

# Continuous periodic solution of a nonlinear pseudo-oscillator equation in which the restoring force is inversely proportional to the dependent variable

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## Abstract

In this paper, we consider a method for the simple exact analytical solution of autonomous nonlinear oscillator equations. While the approach can be used to solve nonlinear oscillator equations with smooth solutions (and we demonstrate this with an application of the approach to an autonomous Duffing equation), our primary interest will be on solving equations with non-smooth yet continuous solutions. To this end, we consider the second-order pseudo-oscillator equation  $yy'' + 1 = 0$  used as a simple model of the path taken by an electron in an electron beam injected into a plasma tube. In recent results of Gadella and Lara, the authors claim the non-existence of periodic solutions to this equation, but actually show that there are no smooth periodic solutions. We show that although there are no smooth solutions to this equation, there do exist a type of continuous periodic solution on the whole problem domain, hence periodic solutions do indeed exist. These periodic solutions can be constructed to have any arbitrary positive period, and the amplitude of these solutions increases as the period is increased. The approach allows one to construct periodic solutions to a variety of nonlinear oscillator equations, even if the solutions are not smooth, and hence could be a useful tool for those interested in physical applications in which nonlinear oscillator models arise.

*Keywords:* nonlinear equations; nonlinear oscillator; periodic solution; exact solution

## 1 Introduction

While the solution of nonlinear oscillator equations is certainly an area that has attracted considerable attention in the applied mathematics and physics literature [1, 2, 3, 4, 5, 6]. While a number of approximate solution methods have been proposed in recent times for the solution of such equations, often overlooked are relatively straightforward approaches that can still give rather good approximations or even exact solutions to the motion of such oscillators. In the present paper, we shall consider the first integral approach, and use it to construct exact solutions to specific nonlinear oscillator equations. One useful feature of the approach is that it does not require a high degree of regularity in the solutions. Therefore, while many analytical approximation methods rely

on smoothness of solutions, a priori, this approach can be used to obtain continuous solutions which fail to be differentiable of some order in the problem domain.

We shall give special consideration to the nonlinear ordinary differential equation

$$yy'' + 1 = 0, \tag{1}$$

which has generated some interest in recent times. Physically, this equation gives a crude model of the path taken by an electron in an electron beam injected into a plasma tube [7]. There have been a number of works claiming to demonstrate the existence of periodic or simply oscillatory solution to (1), by way of applying a variety of analytical methods to obtain approximating solutions [8, 9, 10, 11, 12]. Such solutions would therefore approximate the true dynamics of (1) assuming a periodic solution exists, with some form of error present. In contrast to these results, recently Gadella and Lara [13] claimed to show that there exist no periodic solutions to (1). In their analysis, Gadella and Lara [13] assume that the solutions are relatively smooth (which is the condition needed to apply a the various analytical techniques applied in the literature), and then show that no such solution can exist. However, their non-existence result does rely on the assumption that a solution is rather smooth, and that a resulting Hamiltonian formulation thereby makes sense. Similarly, the note [14] also assumes relatively smooth solutions. This leads one to inquire as to whether a less smooth solution can exist which is periodic, or at the least oscillatory, in nature.

In the present paper, we demonstrate the existence of a periodic and bounded solution to (1) which is  $C^0(\mathbb{R})$ , i.e. continuous on the real line. This solution remains continuous on all of the real line, yet the derivative is not continuous at all points (so the solution is not  $C^1(\mathbb{R})$ ). Solutions are shown to exist for arbitrary positive period  $\tau$ , and the amplitude of such solutions is monotone increasing in the period  $\tau$ . Unlike in other works, the solutions obtained are exact, rather than a type of analytical or numerical approximation.

The approach outlined here can be applied to a number of other autonomous oscillator equations of physical relevance. If smooth solutions exist, then the method can be used to recover them, for appropriate types of equations. In order to demonstrate this on a physically interesting example, we consider the Duffing equation [15] (without damping or forcing), constructing the exact solutions in the periodic case. While the physical interest in (1) can be debated, the Duffing equation has been shown to have applications in various areas of physics, since it can alternately be used to model softening springs [16] or as a model of a periodically forced steel beam deflected between two magnets [17]. These solutions are indeed smooth ( $C^\infty(\mathbb{R})$ ), highlighting the utility of the approach for finding both smooth solutions and continuous yet non-smooth solutions. In light of the large number of papers published on the approximate or numerical solution to such second order nonlinear differential equations, we feel as though the approach outlined here should be considered before more involved yet less accurate approximation approaches are used.

## 2 The exact periodic solution to the pseudo-oscillator equation (1)

### 2.1 The approach

The approach we have selected to exhibit an exact periodic solution may look somewhat complicated, in light of the structure of the fundamental solution, however it is relatively straightforward. To illustrate the approach before applying it to (1), let us consider the simpler oscillator equation

$$y'' + y = 0, \quad y(0) = 1, \quad y'(0) = 0. \tag{2}$$

Then, multiplying the differential equation by  $2y'$  and integrating, we obtain

$$y'^2 + y^2 = 1. \quad (3)$$

Separating variables, we have

$$x - J = \pm \int^y \frac{dq}{\sqrt{1 - q^2}} = \pm \sin^{-1}(|y|), \quad (4)$$

where  $J$  is an integration constant. Note that, as a function,  $\sin^{-1} : [-1, 1] \rightarrow [-\frac{\pi}{2}, \frac{\pi}{2}]$ . Pick the constant  $J = 0$ . Each  $\pm$  branch of the function is bounded. One can match these branches like

$$y(x) = \begin{cases} \vdots \\ \sin(x + \pi) & \text{if } x \in [-\frac{3\pi}{2}, -\frac{\pi}{2}] , \\ \sin(x) & \text{if } x \in [-\frac{\pi}{2}, \frac{\pi}{2}] , \\ \sin(x - \pi) & \text{if } x \in [\frac{\pi}{2}, \frac{3\pi}{2}] , \\ \vdots \end{cases} \quad (5)$$

and this piecewise function is equivalent to  $\sin(x)$ . Therefore,  $y(x) = \sin(x)$  is the solution to (2).

This approach can be applied to obtain periodic solutions of ordinary differential equations on the real line, once the corresponding solutions on smaller intervals are obtained. The approach is particularly useful when the solution is continuous, yet might not have continuous derivatives.

## 2.2 Construction of the fundamental solutions on a bounded domain

Dividing (1) by  $y$  and multiplying the resulting equation by  $2y'$ , we obtain

$$2y'y'' + \frac{2y'}{y} = 0. \quad (6)$$

Integrating (6) with respect to  $x$ , we obtain

$$y'^2 + 2\ln(y) = I, \quad (7)$$

where  $I$  is an integration constant. Now,  $0 \leq y'^2 = I - 2\ln(y)$ , so we must have  $0 < y < \exp(I/2)$ . Separating variables in (7) and integrating gives

$$x - J = \pm \int^y \frac{dq}{\sqrt{I - 2\ln(q)}} = \mp \sqrt{\frac{\pi}{2}} \exp(I/2) \operatorname{erf} \left( \sqrt{\frac{I}{2} - \ln(y)} \right), \quad (8)$$

where  $J$  is another constant of integration and  $\operatorname{erf}$  denotes the error function. We obtain

$$\operatorname{erf} \left( \sqrt{\frac{I}{2} - \ln(y)} \right) = \pm \sqrt{\frac{2}{\pi}} \exp(-I/2) (J - x). \quad (9)$$

Now,  $\operatorname{erf}(u) \in (-1, 1)$  for any  $u \in \mathbb{R}$ . Since  $\sqrt{\frac{I}{2} - \ln(y)} \geq 0$ , we have  $\operatorname{erf} \left( \sqrt{\frac{I}{2} - \ln(y)} \right) \in [0, 1)$ .

Likewise, for  $-\sqrt{\frac{I}{2} - \ln(y)} \leq 0$ , we have  $\operatorname{erf} \left( -\sqrt{\frac{I}{2} - \ln(y)} \right) \in (-1, 0]$ . Therefore, we have two positive branches.

For the first branch,

$$0 < \sqrt{\frac{2}{\pi}} \exp(-I/2)(J - x) < 1, \quad (10)$$

hence

$$J - \sqrt{\frac{\pi}{2}} \exp(I/2) < x < J. \quad (11)$$

Let us denote the interval

$$X_1^+ = \left( J - \sqrt{\frac{\pi}{2}} \exp(I/2), J \right). \quad (12)$$

Then, the branch has domain  $x \in X^+$ . Meanwhile, for the second branch, we should have the domain

$$X_1^- = \left( J, J + \sqrt{\frac{\pi}{2}} \exp(I/2) \right). \quad (13)$$

Consider the set

$$X_1 = X_1^+ \cup X_1^- = \left( J - \sqrt{\frac{\pi}{2}} \exp(I/2), J + \sqrt{\frac{\pi}{2}} \exp(I/2) \right). \quad (14)$$

On this domain, the inversion of erf in (9) is well-defined (see [18] for a mathematical discussion of this point), so the positive branch is given uniquely by  $y(x) = y_+(x)$ , where

$$y_+(x) = \exp \left( \frac{I}{2} - \left\{ \operatorname{erf}^{-1} \left[ \sqrt{\frac{2}{\pi}} \exp(-I/2)(J - x) \right] \right\}^2 \right), \quad (15)$$

for  $x \in X_1$ . This solution has the properties

$$y_+ \left( J - \sqrt{\frac{\pi}{2}} \exp(I/2) \right) = 0 = y_+ \left( J + \sqrt{\frac{\pi}{2}} \exp(I/2) \right) \quad (16)$$

and

$$y_+(J) = \exp(I/2). \quad (17)$$

Meanwhile, had we started from (6) and assumed a negative solution, we would note that a valid negative solution is  $y_-(x) = -y_+(x)$ . Since we are free to pick the constants  $I$  and  $J$ , let us consider a negative solution

$$y_-(x) = -\exp \left( \frac{\hat{I}}{2} - \left\{ \operatorname{erf}^{-1} \left[ \sqrt{\frac{2}{\pi}} \exp(-\hat{I}/2)(\hat{J} - x) \right] \right\}^2 \right), \quad (18)$$

where  $\hat{I}$  and  $\hat{J}$  are the arbitrary constants of integration. This solution has the properties

$$y_+ \left( \hat{J} - \sqrt{\frac{\pi}{2}} \exp(\hat{I}/2) \right) = 0 = y_+ \left( \hat{J} + \sqrt{\frac{\pi}{2}} \exp(\hat{I}/2) \right) \quad (19)$$

and

$$y_+(\hat{J}) = -\exp(\hat{I}/2). \quad (20)$$

### 2.3 Construction of a solution over a single period

In order to construct a solution over a single period, let us pick the constants in (15) and (18) so that

$$I = \hat{I}, \quad J = \sqrt{\frac{\pi}{2}} \exp(I/2), \quad \hat{J} = 3\sqrt{\frac{\pi}{2}} \exp(\hat{I}/2). \quad (21)$$

Consider then the function

$$\hat{y}(x) = \begin{cases} y_+(x) & \text{if } 0 \leq x < 2\sqrt{\frac{\pi}{2}} \exp(I/2), \\ y_-(x) & \text{if } 2\sqrt{\frac{\pi}{2}} \exp(I/2) \leq x < 4\sqrt{\frac{\pi}{2}} \exp(I/2). \end{cases} \quad (22)$$

The function  $\hat{y}(x)$  is continuous and is a solution of (1) on the interval  $0 \leq x < 4\sqrt{\frac{\pi}{2}} \exp(I/2)$ . Let us now pick

$$I = 2 \ln \left( \frac{\tau}{2\sqrt{2\pi}} \right). \quad (23)$$

Then,

$$y_+(x) = \frac{\tau}{2\sqrt{2\pi}} \exp \left( - \left\{ \operatorname{erf}^{-1} \left( 1 - \frac{4x}{\tau} \right) \right\}^2 \right) \quad (24)$$

and

$$y_-(x) = -\frac{\tau}{2\sqrt{2\pi}} \exp \left( - \left\{ \operatorname{erf}^{-1} \left( 3 - \frac{4x}{\tau} \right) \right\}^2 \right), \quad (25)$$

and with this we have a solution over  $x \in [0, \tau)$ , defined by

$$\hat{y}(x) = \begin{cases} y_+(x) & \text{if } 0 \leq x < \frac{\tau}{2}, \\ y_-(x) & \text{if } \frac{\tau}{2} \leq x < \tau. \end{cases} \quad (26)$$

Note that  $\hat{y}(x)$  has properties

$$\hat{y}(0) = \hat{y}\left(\frac{\tau}{2}\right) = \hat{y}(\tau) = 0, \quad \hat{y}'\left(\frac{\tau}{4}\right) = \frac{\tau}{2\sqrt{2\pi}}, \quad \text{and} \quad \hat{y}'\left(\frac{3\tau}{4}\right) = -\frac{\tau}{2\sqrt{2\pi}}. \quad (27)$$

As such, a solution with period  $\tau$  has amplitude  $A = \frac{\tau}{2\sqrt{2\pi}}$ .

### 2.4 Construction of a continuous periodic solution over the real line $\mathbb{R}$

We now use the solution  $\hat{y}(x)$  defined by (26) in order to construct a continuous and periodic solution to (1) over the real line. To begin with, note that  $\hat{y}(x)$  is defined over the interval  $x \in [0, \tau)$  by construction. If we consider the function  $\hat{y}(x + k\tau)$  for some integer  $k \in \mathbb{Z}$ , then  $\hat{y}(x + k\tau)$  is defined over the region  $x \in [-k\tau, -(k-1)\tau)$ .

Let  $n \in \mathbb{Z}$  be some integer. On any interval of the form  $[n\tau, (n+1)\tau)$ , we have a solution to (1) of the form  $\hat{y}(x - n\tau)$ . Furthermore,  $\hat{y}(n\tau) = 0$  for any such integer  $n$ . This means that we may match solutions on adjacent intervals. Hence, we may match the solution  $\hat{y}(x - n\tau)$  valid on  $[n\tau, (n+1)\tau)$  to a solution  $\hat{y}(x - (n+1)\tau)$  valid on  $[(n+1)\tau, (n+2)\tau)$ . The result is then a solution

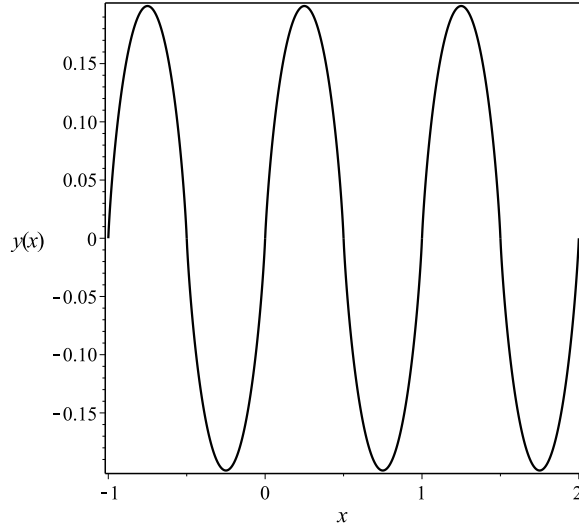


Figure 1: Plot of a solution  $y(x)$  to equation (1) defined piecewise as in (28). We pick  $\tau = 1$ , which gives a representative solution of period one with amplitude  $A = \frac{1}{2\sqrt{2\pi}} = 0.19946$ .

which is continuous over  $[n\tau, (n+2)\tau)$  with period  $\tau$ . Continuing in this manner, we may consider the piecewise defined solution

$$y(x) = \begin{cases} \vdots \\ \hat{y}(x + n\tau) & \text{if } x \in [-n\tau, -(n-1)\tau), \\ \vdots \\ \hat{y}(x + \tau) & \text{if } x \in [-\tau, 0), \\ \hat{y}(x) & \text{if } x \in [0, \tau), \\ \hat{y}(x - \tau) & \text{if } x \in [\tau, 2\tau), \\ \vdots \\ \hat{y}(x - n\tau) & \text{if } x \in [n\tau, (n+1)\tau), \\ \vdots \end{cases} \quad (28)$$

which extends  $\hat{y}(x)$  to the whole real line. The function  $y(x)$  given in (28) is indeed a solution to (1). Furthermore, this function is bounded and continuous over the real line, and has period  $\tau$  and amplitude  $A = \frac{\tau}{2\sqrt{2\pi}}$ . We plot a representative solution, corresponding to  $\tau = 1$ , in Figure 1.

## 2.5 Properties of the solution (28)

Let us restrict our focus to (28) on the interval  $[0, \tau/2]$ ; similar comments will hold on the other sub-intervals. On  $[0, \tau/2]$ , we have

$$y(x) = \frac{\tau}{2\sqrt{2\pi}} \exp \left( - \left\{ \operatorname{erf}^{-1} \left( 1 - \frac{4x}{\tau} \right) \right\}^2 \right), \quad (29)$$

$$y'(x) = 2\sqrt{2}\operatorname{erf}^{-1} \left( 1 - \frac{4x}{\tau} \right), \quad (30)$$

$$y''(x) = -\frac{2\sqrt{2\pi}}{\tau} \exp \left( \left\{ \operatorname{erf}^{-1} \left( 1 - \frac{4x}{\tau} \right) \right\}^2 \right). \quad (31)$$

We should make mention of the fact that we have used

$$\frac{d}{du} \operatorname{erf}^{-1}(u) = \frac{\sqrt{\pi}}{2} \exp \left( \left\{ \operatorname{erf}^{-1}(u) \right\}^2 \right). \quad (32)$$

Then, we clearly see that

$$yy'' = -1, \quad (33)$$

even on the boundary of the interval. Hence the solution does indeed satisfy (1) on  $[0, \tau/2]$ . By similar arguments, it is simple to show that (28) satisfies (1) on the real line. Note that  $y'$  does not remain bounded on  $[0, \tau/2]$ . So, the solution (28) has unbounded first derivative  $y'$  at countably many points over  $\mathbb{R}$ , and the solution cannot be  $C^1(\mathbb{R})$ . Hence, we have obtained a solution which is  $C^0(\mathbb{R})$ . As we observe from the derivations above, that  $y$  is only  $C^0(\mathbb{R})$  does not interfere with the fact that  $y$  is indeed a solution of (1). Indeed, to satisfy (1), a function simply must have the property that  $yy'' = -1$ , and  $y'$  nor  $y''$  enters the equation (1) alone. Hence, a solution exists by virtue of the fact that (1) involves the product  $yy''$  rather than just  $y''$ . This is an interesting situation in which the nonlinearity of the equation has actually helped us, as the product  $yy''$  is far better behaved than the single term  $y''$ .

Another way to look at equation (1) is

$$y'' + \frac{1}{y} = 0. \quad (34)$$

The solution we have constructed has the property that  $y''(x) \equiv -\frac{1}{y(x)}$ , hence this equation is identically satisfied for all  $x$ . In particular, if  $y''$  develops a singularity in  $x$ , then so does  $\frac{1}{y}$ , and each will have a singularity of the exact same strength, but with opposing signs. Therefore, the equation will still be satisfied in the limit where this singularity is approached.

## 3 The Duffing equation as an example giving smooth solutions

We have thus far demonstrated that the  $C^0(\mathbb{R})$  solutions to the nonlinear oscillator equation (1) can be constructed by means of a first integral and then an appropriate matching of local solutions over the real line. In the present section, we demonstrate that for appropriate problems, the same approach can be used to obtain better behaved solutions, such as smooth ( $C^\infty(\mathbb{R})$ ) solutions,

provided they exist. In order to do this, we will take a useful example from the physics literature, the Duffing equation [15].

The Duffing equation we shall consider takes the form

$$y'' + y + 2\epsilon y^3 = 0, \quad (35)$$

where  $\epsilon$  is a parameter. For the sake of perturbation analyses,  $\epsilon$  is often taken to be small. Here, since we are effectively matching exact solutions, we simply take  $\epsilon$  to be a positive parameter. The factor of two is present simply for scaling, and can be absorbed into either the function  $y$  or the parameter  $\epsilon$ . Duffing-type equations have been studied through a variety of methods, both numerical and analytical [19, 20, 21, 22, 23, 24, 25, 26]. While a variety of behaviors are possible, we shall focus our attention on periodic solutions.

A first integral of (35) is found to be

$$y'^2 + y^2 + \epsilon y^4 = I, \quad (36)$$

where  $I$  is an integration constant. Separating variables and integrating, we arrive at

$$x - J = \pm \int^y \frac{dq}{\sqrt{I - q^2 - \epsilon q^4}}. \quad (37)$$

Yet, we know that

$$\int^y \frac{dq}{\sqrt{I - q^2 - \epsilon q^4}} = \sqrt{\frac{2}{1 + \sqrt{1 + 4\epsilon I}}} F\left(\sqrt{\frac{1 + \sqrt{1 + 4\epsilon I}}{2I}} q, \sqrt{\frac{\sqrt{1 + 4\epsilon I} - (1 + 2\epsilon I)}{2\epsilon I}}\right), \quad (38)$$

where  $F(u, \kappa)$  is the elliptic integral of the first kind [27] which satisfies the inversion relation  $F(u, \kappa) = v \Rightarrow u = \text{sn}(v, \kappa)$ , where  $\text{sn}$  denotes the Jacobi  $\text{sn}$  elliptic function [27].

Note that  $\sqrt{1 + 4\epsilon I} < 1 + 2\epsilon I$  for  $\epsilon I > 0$ . Therefore, we have that

$$x - J = \sqrt{\frac{2}{1 + \sqrt{1 + 4\epsilon I}}} F\left(\sqrt{\frac{1 + \sqrt{1 + 4\epsilon I}}{2I}} y(x), \sqrt{\frac{1 + 2\epsilon I - \sqrt{1 + 4\epsilon I}}{2\epsilon I}} i\right). \quad (39)$$

Now, unlike in the previous section, the inversion is not from a compact set to a compact set, but rather from a compact set (delimiting the finite amplitude of the solution) to the real line. Therefore, we will not have to match local solutions. Note, however, that the matching procedure will work here if applied, and we would find that at each of the point in the domain at which the local solutions are matched, the solutions will remain smooth. Therefore, the resulting solutions are  $C^\infty(\mathbb{R})$ .

Performing the inversion, we have

$$y(x) = \sqrt{\frac{2I}{1 + \sqrt{1 + 4\epsilon I}}} \text{sn}\left(\sqrt{\frac{1 + \sqrt{1 + 4\epsilon I}}{2I}}(x - J), \sqrt{\frac{1 + 2\epsilon I - \sqrt{1 + 4\epsilon I}}{2\epsilon I}} i\right). \quad (40)$$

Since the elliptic modulus  $\kappa$  is given by

$$\kappa = \sqrt{\frac{1 + 2\epsilon I - \sqrt{1 + 4\epsilon I}}{2\epsilon I}} i \quad (41)$$



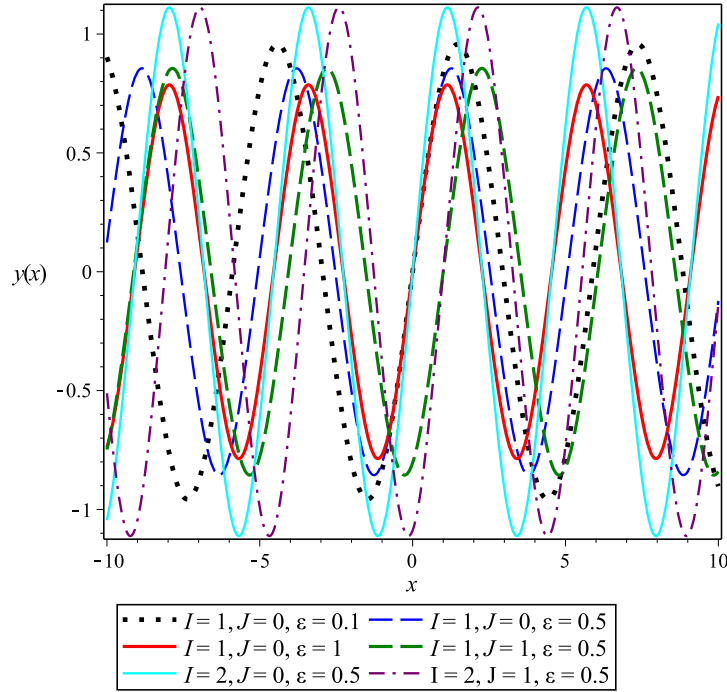


Figure 2: Plot of exact solutions  $y(x)$  to the Duffing equation (35) defined as in (40). Given that the other parameters are fixed, an increase in  $\epsilon$  will decrease the amplitude of the solution, while an increase in the integration constant  $I$  will result in an increase in the amplitude of the solution. Similar results hold for the period of the oscillating solutions. On the other hand, the parameter  $J$  simply gives a phase shift.

we are going to obtain periodic solutions. (If the elliptic modulus was real and equal to one, we would obtain a tanh profile, hence the solutions would not have finite period.) Note that the parameters  $I$  and  $J$  scale and shift the location of solutions, respectively. In particular,  $I$  can be chosen to modify the amplitude and period of solutions, while  $J$  can be chosen to modify the phase. We demonstrate this in Figure 2 for various choices of these parameters.

Note that both  $\epsilon$  and  $I$  determine the amplitude and period of solutions. The amplitude decreases with an increase in  $\epsilon$ , whereas the amplitude increases with an increase in the integration constant  $I$ . The period increases with amplitude, so both a decrease in  $\epsilon$  or an increase in  $I$  will increase the period of the solutions. Meanwhile, the second integration constant,  $J$ , simply gives a phase shift.

With this, we have obtained periodic solutions of the Duffing equation (35) for sufficiently bounded amplitude solutions, by use of a first integral and separation of variables. Unlike the solutions of the previous section, here our solutions are smooth which means that we do not have to go through the added effort of matching local solutions.

## 4 Discussion

Our results show that periodic solutions do indeed exist for (1). The reason this was not picked up in the analysis present in [13] was that that analysis assumed that a condition for such periodicity was that  $y'$  was also continuous. Indeed, this requirement was needed in order to justify the equivalence of (1) with the dynamical system

$$\frac{dy}{dx} = z, \quad \frac{dz}{dx} = -\frac{1}{y}, \quad (42)$$

which is what was done in [13]. Therefore, the analysis of [13] (and also [14]) rules out sufficiently smooth periodic solutions of (1), namely  $C^1(\mathbb{R})$  solutions. However, as our analysis shows, periodic solutions in  $C^0(\mathbb{R})$  are not only possible, but they do exist and can be constructed.

Of course, it is possible to have periodic solutions to a differential equation that fail to be continuous at any level. Indeed, the differential equation  $y' = 1 + y^2$  has the periodic solution  $\tan(x)$ , which is periodic with period  $\pi$ , yet fails to even be continuous over the real line.

The solutions obtained here for equation (1) are confined to being  $C^0(\mathbb{R})$ . This means that analytical approaches which assume arbitrarily many smooth derivatives will produce erroneous results near points on the real line for which  $y'$  fails to be continuous. Therefore, while on the interior of a region defined within a single period of the function  $y$  analytical methods may be useful, on the whole real line the utility of such methods will be limited. All of these analytical approximations in the literature therefore constitute approximations of a  $C^0(\mathbb{R})$  function by smooth  $C^\infty(\mathbb{R})$  functions.

We also demonstrate the approach can be used to obtain smooth solutions, provided they exist. We choose the Duffing equation for this purpose, as the Duffing equation has clear and relevant physical applications. In the case of the Duffing equation where smooth solutions do exist, one can bypass the matching of local solutions and obtain through an inversion a global solution. (However, if one were to match the local solutions, one would find that the resulting global solution created from these matchings was globally smooth.) Therefore, the approach we outline here can be useful for obtaining exact solutions to a variety of physically relevant autonomous oscillator equations that appear throughout the physics and engineering literature. The primary requirements for this approach to work are that:

- (i) the oscillator equation has a first integral;
- (ii) this first integral can be separated;
- (iii) the separated equation yields an implicit relation for the dependent variable, which may be inverted to recover a local (or, global) solution.

If these requirements hold, then one may perform the matching illustrated in Section 2 in order to construct a global solution from the local parts. Such a solution is  $C^0(\mathbb{R})$ . For certain classes of problems, one may be able to demonstrate that the solution will be  $C^n(\mathbb{R})$  for  $n > 0$  or even  $C^\infty(\mathbb{R})$  if the solution is analytic. This is interesting, as many approximate analytical solution methods assume that a solution is  $C^\infty(\mathbb{R})$  a priori, and therefore by applying such a method one would possibly obtain a bad fit to the true solution if it is not actually smooth. The method we outline does not suffer from such a difficulty. Therefore, given an oscillator equation satisfying the above conditions (i)-(iii), it may be possible to construct continuous solutions even if there are no smooth solutions (in which case many other approaches, such as analytical approximations, do not yield solutions). This is of interest when solving physical or engineering problems that may not require fully smooth solutions.

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