Analysis of Several Non-linear PDEs in Fluid Mechanics and Differential Geometry

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To my parents, Ying Zhao and Yifeng Li, and my grandfather Bingnian Zhao (1942.04.30 – 2017.03.24), for their unreserved love.

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Abstract

In the thesis we investigate two problems on Partial Differential Equations (PDEs) in differential geometry and fluid mechanics. First, we prove the weak L^p continuity of the Gauss-Codazzi-Ricci (GCR) equations, which serve as a compatibility condition for the isometric immersions of Riemannian and semi-Riemannian manifolds. Our arguments, based on the generalised compensated compactness theorems established via functional and micro-local analytic methods, are intrinsic and global. Second, we prove the vanishing viscosity limit of an incompressible fluid in three-dimensional smooth, curved domains, with the kinematic and Navier boundary conditions. It is shown that the strong solution of the Navier-Stokes equation in H^{r+1} ($r > 5/2$) converges to the strong solution of the Euler equation with the kinematic boundary condition in H^r, as the viscosity tends to zero. For the proof, we derive energy estimates using the special geometric structure of the Navier boundary conditions; in particular, the second fundamental form of the fluid boundary and the vorticity thereon play a crucial role. In these projects we emphasise the linkages between the techniques in differential geometry and mathematical hydrodynamics.

Contents

Chapter 1

Introduction and Motivation

1.1 Introduction

Two questions in differential geometry and fluid dynamics, both formulated in terms of non-linear Partial Differential Equations (PDEs), are studied in this thesis. One is related to the isometric immersions problem of Riemannian and semi-Riemannian manifolds, and the other one is concerned with the limiting behaviour of an incompressible fluid on 3-dimensional smooth regions as the viscosity vanishes, subject to the Navier and kinematic boundary conditions. Both questions above are motivated by applications in physics and engineering (the isometric immersions problem is closely related to the realisation of elastic bodies in the physical space); on the other hand, they both possess underlying geometric structures which play a crucial role in the mathematical analysis. The exploitation of geometric ideas in physical PDE problems is a prevailing theme of the thesis.

1.2 Main contribution of the thesis

1.2.1 Weak continuity of Gauss-Codazzi-Ricci equations

Let us start with the isometric immersions problem. It has been of considerable interest in the development of differential geometry, non-linear analysis and PDEs; *cf.* Han-Hong [65], Nash [97, 98], Nirenberg [102] and Yau [126]. Here, in particular, we are concerned with the global weak continuity of the Gauss-Codazzi-Ricci (GCR) equations, which is a fundamental system of non-linear PDEs in differential geometry, serving as the compatibility equations to ensure the existence of isometric immersions (*cf.* [21, 22, 54, 61, 62, 111]):

Problem 1.1 *Let (M,g) be a given n-dimensional Riemannian manifold, isometrically immersed into the Euclidean space $(\mathbb{R}^{n+k},g_0)$ via $\{f^\epsilon\}\subset W^{2,p}_{\rm loc}$, with $g\in W^{1,p}_{\rm loc}$ (g_0 is the Euclidean metric). If the corresponding second fundamental forms $\{B^\epsilon\}$ and normal connections $\{\nabla^{\perp,\epsilon}\}$ are uniformly bounded in $L^p_{\rm loc}$, $p>2$, do the weak limits of $\{(B^\epsilon,\nabla^{\perp,\epsilon})\}$ still satisfy the Gauss-Codazzi-Ricci (GCR) equations?*

Besides its background in geometric analysis, the weak rigidity problem is also motivated by non-linear elasticity and the calculus of variations; see Ciarlet–Gratie–Mardare [35], Mardare [88], Szopos [116] and the references cited therein.

Let us briefly explain the analytic substance of Problem 1.1. The GCR equations are a system of first-order non-linear PDEs in the second fundamental form B and the normal connection $\nabla^\perp$, which are geometric quantities depending not only on the metric of the underlying manifold M, but also on the way that M sits in the ambient space $\mathbb{R}^{n+k}$. Schematically, the GCR equations can be represented as

$$\partial(B,\nabla^\perp)=(B,\nabla^\perp)\star(B,\nabla^\perp)+\big\{\text{linear terms in }(B,\nabla^\perp)\big\}, \tag{1.1}$$

where ∂ is a first-order differential operator and $\star$ denotes a quadratic operation. Problem 1.1 thus inquires about the weak continuity of Eq. (1.1) in the weak $L^p_{\rm loc}$ topology. The difficulty arises from the quadratic non-linearity $(B,\nabla^\perp)\star(B,\nabla^\perp)$: In general, for the weakly converging sequences of vector fields $v^\epsilon\rightharpoonup v$, $w^\epsilon\rightharpoonup w$ in $L^p_{\rm loc}$, it is *not* true that $v^\epsilon\cdot w^\epsilon\to v\cdot w$ in distributions. Therefore, the answer to Problem 1.1 relies on further compactness information of $\{B^\epsilon,\nabla^{\perp,\epsilon}\}$, which can be deduced from the specific structures of the first-order differential operator on the left-hand side of Eq. (1.1).

In §2 we shall answer Problem 1.1 in the affirmative. More precisely:

Theorem 1.2 *Let (M,g) be an n-dimensional Riemannian manifold with $W^{1,p}$ metric for $p>2$. Let $\{(B^\epsilon,\nabla^{\perp,\epsilon})\}$ be a sequence of L^p solutions to the GCR equations in the following sense:*

(i) *The tensor fields $B^\epsilon:\Gamma(TM)\times\Gamma(TM)\to\Gamma(TM^\perp)$ and the affine connections $\nabla^{\perp,\epsilon}:\Gamma(TM)\times\Gamma(TM^\perp)\to\Gamma(TM^\perp)$ are uniformly bounded in $L^p_{\rm loc}$ with*

$$\sup_{\epsilon>0}\big\{\|B^\epsilon\|_{L^p(K)}+\|\nabla^{\perp,\epsilon}\|_{L^p(K)}\big\}\leq C_K \tag{1.2}$$

for a constant C_K on any compact subset $K\Subset M$, independent of ϵ;

(ii) $(B^\epsilon, \nabla^{\perp,\epsilon})$ *solves the GCR equations in the distributional sense. More precisely, for any* $X, Y, Z, W \in \Gamma(TM)$ *and* $\eta, \xi \in \Gamma(TM^\perp)$,

$$\langle B^\epsilon(X,Z), B^\epsilon(Y,W)\rangle - \langle B^\epsilon(Y,Z), B^\epsilon(X,W)\rangle = R(X,Y,Z,W), \tag{1.3}$$
$$\langle [S^\epsilon_\eta, S^\epsilon_\xi]X, Y\rangle - R^\perp(X,Y,\eta,\xi) = 0, \tag{1.4}$$
$$\bar\nabla_Y B^\epsilon(X,Z) - \bar\nabla_X B^\epsilon(Y,Z) = 0 \tag{1.5}$$

in the sense of distributions, where S^ϵ *is the shape operator corresponding to* B^ϵ. *(Note that* $\nabla^{\perp,\epsilon}$ *is implicit in the above equations).*

Then, after passing to a subsequence, $\{(B^\epsilon, \nabla^{\perp,\epsilon})\}$ *converges weakly in* $L^p_{\rm loc}$ *to a pair* $(B, \nabla^\perp)$ *that is still a weak solution of the GCR equations.*

One geometric consequence of Theorem 1.2 is the *weak rigidity of isometric immersions*: If in addition $p > n$, one can prove that given Eqs. (1.3)–(1.5) (the GCR equations), a family of $W^{2,p}_{\rm loc}$ isometric immersions $\{f^\epsilon\}$ can be constructed, with $(B^\epsilon, \nabla^{\perp,\epsilon})$ being the corresponding second fundamental forms and normal connections. Then, the weak solutions $(B, \nabla^\perp)$ are the corresponding second fundamental forms and normal connections of a weak $W^{2,p}_{\rm loc}$ limit f of $\{f^\epsilon\}$.

To prove Theorem 1.2, we exploit the *div-curl structure* hidden in the GCR equations (observed first by Chen-Slemrod-Wang in [31, 32], with computations in local coordinates) and apply the div-curl lemma initiated from the works by Murat [95, 96] and Tartar [118, 119]. Moreover, in order to give an *intrinsic* and *global* proof, we establish an "abstract" div-curl lemma in the setting of functional analysis. Our Theorem 1.3 below generalises the recent work [78] by Kozono-Yaganisawa, which explores an alternative approach to the div-curl lemma, instead of the conventional harmonic analytic and real Hodge-theoretic approaches; *cf.* Coifman-Lions-Meyer-Semmes [40] for the former and Robbin-Rogers-Temple [105] for the latter.

Theorem 1.3 *Let* $\mathcal{H}$ *be a Hilbert space over* $\mathbb{K}\,(=\mathbb{R}$ *or* $\mathbb{C})$, Y *and* Z *be reflexive Banach spaces over* $\mathbb{K}$, *and* $S : \mathcal{H} \to Y$ *and* $T : \mathcal{H} \to Z$ *be bounded linear operators satisfying*

(Op 1) *Orthogonality* ($S^\dagger$ *denotes the adjoint operator of* S, *and similarly for* $T^\dagger$):

$$S \circ T^\dagger = 0, \qquad T \circ S^\dagger = 0; \tag{1.6}$$

(Op 2) *For some Hilbert space* $(\underset{\sim}{\mathcal{H}}; \|\cdot\|_{\underset{\sim}{\mathcal{H}}})$ *so that* $\mathcal{H}$ *embeds compactly into* $\underset{\sim}{\mathcal{H}}$*, there exists a constant* $C>0$ *such that, for any* $h \in \mathcal{H}$*,*

$$\|h\|_{\mathcal{H}} \leq C\Big(\|(Sh, Th)\|_{Y \bigoplus Z} + \|h\|_{\underset{\sim}{\mathcal{H}}}\Big) = C\Big(\|Sh\|_Y + \|Th\|_Z + \|h\|_{\underset{\sim}{\mathcal{H}}}\Big). \quad (1.7)$$

Assume that two sequences $\{u^\epsilon\}, \{v^\epsilon\} \subset \mathcal{H}$ *satisfy the following conditions:*

(Seq 1) $u^\epsilon \rightharpoonup \bar{u}$ *and* $v^\epsilon \rightharpoonup \bar{v}$ *in* $\mathcal{H}$ *as* $\epsilon \to 0$*;*

(Seq 2) $\{Su^\epsilon\}$ *is pre-compact in* Y*, and* $\{Tv^\epsilon\}$ *is pre-compact in* Z*.*

Then

$$\langle u^\epsilon, v^\epsilon \rangle_{\mathcal{H}} \to \langle \bar{u}, \bar{v} \rangle_{\mathcal{H}} \qquad \text{as } \epsilon \to 0.$$

Notice that, in comparison with [78] by Kozono-Yanagisawa, our theorem above works for general Hilbert and Banach spaces; also our Condition (Seq 2) is weaker than the assumption in [78]: The latter requires the boundedness $\{u^\epsilon\}, \{v^\epsilon\}$ in L^2, for the special case $\mathcal{H} = L^2$ and $Y, Z = H^{-1}$.

We note that Theorem 1.3 implies a global, geometric div-curl lemma on Riemannian manifolds, by taking $S = d$ (the exterior derivative on differential forms) and $T = \delta$ (the L^2-adjoint of d). This in turn leads to the proof of Theorem 1.2, which crucially relies on the *ellipticity of the Laplace-Beltrami operator* $\Delta := d\delta + \delta d$. In addition, in the endpoint case $n = p = 2$ one also obtains the weak continuity of GCR equations, thanks to the equi-integrability structure of the Gauss equation (1.3) and a div-curl-type result due to Conti-Dolzmann-Müller ([46]).

As a natural extension of the aforementioned work, we also consider the semi-Riemannian analogue of Problem 1.1:

Problem 1.4 *Let* (M, g) *be a given* n*-dimensional semi-Riemannian manifold, isometrically immersed into the semi-Euclidean space* $(\mathbb{R}^{n+k}, \overline{g_0})$ *via* $\{f^\epsilon\} \subset W^{2,p}_{\rm loc}$*, with* $g \in W^{1,p}_{\rm loc}$*. If the corresponding second fundamental forms* $\{B^\epsilon\}$ *and normal connections* $\{\nabla^{\perp,\epsilon}\}$ *are uniformly bounded in* $L^p_{\rm loc}$*,* $p > 2$*, do the weak limits of* $\{(B^\epsilon, \nabla^{\perp,\epsilon})\}$ *still satisfy the Gauss-Codazzi-Ricci* (*GCR*) *equations?*

Here, $\overline{g_0}$ denotes the semi-Euclidean metric:

$$\overline{g_0} = {\rm diag}(\underbrace{-1, \cdots, -1}_{\nu \text{ times}}, \underbrace{1, \cdots, 1}_{(n-\nu) \text{ times}}; \underbrace{-1, \cdots, -1}_{\tau \text{ times}}, \underbrace{1, \cdots, 1}_{(k-\tau) \text{ times}}),$$

where ν and τ are the *indices* of M and the normal bundle, respectively; see §§3.1–3.2 for details. In passing, we remark that semi-Riemannian manifolds (M,g) include Riemannian manifolds as a special class, by relaxing the positive-definiteness of the metric g to the *non-degeneracy condition*: At each $x \in M$, there exists no $v \in T_xM \setminus \{0\}$ such that $g(v,w)=0$ for all $w \in T_xM$ (*cf.* O'Neill [103]).

Analogously, Problem 1.4 is also answered in the affirmative:

Theorem 1.5 *The results of Theorem 1.2 still hold for semi-Riemannian (M,g).*

We notice that the difficulty immediately arises as Theorem 1.3 is no longer valid: $\Delta = d\delta + \delta d$ is elliptic if and only if (M,g) is Riemannian. To overcome this difficulty, we first use *Cartan's formalism* (*i.e.*, the method of moving frames; see S. S. Chern [33], Spivak [111] and Tenenblat [121]) to translate the GCR equations into the second structural equation:

$$d\mathcal{W} = \mathcal{W} \wedge \mathcal{W}, \tag{1.8}$$

where $\mathcal{W}$ is the connection 1-form incorporating the second fundamental form, normal connection, and the Levi-Civita connection on TM (see §2 for details), and then deduce the weak continuity of Eq. (1.8) intrinsically and globally via the following

Theorem 1.6 *Let M be a semi-Riemannian manifold with metric g and volume form $\mathrm{d}V_g$. Also, let E, F be two real vector bundles over M. Consider a sequence of E-sections $\{u^\epsilon\} \subset L^2_{\mathrm{loc}}(M;E)$, a differential operator $\mathcal{T} \in \mathrm{Diff}^m(M;E,F)$ for some $m \in \mathbb{R}_+$ with the principal symbol $\sigma_m(\mathcal{T}) : T^*M \to \mathrm{Hom}(E;F^{\mathbb{C}})$, and a quadratic polynomial $Q : \Gamma(E) \to \mathbb{R}$. Assume that the following conditions hold:*

(C1) *$u^\epsilon \rightharpoonup u$ weakly in $L^2_{\mathrm{loc}}(M;E)$;*

(C2) *$\{\mathcal{T}u^\epsilon\}$ is pre-compact in $H^{-m}_{\mathrm{loc}}(M;F)$;*

(C3) *$Q(s)=0$ for all $s \in \Lambda_{\mathcal{T}}$, where the* operator cone *of $\mathcal{T}$ is defined by*

$$\Lambda_{\mathcal{T}} := \Big\{ s \in \Gamma(E) : \sigma_m(\mathcal{T})(\xi)(s) = 0 \in \Gamma(F^{\mathbb{C}}) \text{ for some } \xi \in T^*M \setminus \{0\} \Big\}. \tag{1.9}$$

Then we have

$$\lim_{\epsilon \to 0} \int_M (Q \circ u^\epsilon)\psi \,\mathrm{d}V_g = \int_M (Q \circ u)\psi \,\mathrm{d}V_g, \qquad \text{for any } \psi \in C_c^\infty(M). \tag{1.10}$$

Theorem 1.6 generalises the quadratic theorem of Tartar (Corollary 13, [118]). A key feature of the theorem is that the differential operator $\mathcal{T}$ on bundles (where $F^{\mathbb{C}}$ denotes the complexified bundle $F \otimes_{\mathbb{R}} \mathbb{C}$), the principal symbol $\sigma_m(\mathcal{T})$, the quadratic Q and the Sobolev spaces can all be defined globally, without referring to local coordinates. Heuristically, the obstruction to the distributional convergence of $\{Q \circ u^\epsilon\}$ occurs only in the high-frequency regime on the Fourier side; by the construction of the *operator cone* $\Lambda_{\mathcal{T}}$, such obstruction is forbidden.

In addition, we remark that the weak continuity of GCR equations in Theorem 1.5 also implies the weak rigidity of $W^{2,p}_{\rm loc}$ isometric immersions, as in the Riemannian case. In the case of immersions of degenerating hypersurfaces (which cannot be isometric), the weak rigidity theorem is also proved, in the presence of a "rigging vector field" ([80, 81, 108]). Moreover, some linkages have been established between the global compensated compactness result (Theorem 1.6) and the weak continuity of quasi-linear wave equations with the well-known *null condition*, which is introduced by Klainerman and Christodoulou in [75, 34] and plays a crucial role in the small data global existence theory in 3 spatial dimensions. The crucial observation is that the light cone of the Minkowski space embeds into the operator cone $\Lambda_{\mathcal{T}}$ in Theorem 1.6, whenever the non-linear part of the relevant differential operators satisfies the null condition.

To conclude our introduction for the first project, let us emphasise that Problems 1.1 and 1.4 are answered *globally* and *intrinsically* by Theorems 1.3 and 1.6, which are new and "abstract" compensated compactness results. Historically, the theory of compensated compactness has found profound applications in gas dynamical PDEs and conservation laws; *cf.* Dafermos [48], DiPerna-Majda [50], and Ding-Chen-Luo [49] and the references cited therein. In this thesis we offer some generalisations of the theory, adaptable for applications to the differential geometric problems with global and intrinsic nature. Such methods may lead to further weak continuity/weak rigidity results of other non-linear PDEs, with non-linearities possessing certain structures reflecting the phenomenon of *cancellation*, *e.g.*, Condition (Op 1) of Theorem 1.3 and Condition (C3) of Theorem 1.6.

1.2.2 Vanishing viscosity limit of the 3D incompressible fluid with Navier boundary condition

Our second project is about a classical topic in mathematical hydrodynamics, known as the *vanishing viscosity limit* (or the *inviscid limit*) problem. Consider the Cauchy problem of the 3D incompressible Navier-Stokes equation with viscosity coefficient $\nu > 0$:

$$\begin{cases} \partial_t u^\nu + (u^\nu \cdot \nabla) u^\nu - \nu \Delta u^\nu + \nabla p^\nu = 0 \qquad \text{in } [0,T] \times \Omega, \\ \nabla \cdot u^\nu = 0 \qquad \text{in } [0,T] \times \Omega, \\ u|_{t=0} = u_0 \qquad \text{on } \Omega. \end{cases} \tag{1.11}$$

Here, the vector field $u^\nu : \Omega \to \mathbb{R}^3$ is the velocity of the fluid and the scalar field $p^\nu : \Omega \to \mathbb{R}$ is the pressure, both of which depend on ν. Throughout $\Omega \subset \mathbb{R}^3$ is a bounded open smooth domain in $\mathbb{R}^3$.

In our work we impose the *kinematic boundary condition*, namely

$$u^\nu \cdot \mathbf{n} = 0 \qquad \text{on } \partial\Omega \times [0,T], \tag{1.12}$$

as well as the *Navier boundary condition.* The latter was first written down by Navier in the classical work [99], and then reappeared in [93] by Maxwell. It requires that *the tangential component of* $u^\nu|_{\partial\Omega}$ *to be proportional* (*by a constant* ζ, *known as the* slip length) *to the tangential component of the normal vector field of the Cauchy stress tensor.* The Navier boundary condition has received considerable attention in both the physics and mathematics community recently; *cf.* Einzel-Panzer-Liu [53] and Qian-Wang-Sheng [104] for the discussion on the physical models, as well as Jäger-Mikelič [68, 69], Lopes Filho-Nussenzveig Lopes-Planas [84], Iftimie-Planas [66], Iftimie-Raugel-Sell [67], Masmoudi-Rousset [92], Neustupa-Penel [100, 101] and Xiao-Xin [124], among others, for the mathematical treatment.

To derive the mathematical formulation of the Navier boundary condition, let us first introduce the rate-of-strain tensor:

$$\mathbb{D}u := \frac{1}{2}(\nabla u + \nabla^\top u), \tag{1.13}$$

which can be expressed in local coordinates as

$$(\mathbb{D}u)_{ij} = \frac{1}{2}\big(\nabla_i u^j + \nabla_j u^i\big) \qquad \text{for } i,j \in \{1,2,3\}.$$

Then, the Navier boundary condition reads:

$$u \cdot \tau = -2\zeta\, \mathbb{D}(\tau, \mathbf{n}) \qquad \text{for any } \tau \in T(\partial\Omega) \text{ on } \partial\Omega \times [0,T], \tag{1.14}$$

where $\mathbb{D}u(\tau, \mathbf{n}) = \tau^\top \cdot \mathbb{D}u \cdot \mathbf{n}$ is defined by the matrix multiplication.

The key issue for establishing the existence of strong solution (in H^{r+1}) and the vanishing viscosity limit (in H^r) is to derive the *higher-order energy estimates* of the velocities u^ν in H^{r+1}. For this purpose, we need to estimate the L^2 norm of H^{r+1}

derivatives of u^ν, which is bounded by their iterated curls, as u^ν are divergence free (justified in §4.2). So, it is helpful to *reformulate the Navier boundary condition in terms of the vorticities* $\omega^\nu = \nabla \times u^\nu$.

This is done by Chen-Osborne-Qian [29] and Chen-Qian [30] (and will be sketched in §4.1), in which the Navier boundary condition is shown to be equivalent to the following identity involving only geometric quantities:

$$\omega^\nu \cdot \tau = -\frac{1}{\zeta}(\mathcal{R}u^\nu)\cdot\tau + 2\mathcal{R}\big(\mathcal{S}(u^\nu)\big)\cdot\tau \qquad \text{on } \partial\Omega \times [0,T]. \tag{1.15}$$

Here $\tau \in T(\partial\Omega)$ is an arbitrary tangential vector field on $\partial\Omega$, $\mathcal{S} : T(\partial\Omega) \to T(\partial\Omega)$ is the shape operator corresponding to the second fundamental form II of $\partial\Omega$, namely

$$\mathcal{S}u := -\nabla_u \mathbf{n} \tag{1.16}$$

for the unit normal $\mathbf{n} : \partial\Omega \to \mathbb{S}^2$, and $\mathcal{R}$ denotes the anti-clockwise rotation of a vector field in $T(\partial\Omega)$ by $\pi/2$. In local coordinates on $\partial\Omega$ one has

$$\mathcal{R}(V^1, V^2)^\top = (-V^2, V^1)^\top \Longleftrightarrow \mathcal{R} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}; \tag{1.17}$$

in fact, $\mathcal{R}$ can be defined globally and intrinsically too:

$$\mathcal{R}V = (*(V^\sharp))^\flat \qquad \text{for } V \in T(\partial\Omega), \tag{1.18}$$

where $*$ is the Hodge star for differential forms on $\partial\Omega$, $\sharp$ is the canonical isomorphism between vector fields and differential one-forms, and $\flat$ is the inverse of $\sharp$. The punchline is that the Navier boundary condition admits a natural, global formulation, purely in terms of geometric quantities on the embedded regular surface $\partial\Omega$. This formulation helps us to treat the boundary terms during the integration by parts.

Now, let us consider the Euler equation describing the motion of the inviscid incompressible fluid:

$$\begin{cases} \partial_t u + (u\cdot\nabla)u + \nabla p = 0 \qquad \text{in } [0,T]\times\Omega, \\ \nabla\cdot u = 0 \qquad \text{in } [0,T]\times\Omega, \\ u|_{t=0} = u_0 \qquad \text{on } \Omega, \end{cases} \tag{1.19}$$

with the *no-penetration* (*kinematic*) *boundary condition*:

$$u\cdot\mathbf{n} = 0 \qquad \text{on } [0,T]\times\partial\Omega. \tag{1.20}$$

The initial data u_0 coincides with that of the Navier-Stokes equation. Our question about the vanishing viscosity limit is as follows:

Problem 1.7 *As $\nu \to 0^+$, do the strong solutions u^ν (if they exist) converge to u in suitable topologies?*

In the thesis we first prove the local (in-time) existence of strong solutions u^ν to the Navier-Stokes equation (Eqs. (1.11), (1.12) and (1.15)) in the Sobolev spaces H^{r+1}, provided that $r > 5/2$. In fact, if the solution of the Euler equation (Eqs. (1.19) and (1.20)) in the same regularity class exists upto time $T_\star$, then the lifespan of the corresponding Navier-Stokes equation must be greater than or equal to $T_\star$. (The above statement is proved by adapting P. Constantin's work [41] on $\mathbb{R}^3$ or $\mathbb{T}^3$.) Hence, under the assumption that $\partial\Omega$ is smooth (indeed, we need the boundedness of the second fundamental form in C^r), the inviscid limit holds in H^r. Here and throughout, when we write H^r on its own, it denotes the regularity in the *spatial* variable.

More precisely, let $K_2(\Omega) := \{u \in L^2(\Omega; \mathbb{R}^3) : \nabla \cdot u = 0\}$, namely the space of divergence-free L^2 vector fields on Ω; we prove that

Theorem 1.8 *Let $u \in C([0, T_\star); H^{r+1}(\Omega; \mathbb{R}^3) \cap K_2(\Omega))$ be the unique strong solution of the incompressible Euler equation* (1.19) *subject to the no-penetration boundary condition* (1.20)*, $r > 5/2$. Then there exists some $\nu_\star = \nu_\star\Big(T_\star, \int_0^{T_\star} \|u(t,\cdot)\|^2_{H^{r+1}(\Omega)}\,\mathrm{d}t\Big)$, such that whenever $0 < \nu \leq \nu_\star$, the strong solution u^ν of the Navier-Stokes equation* (1.11) *with Navier and kinematic boundary conditions* (1.15) *and* (1.12) *exists in $C([0, T_\star); H^{r+1}(\Omega; \mathbb{R}^3) \cap K_2(\Omega))$. Moreover, there exists a constant C depending only on ν^0, $T_\star$ and $\|u\|_{L^2([0,T_\star);H^{r+1}(\Omega))}$ and $\|\mathrm{II}\|_{C^r(\partial\Omega)}$, such that*

$$\sup_{0 \leq t < T_\star} \|u^\nu(t,\cdot) - u(t,\cdot)\|_{H^r(\Omega)} \leq C\nu^{1/3}. \tag{1.21}$$

In particular, as $\nu \to 0^+$, u^ν converges to u in $H^r(\Omega)$ uniformly in time.

Let us comment on the history of Problem 1.7.

First of all, it is well-known that the strong solution of the Navier-Stokes equation with no-slip boundary condition (*i.e.*, the Dirichlet condition $u|_{\partial\Omega} = 0$) does not converge in strong topologies to that of the Euler equation with no-penetration boundary condition. Indeed, *boundary layers* are developed according to the theory of Prandtl, which inhibits the strong inviscid limits; *cf.* [3, 59, 91, 114, 74] and the references cited therein. If, in contrast, we consider a fluid domain with no boundary (*i.e.,* on $\mathbb{R}^3$ or $\mathbb{T}^3$), the vanishing viscosity limit holds; see Constantin [41].

On the other hand, for the kinematic and Navier boundary conditions, if the domain of the fluid is *flat* (*e.g.* the half space $\{x_3 \geq 0\}$), various previous papers showed that the solutions of the Navier-Stokes equation in X (a Banach space such as H^s, $W^{k,p}$, etc.) converge in X' to the Euler solutions as $\nu \to 0^+$ (uniformly in time), where X compactly embeds into X'. In other words, no boundary layers are developed in X', which has strictly coarser topology than X; *cf.* Xiao-Xin [124], Beirão da Veiga-Crispo [12, 13], Berselli-Spirito [19], Clopeau-Mikelič-Robert [36], Bellout-Neustupa-Penel [16] and many others. The flatness of the fluid domain is crucial in these works: In this case the Navier boundary condition becomes

$$\omega^\nu \times \mathbf{n} = 0 \qquad \text{on } \partial\Omega \times [0, T]. \tag{1.22}$$

This is commonly known as the modified "slip-type" or "Navier-type" boundary conditions, which has been first proposed by Bardos [9] and Solonnikov-Ščadilov [110] to analyse the stationary linearised Navier-Stokes equations. Subject to Eq. (1.22), the energy estimates employed to derive the inviscid limits in the aforementioned works are enormously simplified.

Now let us discuss the "real" Navier boundary condition, namely Eq. (1.15), on a regular 3D fluid domain Ω. In this direction, Chen-Qian [30] establishes the existence of weak solutions in H^1 and the inviscid limit in $L^\infty_t L^2_x$, provided that the Euler equation has a solution in H^2, and that the Navier-Stokes equations has a uniform lifespan independent of ν. Similar results are proved by Iftimie-Planas via computations in local coordinates in [66]. Also, Neustupa-Penel [100, 101] proves the existence of strong solutions in $L^\infty(0, T; H^1(\Omega; \mathbb{R}^3)) \cap L^2(0, T; H^2(\Omega; \mathbb{R}^3))$, for $u_0 \in H^4(\Omega; \mathbb{R}^3) \cap K^2(\Omega)$. Moreover, in [92] Masmoudi-Rousset establishes the existence of strong solutions in $L^\infty(0, T; E^m(\Omega; \mathbb{R}^3)) \cap L^2(0, T; H^{m+1}(\Omega; \mathbb{R}^3))$ for $m > 6$ and the inviscid limit in $L^\infty_t L^2_x$, where the anisotropic co-normal Sobolev space E^m is given by $E^m := \{u \in H^m_{\rm co} : \nabla u \in H^{m-1}_{\rm co}\}$, and $u \in H^m_{\rm co}$ whenever $\sum_{0 \leq |l| \leq m} \|Z_l u\|_{L^2(\Omega)} < \infty$ with $\{Z_l\}$ spanning the space of vector fields tangential to $\partial\Omega$.

In view of the preceding survey, the new features of Theorem 1.8 include (i), the existence of strong solutions in higher-order Sobolev spaces H^{r+1}, $r > 5/2$ with a lifespan independent of ν, and (ii), the convergence of inviscid limits in H^r under the "real" Navier boundary condition (1.15), especially in the presence of the non-vanishing second fundamental form of $\partial\Omega$.

Moreover, we also show the failure of the vanishing viscosity limit of strong solutions under the modified "slip-type" boundary condition (1.22), in arbitrarily short time. This is a re-proof of the results in [14, 15] by Beirão da Veiga-Crispo, and we provide purely geometric and global arguments here, by exploiting a Lie bracket structure hidden in the vorticity equation of the incompressible Navier-Stokes equation (1.11). Roughly speaking, it rules out the hope to use the model problem for flat domains in the analysis of the real problem on curved domains.

Theorem 1.9 *Let $\Omega \subset \mathbb{R}^3$ be a bounded, regular, open set. Let the admissible initial data u_0 be given for the initial-boundary value problems Eqs.* (1.11)(1.22)(1.19) *and* (1.20), *such that the following condition holds:*

$$\omega_0(x_0) \neq 0 \qquad \text{for some } x_0 \in \partial\Omega \text{ such that } K(x_0) \neq 0, \tag{1.23}$$

where K is the Gauss curvature of $\partial\Omega$. Then for arbitrary $\delta > 0$, $p, q \geq 1$ and $s > 1$, the "strong" inviscid limit fails:

$$u^\nu \nrightarrow u \text{ in } L^p([0,\delta]; W^{s,q}(\Omega; \mathbb{R}^3)). \tag{1.24}$$

1.3 Outline of the thesis

In §2 we prove the weak continuity of the Gauss-Codazzi-Ricci Equations on Riemannian manifolds. In this process we also establish the generalised div-curl lemma (Theorem 1.3), and investigate the geometric consequences and the end-point cases too. Our presentation is based on the following paper:

[25] G.-Q. Chen, S. Li, Global weak rigidity of the Gauss-Codazzi-Ricci equations and isometric immersions of Riemannian Manifolds with lower regularity. Submitted to *J. Geom. Anal.*, ArXiv: 1607.06862 (2016).

In §3 we study the analogous problem on semi-Riemannian manifolds, for which the generalised quadratic theorem of the compensated compactness theory (Theorem 1.6) is established. In addition, we discuss the weak rigidity of isometric immersions of semi-Riemannian manifolds, the case of degenerate hypersurfaces, as well as the weak continuity of quasi-linear wave equations with the null condition. Our presentation follows from:

[26] G.-Q. Chen, S. Li, Weak continuity of Gauss-Codazzi-Ricci equations for isometric immersions of semi-Riemannian manifolds. *In Preparation.*

The problems in §§2–3 are introduced in §1.2.1; also see our exposition paper [27] for a survey:

[27] G.-Q. Chen, S. Li, Compensated compactness in Banach spaces and weak rigidity of isometric immersions of manifolds. Accepted by the proceedings for Helge Holden's sixtieth birthday, published by EMS. ArXiv: 1610.01649.

In §4, the uniform existence of strong solutions and the vanishing viscosity limits of the 3D incompressible Navier-Stokes equation will be established, with the Navier and kinematic boundary conditions (see §1.2.2). Our presentation follows from:

[28] G.-Q. Chen, S. Li and Z. Qian, Inviscid limits from Navier-Stokes to Euler equations under kinematic and Navier boundary conditions. *In preparation.*

Upon the completion of the D.Phil. thesis, another research paper of the candidate has been submitted to a scientific journal. In this work we consider the Gauss-Codazzi equations of the isometric immersions of a family of 2-dimensional surfaces into $\mathbb{R}^3$, and the connections with the existence of "wild" solutions to the incompressible Euler equation, as well as several models in compressible fluid dynamics and elasticity. The aim is to provide further linkages between the isometric immersions problem and mathematical hydrodynamics:

A. Acharya, G.-Q. Chen, S. Li, M. Slemrod and D. Wang, Fluids, elasticity, geometry, and the existence of wrinkled solutions. To appear in *Arch. Rat. Mech. Anal.* (2017), Arxiv: 1605.03058.

Chapter 2

Weak Continuity of Gauss-Codazzi-Ricci Equations: The Riemannian Case

2.1 Geometric preliminaries

In this section we start with the geometric notations about manifolds and vector bundles, and then present some basic facts about isometric immersions and the Gauss-Codazzi-Ricci (GCR) equations on Riemannian manifolds.

2.1.1 Manifolds and vector bundles

Throughout this chapter, we denote (M, g) as an n-dimensional Riemannian manifold. By definition, M is a second-countable, Hausdorff topological space with an atlas of local charts $\mathcal{A} = \{(U_\alpha, \phi_\alpha) : \alpha \in \mathcal{I}\}$ such that each $U_\alpha \subset M$ is an open subset, $\phi_\alpha : U_\alpha \to \phi_\alpha(U_\alpha) \subset \mathbb{R}^n$ is a homeomorphism, and the transition functions $\phi_\alpha \circ \phi_\beta^{-1}$ between the overlapping charts U_α and U_β have required regularity.

If each $\phi_\alpha \circ \phi_\beta^{-1}$ can be chosen to have positive-definite Jacobian determinant almost everywhere, then M is said to be orientable. Moreover, M is simply-connected if $\pi_1(M) = \{0\}$, *i.e.*, each loop on M can be continuously deformed to a point. The Riemannian metric g on M is given by a field of inner products: At each point $x \in M$, $g(x) : T_xM \times T_xM \to \mathbb{R}$ is an inner product denoted by

$$g(x)(v, w) = \langle v, w\rangle \qquad \text{for any } v, w \in T_xM,$$

and $g : \Gamma(TM) \times \Gamma(TM) \to \mathbb{R}$ varies smoothly with respect to $x \in M$. We write $\Gamma(TM)$ for the space of vector fields on M with required regularity, and we suppress the dependence on $x \in M$ when using $\langle \cdot, \cdot\rangle$ to denote the inner product associated with the metric. More generally, we always denote by Γ the space of sections of a given vector bundle with required Sobolev regularity (which needs not to be smooth, for the applications to the realisation problem in §2.4).

Given (M, g), an affine connection on TM is a bilinear map $\nabla : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$ such that, for any $f : M \to \mathbb{R}$ and $X, Y, Z \in \Gamma(TM)$,

$$\nabla_{fX} Y = f \nabla_X Y, \qquad \nabla_X (fY) = f \nabla_X Y + X(f) Y.$$

Here and in the sequel, $X(f)$ means the directional derivative of f in the direction of X; *i.e.*, we identify each vector field with a first order differential operator. Moreover, we say that ∇ is compatible with the metric g if

$$Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z),$$

and ∇ is torsion-free if

$$\nabla_X Y - \nabla_Y X = [X, Y],$$

where the Lie bracket given by $[X, Y] = XY - YX$. There exists a unique compatible, torsion-free affine connection ∇ on TM, known as the Levi-Civita connection, where the map ∇ is also called the covariant derivative. As a basic example, consider $(\bar{M}, \bar{g}) = (\mathbb{R}^m, g_0)$ with the Euclidean metric $g_0 = \delta_{ij}$, *i.e.*, the dot product, whose Levi-Civita connection $\bar{\nabla}$ is given by $\bar{\nabla}_X Y := XY$. This is because for a Euclidean coordinate frame $\{\partial_1, \ldots, \partial_m\}$ on $\mathbb{R}^m$ one has $\bar{\nabla}_{\partial_i} \partial_j = 0$ for all $1 \leq i, j \leq m$; hence, for $X = X^i \partial_i$ and $Y = Y^j \partial_j$, it holds that $\bar{\nabla}_X Y = \bar{\nabla}_{X^i \partial_i} Y^j \partial_j = X^i \bar{\nabla}_{\partial_i} (Y^j \partial_j) = X^i \partial_i Y^j \partial_j$. The Einstein summation convention is applied throughout.

Another example pertaining to our construction in §§2–3 is as follows: Let $\bar{\nabla}$ be the Levi–Civita connection on $(\mathbb{R}^m, g_0)$ as above, and let $(M, g) \subset (\mathbb{R}^m, g_0)$ be a submanifold. Then, given a tangential vector field $X \in \Gamma(TM)$ and a normal vector field $\xi \in \Gamma(TM^\perp)$, where $TM^\perp = T\mathbb{R}^m / TM$ is the normal bundle of the submanifold, we can extend them (not relabelled) to $X, \xi \in \Gamma(T\mathbb{R}^m)$, at least locally. Then we set

$$\nabla^\perp_X \xi := \bar{\nabla}_X \xi - \nabla_X \xi. \tag{2.1}$$

Here $\nabla^\perp : \Gamma(TM) \times \Gamma(TM^\perp) \to \Gamma(TM^\perp)$ is the *normal connection* of the submanifold $M \subset \mathbb{R}^m$; it is well–defined, independent of the extensions of X, ξ (see §6 in do Carmo [51]). Moreover, $\nabla^\perp$ also gives the covariant derivatives on the normal bundle $TM^\perp$. For example, consider a tensor $\mathcal{N} : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM^\perp)$; for $X, Y, W \in \Gamma(TM)$, we can define naturally

$$\nabla^\perp_X \mathcal{N}(Y, W) := \bar{\nabla}_X (\mathcal{N}(Y, W)) - \mathcal{N}(\nabla_X Y, W) - \mathcal{N}(Y, \nabla_X W).$$

We use this construction in Theorem 3.1, where $\mathcal{N}$ is the second fundamental form.

Given a manifold M, we say that (E, M, F) is a vector bundle of degree $k \in \mathbb{N}$ over M if there is a surjection $\pi : E \to M$ such that, for any $x \in M$, there exists a local neighbourhood $U \subset M$ containing x so that there is a diffeomorphism $\psi_U : \pi^{-1}(U) \to U \times F$ with $\mathrm{pr}_1 \circ \psi_U = \pi$ on $\pi^{-1}(U)$, where pr_1 is the projection map onto the first coordinate, E and F are also manifolds, and $F \simeq \mathbb{R}^k$ as a vector space isomorphism. Therein, E is called the total space, F is the fibre, M is the base manifold, π is the projection of the bundle, and ψ_U is a local trivialisation. For simplicity, we also say that *E is a vector bundle over M*.

If E_1 and E_2 are both vector bundles over M, we can take the direct sum and the quotient of the bundles, by taking the vector space direct sum and quotient of the fibres. Also, for a vector bundle $(E, M, F = \mathbb{R}^k)$ with projection π, the space of smooth sections is defined by $\Gamma(E) := \{s \in C^\infty(M; E) : \pi \circ s = \mathrm{id}_M\}$. We can define the affine connection $\nabla^E : \Gamma(TM) \times \Gamma(E) \to \Gamma(E)$, by linearity and the Leibniz rule.

As a primary example, consider $E = TM$, the tangent bundle over M. Then π is the projection onto the base point in M, and $\Gamma(TM)$ is the space of smooth vector fields, agreeing with the previous notation. Moreover, ∇^{TM} is precisely the Levi-Civita connection. Another example is the cotangent bundle T^*M over M, whose fibres are the dual spaces of the fibres of TM.

As another example, consider $E_1 = T^*M \otimes T^*M \otimes \cdots \otimes T^*M$, the tensor product of q copies of T^*M, for $q = 0, 1, 2, \ldots$. This is the (covariant) q–tensor algebra over M, which can be viewed as the space of q-linear maps on TM: Any $\phi \in E_1$ is a map $\phi : TM \otimes \ldots \otimes TM \to \mathbb{R}$, which is linear is each of the component TM. Now let $E_2 \subset E_1$ be the subspace of all the alternating q-tensors on TM, *i.e.*, the q-linear maps that change sign when any pair of its indices $\{i, j\} \subset \{1, \ldots, q\}$ gets swapped. By convention, we write $E_2 =: \bigwedge^q T^*M$, known as the alternating q–algebra. Moreover, the space of sections $\Omega^q(M) := \Gamma(\bigwedge^q T^*M)$ is known as the *differential q–forms.* A general element $\alpha \in \Omega^q(M)$ can be written as a linear combination of alternating forms $\xi_1 \wedge \cdots \wedge \xi_q$, where $\{\xi_j\}_{j=1}^q$ is a q-tuple of linearly independent 1–forms on M. If $\dim(M) = n$, $\Omega^q(M) = \{0\}$ for $q \geq n + 1$, so one always restricts to $0 \leq q \leq n$.

Next, we recall four operations on the space of differential forms. The first is the *exterior derivative* $d : \Omega^q(M) \to \Omega^{q+1}(M)$, which satisfies $d^2 = 0$. The second is the *Hodge star* $* : \bigwedge^q T^*M \to \bigwedge^{n-q} T^*M$, which can also be regarded as $* : \Omega^q(M) \to \Omega^{n-q}(M)$. In local coordinates, if $dx^1 \wedge \cdots \wedge dx^n$ is a volume form on M, and $\omega = f(x) dx^{i_1} \wedge \cdots \wedge dx^{i_q}$ for some coefficient f locally defined on M, then $*\omega := f(x) dx^{i_{q+1}} \wedge \cdots \wedge dx^{i_n}$, where the n–tuple $(i_1, \ldots, i_q, i_{q+1}, \ldots, i_n)$ is an even permutation of $(1, 2, \ldots, n)$, *i.e.*, it can be transformed into $(1, 2, \ldots, n)$ by taking the transpositions of an even number of pairs of indices. If such $(i_{q+1}, \ldots, i_n)$ does not exist, then $*\omega := 0$. The Hodge star is a vector bundle isomorphism satisfying $** = (-1)^{q(n-q)}$ when M is orientable. The third is a natural product on differential forms: For $\alpha \in \Omega^q(M)$ and $\beta \in \Omega^r(M)$, we can define the *wedge product* $\alpha \wedge \beta \in \Omega^{q+r}(M)$. The fourth is the *covariant derivative* $\nabla : \Omega^q(M) \to \Omega^{q+1}(M)$: For $\alpha \in \Omega^q(M)$, define $\nabla\alpha \in \Omega^{q+1}(M)$ via the Leibniz rule:

$$\begin{aligned}(\nabla\alpha)(X, Y_1, \ldots, Y_q) &\equiv \nabla_X \alpha(Y_1, \ldots, Y_q) \\ &:= X\big(\alpha(Y_1, \ldots, Y_q)\big) - \alpha(\nabla_X Y_1, \ldots, Y_q) - \cdots - \alpha(Y_1, \ldots, \nabla_X Y_q), \end{aligned} \tag{2.2}$$

for any $X, Y_1, \ldots, Y_q \in \Gamma(TM)$.

There is a natural isomorphism between TM and T^*M, by identifying canonically the fibres of TM with the duals. It induces a canonical isomorphism $\sharp : \Omega^1(M) \to \Gamma(TM)$, for which we write $\sharp^{-1} =: \flat$. Note that, if a vector field and its corresponding 1–form in local coordinates are written by the Einstein summation convention, $\sharp, \flat$ amounts to raising (lowering, respectively) the indices of the coefficients. Clearly, they extend to the isomorphisms between $T^*M \otimes \cdots \otimes T^*M$ (*i.e.*, covariant tensors) and $TM \otimes \cdots \otimes TM$ (*i.e.*, contravariant tensors).

For instance, consider the covariant derivative $\nabla : \Omega^q(M) \to \Omega^{q+1}(M)$ defined above. For $q = 1$ and $\alpha \in \Omega^1(M)$ we set $X := \alpha^\sharp \in \Gamma(TM)$; since $\nabla_Y X \in \Gamma(TM)$ for any $Y \in \Gamma(TM)$, we can view $\nabla X = \nabla\alpha^\sharp \in \Gamma(TM \otimes TM)$, *i.e.*, $(\nabla\alpha^\sharp)^\flat \in \Omega^2(M)$. This example shows that, via the $\sharp$ and $\flat$, the covariant derivative ∇ on $\Omega^q(M)$ generalises the Levi-Civita connection.

Now, let us briefly review how to integrate differential n–forms on n–dimensional manifolds. On any orientable n-dimensional Riemannian manifold (M, g), there is a natural n–form $\mathrm{d}V_g \in \Omega^n(M)$, called the *volume form*, which satisfies $*1 = \mathrm{d}V_g$ and $*\mathrm{d}V_g = 1$. Let $\mathcal{A} = \{(U_\alpha, \phi_\alpha) : \alpha \in \mathcal{I}\}$ be an atlas for M as before. Then, there exists a locally finite C^∞–partition of unity $\{\rho_\alpha : \alpha \in \mathcal{I}\}$ subordinate to $\mathcal{A}$, *i.e.*, $\sum_{\alpha\in\mathcal{I}} \rho_\alpha = 1$, $0 \le \rho_\alpha \le 1$, and $supp(\rho_\alpha) \Subset U_\alpha$. We define the integration of $\omega \in \Omega^n(M)$ over M by

$$\int_M \omega := \sum_{\alpha\in\mathcal{I}} \int_{\mathbb{R}^n} \rho_\alpha (\phi_\alpha^{-1})^* \omega \mathbb{1}_{\phi_\alpha(U_\alpha)} \sqrt{\det g}\, \mathrm{d}x_1 \cdots \mathrm{d}x_n, \tag{2.3}$$

where ϕ^* denotes the pullback under ϕ. The integration of the function ϕ on M is defined as the integration of its Hodge dual, *i.e.* $\int_M \phi := \int_M \phi \, \mathrm{d}V_g$.

We then define the *Sobolev spaces* $W^{k,p}(M;\bigwedge^q T^*M)$ on an n-dimensional Riemannian manifold (M,g), which generalise $W^{k,p}(\mathbb{R}^n)$ for $k\in\mathbb{Z}$, $1\le p\le\infty$. First, $\alpha,\beta\in\Omega^q(M)$, the metric g induces an inner product $g(\alpha,\beta)=\langle\alpha,\beta\rangle$ via

$$\langle\alpha,\beta\rangle\,\mathrm{d}V_g := \alpha\wedge *\beta. \tag{2.4}$$

Eq. (2.4) is an equality of n–forms on M. Then, for alternating contravariant q–tensor fields σ_1 and σ_2, we set $\langle\sigma_1,\sigma_2\rangle := \langle\sigma_1^\flat,\sigma_2^\flat\rangle$. Next, define the L^p norm of $\alpha\in\Omega^q(M)$ by

$$\|\alpha\|_{L^p} := \Big(\int_M \big[*(\alpha\wedge*\alpha)\big]^{\frac{p}{2}}\,\mathrm{d}V_g\Big)^{\frac{1}{p}} = \Big(\int_M \langle\alpha,\alpha\rangle^{\frac{p}{2}}\,\mathrm{d}V_g\Big)^{\frac{1}{p}}. \tag{2.5}$$

Moreover, for $\alpha\in\Omega^q(M)$, set

$$\|\alpha\|_{W^{k,p}} := \Big(\sum_{j=0}^k \|\nabla^j\alpha\|_{L^p}^p\Big)^{\frac{1}{p}} = \Big(\sum_{j=0}^k \|\underbrace{\nabla\circ\cdots\circ\nabla}_{j\text{ times}}\alpha\|_{L^p}^p\Big)^{\frac{1}{p}}. \tag{2.6}$$

We denote by $W^{k,p}(M;\bigwedge^q T^*M)$ the completion of the space of compactly supported q–forms with respect to $\|\cdot\|_{W^{k,p}}$. Notice that, for any contravariant tensor field X, the following holds:

$$\|X\|_{W^{k,p}} := \|X^\flat\|_{W^{k,p}}.$$

In fact, $W^{k,p}(M;E)$ can be defined for an arbitrary bundle E, and we can define the $W^{k,p}$ connection ∇^E on bundle E. This is possible since the moduli space of connections is an affine space modelled over $\bigotimes^\bullet TM\otimes\bigotimes^\bullet T^*M\otimes E$ (*cf.* Jost [71] for detailed constructions). A key feature of this definition lies in its *intrinsic* nature, since the local coordinates on M are not needed to define $W^{k,p}(M;\bigwedge^q T^*M)$. As usual, we denote $H^k(M;\bigwedge^q T^*M) := W^{k,2}(M;\bigwedge^q T^*M)$.

2.1.2 Isometric immersions

We are concerned with the isometric immersions of an n–dimensional Riemannian manifold (M,g) into the Euclidean spaces. A map $f:(M,g)\hookrightarrow(\mathbb{R}^{n+k},g_0)$ is an isometric immersion if the differential $df: TM\to T\mathbb{R}^{n+k}$ is everywhere injective, and

$$g_0(f(x))(df_x(X),df_x(Y)) = g(x)(X,Y) \tag{2.7}$$

for every $x\in M$ and $X,Y\in\Gamma(TM)$. As g_0 is the Euclidean inner product, Eq. (2.7) reads

$$\langle df_x(X),df_x(Y)\rangle = df_x(X)\cdot df_x(Y) = \langle X,Y\rangle, \tag{2.8}$$

or equivalently,

$$D^\top f\cdot Df = g \tag{2.9}$$

as an equation on $O(n)$, the group of symmetric $n \times n$ matrices.

We observe that Eqs. (2.8) and (2.9) are a first-order non-linear system of PDEs of f. It is over-determined, determined or underdetermined for $n+k < J_n$, $n+k = J_n$ or $n+k > J_n$ respectively, where

$$J_n = \dim O(n) = \frac{n(n+1)}{2} \tag{2.10}$$

is known as the *Janet dimension.* The general existence theorem of isometric immersions/embeddings is established by Nash ([97, 98]) for large enough codimensions. In general, these first-order systems are of mixed type; in the special case of the isometric immersions of a 2D surface into $\mathbb{R}^3$, the type of the equations is classified completely by the sign of the Gauss curvature. We refer the readers to the monograph [65] by Han-Hong for details.

2.1.3 Gauss-Codazzi-Ricci (GCR) equations

Nash directly tackled Eq. (2.9) in his seminal work [97]. By exploiting the "Nash wrinkles", he proved that in an arbitrarily small C^0 neighbourhood of any "short" immersion of a given Riemannian manifold, there exists a C^1 isometric immersion, provided that the co–dimension is no less than 2. In this construction, the second derivatives of the C^1 isometric immersion generically blow up at infinitely many points, hence the second fundamental form and the normal connections (which involve one derivative of g) as well as the Riemann curvature (which involves two derivatives of g) are not well defined for Nash's isometric immersions.

On the other hand, let us require more regularity than the scenario in [97]: Suppose that the second fundamental forms and normal connections are L^p_{loc} functions for $p > 2$, and the Riemann curvature tensor $R \in W^{-1,p}_{\mathrm{loc}}$. In this case R is a distribution, so we can talk about the following *compatibility condition* of an isometric immersion $f : (M, g) \to (\mathbb{R}^{n+k}, g_0)$, in the sense of distributions:

$$\begin{aligned} 0 &= \text{Curvature of } \mathbb{R}^{n+k} \\ &\iff \begin{cases} \text{curvature in (tangential, tangential) direction} = 0, \\ \text{curvature in (tangential, normal) direction} = 0, \\ \text{curvature in (normal, normal) direction} = 0. \end{cases} \end{aligned}$$

These three schematic equations above are the *Gauss*, *Codazzi* and *Ricci* equations; see §6 in do Carmo [51]. In other words, the GCR equations are derived from the orthogonal splitting of tangent spaces along the isometric immersion $f : (M, g) \hookrightarrow$

$(\mathbb{R}^{n+k}, g_0)$: The tangent spaces satisfy $T_x\mathbb{R}^{n+k} \cong T_xM \bigoplus T_xM^\perp$ for each $x \in M$, where $T_xM^\perp$ is the fibre of the *normal bundle* $TM^\perp$ at point x, and $TM^\perp$ is defined as the quotient vector bundle $T\mathbb{R}^{n+k}/TM$. Here and hereafter, we obey the widely adopted convention to identify TM with $T(fM)$.

In the rest of this chapter we will use Latin letters $X, Y, Z, \ldots$ to denote tangential vector fields, *i.e.* elements in $\Gamma(TM)$, and Greek letters $\xi, \eta, \zeta, \ldots$ to denote normal vector fields, *i.e.*, elements in $\Gamma(TM^\perp)$. We also adopt the convention that the geometric quantities with an overhead bar are associated with the total space $\mathbb{R}^{n+k}$, while the quantities with superscript $\perp$ are associated with the normal bundle $TM^\perp$.

Moreover, we define a symmetric bilinear form $B : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM^\perp)$, known as the *second fundamental form*, by

$$B(X, Y) := \bar{\nabla}_X Y - \nabla_X Y. \tag{2.11}$$

With a slight abuse of notations, we can view $B : \Gamma(TM) \times \Gamma(TM) \times \Gamma(TM^\perp) \to \mathbb{R}$ as

$$B(X, Y, \xi) := \langle B(X, Y), \xi\rangle \qquad \text{for } X, Y \in \Gamma(TM) \text{ and } \xi \in \Gamma(TM^\perp).$$

Furthermore, for $\xi \in \Gamma(TM^\perp)$, define the shape operator $S_\xi : \Gamma(TM) \to \Gamma(TM)$ by

$$S_\xi X := -\bar{\nabla}_X \xi + \nabla^\perp_X \xi. \tag{2.12}$$

It can also be defined by contracting B:

$$B(X, Y, \xi) =: \langle S_\xi X, Y\rangle. \tag{2.13}$$

In addition, the *Riemann curvature* on M is a rank–4 tensor $R : \Gamma(TM) \times \Gamma(TM) \times \Gamma(TM) \times \Gamma(TM) \to \mathbb{R}$, defined by

$$R(X, Y, Z, W) := \langle \nabla_X \nabla_Y Z, W\rangle - \langle \nabla_Y \nabla_X Z, W\rangle + \langle \nabla_{[X,Y]} Z, W\rangle. \tag{2.14}$$

In fact, for any vector bundle E over M with affine connection ∇^E, we can define the Riemann curvature on E, $R^E : \Gamma(TM) \times \Gamma(TM) \times \Gamma(E) \times \Gamma(E) \to \mathbb{R}$, by

$$R^E(X, Y, s_1, s_2) := \langle [\nabla^E_X, \nabla^E_Y] s_1, s_2\rangle - \langle \nabla^E_{[X,Y]} s_1, s_2\rangle \tag{2.15}$$

for $X, Y \in \Gamma(TM)$ and $s_1, s_2 \in \Gamma(E)$. For our purpose, we consider three bundles over M, namely (TM, ∇), $(TM^\perp, \nabla^\perp)$, and $(T\mathbb{R}^{n+k}, \bar{\nabla})$. Let us denote their Riemann curvatures by $R, R^\perp$, and $\bar{R}$, respectively.

With the geometric notations above, we can now explicitly write out the GCR equations, which express the flatness of the Euclidean space $(\mathbb{R}^{n+k}, g_0)$. In §6 of do Carmo [51] they are expressed in the following compact forms:

Theorem 2.1 (GCR Equations) *Assume that $f : (M, g) \hookrightarrow (\mathbb{R}^{n+k}, g_0)$ is an isometric immersion. Then the following Gauss-Codazzi-Ricci equations are satisfied:*

$$\langle B(X,Z), B(Y,W)\rangle - \langle B(X,W), B(Y,Z)\rangle = R(X,Y,Z,W), \tag{2.16}$$
$$\bar{\nabla}_Y B(X,Z,\eta) = \bar{\nabla}_X B(Y,Z,\eta), \tag{2.17}$$
$$\langle [S_\eta, S_\xi]X, Y\rangle = R^\perp(X,Y,\eta,\xi). \tag{2.18}$$

Here $\bar{\nabla}_Y B(X,Z) := YB(X,Z,\eta) - B(\bar{\nabla}_Y X, Z, \eta) - B(X, \bar{\nabla}_Y Z, \eta) - B(X, Z, \bar{\nabla}_Y \eta)$ is defined via the Leibniz rule, and similarly for $\bar{\nabla}_X B(Y,Z)$.

Moreover, for the convenience of subsequent development, it is helpful to recast the Codazzi and Ricci equations in a less concise manner:

Theorem 2.2 *The following equations are equivalent to the GCR system:*

$$\langle B(X,Z), B(Y,W)\rangle - \langle B(X,W), B(Y,Z)\rangle = R(X,Y,Z,W), \tag{2.19}$$

$$\begin{aligned} XB(Y,Z,\eta) - YB(X,Z,\eta) = B([X,Y],Z,\eta) &- B(X,\nabla_Y Z,\eta) - B(X,Z,\nabla^\perp_Y \eta) \\ &+ B(Y,\nabla_X Z,\eta) + B(Y,Z,\nabla^\perp_X \eta), \end{aligned} \tag{2.20}$$

$$\begin{aligned} X\langle \nabla^\perp_Y \xi, \eta\rangle - Y\langle \nabla^\perp_X \xi, \eta\rangle = \langle \nabla^\perp_{[X,Y]}\xi, \eta\rangle &- \langle \nabla^\perp_X \xi, \nabla^\perp_Y \eta\rangle + \langle \nabla^\perp_Y \xi, \nabla^\perp_X \eta\rangle \\ &+ B(\bar{\nabla}_X \xi - \nabla^\perp_X \xi, Y, \eta) - B(\bar{\nabla}_X \eta - \nabla^\perp_X \eta, Y, \xi) \end{aligned} \tag{2.21}$$

for any tangential vector fields $X, Y, Z, W \in \Gamma(TM)$ and normal vector fields $\xi, \eta \in \Gamma(TM^\perp)$.

Proof. The Gauss equation takes the same form as in Theorem 2.1. For the Codazzi equation, using the Leibniz rule we get

$$\begin{aligned} \bar{\nabla}_X B(Y,Z,\eta) &= X\langle B(Y,Z),\eta\rangle - \langle B(\nabla_X Y, Z),\eta\rangle - \langle B(Y,\nabla_X Z),\eta\rangle - \langle B(Y,Z), \nabla^\perp_X \eta\rangle \\ &= \langle \nabla^\perp_X B(Y,Z),\eta\rangle - \langle B(\nabla_X Y, Z),\eta\rangle - \langle B(Y,\nabla_X Z),\eta\rangle, \end{aligned}$$

as well as the analogous expression for $\bar{\nabla}_Y B(X,Z,\eta)$:

$$\bar{\nabla}_Y B(X,Z,\eta) = \langle \nabla^\perp_Y B(X,Z),\eta\rangle - \langle B(\nabla_Y X, Z),\eta\rangle - \langle B(X,\nabla_Y Z),\eta\rangle.$$

Equating these two expressions by Eq. (2.17) and noticing that $\nabla_X Y - \nabla_Y X = [X,Y]$, we obtain Eq. (2.20).

For the Ricci equation, the right–hand side of Eq. (2.18) can be expanded as

$$\begin{aligned}R^{\perp}(X,Y,\eta,\xi) &= \langle \nabla^{\perp}_X\nabla^{\perp}_Y\eta,\xi\rangle - \langle \nabla^{\perp}_Y\nabla^{\perp}_X\eta,\xi\rangle + \langle \nabla^{\perp}_{[X,Y]}\eta,\xi\rangle\\ &= X\langle \nabla^{\perp}_Y\eta,\xi\rangle - \langle \nabla^{\perp}_Y\eta,\nabla^{\perp}_X\xi\rangle - Y\langle \nabla^{\perp}_X\eta,\xi\rangle + \langle \nabla^{\perp}_X\eta,\nabla^{\perp}_Y\xi\rangle + \langle \nabla^{\perp}_{[X,Y]}\eta,\xi\rangle,\end{aligned}$$

by the definition of $R^{\perp}$ and the Leibniz rule. Moreover, the left–hand side of Eq. (2.18) equals $\langle S_\eta S_\xi X, Y\rangle - \langle S_\xi S_\eta X, Y\rangle$, where

$$\langle S_\eta S_\xi X, Y\rangle = B(S_\xi X, Y, \eta) = -B(\bar{\nabla}_X\xi - \nabla^{\perp}_X\xi, Y, \eta)$$

thanks to the definition of S; similarly,

$$\langle S_\xi S_\eta X, Y\rangle = -B(\bar{\nabla}_X\eta - \nabla^{\perp}_X\eta, Y, \xi).$$

Thus we obtain Eq. (2.21), which completes the proof. ■

From the PDE perspectives, Eqs. (2.19)–(2.21) in Theorem 2.2 form a system of first-order non-linear equations of mixed type. The left-hand sides of Eqs. (2.20)–(2.21) can be regarded as the *principal parts* (by identifying vector fields with first-order differential operators), while the non-linear terms on the right-hand sides are of the zero-th order. The non-linear terms are *quadratic* in the form of $B\otimes B$, $\nabla^{\perp}\otimes\nabla^{\perp}$, and $B\otimes\nabla^{\perp}$. Thus, the GCR equations can be schematically represented as Eq. (1.1), reproduced below:

$$\partial(B,\nabla^{\perp}) = (B,\nabla^{\perp})\star(B,\nabla^{\perp}) + \{\text{linear terms in } (B,\nabla^{\perp})\}.$$

Here ∂ is a first-order differential operator, and $\star$ denotes a quadratic operation.

2.2 A compensated compactness theorem on Banach spaces

In this section, we first formulate and prove a general abstract compensated compactness theorem in the framework of functional analysis (*i.e.*, Theorem 1.3). As a special case, it implies a global intrinsic version of the div-curl lemma on Riemannian manifolds, which generalises the well-known classical versions in $\mathbb{R}^n$ due to Murat [95] and Tartar [118]. This will serve as a basic tool for the subsequent development.

2.2.1 Compensated compactness via functional analysis

In this section, for a normed vector space X with dual space X^*, we use $\langle\cdot,\cdot\rangle_X$ to denote the duality pairing of (X, X^*). Let $\mathcal{H}$ be a Hilbert space over field $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$ such that $\mathcal{H}=\mathcal{H}^*$. Let Y and Z be two Banach spaces over $\mathbb{K}$ with their dual spaces Y^* and Z^*, respectively. In what follows, we consider two bounded linear operators $S:\mathcal{H}\to Y$, $T:\mathcal{H}\to Z$, and their *adjoint operators* $S^\dagger: Y^*\to\mathcal{H}$ and $T^\dagger: Z^*\to\mathcal{H}$, respectively.

Furthermore, the following conventional notations are adopted: For any normed vector spaces X, X_1, and X_2, we write $\{s^\epsilon\}\subset X$ for a sequence $\{s^\epsilon\}$ in X as a subset, and $X_1\Subset X_2$ for a compact embedding between the normed vector spaces. We use $\|\cdot\|_X$ to denote the norm in X. Also, we use $\to$ to denote the strong convergence of sequences and $\rightharpoonup$ for the weak convergence. Furthermore, we denote the closed unit ball of space X by $\bar{B}_X := \{x\in X : \|x\|\le 1\}$, the open unit ball by $B_X := \{x\in X : \|x\|<1\}$. Moreover, for a linear operator $L: X_1\to X_2$, its kernel is written as $\ker(L)\subset X_1$, and its range is denoted by $\mathrm{ran}(L)\subset X_2$. Finally, for $X_1\subset X$ as a vector subspace, its *annihilator* is defined as $X_1^\perp := \{f\in X^* : f(x)=0 \text{ for all } x\in X_1\}$.

To formulate the compensated compactness theorem in the general functional-analytic framework, we first introduce the following two operators: Define

$$\begin{cases} S\oplus T:\mathcal{H}\to Y\bigoplus Z, & (S\oplus T)h := (Sh, Th) \quad \text{for } h\in\mathcal{H};\\ S^\dagger\vee T^\dagger : (Y\bigoplus Z)^*\cong Y^*\bigoplus Z^*\to\mathcal{H}, & (S^\dagger\vee T^\dagger)(a,b) := S^\dagger a + T^\dagger b \text{ for } a\in Y^*,\, b\in Z^*.\end{cases}$$

Here and throughout, the Banach space $Y\bigoplus Z$ is endowed with norm $\|(y,z)\|_{Y\bigoplus Z} := \|y\|_Y + \|z\|_Z$. Notice that, for any $a\in Y^*, b\in Z^*$, and $h\in\mathcal{H}$,

$$\langle h, (S^\dagger\vee T^\dagger)(a,b)\rangle = \langle h, S^\dagger a + T^\dagger b\rangle = \langle Sh, a\rangle_Y + \langle Th, b\rangle_Z = \langle (S\oplus T)h, (a,b)\rangle_{Y\bigoplus Z}.$$

Hence $S^\dagger\vee T^\dagger$ is in fact the adjoint of $S\oplus T$, namely $(S\oplus T)^\dagger = S^\dagger\vee T^\dagger$.

In the setting above, we are concerned with the following question: Let $\{u^\epsilon\}$, $\{v^\epsilon\}\Subset\mathcal{H}$ be two bounded sequences so that there exist $\bar{u},\bar{v}\in\mathcal{H}$ such that $u^\epsilon\rightharpoonup\bar{u}$ and $v^\epsilon\rightharpoonup\bar{v}$ in $\mathcal{H}$ as $\epsilon\to 0$ after passing to subsequences. Under which conditions does the following hold:

$$\langle u^\epsilon, v^\epsilon\rangle_{\mathcal{H}}\to\langle\bar{u},\bar{v}\rangle_{\mathcal{H}} \qquad \text{as } \epsilon\to 0?$$

The goal of this subsection is to provide a sufficient condition for the convergence result as above. Roughly speaking, it requires the existence of a “nice” pair of bounded linear operators $S:\mathcal{H}\to Y$ and $T:\mathcal{H}\to Z$ such that S and T are orthogonal to each other, and $S\oplus T$ gains certain compactness/regularity. More precisely, we prove

Theorem 2.3 (Theorem 1.3 in Introduction) *Let $\mathcal{H}$ be a Hilbert space over $\mathbb{K}$, Y and Z be reflexive Banach spaces over $\mathbb{K}$, and $S : \mathcal{H} \to Y$ and $T : \mathcal{H} \to Z$ be bounded linear operators satisfying*

(Op 1) *Orthogonality:*

$$S \circ T^{\dagger} = 0, \qquad T \circ S^{\dagger} = 0; \tag{2.22}$$

(Op 2) *For some Hilbert space $(\underset{\sim}{\mathcal{H}}; \|\cdot\|_{\underset{\sim}{\mathcal{H}}})$ so that $\mathcal{H}$ embeds compactly into $\underset{\sim}{\mathcal{H}}$, there exists a constant $C > 0$ such that, for any $h \in \mathcal{H}$,*

$$\|h\|_{\mathcal{H}} \leq C\Big(\|(Sh, Th)\|_{Y \bigoplus Z} + \|h\|_{\underset{\sim}{\mathcal{H}}}\Big) = C\Big(\|Sh\|_{Y} + \|Th\|_{Z} + \|h\|_{\underset{\sim}{\mathcal{H}}}\Big). \tag{2.23}$$

Assume that two sequences $\{u^{\epsilon}\}, \{v^{\epsilon}\} \subset \mathcal{H}$ satisfy the following conditions:

(Seq 1) *$u^{\epsilon} \rightharpoonup \bar{u}$ and $v^{\epsilon} \rightharpoonup \bar{v}$ in $\mathcal{H}$ as $\epsilon \to 0$;*

(Seq 2) *$\{Su^{\epsilon}\}$ is pre-compact in Y, and $\{Tv^{\epsilon}\}$ is pre-compact in Z.*

Then

$$\langle u^{\epsilon}, v^{\epsilon} \rangle_{\mathcal{H}} \to \langle \bar{u}, \bar{v} \rangle_{\mathcal{H}} \qquad \text{as } \epsilon \to 0.$$

Before proving the theorem, let us remark that in the literature, three distinct approaches have been employed to prove the div–curl lemma and its variants.

The first approach, developed by Murat and Tartar in [95, 118], is based on harmonic analysis. It is observed that the first–order differential constraints, namely the pre–compactness of $\{\operatorname{div} u^{\epsilon}\}$ and $\{\operatorname{curl} v^{\epsilon}\}$ in H^{-1}, lead to the decay properties of $\{\langle u^{\epsilon}, v^{\epsilon} \rangle\}$ in the high Fourier frequency region. Coifman-Lions-Meyer-Semmes in [40] extended this lemma by combining the exploitation of this observation with further techniques in harmonic analysis, including Hardy space estimates, as well as the commutator estimates of BMO functions and Riesz transforms.

The second approach is based on the Hodge decomposition (*cf.* Theorem 2.4 below). Robbin-Rogers-Temple in [105] observed that, by writing

$$u^{\epsilon} = \Delta\Delta^{-1}u^{\epsilon} = (\operatorname{grad} \circ \operatorname{div} - \operatorname{curl} \circ \operatorname{curl})\Delta^{-1}u^{\epsilon} \qquad \text{on } \mathbb{R}^3, \tag{2.24}$$

$\{u^{\epsilon}\}$ can be decomposed into a weakly convergent part and a strongly convergent part, and similarly for $\{v^{\epsilon}\}$ (also see the exposition in Evans [56]). For this, the advantage can be taken of the first–order differential constraints, the commutativity of the Green operator Δ^{-1} on $\mathbb{R}^3$ with divergence, gradient, and curl, and most crucially, the ellipticity of Δ, so that, for $\{\langle u^{\epsilon}, v^{\epsilon} \rangle\}$, the pairing of the weakly convergent parts pass to the limits via integration by parts, and the pairings of the other terms can be dealt with directly.

The third approach is functional analytic. In this direction, Kozono-Yanagisawa [78] obtained a div–curl lemma using functional analytic results on $L^2(\mathbb{R}^n)$, as well as a geometric version, with emphasis on the weak convergence of vector fields up to the boundary of the domain or compact Riemmanian manifold, for which requires the divergence and curl of the vector fields are bounded in L^2. One of our key observations is that, for the "usual" div–curl lemmas, the specific geometry of Euclidean spaces or manifolds plays no essential role. Based on this observation, we have formulated and established a general compensated compactness theorem through bounded linear operators on Banach spaces. Our proof is based on – and serves as an extension of – the functional analytic arguments in Kozono-Yanagisawa [78]. In particular, our theorem is formulated in terms of general spaces $\mathcal{H}, Y, Z$ instead of L^2 and H^{-1}, and our assumption (Seq2) is weaker than that in [78]. The crucial tool we introduce is the "generalised Laplacian" $\cancel{\Delta}$; see Step 4 in the proof below.

Proof of Theorem 1.3. We divide the proof into eight steps.

1. In order to show that $S \oplus T : \mathcal{H} \to Y \bigoplus Z$ has a finite-dimensional kernel, we consider the following subset of $\mathcal{H}$:

$$E = j^{-1}\big(j[\ker(S \oplus T)] \cap \bar{B}_{\underset{\sim}{\mathcal{H}}}\big),$$

where $j : \mathcal{H} \hookrightarrow \underset{\sim}{\mathcal{H}}$ is a compact embedding between the Hilbert spaces.

Suppose that $j(E) \subset \underset{\sim}{\mathcal{H}}$ is compact. First notice that $j(E)$ is the closed unit ball of $j[\ker(S \oplus T)]$, which is a Banach space, due to the closedness of the kernel. It then follows from the classical Riesz lemma that $j[\ker(S \oplus T)]$ is finite-dimensional. Since j is an embedding, we can conclude that $\dim \ker(S \oplus T) = \dim(j[\ker(S \oplus T)]) < \infty$.

To prove the compactness of $j(E)$, take any $h \in E$ and consider the following estimate deduced from condition (Op 2):

$$\|h\|_{\mathcal{H}} \leq C(\|Sh\|_Y + \|Th\|_Z + \|j(h)\|_{\underset{\sim}{\mathcal{H}}}) \leq C. \tag{2.25}$$

Hence, the boundedness of E in $\mathcal{H}$ and the compactness of $j : \mathcal{H} \to \underset{\sim}{\mathcal{H}}$ imply that $j(E)$ is compact. Hence the first step is complete.

2. We now show that $S \oplus T$ is a closed-ranged operator, *i.e.*, $\mathrm{ran}(S \oplus T) \subset Y \bigoplus Z$ is a closed subspace. To this end, we first prove the existence of a constant $\epsilon_0 > 0$ such that, for all $h \in \mathcal{H}$,

$$\|(S \oplus T)h\|_{Y \bigoplus Z} \geq \epsilon_0 \|j(h)\|_{\underset{\sim}{\mathcal{H}}}. \tag{2.26}$$

In fact, if the inequality were false, then, for any $\mu \in (0,1)$, we could find $\{h^\mu\} \subset [\ker(S \oplus T)]^\perp$ and $\|j(h^\mu)\|_{\underset{\sim}{\mathcal{H}}} = 1$ such that

$$\|(S \oplus T)h^\mu\|_{Y \bigoplus Z} \leq \mu$$

and

$$\operatorname{dist}\big(h^\mu, \ker(S \oplus T)\big) \geq \hat{\epsilon}_0$$

for some $\hat{\epsilon}_0 > 0$. Such a choice of $\{h^\mu\}$ is possible, owing to the finite-dimensionality of $S \oplus T$. Now, plugging $\{h^\mu\}$ into condition (Op 2), we have

$$\|h^\mu\|_{\mathcal{H}} \leq C\big(\|(S \oplus T)h^\mu\|_{Y \bigoplus Z} + \|j(h^\mu)\|_{\underset{\sim}{\mathcal{H}}}\big) \leq C(1+\mu) \leq 2C.$$

It follows from the compactness of j that $\{j(h^\mu)\}$ is pre-compact in $\underset{\sim}{\mathcal{H}}$. Let $j(h)$ be a limit point of $\{j(h^\mu)\}$ in $j(\mathcal{H}) \subset \underset{\sim}{\mathcal{H}}$. Then one must have $h \in \ker(S \oplus T)$, in view of $\|(S \oplus T)h^\mu\|_{Y \bigoplus Z} \leq \mu$. However, this contradicts with the fact that $\{h^\mu\}$ were chosen to have a distance at least $\hat{\epsilon}_0$ from $\ker(S \oplus T)$. Therefore, we have established estimate (2.26).

To proceed, consider a sequence $\{h^\mu\} \subset \mathcal{H}$ such that $(S \oplus T)h^\mu \to w$ for some $w \in Y \bigoplus Z$. Our goal is to show that $w \in \operatorname{ran}(S \oplus T)$. Indeed, by the projection theorem for Hilbert spaces, we can decompose $h^\mu = k^\mu + r^\mu$ with $k^\mu \in \ker(S \oplus T)$ and $r^\mu \in [\ker(S \oplus T)]^\perp$ (which is a closed subspace of $\mathcal{H}$). Then estimate (2.26) gives

$$\|(S \oplus T)(h^{\mu_1} - h^{\mu_2})\|_{Y \bigoplus Z} = \|(S \oplus T)(r^{\mu_1} - r^{\mu_2})\|_{Y \bigoplus Z} \geq \epsilon_0 \|j(r^{\mu_1} - r^{\mu_2})\|_{\underset{\sim}{\mathcal{H}}}. \tag{2.27}$$

As a consequence, $\{j(r^\mu)\}$ is a Cauchy sequence in $j(\mathcal{H})$, which implies that there exists $j(r) \in j(\mathcal{H})$ such that $j(r^\mu) \to j(r)$ in $j(\mathcal{H})$ as $\mu \to 0$. Since j is an embedding, it follows that $r^\mu \to r$ in $\mathcal{H}$. Therefore, by the closed graph theorem of Banach spaces, $(S \oplus T)h^\mu = (S \oplus T)r^\mu \to (S \oplus T)r$. It now follows that $w = (S \oplus T)r$ so that $S \oplus T$ has closed range.

3. As an immediate corollary, we can obtain the following decomposition of $\mathcal{H}$:

$$\mathcal{H} = \ker(S \oplus T) \bigoplus \operatorname{ran}(S^\dagger \vee T^\dagger). \tag{2.28}$$

Here, $\bigoplus$ denotes the topological direct sum of Banach spaces, and the direct summands are orthogonal as Hilbert spaces. Indeed, by the projection theorem, $\mathcal{H} = \ker(S \oplus T) \bigoplus [\ker(S \oplus T)]^\perp$. On the other hand,

$$[\ker(S \oplus T)]^\perp = \overline{\operatorname{ran}[(S \oplus T)^\dagger]} = \overline{\operatorname{ran}(S^\dagger \vee T^\dagger)}.$$

We recall the closed range theorem in Banach spaces, which states that a bounded linear operator is closed-ranged if and only if its adjoint operator is closed-ranged (*cf.* [58]). Hence, we deduce from the preceding equalities that

$$[\ker(S \oplus T)]^{\perp} = \mathrm{ran}(S^{\dagger} \vee T^{\dagger}).$$

The decomposition in Eq. (2.28) now follows immediately.

With this decomposition, we observe that it now suffices to prove Theorem 1.3 for *surjective* operators S and T. Indeed, all the conditions in (Op 1)–(Op 2) and (Seq 1)–(Seq 2) continue to hold when $Y \bigoplus Z$ is replaced by $\mathrm{ran}(S \oplus T)$, which has been proved to be a closed subspace of $Y \bigoplus Z$. Here we have used the fact that closed subspaces of reflexive Banach spaces are still reflexive. Therefore, in the sequel, we always assume $\mathrm{ran}(S \oplus T) = Y \bigoplus Z$ without loss of generality.

4. Now, we introduce the following operator $\not{\Delta} : Y^* \bigoplus Z^* \to Y \bigoplus Z$:

$$\not{\Delta} := (S \oplus T) \circ (S^{\dagger} \vee T^{\dagger}) = SS^{\dagger} \oplus TT^{\dagger}. \tag{2.29}$$

For this *generalised Laplacian*, we also prove that it has finite-dimensional kernel and is close-ranged (in fact, surjective, in view of the reduction at the end of Step 3).

Indeed, we can find the range of $\not{\Delta}$ as follows:

$$\begin{aligned}
\mathrm{ran}(\not{\Delta}) &= \mathrm{ran}\Big((S \oplus T)|_{\mathrm{ran}(S^{\dagger} \vee T^{\dagger})}\Big) \\
&= \mathrm{ran}\Big((S \oplus T)|_{[\ker(S \oplus T)]^{\perp}}\Big) \\
&= \mathrm{ran}\Big((S \oplus T)|_{[\ker(S \oplus T)]^{\perp} \bigoplus \ker(S \oplus T)}\Big) \\
&= \mathrm{ran}(S \oplus T) = Y \bigoplus Z,
\end{aligned} \tag{2.30}$$

where the decomposition in Eq. (2.28) has been used in the second equality.

On the other hand, notice that

$$\ker(\not{\Delta}) = \ker\left(S \oplus T|_{\mathrm{ran}(S^{\dagger} \vee T^{\dagger})}\right) \bigoplus \ker(S^{\dagger} \vee T^{\dagger}). \tag{2.31}$$

In this expression, the first direct summand equals $\ker(S \oplus T)$, by a similar argument as in Eq. (2.36). For the second summand, a standard result in functional analysis gives $\ker(S^{\dagger} \vee T^{\dagger}) = [\mathrm{ran}(S \oplus T)]^{\perp}$, which is assumed to be $\{0\}$ at the end of Step 3. It follows that $\ker(\not{\Delta}) = \ker(S \oplus T)$, which is finite-dimensional by Step 1.

To summarise, $\not{\Delta}$ is a close-ranged operator with finite dimensional kernel; without loss of generality, we may assume $\not{\Delta}$ to be surjective.

5. We now show a crucial result concerning $\not\Delta$:

($\clubsuit$) For each $w \in \operatorname{ran}(\not\Delta)$, there exists $\xi \in \not\Delta^{-1}\{w\}$ such that $\|\xi\|_{Y^* \bigoplus Z^*} \leq M\|w\|_{Y \bigoplus Z}$, where M is a universal constant.

First, applying the open mapping theorem in Banach spaces to the operator $\not\Delta : Y^* \bigoplus Z^* \to \operatorname{ran}(\not\Delta) = Y \bigoplus Z$, we can find a constant $\delta > 0$ and an element $w_0 \in \operatorname{ran}(\not\Delta)$ such that $w_0 + \delta B_{Y \bigoplus Z} \subset \not\Delta[B_{Y^* \bigoplus Z^*}]$.

From this inclusion, we can establish the following:

$$\delta \bar{B}_{Y \bigoplus Z} \subset \not\Delta(\bar{B}_{Y^* \bigoplus Z^*}). \tag{2.32}$$

For any $v_0 \in \delta \bar{B}_{Y \bigoplus Z}$, we can write v_0 as a convex combination of elements in $w_0 + \delta B_{Y \bigoplus Z}$, *e.g.*, $v_0 = \frac{1}{2}\big((v_0 + w_0) + (v_0 - w_0)\big)$. Observe that $\not\Delta[B_{Y^* \bigoplus Z^*}]$ is a convex set, as $\not\Delta$ is a bounded linear operator and $B_{Y^* \bigoplus Z^*}$ is convex. Since $w_0 + \delta B_{Y \bigoplus Z}$ lies in $\not\Delta[B_{Y^* \bigoplus Z^*}]$, we conclude that $v_0 \in \not\Delta[B_{Y^* \bigoplus Z^*}]$. So Eq. (2.32) follows. As a consequence, given any $w \in \operatorname{ran}(\not\Delta) = Y \bigoplus Z$, there exists $\eta \in \bar{B}_{Y^* \bigoplus Z^*}$ such that $\delta \frac{w}{\|w\|} = \not\Delta \eta$. Define $\xi := \frac{\|w\|}{\delta}\eta$, which yields

$$\|\xi\|_{Y^* \bigoplus Z^*} \leq \delta^{-1}\|w\|_{Y \bigoplus Z}, \qquad \not\Delta\xi = w.$$

Now, the proof for ($\clubsuit$) is completed by choosing $M = \delta^{-1}$.

6. Now, we employ the decomposition of $\mathcal{H}$ in Eq. (2.28) to decompose sequences $\{u^\epsilon\}$ and $\{v^\epsilon\}$, and the weak limits $\bar{u}$ and $\bar{v}$. In the sequel, we denote the canonical projection of $\mathcal{H}$ onto the first factor by $\pi_1 : \mathcal{H} = \ker(S \oplus T) \bigoplus \operatorname{ran}(S^\dagger \vee T^\dagger) \to \ker(S \oplus T)$. By Step 1, π_1 is a finite-rank (hence compact) operator.

Employing such a decomposition, we can write

$$\begin{cases} u^\epsilon = \pi_1 u^\epsilon + S^\dagger a^\epsilon + T^\dagger b^\epsilon, \\ v^\epsilon = \pi_1 v^\epsilon + S^\dagger \tilde{a}^\epsilon + T^\dagger \tilde{b}^\epsilon, \\ \bar{u} = \pi_1 \bar{u} + S^\dagger a + T^\dagger b, \\ \bar{v} = \pi_1 \bar{v} + S^\dagger \tilde{a} + T^\dagger \tilde{b}, \end{cases} \tag{2.33}$$

for some $a, \tilde{a}, a^\epsilon, \tilde{a}^\epsilon \in Y^*$ and $b, \tilde{b}, b^\epsilon, \tilde{b}^\epsilon \in Z^*$. Moreover, applying the orthogonality condition (Op 1), we have

$$\begin{cases} \langle u^\epsilon, v^\epsilon \rangle_{\mathcal{H}} = \langle \pi_1 u^\epsilon, \pi_1 v^\epsilon \rangle_{\mathcal{H}} + \langle S^\dagger a^\epsilon, S^\dagger \tilde{a}^\epsilon \rangle_{\mathcal{H}} + \langle T^\dagger b^\epsilon, T^\dagger \tilde{b}^\epsilon \rangle_{\mathcal{H}}, \\ \langle \bar{u}, \bar{v} \rangle_{\mathcal{H}} = \langle \pi_1 \bar{u}, \pi_1 \bar{v} \rangle_{\mathcal{H}} + \langle S^\dagger a, S^\dagger \tilde{a} \rangle_{\mathcal{H}} + \langle T^\dagger b, T^\dagger \tilde{b} \rangle_{\mathcal{H}}. \end{cases}$$

Since $\{u^\epsilon\}$ and $\{v^\epsilon\}$ are weakly convergent by assumption (Seq 1) and π_1 is a compact operator, we have $\langle \pi_1 u^\epsilon, \pi_1 v^\epsilon\rangle \to \langle \pi_1 \bar{u}, \pi_1 \bar{v}\rangle$. It thus remains to establish

$$\langle S^\dagger a^\epsilon, S^\dagger \tilde{a}^\epsilon\rangle_{\mathcal{H}} + \langle T^\dagger b^\epsilon, T^\dagger \tilde{b}^\epsilon\rangle_{\mathcal{H}} \to \langle S^\dagger a, S^\dagger \tilde{a}\rangle_{\mathcal{H}} + \langle T^\dagger b, T^\dagger \tilde{b}\rangle_{\mathcal{H}}. \tag{2.34}$$

7. To prove (2.34), our starting point is to observe that its left-hand side can be rewritten as follows:

$$\begin{aligned} &\langle S^\dagger a^\epsilon, S^\dagger \tilde{a}^\epsilon\rangle_{\mathcal{H}} + \langle T^\dagger b^\epsilon, T^\dagger \tilde{b}^\epsilon\rangle_{\mathcal{H}} \\ &= \langle SS^\dagger a^\epsilon, \tilde{a}^\epsilon\rangle_Y + \langle TT^\dagger \tilde{b}^\epsilon, b^\epsilon\rangle_Z = \big\langle \measuredangle(a^\epsilon, \tilde{b}^\epsilon), (\tilde{a}^\epsilon, b^\epsilon)\big\rangle_{Y\bigoplus Z}. \end{aligned} \tag{2.35}$$

On the other hand, let us apply S to u^ϵ and T to v^ϵ in (2.33). Using the definition of π_1 and the orthogonality condition (Op 1), we immediately find that

$$Su^\epsilon = SS^\dagger a^\epsilon, \qquad Tv^\epsilon = TT^\dagger \tilde{b}^\epsilon, \tag{2.36}$$

i.e., $\measuredangle(a^\epsilon, \tilde{b}^\epsilon) = (Su^\epsilon, Tv^\epsilon)$. As $\{Su^\epsilon\} \subset Y$ and $\{Tv^\epsilon\} \subset Z$ are pre-compact by (Seq 2), it suffices to show the boundedness of $\{(\tilde{a}^\epsilon, b^\epsilon)\} \subset Y^* \bigoplus Z^*$ to conclude (2.34). To see this, assuming the boundedness one can deduce that

$$\begin{aligned} &\Big|\big\langle \measuredangle(a^\epsilon, \tilde{b}^\epsilon), (\tilde{a}^\epsilon, b^\epsilon)\big\rangle_{Y\bigoplus Z} - \big\langle \measuredangle(a, \tilde{b}), (\tilde{a}, b)\big\rangle_{Y\bigoplus Z}\Big| \\ &\leq \Big|\big\langle \measuredangle(a^\epsilon - a, \tilde{b}^\epsilon - \tilde{b}), (\tilde{a}^\epsilon, b^\epsilon)\big\rangle_{Y\bigoplus Z}\Big| + \Big|\big\langle \measuredangle(a, \tilde{b}), (\tilde{a}^\epsilon - \tilde{a}, b^\epsilon - b)\big\rangle_{Y\bigoplus Z}\Big| \\ &\leq \Big|\big\langle (Su^\epsilon - S\bar{u}, Tv^\epsilon - T\bar{v}), (\tilde{a}^\epsilon, b^\epsilon)\big\rangle_{Y\bigoplus Z}\Big| + \Big|\big\langle \measuredangle(a, \tilde{b}), (\tilde{a}^\epsilon - \tilde{a}, b^\epsilon - b)\big\rangle_{Y\bigoplus Z}\Big| \\ &=: \mathrm{I}^\epsilon + \mathrm{II}^\epsilon. \end{aligned}$$

For I^ϵ, since $\|(\tilde{a}^\epsilon, b^\epsilon)\|_{Y^*\bigoplus Z^*} \leq M$, we can use the pre-compactness of $\{Su^\epsilon\} \subset Y$ and $\{Tv^\epsilon\} \subset Z$ in assumption (Seq 1) to conclude that

$$\mathrm{I}^\epsilon \leq M\Big\|\big(S(u^\epsilon - \bar{u}), T(v^\epsilon - \bar{v})\big)\Big\|_{Y\bigoplus Z} \to 0.$$

For II^ϵ, we need to invoke the reflexivity of Y and Z. As a Banach space is reflexive if and only if its dual space is reflexive, we know that $Y^* \bigoplus Z^*$ is reflexive. Then the bounded sequence $\{(\tilde{a}^\epsilon, b^\epsilon)\}$ is weakly pre-compact, by Theorem 3.31 in [58]. Moreover, Theorem 4.47 (the Eberlein-Šmulian theorem) in [58] implies the weak convergence of $\{(\tilde{a}^\epsilon, b^\epsilon)\}$. Hence, viewing $\measuredangle(a, \tilde{b})$ as an element in $(Y \bigoplus Z)^{**}$, we conclude that $\mathrm{II}^\epsilon \to 0$.

8. From the above arguments, it remains to prove the boundedness of $\{(\tilde{a}^\epsilon, b^\epsilon)\} \subset Y^* \bigoplus Z^*$. We further remark that $(\tilde{a}^\epsilon, b^\epsilon)$ can be chosen *modulo* $\ker(\not\Delta)$: More precisely, it is enough to exhibit one particular representative $(\tilde{a}^\epsilon, b^\epsilon)$ in $\not\Delta^{-1}\{(Sv^\epsilon, Tu^\epsilon)\}$ such that $\|(\tilde{a}^\epsilon, b^\epsilon)\|_{Y^* \bigoplus Z^*} \leq C$, where C is independent of ϵ. To see this, notice that

$$\langle S^\dagger a^\epsilon, S^\dagger \tilde{a}^\epsilon \rangle_{\mathcal{H}} + \langle T^\dagger b^\epsilon, T^\dagger \tilde{b}^\epsilon \rangle_{\mathcal{H}} = \big\langle \not\Delta(\tilde{a}^\epsilon, b^\epsilon), (a^\epsilon, \tilde{b}^\epsilon) \big\rangle_{Y \bigoplus Z} = \big\langle (Sv^\epsilon, Tu^\epsilon), (a^\epsilon, \tilde{b}^\epsilon) \big\rangle_{Y \bigoplus Z},$$

analogous to Eqs. (2.35)–(2.36). Therefore, we have the freedom to choose a representative $(\tilde{a}^\epsilon, b^\epsilon)$ in the co-set $(Sv^\epsilon, Tu^\epsilon) + \ker(\not\Delta)$, without changing the expression on the left-hand side of Eq. (2.34).

For this purpose, let us invoke the key estimate (♣) proved in Step 5 to find a universal constant $0 < M < \infty$ satisfying

$$\|(\tilde{a}^\epsilon, b^\epsilon)\|_{Y^* \bigoplus Z^*} \leq M \|(Sv^\epsilon, Tu^\epsilon)\|_{Y \bigoplus Z}.$$

Then it suffices to prove the uniform boundedness of $\{(Sv^\epsilon, Tu^\epsilon)\}$ in $Y \bigoplus Z$.

Indeed, as $u^\epsilon \rightharpoonup \bar{u}$ and $v^\epsilon \rightharpoonup \bar{v}$ according to assumption (Seq 1), we know that $\{u^\epsilon\}$ and $\{v^\epsilon\}$ are bounded in $\mathcal{H}$, by using the weak lower semi-continuity of $\|\cdot\|_{\mathcal{H}}$. By the reflexivity of Hilbert spaces (due to the Riesz representation theorem), $\{u^\epsilon\}$ and $\{v^\epsilon\}$ are weakly pre-compact in $\mathcal{H}$. Next, by standard results in functional analysis, the continuous linear operator $S \oplus T : \mathcal{H} \to Y \bigoplus Z$ with respect to the strong topologies is also continuous when both $\mathcal{H}$ and $Y \bigoplus Z$ are endowed with the weak topologies (see [58] for details). As continuous mappings take pre-compact sets to pre-compact sets, $\{(Sv^\epsilon, Tu^\epsilon)\}$ is weakly pre-compact in $Y \bigoplus Z$; equivalently, we have established the uniform boundedness of $\{(Sv^\epsilon, Tu^\epsilon)\}$ in the strong topology of $Y \bigoplus Z$, because the Banach spaces Y and Z are reflexive.

Therefore, we have proved that

$$\|(\tilde{a}^\epsilon, b^\epsilon)\|_{Y^* \bigoplus Z^*} \leq M \sup_{\epsilon > 0} \|(Sv^\epsilon, Tu^\epsilon)\|_{Y \bigoplus Z} \leq M' < \infty, \tag{2.37}$$

from which the convergence in (2.34) follows. This completes the proof. ■

Let us close this subsection with two remarks on Theorem 1.3. First, in terms of the applications, Y and Z are usually Hilbert spaces $H^s = W^{s,2}$ for $s \in \mathbb{Z}$, or S and T are known to be Fredholm operators. In such cases, our arguments can be essentially simplified. Second, the reflexivity of Y and Z is necessary for Theorem 1.3.

Remark 2.2.1 *If Y and Z are Hilbert spaces, then $\not\Delta : Y^* \bigoplus Z^* \cong Y \bigoplus Z \to Y \bigoplus Z$ is self-adjoint: $\not\Delta^\dagger = (SS^\dagger)^\dagger \oplus (TT^\dagger)^\dagger = SS^\dagger \oplus TT^\dagger = \not\Delta$. Thus, we can decompose*

$$Y \bigoplus Z = \ker(\not\Delta) \bigoplus \mathrm{ran}(\not\Delta) \tag{2.38}$$

(*cf. Proposition* 7.32 *in [58]*) *so that the close-rangedness of* $\mathbb{\Delta}$ *and the crucial estimate* ($\clubsuit$) *are automatically verified. As a consequence, after establishing the finite-dimensionality of* $\ker(\mathbb{\Delta})$ *as in the first half of Step* 1, *we can proceed to Step* 5 *in the proof of Theorem* 1.3. *On the other hand, if* S *and* T *are given* a priori *to have finite dimensional kernels and co-kernels, then* $S \oplus T$ *is a Fredholm operator* (*whose range is automatically closed*). *Again, we can directly proceed to Step* 5.

Remark 2.2.2 *The assumption of reflexivity of* Y *and* Z *is indispensable. One counter-example for non-reflexive* Y *and* Z *is the "Fakir's carpet" in Conti-Dolzmann-Müller [46]* (*a more extensive variant also appeared in DiPerna-Majda [50]*). *Let us consider the vector fields on* $\Omega := (0,1)^3$*:*

$$u^\epsilon(x) = v^\epsilon(x) = \Big(\sqrt{m}\sum_{j=1}^{m} \chi_{[\frac{j}{m},\frac{j}{m}+\frac{1}{m^2}]}, 0, 0\Big)^\top, \qquad m = \lfloor\frac{1}{\epsilon}\rfloor, \tag{2.39}$$

the operators $S = \mathrm{div}$, $T = \mathrm{curl}$, *and the spaces* $\mathcal{H} = L^2(\Omega;\mathbb{R}^3)$, $Y = W^{-1,1}(\Omega)$, $Z = W^{-1,1}(\Omega;\mathbb{R}^3)$. *Then, for any test function* $\psi \in W^{1,\infty}_0(\Omega)$ *and any test vector field* $\phi = (\phi^1,\phi^2,\phi^3)^\top \in W^{1,\infty}_0(\Omega;\mathbb{R}^3)$, *we have*

$$\begin{aligned}\Big|\int_\Omega Su^\epsilon(x)\psi(x)\,\mathrm{d}x\Big| &= \Big|\sqrt{m}\sum_{j=1}^{m}\int_{j/m}^{j/m+1/m^2}\frac{\partial\psi}{\partial x^1}(x)\,\mathrm{d}x\Big|\\ &\leq \|\psi\|_{W^{1,\infty}(\Omega)}\frac{1}{\sqrt{m}} \longrightarrow 0 \qquad \textit{as } m\to\infty \Longleftrightarrow \epsilon\to 0,\end{aligned}$$

and similarly

$$\begin{aligned}\Big|\int_\Omega Tv^\epsilon(x)\cdot\phi(x)\,\mathrm{d}x\Big| &= \Big|\int_\Omega v^\epsilon(x)\cdot\mathrm{curl}\,\phi(x)\,\mathrm{d}x\Big|\\ &= \Big|\sqrt{m}\sum_{j=1}^{m}\int_{j/m}^{j/m+1/m^2}\Big(\frac{\partial\phi^3}{\partial x^2}-\frac{\partial\phi^2}{\partial x^3}\Big)\,\mathrm{d}x\Big|\\ &\leq \|\phi\|_{W^{1,\infty}(\Omega;\mathbb{R}^3)}\frac{1}{\sqrt{m}} \longrightarrow 0 \qquad \textit{as } m\to\infty \Longleftrightarrow \epsilon\to 0.\end{aligned}$$

Therefore, $Su^\epsilon \rightharpoonup 0$ *in* Y *and* $Tv^\epsilon \rightharpoonup 0$ *in* Z. *However, it is straightforward to check that* $u^\epsilon \rightharpoonup 0$, $v^\epsilon \rightharpoonup 0$ *in* $\mathcal{H}$, *but* $\langle u^\epsilon, v^\epsilon\rangle \rightharpoonup (1,0,0)^\top$*; here* $\langle\cdot,\cdot\rangle$ *denotes the* L^2 *inner product. The failure of the weak convergence is related to the phenomenon of "concentration" in fluid mechanics and non-linear elasticity.*

2.2.2 Intrinsic div-curl lemma on Riemannian manifolds

Now we adapt the general functional-analytic theorem, Theorem 1.3, to the geometric settings. Our aim is to obtain a global intrinsic div-curl lemma on Riemannian manifolds (Theorem 2.5), independent of local coordinates, which will be applied to analyse the global weak rigidity of isometric immersions of Riemannian manifolds in the subsequent development. Let us first introduce several geometric quantities.

First of all, it is well-known that the *divergence* of a vector field is globally defined for an arbitrary dimensional orientable manifold M:

$$\mathrm{div}X := *d*(X^\flat) \equiv *(\mathcal{L}_X \mathrm{d}V_g) \qquad \text{for any } X \in \Gamma(TM), \tag{2.40}$$

where $*$ is the Hodge star, $\mathcal{L}$ is the Lie derivative, and $\mathrm{d}V_g$ is the volume form with respect to the metric g. The *gradient* can also be defined intrinsically:

$$\mathrm{grad}f := (df)^\sharp. \tag{2.41}$$

The notion of *curl* is more subtle. Our ordinary definition for *curl*, if we require to be intrinsic, is only well-defined on three-dimensional manifolds. This is because one needs to identify canonically $\Omega^2(M)$ with $\Gamma(TM)$, which is only valid in three dimensions via the Hodge duality. Physically, $\mathrm{curl}(X)$ measures the rotation of the flow generated by X pointing to the rotation axis, where the direction of the rotation axis is unambiguous in three dimensions. For $X \in \Gamma(TM)$ with $\dim M = 3$,

$$\mathrm{curl}X := [*d(X^\flat)]^\sharp, \tag{2.42}$$

where $\sharp \circ *$ identifies the 2–form $dX^\flat$ with the vector field. On an arbitrarily dimensional manifold M, we can define the *generalised curl* just as the 2–form, without pulling back to vector fields:

$$\mathrm{curl}X := d(X^\flat). \tag{2.43}$$

The underlying reason for introducing the generalised curl is the algebra isomorphism $\bigwedge^2(T^*M) \cong \mathfrak{so}(n)$. In this view, $\mathrm{curl}(X)$ is a field of anti-symmetric matrices which can be naturally interpreted as rotations, thanks to the structure of $\mathfrak{so}(n)$.

Next, we write $\delta : \Omega^q(M) \to \Omega^{q-1}(M)$ to be the L^2-adjoint of d:

$$\int_M \delta\alpha \wedge *\beta = \int_M \alpha \wedge *d\beta \qquad \text{for } \alpha \in \Omega^q(M) \text{ and } \beta \in \Omega^{q-1}(M). \tag{2.44}$$

To emphasise the dependence on metric g, it can also be expressed as

$$\int_M \langle \delta\alpha, \beta\rangle \,\mathrm{d}V_g = \int_M \langle \alpha, d\beta\rangle \,\mathrm{d}V_g \qquad \text{for } \alpha \in \Omega^q(M) \text{ and } \beta \in \Omega^{q-1}(M). \tag{2.45}$$

Equivalently, we can define $\delta := (-1)^{n(q+1)+1} * d *$, where $*$ is the Hodge star. Hence, in the definition of divergence (2.40), $\mathrm{div}X$ is simply $\delta X^\flat$ modulo a sign. Therefore, we always regard δ as the *divergence* and d as the *curl*. Moreover, the Laplace-Beltrami operator $\Delta : \Omega^q(M) \to \Omega^q(M)$ on manifold M is given by

$$\Delta := d \circ \delta + \delta \circ d. \tag{2.46}$$

We also denote by $\mathrm{Har}^q(M) := \ker(\Delta)$, the space of harmonic q–forms.

One fundamental result concerning the Laplace-Beltrami operator is the Hodge decomposition theorem (*cf.* Warner [122]):

Theorem 2.4 (Hodge Decomposition) *Let M be a closed orientable Riemannian manifold. For each integer q with $0 \leq q \leq n$, the following orthogonal decomposition holds:*

$$\Omega^q(M) = \mathrm{Har}^q(M) \bigoplus \mathrm{Im}(\Delta) = \mathrm{Har}^q(M) \bigoplus d\Omega^{q-1}(M) \bigoplus \delta\Omega^{q+1}(M), \tag{2.47}$$

where the orthogonality is taken with respect to the L^2 inner product on $\Omega^q(M)$. Moreover, $\dim_{\mathbb{R}}(\mathrm{Har}^q(M)) = \dim_{\mathbb{R}}(H^q_{\mathrm{dR}}(M;\mathbb{R})) < \infty$ for closed manifolds, where $H^q_{\mathrm{dR}}(M;\mathbb{R})$ is the q^{th} de Rham cohomology group of M.

In view of Theorem 2.4, we can define the solution (Green) operator to the Laplace-Beltrami on closed manifolds. Let $\pi_H : \Omega^q(M) \to \mathrm{Har}^q(M)$ stand for the canonical projection. Then, for $\alpha \in \Omega^q(M)$, we set

$$G(\alpha) = \Delta^{-1}\big(\alpha - \pi_H \alpha\big). \tag{2.48}$$

It can be shown that, if T is a linear operator commuting with Δ, then it also commutes with G. In particular, d, δ, Δ and $*$ commute with G. Moreover, Δ and G are bounded linear operators on the Sobolev spaces: $\Delta : W^{k+2,p}(M; \bigwedge^q T^*M) \to W^{k,p}(M; \bigwedge^q T^*M)$ and $G : W^{k,p}(M; \bigwedge^q T^*M) \to W^{k+2,p}(M; \bigwedge^q T^*M)$ are continuous for each $k \in \mathbb{Z}$, $p \in (1, \infty)$ and $0 \leq q \leq n$.

As a direct corollary of Theorem 1.3, we obtain the following geometric div–curl lemma. This is the result employed in §2.3 below that leads to the weak continuity of the GCR equations.

Theorem 2.5 (Geometric Div-Curl Lemma; *cf.* [78, 105]) *Let (M, g) be an n-dimensional Riemannian manifold. Let $\{\omega^\epsilon\}, \{\tau^\epsilon\} \subset L^2_{\mathrm{loc}}(M; \bigwedge^q T^*M)$ be two families of differential q–forms, for $0 \leq q \leq n$, such that*

(i) *$\omega^\epsilon \rightharpoonup \overline{\omega}$ weakly in L^2, and $\tau^\epsilon \rightharpoonup \overline{\tau}$ weakly in L^2;*

(ii) *there are compact subsets K_d, K_δ of the corresponding Sobolev spaces, such that*

$$\begin{cases} \{d\omega^\epsilon\} \subset K_d \Subset H^{-1}_{\mathrm{loc}}(M; \bigwedge^{q+1} T^*M), \\ \{\delta\tau^\epsilon\} \subset K_\delta \Subset H^{-1}_{\mathrm{loc}}(M; \bigwedge^{q-1} T^*M). \end{cases}$$

Then $\langle \omega^\epsilon, \tau^\epsilon \rangle$ converges to $\langle \overline{\omega}, \overline{\tau} \rangle$ in the sense of distributions:

$$\int_M \langle \omega^\epsilon, \tau^\epsilon \rangle \psi \, \mathrm{d}V_g \longrightarrow \int_M \langle \overline{\omega}, \overline{\tau} \rangle \psi \, \mathrm{d}V_g \qquad \text{for any } \psi \in C_c^\infty(M).$$

Proof. First of all, we can reduce the statement of the theorem only for compact manifolds. Indeed, fix any $\psi \in C_c^\infty(M)$. It suffices to prove for $\psi \geq 0$; otherwise, we may decompose it as $\psi = \psi^+ - \psi^-$ and approximate $\psi^\pm$ by C^∞ non-negative functions respectively. Then, without loss of generality, we can always replace $(\omega^\epsilon, \tau^\epsilon)$ by $(\sqrt{\psi}\omega^\epsilon, \sqrt{\psi}\tau^\epsilon)$, and replace M by any compact submanifold of M containing the support of ψ. Thus we may drop the test function ψ and the subscripts "loc" in the function spaces. More precisely, it suffices to take M as a closed Riemannian manifold and establish the convergence:

$$\int_M \langle \omega^\epsilon, \tau^\epsilon \rangle \, \mathrm{d}V_g \longrightarrow \int_M \langle \overline{\omega}, \overline{\tau} \rangle \, \mathrm{d}V_g \qquad \text{as } \epsilon \to 0,$$

provided that $\{\omega^\epsilon\}, \{\tau^\epsilon\} \subset L^2(M; \bigwedge^q T^*M)$, $\{d\omega^\epsilon\} \subset K_d \Subset H^{-1}(M; \bigwedge^{q+1} T^*M)$, and $\{\delta\tau^\epsilon\} \subset K_\delta \Subset H^{-1}(M; \bigwedge^{q-1} T^*M)$.

Now we are in the position of applying Theorem 1.3. For this purpose, we take $\mathcal{H} = L^2(M; \bigwedge^q T^*M)$, $Y = H^{-1}(M; \bigwedge^{q+1} T^*M)$, $Z = H^{-1}(M; \bigwedge^{q-1} T^*M)$, $\underset{\sim}{\mathcal{H}} = H^{-1}(M; \bigwedge^q T^*M)$, $S = d$, and $T = \delta$. In this setting, $\{\omega^\epsilon\}$ and $\{\tau^\epsilon\}$ play the role of $\{u^\epsilon\}$ and $\{v^\epsilon\}$, respectively. Conditions (Seq 1)–(Seq 2) of Theorem 1.3 correspond precisely to conditions (i)–(ii), and condition (Op 1) of Theorem 1.3 is verified by the cohomology chain condition $d \circ d = 0$ and $\delta \circ \delta = 0$.

We are left with checking (Op 2), *i.e.*, for any $\alpha \in L^2(M; \bigwedge^q T^*M)$, there exists a constant $C > 0$ such that

$$\begin{aligned} &\|\alpha\|_{L^2(M; \bigwedge^q T^*M)} \\ &\leq C\Big(\|d\alpha\|_{H^{-1}(M; \bigwedge^{q+1} T^*M)} + \|\delta\alpha\|_{H^{-1}(M; \bigwedge^{q-1} T^*M)} + \|\alpha\|_{H^{-1}(M; \bigwedge^q T^*M)} \Big). \end{aligned} \tag{2.49}$$

The proof relies crucially on the Hodge decomposition, Theorem 2.4, as well as the standard elliptic estimate for the Laplace-Beltrami operator:

$$\|\omega\|_{L^2(M; \bigwedge^q T^*M)} \leq C\Big(\|\Delta\omega\|_{H^{-2}(M; \bigwedge^q T^*M)} + \|\omega\|_{H^{-2}(M; \bigwedge^q T^*M)} \Big) \tag{2.50}$$

for arbitrary $\omega \in L^2(M; \bigwedge^q T^*M)$.

Now let us prove Eq. (2.49). Given $\alpha \in L^2(M; \bigwedge^q T^*M)$, we define $\beta := G\alpha$, which is equivalent to the decomposition: $\alpha = \pi_H\alpha + \Delta\beta$. Applying the elliptic estimate (2.50) to $\omega := \pi_H\alpha$ immediately, one obtains

$$\|\pi_H\alpha\|_{L^2(M;\bigwedge^q T^*M)} \leq C\|\pi_H\alpha\|_{H^{-2}(M;\bigwedge^q T^*M)},$$

where $C > 0$ is a universal constant. As $H^{-2}(M; \bigwedge^q T^*M)$ is a Hilbert space and π_H is a projection, the Pythagorean law gives $\|\pi_H\alpha\|_{H^{-2}(M;\bigwedge^q T^*M)} \leq \|\alpha\|_{H^{-2}(M;\bigwedge^q T^*M)}$. Thus, in view of the compact embedding $H^{-1}(M; \bigwedge^q T^*M) \hookrightarrow H^{-2}(M; \bigwedge^q T^*M)$ due to the Rellich lemma, we conclude that

$$\|\pi_H\alpha\|_{L^2(M;\bigwedge^q T^*M)} \leq C\|\alpha\|_{H^{-1}(M;\bigwedge^q T^*M)}. \tag{2.51}$$

Finally, we can bound $\Delta\beta$ in L^2. It follows from the following estimates:

$$\begin{aligned}
&\|\Delta\beta\|_{L^2(M;\bigwedge^q T^*M)} \\
&= \|\Delta(G\alpha)\|_{L^2(M;\bigwedge^q T^*M)} \\
&= \|G(\Delta\alpha)\|_{L^2(M;\bigwedge^q T^*M)} \\
&\leq C\Big(\|\Delta\alpha\|_{H^{-2}(M;\bigwedge^q T^*M)} + \|G(\Delta\alpha)\|_{H^{-2}(M;\bigwedge^q T^*M)}\Big) \\
&\leq C\Big(\|(d\delta + \delta d)\alpha\|_{H^{-2}(M;\bigwedge^q T^*M)} + \|\alpha - \pi_H(\alpha)\|_{H^{-2}(M;\bigwedge^q T^*M)}\Big) \\
&\leq C\Big(\|\delta\alpha\|_{H^{-1}(M;\bigwedge^{q-1} T^*M)} + \|d\alpha\|_{H^{-1}(M;\bigwedge^{q+1} T^*M)} + \|\alpha\|_{H^{-2}(M;\bigwedge^q T^*M)}\Big) \\
&\leq C\Big(\|\delta\alpha\|_{H^{-1}(M;\bigwedge^{q-1} T^*M)} + \|d\alpha\|_{H^{-1}(M;\bigwedge^{q+1} T^*M)} + \|\alpha\|_{H^{-1}(M;\bigwedge^q T^*M)}\Big),
\end{aligned} \tag{2.52}$$

where we have used the commutativity of G and Δ in the third line, the elliptic estimate in the fourth line with $\omega = G\alpha$ in Eq. (2.50), the definition of Δ and G in the fifth line, the Pythagorean theorem in the Hilbert space $H^{-2}(M; \bigwedge^q T^*M)$ in the sixth line, and the Rellich lemma in the final line. Therefore, the estimate in (2.49) has been obtained, and the proof is now complete. ■

As a remark, Theorem 2.5 is a generalised version of the main theorem in Robbin-Roger-Temple [105] to higher dimensions, and has been obtained previously by Kozono-Yanagisawa [78]. In fact, [78] established a more general geometric div-curl lemma on Riemannian manifolds, which applies to two sequences $\{\omega^\epsilon\}$ and $\{\tau^\epsilon\}$ lying in L^r_{loc} and L^s_{loc} ($1 < r, s < \infty$, $\frac{1}{r} + \frac{1}{s} = 1$); also see the Appendix of Chen-Li [25]. We leave it as an open problem to find a purely functional analytic formulation of this result.

Theorem 2.6 *Let (M, g) be an n-dimensional Riemannian manifold. Let $\{\omega^\epsilon\} \subset L^r_{\rm loc}(M; \bigwedge^q T^*M)$ and $\{\tau^\epsilon\} \subset L^s_{\rm loc}(M; \bigwedge^q T^*M)$ be two families of differential q–forms, for $0 \le q \le n$, $1 < r, s < \infty$, and $\frac{1}{r} + \frac{1}{s} = 1$. Suppose that*

(i) *$\omega^\epsilon \rightharpoonup \overline{\omega}$ weakly in L^r, and $\tau^\epsilon \rightharpoonup \overline{\tau}$ weakly in L^s as $\epsilon \to 0$;*

(ii) *There are compact subsets K_d, K_δ of the corresponding Sobolev spaces, such that*

$$\begin{cases} \{d\omega^\epsilon\} \subset K_d \Subset W^{-1,r}_{\rm loc}(M; \bigwedge^{p+1} T^*M), \\ \{\delta\tau^\epsilon\} \subset K_\delta \Subset W^{-1,s}_{\rm loc}(M; \bigwedge^{q-1} T^*M). \end{cases}$$

Then $\langle\omega^\epsilon, \tau^\epsilon\rangle$ converges to $\langle\overline{\omega}, \overline{\tau}\rangle$ in the sense of distributions: For any $\psi \in C^\infty_c(M)$,

$$\int_M \langle\omega^\epsilon, \tau^\epsilon\rangle \psi \, {\rm d}V_g \longrightarrow \int_M \langle\overline{\omega}, \overline{\tau}\rangle \psi \, {\rm d}V_g \qquad \text{as } \epsilon \to 0.$$

The proof can be found in the Appendix of [25].

2.3 Proof of weak continuity of GCR equations

2.3.1 Global weak rigidity theorem

Our main result in this section is the following:

Theorem 2.7 (Theorem 1.2 in §1.2.1) *Let (M, g) be an n-dimensional Riemannian manifold with $W^{1,p}_{\rm loc}$ metric for $p > 2$. Let a sequence of operators $\{(B^\epsilon, \nabla^{\perp,\epsilon})\}$ satisfy*

(i) *The tensor fields $B^\epsilon : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM^\perp)$ and the affine connections $\nabla^{\perp,\epsilon} : \Gamma(TM) \times \Gamma(TM^\perp) \to \Gamma(TM^\perp)$ are uniformly bounded in $L^p_{\rm loc}$ with*

$$\sup_{\epsilon>0} \left\{ \|B^\epsilon\|_{L^p(K)} + \|\nabla^{\perp,\epsilon}\|_{L^p(K)} \right\} \le C_K \tag{2.53}$$

for a constant C_K on any compact subset $K \Subset M$, independent of ϵ;

(ii) *$(B^\epsilon, \nabla^{\perp,\epsilon})$ are solutions of the GCR equations in the following sense: For any $X,Y,Z,W \in \Gamma(TM)$ and $\eta, \xi \in \Gamma(TM^\perp)$,*

$$\langle B^\epsilon(Y, W), B^\epsilon(X, Z)\rangle - \langle B^\epsilon(X, W), B^\epsilon(Y, Z)\rangle = R(X, Y, Z, W), \tag{2.54}$$

$$\langle [S^\epsilon_\eta, S^\epsilon_\xi] X, Y\rangle = R^\perp(X, Y, \eta, \xi), \tag{2.55}$$

$$\bar{\nabla}_Y B^\epsilon(X, Z) - \bar{\nabla}_X B^\epsilon(Y, Z) = 0 \tag{2.56}$$

in the sense of distributions, where S^ϵ is the shape operator corresponding to B^ϵ (Note that $\nabla^{\perp,\epsilon}$ is implicit in the above equations).

Then, after passing to a subsequence, $\{(B^\epsilon, \nabla^{\perp,\epsilon})\}$ converges weakly in $L^p_{\rm loc}$ to a pair $(B, \nabla^\perp)$ that is still a weak solution of the GCR equations (2.16)–(2.18).

This result can be regarded as a global version on Riemannian manifolds of Theorem 3.3 in Chen-Slemrod-Wang [31]. In this paper, both the statement and the proof for this weak rigidity theorem are global, intrinsic, independent of the local coordinates of the Riemannian manifolds, which offers further geometric insights into the GCR equations.

Remark 2.3.1 *Theorem* 1.2 *for the exact solutions can be extended to the weak continuity of approximate solutions $(B^\epsilon, \nabla^{\perp,\epsilon})$ of the GCR equations. More precisely, instead of* (2.54)–(2.56) *in* (ii)*, let $(B^\epsilon, \nabla^{\perp,\epsilon})$ solve the following approximate GCR equations in the distributional sense: For any $X, Y, Z, W \in \Gamma(TM)$ and $\eta, \xi \in \Gamma(TM^\perp)$,*

$$\langle B^\epsilon(X,W), B^\epsilon(Y,Z)\rangle - \langle B^\epsilon(Y,W), B^\epsilon(X,Z)\rangle + R(X,Y,Z,W) = O^\epsilon_1, \tag{2.57}$$

$$\langle [S^\epsilon_\eta, S^\epsilon_\xi]X, Y\rangle - R^\perp(X,Y,\eta,\xi) = O^\epsilon_2, \tag{2.58}$$

$$\bar\nabla_Y B^\epsilon(X,Z) - \bar\nabla_X B^\epsilon(Y,Z) = O^\epsilon_3, \tag{2.59}$$

such that $\lim_{\epsilon\to 0} O^\epsilon_i = 0$ in $W^{-1,r}_{\rm loc}$ for some $r > 1$. Then, after passing to a subsequence, $\{(B^\epsilon, \nabla^{\perp,\epsilon})\}$ converges weakly in L^p to a pair $(B, \nabla^\perp)$ that is a weak solution of the GCR equations (2.16)–(2.18).

2.3.2 First formulation: Identification of tensor fields with special div-curl structures

Now we begin the proof of Theorem 1.2. First we seek for tensor fields with special *div-curl structure* for the GCR equations on manifolds.

Recall our previous convention: $X, Y, Z, \cdots \in \Gamma(TM)$ and $\xi, \eta, \beta, \cdots \in \Gamma(TM^\perp)$. For each fixed (Z, η, ξ), define the tensor fields $V^{(B)}_{Z,\eta}, V^{(\nabla^\perp)}_{\xi,\eta} : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$ and 1–forms $\Omega^{(B)}_{Z,\eta}, \Omega^{(\nabla^\perp)}_{\xi,\eta} \in \Omega^1(M) = \Gamma(T^*M)$ by

$$V^{(B)}_{Z,\eta}(X,Y) := B(X,Z,\eta)Y - B(Y,Z,\eta)X, \tag{2.60}$$

$$V^{(\nabla^\perp)}_{\xi,\eta}(X,Y) := \langle \nabla^\perp_Y \xi, \eta\rangle X - \langle \nabla^\perp_X \xi, \eta\rangle Y, \tag{2.61}$$

$$\Omega^{(B)}_{Z,\eta} := -B(\bullet, Z, \eta), \tag{2.62}$$

$$\Omega^{(\nabla^\perp)}_{\xi,\eta} := \langle \nabla^\perp_\bullet \xi, \eta\rangle. \tag{2.63}$$

To avoid further notations, we denote the vector fields canonically isomorphic to $\Omega^{(B)}_{Z,\eta}$ and $\Omega^{(\nabla^\perp)}_{\xi,\eta}$ (via $\sharp$ and $\flat$) by the same symbols.

The motivation for introducing V and Ω is as follows: The 1–forms $(\Omega^{(B)}, \Omega^{(\nabla^\perp)})$ are simply contractions of $(B, \nabla^\perp)$, and the vector fields $(V^{(B)}, V^{(\nabla^\perp)})$, being anti–symmetric in the arguments $(X, Y) \in TM \otimes TM$, can be viewed as the *rate of rotations* of $\Omega^{(B)}_{Z,\eta}$ and $\Omega^{(\nabla^\perp)}_{\xi,\eta}$ in the 2–planes generated by (X, Y).

Our first formulation concerns the *divergence* of V in Eqs. (2.60)–(2.61) and the *curl* (as 2-forms) of Ω in Eqs. (2.62)–(2.63):

Lemma 2.3.1 (First Formulation) *The divergence of V and the curl of Ω can be reformulated as follows:*

$$\begin{aligned}\mathrm{div}\big(V^{(B)}_{Z,\eta}(X,Y)\big) &= YB(X,Z,\eta) - XB(Y,Z,\eta)\\ &\quad + B(X,Z,\eta)\mathrm{div}Y - B(Y,Z,\eta)\mathrm{divX},\end{aligned} \tag{2.64}$$

$$\begin{aligned}\mathrm{div}\big(V^{(\nabla^\perp)}_{\xi,\eta}(X,Y)\big) &= -Y\langle\nabla^\perp_X\xi,\eta\rangle + X\langle\nabla^\perp_Y\xi,\eta\rangle\\ &\quad + \langle\nabla^\perp_Y\xi,\eta\rangle\mathrm{div}X - \langle\nabla^\perp_X\xi,\eta\rangle\mathrm{div}Y,\end{aligned} \tag{2.65}$$

$$d\big(\Omega^{(B)}_{Z,\eta}\big)(X,Y) = YB(X,Z,\eta) - XB(Y,Z,\eta) + B([X,Y],Z,\eta), \tag{2.66}$$

$$d\big(\Omega^{(\nabla^\perp)}_{\xi,\eta}\big)(X,Y) = -Y\langle\nabla^\perp_X\xi,\eta\rangle + X\langle\nabla^\perp_Y\xi,\eta\rangle - \langle\nabla^\perp_{[X,Y]}\xi,\eta\rangle. \tag{2.67}$$

Proof. To show the first two identities, we use Eq. (2.40) to express the divergence in terms of the Lie derivative, which is further computed from Cartan's formula:

$$\mathcal{L}_X = d\circ\iota_X + \iota_X\circ d, \tag{2.68}$$

where ι is the interior multiplication. Indeed, in local coordinates let us write $X = X^i\partial_i$, $Y = Y^j\partial_j$ and $dV_g = dx^1\wedge\ldots\wedge dx^n$. Then

$$\begin{cases}\iota_X\, dV_g = (-1)^i X^i dx^1\wedge\ldots\wedge\widehat{dx^i}\wedge\ldots\wedge dx^n,\\ \iota_Y\, dV_g = (-1)^j Y^j dx^1\wedge\ldots\wedge\widehat{dx^j}\wedge\ldots\wedge dx^n,\end{cases}$$

where $\widehat{[\cdots]}$ denotes the omission of the corresponding term. (In the above equation only, we temporarily disregard the Einstein summation convention). As a result,

$$*\Big(d\big(\iota_X\, dV_g\big)\Big) = \mathrm{div}\, X, \qquad *\Big(d\big(\iota_Y\, dV_g\big)\Big) = \mathrm{div}\, Y, \tag{2.69}$$

as $\partial_i X^i = \mathrm{div}\, X$ and $*dV_g = 1$. On the other hand, for any $f : M \to \mathbb{R}$ we have

$$\begin{cases}*\Big(df\wedge(\iota_Y\, dV_g)\Big) = *(\partial_j f Y^j dV_g) = Yf,\\ *\Big(df\wedge(\iota_X\, dV_g)\Big) = *(\partial_i f X^i dV_g) = Xf,\end{cases} \tag{2.70}$$

where the vector fields X, Y are identified with the directional derivatives, as remarked in §2.2.1. Therefore, we conclude that

$$\begin{aligned}
\operatorname{div} V_{Z,\eta}^{(B)}(X,Y) &= *\Big(\mathcal{L}_{V_{Z,\eta}^{(B)}(X,Y)} dV_g\Big) \\
&= *\Big(d\iota_{V_{Z,\eta}^{(B)}(X,Y)}\, dV_g\Big) \\
&= *\Big\{d\Big(B(X,Z,\eta)\iota_Y\, dV_g - B(Y,Z,\eta)\iota_X\, dV_g\Big)\Big\} \\
&= *\Big\{d\Big(B(X,Z,\eta)\wedge(\iota_Y dV_g)\Big)\Big\} + B(X,Z,\eta) * \Big(d\big(\iota_Y\, dV_g\big)\Big) \\
&\qquad - *\Big\{d\Big(B(Y,Z,\eta)\wedge(\iota_X dV_g)\Big)\Big\} - B(Y,Z,\eta) * \Big(d\big(\iota_X\, dV_g\big)\Big) \\
&= YB(X,Z,\eta) - XB(Y,Z,\eta) + B(X,Z,\eta)\operatorname{div} Y - B(Y,Z,\eta)\operatorname{div} X.
\end{aligned}$$

In the first line we use Cartan's formula (2.68), in the second $d(dV_g) = 0$, in the third the definition of $V^{(B)}$, in the fourth the definition of d, and in the last line, we apply Eqs. (2.69)(2.70); hence Eq. (2.64) is established. Moreover, the proof for Eq. (2.65) is analogous.

For the *generalised curl* (*i.e.*, d), recall from §2.2.1 the identity:

$$d\alpha(X,Y) := X\alpha(Y) - Y\alpha(X) - \alpha([X,Y]) \qquad \text{for } \alpha \in \Omega^1(M).$$

Applying the above to $\alpha = \Omega_{Z,\eta}^{(B)}$, we get

$$\begin{aligned}
d\Omega_{Z,\eta}^{(B)}(X,Y) &= X\Omega_{Z,\eta}^{(B)}(Y) - Y\Omega_{Z,\eta}^{(B)}(X) - \Omega_{Z,\eta}^{(B)}([X,Y]) \\
&= -XB(Y,Z,\eta) + YB(X,Z,\eta) + B([X,Y],Z,\eta),
\end{aligned}$$

which is Eq. (2.66). The proof of Eq. (2.67) is analogous, and the proof is complete. ■

As a remark, it is crucial to recognise that $\delta\big(V_{Z,\eta}^{(B)}(X,Y)\big)$ and $d\big(\Omega_{Z,\eta}^{(B)}\big)(X,Y)$, as well as $\delta\big(V_{\xi,\eta}^{(\nabla^\perp)}(X,Y)\big)$ and $d\big(\Omega_{\xi,\eta}^{(\nabla^\perp)}\big)(X,Y)$, are essentially the same: They only differ by a zero-th order term involving the Lie bracket $[X,Y]$. This observation turns out to be crucial for deducing Theorem 1.2.

2.3.3 Second formulation: Div-curl structure of GCR equations

We now express the GCR equations in another form, which is suitable for applying the intrinsic div-curl lemma, *i.e.*, Theorem 2.5. To achieve this, we employ the geometric quantities $V^{(B)}, \Omega^{(B)}, V^{(\nabla^\perp)}$, and $\Omega^{(\nabla^\perp)}$, in the reduced GCR equations (2.19)–(2.21) in Theorem 2.2 to obtain

Lemma 2.3.2 (Second Formulation) *The Gauss, Codazzi, and Ricci equations are equivalent to the following equations, respectively:*

$$\sum_{\eta}\langle V^{(B)}_{W,\eta}(X,Y),\Omega^{(B)}_{Z,\eta}\rangle = -R(X,Y,Z,W), \tag{2.71}$$

$$d(\Omega^{(B)}_{Z,\eta})(X,Y)+\sum_{\beta}\langle V^{(\nabla^{\perp})}_{\eta,\beta}(X,Y),\Omega^{(B)}_{Z,\beta}\rangle + E(B)=0, \tag{2.72}$$

$$d(\Omega^{(\nabla^{\perp})}_{\xi,\eta})(X,Y)+\sum_{\beta}\langle V^{(\nabla^{\perp})}_{\eta,\beta}(X,Y),\Omega^{(\nabla^{\perp})}_{\xi,\beta}\rangle = \sum_{Z}\langle V^{(B)}_{Z,\xi}(X,Y),\Omega^{(B)}_{Z,\eta}\rangle, \tag{2.73}$$

where $E(B):=B(Y,\nabla_X Z,\eta)-B(X,\nabla_Y Z,\eta)$ *is linear in* B*, and all the summations are at most countable and locally finite.*

Proof. We divide the proof into four steps; see Chen-Li [25] for details.

1. We start with Eq. (2.71). Consider the following spanning set of normal vector fields:

$$\mathcal{S}_1 := \big\{\{\eta_j\}_{j=1}^{k}\subset\Gamma(TM^{\perp}) : |\eta_j|=1, \mathrm{span}\{\eta_j\}_{j=1}^{k}=\Gamma(TM^{\perp})\big\}, \tag{2.74}$$

where $n+k$ is the dimension of the target space of the isometric immersion.

Since metric g is in $W^{1,p}_{\mathrm{loc}}$ on M, and the second fundamental form and the normal connection are well-defined in L^p_{loc}, then $\mathcal{S}_1$ exists *a.e.* on M. In view of the remark above, we can make the following computation in the distributional sense:

$$\begin{aligned}\langle B(X,W),B(Y,Z)\rangle &- \langle B(Y,W),B(X,Z)\rangle\\ &=\sum_{\eta\in\mathcal{S}_1}\big(B(X,W,\eta)B(Y,Z,\eta)-B(Y,W,\eta)B(X,Z,\eta)\big)\\ &=\sum_{\eta\in\mathcal{S}_1}B\big(B(X,W,\eta)Y-B(Y,W,\eta)X,Z,\eta\big)\\ &=\sum_{\eta\in\mathcal{S}_1}\langle V^{(B)}_{W,\eta}(X,Y),\Omega^{(B)}_{Z,\eta}\rangle,\end{aligned} \tag{2.75}$$

where the last equality follows from the definition of $V^{(B)}$ and $\Omega^{(B)}$. Combining the above computation with (2.19) yields Eq. (2.71).

2. Next, we establish Eq. (2.72). Let us start from the Codazzi equation (2.20) in the form of Theorem 2.2:

$$\begin{aligned}&XB(Y,Z,\eta)-YB(X,Z,\eta)\\ &=B([X,Y],Z,\eta)-B(X,\nabla_Y Z,\eta)-B(X,Z,\nabla^{\perp}_Y\eta)+B(Y,\nabla_X Z,\eta)+B(Y,Z,\nabla^{\perp}_X\eta).\end{aligned}$$

By Eq. (2.62) in the first formulation, we can transform the above equation to the following:

$$d\big(\Omega^{(B)}_{Z,\eta}\big)(X,Y) - B(X,\nabla_Y Z,\eta) - B(X,Z,\nabla^\perp_Y\eta) + B(Y,\nabla_X Z,\eta) + B(Y,Z,\nabla^\perp_X\eta) = 0. \tag{2.76}$$

Moreover, observe that

$$\begin{aligned} B(Y,Z,\nabla^\perp_X\eta) - B(X,Z,\nabla^\perp_Y\eta) &= \sum_{\beta\in\mathcal{S}_1} \big(B(Y,Z,\beta)\langle\nabla^\perp_X\eta,\beta\rangle - B(X,Z,\beta)\langle\nabla^\perp_Y\eta,\beta\rangle\big) \\ &= \sum_{\beta\in\mathcal{S}_1} B(\langle\nabla^\perp_X\eta,\beta\rangle Y - \langle\nabla^\perp_Y\eta,\beta\rangle X, Z,\beta) \\ &=: \sum_{\beta\in\mathcal{S}_1} \langle V^{(\nabla^\perp)}_{\eta,\beta}(X,Y), \Omega^{(B)}_{Z,\beta}\rangle. \end{aligned} \tag{2.77}$$

Thus, by putting together Eqs. (2.76)–(2.77), we arrive at Eq. (2.72). It expresses the Codazzi equation in terms of V and Ω.

3. Finally, we prove Eq. (2.73). The Ricci equation (2.21) in Theorem 2.2 reads

$$\begin{aligned} &X\langle\nabla^\perp_Y\xi,\eta\rangle - Y\langle\nabla^\perp_X\xi,\eta\rangle \\ &= \langle\nabla^\perp_{[X,Y]}\xi,\eta\rangle - \langle\nabla^\perp_X\xi,\nabla^\perp_Y\eta\rangle + \langle\nabla^\perp_Y\xi,\nabla^\perp_X\eta\rangle \\ &\quad + B(\bar\nabla_X\xi - \nabla^\perp_X\xi, Y,\eta) - B(\bar\nabla_X\eta - \nabla^\perp_X\eta, Y,\xi). \end{aligned}$$

In light of Eq. (2.67) in the first formulation, we have

$$\begin{aligned} &d(\Omega^{(B)}_{\xi,\eta})(X,Y) + \langle\nabla^\perp_X\xi,\nabla^\perp_Y\eta\rangle - \langle\nabla^\perp_Y\xi,\nabla^\perp_X\eta\rangle \\ &= B(\bar\nabla_X\xi - \nabla^\perp_X\xi, Y,\eta) - B(\bar\nabla_Y\eta - \nabla^\perp_Y\eta, X,\xi). \end{aligned} \tag{2.78}$$

The last two terms on the left-hand side can be expressed as

$$\begin{aligned} \langle\nabla^\perp_X\xi,\nabla^\perp_Y\eta\rangle - \langle\nabla^\perp_Y\xi,\nabla^\perp_X\eta\rangle &= \sum_{\beta\in\mathcal{S}_1} \big(\langle\nabla^\perp_X\xi,\beta\rangle\langle\beta,\nabla^\perp_Y\eta\rangle - \langle\nabla^\perp_Y\xi,\beta\rangle\langle\beta,\nabla^\perp_X\eta\rangle\big) \\ &= \sum_{\beta\in\mathcal{S}_1} \langle\nabla^\perp_{\langle\nabla^\perp_Y\eta,\beta\rangle X - \langle\nabla^\perp_X\eta,\beta\rangle Y}\xi,\beta\rangle \\ &=: \sum_{\beta\in\mathcal{S}_1} \langle V^{(\nabla^\perp)}_{\eta,\beta}(X,Y), \Omega^{(\nabla^\perp)}_{\xi,\beta}\rangle, \end{aligned} \tag{2.79}$$

by using the definition of $V^{(\nabla^\perp)}$ and $\Omega^{(\nabla^\perp)}$ in Eqs. (2.61) and (2.63).

To deal with the right-hand side of Eq. (2.78), we temporarily assume the existence of the tangential spanning set of unit vector fields:

$$\mathcal{S}_2 := \big\{\{Z_j\}_{j=1}^n \subset \Gamma(TM) \,:\, |Z_j| \equiv 1,\ \mathrm{span}\{Z_j\}_{j=1}^n = \Gamma(TM)\big\}. \tag{2.80}$$

In this case, one can compute the right-hand side of equation (2.78):

$$
\begin{aligned}
&B(\bar{\nabla}_X\xi - \nabla^{\perp}_X\xi, Y, \eta) - B(\bar{\nabla}_X\eta - \nabla^{\perp}_X\eta, Y, \xi) \\
&= \sum_{Z\in\mathcal{S}_2} \big(B(\langle\bar{\nabla}_X\xi - \nabla^{\perp}_X\xi, Z\rangle Z, Y, \eta) - B(\langle\bar{\nabla}_X\eta - \nabla^{\perp}_X\eta, Z\rangle Z, Y, \xi)\big) \\
&= \sum_{Z\in\mathcal{S}_2} \big(B(Z, X, \eta)B(Z, Y, \xi) - B(Z, X, \xi)B(Z, Y, \eta)\big). \qquad (2.81)
\end{aligned}
$$

Indeed, to obtain the last equality, recall that B is symmetric in the first two arguments. Then, using Eqs. (2.12)–(2.13), we have

$$
\langle\bar{\nabla}_X\xi - \nabla^{\perp}_X\xi, Z\rangle = -\langle S_\xi X, Z\rangle = -B(X, Z, \xi),
$$

where S is the shape operator. The other term follows similarly.

On the other hand, by Eqs. (2.60) and (2.62), we have

$$
\begin{aligned}
B(Z, X, \eta)B(Z, Y, \xi) - B(Z, X, \xi)B(Z, Y, \eta) &= -B(B(X, Z, \xi)Y - B(Y, Z, \xi)X, Z, \xi) \\
&=: \langle V^{(B)}_{Z,\xi}(X, Y), \Omega^{(B)}_{Z,\eta}\rangle. \qquad (2.82)
\end{aligned}
$$

Thus, putting Eqs. (2.78)–(2.79) and (2.81)–(2.82) together, we obtain Eq. (2.73).

4. Unfortunately we cannot take the existence of $\mathcal{S}_2$ for granted. For example, on $M = \mathbb{S}^{2m}$ the *Hairy Ball theorem* shows that any $Z \in \Gamma(TM)$ must vanish at some point. To overcome this difficulty, we may resort to a partition of unity argument. Let $\mathcal{A} = \{U_\alpha : \alpha \in \mathcal{I}\}$ be an atlas for M, and let $\{\rho_\alpha : \alpha \in \mathcal{I}\}$ be a smooth partition of unity subordinate to $\mathcal{A}$. On each U_α, the spanning set in the form of $\mathcal{S}_2$ exists, since U_α is diffeomorphic to $\mathbb{R}^n$. We write

$$
\mathcal{S}_2^\alpha = \big\{\{Z_j^\alpha\}_{j=1}^n \subset \Gamma(TU_\alpha) : |Z_j^\alpha| \equiv 1, \operatorname{span}\{Z_j^\alpha\}_{j=1}^n = \Gamma(TU_\alpha)\big\}.
$$

Define

$$
\widetilde{\mathcal{S}_2} = \big\{Z := \sum_{\alpha\in\mathcal{I}} \rho_\alpha Z^\alpha \mathbb{1}_{\operatorname{supp}(\rho_\alpha)} : Z^\alpha \in \mathcal{S}_2^\alpha, \alpha \in \mathcal{I}\big\}. \qquad (2.83)
$$

Then we can take $\widetilde{\mathcal{S}_2}$ in place of $\mathcal{S}_2$ so that all the preceding arguments for the Ricci equation pass through. This completes the proof. ■

2.3.4 Completion of the proof of Theorem 1.2

We can now prove the global weak rigidity theorem, Theorem 1.2, for the GCR equations on Riemannian manifolds. Indeed, the main ingredients of the proof have already been provided in the previous two formulations, which express the GCR equations in the form suitable for employing Theorem 2.5.

Proof. Our arguments consist of three steps.

1. By taking the orientable double cover when necessary, we may assume that M is orientable. Then we can employ Theorem 2.5 on M. Moreover, by Theorems 2.1–2.2, $(B^\epsilon, \nabla^{\perp,\epsilon})$ are solutions to Eqs. (2.19)–(2.21) in the distributional sense.

2. Consider our second formulation. The zero-th order terms of Eqs. (2.71)–(2.73) contain linear combinations of the following quadratic nonlinear forms:

$$\begin{aligned} &\langle V^{(B^\epsilon)}_{W,\eta}(X,Y), \Omega^{(B^\epsilon)}_{Z,\eta}\rangle, \quad \langle V^{(\nabla^{\perp,\epsilon})}_{\eta,\beta}(X,Y), \Omega^{(\nabla^{\perp,\epsilon})}_{\xi,\beta}\rangle, \\ &\langle V^{(B^\epsilon)}_{Z,\xi}(X,Y), \Omega^{(B^\epsilon)}_{Z,\eta}\rangle, \quad \langle V^{(\nabla^{\perp,\epsilon})}_{\eta,\beta}(X,Y), \Omega^{(B^\epsilon)}_{Z,\beta}\rangle, \end{aligned} \tag{2.84}$$

and the linear term $E(B^\epsilon)$ in B^ϵ. Clearly, $E(B^\epsilon) \to E(B)$ in $\mathcal{D}'$, owing to the linearity of B and the uniform L^p-boundedness of $\{B^\epsilon\}$.

For the four terms in (2.84), by the hypotheses of Theorem 1.2, we find that

$$\{V^{(B^\epsilon)}(X,Y), \Omega^{(B^\epsilon)}, V^{(\nabla^{\perp,\epsilon})}(X,Y), \Omega^{(\nabla^{\perp,\epsilon})}\}$$

are uniformly bounded in L^p_{loc}. As a consequence, the four quadratic terms in (2.84) are uniformly bounded in $L^{p/2}_{\mathrm{loc}}$ for $p > 2$.

Then, by Eqs. (2.72)–(2.73), we know that $d(\Omega^{(\nabla^{\perp,\epsilon})}_{\xi,\eta})(X,Y)$ and $d(\Omega^{(B^\epsilon)}_{Z,\eta})(X,Y)$ are uniformly bounded in $L^{p/2}_{\mathrm{loc}}$, so that they are compact at least in $W^{-1,p'}_{\mathrm{loc}}$ for some $p' \in (1,2)$ by the Sobolev embedding. On the other hand, since $\Omega^{(\nabla^{\perp,\epsilon})}_{\xi,\eta}$ and $\Omega^{(B^\epsilon)}_{Z,\eta}$ are uniformly bounded in $L^p_{\mathrm{loc}}, p > 2$, $d(\Omega^{(\nabla^{\perp,\epsilon})}_{\xi,\eta})(X,Y)$ and $d(\Omega^{(B^\epsilon)}_{Z,\eta})(X,Y)$ are uniformly bounded in $W^{-1,p}_{\mathrm{loc}}, p > 2$. Then, by Lemma 2.3.3 below, we conclude that

$$\big\{d(\Omega^{(\nabla^{\perp,\epsilon})}_{\xi,\eta})(X,Y),\ d(\Omega^{(B^\epsilon)}_{Z,\eta})(X,Y)\big\} \qquad \text{are pre-compact in } H^{-1}_{\mathrm{loc}}(M).$$

Lemma 2.3.3 (Interpolation Lemma) *Let $O \subset \mathbb{R}^n$ be an open bounded set. Let $1 < p' \le 2 < p < \infty$. Assume that $\{f^\epsilon\}$ is bounded in $W^{-1,p}(O)$ and is pre–compact in $W^{-1,p'}(O)$. Then $\{f^\epsilon\}$ is pre–compact in $H^{-1}(O)$.*

This lemma is due to an earlier result of F. Murat (*cf.* Tartar [118], Evans [56]). It has been applied to prove the global existence of weak solutions of 1D conservation law (*cf.* Ding-Chen-Luo [49]) and 1D nonlinear elastodynamics (*cf.* Lin [82]).

Furthermore, Eqs. (2.64)–(2.67) of the first formulation lead to:

$$\begin{aligned} \mathrm{div}\big(V^{(B^\epsilon)}_{Z,\eta}(X,Y)\big) &= d(\Omega^{(\nabla^{\perp,\epsilon})}_{Z,\eta})(X,Y) - B^\epsilon([X,Y],Z,\eta) \\ &\quad + B^\epsilon(X,Z,\eta)\mathrm{div}Y - B^\epsilon(Y,Z,\eta)\mathrm{div}X, \end{aligned} \tag{2.85}$$

$$
\begin{aligned}
\mathrm{div}\big(V^{(\nabla^{\perp,\epsilon})}_{\xi,\eta}(X,Y)\big) = d(\Omega^{(\nabla^{\perp,\epsilon})}_{\xi,\eta})(X,Y) + \langle \nabla^{\perp,\epsilon}_{[X,Y]}\xi, \eta\rangle \\
+ \langle \nabla^{\perp,\epsilon}_{Y}\xi, \eta\rangle \mathrm{div}X - \langle \nabla^{\perp,\epsilon}_{X}\xi, \eta\rangle \mathrm{div}Y.
\end{aligned} \tag{2.86}
$$

Since $(B^\epsilon, \nabla^{\perp,\epsilon})$ are uniformly bounded in $L^p_{\mathrm{loc}}, p > 2$, again by the Sobolev embeddings, they are compact in H^{-1}_{loc}. Since the divergence operator equals δ (the adjoint of d) modulo a sign, we conclude that

$$
\big\{\delta\big(V^{(B^\epsilon)}_{Z,\eta}(X,Y)\big),\ \delta\big(V^{(\nabla^{\perp,\epsilon})}_{\xi,\eta}(X,Y)\big)\big\} \qquad \text{are pre-compact in } H^{-1}_{\mathrm{loc}}(M).
$$

In this step the remark following Lemma 2.3.1 has been used.

3. Since $(B^\epsilon, \nabla^{\perp,\epsilon})$ are uniformly bounded in $L^p_{\mathrm{loc}}, p > 2$, after passing to subsequences, they converge to some $(B, \nabla^\perp)$ weakly in L^p_{loc}. Combining the arguments in Step 2 with Theorem 2.5, we obtain the following convergence in the distributional sense:

$$
\begin{aligned}
&\langle V^{(B^\epsilon)}_{W,\eta}(X,Y), \Omega^{(B^\epsilon)}_{Z,\eta}\rangle \longrightarrow \langle V^{(B)}_{W,\eta}(X,Y), \Omega^{(B)}_{Z,\eta}\rangle,\\
&\langle V^{(\nabla^{\perp,\epsilon})}_{\eta,\beta}(X,Y), \Omega^{(\nabla^{\perp,\epsilon})}_{\xi,\beta}\rangle \longrightarrow \langle V^{(\nabla^{\perp})}_{\eta,\beta}(X,Y), \Omega^{(\nabla^{\perp})}_{\xi,\beta}\rangle,\\
&\langle V^{(B^\epsilon)}_{Z,\xi}(X,Y), \Omega^{(B^\epsilon)}_{Z,\eta}\rangle \longrightarrow \langle V^{(B)}_{Z,\xi}(X,Y), \Omega^{(B)}_{Z,\eta}\rangle,\\
&\langle V^{(\nabla^{\perp,\epsilon})}_{\eta,\beta}(X,Y), \Omega^{(B^\epsilon)}_{Z,\beta}\rangle \longrightarrow \langle V^{(\nabla^{\perp})}_{\eta,\beta}(X,Y), \Omega^{(B)}_{Z,\beta}\rangle.
\end{aligned}
$$

Therefore, we have passed to the limits as $\epsilon \to 0$ in Eqs. (2.71)–(2.73) of the second formulation, namely Lemma 2.3.2. Moreover, in the same lemma it is shown that the second formulation and the GCR equations are equivalent. Thus the weak limit $(B, \nabla^\perp)$ is still a weak solution of the GCR system, which completes the proof. ■

To conclude this section, we remark that Eqs. (2.1)–(2.3) in Chen–Slemrod–Wang [31], which are the Gauss, Codazzi, and Ricci equations in local coordinates, can be seen directly from our global formulations in Theorems 2.1–2.2. Indeed, write $\{\partial_i : 1 \leq i \leq n\}$ as a local coordinate system on the tangent bundle TM, and $\{\partial_\alpha : n+1 \leq \alpha \leq n+k\}$ for a local coordinate system on the normal bundle $TM^\perp$. To write the GCR equations in the local coordinates, we define

$$
h^\alpha_{ij} := \langle B(\partial_i, \partial_j), \partial_\alpha\rangle, \qquad \kappa^\alpha_{i\beta} := \langle \nabla^\perp_{\partial_\alpha}\partial_i, \partial_\beta\rangle, \tag{2.87}
$$

and substitute the unit vector fields ∂_i (or ∂_α) in place of X (or ξ, respectively), in the GCR equations in Theorems 2.1–2.2. We solve for $\{h^\alpha_{ij}, \kappa^\alpha_{i\beta}\}$, namely the second fundamental form and the normal affine connection written coordinate-wise. In this way we recover Eqs. (2.1)–(2.3) in [31]:

$$
h^\alpha_{ik}h^\alpha_{jl} - h^\alpha_{il}h^\alpha_{jk} = R_{ijkl}, \tag{2.88}
$$

$$\frac{\partial h^\alpha_{lj}}{\partial x^k} - \frac{\partial h^\alpha_{kj}}{\partial x^l} + \Gamma^m_{lj} h^\alpha_{km} - \Gamma^m_{kj} h^\alpha_{lm} + \kappa^\alpha_{k\beta} h^\beta_{lj} - \kappa^\alpha_{l\beta} h^\beta_{kj} = 0, \tag{2.89}$$

$$\frac{\partial \kappa^\alpha_{l\beta}}{\partial x^k} - \frac{\partial \kappa^\alpha_{k\beta}}{\partial x^l} - g^{mn}\big(h^\alpha_{ml} h^\beta_{kn} - h^\alpha_{mk} h^\beta_{ln}\big) + \kappa^\alpha_{k\gamma}\kappa^\gamma_{l\beta} - \kappa^\alpha_{l\gamma}\kappa^\gamma_{k\beta} = 0, \tag{2.90}$$

where $R_{ijkl} := R(\partial_i, \partial_j, \partial_k, \partial_l)$, and the Christoffel symbols are defined as usual by $\Gamma^k_{ij} := \frac{1}{2} g^{kl}\big(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}\big)$.

2.4 Realisation of isometric immersions from GCR Equations: Riemannian case

In §2.3, we have proved the global weak continuity of the GCR equations on Riemannian manifolds with lower regularity. The next natural question is about its geometric implications. In particular, we wish to translate the weak continuity of the PDEs (Gauss-Codazzi-Ricci, established in §2.3) to the weak rigidity of the corresponding isometric immersions.

More precisely, the question is as follows: Given a sequence of isometric immersions $\{f^\epsilon\} : (M, g) \to (\mathbb{R}^{n+k}, g_0)$ locally uniformly bounded in $W^{2,p}$, we know that the corresponding second fundamental forms and normal connections, denoted by $(B^\epsilon, \nabla^{\perp,\epsilon})$, satisfy the GCR equations in the sense of distributions, and they are weakly pre–compact in L^p_{loc}. Say that $(\bar{B}, \overline{\nabla^\perp})$ is a weak limit of $\{(B^\epsilon, \nabla^{\perp,\epsilon})\}$ in L^p_{loc}. By §2.3, $(\bar{B}, \overline{\nabla^\perp})$ also satisfies the GCR equations in the sense of distributions. Now, we ask whether a weak limit $\bar{f}$ of $\{f^\epsilon\}$ is still an isometric immersion of (M, g). This amounts to the following problem: Given $(B, \nabla^\perp) \in L^p_{\text{loc}}$ satisfying the GCR equations on (M, g) in the sense of distributions, can we construct an isometric immersion $f : (M, g) \hookrightarrow (\mathbb{R}^{n+k}, g_0)$ such that $(B, \nabla^\perp)$ coincide with the second fundamental form and the normal connection of f? As remarked in §1.2.1, this is related to the *realisation problem* in non-linear elasticity.

2.4.1 Realisation problem for Riemannian manifolds

The realisation problem is answered in the affirmative by the following result, first proved by Mardare [87, 88] for immersed 2D surfaces in $\mathbb{R}^3$, and then by Szopos [116] for arbitrary dimensions and codimensions. Our aim is to give a re–proof, which adopts the classical arguments by K. Tenenblat [121] using the *Cartan formalism* in exterior differential geometry. This method may appear more natural from the geometric perspectives, and it simplifies the arguments in [87, 88, 116] which involve finding several matrices of large sizes.

In what follows we require $p > n$ to guarantee that the differential $df \in W^{1,p}_{\mathrm{loc}} \hookrightarrow C^0_{\mathrm{loc}}$, *i.e.*, the immersions are understood in the classical sense *a.e.*

Theorem 2.8 (The Realisation Theorem, Riemannian Case) *Let (M, g) be an n-dimensional, simply-connected Riemannian manifold with the metric $g \in W^{1,p}_{\mathrm{loc}}, p > n$, and let $(E, M, \mathbb{R}^k)$ be a vector bundle over M. Assume that E has a $W^{1,p}_{\mathrm{loc}}$ metric g^E and an L^p_{loc} connection ∇^E such that ∇^E is compatible with the metric g^E. Moreover, suppose that there exists an L^p_{loc} tensor field $S : \Gamma(E) \times \Gamma(TM) \to \Gamma(TM)$ satisfying*

$$\langle X, S_\eta(Y)\rangle - \langle S_\eta(X), Y\rangle = 0$$

with the corresponding L^p_{loc}–tensor field $B : \Gamma(TM) \times \Gamma(TM) \to \Gamma(E)$ defined by

$$\langle B(X, Y), \eta\rangle = -\langle S_\eta(X), Y\rangle.$$

Assume that the GCR equations on E (as in Theorem 2.1, with $R^\perp$ replaced by R^E) are satisfied in the sense of distributions. Then, there exists an isometric immersion $f \in W^{2,p}_{\mathrm{loc}}(M; \mathbb{R}^{n+k})$ such that $TM^\perp$ is identified with E, and B, ∇^E coincide with the second fundamental form and the normal connection associated with f.

2.4.2 Cartan formalism

We now introduce a useful tool, the Cartan formalism, for isometric immersions. We sketch some results directly pertaining to isometric immersions. For a local isometric immersion $f : U \subset M^n \hookrightarrow \mathbb{R}^{n+k}$, we adopt the index convention as in [121]:

$$1 \le i, j \le n; \qquad 1 \le a, b, c, d, e \le n + k; \qquad n + 1 \le \alpha, \beta, \gamma \le n + k.$$

On a local chart $U \subset M$ where the vector bundle E is trivialised (*i.e.*, $E|_U$ is diffeomorphic to $U \times \mathbb{R}^k$), let $\{\omega^i\} \subset \Omega^1(U)$ be an orthonormal co-frame dual to the orthonormal frame $\{\partial_i\} \subset \Gamma(TU)$. The latter is called a moving frame adapted to U. The Cartan formalism is thus also known as the *method of moving frames*. In this theory, it is well-known that the GCR equations are equivalent to the following two systems, known as the *first and second structural equations* (*cf.* [111] by Spivak):

$$d\omega^i = \sum_j \omega^j \wedge \omega^i_j, \tag{2.91}$$

$$d\omega^a_b = -\sum_c \omega^c_b \wedge \omega^a_c. \tag{2.92}$$

Here, the 1–forms $\{\omega^a_b\}$ are defined as follows: Let $\{\eta_{n+1}, \ldots, \eta_{n+k}\} \subset \Gamma(E)$ be an orthonormal basis for fibre $\mathbb{R}^k$ of bundle E over the trivialized chart U. Then we set

$$\omega^i_j(\partial_k) := \langle \nabla_{\partial_k}\partial_j, \partial_i\rangle, \tag{2.93}$$

$$\omega^i_\alpha(\partial_j) = -\omega^\alpha_i(\partial_j) := \langle B(\partial_i, \partial_j), \eta_\alpha\rangle, \tag{2.94}$$

$$\omega^\alpha_\beta(\partial_j) := \langle \nabla^E_{\partial_j}\eta_\alpha, \eta_\beta\rangle. \tag{2.95}$$

The structural equations (2.91)–(2.92), together with the definitions for the connection form in local coordinates, *i.e.*, Eqs. (2.93)–(2.95), are referred to as the Cartan formalism.

It is both convenient and conceptually important to introduce the following shorthand notations: Write

$$\mathcal{W} = \{\omega^a_b\} \in \Omega^1(U; \mathfrak{so}(n+k)), \tag{2.96}$$

which is a field of 1–form–valued anti-symmetric matrices (or equivalently, the field of anti-symmetric matrix-valued 1–forms). Heuristically, $\mathcal{W}$ packages together the second fundamental form and normal connection. We also write

$$w = (\omega^1, \ldots, \omega^n, 0, \ldots, 0) \in \Omega^1(U; \mathbb{R}^{n+k}). \tag{2.97}$$

As a result, the structural equations can be written as

$$dw = w \wedge \mathcal{W}, \qquad d\mathcal{W} + \mathcal{W} \wedge \mathcal{W} = 0, \tag{2.98}$$

where operator $\wedge$ between two matrix-valued differential forms denotes the wedge product of differential forms together with matrix multiplication. In the sequel, we always adhere to the practice of writing (matrix) Lie group-valued PDEs in short-handed forms as in (2.98).

We remark that the structure equations, as well as various other Lie group-valued equations we will introduce below, are *intrinsic*. That is, these equations are independent of the moving frames, hence are natural in coordinate-free notations. For more details on the Cartan formalism, see [111, 33].

2.4.3 Proof of Theorem 2.8, the realisation theorem

With the Cartan formalism introduced, we are now in the position to prove the main result of this section, *i.e.*, Theorem 2.8. Our proof is intrinsic and global in nature, *i.e.*, covariant with respect to change of local frames.

We divide the proof into seven steps.

1. First, recall that $\mathcal{W} = \{\omega^a_b\} \in \Omega^1(U; \mathfrak{so}(n+k))$ and $w = (\omega^1, \ldots, \omega^n, 0, \ldots, 0) \in \Omega^1(U; \mathbb{R}^{n+k})$. In order to find an isometric immersion based upon the structural equations (2.91)–(2.92), we first solve for an affine map A (taking ∂_j to η_α), which essentially consists of the components of the normal affine connection, and then the isometric immersion f is constructed from A.

We first carry out such constructions locally. More precisely, given $x_0 \in U$, we find an open set V with $x_0 \in V \subset U$ and a field of symmetric matrices $A = \{A^a_b\} : V \to O(n+k)$, satisfying the first-order PDE system:

$$\begin{cases} \mathcal{W} = dA \cdot A^\top & \text{in } V, \\ A(x_0) = A_0, \end{cases} \tag{2.99}$$

where $A^\top$ is the transpose of matrix A. It means that $\mathcal{W}(X|_x) = dA(X|_x) \cdot A(x)$ for $x \in V$ and $X \in \Gamma(TV)$, where the dot is the matrix multiplication.

Now, under the assumption that Eq. (2.99) is satisfied, we next solve for function $f : V \to \mathbb{R}^{n+k}$ such that

$$\begin{cases} df = w \cdot A & \text{in } V, \\ f(x_0) = f_0, \end{cases} \tag{2.100}$$

i.e., $df_x(X|_x) = w(X|_x) \cdot A(x)$ for $x \in V$ and $X \in \Gamma(TV)$. Following [87, 88], we call (2.99) the Pfaff system and (2.100) the Poincaré system. The names come from the theory of integrable systems and the Poincaré lemma in cohomology, respectively.

2. In the C^∞ category, to establish the solvability of PDEs in form (2.99)–(2.100), which can be viewed as complete integrability conditions, we only need to check the *involutiveness*, in view of the Frobenius theorem. This constitutes the arguments in [121]. In the lower regularity case, we seek for the correct analogue to the Frobenius theorem. The following lemma can serve for this purpose:

Lemma 2.4.1 (Mardare [86, 87, 88]) *The following solubility criteria hold for two types of first-order matrix Lie group-valued PDE systems:*

(i) (The Pfaff system, Theorem 7 of [87])*: Let $U \subset \mathbb{R}^n$ be a simply-connected open set, $x_0 \in U$, and $M_0 \in \mathfrak{gl}(n; \mathbb{R})$. Then the following system:*

$$\frac{\partial M}{\partial x^i} = Q_i \cdot M, \quad i = 1, 2, \ldots, n, \qquad M(x_0) = M_0, \tag{2.101}$$

with the matrix fields $Q_i \in L^p_{\mathrm{loc}}(U; \mathfrak{gl}(n; \mathbb{R}))$ for $i = 1, 2, \ldots, n$, and $p > n$, has a unique solution $M \in W^{1,p}_{\mathrm{loc}}(U; \mathfrak{gl}(n; \mathbb{R}))$ if and only if the following compatibility condition holds:

$$\frac{\partial Q_i}{\partial x^j} - \frac{\partial Q_j}{\partial x^i} = [Q_i, Q_j] \qquad \text{in } \mathcal{D}' \quad \text{for each } i, j = 1, 2, \ldots, n. \tag{2.102}$$

(ii) (The Poincaré system, Theorem 6.5 of [88]): *Let $U \subset \mathbb{R}^n$ be a simply-connected open set, $x_0 \in U$, and $\psi_0 \in \mathbb{R}$. Then the following system:*

$$\frac{\partial \psi}{\partial x^i} = \phi_i, \quad i = 1, 2, \ldots, n, \qquad \psi(x_0) = \psi_0, \tag{2.103}$$

with $\phi_i \in L^p_{\mathrm{loc}}(U)$ for $i = 1, 2, \ldots, n$, and $1 \leq p \leq \infty$, has a unique solution $\psi \in W^{1,p}_{\mathrm{loc}}(U)$ if and only if the following compatibility condition holds:

$$\frac{\partial \phi_i}{\partial x^j} - \frac{\partial \phi_j}{\partial x^i} = 0 \qquad \text{in } \mathcal{D}' \quad \text{for each } i, j = 1, 2, \ldots, n. \tag{2.104}$$

Here and throughout, $\mathfrak{gl}(n; \mathbb{R})$ denotes the vector space of $n \times n$ real matrices, *i.e.*, the Lie algebra of the general linear group. We also remark that the uniqueness in (ii) is proved by L. Schwartz [115].

Thanks to Lemma 2.4.1, solving for systems (2.99) and (2.100) can be reduced to checking the compatibility conditions in the form of Eqs. (2.102) and (2.104), provided that the regularity assumptions are satisfied.

3. Let us first solve for the Pfaff system (2.99). Performing contraction with the moving frame $\{\partial_a\}$, one has

$$dA(\partial_a) = \frac{\partial A}{\partial x^a} = \mathcal{W}(\partial_a) \cdot A, \tag{2.105}$$

which is a Pfaff system on $\mathfrak{gl}(n + k; \mathbb{R})$. By the assumptions $g, g^E \in W^{1,p}_{\mathrm{loc}}$ and $B, \nabla^E \in L^p_{\mathrm{loc}}$, so that $\mathcal{W} \in L^p_{\mathrm{loc}}$ with $p > n$. This verifies the regularity hypotheses in Lemma 2.4.1 (i).

The compatibility criterion (2.102) can be written in local coordinates as

$$\partial_b\big(\omega^c_d(\partial_a)\big) - \partial_a\big(\omega^c_d(\partial_b)\big) = \sum_e [\omega^c_e(\partial_a), \omega^e_d(\partial_b)]. \tag{2.106}$$

Recall formula (2.2): As a special case, for any 1–form α and vector fields (X, Y),

$$d\alpha(X, Y) = X\alpha(Y) - Y\alpha(X) - \alpha([X, Y]). \tag{2.107}$$

Thus, taking $\alpha = \omega^c_d$, $X = \partial_b$, and $Y = \partial_a$, we have

$$d\omega^c_d(\partial_a, \partial_b) = \partial_b\big(\omega^c_d(\partial_a)\big) - \partial_a\big(\omega^c_d(\partial_b)\big). \tag{2.108}$$

Moreover, the Lie bracket on the right-hand side of Eq. (2.106) can be written as the wedge-product of 1–forms:

$$\sum_e [\omega^c_e(\partial_a), \omega^e_d(\partial_b)] = \sum_e \big(\omega^c_e \wedge \omega^e_d\big)(\partial_a, \partial_b).$$

Combining these two observations, the compatibility criterion for the Pfaff system (2.99) is equivalent to the following:

$$d\omega^c_d(\partial_a, \partial_b) = \sum_e \big(\omega^e_d \wedge \omega^c_e\big)(\partial_a, \partial_b).$$

This is precisely the second structural equation (2.92). Thus, by Lemma 2.4.1(i), we can find $A \in W^{1,p}_{\mathrm{loc}}(V; \mathfrak{gl}(n+k;\mathbb{R}))$ satisfying Eq. (2.99). Using equation $\mathcal{W} = dA \cdot A^\top$ and the fact that $\mathcal{W} \in \mathfrak{so}(n+k)$, we deduce that A maps into $O(n+k)$.

4. Next, we solve for immersion f from the Poincaré system (2.100). Since $w \cdot A \in W^{1,p}_{\mathrm{loc}}$ with $p > n$, the regularity assumption in Lemma 2.4.1(ii) is satisfied, both for $w \cdot A$ and its first derivatives. Thus, if we verify the compatibility condition in the form of Eq. (2.104), we can establish the existence of the isometric immersion $f \in W^{2,p}_{\mathrm{loc}}$, thanks to Lemma 2.4.1.

Now, observe that the compatibility condition for the Poincaré system (2.100) is equivalent to

$$\partial_b\big(\omega(\partial_a) \cdot A\big) - \partial_a\big(\omega(\partial_b) \cdot A\big) = 0$$

for each index $1 \le a, b \le n+k$. Invoking Eq. (2.107) once more, we can express

$$d(w \cdot A)(\partial_a, \partial_b) - \partial_b\big(\omega(\partial_a) \cdot A\big) + \partial_a\big(\omega(\partial_b) \cdot A\big) = 0.$$

Therefore, it suffices to show that

$$d(w \cdot A) = 0. \tag{2.109}$$

Indeed, using the Pfaff system (2.99) solved above, we have

$$d(w \cdot A) = dw \cdot A - w \wedge dA = dw \cdot A - w \wedge (\mathcal{W} \cdot A). \tag{2.110}$$

However, the first structural equation (2.91) can be expressed as

$$dw^i = \sum_a \omega^a \wedge \omega^i_a = \sum_j \omega^j \wedge \omega^i_j,$$

since $w_\beta = 0$ for any $n+1 \le \beta \le n+k$ by construction. Then

$$dw \cdot A = w \wedge (\mathcal{W} \cdot A). \tag{2.111}$$

Putting Eqs. (2.110)–(2.111) together, one can deduce Eq. (2.109), which is equivalent to the compatibility condition for the Poincaré system (2.100). As a consequence, we find that $f \in W^{2,p}_{\mathrm{loc}}(V; \mathbb{R}^{n+k})$ solves Eq. (2.100).

5. With the solution to the Pfaff system (2.99) and the Poincaré systems (2.100) associated with our isometric immersions problem, it remains to check that f is indeed the immersion for which we are seeking. To this end, we can define a new local moving frame $\{e_1, \ldots, e_{n+k}\}$ by

$$e_i := df(\partial_i), \qquad e_\alpha := \sum_b A_b^\alpha \partial_b. \tag{2.112}$$

Then, following precisely the same arguments as those in pages 32–35 in [121], we can show that such a construction gives the correct normal bundle metric, normal affine connection, and second fundamental form.

Moreover, we observe that $df = w \cdot A$ from Eq. (2.100), $A \in O(n+k)$ is a field of nonsingular matrices, and $\{\omega^1, \ldots, \omega^n\}$ are linearly independent. Since $A \in W^{1,p}_{\text{loc}}$ and $p > n$, Morrey's embedding $W^{1,p} \hookrightarrow C^0$ shows that $df \neq 0$ *a.e.* In other words, f is indeed an immersion modulo a set of measure zero. The almost everywhere uniqueness of f follows from Lemma 2.4.1 and the $\mathbb{R}^{n+k} \rtimes O(n+k)$–symmetry of the Euclidean space.

6. It remains to globalise our arguments for simply-connected manifolds. This follows from a standard monodromy argument: Given any two points $x \neq y \in M$, we connect them by a continuous curve (again since $g \in W^{1,p}_{\text{loc}} \hookrightarrow C^0_{\text{loc}}$ for $p > n$), denoted by $\gamma : [0,1] \mapsto M$ with $\gamma(0) = x$ and $\gamma(1) = y$. Let f be the $W^{2,p}_{\text{loc}}$–isometric immersion in a neighbourhood of x, whose existence is guaranteed by the preceding steps of the same proof. Cover the curve $\gamma([0,1])$ by finitely many charts $\{V_1, \ldots, V_N\}$. By the uniqueness statements in Lemma 2.4.1, we can extend the isometric immersion f to $\bigcup_{i=1}^N V_i$, especially including a neighbourhood of y. Thus, it suffices to verify that the extension of f is independent of the choice of γ. Indeed, if $\eta : [0,1] \mapsto M$ is another continuous curve connecting x and y, by concatenating γ with η, we can obtain a loop $L \subset M$. As M is simply-connected, the restriction $f|_L$ is homotopic to a constant map so that $(f \circ \gamma)(1) = (f \circ \eta)(1)$. In this way, we have verified that f can be extended to a global isometric immersion of M into $\mathbb{R}^{n+k}$, provided that M is simply-connected. This completes the proof.

As a corollary, one can deduce the weak rigidity of isometric immersions. The proof is straightforward and we refer to Corollary 5.2 in [25] for details.

Corollary 2.4.1 (Weak rigidity of isometric immersions, Riemannian case) *Let M be an n-dimensional simply-connected Riemannian manifold with $W^{1,p}_{\text{loc}}$ metric g for $p > n$. Suppose that $\{f^\epsilon\}$ is a family of isometric immersions of M into*

$\mathbb{R}^{n+k}$, *uniformly bounded in* $W^{2,p}_{\rm loc}(M;\mathbb{R}^{n+k})$, *whose second fundamental forms and normal connections are* $\{B^\epsilon\}$ *and* $\{\nabla^{\perp,\epsilon}\}$ *respectively. Then, after passing to subsequences,* $\{f^\epsilon\}$ *converges to* $\bar f$ *weakly in* $W^{2,p}_{\rm loc}$ *which is still an isometric immersion* $\bar f : (M,g) \hookrightarrow \mathbb{R}^{n+k}$. *Moreover, the corresponding second fundamental form* $\bar B$ *is a limit point of* $\{B^\epsilon\}$, *and the corresponding normal connection* $\overline{\nabla^\perp}$ *is a limit point of* $\{B^\epsilon\}$, *both taken in the* $L^p_{\rm loc}$ *topology.*

2.4.4 The end-point case: $n = p = 2$

Our main geometric result established in §2.4, *i.e.* Corollary 2.4.1, deals with the Riemannian manifolds with $W^{1,p}_{\rm loc}$ metrics for $p > n$. Even for the weak rigidity of the GCR equations, we still require $p > 2$, as in §2.3. Therefore, $n = p = 2$ becomes a critical case, from the perspectives of both geometry and PDE. This is what we investigate in this subsection. We focus on a 2-dimensional manifold M with some Riemannian metric $g \in H^1_{\rm loc}$.

The difficulty lies in the insufficiency of regularity for certain Sobolev embeddings. Notice that, for $n = p = 2$, the second fundamental forms and normal connections $(B^\epsilon, \nabla^{\perp,\epsilon})$ are only uniformly bounded in $L^2_{\rm loc}$, so the quadratic nonlinearities in the GCR equations are at most bounded in $L^1_{\rm loc}$. However, $L^1_{\rm loc}$ cannot be embedded into $H^{-1}_{\rm loc}$. This can be seen via the dual spaces, since $H^1(\mathbb{R}^2)$ is only continuously, but not compactly, embedded into $\mathrm{BMO}(\mathbb{R}^2)$, which is the space of functions of bounded mean oscillations; moreover, $\mathrm{BMO}(\mathbb{R}^2)$ is strictly contained in $L^\infty(\mathbb{R}^2)$. A standard example for $f \in H^1(\mathbb{R}^2) \setminus L^\infty(\mathbb{R}^2)$ is $f(x) = \log\log(|x|^{-1})\mathbb{1}_{B_1(0)}$.

However, if the codimension of the immersion is 1, *i.e.*, the surface M is immersed into $\mathbb{R}^3$, we can still obtain the weak rigidity. For this purpose, we need the corresponding critical case of Theorem 2.5, in which $\{d\omega^\epsilon\}$ and $\{\delta\tau^\epsilon\}$ are contained in compact subsets of $W^{-1,1}_{\rm loc}$. This has been treated by Conti-Dolzmann-Müller in [46].

We now state and prove a slight variant of the critical case result in [46], formulated for global differential forms on the Riemannian manifolds. This can be achieved by combining the global arguments on the manifolds developed above with the arguments in [46]; thus, its proof will be sketched briefly.

Theorem 2.9 *Let* (M,g) *be an* n*-dimensional Riemannian manifold. Let* $\{\omega^\epsilon\}$, $\{\tau^\epsilon\} \subset L^2_{\rm loc}(M;\bigwedge^q T^*M)$ *be two families of differential* q*–forms,* $0 \le q \le n$. *Suppose that*

(i) $\omega^\epsilon \rightharpoonup \overline{\omega}$ *weakly in* L^2, *and* $\tau^\epsilon \rightharpoonup \overline{\tau}$ *weakly in* L^2*;*

(ii) *There are compact subsets K_d, K_δ of the corresponding Sobolev spaces, such that*

$$\begin{cases}\{d\omega^\epsilon\} \subset K_d \Subset W^{-1,1}_{\rm loc}(M; \bigwedge^{q+1} T^*M),\\ \{\delta\tau^\epsilon\} \subset K_\delta \Subset W^{-1,1}_{\rm loc}(M; \bigwedge^{q-1} T^*M);\end{cases}$$

(iii) $\{\langle\omega^\epsilon, \tau^\epsilon\rangle\}$ *is equi-integrable.*

Then $\langle\omega^\epsilon, \tau^\epsilon\rangle$ converges to $\langle\overline{\omega}, \overline{\tau}\rangle$ in the sense of distributions:

$$\int_M \langle\omega^\epsilon, \tau^\epsilon\rangle\psi \,{\rm d}V_g \longrightarrow \int_M \langle\overline{\omega}, \overline{\tau}\rangle\psi \,{\rm d}V_g \qquad \textit{for any } \psi \in C^\infty_c(M).$$

Proof. First of all, let us make two reductions: First, as in the proof for Theorem 2.5, it suffices to prove for compact orientable manifold M; Second, without loss of generality, we may take $\bar{\omega} = \bar{\tau} = 0$.

Next we perform a truncation to $\{\omega^\epsilon\}$ and $\{\tau^\epsilon\}$. By assumption (i), the sequence $\{|\omega^\epsilon|^2\}$ is weakly convergent in L^1. Hence, applying Chacon's biting lemma (*cf.* Ball-Murat [7]), one can find subsets $K_\epsilon \subset M$ such that $|K_\epsilon|_g := \int_M \mathbb{1}_{K_\epsilon}\,{\rm d}V_g \to 0$ and $\{|\omega^\epsilon|^2\mathbb{1}_{M\setminus K_\epsilon}\}$ is equi-integrable. We define the truncated differential forms $\tilde{\omega}^\epsilon := \omega^\epsilon\mathbb{1}_{M\setminus K_\epsilon}$. Similarly, we take $\tilde{K}_\epsilon \subset M$ such that $|\tilde{K}_\epsilon|_g \to 0$ and $\{|\tau^\epsilon|^2\mathbb{1}_{M\setminus\tilde{K}_\epsilon}\}$ is equi-integrable to obtain the truncated forms $\tilde{\tau}^\epsilon := \tau^\epsilon\mathbb{1}_{M\setminus\tilde{K}_\epsilon}$.

Now let us measure the L^1 difference of ω^ϵ and $\tilde{\omega}^\epsilon$. By Cauchy-Schwarz,

$$\|\tilde{\omega}^\epsilon - \omega^\epsilon\|_{L^1(M;\bigwedge^q T^*M)} = \|\omega^\epsilon\|_{L^1(K_\epsilon)} \leq \sqrt{|R_\epsilon|_g}\|\omega^\epsilon\|_{L^2(M;\bigwedge^q T^*M)} \to 0,$$

since $\{\omega^\epsilon\}$ is uniformly bounded in L^2 owing to the assumption (i). Moreover, by (ii), $\|d\tilde{\omega}^\epsilon\|_{W^{-1,1}(M;\bigwedge^{q+1} T^*M)} \to 0$. Analogously, we find that $\|\tilde{\tau}^\epsilon - \tau^\epsilon\|_{L^1(M;\bigwedge^q T^*M)} \to 0$ and $\|\delta\tilde{\tau}^\epsilon\|_{W^{-1,1}(M;\bigwedge^{q-1} T^*M)} \to 0$. Moreover, we note that

$$\langle\omega^\epsilon, \tau^\epsilon\rangle - \langle\tilde{\omega}^\epsilon, \tilde{\tau}^\epsilon\rangle \to 0, \tag{2.113}$$

in view of the equi-integrability assumption (iii) and $|K_\epsilon \cup \tilde{K}_\epsilon|_g \to 0$. By the Dunford-Pettis theorem and the assumption (iii), $\{\langle\omega^\epsilon, \tau^\epsilon\rangle\}$ is weakly pre-compact in L^1. Hence, Eq. (2.113) implies that $\{\langle\omega^\epsilon, \tau^\epsilon\rangle\}$ and $\{\langle\tilde{\omega}^\epsilon, \tilde{\tau}^\epsilon\rangle\}$ have the same distributional limits.

Thus, it remains to show that $\langle\tilde{\omega}^\epsilon, \tilde{\tau}^\epsilon\rangle \to 0$ in the distributional sense. In fact, by the Lipschitz truncation argument in [46], $d\tilde{\omega}^\epsilon$ and $\delta\tilde{\tau}^\epsilon$ converge to 0 in $H^{-1}_{\rm loc}$, after passing to subsequences. Therefore, we can conclude the proof from the intrinsic div-curl lemma, *i.e.*, Theorem 2.5. ∎

Remark 2.4.1 *In the proof of Theorem* 2.9*, the technique of Lipschitz truncation has played an important role, which reduces the div-curl lemma from the critical case to the usual case, where the target spaces are reflexive* (*cf.* *Theorem* 2.5)*. Such a technique depends explicitly on the geometry of the underlying manifolds.*

We remark that the endpoint case of the intrinsic div-curl lemma, *i.e.*, Theorem 2.9, can also be generalised in a similar manner as for Theorem 2.6:

Theorem 2.10 *Let (M,g) be an n-dimensional Riemannian manifold. Let $\{\omega^\epsilon\} \subset L^r_{\rm loc}(M;\bigwedge^q T^*M)$ and $\{\tau^\epsilon\} \subset L^s_{\rm loc}(M;\bigwedge^q T^*M)$ be two families of differential q–forms, for $0 \leq q \leq n$, $1 < r, s < \infty$, and $\frac{1}{r}+\frac{1}{s}=1$. Suppose that*

(i) *$\omega^\epsilon \rightharpoonup \overline{\omega}$ weakly in L^r, and $\tau^\epsilon \rightharpoonup \overline{\tau}$ weakly in L^s as $\epsilon \to 0$;*

(ii) *There are compact subsets K_d, K_δ of the corresponding Sobolev spaces, such that*
$$\begin{cases}\{d\omega^\epsilon\} \subset K_d \Subset W^{-1,1}_{\rm loc}(M;\bigwedge^{q+1} T^*M),\\ \{\delta\tau^\epsilon\} \subset K_\delta \Subset W^{-1,1}_{\rm loc}(M;\bigwedge^{q-1} T^*M);\end{cases}$$

(iii) *$\{\langle\omega^\epsilon,\tau^\epsilon\rangle\}$ is equi-integrable.*

Then $\langle\omega^\epsilon,\tau^\epsilon\rangle$ converges to $\langle\overline{\omega},\overline{\tau}\rangle$ in the sense of distributions:
$$\int_M \langle\omega^\epsilon,\tau^\epsilon\rangle\psi\,\mathrm{d}V_g \longrightarrow \int_M \langle\overline{\omega},\overline{\tau}\rangle\psi\,\mathrm{d}V_g \qquad \textit{for any } \psi \in C^\infty_c(M).$$

The proof of Theorem 2.10 follows directly by combining the argument for Theorem 2.6 in the appendix with the proof for Theorem 2.9.

After introducing the intrinsic div-curl lemmas in the critical case, we can now establish the global weak rigidity of isometric immersions for a surface into $\mathbb{R}^3$:

Theorem 2.11 *Let M be a 2-dimensional, simply-connected surface, and let g be a metric in $H^1_{\rm loc}$. If $\{f^\epsilon\}$ is a family of $H^2_{\rm loc}$ isometric immersions of M into $\mathbb{R}^3$ such that the corresponding second fundamental forms $\{B^\epsilon\}$ are uniformly L^2 bounded. Then, after passing to subsequences, $\{f^\epsilon\}$ converges to $\bar{f}$ weakly in $H^2_{\rm loc}$ which is still an isometric immersion $\bar{f} : (M,g) \hookrightarrow \mathbb{R}^3$. Moreover, the corresponding second fundamental form $\bar{B}$ is a weak limit of $\{B^\epsilon\}$ in $L^2_{\rm loc}$.*

Proof. We divide the proof into three steps.

1. Since the codimension of the immersions $\{f^\epsilon\}$ is 1, the normal affine connections are trivial. This is because, in the Ricci equation:
$$\langle[S_\eta,S_\xi]X,Y\rangle = R^\perp(X,Y,\eta,\xi),$$

we can only take the normal vector fields η and ξ to be linearly dependent, as the fibre for the normal bundle is simply $\mathbb{R}$. Then both sides are equal to zero, by the definition of $R^{\perp}$ and the Lie bracket. Now we look at the Codazzi equation. In the second formulation (§2.3.3), we have obtained Eq. (2.72), which is

$$0 = d(\Omega^{(B)}_{Z,\eta})(X,Y) + \sum_{\beta} \langle V^{(\nabla^{\perp})}_{\eta,\beta}(X,Y), \Omega^{(B)}_{Z,\beta}\rangle + E(B).$$

However, since $\nabla^{\perp}$ is trivial, the equation is reduced to

$$d(\Omega^{(B)}_{Z,\eta})(X,Y) = -E(B), \tag{2.114}$$

where $E(B)$ is linear in B. We notice that the Codazzi equation in the critical case becomes linear, as the quadratic nonlinearities are absent, so that we can pass to the weak limits.

2. It remains to prove the weak rigidity of the Gauss equation:

$$\langle B^{\epsilon}(Y,W), B^{\epsilon}(X,Z)\rangle - \langle B^{\epsilon}(X,W), B^{\epsilon}(Y,Z)\rangle = R(X,Y,Z,W).$$

By assumption, $\{B^{\epsilon}\}$ is uniformly bounded in $L^2_{\rm loc}$, so that $R \in L^1_{\rm loc}(M; \bigotimes^4 T^*M)$ is a fixed function, by the Cauchy–Schwarz inequality. It follows that

$$\begin{aligned}&\int_M \Big|\langle B^{\epsilon}(Y,W), B^{\epsilon}(X,Z)\rangle - \langle B^{\epsilon}(X,W), B^{\epsilon}(Y,Z)\rangle\Big| \mathbb{1}_A \, {\rm d}V_g \\ &\leq \int_M \mathbb{1}_A |R(X,Y,Z,W)| \, {\rm d}V_g \to 0 \qquad \text{as } |A|_g \to 0,\end{aligned}$$

i.e., the set of quadratic non-linearities $\{\langle B^{\epsilon}(Y,T), B^{\epsilon}(X,Z)\rangle - \langle B^{\epsilon}(X,T), B^{\epsilon}(Y,Z)\rangle\}$ is equi-integrable. Here it is crucial that R is a locally integrable function, not just a Radon measure.

3. Now, we invoke again the arguments in §2.3 to analyse the div-curl structure of the Gauss equation. First, recall our definition for the tensor field $V^{(B)}_{Z,\eta} : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$ and the 1–form $\Omega^{(B)}_{Z,\eta} \in \Omega^1(M) = \Gamma(T^*M)$. We set

$$\begin{cases} V^{(B)}_{Z,\eta}(X,Y) := B(X,Z,\eta)Y - B(Y,Z,\eta)X, \\ \Omega^{(B)}_{Z,\eta} := -B(\bullet, Z, \eta). \end{cases}$$

Then the first formulation in §2.3.2 leads to

$$\begin{cases} {\rm div}\big(V^{(B)}_{Z,\eta}(X,Y)\big) = YB(X,Z,\eta) - XB(Y,Z,\eta), \\ d\big(\Omega^{(B)}_{Z,\eta}\big)(X,Y) = YB(X,Z,\eta) - XB(Y,Z,\eta) + B([X,Y],Z,\eta), \end{cases} \tag{2.115}$$

and the second formulation in §2.3.3 enables us to recast the Gauss equation into the following form:

$$\sum_\eta \langle V^{(B)}_{W,\eta}(X,Y), \Omega^{(B)}_{Z,\eta}\rangle = R(X,Y,Z,W).$$

Since $\{B^\epsilon\}$ is uniformly bounded in $L^2_{\rm loc}$, we deduce from Eq. (2.115) that $\{\delta V^{(B^\epsilon)}\}$ and $\{d\Omega^{(B^\epsilon)}\}$ are pre-compact in $W^{-1,1}_{\rm loc}$. Now we can employ the critical case of our intrinsic div-curl lemma, Theorem 2.10, to conclude that

$$\begin{aligned}&\langle B^\epsilon(X,T), B^\epsilon(Y,Z) - \langle B^\epsilon(Y,T), B^\epsilon(X,Z)\rangle \\ &\qquad\longrightarrow \langle \overline{B}(X,T), \overline{B}(Y,Z) - \langle \overline{B}(Y,T), \overline{B}(X,Z)\rangle \end{aligned}\tag{2.116}$$

in distributions, where $\overline{B}$ is the $L^2_{\rm loc}$ weak limit of $\{B^\epsilon\}$. Thus we conclude the weak rigidity of the Gauss equation, and the proof is complete. ∎

2.5 Concluding remarks

In §2 we have established the weak continuity of GCR equations on Riemannian manifolds for $W^{2,p}_{\rm loc}$ isometric immersions, with $p > 2$ (Theorem 1.2). Moreover, if in addition $p > n$, *i.e.*, if the immersion is understood in the classical sense modulo a null set, then we can translate the above weak continuity result of PDEs to the weak rigidity of isometric immersions, via the realisation theorem (Theorem 2.8). This addresses Problem 1.1 in the affirmative.

We should notice that $W^{2,p}_{\rm loc}$ is not the optimal space to directly pass to the limits in the defining equation for isometric immersions, namely $D^\top f \cdot Df = g$ on $O(n)$. In view of Nash's result in the C^1 isometric embeddings paper [97], one would conjecture that this equation is weakly continuous in $W^{1,q}_{\rm loc}$ for some $q > 1$. To be more precise, in this work we are investigating the "curvature–preserving" isometric immersions: We wish to pass to the limits in the GCR equations, hence require the Riemann curvature R to make sense in distributions. This can be improved to $R \in L^{p/2}_{\rm loc}$, due to the Gauss equation (2.19).

Moreover, it should be emphasised that our main result, Theorem 1.2, is proved using the intrinsic div-curl lemma (Theorems 1.3, 2.6). Such theorems depend crucially on the ellipticity of the Laplace-Beltrami operator $\Delta = d\delta + \delta d$, which holds if and only if the underlying manifold is Riemannian. In other words, the arguments in §2 are *not* valid in the semi-Riemannian case. Therefore, to tackle Problem 1.4 (which is the main content of §3 below) requires completely distinct techniques.

Chapter 3

Weak Continuity of Gauss-Codazzi-Ricci Equations: The Semi-Riemannian Case

In this chapter we study the semi-Riemannian counterpart of the project in §2, which concerns the weak continuity of the Gauss-Codazzi-Ricci (GCR) equations and the corresponding isometric immersions of semi-Riemannian manifolds.

As stated in §2.1, in the classical work [98] J. Nash established the existence of isometric embeddings of Riemannian manifolds with C^k metric ($k \geq 3$) into the Euclidean spaces of large enough co-dimensions. The analogous problem for semi-Riemannian manifolds, *i.e.*, when the metric tensors are not necessarily positive-definite, is posed as a natural extension. In [38], by adapting Nash's arguments C. J. S. Clarke proved the existence theorem of isometric embeddings of C^k semi-Riemannian manifolds for semi-Euclidean spaces, with extra restrictions on the signature. However, despite such general existence results, the analysis for semi-Riemannian isometric immersions turns out to be more difficult than the Riemannian case, due to the indefiniteness of the metric. Primarily, the Laplace-Beltrami operator is no longer elliptic, which forbids the application of standard elliptic PDE techniques. We refer to Goenner [61] and Greene [62] for the earlier mathematical works.

Meanwhile, the above problem also attracts enormous attention from the community of theoretical physics. The manners in which our four-dimensional spacetime is immersed or embedded into higher dimensions correspond to the cosmological models, designed to explain the properties of our current universe and the plausible machinery for its interactions with other universes; *cf.* Mars-Senovilla ([89, 90]).

Recently, in [17, 18] Bernal and Sánchez proved that a generic non-globally hyperbolic Lorentzian manifold (namely a *space-time*) cannot be globally isometric embedded into any Minkowski space, regardless of the co-dimensions. The obstruction arises from the Lorentzian geometry, and in particular, the non-existence of "temporal functions". Furthermore, instead of restricting only to smooth semi-Riemannian manifolds, one needs to take into consideration semi-Riemannian manifolds *with lower regularity*, *e.g.*, space-times with edges, cusps and discontinuities. Such scenarios arise naturally in the thin-shell model for gravitational source and the junction condition for glueing disjoint space-times; *cf.* the physics papers by Clarke [39], Geroch [60] and Babrabès-Isreal [5] for details.

Motivated by both the mathematical and physical works, let us analyse the isometric immersions of semi-Riemannian manifolds with lower regularity. As in §2 our analysis is carried out based on the GCR equations, which are the compatibility equations for the *curvature* of the isometric immersions (involving two derivatives of the metric).

3.1 Geometric preliminaries

To begin with, we review some basics of the submanifold theory in semi-Riemannian geometry, focusing on the derivation of the GCR equations. Throughout we closely follow [103] by O'Neill, while explaining several *ad hoc* constructions in the framework of vector bundles. As a remark on notations, in §3 we denote the second fundamental form by II as in [103], instead of B as in do Carmo [51] and §2 above.

3.1.1 Semi-Riemannian submanifolds

Let M be an n-dimensional topological/differentiable manifold. It is said to be *semi-Riemannian* if there exists a symmetric, non-degenerate 2-form field g on the tangent bundle TM with constant index. Here, g is *non-degenerate* on M if for each $x \in M$, there exists no $v \in T_xM \setminus \{0\}$ such that $g(v,w) = 0$ for every $w \in T_xM$; the *index* of g on T_xM is defined as

$$\begin{aligned}\mathrm{Ind}(g; T_xM) := \max\Big\{ &\dim V : V \subset T_xM \text{ is a vector subspace,} \\ &\text{such that } g|_V \text{ is negative definite}\Big\}.\end{aligned} \tag{3.1}$$

Clearly, if M is connected, then $\mathrm{Ind}(g; T_pM)$ is constant for all $p \in M$, which shall be written as $\mathrm{Ind}(g)$ in the sequel. Also, g is known as a *semi-Riemannian metric*.

By employing the Gram-Schmidt process to a subset $U \subset M$, we can find a local orthonormal basis $\{e_i\}_1^n \subset TU$, so that g is diagonalised:

$$g = \{g_{ij}\} = \delta_{ij}\epsilon_j|g_{ij}| \qquad \text{for each } i, j \in \{1, 2, \dots, n\}. \tag{3.2}$$

Here $\epsilon = (\epsilon_1, \dots, \epsilon_n)^\top \in \{-1, 1\}^n$ is called the *signature* of the semi-Riemannian metric g. Notice that as g is non-degenerate, it has only non-zero entries on the diagonal, so $\mathrm{Ind}(g)$ equals to the number of -1's in ϵ. For simplicity, from now on the semi-Riemannian manifold (M, g) is always taken to be connected, and we call $\mathrm{Ind}(g)$ the *index of* M, if g is fixed.

Now, let $(\bar{M}, \bar{g})$ be a given semi-Riemannian manifold, and let M be a submanifold via the embedding $\iota : M \hookrightarrow (\bar{M}, \bar{g})$, *i.e.* ι is injective and $d\iota : TM \to T\bar{M}$ is also injective. We say that $(M, \iota^*\bar{g})$ is a *semi-Riemannian submanifold* of $(\bar{M}, \bar{g})$ if $\iota^*\bar{g}$ is non-degenerate on M, where $\iota^*\bar{g}$ denotes the pullback of $\bar{g}$, defined by

$$(\iota^*\bar{g})_x(v, w) := \bar{g}_{\iota(x)}\big(d\iota(v), d\iota(w)\big) \qquad \text{for each } x \in M \text{ and } v, w \in T_xM. \tag{3.3}$$

For the ι above we consider $\iota^*T\bar{M}$, the *pullback bundle* of $T\bar{M}$ via ι. It has base manifold M and fibre $T_{\iota(x)}\bar{M}$ at each $x \in M$, so that $\Gamma(\iota^*T\bar{M})$, the space of sections of the bundle $\iota^*T\bar{M}$, consists of the *vector fields in* $T\bar{M}$ *defined along* M. In particular, its fibre at $x \in M$ decomposes as follows:

$$\iota^*T\bar{M}|_x := T_{\iota(x)}\bar{M} = d_x\iota(T_xM) \bigoplus [d_x\iota(T_xM)]^\perp \cong T_xM \bigoplus [d_x\iota(T_xM)]^\perp, \tag{3.4}$$

whenever ι is a *local immersion*, *i.e.*, $d\iota$ is injective in some neighbourhood of $x \in M$. Here the direct sum is taken with respect to the bilinear form $\bar{g}$ on $T_{\iota(x)}\bar{M}$:

$$\big[d_x\iota(T_xM)\big]^\perp := \big\{v \in T_{\iota(x)}\bar{M} : \bar{g}_{\iota(x)}\big(v, d_x\iota(w)\big) = 0 \text{ for all } w \in T_xM\big\}.$$

Eq. (3.4) is a special case of Lemma 23 on p49 of [103], which is proved by a straightforward dimension-counting. We remark that it only holds when $\iota^*\bar{g}$ is non-degenerate, *i.e.*, M is immersed into $(\bar{M}, \bar{g})$ as a semi-Riemannian submanifold. In this case, TM and $\iota^*T\bar{M}$ are both vector bundles over M and $TM \subset \iota^*T\bar{M}$, so the quotient bundle is well-defined:

Definition 3.1.1 *The normal bundle of the isometric immersion $\iota : M \hookrightarrow \bar{M}$ is the following quotient bundle over M:*

$$TM^\perp := \frac{\iota^*T\bar{M}}{TM}. \tag{3.5}$$

Thus, in view of Eq. (3.4), the fibre of $TM^\perp$ at $x \in M$ (denoted as $T_xM^\perp$) is isomorphic to $[d_x\iota(T_xM)]^\perp$, hence we have the following vector space isomorphism:

$$T_{\iota(x)}\bar{M} \cong T_xM \bigoplus T_xM^\perp. \tag{3.6}$$

The canonical projections of $T_{\iota(x)}\bar{M}$ onto the first and second factors are called the *tangential* and *normal projections*, denoted as in [103] by the following:

$$\tan : \iota^*T\bar{M}|_x \to T_xM, \qquad \mathrm{nor} : \iota^*T\bar{M}|_x \to T_xM^\perp. \tag{3.7}$$

Moreover, these projections naturally induce the projections of vector fields:

$$\tan : \Gamma(\iota^*T\bar{M}) \to \Gamma(TM), \qquad \mathrm{nor} : \Gamma(\iota^*T\bar{M}) \to \Gamma(TM^\perp), \tag{3.8}$$

as well as the projections of vector fields with Sobolev regularities:

$$\tan : W^{k,p}(M;\iota^*T\bar{M}) \to W^{k,p}(M;TM); \qquad \mathrm{nor} : W^{k,p}(M;\iota^*T\bar{M}) \to W^{k,p}(M;TM^\perp), \tag{3.9}$$

for any $p \in [1,\infty]$ and $k \in \mathbb{Z}$. Here and in the sequel, $\Gamma(E)$ for a vector bundle E denotes the space of sections of E, *i.e.*, $s : M \to E$ such that $\pi \circ s = \mathrm{id}_M$, where $\pi : E \to M$ is the projection of the bundle E onto the base manifold.

Having discussed the orthogonal splitting of tangent and normal bundles under isometric immersions between semi-Riemannian manifolds, *cf.* Eqs. (3.4) and (3.6), we now investigate the *splitting of connections*. Before doing so, let us specify two conventions on notations throughout this paper:

Convention 3.1.2 *We always write the* tangential *vector fields as $X, Y, Z, W, \ldots \in \Gamma(TM)$ and the* normal *vector fields as $\xi, \eta, \zeta, \ldots \in \Gamma(TM^\perp)$. For a generic vector field not necessarily tangential or normal,* i.e. *an element in $\Gamma(T\bar{M})$ or $\Gamma(\iota^*T\bar{M})$, we use the letters U, V, W. Finally, for a given bundle E different from TM, $TM^\perp$ and $\iota^*T\bar{M}$, we write $\alpha, \beta, \ldots \in \Gamma(E)$.*

Convention 3.1.3 *Given an isometric immersion $f : M \to \bar{M}$, we write $\{\partial_a\}$, S, g, ∇, R, … for the geometric quantities on M, and $\{\bar{\partial}_a\}$, $\bar{S}$, $\bar{g}_0$, $\bar{\nabla}$, $\bar{R}$, … for those defined on $\bar{M}$.*

Now, take a semi-Riemannian manifold $(\bar{M}, \bar{g})$ and let $\iota : M \hookrightarrow \bar{M}$ be a semi-Riemannian submanifold. Levi-Civita's theorem says that there exists a unique affine connection $\bar{\nabla} : \Gamma(T\bar{M}) \times \Gamma(T\bar{M}) \to \Gamma(T\bar{M})$ which is metric-compatible and torsion-free. More precisely, the following conditions hold for any $\varphi : M \to \mathbb{R}$ and $U, V, W \in \Gamma(T\bar{M})$:

1. Affine: $\bar{\nabla}_{\varphi V} W = \varphi \bar{\nabla}_V W$ and $\bar{\nabla}_V(\varphi W) = V(\varphi)W + \varphi \bar{\nabla}_V W$;

2. Compatible with metric: $U\bar{g}(V, W) = \bar{g}(\bar{\nabla}_U V, W) + \bar{g}(V, \bar{\nabla}_U W)$;

3. Torsion-free: $\bar{\nabla}_V W - \bar{\nabla}_W V = [V, W]$.

Now, by the theory of fibre bundles (*cf.* [112]), the embedding $\iota : M \hookrightarrow \bar{M}$ induces the *pullback connection* $\iota^*\bar{\nabla} : \Gamma(TM) \times \Gamma(\iota^* T\bar{M}) \to \Gamma(\iota^* T\bar{M})$ on the pullback bundle $\iota^* T\bar{M}$, defined as

$$(\iota^*\bar{\nabla})_X(\iota^*\alpha) = \iota^*\big(\bar{\nabla}_{d\iota(X)}\alpha\big) \qquad \text{for any } \alpha \in \Gamma(T\bar{M}), X \in \Gamma(TM). \tag{3.10}$$

Hence for a vector field $V \in \Gamma(T\bar{M})$ along M, namely $V \in \Gamma(\iota^* T\bar{M})$, we have

$$(\iota^*\bar{\nabla})_X V = \iota^*(\bar{\nabla}_{d\iota(X)} d\iota(V)) = \bar{\nabla}_{d\iota(X)} d\iota(V), \tag{3.11}$$

where $d\iota(X)$ and $d\iota(V)$ can be viewed as the local extensions of $X \in \Gamma(TM)$ and $V \in \Gamma(\iota^* T\bar{M})$ to the vector fields in $\Gamma(T\bar{M})$.

To simplify the notations, in the sequel we adhere to the convention as follows. It is slightly abusive, as all the pullbacks by the immersion (ι^*) are dropped systematically; however, it is adopted by the standard geometric texts [103, 51, 121] and many others. In fact, this amounts to viewing M as a subset of $\bar{M}$, and ι as the identity map from M to its image.

Convention 3.1.4 *Let $\iota : (M, g) \hookrightarrow (\bar{M}, \bar{g})$ be an isometric immersion of semi-Riemannian submanifolds. We replace $(\iota^* T\bar{M}, \iota^*\bar{\nabla})$ with the notation $(T\bar{M}, \bar{\nabla})$.*

In this way, let us consider the following decomposition:

$$\bar{\nabla}_X V = \tan[\bar{\nabla}_X(\tan V)] + \tan[\bar{\nabla}_X(\mathrm{nor} V)] + \mathrm{nor}[\bar{\nabla}_X(\tan V)] + \mathrm{nor}[\bar{\nabla}_X(\mathrm{nor} V)],$$

where $X \in \Gamma(TM)$, $V \in \Gamma(T\bar{M})$. Then,

Definition 3.1.5 *Given an isometric immersion $\iota : (M, g) \hookrightarrow (\bar{M}, \bar{g})$, the* tangential connection *$\nabla : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$, the* second fundamental form *$\mathrm{II} : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM^\perp)$, the* shape operator *(associated to II) $S : \Gamma(TM) \times \Gamma(TM^\perp) \to \Gamma(TM)$ and the* normal connection *$\nabla^\perp : \Gamma(TM) \times \Gamma(TM^\perp) \to \Gamma(TM^\perp)$ are defined as follows:*

$$\begin{cases} \nabla_X Y := \tan \bar{\nabla}_X Y, & \mathrm{II}(X, Y) := \mathrm{nor}\, \bar{\nabla}_X Y, \\ S_\xi X := -\tan \bar{\nabla}_X \xi, & \nabla^\perp_X \xi := \mathrm{nor}\, \bar{\nabla}_X \xi, \end{cases} \tag{3.12}$$

for $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(TM^\perp)$.

We notice here that ∇ is the Levi-Civita connection on $(M, \iota^*\bar{g})$ whenever $\bar{\nabla}$ is Levi-Civita, and that II and S are "essentially the same operator" in the following sense:

$$\bar{g}(\mathrm{II}(X,Y),\xi) = \bar{g}(S_\xi X, Y).$$

In addition, II is symmetric (equivalently, S_ξ is self-adjoint) on $\Gamma(TM)$.

3.1.2 Gauss-Codazzi-Ricci equations for semi-Riemannian submanifolds

Convention 3.1.6 *We write $\langle\cdot,\cdot\rangle$ for $\bar{g}(\cdot,\cdot), g(\cdot,\cdot)$ and any other semi-Riemannian metrics, unless we need to emphasise the metric. For example, at times we denote by $\langle\cdot,\cdot\rangle_E$ the metric on a vector bundle E over M.*

For the purpose of deriving the GCR equations, we also need to introduce the *Riemann curvature*. We again do so from the perspective of vector bundles. Suppose that (M,g) is an n-dimensional semi-Riemannian manifold of index ν, and E is a vector bundle over M with fibres $F \cong \mathbb{R}^k_\tau$, the *semi-Euclidean space $\mathbb{R}^k$ with index ν*. Let ∇^E be an affine connection on the bundle E, *i.e.*, a linear map $\nabla : \Gamma(TM) \times \Gamma(E) \to \Gamma(E)$ satisfying $\nabla^E_{\phi X}\alpha = \phi\nabla^E_X\alpha$ and $\nabla^E_X(\phi\alpha) = X(\phi)\alpha + \phi\nabla^E_X\alpha$, for any $\phi : M \to \mathbb{R}$ with required regularity. This can be more compactly written as

$$\nabla^E(\phi\alpha) = \phi\nabla^E\alpha + d\phi \otimes \alpha,$$

where we view $\nabla^E : \Gamma(E) \to \Omega^1(E) := \Gamma(E \otimes T^*M)$, the space of differential 1-forms on the bundle E. Then, the Riemann curvature tensor is given by $R^E : \Gamma(TM) \times \Gamma(TM) \to \Gamma(\mathrm{End}E)$, where $\mathrm{End}E$ is the *endomorphism bundle*, *i.e.* the vector bundle over M with fibres at each point being $\mathfrak{gl}(F)$, the group of linear transforms from F to itself:

$$R^E(X,Y) := [\nabla^E_X, \nabla^E_Y] - \nabla^E_{[X,Y]}. \tag{3.13}$$

Note that for each $\alpha \in \Gamma(E)$, we have $R^E(X,Y,\alpha) \in \Gamma(E)$. It is also customary to write R^E as a (0,4)-tensor field, *i.e.*,

$$R^E(X,Y,\alpha,\beta) := \langle R^E(X,Y,\alpha),\beta\rangle_E,$$

for any $X, Y \in \Gamma(TM)$ and $\alpha, \beta \in \Gamma(E)$.

Recall that in §2.1 the orthogonal splitting of Levi-Civita connections along the orthogonal splitting of the tangent bundles in Eq. (3.6) is investigated. Now, with the Riemann curvature defined above, one can also investigate the splitting of Riemann curvatures along Eq. (3.6). For this purpose, notice that given an isometric immersion $f:(M,g)\to(\bar{M},\bar{g})$, three vector bundles over M appear naturally: $E=TM$, $TM^\perp$ and $f^*T\bar{M}\equiv T\bar{M}$ (as in Convention 3.1.2, with f in place of ι). We also write

$$\nabla=\nabla^{TM}(\text{the Levi-Civita connection on } M),\qquad \bar{\nabla}=\nabla^{T\bar{M}},\qquad \nabla^\perp=\nabla^{TM^\perp},$$

as well as

$$R=R^{TM},\qquad \bar{R}=R^{T\bar{M}},\qquad R^\perp=R^{TM^\perp}.$$

In the special case $\bar{M}=\mathbb{R}^{n+k}_{\nu+\tau}$ and $\mathrm{Ind}(\bar{M})=\mathrm{Ind}(M)+\mathrm{Ind}(\mathbb{R}^k)$, we have $\bar{R}(X,Y)\in\Gamma(\mathrm{End}T\bar{M})$ constantly vanishing for any $X,Y\in\Gamma(TM)$. Therefore,

$$\bar{R}(X,Y,W,Z)=0;\qquad \bar{R}(X,Y,W,\xi)=0;\qquad \bar{R}(X,Y,\xi,\eta)=0,\tag{3.14}$$

for arbitrary $W,Z\in\Gamma(TM)$ and $\xi,\eta\in\Gamma(TM^\perp)$. Applying the projections *tan* and *nor* to Eq. (3.14) and expressing them via $R,R^\perp,\mathrm{II},S$ and ∇ as in Definition 3.1.5, we immediately deduce the following (*cf.* [103] and the Appendix of [26]):

Theorem 3.1 *The following three equations are equivalent to Eq.* (3.14)*: For arbitrary $X,Y,W,Z\in\Gamma(TM)$ and $\xi,\eta\in\Gamma(TM^\perp)$, we have*

$$R(X,Y,W,Z)=\langle\mathrm{II}(X,W),\mathrm{II}(Y,Z)\rangle-\langle\mathrm{II}(X,Z),\mathrm{II}(Y,W)\rangle,\tag{3.15}$$

$$\nabla^\perp_X\mathrm{II}(Y,W)=\nabla^\perp_Y\mathrm{II}(X,W),\tag{3.16}$$

$$R^\perp(X,Y,\xi,\eta)=-\langle[S_\xi,S_\eta]X,Y\rangle.\tag{3.17}$$

Here, the covariant derivative of II *in Eq.* (3.16) *is defined via the Leibniz rule:*

$$\nabla^\perp_X\mathrm{II}(Y,W)=X(\mathrm{II}(Y,W))-\mathrm{II}(\nabla_XY,W)-\mathrm{II}(Y,\nabla_XW).$$

In the above, Eqs. (3.15)–(3.17) are named after Gauss, Codazzi and Ricci, respectively; we refer to this system as the GCR equations in the sequel. Let us conclude this section by making three remarks on the GCR equations:

1. First of all, although the GCR equations in Theorem 3.1 take the same form as in the Riemannian-case (see Theorem 2.1), they are *essentially different.* The definition of II and $\langle\cdot,\cdot\rangle$ already incorporates the signature of the metric, which destroys the div-curl structures identified in §2.3.3 for the Riemannian GCR equations and makes the system more difficult to analyse.

2. The GCR equations form a first-order non-linear PDE system, with g (hence ∇ and R) as given and $(\mathrm{II}, \nabla^\perp)$ as unknowns. The non-linear terms in this system take the form of $\mathrm{II} \otimes \mathrm{II}$, $\mathrm{II} \otimes \nabla^\perp$ or $\nabla^\perp \otimes \nabla^\perp$, so they are known as the *quadratic non-linearities.*

3. Finally, let us observe that the GCR equations can be generalised to any vector bundle E in place of the normal bundle $TM^\perp$. Indeed, since the Riemann curvature is defined for any bundle E (denoted by R^E), so for any symmetric tensor $\mathrm{II} : \Gamma(TM) \times \Gamma(TM) \to \Gamma(E)$ and $S : \Gamma(E) \times \Gamma(TM) \to \Gamma(TM)$ given by $\langle S_\alpha X, Y \rangle = \langle \mathrm{II}(X,Y), \alpha \rangle$ for $X, Y \in \Gamma(TM)$ and $\alpha \in \Gamma(E)$, the GCR equations in Theorem 3.1 are still well-defined for $X, Y, W, Z \in \Gamma(TM)$ and $\xi, \eta \in \Gamma(E)$, wherein we replace $R^\perp$ by R^E in Eq. (3.17). Such equations are termed as "*GCR equations on bundle E*".

3.2 Realisation of isometric immersions from GCR equations: semi-Riemannian case

In this section we address the "realisation problem": Given an n-dimensional semi-Riemannian manifold (M, g) of lower regularity satisfying the GCR equations (Theorem 3.1) in the distributional sense, can we construct an isometric immersion $f : (M, g) \hookrightarrow (\mathbb{R}^{n+k}, g_0)$, where g_0 is the Euclidean metric?

First of all, we notice that two additional necessary conditions emerge naturally, in contrast to the Riemannian case:

(N1) The resulting map f must be an immersion of M (assume it is connected) as a semi-Riemannian submanifold;

(N2) The indices of the manifold M and its normal bundle $TM^\perp = f^*T\mathbb{R}^{n+k}/TM$ (Convention 3.1.2) add up to the index of the target semi-Euclidean space:

$$\mathrm{Ind}(M) + \mathrm{Ind}(T_x M^\perp) = \mathrm{Ind}(\mathbb{R}^{n+k}) \qquad \text{for each } x \in M. \tag{3.18}$$

Here, (N1) holds because f is an isometry, *i.e.*, $f^*g_0 = g$, and by definition a semi-Riemannian metric is non-degenerate. For example, it rules out the possibility that a semi-Riemannian manifold is isometrically embedded into the *lightcone* of the Minkowski spaces. (N2) is a consequence of (N1) together with the direct sum decomposition in Eq. (3.6). Therefore, in the rest of the paper, we fix the target semi-Euclidean metric g_0 to be $\overline{g_0}$, defined as follows:

$$\overline{g_0} = \mathrm{diag}(\underbrace{-1, \cdots, -1}_{\nu \text{ times}}, \underbrace{1, \cdots, 1}_{(n-\nu) \text{ times}}; \underbrace{-1, \cdots, -1}_{\tau \text{ times}}, \underbrace{1, \cdots, 1}_{(k-\tau) \text{ times}}), \tag{3.19}$$

and $\mathrm{Ind}(M) = \nu$. Also we write the corresponding semi-Euclidean space as $\mathbb{R}^{n+k}_{\nu+\tau}$.

3.2.1 Statement of the realisation theorem

The main result of this section, Theorem 3.2, answers the realisation problem in the affirmative on semi-Riemannian submanifolds with weaker regularity, provided that (N1) and (N2) are fulfilled.

The C^∞ version of this result appears to be a "folklore" theorem assumed by researchers in submanifold theory (*cf.* the survey in [24] by B.-Y. Chen); however, to our knowledge, a complete proof does not seem to exist in the literature, especially for the weak regularity case. Closely related results include the paper [55] by Eschenburg-Tribuzy on the existence and uniqueness of isometric immersions of smooth manifolds into affine homogeneous spaces, and the work [81] by Lefloch-Mardare on the existence and uniqueness of isometric immersions of semi-Riemannian manifolds with weak regularity into semi-Euclidean spaces of co-dimensions 0 or 1. The author feels it useful to give a complete, detailed proof of Theorem 3.2. The proof adapts the arguments by Tenenblat for the smooth, Riemannian case in [121] (also see §2.4 above), which employ the Cartan formalism as the central tool.

Theorem 3.2 *Let (M, g) be an n-dimensional simply-connected semi-Riemannian manifold with metric $g \in W^{1,p}_{\mathrm{loc}}(M; O(\nu, n-\nu))$ for $p > n$ and $\nu = \mathrm{Ind}(M) \in \{0, 1, \cdots, n\}$. Suppose that E is a bundle over M with fibre $F = \mathbb{R}^k_\tau$, bundle metric $g^E \in W^{1,p}_{\mathrm{loc}}(M; O(\tau, k-\tau))$, and bundle connection $\nabla^E \in L^p_{loc}(M; T^*M \bigotimes \mathrm{End}E)$ compatible with g^E. Let $\mathrm{II} \in L^p_{\mathrm{loc}}(M; \mathrm{Sym}^2 T^*M \otimes E)$ be a symmetric two-tensor, and let S be defined by $g(S_\alpha X, Y) = g^E(\mathrm{II}(X,Y), \alpha)$ for any $X, Y \in \Gamma(TM)$ and $\alpha \in \Gamma(E)$.*

Assume that the GCR equations on E hold in the sense of distributions. Then, there exists a $W^{2,p}_{\mathrm{loc}}$ isometric immersion $f : (M, g) \hookrightarrow (\bar{M} = \mathbb{R}^{n+k}_{\nu+\tau}, \overline{g_0})$, so that the normal bundle $TM^\perp := f^*T\bar{M}/TM$, the second fundamental form and the shape operator induced by f are identified with E, II and S, respectively. f is unique modulo the congruences in $(\bar{M}, \overline{g_0})$.

When g, ∇^E, g^E, II are smooth, an isometric immersion $f \in C^\infty(M; \bar{M})$ exists.

Let us make three remarks on the statement of the above theorem:

- ∇^E is said to be *compatible* with g^E if, for any $X \in \Gamma(TM)$ and $\alpha, \beta \in \Gamma(E)$, one has
$$Xg^E(\alpha,\beta) = g^E(\nabla^E_X\alpha,\beta) + g^E(\alpha,\nabla^E_X\beta). \tag{3.20}$$
For example, the Levi-Civita connection on M is compatible with g. As in convention 3.1.6, the preceding condition is expressed as
$$X\langle\alpha,\beta\rangle = \langle\nabla^E_X\alpha,\beta\rangle + \langle\alpha,\nabla^E_X\beta\rangle,$$
namely that ∇^E satisfies the Leibniz rule with respect to the bundle metric.

- For a bundle E over M, Sym^2E^* denotes the space of symmetric 2-tensors defined on E, *i.e.*, each $M \in \Gamma(\mathrm{Sym}^2E^*)$ satisfies $M(\alpha,\beta) = M(\beta,\alpha)$ for any $\alpha, \beta \in \Gamma(E)$. As a cautionary remark, a semi-Riemannian metric g on M generally does not lie in $\Gamma(\mathrm{Sym}^2T^*M)$. Instead, $g \in \Gamma(O(\nu, n-\nu))$ since $g_{ij}\epsilon_j = g_{ji}\epsilon_i$ (see below for details).

- We need the standard notations for Lie groups. Let $\mathfrak{gl}(n;\mathbb{R})$ denote the space of $n \times n$ real matrices. Then we define the *semi-orthogonal group of* $\mathbb{R}^n_\nu$ as
$$O(\nu, n-\nu) := \{B \in \mathfrak{gl}(n;\mathbb{R}) : B(v,w) = \widetilde{\epsilon_{n,\nu}}v \cdot w \text{ for all } v, w \in T\mathbb{R}^n_\nu\}, \tag{3.21}$$
where the signature matrix is given by
$$\widetilde{\epsilon_{n,\nu}} = \mathrm{diag}(\underbrace{-1,\cdots,-1}_{\nu \text{ times}}, \underbrace{1,\cdots,1}_{n-\nu \text{ times}}). \tag{3.22}$$
In other words, $O(\nu, n-\nu)$ is the group of linear isometries from $\mathbb{R}^n_\nu$ to itself. Similarly, $O(\tau, k-\tau)$ is defined likewise, with the signature matrix
$$\widetilde{\epsilon_{k,\tau}} = \mathrm{diag}(\underbrace{-1,\cdots,-1}_{\tau \text{ times}}, \underbrace{1,\cdots,1}_{k-\tau \text{ times}}). \tag{3.23}$$
Also, we notice that the following block direct sum of matrices holds:
$$\overline{g_0} = \widetilde{\epsilon_{n,\nu}} \bigoplus \widetilde{\epsilon_{k,\tau}}.$$

For the later development we denote the Lie algebra of $O(n, n-\nu)$ by $\mathfrak{o}(n, n-\nu)$.

3.2.2 Cartan formalism

To prove the main realisation Theorem 3.2, let us also utilise the Cartan formalism as in §2.4.2 for the Riemannian case. Here more delicate constructions are required to account for the non-trivial signatures. The Cartan formalism has been adopted to prove the existence and uniqueness of immersions of a smooth manifold into affine homogeneous spaces by Eschenburg-Tribuzy in [55].

First, let us impose the following:

Convention 3.2.1 *From now on, the superscripts and subscripts obey the rule:*

$$1 \leq i, j, k, l, s, t \leq n; \qquad n+1 \leq \alpha, \beta, \gamma \leq n+k; \qquad 1 \leq a, b, c, e \leq n+k.$$

In view of the above convention, throughout we employ $\{\partial_1, \ldots, \partial_n\} \subset \Gamma(TM)$ to denote a *frame field* for M, *i.e.*, at each point x on M, $\{\partial_i(x)\}_1^n$ forms an orthonormal basis for the tangent space T_xM. We recall that, in the semi-Riemannian setting, "orthonormal" means:

$$\langle \partial_i, \partial_j \rangle = \delta^i_j \epsilon_i \qquad \text{for all } i, j \in \{1, \ldots, n\}. \tag{3.24}$$

Also, we write $\{\theta^1, \ldots, \theta^n\} \subset \Gamma(T^*M)$ for the *coframe field* of $\{\partial_i\}_1^n$, namely

$$\theta^i(\partial_j) = \delta^i_j. \tag{3.25}$$

Similarly, one also takes $\{\partial_{n+1}, \ldots, \partial_{n+k}\} \subset \Gamma(E)$ to be a frame field for E, *i.e.*, orthonormal with respect to the bundle metric g^E, and sets $\{\theta^{n+1}, \ldots, \theta^{n+k}\} \subset \Gamma(E^*)$ to be its co-frame field.

Then, we introduce the *connection one-forms*:

Definition 3.2.2 *Let (M, g) be a semi-Riemannian manifold, and let E be a vector bundle over M with bundle metric g^E. The connection one-form $\mathcal{W}$ is a one-form-valued $(n+k) \times (n+k)$ matrix field, namely*

$$\mathcal{W} = \{\omega^a_b\} \in \Gamma(\mathfrak{gl}(n+k; \mathbb{R}) \bigotimes T^*M), \tag{3.26}$$

defined via the following equations (with the indices obeying Convention 3.2.1):

$$\begin{cases} \omega^i_j(\partial_l) := \theta^j(\nabla_{\partial_l}\partial_i) = \epsilon_j \langle \nabla_{\partial_l}\partial_i, \partial_j \rangle, \\ \omega^i_\alpha(\partial_j) := \theta^\alpha\big(\mathrm{II}(\partial_i, \partial_j)\big) = \epsilon_\alpha \langle \mathrm{II}(\partial_i, \partial_j), \partial_\alpha \rangle, \\ \omega^\alpha_\beta(\partial_i) := \theta^\beta\big(\nabla^E_{\partial_i}\partial_\alpha\big) = \epsilon_\beta \langle \nabla^E_{\partial_i}\partial_\alpha, \partial_\beta \rangle, \\ \omega^\alpha_i := -\epsilon_i \epsilon_\alpha \omega^i_\alpha. \end{cases} \tag{3.27}$$

As a remark, in our case $\Gamma(\mathfrak{gl}(n+k;\mathbb{R})\bigotimes T^*M)\cong\Gamma(\mathfrak{gl}(TM\bigoplus E)\bigotimes T^*M)$, which is defined as $\Omega^1(\mathfrak{gl}(n+k;\mathbb{R}))$, *i.e.*, the space of one-forms on $\mathfrak{gl}(n+k;\mathbb{R})$. In general, for a Lie algebra $\mathfrak{g}$, the space of $\mathfrak{g}$-*valued differential k-forms* is written as

$$\Omega^k(\mathfrak{g}) := \Gamma(\bigwedge^k T^*M \bigotimes \mathfrak{g}). \tag{3.28}$$

Our key result of the Cartan formalism is:

Proposition 3.3 *If the Gauss-Codazzi-Ricci equations (cf. Theorem 3.1) are satisfied, then the connection one-form $\mathcal{W}$ solves the* second structural equation *of the Cartan formalism:*

$$d\mathcal{W} = \mathcal{W}\wedge\mathcal{W}. \tag{3.29}$$

This equation is understood as follows: We view the exterior differential d as acting only on the T^*M factor if $\mathcal{W}\in\Omega^1(\mathfrak{g})$, where $\mathfrak{g}=\mathfrak{gl}(n+k;\mathbb{R})$ in Eq. (3.28). Then $d\mathcal{W}\in\Omega^2(\mathfrak{g})$ with

$$d\mathcal{W}(U,V) := U(\mathcal{W}(V)) - V(\mathcal{W}(U)) + \mathcal{W}([U,V]) \qquad \text{for all } U,V\in\Gamma(TM).$$

On the other hand, the wedge product on $\Omega^1(\mathfrak{g})$ is given by combining the wedge product on the T^*M factor and the matrix multiplication (denoted by $\cdot$) on the $\mathfrak{g}$ factor in Eq. (3.28), namely

$$(\mathcal{W}\wedge\mathcal{W})(U,V) := \mathcal{W}(U)\cdot\mathcal{W}(V) - \mathcal{W}(V)\cdot\mathcal{W}(U) \qquad \text{for all } U,V\in\Gamma(TM).$$

So Eq. (3.29) is an equation on $\Omega^2(\mathfrak{g})$, the matrix-valued two-forms.

As a prerequisite to the proof of Proposition 3.3 we need the following lemma, which characterises the symmetries of the connection one-form:

Lemma 3.2.1 $\mathcal{W}=\{\omega^a_b\}\in\Omega^1(\mathfrak{o}(\nu+\tau;(n+k)-(\nu+\tau)))$.

The proof of Lemma 3.2.1 is standard and can be found in the Appendix of Chen-Li [26]. It says that $\mathcal{W}$ is a "semi-skew-symmetric" matrix, represented below in the block form due to Eq. (3.27):

$$\{\omega_a^b\}_{1\leq a,b\leq k+n} = \begin{bmatrix}\omega^i_j & \omega^\alpha_i\\ \omega^i_\alpha & \omega^\beta_\alpha\end{bmatrix} = \begin{bmatrix}\theta^j(\nabla_\bullet\partial_i) & S_{\partial_\alpha}\partial_i\\ -S_{\partial_\alpha}\partial_i & \theta^\beta(\nabla^E_\bullet\partial_\alpha)\end{bmatrix}. \tag{3.30}$$

Schematically, we may write

$$\mathcal{W} = \begin{bmatrix}\pm\nabla & \pm\mathrm{II}\\ \pm\mathrm{II} & \pm\nabla^E\end{bmatrix}.$$

Remark 3.2.1 *In the case that* (M,g) *is a Riemannian manifold* (*i.e., the setting of* §*2 above*), *we can express the vector fields* $V^{(B)}$, $V^{(\nabla^\perp)}$ *and one-forms* $\Omega^{(B)}, \Omega^{(\nabla^\perp)}$, *defined in Eqs.* (2.60)–(2.63), *which are crucial to the identification of the div-curl structure in disguise in the GCR equations over Riemannian manifolds, in terms of the connection one-form* $\mathcal{W}$. *Indeed, we have*

$$\begin{cases}\Omega^{(B)}_{\partial_i,\partial_\alpha} = \omega^i_\alpha,\\ \Omega^{(\nabla^\perp)}_{\partial_\alpha,\partial_\beta} = \omega^\alpha_\beta,\\ V^{(B)}_{\partial_i,\partial_\alpha}(\partial_j,\partial_k) = \omega^i_\alpha(\partial_j)\partial_k - \omega^i_\alpha(\partial_k)\partial_j,\\ V^{(\nabla^\perp)}_{\partial_\alpha,\partial_\beta}(\partial_j,\partial_k) = \omega^\alpha_\beta(\partial_j)\partial_k - \omega^\alpha_\beta(\partial_k)\partial_j.\end{cases} \tag{3.31}$$

Proof of Proposition 3.3. The proof is adapted from the notes [33] by S.-S. Chern, which is divided into four steps.

1. We begin by observing that the definition of the connection one-form $\mathcal{W}$, *i.e.*, Eq. (3.27), gives us

$$\nabla_{\partial_i}\partial_j = \sum_l \omega^l_j(\partial_i)\partial_l, \qquad \mathrm{II}(\partial_i,\partial_j) = \sum_\alpha \omega^i_\alpha(\partial_j)\partial_\alpha, \qquad \nabla^E_{\partial_i}\partial_\alpha = \sum_\beta \omega^\alpha_\beta(\partial_i)\partial_\beta. \tag{3.32}$$

From here, we immediately derive the following for the shape operator S:

$$S_{\partial_i}\partial_\alpha = \sum_j \epsilon_j\langle S_{\partial_i}\partial_\alpha,\partial_j\rangle\partial_j = \sum_j \epsilon_j\langle \mathrm{II}(\partial_i,\partial_j),\partial_\alpha\rangle\partial_j = \sum_j \epsilon_j\epsilon_\alpha\omega^i_\alpha(\partial_j)\partial_j. \tag{3.33}$$

2. Next, the Gauss equation (3.15) is equivalent to

$$R(\partial_i,\partial_j,\partial_k) = S_{\partial_i}\mathrm{II}(\partial_j,\partial_k) - S_{\partial_j}\mathrm{II}(\partial_i,\partial_k). \tag{3.34}$$

Applying the symmetry of II twice (in the first and third lines below), we obtain:

$$\begin{aligned} S_{\partial_i}\mathrm{II}(\partial_j,\partial_k) - S_{\partial_j}\mathrm{II}(\partial_i,\partial_k) &= S_{\partial_i}\mathrm{II}(\partial_k,\partial_j) - S_{\partial_j}\mathrm{II}(\partial_k,\partial_i)\\ &= S_{\partial_i}\Big(\sum_\alpha \omega^k_\alpha(\partial_j)\partial_\alpha\Big) - S_{\partial_j}\Big(\sum_\alpha \omega^k_\alpha(\partial_i)\partial_\alpha\Big)\\ &= \sum_\alpha\sum_l \epsilon_\alpha\epsilon_l\big(\omega^k_\alpha(\partial_j)\omega^l_\alpha(\partial_i) - \omega^k_\alpha(\partial_i)\omega^l_\alpha(\partial_j)\big)\partial_l\\ &= \sum_\alpha\sum_l (\omega^k_\alpha\wedge\omega^\alpha_l)(\partial_i,\partial_j)\partial_l, \end{aligned}$$

where the last line follows by Lemma 3.2.1. On the other hand, the Riemann curvature of the Levi-Civita connection on TM is computed directly from the definition:

$$\begin{aligned} &R(\partial_i,\partial_j,\partial_k)\\ :=&\nabla_i\nabla_j\partial_k - \nabla_j\nabla_i\partial_k + \nabla_{[\partial_i,\partial_j]}\partial_k \end{aligned}$$

$$
\begin{aligned}
&= \sum_l \Big\{\partial_i\big(\omega_k^l(\partial_j)\big)\partial_l - \partial_j\big(\omega_k^l(\partial_i)\big)\partial_l + \\
&\qquad\qquad + \omega_k^l([\partial_i,\partial_j]\partial_l) + \omega_k^l(\partial_j)\sum_s \omega_l^s(\partial_i)\partial_s - \omega_k^l(\partial_i)\sum_s \omega_l^s(\partial_j)\partial_s\Big\} \\
&= \sum_s \big\{d\omega_k^s - \sum_l \omega_k^l \wedge \omega_l^s\big\}(\partial_i,\partial_j)\partial_s.
\end{aligned}
$$

Equating these computations via Eq. (3.34), we conclude that $d\omega_k^s = \sum_b \omega_b^k \wedge \omega_s^b$.

3. Applying the same argument to $R^E(\partial_i, \partial_j, \partial_\gamma)$ and utilising the Ricci equation (3.17), one deduces that $d\omega_\beta^\alpha = \sum_b \omega_b^\alpha \wedge \omega_\beta^b$. The details are omitted here.

4. Finally, starting with the Codazzi equation (3.16) we obtain:

$$
\begin{aligned}
0 &= \bar{\nabla}_{\partial_i}\mathrm{II}(\partial_j,\partial_k) - \bar{\nabla}_{\partial_j}\mathrm{II}(\partial_i,\partial_k) \\
&= \sum_\gamma \partial_i\big(\omega_\gamma^j(\partial_k)\big)\partial_\gamma + \sum_\beta \omega_\beta^j(\partial_k)\sum_\gamma \omega_\gamma^\beta(\partial_i)\partial_\gamma - \sum_\gamma \partial_j\big(\omega_\gamma^i(\partial_k)\big)\partial_\gamma - \sum_\beta \omega_\beta^i(\partial_k)\sum_\gamma \omega_\gamma^\beta(\partial_j)\partial_\gamma \\
&= \sum_\gamma \Big\{\partial_i\big(\omega_\gamma^k(\partial_j)\big) - \partial_j\big(\omega_\gamma^k(\partial_i)\big) - \sum_\beta \Big(\omega_\beta^k(\partial_i)\omega_\gamma^\beta(\partial_j) - \omega_\beta^k(\partial_j)\omega_\gamma^\beta(\partial_i)\Big)\Big\}\partial_\gamma \\
&= \sum_\gamma \Big\{d\omega_\gamma^k(\partial_i,\partial_j) - \omega_\gamma^k[\partial_i,\partial_j] - \sum_\beta \omega_\beta^k \wedge \omega_\gamma^\beta(\partial_i,\partial_j)\Big\}\partial_\gamma, \qquad (3.35)
\end{aligned}
$$

where in the penultimate line we use the self-adjointness of II, *i.e.*, $\omega_\alpha^i(\partial_j) = \omega_\alpha^j(\partial_i)$; the final line holds by the definition of $d\omega_\gamma^k$ and $\omega_\beta^k \wedge \omega_\gamma^\beta$. To compute the Lie bracket term in the last line of Eq. (3.35), let us make use of the torsion-free condition of the affine connection:

$$
\begin{aligned}
\sum_\gamma \omega_\gamma^k[\partial_i,\partial_j]\partial_\gamma &= \sum_\gamma \omega_\gamma^k(\nabla_{\partial_i}\partial_j - \nabla_{\partial_j}\partial_i)\partial_\gamma \\
&= \sum_\gamma\sum_l \omega_\gamma^k\big(\omega_j^l(\partial_i)\partial_l - \omega_i^l(\partial_j)\partial_l\big)\partial_\gamma \\
&= \sum_\gamma\sum_l \big(\omega_j^l(\partial_i)\omega_\gamma^l(\partial_k) - \omega_i^l(\partial_j)\omega_\gamma^l(\partial_k)\big)\partial_\gamma \\
&= \sum_\gamma\sum_l \omega_l^k \wedge \omega_\gamma^l(\partial_i,\partial_j)\partial_\gamma,
\end{aligned}
$$

again due to the symmetry properties of the second fundamental form and $W \wedge W$. Thus, substituting it back to Eq. (3.35) gives us $d\omega_\gamma^k = \sum_b \omega_b^k \wedge \omega_\gamma^b$.

To conclude, putting together Steps 1-3, we obtain that

$$
d\omega_c^a = \sum_b \omega_b^a \wedge \omega_c^b.
$$

Moreover, as an equation on $\Omega^2(\mathfrak{o}(\nu+\tau,(n+k)-(\nu+\tau)))$, Eq. (3.29) is independent of the choice of moving frames. So the proof is complete. ■

Before moving on to proving Theorem 3.2, let us remark that the following identity always holds, as an equation on $\Omega^1(\mathfrak{gl}(n;\mathbb{R}))$, where $\mathfrak{gl}(n;\mathbb{R})$ is the vector space of $n \times n$ real matrices:

$$d\theta = \theta \wedge \mathcal{W}. \tag{3.36}$$

This is known as the *first structural equation* of Cartan formalism, and is equivalent to the torsion-free property of the connection ∇. As this property is independent of metrics, regardless of Riemannian or semi-Riemannian, it does not provide additional information to the isometric immersions.

3.2.3 Proof of Theorem 3.2, the realisation theorem

Now we prove our main theorem in this section. We first establish the C^∞ case, for which we directly apply the Frobenius theorem (namely, the equivalence of involutive and completely integrable distributions). In the lower regularity case we only need to modify the previous proof by replacing the Frobenius theorem argument with an analogous existence theorem for certain first-order PDE systems with Sobolev coefficients.

Proof of Theorem 3.2. We divide our arguments into seven steps. As remarked at the beginning of §3.2.1, our proof is a variant of Tenenblat [121] and Eschenburg-Tribuzy [55], and to the knowledge of the author, is new to the literature.

1. We first prove the above statement in the C^∞ case. For this purpose, let us start with solving a *Pfaff system* for the bundle connection A on $TM \bigoplus E$. More precisely, we show that for any $x_0 \in M$, the following first-order PDE initial value problem has a solution $A \in C^\infty(U; O(\nu+\tau,(n+k)-(\nu+\tau)))$ in some neighbourhood U of x_0:

$$\mathcal{W} = dA \cdot A^{-1}, \qquad A(0) = A(x_0) \in O(\nu+\tau,(n+k)-(\nu+\tau)). \tag{3.37}$$

Indeed, without loss of generality, we assume that $x_0 = 0$ in the local coordinate $\{\partial_i\}_1^n$. Also, let us take $Z = \{Z^a_b\}$ as the canonical frame field on $\mathfrak{gl}(n+k;\mathbb{R}) \cong \mathbb{R}^{(n+k)^2}$, with signature inherited from $\bar{M} = \mathbb{R}^{n+k}_{\nu+\tau}$. For example, $Z := \overline{g_0}$ is one suitable choice. Motivated by K. Tenenblat [121] we consider the following map:

$$\begin{aligned}\Lambda^{(x,Z)} : T_xM \times T_ZO(\nu+\tau,(n+k)-(\nu+\tau)) &\longrightarrow T_ZO(\nu+\tau,(n+k)-(\nu+\tau)) \\ (X,\mathfrak{m}) \quad &\longmapsto \quad d_xZ(\mathfrak{m}) + Z\cdot W(X)|_x,\end{aligned}$$

which is abbreviated in the sequel as follows:

$$\Lambda^{(x,Z)} = dZ - \mathcal{W}\cdot Z. \tag{3.38}$$

Making use of the characterisation of tangent spaces of semi-orthogonal group and its Lie algebra, $\Lambda^{(x,Z)}$ is well-defined. Indeed,

$$
\begin{aligned}
&Z\overline{g_0}\big[\Lambda^{(x,Z)}(X,\mathfrak{m})\big]^\top + \Lambda^{(x,Z)}(X,\mathfrak{m})\overline{g_0}Z^\top \\
=& Z\overline{g_0}\big[dZ(\mathfrak{m})^\top - Z^\top \mathcal{W}(X)^\top\big] + \big[dZ(\mathfrak{m}) - \mathcal{W}(X)Z\big]\overline{g_0}Z^\top \\
=& -\big[\overline{g_0}\mathcal{W}(X)^\top + \mathcal{W}(X)\overline{g_0}\big] + \big[Z\overline{g_0}dZ(\mathfrak{m})^\top + dZ(\mathfrak{m})\overline{g_0}Z^\top\big] \\
=& 0,
\end{aligned}
$$

thanks to $dZ(\mathfrak{m}) \in T_Z O(\nu+\tau,(n+k)-(\nu+\tau))$ and $\mathcal{W}(X) \in \mathfrak{o}(\nu+\tau,(n+k)-(\nu+\tau))$.

Next, we define the following *distribution*, in the Frobenius sense:

$$
\mathcal{D}^{(x,Z)} := \ker(\Lambda^{(x,Z)}) \subset \big[T_x M \times T_Z O(\nu+\tau,(n+k)-(\nu+\tau))\big]. \tag{3.39}
$$

Our goal is to show that it is *completely integrable.* Assuming so, we can find the unique maximal integral submanifold (*i.e.* the leaf) in some neighbourhood of x_0. Also, clearly

$$
\mathcal{D}^{(0,Z)} \cap \big[\{0\} \bigoplus T_Z O(\nu+\tau,(n+k)-(\nu+\tau))\big] = \{0\}, \tag{3.40}
$$

i.e., the distribution is transversal to the TM factor at the point $x_0 = 0$, because

$$
\Lambda^{(x,Z)}(0,\mathfrak{m}) = \mathfrak{m}.
$$

In view of the classical implicit function theorem, $\mathcal{D}^{(x,Z)}$ is locally a graph of a smooth function A from TU to $TO(\nu+\tau,(n+k)-(\nu+\tau))$, with x lies in U, an open neighbourhood of x_0. Such A solves the Pfaff system (3.37) in view of the definition of $\Lambda^{(x,Z)}$.

2. It now remains to prove the complete integrability of the distribution $\mathcal{D}^{(x,Z)}$. By the *Frobenius Theorem* we show that $\mathcal{D}^{(x,Z)}$ is *involutive*, namely for any $(X_i,\mathfrak{m}_i) \in \mathcal{D}^{(x,Z)}$, $i=1,2$, the commutator stays in $\mathcal{D}^{(x,Z)}$, *i.e.* $\Lambda^{(x,Z)}[(X_1,\mathfrak{m}_1),(X_2,\mathfrak{m}_2)] = 0$.

Indeed, employing the following formula for exterior differential:

$$
d\Lambda^{(x,Z)}(s_1,s_2) = s_1(\Lambda^{(x,Z)}s_2) - s_2(\Lambda^{(x,Z)}s_1) - \Lambda^{(x,Z)}[s_1,s_2]
$$

for $s_1, s_2 \in T_x U \bigoplus T_Z O(\nu+\tau,(n+k)-(\nu+\tau))$, it suffices to show that

$$
d\Lambda^{(x,Z)}\big((X_1,\mathfrak{m}_1),(X_2,\mathfrak{m}_2)\big) = 0. \tag{3.41}
$$

To this end, let us compute $d\Lambda^{(x,Z)}$: Since $\Lambda = dZ - \mathcal{W}\cdot Z$, we have

$$
d\Lambda = d(dZ) - d\mathcal{W}\cdot Z + \mathcal{W}\wedge(dZ)
$$

$$= -\mathcal{W} \wedge \mathcal{W} \cdot Z + \mathcal{W} \wedge (\Lambda + \mathcal{W} \cdot Z) = \mathcal{W} \wedge \Lambda,$$

where for the second equality we use the second structural equation (3.29), together with the definition of Λ in Eq. (3.38). As $(X_i, \mathfrak{m}_i) \in \mathcal{D}^{(x,Z)}$ for $i = 1, 2$, we thus have:

$$\begin{aligned}
& d\Lambda^{(x,Z)}\big((X_1, \mathfrak{m}_1), (X_2, \mathfrak{m}_2)\big) \\
=& \mathcal{W}|_x \wedge \Lambda^{(x,Z)}\big((X_1, \mathfrak{m}_1), (X_2, \mathfrak{m}_2)\big) \\
=& \mathcal{W}(X_1, \mathfrak{m}_1)|_x \Lambda^{(x,Z)}(X_2, \mathfrak{m}_2) - \mathcal{W}(X_2, \mathfrak{m}_2)|_x \cdot \Lambda^{(x,Z)}(X_1, \mathfrak{m}_1) = 0.
\end{aligned}$$

This completes the proof for Eq. (3.41), and hence the Pfaff system (3.37) is solvable.

3. Now, define

$$\underset{\sim}{\Theta} = (\theta^1, \ldots, \theta^n, 0, \ldots, 0)^\top \in \Omega^1(\mathbb{R}^{n+k}), \tag{3.42}$$

and for $x_0 \in M$ let us consider the *Poincaré* system:

$$df = \underset{\sim}{\Theta} \cdot A, \qquad f(x_0) = f_0, \tag{3.43}$$

where $f_0 \in C^\infty(M; \bar{M})$, $d_{x_0} f_0(v) \neq 0$ for $v \in T_{x_0}M \setminus \{0\}$. Suppose this system is solvable; then, as A takes values in $O(\nu + \tau, (n+k) - (\nu + \tau))$, we have $\det(A) = \pm 1$, so A is invertible. It thus follows by Eq. (3.43) that the linear map df has rank n, hence f is indeed an immersion.

To solve for f from Eq. (3.43) is equivalent to demonstrate that $\underset{\sim}{\Theta} \cdot A$ is an exact one-form. For M simply-connected, by *Poincaré's Lemma* it is sufficient to verify $d(\underset{\sim}{\Theta} \cdot A) = 0$, *i.e.* it is a closed one-form. Indeed,

$$d(\underset{\sim}{\Theta} \cdot A) = d\underset{\sim}{\Theta} \cdot A - \underset{\sim}{\Theta} \wedge dA, \tag{3.44}$$

and we can compute the second term by $-\underset{\sim}{\Theta} \wedge dA = -\underset{\sim}{\Theta} \wedge \mathcal{W} \cdot A$, thanks to the Pfaff system (3.37) solved in Steps 2–3 of the same prove. So it remains to establish that

$$d\underset{\sim}{\Theta} = \underset{\sim}{\Theta} \wedge \mathcal{W}.$$

However, this is the first structural equation (3.36). Thus we complete the proof for the solvability of the initial value problem for the Poincaré system (3.43).

4. Having obtained the immersion f from the Poincaré system, let us identify the normal bundle $TM^\perp := f^* T\bar{M}/TM$ with the given bundle E, and identify the second fundamental form induced by f with the given symmetric tensor field II. Moreover, we can deduce the uniqueness of the local immersion upto rigid motions of $\mathbb{R}^{n+k}_{\nu+\tau}$, *i.e.* modulo the actions by the semi-Riemannian congruence group $\mathbb{R}^{n+k} \rtimes O(\nu + \tau, (n + k) - (\nu + \tau))$.

4(a). First of all, define an orthonormal frame $\{\bar{\partial}_a\}_1^{n+k}$ on $T\bar{M}$ via the maps f and A solved by the Pfaff and Poincaré systems. Let us denote by $\left\{\frac{\partial}{\partial Z^a}\right\}_1^{n+k}$ the canonical orthonormal basis on $\bar{M} = \mathbb{R}^{n+k}_{\nu+\tau}$ with respect to $\overline{g_0} = \widetilde{\epsilon_{n\,\nu}} \bigoplus \widetilde{\epsilon_{k\,\tau}}$. In this basis we set

$$\begin{cases} \bar{\partial}_i := df(\partial_i), \\ \bar{\partial}_\alpha := \sum_b A^\alpha_b \frac{\partial}{\partial Z^b}. \end{cases} \tag{3.45}$$

In Eq. (3.45), the definition of $\{\bar{\partial}_a\}$ is independent of the choice of bases on $\bar{M}$: Recall that for each $x \in M$, $A(x)$ lies in $O(\nu+\tau, (n+k)-(\nu+\tau)) \subset \mathfrak{gl}(n+k;\mathbb{R}) \cong \operatorname{End} T\bar{M}$, the group of linear transformations on $T\bar{M}$. Using a further identification $\operatorname{End} T\bar{M} \cong T^*\bar{M} \bigotimes T\bar{M}$, let us view $A(x)$ as a linear map from $T\bar{M}$ to itself. From this perspective $\bar{\partial}_\alpha$ coincides with A^α, *i.e.*, the normal component of A as a $T\bar{M}$-valued function defined on $M \times T\bar{M}$. Thus Eq. (3.45) is equivalent to

$$\bar{\partial} = (df)^\sharp \oplus \mathrm{nor} A, \tag{3.46}$$

where $\sharp : T^*\bar{M} \to T\bar{M}$ is the canonical bundle isomorphism turning a one-form into the corresponding vector field. This gives us an intrinsic definition of the frame $\{\bar{\partial}_a\}$.

Now we verify that $\{\bar{\partial}_a\}$ is indeed an orthonormal frame. First, using the Poincaré system (3.43) defining f, together with a characterisation of the semi-orthogonal group: A matrix $B \in O(\nu, n-\nu)$ if and only if its transpose $B^\top$ takes the form

$$B^\top = \widetilde{\epsilon_{n,\nu}} B^{-1} \widetilde{\epsilon_{n,\nu}}, \tag{3.47}$$

we have

$$\begin{aligned} \overline{g_0}(\bar{\partial}_i, \bar{\partial}_j) &= \overline{g_0}\big(df(\partial_i), df(\partial_j)\big) = \overline{g_0}\big((\underset{\sim}{\Theta}(\partial_i)A, \underset{\sim}{\Theta}(\partial_j)A\big) \\ &= \underset{\sim}{\Theta}(\partial_i)A^\top \overline{g_0} A \underset{\sim}{\Theta}(\partial_j)^\top = \underset{\sim}{\Theta}(\partial_i)\overline{g_0}\underset{\sim}{\Theta}(\partial_j)^\top \\ &= (\widetilde{\epsilon_{n,\nu}})^i_j := \epsilon_i \delta^i_j. \end{aligned}$$

Also, using the shorthand notations in Eq. (3.46) one obtains

$$\overline{g_0}(\bar{\partial}_\alpha, \bar{\partial}_\beta) = [\overline{g_0}(\mathrm{nor}A, \mathrm{nor}A)]^\alpha_\beta = \mathrm{nor}(A^\top \overline{g_0} A)^\alpha_\beta = (\widetilde{\epsilon_{k,\tau}})^\alpha_\beta = \epsilon_\alpha \delta^\alpha_\beta.$$

Finally, again by the Poincaré system (3.43), we find that

$$\overline{g_0}(\partial_i, \partial_\alpha) = \overline{g_0}(\underset{\sim}{\Theta} \cdot A(\partial_i), A^\alpha) = (\underset{\sim}{\Theta}(\partial_i))^\top [A^\top \bar{g}_0 A]^\alpha = (\overline{g_0})^\alpha_i \equiv 0,$$

since $\overline{g_0}$ is a diagonal matrix. Thus we have verified the orthonormality of $\{\bar{\partial}_a\}$.

4(b). Next, we identify the normal bundle induced by f, written as $TM^{\perp} := f^{*}T\bar{M}/TM$ (*cf.* Convention 3.1.2), with the prescribed vector bundle E. For this purpose, we define the following *identification map* on a trivialised chart $U \subset M$:

$$\begin{aligned}\mathcal{I} : (U \subset M) \times (F \cong \mathbb{R}^{k}) &\longrightarrow \bar{M}\\ (x, \xi) \qquad &\mapsto \quad f(x) + \sum_{\beta} \xi_{\beta} \bar{\partial}_{\beta}.\end{aligned}$$

Let us notice that $d\mathcal{I} : TU \bigoplus TF \to T\bar{M}$ coincides with $(df + \mathrm{nor}A)$; equivalently, one can write $d\mathcal{I}(\partial_a) = \bar{\partial}_a$ for each $a \in \{1, \ldots, n+k\}$, and in particular $d\mathcal{I}(TU) \subset TM$ and $d\mathcal{I}(F) \subset TM^{\perp}$, namely that the identification map $\mathcal{I}$ preserves the horizontal and vertical subspaces of the vector bundles $TM \bigoplus E$ and $T\bar{M}$. Moreover, as f is an immersion (justified in Step 4), we deduce that $\mathcal{I}$ is a diffeomorphism, by shrinking the chart U if necessary. Thus we have obtained an identification of E with $TM^{\perp}$ in the trivialised local charts.

In addition, by the construction of the moving frame $\{\bar{\partial}_a\}$ on $T\bar{M}$ in Eq. (3.45), we have

$$\begin{aligned}&\mathcal{I}^{*}\overline{g_0}\Big(\sum_i U^i \partial_i + \sum_{\alpha} U^{\alpha} \partial_{\alpha}, \sum_j V^j \partial_j + \sum_{\beta} V^{\beta} \partial_{\beta}\Big)\\ &= \overline{g_0}\Big(\sum_i U^i df(\partial_i), \sum_j V^j df(\partial_j)\Big) + \overline{g_0}\Big(\sum_{\alpha} U^{\alpha} \bar{\partial}_{\alpha}, \sum_{\beta} V^{\beta} \bar{\partial}_{\beta}\Big)\\ &= \sum_i \sum_j U^i V^j \overline{g_0}(\bar{\partial}_i, \bar{\partial}_j) + \sum_{\alpha} \sum_{\beta} U^{\alpha} V^{\beta} \overline{g_0}(\bar{\partial}_{\alpha}, \bar{\partial}_{\beta})\\ &= g(U|_{TM}, V|_{TM}) + g^{E}(U|_E + V|_E),\end{aligned}$$

for any $U, V \in \Gamma(T, \bar{M})$. Thus $\mathcal{I}$ is an isometry between $TU \bigoplus E$ and $T\bar{M}$, *i.e.*,

$$\mathcal{I}^{*}\overline{g_0} = g \bigoplus g^{E}, \tag{3.48}$$

as block-direct sum of matrices. In particular, it shows that f is a local isometric immersion.

4(c). Now, let us identify the second fundamental form and the normal connection induced by f with II and ∇^{E}, respectively. For this purpose, we write out the Cartan formalism for the isometric immersion f.

Indeed, given $f : (V \subset M, g) \to (\mathbb{R}^{n+k}, \overline{g_0})$ as above (already established as an isometric immersion), we write $\bar{\theta} = (\bar{\theta}^1, \ldots, \bar{\theta}^{n+k}) \in \Omega^1(\mathbb{R}^{n+k}) \cong C^{\infty}(\bar{M}; T^{*}\bar{M} \bigotimes T^{*}\bar{M})$ as the co-frame of $\{\bar{\partial}_a\}_1^{n+k}$, we can express the Gauss-Codazzi-Ricci equations corresponding to f equivalently as the second structural equation with respect to $\bar{\partial}$ or $\bar{\theta}$. Indeed, repeating the constructions in §4.2 for f and the normal bundle $TM^{\perp} := \frac{f^{*}T\bar{M}}{TM}$, one easily obtains

$$\bar{\mathcal{W}} = \begin{bmatrix} \bar{\omega}^i_j & \bar{\omega}^\alpha_i \\ \bar{\omega}^i_\alpha & \bar{\omega}^\beta_\alpha \end{bmatrix} = \begin{bmatrix} \bar{\theta}^j(\tan\bar{\nabla}_\bullet \bar{\partial}_i) & \bar{S}_{\bar{\partial}_\alpha}\bar{\partial}_i \\ -\bar{S}_{\bar{\partial}_\alpha}\bar{\partial}_i & \bar{\theta}^\beta(\nabla^\perp_\bullet \bar{\partial}_\alpha) \end{bmatrix}, \tag{3.49}$$

which is an analogue of Eq. (3.30). Here,

$$\begin{aligned} \bar{\mathcal{W}} = \{\bar{\omega}^a_b\} &\in \Omega^1(\mathfrak{o}(\nu+\tau,(n+k)-(\nu+\tau))) \\ &= C^\infty(\bar{M}; T^*\bar{M} \bigotimes \mathfrak{o}(\nu+\tau,(n+k)-(\nu+\tau))) \end{aligned}$$

is the connection one-form on $T\bar{M}$ with respect to the Levi-Civita connection on $\bar{M}$, $\bar{S}$ is the shape operator associated to f, and $\nabla^\perp$ is the projection of $\bar{\nabla}$ onto the normal bundle $TM^\perp$. Also, by the torsion-free condition of $\bar{\nabla}$, the first structural equation (3.36) holds:

$$d\bar{\theta} = \bar{\theta} \wedge \bar{\mathcal{W}}. \tag{3.50}$$

Therefore, by comparing the coordinate-wise representations of the connection one-forms $\mathcal{W}$ and $\bar{\mathcal{W}}$, *i.e.* Eqs. (3.30) and (3.49), it is enough to establish

$$\mathcal{I}^*\bar{\mathcal{W}} = \mathcal{W}, \tag{3.51}$$

in order to identify $(\bar{S}, \nabla^\perp)$ with the given quantities (S, ∇^E).

To this end, we pullback the first structural equation (3.50) under $\mathcal{I}$. Then, on the one hand

$$\mathcal{I}^*(d\bar{\theta}) = d(\mathcal{I}^*\bar{\theta}) = d(f^*\bar{\theta}) = d\underset{\sim}{\Theta} = \underset{\sim}{\Theta} \wedge \mathcal{W}, \tag{3.52}$$

where we have used in order (a), the commutativity of pullback and exterior differential; (b), the fact that $\mathcal{I}$ respects the orthogonal splitting of $T\bar{M}$ and $TM \bigoplus E$; (c), the duality of $df(\partial_i) = \bar{\partial}_i$; and (d), the first structural equation (3.36) on $TM \bigoplus E$. On the other hand, notice that

$$\mathcal{I}^*(\bar{\theta} \wedge \bar{W}) = \mathcal{I}^*\bar{\theta} \wedge \mathcal{I}^*\bar{W} = \underset{\sim}{\Theta} \wedge \mathcal{I}^*\bar{\mathcal{W}}, \tag{3.53}$$

thanks to the distributivity of the pullback operation with respect to the wedge product, and that $\mathcal{I}^*\bar{\theta} = f^*\bar{\theta} = \underset{\sim}{\Theta}$ as above. Therefore, Eq. (3.51) immediately follows from the preceding two equations.

4(d). Finally we prove the uniqueness of local isometric immersions up to rigid motions of the semi-Euclidean space. It is a direct consequence of the arguments in Step 3. Indeed, if $f' : (V, g) \to (\bar{M}, \overline{g_0})$ is another isometric immersion on V (a trivialised local chart) with $f'(q')$ given, then for any local frame $\{\partial'_a\}$ we can take a rigid motion that transforms q to q' and $\{\bar{\partial}_a\}$ to $\{\bar{\partial}'_a\}$, *i.e.*, a translation composed with an element of $O(\nu+\tau,(n+k)-(\nu+\tau))$. Then the argument follows from the uniqueness of Pfaff system (which in turn is based upon the uniqueness of the maximal integral submanifold found by the Frobenius theorem), as well as the uniqueness of Poincaré system up to an additive constant.

Therefore, putting together steps $4(a)$–$4(d)$, we conclude the realisation of the isometric immersions from the GCR equations in the C^∞ case.

5. Now we prove for the lower regularity case, *i.e.*, when $g \in W^{1,p}_{\rm loc}(M, O(\nu, n-\nu))$. For this purpose, we observe that all the arguments in Step 4 remain valid, provided that each equality is interpreted in the sense of distributions. As a result, if the Pfaff and Poincaré systems with $g \in W^{1,p}_{\rm loc}(M, O(\nu, n-\nu))$ and $\mathcal{W} \in L^p_{\rm loc}(U \subset M; \mathfrak{o}(\nu+\tau,(n+k)-(\nu+\tau)))$ are solved, *i.e.*, the bundle connection A and the immersion f exist in the following spaces:

$$\begin{cases} A \in W^{1,p}_{\rm loc}(U \subset M; O(\nu+\tau,(n+k)-(\nu+\tau))), \\ f \in W^{2,p}_{\rm loc}(M; \bar{M}), \end{cases}$$

such that $\operatorname{rank}(df) = n$, and Eqs. (3.37)–(3.43) are fulfilled, then f is indeed an $W^{2,p}_{\rm loc}$ isometric immersion by construction.

The solubility of the Poincaré system with the corresponding Sobolev coefficients is easy to establish. Suppose we are given $A \in W^{1,p}_{\rm loc}(U \subset M; O(\nu+\tau,(n+k)-(\nu+\tau)))$ and want to solve for $df = \underset{\sim}{\Theta} \cdot A$ in $W^{2,p}_{\rm loc}(M; \bar{M})$. Since all the results are stated in local Sobolev spaces, it suffices to assume that U is a smooth bounded open subset of $\mathbb{R}^n$. In this setting, let $J_\epsilon \in C^\infty(\mathbb{R}^n)$ be the standard mollifier, and set

$$\Theta^\epsilon := J_\epsilon * (\underset{\sim}{\Theta} \cdot A). \tag{3.54}$$

It follows that

$$\Theta^\epsilon \longrightarrow \underset{\sim}{\Theta} \cdot A \qquad \text{in } W^{1,p}(U; \bar{M}) \text{ as } \epsilon \to 0^+, \tag{3.55}$$

hence $\{\Theta^\epsilon\}$ is uniformly bounded in $W^{1,p}$. Now, Θ^ϵ is a smooth closed one-form (*cf.* Step 3) for each $\epsilon > 0$, so we again invoke the solvability of the Poincaré's system to find $f^\epsilon \in C^\infty(U; \bar{M})$ such that $df^\epsilon = \Theta^\epsilon$. By adding a constant we may assume that $\fint_U f^\epsilon \,\mathrm{d}x = 0$. Then, the Poincaré inequality gives us:

$$\|f^\epsilon\|_{W^{2,p}(U;\bar{M})} \leq C\big(\|f^\epsilon\|_{W^{1,p}(U;\bar{M})} + \|\Theta^\epsilon\|_{W^{1,p}(U;\bar{M})}\big).$$

Hence, thanks to the compactness of the embedding $W^{2,p}(U;\bar{M}) \hookrightarrow W^{1,p}(U;\bar{M})$ (Rellich–Kondrachov Lemma) and the uniform boundedness of $\{\Theta^\epsilon\} \subset W^{1,p}(U;\bar{M})$, we can immediately arrive at the uniform bound

$$\|f^\epsilon\|_{W^{2,p}(U;\bar{M})} \leq C_0 < \infty. \tag{3.56}$$

Therefore, there exists a limiting function $\bar{f}$ so that $f^\epsilon \to \bar{f}$ in $W^{2,p}(U; \bar{M})$ up to subsequences, and it satisfies $d\bar{f} = \underset{\sim}{\Theta} \cdot A$.

6. On the other hand, the Pfaff system with little regularity is more difficult to tackle. In particular, the Frobenius Theorem cannot be directly applied, since we need at least C^1 regularity to make it work; moreover, we cannot apply a direct mollification argument, since the compatibility condition (*i.e.*, the second structural equation $d\mathcal{W} = \mathcal{W} \wedge \mathcal{W}$) contains a quadratic non-linear term.

In this case the lemma by Mardare in [88] (Lemma 2.4.1) exactly serves for our purpose. Therein we take

$$\mathfrak{S} = \mathcal{W} \in L^p_{\text{loc}}(U; T^*M \bigotimes \mathfrak{o}(\nu + \tau, (n+k) - (\nu + \tau))), \qquad \mathfrak{S}_i = \mathcal{W}(\partial_i).$$

Moreover, as Lemma 2.4.1 is formulated for $\Omega \subset \mathbb{R}^n$, correspondingly we take $U \subset M$ as a trivialised local chart, namely that the bundle E can be regarded as $U \times \mathbb{R}^k$ over U. Hence, on U, we assume without loss of generality that $[\partial_i, \partial_j] = 0$; it gives us

$$\partial_i \mathfrak{S}_j - \partial_j \mathfrak{S}_i = \partial_i\big(\mathcal{W}(\partial_j)\big) - \partial_j\big(\mathcal{W}(\partial_i)\big) + \mathcal{W}([\partial_i, \partial_j]) = d\mathcal{W}(\partial_i, \partial_j).$$

However, on the other hand we have:

$$[\mathfrak{S}_i, \mathfrak{S}_j] = \mathcal{W}(\partial_i) \cdot \mathcal{W}(\partial_j) - \mathcal{W}(\partial_j) \cdot \mathcal{W}(\partial_i) = \mathcal{W} \wedge \mathcal{W}(\partial_i, \partial_j).$$

So the compatibility condition in Lemma 2.4.1 is verified by the second structural Eq. (3.29). So the Pfaff system (3.37) with Sobolev coefficients is uniquely solvable on local charts. Therefore, combining the arguments in Steps 5–6, one concludes the existence of local isometric immersions in the little regularity case, provided that the GCR equations hold in distributions.

7. Finally we deduce the global existence of isometric immersions, based on the local result established in Steps 2–7 of the same proof. In fact, it can be copied verbatim from the standard monodromy arguments in Step 6 of §2.4.3, hence is omitted here. Combining the arguments in Steps 1–7 above, we now complete the proof. ■

3.3 Generalised quadratic theorems of compensated compactness

Our main tool to establish the weak continuity of GCR equations is a generalised quadratic theorem in the theory of compensated compactness *à la* L. Tartar. First of all, we recall Tartar's original result (Corollary 13 in [119]):

Proposition 3.4 (Tartar's quadratic theorem) *Let $\{u^\epsilon\}$ be a family of functions in $L^2(\mathbb{R}^K;\mathbb{R}^J)$, and A_{ijk} constants for $1 \leq i \leq I$, $1 \leq j \leq J$ and $1 \leq k \leq K$. Let $Q : \mathbb{R}^J \to \mathbb{R}$ be a quadratic form, namely a 2-homogeneous polynomial. Suppose that*

(T1) *$u^\epsilon \rightharpoonup u$ weakly in $L^2(\mathbb{R}^K;\mathbb{R}^J)$;*

(T2) *$\{\sum_{j=1}^J \sum_{k=1}^K A_{ijk}\nabla_k u^\epsilon_j\}_{i=1}^I$ is pre-compact in $H^{-1}_{\rm loc}(\mathbb{R}^K;\mathbb{R}^I)$;*

(T3) *$Q(\lambda) = 0$ for all $\lambda \in \Lambda$, where the* operator cone Λ *is defined by*

$$\Lambda := \Big\{\lambda \in \mathbb{R}^J : \sum_{j=1}^J\sum_{k=1}^K A_{ijk}\lambda_j\xi_k = 0 \text{ for all } 1 \leq i \leq I \text{ and for some } \xi \in \mathbb{R}^K\backslash\{0\}\Big\}.$$

Then $Q(u^\epsilon) \to Q(u)$ in the sense of distributions.

Here, the condition (T2) is known as the *first-order differential constraints* on $\{u^\epsilon\}$ (in which ∇_k denotes the k-th partial derivative on $\mathbb{R}^K$). Heuristically it says that certain constant-coefficient linear combinations of the first-order derivatives of u^ϵ, *i.e.* a directional derivative of u^ϵ in some prescribed directions, is pre-compactness in $H^{-1}_{\rm loc}$. The condition (T3) requires the quadratic form Q to vanishing on the cone Λ. This is desirable since the Fourier transform of the quantities in (T2) is zero on Λ, or loosely speaking, "Λ cannot detect the first-order differential constraints on the Fourier side". (T2)(T3) together promote the weak-star L^∞ convergence of $\{Q(u^\epsilon)\}$ (whose topology is not even metrisable), which follows from (T1) and Cauchy-Schwarz, to a distributional, subsequential convergence.

We generalise Proposition 3.4 to vector bundles over semi-Riemannian manifolds. To do so, we first isolate the key analytic ingredients in §3.3.1, by establishing a compensated compactness theorem on locally abelian topological groups. Then in §3.3.2 we prove our main theorem, which shall be used in §3.4 to establish the weak continuity of GCR equations.

3.3.1 A result on locally compact abelian (LCA) groups

In this subsection we prove a compensated compactness theorem on LCA groups, as a modification of Tartar's quadratic theorem in [118]. Our motivation is to identify the structures and estimates essential to Tartar's result, and thus generalise it in the setting of abstract harmonic analysis.

Let us begin with recalling the basics of abstract harmonic analysis. We refer the readers to Loomis [83] for details.

First of all, a *topological group* G is a group with a topology, in which the group operation and the inverse are continuous. If the group G is abelian, and the topology is Hausdorff and locally compact, we say G is a *locally compact abelian group*, abbreviated as *LCA group* in the sequel. For any LCA group G there exists an invariant Radon measure μ_G, unique up to multiplicative constants, known as the *Haar measure*. The L^p norm $(1 \leq p < \infty)$ for $u : G \to \mathbb{C}$ can thus be defined as follows:

$$\|u\|_{L^p(G)} := \left\{ \int_G |u(g)|^p \, \mathrm{d}\mu_G(g) \right\}^{1/p}. \tag{3.57}$$

Secondly, given any LCA group G, its *group of characters* $\hat{G} := \mathrm{Hom}(G; \mathbb{R}/\mathbb{Z})$ is also an LCA group, endowed with the local-uniform topology of any non-trivial Haar measure (which is the weakest topology making each element of $\hat{G}$ continuous). It is also known as the *dual* of G, due to the following *Pontryagin duality theorem*: G is canonically isomorphic to $\hat{\hat{G}}$. Then, for $u \in L^1(G)$ we can define its *Fourier transform* $\hat{u} : \hat{G} \to \mathbb{C}$ by

$$\hat{u}(\xi) := \int_G u(g) e^{-2\pi i \xi(g)} \, \mathrm{d}\mu_G(g). \tag{3.58}$$

Here $\xi(g)$ is given by the duality pairing of $\hat{G}$ and G. Throughout we write $0 \in \hat{G}$ as the group identity; this is in agreement with the definition $\hat{G} := \mathrm{Hom}(G; \mathbb{R}/\mathbb{Z})$, which is the group of additive characters.

Thirdly, the *Plancherel formula* extends to the general LCA groups:

$$\|u\|_{L^2(G)} = \|\hat{u}\|_{L^2(\hat{G})} \qquad \text{for all } u \in L^2(G), \tag{3.59}$$

with the Haar measures μ_G and $\mu_{\hat{G}}$ suitably normalised. In other words, the Fourier transform defined in Eq. (3.58) is an isometry between $L^2(G)$ and $L^2(\hat{G})$. Notice that all the constructions up to now extend naturally to vector-valued functions $u : G \to \mathbb{C}^I$, $I \geq 1$.

Fourthly, we say that $\mathcal{T} : L^2(G) \to L^2(G)$ is a *multiplier operator* if

$$\widehat{\mathcal{T}u}(\xi) = m(\xi)\hat{u}(\xi) \qquad \text{for some } m : \hat{G} \to \mathbb{C}. \tag{3.60}$$

Here, m is known as the *Fourier multiplier* of $\mathcal{T}$. More generally, for $\mathcal{T} : L^2(G; \mathbb{C}^J) \to L^2(G; \mathbb{C}^I)$ $(I, J \geq 1)$, the multiplier is a mapping

$$m : \hat{G} \to \mathrm{Mat}(I \times J; \mathbb{C}) \cong (\mathbb{C}^J)^* \bigotimes \mathbb{C}^I. \tag{3.61}$$

Namely, for each $\xi \in \hat{G}$, $m(\xi)$ is a linear operator from $\mathbb{C}^J$ to $\mathbb{C}^I$ (equivalently, an $I \times J$ matrix). In the sequel, for any matrix $M \in \mathrm{Mat}(I \times J; \mathbb{C})$ we use $|M| := \sqrt{\sum_{i=1}^I \sum_{j=1}^J |M_{ij}|^2}$ to denote its Hilbert-Schmidt norm.

In this context, we say that $Q : \mathbb{C}^N \to \mathbb{C}$ is a *quadratic polynomial* if Q is a Hermitian 2-form on $\mathbb{C}^N$, *i.e.*, $Q = \{Q_{jk}\}$ as a complex $(n \times n)$ matrix satisfies:

$$Q_{jk} = \overline{Q_{kj}} \qquad \text{for each } j, k = 1, 2, \ldots, N. \tag{3.62}$$

Equivalently, it means:

$$Q(\lambda) = \sum_{j,k=1}^{N} Q_{jk}\lambda^j\overline{\lambda^k} \qquad \text{for } \lambda = (\lambda^1, \ldots, \lambda^N) \in \mathbb{C}^N \text{ and constants } Q_{jk} \in \mathbb{C}. \tag{3.63}$$

Now we can formulate our generalised quadratic theorem on LCA groups:

Theorem 3.5 *Let G be an LCA group with Haar measure μ_G. We consider a sequence $\{u^\epsilon\}$ in $L^2_c(G;\mathbb{C}^J)$, a Fourier multiplier operator $\mathcal{T} : L^2(G;\mathbb{C}^J) \to H^{-s}(G;\mathbb{C}^I)$ with multiplier $m : \hat{G} \to \mathrm{Mat}(I \times J;\mathbb{C})$ for some $s \in \mathbb{R}_+$, and a quadratic polynomial $Q : \mathbb{C}^J \to \mathbb{C}$. Assume that*

(C1) $u^\epsilon \rightharpoonup u$ *weakly in* $L^2(G;\mathbb{C}^J)$.

(C2) *The end of $\hat{G}$ retracts nicely onto a compact set: More precisely, for a large enough compact set $\Xi \Subset \hat{G}$ containing 0, there exist another compact set $\mathcal{K} \Subset \hat{G} \setminus \{0\}$ and a continuous, surjective map $\Phi : \hat{G} \setminus \Xi \to \mathcal{K}$, such that $\{(\Phi^* m)\hat{u^\epsilon}\}$ is pre-compact in $L^2(\hat{G} \setminus \Xi;\mathbb{C}^J)$.*

(C3) $Q(\lambda) = 0$ *for all $\lambda \in \Lambda_{\mathcal{T}}$, where $\Lambda_{\mathcal{T}}$ (the* cone *of $\mathcal{T}$) is defined by*

$$\Lambda_{\mathcal{T}} := \big\{\lambda \in \mathbb{C}^J : m(\xi)(\lambda) = 0 \textit{ for some } \xi \in \hat{G} \setminus \{0\}\big\}. \tag{3.64}$$

Then we have

$$\lim_{\epsilon \to 0} \int_G (Q \circ u^\epsilon)(g)\, \mathrm{d}\mu_G(g) = \int_G (Q \circ u)(g)\, \mathrm{d}\mu_G(g). \tag{3.65}$$

Here, the *pullback* of m under Φ, *i.e.*, the map $\Phi^* m : \hat{G} \setminus \Xi \to [0, \infty]$, is given by

$$\Phi^* m(\xi) := m\big(\Phi(\xi)\big).$$

Also, in the definition of $\Lambda_{\mathcal{T}}$ (Eq. (3.64)), we view $m : \hat{G} \to (\mathbb{C}^J)^* \bigotimes \mathbb{C}^I$ (*i.e.*, $m(\xi)$ is an operator from $\mathbb{C}^J$ to $\mathbb{C}^I$), hence $m(\xi)(\lambda) \in \mathbb{C}^I$. According to this interpretation, another characterisation of the cone is at hand:

$$\Lambda_{\mathcal{T}} = \bigcup_{\xi \in \hat{G} \setminus \{0\}} \ker[m(\xi)].$$

Before giving the proof, we point out that our strategy follows from Tartar's original work [118], in which separate estimates are derived in the Fourier space $\hat{G}$ for the *low-frequency region* (*i.e.*, in a compact set Ξ around 0) and the *high-frequency region* (*i.e.*, in the non-compact set $\hat{G}\backslash\Xi$). The assumptions (C2)–(C3) are required only for controlling the high-frequency region. Notice that the high-frequency region always exists unless $\hat{G}$ is compact, which is equivalent to the condition that G is discrete, for which Theorem 3.5 trivially holds.

Proof. We divide the proof into five steps. The arguments are a modification of the proof of Corollary 13 in [118] by Tartar.

1. First, notice that by substituting u^ϵ with $u^\epsilon - u$, it suffices to assume $u \equiv 0$. Indeed, an expansion of $(u^\epsilon - u)$ gives us

$$\begin{aligned} Q(u^\epsilon - u) &= \sum_{j=1}^{J}\sum_{l=1}^{J}\Big(Q_{jl}(u^\epsilon)^j\overline{(u^\epsilon)^l} + Q_{jl}u^j\overline{u^l} - Q_{jl}(u^\epsilon)^j\overline{u^l} - Q_{jl}u^j\overline{(u^\epsilon)^l}\Big) \\ &=: Q(u^\epsilon) + Q(u) - \sum_{j=1}^{J}\sum_{l=1}^{J}\big(Q_{jl}(u^\epsilon)^j\overline{u^l} + Q_{jl}u^j\overline{(u^\epsilon)^l}\big) \end{aligned} \tag{3.66}$$

and $u^\epsilon \rightharpoonup u$ weakly in L^2 implies that

$$\int_G \Big\{\sum_{j=1}^{J}\sum_{l=1}^{J}\big(Q_{jl}(u^\epsilon)^j\overline{u^l} + Q_{jl}u^j\overline{(u^\epsilon)^l}\big)\Big\}\,\mathrm{d}\mu_G \longrightarrow -2\int_G Q(u)\,\mathrm{d}\mu_G \qquad \text{as } \epsilon \to 0. \tag{3.67}$$

Thus, we have

$$\int_G Q(u^\epsilon - u)\,\mathrm{d}\mu_G \longrightarrow \int_G Q(u^\epsilon)\,\mathrm{d}\mu_G - \int_G Q(u)\,\mathrm{d}\mu_G \qquad \text{as } \epsilon \to 0, \tag{3.68}$$

so $u \equiv 0$ is assumed from now on.

2. Next, as $u^\epsilon \in L^2_c(G;\mathbb{C}^J)$ implies $u^\epsilon \in L^1(G;\mathbb{C}^J)$, by the *Riemann-Lebesgue lemma* on LCA groups (*cf.* Exercise 11 in Tao's notes [117]) we can find a compact set $\Xi \Subset \hat{G}$ such that $|\hat{u}^\epsilon(\xi)| \leq \alpha$ for $\xi \in \hat{G}\setminus\Xi$, for each $\alpha > 0$. In particular, $\sup\{|\hat{u}^\epsilon(\xi)| : \xi \in \hat{G}\} \leq M$. On the other hand, for any $\phi \in L^2(G;\mathbb{C}^J)$, by Plancherel formula there holds

$$\Big|\int_G u^\epsilon\phi\,\mathrm{d}\mu_G\Big| = \Big|\int_{\hat{G}} \hat{u^\epsilon}\hat{\phi}\,\mathrm{d}\mu_{\hat{G}}\Big|,$$

which converges to zero as $\epsilon \to 0$ by the assumption (C1). Thus, choosing

$$\hat{\phi} = \overline{\mathrm{sgn}(\hat{u}^\epsilon)}\cdot\chi_\Xi = \frac{\overline{\hat{u}^\epsilon}}{|\hat{u}^\epsilon|}\cdot\chi_\Xi, \tag{3.69}$$

we obtain that

$$\int_{\Xi} |\hat{u}^\epsilon|^2 \, \mathrm{d}\mu_{\hat{G}} \leq M \int_{\Xi} |\hat{u}^\epsilon| \, \mathrm{d}\mu_{\hat{G}} = M \Big| \int_{\hat{G}} \hat{u}^\epsilon \hat{\phi} \, \mathrm{d}\mu_{\hat{G}} \Big| \longrightarrow 0.$$

Hence for the quadratic polynomial Q one immediately deduces

$$\int_{\Xi} |Q \circ \hat{u}^\epsilon| \, \mathrm{d}\mu_{\hat{G}} \longrightarrow 0. \tag{3.70}$$

For the subsequent development, notice that we have the freedom of enlarging Ξ: It can be chosen as any large enough (with respect to $\mu_{\hat{G}}$) compact subset of $\hat{G}$ containing 0.

3. In this step we establish the following *claim*: Given any $\delta > 0$ and any compact subset $\mathcal{K} \Subset \hat{G}$ such that $0 \notin \mathcal{K}$, there exists a constant $C_{\delta,\mathcal{K}} \in (0, \infty)$ such that for any $\lambda \in \mathbb{C}^J$ and $\eta \in \mathcal{K}$, we have

$$\mathrm{Re}\{Q(\lambda)\} \geq -\delta|\lambda|^2 - C_{\delta,\mathcal{K}}|m(\eta)(\lambda)|^2, \tag{3.71}$$

provided that $\mathrm{Re}(Q) \geq 0$ on Λ_T. Meanwhile, under the same conditions for δ and $\mathcal{K}$,

$$\mathrm{Im}\{Q(\lambda)\} \geq -\delta|\lambda|^2 - C_{\delta,\mathcal{K}}|m(\eta)(\lambda)|^2 \tag{3.72}$$

when $\mathrm{Im}(Q) \geq 0$ on Λ_T. (Notice that such $\mathcal{K}$ indeed exists, as $\hat{G}$ is endowed with a locally compact topology).

Indeed, observe that for $\lambda = 0$ the *claim* holds trivially. For $\lambda \neq 0$ let us prove by contradiction. If the statement were false, there would exist $\delta_0 > 0$ such that for each $n \in \mathbb{N}$, one can find $\lambda_n \in \mathbb{C}^J$ and $\eta_n \in \mathcal{K}$ satisfying

$$\mathrm{Re}\{Q(\lambda_n)\} < -\delta_0|\lambda_n|^2 - n|m(\eta_n)(\lambda_n)|^2. \tag{3.73}$$

This inequality is 2-homogeneous in λ_n, and in particular it is invariant under the scaling $\lambda_n \mapsto c\lambda_n$ for any $c \in \mathbb{C} \setminus \{0\}$. Thus, without loss of generality we may require $|\lambda_n| = 1$ for all n, hence $\{\lambda_n\}$ converges to some $\lambda_\infty \in \mathbb{C}^J$ of norm 1, after passing to subsequences.

In this case, note that $|\mathrm{Re}(Q(\lambda_n))|$ is bounded uniformly in n (say by C_0), hence

$$n|m(\eta_n)(\lambda_n)|^2 \leq C_0 - \delta_0.$$

This forces $|m(\eta_\infty)(\lambda_\infty)| = 0$, where $\eta_\infty \in \mathcal{K}$ is a limit of $\{\eta_n\}$, after passing to a further subsequence if necessary. (Indeed, the subsequential convergence is guaranteed by the fact that $\hat{G}$ is Hausdorff, which is a part of the definition of LCA groups.) By the assumption on $\mathcal{K}$ we have $\eta_\infty \neq 0$, so by the definition of the cone (Eq. (3.64)), $\lambda_\infty \in \Lambda_{\mathcal{T}}$. However, this gives us

$$\mathrm{Re}\{Q(\lambda_\infty)\} \leq -\delta_0,$$

which contradicts the assumption that $\mathrm{Re}(Q) \geq 0$ on $\Lambda_{\mathcal{T}}$. Hence the *claim* is proved for $\mathrm{Re}(Q)$. The arguments for $\mathrm{Im}(Q)$ are exactly the same, thus are omitted here.

4. Now, employing the *claim* in the preceding step, let us establish the following: Whenever $\mathrm{Re}(Q) \geq 0$ on $\Lambda_{\mathcal{T}}$, there holds

$$\liminf_{\epsilon\to 0} \int_{\hat{G}\setminus\Xi} \mathrm{Re}(Q \circ \hat{u}^\epsilon)\,\mathrm{d}\mu_{\hat{G}} \geq 0. \tag{3.74}$$

Meanwhile, for $\mathrm{Im}(Q) \geq 0$ on $\Lambda_{\mathcal{T}}$, we have

$$\liminf_{\epsilon\to 0} \int_{\hat{G}\setminus\Xi} \mathrm{Im}(Q \circ \hat{u}^\epsilon)\,\mathrm{d}\mu_{\hat{G}} \geq 0. \tag{3.75}$$

To prove this statement, we invoke the assumption (C2) on the Fourier multiplier. As $\{[\Phi^* m]\hat{u}^\epsilon\}$ is pre-compact in $L^2(\hat{G};\mathbb{C}^J)$ and $\{\hat{u}^\epsilon\}$ converges to zero weakly in L^2 (by the Plancherel formula), we have

$$\int_{\hat{G}\setminus\Xi} \Big|m\big(\Phi(\xi)\big)\hat{u}^\epsilon(\xi)\Big|^2\,\mathrm{d}\mu_{\hat{G}}(\xi) \longrightarrow 0.$$

Then, let us take $\eta = \Phi(\xi) \in \mathcal{K}$ and $\lambda = \hat{u}^\epsilon(\xi) \in \mathbb{C}^J$ in Eq. (3.71) in Step 2. It shows that for each $\delta > 0$, there exists $0 < C_{\delta,\mathcal{K}} < \infty$ such that

$$\mathrm{Re}\big(Q \circ \hat{u}^\epsilon(\xi)\big) > -\delta|\hat{u}^\epsilon(\xi)|^2 - C_{\delta,\mathcal{K}}\Big|m\big(\Phi(\xi)\big)\hat{u}^\epsilon(\xi)\Big|^2 \qquad \text{for } \xi \in \hat{G}\setminus\Xi.$$

Hence, by integrating over $\hat{G}\setminus\Xi$ and sending $\epsilon \to 0$, one obtains

$$\liminf_{\epsilon\to 0} \int_{\hat{G}\setminus\Xi} \mathrm{Re}(Q \circ \hat{u}^\epsilon)\,\mathrm{d}\mu_{\hat{G}} \geq -\delta \sup_{\epsilon\geq 0} \|\hat{u}^\epsilon\|^2_{L^2(\hat{G}\setminus\Xi)} \geq -\delta M, \tag{3.76}$$

where $M < \infty$ a universal constant. Here we use the pre-compactness of $\{\hat{u}^\epsilon\}$ in $L^2(\hat{G};\mathbb{C}^J)$, which is implied by the assumption (C1) and the Plancherel formula. As $\delta > 0$ is arbitrary, Eq. (3.74) is proved. The proof for the imaginary part, *i.e.*, Eq. (3.75), holds analogously.

5. Finally, to conclude the theorem, note that by changing $Q \mapsto -Q$ in Eq. (3.74) the following inequality holds:

$$\limsup_{\epsilon\to 0} \int_{\hat{G}\setminus\Xi} \mathrm{Re}(Q \circ \hat{u}^\epsilon)\,\mathrm{d}\mu_{\hat{G}} \leq 0 \qquad \text{for } \mathrm{Re}(Q) \leq 0 \text{ on } \Lambda_{\mathcal{T}}. \tag{3.77}$$

By the hypothesis (C3), *i.e.*, $\mathrm{Re}(Q) = 0$ on $\Lambda_{\mathcal{T}}$, this inequality together with (3.74) verifies the assertion outside a compact set Ξ, *i.e.*, $\lim_{\epsilon\to 0} \int_{\hat{G}\setminus\Xi} \mathrm{Re}(Q \circ \hat{u}^\epsilon)\, \mathrm{d}\mu_{\hat{G}} = 0$. Moreover, in Step 1 the same result on Ξ has been established in Eq. (3.70). Hence, in view of the Plancherel formula, one obtains that

$$\lim_{\epsilon\to 0} \int_G \mathrm{Re}(Q \circ u^\epsilon)\, \mathrm{d}\mu_G = 0. \tag{3.78}$$

As in Steps 2–3, the analogous statement for $\mathrm{Im}(Q \circ u^\epsilon)$ can be established similarly. So the proof is now complete. ■

Let us discuss some examples to the quadratic theorem on LCA groups (Theorem 3.5). First of all, Tartar's result (Proposition 3.4) can be deduced from Theorem 3.5: We take $G = \mathbb{R}^K$,

$$\mathcal{T}u = \Big(\sum_{j=1}^{J}\sum_{k=1}^{K} A_{ijk}\nabla_k u^j\Big)_{i=1}^{I} \qquad \text{with constants } A_{ijk} \in \mathbb{R},$$

and define the deformation retraction $\Phi : \mathbb{R}^K \setminus B_1 \to \overline{B_1} \setminus B_{\sqrt{2}/2}$ by

$$\Phi(\xi) = \frac{\xi}{\sqrt{1+|\xi|^2}}.$$

Then, the Fourier multiplier of $\mathcal{T}$ is $m(\xi) = \sum_{k=1}^{K}(-2\pi i)A_{ijk}\xi^k$, while the cone of $\mathcal{T}$ is

$$\Lambda_{\mathcal{T}} := \Big\{\lambda \in \mathbb{R}^J : \sum_{j=1}^{J}\sum_{k=1}^{K} A_{ijk}\xi^k\lambda^j = 0 \text{ for some } \xi \in \mathbb{R}^k \setminus \{0\}\Big\}.$$

Moreover,

$$\Big\|\Phi^* m(\xi)\hat{u}^\epsilon(\xi)\mathbb{1}_{\mathbb{R}^K\setminus B_1}\Big\|_{L^2(\mathbb{R}^K)} = \int_{|\xi|\geq 1} \frac{(-2\pi i)\sum_{j,k} A_{ijk}\xi^k(\hat{u}^\epsilon)^j(\xi)}{\sqrt{1+|\xi|^2}}\, \mathrm{d}\xi = \|Tu^\epsilon\|_{H^{-1}(\mathbb{R}^K\setminus B_1)},$$

thanks to the Fourier characterisation of H^s spaces. By cutting off u with a test function of compact support in G, Proposition 3.4 is immediately recovered.

Next, let us relax the first-order differential constraints in Proposition 3.4: Consider P, a degree-d homogeneous polynomial in $x_1, \ldots, x_K$, $\nabla = (\nabla_1, \ldots, \nabla_K)$ the gradient on $\mathbb{C}^K$, and $P(\nabla)$ the corresponding order-d differential operator. Let $u : \mathbb{C}^K \to \mathbb{C}^J$ be any L^2_{loc} map, and let

$$\mathcal{T}u(x) = A \cdot [P(\nabla)](u), \qquad A \in \mathrm{Mat}(I \times J; \mathbb{C}) \text{ is a constant matrix.}$$

Then, the retraction $\Phi : \mathbb{C}^K \setminus B_1 \to \overline{B_1} \setminus B_{\sqrt{2}/2}$ given by

$$\Phi(\xi) := \frac{P(-2\pi i\xi)}{(1+|\xi|^2)^{d/2}}$$

verifies the assumption (C2). So, by a standard cut-off argument, we obtain:

Corollary 3.3.1 *Let $\{u^\epsilon\} \subset L^2_{\mathrm{loc}}(\mathbb{C}^K;\mathbb{C}^J)$ be a family of functions converging weakly to u in L^2_{loc}. Let $P = \sum_{i_1+\ldots+i_K=d}\Big(\alpha_{i_1,\ldots,i_K} x_1^{i_1}x_2^{i_2}\cdots x_K^{i_K}\Big)$ be a homogeneous polynomial of degree d, and $\mathcal{T}u = A\cdot[P(\nabla)](u)$ for some constant matrix $A \in \mathrm{Mat}(I\times J;\mathbb{C})$, namely:*

$$\mathcal{T}u(x) := \sum_{j=1}^{J}\Big\{\sum_{i_1+\ldots+i_K=d} A_{i;i_1,\ldots,i_K;j}\nabla_1^{i_1}\nabla_2^{i_2}\cdots\nabla_K^{i_K}u_j(x_1,\ldots,x_K)\Big\}$$

for $i_1,\ldots,i_K \in \mathbb{Z}_+$, constants $A_{i;i_1,\ldots,l_K;j}\in\mathbb{C}$, and $i = 1,2,\ldots,I$.

Suppose that $\{\mathcal{T}u^\epsilon\}$ is pre-compact in $H^{-d}_{\mathrm{loc}}(\mathbb{C}^K;\mathbb{C}^L)$, and assume that the quadratic polynomial $Q:\mathbb{C}^J\to\mathbb{C}$ is vanishing on the cone $\Lambda_{\mathcal{T}}$, defined as follows:

$$\Lambda_{\mathcal{T}} := \Big\{\lambda\in\mathbb{C}^J : \sum_{j=1}^{J}\sum_{i_1+\ldots+i_K=d} A_{i;i_1,\ldots,i_K;j}(\xi_1)^{i_1}\cdots(\xi_K)^{i_K}\lambda^j = 0$$
$$\text{for some } \xi=(\xi_1,\ldots,\xi_K)\neq 0 \text{ and each } i=1,2,\ldots,I\Big\}.$$

Then $\mathrm{Re}(Q\circ u^\epsilon)$ converges to $\mathrm{Re}(Q\circ u)$ on $\mathbb{C}^K$ in the sense of distributions.

Still more generally, the differential operator $\mathcal{T}$ does not need to be homogeneous, nor must it have integral orders: Suppose that $\mathcal{T}$ is a differential operator of order $m\in\mathbb{R}_+$, let us denote all the components of $\mathcal{T}$ with order below m by $\mathcal{T}_{\mathrm{low}}$. Then, for $\{u^\epsilon\}$ bounded in $L^2_{\mathrm{loc}}(\mathbb{C}^K;\mathbb{C}^J)$, the Rellich's lemma ($H^s$ embeds into H^r compactly if $s>r$) shows that

$$\{\mathcal{T}_{\mathrm{low}}(u^\epsilon)\} \qquad \text{is pre-compact in } H^{-m}_{\mathrm{loc}}.$$

So it suffices to only consider $(\mathcal{T}-\mathcal{T}_{\mathrm{low}})$, which is again an m-homogeneous differential operator. Therefore, we arrive at the following

Corollary 3.3.2 *Let $\{u^\epsilon\}\subset L^2(\mathbb{C}^K;\mathbb{C}^J)$ be a sequence of functions converging weakly to u in L^2. Let $B(x) = \sum_{\beta_1+\ldots+\beta_K\le m}\Big(\alpha_{\beta_1,\ldots,\beta_K}x_1^{\beta_1}x_2^{\beta_2}\cdots x_K^{\beta_K}\Big)$ be a polynomial of order m, and $\mathcal{T}u = A\cdot[B(\nabla)](u)$ for some constant matrix $A\in\mathrm{Mat}(I\times J;\mathbb{C})$, namely:*

$$\mathcal{T}u(x) := \sum_{j=1}^{J}\Big\{\sum_{\beta_1+\ldots+\beta_K\le m} A_{i;\beta_1,\ldots,\beta_K;j}\nabla_1^{\beta_1}\nabla_2^{\beta_2}\cdots\nabla_K^{\beta_K}u_j(x_1,\ldots,x_K)\Big\}$$

for $\beta_1,\ldots,\beta_K\in\mathbb{R}_+$, constants $A_{i;\beta_1,\ldots,\beta_K;j}\in\mathbb{C}$, and $i=1,2,\ldots,I$. Suppose that $\{\mathcal{T}u^\epsilon\}$ is pre-compact in $H^{-m}_{\mathrm{loc}}(\mathbb{C}^K;\mathbb{C}^I)$, and assume that the quadratic polynomial $Q:\mathbb{C}^J\to\mathbb{C}$ is vanishing on the cone

$$\Lambda_{\mathcal{T}} := \Big\{\lambda\in\mathbb{C}^J : \sum_{j=1}^{J}\sum_{\beta_1+\ldots+\beta_K=m} A_{i;\beta_1,\ldots,\beta_K;j}(\xi_1)^{\beta_1}\cdots(\xi_K)^{\beta_K}\lambda^j = 0$$
$$\text{for some } \xi=(\xi_1,\ldots,\xi_K)\neq 0 \text{ and each } i=1,2,\ldots,I\Big\}.$$

Then, $\mathrm{Re}(Q\circ u^\epsilon)$ converges to $\mathrm{Re}(Q\circ u)$ on $\mathbb{C}^K$ in the sense of distributions.

3.3.2 Intrinsic quadratic theorem via micro-local analysis

Our major generalised quadratic theorem concerns the weakly convergent L^2 sections of a vector bundle E over a semi-Riemannian manifold M. In order to formulate it globally and intrinsically, two difficulties immediately arise:

1. Being endowed with a semi-Riemannian metric, M is a *real* manifold. However, our theorem has to be formulated over the complex $\mathbb{C}$, since the proof is based on Fourier analysis, which brings about the factor $i = \sqrt{-1}$.

2. More importantly, we notice that the conditions (C2)–(C3) in quadratic theorem on LCA groups, *i.e.*, Theorem 3.5 are proposed in terms of Fourier multiplier operators. However, the Fourier transform cannot be defined globally on a generic semi-Riemannian manifold, not to mention the Fourier multiplier operators. In fact, for $u \in L^2(M; E)$, the best one can do is to define
$$\hat{u}(x, \xi) := \int_M u(y) e^{-2\pi i \langle \exp_x^{-1}(y), \xi\rangle} \, \mathrm{d}V_g(y) \qquad \text{for } x \in M, \xi \in T_x^* M, \tag{3.79}$$
where $\exp_x : T_x M \to M$ is the exponential map on the manifold, $\langle \cdot, \cdot \rangle$ is the paring of TM and T^*M given by the metric g, and $\mathrm{d}V_g$ is the volume form of g. Indeed, it is only well-defined at $x \in M$ *up to the first conjugation point of* x, for which $\exp_x^{-1}$ can be specified unambiguously. However, in non-linear analysis, the compensated compactness theorems are known to be invariant under localisation.

These considerations call for a quadratic theorem on real manifolds, for which the differential constraints (*i.e.*, the operator $\mathcal{T}$ and the cone $\Lambda_{\mathcal{T}}$ in (C2)–(C3) of Theorem 3.5) are formulated globally and intrinsically. For this purpose we introduce three new ingredients:

- The (principal) symbol of a differential operator;
- A quadratic polynomial defined globally on vector bundles;
- Complexifications of vector bundles and quadratic polynomials.

Principal Symbols. We only collect some basic facts here and refer to P. Albin [2] for details.

For an arbitrary differential operator $\mathcal{T}$ of order m that maps E-sections to F-sections, namely $\mathcal{T} : \Gamma(E) \to \Gamma(F)$, let us denote it by $\mathcal{T} \in \mathrm{Diff}^m(M; E, F)$. It is a crucial observation in micro-local analysis that $\sigma_m(\mathcal{T})$, the *principal symbol* of $\mathcal{T}$, can be defined intrinsically. Indeed, for any $\xi \in T^*_x M$, one may choose a function $f \in C^\infty(M)$ such that $d_x f = \xi$, and then set

$$\sigma_m(\mathcal{T})(x,\xi) := \lim_{t\to\infty} \frac{[e^{-2\pi i t f} \circ \mathcal{T} \circ e^{2\pi i t f}](x)}{t^m}. \tag{3.80}$$

It is easy to check that $\sigma_m(\mathcal{T})(x,\xi) \in \mathrm{Hom}(E_x, F_x)$ for the given ξ, and that the definition is independent of the choice of f. Here and throughout $E_x \cong \mathbb{R}^J$, $F_x \cong \mathbb{R}^I$ denote the fibre of E and F at the point $x \in M$, respectively, and $\mathrm{Hom}(E_x, F_x)$ denotes the space of vector space homomorphisms from E_x to F_x. Moreover, σ_m is a homogeneous polynomial of order m on each fibre of T^*M, *i.e.*,

$$\sigma_m(\mathcal{T})(x,\lambda\xi) = |\lambda|^m \sigma_m(\mathcal{T})(x,\xi) \qquad \text{for all } x \in M, \xi \in T^*_x M \text{ and } \lambda \in \mathbb{C}. \tag{3.81}$$

More abstractly, denoting $\mathcal{P}_l(V,W)$ as the vector space of degree-l homogeneous polynomials between the vector bundles V and W, the principal symbol map σ_m defines the following vector space homomorphism:

$$\sigma_m : \mathrm{Diff}^m(M;E,F) \to \mathcal{P}_m(T^*M; \mathrm{Hom}(E;F^{\mathbb{C}})). \tag{3.82}$$

Here $F^{\mathbb{C}} := F \bigotimes_{\mathbb{R}} \mathbb{C}$ is the complexified vector bundle, which is necessary since $i = \sqrt{-1}$ appears in the definition of σ_m in Eq. (3.80). We adopt this abstract language in order to emphasise the global, intrinsic nature of the principal symbol.

For the application in §5 let us discuss a key example, that is, the exterior differential operator $\mathcal{T} = d : \bigwedge^q T^*M \to \bigwedge^{q+1} T^*M$. In fact, we have

$$d \in \mathrm{Diff}^1(M; \bigwedge^q T^*M; \bigwedge^{q+1} T^*M),$$

whose principal symbol $\sigma_1(d)$ is given by

$$[\sigma_1(d)(\xi)](\omega) = -2\pi i \xi \wedge \omega \qquad \text{for } \xi \in T^*M,\ \omega \in \bigwedge^q T^*M.$$

Due to the presence of $i = \sqrt{-1}$, we view the exterior algebra in the range of d as being complexified, *i.e.*, for each $\xi \in T^*M$ we have $\sigma_1(d)(\xi) \in \mathcal{P}_1(\bigwedge^q T^*M; \bigwedge^{q+1} T^*M \bigotimes \mathbb{C})$. In this case, notice that $\sigma_1(d)(\xi) = [-2\pi i \xi \wedge]$, which is indeed a 1-homogeneous polynomial of operators from q-tensors to complexfied $(q+1)$-tensors.

Intrinsic Formulation of Quadratic Polynomials. Now we need to explain what it means by a quadratic polynomial on a vector bundle E:

Definition 3.3.1 *Let E be a vector bundle over a real manifold M. A map $Q : \Gamma(E) \to \mathbb{C}$ is a quadratic polynomial on E if it factors as follows:*

$$Q : \Gamma(E) \xrightarrow{j} \Gamma(E \bigotimes E) \xrightarrow{\mathbf{q}} \mathbb{C},$$

where $j(s) = (s, s)$ is the natural inclusion of the diagonal, and $\mathbf{q} \in \Gamma(\mathrm{Hom}(E \otimes E; \mathbb{C}))$ is conjugate 1-homogeneous in each argument, i.e.,

$$\mathbf{q}(\lambda s_1, s_2) = \lambda \mathbf{q}(s_1, s_2) \ \text{ and } \ \mathbf{q}(s_1, \mu s_2) = \overline{\mu}\mathbf{q}(s_1, s_2) \text{ for all } s_1, s_2 \in \Gamma(E) \text{ and } \lambda, \mu \in \mathbb{C}.$$

In this case, we write $Q \in \mathcal{P}_2(E; \mathbb{C})$.

We notice that such constructions remain valid for $\mathbb{C}$ replaced by $\mathbb{R}$. In this case, say Q is a real quadratic polynomial on E. It follows from the definition that any quadratic polynomial Q is 2-homogeneous, namely

$$Q(\lambda s) = |\lambda|^2 Q(s) \text{ for all } s \in \Gamma(E) \text{ and } \lambda \in \mathbb{C}. \tag{3.83}$$

Moreover, suppose that $U \subset M$ is a trivialised chart for the vector bundle E of degree J, *i.e.*, there exists a diffeomorphism $\Phi : E \supset \pi^{-1}(U) \xrightarrow{\sim} U \times \mathbb{C}^J$. Then, for $s = \Phi^{-1}(x, z) \in U \times \mathbb{C}^J$, the value of the quadratic polynomial Q at s is given by

$$Q(s) = \sum_{j,k=1}^{J} Q_{jk}(x) z^j \overline{z^k} \qquad \text{with } Q_{jk} \in C^\infty(U). \tag{3.84}$$

Thus the local representation of Q is obtained. It is compatible with the characterisation of quadratic polynomials on $\mathbb{C}^J$ in Eq. (3.63), which is globally trivialised.

Sobolev Norms over Semi-Riemannian Manifolds. The Sobolev norms can be defined globally an intrinsically on a (connected) semi-Riemannian manifold M. Let (E, g^E) be a vector bundle over (M, g), and consider $1 \leq p < \infty$, $k \in \mathbb{Z}$. For each section $\alpha : M \to E$ we wish to define $\|\alpha\|_{W^{k,p}(M,g;E,g^E)}$, which is abbreviated as $\|\alpha\|_{W^{k,p}(M;E)}$ in the sequel.

1. First, let us integrate a function $f : (M, g) \to \mathbb{R}$. This is the same as integrating on Riemannian manifolds, except that we take the absolute value of $\det g$. More precisely, consider an atlas $\{U_\alpha\}_{\alpha \in \mathcal{I}}$ of coordinate charts on M, and the diffeomorphisms $\{\phi_\alpha\}_{\alpha \in \mathcal{I}}$ mapping each U_α to $\mathbb{R}^n_\nu$, where $\nu = \mathrm{Ind}(M)$. Moreover, let $\{\rho_\alpha\}_{\alpha \in \mathcal{I}}$ be a partition-of-unity subordinate to the atlas. Then

$$\int_M f \, \mathrm{d}V_g := \sum_{\alpha \in \mathcal{I}} \int_{\mathbb{R}^n} \left\{ \rho_\alpha(x) \big[(\phi_\alpha^{-1})^* f(x) \big] \mathbb{1}_{\phi_\alpha(U_\alpha)}(x) \sqrt{\big| \det g(x) \big|} \right\} \mathrm{d}x^1 \ldots \mathrm{d}x^n. \tag{3.85}$$

Moreover, let $\epsilon = (\epsilon^1, \ldots, \epsilon^n)^\top$ be the signature of g, and we take

$$\tilde{\epsilon} := \mathrm{diag}(\epsilon^1, \ldots, \epsilon^n) \tag{3.86}$$

as the diagonal matrix associated to ϵ. Then $|\det g| = \det(\tilde{\epsilon}g)$, where $(\tilde{\epsilon}g)$ is understood as the matrix multiplication. Hence,

$$\int_M f\,\mathrm{d}V_g := \sum_{\alpha\in\mathcal{I}} \int_{\mathbb{R}^n} \left\{\rho_\alpha(x)\big[(\phi_\alpha^{-1})^* f(x)\big]\mathbb{1}_{\phi_\alpha(U_\alpha)}(x)\sqrt{\det(\tilde{\epsilon}g)(x)}\right\} \mathrm{d}x^1\ldots\mathrm{d}x^n. \tag{3.87}$$

2. Next, we define $\|\alpha\|_{L^p(M,g;E,g^E)} \equiv \|\alpha\|_{L^p(M;E)}$. As above we write $\widetilde{\epsilon^E}$ as the diagonal matrix associated to the signature of g^E, the bundle metric. Then we define:

$$\|\alpha\|_{L^p(M;E)} := \left\{\int_M \left|g^E(\alpha,\alpha)\right|^{\frac{p}{2}}\mathrm{d}V_g\right\}^{\frac{1}{p}} = \left\{\int_M \left(\widetilde{\epsilon^E} g^E(\alpha,\alpha)\right)^{\frac{p}{2}}\mathrm{d}V_g\right\}^{\frac{1}{p}}. \tag{3.88}$$

This formula is well-defined: We integrate the function $f = |g^E(\alpha,\alpha)|$ over the manifold M as in the first item above.

3. Finally, let ∇^E be an affine connection on the bundle E, which is equivalent to a covariant derivative on E. Then we set:

$$\|\alpha\|_{W^{k,p}(M;E)} := \left\{\sum_{j=0}^k \left(\Big\|\underbrace{\nabla^E\circ\nabla^E\circ\ldots\circ\nabla^E}_{j\text{ times}}\alpha\Big\|_{L^p(M;E)}\right)^p\right\}^{\frac{1}{p}}. \tag{3.89}$$

In this way we have defined the Sobolev norms on sections of vector bundles over semi-Riemannian manifolds. This definition is *intrinsic*, as it is independent of the choice of atlas $\{U_\alpha\}$ and the partition-of-unity. As usual, the corresponding Sobolev spaces are defined as the completion of smooth sections with respect to the $W^{k,p}$ norms:

$$W^{k,p}(M;E) := \overline{C^\infty(M;E)}^{\|\cdot\|_{W^{k,p}(M;E)}}, \tag{3.90}$$

and α is in the Frechét space $W^{k,p}_{\mathrm{loc}}(M;E)$ if and only if there exists a compact subset of M, such that the restriction of α on this set is in $W^{k,p}(M;E)$.

Our main result of this subsection is Theorem 1.6 in §1.2.1:

Theorem 3.6 (Theorem 1.6 in §1.2.1) *Let M be a semi-Riemannian real manifold with the metric g and volume form $\mathrm{d}V_g$. Also, let E, F be two real vector bundles over M. We consider a sequence of E-sections $\{u^\epsilon\} \subset L^2_{\mathrm{loc}}(M;E)$, a differential operator $\mathcal{T} \in \mathrm{Diff}^m(M;E,F)$ for some $m \in \mathbb{R}_+$ with the principal symbol*

$\sigma_m(\mathcal{T}) : T^*M \to \mathrm{Hom}(E;F^{\mathbb{C}})$, and a quadratic polynomial $Q : \Gamma(E) \to \mathbb{R}$. Assume that the following conditions hold:

(C1') *$u^\epsilon \rightharpoonup u$ weakly in $L^2_{\rm loc}(M;E)$;*

(C2') *$\{\mathcal{T}u^\epsilon\}$ is pre-compact in $H^{-m}_{\rm loc}(M;F)$;*

(C3') *$Q(s) = 0$ for all $s \in \Lambda_{\mathcal{T}}$, where the* cone *of $\mathcal{T}$ is defined by*

$$\Lambda_{\mathcal{T}} := \big\{ s \in \Gamma(E) : \sigma_m(\mathcal{T})(\xi)(s) = 0 \in \Gamma(F^{\mathbb{C}}) \text{ for some } \xi \in T^*M \backslash \{0\} \big\}. \tag{3.91}$$

Then we have

$$\lim_{\epsilon \to 0} \int_M (Q \circ u^\epsilon)\psi \, \mathrm{d}V_g = \int_M (Q \circ u)\psi \, \mathrm{d}V_g, \qquad \text{for any } \psi \in C^\infty_c(M). \tag{3.92}$$

Before presenting the proof, let us make a few comments on Theorem 1.6:

1. Theorem 1.6 is formulated globally and intrinsically on the manifold M, as the symbol σ_m, the cone $\Lambda_{\mathcal{T}}$, as well as the Sobolev spaces $H^\bullet$ of sections, are all defined without referring to local coordinates. In addition, we remark that σ_m is defined only using the differentiable structure of M, but not the Riemannian or semi-Riemannian structure. Therefore, the cone $\Lambda_{\mathcal{T}}$ in (C3') depends only on the *algebraic* properties of $\mathcal{T}$.

2. In the theorem we denote the target space of the symbol $\sigma_m(\mathcal{T})$ by $\mathrm{Hom}(E;F^{\mathbb{C}})$, which is understood as the vector bundle of $\mathbb{R}$-bundle homomorphisms from E to the *complexification* of F, *i.e.*, $F^{\mathbb{C}} := F \otimes \mathbb{C}$. One also commonly writes

$$\sigma_m(\mathcal{T}) \in \Gamma\big(TM \bigotimes \mathrm{Hom}(E;F^{\mathbb{C}})\big). \tag{3.93}$$

3. The following lemma concerns the *naturality* of the principal symbol under the action of diffeomorphism group. It is crucial for the proof of Theorem 1.6:

Lemma 3.3.1 *Let $\mathcal{O}, \tilde{\mathcal{O}}$ be open subsets of $\mathbb{R}^n$, and let $F : \mathcal{O} \to \tilde{\mathcal{O}}$ be a diffeomorphism. Then, for $P \in \mathrm{Diff}^m(\mathcal{O})$, we have $F_*P \in \mathrm{Diff}^m(\tilde{\mathcal{O}})$ too.*

Moreover, the principal symbols of P and F_*P, i.e., $\sigma_m(F_*P)$ and $\sigma_m(P)$ respectively, are related as follows:

$$\sigma_m(F_*P)\big(F(x),\xi\big) = \sigma_m(P)\big(x,[d_xF]^\top(\xi)\big) \qquad \text{for each } x \in \mathcal{O} \text{ and } \xi \in (T_x\mathbb{R}^n)^*. \tag{3.94}$$

Here, F_*P denotes the pushforward *of P under F:*

$$(F_*P)(\varphi) = P(\varphi \circ F) \circ F^{-1} \qquad \text{for all } \varphi \in C^\infty_c(\tilde{\mathcal{O}}). \tag{3.95}$$

In fact, Lemma 3.3.1 is a special case of Theorem 20 in the notes by P. Albin ([2]): the first assertion holds for general pseudo-differential operators, and the second assertion holds for pseudo-differential operators with classical total symbols.

The strategy of the proof is as follows: First of all, making use of the partition-of-unity, together with the commutator estimate of $\mathcal{T} \in \mathrm{Diff}^m(M; E, F)$ and a multiplication operator, we reduce the theorem to a local problem on one single chart of the manifold. Next, thanks to Lemma 3.3.1, we can flatten the local chart to $\mathbb{R}^n$; it is an LCA group, so our hope is to apply Theorem 3.5. This cannot be achieved directly due to the non-trivial semi-Riemannian metrics on the manifold and the bundles. Nevertheless, exploiting the quadratic structure of Q, the *signature* of the semi-Riemannian metrics do not affect the proof. Therefore, locally we may regard the metrics as "close" to the Euclidean metrics, and the arguments follow from that for Theorem 3.5, since Euclidean spaces are LCA groups.

Proof of Theorem 1.6. We divide the arguments into eight steps.

1. At the beginning of the proof we justify the following two reductions: First, it suffices to prove for $u = 0$. The argument is the same as in Step 1 of the proof of Theorem 3.5. Second, we can *localise* the statement to each chart of the differentiable manifold M. To fix the notations, let $\{U_\alpha\}_{\alpha\in\mathcal{I}}$ be an atlas of the differentiable manifold M; we *claim* that it suffices to prove Theorem 1.6 for the sequences $\{u^\epsilon\}$ supported on one single U_α.

For this purpose, let us take any $\psi \in C_c^\infty(M)$ and consider the following identity:

$$\mathcal{T}(\psi u^\epsilon) = \psi \mathcal{T} u^\epsilon + [\mathcal{T}, \psi] u^\epsilon. \tag{3.96}$$

Here $[\mathcal{T}, \psi]$ denotes the commutator of $\mathcal{T}$ and the operator of multiplication by ψ.

Clearly, $[\mathcal{T}, \psi]$ is a differential operator of order not exceeding $(m-1)$. As $\{u^\epsilon\}$ is pre-compact (hence uniformly bounded) in L^2_{loc}, we know that $\{[\mathcal{T}, \psi]u^\epsilon\}$ is uniformly bounded in H^{-m+1}_{loc}, which is compactly embedded in H^{-m}_{loc} by the Rellich lemma. Moreover, by the condition (C2'), $\{\psi \mathcal{T} u^\epsilon\}$ is also pre-compact in H^{-m}_{loc}. Hence the same holds for $\{\mathcal{T}(\psi u^\epsilon)\}$. In addition, the transition function $\varphi_{\alpha,\beta}$ between any two overlapping charts U_α and U_β is a diffeomorphism, so $\mathcal{T}|_{U^\alpha}$ and $\mathcal{T}|_{U^\beta}$ both have the principal symbols of order m, which are m-homogeneous polynomials in the fibre of the cotangent bundle T^*M. Indeed, they differ only by a multiplicative factor controlled by the Lipschitz norm of $\varphi_{\alpha,\beta}$, which is bounded uniformly on M for all

$\alpha, \beta \in \mathcal{I}$. Up to now, we justified that the assumptions of the theorem are invariant under the operation $u^\epsilon \mapsto \psi u^\epsilon$, where $\psi \in C_c^\infty(M)$ is an arbitrary test function.

It remains to establish the "local-to-global" result, *i.e.*, if the assertion holds for $\{u^\epsilon\}$ supported in each chart, then it also holds for arbitrary $\{u^\epsilon\}$. To this end we let $\{\phi_\alpha\}_{\alpha\in\mathcal{I}}$ be a partition-of-unity subordinate to the atlas $\{U_\alpha\}_{\alpha\in\mathcal{I}}$, *i.e.*, $0 \leq \phi_\alpha \leq 1$ and $\phi_\alpha \in C_c^\infty(U_\alpha)$ for each $\alpha \in \mathcal{I}$, as well as $\sum_{\alpha\in\mathcal{I}} \phi_\alpha = 1$ on M. Moreover, we can find $0 \leq \psi_\alpha \leq 1$ and $\psi_\alpha \in C_c^\infty(U_\alpha)$, such that $\phi_\alpha = \psi_\alpha^2$ for each $\alpha \in \mathcal{I}$.

To proceed, suppose that Theorem 1.6 is proved for the sequence $\{\psi_\alpha u^\epsilon\} \subset L^2(U_\alpha; E)$ for each $\alpha \in \mathcal{I}$, with $\psi_\alpha u^\epsilon \rightharpoonup \psi_\alpha u$ in $L^2(U_\alpha; E)$ along some subsequence $\{\psi_\alpha u^{\epsilon_i}\}_{i\in\mathcal{I}_1\subset\mathcal{I}}$. Then, for a neighbouring chart U_β, *i.e.*, $U_\alpha \cap U_\beta \neq \emptyset$, we can select a further subsequence $\{\psi_\beta u^{\epsilon_j}\}_{j\in\mathcal{I}_2} \subset \{\psi_\alpha u^{\epsilon_i}\}_{i\in\mathcal{I}_1}$ such that $\mathcal{I}_2 \subset \mathcal{I}_1$ and $\{\psi_\beta u_\beta^\epsilon\}$ converges weakly in $L^2(U_\beta; E)$ to some $\psi_\beta \tilde{u}$. However, due to the uniqueness of subsequential weak limits, we have

$$u = \tilde{u} \qquad \text{on } U_\alpha \cap U_\beta.$$

Hence, we can write $\psi_\beta \tilde{u}$ as $\psi_\beta u$ without ambiguity, according to the following interpretation: the function u, previously defined only on U_α, is now extended to the domain $U_\alpha \cup U_\beta$.

Now, as M is second-countable (which is part of the definition of differentiable manifolds), we can take the indexing set $\mathcal{I}$ for the atlas to be at most countable. Thus, performing a diagonalisation process to the arguments in the preceding paragraph, we obtain a subsequence $\{u^\epsilon\}$ (not relabelled) and a function $u \in L^2_{\rm loc}(M; E)$ defined on the manifold M, such that

$$\psi_\alpha u^\epsilon \rightharpoonup \psi_\alpha u \qquad \text{for each } \alpha \in \mathcal{I}.$$

Therefore, for any test function $\psi \in C_c^\infty(M)$ we can pass to the limits:

$$\begin{aligned}
\lim_{\epsilon\to 0} \int_M (Q \circ u^\epsilon)(x)\psi(x)\,\mathrm{d}V_g(x) &= \sum_{\alpha\in\mathcal{I}} \lim_{\epsilon\to 0} \int_M \psi_\alpha^2(x)\,(Q \circ u^\epsilon)(x)\psi(x)\,\mathrm{d}V_g(x) \\
&= \sum_{\alpha\in\mathcal{I}} \lim_{\epsilon\to 0} \int_M Q\big(\psi_\alpha(x)u^\epsilon(x)\big)\psi(x)\,\mathrm{d}V_g(x) \\
&= \sum_{\alpha\in\mathcal{I}} \int_M Q\big(\psi_\alpha(x)u(x)\big)\psi(x)\,\mathrm{d}V_g(x) \\
&= \sum_{\alpha\in\mathcal{I}} \int_M \psi_\alpha^2(x)\,(Q \circ u)(x)\psi(x)\,\mathrm{d}V_g(x) \\
&= \int_M (Q \circ u)(x)\psi(x)\,\mathrm{d}V_g(x).
\end{aligned} \tag{3.97}$$

In the first and the last lines we use $\sum_{\alpha\in\mathcal{I}}\psi_\alpha^2 = 1$ on M, while in the second and the fourth lines we use the quadratic structure of Q. Moreover, the order of summation over $\alpha \in \mathcal{I}$ can be interchanged with the limit and the integration, because the partition-of-unity is locally finite. Thus, the localisation argument is complete due to Eq. (3.97).

2. From now on, $\{u^\epsilon\}$ is assumed to be supported on a single chart $U_\alpha \subset M$. In this step let us "*flatten the chart*" by transforming U_α to $\mathbb{R}^n$ via the coordinate map. First, we observe that there is no loss of generality to assume that the vector bundles E, F are trivialised on U_α: one simply takes refinements of the atlas $\{U_\alpha\}_{\alpha\in\mathcal{I}}$ if necessary. Now, by basic manifold theory, there exists a diffeomorphism $F_\alpha : U_\alpha \xrightarrow{\sim} \mathbb{R}^n$ so that

$$(F_\alpha)_*\mathcal{T} \in \mathrm{Diff}^m(\mathbb{R}^n; \mathbb{R}^n \times \mathbb{R}^J, \mathbb{R}^n \times \mathbb{R}^I). \tag{3.98}$$

We always assume that E, F have typical fibres $\mathbb{R}^J$ and $\mathbb{R}^I$, respectively. Moreover, Lemma 3.3.1 implies that

$$\sigma_m\big((F_\alpha)_*\mathcal{T}\big)\big(F_\alpha(x),\zeta\big) = \sigma_m(\mathcal{T})\big(x, [d_xF_\alpha]^\top(\zeta)\big) \tag{3.99}$$

for all $x \in U_\alpha$ and $\zeta \in \mathbb{R}^J$. Noticing that ζ and $[d_xF_\alpha]^\top(\zeta)$ are simultaneously non-vanishing in Eq. (3.99), since F_α is a diffeomorphism, one concludes

$$\Lambda_{\mathcal{T}} = \Lambda_{(F_\alpha)_*\mathcal{T}}, \tag{3.100}$$

i.e., the cones of $\mathcal{T}$ and $(F_\alpha)_*\mathcal{T}$ coincide.

Therefore, it suffices to prove the theorem with $\{dF_\alpha(\psi_\alpha u^\epsilon)\}$ and $(F_\alpha)_*\mathcal{T}$ in place of $\{u^\epsilon\}$ and $\mathcal{T}$, respectively. Here $\{\psi_\alpha : \alpha \in \mathcal{I}\}$ is the partition-of-unity subordinate to the atlas $\{U_\alpha : \alpha \in \mathcal{I}\}$ as in Step 1. In addition, by the paracompactenss of topological manifolds one can assume ψ_α to be supported in a compact subset of U_α for each $\alpha \in \mathcal{I}$. Thus, in the sequel we take $dF_\alpha(\psi_\alpha u^\epsilon)$ to be compactly supported in $\mathbb{R}^n$, and identity it with the map on the whole of $\mathbb{R}^n$, obtained via extension-by-zero. To simplify the notations, we still label $\{dF_\alpha(\psi_\alpha u^\epsilon)\}$ as $\{u^\epsilon\}$. Thus we reduce to the case $M = \mathbb{R}^n$, which is an LCA group as in Theorem 3.5.

3. Thanks to the localisation and flattening arguments in Steps 1–2, from now on we assume $\{u^\epsilon\} \subset L^2_c(\mathbb{R}^n, \mathbb{R}^n \times \mathbb{R}^J)$ and $\mathcal{T} \in \mathrm{Diff}^m(\mathbb{R}^n, \mathbb{R}^n \times \mathbb{R}^J; \mathbb{R}^n \times \mathbb{R}^I)$. To simplify the notations, let us still write $E = \mathbb{R}^n \times \mathbb{R}^J$, $F = \mathbb{R}^n \times \mathbb{R}^I$ and denote the metric on $\mathbb{R}^n$ by g with an abuse of notations, *i.e.*, assuming that $M = \mathbb{R}^n$, and that the bundles E, F are globally trivialised.

To begin with, recall that the L^p norm of $u : \mathbb{R}^n \to E$ is defined as

$$\begin{aligned} \|u\|_{L^p(\mathbb{R}^n;E)} &:= \Big(\int_{\mathbb{R}^n} |u|^2_{g^E} \,\mathrm{d}V_g \Big)^{\frac{1}{p}} \\ &= \Big(\int_{\mathbb{R}^n} \Big\{ \sum_{j=1}^{J} \sum_{k=1}^{J} \epsilon_k g^E_{jk}(x) u^j(x) u^k(x) \Big\}^{\frac{p}{2}} \sqrt{|\det\, g(x)|}\,\mathrm{d}x \Big)^{\frac{1}{p}}. \end{aligned} \tag{3.101}$$

Here, g^E is the bundle metric on E, the indices $1 \leq j,k \leq J$ are for the fibre of E, and $\epsilon_k \in \{\pm 1\}$ is the *signature* of the k-th component of g^E, such that $h_{jk} := \epsilon_k g^E_{jk}$ becomes positive definite. We choose a coordinate system in which g^E is diagonalised:

$$\begin{aligned} &g^E = \operatorname{diag}(\lambda_1, \ldots, \lambda_\tau; \lambda_{\tau+1}, \ldots, \lambda_J), \\ &\text{where } \lambda_j < 0 \text{ for } 1 \leq j \leq \tau, \text{ and } \lambda_j > 0 \text{ for } \tau+1 \leq j \leq J. \end{aligned}$$

Correspondingly, we have $\epsilon_1 = \cdots = \epsilon_\tau = -1$ and $\epsilon_{\tau+1} = \cdots = \epsilon_J = 1$, where τ is the index of g. In the sequel we always write $\{u^\mu\}$ in place of $\{u^\epsilon\}$, to avoid possible confusions between the superscript ϵ and the signature ϵ.

Now, define a new sequence of sections $\{v^\mu\} \subset L^2(\mathbb{R}^n; E)$ by components:

$$(v^\mu)^j := \sqrt{\epsilon_j \lambda_j}(u^\mu)^j |\det\, g|^{\frac{1}{4}} \qquad \text{for each } j = 1, 2, \ldots J. \tag{3.102}$$

Namely, we write $v^\mu = ((v^\mu)^1, \ldots, (v^\mu)^J)^\top$. By this definition v^μ depends on g^E, g and u^μ, and it verifies the following identity

$$\|v^\mu\|^2_{L^2(\mathbb{R}^n,\, g_0; E)} \equiv \sum_{j=1}^{J} \int_{\mathbb{R}^n} \{(v^\mu)^j(x)\}^2 \,\mathrm{d}x = \|u^\mu\|^2_{L^2(\mathbb{R}^n,\, g; E)} \qquad \text{for each } \mu, \tag{3.103}$$

where g_0 denotes the *Euclidean* metric on $\mathbb{R}^n$. So, by the condition (C1') we know that $\{v^\mu\}$ is uniformly bounded in L^2 with respect to g_0. Moreover, $\operatorname{supp}(v^\mu) \subset \operatorname{supp}(u^\mu)$ for each μ, so all the terms of $\{v^\mu\}$ are supported on a common compact set. At this stage, we can apply the arguments from Step 1 in the proof of Theorem 3.5 to the sequence $\{v^\mu\}$, which yields:

$$\lim_{\mu \to 0} \int_{|\xi| \leq K} |\hat{v}^\mu(\xi)|^2 \,\mathrm{d}\xi = 0. \tag{3.104}$$

Here $K > 0$ is a constant to be determined later.

As a remark, the norm on ξ is also taken with respect the Euclidean metric, as it is the metric induced by g_0 on the cotangent bundle T^*M.

4. Next we control the high-frequency region of $\{v^\mu\}$. For $j = 1, 2, \ldots, J$, set

$$\chi^j = \chi^j(g) := |\det\, g|^{\frac{1}{4}} \sqrt{\epsilon_j \lambda_j}; \tag{3.105}$$

then $(v^\mu)^j = \chi^j (u^\mu)^j$ for each j. Notice that $\chi^j > 0$ strictly, by the non-degeneracy of the metrics g, g^E. Writing $u^\mu = \sum_{j=1}^J (u^\mu)^j \partial_j$ and similarly for v^μ in local coordinates, by the linearity of the differential operator $\mathcal{T}$, we have

$$\begin{aligned}
\mathcal{T} v^\mu &= \mathcal{T}\Big\{ \sum_{j=1}^J \chi^j (u^\mu)^j \partial_j \Big\} \\
&= \sum_{j=1}^J \chi^j \cdot \mathcal{T}\big[(u^\mu)^j \partial_j\big] + \sum_{j=1}^J [\mathcal{T}, \chi^j](u^\mu)^j \partial_j =: \mathrm{I}^\mu + \mathrm{II}^\mu.
\end{aligned}$$

In II^μ, $[\mathcal{T}, \chi^j]$ is the commutator between $\mathcal{T}$ and the multiplication operator by χ^j.

We now argue that $\{\mathcal{T} v^\mu\}$ is pre-compact in $H^{-m}(\mathbb{R}^n, g_0; F)$. First of all, this sequence is compactly supported, by the construction of $\{v^\mu\}$ and the locality of the differential operator $\mathcal{T}$. So we neglect the subscript "*loc*" for the corresponding Sobolev spaces. By explicitly writing out g_0 in the subscript, we emphasise that $M = \mathbb{R}^n$ is equipped with the Euclidean metric. To this end, it shall be established that both $\{\mathrm{I}^\mu\}$ and $\{\mathrm{II}^\mu\}$ are pre-compact in $H^{-m}(\mathbb{R}^n, g_0; F)$.

For I^μ, first we compute as follows:

$$\begin{aligned}
\|\mathrm{I}^\mu\|_{H^{-m}(\mathbb{R}^n, g_0; F)} &\leq \Big(\sup_{1 \leq j \leq J} \|\chi^j - 1\|_{L^\infty(\mathbb{R}^n)} \Big) \|Tu^\mu\|_{H^{-m}(\mathbb{R}^n, g_0; F)} \\
&\leq \Big(1 + \|g^E\|^{1/2}_{L^\infty(E)} \| \det\, g\|^{1/4}_{L^\infty(M)} \Big) \|Tu^\mu\|_{H^{-m}(\mathbb{R}^n, g_0; F)}.
\end{aligned} \tag{3.106}$$

Next, we show that $\|Tu^\mu\|_{H^{-m}(\mathbb{R}^n, g_0; F)}$ can be related to $\|Tu^\mu\|_{H^{-m}(\mathbb{R}^n, g; F)}$, whose pre-compactness is assumed by the condition (C2'). For this purpose one needs to invoke the *Fourier characterisation of Sobolev norms* $\|\cdot\|_{H^{-m}(\mathbb{R}^n, g_0; F)}$ and $\|\cdot\|_{H^{-m}(\mathbb{R}^n, g; F)}$. Here, since in Steps 1–2 we already localised the sequence $\{u^\mu\}$ to a chart U_α of M on which E and F are trivialised, so $g|_{U_\alpha}$ has no self-intersecting geodesics, provided that U_α is contained a geodesic normal neighbourhood. This can be assumed by shrinking U_α if necessary. Then, the pushed-forward metric $(F_\alpha)_* g$ – which is still labelled as g from Step 2 onward – satisfies the same property on $\mathbb{R}^n = M$, hence $\|\cdot\|_{H^{-m}(\mathbb{R}^n, g; F)}$ can be defined globally via Fourier transforms unambiguously.

In this way we obtain the following:

$$\begin{aligned}
\|\mathcal{T} u^\mu\|_{H^{-m}(\mathbb{R}^n, g_0; F)} &= \int_{\mathbb{R}^n} \frac{|\widehat{\mathcal{T} u^\mu}(\xi)|^2_{g^F}}{(1 + |\xi|^2)^{m/2}} \, \mathrm{d}\xi \\
&= \int_{\mathbb{R}^n} \frac{|\widehat{\mathcal{T} u^\mu}(\xi)|^2_{g^F}}{(1 + |\xi|_g^2)^{m/2}} \sqrt{|\det\, g|} \Big\{ \frac{(1 + |\xi|_g^2)^{m/2}}{(1 + |\xi|^2)^{m/2}} \frac{1}{\sqrt{|\det\, g|}} \Big\} \, \mathrm{d}\xi \\
&\leq C \int_{\mathbb{R}^n} \frac{|\widehat{\mathcal{T} u^\mu}(\xi)|^2_{g^F}}{(1 + |\xi|_g^2)^{m/2}} \sqrt{|\det\, g|} \, \mathrm{d}\xi
\end{aligned}$$

$$=: C\|\mathcal{T}u^\mu\|_{H^{-m}(\mathbb{R}^n,g;F)}, \tag{3.107}$$

where C only depends on m, $\|g\|_{L^\infty}(M)$, and $\inf_M|\det g|$. This together with Eq. (3.106) gives us

$$\|\mathrm{I}^\mu\|_{H^{-m}(\mathbb{R}^n,g_0;F)} \leq C'\|\mathcal{T}u^\mu\|_{H^{-m}(\mathbb{R}^n,g;F)}, \tag{3.108}$$

where C' depends only g, g^E and m, but not on μ. In view of (C2'), so $\{\mathrm{I}^\mu\}$ is pre-compact in $H^{-m}(\mathbb{R}^n, g; F)$.

Now let us turn to $\{\mathrm{II}^\mu\}$. Since $\mathcal{T} \in \mathrm{Diff}^m(M; E, F)$ and χ^j is a multiplication operator, so $[\mathcal{T}, \chi^j] \in \mathrm{Diff}^r(M; E, F)$ for $r \leq m-1$. By the assumption (C1'), $\{u^\mu\}$ is bounded in $L^2(M, g; E)$, and hence $\{\mathrm{II}^\mu\}$ is pre-compact in $H^{-m}(\mathbb{R}^n, g; F)$ due to the Rellich Lemma. Again, by the estimates in Eq. (3.107), $\{\mathrm{II}^\mu\}$ is also pre-compact in $H^{-m}(\mathbb{R}^n, g; F)$. Therefore, $\{\mathcal{T}v^\mu\}$ is pre-compact in $H^{-m}(\mathbb{R}^n, g_0; F)$, which leads to

$$\mathcal{T}v^\mu \longrightarrow 0 \qquad \text{strongly in } H^{-m}(\mathbb{R}^n, g_0; F). \tag{3.109}$$

This is because $u^\mu \rightharpoonup 0$ in L^2; see Step 1 of the same proof. Here $\mathbb{R}^n$ is endowed with the Euclidean metric g_0, and F has the bundle metric g^F.

5. Now we estimate the Euclidean L^2 norm of $\widehat{\mathcal{T}v^\mu}$ on $\{|\xi| \geq 1\}$. Here, $\widehat{\mathcal{T}v^\mu}$ is the standard Fourier transform on Euclidean spaces, namely

$$\widehat{\mathcal{T}v^\mu}(\xi) := \int_{\mathbb{R}^n} \mathcal{T}v^\mu(x)e^{-2\pi i\xi\cdot x}\,\mathrm{d}x.$$

Indeed, since $\mathcal{T} \in \mathrm{Diff}^m(M; E, F)$, by localisation and flattening in Steps 1–2 we have:

$$\mathcal{T} = \sum_{|\alpha|\leq m} a_\alpha(x)\partial_x^\alpha,$$

and the principal symbol is given by

$$\sigma_m(\mathcal{T})(x,\xi) = (-2\pi i)^m \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha,$$

cf. §3 in [2]. Combining with the lower order terms, we get

$$\widehat{\mathcal{T}v^\mu}(\xi) = (-2\pi i)^m \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha\hat{v^\mu}(\xi) + \sum_{|\beta|\leq m-1} b_\beta(x,\xi)\xi^\beta\hat{v^\mu}(\xi), \tag{3.110}$$

where $|a_\alpha(x)|, |b_\beta(x,\xi)| \leq C_0$ for all $x \in M, \xi \in T_x^*M$ and each α, β. Then

$$\|\mathcal{T}v^\mu\|^2_{H^{-m}(\{|\xi|\geq K\})}$$

$$
\begin{aligned}
&:= \int_{|\xi|\geq K} \frac{\left|(-2\pi i)^m \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha \hat{v^\mu}(\xi) + \sum_{|\beta|\leq m-1} b_\beta(x,\xi)\xi^\beta \hat{v^\mu}(\xi)\right|^2}{(1+|\xi|^2)^m}\,\mathrm{d}\xi \\
&\geq C_1 \Big\{ \sum_{|\alpha|=m} \int_{|\xi|\geq K} \frac{|a_\alpha(x)|^2|\xi|^{2m}}{(1+|\xi|^2)^m} |\hat{v^\mu}(\xi)|^2\,\mathrm{d}\xi \Big\} - C_2 \Big\{ \sum_{|\gamma|\leq 2m-1} \int_{|\xi|\geq K} \frac{|\xi|^\gamma}{(1+|\xi|^2)^m} |\hat{v^\mu}(\xi)|^2\,\mathrm{d}\xi \Big\},
\end{aligned}
$$

where C_1 depends only on m, while $C_2 = C_2(\sup_x |b_\beta(x)|, m)$. This is obtained by expanding the quadratic in the second line and separating the highest order term from the other terms. Now, choosing $K \geq 1$ so large that the second term is majorised by the first term in the last line, we have

$$
\|\mathcal{T}v^\mu\|^2_{H^{-m}(\{|\xi|\geq K\})} \geq C_3 \Big\{ \sum_{|\alpha|=m} \int_{|\xi|\geq K} \frac{|a_\alpha(x)|^2|\xi|^{2m}}{(1+|\xi|^2)^m} |\hat{v^\mu}(\xi)|^2\,\mathrm{d}\xi \Big\}, \tag{3.111}
$$

which converges to 0 by Eq. (3.109). Here C_3 depends on C_1, C_2 as well as K.

6. In this step let us complexify Q and $\Lambda_\mathcal{T}$. First, we view $Q : \Gamma(E) \to \mathbb{R}$ as a complex quadratic polynomial $Q^\mathbb{C} : \Gamma(E^\mathbb{C}) \to \mathbb{C}$, given by the following expression in local coordinates:

$$
Q^\mathbb{C}(x,z) := \sum_{j,k} Q_{jk}(x) z^j \overline{z^k} \qquad \text{for } x \in M, z \in E^\mathbb{C}_x,
$$

where $E^\mathbb{C}_x \cong \mathbb{C}^J$ is the fibre of the complexified bundle $E^\mathbb{C} := E \bigotimes \mathbb{C}$ at the point x. Thus, $Q^\mathbb{C}(s) = Q(s)$ for real $s \in \Gamma(E)$. Moreover, we define the *complexified cone* by

$$
\Lambda^\mathbb{C}_\mathcal{T} := \Lambda_\mathcal{T} + i\Lambda_\mathcal{T} = \{s^\mathbb{C} := s + is \text{ for } s \in \Lambda_\mathcal{T}\}. \tag{3.112}
$$

Now, let us compute $Q^\mathbb{C}(\zeta)$ for $\zeta = s + ir$, where $s, r \in \Gamma(E)$ are real: indeed,

$$
Q^\mathbb{C}(s+ir) = Q(s) + Q(r) + i\big\{\mathbf{q}(r,s) + \mathbf{q}(s,r)\big\},
$$

where $\mathbf{q}(r,s)_x := \sum_{j,k} Q_{jk}(x) r^j \overline{s^k}$ and $Q(s) = \mathbf{q}(s,s)$ as before. In particular, we have

$$
Q^\mathbb{C}(s^\mathbb{C}) = 2Q(s) + 2iQ(s) = 2Q(s)^\mathbb{C},
$$

and hence for $s^\mathbb{C} = s + is \in \Lambda^\mathbb{C}_\mathcal{T}$ the following hold:

1. $Q(s) >, =$ or < 0 if and only if $\mathrm{Re}\{Q^\mathbb{C}(s^\mathbb{C})\} >, =$ or < 0 (respectively);

2. $s \in \Lambda_\mathcal{T}$ if and only if $s^\mathbb{C} \in \Lambda^\mathbb{C}_\mathcal{T}$;

3. For any $\psi \in C^\infty_c(M)$ and $(u^\mu)^\mathbb{C} := u^\mu + iu^\mu$, we have

$$
\lim_{\mu\to 0} \int_M (Q \circ u^\mu)\psi\,\mathrm{d}V_g = 0 \quad \text{if and only if} \quad \lim_{\mu\to 0} \int_M \mathrm{Re}\{Q^\mathbb{C} \circ (u^\mu)^\mathbb{C}\}\psi\,\mathrm{d}V_g = 0.
$$

7. Now we adapt the arguments in Steps 2–3 in the proof of Theorem 3.5. First, a pointwise inequality is established: For each $\delta > 0$ and any compact set $\mathcal{K} \Subset T^*M \setminus \{0\}$, there is a constant $C_{\delta,\mathcal{K}} \in (0,\infty)$ such that

$$\mathrm{Re}\{Q^{\mathbb{C}}(s^{\mathbb{C}})\} \geq -\delta\Big||s^{\mathbb{C}}|_{g^{E,\mathbb{C}}}\Big|^2 - C_{\delta,\mathcal{K}}\Big||\sigma_m(T)(\eta)(s^{\mathbb{C}})\Big|_{g^{F,\mathbb{C}}}\Big|^2 \qquad \text{for each } \eta \in \mathcal{K}, s \in \Gamma(E), \tag{3.113}$$

provided that $\mathrm{Re}(Q) \geq 0$ on $\Lambda_{\mathcal{T}}$. Here $g^{E,\mathbb{C}}$ is the *complexified bundle metric on E*, obtained according to the same rule for $Q \mapsto Q^{\mathbb{C}}$, by viewing g^E as a quadratic form on each fibre (*i.e.*, a vector space) of E, and similarly for $g^{F,\mathbb{C}}$.

Indeed, noticing that Eq. (3.113) is 2-homogeneous in $s^{\mathbb{C}}$, thus the scaling $s^{\mathbb{C}} \mapsto \lambda s^{\mathbb{C}}$ by any $\lambda \in \mathbb{C}$ leaves it invariant. In particular, it is independent of the signatures of the semi-Riemannian bundle metrics g^E and g^F. Moreover, the cone $\Lambda_{\mathcal{T}}$ in (C3') is completely determined by $\mathcal{T}$, which is independent of the metrics g, g^E, g^F and the sequences $\{u^\mu\}$, $\{v^\mu\}$. So the proof of the analogous result in Theorem 3.5 remains valid (*cf.* Step 2 therein).

Now, we integrate Eq. (3.113) over $\{|\xi| \geq K\}$, with $K \geq 1$ specified at the end of Step 5 in this proof, $s^{\mathbb{C}} = (\hat{v^\mu})^{\mathbb{C}}$ and $\eta := \frac{\xi^{2m}}{(1+|\xi|^2)^m}$. Then one easily estimates

$$2^{-m} = \frac{|\xi|^{2m}}{(2|\xi|^2)^m} \leq |\eta| \leq 1 \qquad \text{for all } |\xi| \geq K. \tag{3.114}$$

Here, we remark that it is crucial for the sequence $\{v^\mu\}$ to be taken on $M = \mathbb{R}^n$ with respect to the Euclidean metric (*cf.* Step 3 of the same proof). In this case, the metric induced on the cotangent bundle T^*M is also Euclidean, hence $|\xi| \neq 0$ for all $\xi \in T^*M \setminus \{0\}$.

To proceed, $\eta \in \mathcal{K} := \{2^{-m} \leq |\xi| \leq 1\}$ is indeed compact in $T^*M \setminus \{0\}$, so

$$\begin{aligned}&\int_{|\xi|\geq K} \mathrm{Re}\{Q^{\mathbb{C}}(\hat{v^\mu})^{\mathbb{C}}\}\,\mathrm{d}\xi \\ &\geq -\delta\|v^\epsilon\|^2_{L^2(\mathbb{R}^N;E)} - C_{\delta,\mathcal{K}}\Big\{\sum_{|\alpha|=m}\int_{|\xi|\geq K}\frac{|a_\alpha(x)|^2|\xi|^{2m}}{(1+|\xi|^2)^m}|\hat{v^\mu}(\xi)|^2\,\mathrm{d}\xi\Big\},\end{aligned} \tag{3.115}$$

where the last term on the right-hand side tends to zero as $\mu \to 0$, *cf.* Step 5. Therefore,

$$\int_{|\xi|\geq K} \mathrm{Re}\{Q^{\mathbb{C}}(\hat{v^\mu})^{\mathbb{C}}\}\,\mathrm{d}\xi \geq -\delta\|v^\epsilon\|^2_{L^2(\mathbb{R}^n;E)}$$

for arbitrary $\delta > 0$, which implies that the left-hand side in non-negative. Applying the same argument for $-Q$ in place of Q, thanks to the condition (C3') and Step 6 in this proof, we finally arrive at the following:

$$\lim_{\mu\to 0}\int_{|\xi|\geq K} |\hat{v}^\mu(\xi)|^2\,\mathrm{d}\xi = 0. \tag{3.116}$$

Thus, combining Eqs. (3.104) and (3.116) and employing the Plancherel formula, one obtains:

$$\lim_{\mu\to 0}\int_{\mathbb{R}^n} |\hat{v^\mu}(\xi)|^2\,\mathrm{d}\xi = \lim_{\mu\to 0}\int_{\mathbb{R}^n} |v^\mu(\xi)|^2\,\mathrm{d}x = 0. \tag{3.117}$$

8. Finally, we notice that Eq. (3.117) is equivalent to the following:

$$\lim_{\mu\to 0}\|u^\mu\|_{L^2(M,g;E)} := \lim_{\mu\to 0}\Big\{\int_M \sum_{j=1}^J\sum_{k=1}^J \epsilon_k g^E_{jk}(u^\mu)^j(u^\mu)^k\sqrt{|\det g^E|}\,\mathrm{d}V_g\Big\}^{\frac{1}{2}} = 0, \tag{3.118}$$

which follows from the definition of v^μ: see Eq. (3.103). As Q is quadratic, it implies

$$\lim_{\mu\to 0}\int_M (Q\circ u^\mu)\,\mathrm{d}V_g = 0. \tag{3.119}$$

Moreover, we recall from Step 1 that the assertion of Theorem 1.6 is invariant under localisations, *i.e.*, multiplication by test functions $\psi \in C_c^\infty(M)$. Therefore, we can now conclude that $\{Q\circ u^\mu\}$ converges in distributions to $Q\circ u$.

This completes the proof. ■

As a concluding remark, we emphasise that the non-degeneracy condition $\det g \neq 0$ is crucial to the proof. We need it in Eq. (3.107) to compare the H^{-m} norm of $(\mathcal{T}\hat{u}^\mu)$ taken with respect to g and the Euclidean metric g_0.

3.4 Proof of weak continuity of GCR equations

Now let us prove the weak continuity of the GCR equations, and thus the weak rigidity of isometric immersions in the semi-Riemannian setting. With the preparation of previous sections, the weak rigidity problem can be settled naturally – Our strategy is to translate the weak continuity of the GCR equations to the weak continuity of the (second) structural equation of the Cartan formalism, and then apply the global, intrisic quadratic theorem in §3.3.

Theorem 3.7 *Let (M,g) be a simply-connected semi-Riemannian manifold of dimension n, with $\mathrm{Ind}(M)=\nu$ and $g\in W^{1,p}_{\mathrm{loc}}(M,O(\nu,n-\nu))$. Let $\{f^\epsilon\}\subset W^{2,p}_{\mathrm{loc}}(M;\mathbb{R}^{n+k})$ be a family of isometric immersions of semi-Riemannian submanifolds, with the second fundamental forms $\{\mathrm{II}^\epsilon\}$ and normal connections $\{\nabla^{\perp,\epsilon}\}$ satisfying the Gauss-Codazzi-Ricci equations (3.15)–(3.17). Assume that $p>n$ and $\{f^\epsilon\}$ is uniformly bounded in $W^{2,p}_{\mathrm{loc}}$, and that $\mathbb{R}^{n+k}$ is endowed with the semi-Euclidean metric $\overline{g_0}$ as in Eq. (3.19).*

Then, after passing to a subsequence, $\{f^\epsilon\}$ converges in distributions to an isometric immersion $f \in W^{2,p}_{\rm loc}(M;\mathbb{R}^{n+k})$. Moreover, the second fundamental form and the normal connection of f are weak $L^p_{\rm loc}$ limits of $\{\mathrm{II}^\epsilon\}$ and $\{\nabla^{\perp,\epsilon}\}$, respectively, and they satisfy the GCR equations.

Proof. We divide the arguments into four steps.

1. First of all, in view of the realisation Theorem 3.2, it suffices to pass to the weak limits in the second structural equation:

$$d\mathcal{W}^\epsilon = \mathcal{W}^\epsilon \wedge \mathcal{W}^\epsilon, \tag{3.120}$$

where $\mathcal{W}^\epsilon \in L^p_{\rm loc}(M; T^*M \bigotimes \mathfrak{o}(\nu+\tau, (n+k)-(\nu+\tau)))$ is the connection one-form associated with f^ϵ, defined as in Eq. (3.27). Indeed, as $\{f^\epsilon\}$ is uniformly bounded in $W^{2,p}_{\rm loc}$, it follows that $\{\mathcal{W}^\epsilon\}$ is uniformly bounded in $L^p_{\rm loc}$. Hence, $\{\mathcal{W}^\epsilon\}$ converges weakly in $L^p_{\rm loc}$ to some $\mathcal{W} \in L^p_{\rm loc}(M; T^*M \bigotimes \mathfrak{o}(\nu+\tau, (n+k)-(\nu+\tau)))$.

Assume that Eq. (3.120) is weakly continuous with respect to $\mathcal{W}^\epsilon \rightharpoonup \mathcal{W}$. Then $d\mathcal{W} = \mathcal{W} \wedge \mathcal{W}$, thus by Theorem 3.2 we can construct a $W^{2,p}_{\rm loc}$ isometric immersion $f : (M,g) \hookrightarrow (\mathbb{R}^{n+k}, \overline{g_0})$ with connection one-form $\mathcal{W}$. However, such f is locally unique up to a rigid motion in $\mathbb{R}^{n+k}_{\nu+\tau}$, so by the uniqueness of subsequential limits in the weak topology of $W^{2,p}$, we have $f^\epsilon \rightharpoonup f$ weakly in $W^{2,p}_{\rm loc}$ modulo subsequences.

In the sequel, for brevity we write $\mathfrak{h} = \mathfrak{o}(\nu+\tau, (n+k)-(\nu+\tau))$.

2. Let us take an arbitrary test differential form $\varphi \in C^\infty_c(M; \bigwedge^{n-2} T^*M)$ and consider the equation

$$d\mathcal{W}^\epsilon \wedge \varphi = \mathcal{W}^\epsilon \wedge (\mathcal{W}^\epsilon \wedge \varphi) = \langle \star\mathcal{W}^\epsilon, \mathcal{W}^\epsilon \wedge \varphi \rangle. \tag{3.121}$$

We notice that φ has no $\mathfrak{h}$-component in its range. Here $\star : \bigwedge^j T^*M \to \bigwedge^{n-j} T^*M$ is the *Hodge-star* operator, which is a vector bundle isomorphism by definition. In this proof we also use $\star$ to denote its natural extension $\star : \bigwedge^j T^*M \bigotimes \mathfrak{h} \to \bigwedge^{n-j} T^*M \bigotimes \mathfrak{h}$, given by $\star(\omega \otimes A) := \star\omega \otimes A$ for $\omega \in \bigwedge^j T^*M$ and $A \in \mathfrak{h}$. In other words, we do not distinguish between $\star$ and $(\star \otimes \mathrm{id}_{\mathfrak{h}})$.

3. To establish the weak continuity of Eq. (3.121), let us first figure out the differential constraints of this equation. As $d\mathcal{W}^\epsilon = \mathcal{W}^\epsilon \wedge \mathcal{W}^\epsilon$, by Hölder's inequality we have $\mathcal{W}^\epsilon \wedge \mathcal{W}^\epsilon \in L^{p/2}_{\rm loc}(U; \bigwedge^2 T^*M \bigotimes \mathfrak{h})$. We thus obtain the following compact Sobolev embedding:

$$L^{p/2}_{\rm loc}(U; \bigwedge^2 T^*M \bigotimes \mathfrak{h}) \hookrightarrow W^{-1,q}_{\rm loc}(U; \bigwedge^2 T^*M \bigotimes \mathfrak{h}) \qquad \text{for any } q < \frac{pn}{2n-p},$$

provided that $p < 2n$. For $p \geq 2n$, we first embed

$$L^{p/2}_{\rm loc}(U;\bigwedge^2 T^*M\bigotimes\mathfrak{h}) \hookrightarrow L^{\tilde{p}/2}_{\rm loc}(U;\bigwedge^2 T^*M\bigotimes\mathfrak{h}) \qquad \text{where } n<\tilde{p}<2n,$$

then again compactly embed it into $W^{-1,q}_{\rm loc}$ as above. Choosing $1<q<2$, we find that $\{d\mathcal{W}^\epsilon\}$ is pre-compact in $W^{-1,q}_{\rm loc}(U;\bigwedge^2 T^*M\otimes\mathfrak{h})$ for any $1<q<2$. On the other hand, $\{\mathcal{W}^\epsilon\}$ is pre-compact in the L^p norm, hence $\{d\mathcal{W}^\epsilon\}$ is pre-compact in $W^{-1,p}_{\rm loc}(U;\bigwedge^2 T^*M\otimes\mathfrak{h})$ for $p>n\geq 2$. Thus, by interpolation we deduce that $\{d\mathcal{W}^\epsilon\}$ is pre-compact in $H^{-1}_{\rm loc}(U;\bigwedge^2 T^*M\otimes\mathfrak{h})$. Furthermore, as the identity below holds by the super-commutativity of d:

$$d(\mathcal{W}^\epsilon\wedge\varphi) = d\mathcal{W}^\epsilon\wedge\varphi - \mathcal{W}^\epsilon\wedge d\varphi,$$

it follows that

$$\{d(\mathcal{W}^\epsilon\wedge\varphi)\} \quad \text{is pre-compact in } H^{-1}_{\rm loc}(U;\bigwedge^n T^*M\bigotimes\mathfrak{h}). \tag{3.122}$$

Second, for the other term on the right-hand side of Eq. (3.121), we recall that the L^2-adjoint of d, denoted by $\delta:\bigwedge^j T^*M\to\bigwedge^{j-1}T^*M$ for $1\leq j\leq n$, is related to d by

$$\delta = (-1)^{j(n-j)+1}\star d\star. \tag{3.123}$$

Also, $\star$ extends to an isometric isomorphism $\star: L^q(U;\bigwedge^j T^*M)\xrightarrow{\sim} L^q(U;\bigwedge^{n-j}T^*M)$ for each $0\leq j\leq n$, and for M with signature ν we have

$$\star\star = (-1)^{j(n-j)+\nu}\mathrm{id}_{\bigwedge^j T^*M}. \tag{3.124}$$

Thus the second differential constraint reads:

$$\{\delta\star\mathcal{W}^\epsilon\} \quad \text{is pre-compact in } H^{-1}_{\rm loc}(U;\mathfrak{h}). \tag{3.125}$$

As a summary, writing

$$\begin{cases} V^\epsilon := \mathcal{W}^\epsilon\wedge\varphi\in L^p_{\rm loc}(U;\bigwedge^{n-1}T^*M\otimes\mathfrak{h}),\\ Z^\epsilon := \star\mathcal{W}^\epsilon\in L^p_{\rm loc}(U;\bigwedge^{n-1}T^*M\otimes\mathfrak{h})\end{cases} \tag{3.126}$$

in Eq. (3.121), it suffices to establish the following general *claim*:

Let $\{V^\epsilon\}$ be a sequence of $(n-1)$-forms so that $\{dV^\epsilon\}$ is pre-compact in $H^{-1}_{\rm loc}$ and let $\{Z^\epsilon\}$ be a family of $(n-1)$-differential forms so that $\{\delta Z^\epsilon\}$ is pre-compact in $H^{-1}_{\rm loc}$. Then $\{\langle V^\epsilon, Z^\epsilon\rangle\}$ converges to $\langle V,Z\rangle$ in distributions. Here, V,Z denote the limiting points of $\{V^\epsilon\}$ and $\{Z^\epsilon\}$ in the weak $L^p_{\rm loc}$ topology, respectively.

4. We prove the above *claim* via the geometric quadratic theorem established previously, *i.e.*, Theorem 1.6. For this purpose, the key is to specify the operator $\mathcal{T}$ and its domain and image vector bundles E, F as in Theorem 1.6. Indeed, we take

$$\begin{cases} E := [\bigwedge^{n-1} T^*M \bigotimes \mathfrak{h}] \bigoplus [\bigwedge^{n-1} T^*M \bigotimes \mathfrak{h}], \\ F := [\bigwedge^{n} T^*M \bigotimes \mathfrak{h}] \bigoplus [\bigwedge^{n-2} T^*M \bigotimes \mathfrak{h}], \end{cases} \tag{3.127}$$

and the bundle operator $\mathcal{T} : E \to F$ is given by

$$\mathcal{T} := d \oplus \delta. \tag{3.128}$$

In this setting, the operator cone is given by

$$\Lambda_{\mathcal{T}} = \Big\{ (\mu, \lambda)^\top \in \Gamma(E) : [\sigma_1(d)(\xi)](\mu) = 0 \text{ and } [\sigma_1(\delta)(\xi)](\lambda) = 0 \\ \text{simultaneously for some } \xi \in T^*M \setminus \{0\} \Big\}. \tag{3.129}$$

Here we use the identity $\sigma_1(d \oplus \delta) = \sigma_1(d) \oplus \sigma_1(\delta)$, which is an equation as first-order homogeneous polynomials mapping the cotangent bundle to the hom-bundle from E to $F^{\mathbb{C}}$, namely $\mathcal{P}_1(T^*M; \mathrm{Hom}(E; F^{\mathbb{C}}))$.

To proceed, it is a well-known result in pseudo-differential operator theory that the principal symbols of d and δ have global and intrinsic representations (*cf.* §3.1, [2]). Indeed, we have

$$\sigma_1(d)(\xi) = -(2\pi i)\xi\wedge; \qquad \sigma_1(\delta)(\xi) = (2\pi i)\iota_{\xi^\sharp}, \tag{3.130}$$

where $\xi^\sharp$ is the element of the tangent bundle TM canonically isomorphic to ξ (which can be obtained by raising the indices in local coordinates), and ι_X is the *interior multiplication* of a differential form by the vector field $X \in \Gamma(TM)$, *i.e.* contraction with X. Thus, the cone is further characterised as

$$\Lambda_{\mathcal{T}} = \Big\{ (\mu, \lambda)^\top \in \Gamma(E) : \xi \wedge \mu = 0 \text{ and } \iota_{\xi^\sharp}(\lambda) = 0 \\ \text{simultaneously for some } \xi \in T^*M \setminus \{0\} \Big\}. \tag{3.131}$$

On the other hand, $\xi \wedge \mu = 0$ if and only if $\mu = \xi \wedge \tilde{\mu} \otimes A$ for some $A \in \mathfrak{h}$ and $\tilde{\mu} \in \bigwedge^{n-2} T^*M$. Meanwhile, $\iota_{\xi^\sharp}(\lambda) = 0$ if and only if $\tilde{\lambda}$ and ξ spans orthogonal subspaces in T^*M, so that $\lambda = \tilde{\lambda} \otimes B$ for $B \in \mathfrak{h}$. Let us now define the quadratic polynomial $Q : \Gamma(E) \to \mathbb{R}$ by

$$Q((\mu, \lambda)^\top) := \langle \mu, \lambda \rangle, \tag{3.132}$$

where $\langle \cdot, \cdot \rangle$ on the right-hand side is the combination of the inner product on $\bigwedge^{n-1} T^*M$ and the matrix product on $\mathfrak{h}$. Thus, for $(\mu, \lambda)^\top \in \Lambda_{\mathcal{T}}$, we have

$$Q((\mu, \lambda)^\top) = \big\langle \xi \wedge \tilde{\mu} \otimes A, \tilde{\lambda} \otimes B \big\rangle = \langle \xi \wedge \tilde{\mu}, \tilde{\lambda} \rangle \otimes (A \cdot B),$$

where $\cdot$ denotes the matrix multiplication.

Now, let us *claim* that $\langle \xi \wedge \tilde{\mu}, \tilde{\lambda} \rangle = 0$. Indeed, recall that the inner product $\langle \cdot, \cdot \rangle$ on $\bigwedge^{n-1} T^*M$ is induced from the inner product on T^*M by the following rule: For $\{\theta^{i_1}, \ldots, \theta^{i_{n-1}}\}$ and $\{\theta^{j_1}, \ldots, \theta^{j_{n-1}}\}$ two $(n-1)$-tuples of basic elements in the cotangent bundle T^*M, there holds

$$\big\langle \theta^{i_1} \wedge \ldots \wedge \theta^{i_{n-1}}, \theta^{j_1} \wedge \ldots \wedge \theta^{j_{n-1}} \big\rangle := \det\big(\langle \theta^{i_k}, \theta^{j_l} \rangle_{1 \leq k, l \leq n-1} \big). \tag{3.133}$$

In particular, if some θ^{i_k} is orthogonal to θ^{j_l} in T^*M, then the right-hand side of Eq. (3.133) is zero. Here, the arbitrary basic elements ξ and $\tilde{\lambda}$ are indeed orthogonal, so we conclude that $\langle \xi \wedge \tilde{\mu}, \tilde{\lambda} \rangle = 0$. Thus, the quadratic polynomial Q indeed vanishes on the cone $\Lambda_{\mathcal{T}}$, and the *claim* immediately follows.

In view of the above arguments, the conditions (C1')–(C3') in Theorem 1.6 are verified. An application of this theorem gives us

$$Q((V^\epsilon, Z^\epsilon)^\top) := \langle \mathcal{W}^\epsilon \wedge \varphi, \star \mathcal{W}^\epsilon \rangle \longrightarrow \langle \bar{\mathcal{W}} \wedge \varphi, \star \bar{\mathcal{W}} \rangle \qquad \text{in distributions.} \tag{3.134}$$

However, using the relation between semi-Riemannian metric and the Hodge star, the expressions for $\star\star$, and the super-commutativity of the wedge product, one easily deduces that

$$\begin{aligned} \langle \mathcal{W}^\epsilon \wedge \varphi, \star \mathcal{W}^\epsilon \rangle &= (-1)^\nu \mathcal{W}^\epsilon \wedge \varphi \wedge \star\star\, \mathcal{W}^\epsilon \\ &= (-1)^{2\nu + n - 1} \mathcal{W}^\epsilon \wedge \varphi \wedge \mathcal{W}^\epsilon \\ &= (-1)^{2\nu + 2(n-1)} \mathcal{W}^\epsilon \wedge \mathcal{W}^\epsilon \wedge \varphi \\ &= \mathcal{W}^\epsilon \wedge \mathcal{W}^\epsilon \wedge \varphi. \end{aligned} \tag{3.135}$$

Therefore, the previous convergence result is equivalent to:

$$\mathcal{W}^\epsilon \wedge \mathcal{W}^\epsilon \wedge \varphi \longrightarrow \mathcal{W} \wedge \mathcal{W} \wedge \varphi \qquad \text{in distributions.} \tag{3.136}$$

As the test form φ is arbitrary, the proof is now complete. ■

Remark 3.4.1 *As in the Riemannian case in §2, Theorem 3.7 on the weak continuity of GCR equations holds for $p > 2$, regardless of n. We only state the theorem for $p > n$ in order to ensure that f is an immersion (namely, df is injective) almost everywhere. The proof holds the same for the more general case $p > 2$.*

Remark 3.4.2 *To conclude the section, let us remark that the main results of §2 above, including the weak rigidity of Gauss-Codazzi-Ricci equations (Theorem 1.2) and the realisation theorem of Riemannian manifolds (Theorem 2.8), follow from the results in §3. This is because that every Riemannian manifold is a semi-Riemannian manifold with positive-definite metric. The results in §2 are proved by exploiting the Riemannian structures (e.g., the positivity of the Laplace-Beltrami operator, and the div-curl structure of the GCR equations) on Riemannian manifolds. In contrast, the arguments in §3 rely on the generalised quadratic theorem and the second structural equation of the Cartan formalism, which are more general tools independent of the Riemannian structures.*

3.5 Discussions and remarks

The goal of this final section is two-fold: On the one hand, we discuss the weak rigidity of general immersed hypersurfaces, *i.e.*, a topological submanifold of co-dimension 1 with possibly degenerate induced metric. On the other hand, we establish the weak continuity (weak stability) of quasi-linear wave equations verifying the *null condition* due to Christodoulou and Klainerman, via the intrinsic compensated compactness result (Theorem 1.6).

3.5.1 Weak rigidity of general immersed hypersurfaces

It is remarked in §3.2 (*cf.* Condition (N1) for Theorem 3.2) that, if the metric is degenerate on a hypersurface Σ, then Σ cannot be obtained via an isometric immersion of any semi-Riemannian manifold. Nevertheless, such degenerate scenarios occur naturally in physics.

One primary example is the *light-cone* $\Lambda = \{(t, x_1, x_2, x_3) \in \mathbb{R}^4 : 0 = -t^2 + x_1^2 + x_2^2 + x_3^2\}$ of the Minkowski spacetime $(\mathbb{R}^{3+1}, \mathfrak{m} = \mathrm{diag}(-1, 1, 1, 1))$: Although for any $x, v \in \Lambda$, $g_x(v, w) \neq 0$ for all time-like vectors w in the tangent space at point x, we have $g_x(v, \cdot) \equiv 0$ on $T_x\Lambda$. Here Λ is known as a *null hypersurface.* In addition, the stationary limit surface of Kerr's vacuum solution is everywhere timelike except at points on the axis, where it is null and tangent to the horizon (*cf.* [89]). A more recent example in [90] considers the glueing of two Anti-de-Sitter (AdS) 5-dimensional spacetimes with different cosmological constants along a general hypersurface $\Sigma = \Sigma^E \sqcup \Sigma^{\mathrm{null}} \sqcup \Sigma^L$, where Σ^{null} is 3-dimensional, such that the restriction of the metric is time-like on Σ^L, space-like on Σ^E and null on S. If the coordinate system is suitably chosen, Σ^{null} may lie in the hypersurface of the form $\{t = t_0\}$. This example gives a possible model for the transition between two distinct AdS universes across the "brane" Σ, in which Σ^{null} corresponds to the big-bang singularity.

Therefore, motivated by the physical applications above, a treatment for the realisation problem and the weak rigidity of general hypersurfaces is desired. Let us notice that the constructions in §2.2, in particular the derivation of GCR equations, fail in this case: No orthogonal decomposition of tangent spaces as in Eq. (3.4) is admitted.

To overcome this difficulty, we employ the construction of "*rigging vector fields*"; *cf.* [89, 80, 81] as well as the classical work [108] by Schouten. The idea is as follows: Consider the hypersurface via the local embedding $\iota : \Sigma \hookrightarrow (\bar{M}, \bar{g})$; if $\iota^*\bar{g}$ is null, let us find a non-vanishing vector field $l \in \Gamma(\iota^* T\bar{M})$ along Σ, so that

$$T_{\iota(x)}\bar{M} \cong T_x\Sigma \bigoplus \operatorname{span}\{l_x\}. \tag{3.137}$$

Thus, we can derive the Gauss-Codazzi equations (for hypersurfaces, Ricci equation is always trivial) from the orthogonal decomposition (3.137). However, technicalities are unavoidable because the rigging l *never* coincides with the normal vector field, whenever Σ is null – this leads to three Codazzi equations instead of one.

The first main result in this section is stated as follows. From now on, α always denotes a co-vector field, namely an element of $\Gamma(T^*\Sigma)$.

Theorem 3.8 *Let $\iota : \Sigma \hookrightarrow (\mathbb{R}^{n+1}, \overline{g_0})$ be a $W^{2,p}_{\text{loc}}$ immersion of a simply-connected general hypersurface, where $p > n$, and the pulled-back tensor $\iota^*\overline{g_0}$ is allowed to degenerate on Σ. Let the normal one-form of Σ to be $\mathfrak{n} \in \Gamma(\iota^* T^*\mathbb{R}^{n+1})$. Moreover, assume that $l \in \Gamma(T\mathbb{R}^{n+1})$ is a rigging vector field,* i.e., *$\mathfrak{n}(l) = 1$ everywhere on Σ. Take $\{e_i\}_{i=1}^n \subset \Gamma(T\Sigma)$ as an orthonormal frame on Σ, and $\{\theta^i\}_{i=1}^n \subset \Gamma(T^*\Sigma)$ as its co-frame. Furthermore, we define the tensor fields $K \in W^{1,p}_{\text{loc}}(\Sigma; \bigwedge^2 T^*\Sigma)$ and $\Psi \in W^{1,p}_{\text{loc}}(\Sigma; T\Sigma \bigotimes T^*\Sigma) = W^{1,p}_{\text{loc}}(\Sigma; \operatorname{End} T\Sigma)$ by*

$$\begin{cases} K := \bar{\nabla}\mathfrak{n}, \\ \Psi := \bar{\nabla} l, \end{cases} \tag{3.138}$$

which more precisely means that

$$\begin{cases} K(X,Y) = \bar{\nabla}\mathfrak{n}(X,Y), \\ \Psi(\alpha, X) := \alpha(\bar{\nabla}_X l), \end{cases} \tag{3.139}$$

for each $X, Y \in \Gamma(T\Sigma)$ and $\alpha \in \Gamma(T^*\Sigma)$. We also define $\psi \in W^{1,p}_{\text{loc}}(\Sigma; T^*\Sigma)$ by

$$\psi(X) := \mathfrak{n}(\bar{\nabla}_X l). \tag{3.140}$$

Then the following equations hold in the distributional sense:

$$0 = \alpha\big(R(X,Y)Z\big) - K(Y,Z)\Psi(\alpha,X) + K(X,Z)\Psi(\alpha,Y), \tag{3.141}$$

$$\begin{aligned} 0 = &-K(X,\nabla_Y Z) + K(Y,\nabla_X Z) - K([X,Y],Z) - XK(Y,Z) \\ &+ YK(X,Z) + K(Y,Z)\psi(X) - K(X,Z)\psi(Y), \end{aligned} \tag{3.142}$$

$$\begin{aligned} 0 = &X\Psi(\alpha,Y) - Y\Psi(\alpha,X) + \Psi(\alpha,[X,Y]) + \psi(Y)\Psi(\alpha,X) \\ &- \psi(X)\Psi(\alpha,Y) + \sum_{i=1}^{n}\Big\{\Psi(\theta^i,Y)\alpha(\nabla_X e_i) - \Psi(\theta^i,X)\alpha(\nabla_Y e_i)\Big\}, \end{aligned} \tag{3.143}$$

$$0 = X\psi(Y) - Y\psi(X) + \psi([X,Y]) + \sum_{i=1}^{n}\Big\{K(e_i,Y)\Psi(\theta^i,X) - K(e_i,X)\Psi(\theta^i,Y)\Big\}, \tag{3.144}$$

for $X,Y,Z \in \Gamma(T\Sigma)$, $\alpha \in \Gamma(T^*\Sigma)$ and $\alpha(l) = 0$; R is the Riemann curvature of Σ.

Conversely, if Eqs. (3.141)–(3.144) *hold in distributions for $K \in W^{1,p}_{\mathrm{loc}}(\Sigma; \bigwedge^2 T^*\Sigma)$, $\Psi \in W^{1,p}_{\mathrm{loc}}(\Sigma; \mathrm{End}\, T\Sigma)$ and $\psi \in W^{1,p}_{loc}(\Sigma; T^*\Sigma)$, then one can find an immersion $\iota \in W^{2,p}_{\mathrm{loc}}(\Sigma; \mathbb{R}^{n+1})$ and a rigging vector field $l \in \Gamma(T\mathbb{R}^{n+1})$, such that $K = \bar{\nabla}\mathfrak{n}$, $\Psi = \bar{\nabla} l$ and $\psi(X) = \mathfrak{n}(\bar{\nabla}_X l)$.*

In the statement of the theorem Eqs. (3.141)–(3.144) are known as the *Gauss, Codazzi-1, 2, and 3 equations* of the general hypersurface Σ, respectively. As in the physics literature (*cf.* Shouten [108], Clarke [39], Mars–Senovilla [89] and the references cited therein), the geometric quantities $\{K, \Psi, \psi\}$ are interpreted as the *intrinsic, extrinsic and normal second fundamental forms* of Σ, respectively. If the metric $\overline{g_0}$ is *Lorentzian*, *i.e.*, of signature $\{-1, +1, \ldots, +1\}$, the rigging vector field l can be chosen as time-like, whose trajectory thus corresponds to the worldline of an observer. On the other hand, if $\overline{g_0}|_\Sigma$ is non-degenerate, then l can be chosen as the unit normal vector field, and Eqs. (3.141)–(3.144) reduce to the usual Gauss-Codazzi equations in §2.2.

Before proving the theorem, we remark that it is also demonstrated in [81] via calculations in local coordinates. Our proof below, being global and intrinsic in nature, both helps clarify the geometric meanings of the various types of second fundamental forms $\{K, \Psi, \psi\}$, and serves as a crucial step in establishing the weak rigidity result, *i.e.*, Theorem 3.9.

Proof of Theorem 3.8. The proof consists of three steps. We emphasise the difference between the case of general hypsersurfaces and the case of semi-Riemannian submanifolds (Theorem 3.2), and the parallel arguments are only briefly sketched.

1. First of all, let us deduce the Gauss and Codazzi-1, 2, 3 equations (3.141)–(3.144) from the immersion ι. As in §3.1.2, these equations are obtained by expressing the flatness of $\mathbb{R}^{n+1}$, namely that the Riemann curvature $\bar{R} = 0$, with respect to the orthogonal splitting $T_x\mathbb{R}^{n+1} \cong T_x\Sigma \bigoplus \mathrm{span}\,(l_x)$. Indeed, from the definition of Riemann curvature we get

$$\begin{aligned}\bar{R}(X,Y)Z &= R(X,Y)Z - K(X,\bar{\nabla}_Y Z) - K(X,\bar{\nabla}_Y Z)l + K(Y,\bar{\nabla}_X Z)l \\ &\quad - \bar{\nabla}_X\big(K(Y,Z)l\big) + \bar{\nabla}_Y\big(K(X,Z)l\big) + K([X,Y],Z)l. \end{aligned} \tag{3.145}$$

The detailed computation can be found in §3 of [89], with a slightly different sign convention for $\bar{R}$. The Gauss equation is obtained by contracting with α: Due to $\alpha(l) = 0$, we have

$$\begin{aligned} 0 &= \alpha\big(\bar{R}(X,Y),Z\big) \\ &= \alpha\big(R(X,Y)Z\big) - \alpha\big(\bar{\nabla}_X(K(Y,Z)l)\big) + \alpha\big(\bar{\nabla}_Y(K(X,Z)l)\big), \end{aligned}$$

which leads to (3.141) by definition of Ψ.

To get the Codazzi-1 equation we consider $0 = \mathfrak{n}(R(X,Y)Z)$, where $\mathfrak{n}$ is the normal one-form. Invoking again Eq. (3.145) for $\bar{R}$, one has:

$$\begin{aligned}\mathfrak{n}\big(\bar{R}(X,Y)Z\big) &= -K(X,\nabla_Y Z) + K(Y,\nabla_X Z) - K([X,Y],Z) \\ &\quad - \mathfrak{n}\big(\bar{\nabla}_X(K(Y,Z)l)\big) + \mathfrak{n}\big(\bar{\nabla}_Y(K(X,Z)l)\big).\end{aligned}$$

But the Leibniz rule of the connection gives us

$$\mathfrak{n}\big(\bar{\nabla}_Y(K(X,Z)l)\big) = YK(X,Z)\mathfrak{n}(l) - K(X,Z)\mathfrak{n}(\nabla_Y l),$$

which, together with $\mathfrak{n}(l) = 1$ and the definition for ψ, implies (3.142).

Next we contract $\bar{R}(X,Y)l = \bar{\nabla}_X\bar{\nabla}_Y l - \bar{\nabla}_Y\bar{\nabla}_X l + \bar{\nabla}_{[X,Y]}l$ with α. To wit,

$$\bar{\nabla}_X\bar{\nabla}_Y l = \sum_{i=1}^n X\Psi(\theta^i,Y)e_i + X\psi(Y)l + \sum_{i=1}^n \Psi(\theta^i,Y)\bar{\nabla}_X e_i + \psi(Y)\bar{\nabla}_X l,$$

where the following key identities is used:

$$\begin{cases}\bar{\nabla}_X Y = \nabla_X Y - K(X,Y)l, \\ \bar{\nabla}_X l = \sum_{i=1}^n \Psi(\theta^i,X)e_i + \psi(X)l.\end{cases} \tag{3.146}$$

From here one obtains that

$$\begin{aligned}\bar\nabla_X\bar\nabla_Y l = &\sum_{i=1}^n X\Psi(\theta^i,Y)e_i + X\psi(Y)l + \sum_{i=1}^n \Psi(\theta^i,Y)\nabla_X e_i\\ &-\sum_{i=1}^n \Psi(\theta^i,Y)K(e_i,X)l + \sum_{i=1}^n \psi(Y)\psi(\theta^i,X)e_i + \psi(X)\psi(Y)l,\end{aligned}$$

as well as a similar expression for $\bar\nabla_Y\bar\nabla_X l$ by interchanging X and Y:

$$\begin{aligned}\bar\nabla_Y\bar\nabla_X l = &\sum_{i=1}^n Y\Psi(\theta^i,X)e_i + Y\psi(X)l + \sum_{i=1}^n \Psi(\theta^i,Y)\nabla_Y e_i\\ &-\sum_{i=1}^n \Psi(\theta^i,X)K(e_i,Y)l + \sum_{i=1}^n \psi(X)\psi(\theta^i,Y)e_i + \psi(Y)\psi(X)l,\end{aligned}$$

Thus, contracting with $\alpha \in \Omega^1(\Sigma)$ and noting that $\alpha(l)=0$, the Codazzi-2 equation (3.143) follows.

Finally, the Codazzi-3 equation (3.144) is obtained by contracting $\bar R(X,Y)l$ with the normal one-form $\mathfrak{n}$: Similarly to the above computations, we have

$$\begin{aligned}\mathfrak{n}(\bar\nabla_X\bar\nabla_Y l) &= \mathfrak{n}\Big[\bar\nabla_X\Big(\sum_{i=1}^n \theta^i(\bar\nabla_Y l)e_i + \mathfrak{n}(\nabla_Y l)l\Big)\Big]\\ &= X\psi(Y) + \sum_{i=1}^n \theta^i(\bar\nabla_Y l)\mathfrak{n}(\bar\nabla_X e_i) + \psi(Y)\psi(X)\\ &= X\psi(Y) + \psi(Y)\psi(X) - \sum_{i=1}^n K(e_i,X)\Psi(\theta^i,Y),\end{aligned}$$

thanks to another identity

$$\mathfrak{n}(\bar\nabla_X Y) = -\mathfrak{n}\big(K(Y,X)l\big) = -K(Y,X). \tag{3.147}$$

Therefore, computing for $\mathfrak{n}(\bar\nabla_Y\bar\nabla_X l)$ in the similar manner, we have

$$\mathfrak{n}(\bar\nabla_Y\bar\nabla_X l) = Y\psi(X) + \psi(X)\psi(Y) - \sum_{i=1}^n K(e_i,Y)\Psi(\theta^i,X),$$

so Eq. (3.144) follows. Furthermore, observe that the above computations still hold in distributions for the immersion with lower regularity, *i.e.*, $\iota \in W^{2,p}_{\mathrm{loc}}(\Sigma;\mathbb{R}^{n+1})$. This proves the first part of the theorem.

2. Now let us tackle the "realisation problem", namely, finding the immersion ι from Eqs. (3.141)–(3.144). As in the semi-Riemannian submanifolds case, the key is to verify the second structural equation (3.29) for a suitable connection one-form. For this purpose we invoke the identity of differential forms:

$$d\beta(X,Y) = X\beta(Y) - Y\beta(X) + \beta([X,Y]),$$

for any $\beta \in \Gamma(T^*\Sigma)$. Thus, one rewrites the three Codazzi equations as follows:

$$\begin{cases} dK(X,Y,Z) = K(Y,\nabla_X Z) - K(X,\nabla_Y Z) + \psi(X)K(Y,Z) - K(X,Z)\psi(Y), \\ d\Psi(\alpha,X,Y) = \psi(X)\Psi(\alpha,Y) - \psi(Y)\Psi(\alpha,X) \\ \qquad\qquad + \sum_{i=1}^n \Big[\Psi(\theta^i,X)\alpha(\nabla_Y e_i) - \Psi(\theta^i,Y)\alpha(\nabla_X e_i)\Big], \\ d\psi(X,Y) = \sum_{i=1}^n \Big[\Psi(\theta^i,Y)K(X,e_i) - \Psi(\theta^i,X)K(Y,e_i)\Big]. \end{cases} \tag{3.148}$$

Now, we define the connection one form $\mathcal{W}_\Sigma \in W^{1,p}_{\text{loc}}(\Sigma; T^*\Sigma \bigotimes \mathfrak{gl}(n+1;\mathbb{R}))$ as

$$\mathcal{W}_\Sigma = \begin{bmatrix} \Gamma & \Psi \\ K & \psi \end{bmatrix}, \tag{3.149}$$

which more precisely means the following in local coordinates:

$$\mathcal{W}_\Sigma = \begin{bmatrix} \Gamma^k_{ij}\theta^k & \Psi(\theta^i,\bullet) \\ K(e_i,\bullet) & \psi(\bullet). \end{bmatrix} \tag{3.150}$$

Here, as usual, the Christoffel symbols are defined via $\nabla_{e_i} e_j = \sum_{k=1}^n \Gamma^k_{ij} e_k$, and computed by $\Gamma^k_{ij} = \frac{1}{2} g^{kl}(\partial_i g_{jl} + \partial_j g_{li} - \partial_l g_{ij})$. The block-matrix representation of $\mathcal{W}$ in Eq. (3.149) is interpreted as identifying

$$\begin{cases} \Gamma = \Gamma^k_{ij}\theta^k \in W^{1,p}_{\text{loc}}(\Sigma; T^*\Sigma \bigotimes \mathfrak{gl}(n;\mathbb{R})), \\ \Psi, K \in W^{1,p}_{\text{loc}}(\Sigma; T^*\Sigma \bigotimes \mathbb{R}^n), \\ \psi \in W^{1,p}_{\text{loc}}(\Sigma; T^*\Sigma). \end{cases} \tag{3.151}$$

Thus, we can recast the Gauss equation (3.141) and the Codazzi equations in the form of (3.148) into the following schematic equalities:

$$\begin{cases} dK = K\Gamma - \Gamma K - K\psi + \psi K = K \wedge \Gamma - K \wedge \psi, \\ d\Psi = \Gamma\Psi - \Psi\Gamma + \Psi\psi - \psi\Psi = \Gamma \wedge \Psi - \Psi \wedge \psi, \\ d\psi = K\Psi - \Psi K = K \wedge \Psi, \end{cases} \tag{3.152}$$

where the juxtaposition of matrices (*e.g.* $K\Gamma$) denotes the matrix multiplication, and $\wedge$ is an intertwining of the wedge product on differential forms and matrix multiplication.

On the other hand, simple manipulations on block matrices give us:

$$\mathcal{W}_\Sigma \wedge \mathcal{W}_\Sigma = \begin{bmatrix} \Gamma\wedge\Gamma + \Psi\wedge K & \Gamma\wedge\Psi + \psi\wedge\Psi \\ K\wedge\Gamma + K\wedge\psi & K\wedge\Psi \end{bmatrix}. \tag{3.153}$$

In this notation the Riemann curvature is given by

$$R = d\Gamma - \Gamma\wedge\Gamma \in L^p_{\text{loc}}(\Sigma; \bigwedge^2 T^*\Sigma \bigotimes \mathfrak{gl}(n;\mathbb{R})). \tag{3.154}$$

So the preceding two equations yield

$$d\mathcal{W}_\Sigma - \mathcal{W}_\Sigma \wedge \mathcal{W}_\Sigma = 0, \tag{3.155}$$

i.e., the second structural equation as in Eq. (3.29). Thus, invoking again Lemma 2.4.1 due to Mardare ([88]), we obtain a local solution $A \in W^{1,p}_{\mathrm{loc}}(U \subset \Sigma; \mathfrak{gl}(n+1, \mathbb{R}))$ to the following Pfaff system

$$dA = \mathcal{W}_\Sigma \cdot A,$$

where $U \subset \Sigma$ is an open neighbourhood.

3. Now we are in the position to solve for the local isometric immersion $\iota : \Sigma \hookrightarrow \mathbb{R}^{n+1}$, via the Poincaré system

$$d\iota = \tilde{\theta} \cdot A,$$

where $\tilde{\theta} = (\theta^1, \ldots, \theta^n, 0)^\top : U \subset \Sigma \to \mathbb{R}^{n+1} \bigotimes T^*\Sigma$ is the $\mathbb{R}^{n+1}$-valued differential one-form. As before, this equation is solvable if and only if the first structural equation $d\tilde{\theta} = \tilde{\theta} \wedge \mathcal{W}_\Sigma$ is satisfied. In Step 4 of the proof of Theorem 3.2, we see that the first structural equation holds whenever the tangentially projected connection $\nabla = \iota^* \bar{\nabla}$ is torsion-free. Here, as $K(X, Y) = K(Y, X)$ ([80, 89]), the torsion-free condition is verified, which leads to the existence of solution $\iota \in W^{2,p}_{\mathrm{loc}}(U; \mathbb{R}^{n+1})$. The remaining of the proof follows from the proof (Steps 6, 8, in particular) of Theorem 3.2. ■

In the proof above, it is crucial to establish the equivalence of the Gauss and Codazzi–1, 2, 3 equations with Eq. (3.155), namely the second structural equation for $\mathcal{W}_\Sigma$, which is defined in Eq. (3.149) in terms of the Christoffel symbol Γ and the intrinsic, extrinsic and normal second fundamental forms $\{K, \Psi, \psi\}$. Therefore, again by invoking the generalised quadratic theorem (Theorem 1.6) and establishing the weak continuity of $d\mathcal{W}_\Sigma = \mathcal{W}_\Sigma \wedge \mathcal{W}_\Sigma$, we arrive at the weak rigidity theorem for the general hypersurfaces:

Theorem 3.9 *Let (Σ, g) be a simply-connected n-dimensional hypersurface of semi-Euclidean space $\mathbb{R}^{n+1}$, with $\mathrm{Ind}(\Sigma) = \nu$, $g \in W^{1,p}_{\mathrm{loc}}(\Sigma, O(\nu, n-\nu))$ and $p > n$. Let $\{f^\epsilon\}$ be a family of immersions of semi-Riemannian submanifolds uniformly bounded in $W^{2,p}_{\mathrm{loc}}(\Sigma; \mathbb{R}^{n+k})$, and $\{l^\epsilon\}$ be an associated family of rigging vector fields uniformly bounded in $W^{1,p}_{\mathrm{loc}}(\Sigma; T\Sigma)$. Let $\{K^\epsilon, \Psi^\epsilon, \psi^\epsilon\}$ be the corresponding intrinsic, extrinsic and normal second fundamental forms.*

Then, after passing to a subsequence, $\{f^\epsilon\}$ converges in distributions to an immersion $f \in W^{2,p}_{\mathrm{loc}}(\Sigma; \mathbb{R}^{n+1})$. Moreover, its intrinsic, extrinsic and normal second fundamental forms are weak limits in L^p_{loc} of $\{K^\epsilon\}$, $\{\Psi^\epsilon\}$ and $\{\psi^\epsilon\}$, respectively.

Proof. In view of Eq. (3.153), all the entries of the two-form-valued matrix $\mathcal{W}_\Sigma \wedge \mathcal{W}_\Sigma$ are linear combinations of the quadratic forms in Γ, Ψ, K and ψ, each of which lies in $W^{1,p}_{\rm loc}$. Thus $\mathcal{W}_\Sigma \wedge \mathcal{W}_\Sigma \in W^{1,p/2}_{\rm loc}$ by Cauchy-Schwarz inequality, which is compactly embedded in $W^{-1,q}_{\rm loc}$ for some $1 < q < 2$ by Sobolev embedding, as computed in Step 3 in the proof of Theorem 3.7. On the other hand, $\mathcal{W}_\Sigma \in W^{1,p}_{\rm loc}$ implies that $d\mathcal{W}_\Sigma \in L^p_{\rm loc}$, which is compactly embedded into $W^{-1,p}_{\rm loc}$ due to the Rellich lemma. Equating these two terms via Eq. (3.153) and interpolating the Sobolev spaces, we deduce that $\{d\mathcal{W}^\epsilon_\Sigma\}$ is pre-compact in $H^{-1}_{\rm loc}$.

Therefore, with the above pre-compactness result established, the proof proceeds as for Theorem 3.7: We establish the weak continuity of the structural equation $d\mathcal{W}_\Sigma = \mathcal{W}_\Sigma \wedge \mathcal{W}_\Sigma$, and in view of the realisation theorem for general hypersurfaces (*i.e.*, Theorem 3.8) it implies the existence of the limiting immersion f, together with a rigging vector field l, from which the intrinsic, extrinsic and normal second fundamental forms $\{K, \Psi, \psi\}$ are well-defined. After passing to subsequences if necessary, $\{K^\epsilon, \Psi^\epsilon, \psi^\epsilon\}$ converges in the weak local L^p topology to $\{K, \Psi, \psi\}$ due to the uniqueness of weak limits.

Hence the proof is now complete. ∎

3.5.2 Weak continuity of quasi-linear wave equations

Before concluding §3, let us give an application of the generalised quadratic theorem of compensated compactness, *i.e.*, Theorem 1.6, to the weak continuity/stability of a special class of nonlinear wave equations:

$$\begin{cases} \Box_{\mathfrak{m}}\phi^I = F^I(\phi, \partial\phi) \qquad \text{for all } I \in \{1,2,\ldots,N\}, \\ (\phi, \partial_t\phi)\Big|_{t=0} = \epsilon(f,g), \end{cases} \tag{3.156}$$

where $\Box_{\mathfrak{m}} := -\partial_{tt} + \Delta$ is the d'Alembertian. The system is posed on $(\mathbb{R}^{3+1}, \mathfrak{m})$, where $\mathfrak{m} = \mathrm{diag}(-1,1,1,1)$ is the Minkowski metric, $(f,g) \in C^\infty_c(\mathbb{R}^3; \mathbb{R}^N \times \mathbb{R}^N)$ is the initial data, and $F = \{F^I\}_{1\leq I\leq N}$ is the source term. We are solving for $\phi = \{\phi^I\}_{1\leq I\leq N} : \mathbb{R}^{3+1} \to \mathbb{R}^N$. The source F depends on up to the first order of ϕ.

A classical result due to Klainerman ([75]) and Christodoulou ([34]) is as follows: There exists $\epsilon_0 = \epsilon_0(f,g) > 0$ such that Eq. (3.156) has a unique solution $\phi \in C^\infty_c([0,\infty) \times \mathbb{R}^3; \mathbb{R}^N)$ for every $0 \leq \epsilon < \epsilon_0$, provided that F satisfies the *null condition*:

$$\begin{cases} F^I(0) = 0, \qquad \partial F^I(0) = 0, \\ Q_{F^I}(\partial u) = \sum_{J,K=1}^N \sum_{\mu,\nu=0}^3 A^{\mu\nu}_{IJK}(\partial_\mu u^J)(\partial_\nu u^K) \quad \text{for each } I \in \{1,2,\ldots,N\}, \text{ where} \\ \sum_{\mu,\nu=0}^3 A^{\mu\nu}_{IJK}\xi_\mu\xi_\nu = 0 \quad \text{for any null co-vector } \xi \in T^*\mathbb{R}^{3+1} \text{ and } I,J,K \in \{1,2,\ldots,N\}. \end{cases}$$

Here Q_{F^I} denotes the quadratic part in the Taylor expansion of F^I about zero, namely

$$Q_{F^I}(z) := \sum_{|\alpha|=2} \frac{\big(\partial_\alpha F^I\big)(0) z^\alpha}{\alpha!} \qquad \text{for all } z \in \mathbb{R}^N \tag{3.157}$$

in the multi-index notations. In particular, it is part of the definition for the null condition that Q_{F^I} depends only on ∂u, where ∂ denotes the total space-time derivative. We say that $\xi \in T^*\mathbb{R}^{3+1}$ is a null co-vector if and only if $\mathfrak{m}^{\mu\nu}\xi_\mu\xi_\nu = 0$.

For our purpose, let us take the following bundle of type–$(1,1)$ tensors:

$$E = T^*\mathbb{R}^{3+1} \bigotimes T\mathbb{R}^N. \tag{3.158}$$

Then, define for each $I \in \{1, 2, \ldots, N\}$ the bundle operator $\mathcal{T}_I \in \mathrm{Hom}(E; \mathbb{R})$:

$$\mathcal{T}_I\, s := \sum_{J,K=1}^{N} \sum_{\mu,\nu=0}^{3} A^{\mu\nu}_{IJK} (\partial_\nu s^J_\mu) \theta^K, \tag{3.159}$$

in which $\{\theta^K\} \subset T^*\mathbb{R}^N$ is the co-vector basis dual to $\{\partial_K\}$. The associated operator cone is

$$\begin{aligned} \Lambda_{\mathcal{T}_I} := \Big\{ \lambda \in T^*\mathbb{R}^{3+1} \bigotimes T\mathbb{R}^N : \sum_{\mu,\nu=0}^{3} A^{\mu\nu}_{IJK} \lambda^J_\mu s^K_\nu = 0 \\ \text{for some non-zero section } s \in \Gamma(T^*\mathbb{R}^{3+1} \bigoplus T\mathbb{R}^N) \setminus \{0\} \Big\}. \end{aligned} \tag{3.160}$$

We notice that for each null co-vector $\lambda \in T^*\mathbb{R}^{3+1}$, if we identify it with $\lambda \otimes \mathrm{id} \in T^*\mathbb{R}^{3+1} \otimes T\mathbb{R}^N$ (where id is the tautological tensor on $T\mathbb{R}^N$), then it lies in $\Lambda_{\mathcal{T}_I}$. In other words, the *null cone* of the space-time $(\mathbb{R}^{3+1}, \mathfrak{m})$ can be viewed as a subset of the operator cone $\Lambda_{\mathcal{T}_I}$ for every $I \in \{1, 2, \ldots, N\}$. Furthermore, for each $I \in \{1, 2, \ldots, N\}$, consider the quadratic form as follows:

$$Q_{F^I}(s) := \sum_{J,K=1}^{N} \sum_{\mu,\nu=0}^{3} A^{\mu\nu}_{IJK} s^J_\nu s^K_\mu \qquad \text{for } s = \{s^J_\mu\}_{1\le J\le N, 0\le\mu\le 3} \in \Gamma(E). \tag{3.161}$$

It can be defined intrinsically on $\Gamma(E)$, and it is easy to check that Q_{F_I} agrees with the quadratic part of the source term F_I.

Now, applying Theorem 1.6 to $\{\partial\phi^\epsilon\} \subset L^2_{\mathrm{loc}}(\mathbb{R}^{3+1}; T^*\mathbb{R}^{3+1} \otimes T\mathbb{R}^N)$, we immediately obtain the following "compensated compactness framework", which enables us to verify the H^1_{loc} weak stability of Eq. (3.156). For this purpose, one only needs to pass to the limits in the source term $F^I(\phi^\epsilon, \partial\phi^\epsilon)$, as the left-hand side of the equation is linear in ϕ^ϵ.

Proposition 3.10 *Let $\{\phi^\epsilon\}$ be a family in $H^1_{\mathrm{loc}}(\mathbb{R}^{3+1}, \mathbb{R}^N)$. Assume that the source term $F^{I,\epsilon} := F^I(\phi^\epsilon, \partial\phi^\epsilon)$ satisfies the null condition for each $\epsilon > 0$, and that*

1. *$\phi^\epsilon \rightharpoonup \phi$ weakly in H^1_{loc};*
2. *The family $\left\{\sum_{J=1}^N \sum_{\mu,\nu=0}^3 A^{\mu\nu}_{IJK}\partial_\mu\partial_\nu(\phi^\epsilon)^J\right\}$ is pre-compact in $H^{-1}_{\mathrm{loc}}(\mathbb{R}^{3+1})$ for all $I, K \in \{1, 2, \ldots, N\}$;*
3. *$Q_{F^{I,\epsilon}}(s) = 0$ for any $s \in \Lambda_{\mathcal{T}^{I,\epsilon}}$. Here, the operator cone $\Lambda_{\mathcal{T}^{I,\epsilon}}$ is defined according to Eqs.* (3.160) *and* (3.161)*.*

Then $F^I(\phi, \partial\phi)$ satisfies the null condition in the sense of distributions. In particular, a necessary condition for (2) above is that $Q_{F^{I,\epsilon}}(\xi \otimes \mathrm{id}) = 0$ for any null co-vector $\xi \in T^*\mathbb{R}^{3+1}$.

As a consequence, if Eq. (3.156) *admits a weak solution in the distributional sense under the null condition, then for a family $\{\phi^\epsilon\} \subset H^1_{\mathrm{loc}}(\mathbb{R}^{3+1}, \mathbb{R}^N)$ of weak solutions, the weak H^1 limit ϕ is also a solution.*

The above proposition shows that the quasi-linear wave equation with null condition in $3+1$ dimensions is weakly continuous, or weakly stable in the physical terms, in the function space H^1_{loc}. However, it is well-known (*cf.* Rodnianski [106]) that the Einstein equations fail to satisfy the null conditions, even in the vacuum or scalar field cases. We shall leave the weak continuity of the Einstein equations and other physical/geometric PDEs for future investigations.

Chapter 4

Vanishing Viscosity Limit of the 3D Incompressible Fluid with Navier Boundary Condition

This chapter is devoted to the third and final project in the thesis: The vanishing viscosity limit (or inviscid limit) of a 3D incompressible fluid in a smooth region with the "physical" – Navier and kinematic – boundary conditions.

4.1 Set-up of the problem

Throughout we take $\Omega \subset \mathbb{R}^3$ to be a bounded open smooth domain in $\mathbb{R}^3$. In this case, the boundary $\partial\Omega$ is an embedded oriented 2-dimensional regular surface. The Cauchy problem in $[0,T] \times \Omega$ of the Navier-Stokes equation is as follows:

$$\begin{cases} \partial_t u^\nu + (u^\nu \cdot \nabla) u^\nu - \nu \Delta u^\nu + \nabla p^\nu = 0 \qquad \text{in } [0,T] \times \Omega, \\ \nabla \cdot u^\nu = 0 \qquad \text{in } [0,T] \times \Omega, \\ u|_{t=0} = u_0 \qquad \text{on } \Omega. \end{cases} \tag{4.1}$$

Recall that the boundary of the domain of fluid, $\partial\Omega$, is a regular surface embedded in $\mathbb{R}^3$. We denote its second fundamental form by $\mathrm{II} : T(\partial\Omega) \times T(\partial\Omega) \to \mathbb{R}$, where $T(\partial\Omega)$ is the tangent bundle of $\partial\Omega$. Hence, writing $\mathbf{n} \in T(\partial\Omega)^\perp$ as the outward unit normal (viewed as the Gauss map $\mathbf{n} : \partial\Omega \to \mathbb{S}^2$), we have

$$\mathrm{II} = -\nabla \mathbf{n}. \tag{4.2}$$

The trace of the second fundamental form is the *mean curvature*, denoted by

$$H := \mathrm{tr}(\mathrm{II}). \tag{4.3}$$

In addition, let us take $\{e_1, e_2, e_3\}$ to be the orthonormal frame such that $e_1, e_2 \in T(\partial\Omega)$ and $e_3 = \mathbf{n}$. Then we have the local expression

$$\mathrm{II}(u, v) = \sum_{\alpha=1}^{2} \sum_{\beta=1}^{2} \mathrm{II}_{\alpha\beta} u^\alpha v^\beta. \tag{4.4}$$

We also consider the shape operator $\mathcal{S} : T(\partial\Omega) \to T(\partial\Omega)$:

$$\mathcal{S}(u) := -\nabla_u \mathbf{n}, \tag{4.5}$$

where ∇_u means the directional derivative in the direction of u:

Now we are at the stage of defining the *kinematic and Navier boundary conditions*. Adopting the description in the previous work [29, 30] by Chen and Qian, we have

$$\begin{cases} u^\nu \cdot \mathbf{n} = 0, \\ \omega^\nu \cdot \tau = -\frac{1}{\zeta}(\mathcal{R}u^\nu) \cdot \tau + 2\mathcal{R}\big(\mathcal{S}(u^\nu)\big) \cdot \tau \qquad \text{on } \partial\Omega \times [0, T], \end{cases} \tag{4.6}$$

where

$$\omega^\nu := \nabla \times u^\nu \tag{4.7}$$

is the *vorticity* of the fluid, $\tau \in T(\partial\Omega)$ is an arbitrary tangential vector field on the boundary $\partial\Omega$, and the constant $\zeta > 0$ is known as the *slip length* of the fluid. Recall from §1.2.3 that, for any 2D vector field $V = (V^1, V^2)^\top$, we define a zero-th order operator $\mathcal{R}$ globally and intrinsically as follows:

$$\mathcal{R}V = (*(V^\sharp))^\flat = (-V^2, V^1)^\top, \tag{4.8}$$

where $\sharp$ is the canonical isomorphism between vector fields and differential one-forms, and $\flat$ is its inverse (the same as in §2). In local coordinates on $\partial\Omega$, $\mathcal{R}$ takes the form of the matrix multiplication:

$$\mathcal{R} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \tag{4.9}$$

i.e., the anti-clockwise rotation by $\pi/2$.

Before further development, let us first remark that our second equation in (4.6) is equivalent to the classical Navier boundary condition, proposed by Navier in [99]. Here for simplicity we drop the superscript ν temporarily. Recall that the classical version of the Navier boundary condition reads

$$u \cdot \tau = -2\zeta\, \mathbb{D}u(\tau, \mathbf{n}) \qquad \text{on } \partial\Omega \times [0, T], \text{ for any } \tau \in T(\partial\Omega), \tag{4.10}$$

where the rate-of-strain tensor $\mathbb{D}u$ is defined by

$$\mathbb{D}u := \frac{1}{2}\Big(\nabla u + \nabla^\top u\Big), \tag{4.11}$$

namely

$$(\mathbb{D}u)_{ij} = \frac{1}{2}\Big(\nabla_i u^j + \nabla_j u^i\Big), \qquad 1 \leq i, j \leq 3$$

in local coordinates, with $\mathbb{D}u(\tau, \mathbf{n}) := \tau^\top \cdot \mathbb{D}u \cdot \mathbf{n}$ denoting the matrix multiplication. In local coordinates, suppose that $\{e_1, e_2, e_3\}$ is an orthonormal moving frame adapted to $\partial\Omega$, with $e_3 = \mathbf{n}$. Then Eq. (4.10) is equivalent to the following:

$$\begin{cases} u^1 = -\zeta(\nabla_3 u^1 + \nabla_1 u^3), \\ u^2 = -\zeta(\nabla_3 u^2 + \nabla_2 u^3), \qquad \text{on } \partial\Omega \times [0, T]. \end{cases} \tag{4.12}$$

First proposed by Navier in 1816 (*cf.* [99]), the Navier boundary condition requires that the tangential component of the velocity field is proportional to that of the normal vector field of the Cauchy stress tensor. The proportionality constant $\zeta > 0$ is known as the *slip length*. Physically, the Navier boundary condition can be induced by the effects of free capillary boundaries, perforated boundaries, or exterior electric fields; *cf.* Achdou-Pironneau-Valentin [1], Bänsch [8], Beavers-Joseph [11], Einzel-Panzer-Liu [53], Maxwell [93], Jäger-Mikelič [68, 69] and the references cited therein.

One main issue of this chapter is to derive the higher-order energy estimates of the velocity u. As shown in §5.2 below, the H^r norm of u is estimated purely by the L^2 norm of the r-th iterated *curls* of u (*cf.* Theorem 4.2). So let us seek for the boundary condition with respect to the vorticity $\omega = \nabla \times u$.

For this purpose, note that in the local frame $\{e_1, e_2, e_3\}$ as above, we have

$$\omega = \begin{bmatrix} \nabla_2 u^3 - \nabla_3 u^2 \\ \nabla_3 u^1 - \nabla_1 u^3 \\ \nabla_1 u^2 - \nabla_2 u^1 \end{bmatrix}.$$

The Navier boundary condition (4.10) gives us

$$\nabla_k u^3 + \nabla_3 u^k = -\frac{1}{\zeta} u^k \qquad \text{for } k \in \{1, 2\}; \tag{4.13}$$

on the other hand, $\nabla_k u^3$ can be computed as follows:

$$\nabla_k u^3 = \nabla_k (u \cdot \mathbf{n}) = \frac{\partial}{\partial x^3}(u \cdot \mathbf{n}) + \sum_{j=1}^{3} \Gamma^3_{kj} u^j, \tag{4.14}$$

where $\Gamma^k_{ij} = \frac{1}{2} g^{kl}(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij})$ are the Christoffel symbols. Observe also that

$$\begin{aligned} \mathrm{II}_{jk} &= \mathrm{II}(\mathbf{e}_j, \mathbf{e}_k) \cdot \mathbf{n} = -\nabla_j \mathbf{e}_k \cdot \mathbf{n} \\ &= -\sum_{l=1}^{3} \Gamma^l_{jk} \mathbf{e}_l \cdot \mathbf{n} = -\Gamma^3_{jk}. \end{aligned} \tag{4.15}$$

Hence, by collecting Eqs. (4.13)–(4.15) we obtain:

$$\nabla_k u^3 - \nabla_3 u^k = 2\frac{\partial}{\partial x^k}(u \cdot \mathbf{n}) - 2\sum_{j=1}^{3} \mathrm{II}_{jk} u^j + \frac{1}{\zeta} u^k. \tag{4.16}$$

Finally, in view of the kinematic boundary condition (the first equation in (4.6)) we have $u \cdot \mathbf{n} = 0$ on $\partial\Omega$; then by taking $k = 1, 2$ respectively, we recover the second equation in (4.6). Hence, in this chapter it is referred to as *the Navier boundary condition.*

Our goal in this chapter is to send $\nu \to 0^+$ and investigate if the solution u^ν converges, in suitable norms, to the solution of the Euler equation describing the motion of incompressible, inviscid fluids:

$$\begin{cases} \partial_t u + (u \cdot \nabla)u + \nabla p = 0 \qquad \text{in } [0,T] \times \Omega, \\ \nabla \cdot u = 0 \qquad \text{in } [0,T] \times \Omega, \\ u|_{t=0} = u_0 \qquad \text{on } \Omega. \end{cases} \tag{4.17}$$

(See Problem 1.7 in §1.2.3). For the Euler equation, it is natural to impose the *no-penetration* (namely, kinematic) boundary condition:

$$u \cdot \mathbf{n} = 0 \qquad \text{on } [0,T] \times \partial\Omega. \tag{4.18}$$

The system (4.17)(4.18) has a unique solution in $C([0,T_\star); H^{r+1}(\Omega;\mathbb{R}^3))$ for some $T_\star > 0$ depending on the initial data u_0 and the domain Ω, provided that $u_0 \in H^{r+1}(\Omega;\mathbb{R}^3)$ is divergence–free with $r > 5/2$; *cf.* the classical work by Ebin-Marsden [52].

The problem of vanishing inviscid limits has been a central topic in mathematical hydrodynamics (*cf.* the survey [42] by Constantin). In 1975, Swann [114] proved that for $\Omega = \mathbb{R}^3$, if the initial vorticity is in $H^{3+\delta}$, divergence–free and vanishing at spatial infinity, and the right–hand side of the vorticity equation lies in $C([0,T); H^2(\mathbb{R}^3))$ for some small T, then the Navier–Stokes equation with zero boundary condition has a unique strong solution, and the vanishing viscosity limit holds in $L^6 \cap \dot{H}^1$. Later, in 1986 Constantin [41] showed that for $\Omega = \mathbb{R}^3$, if the Euler equation with initial data $v_0 \in H^{m+2}(\mathbb{R}^3)$, $m \geq 3$ has a strong solution in $X = C([0,T]; H^m(\mathbb{R}^3))$ up to time T, then there exists $\nu_\star = \nu_\star(T, v_0)$ such that the corresponding Navier-Stokes equation for any $\nu \leq \nu_\star$ also has a strong solution in X, and the vanishing viscosity limit holds in H^m. In fact, for $\Omega = \mathbb{R}^d$, $d = 2$ or 3, for any $s > d/2+1$ and initial data $v_0 \in H^s$, the convergence can be obtained in H^s norm; *cf.* Masmoudi [91]. Moreover, in Constantin-Wu [44], vanishing viscosity limits are also proved on $\Omega = \mathbb{R}^2$ for the initial vorticity in $L^1(\mathbb{R}^2) \cap L^\infty_c(\mathbb{R}^2)$.

On the other hand, in the case that Ω is a bounded domain with boundary and the Navier-Stokes equation is equipped with the Dirichlet boundary condition, the vanishing viscosity limit in general fails: This is due to the formation of *boundary layers*, in which the Prandtl equations serve as a candidate for matching the Navier-Stokes and Euler equations; see *e.g.*, Alexandre-Wang-Xu-Yang [3], Gérard-Varet and Dormy [59], and the references cited therein. In contrast, when the Navier and kinematic boundary conditions are imposed to the Navier-Stokes equation, the vanishing viscosity limit can be established in the affirmative. In 2007, Xiao-Xin [124] proved that for the initial data in H^3 on a 3D flat domain, there exists $T_\star > 0$ such that the vanishing viscosity limit holds in $C([0,T_0];H^2) \cap L^p(0,T_0;H^3)$, for all $1 \leq p < \infty$. Various convergence results of this kind for a "modified Navier–type slip boundary condition" (first proposed by Bardos [9], which agrees with the Navier condition if and only if the domain is a part of the flat half–space) have been established, in $W^{k,p}$, H^s or L^p spaces and on 2D or 3D spatial domains; *cf.* Xiao-Xin [124], Beirão da Veiga-Crispo [12, 13], Bellout-Neustupa-Penel [16], Berselli-Spirito [19], Chen-Osborne-Qian [29], Clopeau-Mikelič-Robert [36], Kelliher [74], Wang-Xin-Zang [125] and Zhong [127], as well as the references cited therein.

Moreover, for the "true" Navier boundary conditions, Chen-Qian [30] and Iftimie-Planas [66] obtained the vanishing viscosity limit in $L^\infty_t L^2_x$ on smooth domains $\Omega \subset \mathbb{R}^3$ and $\mathbb{R}^d, d \geq 2$, provided that strong solutions exist in H^2 and $H^{d/2+1+\epsilon}$, respectively; also see the related results by Ifitimie-Raugel-Sell [67] on a 3D thin domain, and by Lopes Filho-Nussenzveig Lopes-Planas [84] on 2D domains. In addition, by computations in local coordinates Neustupa-Penel [100, 101] proved the convergence in $L^\infty(0,T_\star;H^1) \cap L^2(0,T_\star;H^2)$, provided that the initial data is in H^4, where $T_\star > 0$ is a constant depending only on Ω and the initial data. Furthermore, using the geometric vector field approach, Masmoudi-Rousset [92] established the existence of strong solutions in $L^\infty(0,T;E^m(\Omega;\mathbb{R}^3)) \cap L^2(0,T;H^{m+1}(\Omega;\mathbb{R}^3))$ for $m > 6$ and the inviscid limit in $L^\infty_t L^2_x$, where the anisotropic co-normal Sobolev space E^m is given by $E^m := \{u \in H^m_{\rm co} : \nabla u \in H^{m-1}_{\rm co}\}$, and $u \in H^m_{\rm co}$ whenever $\sum_{0\leq|l|\leq m} \|Z_l u\|_{L^2(\Omega)} < \infty$ with $\{Z_l\}$ spanning the space of vector fields tangential to $\partial\Omega$.

Our main result is Theorem 1.8 in §1.2.3, reproduced below:

Theorem 4.1 (Theorem 1.8 in §1.2.3) *Let $u \in C([0,T_\star);H^{r+1}(\Omega;\mathbb{R}^3) \cap K_2(\Omega))$ be the unique strong solution of the incompressible Euler equation* (1.19) *subject to the no-penetration boundary condition* (1.20)*, $r > 5/2$. Then there exists some $\nu_\star = \nu_\star\Big(T_\star, \int_0^{T_\star} \|u(t,\cdot)\|^2_{H^{r+1}(\Omega)}\,{\rm d}t\Big)$, such that whenever $0 < \nu \leq \nu_\star$, the strong solution u^ν of the Navier-Stokes equation* (1.11) *with Navier and kinematic boundary* *conditions* (1.15)(1.12) *exists in* $C([0,T_\star); H^{r+1}(\Omega;\mathbb{R}^3)\cap K_2(\Omega))$. *Moreover, there exists a constant* C *depending only on* $\nu_\star$, $T_\star$ *and* $\|u\|_{L^2([0,T_\star);H^{r+1}(\Omega))}$ *and* $\|\mathrm{II}\|_{C^r(\partial\Omega)}$, *such that*

$$\sup_{0\leq t<T_\star}\|u^\nu(t,\cdot)-u(t,\cdot)\|_{H^r(\Omega)}\leq C\nu^{1/3}. \tag{4.19}$$

In particular, as $\nu\to 0^+$, u^ν *converges to* u *in* $H^r(\Omega)$ *uniformly in time.*

Here and throughout, we write $H^r(\Omega;\mathbb{R}^3)=W^{2,r}(\Omega;\mathbb{R}^3)$ for the Sobolev space of vector fields $\phi:\Omega\to\mathbb{R}^3$: In the multi-index notation

$$\|\phi\|_{H^r(\Omega)}:=\Big(\sum_{0\leq|\alpha|\leq r}\int_\Omega|\nabla^\alpha\phi|^2\,\mathrm{d}x\Big)^{1/2}<\infty. \tag{4.20}$$

Moreover,

$$K_2(\Omega):=\{v\in L^2(\Omega;\mathbb{R}^3):\operatorname{div}v=0\}. \tag{4.21}$$

Also, we write $\nabla^{[s]}$ to denote a generic differential operator $(\nabla_{i_1}\nabla_{i_2}\cdots\nabla_{i_s})$ for order $s\geq 1$. The Einstein summation convention is observed throughout, where for the indices we write $i_1,i_2,\ldots,j,k,l,\ldots\in\{1,2,3\}$ and $\alpha,\beta,\gamma,\delta,\ldots\in\{1,2\}$. The angular brackets $\langle\cdot,\cdot\rangle$ denotes the Euclidean inner product of two vectors in $\mathbb{R}^3$. Furthermore, we write $f\lesssim g$ if $|f|\leq C|g|$ for a generic constant C depends only on r, $\|\mathrm{II}\|_{C^{r-1}(\bar\omega)}$ and ζ, and we write $f\simeq g$ whenever $f\lesssim g$ and $g\lesssim f$. $\mathcal{H}^2$ denotes the 2-dimensional Hausdorff measure. Finally, $\mathbf{curl}^r:=\mathbf{curl}\circ\ldots\circ\mathbf{curl}$ means the composition of r curl operations.

To prove Theorem 1.8 above, motivated by the references cited above [114, 41, 43, 19, 16, 74, 91, 92, 29, 30, 84, 100, 101, 66, 67, 124, 125, 127], we consider the evolution equation for the difference of the solutions of the Navier-Stokes and Euler equations. We derive the inviscid limits by energy estimates and the Gronwall inequality, for which the specific structure of the Navier boundary condition plays an essential role. In this process we establish that the lifespan of the strong solution of the Navier-Stokes equation is no shorter than that of the Euler equation, whenever the viscosity is sufficiently small. The formulation and the proof of this result is adapted from the classical work [41] by Constantin.

4.2 A div-curl estimate

In this section we show that the H^{r+1} norm of a divergence-free vector field is equivalent to the sum of the L^2 norms of its iterated curls up to the $(r+1)$-th order. It is a variant of the well-known div-curl estimate due to Caldéron-Zygmund for divergence-free vector fields:

Theorem 4.2 *Let $u \in H^{r+1}(\Omega;\mathbb{R}^3) \cap K_2(\Omega)$ for $r \geq 0$ satisfy the kinematic and Navier boundary conditions,* i.e., *Eq.* (4.6)*. Then there exists a universal constant $M = M(r,\Omega) > 0$ such that*

$$\|\nabla^{r+1}u\|^2_{L^2(\Omega)} \leq M\big(\sum_{l=0}^{r+1} \|\mathbf{curl}^l\, u\|^2_{L^2(\Omega)}\big). \tag{4.22}$$

Here and in the sequel, t is always suppressed when we only consider the spatial regularities. In the sequel the following *Sobolev trace theorem* is frequently used:

Lemma 4.2.1 (Theorem 5.36 in [4]) *Let Ω be a domain in $\mathbb{R}^n$ satisfying the uniform C^m-regularity condition, and suppose there exists a (m,p)-extension operator for Ω. Also suppose that*

$$mp < n \qquad \textit{and} \qquad p \leq q \leq p^* := \frac{(n-1)p}{n-mp}. \tag{4.23}$$

Then, we have the continuous embedding $W^{m,p}(\Omega) \hookrightarrow L^q(\partial\Omega)$.

In particular, $H^1(\Omega) \hookrightarrow L^q(\partial\Omega)$ for any $q \in [2,4]$ for the regular domain $\Omega \subset \mathbb{R}^3$.

Proof of Theorem 4.2. We prove the theorem by induction on r; the arguments are divided into seven steps.

1. First we establish the base case $r = 0$. Indeed, in view of the following identity (see Eq. (3.3) in Chen-Qian [30]):

$$\|\nabla u\|^2_{L^2(\Omega)} = \|\nabla \times u\|^2_{L^2(\Omega)} + \|\nabla\cdot u\|^2_{L^2(\Omega)} - \int_{\partial\Omega} (\nabla\cdot u)\langle u, \mathbf{n}\rangle \,\mathrm{d}\mathcal{H}^2 + \int_{\partial\Omega} \langle u\cdot\nabla u, \mathbf{n}\rangle \,\mathrm{d}\mathcal{H}^2,$$

for incompressible velocity field satisfying the kinematic boundary condition we have:

$$\|\nabla u\|^2_{L^2(\Omega)} = \|\nabla \times u\|^2_{L^2(\Omega)} + \int_{\partial\Omega} \mathrm{II}(u,u)\,\mathrm{d}\mathcal{H}^2. \tag{4.24}$$

Here we utilise the definition of the second fundamental form $\mathrm{II} := -\nabla\mathbf{n}$. As $\|\mathrm{II}\|_{L^\infty(\partial\Omega)} < \infty$, we bound

$$\Big|\int_{\partial\Omega} \mathrm{II}(u,u)\,\mathrm{d}\mathcal{H}^2\Big| \leq \|\mathrm{II}\|_{L^\infty(\partial\Omega)}\|u\|^2_{L^2(\partial\Omega)} \leq \epsilon\|\nabla u\|^2_{L^2(\Omega)} + \frac{C}{\epsilon}\|u\|^2_{L^2(\Omega)}, \tag{4.25}$$

thanks to the Sobolev trace inequality and Young's inequality. Thus, the case for $r = 0$ follows immediately by choosing ϵ suitably small.

2. In the sequel let us assume that the result holds for $r \geq 0$ and prove it for $(r+1)$. First of all we apply integration by parts twice to get:

$$\|\nabla^{r+1}u\|^2_{L^2(\Omega)}$$

$$
\begin{aligned}
&= \int_\Omega \big(\partial_{i_1}\cdots\partial_{i_{r+1}}u^k\big)\big(\partial_{i_1}\cdots\partial_{i_{r+1}}u^k\big)\,\mathrm{d}x\\
&= \int_\Omega \partial_{i_1}\Big\{\big(\partial_{i_2}\cdots\partial_{i_{r+1}}u^k\big)\big(\partial_{i_1}\cdots\partial_{i_{r+1}}u^k\big)\Big\}\,\mathrm{d}x\\
&- \int_\Omega \partial_{i_2}\Big\{\big(\partial_{i_2}\cdots\partial_{i_{r+1}}u^k\big)\big(\Delta\partial_{i_3}\cdots\partial_{i_{r+1}}u^k\big)\Big\}\,\mathrm{d}x\\
&+ \int_\Omega \big(\Delta\partial_{i_3}\cdots\partial_{i_{r+1}}u^k\big)\big(\Delta\partial_{i_3}\cdots\partial_{i_{r+1}}u^k\big)\Big\}\,\mathrm{d}x =: I+J+K.
\end{aligned}
\tag{4.26}
$$

Using the divergence theorem, the above three integrals are expressed as follows:

$$
\begin{cases}
I = \frac{1}{2}\int_{\partial\Omega}\partial_{\mathbf{n}}|\nabla^r u|^2\,\mathrm{d}\mathcal{H}^2,\\
J = \int_{\partial\Omega}\big(\partial_{i_2}\cdots\partial_{i_{r+1}}u^k\big)\big(\Delta\partial_{i_3}\cdots\partial_{i_{r+1}}u^k\big)\langle\nabla_{i_2},\mathbf{n}\rangle\,\mathrm{d}\mathcal{H}^2,\\
K = \int_\Omega |\nabla^{r-1}\psi|^2\,\mathrm{d}x.
\end{cases}
\tag{4.27}
$$

Here $\psi = \mathbf{curl}\,\omega = -\Delta u$ is the stream function.

3. Now let us bound the surface integral I. We introduce a local moving frame $\{e_1,e_2,e_3\}$ on the surface $\partial\Omega$, such that $e_1,e_2\in T(\partial\Omega)$ and $e_3=\mathbf{n}$. Thus

$$
\begin{aligned}
I &= \int_{\partial\Omega}\big(\nabla_{i_1}\cdots\nabla_{i_r}u^k\big)\big(\nabla_{i_1}\cdots\nabla_{i_r}\nabla_3 u^k\big)\,\mathrm{d}\mathcal{H}^2\\
&+ \int_{\partial\Omega}\big(\nabla_{i_1}\cdots\nabla_{i_r}u^k\big)\big([\nabla_3,\nabla_{i_1}\cdots\nabla_{i_r}]u^k\big)\,\mathrm{d}\mathcal{H}^2 =: I^1+I^2,
\end{aligned}
\tag{4.28}
$$

where $[\cdot,\cdot]$ denotes the commutator. Since the commutator is of lower order, the second term in the integrand of I^2 is schematically represented as $\nabla^{[r-1]}u^k$. More precisely, by the Ricci identity

$$
\nabla_i\nabla_j V^k - \nabla_j\nabla_i V^k = \sum_l C^{kl}_{ij}V_l \tag{4.29}
$$

for any vector field $V\in T\mathbb{R}^3$ and some constants C^{kl}_{ij}, each time we exchange ∇_3 with ∇_{i_j} a zero-th order term is obtained, so by Leibniz rule one deduces:

$$
[\nabla_3,\nabla_{i_1}\cdots\nabla_{i_r}]u^k \simeq \nabla^{[r-1]}u^k. \tag{4.30}
$$

Thus, the Cauchy-Schwarz inequality gives us

$$
\begin{aligned}
|I^2| &\lesssim \|u\|^2_{H^r(\partial\Omega)} + \|u\|^2_{H^{r-1}(\partial\Omega)}\\
&\lesssim \epsilon\|\nabla^{r+1}u\|^2_{L^2(\Omega)} + (1+\frac{1}{\epsilon})\|u\|^2_{H^r(\Omega)},
\end{aligned}
\tag{4.31}
$$

where the second line follows from the Sobolev trace embedding $H^{r+1}(\Omega)\hookrightarrow H^r(\partial\Omega)$ for $r\geq 0$, together with the interpolation inequalities.

4. To bound I^1 we make crucial use of the kinematic and Navier boundary conditions, *i.e.*, Eq. (4.6). First let us rewrite it in the local frame $\{e_1, e_2, e_3\}$ as follows:

$$\begin{cases} u^3 = 0, \\ \nabla_3 u^\beta = 2\mathrm{II}_{\alpha\beta} u^\alpha - \frac{1}{\zeta} u^\beta \qquad \text{for } \beta \in \{1,2\}. \end{cases} \tag{4.32}$$

Here $\nabla_\alpha u^3 \equiv 0$, so $\omega^1 = -\nabla_3 u^2$ and $\omega^2 = \nabla_1 \omega^3$. Moreover, from the incompressibility condition ($\nabla \cdot u = 0$) the following identities are easily deduced:

$$\begin{cases} \nabla_3 u^3 = -\nabla_\alpha u^\alpha, \\ \nabla_3 \nabla_3 u^\alpha = -\psi^\alpha - \nabla_\beta \nabla_\beta u^\alpha. \end{cases} \tag{4.33}$$

The key to the above Eqs. (4.32) and (4.33) is that the *normal derivatives* (∇_3) of the normal components can be replaced by tangential derivatives, and the normal derivatives of the tangential components can be replaced by lower order terms.

5. In this step we estimate I^1. For simplicity, we introduce the short-hand notations

$$\nabla^{(r-3)} A \cdot \nabla^{(r-3)} B := \big(\nabla_{i_1} \cdots \nabla_{i_{r-3}} A\big) \cdot \big(\nabla_{i_1} \cdots \nabla_{i_{r-3}} B\big), \tag{4.34}$$

for any functions A, B sufficiently regular. Then, let us split I^1 into six terms

$$I^1 = I^{1,1} + I^{1,2} + I^{1,3} + I^{1,4} + I^{1,5} + I^{1,6},$$

where

$$\begin{cases} I^{1,1} = \int_{\partial\Omega} \big(\nabla^{(r-3)} \nabla_\alpha \nabla_\beta u^3\big) \cdot \big(\nabla^{(r-3)} \nabla_\alpha \nabla_\beta \nabla_3 u^3\big) \, \mathrm{d}\mathcal{H}^2, \\ I^{1,2} = \int_{\partial\Omega} \big(\nabla^{(r-3)} \nabla_\alpha \nabla_\beta u^\gamma\big) \cdot \big(\nabla^{(r-3)} \nabla_\alpha \nabla_\beta \nabla_3 u^\gamma\big) \, \mathrm{d}\mathcal{H}^2, \\ I^{1,3} = \int_{\partial\Omega} \big(\nabla^{(r-3)} \nabla_\alpha \nabla_3 u^3\big) \cdot \big(\nabla^{(r-3)} \nabla_\alpha \nabla_3 \nabla_3 u^3\big) \, \mathrm{d}\mathcal{H}^2, \\ I^{1,4} = \int_{\partial\Omega} \big(\nabla^{(r-3)} \nabla_\alpha \nabla_3 u^\gamma\big) \cdot \big(\nabla^{(r-3)} \nabla_\alpha \nabla_3 \nabla_3 u^\gamma\big) \, \mathrm{d}\mathcal{H}^2, \\ I^{1,5} = \int_{\partial\Omega} \big(\nabla^{(r-3)} \nabla_3 \nabla_3 u^3\big) \cdot \big(\nabla^{(r-3)} \nabla_3 \nabla_3 \nabla_3 u^3\big) \, \mathrm{d}\mathcal{H}^2, \\ I^{1,6} = \int_{\partial\Omega} \big(\nabla^{(r-3)} \nabla_3 \nabla_3 u^\gamma\big) \cdot \big(\nabla^{(r-3)} \nabla_3 \nabla_3 \nabla_3 u^\gamma\big) \, \mathrm{d}\mathcal{H}^2. \end{cases} \tag{4.35}$$

In the sequel we estimate these terms one by one.

- First of all, we have $I^{1,1} = 0$, as $\nabla_\beta u^3 \equiv 0$.

- To estimate $I^{2,2}$, we first notice that

$$\begin{aligned} \big|\nabla^{(r-3)} \nabla_\alpha \nabla_\beta \nabla_3 u^\gamma\big| &= \Big|\nabla^{(r-3)} \nabla_\alpha \nabla_\beta \big\{2\mathrm{II}_{\gamma\beta} u^\beta - \frac{1}{\zeta} u^\gamma\big\}\Big| \\ &= C|\nabla^{[r-1]} u| + \{\text{l.o.t.}\}, \end{aligned} \tag{4.36}$$

where C depends on $\|\mathrm{II}\|_{C^{r-1}(\partial\Omega)}$ and ζ^{-1}, and $\{\text{l.o.t.}\}$ contains derivatives of u of order less then or equal to $(r-2)$. Next, considering the cases $\alpha=\beta$ and $\alpha\neq\beta$ separately, we deduce that

$$\begin{aligned}I^{1,2} &= \int_{\partial\Omega}\left(\nabla^{(r-3)}\Delta u^\gamma\right)\cdot\left(\nabla^{(r-3)}\Delta\nabla_3 u^\gamma\right)\mathrm{d}\mathcal{H}^2\\&\quad+2\int_{\partial\Omega}\left(\nabla^{(r-3)}\nabla_1\nabla_2 u^\gamma\right)\cdot\left(\nabla^{(r-3)}\nabla_1\nabla_2\nabla_3 u^\gamma\right)\mathrm{d}\mathcal{H}^2.\end{aligned}\tag{4.37}$$

For the first term, again by the Ricci identity we write

$$\nabla^{[r-3]}\Delta\nabla_3 u^\gamma=\nabla^{(r-3)}\nabla_3\Delta u^\gamma+\nabla^{[r-3]}(\Delta u^\gamma)=-\nabla^{[r-3]}\nabla_3\psi^\gamma-\nabla^{[r-1]}u^\gamma,\tag{4.38}$$

and we treat the second term as in Eq. (4.36) above. Thus we obtain

$$\begin{aligned}|I^{1,2}| &\lesssim \left|\int_{\partial\Omega}\left(\nabla^{[r-3]}\nabla_3\psi^\gamma\right)\cdot\left(\nabla^{[r-1]}u^\gamma\right)\mathrm{d}\mathcal{H}^2\right|+\int_{\partial\Omega}|\nabla^{[r-1]}u|^2\,\mathrm{d}\mathcal{H}^2\\&\lesssim \|u\|^2_{H^{r-1}(\partial\Omega)}+\|\nabla^{[r-3]}(\nabla\times\psi)\|^2_{L^2(\partial\Omega)}\end{aligned}\tag{4.39}$$

by Cauchy-Schwarz. But by the trace and interpolation inequalities we have

$$\|\nabla^{[r-3]}(\nabla\times\psi)\|^2_{L^2(\partial\Omega)}\lesssim\epsilon\|\nabla^{r+1}u\|^2_{H^{r+1}(\Omega)}+\frac{1}{\epsilon}\|u\|^2_{H^r(\Omega)},$$

so one estimates

$$|I^{1,2}|\lesssim\epsilon\|\nabla^{r+1}u\|^2_{H^{r+1}(\Omega)}+\frac{1}{\epsilon}\|u\|^2_{H^r(\Omega)}.\tag{4.40}$$

- For $I^{1,3}$, again by Eq. (4.33), the Ricci identity, the boundary condition (4.32) as well as the trace and interpolation inequalities, we have

$$\begin{aligned}|I^{1,3}| &= \left|\int_{\partial\Omega}\left(\nabla^{(r-3)}\nabla_\alpha\nabla_\beta u^\beta\right)\left(\nabla^{(r-3)}\nabla_\alpha\nabla_3\nabla_\gamma u^\gamma\right)\mathrm{d}\mathcal{H}^2\right|\\&\simeq\left|\int_{\partial\Omega}\left(\nabla^{(r-3)}\nabla_\alpha\nabla_\beta u^\beta\right)\left(\nabla^{(r-3)}\nabla_\alpha\nabla_\gamma\nabla_3 u^\gamma+\nabla^{[r-1]}u\right)\mathrm{d}\mathcal{H}^2\right|\\&\simeq\left|\int_{\partial\Omega}\left(\nabla^{(r-3)}\nabla_\alpha\nabla_\beta u^\beta\right)\left(\nabla^{(r-3)}\nabla_\alpha\nabla_\gamma\left\{2\mathrm{II}_{\delta\gamma}u^\delta-\frac{1}{\zeta}u^\gamma\right\}+\nabla^{[r-1]}u\right)\mathrm{d}\mathcal{H}^2\right|\\&\lesssim\|u\|^2_{H^{r-1}(\partial\Omega)}\lesssim\|u\|^2_{H^r(\Omega)}.\end{aligned}\tag{4.41}$$

- The treatment for $I^{1,4}$ is similar to the above for $I^{1,3}$:

$$\begin{aligned}|I^{1,4}| &= \left|\int_{\partial\Omega}\left(\nabla^{(r-3)}\nabla_\alpha\left\{2\mathrm{II}_{\beta\gamma}u^\beta-\frac{1}{\zeta}u^\gamma\right\}\right)\left(\nabla^{(r-3)}\nabla_\alpha\left\{-\psi^\alpha-\nabla_\delta\nabla_\delta u^\gamma\right\}\right)\mathrm{d}\mathcal{H}^2\right|\\&\lesssim\left|\int_{\partial\Omega}\left(\nabla^{[r-2]}u\right)\left(\nabla^{[r]}u\right)\mathrm{d}\mathcal{H}^2\right|\\&\lesssim\|u\|^2_{H^r(\partial\Omega)}\lesssim\epsilon\|\nabla^{r+1}u\|^2_{L^2(\Omega)}+\frac{1}{\epsilon}\|u\|^2_{H^r(\Omega)}.\end{aligned}\tag{4.42}$$

- For $I^{1,5}$, let us first substitute in Eq. (4.32) to derive

$$I^{1,5} = \int_{\partial\Omega} \big(\nabla^{(r-3)}\nabla_3\nabla_\beta u^\beta\big)\big(\nabla^{(r-3)}\nabla_3\nabla_3\nabla_\alpha u^\alpha\big)\,\mathrm{d}\mathcal{H}^2. \tag{4.43}$$

Then, applying the Ricci identity once to the first term and twice to the second term in the integrand, we have

$$\begin{aligned}
|I^{1,5}| &\lesssim \Big|\int_{\partial\Omega} \big(\nabla^{(r-3)}\nabla_\beta\nabla_3 u^\beta + \nabla^{[r-2]}u\big)\big(\nabla^{(r-3)}\nabla_\alpha\nabla_3\nabla_3 u^\alpha + \nabla^{[r-1]}u\big)\,\mathrm{d}\mathcal{H}^2\Big| \\
&\simeq \Big|\int_{\partial\Omega} \Big(\nabla^{(r-3)}\nabla_\beta\big\{2\mathrm{II}_{\beta\gamma}u^\gamma - \frac{1}{\zeta}u^\gamma\big\} + \nabla^{[r-2]}u\Big) \\
&\qquad \times \Big(\nabla^{(r-3)}\nabla_\alpha\big\{-\psi^\alpha - \nabla_\delta\nabla_\delta u^\alpha\big\} + \nabla^{[r-1]}u\Big)\,\mathrm{d}\mathcal{H}^2\Big| \\
&\lesssim \Big|\int_{\partial\Omega} \big(\nabla^{[r-2]}u\big)\big(\nabla^{[r]}u\big)\,\mathrm{d}\mathcal{H}^2\Big| \lesssim \epsilon\|\nabla^{r+1}u\|^2_{L^2(\Omega)} + \frac{1}{\epsilon}\|u\|^2_{H^r(\Omega)},
\end{aligned} \tag{4.44}$$

where the last line follows analogously to the final inequality in Eq. (4.42).

- Finally, for $I^{1,6}$, using Eqs. (4.32) and (4.33) one gets:

$$\begin{aligned}
|I^{1,6}| &= \Big|\int_{\partial\Omega} \Big(\nabla^{(r-3)}\big\{-\psi^\alpha - \nabla_\beta\nabla_\beta u^\alpha\big\}\Big)\Big(\nabla^{(r-3)}\nabla_3\nabla_3\big\{2\mathrm{II}_{\alpha\gamma}u^\gamma - \frac{1}{\zeta}u^\gamma\big\}\Big)\,\mathrm{d}\mathcal{H}^2\Big| \\
&\lesssim \|\nabla^{[r-1]}u\|^2_{L^2(\partial\Omega)} \lesssim \|u\|^2_{H^r(\Omega)}.
\end{aligned} \tag{4.45}$$

Hence, combining Eqs. (4.40)(4.41)(4.42)(4.44) and (4.45), the term I^1 is estimated by

$$|I^1| \lesssim \epsilon\|\nabla^{r+1}u\|^2_{L^2(\Omega)} + \frac{1}{\epsilon}\|u\|^2_{H^r(\Omega)}. \tag{4.46}$$

6. Now let us derive the estimates for the J term. In fact, J differs from I only by lower order terms, hence the estimates follow immediately. More precisely, we notice that

$$J = \int_{\partial\Omega} \big(\nabla_{i_2}\nabla^{(r-3)}\nabla_{i_{r+1}}u^k\big)\big(\Delta\nabla^{(r-3)}\nabla_{i_{r+1}}u^k\big)\langle\nabla_{i_2},\mathbf{n}\rangle\,\mathrm{d}\mathcal{H}^2, \tag{4.47}$$

where we relabel $\nabla^{(r-3)} = \nabla_{i_3}\cdots\nabla_{i_{r+1}}$ as before. Then, invoking the Ricci identity again, it follows that

$$\begin{aligned}
J &\lesssim \int_{\partial\Omega} \big(\nabla^{(r-3)}\nabla_{i_2}\nabla_{i_{r+1}}u^k + \nabla^{[r-2]}u\big)\big(\nabla^{(r-3)}\nabla_{i_{r+1}}\Delta u^k + \nabla^{[r-1]}u\big)\,\mathrm{d}\mathcal{H}^2 \\
&= \int_{\partial\Omega} \big(\nabla^{(r-3)}\nabla_{i_2}\nabla_{i_{r+1}}u^k\big)\big(\nabla^{(r-3)}\nabla_{i_{r+1}}\Delta u^k\big)\,\mathrm{d}\mathcal{H}^2 + \int_{\partial\Omega}\big(\nabla^{[r-2]}u\big)\big(\nabla^{[r-1]}u\big)\,\mathrm{d}\mathcal{H}^2 \\
&+ \int_{\partial\Omega} \big(\nabla_{i_{r+1}}\nabla^{(r-3)}\Delta u^k\big)\big(\nabla^{[r-2]}u\big)\,\mathrm{d}\mathcal{H}^2 + \int_{\partial\Omega}\big(\nabla^{[r-1]}u\big)\big(\nabla^{[r-1]}u\big)\,\mathrm{d}\mathcal{H}^2 \\
&=: J^1 + J^2 + J^3 + J^4.
\end{aligned} \tag{4.48}$$

Here, by the trace, interpolation and Young's inequalities, once again we have:

$$|J^2|, |J^4| \lesssim \|\nabla^{[r-1]}u\|_{L^2(\partial\Omega)}^2 \lesssim \epsilon\|\nabla^{r+1}u\|_{L^2(\Omega)}^2 + \frac{1}{\epsilon}\|u\|_{H^r(\Omega)}^2. \tag{4.49}$$

Also, J^1 has the same decomposition as I^1 into $I^{1,1}, \ldots, I^{1,6}$, so by Step 4 we conclude that

$$|J^1| \lesssim \epsilon\|\nabla^{r+1}u\|_{L^2(\Omega)}^2 + \frac{1}{\epsilon}\|u\|_{H^r(\Omega)}^2. \tag{4.50}$$

In the end, J^3 is estimated via integration by parts again: As $\partial\Omega$ is a 2-dimensional surface without boundary, the divergence theorem immediately yields that

$$J^3 = -\int_{\partial\Omega} \big(\nabla^{(r-3)}\Delta u^k\big)\big(\nabla_{i_{r+1}}\nabla^{[r-2]}u\big)\,\mathrm{d}\mathcal{H}^2 \simeq \int_{\partial\Omega} \big(\nabla^{[r-1]}u\big)\big(\nabla^{[r-1]}u\big)\,\mathrm{d}\mathcal{H}^2, \tag{4.51}$$

thus it verifies the same estimate for J^4.

7. Finally, putting together the estimates for I, J, K in Steps 1–5, we arrive at

$$\|\nabla^{r+1}u\|_{L^2(\Omega)}^2 \lesssim \epsilon\|\nabla^{r+1}u\|_{L^2(\Omega)}^2 + \frac{1}{\epsilon}\|u\|_{H^r(\Omega)}^2 + \int_\Omega |\nabla^{r-1}\psi|^2\,\mathrm{d}x. \tag{4.52}$$

Let us choose ϵ sufficiently small, so that

$$\|\nabla^{r+1}u\|_{L^2(\Omega)}^2 \lesssim \|u\|_{H^r(\Omega)}^2 + K \qquad \Big(\text{where } K := \int_\Omega |\nabla^{r-1}\psi|^2\,\mathrm{d}x \text{ as before}\Big). \tag{4.53}$$

The first term on the right-hand side, $\|u\|_{H^r(\Omega)}^2$, is bounded by $\sum_{l=0}^r \|\mathbf{curl}^l\, u\|_{L^2(\Omega)}^2$ up to a multiplicative constant, thanks to the induction hypothesis.

Now, it remains to show that K is bounded by the L^2 norm of iterated curls: This is achieved by iterating the constructions in Step 1. Indeed, relabelling the indices yields that

$$K = \int_\Omega \big(\Delta\partial_{i_1}\cdots\partial_{i_{r-1}}u^k\big)\big(\Delta\partial_{i_1}\cdots\partial_{i_{r-1}}u^k\big)\,\mathrm{d}x.$$

Then, as in Step 1, we integrate by parts twice to compute as follows:

$$\begin{aligned}
K &= \int_{\partial\Omega} \big(\Delta\partial_{i_2}\cdots\partial_{i_{r-1}}u^k\big)\big(\Delta\partial_{i_1}\cdots\partial_{i_{r-1}}u^k\big)\langle\partial_{i_1},\mathbf{n}\rangle\,\mathrm{d}\mathcal{H}^2 \\
&\quad - \int_{\partial\Omega} \big(\Delta\partial_{i_2}\cdots\partial_{i_{r-1}}u^k\big)\big(\Delta\Delta\partial_{i_3}\cdots\partial_{i_{r-1}}u^k\big)\langle\partial_{i_2},\mathbf{n}\rangle\,\mathrm{d}\mathcal{H}^2 \\
&\quad + \int_\Omega \big(\Delta\Delta\partial_{i_3}\cdots\partial_{i_{r-1}}u^k\big)\big(\Delta\Delta\partial_{i_3}\cdots\partial_{i_{r-1}}u^k\big)\,\mathrm{d}x \\
&= \frac{1}{2}\int_{\partial\Omega} \partial_{\mathbf{n}}\big|\nabla^{r-2}\Delta u\big|^2\,\mathrm{d}\mathcal{H}^2 - \int_{\partial\Omega} \big(\Delta\partial_{i_2}\cdots\partial_{i_{r-1}}u^k\big)\big(\Delta\Delta\partial_{i_3}\cdots\partial_{i_{r-1}}u^k\big)\langle\partial_{i_2},\mathbf{n}\rangle\,\mathrm{d}\mathcal{H}^2 \\
&\quad + \int_\Omega \big|\Delta\Delta\nabla^{r-3}u\big|^2\,\mathrm{d}x =: \tilde{I} + \tilde{J} + \tilde{K}.
\end{aligned} \tag{4.54}$$

It is crucial here that the flat gradient ∂_{i_l} and flat Laplacian Δ on $\mathbb{R}^3$ commute. Now, as $\tilde{I}$ and $\tilde{J}$ are obtained from I and J, respectively, by taking the trace over a pair of indices, so they satisfy the same estimates (which are given in Step 5 above). Repeating this process for finitely many times, we refine the estimate (4.53) as follows:

$$\|\nabla^{r+1}u\|^2_{L^2(\Omega)} \lesssim \sum_{l=0}^{r} \|\mathbf{curl}^l\, u\|^2_{L^2(\Omega)} + \begin{cases} \int_\Omega \left|\Delta^{\frac{r+1}{2}} u\right|^2 \mathrm{d}x, & \text{if } r \text{ is odd;} \\ \int_\Omega \left|\Delta^{\frac{r}{2}} \nabla u\right|^2 \mathrm{d}x, & \text{if } r \text{ is even.} \end{cases} \tag{4.55}$$

To conclude the proof, we notice that for the divergence-free vector field u, $\Delta u = -curl^2 u$. Thus, Eq. (4.55) gives the desired estimate for odd r. On the other hand, for even r we apply Eq. (4.24) in Step 1 of the same proof to the divergence-free vector field $(\Delta^{\frac{r}{2}} u)$, deducing that

$$\int_\Omega \left|\Delta^{\frac{r}{2}} \nabla u\right|^2 \mathrm{d}x = \int_\Omega \left|\nabla \Delta^{\frac{r}{2}} u\right|^2 \mathrm{d}x = \int_\Omega \left|\mathbf{curl}(\Delta^{\frac{r}{2}} u)\right|^2 \mathrm{d}x + \int_{\partial\Omega} \mathrm{II}(\Delta^{\frac{r}{2}} u, \Delta^{\frac{r}{2}} u)\, \mathrm{d}\mathcal{H}^2. \tag{4.56}$$

Here we need the commutativity of divergence, gradient and curl. For the first term on the right-hand side we have $\mathbf{curl}(\Delta^{\frac{r}{2}} u) = (-1)^{\frac{r}{2}} \mathbf{curl}^{r+1} u$, while for the second term we have

$$\int_{\partial\Omega} \mathrm{II}(\Delta^{\frac{r}{2}} u, \Delta^{\frac{r}{2}} u)\, \mathrm{d}\mathcal{H}^2 \lesssim \|\Delta^{\frac{r}{2}} u\|^2_{L^2(\partial\Omega)} \simeq \|\nabla^r u\|^2_{L^2(\partial\Omega)} \lesssim \epsilon' \|\nabla^{r+1}\|^2_{L^2(\Omega)} + \frac{1}{\epsilon'} \|u\|^2_{H^r(\Omega)}, \tag{4.57}$$

again by the boundedness of the second fundamental form, as well as the trace and interpolation inequalities. The proof is completed by choosing ϵ' sufficiently small. ■

Before concluding the section, we emphasise that Theorem 4.2 is independent of the Navier-Stokes equation (4.1). It is a general property of divergence-free vector fields satisfying the kinematic and Navier boundary conditions.

4.3 Energy estimate in H^r

In the current section let us derive the higher-order energy estimates. We show that the solution is in the spatial Sobolev space H^r ($r \geq 2$), provided that the initial data lies in the same space. This allows us to prove the existence of strong solution with spatial regularity H^r.

4.3.1 Local existence of weak solutions

Our starting point is the existence of *weak solutions* of the Navier-Stokes equation (4.1), under the kinematic and Navier boundary conditions. This is established via a Galerkin approximation scheme in [30], and is summarised as follows.

Remark 4.3.1 *Throughout in §5.3 let us drop the superscript ν in the solution u^ν to the Navier-Stokes equations, because we do not deal with the inviscid limits here.*

To begin with, we recall the following vector space direct sum

$$L^2(\Omega;\mathbb{R}^3) = K_2(\Omega) \bigoplus G_2(\Omega) \tag{4.58}$$

for the Hodge decomposition (the same as in §2), where $K_2(\Omega) := \big\{u \in L^2(\Omega;\mathbb{R}^3) : \mathrm{div}(u) = 0\big\}$ as before, and $G_2(\Omega) := \big\{u \in L^2(\Omega;\mathbb{R}^3) : \mathrm{curl}(u) = 0\big\}$. Letting $\mathbb{P}_\infty$ be the projection onto the first factor, we introduce the *Stokes operator*:

$$S := \mathbb{P}_\infty \circ \Delta, \tag{4.59}$$

where Δ is the flat Laplacian on $\mathbb{R}^3$. It is shown in §4 of [30] that S is densely defined on $K_2(\Omega)$ with a compact resolvant. Thus it has a discrete spectrum $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \ldots \downarrow -\infty$, and the corresponding eigenfunctions $\{a_n\}$ form a complete orthonormal basis of $K_2(\Omega)$. Now we look at the graded chain of finite-dimensional Hilbert spaces:

$$K_2(\Omega) \supset \ldots \supset V_N := \bigoplus_{j=1}^{N} \mathbb{R}a_j \supset V_{N-1} \supset \ldots \supset V_2 \supset V_1, \tag{4.60}$$

and denote by $\mathbb{P}_N : K_2(\Omega) \to V_N$ the canonical projection, *i.e.*,

$$(\mathbb{P}_N u)(t,x) := \sum_{j=1}^{N} a_j(x) \int_\Omega \big\langle a_j(y), u(t,y)\big\rangle \,\mathrm{d}y. \tag{4.61}$$

Thus $\mathbb{P}_\infty$ is indeed the L^2-limit of $\mathbb{P}_N$, as N tends to ∞.

In §5 of [30] the *weak formulation* for the Navier-Stokes equation with the Navier and kinematic boundary conditions is introduced as follows:

Definition 4.3.1 *For $T > 0$, we say $u \in L^2([0,T]; H^1(\Omega;\mathbb{R}^3))$ is a weak solution of the Cauchy problem of the Navier-Stokes equation* (4.1) *subject to the Navier and kinematic boundary conditions* (*Eq.* (4.6))*, provided that*

1. *$u(t,\cdot) \in K_2(\Omega)$ for each $t \in (0,T)$;*

2. *For each $\phi \in C^\infty([0,T]\times\Omega)$, $\phi(t,\cdot) \in K_2(\Omega)$, there holds:*

$$\begin{aligned}
&\int_\Omega \langle u(T,\cdot), \phi(T,\cdot)\rangle \,\mathrm{d}x \\
&= \int_\Omega \langle u_0(x), \phi(0,x)\rangle \,\mathrm{d}x + \int_0^T \int_\Omega \langle u, \partial_t\phi\rangle - \big\langle \mathbf{curl}\, u, (u\times\phi + \nu\,\mathbf{curl}\,\phi)\big\rangle \,\mathrm{d}x\,\mathrm{d}t \\
&-\frac{\nu}{\zeta}\int_0^T \int_{\partial\Omega} \langle u, \phi\rangle \,\mathrm{d}\mathcal{H}^2\,\mathrm{d}t + 2\nu \int_0^T \int_{\partial\Omega} \mathrm{II}(u,\phi)\,\mathrm{d}\mathcal{H}^2\,\mathrm{d}t;
\end{aligned} \tag{4.62}$$

3. The energy inequality is fulfilled:

$$\|u(T,\cdot)\|^2_{L^2(\Omega)} + 2\nu\int_0^T \|\nabla u(t,\cdot)\|^2_{L^2(\Omega)}\,\mathrm{d}t \\ + 2\nu\int_0^T\int_{\partial\Omega}\big(\frac{1}{\zeta}|u|^2 - \mathrm{II}(u,u)\big)\,\mathrm{d}\mathcal{H}^2\,\mathrm{d}t \leq \|u_0\|^2_{L^2(\Omega)}. \tag{4.63}$$

Then, by solving the projected equations obtained via taking $\mathbb{P}_N$ to Eq. (4.1) and deriving *a priori* estimates for the finite-dimensional approximate solutions $\{u_N\} \subset L^2([0,T]; H^1(\Omega;\mathbb{R}^3))$ uniformly in N, we are able to deduce the existence of weak solutions via a compactness argument. This is known as the Galerkin approximation, which relies crucially on the spectral analysis of the Stokes operator $S = \mathbb{P}_\infty \circ \Delta$.

More precisely, the following result is obtained:

Lemma 4.3.1 (Theorem 5.1 in [30]) *For any $u_0 \in K_2(\Omega)$ and $T > 0$, there exists a weak solution $u \in L^2([0,T]; K_2(\Omega))$ to the Cauchy problem in Eqs. (4.1) and (4.6). In fact, such u can be obtained as a weak subsequential limit of the family $\{u_N\}$ of finite-dimensional approximate solutions.*

4.3.2 Local existence of the strong solutions

Now, taking the weak solution u to the Navier-Stokes equations constructed by the Galerkin approximation scheme in Lemma 4.3.1 above, let us derive the *a priori* estimate for the higher-order energy of u in the Sobolev spaces H^r with $r \geq 2$. Indeed, the case $r = 2$ is proved in Theorem 5.3 of [30]. The higher order energy estimate is proved by induction on r, for which purpose the reduction of order of differentiations in the boundary terms is essential. This is achieved by exploiting by the kinematic and Navier boundary conditions. In particular, we need to explore the role of the curl operator, the rotation matrix $\mathcal{R}$ and the shape operator $\mathcal{S}$ (see §5.1).

Our main theorem of the section is as follows:

Theorem 4.3 *Let $u_0 \in H^r(\Omega;\mathbb{R}^3) \cap K_2(\Omega)$ for some $r \geq 2$. Then, there exists some $T_\star > 0$, such that the weak solution $u \in L^2([0,T_\star]; K_2(\Omega))$ satisfies*

$$\sup_{0\leq t\leq T_\star}\Big(\|u(t,\cdot)\|_{H^r(\Omega)} + \|\partial_t u(t,\cdot)\|_{H^{r-2}(\Omega)}\Big) \leq C, \tag{4.64}$$

where the constant $C > 0$ depends only on ζ, ν, $\|\mathrm{II}\|_{C^{r-1}(\partial\Omega)}$ and $\|u_0\|_{H^r(\Omega)}$. As a consequence, there exists a unique strong solution $u \in C([0,T_\star); H^r(\Omega;\mathbb{R}^3)) \cap C^1([0,T_\star); H^{r-2}(\Omega;\mathbb{R}^3))$.

Proof. We divide the arguments in six steps: In Step 1 we set up the equations for the energy estimate. Then, in Steps 2–5 we control $\|u(t,\cdot)\|_{H^r(\Omega)}$ and specify the lifespan $T_\star$. Finally, in Step 6 we derive the energy estimate for $\partial_t u$.

1. We first deduce the evolution equation for the iterated curls of the velocity field u. For this purpose, let us apply the divergence-free projection $\mathbb{P}_\infty : L^2(\Omega;\mathbb{R}^3) \to K_2(\Omega)$ to the Navier-Stokes equation (4.1) to get

$$\partial_t u - \nu\Delta u + \mathbb{P}_\infty(u\cdot\nabla u) = 0. \tag{4.65}$$

However, in 3 spatial dimensions we have the vectorial identity

$$u\cdot\nabla u = \frac{1}{2}\nabla(|u|^2) - u\times\omega,$$

hence the projected Navier-Stokes equation (4.65) is equivalent to

$$\partial_t u - \nu\Delta u + \mathbb{P}_\infty(u\times\omega) = 0. \tag{4.66}$$

For brevity of presentation, let us introduce the abbreviation as follows:

$$\Psi := \mathbb{P}_\infty(u\times\omega). \tag{4.67}$$

Then, taking iterated curls to Eq. (4.66), one obtains the evolution equation

$$\partial_t \mathbf{q}_r - \nu\Delta\mathbf{q}_r + \mathbf{curl}^r\,\Psi = 0. \tag{4.68}$$

Here and in the sequel we denote

$$\mathbf{q}_r := \mathbf{curl}^r\, u. \tag{4.69}$$

To derive the energy estimate, we multiply $\mathbf{q}_r$ to Eq. (4.68) and integrate over Ω:

$$0 = \frac{1}{2}\frac{d}{dt}\int_\Omega |\mathbf{q}_r|^2\,\mathrm{d}x - \nu\int_\Omega\langle\mathbf{q}_r,\Delta\mathbf{q}_r\rangle\,\mathrm{d}x + \int_\Omega\langle\mathbf{q}_r,\mathbf{curl}^r\,\Psi\rangle\,\mathrm{d}x, \tag{4.70}$$

and integrate the last two terms by parts. For the second term we have

$$\int_\Omega\langle\mathbf{q}_r,\Delta\mathbf{q}_r\rangle\,\mathrm{d}x = \int_{\partial\Omega}\langle(\mathbf{q}_r\cdot\nabla)\mathbf{q}_r,\mathbf{n}\rangle\,\mathrm{d}\mathcal{H}^2 - \int_\Omega|\nabla\mathbf{q}_r|^2\,\mathrm{d}x.$$

For the final term, noticing that for any 3-dimensional vector fields V and W, there holds (with the Einstein summation convention):

$$\begin{aligned}\int_\Omega\langle V,\mathbf{curl}\,W\rangle\,\mathrm{d}x &= \int_\Omega V^k\epsilon^{ijk}\partial_i W^j\,\mathrm{d}x\\ &= \int_{\partial\Omega}\epsilon^{ijk}V^kW^j\langle\partial_i,\mathbf{n}\rangle\,\mathrm{d}\mathcal{H}^2 - \int_\Omega\epsilon^{ijk}W^j(\partial_iV^k)\,\mathrm{d}x\end{aligned}$$

$$= \int_{\partial\Omega} \langle W \times V, \mathbf{n} \rangle \,\mathrm{d}\mathcal{H}^2 + \int_\Omega \langle \mathbf{curl}\, V, W \rangle \,\mathrm{d}x. \tag{4.71}$$

As a result,

$$\int_\Omega \langle \mathbf{q}_r, \mathbf{curl} \circ \mathbf{curl}^{r-1}\Psi \rangle \,\mathrm{d}x = \int_{\partial\Omega} \langle \mathbf{curl}^{r-1}\, \Psi \times \mathbf{q}_r, \mathbf{n} \rangle \,\mathrm{d}\mathcal{H}^2 + \int_\Omega \langle \mathbf{curl}\, \mathbf{q}_r, \mathbf{curl}^{r-1}\, \Psi \rangle \,\mathrm{d}x,$$

so Eq. (4.70) is written as

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega |\mathbf{q}_r|^2 \,\mathrm{d}x + \nu \int_\Omega |\nabla \mathbf{q}_r|^2 \,\mathrm{d}x \\
&= \nu \int_{\partial\Omega} \langle (\mathbf{q}_r \cdot \nabla)\mathbf{q}_r, \mathbf{n} \rangle \,\mathrm{d}\mathcal{H}^2 + \int_{\partial\Omega} \langle \mathbf{curl}^{r-1}\, \Psi \times \mathbf{q}_r, \mathbf{n} \rangle \,\mathrm{d}\mathcal{H}^2 + \int_\Omega \langle \mathbf{curl}\, \mathbf{q}_r, \mathbf{curl}^{r-1}\, \Psi \rangle \,\mathrm{d}x \\
&= I + J + K.
\end{aligned} \tag{4.72}$$

Our task is to estimate each of the I, J, K terms. As the $r = 2$ case is established in Theorem 5.3 of [30], in the sequel we assume that the result holds for $(r-1)$ and prove it for r, where $r > 5/2$.

2. For the I term in Eq. (4.72), observe that

$$I = \frac{\nu}{2} \int_{\partial\Omega} \partial_{\mathbf{n}} |\mathbf{q}_r|^2 \,\mathrm{d}\mathcal{H}^2, \tag{4.73}$$

which has been treated in the proof of Theorem 4.2. Indeed, it coincides with the I term in Eq. (4.27) up to a constant ν. Utilising the estimates in Steps 2–4 of the proof therein,

$$|I| \lesssim \epsilon \|\nabla \mathbf{q}_r\|^2_{L^2(\Omega)} + \frac{1}{\epsilon}\|u\|^2_{H^r(\Omega)}. \tag{4.74}$$

3. To prove for the K term in Eq. (4.72) we first notice that, by Cauchy-Schwarz and Young's inequality, there holds:

$$\begin{aligned}
|K| &\leq \|\mathbf{curl}\, \mathbf{q}_r\|_{L^2(\Omega)} \|\mathbf{curl}^{r-1}\, \Psi\|_{L^2(\Omega)} \\
&\lesssim \epsilon \|\nabla \mathbf{q}_r\|^2_{L^2(\Omega)} + \frac{1}{\epsilon}\|\mathbf{curl}^{r-1}\, (\mathbb{P}_\infty(u \times \omega))\|_{L^2(\Omega)}.
\end{aligned} \tag{4.75}$$

Since $H^s(\Omega)$ for $s > 3/2$ is a Banach algebra on $\mathbb{R}^3$ and $\mathbb{P}_\infty$ is a bounded linear operator, we have

$$\|\mathbf{curl}^{r-1}\, (\mathbb{P}_\infty(u \times \omega))\|_{L^2(\Omega)} \lesssim \|u\|^2_{H^r(\Omega)}. \tag{4.76}$$

This gives us the estimate for K.

4. Now it remains to control the boundary term J in Eq. (4.72):

$$J := \int_{\partial\Omega} \langle \mathbf{curl}^{r-1}(\mathbb{P}_\infty(u \times \omega)) \times \mathbf{q}_r, \mathbf{n} \rangle \,\mathrm{d}\mathcal{H}^2. \tag{4.77}$$

It is crucial to reduce the order of differentiation with the help of the boundary conditions, *i.e.*, Eq. (4.6). For this purpose let us establish the following *identity*:

$$\begin{aligned}&\pi\big\{\mathbf{curl}^k(\mathbb{P}_\infty(u\times\omega)\big\}\\ &=-\frac{1}{\zeta}\mathcal{R}\circ\pi\big\{\mathbf{curl}^{k-1}(\mathbb{P}_\infty(u\times\omega))\big\}+2\mathcal{R}\circ\pi\big\{\mathbf{curl}^{k-1}\,\mathcal{S}(\pi\circ\mathbb{P}_\infty(u\times\omega))\big\}.\end{aligned}\tag{4.78}$$

We recall that $\mathcal{R}$ is the orthogonal matrix rotating in the (x,y)-plane anti-clockwise by $\pi/2$, $\mathcal{S}$ is the shape operator corresponding to the second fundamental form II, and the operator π denotes the projection onto tangential components of a vector field,

$$\pi(V):=V-\langle V,\mathbf{n}\rangle,\tag{4.79}$$

viewed either as a 2-dimensional vector or a 3-dimensional vector with zero x_3 component. Here one only needs to consider the tangential components, as $\langle\mathbf{n}\times\mathbf{q}_r,\mathbf{n}\rangle\equiv 0$.

The above *identity* (4.78) is proved by induction. The base step $k=1$ is shown in the computations preceding Eq. (5.21) in [30]. Now we assume that the result holds for k; by the induction hypothesis, we have

$$\begin{aligned}&\pi\big\{\mathbf{curl}^{k+1}\,(\mathbb{P}_\infty(u\times\omega))\big\}\\ &=\pi\big\{\mathbf{curl}\circ\mathbf{curl}^k\,(\mathbb{P}_\infty(u\times\omega))\pi\big\}\\ &=\pi\circ\mathbf{curl}\Big\{-\frac{1}{\zeta}\mathcal{R}\circ\pi\Big(\mathbf{curl}^{k-1}\,\big(\mathbb{P}_\infty(u\times\omega)\big)\Big)+2\mathcal{R}\circ\pi\Big(\mathbf{curl}^{k-1}\,\mathcal{S}\,\big(\pi\circ\mathbb{P}_\infty(u\times\omega)\big)\Big)\Big\}\\ &=:\pi\circ\mathbf{curl}\big\{-\frac{1}{\zeta}\mathcal{R}\circ\pi(\mathbf{curl}^{k-1}\,\Psi)+2\mathcal{R}\circ\pi\circ\mathbf{curl}^{k-1}(\mathcal{S}\circ\pi\,\Psi)\big\},\end{aligned}\tag{4.80}$$

where we recall the short-hand notation Ψ in Eq. (4.67). Here and throughout, for a 2-dimensional vector field $W=(W^1,W^2)^\top$ (*e.g.*, $W=\mathcal{S}\circ\pi\Psi$), we define its curl as $\mathbf{curl}(W):=\mathbf{curl}(W^1,W^2,0)^\top$. It then suffices to check that

$$\pi\circ\mathbf{curl}\circ\mathcal{R}\circ\pi(V)=\mathcal{R}\circ\pi\circ\mathbf{curl}(V)\qquad\text{for any } V\in T(\partial\Omega).\tag{4.81}$$

From here, Eq. (4.80) immediately implies that

$$\pi\big\{\mathbf{curl}^{k+1}(\mathbb{P}_\infty(u\times\omega))\big\}=-\frac{1}{\zeta}\mathcal{R}\circ\pi\circ\mathbf{curl}^k\,(\Psi)+2\mathcal{R}\circ\pi\circ\mathbf{curl}^k\,(\mathcal{S}\circ\pi\,\Psi).\tag{4.82}$$

Indeed, we observe that

$$\pi\circ\mathbf{curl}\circ\mathcal{R}\circ\pi\begin{bmatrix}V^1\\V^2\\V^3\end{bmatrix}=\pi\circ\mathbf{curl}\begin{bmatrix}-V^2\\V^1\\0\end{bmatrix}=\pi\begin{bmatrix}-\nabla_3V^1\\-\nabla_3V^2\\0\end{bmatrix}=\begin{bmatrix}-\nabla_3V^1\\-\nabla_3V^2\end{bmatrix},$$

as well as

$$\mathcal{R}\circ\pi\circ\mathbf{curl}\begin{bmatrix}V^1\\V^2\\V^3\end{bmatrix}=\mathcal{R}\circ\pi\begin{bmatrix}\nabla_2V^3-\nabla_3V^2\\\nabla_3V^1-\nabla_1V^3\\\nabla_1V^2-\nabla_2V^1\end{bmatrix}=\mathcal{R}\begin{bmatrix}-\nabla_3V^2\\\nabla_3V^1\end{bmatrix}=\begin{bmatrix}-\nabla_3V^1\\-\nabla_3V^2\end{bmatrix},$$

since $V^3=0$ for tangential vector fields $V\in T(\partial\Omega)$. Hence Eq. (4.81) is proved, and the *identity* in Eq. (4.78) follows by induction.

Now, in view of the above identity, J can be expressed as follows:

$$\begin{aligned}J=\int_{\partial\Omega}\Big\langle\Big\{-\frac{1}{\zeta}\mathcal{R}\circ\pi\big(\mathbf{curl}^{r-2}\left(\mathbb{P}_\infty(u\times\omega)\right)\big)\\
+2\mathcal{R}\circ\pi\circ\mathbf{curl}^{r-2}\Big(\mathcal{S}\circ\pi\circ\mathbb{P}_\infty(u\times\omega)\Big)\Big\}\times\mathbf{q}_r,\mathbf{n}\Big\rangle\,\mathrm{d}\mathcal{H}^2.\end{aligned}\tag{4.83}$$

The crucial observation is that only the derivatives upto the $(r-1)$-th order of u are involved. This is because $\mathcal{S}$ has bounded norm in C^{r-1} due to the assumption of bounded extrinsic geometry, and $\mathbb{P}_\infty$, π and $\mathcal{R}$ are all smooth operators with the operator norm bounded by 1. Thus we arrive at the following estimates:

$$\begin{aligned}|J|&\lesssim\|\mathbf{curl}^{r-2}(u\times\omega)\|_{L^2(\partial\Omega)}\|\mathbf{q}_r\|_{L^2(\partial\Omega)}\\
&\lesssim\|u\|^2_{H^{r-1}(\partial\Omega)}\|u\|_{H^r(\partial\Omega)}\\
&\lesssim\Big(\epsilon\|u\|^2_{H^r(\Omega)}+\|u\|^2_{H^{r-1}(\Omega)}\Big)\Big(\epsilon\|u\|_{H^{r+1}(\Omega)}+\|u\|_{H^r(\Omega)}\Big)\\
&\simeq\epsilon^2\|u\|^2_{H^r(\Omega)}\|u\|_{H^{r+1}(\Omega)}+\epsilon\|u\|^3_{H^r(\Omega)}+\epsilon\|u\|^2_{H^{r-1}(\Omega)}\|u\|_{H^{r+1}(\Omega)}+\|u\|^2_{H^{r-1}(\Omega)}\|u\|_{H^r(\Omega)}\\
&\lesssim(\epsilon^2+\epsilon)\|u\|^2_{H^{r+1}(\Omega)}+\|u\|^4_{H^r(\Omega)}.\end{aligned}\tag{4.84}$$

In the above, the first line follows from Cauchy-Schwarz, the second line follows from the argument as for Eq. (4.76), the third line holds by Sobolev trace inequality, and the final line follows by interpolation and Young's inequality.

5. Now, combining the estimates in Steps 2–4 for I, J and K (in particular, Eqs. (4.74)(4.75)(4.76) and (4.84)), Eq. (4.72) becomes:

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega|\mathbf{q}_r|^2\,\mathrm{d}x+\nu\int_\Omega|\nabla\mathbf{q}_r|^2\,\mathrm{d}x\\
&\lesssim\epsilon\|\nabla^{r+1}u\|^2_{L^2(\Omega)}+\frac{1}{\epsilon}\|u\|^2_{H^r(\Omega)}\\
&+\epsilon\|\nabla\mathbf{q}_r\|^2_{L^2(\Omega)}+\frac{1}{\epsilon}\|u\|^2_{H^r(\Omega)}+(\epsilon+\epsilon^2)\|u\|^2_{H^{r+1}(\Omega)}+\|u\|^4_{H^r(\Omega)}.\end{aligned}\tag{4.85}$$

Here, in light of Theorem 4.2 we have $\|\nabla^{r+1}u\|_{L^2(\Omega)}\simeq\|\nabla\mathbf{q}_r\|_{L^2(\Omega)}\simeq\|\mathbf{curl}^{r+1}u\|_{L^2(\Omega)}$, and similarly $\|\mathbf{q}_r\|_{L^2(\Omega)}\lesssim\|u\|_{H^r(\Omega)}$. So, choosing ϵ suitably small in comparison with ν and considering the energy at the r-th order, namely

$$E_r:=\|\mathbf{q}_r\|^2_{L^2(\Omega)},\tag{4.86}$$

we obtain the following differential inequality:

$$E_r'(t) \leq E_r'(t) + \nu E_{r+1} \leq M E_r(t) + M E_r(t)^2, \tag{4.87}$$

where M depends on ν, ζ and $\|\mathrm{II}\|_{C^{r-1}(\Omega)}$.

To proceed, consider the auxiliary ODE:

$$\begin{cases} A'(t) = M(A(t) + A(t)^2), \\ A(0) = E_r(0) + \eta, \end{cases} \tag{4.88}$$

for arbitrary $\eta > 0$. It is solved explicitly by

$$A(t) = \frac{(\eta + E_r(0))e^{Mt}}{1 - (\eta + E_r(0))(e^{Mt} - 1)},$$

hence for any $t > 0$ before the blowup time

$$T_\star = \frac{1}{M} \log\Big(1 + \frac{1}{\eta + E_r(0)}\Big) > 0 \tag{4.89}$$

we have $A(t) < \infty$. Comparing the differential inequality (4.87) with the ODE (4.88), we find that $E_r(t) \leq A(t)$ for all $0 \leq t < T_\star$. In particular, as $\delta > 0$ is arbitrary, the upper bound for A (hence for E_r) is controlled by $E_r(0) := \|u_0\|^2_{H^r(\Omega)}$ and M. This proves

$$\sup_{0 \leq t < T_\star} E_r(t) \leq C. \tag{4.90}$$

6. It remains to derive the L^2 estimate for $\partial_t \mathbf{q}_r$. To this end, we take ∂_t to Eq. (4.68), multiply by $\partial_t \mathbf{q}_r$ and then integrate over Ω, which leads to

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\partial_t \mathbf{q}_r|^2 \,\mathrm{d}x - \nu \int_\Omega \langle \Delta \partial_t \mathbf{q}_r, \partial_t \mathbf{q}_r \rangle \,\mathrm{d}x + \int_\Omega \langle \mathbf{curl}^r\, \partial_t \Psi, \partial_t \mathbf{q}_r \rangle \,\mathrm{d}x = 0. \tag{4.91}$$

Applying integration by parts and the divergence theorem to the last two terms (*cf.* Eq. (4.71) for curl), we arrive at

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_\Omega |\partial_t \mathbf{q}_r|^2 \,\mathrm{d}x + \frac{\nu}{2}\int_\Omega |\nabla \partial_t \mathbf{q}_r|^2 \,\mathrm{d}x \\ =&\, \nu \int_{\partial\Omega} \big\langle (\nabla \partial_t \mathbf{q}_r \cdot \partial_t \mathbf{q}_r), \mathbf{n} \big\rangle \,\mathrm{d}\mathcal{H}^2 + \int_{\partial\Omega} \langle \mathbf{curl}^{r-1}\, \partial_t \Psi \times \partial_t \mathbf{q}_r, \mathbf{n} \rangle \,\mathrm{d}\mathcal{H}^2 \\ &+ \int_\Omega \langle \mathbf{curl}^{r-1}\, \partial_t \Psi, \mathbf{curl}\, \partial_t \mathbf{q}_r \rangle \,\mathrm{d}x =: \tilde{I} + \tilde{J} + \tilde{K}. \end{aligned} \tag{4.92}$$

Now, it is crucial to observe the following: Eq. (4.72) only differs from Eq. (4.92) by the time derivatives. More precisely, if we change the variables $(\partial_t \mathbf{q}_r, \partial_t \Psi)$ in the terms $(\tilde{I}, \tilde{J}, \tilde{K})$ in Eq. (4.92) to $(\mathbf{q}_r, \Psi)$, the terms (I, J, K) in Eq. (4.72) are immediately recovered. Furthermore, as the spatial derivatives commute with ∂_t, the integration by parts arguments in Steps 2–5 above all carry through. Therefore, given that $\|\partial_t u(t,\cdot)\|^2_{L^2(\Omega)} \leq C$ for $u_0 \in H^2(\Omega;\mathbb{R}^3)$ which is established in Theorem 5.3, [30], we repeat the arguments above for $(\partial_t \mathbf{q}_r, \partial_t \Psi)$ to deduce that

$$\sup_{0\leq t<T_\star} \|\partial_t u(t,\cdot)\|^2_{H^{r-2}(\Omega)} \leq C, \tag{4.93}$$

where C depends on $\|u_0\|^2_{H^r(\Omega)}$, ζ, ν, $\|\mathrm{II}\|_{C^{r-1}(\Omega)}$, and $T_\star$ is the same blowup time as in Step 5. Hence, combining Eqs. (4.90) and (4.93), we conclude the proof. ■

As a corollary, if the initial energy $E_r(0)$ of the fluid is uniformly bounded for all $r \in \mathbb{N}$, Eq. (4.89) implies that there exists a uniform life span $T_\star > 0$ for all the "levels" of the kinetic energy. Therefore, in view of $C^\infty(\Omega;\mathbb{R}^3) = \bigcap_{r\in\mathbb{N}} H^r(\Omega;\mathbb{R}^3)$ we arrive at the following

Corollary 4.3.1 *Let $u_0 \in C^\infty(\Omega;\mathbb{R}^3)$ be a divergence-free vector field, and let the domain $\Omega \subset \mathbb{R}^3$ have smooth second fundamental form. Then there exists $T_\star > 0$ such that the Cauchy problem of the Navier-Stokes equation* (4.1) *has a smooth solution $u \in C^1(0,T_\star;C^\infty(\Omega;\mathbb{R}^3))$ satisfying the kinematic and Navier boundary conditions in Eq.* (4.6)*.*

4.4 Vanishing viscosity limits

Now we prove the main result of this project, *i.e.*, Theorem 1.8 in §1.2.3. The crucial ingredient is an energy estimate of $(u^\nu - u)$, which is uniform in the viscosity ν.

Proof of Theorem 1.8. We divide the arguments into seven steps.

1. First of all, let us define

$$v^\nu := u^\nu - u, \qquad P^\nu = p^\nu - p. \tag{4.94}$$

Our goal is to show that, uniformly in ν, v^ν converges to zero in the H^r norm. First, subtracting the Euler equation (4.17) from the Navier-Stokes (4.1) yields that

$$\begin{cases} \partial_t v^\nu + \big\{(u+v^\nu)\cdot\nabla\big\}v^\nu - \nu\Delta v^\nu - (v^\nu\cdot\nabla)u - \nu\Delta u + \nabla P^\nu = 0, \\ \nabla\cdot v^\nu = 0. \end{cases} \tag{4.95}$$

Next, introducing the following abbreviation

$$\mathbf{V}^\nu_r := \mathbf{curl}^r\, v^\nu \in H^1(\Omega;\mathbb{R}^3), \tag{4.96}$$

and noticing that $\mathbf{curl}(\nabla P^\nu)=0$, we have

$$\partial_t \mathbf{V}^\nu_r + \mathbf{curl}^r \left\{\big((u+v^\nu)\cdot\nabla\big)v^\nu\right\} - \mathbf{curl}^r \left\{(v^\nu\cdot\nabla)u\right\} - \nu\Delta\mathbf{V}^\nu_r - \nu\Delta\mathbf{curl}^r\, u = 0. \tag{4.97}$$

In view of Theorem 4.2, in order to bound v^ν in H^r, it suffices to bound the L^2 norm of $\mathbf{V}^\nu_r$. To do so, let us multiply $\mathbf{V}^\nu_r$ to Eq. (4.97) and integrate over Ω:

$$\begin{aligned}0 = {}& \frac{1}{2}\frac{d}{dt}\int_\Omega |\mathbf{V}^\nu_r|^2\,\mathrm{d}x + \int_\Omega \mathbf{V}^\nu_r\cdot\mathbf{curl}^r\left\{\big((u+v^\nu)\cdot\nabla\big)v^\nu\right\}\mathrm{d}x\\ &- \int_\Omega \mathbf{V}^\nu_r\cdot\mathbf{curl}^r\left\{(v^\nu\cdot\nabla)u\right\}\mathrm{d}x - \nu\int_\Omega \mathbf{V}^\nu_r\cdot\Delta\mathbf{V}^\nu_r\,\mathrm{d}x - \nu\int_\Omega \mathbf{V}^\nu_r\cdot\Delta\mathbf{curl}^r\,u\,\mathrm{d}x.\end{aligned}\tag{4.98}$$

Using integration by parts and the divergence theorem, the fourth term on the right-hand side of Eq. (4.98) becomes

$$\begin{aligned}-\nu\int_\Omega \mathbf{V}^\nu_r\cdot\Delta\mathbf{V}^\nu_r\,\mathrm{d}x &= -\nu\int_{\partial\Omega}(\mathbf{V}^\nu_r\cdot\nabla\mathbf{V}^\nu_r)\cdot\mathbf{n}\,\mathrm{d}\mathcal{H}^2 + \nu\int_\Omega|\nabla\mathbf{V}^\nu_r|^2\,\mathrm{d}x\\ &= \frac{\nu}{2}\int_{\partial\Omega}\mathrm{II}(\mathbf{V}^\nu_r,\mathbf{V}^\nu_r)\,\mathrm{d}\mathcal{H}^2 + \nu\int_\Omega|\nabla\mathbf{V}^\nu_r|^2\,\mathrm{d}x,\end{aligned}$$

since $\mathrm{II} = -\nabla\mathbf{n}$. Meanwhile, the fifth term on the right-hand side of Eq. (4.98) satisfies

$$-\nu\int_\Omega \mathbf{V}^\nu_r\cdot\Delta\mathbf{curl}^r\,u\,\mathrm{d}x = -\nu\int_{\partial\Omega}\langle\mathbf{V}^\nu_r\cdot\nabla(\mathbf{curl}^r\,u),\mathbf{n}\rangle\,\mathrm{d}\mathcal{H}^2 + \nu\int_\Omega(\nabla\mathbf{V}^\nu_r):(\nabla\mathbf{curl}^r\,u)\,\mathrm{d}x,$$

where $A:B=\sum_{i,j=1}^3 A_{ij}B_{ji}$ for (3×3) matrices A and B. Therefore, the following identity is derived from Eq. (4.98):

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega|\mathbf{V}^\nu_r|^2\,\mathrm{d}x + \nu\int_\Omega|\nabla\mathbf{V}^\nu_r|^2\,\mathrm{d}x\\ &= -\frac{\nu}{2}\int_{\partial\Omega}\mathrm{II}(\mathbf{V}^\nu_r,\mathbf{V}^\nu_r)\,\mathrm{d}\mathcal{H}^2\\ &\quad-\nu\int_\Omega(\nabla\mathbf{V}^\nu_r):(\nabla\mathbf{curl}^r\,u)\,\mathrm{d}x - \int_\Omega\left\langle\mathbf{V}^\nu_r,\,\mathbf{curl}^r\left\{\big((u+v^\nu)\cdot\nabla\big)v^\nu\right\}\right\rangle\mathrm{d}x\\ &\quad+\nu\int_{\partial\Omega}\langle\mathbf{V}^\nu_r\cdot\nabla(\mathbf{curl}^r\,u),\mathbf{n}\rangle\,\mathrm{d}\mathcal{H}^2 + \int_\Omega\left\langle\mathbf{V}^\nu_r,\,\mathbf{curl}^r\left\{(v^\nu\cdot\nabla)u\right\}\right\rangle\mathrm{d}x\\ &=: \mathcal{V}^1+\mathcal{V}^2+\mathcal{V}^3+\mathcal{V}^4+\mathcal{V}^5.\end{aligned}\tag{4.99}$$

In the following steps let us estimate each of the terms $\mathcal{V}^1$–$\mathcal{V}^5$.

2. The bound for $\mathcal{V}^1$ is straightforward: As the second fundamental form is bounded in $C^{r-1}(\partial\Omega)$, we have

$$|\mathcal{V}^1| \lesssim \nu\|\mathbf{V}^\nu_r\|^2_{L^2(\partial\Omega)} \simeq \nu\|v^\nu\|^2_{H^r(\partial\Omega)} \lesssim \epsilon\nu\|\mathbf{curl}^{r+1}\,v^\nu\|^2_{L^2(\Omega)} + \frac{\nu}{\epsilon}\|v^\nu\|^2_{H^r(\Omega)}, \tag{4.100}$$

thanks to Cauchy-Schwarz, Theorem 4.2, Young's inequality and the Sobolev trace inequality. In addition, the bound for $\mathcal{V}^2$ is also easy:

$$|\mathcal{V}^2| \lesssim \nu\|\nabla\mathbf{V}^\nu_r\|_{L^2(\Omega)}\|\nabla\mathbf{curl}^r\,u\|_{L^2(\Omega)}$$

$$\simeq \nu \|\mathbf{curl}^{r+1} v^\nu\|^2_{L^2(\Omega)} \|u\|_{H^{r+1}(\Omega)}$$
$$\lesssim \epsilon\nu \|\mathbf{curl}^{r+1} v^\nu\|^2_{L^2(\Omega)} + \frac{\nu}{\epsilon}\|u\|^2_{H^{r+1}(\Omega)}, \tag{4.101}$$

again using Cauchy-Schwarz and Young's inequalities.

3. Next let us bound $\mathcal{V}^3 := -\int_\Omega \left\langle \mathbf{V}^\nu_r,\, \mathbf{curl}^r \left\{\big((u+v^\nu)\cdot\nabla\big)v^\nu\right\}\right\rangle \mathrm{d}x$ by applying a commutator estimate in the spirit of Kato-Ponce [73] to the non-linear convective terms. For this purpose, let us introduce the following abbreviation for the operator:

$$\mathcal{T} = \mathcal{T}(u, v^\nu) := (u + v^\nu)\cdot\nabla. \tag{4.102}$$

Then, $\mathcal{V}^3$ can be written as

$$\begin{aligned}\mathcal{V}^3 &= -\int_\Omega \langle \mathbf{V}^\nu_r, \mathbf{curl}^r \circ \mathcal{T}(v^\nu)\rangle \,\mathrm{d}x\\ &= -\int_\Omega \langle \mathbf{V}^\nu_r, \mathcal{T}(\mathbf{V}^\nu_r)\rangle \,\mathrm{d}x - \int_\Omega \langle \mathbf{V}^\nu_r, [\mathbf{curl}^r, \mathcal{T}]v^\nu\rangle \,\mathrm{d}x,\end{aligned} \tag{4.103}$$

where

$$[\mathbf{curl}^r, \mathcal{T}] = \mathbf{curl}^r \circ \mathcal{T} - \mathcal{T} \circ \mathbf{curl}^r$$

is the commutator of the differential operators. We note that the first term on the right-hand side of Eq. (4.103) equals

$$-\int_\Omega \langle \mathbf{V}^\nu_r, \mathcal{T}(\mathbf{V}^\nu_r)\rangle \,\mathrm{d}x = -\frac{1}{2}\int_{\partial\Omega} |\mathbf{V}^\nu_r|^2 (u+v^\nu)\cdot \mathbf{n}\,\mathrm{d}\mathcal{H}^2 = 0, \tag{4.104}$$

thanks to the divergence theorem and the incompressibility of u and v^ν, as well as the kinematic boundary conditions for u and v^ν.

For the second term on the right-hand side of Eq. (4.103), as $\mathcal{T}$ is a first-order differential operator, $[\mathbf{curl}^r, \mathcal{T}]$ is of order $\leq r$, with the coefficients involving the derivatives of u and v^ν. More precisely, we have

$$\Big|[\mathbf{curl}^r, \mathcal{T}]v^\nu\Big| \lesssim \sum_{l=1}^{r} \Big|\nabla^{[l]}u + \nabla^{[l]}v^\nu\Big|\Big|\nabla^{[r+1-l]}v^\nu\Big|, \tag{4.105}$$

by directly applying the Leibniz rule. Here the schematic symbol $\nabla^{[l]}$ denotes, as before, the derivatives up to the l-th order. Next, utilising the interpolation inequalities we obtain:

$$\begin{aligned}|\mathcal{V}^3| &= \Big|\int_\Omega \langle \mathbf{V}^\nu_r, [\mathbf{curl}^r, \mathcal{T}]v^\nu\rangle \,\mathrm{d}x\Big|\\ &\lesssim \sum_{l=1}^{r}\int_\Omega \Big\{|\nabla^{[r]}v^\nu||\nabla^{[l]}u||\nabla^{[r+1-l]}v^\nu| + |\nabla^{[r]}v^\nu||\nabla^{[l]}v^\nu||\nabla^{[r+1-l]}v^\nu|\Big\}\,\mathrm{d}x\\ &\lesssim \|v^\nu\|^2_{H^r(\Omega)}\|\nabla u\|_{L^\infty(\Omega)} + \|v^\nu\|_{H^r(\Omega)}\|u\|_{H^r(\Omega)}\|\nabla v^\nu\|_{L^\infty} + \|v^\nu\|^2_{H^r(\Omega)}\|\nabla v^\nu\|_{L^\infty(\Omega)}.\end{aligned} \tag{4.106}$$

Therefore, in view the Sobolev embedding $H^r \hookrightarrow L^\infty$ in 3-dimensions for $r > 2.5$, we have

$$|\mathcal{V}^3| \lesssim \Big(1 + \|u\|_{H^r(\Omega)}\Big)\|v^\nu\|^2_{H^r(\Omega)} + \|v^\nu\|^3_{H^r(\Omega)}. \tag{4.107}$$

In particular, we observe that the above upper bound for $\mathcal{V}^3$ involves only the derivatives up to the r-th order of v^ν.

4. Next, to treat the fourth term $\mathcal{V}^4 := \nu \int_{\partial\Omega} \langle \mathbf{V}^\nu_r \cdot \nabla(\mathbf{curl}^r\, u), \mathbf{n}\rangle \, \mathrm{d}\mathcal{H}^2$, it is crucial to take into account the Navier boundary condition. Indeed, using Einstein's summation convention and integrating by parts on $\partial\Omega$, we first obtain that

$$\begin{aligned}
\mathcal{V}^4 &= \nu \int_{\partial\Omega} (\mathbf{V}^\nu_r)^i \nabla_j (\mathbf{curl}^r\, u)^i \mathbf{n}^j \, \mathrm{d}\mathcal{H}^2 \\
&= -\nu \int_{\partial\Omega} \nabla_j (\mathbf{V}^\nu_r)^i (\mathbf{curl}^r\, u)^i \mathbf{n}^j \, \mathrm{d}\mathcal{H}^2 + \nu \int_{\partial\Omega} H \langle \mathbf{V}^\nu_r, \mathbf{curl}^r\, u\rangle \, \mathrm{d}\mathcal{H}^2 =: \mathcal{V}^{4,1} + \mathcal{V}^{4,2},
\end{aligned} \tag{4.108}$$

where $H = -\nabla_j \mathbf{n}^j$ is the mean curvature of the boundary. Here, by the boundedness of the second fundamental form, the second term is bounded by

$$\begin{aligned}
|\mathcal{V}^{4,2}| &\lesssim \nu(\|v^\nu\|^2_{H^r(\partial\Omega)} + \|u\|^2_{H^r(\partial\Omega)}) \\
&\lesssim \epsilon\nu(\|v^\nu\|^2_{H^{r+1}(\Omega)} + \|u\|^2_{H^{r+1}(\Omega)}) + \frac{\nu}{\epsilon}(\|v^\nu\|^2_{H^r(\Omega)} + \|u\|^2_{H^r(\Omega)}).
\end{aligned} \tag{4.109}$$

For $\mathcal{V}^{4,1}$, let us add and subtract $\nabla_i (\mathbf{V}^\nu_r)^j$ from the integrand to get

$$\begin{aligned}
\mathcal{V}^{4,1} &= -\nu \int_{\partial\Omega} \Big\{\nabla_j (\mathbf{V}^\nu_r)^i - \nabla_i (\mathbf{V}^\nu_r)^j\Big\} (\mathbf{curl}^r\, u)^i \mathbf{n}^j \, \mathrm{d}\mathcal{H}^2 \\
&\quad + \nu \int_{\partial\Omega} \nabla_i (\mathbf{V}^\nu_r)^j (\mathbf{curl}^r\, u)^i \mathbf{n}^j \, \mathrm{d}\mathcal{H}^2 =: \mathcal{V}^{4,1}_{\mathrm{a}} + \mathcal{V}^{4,1}_{\mathrm{s}}.
\end{aligned} \tag{4.110}$$

Here, by the incompressibility condition the integral vanishes for $i = j$, so we only consider $i \neq j$ in the summation. In this case, denoting by k the index in $\{1, 2, 3\}$ different from i, j, we have

$$\mathcal{V}^{4,1}_{\mathrm{a}} = \sigma\nu \int_{\partial\Omega} (\mathbf{curl}^{r+1}\, v^\nu)^k (\mathbf{curl}^r\, u)^i \mathbf{n}^j \, \mathrm{d}\mathcal{H}^2, \tag{4.111}$$

where $\sigma \in \{1, -1\}$ is a sign. To proceed, let us establish the following *claim*: Let $\pi : T\mathbb{R}^3 \to T(\partial\Omega)$ be the tangential projection as before. Then the iterated curls of v^ν satisfy the following Navier-type boundary condition:

$$\pi \circ \mathbf{curl}^{r+1}\, v^\nu = -\frac{1}{\zeta}\mathcal{R} \circ \pi(\mathbf{curl}^r v^\nu) + 2\mathcal{R} \circ \pi \circ \mathbf{curl}^r \circ \mathcal{S} \circ \pi(v^\nu) \qquad \text{on } \partial\Omega. \tag{4.112}$$

The proof of the above *claim* follows by induction on r, similar to the arguments in Step 4 in the proof of Theorem 4.3. Indeed, for $r = 0$ it reduces to the Navier boundary condition (4.6), so we assume that the result holds for r and prove it for $(r+1)$. Taking a moving frame $\{\nabla_1, \nabla_2, \nabla_3\}$ adapted to the surface $\partial\Omega$, such that $\nabla_1, \nabla_2 \in T(\partial\Omega)$ and $\nabla_3 = \mathbf{n}$, by the induction hypothesis we have

$$\begin{aligned}
\pi(\mathbf{curl}^{r+1} v^\nu) &= \pi \circ \mathbf{curl}\Big(-\frac{1}{\zeta}\mathcal{R} \circ \pi(\mathbf{curl}^{r-1} v^\nu) + 2\mathcal{R} \circ \pi \circ \mathbf{curl}^{r-1} \circ \mathcal{S} \circ \pi(v^\nu)\Big) \\
&= \begin{bmatrix} -\nabla_3\Big(-\frac{1}{\zeta}\mathcal{R} \circ \pi(\mathbf{curl}^{r-1} v^\nu) + 2\mathcal{R} \circ \pi \circ \mathbf{curl}^{r-1} \circ \mathcal{S} \circ \pi(v^\nu)\Big)^2 \\ \nabla_3\Big(-\frac{1}{\zeta}\mathcal{R} \circ \pi(\mathbf{curl}^{r-1} v^\nu) + 2\mathcal{R} \circ \pi \circ \mathbf{curl}^{r-1} \circ \mathcal{S} \circ \pi(v^\nu)\Big)^1 \end{bmatrix}.
\end{aligned}$$

As $\mathcal{R}((V^1, V^2)^\top) = (-V^2, V^1)^\top$ for any two-dimensional vectors V, the above equalities give us

$$\begin{aligned}
\pi(\mathbf{curl}^{r+1} v^\nu) &= \begin{bmatrix} -\nabla_3\Big(-\frac{1}{\zeta}(\mathbf{curl}^{r-1} v^\nu)^1 + 2(\mathbf{curl}^{r-1} \circ \mathcal{S} \circ \pi(v^\nu))^1\Big) \\ -\nabla_3\Big(-\frac{1}{\zeta}(\mathbf{curl}^{r-1} v^\nu)^2 + 2(\mathbf{curl}^{r-1} \circ \mathcal{S} \circ \pi(v^\nu))^2\Big) \end{bmatrix} \\
&= -\frac{1}{\zeta}\Big\{ -\nabla_3(\mathbf{curl}^{r-1} v^\nu)\Big\} - 2\nabla_3\Big\{\mathbf{curl}^{r-1} \circ \mathcal{S} \circ \pi(v^\nu)\Big\} \\
&= -\frac{1}{\zeta}\Big\{\mathcal{R} \circ \pi \circ \mathbf{curl}\,(\mathbf{curl}^{r-1} v^\nu)\Big\} + 2\Big\{\mathcal{R} \circ \pi \circ \mathbf{curl}\,(\mathbf{curl}^{r-1} \circ \mathcal{S} \circ \pi(v^\nu))\Big\} \\
&= -\frac{1}{\zeta}\Big\{\mathcal{R} \circ \pi \circ \mathbf{curl}^r\,(v^\nu)\Big\} + 2\Big\{\mathcal{R} \circ \pi \circ \mathbf{curl}^r \circ \mathcal{S} \circ \pi(v^\nu)\Big\}. \qquad (4.113)
\end{aligned}$$

Hence Eq. (4.112) is proved.

This *claim* shows that the iterated curls of v^ν can be expressed on the boundary $\partial\Omega$ by derivatives of v^ν upto the r-th order, together with the bounded operators $\mathcal{R}$, $\mathcal{S}$ and π. Thus, Eq. (4.111) becomes

$$\begin{aligned}
\mathcal{V}^{4,1}_{\mathrm{a}} &= \sigma\nu \int_{\partial\Omega} \Big\{ -\frac{1}{\zeta}[\mathcal{R} \circ \pi(\mathbf{curl}^r v^\nu)]^k (\mathbf{curl}^r u)^i \mathbf{n}^j \Big\}\, \mathrm{d}\mathcal{H}^2 \\
&\quad + 2\sigma\nu \int_{\partial\Omega} \Big\{[\mathcal{R} \circ \pi \circ \mathbf{curl}^r \circ \mathcal{S} \circ \pi(v^\nu)]^k (\mathbf{curl}^r u)^i \mathbf{n}^j\Big\}\, \mathrm{d}\mathcal{H}^2.
\end{aligned}$$

Here the non-zero contributions only come from the tangential components of $\mathbf{curl}^{r+1} v^\nu$, and $\mathbf{n}^j = 0$ unless $j = 3$ in the moving frame $\{\nabla_1, \nabla_2, \nabla_3\}$, which forces $k \in \{1, 2\}$. Thus we arrive at the following estimate:

$$\begin{aligned}
|\mathcal{V}^{4,1}_{\mathrm{a}}| &\lesssim \nu\Big(\|v^\nu\|^2_{H^r(\partial\Omega)} + \|u\|^2_{H^r(\partial\Omega)}\Big) \\
&\lesssim \nu\Big(\epsilon\|\mathbf{curl}^{r+1} v^\nu\|^2_{H^{r+1}(\Omega)} + \|v^\nu\|^2_{H^r(\Omega)} + \epsilon\|u\|^2_{H^{r+1}(\Omega)} + \|u\|^2_{H^r(\Omega)}\Big). \qquad (4.114)
\end{aligned}$$

On the other hand, for $\mathcal{V}^{4,1}_{\mathrm{s}}$ let us integrate by parts once more to get

$$\mathcal{V}^{4,1}_{\mathrm{s}} = -\nu \int_{\partial\Omega} (\mathbf{V}^\nu_r)^j (\mathbf{curl}^r u)^i \nabla_i \mathbf{n}^j \, \mathrm{d}\mathcal{H}^2 = \nu \int_{\partial\Omega} \mathrm{II}(\mathbf{V}^\nu_r, \mathbf{curl}^r u)\, \mathrm{d}\mathcal{H}^2. \qquad (4.115)$$

This is because $\operatorname{div} \circ \mathbf{curl}^r u = \mathbf{curl}^r \circ \operatorname{div} u = 0$, and $\mathrm{II} = -\nabla \mathbf{n}$ by the definition of the second fundamental form. By assumption there holds $\|\mathrm{II}\|_{L^\infty(\partial\Omega)} \leq C$, so

$$
\begin{aligned}
|\mathcal{V}_{\mathrm{s}}^{4,1}| &\lesssim \nu \|\mathbf{V}_r^\nu\|_{L^2(\partial\Omega)} \|\mathbf{curl}^r u\|_{L^2(\partial\Omega)} \\
&\lesssim \epsilon\nu \Big(\|\mathbf{curl}^{r+1}\|_{L^2(\Omega)}^2 + \|u\|_{H^{r+1}(\Omega)}^2 \Big) + \frac{\nu}{\epsilon} \Big(\|v^\nu\|_{H^r(\Omega)}^2 + \|u\|_{H^r(\Omega)}^2 \Big).
\end{aligned} \tag{4.116}
$$

Thus, putting together Eqs. (4.108)(4.114)(4.116) and (4.109), we conclude that

$$
|\mathcal{V}^4| \lesssim \epsilon\nu \Big(\|\mathbf{curl}^{r+1} v^\nu\|_{L^2(\Omega)}^2 + \|u\|_{H^{r+1}(\Omega)}^2 \Big) + \frac{\nu}{\epsilon} \Big(\|v^\nu\|_{H^r(\Omega)}^2 + \|u\|_{H^r(\Omega)}^2 \Big). \tag{4.117}
$$

5. The bound for $\mathcal{V}^5 := \int_\Omega \big\langle \mathbf{curl}^r v^\nu, \mathbf{curl}^r (v^\nu \cdot \nabla u) \big\rangle \,\mathrm{d}x$ proceeds by the Leibniz rule:

$$
\begin{aligned}
|\mathcal{V}^5| &\lesssim \int_\Omega |\mathbf{V}_r^\nu| \times \Big(\sum_{l=0}^r |\nabla^{[l]} v^\nu| |\nabla^{[r-l+1]} u| \Big) \,\mathrm{d}x \\
&\lesssim \int_\Omega |\nabla^{[r]} v^\nu| \Big(|\nabla^r v^\nu| |\nabla u| + |v^\nu| |\nabla^{r+1} u| \Big) \,\mathrm{d}x \\
&\lesssim \|v^\nu\|_{H^r(\Omega)}^2 \|\nabla u\|_{L^\infty(\Omega)} + \|v^\nu\|_{H^r(\Omega)} \|v^\nu\|_{L^\infty(\Omega)} \|u\|_{H^{r+1}(\Omega)} \\
&\lesssim \|v^\nu\|_{H^r(\Omega)}^2 \|u\|_{H^{\frac{5}{2}+\delta}(\Omega)} + \|v^\nu\|_{H^r(\Omega)} \|v^\nu\|_{H^{\frac{3}{2}+}(\Omega)} \|u\|_{H^{r+1}(\Omega)},
\end{aligned}
$$

where the last two lines hold by interpolation and the Sobolev embedding $H^2(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3)$, and δ denotes an arbitrarily small positive number. Thus, one obtains that

$$
|\mathcal{V}^5| \lesssim \|v^\nu\|_{H^r(\Omega)}^2 \|u\|_{H^{r+1}(\Omega)}, \tag{4.118}
$$

whenever $r \geq 2$.

6. Therefore, collecting Eqs. (4.100)(4.101)(4.107)(4.117) and (4.118), we arrive at

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt} \int_\Omega |\mathbf{V}_r^\nu|^2 \,\mathrm{d}x + \nu \int_\Omega |\nabla \mathbf{V}_r^\nu|^2 \,\mathrm{d}x \\
\lesssim\, &\epsilon\nu \|\mathbf{curl}^{r+1} v^\nu\|_{L^2(\Omega)}^2 + \Big(\epsilon\nu + \frac{\nu}{\epsilon} + \|u\|_{H^{r+1}(\Omega)} \Big) \|v^\nu\|_{H^r(\Omega)}^2 \\
&+ \|v^\nu\|_{H^r(\Omega)}^3 + \Big(\epsilon\nu + \frac{\nu}{\epsilon} \Big) \|u\|_{H^{r+1}(\Omega)}^2.
\end{aligned} \tag{4.119}
$$

Thus, we can choose $\epsilon > 0$ to be so small that

$$
\begin{aligned}
&\frac{d}{dt} \int_\Omega |\mathbf{V}_r^\nu|^2 \,\mathrm{d}x + \nu \|\mathbf{curl}^{r+1} v^\nu\|_{L^2(\Omega)}^2 \\
\leq\, &C_1 \Big(1 + \|u\|_{H^{r+1}(\Omega)} \Big) \|v^\nu\|_{H^r(\Omega)}^2 + \|v^\nu\|_{H^r(\Omega)}^3 + C_2 \nu \|u\|_{H^{r+1}(\Omega)}^2,
\end{aligned} \tag{4.120}
$$

where the constants C_1, C_2 depend on ν_0 and ϵ. Here it is crucial to choose ϵ that depends only on $\|\mathrm{II}\|_{C^r(\partial\Omega)}$; in particular, it is *independent of* ν.

7. The arguments in this final step are adapted from P. Constantin [41], in which the analogous result is shown for $\Omega = \mathbb{R}^3$ or $\mathbb{T}^3$. For any fixed $T \in (0, T_\star)$, we set

$$\begin{cases} \alpha(t) := \exp\big\{ -C_1 t - C_1 \int_0^t \|u(s,\cdot)\|_{H^{r+1}(\Omega)}\,\mathrm{d}s \big\}, \\ F(t) := C_2\alpha(t)\|u(t,\cdot)\|^2_{H^{r+1}(\Omega)}, \\ G := C_1[\alpha(T)]^{-1} = \text{const.}, \\ \Phi(t) := \alpha(t)\|v^\nu(t,\cdot)\|^2_{H^r(\Omega)}. \end{cases} \tag{4.121}$$

Then, by Steps 1–6, Φ satisfies the ordinary differential inequality

$$\Phi'(t) \leq \nu F(t) + G\Phi(t)^{3/2} \qquad \text{on } [0,T]. \tag{4.122}$$

For some parameter $\beta > 0$ to be chosen later, we divide Eq. (4.122) by $(1+\beta\Phi(t))^{3/2}$ to get

$$\frac{\Phi'(t)}{(1+\beta\Phi(t))^{3/2}} \leq \nu F(t) + \frac{G}{\beta^{3/2}}.$$

Thus, by integrating from 0 to $t \in [0,T]$ we obtain the estimate as follows:

$$\frac{2}{\beta}\Big(1 - \frac{1}{\sqrt{1+\beta\Phi(t)}}\Big) \leq \nu \int_0^T F(t)\,\mathrm{d}t + \frac{GT}{\beta^{3/2}},$$

namely that

$$\frac{1}{\sqrt{1+\beta\Phi(t)}} \geq 1 - \frac{\beta\nu}{2}\int_0^T F(t)\,\mathrm{d}t - \frac{GT}{2\sqrt{\beta}}. \tag{4.123}$$

Notice that the right-hand side of Eq. (4.123) is maximised when the two negative terms are equal, *i.e.*,

$$\beta = \Big(\frac{GT}{\nu\int_0^T F(t)\,\mathrm{d}t}\Big)^{2/3}. \tag{4.124}$$

Then the right-hand side equals $1 - \frac{\left(\nu\int_0^T F(t)\,\mathrm{d}t\right)^{1/3}}{(GT)^{2/3}}$. It is greater than or equal to 1/2 if and only if $\nu \leq \frac{G^2T^2}{8\int_0^T F(t)\,\mathrm{d}t}$. On the other hand, by Eq. (4.123) $\frac{1}{\sqrt{1+\beta\Phi(t)}} \geq 1/2$ for all $0 \leq t \leq T$, namely that $\Phi(t) \leq 3/\beta$ on $[0,T]$.

In summary, we have established the following: If we set

$$\nu_\star := \sup_{T\in(0,T_\star)}\Big(\frac{G^2T^2}{8\int_0^T F(t)\,\mathrm{d}t}\Big), \tag{4.125}$$

then, whenever $0 < \nu \leq \nu_\star$, we have

$$\Phi(t) \leq 3\Big(\frac{\int_0^T F(t)\,\mathrm{d}t}{GT}\Big)^{2/3}\nu^{2/3}, \tag{4.126}$$

which is equivalent to

$$\|u^\nu(t,\cdot) - u(t,\cdot)\|^2_{H^r(\Omega)}$$

$$\leq \left(3 \frac{\left[\int_0^T F(t)\,\mathrm{d}t \right]^{2/3} \exp\left\{ C_1 T + C_1 \int_0^T \|u(s,\cdot)\|_{H^{r+1}(\Omega)}\,\mathrm{d}s \right\}}{(GT)^{2/3}} \right) \times \nu^{2/3}. \tag{4.127}$$

This holds for any $t \in [0,T]$, where T is an arbitrary number in $(0, T_\star)$. Hence the Navier-Stokes solution ν does not blow up, provided that $0 \leq \nu \leq \nu_\star$. Thus we conclude the proof by taking the supremum over $[0, T_\star)$. ■

In passing, we remark that the positive *uniform* lifespan $T_\star$ for $\{v^\nu\}$ does not directly follow from our proof of Theorem 4.3. In fact, the constant M is proportional to ν^{-1} in Eq. (4.87); hence the lifespan for the Navier-Stokes equations with viscosity ν is proportional to $1/M$. In Step 7 of the proof above, we proved that the Navier-Stokes solutions do not blow up in H^{r+1} before the maximal lifespan of the Euler solutions, subject to the kinematic and Navier boundary conditions. As remarked before, this result may be compared with Constantin [41].

4.5 Discussions on the modified slip-type boundary condition

In §5.1 a modified version of the Navier boundary condition, originally introduced by Bardos ([9]) and Solonnikov-Ščadilov ([110]), has been briefly discussed. Physically it describes the phenomenon that the tangential part of the normal vector field of the Cauchy stress tensor is uniformly vanishing, and it agrees with the Navier boundary condition if and only if the boundary $\partial\Omega$ is flat. Together with the kinematic boundary condition, we have

$$\begin{cases} u^\nu \cdot \mathbf{n} = 0 & \text{on } \partial\Omega, \\ \omega^\nu \times \mathbf{n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.128}$$

This is referred to as the so-called *slip-type boundary condition* (or the Navier-type slip boundary condition) in the sequel. In the two papers [14, 15] by Beirão da Veiga-Crispo, the vanishing viscosity limit is analysed for the Navier-Stokes equations subject to the boundary conditions in Eq. (4.128), and the Euler equations subject to the no-penetration boundary condition in Eq. (4.18). They showed that the vanishing viscosity limit in general fails for the strong solutions, whenever the domain is non-flat; *cf.* Theorem 1.9 in §1.2.3, reproduced below:

Theorem 4.4 (Theorem 1.9 in §1.2.3. Beirão da Veiga-Crispo [14, 15]) *Let $\Omega \subset \mathbb{R}^3$ be a bounded, regular, open connected set. Let the admissible initial data u_0*

be given for the initial-boundary value problems Eqs. (4.1)(4.6)(4.17) *and* (4.18)*, such that the following condition holds:*

$$\omega_0(x_0) \neq 0 \qquad \text{for some } x_0 \in \partial\Omega \text{ such that } K(x_0) \neq 0. \tag{4.129}$$

Then, for arbitrary $\delta > 0$, $p, q \geq 1$ and $s > 1$, the "strong" inviscid limit fails:

$$u^\nu \nrightarrow u \text{ in } L^p([0,\delta]; W^{s,q}(\Omega;\mathbb{R}^3)). \tag{4.130}$$

Here and throughout K denotes the Gauss curvature of $\partial\Omega$; also, the following definition is introduced (*cf.* Definitions 2.1, 2.3 in [15]):

Definition 4.5.1 *We say that*

1. *$u_0 \in C^\infty(\Omega;\mathbb{R}^3)$ is an admissible initial data if $\nabla \cdot u_0 = 0$ in the closure $\overline{\Omega}$, as well as $u_0 \cdot \mathbf{n} = 0$ and $\omega_0 \times \mathbf{n} = 0$ on $\partial\Omega$;*
2. *The inviscid limit $u^\nu \to u$ holds "strongly" in $L^p([0,T]; W^{s,q}(\Omega;\mathbb{R}^3))$ for some $T > 0$, $p, q \geq 1$ and $s > 1$ if the convergence holds with respect to the strong topology on $L^p([0,T]; W^{s,q}(\Omega;\mathbb{R}^3))$.*

In particular, notice that if $u^\nu \to u$ strongly, then $\omega^\nu \times n = 0$ on $[0,T] \times \partial\Omega$ implies $\omega \times n = 0$ on $[0,T] \times \partial\Omega$. This is termed as the "*persistence property*" in [14, 15]. The non-convergence result (Theorem 1.9) is proved in the presence of such property, first by considering a special example on $\mathbb{S}^2$ ([14]) and then via computations of the principal curvatures on $\partial\Omega$ in local coordinates ([15]).

We obtain an alternate proof of Theorem 1.9, which avoids computations in local coordinates on $\partial\Omega$ as in [14, 15]. This offers a new, *global* perspective for the above theorem, of which the proof makes essential use of the structure of Lie derivatives.

Proof of Theorem 1.9. First of all, following the original proof in [14, 15], we consider the inviscid vorticity equation, which is obtained by taking the curl of the Euler equation (4.17):

$$\partial_t \omega + (u \cdot \nabla)\omega - (\omega \cdot \nabla) u = 0. \tag{4.131}$$

Then, taking the cross product with the outer unit normal $\mathbf{n} : \partial\Omega \to \mathbb{S}^2$, one obtains

$$\partial_t(\omega \times \mathbf{n}) + \big\{(u \cdot \nabla)\omega - (\omega \cdot \nabla) u\big\} \times \mathbf{n} = 0. \tag{4.132}$$

It is observed by Xiao-Xin (*cf.* Corollary 8.3 in [124]) that a necessary condition for the "strong" inviscid limit in the time interval $(0,\delta)$ is that

$$\omega \times \mathbf{n} \equiv 0 \qquad \text{on } (0,\delta) \times \partial\Omega, \tag{4.133}$$

namely that the persistence property is verified (Definition 4.5.1). Thus, if the "strong" inviscid limit were valid, then Eq. (4.132) would imply that $\{(u\cdot\nabla)\omega - (\omega\cdot\nabla)u\}\times\mathbf{n}=0$ for all time. In particular, sending $t\to 0^+$, the following condition must be fulfilled:

$$\big\{(u_0\cdot\nabla)\omega_0 - (\omega_0\cdot\nabla)u_0\big\}\times\mathbf{n}=0 \qquad \text{on } \partial\Omega. \tag{4.134}$$

Now here comes our crucial observation: The expression in the bracket coincides with the Lie bracket of the vector fields $u_0,\omega_0\in T\mathbb{R}^3$, namely that

$$(u_0\cdot\nabla)\omega_0 - (\omega_0\cdot\nabla)u_0 = [u_0,\omega_0]. \tag{4.135}$$

To prove Eq. (4.135), let $V=\sum_{i=1}^3 V^i\partial_i$ and $W=\sum_{j=1}^3 W^j\partial_j$ be two smooth vector fields in $T\mathbb{R}^3$, where $\{\partial_1,\partial_2,\partial_3\}$ denotes the canonical Euclidean frame; the vector fields are identified with the first-order differential operators. Thus, the Lie bracket of V,W can be computed as follows:

$$\begin{aligned}[V,W] &= VW-WV\\ &= \Big(V^i\partial_i(W^j\partial_j) - W^j\partial_j(V^i\partial_i)\Big)\\ &= (V\cdot\nabla)W-(W\cdot\nabla)V.\end{aligned}$$

This verifies the Eq. (4.135).

To proceed, we recall that the Lie bracket of two vector fields on a differentiable manifold equals the *Lie derivative* $\mathcal{L}$ of one vector field along the other, *i.e.*,

$$[u_0,\omega_0]=\mathcal{L}_{u_0}\omega_0. \tag{4.136}$$

This result is standard in the differentiable manifold theory (*cf.* [51]). Furthermore, the Lie derivative at a point $x_0\in\partial\Omega$ is given by

$$\mathcal{L}_{u_0}\omega_0(x_0) := \lim_{s\to 0^+}\frac{(\theta_s)^*\big\{\omega_0\big(\theta_s(x_0)\big)\big\}-\omega_0(x_0)}{s}, \tag{4.137}$$

where $\{\theta_s(x):s\geq 0, x\in\mathbb{R}^3\}$ is the one-parameter subgroup determined by the following ODE:

$$\begin{cases}\frac{d}{ds}\theta_s(x)=u_0\big(\theta_s(x)\big) \qquad \text{for all } s\geq 0,\ x\in\mathbb{R}^3,\\ \theta_0(x)=x \qquad \text{for all } x\in\mathbb{R}^3.\end{cases} \tag{4.138}$$

In other words, the trajectory $\{\theta_s(x)\}_{s\geq 0}$ is the integral curve of the vector field u_0 emanating from x. Here $(\theta_s)^*$ denotes the pullback operation under the map θ_s.

By the admissibility of initial data (Definition 4.5.1), $\omega_0 \in T(\partial\Omega)^{\perp}$ is orthogonal to the boundary since $\omega_0 \times \mathbf{n} = 0$, and $u_0 \in T(\partial\Omega)$ is tangential to the boundary because of the kinematic boundary condition $u_0 \cdot \mathbf{n} = 0$. The expression on the right-hand side of Eq. (4.136) is well-defined since ω_0 is a vector field defined along the manifold $\partial\Omega$. In geometric terminologies it means that $\omega_0 \in \iota^* T\mathbb{R}^3$, where $\iota : \partial\Omega \hookrightarrow \mathbb{R}^3$ is the embedding of Riemannian submanifold, and $\iota^* T\mathbb{R}^3$ is the pullback vector bundle. This enables us to take the Lie derivative of ω_0 along any vector field (*e.g.*, u_0) tangent to $\partial\Omega$.

Now, by the assumptions of the theorem, there is a point $x_0 \in \partial\Omega$ such that $\omega_0(x_0) \neq 0$ and $K(x_0) \neq 0$. Due to the non-vanishing curvature, there exists some small neighbourhood $U \subset \partial\Omega$ of x_0, such that every smooth curve $\gamma : (-\delta, \delta) \to \partial\Omega$ satisfying $\gamma(0) = x_0$ is not a straight line segment in $\mathbb{R}^3$. In addition, as the vorticity ω_0 is non-vanishing at x_0, we have

$$\langle u_0, \dot{\gamma} \rangle \neq 0 \qquad \text{on } U. \tag{4.139}$$

On the other hand, using the definition of the Lie derivative in terms of the integral curve, namely Eq. (4.137), we have

$$\mathcal{L}_{\dot{\gamma}} \omega_0 \neq 0 \qquad \text{on } T(\partial\Omega). \tag{4.140}$$

This is because the parallel-transport of ω_0 along γ cannot be obtained by a Euclidean translation, and hence

$$(\theta_s)^* \big\{ \omega_0\big(\theta_s(x_0)\big) \big\} - \omega_0(x_0) \neq 0 \tag{4.141}$$

in Eq. (4.137). Therefore, we conclude that $\mathcal{L}_{\langle u_0, \dot{\gamma} \rangle \dot{\gamma}} \omega_0 \neq 0$ on $T(\partial\Omega)$, so

$$\mathcal{L}_{u_0} \omega_0(x_0) \times \mathbf{n}(x_0) \neq 0 \qquad \text{in } \mathbb{R}^3. \tag{4.142}$$

This contradicts Eq. (4.134), hence it completes the proof. ■

Chapter 5

Conclusions and Perspectives

In this thesis we have studied two problems:

1. The weak continuity of the Gauss-Codazzi-Ricci (GCR) equations, as well as the corresponding isometric immersions, of both Riemannian and semi-Riemannian manifolds;

2. The vanishing viscosity limits of strong solutions in H^r of the 3D incompressible Navier-Stokes equations in curved fluid domains, under the kinematic and Navier boundary conditions.

For the project regarding the GCR equations, the crucial observation is the div-curl structure "in disguise", which, in fact, is manifested in the second structural equation of the Cartan formalism: $d\mathcal{W} = \mathcal{W} \wedge \mathcal{W}$. For this equation, even if the sign of $\mathcal{W}$-entries is arbitrary (*i.e.*, the semi-Riemannian case), we can still establish *globally* and *intrinsically* the weak continuity using the generalised quadratic theorem. This is because the principal symbol of d and its adjoint δ (hence the operator cone of $d \oplus \delta$) can be computed explicitly. It implies the div-curl lemma in the Riemannian case, in which $\Delta = d\delta + \delta d$ is elliptic.

On the other hand, for the "rough" isometric immersions of which the curvature blows up at many points – *e.g.*, Nash's wrinkled solutions for C^1 isometric immersions in [97] – the requirement of preserving the GCR equations (namely, the curvature) is too stringent. It is of independent interest to study only the weak continuity of the matrix-valued equation $D^\top f \cdot Df = g$. Similar to the GCR equations, this is a first-order system of PDEs with quadratic non-linear terms; hence the natural space for the weak continuity problem is $f \in W^{1,p}_{\mathrm{loc}}$. In view of the rigidity/non-rigidity results in isometric immersions (see Nash [97, 98], Gromov [63] and Conti-De Lellis-Székelyhidi [45]), we conjecture that there is a critical index $p_\star \in (1,\infty)$ such that when $p > p_\star$, this equation is $W^{1,p}_{\mathrm{loc}}$-weakly continuous, and for $p < p_\star$ it is not. The solution to this problem may require more delicate techniques than the compensated compactness theorems established and investigated in §§2–3 of the thesis.

Finally, in view of the project on the vanishing viscosity limit, it should be emphasised that the *geometry* of the fluid boundary $\partial\Omega$ helps with the energy estimates, in the case of the Navier boundary condition. In other words, the Navier boundary condition is "geometrically natural" (*i.e.,* covariant). This suggests that a similar result to Theorem 1.8 may be formulated and proved in arbitrary dimensions. Moreover, we remove the hypotheses on the "uniform (in viscosity) lifespan" of the Navier-Stokes solutions, by showing that if the Euler equation with the no-penetration boundary condition does not breakdown before the time $T_\star$, then the Navier-Stokes do not breakdown before $T_\star$, as long as the viscosity is smaller than a number dependent only on $T_\star$ and the initial data.

On the other hand, unfortunately, there is a gap between the existence result and the vanishing viscosity limit in terms of the regularity issues. More precisely, we obtain the existence of strong solutions in H^{r+1} ($r > 5/2$), but we can only deduce the inviscid limits in H^r. This leaves open the possibility of the formation of boundary layers in the top-order of regularity, namely in H^{r+1}. To tackle this problem possibly requires much more advanced techniques than the direct energy estimates in §5.

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