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On the Bohm criterion in the presence of a magnetic field

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ABSTRACT

The present paper deals with a particular case involving a magnetic field, namely that of a positive column situated in an axial magnetic field, and it is shown that the Bohm criterion still holds for the normal component of the ion velocity at the plasma boundary. This result is of considerable interest because the Boltzmann relation for the electron density has not been assumed. It can be recalled that the criterion was originally derived by Bohm using the Boltzmann relation for the electron density. In the present case a Boltzm ann gradient conditionhas been found at the plasma boundary which leads to the usual Bohm velocity. Some insight can be gained by considering the ion and electron densities plotted against potential, for both the plasma and the sheath regions. It would appear that the result is a general one applicable to other situations.

1. Introduction

A considerable literature exists on the boundary of a plasma, see the review article by Riemann [1], but it is largely concerned with plasmas in the absence of a magnetic field. The Bohm criterion [2], which states that the ion velocity at the sheath edge is equal to or greater than the ion sound speed, is valid in that case. The criterion is derived using a two-scale model in which the Debye distance is infinitesimally small compared with the width of the plasma. This model yields an excellent approximation for the ion velocity on entering the sheath. It is of considerable interest to know whether it is still valid in the presence of a magnetic field, since the electron energy distribution is no longer Maxwellian in the presence of an appreciable electron current. The derivation of the criterion depends on the assumption that the electron density is described by the Boltzmann relation. Early work on the plasma-sheath transition in the presence of a magnetic field was carried out by Allen, Boschi and Magistrelli at Frascati [3,4]. The case considered was the pinch effect, the work involving both experiment and theory. The experimental work was carried out using a low pressure mercury arc, following pioneering work by Thonemann and Cowhig [5]. The theory has been reviewed elsewhere [6] and the conclusion reached was that the Bohm criterion was still valid, even though the electrons were not assumed to have a Maxwellian distribution of velocities.

The present paper deals with a particular case, that of a positive column situated in an axial magnetic field. Interest in this classical problem [7] was renewed by the work of Sternberg et al. [8]. These authors found that the velocity of positive ions at the plasma boundary was the ion acoustic velocity, as predicted by Bohm [2], but rather surprisingly did not comment on this fact. The subject was taken up by Allen [6] who pointed out that Sternberg et al. had omitted certain terms in the momentum equations for both electrons and ions. The principal result in this second paper was that the Bohm criterion was satisfied and that the quantity $\mathrm{d}n/\mathrm{d}\phi$ at the plasma boundary corresponded to that given by the Boltzmann relation. The paper, however, did not contain numerical solutions for the radial variations of electron density and the radial and azimuthal velocities of both electrons and ions. Furthermore the variation in the magnetic field strength due to the azimuthal currents had been neglected. The present paper contains a complete description of the plasma column, including the diamagnetic effect due to the azimuthal electron currents. The principal result, however, is that the Bohm criterion is satisfied in this case. The quantity $\mathrm{d}n/\mathrm{d}\phi$ approaches the Boltzmann value at the plasma boundary.

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2. Positive column in an axial magnetic field

In the presence of an axial magnetic field an appreciable electron current is set up in the azimuthal direction. This is due to the Lorentz force given by the (small) radial electron current flowing across the magnetic field. Early work on magnetic field effects on a plasma column has been summarized by Franklin [7] and a more recent treatment is that of Sternberg et al. [8]. In what follows the parameter range considered by the latter authors is considered, but the previously missing terms in the acceleration, pointed out by Allen [6], have been included. In addition the radial variation in the magnetic field due to the azimuthal current has been included; this had not been taken into account by either Sternberg or Allen. The model assumes both quasineutrality and cold ions. The latter assumption was introduced by Woods [9], who employed a technique which ensured that the ion momentum equation was satisfied. The application in his case was to the propagation of ion acoustic waves in a positive column.

The radial ion continuity equation for this model is

$$\frac{\mathrm{d}}{\mathrm{d}r}(rnv_r) = rZn \tag{1}$$

where n and v_r are the plasma density and radial velocity and Z is the electron ionization frequency The momentum equations for the ions, which carry a single positive elementary charge e, are

$$m_i n\left(v_r\frac{\mathrm{d}v_r}{\mathrm{d}r} - \frac{v_{i\theta}^2}{r}\right) + en\frac{\mathrm{d}\phi}{\mathrm{d}r} + m_i Znv_r - env_{i\theta}B + nF_{ir} = 0 \tag{2}$$

and

$$m_i n\left(v_r\frac{\mathrm{d}v_{i\theta}}{\mathrm{d}r} + \frac{v_r v_{i\theta}}{r}\right) + m_i Znv_{i\theta} + env_r B + F_{i\theta} = 0 \tag{3}$$

for the radial and azimuthal directions respectively, where B is the magnetic field strength, ϕ is the electrostatic potential, and m_i, $v_{i\theta}$ and F_i are the ion mass, azimuthal velocity and friction forces. The momentum equations for the electrons are

$$m_e n\left(v_r\frac{\mathrm{d}v_r}{\mathrm{d}r} - \frac{v_{e\theta}^2}{r}\right) + k_B T_e\frac{\mathrm{d}n}{\mathrm{d}r} - en\frac{\mathrm{d}\phi}{\mathrm{d}r} + m_e Znv_r + env_{e\theta}B + nF_{er} = 0 \tag{4}$$

and

$$m_e n\left(v_r\frac{\mathrm{d}v_{e\theta}}{\mathrm{d}r} + \frac{v_r v_{e\theta}}{r}\right) + m_e Znv_{e\theta} - env_r B + F_{e\theta} = 0 \tag{5}$$

where electron quantities corresponding to the previously-described ion quantities are denoted with a subscripted e, k_B is Boltzmann's constant and T_e is the electron temperature. Finally the variation in magnetic field is accounted for by including Ampère's law,

$$\frac{\mathrm{d}B}{\mathrm{d}r} = -\mu_0 en(u_{i\theta} - u_{e\theta}), \tag{6}$$

where μ_0 is the permeability of free space.

The ion sound speed, $c_s = (k_B T_e/m_i)^{1/2}$, outer plasma radius a, on-axis plasma density n_0 and ion cyclotron frequency, $\omega_{ci} = eB/m_i$ are introduced so that Eqs. (1)–(6) can be written in terms of the dimensionless variables

$$R = \frac{r}{a}, \quad U = \frac{v}{c_s}, \quad N = \frac{n}{n_0}, \quad \Phi = \frac{e\phi}{k_B T_e},$$
$$S = \frac{aZ}{c_s}, \quad f = \frac{aF}{m_i c_s^2}, \quad b_i = \frac{a\omega_{ci}}{c_s}. \tag{7}$$

Eqs. (1)–(6) are then rearranged under the assumptions that $m_i \gg m_e$ and the electron friction force F_{er} is negligible, yielding

$$\frac{\mathrm{d}U_R}{\mathrm{d}R} = S\left(\frac{1+U_R^2}{1-U_R^2}\right) + \frac{U_R}{1-U_R^2}\left[b_i(U_{e\theta} - U_{i\theta}) - \frac{1+U_{i\theta}^2 + m_e U_{e\theta}^2/m_i}{R} + f_{ir}\right], \tag{8}$$

$$\frac{\mathrm{d}U_{i\theta}}{\mathrm{d}R} = -\frac{SU_{i\theta}}{U_R} - b_i - \frac{U_{i\theta}}{R} - \frac{f_{i\theta}}{U_R}, \tag{9}$$

$$\frac{\mathrm{d}U_{e\theta}}{\mathrm{d}R} = -\frac{SU_{e\theta}}{U_R} + \frac{m_i}{m_e}b_i - \frac{U_{e\theta}}{R} - \frac{f_{e\theta}}{U_R}, \tag{10}$$

$$\frac{\mathrm{d}N}{\mathrm{d}R} = \frac{-N}{1-U_R^2}\left[2U_R S + b_i(U_{e\theta} - U_{i\theta}) - \frac{U_R^2 + U_{i\theta}^2 + m_e U_{e\theta}^2/m_i}{R} + f_{ir}\right], \tag{11}$$

$$\frac{\mathrm{d}\Phi}{\mathrm{d}R} = \frac{-1}{1-U_R^2}\left[2U_R S + b_i(U_R^2 U_{e\theta} - U_{i\theta}) - \frac{U_R^2 + U_{i\theta}^2 + m_e U_R^2 U_{e\theta}^2/m_i}{R} + f_{ir}\right], \tag{12}$$

$$\frac{\mathrm{d}b_i}{\mathrm{d}R} = \frac{\mu_0 e^2 a^2 n_0}{m_i}N(U_{e\theta} - U_{i\theta}). \tag{13}$$

These equations differ from those used in [8] as the complete azimuthal acceleration terms are used, the $m_e U_{e\theta}^2/m_i$ terms have been retained, and Ampère's law is included.

The equations are completed with the dimensionless friction terms, given by

$$f_{ir} = \frac{\pi}{2}\alpha_i U_R\sqrt{U_R^2 + U_{i\theta}^2},$$
$$f_{i\theta} = \frac{\pi}{2}\alpha_i U_{i\theta}\sqrt{U_R^2 + U_{i\theta}^2},$$
$$f_{e\theta} = \alpha_e U_{e\theta}, \tag{14}$$

where the ion and electron collision parameters are, in terms of their mean free paths λ,

$$\alpha_i = \frac{a}{\lambda_i} \quad \text{and} \quad \alpha_e = \frac{a}{\lambda_e}\sqrt{\frac{m_i}{m_e}}. \tag{15}$$

It is immediately seen that this system of equations has two singular points: a regular singular point at the origin, which is a consequence of the cylindrical coordinates, and an irregular singular point when $U_R = 1$; this is identified as the plasma boundary in the context of the two-scale model employed here *i.e.* the Debye distance is taken to be infinitesimally small compared with the radius of the plasma. The equations must be solved in the region between these points, with the boundary conditions $U_R = U_{i\theta} = U_{e\theta} = \Phi = 0$ and $N = 1$ at the centre of the column and $U_R = 1$ and $b_i = b_{i0}$ at its edge, where b_{i0} is the normalized magnitude of the applied magnetic field.

3. Numerical solutions of the equations

Eqs. (8)–(13) are solved using a shooting method, starting at the centre of the column, with the first step made using a Taylor series expansion to nullify the regular singular point at the origin. The normalized ionization frequency S and axial magnetic field $b_i(0)$ are varied using an iterative procedure until the solution, which

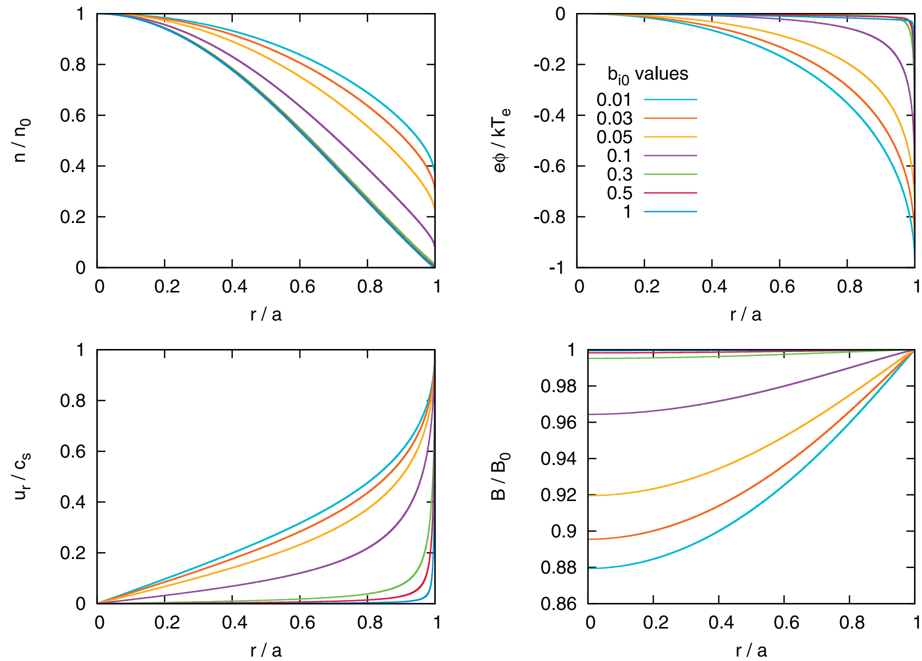

Fig. 1. The radial variation of plasma density, electrostatic potential, radial velocity and magnetic field for an argon plasma column with $\alpha_i = 1$ and $n_0 = 10^{18}$ m^{-3}. The magnitude of the normalized applied magnetic field is varied from $b_{i0} = 0.01$ to $b_{i0} = 1$; the reduction of the magnetic field inside the plasma, known as the diamagnetic effect, is clearly evident.

generates the correct outer boundary values $U_R(1) = 1$ and $b_i(1) = b_{i0}$, is obtained.

The plasma parameters used are based on an experimental study of an inductively-coupled plasma [10], and are the same as those in [8] to allow easy comparisons to be made with their results. The ion species is taken as argon, so the ion-to-electron mass ratio is $m_i/m_e = 73446$. The ion and electron collision parameters are taken as $\alpha_i = 1, 10, 100$ and $\alpha_e = 50\alpha_i^{1/2}$, and the dimensionless applied field b_{i0} is between 0.01 and 1. For a discharge tube of outer radius $a = 3$ cm, this corresponds to a plasma pressure between approximately 0.1 and 10 Pa and a magnetic field of up to 500 Gauss. The electron temperature and plasma density under these conditions are provided by Ref. [10] as $T_e = 1 - 10$ eV and $n = 10^{16} - 10^{19}$ m^{-3}.

The radial profiles of the plasma density, electrostatic potential, radial velocity and magnetic field are displayed in Figs. 1 and 2 for these plasma parameters in the specific cases where $\alpha_i = 1$ and $n_0 = 10^{18}$ or 10^{19} m^{-3}. The inclusion of the all the acceleration terms has not resulted in any appreciable difference from the results found by Sternberg et al. [8]; however, the assumption that the magnetic field remains constant throughout the plasma is evidently incorrect. The plasma exhibits a strong diamagnetic effect and the field strength at the centre of the column is reduced by more than 70% of the applied field in some cases. The enhancement of the diamagnetic effect due to increased plasma density is to be expected from the explicit dependence of Eq. (13) on n_0.

4. Examination of the conditions at the plasma boundary

Dividing Eq. (11) by Eq. (12) yields

$$\frac{dN}{d\Phi} = N\left[2U_R S + b_i(U_{e\theta} - U_{i\theta}) - \frac{U_R^2 + U_{i\theta}^2 + m_e U_{e\theta}^2/m_i}{R} + f_{ir}\right] \times \left[2U_R S + b_i(U_R^2 U_{e\theta} - U_{i\theta}) - \frac{U_R^2 + U_{i\theta}^2 + m_e U_R^2 U_{e\theta}^2/m_i}{R} + f_{ir}\right]^{-1}. \quad (16)$$

Allowing $U_R \to 1$ gives $dN/d\Phi \to N$, so that on reverting to dimensional quantities

$$\frac{dn}{d\phi} \to \frac{ne}{k_B T_e}, \quad (17)$$

which has been termed the Boltzmann gradient condition by Allen [6]. It can be readily shown, as in the Appendix, that this condition leads to the equality form [1] of the well-known Bohm criterion [2].

The radial variation of density near the plasma boundary is given in Fig. 3. The density is also plotted as a function of potential for comparison with that given by the Boltzmann relation for electrons. It is seen that the electron density tends to the Boltzmann value in this region. The reason is that the electric force dominates the magnetic Lorentz force on the electrons as the plasma boundary is approached.

5. Conclusions

The study of the case of a positive column in an axial magnetic field [6] has been completed. It was found that the momentum terms which were missing in a previous study [8] had little effect on the structure of the positive column. The axial magnetic field, on the other hand, was found to be significantly reduced by the azimuthal electron currents, an effect neglected by Sternberg et al. [8]. The principal point of interest in the present paper, however, was to verify that the Bohm criterion was still valid in the presence of a magnetic field in this case. It was not assumed that the electron density was given by the Boltzmann relation, an assumption originally made by Bohm in his derivation of the criterion for sheath formation. In the present work we find that the boundary condition is $dn/d\phi = ne/k_B T_e$, previously termed the Boltzmann gradient condition by Allen [6]. This result leads to the well-known Bohm criterion in its equality form. Fig. 3 illustrates that the electron density tends to the value given by the Boltzmann relation as the plasma boundary is approached. The electric force on the electrons dominates the magnetic Lorentz force in this region. The Bohm criterion would appear to be of general validity, applicable to a variety of situations. In reality the Debye distance is

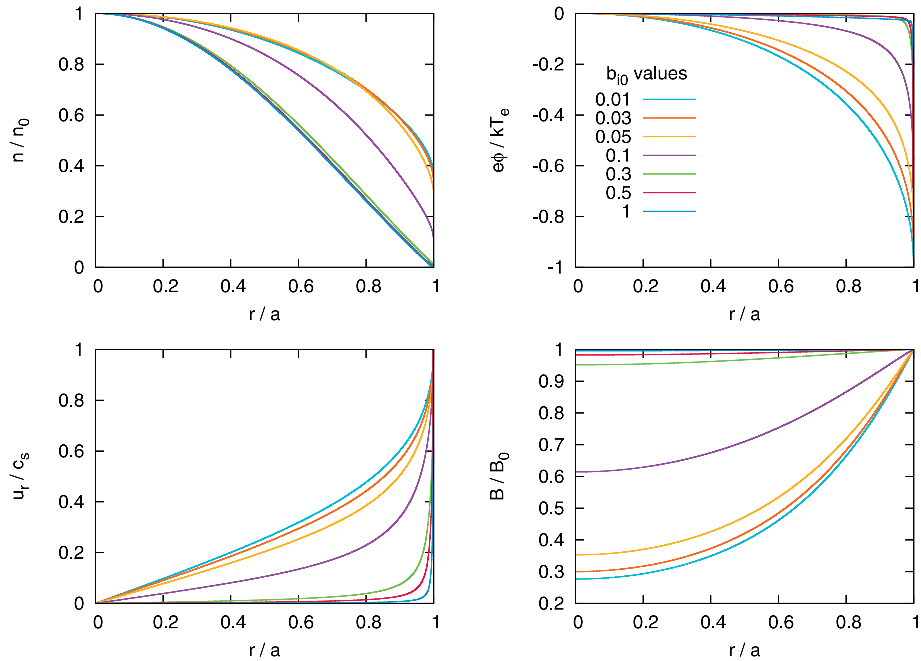

Fig. 2. The radial variation of plasma parameters for the same case as in Fig. 1, except the plasma density at the centre of the column is now $n_0 = 10^{19}$ m^{-3}. The diamagnetic effect is much more pronounced in this denser plasma and weak magnetic fields are almost entirely excluded from the centre of the plasma.

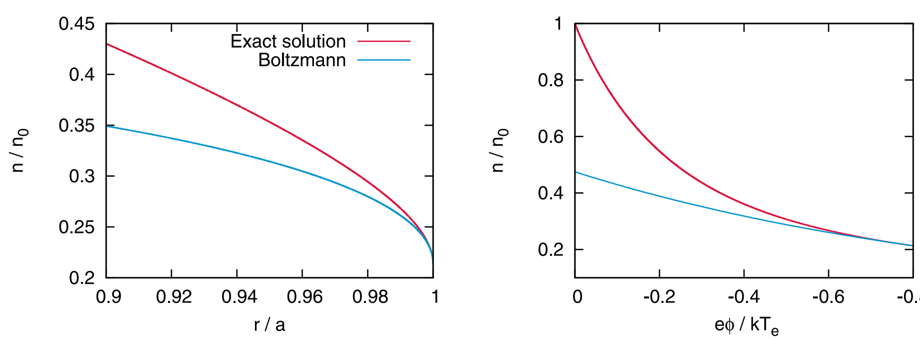

Fig. 3. The variation of density near the plasma boundary for the parameters $\alpha_i = 1$, $b_{i0} = 0.05$ and $n_0 = 10^{18}$ m^{-3} as a function of radius (left) and potential (right). Near the plasma boundary the solution of Eqs. (8)–(13) tends to that given by the Boltzmann relation for electron density, $n = n_s \exp[e(\phi - \phi_s)/k_B T_e]$, where s denotes the sheath-edge value. This illustrates the approach to the Boltzmann gradient condition, Eq. (17).

not infinitesimally small compared with the plasma dimension, as assumed in a two-scale theory. The Bohm criterion, however, can be employed to obtain excellent approximations for the ion velocity, ion current and electric potential at the boundary of a plasma.

Appendix A. A note on the boundary condition

It is of interest to consider the number densities of ions and electrons, in both plasma and sheath, as functions of potential. The sheath can be considered as one-dimensional (plane) since the Debye length is infinitesimally small compared with the plasma dimension. The Boltzmann gradient condition, derived in Section 4, is

$$\left.\frac{\mathrm{d}n_e}{\mathrm{d}\phi}\right|_s = \frac{n_s e}{k_B T_e}. \tag{A.1}$$

In the sheath, using the conservation of both flux and energy for the positive ions, we have

$$n_i = n_s\left(1 - \frac{\phi}{\phi_s}\right)^{-1/2} \tag{A.2}$$

where $e\phi_s$ is the ion energy on entering the sheath and ϕ is measured from the plasma-sheath boundary. Differentiation gives

$$\left.\frac{\mathrm{d}n_i}{\mathrm{d}\phi}\right|_s = \frac{n_s}{2\phi_s}. \tag{A.3}$$

The matching condition at the plasma boundary, first employed by Allen and Thonemann [11], is given by

$$\left.\frac{\mathrm{d}n_e}{\mathrm{d}\phi}\right|_s = \left.\frac{\mathrm{d}n_i}{\mathrm{d}\phi}\right|_s \tag{A.4}$$

so that, using Eqs. (A.1) and (A.3), we can write

$$e\phi_s = \frac{k_B T_e}{2} \tag{A.5}$$

or

$$v_{is} = \sqrt{\frac{k_B T_e}{m_i}} \tag{A.6}$$

In this way we have an explanation of why the Bohm velocity was reached at the plasma boundary, in a situation where the Boltzmann relation for electron density had not been assumed and would not have been valid. Fig. A.4 is a plot of ion and electron densities as functions of potential so that regions of infinitely different length scales can be illustrated on the same diagram. The

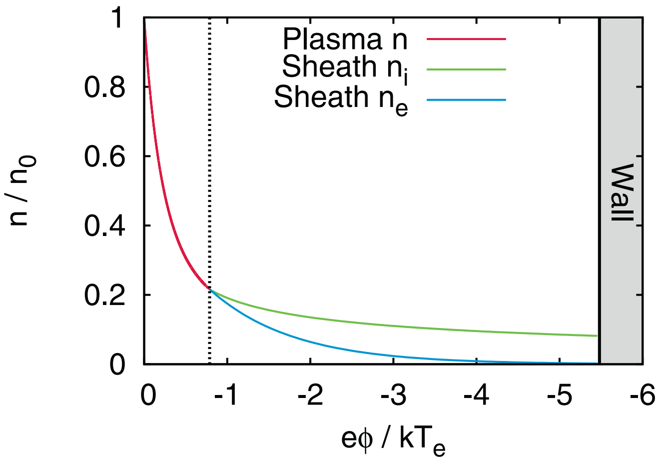

Fig. A4. Ion and electron densities in the plasma and sheath regions for the same conditions as Fig. 3. The potential difference between the centre and the plasma edge is −0.796 while, for argon, the potential drop across the sheath is calculated to be −4.683.

two functions have the same first derivative with respect to potential at the plasma-sheath boundary.

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