is possible to normalise the price of the output to 1 and interpret f as the firm's production function.

In this paper, we present two characterizations of data sets consistent with the hypothesis of profit-maximisation with production complementarities. The first characterization has an intuitive, axiomatic flavour and imposes a condition on the empirical distributions of samples drawn "with replacement" from the set of observations. The formulation of this condition makes crucial use of the notion of *supermodular stochastic dominance*, recently introduced by Meyer and Strulovici (2013), but our overall approach is similar to that adopted in the classical literature on intuitive probability and additive plausibility (see, e.g., Kraft, Pratt, and Seidenberg, 1959 or Scott, 1964), as well as the approach taken by Border (1992) in his characterisation of expected utility, or by Dziewulski (2014) in an axiomatic analysis of time-preference. Our second characterization is in the form of a test that can be easily applied to any empirical study of the hypothesis. We show that a data set is consistent with profit-maximization subject to a supermodular production

function if and only if there is a solution to a system of linear inequalities constructed from the data. The solution, if it exists, can be then be used to explicitly construct a supermodular function obeying (1).[1]

Our paper is not the first to develop a revealed preference characterization of profit-maximizing firm behavior (or what is formally a closely related phenomenon, maximization of a quasilinear utility function). An early discussion can be found in Afriat (1972), though that analysis differs from ours by assuming that output levels are observed. Analyses which do not make that assumption include Rockafellar (1970), Rochet (1987), Brown and Calsamiglia (2007), and Sákovics (2013). The problem studied by these papers is essentially the same as the one we stated, except that they do not require the function f constructed to satisfy (1) to be a supermodular function. In other words, they are concerned with the characterization of data sets consistent with profit-maximization subject to an arbitrary production function f. The first paper that characterizes data sets consistent with profit-maximisation subject to a supermodular production function is Chambers and Echenique (2009), which develops a necessary and sufficient condition on the data set they call *cyclical supermodularity*. However, their results are formulated in the context of production functions defined on a *finite* sub-lattice X of the Euclidean space. We extend their result by allowing X to be non-finite so, in particular, we allow $X = \mathbb{R}^{\ell}_{+}$. Even in the case where X is finite, we show that the issue of finding a supermodular function defined on X obeying (1) is in fact equivalent to that of finding a supermodular function defined on a smaller domain $X' \subset X$ that obeys (1); this result simplifies any empirical test of the hypothesis. We also present a condition that characterizes data sets rationalisable by a supermodular *and* increasing production function, which turns out to be strictly stronger than the characterization of rationalizability by a supermodular (but not necessarily increasing) production function. This result contrasts strikingly with earlier papers showing that rationalization by an arbitrary production function is equivalent to rationalization by a concave, or concave and increasing, production function.

The remainder of this paper is organised as follows. In Section 2 we present the key lattice-theoretical notions used in our analysis, including the notion of supermodular

[1]The overall structure of our result (though, obviously not the content) is similar to Afriat's Theorem, which asserts a threefold equivalence: a data set is consistent with the maximization of a locally non-satiated utility function subject to a budget set if and only if it obeys an intuitive property called the generalized axiom of revealed preference (GARP) and if and only if there is a solution to a family of linear inequalities constructed form the data known as the Afriat inequalities.

stochastic dominance introduced by Meyer and Strulovici (2013). In Section 3 we introduce our axioms as well as the main results of the paper. Our results can also be applied to the problem of rationalising consumer expenditure data with a quasilinear utility function, as in Brown and Calsamiglia (2007) or Sákovics (2013). This is discussed in Section 4. Auxiliary results not presented in the main body of the paper can be found in the Appendix.

2 Preliminaries

Let $\mathbb{R}^\ell$ be the real ℓ-dimensional space endowed with the standard, coordinate-wise partial order $\geq$. Hence, for any two real vectors $x = (x_i)_{i=1}^\ell$ and $y = (y_i)_{i=1}^\ell$ in $\mathbb{R}^\ell$, we have $x \geq y$ if and only if $x_i \geq y_i$, for all $i = 1, \ldots, \ell$. Moreover, for any x and y in $\mathbb{R}^\ell$, we denote their *join* and *meet* by $x \vee y$ and $x \wedge y$ respectively, where $(x \vee y)_i := \max\{x_i, y_i\}$ and $(x \wedge y)_i := \max\{x_i, y_i\}$, for $i = 1, \ldots, \ell$.

A pair $(X, \geq)$ is a sub-lattice of $(\mathbb{R}^\ell, \geq)$, whenever X is a subset of $\mathbb{R}^\ell_+$ and for any two elements x, $y \in X$, both $x \vee y$ and $x \wedge y$ belong to X.

Remark. With a slight abuse of the terminology, throughout this paper we shall say that X is a *lattice* if and only if $(X, \geq)$ is a sub-lattice of $(\mathbb{R}^\ell, \geq)$.

For any lattices X and Y, we say that X dominates Y with respect to the *strong set order* if, for any $x \in X$ and $y \in Y$, we have $x \vee y \in X$ and $x \wedge y \in Y$. We will say that X is a *product lattice*, if it is defined by the Cartesian product of its projections on $\mathbb{R}$. Hence, $X := \times_{i=1}^\ell X_i$, where $X_i := \{y \in \mathbb{R} : y = x_i, \text{ while } (x_j)_{j=1}^\ell \in X\}$, for all $i = 1, \ldots, \ell$. Given any lattice X, we say that function $f : X \to \mathbb{R}$ is *supermodular*, whenever for any x and y in X, we have

$$f(x \vee y) + f(x \wedge y) \geq f(x) + f(y).$$

Suppose that lattice X is finite. For any vector $x \in X$, let $x + e_i$ denote the element y in X such that $y_j = x_j$, for all $j \neq i$, while $y_i := \min\{z \in X_i : z > x_i\}$, where X_i is the i'th projection of set X, $i = 1, \ldots, \ell$. In particular, note that e_i need not be a unit vector. For example, consider a finite lattice $X := \{(1,2), (1,4), (2,2), (2,4)\}$. Denote $x = (1,2)$. Then, we have $x + e_1 + e_2 = (2,2) + e_2 = (1,4) + e_1 = (2,4)$. Hence, even though e_1 is a unit vector, e_2 is not. Moreover, since X is a lattice, whenever $x + e_i$ and $x + e_j$ belong

to X, for some $i \neq j$, then so does x and $x + e_i + e_j$. Finally, note that any vector e_i is defined conditionally on x.

Given a finite lattice X, by Topkis (1978, Theorems 3.1 and 3.2), function $f : X \to \mathbb{R}$ is supermodular if and only if, for any $x \in X$ such that $x + e_i + e_j \in X$, $i \neq j$, we have

$$f(x + e_i + e_j) + f(x) \geq f(x + e_i) + f(x + e_j).$$

Moreover, function f is increasing whenever $f(x + e_i) \geq f(x)$, for any i and $x \in X$ such that $x + e_i \in X$. Finally, throughout this paper we find it convenient to refer to vectors $\varepsilon_x \in \{0, 1\}^{|X|}$, with the entry corresponding to element $x \in X$ equal to 1 and all the remaining coordinates equal to zero.

2.1 Supermodularity and stochastic dominance

In this subsection we present the notion of *supermodular stochastic dominance* introduced in Meyer and Strulovici (2013). Throughout this section assume that X is a finite product lattice with cardinality $|X|$. We begin by defining a class of *elementary lattice transformations*. For any $x \in X$ such that $x + e_i + e_j$ belongs to X, $i \neq j$, let $t^x_{ij} : X \to \mathbb{R}^{|X|}$ denote a function defined by

$$t^x_{ij}(y) := \begin{cases} 1 & \text{if } \; y = x \;\; \text{or} \;\; y = (x + e_i + e_j), \\ -1 & \text{if } \; y = (x + e_i) \;\; \text{or} \;\; y = (x + e_j), \\ 0 & \text{otherwise.} \end{cases}$$

Equivalently, we may represent the elementary lattice transformation in terms of a vector $t^x_{ij} \in \{-1, 0, 1\}^{|X|}$ such that $t^x_{ij} := \varepsilon_{(x+e_i+e_j)} + \varepsilon_x - \varepsilon_{(x+e_i)} - \varepsilon_{(x+e_j)}$, where ε_x is defined as previously. We denote the set of all such functions by T.

Let Δ_X be the set of probability distributions over X. For any μ and $\nu \in \Delta_X$, we say that distribution μ dominates ν with respect to the *supermodular stochastic ordering*, and denote it by $\mu \succeq_S \nu$, if and only if for any supermodular function $f : X \to \mathbb{R}$, we have

$$\sum_{x \in X} f(x)\mu(x) \geq \sum_{x \in X} f(x)\nu(x). \tag{2}$$

Meyer and Strulovici (2013) present a dual characterisation of the above relation, which plays an important role in the remainder of the paper.

Proposition 1. *For any two probability distributions μ, $\nu \in \Delta_X$, we have $\mu \succeq_S \nu$ if and only if there exist some non-negative coefficients $\{\lambda_t\}_{t\in T}$ such that*

$$\mu = \nu + \sum_{t\in T} \lambda_t t,$$

where T is the set of elementary lattice transformations on X.

See Meyer and Strulovici (2013) for the proof. Roughly speaking, we say that distribution μ dominates ν with respect to the supermodular stochastic dominance order, whenever the probability assigned by ν to any two unordered points x and y in its support is "shifted" in an equal value to $x \vee y$ and $x \wedge y$ in distribution μ. In other words, relatively to ν, distribution μ assigns an equally lower probability to any two unordered points x and y, and a respectively higher probability to their join $x \vee y$ and meet $x \wedge y$. We discuss the notion in the following example.

Example 1. Consider lattice $X := \{2, 4, 6\} \times \{1, 3, 5\}$ and two probability distributions μ and ν defined over X as follows. Distribution μ assigns probability $1/4$ to points $(2, 1)$, $(2, 3)$, $(4, 5)$, and $(6, 5)$, while ν assigns probability $1/4$ to points $(4, 1)$ and $(6, 3)$, and $1/2$ to $(2, 5)$. We depict the two distributions in Figure 1.

Note that μ dominates ν with respect to the supermodular stochastic dominance order. In fact, we can represent the former distribution by

$$\mu = \nu + \tfrac{1}{4} t_{12}^{(2,1)} + \tfrac{1}{2} t_{12}^{(2,3)} + \tfrac{1}{4} t_{12}^{(4,3)},$$

where the elementary lattice transformations t_{ij}^x are define as previously. This is to say that we can construct μ by modifying distribution ν using only the elementary lattice transformations. In this case it suffices to shift the probability of $1/4$ from points $(2, 5)$ and $(6, 3)$ to their join and meet $(6, 5)$ and $(2, 3)$, and the probability of $1/4$ from points $(2, 5)$ and $(4, 1)$ to $(4, 5)$ and $(2, 1)$ respectively.

By definition of the elementary lattice transformations, for any two distributions μ and ν in Δ_X, if μ dominates ν with respect to the supermodular dominance order, then all marginal distributions of μ and ν are equal. In particular, whenever Y is the smallest product lattice such that $\text{supp}(\nu) \subseteq Y$, then for any probability distribution $\mu \in \Delta_X$, we have that $\mu \succeq_S \nu$ implies $\text{supp}(\mu) \subseteq Y$.

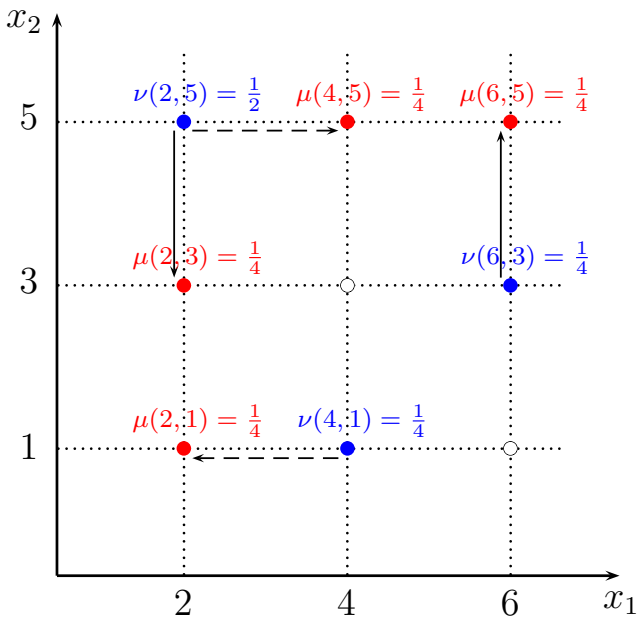


Figure 1: The figure depicts the distributions μ and ν discussed in Example 1. Observe that the probability of $1/4$ assigned to points $(2,5)$ and $(6,3)$ in ν is shifted equally to their join and meet, i.e., $(6,5)$ and $(2,3)$, in distribution μ (depicted by the solid arrows). An analogous shift takes place from points $(2,5)$ and $(4,1)$ to $(2,1)$ and $(4,5)$ (depicted by the dashed arrows). Hence $\mu \succeq_S \nu$.

2.2 Increasing supermodular dominance

The interest of this paper also concerns increasing supermodular functions. First, we define the *elementary monotone transformations* on a finite product lattice X. For any $x \in X$ and $i = 1, \ldots, \ell$ such that $x + e_i \in X$, define $\tau_i^x : X \to \mathbb{R}^{|X|}$ by

$$\tau_i^x(y) := \begin{cases} 1 & \text{if } \ y = x + e_i, \\ -1 & \text{if } \ y = x, \\ 0 & \text{otherwise.} \end{cases}$$

Similarly to the elementary lattice transformations, we can represent function τ_i^x as a vector in $\{-1, 0, 1\}^{|X|}$ defined by $\tau_i^x := \varepsilon_{(x+e_i)} - \varepsilon_x$, where ε_x is specified as previously. Denote the set of all such transformations by $\mathcal{T}$.

For any two μ, $\nu \in \Delta_X$, we say that μ dominates ν with respect to the *increasing* supermodular stochastic ordering, and denote it by $\mu \succeq_{IS} \nu$, if and only if for any increasing and supermodular function $f : X \to \mathbb{R}$, condition (2) holds. As previously, Meyer and Strulovici present a dual characterisation of this notion.

Proposition 2. *For any two probability distributions μ, $\nu \in \Delta_X$, we have $\mu \succeq_{IS} \nu$ if and only if there exist some non-negative coefficients $\{\theta_\tau\}_{\tau\in\mathcal{T}}$ and $\{\lambda_t\}_{t\in T}$ such that*

$$\mu = \nu + \sum_{\tau\in\mathcal{T}} \theta_\tau \tau + \sum_{t\in T} \lambda_t t,$$

where $\mathcal{T}$ and T are the sets of the elementary monotone and lattice transformations on X respectively.

Roughly speaking, the increasing supermodular stochastic order is a combination of the supermodular and the first order stochastic dominance. Take any two probability distributions μ and ν such that $\mu \succeq_{IS} \nu$. By Proposition 2, this implies that, for some non-negative weights $\{\theta_\tau\}_{\tau\in\mathcal{T}}$ and $\{\lambda_t\}_{t\in T}$ we have $\mu = \nu + \sum_{\tau\in\mathcal{T}} \theta_\tau \tau + \sum_{t\in T} \lambda_t t$. Suppose that $\omega := \nu + \sum_{t\in T} \lambda_t t$ is a probability distribution. In particular, this implies that $\mu = \omega + \sum_{\tau\in\mathcal{T}} \theta_\tau \tau$. Hence, there exists a probability distribution ω such that $\omega \succeq_S \nu$, while μ first order stochastically dominates ω. Alternatively, whenever $\omega' := \nu + \sum_{\tau\in\mathcal{T}} \theta_\tau \tau$ is a probability distribution, then it first order stochastically dominates ν, while $\mu \succeq_S \omega'$ (see Meyer and Strulovici, 2013). The above description need not be precise as in general ω and ω' may not be probability distributions.

Example 2. Consider the lattice discussed in Example 1. Suppose that μ assigns probability $1/4$ to points $(2,1)$, $(4,3)$, $(4,5)$, and $(6,5)$, while ν assigns probability $1/4$ to $(4,1)$ and $(6,3)$, and $1/2$ to $(2,5)$. Note that we can express μ by

$$\mu = \nu + \tfrac{1}{4}\tau_1^{(2,3)} + \tfrac{1}{4}t_{12}^{(2,1)} + \tfrac{1}{2}t_{12}^{(2,3)} + \tfrac{1}{4}t_{12}^{(4,3)},$$

where the monotone and lattice transformations τ_i^x and t_{ij}^x are defined as previously. Denote $\omega := \nu + \frac{1}{4}t_{12}^{(2,1)} + \frac{1}{2}t_{12}^{(2,3)} + \frac{1}{4}t_{12}^{(4,3)}$, which is a probability distribution that assigns probability $1/4$ to points $(2,1)$, $(2,3)$, $(4,5)$, and $(6,5)$. Hence, by Example 1, we have $\omega \succeq_S \nu$. Moreover, as $\mu = \omega + \frac{1}{4}\tau_1^{(2,3)}$, distribution μ first order stochastically dominates ω. Analogously, it is possible to construct a distribution $\omega' = \nu + \tau_1^{(2,3)}$, which first order stochastically dominates ν and $\mu \succeq_S \omega'$.

Similarly to the case of supermodular dominance, whenever distribution μ dominates ν with respect to the increasing supermodular dominance order, then any marginal distribution of μ first order stochastically dominates the corresponding marginal distribution of ν. In particular, there exist some product lattices Y_μ and Y_ν such that $\mathrm{supp}(\mu) \subseteq Y_\mu$ and $\mathrm{supp}(\nu) \subseteq Y_\nu$, while Y_μ dominates Y_ν with respect to the strong set order.

3 The main result

Suppose that we observe a finite set of price and factor input data which are elements of $\mathbb{R}^\ell_+$. Therefore, an observation is an ordered pair $(x,p) \in \mathbb{R}^\ell_+ \times \mathbb{R}^\ell_+$, consisting of an input vector x demanded at price p. Denote a *finite* set of observations by $\mathcal{O}$. The corresponding set of observable inputs is $\mathcal{X} := \{x \in \mathbb{R}^\ell_+ : (x,p) \in \mathcal{O} \text{ for some } p\}$. For any set $Y \subseteq \mathbb{R}^\ell_+$ such that $\mathcal{X} \subseteq Y$, we say that function $f : Y \to \mathbb{R}_+$ *rationalises* $\mathcal{O}$ *over* Y, if for any $(x,p) \in \mathcal{O}$, we have

$$f(x) - p \cdot x \geq f(y) - p \cdot y, \text{ for all } y \in Y.$$

We interpret $f(x)$ as the revenue the firm receives when is uses factors x. Alternatively, whenever we consider a price-taking firm producing a single good, it is possible to normalise the price of the output to 1 and interpret f as a production function.

In the remainder of this section we discuss the necessary and sufficient conditions on the set of observations which allow us to rationalise the data by profit-maximisation with production complementarities. In the following subsection we state two axioms characterising the set of observations, which will be central to our argument. Then, in Section 3.2, we present a simplified version of our analysis by constraining our attention to finite sub-lattices of $\mathbb{R}^\ell_+$. Finally, we state the main result of the paper in Section 3.3, in which we extend the notion of rationalisation to an arbitrary sub-lattice of the real space.

3.1 Observable choices and supermodular dominance

Take any product lattice $Y \supseteq \mathcal{X}$ and define the following set of ordered triples:

$$\mathcal{R}_Y := \{(x,y,p) : (x,p) \in \mathcal{O} \text{ and } y \in Y\}.$$

Note that the above set captures the intuition of a directly revealed preference relation over elements in Y. Each triple, consists of two vectors of inputs x and y, and a vector of prices p, such that (x,p) constitutes a single observation, while y belongs to the product lattice Y. Clearly, under the assumption of profit maximisation, whenever we observe (x,p) it implies that, given prices p, acquiring inputs x yielded higher profit than y. Therefore, conditional on p, vector x was "preferred" by the firm to y. Moreover, we expect this condition to hold for any observation (x,p) in $\mathcal{O}$ and any vector $y \in Y$. In

particular, whenever function $f : Y \to \mathbb{R}_+$ rationalises set $\mathcal{O}$ over Y, it must be that $(x, y, p) \in \mathcal{R}_Y$ implies $f(x) - p \cdot x \geq f(y) - p \cdot y$.

We define a *sample* of $\mathcal{R}_Y$ as a finite, indexed collection $\{(x^s, y^s, p^s)\}_{s \in S}$ of elements in $\mathcal{R}_Y$. In particular, we allow for a sample to be constructed "with replacement". That is, a single element of the set may appear more than once in the sample.

Axiom 1 (Supermodular dominance)**.** *For any sample $\{(x^s, y^s, p^s)\}_{s \in S}$ of $\mathcal{R}_Y$ define probability distributions μ and ν in Δ_Y by*

$$\nu(z) := \tfrac{1}{|S|}|\{s \in S : z = x^s\}| \quad \textit{and} \quad \mu(z) := \tfrac{1}{|S|}|\{s \in S : z = y^s\}|.$$

Set of observations $\mathcal{O}$ obeys the supermodular dominance axiom *over Y whenever for any sample of $\mathcal{R}_Y$, $\mu \succeq_S \nu$ implies $\sum_{s \in S} p^s \cdot (x^s - y^s) \leq 0$.*

The set of observations $\mathcal{O}$ obeys the above axiom over a product lattice $Y \supseteq \mathcal{X}$, whenever it is impossible to construct a sample $\{(x^s, y^s, p^s)\}_{s \in S}$ of $\mathcal{R}_Y$ such that the distribution of elements x^s dominates the distribution of y^s with respect to the supermodular stochastic ordering, while the total cost of inputs $\sum_{s \in S} p^s \cdot x^s$ is strictly greater than $\sum_{s \in S} p^s \cdot y^s$.

The above condition becomes more intuitive once we consider a set of observations consisting only of two elements (x^1, p^1) and (x^2, p^2). Consider a sample containing elements $(x^1, x^1 \vee x^2, p^1)$ and $(x^2, x^1 \wedge x^2, p^2)$. Whenever x^1 and x^2 are unordered, the distribution that assigns probability $1/2$ to $x^1 \vee x^2$ and $x^1 \wedge x^2$ dominates the distribution that assigns $1/2$ to x^1 and x^2 with respect to the supermodular stochastic ordering. Hence, for the axiom to hold, it must be that

$$p^1 \cdot x^1 + p^2 \cdot x^2 \leq p^1 \cdot (x^1 \vee x^2) + p^2 \cdot (x^1 \wedge x^2),$$

or equivalently, $p^2 \cdot (x^2 - x^1 \wedge x^2) \leq p^1 \cdot (x^1 \vee x^2 - x^1)$. That is, the difference between the cost of x^2 and $(x^1 \wedge x^2)$ under prices p^2 cannot be strictly greater than the difference in the cost of inputs $x^1 \vee x^2$ and x^1 under prices p^1. Once we consider sets with a greater number of observations, the above requirement becomes more sophisticated, as it has to be verified not only for individual pairs of input vectors, but also for their arbitrary sets. We discuss one such case in the following example.

Example 3. Consider lattice $Y = \{2, 4, 6\} \times \{1, 3, 5\}$. Suppose that we observe three input choices of a firm, equal to $(6, 3)$, $(4, 1)$, and $(2, 5)$, at prices $(1, 1)$, $(1.5, 2.5)$, and

$(2,2)$ respectively. The corresponding set $\mathcal{R}_Y$ consists of all triples (x,y,p), where x is equal to one of the observed choices, p is the price at which x was chosen, while y is any element in Y. We claim that the above observations fail to satisfy the supermodular dominance axiom. Consider the following sample of $\mathcal{R}_Y$:

$$\left\{\begin{array}{c}(x^1,y^1,p^1),\\(x^2,y^2,p^2),\\(x^3,y^3,p^3),\\(x^4,y^4,p^4)\end{array}\right\}=\left\{\begin{array}{c}((6,3),(4,5),(1,1)),\\((4,1),(2,3),(1.5,2.5)),\\((2,5),(2,1),(2,2)),\\((2,5),(6,5),(2,2))\end{array}\right\}.$$

Denote the distribution of x^s in the above collection by ν. Clearly, ν assigns probability $1/4$ to points $(6,3)$ and $(4,1)$, and $1/2$ to $(2,5)$. On the other hand, the distribution of y^s, which we denote by μ, assigns probability $1/4$ to each element in its support, i.e., $(4,5)$, $(2,3)$, $(2,1)$, and $(6,5)$. By Example 1, we know that $\mu \succeq_S \nu$. Moreover, we have $\sum_{s=1}^4 p^s \cdot (x^s - y^s) = 1$, which is strictly greater than 0. Since there exists at least one sample violating the condition stated in the axiom, the set of observations fails to satisfy the supermodular dominance axiom.

Given the notion of increasing supermodular dominance order, we are able to specify a stronger requirement imposed on the set of observations.

Axiom 2 (Increasing supermodular dominance)**.** *For any sample* $\{(x^s,y^s,p^s)\}_{s\in S}$ *of* $\mathcal{R}_Y$ *define probability distributions* μ *and* ν *in* Δ_Y *by*

$$\nu(z) := \tfrac{1}{|S|}|\{s\in S : z = x^s\}| \quad \textit{and} \quad \mu(z) := \tfrac{1}{|S|}|\{s\in S : z = y^s\}|.$$

Set of observations $\mathcal{O}$ *obeys the* increasing supermodular dominance axiom *over* Y *whenever for any sample of* $\mathcal{R}_Y$*,* $\mu \succeq_{IS} \nu$ *implies* $\sum_{s\in S} p^s \cdot (x^s - y^s) \leq 0$*.*

The above condition is similar to the supermodular dominance axiom, apart from the weaker notion of dominance used in the definition of the requirement. Clearly, the second axiom is stronger, as it requires to verify the condition for a larger class of samples of $\mathcal{R}_Y$. Namely, since $\mu \succeq_S \nu$ implies $\mu \succeq_{IS} \nu$, the set of samples that potentially violate the increasing supermodular dominance axiom is greater (by set inclusion) then the set of collections violating the supermodular dominance axiom. In the following example we present one set of observations that obeys the supermodular dominance axiom but fails to satisfy the increasing supermodular dominance axiom.

Example 4. Consider lattice Y discussed in Example 3. Suppose that the set of observations is $\mathcal{O} = \{(x^k, p^k)\}_{k=1}^4$, where $x^1 = x^2 = (2,5)$, $x^3 = (4,1)$, $x^4 = (6,3)$, while $p^1 = (1,2)$, $p^2 = (2,2)$, $p^3 = (2,1)$, and $p^4 = (3,2)$. We do not verify it directly, but it is possible to show that the data set obeys the supermodular dominance axiom.[2] However, it does not satisfy the increasing supermodular dominance axiom. To show this, take the following sample of set $\mathcal{R}_Y$ induced by the set of observations:

$$\left\{ \begin{array}{l} (x^1, y^1, p^1), \\ (x^2, y^2, p^2), \\ (x^3, y^3, p^3), \\ (x^4, y^4, p^4) \end{array} \right\} = \left\{ \begin{array}{l} ((2,5),(2,1),(1,2)), \\ ((2,5),(4,5),(2,2)), \\ ((4,1),(6,5),(2,1)), \\ ((6,3),(4,3),(3,2)) \end{array} \right\}.$$

By Example 2, we know that the distribution of y^s dominates the distribution of x^s with respect to the increasing supermodular dominance order. Moreover, $\sum_{s=1}^4 p^s \cdot (x^s - y^s) = 2$, which is strictly greater than 0. Hence, the increasing supermodular axiom is violated.

The axioms stated above have a similar flavour to the *cancellation law* used in the in the classical literature on intuitive probability and additive plausibility (see, e.g., Kraft, Pratt, and Seidenberg, 1959 or Scott, 1964). As in the above literature, our condition is imposed directly on the samples of elements drawn from the revealed preference relation. Clearly, the requirements are not equivalent, as the framework we discuss is different, however, the line of our argument highly resembles the one applied in those papers.

3.2 Supermodular rationalisation on a finite lattice

In this subsection we show that the supermodular (increasing supermodular) dominance axiom is a necessary and sufficient condition for the observations to be rationalisable over a finite lattice by a supermodular (increasing and supermodular) technology. In particular, we revisit the result by Chambers and Echenique (2009) and show that the cyclical supermodularity requirement, specified in their paper, is equivalent to the supermodular dominance axiom. In the following proposition we show the necessity of the supermodular (increasing supermodular) dominance axiom for rationalisation.

[2] Given Proposition 3, presented in the remainder of the paper, we know that in order to verify the supermodular dominance axiom it is sufficient to determine a supermodular function $f : Y \to \mathbb{R}_+$ that rationalises the set of observations over Y. In fact, there exists at least one such function. For example, function $f(2,1) = 15$, $f(4,1) = 11$, $f(6,1) = f(6,5) = 1$, $f(2,3) = 13$, $f(4,3) = f(2,5) = 9$, and $f(6,3) = f(4,5) = 5$, is both supermodular and rationalises $\mathcal{O}$ over Y.

Proposition 3. *Let $\mathcal{O}$ be rationalisable by a supermodular (increasing and supermodular) function $f : Y \to \mathbb{R}_+$ over Y, for some finite product lattice $Y \supseteq \mathcal{X}$. Then it obeys the supermodular (increasing supermodular) dominance axiom over Y.*

In order to show that the result in the brackets holds, take any increasing, supermodular function $f : Y \to \mathbb{R}_+$ and any sample $\{(x^s, y^s, p^s)\}_{s\in S}$ of $\mathcal{R}_Y$. By definition, for any $s \in S$, we have $f(x^s) - f(y^s) \geq p^s \cdot (x^s - y^s)$. In particular, once we sum up the inequalities with respect to $s \in S$, we obtain

$$\sum_{s\in S} f(x^s) - \sum_{s\in S} f(y^s) \geq \sum_{s\in S} p^s \cdot (x^s - y^s).$$

Given distributions ν and μ, defined as in the increasing supermodular dominance axiom, we have $\sum_{s\in S} f(x^s) = |S| \sum_{x\in Y} f(x)\nu(x)$ while $\sum_{s\in S} f(y^s) = |S| \sum_{x\in Y} f(x)\mu(x)$. Since $\mu \succeq_{IS} \nu$, by the definition of the increasing supermodular stochastic dominance,

$$0 \geq \sum_{s\in S} f(x^s) - \sum_{s\in S} f(y^s) \geq \sum_{s\in S} p^s \cdot (x^s - y^s).$$

Equivalently, we can show the necessity of the axiom using the dual representation of the increasing supermodular dominance order, specified in Proposition 2. Suppose that $\mu \succeq_{IS} \nu$. Therefore, there exist some positive weights $\{\theta_\tau\}_{\tau\in\mathcal{T}}$ and $\{\lambda_t\}_{t\in T}$ such that $\mu = \nu + \sum_{\tau\in\mathcal{T}} \theta_\tau \tau + \sum_{t\in T} \lambda_t t$, where $\mathcal{T}$ and T denote the sets of elementary monotone and lattice transformations respectively. In particular,

$$\begin{aligned}
\sum_{s\in S} f(x^s) - \sum_{s\in S} f(y^s) &= -|S| \sum_{x\in Y} f(x)(\mu(x) - \nu(x)) \\
&= -|S| \sum_{x\in Y} f(x) \left(\sum\nolimits_{\tau\in\mathcal{T}} \theta_\tau \tau(x) + \sum\nolimits_{t\in T} \lambda_t t(x) \right) \\
&= -|S| \sum_{\tau\in\mathcal{T}} \theta_\tau \sum_{x\in Y} f(x)\tau(x) - |S| \sum_{t\in T} \lambda_t \sum_{x\in Y} f(x)t(x).
\end{aligned}$$

Whenever function f is increasing, for any $\tau \in \mathcal{T}$ there is a $y \in Y$ with $y + e_i \in Y$ such that $\sum_{x\in Y} f(x)\tau(x) = f(y + e_i) - f(y) \geq 0$, for some $i = 1, \ldots, \ell$. Similarly, if the function is supermodular, then for any $t \in T$ there is a $y \in Y$ with $y + e_i + e_j \in Y$ such that $\sum_{x\in Y} f(x)t(x) = f(y + e_i + e_j) + f(y) - f(y + e_i) - f(y + e_j) \geq 0$, $i \neq j$. Clearly, these two observations guarantee that $\sum_{s\in S} f(x^s) - \sum_{s\in S} f(y^s) \leq 0$, which by our previous argument implies $\sum_{s\in S} p^s \cdot (x^s - y^s) \leq 0$.

The result outside the brackets can be proven in an analogous way. Whenever $\mu \succeq_S \nu$, then for any supermodular function $f : Y \to \mathbb{R}_+$, we have $\sum_{s\in S} f(x^s) - \sum_{s\in S} f(y^s) \leq 0$.

Hence, it must be that $\sum_{s\in S} p^s \cdot (x^s - y^s) \leq 0$. Similarly, it is possible to show the result using the dual representation of the supermodular stochastic dominance order. Recall that, whenever μ dominates ν in the above sense, we have $\mu = \nu + \sum_{\tau\in\mathcal{T}} \theta_\tau \tau + \sum_{t\in T} \lambda_t t$, where weights $\{\lambda_\tau\}_{\tau\in\mathcal{T}}$ are positive and $\theta_\tau = 0$, for all $\tau \in \mathcal{T}$. Hence, we can apply the same argument as previously.

Showing that the supermodular (increasing supermodular) dominance axiom is at the same time a sufficient condition for rationalisation is more demanding.

Proposition 4. *Suppose that $\mathcal{O}$ obeys the supermodular (increasing supermodular) dominance axiom over some finite product lattice $Y \supseteq \mathcal{X}$. There is a supermodular (increasing and supermodular) function $f : Y \to \mathbb{R}_+$ that rationalises $\mathcal{O}$ over Y.*

Proof. Enumerate the elements of $\mathcal{R}_Y$ such that the set is equal to $\{(x^r, y^r, p^r)\}_{r\in R}$. Let A be a $|\mathcal{T}| \times |Y|$ matrix, in which every row corresponds to a single elementary lattice transformation $\tau \in \mathcal{T}$. Therefore, for any $\tau_i^x \in \mathcal{T}$, the corresponding row is equal to $\varepsilon_{(x+e_i)} - \varepsilon_x$, where ε_x is defined as in Section 2. Similarly, let T be the set of the elementary lattice transformations defined over Y. By B we denote a $|T| \times |Y|$ matrix, in which every row corresponds to a single elementary lattice transformation $t \in T$. Hence, for any $t_{ij}^x \in T$, the corresponding row is equal to $\varepsilon_{(x+e_i+e_j)} + \varepsilon_x - \varepsilon_{(x+e_i)} - \varepsilon_{(x+e_j)}$. Let C be a $|R| \times |Y|$ matrix, where for each $r \in R$, the r'th row is equal to $\varepsilon_{x^r} - \varepsilon_{y^r}$. Finally, by d we denote a vertical vector of length $|R|$, where for each $r \in R$ the corresponding entry is equal to $p^r \cdot (x^r - y^r)$.

Observe that in order to prove the result outside the brackets, it suffices to show that, given the supermodular dominance axiom, there is a vector $f \in \mathbb{R}_+^{|Y|}$ such that $B \cdot f \geq 0$ and $C \cdot f \geq d$. On the other hand, proving the statement in the brackets is equivalent to showing that the increasing supermodular dominance axiom implies the existence of some $f \in \mathbb{R}_+^{|Y|}$ such that $A \cdot f \geq 0$, $B \cdot f \geq 0$, and $C \cdot f \geq d$. We prove only the second case. The result outside the brackets can be proven analogously.

First, we claim that whenever set $\mathcal{O}$ obeys the increasing supermodular dominance axiom, there is some $f \in \mathbb{R}^{|Y|}$ such that $A \cdot f \geq 0$, $B \cdot f \geq 0$, and $C \cdot f \geq d$. By Theorem A.1 (see the Appendix), whenever the system of inequalities has no solution, there are vectors $\theta \in \mathbb{Z}_+^{|\mathcal{T}|}$, $\lambda \in \mathbb{Z}_+^{|T|}$ and $\gamma \in \mathbb{Z}_+^{|R|}$, where either $\theta > 0$, $\lambda > 0$, or $\gamma > 0$, such that $\theta \cdot A + \lambda \cdot B + \gamma \cdot C = 0$, while $\gamma \cdot d > 0$. Take any such vectors and denote $\theta = (\theta_\tau)_{\tau\in\mathcal{T}}$,

$\lambda = (\lambda_t)_{t\in T}$, and $\gamma = (\gamma_r)_{r\in R}$. The above conditions can be rephrased as

$$\sum_{r\in R} \gamma_j \varepsilon_{x^r} + \sum_{\tau\in\mathcal{T}} \theta_\tau \tau + \sum_{t\in T} \lambda_t t = \sum_{r\in R} \gamma_r \varepsilon_{y^r} \quad \text{and} \quad \sum_{r\in R} \gamma_r p^r \cdot (x^r - y^r) > 0,$$

respectively. For any $x \in Y$, let $K(x) := \{r \in R : x = x^r\}$. That is, $K(x)$ is the set of all indices $r \in R$ enumerating the elements of $\mathcal{R}_Y$, for which the corresponding element x^r is equal to x. Analogously, let $L(y) := \{r \in R : y = y^r\}$. Define probability distributions ν and μ, by

$$\nu(x) := \frac{\sum_{k\in K(x)} \gamma_k}{\sum_{r\in R} \gamma_r} \quad \text{and} \quad \mu(y) := \frac{\sum_{l\in L(y)} \gamma_l}{\sum_{r\in R} \gamma_r}.$$[3]

Therefore, probability $\nu(x)$ is equal to the sum of weights γ_r, $r \in R$, for which the corresponding element x^r is equal to x, divided by the sum of all γ_r. Analogously, probability $\mu(y)$ is equal to the normalised sum of weights γ_r, $r \in R$, for which the corresponding element y^r is equal to y. Moreover, we denote $\hat{\theta}\tau = \theta_\tau / \sum_{r\in R} \gamma_r$ and $\hat{\lambda}_t = \lambda_t / \sum_{r\in R} \gamma_r$. The above conditions imply that

$$\nu + \sum_{\tau\in\mathcal{T}} \hat{\theta}_\tau \tau + \sum_{t\in T} \hat{\lambda}_t t = \mu \quad \text{and} \quad \sum_{r\in R} \gamma_r p^r (x^r - y^r) > 0.$$

By Proposition 2, we have $\mu \succeq_{IS} \nu$, while $\sum_{r\in R} \gamma_r p^r (x^r - y^r) > 0$. Construct a sample $\{(x^s, y^s, p^s)\}_{s\in S}$ of $\mathcal{R}_Y$ such that each element (x^r, y^r, p^r) appears γ_r times, for all $r \in R$. Observe that the distributions of x^s and y^s are equal to ν and μ respectively. In addition, we have $\sum_{s\in S} p^s \cdot (x^s - y^s) > 0$. Since $\mu \succeq_{IS} \nu$, this violates the increasing supermodular dominance axiom. Therefore, we conclude that the axiom implies the existence of some $f \in \mathbb{R}^{|Y|}$ such that $A \cdot f \geq 0$, $B \cdot f \geq 0$, and $C \cdot f \geq d$.

In order to show that there is a solution in $\mathbb{R}^{|Y|}_+$, note that by definition of matrices A, B, and C, whenever vector f satisfies the above system of inequalities, then so does $f + a$, where a is a vector in $\mathbb{R}^{|Y|}$ with every entry equal to some constant. Clearly, we can always find an a such that $(f + a) \in \mathbb{R}^{|Y|}_+$. □

Propositions 3 and 4 imply that the supermodular (increasing supermodular) dominance axiom is both a necessary and sufficient for the set of observations to be rationalisable by a supermodular (increasing and supermodular) function $f : Y \to \mathbb{R}_+$ over an arbitrary finite product lattice Y that contains $\mathcal{X}$. In particular, whenever we restrict our attention to product lattices, the result concerning the necessity and sufficiency of the supermodular dominance axiom for rationalisation of the data set is equivalent to

[3] In our notation we assume that $\sum_{i\in\emptyset} a_i = 0$.

Chambers and Echenique (2009, Proposition 2). This implies that the set of observations satisfies our axiom if and only if it is cyclically supermodular (see the above paper for a formal definition of this notion). In other words, we present an alternative characterisation of profit-maximisation with production complamentarities in terms of distributions defined over the observable inputs, rather than their cycles. In addition, by introducing the increasing supermodular dominance axiom, we provide a stricter requirement under which the set of observations is rationalisable by an increasing and supermodular technology. Given Example 4, we know that monotonicity of production with complementarities is testable.

3.3 Supermodular rationalisation on an arbitrary lattice

In this subsection we discuss conditions under which the set of observations can be rationalised over any sub-lattice of the Euclidean space. Before we state the main theorem, we need to introduce some additional notation. Let X^* denote the smallest (by set inclusion) product lattice that contains $\mathcal{X}$. In other words, we define $X^* := \times_{i=1}^{\ell} \mathcal{X}_i$, where $\mathcal{X}_i := \{y \in \mathbb{R} : y = x_i, \text{ where } (x_j)_{j=1}^{\ell} \in \mathcal{X}\}$ is the i'th projection of set $\mathcal{X}$. Consider the main theorem of this paper.

Theorem 1. *Set $\mathcal{O}$ is rationalisable over $\mathbb{R}_+^{\ell}$ by a supermodular (increasing and supermodular) function $f : \mathbb{R}_+^{\ell} \to \mathbb{R}_+$ if and only if it obeys the supermodular (increasing supermodular) dominance axiom over X^*.*

The necessity part of the above theorem is directly implied by Proposition 3. Therefore, in the remainder of the section we focus on the sufficiency of the axioms. Let X_0^* be the smallest (by set inclusion) product lattice containing $\mathcal{X} \cup \{0\}$. In other words, we define $X_0^* := \times_{i=1}^{\ell}(\mathcal{X}_i \cup \{0\})$. Consider the following lemma.

Lemma 1. *Suppose that $\mathcal{O}$ obeys the supermodular (increasing supermodular) dominance axiom over X^*. Then it obeys the axiom over X_0^*.*

Proof. Take any sample $\{(x^s, y^s, p^s)\}_{s \in S}$ of $\mathcal{R}_{X_0^*}$ such that the distribution of y^s, denoted by μ, dominates the distribution of x^s, denoted by ν, with respect to the supermodular stochastic (increasing supermodular) dominance order. Denote the smallest (by set inclusion) product lattice containing $\text{supp}(\nu)$ by Y_ν. Similarly, let Y_μ denote the smallest product lattice containing $\text{supp}(\mu)$. Since $x^s \in \mathcal{X}$, for all $s \in S$, we have $\text{supp}(\nu) \subseteq X^*$. Moreover, if $\mu \succeq_S \nu$ then $Y_\mu = Y_\nu$, while $\mu \succeq_{IS} \nu$ implies that Y_μ dominates Y_ν with

respect to the strong set order. In either case it must be that $Y_\mu \subseteq X^*$. Therefore, whenever $\mathcal{O}$ obeys the supermodular (increasing supermodular) dominance axiom over X^*, it satisfies the axiom over X_0^*. □

Given Proposition 4, the above lemma implies that whenever the set of observations obeys the supermodular (increasing supermodular) dominance axiom over X^*, there exists a supermodular (increasing and supermodular) function $f : X_0^* \to \mathbb{R}_+$, which rationalises the data over X_0^*. In our main argument, we show that any such function has a supermodular (increasing and supermodular) extension to $\mathbb{R}_+^\ell$, which rationalises $\mathcal{O}$ over $\mathbb{R}_+^\ell$. Our argument is constructive. That is, we explicitly define the extension by applying a specific form of multi-linear interpolation between the points of $f(X_0^*)$. Before we prove the general result, we state a lemma that will play a key role in the main proof.

Lemma 2. *Let $X \subseteq \mathbb{R}_+$ be finite and $Y \subseteq \mathbb{R}_+^{\ell-1}$ be a lattice. Suppose that a continuous and (increasing and supermodular) supermodular function $f : X \times Y \to \mathbb{R}$ rationalises $\mathcal{O}$ over $X \times Y$. There exists a continuous and (increasing and supermodular) supermodular extension of function f to $\mathrm{co}(X) \times Y$ that rationalises $\mathcal{O}$ over $\mathrm{co}(X) \times Y$.*

Proof. Take any finite set $X \subseteq \mathbb{R}_+$ and a lattice Y. Suppose that function $f : X \times Y \to \mathbb{R}$ is supermodular. By Topkis (1978, Theorem 3.1), f has increasing differences on $X \times Y$, i.e., for any x', $x \in X$ and y', $y \in Y$, such that $x' \geq x$ and $y' \geq y$, we have

$$f(x', y') - f(x, y') \geq f(x', y) - f(x, y).$$

As $X \subseteq \mathbb{R}_+$, f is trivially supermodular on X, for any $y \in Y$. To see that it is also supermodular on Y, for any $x \in X$, take any y and y' in Y. Since Y is a lattice, both $y \vee y'$ and $y \wedge y'$ belong to Y. Moreover, for any $x \in X$, we have $(x, y) \vee (x, y') = (x, y' \vee y)$, while $(x, y) \wedge (x, y') = (x, y' \wedge y)$. By supermodularity of f, we have

$$\begin{aligned} f(x, y) + f(x, y') &\leq f\big((x, y) \vee (x, y')\big) + f\big((x, y) \wedge (x, y')\big) \\ &= f(x, y' \vee y) + f(x, y' \wedge y), \end{aligned}$$

which proves the claim. Hence, f is separately supermodular on X and Y.

Step 1: We propose an extension of function f to $\mathrm{co}(X) \times Y$. For any element $x \in X$, let $x + e_x := \min\{y \in X : y > x\}$, whenever it exists. Hence, by e_x we denote the distance between x and its immediate successor $x + e_x$ in X. Construct function $g : \mathrm{co}(X) \times Y \to \mathbb{R}$ as follows. For any $x \in X$ such that $x + e_x \in X$, any $z \in [x, x + e_x]$, and $y \in Y$, define

$$g(z, y) \;:=\; \tfrac{z-x}{e_x} f(x + e_x, y) + \big(1 - \tfrac{z-x}{e_x}\big) f(x, y).$$

Clearly, the function is well-defined and continuous. Moreover, whenever f is increasing, then so is g. Finally, for any $(x, y) \in X \times Y$, we have $g(x, y) = f(x, y)$, which makes g a properly defined extension of f.

Step 2: We show that function g is supermodular. By Topkis (1998, Theorem 2.6.2), it suffices to show that function g is (i) supermodular on $\text{co}(X)$, for any $y \in Y$, (ii) supermodular on Y, for any $z \in \text{co}(X)$, and (iii) has increasing differences on $\text{co}(X) \times Y$.

Condition (i) is trivially satisfied. In order to show (ii), take some $x \in X$ such that $x + e_x \in X$, and any $z \in [x, x + e_x]$. Then, for any y, $y' \in Y$,

$$\begin{aligned} g(x,y) + g(x,y') &= \left(\tfrac{z-x}{e_x}\right)\left[f(x+e_x,y) + f(x+e_x,y')\right] \\ &\qquad + \left(1 - \tfrac{z-x}{e_x}\right)\left[f(x,y) + f(x,y')\right] \\ &\leq \left(\tfrac{z-x}{e_x}\right)\left[f(x+e_x,y\wedge y') + f(x+e_x,y\vee y')\right] \\ &\qquad + \left(1 - \tfrac{z-x}{e_x}\right)\left[f(x,y\wedge y') + f(x,y\vee y')\right] \\ &= g(x,y\vee y') + g(x,y\wedge y'), \end{aligned}$$

where the inequality is implied by supermodularity of f over Y and $(z - x)/e_x \in [0, 1]$.

To show (iii), take any z', $z \in \text{co}(X)$ and y', $y \in Y$ such that $z' \geq z$ and $y' \geq y$. By definition, there exist some x and x' in X, where $x' \geq x$, such that $z \in [x, x + e_x]$ and $z' \in [x', x' + e_{x'}]$. Suppose that $x' = x$. Then,

$$\begin{aligned} g(z',y) - g(z,y) &= \left[f(x+e_x,y) - f(x,y)\right]\left(\tfrac{z'-z}{e_x}\right) \\ &\leq \left[f(x+e_x,y') - f(x,y')\right]\left(\tfrac{z'-z}{e_x}\right) \\ &= g(z',y') - g(z,y'), \end{aligned}$$

where the inequality is implied by the increasing differences of function f on $X \times Y$. Whenever $x' > x$, then

$$\begin{aligned} g(z',y) - g(z,y) &= \left[f(x',y) - f(x+e_x,y)\right] + \left(\tfrac{z'-x'}{e_{x'}}\right)\left[f(x'+e_{x'},y) - f(x',y)\right] \\ &\qquad + \left(1 - \tfrac{z-x}{e_x}\right)\left[f(x+e_x,y) - f(x,y)\right] \\ &\leq \left[f(x',y') - f(x+e_x,y')\right] + \left(\tfrac{z'-x'}{e_{x'}}\right)\left[f(x'+e_{x'},y') - f(x',y')\right] \\ &\qquad + \left(1 - \tfrac{z-x}{e_x}\right)\left[f(x+e_x,y') - f(x,y')\right] \\ &= g(z',y') - g(z,y'), \end{aligned}$$

where the inequality is implied by the increasing differences of function f over $X \times Y$ and the fact that $x' > x$ implies $x' \geq x + e_x$. Therefore, function g has increasing differences

over $\text{co}(X) \times Y$. Since conditions (i), (ii), and (iii) are satisfied, by Topkis (1998, Theorem 2.6.2) we conclude that function g is supermodular over its domain.

Step 3: We show that function g rationalises $\mathcal{O}$ over $\text{co}(X) \times Y$. Take any observation $\big((x,y),p\big)$ in $\mathcal{O}$ and any $(z',y') \in \text{co}(X) \times Y$. By definition, there exists some $x' \in X$ with $x' + e_{x'} \in X$ such that $z' \in [x', x' + e_{x'}]$. Since g extends f to $\text{co}(X) \times Y$, we have

$$\begin{aligned}
g(x,y) - p \cdot (x,y) &= f(x,y) - p \cdot (x,y) \\
&= \big(\tfrac{z'-x'}{e_{x'}}\big)\big[f(x,y) - p \cdot (x,y)\big] + \big(1 - \tfrac{z'-x'}{e_{x'}}\big)\big[f(x,y) - p \cdot (x,y)\big] \\
&\geq \big(\tfrac{z'-x'}{e_{x'}}\big)\big[f(x' + e_{x'}, y') - p \cdot (x + e_{x'}, y')\big] \\
&\qquad + \big(1 - \tfrac{z'-x'}{e_{x'}}\big)\big[f(x',y') - p \cdot (x',y')\big] \\
&= \big(\tfrac{z'-x'}{e_{x'}}\big) f(x + e_{x'}, y') + \big(1 - \tfrac{z'-x'}{e_{x'}}\big) f(x',y') - p \cdot (z',y') \\
&= g(z',y') - p \cdot (z',y'),
\end{aligned}$$

where the inequality follows from the fact that f rationalises $\mathcal{O}$ over $X \times Y$. □

Given the preliminary results, we proceed with the argument supporting Theorem 1.

Proof of Theorem 1. By Proposition 4 and Lemma 1, there exists a supermodular (increasing and supermodular) function $f : X_0^* \to \mathbb{R}_+$ that rationalises $\mathcal{O}$ over X_0^*. Moreover, by Lemma 2 and an inductive argument on the subsequent dimensions of set X_0^*, it is possible to show that there exists a supermodular (increasing and supermodular) extension of f to $\text{co}(X_0^*)$ that rationalises $\mathcal{O}$ over its domain.

Denote any such extension by $g : \text{co}(X_0^*) \to \mathbb{R}_+$. Moreover, let $\bar{x}$ be the greatest element of $\text{co}(X_0^*)$. Define function $\bar{f} : \mathbb{R}_+^\ell \to \mathbb{R}_+$, by $\bar{f}(y) := g(y \wedge \bar{x})$. Clearly, $\bar{f}$ extends g to $\mathbb{R}_+^\ell$. Moreover, whenever g is increasing, so is $\bar{f}$. We claim that the function is also (i) supermodular and (ii) rationalises $\mathcal{O}$ over $\mathbb{R}_+^\ell$.

In order to show (i), note that for any y, $z \in \mathbb{R}_+^\ell$, we have

$$\begin{aligned}
\bar{f}(y) + \bar{f}(z) &= g(y \wedge \bar{x}) + g(z \wedge \bar{x}) \\
&\leq g\big((y \wedge \bar{x}) \vee (z \wedge \bar{x})\big) + g\big((y \wedge \bar{x}) \wedge (z \wedge \bar{x})\big) \\
&= g\big((y \vee z) \wedge \bar{x}\big) + g\big((y \wedge z) \wedge \bar{x}\big) \\
&= \bar{f}(y \vee z) + \bar{f}(y \wedge z),
\end{aligned}$$

where the inequality follows from supermodularity of function g.

To prove (ii), take any observation (x, p) in $\mathcal{O}$ and any vector $y \in \mathbb{R}^\ell_+$. Given that g rationalises $\mathcal{O}$ over $\text{co}(X_0^*)$, and for any $x \in X$, we have $x \wedge \bar{x} = x$,

$$\begin{aligned} \bar{f}(x) - p \cdot x &= g(x) - p \cdot x \\ &\geq g(y \wedge \bar{x}) - p \cdot (y \wedge \bar{x}) \\ &\geq g(y \wedge \bar{x}) - p \cdot y \\ &= \bar{f}(y) - p \cdot y, \end{aligned}$$

where the second inequality follows from the fact that $p \in \mathbb{R}^\ell_+$. □

The above argument highly relies on Proposition 4. Given this result, we know that whenever the set of observations obeys the supermodular (increasing supermodular) dominance axiom over X_0^*, there exists a supermodular (increasing and supermodular) production that rationalises $\mathcal{O}$ over X_0^*. The main difficulty of the proof of Theorem 1 was to extend the production function from the restricted domain to $\mathbb{R}^\ell_+$, such that the extension is supermodular (increasing and supermodular) and rationalises the set of observations over the extended domain. To this end, we apply Lemma 2 in order to construct the extension by a multi-linear combination of the points generated by the restricted production function. To make our presentation transparent, we discuss the following example.

Example 5. Consider observations (x^1, p^1) and (x^2, p^2), where $x^1 = (3, 1)$, $p^1 = (2, 1)$, while $x^2 = (1, 2)$ and $p^2 = (1, 1)$. Hence, $\mathcal{X} = \{(3, 1), (2, 1)\}$, while the smallest product lattice containing $\mathcal{X}$ is $X^* = \{(1, 1), (3, 1), (2, 1), (3, 2)\}$. Finally, set X_0^* is equal to $X^* \cup \{(0, 0), (1, 0), (3, 0), (0, 1), (0, 2)\}$. In Figure 2 we plot the lattice as well as an increasing and supermodular function $f : X_0^* \to \mathbb{R}$ that rationalises the set of observations over X_0^*. We claim that there is an extension of f to $\mathbb{R}^2_+$.

Take any point in $\mathbb{R}^2_+$, e.g., $y = (y_1, y_2)$ depicted as in Figure 2. Note that the point belongs to the convex hull of elements $(1, 1)$, $(3, 1)$, $(1, 2)$, and $(3, 2)$. Clearly, it can be expressed as a linear combination of these points in the following way:

$$\begin{aligned} y &= (y_1, y_2) \\ &= \left(3 \cdot \tfrac{3-y_1}{3-1} + 1 \cdot \tfrac{y_1-1}{3-1},\ 2 \cdot \tfrac{2-y_2}{2-1} + 1 \cdot \tfrac{y_2-1}{2-1}\right) \\ &= \tfrac{2-y_2}{2-1} \cdot \left[\tfrac{3-y_1}{3-1} \cdot (3, 2) + \tfrac{y_1-1}{3-1} \cdot (1, 2)\right] + \tfrac{y_2-1}{2-1} \cdot \left[\tfrac{3-y_1}{3-1} \cdot (3, 1) + \tfrac{y_1-1}{3-1} \cdot (1, 1)\right] \\ &= \underbrace{\tfrac{2-y_2}{2-1} \cdot \tfrac{3-y_1}{3-1}}_{\alpha_1} \cdot (3, 2) + \underbrace{\tfrac{2-y_2}{2-1} \cdot \tfrac{y_1-1}{3-1}}_{\alpha_2} \cdot (1, 2) + \underbrace{\tfrac{y_2-1}{2-1} \cdot \tfrac{3-y_1}{3-1}}_{\alpha_3} \cdot (3, 1) + \underbrace{\tfrac{y_2-1}{2-1} \cdot \tfrac{y_1-1}{3-1}}_{\alpha_4} \cdot (1, 1). \end{aligned}$$

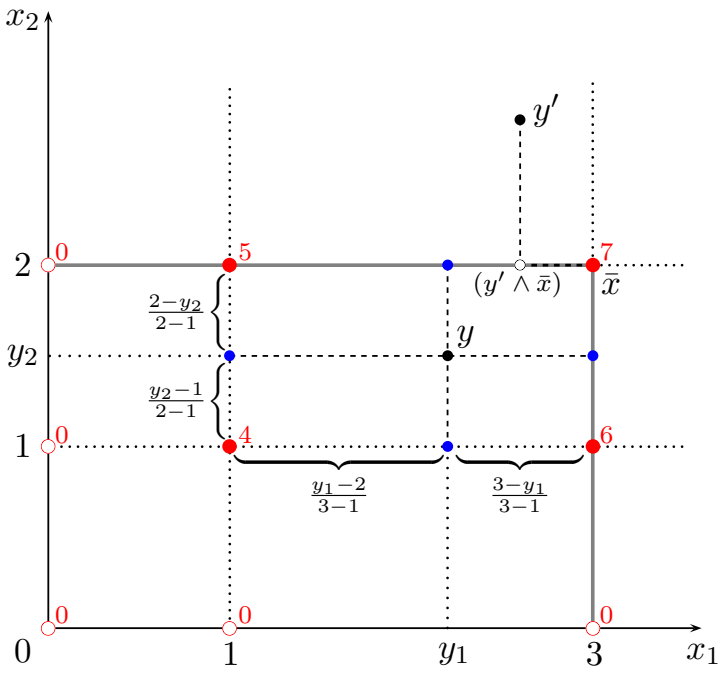


Figure 2: The figure depicts a production function rationalising the set of observations discussed in Example 5. The points in the corresponding lattice X^* are represented by solid red dots, while the void red dots represent the elements of $X_0^* \backslash X^*$. The values of function $f : X_0^* \to \mathbb{R}$ are denoted in red next to the corresponding points in X_0^*. The grey solid line denotes the upper bound of $\mathrm{co}(X_0^*)$.

Given the above weights, we are able to define the value of function f at point y by taking a linear combination of its values at points in X^*. That is,

$$\begin{aligned} f(y) &= \alpha_1 f(3,2) + \alpha_2 f(1,2) + \alpha_3 f(3,1) + \alpha_4 f(1,1) \\ &= \alpha_1 \cdot 7 + \alpha_2 \cdot 5 + \alpha_3 \cdot 6 + \alpha_4 \cdot 4. \end{aligned}$$

Clearly, we can determine the value of the function in an analogous way for any point y that belongs to $\mathrm{co}(X_0^*)$. Whenever a point is not in $\mathrm{co}(X_0^*)$ (e.g., y' depicted in Figure 2) its value is equal to the value of the "closest" point that belongs to $\mathrm{co}(X_0^*)$, i.e., $y' \wedge \bar{x}$, where $\bar{x} := \bigvee \mathrm{co}(X_0^*) = (3,2)$.

In some cases we might be interested in rationalising the set of observations by a production function which is defined over a subset of $\mathbb{R}_+^\ell$ (which might not necessarily be finite). In particular, this might take place when some input factors can only be acquired in integer amounts. The following corollary to Theorem 1 shows that the axioms stated in Section 3.1 remain both necessary and sufficient for this form of rationalisation.

Corollary 1. *Let Y be any lattice such that $X^* \subseteq Y$. Then, set $\mathcal{O}$ is rationalisable over Y by a supermodular (increasing and supermodular) function $f : Y \to \mathbb{R}$ if and only if it obeys the supermodular (increasing supermodular) dominance axiom over X^*. (Note that Y does not have to be a product lattice.)*

The above result follows from the fact that whenever there exists a function rationalising $\mathcal{O}$ over $\mathbb{R}^\ell_+$, the same function rationalises the set over any restricted domain. Moreover, both supermodularity and monotonicity are preserved over the new domain. This implies that the supermodular (increasing supermodular) dominance axiom is a necessary and sufficient condition for rationalising the data set over an arbitrary sub-lattice of the Euclidean space.

Suppose that lattice Y is finite. The above observation implies that, whenever Y contains X^*, in order to rationalise $\mathcal{O}$ over Y, it is sufficient for the set of observations to obey the supermodular (increasing supermodular) dominance axiom over X^*, rather than Y. In other words, Corollary 1 provides a tighter condition than the result discussed in Propositions 3 and 4, as it suffices for our axioms to be verified over a smaller set. In particular, this also refers to Chambers and Echenique (2009, Proposition 2), who require for the cyclical supermodularity condition to be satisfied for any cycle in Y. Since their axiom is equivalent to supermodular dominance over Y, it is enough to check either the cyclical supermodularity condition or the supermodular dominance axiom for the elements of X^* *only*. This plays an important role for the applicability of our results to empirical analysis. Clearly, whenever set Y is much larger than X^*, the above observation allows us to substantially reduce the complexity of the test.

Even though our axioms provide an intuitive condition for rationalisability of the set of observations, checking whether the requirement is satisfied by the data might be cumbersome and hard to apply in an empirical analysis. Fortunately, there exists a more convenient way of implementing the test. Consider the following proposition.

Proposition 5. *Let $Y \supseteq \mathcal{X}$ be a finite product lattice. Set $\mathcal{O}$ obeys the supermodular (increasing supermodular) dominance axiom over Y if and only if there is a sequence (increasing sequence) of numbers $\{f_x\}_{x \in Y}$ such that, for any $x \in Y$ with $x + e_i + e_j$ in Y, $i \neq j$, we have $f_{(x+e_i+e_j)} + f_x \geq f_{(x+e_i)} + f_{(x+e_j)}$, and $(x, y, p) \in \mathcal{R}_Y$ implies $f_x - p \cdot x \geq f_y - p \cdot y$.*

The result states that the supermodular (increasing supermodular) dominance axiom is equivalent to the existence of a solution to a certain system of linear inequalities. It is

clear that any such solution constitutes a supermodular production function that rationalises set $\mathcal{O}$ over Y.[4] Therefore, Proposition 5 is simply a reformulation of Propositions 3 and 4. Note that the above result provides us with a simple method of verifying if $\mathcal{O}$ is rationalisable over $\mathbb{R}^{\ell}_{+}$. By Theorem 1, the set of observations is rationalisable over the Euclidean space if and only if it obeys the supermodular (increasing supermodular) dominance axiom over X^*. By Proposition 5, we can verify the condition by finding a solution to the corresponding system of linear inequalities. As it is well known, such systems can be efficiently solved in a finite number of steps, which we find crucial for the applicability of our result.

4 Related results and alternative applications

In the previous sections we concentrated on the necessary and sufficient conditions under which a set of observations $\mathcal{O}$ can be rationalised by either a supermodular or an increasing and supermodular production function. Clearly, the axioms we discussed are at the same time sufficient for rationalising $\mathcal{O}$ by *any* production function, though they will not generally be necessary. We discuss this issue in greater detail in Section 4.1. The problem of rationalising consumer expenditure data with a quasilinear utility function is formally very similar to that of rationalizing input data from a profit-maximizing firm, so our results can also be applied to the former context. We discuss this in Section 4.2.

4.1 Cyclical monotonicity and supermodular dominance

The necessary and sufficient conditions for rationalisation of input-price data by an arbitrary technology can be obtained directly by the application of Rockafellar (1970, Theorem 24.8). Various versions of the argument can be found, e.g., in Brown and Calsamiglia (2007), Chambers and Echenique (2009), or Sákovics (2013). In order to make our paper self-contained, we present the result below.

Proposition 6. *The following statements are equivalent.*

(i) Set $\mathcal{O}$ is rationalisable by a function $f : \mathbb{R}^{\ell}_{+} \to \mathbb{R}_{+}$ over $\mathbb{R}^{\ell}_{+}$.

[4] Obviously, we can define $f : Y \to \mathbb{R}$, by $f(x) := f_x$. Given this, for any $x \in Y$ with $x + e_i$ and $x + e_j$ in Y, $i \neq j$, we have $f(x + e_i + e_j) + f(x) \geq f(x + e_i) + f(x + e_j)$, which implies supermodularity of the function. Finally, as $(x, y, p) \in \mathcal{R}_Y$ implies $f(x) - p \cdot x \geq f(y) - p \cdot y$, function f rationalises $\mathcal{O}$ over Y.

(ii) The set of observations is cyclically monotone. *That is, for any finite sequence* $\{(x^n, p^n)\}_{n=1}^m$ *in* $\mathcal{O}$*, we have*

$$p^1 \cdot (x^1 - x^2) + p^2 \cdot (x^2 - x^3) + \ldots + p^{m-1} \cdot (x^{m-1} - x^m) + p^m \cdot (x^m - x^1) \leq 0.$$

(iii) Set $\mathcal{O}$ *is rationalisable over* $\mathbb{R}^\ell_+$ *by an increasing, concave function* $f : \mathbb{R}^\ell_+ \to \mathbb{R}_+$.

In order to understand how the above condition is related to the supermodular (increasing supermodular) dominance axiom, take any sample $\{(x^s, y^s, p^s)\}_{s\in S}$ of $\mathcal{R}_{X^*}$, where X^* is the smallest (by set inclusion) product lattice containing $\mathcal{X}$. As previously, let distributions ν and $\mu \in \Delta_{X^*}$ be defined by $\nu(z) := \frac{1}{|S|}|\{s \in S : z = x^s\}|$ and $\mu(z) := \frac{1}{|S|}|\{s \in S : z = y^s\}|$. We claim that set $\mathcal{O}$ is cyclically monotone if and only if for any sample $\{(x^s, y^s, p^s)\}_{s\in S}$ of $\mathcal{R}_{X^*}$ such that $\mu = \nu$, we have $\sum_{s\in S} p^s(x^s - y^s) \leq 0$ (denote the latter condition by $*$).

To see that cyclical monotonicity implies $(*)$, take any $\{(x^s, y^s, p^s)\}_{s\in S}$ such that $\mu = \nu$. In particular, for all $s \in S$, we have $y^s \in \mathcal{X}$, or equivalently, $(y^s, p) \in \mathcal{O}$, for some p. Clearly, it is possible to construct a sequence $\{(x^n, p^n)\}_{n=1}^m$ of length $m = |S|$ such that (x^1, x^2, p^1), (x^2, x^3, p^2), ..., (x^m, x^1, p^m) belong to $\mathcal{R}_{X^*}$.[5] Moreover, by cyclical monotonicity, we have

$$p^1 \cdot (x^1 - x^2) + p^2 \cdot (x^2 - x^3) + \cdots + p^m(x^m - x^1) = \sum_{s\in S} p^s(x^s - y^s) \leq 0$$

In order to show that the opposite implication also holds, take any finite sequence $\{(x^n, p^n)\}_{n=1}^m$ in $\mathcal{O}$. Observe that any such sequence generates a sample $\{(x^n, y^n, p^n)\}_{n=1}^m$, where $y^n = x^{n+1}$ for $n = 1, \ldots, m-1$, and $y^m = x^1$. Clearly the corresponding distributions μ and ν are equal. Moreover, condition $(*)$ implies that $\sum_{n=1}^m p^n \cdot (x^n - y^n) \leq 0$.

Given the above characterisation, it is straightforward to show that the supermodular (increasing supermodular) dominance axiom imposes a stronger condition on the set of observations than cyclical monotonicity. Namely, as any two equivalent distributions are at the same time ordered with respect to $\succeq_S$ ($\succeq_{IS}$), in order to verify whether the set of observations obeys the supermodular (increasing supermodular) dominance axiom we need to check the condition stated in the definition of the axiom for a larger class of samples. That is, not only the samples for which the corresponding distributions are equivalent, but also those for which they are ordered with respect to $\succeq_S$ ($\succeq_{IS}$).

[5]The sequence can be constructed as follows. Take any element (x, y, p) from $\{(x^s, y^s, p^s)\}_{s\in S}$, and denote it (x^1, x^2, p^1). Given that $\mu = \nu$, there exists a triple (x', y', p') in the sample such that $x' = x^2$. Denote $(x^2, x^3, p^2) := (x', y', p')$. Analogously, we can find some (x'', y'', p'') such that $x'' = x^3$, and so on. Since the sample is finite, eventually we reach the final element (x^m, x^1, p^m).

4.2 Supermodularity and quasilinear rationalisation

The results obtained in the previous sections can be easily reinterpreted as a consumer optimisation problem with quasilinear utility function. Suppose that the consumer's preference can be represented by function $v : \mathbb{R}^{\ell}_{+} \times \mathbb{R}_{+} \to \mathbb{R}$, defined by

$$v(x, m) := u(x) + m,$$

where $u : \mathbb{R}^{\ell}_{+} \to \mathbb{R}$ is a utility function defined over ℓ consumption goods, while m denotes the value of the numeraire. Given some initial wealth w, the consumer optimisation problem is

$$\max_{(x,m)} v(x, m), \text{ s.t. } p \cdot x + m \leq w.$$

Clearly, as function v strictly increases with respect to m, the budget constraint is binding at every solution to the above problem. Therefore, without loss of generality, the inequality in the constraint can be replaced with equality.

Suppose that the set of observations is given by $\{(p^k, x^k, m^k)\}_{k \in K}$, where p^k and x^k denote the price and the demand for the consumption goods, while m^k is the chosen level of the numeraire. The set of observations is *rationalisable* by a quasilinear utility function if there is a function $u : \mathbb{R}^{\ell}_{+} \to \mathbb{R}$ such that for any $k \in K$, we have $u(x^k) + m^k \geq u(y) + n$, for all $y \in \mathbb{R}^{\ell}_{+}$ and $n \geq 0$ such that $p^k \cdot x^k + m^k = p^k \cdot y + n$. Equivalently, the set is rationalisable in the above sense whenever

$$u(x^k) - p^k \cdot x^k \geq u(y) - p^k \cdot y,$$

for all $y \in \mathbb{R}^{\ell}_{+}$ such that $p^k \cdot x^k + m^k \geq p^k \cdot y$. The above objective function very closely resembles the profit function discussed in Section 3. In particular, under some additional assumptions, Theorem 1 can be applied directly to the above context.

Given the observation set $\{(p^k, x^k, m^k)\}_{k \in K}$, denote the reduced set of observations by $\mathcal{O} := \{(x^k, p^k)\}_{k \in K}$, and let $\mathcal{X}$ be the set of observable demands $\{x^k\}_{k \in K}$. Moreover, let X^* be the smallest (by set inclusion) product lattice containing $\mathcal{X}$, while

$$\mathcal{R}_{X^*} := \{(x, y, p) : (x, p) \in \mathcal{O} \text{ and } y \in X^*\}.$$

Suppose that the set of observations $\{(p^k, x^k, m^k)\}_{k \in K}$ is such that for all $k \in K$, we have $p^k \cdot x^k + m^k \geq p^k \cdot x^l$, for all $l \in K$. Clearly, by applying Proposition 6, the data set is rationalisable by a quasilinear utility function if and only if it obeys cyclical

monotonicity. Moreover, without loss of generality we may assume that the function is concave and increasing.

In a similar manner we can apply Theorem 1 in order to determine the necessary and sufficient conditions for the set of observations to be rationalisable by a supermodular (increasing and supermodular) quasilinear utility function.

Proposition 7. *Suppose that the set of observations* $\{(p^k, x^k, m^k)\}_{k\in K}$ *is such that for all* $k \in K$*, we have* $p^k \cdot x^k + m^k \geq p^k \cdot y$*, for all* $y \in X^*$*. Then, it is rationalisable by a supermodular (increasing and supermodular) quasilinear utility function if and only if set* $\mathcal{O}$ *obeys the supermodular (increasing supermodular) dominance axiom over* X^**.*

Proof. First, we show ($\Rightarrow$). Take any supermodular (increasing and supermodular) function $u : \mathbb{R}^\ell_+ \to \mathbb{R}$ and any sample $\{(x^s, y^s, p^s)\}_{s\in S}$ of $\mathcal{R}_{X^*}$. Since for all $k \in K$, we have $p^k \cdot x^k + m^k \geq p^k \cdot y$, for all $y \in X^*$, it must be that $u(x^s) - p^s \cdot x^s \geq u(y^s) - p^s \cdot y^s$, for all $s \in S$. Define distributions ν and μ over X^* as in the definition of the axiom. Whenever $\mu \succeq_s \nu$ ($\mu \succeq_{IS} \nu$), then $0 \geq \sum_{s\in S} u(x^s) - \sum_{s\in S} u(y^s) \geq \sum_{s\in S} p^s \cdot (x^s - y^s)$.

In order to show ($\Leftarrow$), note that by Theorem 1, whenever set $\mathcal{O}$ obeys the supermodular (increasing supermodular) dominance axiom over X^*, there exists a supermodular (increasing supermodular) function $u : \mathbb{R}^\ell_+ \to \mathbb{R}$ such that, for any $(x, p) \in \mathcal{O}$, we have $u(x) - p \cdot x \geq u(y) - p \cdot y$, for all $y \in \mathbb{R}^\ell_+$. In particular, the condition holds for any triple (x^k, p^k, m^k) in the set of observations and any $y \in \mathbb{R}^\ell_+$ such that $p^k \cdot x^k + m^k \geq p^k \cdot y$. □

The assumption stated at the beginning of the proposition is crucial for the result to hold. Namely it requires that every element in set X^* is affordable for the agent at every observation. Equivalently, it implies that the value of the numeraire is high enough for the budget constraint to be neglected while rationalising the set of observations. This allows us to treat the consumer optimisation problem in the exact same way as profit-maximisation discussed in Section 3. Once the additional assumption is dropped, the supermodular (increasing supermodular) dominance axiom remains sufficient for the corresponding form of rationalisation, but is no longer necessary. This is due to the fact that, for any observation (x^k, p^k, m^k), condition $u(x^k) - p^k \cdot x^k \geq u(y) - p^k \cdot y$ has to be satisfied only for those $y \in X^*$, for which $p^k \cdot x^k + m^k \geq p^k \cdot y$.

Appendix

In the following section we present a specific version of Farkas' Lemma that plays a substantial role in the proof of our main result.

Theorem A.1 (Farkas' Lemma)**.** *Let A be a $m \times n$ rational matrix and let $b \in \mathbb{R}^m$. Exactly one of the following alternatives is true.*

(i) There exists $x \in \mathbb{R}^n$ such that $A \cdot x \geq b$.

(ii) There exists $\lambda \in \mathbb{Q}^m$, where $\lambda > 0$, such that $\lambda \cdot A = 0$ and $\lambda \cdot b > 0$.

The above result is a special case of the rational version of Farkas' Lemma. However, in the above theorem we allow for vector b to be an element of $\mathbb{R}^m$ rather than $\mathbb{Q}^m$. This variation on the original result is significant for the main argument of the paper. Clearly, without loss of generality, we can substitute $\mathbb{Q}^m$ with $\mathbb{Z}^m$. We prove the result using Fourier-Motzkin's Elimination Theorem. In order to make our paper self-contained, we present a modified version of the theorem below. The original proof can be found in Fourier (1826) or Motzkin (1951).

Lemma A.1 (Fourier-Motzkin's Elimination)**.** *Define $P := \{x \in \mathbb{R}^n : A \cdot x \geq b\}$, where A is a $m \times n$ rational matrix and $b \in \mathbb{R}^m$, and let $\pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ be a projection defined by $\pi(x^1, x^2, \ldots, x^n) := (x^2, \ldots, x^n)$. There exists a positive rational matrix Λ such that*

$$\pi(P) = \{(x_2, \ldots, x_n) \in \mathbb{R}^{n-1} : \Lambda \cdot A \cdot (x_1, x_2, \ldots, x_n) \geq \Lambda \cdot b, \text{ for all } x_1 \in \mathbb{R}\}.$$

Proof. Denote the entries of matrix A by a_{ij}. By definition, set P consists of all the elements x in $\mathbb{R}^n$ that satisfy

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \geq b_i, \text{ for all } i = 1, \ldots, m.$$

Define sets $G := \{i : a_{i1} > 0\}$, $Z := \{i : a_{i1} = 0\}$, and $L := \{i : a_{i1} < 0\}$, and reformulate every inequality $k \in G$ such that

$$x_1 \geq -\frac{a_{k2}}{|a_{k1}|}x_2 - \cdots - \frac{a_{kn}}{|a_{k1}|}x_n - \frac{b_k}{|a_{k1}|}.$$

Moreover, reformulate any inequality in $l \in L$ so that

$$x_1 \leq \frac{a_{l2}}{|a_{l1}|}x_2 + \cdots + \frac{a_{ln}}{|a_{l1}|}x_n + \frac{b_l}{|a_{l1}|},$$

The above reformulation allows us to reduce $|G| + |L|$ inequalities corresponding to the indices in sets G and L to just two inequalities defined by

$$\min_{l\in L}\left\{\frac{a_{l2}}{|a_{l1}|}x_2 + \cdots + \frac{a_{ln}}{|a_{l1}|}x_n + \frac{b_l}{|a_{l1}|}\right\} \geq x_1 \geq \max_{k\in G}\left\{-\frac{a_{k2}}{|a_{k1}|}x_2 - \cdots - \frac{a_{kn}}{|a_{k1}|}x_n - \frac{b_k}{|a_{k1}|}\right\}.$$

Observe that vector $(x_2, \ldots, x_n)$ belongs to projection $\pi(P)$ if and only if there exists some real number x_1 such that $(x_1, x_2, \ldots, x_n) \in P$. This implies that vector $(x_2, \ldots, x_n)$ has to satisfy all the inequalities in Z as well as

$$\min_{l\in L}\left\{\frac{a_{l2}}{|a_{l1}|}x_2 + \cdots + \frac{a_{ln}}{|a_{l1}|}x_n + \frac{b_l}{|a_{l1}|}\right\} \geq \max_{k\in G}\left\{-\frac{a_{k2}}{|a_{k1}|}x_2 - \cdots - \frac{a_{kn}}{|a_{k1}|}x_n - \frac{b_k}{|a_{k1}|}\right\}.$$

Moreover, vector $(x_2, \ldots, x_n)$ satisfies the above inequality if and only if it satisfies $|G||L|$ inequalities such that for any $k \in G$ and $l \in L$, we have

$$\frac{a_{l2}}{|a_{l1}|}x_2 + \cdots + \frac{a_{ln}}{|a_{l1}|}x_n + \frac{b_l}{|a_{l1}|} \geq -\frac{a_{k2}}{|a_{k1}|}x_2 - \cdots - \frac{a_{kn}}{|a_{k1}|}x_n - \frac{b_k}{|a_{k1}|}$$

or equivalently

$$\left(\frac{a_{k2}}{|a_{k1}|} + \frac{a_{l2}}{|a_{l1}|}\right)x_2 + \cdots + \left(\frac{a_{kn}}{|a_{k1}|} + \frac{a_{ln}}{|a_{l1}|}\right)x_n \geq \left(\frac{b_k}{|a_{k1}|} + \frac{b_l}{|a_{l1}|}\right).$$

Define a $|G||L| + |Z|$ by m matrix Λ as follows. For each of the first $|G||L|$ rows, take the corresponding indices $k \in G$ and $l \in L$. Let the k'th entry of that row be equal to $1/|a_{k1}|$, the l'th entry be equal to $1/|a_{l1}|$, and all the remaining entries be equal to zero. Finally, in each of the $|Z|$ remaining rows, let the entry corresponding to element $z \in Z$ be equal to 1, and all the remaining entries be equal to zero.

Clearly, Λ is positive. In addition, since A is rational, so is Λ. In order to conclude the proof, note that, by construction, vector $(x_2, \ldots, x_n)$ solves the above system of inequalities if and only if $(\Lambda \cdot A) \cdot (x_1, x_2, \ldots, x_n) \geq (\Lambda \cdot b)$, for all $x_1 \in \mathbb{R}$. Hence, $(x_2, \ldots, x_n) \in \pi(P)$ if and only if $(\Lambda \cdot A) \cdot (x_1, x_2, \ldots, x_n) \geq (\Lambda \cdot b)$, for any $x_1 \in \mathbb{R}$. □

We proceed with the proof of Theorem A.1.

Proof of Theorem A.1. We begin by showing (i). Suppose that there is no $\lambda \in \mathbb{Q}^m$, $\lambda > 0$, such that $\lambda \cdot A = 0$ and $\lambda \cdot b > 0$. Define set $P := \{x \in \mathbb{R}^n : A \cdot x \geq b\}$, where A and b are specified as in the thesis of the theorem. Hence, $A \cdot x \geq b$, for some $x \in \mathbb{R}^n$ if and only if $x \in P$. Let $\pi^\ell : \mathbb{R}^\ell \to \mathbb{R}^{\ell-1}$ be a projection defined by $\pi^\ell(x_1, x_2, \ldots, x_\ell) = (x_2, \ldots, x_\ell)$, for some $\ell > 1$. Clearly, set P is non-empty if and only if set $\pi^2 \circ \pi^3 \circ \cdots \circ \pi^n(P) \subseteq \mathbb{R}$ is non-empty.

By Lemma A.1, there is a sequence of positive rational matrices $\{\Lambda_i\}_{i=2}^n$ such that

$$\pi^2 \circ \pi^3 \circ \cdots \circ \pi^n(P) = \Big\{x_n \in \mathbb{R} : (\Lambda_2 \cdots \Lambda_n \cdot A) \cdot (x_1, \ldots, x_n) \geq (\Lambda_2 \cdots \Lambda_n \cdot b), \text{ for all } (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}\Big\}.$$

Denote $C := (\Lambda_2 \cdots \Lambda_n \cdot A)$ and $d = (\Lambda_2 \cdots \Lambda_n \cdot b)$. Clearly, matrix C has n columns. Denote the number of rows of the matrix and the dimension of vector d by r. Note that matrix C is rational, with non-zero entries appearing only in the n'th column. We shall denote the entries of the matrix by c_{ij}, and the entries of vector d by d_i, where $i = 1, \ldots, r$ and $j = 1, \ldots, n$.

Define sets $G := \{i : c_{in} > 0\}$, $Z := \{i : c_{in} = 0\}$, and $L := \{i : c_{in} < 0\}$. Observe that, for any $i \in Z$, it must be that $d_i \leq 0$. Otherwise, there would exist a row of matrix $(\Lambda_2 \cdots \Lambda_n)$, denoted by λ, such that $\lambda \cdot A = 0$ and $\lambda \cdot b > 0$. Since λ is positive and rational, this would lead to a contradiction.

Set $\pi^2 \circ \pi^3 \circ \cdots \circ \pi^n(P)$ is non-empty if and only if $\min_{l \in L} d_l/c_{ln} \geq \max_{k \in G} d_k/c_{kn}$. Suppose that $\min_{l \in L} d_l/c_{ln} < \max_{k \in G} d_k/c_{kn}$. Take any $l \in L$ and $k \in K$ such that $d_l/c_{ln} < d_k/c_{kn}$ and construct a vector $\gamma \in \mathbb{Q}^r$ with the l'th and k'th entry equal to $1/|c_{ln}|$ and $1/|c_{kn}|$ respectively, while every other entry is equal to zero. Clearly, $\lambda := \gamma \cdot \Lambda_2 \cdots \Lambda_n \in \mathbb{Q}^m$ is a positive vector. Moreover, by definition, $\lambda \cdot A = 0$ while $\lambda \cdot b > 0$. Contradiction. Therefore, whenever there is no $\lambda \in \mathbb{Q}^m$, $\lambda > 0$, such that $\lambda \cdot A = 0$ and $\lambda \cdot b > 0$, there exists some $x \in \mathbb{R}^n_+$ such that $A \cdot x \geq b$.

In order to prove the second part of the result, suppose that there is some $\lambda \in \mathbb{Q}^m$, $\lambda > 0$, such that $\lambda \cdot A = 0$ and $\lambda \cdot b > 0$. Whenever there exists some $x \in \mathbb{R}^n$ such that $A \cdot x \geq b$, we have $(\lambda \cdot b) > (\lambda \cdot A) \cdot x \geq (\lambda \cdot b)$, which yields a contradiction. □

References

Afriat, S. N. (1972): "Efficiency estimation of production functions," *International Economic Review*, 13(3), 568–598.

Border, K. C. (1992): "Revealed preference, stochastic dominance, and the expected utility hypothesis," *Journal of Economic Theory*, 56, 20–42.

Brown, D. J., and C. Calsamiglia (2007): "The nonparametric approach to applied welfare analysis," *Economic Theory*, 31(1), 183–188.

Chambers, C., and F. Echenique (2009): "Profit maximization and supermodular technology," *Economic Theory*, 40(2), 173–183.

Dziewulski, P. (2014): "Revealed time-preference," MPRA Paper 56596, University Library of Munich, Germany.

Fourier, J. (1826): "Solution d'une question perticulière du calcul des inégalités," *Nouveau Bulletin des Sciences par la Sociéeté philomatique de Paris*, p. 317–319.

Kraft, C. H., J. W. Pratt, and A. Seidenberg (1959): "Intuitive probability on finite sets," *Annals of Mathematical Statistics*, 30, 408–419.

Meyer, M., and B. Strulovici (2013): "The supermodular stochastic ordering," Discussion Papers 1563, Northwestern University, Center for Mathematical Studies in Economics and Management Science.

Milgrom, P., and J. Roberts (1990): "The economics of modern manufacturing: Technology, strategy, and organization," *American Economic Review*, 80(3), 511–28.

——— (1994): "Complementarities and systems: Understanding japanese economic organization," *Estudios Económicos*, 9(1), 3–42.

——— (1995): "Complementarities and fit strategy, structure, and organizational change in manufacturing," *Journal of Accounting and Economics*, 19(2-3), 179–208.

Milgrom, P., and C. Shannon (1994): "Monotone comparative statics," *Econometrica*, 62(1), 157–180.

Motzkin, T. S. (1951): "Two consequences of the transposition theorem on linear inequalities," *Econometrica*, p. 184–185.

Rochet, J. C. (1987): "A necessary and sufficient condition for rationalizability in a quasilinear context," *Journal of Monetary Economics*, 16, 191–200.

Rockafellar, T. R. (1970): *Convex analysis*. Princeton University Press.

Sákovics, J. (2013): "Revealed cardinal preference," *Economic Theory Bulletin*, 1, 39–45.

Scott, D. (1964): "Measurement structures and linear inequalities," *Journal of Mathematical Psychology*, 1, 233–247.

Topkis, D. M. (1978): "Minimazing a submodular function on a lattice," *Operations Research*, 26(2), 305–321.

——— (1995): "Comparative statics of the firm," *Journal of Economic Theory*, 67, 370–401.

——— (1998): *Supermodularity and complementarity*, Frontiers of economic research. Princeton University Press.