

## EXISTENCE AND UNIQUENESS OF ARROW–DEBREU EQUILIBRIA WITH CONSUMPTIONS IN $\mathbf{L}_+^{0*}$

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(Translated by the author)

**Abstract.** We consider an economy where agents' consumption sets are given by the cone  $\mathbf{L}_+^0$  of nonnegative measurable functions and whose preferences are defined by additive utilities satisfying the Inada conditions. We extend to this setting the results of Dana [*J. Math. Econom.*, 22 (1993), pp. 563–579; *Econometrica*, 61 (1993), pp. 953–957] on the existence and uniqueness of Arrow–Debreu equilibria. In the case of existence, our conditions are necessary and sufficient.

**Key words.** Arrow–Debreu equilibrium, Pareto allocation

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**1. Main results.** This paper states necessary and sufficient conditions for the existence of (static) Arrow–Debreu equilibria in the economy where agents' consumption sets are given by the cone  $\mathbf{L}_+^0$  of nonnegative measurable functions and whose preferences are defined by additive utilities satisfying the Inada conditions. For completeness, a version of well-known uniqueness criteria is also provided. The results generalize those in [1], [2]; see Remarks 1.2 and 1.3 for details. They are used in [6] to obtain criteria for the existence of (dynamic) Radner equilibria in an economy with prefixed set of stocks.

Consider an economy with  $M \in \{1, 2, \dots\}$  agents and a state space  $(\mathbf{S}, \mathcal{S}, \mu)$  which is a measure space with a  $\sigma$ -finite measure  $\mu$ . By  $\mathbf{L}^0$  we denote the space of (equivalence classes of) measurable functions with values in  $[-\infty, \infty)$ ;  $\mathbf{L}_+^0$  stands for the cone of nonnegative measurable functions. For  $\alpha \in \mathbf{L}^0$  we use the convention

$$\mathbf{E}[\alpha] \triangleq \int_{\mathbf{S}} \alpha(s) \mu(ds) \triangleq -\infty \quad \text{if} \quad \mathbf{E}[\min(\alpha, 0)] = -\infty.$$

The consumption set of an  $m$ th agent equals  $\mathbf{L}_+^0$ , and this agent's preferences are defined by the additive utility

$$u_m(\alpha^m) \triangleq \mathbf{E}[U_m(\alpha^m)], \quad \alpha^m \in \mathbf{L}_+^0.$$

The utility measurable field  $U_m: [0, \infty) \rightarrow \mathbf{L}^0$  has the following properties.

**ASSUMPTION 1.1.** For every  $s \in \mathbf{S}$  the function  $U_m(\cdot)(s)$  on  $(0, \infty)$  is strictly increasing, strictly concave, continuously differentiable, and satisfies the Inada conditions

$$\lim_{c \rightarrow \infty} U'_m(c)(s) = 0, \quad \lim_{c \downarrow 0} U'_m(c)(s) = \infty.$$

At  $c = 0$ , by continuity,  $U_m(0)(s) = \lim_{c \downarrow 0} U_m(c)(s)$ ; this limit may be  $-\infty$ .

**Remark 1.1.** This framework readily accommodates consumption preferences which are common in mathematical finance. For instance, it includes a continuous-time model defined on a filtered probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbf{P})$ , where the agents consume according to an optional nonnegative process  $\alpha = (\alpha_t)$  of consumption rates, so that  $C_t = \int_0^t \alpha_v dv$  is the

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cumulative consumption up to the time  $t$ . In this case, the state space  $\mathbf{S} = \Omega \times [0, \infty)$ ,  $\mathcal{S}$  is the optional  $\sigma$ -algebra, and  $\mu(ds) = \mu(d\omega, dt) = \mathbf{P}[d\omega] \times dt$ .

By  $\Lambda \in \mathbf{L}_+^0$  we denote the total endowment in the economy. A family  $(\alpha^m)_{m=1, \dots, M}$  of elements of  $\mathbf{L}_+^0$  such that  $\sum_{m=1}^M \alpha^m = \Lambda$  is called an *allocation* of  $\Lambda$ . By  $(\alpha_0^m) \subset \mathbf{L}_+^0$  we denote the *initial allocation* of  $\Lambda$  between the agents.

DEFINITION 1.1. A pair  $(\zeta, (\alpha_1^m)_{m=1, \dots, M})$ , consisting of a measurable function  $\zeta > 0$  (a state price density) and an allocation  $(\alpha_1^m)$  of  $\Lambda$  (an equilibrium allocation) is an Arrow-Debreu equilibrium if

$$\mathbf{E}[\zeta \Lambda] < \infty$$

and, for  $m = 1, \dots, M$ ,

$$(1.1) \quad \begin{aligned} &\mathbf{E}[|U_m(\alpha_1^m)|] < \infty, \\ &\alpha_1^m = \arg \max \{ \mathbf{E}[U_m(\alpha)] : \alpha \in \mathbf{L}_+^0, \mathbf{E}[\zeta(\alpha - \alpha_0^m)] = 0 \}. \end{aligned}$$

This is the main result of the paper.

THEOREM 1.1. Suppose that Assumption 1.1 holds, the total endowment  $\Lambda > 0$ , and the initial allocation  $\alpha_0^m \neq 0$ ,  $m = 1, \dots, M$ . Then an Arrow-Debreu equilibrium exists if and only if there is an allocation  $(\alpha^m)$  of  $\Lambda$  such that

$$(1.2) \quad \mathbf{E}[|U_m(\alpha^m)| + U'_m(\alpha^m)\Lambda] < \infty, \quad m = 1, \dots, M.$$

COROLLARY 1.1. Suppose that Assumption 1.1 holds, the total endowment  $\Lambda > 0$ , the initial allocation  $\alpha_0^m \neq 0$ ,  $m = 1, \dots, M$ , and

$$\mathbf{E}[|U_m(s\Lambda)|] < \infty, \quad s \in (0, 1], \quad m = 1, \dots, M.$$

Then an Arrow-Debreu equilibrium exists.

*Proof.* The concavity of  $U_m$  implies that

$$cU'_m(c) \leq 2\left(U_m(c) - U_m\left(\frac{c}{2}\right)\right), \quad c > 0,$$

and the result follows from Theorem 1.1, where one can take  $\alpha^m = \Lambda/M$ .

The Arrow-Debreu equilibrium is, in general, not unique; see, e.g., [7, Example 15.B.2] which can be easily adapted to our setting. We state a version of the well-known uniqueness criteria.

THEOREM 1.2. Suppose that the conditions of Corollary 1.1 hold and there is an index  $m_0$  such that for  $m \neq m_0$

$$(1.3) \quad \text{the map } c \mapsto cU'_m(c) \text{ of } (0, \infty) \text{ to } \mathbf{L}_+^0 \text{ is nondecreasing.}$$

Then an Arrow-Debreu equilibrium exists and is unique.

The proofs of Theorems 1.1 and 1.2 are given in section 3 and rely on the finite-dimensional characterization of Arrow-Debreu equilibria in Theorem 2.2. Note that all ingredients of the proofs are fairly standard and have already appeared in some form previously.

Remark 1.2. Being necessary and sufficient, condition (1.2) generalizes to the case of  $\mathbf{L}_+^0$ -valued consumptions the criteria in [1], [2]. These papers require the existence of conjugate exponents  $p, q \in [1, \infty]$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ , and measurable functions  $\zeta \in \mathbf{L}_+^q$  and  $\eta \in \mathbf{L}_+^1$  such that  $\Lambda \in \mathbf{L}_+^p$  and the measurable fields  $(U_m)$  are twice continuously differentiable and satisfy

$$U_m(c) \leq \zeta c + \eta, \quad c > 0, \quad m = 1, \dots, M,$$

and

$$\sup_{w \in \Sigma^M} U_c(w, \Lambda) \Lambda \in \mathbf{L}_+^q.$$

Here  $U(w, c)$  is an aggregate utility measurable field (see (2.2) below), and  $\Sigma^M$  is the simplex in  $\mathbf{R}^M$ . The consumption set is restricted to  $\mathbf{L}_+^p$ .

Note that textbook versions of the existence results usually require the total endowment to be uniformly bounded below:

$$\Lambda \geq \text{const.} > 0$$

(see, e.g., [5, Condition 4.2.1], [3, Theorem 7.2.8], and [4, section 10.G]).

**Remark 1.3.** If the utility measurable fields  $(U_m)$  are twice differentiable, then the key condition (1.3) is equivalent to the boundedness by 1 of their relative risk-aversions  $-cU_m''(c)/U_m'(c)$  and, in this form, is well known (see, e.g., [7, Example 17.F.2 and Proposition 17.F.3], Theorem 4.6.1 in [5], and [1], [2]). Note that, contrary to the above references, we allow this condition to fail for one economic agent.

**2. Pareto optimal allocations.** In this section we state the properties of Pareto optimal allocations needed for the proofs of Theorems 1.1 and 1.2.

**DEFINITION 2.1.** An allocation  $(\alpha^m)$  of  $\Lambda$  is Pareto optimal if

$$(2.1) \quad \mathbf{E}[|U_m(\alpha^m)|] < \infty, \quad m = 1, \dots, M,$$

and there is no allocation  $(\beta^m)$  of  $\Lambda$  which dominates  $(\alpha^m)$  in the sense that

$$\begin{aligned} \mathbf{E}[U_m(\beta^m)] &\geq \mathbf{E}[U_m(\alpha^m)], \quad m = 1, \dots, M, \\ \mathbf{E}[U_l(\beta^l)] &> \mathbf{E}[U_l(\alpha^l)] \quad \text{for some } l \in \{1, \dots, M\}. \end{aligned}$$

See Remark 2.1 for a justification of the integrability condition (2.1).

As usual, in the study of complete equilibria, an important role is played by the *aggregate utility measurable field*  $U = U(w, c): \mathbf{R}_+^M / \{0\} \times [0, \infty) \rightarrow \mathbf{L}^0$ :

$$(2.2) \quad U(w, c) \triangleq \sup \left\{ \sum_{m=1}^M w^m U_m(c^m) : c^m \geq 0, c^1 + \dots + c^M = c \right\}.$$

Here  $\mathbf{R}_+^M / \{0\}$  is the nonnegative orthant in  $\mathbf{R}^M$  without the origin:

$$\mathbf{R}_+^M \setminus \{0\} \triangleq \left\{ w \in [0, \infty)^M : \sum_{m=1}^M w^m > 0 \right\}.$$

Due to the 1-homogeneity property of  $U = U(w, c)$  with respect to the weight  $w$ :  $U(yw, c) = yU(w, c)$ ,  $y > 0$ , it is often convenient to restrict its  $w$ -domain to the simplex

$$\Sigma^M \triangleq \left\{ w \in [0, \infty)^M : \sum_{m=1}^M w^m = 1 \right\}.$$

Elementary arguments show that for every  $w \in \mathbf{R}_+^M / \{0\}$  the measurable field  $U(w, \cdot)$  on  $[0, \infty)$  satisfies Assumption 1.1 and that the upper bound in (2.2) is attained at the measurable functions  $\pi^m(w, c)$ ,  $m = 1, \dots, M$ , such that

$$(2.3) \quad w^m U_m'(\pi^m(w, c)) = U_c(w, c) \triangleq \frac{\partial}{\partial c} U(w, c) \quad \text{if } w^m > 0, c > 0,$$

$$(2.4) \quad \pi^m(w, c) = 0 \quad \text{if } w^m = 0 \text{ or } c = 0.$$

These identities readily imply that the measurable fields  $\pi^m: \mathbf{R}_+^M / \{0\} \times [0, \infty) \rightarrow \mathbf{L}_+^0$ ,  $m = 1, \dots, M$ , are continuous and have the following properties:

- (A1) They are 0-homogeneous with respect to  $w$ :  $\pi^m(yw, c) = \pi^m(w, c)$ ,  $y > 0$ ;
- (A2) for  $c > 0$  the measurable field  $\pi^m(\cdot, c) = \pi^m(w^1, \dots, w^M, c)$  is strictly increasing with respect to  $w^m$  and strictly decreasing with respect to  $w^l$ ,  $l \neq m$ ;
- (A3) if  $c > 0$  and  $w_1$  and  $w_2$  are distinct vectors in  $\Sigma^M$ , then for every  $s \in \mathbf{S}$  the vectors  $(\pi^m(w_1, c)(s))$  and  $(\pi^m(w_2, c)(s))$  in  $c\Sigma^M$  are distinct.

We state a version of the well-known parametrization of Pareto optimal allocations by the elements of  $\Sigma^M$ .

**THEOREM 2.1.** *Suppose that Assumption 1.1 holds. Let  $\Lambda > 0$  and let  $(\alpha^m)$  be an allocation of  $\Lambda$  satisfying (2.1). Then  $(\alpha^m)$  is Pareto optimal if and only if there is  $w \in \Sigma^M$  such that*

$$(2.5) \quad \alpha^m = \pi^m(w, \Lambda), \quad m = 1, \dots, M.$$

*Proof.* Assume first that  $(\alpha^m)$  is Pareto optimal. By concavity of the measurable fields  $(U_m)$ , the set

$$C \triangleq \{z \in \mathbf{R}^M : z^m \leq \mathbf{E}[U_m(\beta^m)] \text{ for some allocation } (\beta^m) \text{ of } \Lambda\}$$

is convex and, in view of (2.1), it has a nonempty interior. From the Pareto optimality of  $(\alpha^m)$  we deduce that the point

$$\hat{z}^m \triangleq \mathbf{E}[U_m(\alpha^m)], \quad m = 1, \dots, M,$$

belongs to the boundary of  $C$ . Hence, there is a nonzero  $w \in \mathbf{R}^M$  such that

$$(2.6) \quad \sum_{m=1}^M w^m \hat{z}^m \geq \sum_{m=1}^M w^m z^m, \quad z \in C.$$

Since  $C - [0, \infty)^M = C$ , we obtain that  $w^m \geq 0$ . Then we can normalize  $w$  to be in  $\Sigma^M$ . Observe now that (2.6) can be written as

$$\begin{aligned} \mathbf{E} \left[ \sum_{m=1}^M w^m U_m(\alpha^m) \right] &= \sup \left\{ \mathbf{E} \left[ \sum_{m=1}^M w^m U_m(\beta^m) \right] : (\beta^m) \text{ is an allocation of } \Lambda \right\} \\ &= \mathbf{E}[U(w, \Lambda)], \end{aligned}$$

which readily implies (2.5).

Conversely, if  $(\alpha^m)$  is given by (2.5), then for any allocation  $(\beta^m)$  of  $\Lambda$

$$\sum_{m=1}^M w^m U_m(\beta^m) \leq U(w, \Lambda) = \sum_{m=1}^M w^m U_m(\alpha^m),$$

which yields the Pareto optimality of  $(\alpha^m)$  after we account for the integrability condition (2.1) and recall that  $\alpha^m \triangleq \pi^m(w, \Lambda) = 0$  if  $w^m = 0$ . Theorem 2.1 is proved.

**Remark 2.1.** Without the integrability condition (2.1) in Definition 2.1 the assertion of Theorem 2.1 does not hold. As a counterexample, take  $M = 2$  and select the total endowment and the utility functions so that  $\Lambda > 0$ ,  $U_1 = U_2$ , and

$$\mathbf{E}[|U_i(\Lambda)|] < \infty \quad \text{and} \quad \mathbf{E} \left[ U_i \left( \frac{\Lambda}{2} \right) \right] = -\infty.$$

Then the allocation  $(\pi_1(1/2, \Lambda), \pi_2(1/2, \Lambda)) = (\Lambda/2, \Lambda/2)$  is dominated by  $(0, \Lambda)$  and  $(\Lambda, 0)$ .

After these preparations, we are ready to state the main result of this section.

**THEOREM 2.2.** *Suppose that Assumption 1.1 holds, the total endowment  $\Lambda > 0$ , and the initial allocation  $\alpha_0^m \neq 0$ ,  $m = 1, \dots, M$ . Then  $(\zeta, (\alpha_1^m))$  is an Arrow-Debreu equilibrium if and only if there is  $w \in \text{int } \Sigma^M$ , the interior of  $\Sigma^M$ , and a constant  $z > 0$  such that*

$$(2.7) \quad \zeta = zU_c(w, \Lambda),$$

$$(2.8) \quad \alpha_1^m = \pi^m(w, \Lambda),$$

and

$$(2.9) \quad \mathbf{E} \left[ \sum_{m=1}^M |U_m(\pi^m(w, \Lambda))| + U_c(w, \Lambda)\Lambda \right] < \infty,$$

$$(2.10) \quad \mathbf{E}[U_c(w, \Lambda)(\pi^m(w, \Lambda) - \alpha_0^m)] = 0.$$

In this case, the equilibrium allocation  $(\alpha_1^m)$  is Pareto optimal.

*Remark 2.2.* As Lemma 3.1 below shows, the integrability condition (2.9) holds for some  $w \in \text{int } \Sigma^M$  if and only if it holds for every  $w \in \text{int } \Sigma^M$  and is equivalent to the existence of an allocation  $(\alpha^m)$  of  $\Lambda$  satisfying (1.2).

For the proof of the theorem we need the following lemma.

LEMMA 2.1. Let  $U: [0, \infty) \rightarrow \mathbf{L}^0$  be a measurable field satisfying Assumption 1.1, and let  $\alpha > 0$  and  $\zeta > 0$  be measurable functions such that  $\mathbf{E}[\zeta\alpha] < \infty$ , and let

$$-\infty < \mathbf{E}[U(\alpha)] = \sup\{\mathbf{E}[U(\beta)]: \beta \in \mathbf{L}_+^0, \mathbf{E}[\zeta(\beta - \alpha)] = 0\} < \infty.$$

Then

$$(2.11) \quad \zeta = \frac{\mathbf{E}[\zeta\alpha]}{\mathbf{E}[U'(\alpha)\alpha]} U'(\alpha).$$

*Proof.* Let  $\beta$  be a measurable function such that

$$0 < \beta < \alpha \quad \text{and} \quad \mathbf{E}[U(\alpha - \beta)] > -\infty.$$

For instance, we can choose  $\beta$  so that

$$U(\alpha) - U(\alpha - \beta) = \min(\eta, U(2\alpha) - U(\alpha)),$$

where the measurable function  $\eta > 0$  and  $\mathbf{E}[\eta] < \infty$ . Observe that

$$(2.12) \quad \mathbf{E}[\zeta\beta] < \infty \quad \text{and} \quad \mathbf{E}[U'(\alpha)\beta] < \infty,$$

where the second bound holds because

$$U'(\alpha)\beta \leq U(\alpha) - U(\alpha - \beta).$$

Take a measurable function  $\gamma$  such that

$$|\gamma| \leq \beta \quad \text{and} \quad \mathbf{E}[\zeta\gamma] = 0.$$

From the properties of  $U$  in Assumption 1.1 we deduce that for  $|y| < 1$

$$\begin{aligned} U'\left(\alpha + y \frac{\gamma}{2}\right) \frac{|\gamma|}{2} &\leq U'\left(\alpha - \frac{\beta}{2}\right) \frac{\beta}{2} \\ &\leq U\left(\alpha - \frac{\beta}{2}\right) - U(\alpha - \beta) \leq U(\alpha) - U(\alpha - \beta). \end{aligned}$$

Accounting for the integrability of  $U(\alpha)$  and  $U(\alpha - \beta)$  we obtain that the function

$$f(y) \triangleq \mathbf{E}\left[U\left(\alpha + y \frac{\gamma}{2}\right)\right], \quad y \in (-1, 1),$$

is continuously differentiable and

$$f'(y) = \mathbf{E}\left[U'\left(\alpha + y \frac{\gamma}{2}\right) \frac{\gamma}{2}\right], \quad y \in (-1, 1).$$

As  $\alpha$  is optimal,  $f$  attains its maximum at  $y = 0$ . Therefore,

$$f'(0) = \mathbf{E}[U'(\alpha)\gamma] = 0.$$

Thus, we have found a measurable function  $\beta > 0$  such that (2.12) holds and

$$(2.13) \quad \forall \gamma \in \mathbf{L}^0: \quad |\gamma| \leq \beta \quad \text{and} \quad \mathbf{E}[\zeta\gamma] = 0 \quad \Rightarrow \quad \mathbf{E}[U'(\alpha)\gamma] = 0.$$

This readily implies (2.11). Indeed, if we define the probability measures  $\mathbf{P}$  and  $\mathbf{Q}$  as

$$\frac{d\mathbf{P}}{d\mu} \triangleq \frac{\zeta\beta}{\mathbf{E}[\zeta\beta]} \quad \text{and} \quad \frac{d\mathbf{Q}}{d\mu} \triangleq \frac{U'(\alpha)\beta}{\mathbf{E}[U'(\alpha)\beta]},$$

then for a set  $A \in \mathcal{S}$  the implication (2.13) with  $\gamma = \beta(1_A - \mathbf{P}[A])$  yields that  $\mathbf{P}[A] = \mathbf{Q}[A]$ . Hence,  $\mathbf{P} = \mathbf{Q}$ , and the result follows. Lemma 2.1 is proved.

*Proof of Theorem 2.2.* Assume first that  $(\zeta, (\alpha_1^m))$  is an Arrow-Debreu equilibrium. We claim that  $(\alpha_1^m)$  is a Pareto optimal allocation of  $\Lambda$ . Indeed, if there is an allocation  $(\beta^m)$  of  $\Lambda$  which dominates  $(\alpha_1^m)$  in the sense of Definition 2.1, then for some  $l \in \{1, \dots, M\}$  either

$$\mathbf{E}[U_l(\beta^l)] \geq \mathbf{E}[U_l(\alpha_1^l)] \quad \text{and} \quad \mathbf{E}[\zeta\beta^l] < \mathbf{E}[\zeta\alpha_1^l]$$

or

$$\mathbf{E}[U_l(\beta^l)] > \mathbf{E}[U_l(\alpha_1^l)] \quad \text{and} \quad \mathbf{E}[\zeta\beta^l] \leq \mathbf{E}[\zeta\alpha_1^l],$$

contradicting the optimality of  $\alpha_1^l$  in (1.1).

From Theorem 2.1 we then obtain the representation (2.8) for  $(\alpha_1^m)$  with  $w \in \Sigma^M$ . As  $\alpha_0^m \neq 0$ , we have  $\alpha_1^m \neq 0$ . Hence,  $w \in \text{int } \Sigma^M$  and  $\alpha_1^m = \pi^m(w, \Lambda) > 0$ . Lemma 2.1 and the identities (2.3) now imply (2.7). Finally, properties (2.9) and (2.10) are parts of Definition 1.1.

Conversely, let  $w \in \text{int } \Sigma^M$  be such that the conditions (2.9) and (2.10) hold, and define  $(\zeta, (\alpha_1^m))$  using (2.7) and (2.8). Except for (1.1), all conditions of Definition 1.1 hold trivially. Property (1.1) follows from the inequalities

$$\begin{aligned} U_m(\beta^m) - \frac{1}{w^m} U_c(w, \Lambda) \beta^m &\leq \sup_{c \geq 0} \left\{ U_m(c) - \frac{1}{w^m} U_c(w, \Lambda) c \right\} \\ &= U_m(\pi^m(w, \Lambda)) - \frac{1}{w^m} U_c(w, \Lambda) \pi^m(w, \Lambda), \end{aligned}$$

where  $\beta^m \in \mathbf{L}_+^0$  and, at the last step, we used the definition of  $\pi^m$  in (2.3).

### 3. Proofs of Theorems 1.1 and 1.2.

We begin with some lemmas.

LEMMA 3.1. Suppose that Assumption 1.1 holds, the total endowment  $\Lambda > 0$ , and there is an allocation  $(\alpha^m)$  of  $\Lambda$  satisfying (1.2). Then

$$(3.1) \quad \mathbf{E}[U_m(\pi^m(w, \Lambda))] < \infty, \quad w \in \text{int } \Sigma^M, \quad m = 1, \dots, M,$$

$$(3.2) \quad \mathbf{E} \left[ \sup_{w \in \Sigma^M} U_c(w, \Lambda) \Lambda \right] < \infty.$$

*Proof.* First, take  $w \in \text{int } \Sigma^M$ . From the monotonicity and the concavity of  $U_m$  we deduce that

$$U_m(\pi^m(w, \Lambda)) \leq U_m(\Lambda) \leq U_m(\alpha^m) + U'_m(\alpha^m) \Lambda,$$

and from the definition of  $\pi^m$  that

$$\sum_{m=1}^M w^m U_m(\alpha^m) \leq U(w, \Lambda) = \sum_{m=1}^M w^m U_m(\pi^m(w, \Lambda)).$$

These inequalities readily imply (3.1).

Assume now that  $w \in \Sigma^M$ . Since

$$0 < \Lambda = \sum_{m=1}^M \pi^m(w, \Lambda) = \sum_{m=1}^M \alpha^m,$$

for every  $s \in \mathbf{S}$  there is an index  $m(s)$  such that

$$0 < \pi^{m(s)}(w, \Lambda)(s) \quad \text{and} \quad \alpha^{m(s)}(s) \leq \pi^{m(s)}(w, \Lambda)(s).$$

Accounting for (2.3) and (2.4) we deduce that

$$U_c(w, \Lambda) \leq \max\{w^m U'_m(\alpha^m) : w^m > 0\} \leq \sum_{m=1}^M U'_m(\alpha^m),$$

which implies (3.2).

The following corollary of the Brouwer's fixed point theorem plays a key role. It is well known; see, e.g., [3, Theorem 6.3.6].

LEMMA 3.2. Let  $f_m: \Sigma^M \rightarrow \mathbf{R}$ ,  $m = 1, \dots, M$ , be continuous functions such that

$$(3.3) \quad f_m(w) < 0 \quad \text{if } w^m = 0,$$

$$(3.4) \quad \sum_{m=1}^M f_m(w) = 0, \quad w \in \Sigma^M.$$

Then there is  $\hat{w} \in \text{int } \Sigma^M$  such that

$$(3.5) \quad f_m(\hat{w}) = 0, \quad m = 1, \dots, M.$$

*Proof.* Since

$$g_m(w) \triangleq \frac{w^m + \max(0, -f_m(w))}{1 + \sum_{n=1}^M \max(0, -f_n(w))}, \quad m = 1, \dots, M,$$

are continuous maps of  $\Sigma^M$  into  $\Sigma^M$ , the Brouwer's fixed point theorem implies the existence of  $\hat{w} \in \Sigma^M$  such that

$$\hat{w}^m = g_m(\hat{w}) = \frac{\hat{w}^m + \max(0, -f_m(\hat{w}))}{1 + \sum_{n=1}^M \max(0, -f_n(\hat{w}))}, \quad m = 1, \dots, M.$$

In view of (3.3), we obtain that  $\hat{w} \in \text{int } \Sigma^M$ . From (3.4) we deduce the existence of an index  $l$  such that  $f_l(\hat{w}) \geq 0$ . Then

$$0 < \hat{w}^l = \frac{\hat{w}^l}{1 + \sum_{m=1}^M \max(0, -f_m(\hat{w}))},$$

implying that  $f_m(\hat{w}) \geq 0$ ,  $m = 1, \dots, M$ . Another use of (3.4) yields the result.

*Proof of Theorem 1.1.* The “only if” part follows directly from Theorem 2.2, which, in particular, implies that the inequality (1.2) holds for any equilibrium allocation  $(\alpha_1^m)$ .

Conversely, assume that (1.2) holds for some allocation  $(\alpha^m)$  of  $\Lambda$ . From Lemma 3.1 we obtain the bound (3.2). The dominated convergence theorem and the continuity of the measurable fields  $\pi^m(\cdot, \Lambda)$  and  $U_c(\cdot, \Lambda)$  on  $\Sigma^M$  imply the continuity of the functions

$$f_m(w) \triangleq \mathbf{E}[U_c(w, \Lambda)(\pi^m(w, \Lambda) - \alpha_0^m)], \quad w \in \Sigma^M.$$

In view of (2.4) and as  $\alpha_0^m \neq 0$ , the functions  $(f_m)$  satisfy (3.3), and since

$$\sum_{m=1}^M \alpha_0^m = \sum_{m=1}^M \pi^m(w, \Lambda) = \Lambda,$$

they also satisfy (3.4).

Lemma 3.2 then implies the existence of  $\hat{w} \in \text{int } \Sigma^M$  such that (3.5) holds. The result now follows from Theorem 2.2, where the integrability condition (2.9) for  $w = \hat{w}$  holds because of Lemma 3.1. Theorem 1.1 is proved.

*Proof of Theorem 1.2.* The existence follows from Corollary 1.1. To verify the uniqueness we define the *excess utility* functions

$$\begin{aligned} h_m(w) &\triangleq \mathbf{E}[U'_m(\pi^m(w, \Lambda))(\pi^m(w, \Lambda) - \alpha_0^m)] \\ &= \frac{1}{w^m} \mathbf{E}[U_c(w, \Lambda)(\pi^m(w, \Lambda) - \alpha_0^m)], \quad w \in (0, \infty)^M. \end{aligned}$$

By Lemma 3.1, these functions are well defined and finite. The condition (1.3), the properties (A1) and (A2) for  $\pi^m = \pi^m(w, c)$ , and the fact that the stochastic field  $U'_m = U'_m(c)$  is strictly decreasing imply that

(B1) for every  $m = 1, \dots, M$ , the function  $h_m$  is 0-homogeneous:  $h_m(yw) = h_m(w)$ ,  $w \in (0, \infty)^M$ ,  $y > 0$ ;

(B2) for  $m \neq m_0$ , where  $m_0$  is the index for which (1.3) may not hold, the function  $h_m$  has the *gross-substitute* property:  $h_m = h_m(w^1, \dots, w^M)$  is strictly decreasing with respect to  $w^l$ ,  $l \neq m$ .

Theorem 2.2 and the property (A3) for  $\pi^m = \pi^m(w, c)$  yield the uniqueness if we can show that the equation

$$h_m(w) = 0, \quad m = 1, \dots, M,$$

has no multiple solutions on  $\text{int } \Sigma^M$ . Let  $w_1$  and  $w_2$  be two such solutions,  $w_1 \neq w_2$ . Assume that  $w_1^{m_0} \leq w_2^{m_0}$ ; of course, this does not restrict any generality. Then

$$y \triangleq \max_{m=1, \dots, M} \frac{w_1^m}{w_2^m} > 1,$$

and there is an index  $l_0 \neq m_0$  such that  $yw_2^{l_0} = w_1^{l_0}$ . From (B1) we obtain that

$$h_m(yw_2) = h_m(w_2) = 0, \quad m = 1, \dots, M,$$

while from the gross-substitute property in (B2) we deduce that

$$h_{l_0}(yw_2) < h_{l_0}(w_1) = 0.$$

This contradiction concludes the proof of Theorem 1.2.

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