

EXPONENTIAL ASYMPTOTICS FOR A MODEL PROBLEM OF AN EQUATORIALLY TRAPPED ROSSBY WAVE*

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Abstract. We examine a misleadingly simple linear second-order eigenvalue problem (the Hermite-with-pole equation) that was previously proposed as a model problem of an equatorially trapped Rossby wave. In the singularly perturbed limit representing small latitudinal shear, the eigenvalue contains an exponentially small imaginary part; the derivation of this component requires exponential asymptotics. In this work, by considering the problem in the complex plane, we show that it contains a number of interesting features that were not remarked upon in the original studies of this equation. These include, in particular, the presence of inactive Stokes lines due to the higher-order Stokes phenomenon. Since an understanding of the behavior in the complex plane is often crucial for problems in exponential asymptotics, we hope that our results, as well as the techniques developed, will prove useful when solving more general linear (and even nonlinear) eigenvalue problems involving asymptotics beyond-all-orders.

Key words. exponential asymptotics, beyond-all-orders analysis, Stokes phenomenon

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1. Introduction. The motivation of this work stems from an interesting mathematical model that was proposed by Boyd and Natarov [7] in order to describe equatorially trapped Rossby waves when the mean shear flow is only a function of the latitude. In such cases, the eigenfunctions are modeled by the so-called *Hermite-with-pole* equation

$$(1.1a) \quad \frac{d^2 u}{dz^2} + \left[\frac{1}{z} - \lambda - \left(z - \frac{1}{\epsilon} \right)^2 \right] u = 0,$$

$$(1.1b) \quad u(z) \rightarrow 0 \quad \text{as} \quad z \rightarrow \pm\infty,$$

$$(1.1c) \quad u(0) = 1.$$

Here, ϵ corresponds to the latitudinal shear strength, where the latitude is measured by z . The solution $u(z)$ is the amplitude of a longitudinal traveling wave, and λ is an eigenvalue determined by the boundary condition (1.1c) at $z = 0$.

The geophysical flow that (1.1a) models is an equatorial zonal flow combined with weak latitudinal shear. The physical equations for equatorial Rossby waves may be found by substituting traveling wave ansatzes into the longitudinal, latitudinal, and pressure momentum equations that govern perturbations to the background shear flow. We specify the shear flow to be linear in the latitudinal direction and to have strength ϵ . This yields the system

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$$(1.2a) \quad (\epsilon z - 1 - \lambda)u' - (z - \frac{1}{\epsilon} - \epsilon)v + p = 0,$$

$$(1.2b) \quad (z - \frac{1}{\epsilon})u + p' = 0,$$

$$(1.2c) \quad (\epsilon z - 1 - \lambda)p + u + v' = 0$$

for the solutions $u(z)$, $v(z)$, and $p(z)$ (cf. Boyd [5]). It is then possible to manipulate these into a single nonlinear differential equation for $v(z)$ containing a critical layer instability, as performed, for instance, by Natarov and Boyd [19] and Griffiths [16], yielding

$$(1.3) \quad v'' + \frac{2\epsilon(\lambda + 1 - \epsilon z)}{(\lambda + 1 - \epsilon z)^2 - 1}v' + \left[\frac{2(\epsilon^2 + 1 - \epsilon z)}{(\lambda + 1 - \epsilon z)^2 - 1} - \frac{1}{\lambda + 1 - \epsilon z} + \epsilon z - 1 - \left(z - \frac{1}{\epsilon}\right)^2 \right]v = 0.$$

It appears that Boyd designed the Hermite-with-pole equation (1.1a) to have similar behavior to this; the pole at $z = 0$ in (1.1a) captures the critical layer at $z = (\lambda + 1)/\epsilon$ in (1.3), and the leading-order solution with $\epsilon = 0$ is equatorially trapped due to Gaussian decay as $z \rightarrow \pm\infty$. We note that the Hermite-with-pole equation was also used by Boyd and Natarov [8] to test the numerical application of Hermite-Pade approximants in critical latitude problems.

As it turns out, the associated eigenvalue to (1.1a) is complex-valued; in the limit $\epsilon \rightarrow 0$, the eigenvalue contains an exponentially small imaginary part

$$(1.4) \quad \text{Im}[\lambda] \sim \pm\sqrt{\pi}\left[1 - 2\epsilon\log\epsilon + (\gamma + \log 2)\epsilon\right]\epsilon^{-1/\epsilon^2},$$

which had previously been derived by Boyd and Natarov [7] using matched asymptotics along the real axis and the properties of special functions. Figure 1 compares this analytical prediction to the numerical values obtained by Boyd and Natarov. Since λ is the wavespeed of a traveling-wave solution, $u(z)e^{ik(x-\lambda t)}$, complex values of λ yield solutions that grow without bound in time. One of the aims of our work

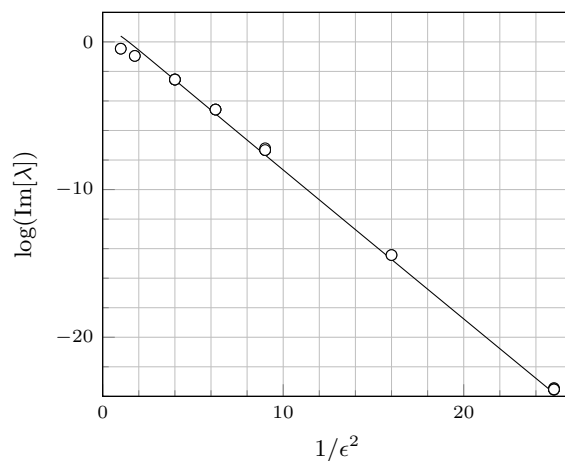


FIG. 1. The imaginary component of the eigenvalue, λ , is shown for the numerical solutions of [7] (circles) and the analytical prediction of $\text{Im}[\lambda] = \sqrt{\pi}[1 - 2\epsilon\log\epsilon + \epsilon(\log 2 + \gamma)]\epsilon^{-1/\epsilon^2}$ (line). Here, $\gamma \approx 0.577$ is the Euler-Mascheroni constant.

is to develop a generalizable framework using exponential asymptotics techniques for the derivation of this exponentially small eigenvalue component.

In their work, Boyd and Natarov [7] note that an asymptotic expansion of $u(z)$ in integer powers of ϵ diverges, and they develop a procedure for approximating $\text{Im}[\lambda]$ with the use of an integral property from Sturm–Liouville theory. Their approach relies upon the asymptotics of special functions and the niceties of the linear differential equation. In contrast, the emphasis of our work here will be on reproducing (1.4) by applying more general techniques of exponential asymptotics to understand the behavior of the solution in the complex plane. Our analysis is more complicated than that in [7], and we do not claim that it is an easier way to approach the problem. On the other hand, an understanding of the behavior in the complex plane is often illuminating, and we will see that our analysis highlights some interesting features of the problem. Moreover, the techniques we use are applicable for more general differential equations, including nonlinear problems, such as the full geophysical problem of the equatorial Kelvin wave instability, which (1.1a) aims to approximate. A discussion of other problems exhibiting similar critical layer instabilities occurs in section 7.1.

For analysis, there is a more convenient form of (1.1a), which is found by shifting

$$(1.5) \quad y = z - \frac{1}{\epsilon},$$

where $y = 0$ corresponds to the equator. Then we have, for $u = u(y)$,

$$(1.6) \quad \frac{d^2 u}{dy^2} + \left[\frac{\epsilon}{1 + \epsilon y} - y^2 \right] u = \lambda u.$$

This intermediary equation contains a turning point at $y = -1/\epsilon$ which we study by rescaling $y = Y/\epsilon$. We then set $u(y) = e^{-y^2/2} \psi(Y)$, which yields the system

$$(1.7a) \quad \epsilon^2 \psi'' - 2Y\psi' + \frac{\epsilon\psi}{1+Y} = (\lambda + 1)\psi,$$

$$(1.7b) \quad e^{-Y^2/2\epsilon^2} \psi(Y) \rightarrow 0 \quad \text{as} \quad Y \rightarrow \pm\infty,$$

$$(1.7c) \quad \psi(0) = 1.$$

In (1.7a) and throughout the paper, we use a prime symbol ($'$) to denote differentiation with respect to Y .

2. A roadmap of the methodology and main results. As it turns out, applying some of the standard techniques of beyond-all-orders asymptotics to the Hermite-with-pole problem (1.7a) throws up a number of nontrivial elements that are not apparent in the approach of Boyd and Natarov. We explain some of these aspects in the context of singularly perturbed linear eigenvalue problems of the form (1.7a), $\mathcal{L}(\psi; \epsilon) = \lambda\psi$, although many of the same ideas apply more generally.

First, asymptotic expansions for $\psi = \psi_0 + \epsilon\psi_1 + \dots$ and $\lambda = \lambda_0 + \epsilon\lambda_1 + \dots$ are sought, but these expansions are divergent and must be optimally truncated. The solution is then expressed as a truncated series with a remainder by considering

$$(2.1) \quad \psi(Y) = \sum_{n=0}^{N-1} \epsilon^n \psi_n(Y) + \mathcal{R}_N(Y),$$

with a similar expression for the eigenvalue, λ . When N is chosen optimally (later shown to be of $O(\epsilon^{-2})$), the remainder $\mathcal{R}_N(Y)$ is exponentially small and satisfies the linear problem

$$(2.2) \quad \mathcal{L}(\mathcal{R}_N; \epsilon) \sim -\epsilon^N \psi_{N-2}''.$$

The remainder, \mathcal{R}_N , will exhibit the Stokes phenomenon, in which its magnitude rapidly varies across certain contours in the complex Y -plane. Indeed, as we shall show, this behavior can be predicted by estimating the growth of the forcing term ψ''_{N-2} . Thus, the late-term behavior of the divergent series, ψ_N with $N \rightarrow \infty$, is required in order to correctly resolve the Stokes phenomenon on the remainder $\mathcal{R}_N(Y)$. This “decoding” of divergence is one of the hallmarks of exponential asymptotics.

The late terms of most “standard” singularly perturbed differential equations exhibit a factorial-power-divergence of the form (cf. [14, 3, 11, 13])

$$(2.3) \quad \psi_n \sim \frac{Q(Y)\Gamma(\beta n + \alpha)}{\chi(Y)^{\beta n + \alpha}} \quad \text{as } n \rightarrow \infty$$

for functions Q and χ (the amplitude and singulant) and constants α and β . Sometimes, however, more complicated factorial/power divergence is seen, and the Hermite-with-pole problem is a case in point. We will see that the divergence is of the form

$$(2.4) \quad \psi_n \sim \begin{cases} \mathcal{S}(Y) \left[L(Y) \log(n) + Q(Y) \right] \frac{\Gamma(\frac{n}{2} + \alpha_0)}{\chi^{n/2 + \alpha_0}} \\ \quad + Q_0^{(\lambda_n)}(Y) \log^2(n) \Gamma\left(\frac{n+1}{2} + \alpha_0\right) & \text{for } n \text{ even,} \\ \underbrace{\mathcal{S}(Y) R(Y) \frac{\Gamma(\frac{n}{2} + \alpha_1)}{\chi^{n/2 + \alpha_1}}}_{\text{HOSP naïve divergence}} + \underbrace{R_1^{(\lambda_n)}(Y) \log(n) \Gamma\left(\frac{n+1}{2} + \alpha_1\right)}_{\lambda_n \text{ divergence}} & \text{for } n \text{ odd.} \end{cases}$$

Here, the singulant, $\chi(Y)$, takes a value of zero at singularities in the early orders of the asymptotic expansion, and $\mathcal{S}(Y)$ is a higher-order Stokes multiplier which takes the values of $\mathcal{S} = 1$ for $\text{Re}[Y] < 0$ and $\mathcal{S} = 0$ for $\text{Re}[Y] > 0$. This change in \mathcal{S} occurs smoothly across a boundary layer (of diminishing width as $n \rightarrow \infty$) surrounding the imaginary axis. The divergence in ψ_n (2.4) is associated with a corresponding divergent eigenvalue

$$(2.5) \quad \lambda_n \sim \begin{cases} \left[\delta_0 \log(n) + \delta_1 \right] \Gamma\left(\frac{n+1}{2} + \alpha_0\right) & \text{for } n \text{ even,} \\ \delta_2 \Gamma\left(\frac{n+1}{2} + \alpha_1\right) & \text{for } n \text{ odd.} \end{cases}$$

Once these late-term components of the solution and eigenvalue are known, a procedure for the derivation of the exponentially small components can be followed.

The divergence of the eigenvalue expansion in (2.5) is directly associated with the existence of an exponentially small imaginary component, as is evident from applying Borel summation to the series $\sum_n \lambda_n \epsilon^n$ (as seen, for example, in Bender and Wu [2] and Dunne and Ünsal [15]). Here we will find this exponentially small component directly, by examining explicitly the effect of the late terms of λ_n on the late terms of ψ_n . We will find the following interesting features.

It is well known (cf. Dingle [14], Berry [3], and Chapman, King, and Adams [11]) that (typically) the divergence of the late terms in a singular perturbation problem is captured by (a sum of) factorial-over-power ansatzes of the form (2.3), each driven by repeated differentiation of a singularity in the early terms (with each factorial-over-power component associated with a corresponding singularity in the Borel plane). In the Hermite-with-pole problem, this divergence is driven by a singularity at $Y = -1$, and the corresponding singulant is $\chi = 1 - Y^2$. However, in addition to $\chi(-1) = 0$, we also find that $\chi(1) = 0$. This suggests that the late terms are singular at $Y = 1$

also, and yet no singularity appears in the early terms there. The situation is very similar to that investigated in Chapman et al. [10] (albeit for a nonlinear equation), and the resolution is the same: the component of the late terms which would be singular is in fact not present near $Y = 1$ but is “switched off” across what is known as a higher-order Stokes line (cf. [18, 21, 4, 12, 17, 20]), which is a result of the Stokes phenomenon in the approximation of ψ_n as $n \rightarrow \infty$.

The component of ψ_n responsible for this switching is exactly that driven by λ_n . This higher-order Stokes phenomenon is also associated with the requirement for an unusual boundary layer in the late-term approximation (2.4). The naïve factorial-over-power divergence (2.4) is unable to satisfy boundary condition (1.7c) at $Y = 0$ due to the functional prefactor growing without bound as $Y \rightarrow 0$, and a boundary layer of vanishing size as $n \rightarrow \infty$ must be introduced. In this region (which is the analogue for the higher-order Stokes phenomenon of an end point or pole coalescing with a saddle for the regular Stokes phenomenon), the two divergences shown in (2.4) interact, and the higher-order Stokes line is born.

It is the explicit resolution of these interactions to give an understanding of the behavior in the complex plane, including a full description of Stokes switchings, that separates our work from the previous work by Boyd and Natarov [7].

3. An initial asymptotic expansion. We begin by considering the asymptotic expansions

$$(3.1) \quad \psi(Y) = \sum_{n=0}^{\infty} \epsilon^n \psi_n(Y) \quad \text{and} \quad \lambda = \sum_{n=0}^{\infty} \epsilon^n \lambda_n.$$

At leading order in (1.7a) we find the solution $\psi_0 = C_0 Y^{-(1+\lambda_0)/2}$, where C_0 is a constant of integration. In general this solution is singular or contains a branch point at $Y = 0$. In order to apply the leading-order boundary condition of $\psi_0(0) = 1$ at the same location, a boundary layer should typically be considered. However, we can verify through an inner-matching procedure that the leading-order eigenvalue is $\lambda_0 = -1$. Then the boundary condition at $Y = 0$ gives $C_0 = 1$, and no boundary-layer theory is required. This yields our leading-order solution of

$$(3.2) \quad \psi_0 = 1 \quad \text{and} \quad \lambda_0 = -1.$$

We emphasize that the singularity at $Y = 0$ in the leading-order solution has been removed by the choice of the eigenvalue, $\lambda_0 = -1$. A similar argument will be applied in subsequent orders to enforce regularity of the solution at $Y = 0$.

At the next order, $O(\epsilon)$, of (1.7a), we find the solution

$$(3.3) \quad \psi_1 = C_1 + \frac{(1 - \lambda_1)}{2} \log(Y) - \frac{1}{2} \log(1 + Y),$$

which contains singularities at both $Y = 0$ and $Y = -1$. To apply the boundary condition $\psi_1(0) = 0$, we require $\lambda_1 = 1$, which then determines the constant of integration as $C_1 = 0$. Thus, our $O(\epsilon)$ solution is

$$(3.4) \quad \psi_1 = -\frac{1}{2} \log(1 + Y) \quad \text{and} \quad \lambda_1 = 1.$$

Note that the above is singular at $Y = -1$. Since successive terms in the asymptotic series for ψ in (3.1) rely on repeated differentiation of previous terms, the logarithmic singularity will result in the divergence of the series for ψ_n as $n \rightarrow \infty$. It is this

divergence that we wish to characterize. Note that in the $n \rightarrow \infty$ limit, on the assumption that ψ_n is divergent, there exists a dominant balance between the two terms $\epsilon^2 \psi''$ and $-2Y\psi'$ of (1.7a). Thus, we must continue to derive additional early orders of the solution until the effects of the $\epsilon^2 \psi''$ term become apparent. Since the singularity at $Y = -1$ in ψ_1 first appears at $O(\epsilon)$, the effects of this term will begin at $O(\epsilon^3)$.

The same procedure is applied at $O(\epsilon^2)$ and $O(\epsilon^3)$, for which we find the solutions

$$(3.5a) \quad \psi_2 = \frac{1}{8} \log^2(1+Y), \quad \lambda_2 = 0,$$

$$(3.5b) \quad \psi_3 = -\frac{Y}{4(1+Y)} - \frac{1}{48} \log^3(1+Y) - \frac{1}{4} \log(1+Y), \quad \lambda_3 = \frac{1}{2}.$$

Note that while the singularities at $Y = -1$ in ψ_1 and ψ_2 were logarithmic, the dominant singularity in ψ_3 is algebraic and of order unity. Typically the order of the singular behavior of successive terms in the asymptotic series would increase linearly in a predictable fashion (see, e.g., the work by Chapman, King, and Adams [11]). This is not the case for our current problem, which can be seen by progressing to the next order, which has the solution

$$(3.6) \quad \psi_4 = -\frac{\log(1+Y)}{8(1+Y)} - \frac{Y}{8(1+Y)} + \frac{\log^4(1+Y)}{384} + \frac{\log^2(1+Y)}{8} \quad \text{and} \quad \lambda_4 = \frac{1}{4}.$$

From (3.5b) and (3.6), we find the singular scalings, as $Y \rightarrow -1$, of

$$(3.7) \quad \psi_3 \sim \frac{1}{4(1+Y)} \quad \text{and} \quad \psi_4 \sim \frac{-\log(1+Y)}{8(1+Y)}.$$

From this, we anticipate that the singular behavior as $Y \rightarrow -1$ of the asymptotic series will proceed in the pairwise fashion of

$$(3.8) \quad \psi_{2k-1} = O\left(\frac{1}{(1+Y)^{k-1}}\right) \quad \text{and} \quad \psi_{2k} = O\left(\frac{\log(1+Y)}{(1+Y)^{k-1}}\right)$$

for integer $k \geq 2$, and hence the order of the algebraic singularity increases every other term. As it turns out, the above form in (3.8), which predicts the behavior of the late-order terms as $Y \rightarrow -1$ and $n \rightarrow \infty$, also hints at the proper ansatz for $n \rightarrow \infty$ in general. In the late-term analysis that follows we will employ separate, divergent predictions for ψ_n , distinguishing between the cases of n even and n odd. The decoupling of the even and odd terms in the expansion as $n \rightarrow \infty$ essentially arises because (1.7a) without the $\epsilon\psi/(1+Y)$ term has a natural expansion in powers of ϵ^2 , but the addition of this term forces an expansion in powers of ϵ ; similar behavior has been observed by Chapman [9].

4. Typical exponential asymptotics and the naïve divergence. The goal of the exponential asymptotics procedure is to predict the exponentially small eigenvalue and eigenfunction solutions. We shall see in section 7 that these exponentially small terms are connected to the divergence of the expansion (3.1).

Our task in this section is to derive the analytical form of the late terms of (3.1) in the limit of $n \rightarrow \infty$. For this, we follow the procedure of introducing an ansatz for the factorial-over-power divergence. However, this ansatz, given in (4.2) below, takes an unusual form due to the inclusion of a $\log(n)$ divergent scaling for even values of n . It is demonstrated in section 4.1.2, through an inner analysis at the singularity, why the divergent ansatz must take this form.

At $O(\epsilon^n)$ in (1.7a), we have

$$(4.1a) \quad \psi''_{n-2} - 2Y\psi'_n - \frac{Y}{1+Y}\psi_{n-1} = \lambda_3\psi_{n-3} + \cdots + \lambda_{n-1}\psi_1 + \lambda_n,$$

and the boundary condition of (1.7c) yields, at $O(\epsilon^n)$,

$$(4.1b) \quad \psi_n(0) = 0.$$

The late-order solutions, ψ_n , will contain a singularity at $Y = -1$ in the manner prescribed by (3.8). Moreover, since subsequent orders are determined by differentiation of earlier terms in the expansion, we anticipate that the divergence of the solution, introduced in (2.4), will be captured by the factorial-over-power ansatz,

$$(4.2) \quad \psi_n \sim \begin{cases} \left[L(Y) \log(n) + Q(Y) \right] \frac{\Gamma(\frac{n}{2} + \alpha_0)}{[\chi(Y)]^{n/2 + \alpha_0}} & \text{for } n \text{ even,} \\ R(Y) \frac{\Gamma(\frac{n}{2} + \alpha_1)}{[\chi(Y)]^{n/2 + \alpha_1}} & \text{for } n \text{ odd.} \end{cases}$$

As we have warned, the analysis to follow is quite involved. In essence, our first task is to derive what we call the *naïve divergence* that appears in (2.4) and above in (4.2). This is performed in section 4.2 by neglecting the late terms of the eigenvalue in the $O(\epsilon^n)$ equation. Before we do this, however, we shall motivate the unusual form of (4.2) in the next section by considering the outer limit of an inner solution at the boundary layer near $Y = -1$.

4.1. Inner problem for the singularity of $Y = -1$. First, we note that the early orders of expansion (3.1) reorder as we approach the singularity at $Y = -1$. Instead of consecutive terms in the outer expansion reordering, those with odd and even powers of ϵ will reorder among themselves. For instance, the reordering occurs between odd terms for $\epsilon^3\psi_3 \sim \epsilon^5\psi_5$ and even terms for $\epsilon^4\psi_4 \sim \epsilon^6\psi_6$. Since $\psi_3 \sim (1+Y)^{-1}$ and $\psi_5 \sim (1+Y)^{-2}$ from the singular behavior introduced in (3.8), we balance $(1+Y)^{-1} \sim \epsilon^2(1+Y)^{-2}$ to find that the width of the boundary layer is of $O(\epsilon^2)$. The same width is found by considering the even reordering. We thus introduce the inner variable, \hat{y} , by setting

$$(4.3) \quad 1 + Y = \epsilon^2 \hat{y},$$

with \hat{y} of $O(1)$ in the inner region. The inner equation may then be derived by substituting for \hat{y} , giving

$$(4.4) \quad \frac{d^2 \hat{\psi}}{d\hat{y}^2} + 2(1 - \epsilon^2 \hat{y}) \frac{d\hat{\psi}}{d\hat{y}} + \frac{\epsilon \hat{\psi}}{\hat{y}} = \epsilon^2(1 + \lambda) \hat{\psi},$$

where we denote the inner solution by $\hat{\psi}$.

4.1.1. Inner limit of the early orders. To motivate the correct form for the inner solution, we take the inner limit of the outer solution by substituting for \hat{y} and expanding as $\epsilon \rightarrow 0$. This yields

$$(4.5) \quad \begin{aligned} \psi_{\text{outer}} \sim & 1 - \epsilon \log(\epsilon) + \epsilon \left[-\frac{\log(\hat{y})}{2} + \frac{1}{4\hat{y}} + \cdots \right] + \frac{\epsilon^2 \log^2(\epsilon)}{2} \\ & + \epsilon^2 \log(\epsilon) \left[\frac{\log(\hat{y})}{2} - \frac{1}{4\hat{y}} + \cdots \right] + \epsilon^2 \left[\frac{\log^2(\hat{y})}{8} - \frac{\log(\hat{y})}{8\hat{y}} + \frac{1}{8\hat{y}} + \cdots \right] + \cdots. \end{aligned}$$

4.1.2. Outer limit of the inner solution. In Appendix B, we solve the inner equation (4.4) by considering an inner solution, motivated by (4.5), of the form $\hat{\psi} = \hat{\psi}_0 + \epsilon \log(\epsilon) \hat{\psi}_{(1,1)} + \epsilon \hat{\psi}_1 + \epsilon^2 \log^2(\epsilon) \hat{\psi}_{(2,2)} + \epsilon^2 \log(\epsilon) \hat{\psi}_{(2,1)} + \epsilon^2 \hat{\psi}_2 + \dots$. We write the inner solution from (B.1) in outer variables by substituting for $\hat{y} = (1+Y)/\epsilon^2$ to give the outer limit (of the first six terms of the inner series) as

$$(4.6) \quad \begin{aligned} \hat{\psi} \sim & 1 - \epsilon \frac{\log(1+Y)}{2} + \epsilon^2 \frac{\log^2(1+Y)}{8} + \sum_{k=1}^{\infty} \frac{\epsilon^{1+2k}}{2} \frac{\Gamma(k)}{[2(1+Y)]^k} \\ & + \sum_{k=1}^{\infty} \frac{\epsilon^{2+2k}}{4} \frac{[4b_k - \log(1+Y)\Gamma(k)]}{[2(1+Y)]^k}. \end{aligned}$$

The divergent constant b_k is determined by the recurrence relation (B.9), which may be solved in the limit of $k \rightarrow \infty$ (as performed in (B.12)) to give $b_k \sim \frac{1}{2}(\log(k) + \gamma)\Gamma(k)$. It is this extra factor of $\log(k)$ in the expansion for b_k that causes the $\log(n)$ in (4.2). In order to compare (4.6) with the late terms of the outer solution at $O(\epsilon^n)$, we substitute $n = 1 + 2k$ for the first sum on the right-hand side of (4.6) and $n = 2 + 2k$ for the second. The $O(\epsilon^n)$ component of this outer limit is then

$$(4.7) \quad \hat{\psi} \sim \begin{cases} \epsilon^n \left[\frac{1}{2} \log(n) + \frac{\gamma - \log(2)}{2} - \frac{1}{4} \log(1+Y) \right] \frac{\Gamma(\frac{n}{2} - 1)}{[2(1+Y)]^{\frac{n}{2}-1}} & \text{for } n \text{ even,} \\ \frac{\epsilon^n}{2} \frac{\Gamma(\frac{n-1}{2})}{[2(1+Y)]^{\frac{n-1}{2}}} & \text{for } n \text{ odd,} \end{cases}$$

where we expanded $b_{n/2-1} \sim \frac{1}{2}[\log(n) + \gamma - \log(2)]\Gamma(\frac{n}{2} - 1)$ for $n \rightarrow \infty$ as in (B.12). Equation (4.7) motivates the slightly unusual form of the factorial-over-power ansatz we had previously introduced in (4.2). We are now ready to return to studying the divergence of the outer solution.

4.2. Divergence of the homogeneous late-term equation. In this section, we derive the *naïve divergence*, which is obtained as a solution to the $O(\epsilon^n)$ equation (4.1a) when the late terms of the eigenvalue are neglected. We thus study the equation

$$(4.8) \quad \psi''_{n-2} - 2Y\psi'_n - \frac{Y}{1+Y}\psi_{n-1} = \lambda_3\psi_{n-3} + \dots,$$

where the lower-order terms on the right-hand side are of orders ψ_{n-4} , ψ_{n-5} , and so forth. Later in section 6, we demonstrate that the late terms of the eigenvalue produce particular solutions that are subdominant as $n \rightarrow \infty$ near the singularity of $Y = -1$ but are crucially responsible for the higher-order Stokes phenomenon.

Substituting the factorial-over-power ansatz (4.2) into the homogeneous equation (4.8), we see that the dominant terms in the equation are of orders $O(\log(n)\Gamma(n/2 + \alpha_0 + 1)/\chi^{n/2+\alpha_0+1})$ for n even and $O(\Gamma(n/2 + \alpha_1 + 1)/\chi^{n/2+\alpha_1+1})$ for n odd. Factoring this dominant behavior out of the equation gives terms of orders $O(1)$, $O(n^{-1})$, etc. for n odd and $O(1)$, $O(\log^{-1}n)$, $O(n^{-1})$, $O(n^{-1}\log^{-1}n)$, etc. for n even. At leading order as $n \rightarrow \infty$, both cases give

$$(4.9) \quad \chi'(\chi' + 2Y) = 0.$$

The singular behavior of ψ_n will be captured by the nontrivial solution, $\chi' = -2Y$. Since we require $\chi(-1) = 0$ in order to match to the inner solution near the singularity from (4.7), we find

$$(4.10) \quad \chi(Y) = 1 - Y^2.$$

Equations for the prefactor functions L , Q , and R are found at the following orders of n in (4.8). Since even and odd components of the divergence now interact, between ψ_n and ψ_{n-1} for instance, it is necessary to specify

$$(4.11) \quad \alpha_1 - \alpha_0 = 1/2$$

in keeping with the different rates of divergence in (4.7). At $O(n^{-1})$ for n even we find an equation for $L(Y)$. Similarly, the $R(Y)$ and $Q(Y)$ equations are found at $O(n^{-1} \log^{-1} n)$ for the cases of n odd and n even, respectively. These equations are

$$(4.12a,b) \quad L'(Y) + \frac{1}{Y}L(Y) = 0, \quad R'(Y) + \frac{1}{Y}R(Y) = 0,$$

$$(4.12c) \quad Q'(Y) + \frac{1}{Y}Q(Y) = \frac{R(Y)}{2(1+Y)} + \frac{2YL(Y)}{1-Y^2},$$

which may be integrated directly to find the solutions

$$(4.13a,b) \quad L(Y) = \frac{\Lambda_L}{Y}, \quad R(Y) = \frac{\Lambda_R}{Y},$$

$$(4.13c) \quad Q(Y) = \frac{\Lambda_Q}{Y} + \frac{\Lambda_R}{2Y} \log(1+Y) - \frac{\Lambda_L}{Y} \log(1-Y^2),$$

where Λ_L , Λ_R , and Λ_Q are constants of integration.

Substitution of solutions (4.13a)–(4.13c) into the ansatz (4.2) gives our divergent prediction for ψ_n , with $n \rightarrow \infty$ as

$$(4.14) \quad \psi_n \sim \begin{cases} \left[\frac{\Lambda_L}{Y} \log(n) + \left(\frac{\Lambda_Q}{Y} + \frac{\Lambda_R}{2Y} \log(1+Y) - \frac{\Lambda_L}{Y} \log(1-Y^2) \right) \right] \frac{\Gamma(\frac{n}{2} + \alpha_0)}{(1-Y^2)^{n/2+\alpha_0}} & \text{for } n \text{ even,} \\ \frac{\Lambda_R}{Y} \frac{\Gamma(\frac{n}{2} + \alpha_0 + \frac{1}{2})}{(1-Y^2)^{n/2+\alpha_0+1/2}} & \text{for } n \text{ odd.} \end{cases}$$

We refer to the late-order form of (4.14) as corresponding to the *naïve divergence*, for which the following two noticeable issues are present:

1. The boundary condition, $\psi_n(0) = 0$, is unable to be satisfied, as our current form is unbounded at $Y = 0$.
2. There are additional locations at which the singulant, $\chi(Y)$, is equal to zero. Since $\chi(Y) = 1 - Y^2$, our late-term expression predicts singularities at both $Y = -1$ and $Y = 1$. This is in contrast to the early orders of the expansion, which are singular at $Y = -1$ only.

The first of these issues will be resolved in section 6.1. There, we demonstrate that as $n \rightarrow \infty$, a boundary layer emerges in the late-order solution near $Y = 0$. This boundary layer is of diminishing width as $n \rightarrow \infty$. A matched asymptotic approach then allows us to develop an inner solution that satisfies the boundary condition of $\psi_n(0) = 0$. Regarding the second issue, the late terms (4.14) in fact switch off across a higher-order Stokes line along the imaginary axis. This is known as the higher-order Stokes phenomenon, which is in fact generated by the singularity discussed in item 1. For a derivation of this phenomenon from the perspective of the divergent series, we refer the reader to [23].

4.3. Determination of the unknown constants. It remains to find values for the constants Λ_L , Λ_R , Λ_Q , and α_0 that appear in the late-term solution for

ψ_n in (4.14). These are determined through matching with the outer limit of the inner solution about the singularity at $Y = -1$ given in (4.7). Expanding the outer solution for ψ_n from (4.14) as $Y \rightarrow -1$, we have

$$(4.15) \quad \psi_n \sim \begin{cases} \left[-\Lambda_L \log(n) + \left(\Lambda_L \log(2) - \Lambda_Q + \left[\Lambda_L - \frac{\Lambda_R}{2} \right] \log(1+Y) \right) \right] \frac{\Gamma(\frac{n}{2} + \alpha_0)}{[2(1+Y)]^{n/2+\alpha_0}} & \text{for } n \text{ even,} \\ -\Lambda_R \frac{\Gamma(\frac{n}{2} + \alpha_0 + \frac{1}{2})}{[2(1+Y)]^{n/2+\alpha_0+1/2}} & \text{for } n \text{ odd.} \end{cases}$$

This form may now be compared to the outer limit of the inner solution in (4.7) to find

$$(4.16) \quad \Lambda_R = -\frac{1}{2}, \quad \Lambda_L = -\frac{1}{2}, \quad \Lambda_Q = -\frac{\gamma}{2}, \quad \alpha_0 = -1,$$

where $\gamma \approx 0.577$ is the Euler–Mascheroni constant.

5. Late-term divergence of the eigenvalue expansion.

5.1. The boundary layer near $Y = 0$. We saw in section 3 that each term in the expansion of the eigenvalue was determined by imposing that the outer solution had no singularity at $Y = 0$. We can find this expansion more readily by considering a local expansion in the vicinity of $Y = 0$. Writing $Y = \epsilon y$, we find

$$\frac{d^2\psi}{dy^2} - 2y \frac{d\psi}{dy} + \frac{\epsilon\psi}{1+\epsilon y} = (\lambda+1)\psi.$$

Expanding in powers of ϵ as usual,

$$\psi = \sum_{n=0}^{\infty} \epsilon^n \psi_n$$

gives

$$(5.1) \quad \frac{d^2\psi_n}{dy^2} - 2y \frac{d\psi_n}{dy} = -\sum_{k=1}^n (-y)^{k-1} \psi_{n-k} + \sum_{k=1}^n \lambda_k \psi_{n-k},$$

where we have expanded

$$\frac{\epsilon}{1+\epsilon y} = \sum_{n=1}^{\infty} \epsilon^n (-y)^{n-1}.$$

We find $\psi_0 = 1$, $\psi_1 = 0$, $\lambda_1 = 1$, and in general,

$$\psi_n = \sum_{m=1}^{n-1} a_{m,n} y^m \quad \text{for } n \geq 2,$$

where the series coefficient satisfies the recurrence relation

$$(5.2) \quad 2ra_{r,n} = (r+2)(r+1)a_{r+2,n} - \sum_{k=1}^{n-r} \lambda_k a_{r,n-k} + \sum_{k=1}^{r+1} (-1)^{k-1} a_{r+1-k,n-k}$$

with $a_{r,n} = 0$ if $r \geq n - 1$ or $r = 0$. It is straightforward to solve (5.2) numerically, stepping down from $r = n - 1$ to $r = 0$ for each n . When $r = 0$, the left-hand side is zero; the fact that the right-hand side must vanish then gives the equation for λ_n , which is

$$\lambda_n = 2a_{2,n}.$$

These numerical solutions are later compared to the divergent prediction for λ_n in Figure 2.

It is not so straightforward to determine the divergence of λ_n as $n \rightarrow \infty$ from (5.2), but we can make some progress by observing that the solution of the homogeneous adjoint to (5.1) is e^{-y^2} . Multiplying by this and integrating gives

$$\begin{aligned} \lambda_n \sqrt{\pi} &= \sum_{k=1}^n \int_{-\infty}^{\infty} e^{-y^2} (-y)^{k-1} \psi_{n-k} dy - \sum_{k=1}^{n-1} \lambda_k \int_{-\infty}^{\infty} e^{-y^2} \psi_{n-k} dy \\ &\sim \frac{(1 - (-1)^n)}{2} \Gamma(n/2) + \dots \end{aligned}$$

When n is odd, this gives

$$(5.3) \quad \lambda_n = \frac{1}{\sqrt{\pi}} \Gamma(n/2).$$

However, when n is even, the first term vanishes, and the correction term is much harder to determine.

5.2. Solution divergence forced by the eigenvalue. We now consider the particular solution of (4.1a) generated by the divergent eigenvalue expansion λ_n . We will see (motivated by (5.3)) that the correct form of the eigenvalue divergence is

$$(5.4) \quad \lambda_n \sim \begin{cases} \left[\delta_0 \log(n) + \delta_1 \right] \Gamma\left(\frac{n-1}{2}\right) & \text{for } n \text{ even,} \\ \delta_2 \Gamma\left(\frac{n}{2}\right) & \text{for } n \text{ odd,} \end{cases}$$

where we expect to find $\delta_2 = 1/\sqrt{\pi}$. We find that this generates a particular solution in ψ_n of the form

$$(5.5) \quad \psi_n(Y) \sim \begin{cases} \left[Q_0^{(\lambda_n)}(Y) \log^2(n) + Q_1^{(\lambda_n)}(Y) \log(n) + Q_2^{(\lambda_n)}(Y) \right] \Gamma\left(\frac{n-1}{2}\right) & \text{for } n \text{ even,} \\ \left[R_1^{(\lambda_n)}(Y) \log(n) + R_2^{(\lambda_n)}(Y) \right] \Gamma\left(\frac{n}{2}\right) & \text{for } n \text{ odd.} \end{cases}$$

Substituting ansatz (5.5) into the $O(\epsilon^n)$ equation (4.1a), we factor out the dominant behavior, which is $\log^2(n) \Gamma((n-1)/2)$ for n even and $\log(n) \Gamma(n/2)$ for n odd. At $O(n^0)$ for n odd and n even, we then find

$$(5.6) \quad R_1^{(\lambda_n)'}(Y) = 0, \quad Q_0^{(\lambda_n)'}(Y) = 0,$$

with solution

$$(5.7) \quad R_1^{(\lambda_n)}(Y) = A_1, \quad Q_0^{(\lambda_n)}(Y) = B_0,$$

where A_1 and B_0 are constants. Next, at $O(\log^{-1}(n))$, for n odd and even, respectively, we find

$$(5.8) \quad R_2^{(\lambda_n)'}(Y) = -\frac{\delta_2}{2Y} \quad \text{and} \quad Q_1^{(\lambda_n)'}(Y) = -\frac{\delta_0}{2Y} - \frac{A_1}{2(1+Y)},$$

with solution

$$(5.9) \quad R_2^{(\lambda_n)}(Y) = A_2 - \frac{\delta_2}{2} \log(Y), \quad Q_1^{(\lambda_n)}(Y) = B_1 - \frac{\delta_0}{2} \log(Y) - \frac{A_1}{2} \log(1+Y),$$

where A_2 and B_1 are constants. At the next order of $O(\log^{-2}(n))$ for n even, we find

$$(5.10) \quad Q_2^{(\lambda_n)'}(Y) = -\frac{\delta_1}{2Y} - \frac{\delta_2}{2Y} \psi_1(Y) - \frac{R_2^{(\lambda_n)}(Y)}{2(1+Y)},$$

with solution

$$(5.11) \quad Q_2^{(\lambda_n)}(Y) = B_2 - \frac{\delta_1}{2} \log(Y) - \frac{A_2}{2} \log(1+Y) + \frac{\delta_2}{4} \log(Y) \log(1+Y).$$

Overall, the divergence of ψ_n is given by combining (4.14) with (5.5) to give

$$(5.12) \quad \psi_n \sim \begin{cases} \left[L(Y) \log(n) + Q(Y) \right] \frac{\Gamma(\frac{n}{2} - 1)}{\chi^{n/2-1}} \\ + \left[Q_0^{(\lambda_n)}(Y) \log^2(n) + Q_1^{(\lambda_n)}(Y) \log(n) + Q_2^{(\lambda_n)}(Y) \right] \Gamma\left(\frac{n-1}{2}\right) & \text{for } n \text{ even,} \\ R(Y) \frac{\Gamma(\frac{n-1}{2})}{\chi^{(n-1)/2}} + \left[R_1^{(\lambda_n)}(Y) \log(n) + R_2^{(\lambda_n)}(Y) \right] \Gamma\left(\frac{n}{2}\right) & \text{for } n \text{ odd.} \end{cases}$$

In the next section, we demonstrate how these divergences interact in a boundary layer near $Y = 0$, justifying the ansatzes (5.4) and (5.5), resolving issues 1 and 2 in section 4.2, and determining the coefficients δ_0 , δ_1 , and δ_2 .

6. The late-term boundary layer at $Y = 0$. Recall that in the early orders of the expansion, each order of the eigenvalue was determined by enforcing the boundary condition at $Y = 0$. However, late-term expansion (5.12) is unbounded at $Y = 0$ and cannot satisfy the condition $\psi_n(0) = 0$. If we continue the expansion (5.12) to higher orders (in $1/n$), we find that the singularity at leading order (for instance, $R_0 \sim Y^{-1}$) forces a stronger singularity at the next order (so that $R_1 \sim Y^{-3}$). Thus, this series reorders as $Y \rightarrow 0$, so that there is a boundary layer in the late-term approximation near $Y = 0$, for which an inner analysis is required. Note the distinction between this boundary layer and that of section 5.1. There the boundary layer was due to a nonuniformity in the expansion of ψ in ϵ and involved rescaling Y with ϵ . Here the boundary layer is due to a nonuniformity in the expansion of ψ_n in n and involves rescaling Y with n .

6.1. Reordering of the late terms as $Y \rightarrow 0$. In order to determine the width of this boundary layer in the late-term solution, we introduce in Appendix A a factorial-over-power ansatz of the form

$$(6.1) \quad \psi_n \sim \begin{cases} \left[L_0(Y) \log(n) + Q_0(Y) + \frac{\log(n)}{n} L_1(Y) + \dots \right] \frac{\Gamma(\frac{n}{2} - 1)}{\chi^{n/2-1}} & \text{for } n \text{ even,} \\ \left[R_0(Y) + \frac{\log(n)}{n} M_1(Y) + \frac{R_1(Y)}{n} + \dots \right] \frac{\Gamma(\frac{n-1}{2})}{\chi^{(n-1)/2}} & \text{for } n \text{ odd.} \end{cases}$$

Here, the leading-order solutions of $L_0(Y)$, $R_0(Y)$, and $Q_0(Y)$ are the same as $L(Y)$, $R(Y)$, and $Q(Y)$ derived previously in (4.13a)–(4.13c). The solutions of $M_1(Y)$, $L_1(Y)$, and $R_1(Y)$ are given in (A.3) and (A.4). For the purpose of observing the reordering of these series near $Y = 0$, it is sufficient to display only their singular behavior here, which is given by

$$(6.2) \quad L_0 \sim \frac{\Lambda_L}{Y}, \quad L_1 \sim \frac{\Lambda_L}{Y^3}, \quad R_0 \sim \frac{\Lambda_R}{Y}, \quad R_1 \sim \frac{\Lambda_R}{Y^3}.$$

The series expansion of ψ_n reorders when two consecutive terms in (6.1) are of the same order as $n \rightarrow \infty$. Since this occurs for $Y = O(n^{-1/2})$, we introduce the inner variable $\bar{y} = n^{1/2}Y$. Substituting this inner variable in the $O(\epsilon^n)$ equation (4.1a) gives the inner equation as

$$(6.3) \quad n \frac{d^2 \bar{\psi}_{n-2}}{d\bar{y}^2} - 2\bar{y} \frac{d\bar{\psi}_n}{d\bar{y}} + \frac{\bar{y}}{n^{1/2}} \left(1 + \frac{\bar{y}}{n^{1/2}}\right)^{-1} \bar{\psi}_{n-1} = \lambda_3 \bar{\psi}_{n-3} + \cdots + \lambda_{n-1} \bar{\psi}_1 + \lambda_n,$$

where $\bar{\psi}_1 = -\frac{1}{2} \log(1 + n^{-1/2}\bar{y}) \sim -\frac{1}{2}n^{-1/2}\bar{y}$.

6.2. Inner limit of the late-term divergence. We now take the inner limit of the outer divergent solution to motivate the correct form for the inner solution. We begin by substituting the inner variable \bar{y} in the naïve divergence (4.14) and taking the limit $n \rightarrow \infty$. For the singularant we find

$$(6.4) \quad (1 - Y^2)^{-n/2} = \left(1 - \frac{\bar{y}^2}{n}\right)^{-n/2} \sim e^{\bar{y}^2/2} \quad \text{as } n \rightarrow \infty.$$

Furthermore, the scaling of $Q(Y) \sim Y^{-1}$ will increase the argument of the gamma function by one half. This yields

$$(6.5) \quad \psi_n \sim \begin{cases} \left[-\frac{\log(n)}{\sqrt{2}} - \frac{\gamma}{\sqrt{2}} \right] \frac{e^{\bar{y}^2/2}}{\bar{y}} \Gamma\left(\frac{n-1}{2}\right) & \text{for } n \text{ even,} \\ -\frac{1}{\sqrt{2}} \frac{e^{\bar{y}^2/2}}{\bar{y}} \Gamma\left(\frac{n}{2}\right) & \text{for } n \text{ odd.} \end{cases}$$

We now take the inner limit of the particular solution generated by the divergent eigenvalue (5.5), which yields

$$(6.6) \quad \psi_n \sim \begin{cases} \left[\left(B_0 + \frac{\delta_0}{4} \right) \log^2(n) + \left(B_1 + \frac{\delta_1}{4} - \frac{\delta_0}{2} \log(\bar{y}) \right) \log(n) \right. \\ \quad \left. + \left(B_2 - \frac{\delta_1}{2} \log(\bar{y}) \right) \right] \Gamma\left(\frac{n-1}{2}\right) & \text{for } n \text{ even,} \\ \left[\left(A_1 + \frac{\delta_2}{4} \right) \log(n) + \left(A_2 - \frac{\delta_2}{2} \log(\bar{y}) \right) \right] \Gamma\left(\frac{n}{2}\right) & \text{for } n \text{ odd.} \end{cases}$$

Altogether, we find for n even

$$(6.7a) \quad \psi_n \sim \left[\left(B_0 + \frac{\delta_0}{4} \right) \log^2(n) + \left(-\frac{e^{\bar{y}^2/2}}{\sqrt{2}\bar{y}} + B_1 + \frac{\delta_1}{4} - \frac{\delta_0}{2} \log(\bar{y}) \right) \log(n) \right. \\ \left. + \left(-\frac{\gamma e^{\bar{y}^2/2}}{\sqrt{2}\bar{y}} + B_2 - \frac{\delta_1}{2} \log(\bar{y}) \right) \right] \Gamma\left(\frac{n-1}{2}\right),$$

and for n odd

$$(6.7b) \quad \psi_n \sim \left[\left(A_1 + \frac{\delta_2}{4} \right) \log(n) + \left(-\frac{e^{\bar{y}^2/2}}{\sqrt{2}\bar{y}} + A_2 - \frac{\delta_2}{2} \log(\bar{y}) \right) \right] \Gamma\left(\frac{n}{2}\right).$$

6.3. An inner solution. We now look for a solution to the inner equation (6.3). Motivated by the form of the inner limit in (6.7), we make the ansatz

$$(6.8) \quad \bar{\psi}_n \sim \begin{cases} \left[\bar{L}(\bar{y}) \log(n) + \bar{Q}(\bar{y}) \right] \Gamma\left(\frac{n-1}{2}\right) & \text{for } n \text{ even,} \\ \bar{R}(\bar{y}) \Gamma\left(\frac{n}{2}\right) & \text{for } n \text{ odd.} \end{cases}$$

Substituting (6.8) and (5.4) into (6.3) and isolating the dominant factorial divergence of $\Gamma(\frac{n}{2})$ for n odd and $\Gamma(\frac{n-1}{2})$ for n even yields at leading order the equations

$$(6.9a-c) \quad \bar{R}'' - \bar{y}\bar{R}' = \frac{\delta_2}{2}, \quad \bar{L}'' - \bar{y}\bar{L}' = \frac{\delta_0}{2}, \quad \bar{Q}'' - \bar{y}\bar{Q}' = \frac{\delta_1}{2}.$$

These three equations all have solutions of a similar form. We will now focus on the equation for \bar{R} and adapt the following results analogously for \bar{L} and \bar{Q} . Integrating (6.9a), we find

$$(6.10) \quad \bar{R}(\bar{y}) = \bar{B}_R + \bar{A}_R \int_0^{\bar{y}} e^{t^2/2} dt + \frac{\delta_2}{2} \int_0^{\bar{y}} e^{t^2/2} \left[\int_0^t e^{-p^2/2} dp \right] dt,$$

with constants of integration \bar{A}_R and \bar{B}_R . We are now able to apply the condition $\bar{\psi}_n(0) = 0$ (resolving issue 1 of section 4.2), which gives $\bar{B}_R = 0$. The remaining constants are determined by matching the outer solution.

We see that in the outer limit of $|\bar{y}| \rightarrow \infty$, (6.9a) itself exhibits the Stokes phenomenon. There is a Stokes line on the imaginary axis, across which the asymptotic behavior of the term proportional to δ_2 changes from

$$\frac{\log(-\bar{y})}{2} + \dots - \frac{\pi^{1/2}}{2^{3/2}} \frac{e^{\bar{y}^2/2}}{\bar{y}} \quad \text{to} \quad \frac{\log(-\bar{y})}{2} + \dots + \frac{\pi^{1/2}}{2^{3/2}} \frac{e^{\bar{y}^2/2}}{\bar{y}}.$$

This is an example of what is known as the higher-order Stokes phenomenon, which is a Stokes phenomenon in the asymptotic approximation of the late terms of the expansion. Additionally, there is a second Stokes line on the real axis, across which the asymptotic behavior of the term proportional to \bar{A}_R picks up an additional constant (the complementary function is just an error function of imaginary argument). Altogether, on the real axis, as $\bar{y} \rightarrow \infty$,

$$(6.11) \quad \bar{R}(\bar{y}) \sim \left[\bar{A}_R + \frac{1}{2} \left(\frac{\pi}{2} \right)^{\frac{1}{2}} \delta_2 \right] \frac{e^{\bar{y}^2/2}}{\bar{y}} + \dots - \frac{\delta_2}{4} \left(\log(2) + \gamma + \log(-\bar{y}^2) + \dots \right),$$

where $\gamma \approx 0.577$ is the Euler–Mascheroni constant, while as $\bar{y} \rightarrow -\infty$,

$$(6.12) \quad \bar{R}(\bar{y}) \sim \left[\bar{A}_R - \frac{1}{2} \left(\frac{\pi}{2} \right)^{\frac{1}{2}} \delta_2 \right] \frac{e^{\bar{y}^2/2}}{\bar{y}} + \dots - \frac{\delta_2}{4} \left(\log(2) + \gamma + \log(-\bar{y}^2) + \dots \right).$$

Similar expressions hold for \bar{L} and \bar{Q} , with different values for the constants of integration.

6.4. Matching. We now match the inner limit of the outer solution (6.7) with the outer limit of the inner solution as $\bar{y} \rightarrow -\infty$ given by (6.12). First, since the inner solution contains no terms of $O(\log n)$ for n odd and $O(\log^2(n))$ for n even, we require

$$(6.13) \quad A_1 = -\frac{\delta_2}{4} \quad \text{and} \quad B_0 = -\frac{\delta_0}{4}.$$

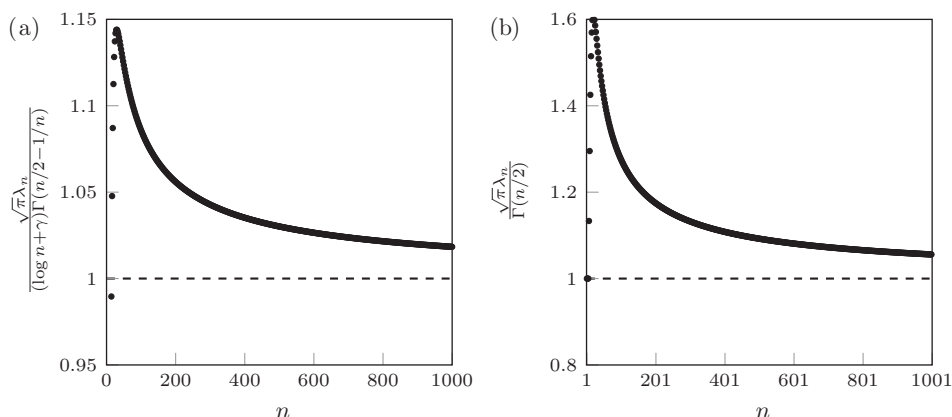


FIG. 2. The coefficient λ_n , numerically calculated by the scheme of section 5.1 is compared to the asymptotic prediction (5.4). Comparison occurs for even n in (a) and odd n in (b).

Next, matching each of the coefficients of $e^{\bar{y}^2/2}/\bar{y}$ as $\bar{y} \rightarrow -\infty$ yields

$$(6.14a) \quad \bar{A}_L - \delta_0 \sqrt{\frac{\pi}{8}} = -\frac{1}{\sqrt{2}}, \quad \bar{A}_Q - \delta_1 \sqrt{\frac{\pi}{8}} = -\frac{\gamma}{\sqrt{2}}, \quad \bar{A}_R - \delta_2 \sqrt{\frac{\pi}{8}} = -\frac{1}{\sqrt{2}}.$$

As $\bar{y} \rightarrow \infty$, we need the coefficients of $e^{\bar{y}^2/2}/\bar{y}$ to be zero, in order that the naïve divergence is not present near the phantom singularity at $Y = +1$ (resolving issue 2 of section 4.2). Thus, matching as $\bar{y} \rightarrow \infty$ gives

$$(6.14b) \quad \bar{A}_L + \delta_0 \sqrt{\frac{\pi}{8}} = 0, \quad \bar{A}_Q + \delta_1 \sqrt{\frac{\pi}{8}} = 0, \quad \bar{A}_R + \delta_2 \sqrt{\frac{\pi}{8}} = 0.$$

Solving (6.14a) and (6.14b) gives

$$(6.15) \quad \delta_0 = \delta_2 = \frac{1}{\sqrt{\pi}}, \quad \delta_1 = \frac{\gamma}{\sqrt{\pi}}, \quad \bar{A}_L = \bar{A}_R = -\frac{1}{\sqrt{8}}, \quad \bar{A}_Q = -\frac{\gamma}{\sqrt{8}},$$

which is consistent with $\delta_0 = 1/\sqrt{\pi}$ from (5.3). In Figure 2 we compare the asymptotic behavior (5.4) with λ_n determined numerically following the procedure described in section 5.1; the agreement validates our predictions for δ_0 , δ_1 , and δ_2 .

7. Stokes smoothing and determination of $\text{Im}[\lambda]$. Having determined the form of the late terms, we now truncate the divergent expansions for the solution and eigenvalue after N terms and study the remainder by writing

$$(7.1) \quad \psi(Y) = \underbrace{\sum_{n=0}^{N-1} \epsilon^n \psi_n(Y)}_{\psi_{\text{reg}}(Y)} + \mathcal{R}_N(Y) \quad \text{and} \quad \lambda = \underbrace{\sum_{n=0}^{N-1} \epsilon^n \lambda_n}_{\lambda_{\text{reg}}} + \lambda_{\text{exp}},$$

where the truncated series are denoted by $\psi_{\text{reg}}(Y)$ and λ_{reg} . We truncate optimally by setting

$$(7.2) \quad N = \frac{2|\chi|}{\epsilon^2} + \rho,$$

where $0 \leq \rho < 1$ ensures that N takes integer values. Substituting into (1.7a) gives

$$(7.3) \quad \epsilon^2 \mathcal{R}_N'' - 2Y \mathcal{R}_N' + \left[\frac{\epsilon}{1+Y} - (1 + \lambda_{\text{reg}}) \right] \mathcal{R}_N = \psi_{\text{reg}} \lambda_{\text{exp}} + \xi_{\text{eq}} + O(\lambda_{\text{exp}} \mathcal{R}_N),$$

where the forcing term ξ_{eq} is of $O(\epsilon^N)$ and defined by

$$(7.4) \quad \xi_{\text{eq}} = (1 + \lambda_{\text{reg}}) \psi_{\text{reg}} - \epsilon^2 \psi_{\text{reg}}'' + 2Y \psi_{\text{reg}}' - \epsilon \frac{\psi_{\text{reg}}}{1+Y}.$$

As $\epsilon \rightarrow 0$,

$$(7.5) \quad \xi_{\text{eq}} \sim -\epsilon^{N+2} \psi_N'' - \epsilon^{N+3} \psi_{N+1}'' + \dots.$$

The procedure now is as follows:

1. Expand (7.5) as $\epsilon \rightarrow 0$, $N \rightarrow \infty$ using (7.2).
2. Write \mathcal{R}_N as a Stokes multiplier $\mathcal{S}(Y)$ multiplied by a homogeneous solution by setting, in this case, $\mathcal{R}_N = \mathcal{S}(Y) \psi_{\text{exp}}$ with

$$\psi_{\text{exp}} = \left(-\frac{1}{2Y} - \frac{\epsilon}{2Y} \left[\log(2/\epsilon^2) + \gamma + \frac{\log(1+Y)}{2} \right] + \dots \right) e^{-(1-Y^2)/\epsilon^2}.$$

3. Localize in a boundary layer near the Stokes lines where $\chi = 1 - Y^2$ is real and positive.
4. Solve for \mathcal{S} to explicitly observe the rapid jump across the Stokes line.

Since these steps are fairly standard (see, e.g., [11]) we omit the details here; the interested reader may refer to the geophysical study for the Kelvin wave problem by Shelton et al. [22], where more details are given. The upshot is that there is a jump in \mathcal{S} of $2\pi i \epsilon$ as the Stokes line $-1 \leq Y < 0$ is crossed, so that a multiple of ψ_{exp} is turned on, as shown in Figure 3.

While $\chi = 1 - Y^2$ is also real and positive on the imaginary axis, the Stokes line there is coincident with the higher-order Stokes line, across which the relevant contribution to ψ_n , including the right-hand side of (7.3), is switched off. The upshot is that the Stokes multiplier is multiplied by $1/2$ on this segment of the Stokes line.

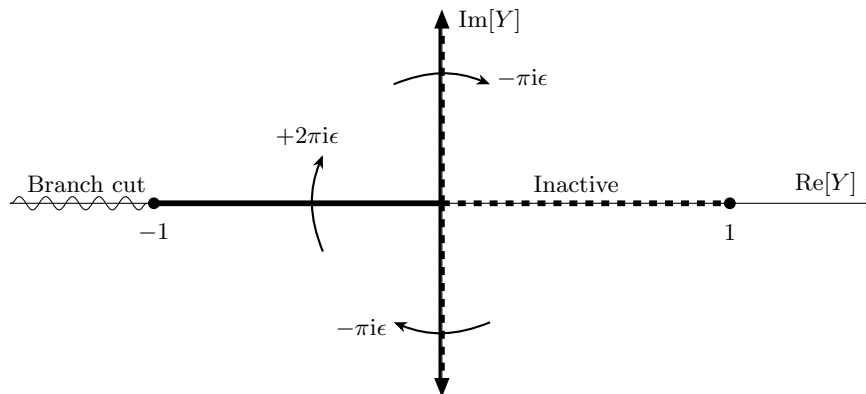


FIG. 3. The Stokes lines generated by the divergent series expansion for our problem are shown (bold). Inactive Stokes lines are shown dashed, and along the imaginary axis, the Stokes line has a multiplier of half the usual value. This inactivity is caused by the higher-order Stokes phenomenon, which switches off the naïve divergence across the imaginary axis.

Finally, on the strip $0 < Y < 1$, the relevant terms in the expansion of ψ_n are no longer present, having been turned off by the higher-order Stokes phenomenon, so that this prospective Stokes line is inactive, and no switching occurs.

The additional term ψ_{exp} switched on across the Stokes lines does not satisfy the decay condition as $Y \rightarrow \infty$. This term is cancelled by an additional contribution to \mathcal{R}_N generated by the forcing term $\psi_{\text{reg}}\lambda_{\text{exp}}$ due to the exponentially small correction to the eigenvalue; indeed, it is this requirement of cancellation which determines λ_{exp} . This additional particular solution satisfies (to two orders in ϵ)

$$(7.6) \quad \epsilon^2 \mathcal{R}_N'' - 2Y \mathcal{R}_N' \sim \lambda_{\text{exp}},$$

which may be solved in terms of special functions to find

$$(7.7) \quad \mathcal{R}_N \sim \frac{\epsilon \lambda_{\text{exp}} \sqrt{\pi}}{2Y} e^{Y^2/\epsilon^2}$$

as $Y \rightarrow \infty$.

We now determine λ_{exp} by imposing that the coefficient of $e^{Y^2/\epsilon^2}/Y$ as $Y \rightarrow \infty$ is zero. First, we note that the decay condition as $Y \rightarrow -\infty$ may be enforced on different Riemann sheets generated by the singularity at $Y = -1$; essentially, as we move from $Y = -\infty$ to $Y = +\infty$ we have to decide whether we pass above or below the point $Y = -1$. If we pass above it, the Stokes switching associated with the base expansion gives $-\pi i \epsilon \psi_{\text{exp}}$ at $Y = \infty$, while passing below it gives $\pi i \epsilon \psi_{\text{exp}}$. This must cancel with the contribution from (7.7), which gives

$$(7.8) \quad \lambda_{\text{exp}} \sim \pm \sqrt{\pi} i [1 - 2\epsilon \log \epsilon + (\gamma + \log 2) \epsilon] e^{-1/\epsilon^2}.$$

These are the complex-conjugate pairs for $\text{Im}[\lambda]$, which correspond to growing and decaying temporal instabilities in the solution. We note that (7.8) is consistent with a direct application of Borel summation to the divergent series (5.4), as we would expect.

7.1. Conclusion and discussion. We have derived the exponentially small component of the eigenvalue,

$$(7.9) \quad \text{Im}[\lambda] \sim \pm \sqrt{\pi} \left[1 - 2\epsilon \log \epsilon + (\gamma + \log 2) \epsilon \right] e^{-1/\epsilon^2},$$

by considering the Stokes phenomenon displayed by the solution, $\psi(Y)$, throughout the complex plane. Since this exponentially small component of λ is imaginary, it corresponds to a growing temporal instability of the solution associated with weak shear and is known as a critical layer instability.

As we noted in section 2, the Hermite-with-pole problem, posed by Boyd and Natarov [7] as a model for weak latitudinal shear of the equatorial Kelvin wave, is an unusually difficult problem in exponential asymptotics.

Some of the issues we have had to confront, such as the differing asymptotic behaviors for even and odd terms in the expansion, arise from an unfortunate choice of model equation, forcing the expansion to proceed in powers of ϵ when it would more naturally proceed in powers of ϵ^2 . Some issues, such as the divergence of the asymptotic series for the eigenvalue, and its associated exponentially small imaginary component, are more generic.

The logarithmic factors of n in the behavior of the late terms are associated with the logarithmic factors of ϵ in the expansion of the imaginary part of the eigenvalue

(7.9). It is not clear to what extent we were just unlucky to have to confront these, although we note that if we had only wanted the leading term in (7.9), we could have avoided most (but not all) of the logarithms by considering only the dominant (i.e., odd n) terms in the expansions of ψ and λ .

The most interesting aspects of the problem have been the phantom singularity in the naïve expansion of the late terms and its resolution via a higher-order Stokes phenomenon driven by the divergent eigenvalue expansion. At the moment it is not clear to us whether this behavior is unusual or generic, but we hope the analysis we have presented will act as a roadmap for similar problems. Although we only considered the higher-order Stokes line in the vicinity of $Y = 0$, it is possible to show that it extends along the whole imaginary axis and to smooth it in a manner similar to the smoothing of regular Stokes lines; the interested reader is referred to [23].

One question of interest is how solutions of the Hermite-with-pole equation compare to the physical Kelvin wave problem (1.3). Numerical values for the temporal growth rate of the full problem were determined by Natarov and Boyd [19], who deformed the contour of integration into the complex plane by the method of Boyd [6] to avoid the critical layer singularity on the real axis. Their results seemed to suggest a growth rate given by $\text{Im}[\lambda] = O(e^{-1/\epsilon^2})$. However, recent analytical and high precision numerical work by Shelton et al. [22] has yielded $\text{Im}[\lambda] \sim \epsilon^3 \frac{1}{4\sqrt{\pi}} e^{-1/\epsilon^2}$. This corrected growth rate was derived by applying the beyond-all-orders methodology developed in this current paper to the physical Kelvin wave problem.

It would be interesting to apply our methodology for the derivation of exponentially small eigenvalues and eigenfunctions to other singularly perturbed eigenvalue problems, which occur frequently across multiple disciplines in applied mathematics. These include critical layer instabilities in hydrodynamics, magnetohydrodynamics, and acoustics (cf. Adam [1]), for which the growth rate (corresponding to the imaginary component of the eigenvalue) is often exponentially small.

Appendix A. Lower-order divergence of the naïve ansatz. As noted in section 4.2, the naïve factorial-over-power solution to the homogeneous late-term equation (4.8) is unable to satisfy the boundary condition at $Y = 0$. This is due to a singularity in the prefactors of the divergent ansatz, $L(Y)$, $R(Y)$, and $Q(Y)$, from (4.13a)–(4.13c). One may consider lower-order terms, as $n \rightarrow \infty$, in the divergence of the homogeneous solution by considering a prefactor of the form (for n odd)

$$(A.1) \quad R(Y) = R_0(Y) + \frac{\log(n)}{n} M_1(Y) + \frac{R_1(Y)}{n} + \cdots,$$

where the subsequent terms in this series will be of $O(n^{-2} \log n)$ and $O(n^{-2})$. We will see that the strength of the singularity in $R_0(Y)$ at $Y = 0$ increases in later orders and thus forces a reordering of the series as $Y \rightarrow 0$.

The method for calculating these lower-order solutions is similar to that briefly presented in section 4.2 for the leading orders. We substitute an ansatz for $\psi_n(Y)$ of the form

$$(A.2) \quad \psi_n \sim \begin{cases} \left[L_0(Y) \log(n) + Q_0(Y) + \frac{\log(n)}{n} L_1(Y) + \cdots \right] \frac{\Gamma(\frac{n}{2} - 1)}{\chi^{n/2-1}} & \text{for } n \text{ even,} \\ \left[R_0(Y) + \frac{\log(n)}{n} M_1(Y) + \frac{R_1(Y)}{n} + \cdots \right] \frac{\Gamma(\frac{n-1}{2})}{\chi^{(n-1)/2}} & \text{for } n \text{ odd} \end{cases}$$

into the homogeneous equation (4.8). Dividing out by the dominant factorial-over-power scaling in the $O(\epsilon^n)$ equation (4.8) yields terms of orders n^0 , $n^{-1} \log(n)$, n^{-1} ,

$n^{-2} \log(n)$, and n^{-2} for odd values of n . The case for even values of n is similar, except for terms of order $\log(n)$ also appearing. Distinct equations are found at each of these orders for the case of n even or n odd.

The first few equations are the same as those considered in section 4.2 and yield the singulant $\chi(Y) = 1 - Y^2$ from (4.10) and prefactors $L_0(Y)$, $R_0(Y)$, and $Q_0(Y)$ from (4.13a)–(4.13c). Equations for $M_1(Y)$ and $L_1(Y)$ are then found at $O(n^{-2} \log(n))$ for odd and even values of n , respectively, which have the solutions

$$(A.3) \quad \begin{aligned} M_1(Y) &= \left[\Lambda_{M_1} + \Lambda_{L_0} \log(1+Y) \right] \frac{(1-Y^2)}{Y}, \\ L_1(Y) &= \left[\Lambda_{L_1} + \frac{\Lambda_{M_1}}{2} \log(1+Y) + \frac{\Lambda_{L_0}}{Y^2} + \frac{\Lambda_{L_0}}{4} \log^2(1+Y) \right] \frac{(1-Y^2)}{Y}, \end{aligned}$$

where Λ_{M_1} and Λ_{L_1} are constants of integration. It remains to determine $R_1(Y)$, whose governing equation will be found at $O(n^{-2})$ when n is odd. This has the solution

$$(A.4) \quad \begin{aligned} R_1(Y) &= \left[\Lambda_{R_1} + \Lambda_{Q_0} \log(1+Y) + \frac{\Lambda_{R_0}}{Y^2} + \frac{\Lambda_{R_0}}{4} \log^2(1+Y) \right. \\ &\quad \left. - \Lambda_{M_1} \log(1-Y^2) - \Lambda_{L_0} \log(1+Y) \log(1-Y^2) \right] \frac{(1-Y^2)}{Y}, \end{aligned}$$

where Λ_{R_1} is a constant of integration.

To conclude, we note that the functional prefactor of a factorial-over-power ansatz for the late-term solution may contain singularities or branch points at locations not seen in the early orders of the expansion. In our case this is the location $Y = 0$. In these instances, it is necessary to consider lower-order terms of the divergent ansatz in order to determine the correct inner-variable scaling for the resultant boundary-layer matching procedure.

Appendix B. Inner solution at the singularity $Y = -1$. Motivated by the inner limit of the outer solution, (4.5), we consider an inner solution of the form

$$(B.1) \quad \hat{\psi}_{\text{inner}}(\hat{y}) = \sum_{n=0}^{\infty} \sum_{m=0}^n \epsilon^n \log^m(\epsilon) \hat{\psi}_{(n,m)}(\hat{y}).$$

Substitution into the inner equation (4.4) yields, at $O(1)$, $O(\epsilon \log(\epsilon))$, and $O(\epsilon^2 \log^2(\epsilon))$,

$$(B.2) \quad \hat{\mathcal{L}}[\hat{\psi}_{(0,0)}] \equiv \frac{d^2 \hat{\psi}_{(0,0)}}{d\hat{y}^2} + 2 \frac{d\hat{\psi}_{(0,0)}}{d\hat{y}} = 0, \quad \hat{\mathcal{L}}[\hat{\psi}_{(1,1)}] = 0, \quad \hat{\mathcal{L}}[\hat{\psi}_{(2,2)}] = 0.$$

These equations have solutions of a similar form given by $\hat{\psi}_{(0,0)}(\hat{y}) = A_{(0,0)} + B_{(0,0)} \exp(-2\hat{y})$, for instance. Matching the $O(1)$, $O(\epsilon \log(\epsilon))$, and $O(\epsilon^2 \log^2(\epsilon))$ components of the inner-limit of ψ_{outer} in (4.5) requires the coefficient of $\exp(-2\hat{y})$ to be zero for each of these solutions. Matching the constant components then yields

$$(B.3) \quad \hat{\psi}_{(0,0)}(\hat{y}) = 1, \quad \hat{\psi}_{(1,1)}(\hat{y}) = -1, \quad \hat{\psi}_{(2,2)}(\hat{y}) = \frac{1}{2}.$$

Next, at $O(\epsilon)$ and $O(\epsilon^2 \log \epsilon)$, we find equations similar to (B.2), with the exception of a forcing term that relies on $\hat{\psi}_{(0,0)}(\hat{y})$ and $\hat{\psi}_{(1,1)}(\hat{y})$, respectively. These equations are found to be

$$(B.4) \quad \hat{\mathcal{L}}[\hat{\psi}_{(1,0)}] = -\frac{1}{\hat{y}} \quad \text{and} \quad \hat{\mathcal{L}}[\hat{\psi}_{(2,1)}] = \frac{1}{\hat{y}},$$

where $\widehat{\mathcal{L}}$ is the linear differential operator defined in (B.2). For brevity, only the exact solution of the first of these is provided here. This has the solution of

$$(B.5) \quad \hat{\psi}_{(1,0)}(\hat{y}) = A_{(1,0)} + B_{(1,0)}e^{-2\hat{y}} - e^{-2\hat{y}} \int_0^{\hat{y}} \log(y)e^{2y} dy.$$

Analogously, for the second equation in (B.4) the exact solution will have constants $A_{(2,1)}$ and $B_{(2,1)}$ and a positive sign (+) in front of the last component of the solution in (B.5). To facilitate matching with the $O(\epsilon)$ outer solution, we take the outer limit of (B.5) as $\hat{y} \rightarrow \infty$, yielding

$$(B.6) \quad \hat{\psi}_{(1,0)}(\hat{y}) \sim -\frac{1}{2} \log(\hat{y}) + \frac{1}{2} \sum_{k=1}^{\infty} \frac{\Gamma(k)}{(2\hat{y})^k} \quad \text{and} \quad \hat{\psi}_{(2,1)}(\hat{y}) \sim \frac{1}{2} \log(\hat{y}) - \frac{1}{2} \sum_{k=1}^{\infty} \frac{\Gamma(k)}{(2\hat{y})^k}.$$

Here we set $A_{(1,0)} = 0$, $B_{(1,0)} = 0$, $A_{(2,1)} = 0$, and $B_{(2,1)} = 0$ to match the $O(\epsilon)$ term of the inner limit of the outer solution from (4.5).

Note that we have been able to construct an exact solution for $\hat{\psi}_{(1,0)}$ and $\hat{\psi}_{(2,1)}$. In general, and typically for nonlinear problems, this is not possible, and an ansatz must be introduced to capture the series expansion of the outer-limit behavior of $\hat{\psi}(\hat{y})$, from which the coefficients of this series, in our case $\Gamma(k)$, would be determined via the solution to a recurrence relation problem. This will be the approach used when considering the $O(\epsilon^2)$ equation,

$$(B.7) \quad \widehat{\mathcal{L}}[\hat{\psi}_{(2,0)}] = \frac{\log(\hat{y})}{2\hat{y}} - \sum_{k=1}^{\infty} \frac{\Gamma(k)}{(2\hat{y})^{k+1}},$$

for which we consider a series expansion as $\hat{y} \rightarrow \infty$ of the form

$$(B.8) \quad \hat{\psi}_{(2,0)}(\hat{y}) = \frac{\log^2(\hat{y})}{8} + \log(\hat{y}) \sum_{k=1}^{\infty} \frac{a_k}{(2\hat{y})^k} + \sum_{k=1}^{\infty} \frac{b_k}{(2\hat{y})^k}.$$

Substitution of series (B.8) into (B.7) yields terms of orders $(2\hat{y})^{-k}$ and $\log(y)(2\hat{y})^{-k}$. Examining the equations which arise at each of these orders yields the following recurrence relations for a_k and b_k :

$$(B.9) \quad \begin{aligned} a_1 &= -\frac{1}{4}, & a_k &= (k-1)a_{k-1}, \\ b_1 &= \frac{1}{4}, & b_k &= (k-1)b_{k-1} + \frac{(2k-1)}{4k} \Gamma(k-1), \end{aligned}$$

where $k \geq 2$. In substituting for $b_k = \Gamma(k)d_k$, the recurrence relation for b_k may be written in a form with a series solution, yielding for $k \geq 2$

$$(B.10) \quad a_k = -\frac{\Gamma(k)}{4} \quad \text{and} \quad b_k = \left[\frac{1}{2} - \frac{1}{4k} + \frac{1}{2} \sum_{j=2}^k \frac{1}{j} \right] \Gamma(k).$$

Thus, as $\hat{y} \rightarrow \infty$, our $O(\epsilon^2)$ inner solution is given by

$$(B.11) \quad \hat{\psi}_{(2,0)}(\hat{y}) = \frac{\log^2(\hat{y})}{8} - \frac{\log(\hat{y})}{4} \sum_{k=1}^{\infty} \frac{\Gamma(k)}{(2\hat{y})^k} + \sum_{k=1}^{\infty} \frac{b_k}{(2\hat{y})^k},$$

where b_k is defined as in (B.10). In section 4.1.2, we use the outer limit of this solution to motivate the correct form for the factorial-over-power ansatz of ψ_n as $n \rightarrow \infty$. Thus, we are also interested in the limit of $k \rightarrow \infty$ of b_k . Expanding b_k given in (B.10) as $k \rightarrow \infty$ yields

$$(B.12) \quad b_k \sim \left[\frac{1}{2} \log(k) + \frac{\gamma}{2} + O(k^{-1}) \right] \Gamma(k),$$

where $\gamma \approx 0.577$ is the Euler–Mascheroni constant.

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