

**A theory of generalized  
Donaldson–Thomas invariants**

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## Abstract

*Donaldson–Thomas invariants*  $DT^\alpha(\tau)$  are integers which ‘count’  $\tau$ -stable coherent sheaves with Chern character  $\alpha$  on a Calabi–Yau 3-fold  $X$ , where  $\tau$  denotes Gieseker stability for some ample line bundle on  $X$ . They are unchanged under deformations of  $X$ . The conventional definition works only for classes  $\alpha$  containing no strictly  $\tau$ -semistable sheaves. Behrend showed that  $DT^\alpha(\tau)$  can be written as a weighted Euler characteristic  $\chi(\mathcal{M}_{\text{st}}^\alpha(\tau), \nu_{\mathcal{M}_{\text{st}}^\alpha(\tau)})$  of the stable moduli scheme  $\mathcal{M}_{\text{st}}^\alpha(\tau)$  by a constructible function  $\nu_{\mathcal{M}_{\text{st}}^\alpha(\tau)}$  we call the ‘Behrend function’.

This book studies *generalized Donaldson–Thomas invariants*  $\bar{DT}^\alpha(\tau)$ . They are rational numbers which ‘count’ both  $\tau$ -stable and  $\tau$ -semistable coherent sheaves with Chern character  $\alpha$  on  $X$ ; strictly  $\tau$ -semistable sheaves must be counted with complicated rational weights. The  $\bar{DT}^\alpha(\tau)$  are defined for all classes  $\alpha$ , and are equal to  $DT^\alpha(\tau)$  when it is defined. They are unchanged under deformations of  $X$ , and transform by a wall-crossing formula under change of stability condition  $\tau$ .

To prove all this we study the local structure of the moduli stack  $\mathfrak{M}$  of coherent sheaves on  $X$ . We show that an atlas for  $\mathfrak{M}$  may be written locally as  $\text{Crit}(f)$  for  $f : U \rightarrow \mathbb{C}$  holomorphic and  $U$  smooth, and use this to deduce identities on the Behrend function  $\nu_{\mathfrak{M}}$ . We compute our invariants  $\bar{DT}^\alpha(\tau)$  in examples, and make a conjecture about their integrality properties. We also extend the theory to abelian categories  $\text{mod-}CQ/I$  of representations of a quiver  $Q$  with relations  $I$  coming from a superpotential  $W$  on  $Q$ , and connect our ideas with Szendrői’s noncommutative Donaldson–Thomas invariants, and work by Reineke and others on invariants counting quiver representations. Our book is closely related to Kontsevich and Soibelman’s independent paper [63].

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## CHAPTER 1

### Introduction

Let  $X$  be a Calabi–Yau 3-fold over the complex numbers  $\mathbb{C}$ , and  $\mathcal{O}_X(1)$  a very ample line bundle on  $X$ . Our definition of Calabi–Yau 3-fold requires  $X$  to be projective, with  $H^1(\mathcal{O}_X) = 0$ . Write  $\text{coh}(X)$  for the abelian category of coherent sheaves on  $X$ , and  $K^{\text{num}}(\text{coh}(X))$  for the numerical Grothendieck group of  $\text{coh}(X)$ . We use  $\tau$  to denote Gieseker stability of coherent sheaves with respect to  $\mathcal{O}_X(1)$ . If  $E$  is a coherent sheaf on  $X$  then the class  $[E] \in K^{\text{num}}(\text{coh}(X))$  is in effect the Chern character  $\text{ch}(E)$  of  $E$  in  $H^{\text{even}}(X; \mathbb{Q})$ .

For  $\alpha \in K^{\text{num}}(\text{coh}(X))$  we form the coarse moduli schemes  $\mathcal{M}_{\text{ss}}^\alpha(\tau), \mathcal{M}_{\text{st}}^\alpha(\tau)$  of  $\tau$ -(semi)stable sheaves  $E$  with  $[E] = \alpha$ . Then  $\mathcal{M}_{\text{ss}}^\alpha(\tau)$  is a projective  $\mathbb{C}$ -scheme whose points correspond to S-equivalence classes of  $\tau$ -semistable sheaves, and  $\mathcal{M}_{\text{st}}^\alpha(\tau)$  is an open subscheme of  $\mathcal{M}_{\text{ss}}^\alpha(\tau)$  whose points correspond to isomorphism classes of  $\tau$ -stable sheaves.

For Chern characters  $\alpha$  with  $\mathcal{M}_{\text{ss}}^\alpha(\tau) = \mathcal{M}_{\text{st}}^\alpha(\tau)$ , following Donaldson and Thomas [20, §3], Thomas [100] constructed a symmetric obstruction theory on  $\mathcal{M}_{\text{st}}^\alpha(\tau)$  and defined the *Donaldson–Thomas invariant* to be the virtual class

$$DT^\alpha(\tau) = \int_{[\mathcal{M}_{\text{st}}^\alpha(\tau)]^{\text{vir}}} 1 \in \mathbb{Z}, \quad (1.1)$$

an integer which ‘counts’  $\tau$ -semistable sheaves in class  $\alpha$ . Thomas’ main result [100, §3] is that  $DT^\alpha(\tau)$  is unchanged under deformations of the underlying Calabi–Yau 3-fold  $X$ . Later, Behrend [3] showed that Donaldson–Thomas invariants can be written as a weighted Euler characteristic

$$DT^\alpha(\tau) = \chi(\mathcal{M}_{\text{st}}^\alpha(\tau), \nu_{\mathcal{M}_{\text{st}}^\alpha(\tau)}), \quad (1.2)$$

where  $\nu_{\mathcal{M}_{\text{st}}^\alpha(\tau)}$  is the *Behrend function*, a constructible function on  $\mathcal{M}_{\text{st}}^\alpha(\tau)$  depending only on  $\mathcal{M}_{\text{st}}^\alpha(\tau)$  as a  $\mathbb{C}$ -scheme. (Here, and throughout, Euler characteristics are taken with respect to cohomology with compact support.)

Conventional Donaldson–Thomas invariants  $DT^\alpha(\tau)$  are only defined for classes  $\alpha$  with  $\mathcal{M}_{\text{ss}}^\alpha(\tau) = \mathcal{M}_{\text{st}}^\alpha(\tau)$ , that is, when there are no strictly  $\tau$ -semistable sheaves. Also, although  $DT^\alpha(\tau)$  depends on the stability condition  $\tau$ , that is, on the choice of very ample line bundle  $\mathcal{O}_X(1)$  on  $X$ , this dependence was not understood until now. The main goal of this book is to address these two issues.

For a Calabi–Yau 3-fold  $X$  over  $\mathbb{C}$  we will define *generalized Donaldson–Thomas invariants*  $\bar{DT}^\alpha(\tau) \in \mathbb{Q}$  for all  $\alpha \in K^{\text{num}}(\text{coh}(X))$ , which ‘count’  $\tau$ -semistable sheaves in class  $\alpha$ . These have the following important properties:

- $\bar{DT}^\alpha(\tau) \in \mathbb{Q}$  is unchanged by deformations of the Calabi–Yau 3-fold  $X$ .
- If  $\mathcal{M}_{\text{ss}}^\alpha(\tau) = \mathcal{M}_{\text{st}}^\alpha(\tau)$  then  $\bar{DT}^\alpha(\tau)$  lies in  $\mathbb{Z}$  and equals the conventional Donaldson–Thomas invariant  $DT^\alpha(\tau)$  defined by Thomas [100].
- If  $\mathcal{M}_{\text{ss}}^\alpha(\tau) \neq \mathcal{M}_{\text{st}}^\alpha(\tau)$  then conventional Donaldson–Thomas invariants  $DT^\alpha(\tau)$  are not defined for class  $\alpha$ . Our generalized invariant  $\bar{DT}^\alpha(\tau)$  may lie in  $\mathbb{Q}$

because strictly semistable sheaves  $E$  make (complicated)  $\mathbb{Q}$ -valued contributions to  $\bar{DT}^\alpha(\tau)$ . We can write the  $\bar{DT}^\alpha(\tau)$  in terms of other invariants  $\hat{DT}^\alpha(\tau)$  which, in the case of ‘generic’  $\tau$ , we conjecture to be integer-valued.

- If  $\tau, \tilde{\tau}$  are two stability conditions on  $\text{coh}(X)$ , there is an explicit change of stability condition formula giving  $\bar{DT}^\alpha(\tilde{\tau})$  in terms of the  $\bar{DT}^\beta(\tau)$ .

These invariants are a continuation of the first author’s programme [49–55].

We begin with three sections of background. Sections 2–3 explain material on constructible functions, stack functions, Ringel–Hall algebras, counting invariants for Calabi–Yau 3-folds, and wall-crossing, from the first author’s series [49–55]. Section 4 explains Behrend’s approach [3] to Donaldson–Thomas invariants as Euler characteristics of moduli schemes weighted by the Behrend function, as in (1.2). We include some new material here, and explain the connection between Behrend functions and the theory of perverse sheaves and vanishing cycles. Our main results are given in §5, including the definition of generalized Donaldson–Thomas invariants  $\bar{DT}^\alpha(\tau) \in \mathbb{Q}$ , their deformation-invariance, and wall-crossing formulae under change of stability condition  $\tau$ .

Sections 6 and 7 give many examples, applications, and generalizations of the theory, with §6 focussing on coherent sheaves on (possibly noncompact) Calabi–Yau 3-folds, and §7 on representations of quivers with superpotentials, in connection with work by many authors on 3-Calabi–Yau categories, noncommutative Donaldson–Thomas invariants, and so on. One new piece of theory is that in §6.2, motivated by ideas in Kontsevich and Soibelman [63, §2.5 & §7.1], we define *BPS invariants*  $\hat{DT}^\alpha(\tau)$  by the formula

$$\bar{DT}^\alpha(\tau) = \sum_{m \geq 1, m|\alpha} \frac{1}{m^2} \hat{DT}^{\alpha/m}(\tau).$$

These are supposed to count ‘BPS states’ in some String Theory sense, and we conjecture that for ‘generic’ stability conditions  $\tau$  we have  $\bar{DT}^\alpha(\tau) \in \mathbb{Z}$  for all  $\alpha$ . An analogue of this conjecture for invariants  $\hat{DT}_Q^d(\mu)$  counting representations of a quiver  $Q$  without relations is proved in §7.6.

Sections 8–13 give the proofs of the main results stated in §5, and we imagine many readers will not need to look at these. In the rest of this introduction we survey §2–§7. Section 1.1 very briefly sketches the set-up of [49–55], which will be explained in §2–§3. Section 1.2 discusses *Behrend functions* from §4, §1.3 outlines the main results in §5, and §1.4–§1.5 summarize the applications and generalizations in §6–§7. Finally, §1.6 explains the relations between our work and the recent paper of Kontsevich and Soibelman [63], which is summarized in [64]. This book is surveyed in [56].

In §4–§7 we give problems for further research, as Questions or Conjectures.

### 1.1. Brief sketch of background from [49–55]

We recall a few important ideas from [49–55], which will be explained at greater length in §2–§3. We work not with coarse moduli schemes, as in [100], but with *Artin stacks*. Let  $X$  be a Calabi–Yau 3-fold over  $\mathbb{C}$ , and write  $\mathfrak{M}$  for the moduli stack of all coherent sheaves  $E$  on  $X$ . It is an Artin  $\mathbb{C}$ -stack.

The ring of ‘stack functions’  $\text{SF}(\mathfrak{M})$  in [50] is basically the Grothendieck group  $K_0(\text{Sta}_{\mathfrak{M}})$  of the 2-category  $\text{Sta}_{\mathfrak{M}}$  of stacks over  $\mathfrak{M}$ . That is,  $\text{SF}(\mathfrak{M})$  is generated

by isomorphism classes  $[(\mathfrak{R}, \rho)]$  of representable 1-morphisms  $\rho : \mathfrak{R} \rightarrow \mathfrak{M}$  for  $\mathfrak{R}$  a finite type Artin  $\mathbb{C}$ -stack, with the relation

$$[(\mathfrak{R}, \rho)] = [(\mathfrak{S}, \rho|_{\mathfrak{S}})] + [(\mathfrak{R} \setminus \mathfrak{S}, \rho|_{\mathfrak{R} \setminus \mathfrak{S}})]$$

when  $\mathfrak{S}$  is a closed  $\mathbb{C}$ -substack of  $\mathfrak{R}$ . But there is more to stack functions than this. In [50] we study different kinds of stack function spaces with other choices of generators and relations, and operations on these spaces. These include projections  $\Pi_n^{\text{vi}} : \text{SF}(\mathfrak{M}) \rightarrow \text{SF}(\mathfrak{M})$  to stack functions of ‘virtual rank  $n$ ’, which act on  $[(\mathfrak{R}, \rho)]$  by modifying  $\mathfrak{R}$  depending on its stabilizer groups.

In [52, §5.2] we define a Ringel–Hall type algebra  $\text{SF}_{\text{al}}(\mathfrak{M})$  of stack functions ‘with algebra stabilizers’ on  $\mathfrak{M}$ , with an associative, non-commutative multiplication  $*$ . In [52, §5.2] we define a Lie subalgebra  $\text{SF}_{\text{al}}^{\text{ind}}(\mathfrak{M})$  of stack functions ‘supported on virtual indecomposables’. In [52, §6.5] we define an explicit Lie algebra  $L(X)$  to be the  $\mathbb{Q}$ -vector space with basis of symbols  $\lambda^\alpha$  for  $\alpha \in K^{\text{num}}(\text{coh}(X))$ , with Lie bracket

$$[\lambda^\alpha, \lambda^\beta] = \bar{\chi}(\alpha, \beta) \lambda^{\alpha+\beta}, \quad (1.3)$$

for  $\alpha, \beta \in K^{\text{num}}(\text{coh}(X))$ , where  $\bar{\chi}(\cdot, \cdot)$  is the Euler form on  $K^{\text{num}}(\text{coh}(X))$ . As  $X$  is a Calabi–Yau 3-fold,  $\bar{\chi}$  is antisymmetric, so (1.3) satisfies the Jacobi identity and makes  $L(X)$  into an infinite-dimensional Lie algebra over  $\mathbb{Q}$ .

Then in [52, §6.6] we define a *Lie algebra morphism*  $\Psi : \text{SF}_{\text{al}}^{\text{ind}}(\mathfrak{M}) \rightarrow L(X)$ , as in §3.4 below. Roughly speaking this is of the form

$$\Psi(f) = \sum_{\alpha \in K^{\text{num}}(\text{coh}(X))} \chi^{\text{stk}}(f|_{\mathfrak{M}^\alpha}) \lambda^\alpha, \quad (1.4)$$

where  $f = \sum_{i=1}^m c_i [(\mathfrak{R}_i, \rho_i)]$  is a stack function on  $M$ , and  $\mathfrak{M}^\alpha$  is the substack in  $\mathfrak{M}$  of sheaves  $E$  with class  $\alpha$ , and  $\chi^{\text{stk}}$  is a kind of stack-theoretic Euler characteristic. But in fact the definition of  $\Psi$ , and the proof that  $\Psi$  is a Lie algebra morphism, are highly nontrivial, and use many ideas from [49, 50, 52], including those of ‘virtual rank’ and ‘virtual indecomposable’. The problem is that the obvious definition of  $\chi^{\text{stk}}$  usually involves dividing by zero, so defining (1.4) in a way that makes sense is quite subtle. The proof that  $\Psi$  is a Lie algebra morphism uses Serre duality and the assumption that  $X$  is a Calabi–Yau 3-fold.

Now let  $\tau$  be a stability condition on  $\text{coh}(X)$ , such as Gieseker stability. Then we have open, finite type substacks  $\mathfrak{M}_{\text{ss}}^\alpha(\tau), \mathfrak{M}_{\text{st}}^\alpha(\tau)$  in  $\mathfrak{M}$  of  $\tau$ -(semi)stable sheaves  $E$  in class  $\alpha$ , for all  $\alpha \in K^{\text{num}}(\text{coh}(X))$ . Write  $\bar{\delta}_{\text{ss}}^\alpha(\tau)$  for the characteristic function of  $\mathfrak{M}_{\text{ss}}^\alpha(\tau)$ , in the sense of stack functions [50]. Then  $\bar{\delta}_{\text{ss}}^\alpha(\tau) \in \text{SF}_{\text{al}}(\mathfrak{M})$ . In [53, §8], we define elements  $\bar{\epsilon}^\alpha(\tau)$  in  $\text{SF}_{\text{al}}(\mathfrak{M})$  by

$$\bar{\epsilon}^\alpha(\tau) = \sum_{\substack{n \geq 1, \alpha_1, \dots, \alpha_n \in K^{\text{num}}(\text{coh}(X)): \\ \alpha_1 + \dots + \alpha_n = \alpha, \tau(\alpha_i) = \tau(\alpha), \text{ all } i}} \frac{(-1)^{n-1}}{n} \bar{\delta}_{\text{ss}}^{\alpha_1}(\tau) * \bar{\delta}_{\text{ss}}^{\alpha_2}(\tau) * \dots * \bar{\delta}_{\text{ss}}^{\alpha_n}(\tau), \quad (1.5)$$

where  $*$  is the Ringel–Hall multiplication in  $\text{SF}_{\text{al}}(\mathfrak{M})$ . Then [53, Th. 8.7] shows that  $\bar{\epsilon}^\alpha(\tau)$  lies in the Lie subalgebra  $\text{SF}_{\text{al}}^{\text{ind}}(\mathfrak{M})$ , a nontrivial result.

Thus we can apply the Lie algebra morphism  $\Psi$  to  $\bar{\epsilon}^\alpha(\tau)$ . In [54, §6.6] we define invariants  $J^\alpha(\tau) \in \mathbb{Q}$  for all  $\alpha \in K^{\text{num}}(\text{coh}(X))$  by

$$\Psi(\bar{\epsilon}^\alpha(\tau)) = J^\alpha(\tau) \lambda^\alpha. \quad (1.6)$$

These  $J^\alpha(\tau)$  are rational numbers ‘counting’  $\tau$ -semistable sheaves  $E$  in class  $\alpha$ . When  $\mathcal{M}_{\text{ss}}^\alpha(\tau) = \mathcal{M}_{\text{st}}^\alpha(\tau)$  we have  $J^\alpha(\tau) = \chi(\mathcal{M}_{\text{st}}^\alpha(\tau))$ , that is,  $J^\alpha(\tau)$  is the naïve Euler characteristic of the moduli space  $\mathcal{M}_{\text{st}}^\alpha(\tau)$ . This is *not* weighted by the



Behrend function  $\nu_{\mathcal{M}_{\text{st}}^\alpha(\tau)}$ , and so in general does not coincide with the Donaldson–Thomas invariant  $DT^\alpha(\tau)$  in (1.3).

As the  $J^\alpha(\tau)$  do not include Behrend functions, they do not count semistable sheaves with multiplicity, and so they will not in general be unchanged under deformations of the underlying Calabi–Yau 3-fold, as Donaldson–Thomas invariants are. However, the  $J^\alpha(\tau)$  do have very good properties under change of stability condition. In [54] we show that if  $\tau, \tilde{\tau}$  are two stability conditions on  $\text{coh}(X)$ , then we can write  $\bar{\epsilon}^\alpha(\tilde{\tau})$  in terms of a (complicated) explicit formula involving the  $\bar{\epsilon}^\beta(\tau)$  for  $\beta \in K^{\text{num}}(\text{coh}(X))$  and the Lie bracket in  $\text{SF}_{\text{al}}^{\text{ind}}(\mathfrak{M})$ .

Applying the Lie algebra morphism  $\Psi$  shows that  $J^\alpha(\tilde{\tau})\lambda^\alpha$  may be written in terms of the  $J^\beta(\tau)\lambda^\beta$  and the Lie bracket in  $L(X)$ , and hence [54, Th. 6.28] we get an explicit transformation law for the  $J^\alpha(\tau)$  under change of stability condition. In [55] we show how to encode invariants  $J^\alpha(\tau)$  satisfying a transformation law in generating functions on a complex manifold of stability conditions, which are both holomorphic and continuous, despite the discontinuous wall-crossing behaviour of the  $J^\alpha(\tau)$ . This concludes our sketch of [49–55].

## 1.2. Behrend functions of schemes and stacks, from §4

Let  $X$  be a  $\mathbb{C}$ -scheme or Artin  $\mathbb{C}$ -stack, locally of finite type, and  $X(\mathbb{C})$  the set of  $\mathbb{C}$ -points of  $X$ . The *Behrend function*  $\nu_X : X(\mathbb{C}) \rightarrow \mathbb{Z}$  is a  $\mathbb{Z}$ -valued locally constructible function on  $X$ , in the sense of [49]. They were introduced by Behrend [3] for finite type  $\mathbb{C}$ -schemes  $X$ ; the generalization to Artin stacks in §4.1 is new but straightforward. Behrend functions are also defined for complex analytic spaces  $X_{\text{an}}$ , and the Behrend function of a  $\mathbb{C}$ -scheme  $X$  coincides with that of the underlying complex analytic space  $X_{\text{an}}$ .

A good way to think of Behrend functions is as *multiplicity functions*. If  $X$  is a finite type  $\mathbb{C}$ -scheme then the Euler characteristic  $\chi(X)$  ‘counts’ points without multiplicity, so that each point of  $X(\mathbb{C})$  contributes 1 to  $\chi(X)$ . If  $X^{\text{red}}$  is the underlying reduced  $\mathbb{C}$ -scheme then  $X^{\text{red}}(\mathbb{C}) = X(\mathbb{C})$ , so  $\chi(X^{\text{red}}) = \chi(X)$ , and  $\chi(X)$  does not see non-reduced behaviour in  $X$ . However, the weighted Euler characteristic  $\chi(X, \nu_X)$  ‘counts’ each  $x \in X(\mathbb{C})$  weighted by its multiplicity  $\nu_X(x)$ . The Behrend function  $\nu_X$  detects non-reduced behaviour, so in general  $\chi(X, \nu_X) \neq \chi(X^{\text{red}}, \nu_{X^{\text{red}}})$ . For example, let  $X$  be the  $k$ -fold point  $\text{Spec}(\mathbb{C}[z]/(z^k))$  for  $k \geq 1$ . Then  $X(\mathbb{C})$  is a single point  $x$  with  $\nu_X(x) = k$ , so  $\chi(X) = 1 = \chi(X^{\text{red}}, \nu_{X^{\text{red}}})$ , but  $\chi(X, \nu_X) = k$ .

An important moral of [3] is that (at least in moduli problems with symmetric obstruction theories, such as Donaldson–Thomas theory) it is better to ‘count’ points in a moduli scheme  $\mathcal{M}$  by the weighted Euler characteristic  $\chi(\mathcal{M}, \nu_{\mathcal{M}})$  than by the unweighted Euler characteristic  $\chi(\mathcal{M})$ . One reason is that  $\chi(\mathcal{M}, \nu_{\mathcal{M}})$  often gives answers unchanged under deformations of the underlying geometry, but  $\chi(\mathcal{M})$  does not. For example, consider the family of  $\mathbb{C}$ -schemes  $X_t = \text{Spec}(\mathbb{C}[z]/(z^2 - t^2))$  for  $t \in \mathbb{C}$ . Then  $X_t$  is two reduced points  $\pm t$  for  $t \neq 0$ , and a double point when  $t = 0$ . So as above we find that  $\chi(X_t, \nu_{X_t}) = 2$  for all  $t$ , which is deformation-invariant, but  $\chi(X_t)$  is 2 for  $t \neq 0$  and 1 for  $t = 0$ , which is not deformation-invariant.

Here are some important properties of Behrend functions:

- (i) If  $X$  is a smooth Artin  $\mathbb{C}$ -stack of dimension  $n \in \mathbb{Z}$  then  $\nu_X \equiv (-1)^n$ .

- (ii) If  $\varphi : X \rightarrow Y$  is a smooth 1-morphism of Artin  $\mathbb{C}$ -stacks of relative dimension  $n \in \mathbb{Z}$  then  $\nu_X \equiv (-1)^n f^*(\nu_Y)$ .
- (iii) Suppose  $X$  is a proper  $\mathbb{C}$ -scheme equipped with a symmetric obstruction theory, and  $[X]^{\text{vir}}$  is the corresponding virtual class. Then

$$\int_{[X]^{\text{vir}}} 1 = \chi(X, \nu_X) \in \mathbb{Z}. \quad (1.7)$$

- (iv) Let  $U$  be a complex manifold and  $f : U \rightarrow \mathbb{C}$  a holomorphic function, and define  $X$  to be the complex analytic space  $\text{Crit}(f) \subseteq U$ . Then the Behrend function  $\nu_X$  of  $X$  is given by

$$\nu_X(x) = (-1)^{\dim U} (1 - \chi(MF_f(x))) \quad \text{for } x \in X, \quad (1.8)$$

where  $MF_f(x)$  is the *Milnor fibre* of  $f$  at  $x$ .

Equation (1.7) explains the equivalence of the two expressions for  $DT^\alpha(\tau)$  in (1.1) and (1.2) above. The Milnor fibre description (1.8) of Behrend functions will be crucial in proving the Behrend function identities (1.10)–(1.11) below.

### 1.3. Summary of the main results in §5

The basic idea behind this whole book is that we should insert the Behrend function  $\nu_{\mathfrak{M}}$  of the moduli stack  $\mathfrak{M}$  of coherent sheaves in  $X$  as a weight in the programme of [49–55] summarized in §1.1. Thus we will obtain weighted versions  $\tilde{\Psi}$  of the Lie algebra morphism  $\Psi$  of (1.4), and  $\tilde{DT}^\alpha(\tau)$  of the counting invariant  $J^\alpha(\tau) \in \mathbb{Q}$  in (1.6). Here is how this is worked out in §5.

We define a modification  $\tilde{L}(X)$  of the Lie algebra  $L(X)$  above, the  $\mathbb{Q}$ -vector space with basis of symbols  $\tilde{\lambda}^\alpha$  for  $\alpha \in K^{\text{num}}(\text{coh}(X))$ , with Lie bracket

$$[\tilde{\lambda}^\alpha, \tilde{\lambda}^\beta] = (-1)^{\bar{\chi}(\alpha, \beta)} \tilde{\chi}(\alpha, \beta) \tilde{\lambda}^{\alpha+\beta},$$

which is (1.3) with a sign change. Then we define a *Lie algebra morphism*  $\tilde{\Psi} : \text{SF}_{\text{al}}^{\text{ind}}(\mathfrak{M}) \rightarrow \tilde{L}(X)$ . Roughly speaking this is of the form

$$\tilde{\Psi}(f) = \sum_{\alpha \in K^{\text{num}}(\text{coh}(X))} \chi^{\text{stk}}(f|_{\mathfrak{M}^\alpha}, \nu_{\mathfrak{M}}) \tilde{\lambda}^\alpha, \quad (1.9)$$

that is, in (1.4) we replace the stack-theoretic Euler characteristic  $\chi^{\text{stk}}$  with a stack-theoretic Euler characteristic weighted by the Behrend function  $\nu_{\mathfrak{M}}$ .

The proof that  $\tilde{\Psi}$  is a Lie algebra morphism combines the proof in [52] that  $\Psi$  is a Lie algebra morphism with the two Behrend function identities

$$\nu_{\mathfrak{M}}(E_1 \oplus E_2) = (-1)^{\bar{\chi}([E_1], [E_2])} \nu_{\mathfrak{M}}(E_1) \nu_{\mathfrak{M}}(E_2), \quad (1.10)$$

$$\begin{aligned} & \int_{\substack{[\lambda] \in \mathbb{P}(\text{Ext}^1(E_2, E_1)): \\ \lambda \Leftrightarrow 0 \rightarrow E_1 \rightarrow F \rightarrow E_2 \rightarrow 0}} \nu_{\mathfrak{M}}(F) d\chi - \int_{\substack{[\lambda'] \in \mathbb{P}(\text{Ext}^1(E_1, E_2)): \\ \lambda' \Leftrightarrow 0 \rightarrow E_2 \rightarrow F' \rightarrow E_1 \rightarrow 0}} \nu_{\mathfrak{M}}(F') d\chi \\ &= (\dim \text{Ext}^1(E_2, E_1) - \dim \text{Ext}^1(E_1, E_2)) \nu_{\mathfrak{M}}(E_1 \oplus E_2), \end{aligned} \quad (1.11)$$

which will be proved in Theorem 5.11. Here in (1.11) the correspondence between  $[\lambda] \in \mathbb{P}(\text{Ext}^1(E_2, E_1))$  and  $F \in \text{coh}(X)$  is that the point  $[\lambda] \in \mathbb{P}(\text{Ext}^1(E_2, E_1))$  lifts to some  $0 \neq \lambda \in \text{Ext}^1(E_2, E_1)$ , which corresponds to a short exact sequence  $0 \rightarrow E_1 \rightarrow F \rightarrow E_2 \rightarrow 0$  in  $\text{coh}(X)$ . The function  $[\lambda] \mapsto \nu_{\mathfrak{M}}(F)$  is a constructible function  $\mathbb{P}(\text{Ext}^1(E_2, E_1)) \rightarrow \mathbb{Z}$ , and the integrals in (1.11) are integrals of constructible functions using Euler characteristic as measure, as in [49].

Proving (1.10)–(1.11) requires a deep understanding of the local structure of the moduli stack  $\mathfrak{M}$ , which is of interest in itself. First we show in §8 using a

composition of Seidel–Thomas twists by  $\mathcal{O}_X(-n)$  for  $n \gg 0$  that  $\mathfrak{M}$  is locally 1-isomorphic to the moduli stack  $\mathfrak{Vect}$  of vector bundles on  $X$ . Then we prove in §9 that near  $[E] \in \mathfrak{Vect}(\mathbb{C})$ , an atlas for  $\mathfrak{Vect}$  can be written locally in the complex analytic topology in the form  $\text{Crit}(f)$  for  $f : U \rightarrow \mathbb{C}$  a holomorphic function on an open set  $U$  in  $\text{Ext}^1(E, E)$ . These  $U, f$  are *not algebraic*, they are constructed using gauge theory on the complex vector bundle  $E$  over  $X$  and transcendental methods. Finally, we deduce (1.10)–(1.11) in §10 using the Milnor fibre expression (1.8) for Behrend functions applied to these  $U, f$ .

We then define *generalized Donaldson–Thomas invariants*  $\bar{DT}^\alpha(\tau) \in \mathbb{Q}$  by

$$\tilde{\Psi}(\bar{\epsilon}^\alpha(\tau)) = -\bar{DT}^\alpha(\tau)\tilde{\lambda}^\alpha, \quad (1.12)$$

as in (1.6). When  $\mathcal{M}_{\text{ss}}^\alpha(\tau) = \mathcal{M}_{\text{st}}^\alpha(\tau)$  we have  $\bar{\epsilon}^\alpha(\tau) = \bar{\delta}_{\text{ss}}^\alpha(\tau)$ , and (1.9) gives

$$\tilde{\Psi}(\bar{\epsilon}^\alpha(\tau)) = \chi^{\text{stk}}(\mathfrak{M}_{\text{st}}^\alpha(\tau), \nu_{\mathfrak{M}_{\text{st}}^\alpha(\tau)})\tilde{\lambda}^\alpha. \quad (1.13)$$

The projection  $\pi : \mathfrak{M}_{\text{st}}^\alpha(\tau) \rightarrow \mathcal{M}_{\text{st}}^\alpha(\tau)$  from the moduli stack to the coarse moduli scheme is smooth of dimension  $-1$ , so  $\nu_{\mathfrak{M}_{\text{st}}^\alpha(\tau)} = -\pi^*(\nu_{\mathcal{M}_{\text{st}}^\alpha(\tau)})$  by (ii) in §1.2, and comparing (1.2), (1.12), (1.13) shows that  $\bar{DT}^\alpha(\tau) = DT^\alpha(\tau)$ . But our new invariants  $\bar{DT}^\alpha(\tau)$  are also defined for  $\alpha$  with  $\mathcal{M}_{\text{ss}}^\alpha(\tau) \neq \mathcal{M}_{\text{st}}^\alpha(\tau)$ , when conventional Donaldson–Thomas invariants  $DT^\alpha(\tau)$  are not defined.

Write  $C(\text{coh}(X)) = \{[E] \in K^{\text{num}}(\text{coh}(X)) : 0 \neq E \in \text{coh}(X)\}$  for the ‘positive cone’ of classes in  $K^{\text{num}}(\text{coh}(X))$  of nonzero objects in  $\text{coh}(X)$ . Then  $\mathcal{M}_{\text{ss}}^\alpha(\tau) = \mathcal{M}_{\text{st}}^\alpha(\tau) = \emptyset$  if  $\alpha \in K^{\text{num}}(\text{coh}(X)) \setminus C(\text{coh}(X))$ , so  $\bar{DT}^\alpha(\tau) = 0$ . By convention the zero sheaf is not (semi)stable, so  $\mathcal{M}_{\text{ss}}^0(\tau) = \emptyset$  and  $\bar{DT}^0(\tau) = 0$ .

Since  $\tilde{\Psi}$  is a Lie algebra morphism, the change of stability condition formula for the  $\bar{\epsilon}^\alpha(\tau)$  in [54] implies a formula for the elements  $-\bar{DT}^\alpha(\tau)\tilde{\lambda}^\alpha$  in  $\tilde{L}(X)$ , and thus a transformation law for the invariants  $\bar{DT}^\alpha(\tau)$ , of the form

$$\begin{aligned} \bar{DT}^\alpha(\tilde{\tau}) = & \sum_{\substack{\text{iso.} \\ \text{classes} \\ \text{of finite} \\ \text{sets } I}} \sum_{\substack{\kappa : I \rightarrow C(\text{coh}(X)) : \\ \sum_{i \in I} \kappa(i) = \alpha}} \sum_{\substack{\text{connected,} \\ \text{simply-} \\ \text{connected} \\ \text{digraphs } \Gamma, \\ \text{vertices } I}} (-1)^{|I|-1} V(I, \Gamma, \kappa; \tau, \tilde{\tau}) \cdot \prod_{i \in I} \bar{DT}^{\kappa(i)}(\tau) \\ & \cdot (-1)^{\frac{1}{2} \sum_{i, j \in I} |\bar{\chi}(\kappa(i), \kappa(j))|} \cdot \prod_{\substack{\text{edges } \bullet \rightarrow \bullet \text{ in } \Gamma}} \bar{\chi}(\kappa(i), \kappa(j)), \end{aligned} \quad (1.14)$$

where  $\bar{\chi}$  is the Euler form on  $K^{\text{num}}(\text{coh}(X))$ , and  $V(I, \Gamma, \kappa; \tau, \tilde{\tau}) \in \mathbb{Q}$  are combinatorial coefficients defined in §3.5.

To study our new invariants  $\bar{DT}^\alpha(\tau)$ , we find it helpful to introduce another family of invariants  $PI^{\alpha, n}(\tau')$ , similar to Pandharipande–Thomas invariants [86]. Let  $n \gg 0$  be fixed. A *stable pair* is a nonzero morphism  $s : \mathcal{O}_X(-n) \rightarrow E$  in  $\text{coh}(X)$  such that  $E$  is  $\tau$ -semistable, and if  $\text{Im } s \subset E' \subset E$  with  $E' \neq E$  then  $\tau([E']) < \tau([E])$ . For  $\alpha \in K^{\text{num}}(\text{coh}(X))$  and  $n \gg 0$ , the moduli space  $\mathcal{M}_{\text{stp}}^{\alpha, n}(\tau')$  of stable pairs  $s : \mathcal{O}_X(-n) \rightarrow X$  with  $[E] = \alpha$  is a fine moduli scheme, which is proper and has a symmetric obstruction theory. We define

$$PI^{\alpha, n}(\tau') = \int_{[\mathcal{M}_{\text{stp}}^{\alpha, n}(\tau')]^{\text{vir}}} 1 = \chi(\mathcal{M}_{\text{stp}}^{\alpha, n}(\tau'), \nu_{\mathcal{M}_{\text{stp}}^{\alpha, n}(\tau')}) \in \mathbb{Z}, \quad (1.15)$$

where the second equality follows from (1.7). By a similar proof to that for Donaldson–Thomas invariants in [100], we find that  $PI^{\alpha, n}(\tau')$  is unchanged under deformations of the underlying Calabi–Yau 3-fold  $X$ .

By a wall-crossing proof similar to that for (1.14), we show that  $PI^{\alpha,n}(\tau')$  can be written in terms of the  $\bar{D}T^\beta(\tau)$  by

$$PI^{\alpha,n}(\tau') = \sum_{\substack{\alpha_1, \dots, \alpha_l \in C(\text{coh}(X)), \\ l \geq 1: \alpha_1 + \dots + \alpha_l = \alpha, \\ \tau(\alpha_i) = \tau(\alpha), \text{ all } i}} \frac{(-1)^l}{l!} \prod_{i=1}^l [(-1)^{\bar{\chi}([\mathcal{O}_X(-n)] - \alpha_1 - \dots - \alpha_{i-1}, \alpha_i)} \bar{\chi}([\mathcal{O}_X(-n)] - \alpha_1 - \dots - \alpha_{i-1}, \alpha_i) \bar{D}T^{\alpha_i}(\tau)], \quad (1.16)$$

for  $n \gg 0$ . Dividing the sum in (1.16) into cases  $l = 1$  and  $l \geq 1$  gives

$$PI^{\alpha,n}(\tau') = (-1)^{P(n)-1} P(n) \bar{D}T^\alpha(\tau) + \{\text{terms in } \prod_{i=1}^l \bar{D}T^{\alpha_i}(\tau), l \geq 2\}, \quad (1.17)$$

where  $P(n) = \bar{\chi}([\mathcal{O}_X(-n)], \alpha)$  is the Hilbert polynomial of  $\alpha$ , so that  $P(n) > 0$  for  $n \gg 0$ . As  $PI^{\alpha,n}(\tau')$  is deformation-invariant, we deduce from (1.17) by induction on rank  $\alpha$  with  $\dim \alpha$  fixed that  $\bar{D}T^\alpha(\tau)$  is also deformation-invariant.

The pair invariants  $PI^{\alpha,n}(\tau')$  are a useful tool for computing the  $\bar{D}T^\alpha(\tau)$  in examples in §6. The method is to describe the moduli spaces  $\mathcal{M}_{\text{stp}}^{\alpha,n}(\tau')$  explicitly, and then use (1.15) to compute  $PI^{\alpha,n}(\tau')$ , and (1.16) to deduce the values of  $\bar{D}T^\alpha(\tau)$ . Our point of view is that the  $\bar{D}T^\alpha(\tau)$  are of primary interest, and the  $PI^{\alpha,n}(\tau')$  are secondary invariants, of less interest in themselves.

#### 1.4. Examples and applications in §6

In §6 we compute the invariants  $\bar{D}T^\alpha(\tau)$  and  $PI^{\alpha,n}(\tau')$  in several examples. One basic example is this: suppose that  $E$  is a rigid,  $\tau$ -stable sheaf in class  $\alpha$  in  $K^{\text{num}}(\text{coh}(X))$ , and that the only  $\tau$ -semistable sheaf in class  $m\alpha$  up to isomorphism is  $mE = \oplus^m E$ , for all  $m \geq 1$ . Then we show that

$$\bar{D}T^{m\alpha}(\tau) = \frac{1}{m^2} \quad \text{for all } m \geq 1. \quad (1.18)$$

Thus the  $\bar{D}T^\alpha(\tau)$  can lie in  $\mathbb{Q} \setminus \mathbb{Z}$ . We think of (1.18) as a ‘multiple cover formula’, similar to the well known Aspinwall–Morrison computation for a Calabi–Yau 3-fold  $X$ , that a rigid embedded  $\mathbb{CP}^1$  in class  $\alpha \in H_2(X; \mathbb{Z})$  contributes  $1/m^3$  to the genus zero Gromov–Witten invariant  $GW_{0,0}(m\alpha)$  of  $X$  in class  $m\alpha$  for all  $m \geq 1$ .

In Gromov–Witten theory, one defines *Gopakumar–Vafa invariants*  $GV_g(\alpha)$  which are  $\mathbb{Q}$ -linear combinations of Gromov–Witten invariants, and are conjectured to be integers, so that they ‘count’ curves in  $X$  in a more meaningful way. For a Calabi–Yau 3-fold in genus  $g = 0$  these satisfy

$$GW_{0,0}(\alpha) = \sum_{m \geq 1, m|\alpha} \frac{1}{m^3} GV_0(\alpha/m),$$

where the factor  $1/m^3$  is the Aspinwall–Morrison contribution. In a similar way, and following Kontsevich and Soibelman [63, §2.5 & §7.1], we define *BPS invariants*  $\hat{D}T^\alpha(\tau)$  to satisfy

$$\bar{D}T^\alpha(\tau) = \sum_{m \geq 1, m|\alpha} \frac{1}{m^2} \hat{D}T^{\alpha/m}(\tau), \quad (1.19)$$

where the factor  $1/m^2$  comes from (1.18). The inverse of (1.19) is

$$\hat{D}T^\alpha(\tau) = \sum_{m \geq 1, m|\alpha} \frac{\text{Mö}(m)}{m^2} \bar{D}T^{\alpha/m}(\tau),$$

where  $\text{Mö}(m)$  is the Möbius function from elementary number theory. We have  $\hat{D}T^\alpha(\tau) = DT^\alpha(\tau)$  when  $\mathcal{M}_{\text{ss}}^\alpha(\tau) = \mathcal{M}_{\text{st}}^\alpha(\tau)$ , so the BPS invariants are also generalizations of Donaldson–Thomas invariants.

A *stability condition*  $(\tau, T, \leq)$ , or  $\tau$  for short, on  $\text{coh}(X)$  is a totally ordered set  $(T, \leq)$  and a map  $\tau : C(\text{coh}(X)) \rightarrow T$  such that if  $\alpha, \beta, \gamma \in C(\text{coh}(X))$  with  $\beta = \alpha + \gamma$  then  $\tau(\alpha) < \tau(\beta) < \tau(\gamma)$  or  $\tau(\alpha) = \tau(\beta) = \tau(\gamma)$  or  $\tau(\alpha) > \tau(\beta) > \tau(\gamma)$ . We call a stability condition  $\tau$  *generic* if for all  $\alpha, \beta \in C(\text{coh}(X))$  with  $\tau(\alpha) = \tau(\beta)$  we have  $\bar{\chi}(\alpha, \beta) = 0$ , where  $\bar{\chi}$  is the Euler form of  $\text{coh}(X)$ . We conjecture that if  $\tau$  is generic, then  $\hat{D}T^\alpha(\tau) \in \mathbb{Z}$  for all  $\alpha \in C(\text{coh}(X))$ . We give evidence for this conjecture, and in §7.6 we prove the analogous result for invariants  $\hat{D}T_Q^d(\mu)$  counting representations of a quiver  $Q$  without relations.

In the situations in §6–§7 in which we can compute invariants explicitly, we usually find that the values of  $PI^{\alpha,n}(\tau')$  are complicated (often involving generating functions with a MacMahon function type factorization), the values of  $DT^\alpha(\tau)$  are simpler, and the values of  $\hat{D}T^\alpha(\tau)$  are simplest of all. For example, for dimension zero sheaves, if  $p = [\mathcal{O}_x] \in K^{\text{num}}(\text{coh}(X))$  is the class of a point sheaf, and  $\chi(X)$  is the Euler characteristic of the Calabi–Yau 3-fold  $X$ , we have

$$1 + \sum_{d \geq 1} PI^{dp,n}(\tau') s^d = \prod_{k \geq 1} (1 - (-s)^k)^{-k \chi(X)},$$

$$\bar{D}T^{dp}(\tau) = -\chi(X) \sum_{l \geq 1, l|d} \frac{1}{l^2}, \quad \text{and} \quad \hat{D}T^{dp}(\tau) = -\chi(X), \quad \text{all } d \geq 1.$$

### 1.5. Extension to quivers with superpotentials in §7

Section 7 studies an analogue of Donaldson–Thomas theory in which the abelian category  $\text{coh}(X)$  of coherent sheaves on a Calabi–Yau 3-fold is replaced by the abelian category  $\text{mod-}\mathbb{C}Q/I$  of representations of a quiver with relations  $(Q, I)$ , where the relations  $I$  are defined using a *superpotential*  $W$  on the quiver  $Q$ . This builds on the work of many authors; we mention in particular Ginzburg [30], Hanany et al. [26, 36–38], Nagao and Nakajima [83–85], Reineke et al. [24, 82, 88–90], Szendrői [99], and Young and Bryan [104, 105].

Categories  $\text{mod-}\mathbb{C}Q/I$  coming from a quiver  $Q$  with superpotential  $W$  share two important properties with categories  $\text{coh}(X)$  for  $X$  a Calabi–Yau 3-fold:

- (a) The moduli stack  $\mathfrak{M}_{Q,I}$  of objects in  $\text{mod-}\mathbb{C}Q/I$  can locally be written in terms of  $\text{Crit}(f)$  for  $f : U \rightarrow \mathbb{C}$  holomorphic and  $U$  smooth.
- (b) For all  $D, E$  in  $\text{mod-}\mathbb{C}Q/I$  we have

$$\bar{\chi}(\mathbf{dim} D, \mathbf{dim} E) = (\dim \text{Hom}(D, E) - \dim \text{Ext}^1(D, E)) - (\dim \text{Hom}(E, D) - \dim \text{Ext}^1(E, D)),$$

where  $\bar{\chi} : \mathbb{Z}^{Q_0} \times \mathbb{Z}^{Q_0}$  is an explicit antisymmetric biadditive form on the group of dimension vectors for  $Q$ .

Using these we can extend most of §1.3 to  $\text{mod-}\mathbb{C}Q/I$ : the Behrend function identities (1.10)–(1.11), the Lie algebra  $\tilde{L}(X)$  and Lie algebra morphism  $\tilde{\Psi}$ , the invariants  $\bar{D}T^\alpha(\tau)$ ,  $PI^{\alpha,n}(\tau')$  and their transformation laws (1.14) and (1.16). We allow the case  $W \equiv 0$ , so that  $\text{mod-}\mathbb{C}Q/I = \text{mod-}\mathbb{C}Q$ , the representations of a quiver  $Q$  without relations.

One aspect of the Calabi–Yau 3-fold case which does not extend is that in  $\text{coh}(X)$  the moduli schemes  $\mathcal{M}_{\text{ss}}^\alpha(\tau)$  and  $\mathcal{M}_{\text{stp}}^{\alpha,n}(\tau')$  are *proper*, but the analogues

in  $\text{mod-}\mathbb{C}Q/I$  are not. Properness is essential for forming virtual cycles and proving deformation-invariance of  $\bar{DT}^\alpha(\tau), PI^{\alpha,n}(\tau')$ . Therefore, the quiver analogues of  $\bar{DT}^\alpha(\tau), PI^{\alpha,n}(\tau')$  will in general not be invariant under deformations of the superpotential  $W$ .

It is an interesting question why such categories  $\text{mod-}\mathbb{C}Q/I$  should be good analogues of  $\text{coh}(X)$  for  $X$  a Calabi–Yau 3-fold. In some important classes of examples  $Q, W$ , such as those coming from the *McKay correspondence* for  $\mathbb{C}^3/G$  for finite  $G \subset \text{SL}(3, \mathbb{C})$ , or from a *consistent brane tiling*, the abelian category  $\text{mod-}\mathbb{C}Q/I$  is 3-Calabi–Yau, that is, Serre duality holds in dimension 3, so that  $\text{Ext}^i(D, E) \cong \text{Ext}^{3-i}(E, D)^*$  for all  $D, E$  in  $\text{mod-}\mathbb{C}Q/I$ . In the general case,  $\text{mod-}\mathbb{C}Q/I$  can be embedded as the heart of a t-structure in a 3-Calabi–Yau triangulated category  $\mathcal{T}$ .

It turns out that our new Donaldson–Thomas type quiver invariants  $\bar{DT}_{Q,I}^{\mathbf{d}}(\mu), \hat{DT}_{Q,I}^{\mathbf{d}}(\mu)$  have not really been considered, but the quiver analogues of pair invariants  $PI^{\alpha,n}(\tau')$ , which we write as  $NDT_{Q,I}^{\mathbf{d},e}(\mu')$ , are known in the literature as *noncommutative Donaldson–Thomas invariants*, and studied in [24, 82, 88–90, 99, 104, 105]. We prove that the analogue of (1.16) holds:

$$NDT_{Q,I}^{\mathbf{d},e}(\mu') = \sum_{\substack{\mathbf{d}_1, \dots, \mathbf{d}_l \in C(\text{mod-}\mathbb{C}Q/I), \\ l \geq 1: \mathbf{d}_1 + \dots + \mathbf{d}_l = \mathbf{d}, \\ \mu(\mathbf{d}_i) = \mu(\mathbf{d}), \text{ all } i}} \frac{(-1)^l}{l!} \prod_{i=1}^l [(-1)^{e \cdot \mathbf{d}_i - \bar{\chi}(\mathbf{d}_1 + \dots + \mathbf{d}_{i-1}, \mathbf{d}_i)} (e \cdot \mathbf{d}_i - \bar{\chi}(\mathbf{d}_1 + \dots + \mathbf{d}_{i-1}, \mathbf{d}_i)) \bar{DT}_{Q,I}^{\mathbf{d}_i}(\mu)]. \quad (1.20)$$

We use computations of  $NDT_{Q,I}^{\mathbf{d},e}(\mu')$  in examples by Szendrői [99] and Young and Bryan [105], and equation (1.20) to deduce values of  $\bar{DT}_{Q,I}^{\mathbf{d}}(\mu)$  and hence  $\hat{DT}_{Q,I}^{\mathbf{d}}(\mu)$  in examples. We find the  $NDT_{Q,I}^{\mathbf{d},e}(\mu')$  are complicated, the  $\bar{DT}_{Q,I}^{\mathbf{d}}(\mu)$  simpler, and the  $\hat{DT}_{Q,I}^{\mathbf{d}}(\mu)$  are very simple; this suggests that the  $\hat{DT}_{Q,I}^{\mathbf{d}}(\mu)$  may be more useful invariants than the  $NDT_{Q,I}^{\mathbf{d},e}(\mu')$ , a better tool for understanding what is really going on in these examples.

For quivers  $Q$  without relations (that is, with superpotential  $W \equiv 0$ ) and for generic slope stability conditions  $\mu$  on  $\text{mod-}\mathbb{C}Q$ , we prove using work of Reineke [88, 90] that the quiver BPS invariants  $\hat{DT}_Q^{\mathbf{d}}(\mu)$  are integers.

### 1.6. Relation to the work of Kontsevich and Soibelman [63]

The recent paper of Kontsevich and Soibelman [63], summarized in [64], has significant overlaps with this book, and with the previously published series [49–55]. Kontsevich and Soibelman also study generalizations of Donaldson–Thomas invariants, but they are more ambitious than us, and work in a more general context — they consider derived categories of coherent sheaves, Bridgeland stability conditions, and general motivic invariants, whereas we work only with abelian categories of coherent sheaves, Gieseker stability, and the Euler characteristic.

The large majority of the research in this book was done independently of [63]. After the appearance of Behrend’s seminal paper [3] in 2005, it was clear to the first author that Behrend’s approach should be integrated with [49–55] to extend Donaldson–Thomas theory, much along the lines of the present book. Within a few months the first author applied for an EPSRC grant to do this, and started work on the project with the second author in October 2006.

When we first received an early version of [63] in April 2008, we understood the material of §5.3–§5.4 below and many of the examples in §6, and had written §12

as a preprint, and we knew we had to prove the Behrend function identities (1.10)–(1.11), but for some months we were unable to do so. Our eventual solution of the problem, in §5.1–§5.2, was rather different to the Kontsevich–Soibelman formal power series approach in [63, §4.4 & §6.3].

There are three main places in this book in which we have made important use of ideas from Kontsevich and Soibelman [63], which we would like to acknowledge with thanks. The first is that in the proof of (1.10)–(1.11) in §10 we localize by the action of  $\{\text{id}_{E_1} + \lambda \text{id}_{E_2} : \lambda \in \mathbb{U}(1)\}$  on  $\text{Ext}^1(E_1 \oplus E_2, E_1 \oplus E_2)$ , which is an idea we got from [63, Conj. 4, §4.4]. The second is that in §6.2 one should define BPS invariants  $\hat{DT}^\alpha(\tau)$ , and they should be integers for generic  $\tau$ , which came from [63, §2.5 & §7.1]. The third is that in §7 one should consider Donaldson–Thomas theory for  $\text{mod-}\mathbb{C}Q/I$  coming from a quiver  $Q$  with arbitrary minimal superpotential  $W$ , rather than only those for which  $\text{mod-}\mathbb{C}Q/I$  is 3-Calabi–Yau, which came in part from [63, Th. 9, §8.1].

Having said all this, we should make it clear that the parallels between large parts of [49–55] and this book on the one hand, and large parts of [63, §§1,2,4,6 & 7] on the other, are really very close indeed. Some comparisons:

- ‘Motivic functions in the equivariant setting’ [63, §4.2] are basically the stack functions of [50].
- The ‘motivic Hall algebra’  $H(\mathcal{C})$  [63, §6.1] is a triangulated category version of Ringel–Hall algebras of stack functions  $\text{SF}(\mathfrak{M}_{\mathcal{A}})$  in [52, §5].
- The ‘motivic quantum torus’  $\mathcal{R}_\Gamma$  [63, §6.2] is basically the algebra  $A(\mathcal{A}, \Lambda, \chi)$  in [52, §6.2].
- The Lie algebra  $\mathfrak{g}_\Gamma$  of [63, §1.4] is our  $\tilde{L}(X)$  in §1.3.
- The algebra morphism  $\Phi : H(\mathcal{C}) \rightarrow \mathcal{R}_\Gamma$  in [63, Th. 8] is similar to the algebra morphism  $\Phi^\Lambda : \text{SF}(\mathfrak{M}_{\mathcal{A}}) \rightarrow A(\mathcal{A}, \Lambda, \chi)$  in [52, §6.2], and our Lie algebra morphism  $\tilde{\Psi}$  in §5.3 should be some kind of limit of their  $\Phi$ .
- Once their algebra morphism  $\Phi$  and our Lie algebra morphism  $\tilde{\Psi}$  are constructed, we both follow the method of [54] exactly to define Donaldson–Thomas invariants and prove wall-crossing formulae for them.
- Our  $\bar{DT}^\alpha(\tau)$  and  $\hat{DT}^\alpha(\tau)$  in §5.3, §6.2 should correspond to their ‘quasi-classical invariants’  $-a(\alpha)$  and  $\Omega(\alpha)$  in [63, §2.5 & §7.1], respectively.

Some differences between our programme and that of [63]:

- Nearly every major result in [63] depends explicitly or implicitly on conjectures, whereas by being less ambitious, we can give complete proofs.
- Kontsevich and Soibelman also tackle issues to do with triangulated categories, such as including  $\text{Ext}^i(D, E)$  for  $i < 0$ , which we do not touch.
- Although our wall-crossing formulae are both proved using the method of [54], we express them differently. Our formulae are written in terms of combinatorial coefficients  $S, U(\alpha_1, \dots, \alpha_n; \tau, \tilde{\tau})$  and  $V(I, \Gamma, \kappa; \tau, \tilde{\tau})$ , as in §3.3 and §3.5. These are not easy to work with; see §13.3 for a computation of  $U(\alpha_1, \dots, \alpha_n; \tau, \tilde{\tau})$  in an example.

By contrast, Kontsevich and Soibelman write their wall-crossing formulae in terms of products in a pro-nilpotent Lie group  $G_V$ . This seems to be an important idea, and may be a more useful point of view than ours. See Reineke [90] for a proof of an integrality conjecture [63, Conj. 1] on factorizations in  $G_V$ , which is probably related to our Theorem 7.29.

- We prove the Behrend function identities (1.10)–(1.11) by first showing that near a point  $[E]$  the moduli stack  $\mathfrak{M}$  can be written in terms of  $\text{Crit}(f)$  for  $f : U \rightarrow \mathbb{C}$  holomorphic and  $U$  open in  $\text{Ext}^1(E, E)$ . The proof uses gauge theory and transcendental methods, and works only over  $\mathbb{C}$ .

Their parallel passages [63, §4.4 & §6.3] work over a field  $\mathbb{K}$  of characteristic zero, and say that the formal completion  $\hat{\mathfrak{M}}_{[E]}$  of  $\mathfrak{M}$  at  $[E]$  can be written in terms of  $\text{Crit}(f)$  for  $f$  a formal power series on  $\text{Ext}^1(E, E)$ , with no convergence criteria. Their analogue of (1.10)–(1.11), [63, Conj. 4], concerns the ‘motivic Milnor fibre’ of the formal power series  $f$ .

- In [50, 52–54] the first author put a lot of effort into the difficult idea of ‘virtual rank’, the projections  $\Pi_n^{\text{vi}}$  on stack functions, the Lie algebra  $\text{SF}_{\text{al}}^{\text{ind}}(\mathfrak{M})$  of stack functions ‘supported on virtual indecomposables’, and the proof [53, Th. 8.7] that  $\bar{e}^\alpha(\tau)$  in (1.5) lies in  $\text{SF}_{\text{al}}^{\text{ind}}(\mathfrak{M})$ . This is very important for us, as our Lie algebra morphism  $\bar{\Psi}$  in (1.9) is defined only on  $\text{SF}_{\text{al}}^{\text{ind}}(\mathfrak{M})$ , so  $DT^\alpha(\tau)$  in (1.12) is only defined because  $\bar{e}^\alpha(\tau) \in \text{SF}_{\text{al}}^{\text{ind}}(\mathfrak{M})$ .

Most of this has no analogue in [63], but they come up against the problem this technology was designed to solve in [63, §7]. Roughly speaking, they first define Donaldson–Thomas invariants  $DT_{\text{vP}}^\alpha(\tau)$  over virtual Poincaré polynomials, which are rational functions in  $t$ . They then want to specialize to  $t = -1$  to get Donaldson–Thomas invariants over Euler characteristics, which should coincide with our  $DT^\alpha(\tau)$ . But this is only possible if  $DT_{\text{vP}}^\alpha(\tau)$  has no pole at  $t = -1$ , which they assume in their ‘absence of poles conjectures’ in [63, §7]. The fact that  $\bar{e}^\alpha(\tau)$  lies in  $\text{SF}_{\text{al}}^{\text{ind}}(\mathfrak{M})$  should be the key to proving these conjectures.

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## CHAPTER 2

# Constructible functions and stack functions

We begin with some background material on Artin stacks, constructible functions, stack functions, and motivic invariants, drawn mostly from [49, 50].

### 2.1. Artin stacks and (locally) constructible functions

*Artin stacks* are a class of geometric spaces, generalizing schemes and algebraic spaces. For a good introduction to Artin stacks see Gómez [31], and for a thorough treatment see Laumon and Moret-Bailly [67]. We make the convention that all Artin stacks in this book are *locally of finite type*, and substacks are *locally closed*. We work throughout over an algebraically closed field  $\mathbb{K}$ . For the parts of the story involving constructible functions, or needing a well-behaved notion of Euler characteristic,  $\mathbb{K}$  must have characteristic zero.

Artin  $\mathbb{K}$ -stacks form a 2-category. That is, we have *objects* which are  $\mathbb{K}$ -stacks  $\mathfrak{F}, \mathfrak{G}$ , and also two kinds of morphisms, 1-morphisms  $\phi, \psi : \mathfrak{F} \rightarrow \mathfrak{G}$  between  $\mathbb{K}$ -stacks, and 2-morphisms  $A : \phi \rightarrow \psi$  between 1-morphisms.

DEFINITION 2.1. Let  $\mathbb{K}$  be an algebraically closed field, and  $\mathfrak{F}$  a  $\mathbb{K}$ -stack. Write  $\mathfrak{F}(\mathbb{K})$  for the set of 2-isomorphism classes  $[x]$  of 1-morphisms  $x : \text{Spec } \mathbb{K} \rightarrow \mathfrak{F}$ . Elements of  $\mathfrak{F}(\mathbb{K})$  are called  $\mathbb{K}$ -points, or *geometric points*, of  $\mathfrak{F}$ . If  $\phi : \mathfrak{F} \rightarrow \mathfrak{G}$  is a 1-morphism then composition with  $\phi$  induces a map of sets  $\phi_* : \mathfrak{F}(\mathbb{K}) \rightarrow \mathfrak{G}(\mathbb{K})$ .

For a 1-morphism  $x : \text{Spec } \mathbb{K} \rightarrow \mathfrak{F}$ , the *stabilizer group*  $\text{Iso}_{\mathfrak{F}}(x)$  is the group of 2-morphisms  $x \rightarrow x$ . When  $\mathfrak{F}$  is an Artin  $\mathbb{K}$ -stack,  $\text{Iso}_{\mathfrak{F}}(x)$  is an *algebraic  $\mathbb{K}$ -group*. We say that  $\mathfrak{F}$  has *affine geometric stabilizers* if  $\text{Iso}_{\mathfrak{F}}(x)$  is an affine algebraic  $\mathbb{K}$ -group for all 1-morphisms  $x : \text{Spec } \mathbb{K} \rightarrow \mathfrak{F}$ .

As an algebraic  $\mathbb{K}$ -group up to isomorphism,  $\text{Iso}_{\mathfrak{F}}(x)$  depends only on the isomorphism class  $[x] \in \mathfrak{F}(\mathbb{K})$  of  $x$  in  $\text{Hom}(\text{Spec } \mathbb{K}, \mathfrak{F})$ . If  $\phi : \mathfrak{F} \rightarrow \mathfrak{G}$  is a 1-morphism, composition induces a morphism of algebraic  $\mathbb{K}$ -groups  $\phi_* : \text{Iso}_{\mathfrak{F}}([x]) \rightarrow \text{Iso}_{\mathfrak{G}}(\phi_*([x]))$ , for  $[x] \in \mathfrak{F}(\mathbb{K})$ .

Next we discuss *constructible functions* on  $\mathbb{K}$ -stacks, following [49].

DEFINITION 2.2. Let  $\mathbb{K}$  be an algebraically closed field of characteristic zero, and  $\mathfrak{F}$  an Artin  $\mathbb{K}$ -stack. We call  $C \subseteq \mathfrak{F}(\mathbb{K})$  *constructible* if  $C = \bigcup_{i \in I} \mathfrak{F}_i(\mathbb{K})$ , where  $\{\mathfrak{F}_i : i \in I\}$  is a finite collection of finite type Artin  $\mathbb{K}$ -substacks  $\mathfrak{F}_i$  of  $\mathfrak{F}$ . We call  $S \subseteq \mathfrak{F}(\mathbb{K})$  *locally constructible* if  $S \cap C$  is constructible for all constructible  $C \subseteq \mathfrak{F}(\mathbb{K})$ .

A function  $f : \mathfrak{F}(\mathbb{K}) \rightarrow \mathbb{Q}$  is called *constructible* if  $f(\mathfrak{F}(\mathbb{K}))$  is finite and  $f^{-1}(c)$  is a constructible set in  $\mathfrak{F}(\mathbb{K})$  for each  $c \in f(\mathfrak{F}(\mathbb{K})) \setminus \{0\}$ . A function  $f : \mathfrak{F}(\mathbb{K}) \rightarrow \mathbb{Q}$  is called *locally constructible* if  $f \cdot \delta_C$  is constructible for all constructible  $C \subseteq \mathfrak{F}(\mathbb{K})$ , where  $\delta_C$  is the characteristic function of  $C$ . Write  $\text{CF}(\mathfrak{F})$  and  $\text{LCF}(\mathfrak{F})$  for the  $\mathbb{Q}$ -vector spaces of  $\mathbb{Q}$ -valued constructible and locally constructible functions on  $\mathfrak{F}$ .

Following [49, Def.s 4.8, 5.1 & 5.5] we define *pushforwards* and *pullbacks* of constructible functions along 1-morphisms.

DEFINITION 2.3. Let  $\mathbb{K}$  have characteristic zero, and  $\mathfrak{F}$  be an Artin  $\mathbb{K}$ -stack with affine geometric stabilizers and  $C \subseteq \mathfrak{F}(\mathbb{K})$  be constructible. Then [49, Def. 4.8] defines the *naïve Euler characteristic*  $\chi^{\text{na}}(C)$  of  $C$ . It is called *naïve* as it takes no account of stabilizer groups. For  $f \in \text{CF}(\mathfrak{F})$ , define  $\chi^{\text{na}}(\mathfrak{F}, f)$  in  $\mathbb{Q}$  by

$$\chi^{\text{na}}(\mathfrak{F}, f) = \sum_{c \in f(\mathfrak{F}(\mathbb{K}) \setminus \{0\})} c \chi^{\text{na}}(f^{-1}(c)).$$

Let  $\mathfrak{F}, \mathfrak{G}$  be Artin  $\mathbb{K}$ -stacks with affine geometric stabilizers, and  $\phi : \mathfrak{F} \rightarrow \mathfrak{G}$  a 1-morphism. For  $f \in \text{CF}(\mathfrak{F})$ , define  $\text{CF}^{\text{na}}(\phi)f : \mathfrak{G}(\mathbb{K}) \rightarrow \mathbb{Q}$  by

$$\text{CF}^{\text{na}}(\phi)f(y) = \chi^{\text{na}}(\mathfrak{F}, f \cdot \delta_{\phi_*^{-1}(y)}) \quad \text{for } y \in \mathfrak{G}(\mathbb{K}),$$

where  $\delta_{\phi_*^{-1}(y)}$  is the characteristic function of  $\phi_*^{-1}(\{y\}) \subseteq \mathfrak{F}(\mathbb{K})$  on  $\mathfrak{F}(\mathbb{K})$ . Then  $\text{CF}^{\text{na}}(\phi) : \text{CF}(\mathfrak{F}) \rightarrow \text{CF}(\mathfrak{G})$  is a  $\mathbb{Q}$ -linear map called the *naïve pushforward*.

Now suppose  $\phi$  is *representable*. Then for any  $x \in \mathfrak{F}(\mathbb{K})$  we have an injective morphism  $\phi_* : \text{Iso}_{\mathfrak{F}}(x) \rightarrow \text{Iso}_{\mathfrak{G}}(\phi_*(x))$  of affine algebraic  $\mathbb{K}$ -groups. The image  $\phi_*(\text{Iso}_{\mathfrak{F}}(x))$  is an affine algebraic  $\mathbb{K}$ -group closed in  $\text{Iso}_{\mathfrak{G}}(\phi_*(x))$ , so the quotient  $\text{Iso}_{\mathfrak{G}}(\phi_*(x))/\phi_*(\text{Iso}_{\mathfrak{F}}(x))$  exists as a quasiprojective  $\mathbb{K}$ -variety. Define a function  $m_\phi : \mathfrak{F}(\mathbb{K}) \rightarrow \mathbb{Z}$  by  $m_\phi(x) = \chi(\text{Iso}_{\mathfrak{G}}(\phi_*(x))/\phi_*(\text{Iso}_{\mathfrak{F}}(x)))$  for  $x \in \mathfrak{F}(\mathbb{K})$ . For  $f \in \text{CF}(\mathfrak{F})$ , define  $\text{CF}^{\text{stk}}(\phi)f : \mathfrak{G}(\mathbb{K}) \rightarrow \mathbb{Q}$  by

$$\text{CF}^{\text{stk}}(\phi)f(y) = \chi^{\text{na}}(\mathfrak{F}, m_\phi \cdot f \cdot \delta_{\phi_*^{-1}(y)}) \quad \text{for } y \in \mathfrak{G}(\mathbb{K}).$$

An alternative definition is

$$\text{CF}^{\text{stk}}(\phi)f(y) = \chi(\mathfrak{F} \times_{\phi, \mathfrak{G}, y} \text{Spec } \mathbb{K}, \pi_{\mathfrak{F}}^*(f)) \quad \text{for } y \in \mathfrak{G}(\mathbb{K}),$$

where  $\mathfrak{F} \times_{\phi, \mathfrak{G}, y} \text{Spec } \mathbb{K}$  is a  $\mathbb{K}$ -scheme (or algebraic space) as  $\phi$  is representable, and  $\chi(\dots)$  is the Euler characteristic of this  $\mathbb{K}$ -scheme weighted by  $\pi_{\mathfrak{F}}^*(f)$ . These two definitions are equivalent as the projection  $\pi_1 : \mathfrak{F} \times_{\phi, \mathfrak{G}, y} \text{Spec } \mathbb{K} \rightarrow \mathfrak{F}$  induces a map on  $\mathbb{K}$ -points  $(\pi_1)_* : (\mathfrak{F} \times_{\phi, \mathfrak{G}, y} \text{Spec } \mathbb{K})(\mathbb{K}) \rightarrow \phi_*^{-1}(y) \subset \mathfrak{F}(\mathbb{K})$ , and the fibre of  $(\pi_1)_*$  over  $x \in \phi_*^{-1}(y)$  is  $(\text{Iso}_{\mathfrak{G}}(\phi_*(x))/\phi_*(\text{Iso}_{\mathfrak{F}}(x)))(\mathbb{K})$ , with Euler characteristic  $m_\phi(x)$ . Then  $\text{CF}^{\text{stk}}(\phi) : \text{CF}(\mathfrak{F}) \rightarrow \text{CF}(\mathfrak{G})$  is a  $\mathbb{Q}$ -linear map called the *stack pushforward*. If  $\mathfrak{F}, \mathfrak{G}$  are  $\mathbb{K}$ -schemes then  $\text{CF}^{\text{na}}(\phi), \text{CF}^{\text{stk}}(\phi)$  coincide, and we write them both as  $\text{CF}(\phi) : \text{CF}(\mathfrak{F}) \rightarrow \text{CF}(\mathfrak{G})$ .

Let  $\theta : \mathfrak{F} \rightarrow \mathfrak{G}$  be a finite type 1-morphism. If  $C \subseteq \mathfrak{G}(\mathbb{K})$  is constructible then so is  $\theta_*^{-1}(C) \subseteq \mathfrak{F}(\mathbb{K})$ . It follows that if  $f \in \text{CF}(\mathfrak{G})$  then  $f \circ \theta_*$  lies in  $\text{CF}(\mathfrak{F})$ . Define the *pullback*  $\theta^* : \text{CF}(\mathfrak{G}) \rightarrow \text{CF}(\mathfrak{F})$  by  $\theta^*(f) = f \circ \theta_*$ . It is a linear map.

Here [49, Th.s 4.9, 5.4, 5.6 & Def. 5.5] are some properties of these.

THEOREM 2.4. Let  $\mathbb{K}$  have characteristic zero,  $\mathfrak{E}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H}$  be Artin  $\mathbb{K}$ -stacks with affine geometric stabilizers, and  $\beta : \mathfrak{F} \rightarrow \mathfrak{G}, \gamma : \mathfrak{G} \rightarrow \mathfrak{H}$  be 1-morphisms. Then

$$\text{CF}^{\text{na}}(\gamma \circ \beta) = \text{CF}^{\text{na}}(\gamma) \circ \text{CF}^{\text{na}}(\beta) : \text{CF}(\mathfrak{F}) \rightarrow \text{CF}(\mathfrak{H}), \quad (2.1)$$

$$\text{CF}^{\text{stk}}(\gamma \circ \beta) = \text{CF}^{\text{stk}}(\gamma) \circ \text{CF}^{\text{stk}}(\beta) : \text{CF}(\mathfrak{F}) \rightarrow \text{CF}(\mathfrak{H}), \quad (2.2)$$

$$(\gamma \circ \beta)^* = \beta^* \circ \gamma^* : \text{CF}(\mathfrak{H}) \rightarrow \text{CF}(\mathfrak{F}), \quad (2.3)$$

supposing  $\beta, \gamma$  representable in (2.2), and of finite type in (2.3). If

$$\begin{array}{ccc}
 \mathfrak{E} & \xrightarrow{\eta} & \mathfrak{G} \\
 \downarrow \theta & & \downarrow \psi \\
 \mathfrak{F} & \xrightarrow{\phi} & \mathfrak{H}
 \end{array}
 \quad \begin{array}{l}
 \text{is a Cartesian square with} \\
 \eta, \phi \text{ representable and} \\
 \theta, \psi \text{ of finite type, then} \\
 \text{the following commutes:}
 \end{array}
 \quad \begin{array}{ccc}
 \mathrm{CF}(\mathfrak{E}) & \xrightarrow{\mathrm{CF}^{\mathrm{stk}}(\eta)} & \mathrm{CF}(\mathfrak{G}) \\
 \uparrow \theta^* & & \uparrow \psi^* \\
 \mathrm{CF}(\mathfrak{F}) & \xrightarrow{\mathrm{CF}^{\mathrm{stk}}(\phi)} & \mathrm{CF}(\mathfrak{H}).
 \end{array}
 \quad (2.4)$$

As discussed in [49, §3.3], equation (2.2) is *false* for  $\mathbb{K}$  of positive characteristic, so constructible function methods tend to fail in positive characteristic.

## 2.2. Stack functions

*Stack functions* are a universal generalization of constructible functions introduced in [50, §3]. Here [50, Def. 3.1] is the basic definition.

DEFINITION 2.5. Let  $\mathbb{K}$  be an algebraically closed field, and  $\mathfrak{F}$  be an Artin  $\mathbb{K}$ -stack with affine geometric stabilizers. Consider pairs  $(\mathfrak{R}, \rho)$ , where  $\mathfrak{R}$  is a finite type Artin  $\mathbb{K}$ -stack with affine geometric stabilizers and  $\rho : \mathfrak{R} \rightarrow \mathfrak{F}$  is a 1-morphism. We call two pairs  $(\mathfrak{R}, \rho)$ ,  $(\mathfrak{R}', \rho')$  *equivalent* if there exists a 1-isomorphism  $\iota : \mathfrak{R} \rightarrow \mathfrak{R}'$  such that  $\rho' \circ \iota$  and  $\rho$  are 2-isomorphic 1-morphisms  $\mathfrak{R} \rightarrow \mathfrak{F}$ . Write  $[(\mathfrak{R}, \rho)]$  for the equivalence class of  $(\mathfrak{R}, \rho)$ . If  $(\mathfrak{R}, \rho)$  is such a pair and  $\mathfrak{S}$  is a closed  $\mathbb{K}$ -substack of  $\mathfrak{R}$  then  $(\mathfrak{S}, \rho|_{\mathfrak{S}})$ ,  $(\mathfrak{R} \setminus \mathfrak{S}, \rho|_{\mathfrak{R} \setminus \mathfrak{S}})$  are pairs of the same kind.

Define  $\underline{\mathrm{SF}}(\mathfrak{F})$  to be the  $\mathbb{Q}$ -vector space generated by equivalence classes  $[(\mathfrak{R}, \rho)]$  as above, with for each closed  $\mathbb{K}$ -substack  $\mathfrak{S}$  of  $\mathfrak{R}$  a relation

$$[(\mathfrak{R}, \rho)] = [(\mathfrak{S}, \rho|_{\mathfrak{S}})] + [(\mathfrak{R} \setminus \mathfrak{S}, \rho|_{\mathfrak{R} \setminus \mathfrak{S}})]. \quad (2.5)$$

Define  $\mathrm{SF}(\mathfrak{F})$  to be the  $\mathbb{Q}$ -vector space generated by  $[(\mathfrak{R}, \rho)]$  with  $\rho$  representable, with the same relations (2.5). Then  $\mathrm{SF}(\mathfrak{F}) \subseteq \underline{\mathrm{SF}}(\mathfrak{F})$ .

Elements of  $\underline{\mathrm{SF}}(\mathfrak{F})$  will be called *stack functions*. We write stack functions either as letters  $f, g, \dots$ , or explicitly as sums  $\sum_{i=1}^m c_i [(\mathfrak{R}_i, \rho_i)]$ . If  $[(\mathfrak{R}, \rho)]$  is a generator of  $\underline{\mathrm{SF}}(\mathfrak{F})$  and  $\mathfrak{R}^{\mathrm{red}}$  is the reduced substack of  $\mathfrak{R}$  then  $\mathfrak{R}^{\mathrm{red}}$  is a closed substack of  $\mathfrak{R}$  and the complement  $\mathfrak{R} \setminus \mathfrak{R}^{\mathrm{red}}$  is empty. Hence (2.5) implies that

$$[(\mathfrak{R}, \rho)] = [(\mathfrak{R}^{\mathrm{red}}, \rho|_{\mathfrak{R}^{\mathrm{red}}})].$$

Thus, the relations (2.5) destroy all information on nilpotence in the stack structure of  $\mathfrak{R}$ . In [50, Def. 3.2] we relate  $\mathrm{CF}(\mathfrak{F})$  and  $\mathrm{SF}(\mathfrak{F})$ .

DEFINITION 2.6. Let  $\mathfrak{F}$  be an Artin  $\mathbb{K}$ -stack with affine geometric stabilizers, and  $C \subseteq \mathfrak{F}(\mathbb{K})$  be constructible. Then  $C = \coprod_{i=1}^n \mathfrak{R}_i(\mathbb{K})$ , for  $\mathfrak{R}_1, \dots, \mathfrak{R}_n$  finite type  $\mathbb{K}$ -substacks of  $\mathfrak{F}$ . Let  $\rho_i : \mathfrak{R}_i \rightarrow \mathfrak{F}$  be the inclusion 1-morphism. Then  $[(\mathfrak{R}_i, \rho_i)] \in \mathrm{SF}(\mathfrak{F})$ . Define  $\bar{\delta}_C = \sum_{i=1}^n [(\mathfrak{R}_i, \rho_i)] \in \mathrm{SF}(\mathfrak{F})$ . We think of this stack function as the analogue of the characteristic function  $\delta_C \in \mathrm{CF}(\mathfrak{F})$  of  $C$ . When  $\mathbb{K}$  has characteristic zero, define a  $\mathbb{Q}$ -linear map  $\iota_{\mathfrak{F}} : \mathrm{CF}(\mathfrak{F}) \rightarrow \mathrm{SF}(\mathfrak{F})$  by  $\iota_{\mathfrak{F}}(f) = \sum_{0 \neq c \in f(\mathfrak{F}(\mathbb{K}))} c \cdot \bar{\delta}_{f^{-1}(c)}$ . Define  $\mathbb{Q}$ -linear  $\pi_{\mathfrak{F}}^{\mathrm{stk}} : \mathrm{SF}(\mathfrak{F}) \rightarrow \mathrm{CF}(\mathfrak{F})$  by

$$\pi_{\mathfrak{F}}^{\mathrm{stk}}\left(\sum_{i=1}^n c_i [(\mathfrak{R}_i, \rho_i)]\right) = \sum_{i=1}^n c_i \mathrm{CF}^{\mathrm{stk}}(\rho_i) 1_{\mathfrak{R}_i},$$

where  $1_{\mathfrak{R}_i}$  is the function 1 in  $\mathrm{CF}(\mathfrak{R}_i)$ . Then [50, Prop. 3.3] shows  $\pi_{\mathfrak{F}}^{\mathrm{stk}} \circ \iota_{\mathfrak{F}}$  is the identity on  $\mathrm{CF}(\mathfrak{F})$ . Thus,  $\iota_{\mathfrak{F}}$  is injective and  $\pi_{\mathfrak{F}}^{\mathrm{stk}}$  is surjective. In general  $\iota_{\mathfrak{F}}$  is far from surjective, and  $\underline{\mathrm{SF}}, \mathrm{SF}(\mathfrak{F})$  are much larger than  $\mathrm{CF}(\mathfrak{F})$ .

The operations on constructible functions in §2.1 extend to stack functions.

DEFINITION 2.7. Define *multiplication* ‘ $\cdot$ ’ on  $\underline{\mathrm{SF}}(\mathfrak{F})$  by

$$[(\mathfrak{R}, \rho)] \cdot [(\mathfrak{G}, \sigma)] = [(\mathfrak{R} \times_{\rho, \mathfrak{F}, \sigma} \mathfrak{G}, \rho \circ \pi_{\mathfrak{R}})]. \quad (2.6)$$

This extends to a  $\mathbb{Q}$ -bilinear product  $\underline{\mathrm{SF}}(\mathfrak{F}) \times \underline{\mathrm{SF}}(\mathfrak{F}) \rightarrow \underline{\mathrm{SF}}(\mathfrak{F})$  which is commutative and associative, and  $\mathrm{SF}(\mathfrak{F})$  is closed under ‘ $\cdot$ ’. Let  $\phi : \mathfrak{F} \rightarrow \mathfrak{G}$  be a 1-morphism of Artin  $\mathbb{K}$ -stacks with affine geometric stabilizers. Define the *pushforward*  $\phi_* : \underline{\mathrm{SF}}(\mathfrak{F}) \rightarrow \underline{\mathrm{SF}}(\mathfrak{G})$  by

$$\phi_* : \sum_{i=1}^m c_i[(\mathfrak{R}_i, \rho_i)] \mapsto \sum_{i=1}^m c_i[(\mathfrak{R}_i, \phi \circ \rho_i)].$$

If  $\phi$  is representable then  $\phi_*$  maps  $\mathrm{SF}(\mathfrak{F}) \rightarrow \mathrm{SF}(\mathfrak{G})$ . For  $\phi$  of finite type, define *pullbacks*  $\phi^* : \underline{\mathrm{SF}}(\mathfrak{G}) \rightarrow \underline{\mathrm{SF}}(\mathfrak{F})$ ,  $\phi^* : \mathrm{SF}(\mathfrak{G}) \rightarrow \mathrm{SF}(\mathfrak{F})$  by

$$\phi^* : \sum_{i=1}^m c_i[(\mathfrak{R}_i, \rho_i)] \mapsto \sum_{i=1}^m c_i[(\mathfrak{R}_i \times_{\rho_i, \mathfrak{G}, \phi} \mathfrak{F}, \pi_{\mathfrak{F}})]. \quad (2.7)$$

The *tensor product*  $\otimes : \underline{\mathrm{SF}}(\mathfrak{F}) \times \underline{\mathrm{SF}}(\mathfrak{G}) \rightarrow \underline{\mathrm{SF}}(\mathfrak{F} \times \mathfrak{G})$  or  $\mathrm{SF}(\mathfrak{F}) \times \mathrm{SF}(\mathfrak{G}) \rightarrow \mathrm{SF}(\mathfrak{F} \times \mathfrak{G})$  is

$$\left( \sum_{i=1}^m c_i[(\mathfrak{R}_i, \rho_i)] \right) \otimes \left( \sum_{j=1}^n d_j[(\mathfrak{G}_j, \sigma_j)] \right) = \sum_{i,j} c_i d_j [(\mathfrak{R}_i \times \mathfrak{G}_j, \rho_i \times \sigma_j)]. \quad (2.8)$$

Here [50, Th. 3.5] is the analogue of Theorem 2.4.

THEOREM 2.8. Let  $\mathfrak{E}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H}$  be Artin  $\mathbb{K}$ -stacks with affine geometric stabilizers, and  $\beta : \mathfrak{F} \rightarrow \mathfrak{G}$ ,  $\gamma : \mathfrak{G} \rightarrow \mathfrak{H}$  be 1-morphisms. Then

$$\begin{aligned} (\gamma \circ \beta)_* &= \gamma_* \circ \beta_* : \underline{\mathrm{SF}}(\mathfrak{F}) \rightarrow \underline{\mathrm{SF}}(\mathfrak{H}), & (\gamma \circ \beta)_* &= \gamma_* \circ \beta_* : \mathrm{SF}(\mathfrak{F}) \rightarrow \mathrm{SF}(\mathfrak{H}), \\ (\gamma \circ \beta)^* &= \beta^* \circ \gamma^* : \underline{\mathrm{SF}}(\mathfrak{H}) \rightarrow \underline{\mathrm{SF}}(\mathfrak{F}), & (\gamma \circ \beta)^* &= \beta^* \circ \gamma^* : \mathrm{SF}(\mathfrak{H}) \rightarrow \mathrm{SF}(\mathfrak{F}), \end{aligned}$$

for  $\beta, \gamma$  representable in the second equation, and of finite type in the third and fourth. If  $f, g \in \underline{\mathrm{SF}}(\mathfrak{G})$  and  $\beta$  is finite type then  $\beta^*(f \cdot g) = \beta^*(f) \cdot \beta^*(g)$ . If

$$\begin{array}{ccc} \mathfrak{E} & \xrightarrow{\eta} & \mathfrak{G} \\ \downarrow \theta & & \downarrow \psi \\ \mathfrak{F} & \xrightarrow{\phi} & \mathfrak{H} \end{array} \quad \begin{array}{l} \text{is a Cartesian square with} \\ \theta, \psi \text{ of finite type, then} \\ \text{the following commutes:} \end{array} \quad \begin{array}{ccc} \underline{\mathrm{SF}}(\mathfrak{E}) & \xrightarrow{\eta_*} & \underline{\mathrm{SF}}(\mathfrak{G}) \\ \uparrow \theta^* & & \uparrow \psi^* \\ \underline{\mathrm{SF}}(\mathfrak{F}) & \xrightarrow{\phi_*} & \underline{\mathrm{SF}}(\mathfrak{H}). \end{array}$$

The same applies for  $\mathrm{SF}(\mathfrak{E}), \dots, \mathrm{SF}(\mathfrak{H})$  if  $\eta, \phi$  are representable.

In [50, Prop. 3.7 & Th. 3.8] we relate pushforwards and pullbacks of stack and constructible functions using  $\iota_{\mathfrak{F}}, \pi_{\mathfrak{F}}^{\mathrm{stk}}$ .

THEOREM 2.9. Let  $\mathbb{K}$  have characteristic zero,  $\mathfrak{F}, \mathfrak{G}$  be Artin  $\mathbb{K}$ -stacks with affine geometric stabilizers, and  $\phi : \mathfrak{F} \rightarrow \mathfrak{G}$  be a 1-morphism. Then

- (a)  $\phi^* \circ \iota_{\mathfrak{G}} = \iota_{\mathfrak{F}} \circ \phi^* : \mathrm{CF}(\mathfrak{G}) \rightarrow \mathrm{SF}(\mathfrak{F})$  if  $\phi$  is of finite type;
- (b)  $\pi_{\mathfrak{G}}^{\mathrm{stk}} \circ \phi_* = \mathrm{CF}^{\mathrm{stk}}(\phi) \circ \pi_{\mathfrak{F}}^{\mathrm{stk}} : \mathrm{SF}(\mathfrak{F}) \rightarrow \mathrm{CF}(\mathfrak{G})$  if  $\phi$  is representable; and
- (c)  $\pi_{\mathfrak{F}}^{\mathrm{stk}} \circ \phi^* = \phi^* \circ \pi_{\mathfrak{G}}^{\mathrm{stk}} : \mathrm{SF}(\mathfrak{G}) \rightarrow \mathrm{CF}(\mathfrak{F})$  if  $\phi$  is of finite type.

In [50, §3] we extend all the material on  $\underline{\mathrm{SF}}, \mathrm{SF}(\mathfrak{F})$  to *local stack functions*  $\underline{\mathrm{LSF}}, \mathrm{LSF}(\mathfrak{F})$ , the analogues of locally constructible functions. The main differences are in which 1-morphisms must be of finite type.

### 2.3. Operators $\Pi^\mu$ and projections $\Pi_n^{\text{vi}}$

We will need the following standard notation and facts about algebraic  $\mathbb{K}$ -groups and tori, which can be found in Borel [10]. Throughout  $\mathbb{K}$  is an algebraically closed field and  $G$  is an affine algebraic  $\mathbb{K}$ -group.

- Write  $\mathbb{G}_m$  for  $\mathbb{K} \setminus \{0\}$  as a  $\mathbb{K}$ -group under multiplication.
- By a *torus* we mean an algebraic  $\mathbb{K}$ -group isomorphic to  $\mathbb{G}_m^k$  for some  $k \geq 0$ . A *subtorus* of  $G$  means a  $\mathbb{K}$ -subgroup of  $G$  which is a torus.
- A *maximal torus* in  $G$  is a subtorus  $T^G$  contained in no larger subtorus  $T$  in  $G$ . All maximal tori in  $G$  are conjugate by Borel [10, Cor. IV.11.3]. The *rank*  $\text{rk } G$  is the dimension of any maximal torus. A maximal torus in  $\text{GL}(k, \mathbb{K})$  is the subgroup  $\mathbb{G}_m^k$  of diagonal matrices.
- Let  $T$  be a torus and  $H$  a closed  $\mathbb{K}$ -subgroup of  $T$ . Then  $H$  is isomorphic to  $\mathbb{G}_m^k \times K$  for some  $k \geq 0$  and finite abelian group  $K$ .
- If  $S$  is a subset of  $T^G$ , define the *centralizer* of  $S$  in  $G$  to be  $C_G(S) = \{\gamma \in G : \gamma s = s\gamma \ \forall s \in S\}$ , and the *normalizer* of  $S$  in  $G$  to be  $N_G(S) = \{\gamma \in G : \gamma^{-1}S\gamma = S\}$ . They are closed  $\mathbb{K}$ -subgroups of  $G$  containing  $T^G$ , and  $C_G(S)$  is normal in  $N_G(S)$ .
- The quotient group  $W(G, T^G) = N_G(T^G)/C_G(T^G)$  is called the *Weyl group* of  $G$ . As in [10, IV.11.19] it is a finite group, which acts on  $T^G$ .
- Define the *centre* of  $G$  to be  $C(G) = \{\gamma \in G : \gamma\delta = \delta\gamma \ \forall \delta \in G\}$ . It is a closed  $\mathbb{K}$ -subgroup of  $G$ .
- An algebraic  $\mathbb{K}$ -group  $G$  is called *special* if every principal  $G$ -bundle locally trivial in the étale topology is also locally trivial in the Zariski topology. Properties of special  $\mathbb{K}$ -groups can be found in [16, §§1.4, 1.5 & 5.5] and [50, §2.1]. Special  $\mathbb{K}$ -groups are always affine and connected. Products of special groups are special.
- $\mathbb{G}_m^k$  and  $\text{GL}(k, \mathbb{K})$  are special for all  $k \geq 0$ .

Now we define some linear maps  $\Pi^\mu : \underline{\text{SF}}(\mathfrak{F}) \rightarrow \underline{\text{SF}}(\mathfrak{F})$ .

DEFINITION 2.10. A *weight function*  $\mu$  is a map

$$\mu : \{\mathbb{K}\text{-groups } \mathbb{G}_m^k \times K, k \geq 0, K \text{ finite abelian, up to isomorphism}\} \longrightarrow \mathbb{Q}.$$

For any Artin  $\mathbb{K}$ -stack  $\mathfrak{F}$  with affine geometric stabilizers, we will define linear maps  $\Pi^\mu : \underline{\text{SF}}(\mathfrak{F}) \rightarrow \underline{\text{SF}}(\mathfrak{F})$  and  $\Pi^\mu : \text{SF}(\mathfrak{F}) \rightarrow \text{SF}(\mathfrak{F})$ . Now  $\underline{\text{SF}}(\mathfrak{F})$  is generated by  $[(\mathfrak{R}, \rho)]$  with  $\mathfrak{R}$  1-isomorphic to a quotient  $[X/G]$ , for  $X$  a quasiprojective  $\mathbb{K}$ -variety and  $G$  a special algebraic  $\mathbb{K}$ -group, with maximal torus  $T^G$ .

Let  $\mathcal{S}(T^G)$  be the set of subsets of  $T^G$  defined by Boolean operations upon closed  $\mathbb{K}$ -subgroups  $L$  of  $T^G$ . Given a weight function  $\mu$  as above, define a measure  $d\mu : \mathcal{S}(T^G) \rightarrow \mathbb{Q}$  to be additive upon disjoint unions of sets in  $\mathcal{S}(T^G)$ , and to satisfy  $d\mu(L) = \mu(L)$  for all algebraic  $\mathbb{K}$ -subgroups  $L$  of  $T^G$ . Define

$$\begin{aligned} \Pi^\mu([( \mathfrak{R}, \rho)]) &= \\ \int_{t \in T^G} \frac{|\{w \in W(G, T^G) : w \cdot t = t\}|}{|W(G, T^G)|} & [([X^{\{t\}}/C_G(\{t\})], \rho \circ \iota^{\{t\}})] d\mu. \end{aligned} \quad (2.9)$$

Here  $X^{\{t\}}$  is the subvariety of  $X$  fixed by  $t$ , and  $\iota^{\{t\}} : [X^{\{t\}}/C_G(\{t\})] \rightarrow [X/G]$  is the obvious 1-morphism of Artin stacks.

The integrand in (2.9), regarded as a function of  $t \in T^G$ , is a constructible function taking only finitely many values. The level sets of the function lie in  $\mathcal{S}(T^G)$ , so they are measurable w.r.t.  $d\mu$ , and the integral is well-defined.

If  $\mathfrak{R}$  has abelian stabilizer groups, then  $\Pi^\mu([\mathfrak{R}, \rho])$  simply weights each point  $r$  of  $\mathfrak{R}$  by  $\mu(\text{Iso}_{\mathfrak{R}}(r))$ . But if  $\mathfrak{R}$  has nonabelian stabilizer groups, then  $\Pi^\mu([\mathfrak{R}, \rho])$  replaces each point  $r$  with stabilizer group  $G$  by a  $\mathbb{Q}$ -linear combination of points with stabilizer groups  $C_G(\{t\})$  for  $t \in T^G$ , where the  $\mathbb{Q}$ -coefficients depend on the values of  $\mu$  on subgroups of  $T^G$ . Then [50, Th.s 5.11 & 5.12] shows:

**THEOREM 2.11.** *In the situation above,  $\Pi^\mu([\mathfrak{R}, \rho])$  is independent of the choices of  $X, G, T^G$  and 1-isomorphism  $\mathfrak{R} \cong [X/G]$ , and  $\Pi^\mu$  extends to unique linear maps  $\Pi^\mu : \underline{\text{SF}}(\mathfrak{F}) \rightarrow \underline{\text{SF}}(\mathfrak{F})$  and  $\Pi^\mu : \text{SF}(\mathfrak{F}) \rightarrow \text{SF}(\mathfrak{F})$ .*

**THEOREM 2.12.** (a)  $\Pi^1$  defined using  $\mu \equiv 1$  is the identity on  $\underline{\text{SF}}(\mathfrak{F})$ .

- (b) If  $\phi : \mathfrak{F} \rightarrow \mathfrak{G}$  is a 1-morphism of Artin  $\mathbb{K}$ -stacks with affine geometric stabilizers then  $\Pi^\mu \circ \phi_* = \phi_* \circ \Pi^\mu : \underline{\text{SF}}(\mathfrak{F}) \rightarrow \underline{\text{SF}}(\mathfrak{G})$ .
- (c) If  $\mu_1, \mu_2$  are weight functions as in Definition 2.10 then  $\mu_1 \mu_2$  is also a weight function and  $\Pi^{\mu_2} \circ \Pi^{\mu_1} = \Pi^{\mu_1} \circ \Pi^{\mu_2} = \Pi^{\mu_1 \mu_2}$ .

**DEFINITION 2.13.** For  $n \geq 0$ , define  $\Pi_n^{\text{vi}}$  to be the operator  $\Pi^{\mu_n}$  defined with weight  $\mu_n$  given by  $\mu_n([H]) = 1$  if  $\dim H = n$  and  $\mu_n([H]) = 0$  otherwise, for all  $\mathbb{K}$ -groups  $H \cong \mathbb{G}_m^k \times K$  with  $K$  a finite abelian group.

Here [50, Prop. 5.14] are some properties of the  $\Pi_n^{\text{vi}}$ .

**PROPOSITION 2.14.** *In the situation above, we have:*

- (i)  $(\Pi_n^{\text{vi}})^2 = \Pi_n^{\text{vi}}$ , so that  $\Pi_n^{\text{vi}}$  is a projection, and  $\Pi_m^{\text{vi}} \circ \Pi_n^{\text{vi}} = 0$  for  $m \neq n$ .
- (ii) For all  $f \in \underline{\text{SF}}(\mathfrak{F})$  we have  $f = \sum_{n \geq 0} \Pi_n^{\text{vi}}(f)$ , where the sum makes sense as  $\Pi_n^{\text{vi}}(f) = 0$  for  $n \gg 0$ .
- (iii) If  $\phi : \mathfrak{F} \rightarrow \mathfrak{G}$  is a 1-morphism of Artin  $\mathbb{K}$ -stacks with affine geometric stabilizers then  $\Pi_n^{\text{vi}} \circ \phi_* = \phi_* \circ \Pi_n^{\text{vi}} : \underline{\text{SF}}(\mathfrak{F}) \rightarrow \underline{\text{SF}}(\mathfrak{G})$ .
- (iv) If  $f \in \underline{\text{SF}}(\mathfrak{F})$ ,  $g \in \underline{\text{SF}}(\mathfrak{G})$  then  $\Pi_n^{\text{vi}}(f \otimes g) = \sum_{m=0}^n \Pi_m^{\text{vi}}(f) \otimes \Pi_{n-m}^{\text{vi}}(g)$ .

Very roughly speaking,  $\Pi_n^{\text{vi}}$  projects  $[(\mathfrak{R}, \rho)] \in \underline{\text{SF}}(\mathfrak{F})$  to  $[(\mathfrak{R}_n, \rho)]$ , where  $\mathfrak{R}_n$  is the  $\mathbb{K}$ -substack of points  $r \in \mathfrak{R}(\mathbb{K})$  whose stabilizer groups  $\text{Iso}_{\mathfrak{R}}(r)$  have rank  $n$ , that is, maximal torus  $\mathbb{G}_m^n$ . Unfortunately, it is more complicated than this. The right notion is not the actual rank of stabilizer groups, but the *virtual rank*. We treat  $r \in \mathfrak{R}(\mathbb{K})$  with nonabelian stabilizer group  $G = \text{Iso}_{\mathfrak{R}}(r)$  as a linear combination of points with ‘virtual ranks’ in the range  $\text{rk } C(G) \leq n \leq \text{rk } G$ . Effectively this *abelianizes stabilizer groups*, that is, using virtual rank we can treat  $\mathfrak{R}$  as though its stabilizer groups were all abelian, essentially tori  $\mathbb{G}_m^n$ .

## 2.4. Stack function spaces $\underline{\text{SF}}, \text{SF}(\mathfrak{F}, \chi, \mathbb{Q})$

In [50, §4] we extend *motivic* invariants of quasiprojective  $\mathbb{K}$ -varieties, such as Euler characteristics, virtual Poincaré polynomials, and virtual Hodge polynomials, to Artin stacks. Then in [50, §4–§6] we define several different classes of stack function spaces ‘twisted by motivic invariants’. This is a rather long, complicated story, which we will not explain. Instead, we will discuss only the spaces  $\underline{\text{SF}}, \text{SF}(\mathfrak{F}, \chi, \mathbb{Q})$  ‘twisted by the Euler characteristic’ which we need later.

Throughout this section  $\mathbb{K}$  is an algebraically closed field of characteristic zero. We continue to use the notation on algebraic  $\mathbb{K}$ -groups in §2.3. Here is some more notation, [50, Def.s 5.5 & 5.16].

**DEFINITION 2.15.** Let  $G$  be an affine algebraic  $\mathbb{K}$ -group with maximal torus  $T^G$ . If  $S \subset T^G$  then  $Q = T^G \cap C(C_G(S))$  is a closed  $\mathbb{K}$ -subgroup of  $T^G$  containing  $S$ . As  $S \subseteq Q$  we have  $C_G(Q) \subseteq C_G(S)$ . But  $Q$  commutes with  $C_G(S)$ , so  $C_G(S) \subseteq C_G(Q)$ . Thus  $C_G(S) = C_G(Q)$ . So  $Q = T^G \cap C(C_G(Q))$ , and  $Q$  and  $C_G(Q)$  determine each other, given  $G, T^G$ . Define  $\mathcal{Q}(G, T^G)$  to be the set of closed  $\mathbb{K}$ -subgroups  $Q$  of  $T^G$  such that  $Q = T^G \cap C(C_G(Q))$ .

In [50, Lem. 5.6] we show that  $\mathcal{Q}(G, T^G)$  is finite and closed under intersections, with maximal element  $T^G$  and minimal element  $Q_{\min} = T^G \cap C(G)$ .

An affine algebraic  $\mathbb{K}$ -group  $G$  is called *very special* if  $C_G(Q)$  and  $Q$  are special for all  $Q \in \mathcal{Q}(G, T^G)$ , for any maximal torus  $T^G$  in  $G$ . Then  $G$  is special, as  $G = C_G(Q_{\min})$ . In [50, Ex. 5.7 & Def. 5.16] we compute  $\mathcal{Q}(G, T^G)$  for  $G = \text{GL}(k, \mathbb{K})$ , and deduce that  $\text{GL}(k, \mathbb{K})$  is very special.

We can now define the spaces  $\underline{\text{SF}}, \text{SF}(\mathfrak{F}, \chi, \mathbb{Q})$ , [50, Def.s 5.17 & 6.8].

**DEFINITION 2.16.** Let  $\mathfrak{F}$  be an Artin  $\mathbb{K}$ -stack with affine geometric stabilizers. Consider pairs  $(\mathfrak{R}, \rho)$ , where  $\mathfrak{R}$  is a finite type Artin  $\mathbb{K}$ -stack with affine geometric stabilizers and  $\rho : \mathfrak{R} \rightarrow \mathfrak{F}$  is a 1-morphism, with equivalence of pairs as in Definition 2.5. Define  $\underline{\text{SF}}(\mathfrak{F}, \chi, \mathbb{Q})$  to be the  $\mathbb{Q}$ -vector space generated by equivalence classes  $[(\mathfrak{R}, \rho)]$  as above, with the following relations:

- (i) Given  $[(\mathfrak{R}, \rho)]$  as above and  $\mathfrak{S}$  a closed  $\mathbb{K}$ -substack of  $\mathfrak{R}$  we have  $[(\mathfrak{R}, \rho)] = [(\mathfrak{S}, \rho|_{\mathfrak{S}})] + [(\mathfrak{R} \setminus \mathfrak{S}, \rho|_{\mathfrak{R} \setminus \mathfrak{S}})]$ , as in (2.5).
- (ii) Let  $\mathfrak{R}$  be a finite type Artin  $\mathbb{K}$ -stack with affine geometric stabilizers,  $U$  a quasiprojective  $\mathbb{K}$ -variety,  $\pi_{\mathfrak{R}} : \mathfrak{R} \times U \rightarrow \mathfrak{R}$  the natural projection, and  $\rho : \mathfrak{R} \rightarrow \mathfrak{F}$  a 1-morphism. Then  $[(\mathfrak{R} \times U, \rho \circ \pi_{\mathfrak{R}})] = \chi([U])([\mathfrak{R}, \rho])$ .

Here  $\chi(U) \in \mathbb{Z}$  is the Euler characteristic of  $U$ . It is a *motivic invariant* of  $\mathbb{K}$ -schemes, that is,  $\chi(U) = \chi(V) + \chi(U \setminus V)$  for  $V \subset U$  closed.

- (iii) Given  $[(\mathfrak{R}, \rho)]$  as above and a 1-isomorphism  $\mathfrak{R} \cong [X/G]$  for  $X$  a quasiprojective  $\mathbb{K}$ -variety and  $G$  a very special algebraic  $\mathbb{K}$ -group acting on  $X$  with maximal torus  $T^G$ , we have

$$[(\mathfrak{R}, \rho)] = \sum_{Q \in \mathcal{Q}(G, T^G)} F(G, T^G, Q) [(X/Q), \rho \circ \iota^Q], \quad (2.10)$$

where  $\iota^Q : [X/Q] \rightarrow \mathfrak{R} \cong [X/G]$  is the natural projection 1-morphism.

Here  $F(G, T^G, Q) \in \mathbb{Q}$  are a system of rational coefficients with a complicated definition in [50, §6.2], which we will not repeat. In [50, §6.2] we derive an inductive formula for computing them when  $G = \text{GL}(k, \mathbb{K})$ .

Similarly, define  $\text{SF}(\mathfrak{F}, \chi, \mathbb{Q})$  to be the  $\mathbb{Q}$ -vector space generated by  $[(\mathfrak{R}, \rho)]$  with  $\rho$  representable, and relations (i)–(iii) as above. Then  $\text{SF}(\mathfrak{F}, \chi, \mathbb{Q}) \subset \underline{\text{SF}}(\mathfrak{F}, \chi, \mathbb{Q})$ . Define projections  $\bar{\Pi}_{\mathfrak{F}}^{\chi, \mathbb{Q}} : \underline{\text{SF}}(\mathfrak{F}) \rightarrow \underline{\text{SF}}(\mathfrak{F}, \chi, \mathbb{Q})$  and  $\text{SF}(\mathfrak{F}) \rightarrow \text{SF}(\mathfrak{F}, \chi, \mathbb{Q})$  by  $\bar{\Pi}_{\mathfrak{F}}^{\chi, \mathbb{Q}} : \sum_{i \in I} c_i [(\mathfrak{R}_i, \rho_i)] \mapsto \sum_{i \in I} c_i [(\mathfrak{R}_i, \rho_i)]$ .

Define *multiplication* ‘ $\cdot$ ’, *pushforwards*  $\phi_*$ , *pullbacks*  $\phi^*$ , and *tensor products*  $\otimes$  on the spaces  $\underline{\text{SF}}, \text{SF}(*, \chi, \mathbb{Q})$  as in Definition 2.7, and *projections*  $\Pi_n^i$  as in §2.3. The important point is that (2.6)–(2.9) are compatible with the relations defining  $\underline{\text{SF}}, \text{SF}(*, \chi, \mathbb{Q})$ , or they would not be well-defined. This is proved in [50, Th.s 5.19 & 6.9], and depends on deep properties of the  $F(G, T^G, Q)$ .



Here [50, Prop.s 5.21 & 5.22 & §6.3] is a useful way to represent these spaces.

PROPOSITION 2.17.  $\underline{\mathrm{SF}}, \bar{\mathrm{SF}}(\mathfrak{F}, \chi, \mathbb{Q})$  are spanned over  $\mathbb{Q}$  by elements  $[(U \times [\mathrm{Spec} \mathbb{K}/T], \rho)]$ , for  $U$  a quasiprojective  $\mathbb{K}$ -variety and  $T$  an algebraic  $\mathbb{K}$ -group isomorphic to  $\mathbb{G}_m^k \times K$  for  $k \geq 0$  and  $K$  finite abelian.

Suppose  $\sum_{i \in I} c_i [(U_i \times [\mathrm{Spec} \mathbb{K}/T_i], \rho_i)] = 0$  in  $\underline{\mathrm{SF}}(\mathfrak{F}, \chi, \mathbb{Q})$  or  $\bar{\mathrm{SF}}(\mathfrak{F}, \chi, \mathbb{Q})$ , where  $I$  is finite set,  $c_i \in \mathbb{Q}$ ,  $U_i$  is a quasiprojective  $\mathbb{K}$ -variety, and  $T_i$  is an algebraic  $\mathbb{K}$ -group isomorphic to  $\mathbb{G}_m^{k_i} \times K_i$  for  $k_i \geq 0$  and  $K_i$  finite abelian, with  $T_i \not\cong T_j$  for  $i \neq j$ . Then  $c_j [(U_j \times [\mathrm{Spec} \mathbb{K}/T_j], \rho_j)] = 0$  for all  $j \in I$ .

In this representation, the operators  $\Pi_n^i$  of §2.3 are easy to define: we have

$$\Pi_n^i([(U \times [\mathrm{Spec} \mathbb{K}/T], \rho)]) = \begin{cases} [(U \times [\mathrm{Spec} \mathbb{K}/T], \rho)], & \dim T = n, \\ 0, & \text{otherwise.} \end{cases}$$

Proposition 2.17 says that a general element  $[(\mathfrak{R}, \rho)]$  of  $\underline{\mathrm{SF}}, \bar{\mathrm{SF}}(\mathfrak{F}, \chi, \mathbb{Q})$ , whose stabilizer groups  $\mathrm{Iso}_{\mathfrak{R}}(x)$  for  $x \in \mathfrak{R}(\mathbb{K})$  are arbitrary affine algebraic  $\mathbb{K}$ -groups, may be written as a  $\mathbb{Q}$ -linear combination of elements  $[(U \times [\mathrm{Spec} \mathbb{K}/T], \rho)]$  whose stabilizer groups  $T$  are of the form  $\mathbb{G}_m^k \times K$  for  $k \geq 0$  and  $K$  finite abelian. That is, by working in  $\underline{\mathrm{SF}}, \bar{\mathrm{SF}}(\mathfrak{F}, \chi, \mathbb{Q})$ , we can treat all stabilizer groups as if they are abelian. Furthermore, although  $\underline{\mathrm{SF}}, \bar{\mathrm{SF}}(\mathfrak{F}, \chi, \mathbb{Q})$  forget information about nonabelian stabilizer groups, they do remember the difference between abelian stabilizer groups of the form  $\mathbb{G}_m^k \times K$  for finite  $K$ .

In [50, Prop. 6.11] we completely describe  $\underline{\mathrm{SF}}, \bar{\mathrm{SF}}(\mathrm{Spec} \mathbb{K}, \chi, \mathbb{Q})$ .

PROPOSITION 2.18. Define a commutative  $\mathbb{Q}$ -algebra  $\Lambda$  with basis isomorphism classes  $[T]$  of  $\mathbb{K}$ -groups  $T$  of the form  $\mathbb{G}_m^k \times K$ , for  $k \geq 0$  and  $K$  finite abelian, with multiplication  $[T] \cdot [T'] = [T \times T']$ . Define  $i_\Lambda : \Lambda \rightarrow \underline{\mathrm{SF}}(\mathrm{Spec} \mathbb{K}, \chi, \mathbb{Q})$  by  $\sum_i c_i [T_i] \mapsto \sum_i c_i [(\mathrm{Spec} \mathbb{K}/T_i)]$ . Then  $i_\Lambda$  is an algebra isomorphism. It restricts to an isomorphism  $i_\Lambda : \mathbb{Q}[\{1\}] \rightarrow \underline{\mathrm{SF}}(\mathrm{Spec} \mathbb{K}, \chi, \mathbb{Q}) \cong \mathbb{Q}$ .

Proposition 2.18 shows that the relations Definition 2.16(i)–(iii) are well chosen, and in particular, the coefficients  $F(G, T^G, Q)$  in (2.10) have some beautiful properties. If the  $F(G, T^G, Q)$  were just some random numbers, one might expect relation (iii) to be so strong that  $\underline{\mathrm{SF}}(\mathfrak{F}, \chi, \mathbb{Q})$  would be small, or even zero, for all  $\mathfrak{F}$ . But  $\underline{\mathrm{SF}}(\mathrm{Spec} \mathbb{K}, \chi, \mathbb{Q})$  is large, and easily understood.

## CHAPTER 3

### Background material from [51–54]

Next we review material from the first author’s series of papers [51–54].

#### 3.1. Ringel–Hall algebras of an abelian category

Let  $\mathcal{A}$  be a  $\mathbb{K}$ -linear abelian category. We define the *Grothendieck group*  $K_0(\mathcal{A})$ , the *Euler form*  $\bar{\chi}$ , and the *numerical Grothendieck group*  $K^{\text{num}}(\mathcal{A})$ .

DEFINITION 3.1. Let  $\mathcal{A}$  be an abelian category. The *Grothendieck group*  $K_0(\mathcal{A})$  is the abelian group generated by all isomorphism classes  $[E]$  of objects  $E$  in  $\mathcal{A}$ , with the relations  $[E] + [G] = [F]$  for each short exact sequence  $0 \rightarrow E \rightarrow F \rightarrow G \rightarrow 0$  in  $\mathcal{A}$ . In many interesting cases such as  $\mathcal{A} = \text{coh}(X)$ , the Grothendieck group  $K_0(\mathcal{A})$  is very large, and it is useful to replace it by a smaller group. Suppose  $\mathcal{A}$  is  $\mathbb{K}$ -linear for some algebraically closed field  $\mathbb{K}$ , and that  $\text{Ext}^*(E, F)$  is finite-dimensional over  $\mathbb{K}$  for all  $E, F \in \mathcal{A}$ . The *Euler form*  $\bar{\chi} : K_0(\mathcal{A}) \times K_0(\mathcal{A}) \rightarrow \mathbb{Z}$  is a biadditive map satisfying

$$\bar{\chi}([E], [F]) = \sum_{i \geq 0} (-1)^i \dim \text{Ext}^i(E, F) \quad (3.1)$$

for all  $E, F \in \mathcal{A}$ . We use the notation  $\bar{\chi}$  rather than  $\chi$  for the Euler form, because  $\chi$  will be used often to mean Euler characteristic or weighted Euler characteristic. The *numerical Grothendieck group*  $K^{\text{num}}(\mathcal{A})$  is the quotient of  $K_0(\mathcal{A})$  by the (two-sided) kernel of  $\bar{\chi}$ , that is,  $K^{\text{num}}(\mathcal{A}) = K_0(\mathcal{A})/I$  where  $I = \{\alpha \in K_0(\mathcal{A}) : \bar{\chi}(\alpha, \beta) = \bar{\chi}(\beta, \alpha) = 0 \text{ for all } \beta \in K_0(\mathcal{A})\}$ . Then  $\bar{\chi}$  on  $K_0(\mathcal{A})$  descends to a biadditive Euler form  $\bar{\chi} : K^{\text{num}}(\mathcal{A}) \times K^{\text{num}}(\mathcal{A}) \rightarrow \mathbb{Z}$ .

If  $\mathcal{A}$  is 3-Calabi–Yau then  $\bar{\chi}$  is antisymmetric, so the left and right kernels of  $\bar{\chi}$  on  $K_0(\mathcal{A})$  coincide, and  $\bar{\chi}$  on  $K^{\text{num}}(\mathcal{A})$  is nondegenerate. If  $\mathcal{A} = \text{coh}(X)$  for  $X$  a smooth projective  $\mathbb{K}$ -scheme of dimension  $m$  then Serre duality implies that  $\bar{\chi}(E, F) = (-1)^m \bar{\chi}(F, E \otimes K_X)$ . Thus, again, the left and right kernels of  $\bar{\chi}$  on  $K_0(\text{coh}(X))$  are the same, and  $\bar{\chi}$  on  $K^{\text{num}}(\text{coh}(X))$  is nondegenerate.

Our goal is to associate a *Ringel–Hall algebra*  $\text{SF}_{\text{al}}(\mathfrak{M}_{\mathcal{A}})$  to  $\mathcal{A}$ . To do this we will need to be able to do algebraic geometry in  $\mathcal{A}$ , in particular, to form moduli  $\mathbb{K}$ -stacks of objects and exact sequences in  $\mathcal{A}$  and 1-morphisms between them. This requires some extra data, described in [51, Assumptions 7.1 & 8.1].

ASSUMPTION 3.2. Let  $\mathbb{K}$  be an algebraically closed field and  $\mathcal{A}$  a  $\mathbb{K}$ -linear abelian category with  $\text{Ext}^i(E, F)$  finite-dimensional  $\mathbb{K}$ -vector spaces for all  $E, F$  in  $\mathcal{A}$  and  $i \geq 0$ . Let  $K(\mathcal{A})$  be the quotient of the Grothendieck group  $K_0(\mathcal{A})$  by some fixed subgroup. Usually we will take  $K(\mathcal{A}) = K^{\text{num}}(\mathcal{A})$ , the numerical Grothendieck group from Definition 3.1. Suppose that if  $E \in \mathcal{A}$  with  $[E] = 0$  in  $K(\mathcal{A})$  then  $E \cong 0$ . From §3.2 we will also assume  $\mathcal{A}$  is *noetherian*.

To define moduli stacks of objects or configurations in  $\mathcal{A}$ , we need some *extra data*, to tell us about algebraic families of objects and morphisms in  $\mathcal{A}$ , parametrized

by a base scheme  $U$ . We encode this extra data as a *stack in exact categories*  $\mathfrak{F}_{\mathcal{A}}$  on the *category of  $\mathbb{K}$ -schemes*  $\text{Sch}_{\mathbb{K}}$ , made into a *site* with the *étale topology*. The  $\mathbb{K}, \mathcal{A}, K(\mathcal{A}), \mathfrak{F}_{\mathcal{A}}$  must satisfy some complex additional conditions [51, Assumptions 7.1 & 8.1], which we do not give.

Examples of data satisfying Assumption 3.2 are given in [51, §9–§10]. These include  $\mathcal{A} = \text{coh}(X)$ , the abelian category of coherent sheaves on a smooth projective  $\mathbb{K}$ -scheme  $X$ , with  $K(\mathcal{A}) = K^{\text{num}}(\text{coh}(X))$ , and  $\mathcal{A} = \text{mod-}\mathbb{K}Q/I$ , the abelian category of  $\mathbb{K}$ -representations of a quiver  $Q = (Q_0, Q_1, b, e)$  with relations  $I$ , with  $K(\mathcal{A}) = \mathbb{Z}^{Q_0}$ , the lattice of dimension vectors for  $Q$ .

Suppose Assumption 3.2 holds. We will use the following notation:

- Define the ‘positive cone’  $C(\mathcal{A})$  in  $K(\mathcal{A})$  to be

$$C(\mathcal{A}) = \{[E] \in K(\mathcal{A}) : 0 \not\cong E \in \mathcal{A}\} \subset K(\mathcal{A}). \quad (3.2)$$

- Write  $\mathfrak{M}_{\mathcal{A}}$  for the moduli stack of objects in  $\mathcal{A}$ . It is an Artin  $\mathbb{K}$ -stack, locally of finite type. Elements of  $\mathfrak{M}_{\mathcal{A}}(\mathbb{K})$  correspond to isomorphism classes  $[E]$  of objects  $E$  in  $\mathcal{A}$ , and the stabilizer group  $\text{Iso}_{\mathfrak{M}_{\mathcal{A}}}([E])$  in  $\mathfrak{M}_{\mathcal{A}}$  is isomorphic as an algebraic  $\mathbb{K}$ -group to the automorphism group  $\text{Aut}(E)$ .
- For  $\alpha \in C(\mathcal{A})$ , write  $\mathfrak{M}_{\mathcal{A}}^{\alpha}$  for the substack of objects  $E \in \mathcal{A}$  in class  $\alpha$  in  $K(\mathcal{A})$ . It is an open and closed  $\mathbb{K}$ -substack of  $\mathfrak{M}_{\mathcal{A}}$ .
- Write  $\mathfrak{E}\mathfrak{r}\mathfrak{a}\mathfrak{c}\mathfrak{t}_{\mathcal{A}}$  for the moduli stack of short exact sequences  $0 \rightarrow E_1 \rightarrow E_2 \rightarrow E_3 \rightarrow 0$  in  $\mathcal{A}$ . It is an Artin  $\mathbb{K}$ -stack, locally of finite type.
- For  $j = 1, 2, 3$  write  $\pi_j : \mathfrak{E}\mathfrak{r}\mathfrak{a}\mathfrak{c}\mathfrak{t}_{\mathcal{A}} \rightarrow \mathfrak{M}_{\mathcal{A}}$  for the 1-morphism of Artin stacks projecting  $0 \rightarrow E_1 \rightarrow E_2 \rightarrow E_3 \rightarrow 0$  to  $E_j$ . Then  $\pi_2$  is *representable*, and  $\pi_1 \times \pi_3 : \mathfrak{E}\mathfrak{r}\mathfrak{a}\mathfrak{c}\mathfrak{t}_{\mathcal{A}} \rightarrow \mathfrak{M}_{\mathcal{A}} \times \mathfrak{M}_{\mathcal{A}}$  is of *finite type*.

In [52] we define *Ringel–Hall algebras*, using stack functions.

DEFINITION 3.3. Suppose Assumption 3.2 holds. Define bilinear operations  $*$  on the stack function spaces  $\underline{\text{SF}}, \text{SF}(\mathfrak{M}_{\mathcal{A}})$  and  $\underline{\text{SF}}, \text{SF}(\mathfrak{M}_{\mathcal{A}}, \chi, \mathbb{Q})$  by

$$f * g = (\pi_2)_* ((\pi_1 \times \pi_3)^*(f \otimes g)), \quad (3.3)$$

using pushforwards, pullbacks and tensor products in Definition 2.7. They are well-defined as  $\pi_2$  is representable, and  $\pi_1 \times \pi_3$  is of finite type. By [52, Th. 5.2] this  $*$  is *associative*, and makes  $\underline{\text{SF}}, \text{SF}(\mathfrak{M}_{\mathcal{A}}), \underline{\text{SF}}, \text{SF}(\mathfrak{M}_{\mathcal{A}}, \chi, \mathbb{Q})$  into noncommutative  $\mathbb{Q}$ -algebras, with identity  $\bar{\delta}_{[0]}$ , where  $[0] \in \mathfrak{M}_{\mathcal{A}}$  is the zero object. We call them *Ringel–Hall algebras*, as they are a version of the Ringel–Hall method for defining algebras from abelian categories. The natural inclusions and projections  $\bar{\Pi}_{\mathfrak{M}_{\mathcal{A}}}^{\chi, \mathbb{Q}}$  between these spaces are algebra morphisms.

As these algebras are inconveniently large for some purposes, in [52, Def. 5.5] we define subalgebras  $\text{SF}_{\text{al}}(\mathfrak{M}_{\mathcal{A}}), \bar{\text{SF}}_{\text{al}}(\mathfrak{M}_{\mathcal{A}}, \chi, \mathbb{Q})$  using the algebra structure on stabilizer groups in  $\mathfrak{M}_{\mathcal{A}}$ . Suppose  $[(\mathfrak{R}, \rho)]$  is a generator of  $\text{SF}(\mathfrak{M}_{\mathcal{A}})$ . Let  $r \in \mathfrak{R}(\mathbb{K})$  with  $\rho_*(r) = [E] \in \mathfrak{M}_{\mathcal{A}}(\mathbb{K})$ , for some  $E \in \mathcal{A}$ . Then  $\rho$  induces a morphism of stabilizer  $\mathbb{K}$ -groups  $\rho_* : \text{Iso}_{\mathfrak{R}}(r) \rightarrow \text{Iso}_{\mathfrak{M}_{\mathcal{A}}}([E]) \cong \text{Aut}(E)$ . As  $\rho$  is representable this is *injective*, and induces an isomorphism of  $\text{Iso}_{\mathfrak{R}}(r)$  with a  $\mathbb{K}$ -subgroup of  $\text{Aut}(E)$ . Now  $\text{Aut}(E) = \text{End}(E)^{\times}$  is the  $\mathbb{K}$ -group of invertible elements in a *finite-dimensional  $\mathbb{K}$ -algebra*  $\text{End}(E) = \text{Hom}(E, E)$ . We say that  $[(\mathfrak{R}, \rho)]$  *has algebra stabilizers* if whenever  $r \in \mathfrak{R}(\mathbb{K})$  with  $\rho_*(r) = [E]$ , the  $\mathbb{K}$ -subgroup  $\rho_*(\text{Iso}_{\mathfrak{R}}(r))$  in  $\text{Aut}(E)$  is the  $\mathbb{K}$ -group  $A^{\times}$  of invertible elements in a  $\mathbb{K}$ -subalgebra  $A$  in  $\text{End}(E)$ . Write  $\text{SF}_{\text{al}}(\mathfrak{M}_{\mathcal{A}}), \bar{\text{SF}}_{\text{al}}(\mathfrak{M}_{\mathcal{A}}, \chi, \mathbb{Q})$  for the subspaces of  $\text{SF}(\mathfrak{M}_{\mathcal{A}}), \bar{\text{SF}}(\mathfrak{M}_{\mathcal{A}}, \chi, \mathbb{Q})$  spanned over  $\mathbb{Q}$  by  $[(\mathfrak{R}, \rho)]$  with algebra stabilizers. Then

[52, Prop. 5.7] shows that  $\mathrm{SF}_{\mathrm{al}}(\mathfrak{M}_{\mathcal{A}}), \bar{\mathrm{SF}}_{\mathrm{al}}(\mathfrak{M}_{\mathcal{A}}, \chi, \mathbb{Q})$  are *subalgebras* of the Ringel–Hall algebras  $\mathrm{SF}(\mathfrak{M}_{\mathcal{A}}), \bar{\mathrm{SF}}(\mathfrak{M}_{\mathcal{A}}, \chi, \mathbb{Q})$ .

Now [52, Cor. 5.10] shows that  $\mathrm{SF}_{\mathrm{al}}(\mathfrak{M}_{\mathcal{A}}), \bar{\mathrm{SF}}_{\mathrm{al}}(\mathfrak{M}_{\mathcal{A}}, \chi, \mathbb{Q})$  are closed under the operators  $\Pi_n^{\mathrm{vi}}$  on  $\mathrm{SF}(\mathfrak{M}_{\mathcal{A}}), \bar{\mathrm{SF}}(\mathfrak{M}_{\mathcal{A}}, \chi, \mathbb{Q})$  defined in §2.3. In [52, Def. 5.14] we define  $\mathrm{SF}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{M}_{\mathcal{A}}), \bar{\mathrm{SF}}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{M}_{\mathcal{A}}, \chi, \mathbb{Q})$  to be the subspaces of  $f$  in  $\mathrm{SF}_{\mathrm{al}}(\mathfrak{M}_{\mathcal{A}})$  and  $\bar{\mathrm{SF}}_{\mathrm{al}}(\mathfrak{M}_{\mathcal{A}}, \chi, \mathbb{Q})$  with  $\Pi_1^{\mathrm{vi}}(f) = f$ . We think of  $\mathrm{SF}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{M}_{\mathcal{A}}), \bar{\mathrm{SF}}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{M}_{\mathcal{A}}, \chi, \mathbb{Q})$  as stack functions ‘supported on virtual indecomposables’. This is because if  $E \in \mathcal{A}$  then  $\mathrm{rk} \mathrm{Aut}(E)$  is the number of indecomposable factors of  $E$ , that is,  $\mathrm{rk} \mathrm{Aut}(E) = r$  if  $E \cong E_1 \oplus \cdots \oplus E_r$  with  $E_i$  nonzero and indecomposable in  $\mathcal{A}$ . But  $\Pi_1^{\mathrm{vi}}$  projects to stack functions with ‘virtual rank’ 1, and thus with ‘one virtual indecomposable factor’.

In [52, Th. 5.18] we show  $\mathrm{SF}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{M}_{\mathcal{A}}), \bar{\mathrm{SF}}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{M}_{\mathcal{A}}, \chi, \mathbb{Q})$  are closed under the Lie bracket  $[f, g] = f * g - g * f$  on  $\mathrm{SF}_{\mathrm{al}}(\mathfrak{M}_{\mathcal{A}}), \bar{\mathrm{SF}}_{\mathrm{al}}(\mathfrak{M}_{\mathcal{A}}, \chi, \mathbb{Q})$ . Thus,  $\mathrm{SF}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{M}_{\mathcal{A}}), \bar{\mathrm{SF}}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{M}_{\mathcal{A}}, \chi, \mathbb{Q})$  are *Lie subalgebras* of  $\mathrm{SF}_{\mathrm{al}}(\mathfrak{M}_{\mathcal{A}}), \bar{\mathrm{SF}}_{\mathrm{al}}(\mathfrak{M}_{\mathcal{A}}, \chi, \mathbb{Q})$ . The projection  $\bar{\Pi}_{\mathfrak{M}_{\mathcal{A}}}^{\chi, \mathbb{Q}} : \mathrm{SF}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{M}_{\mathcal{A}}) \rightarrow \bar{\mathrm{SF}}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{M}_{\mathcal{A}}, \chi, \mathbb{Q})$  is a Lie algebra morphism.

As in [52, Cor. 5.11], the first part of Proposition 2.17 simplifies to give:

**PROPOSITION 3.4.**  *$\bar{\mathrm{SF}}_{\mathrm{al}}(\mathfrak{M}_{\mathcal{A}}, \chi, \mathbb{Q})$  is spanned over  $\mathbb{Q}$  by elements of the form  $[(U \times [\mathrm{Spec} \mathbb{K}/\mathbb{G}_m^k], \rho)]$  with algebra stabilizers, for  $U$  a quasiprojective  $\mathbb{K}$ -variety and  $k \geq 0$ . Also  $\bar{\mathrm{SF}}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{M}_{\mathcal{A}}, \chi, \mathbb{Q})$  is spanned over  $\mathbb{Q}$  by  $[(U \times [\mathrm{Spec} \mathbb{K}/\mathbb{G}_m], \rho)]$  with algebra stabilizers, for  $U$  a quasiprojective  $\mathbb{K}$ -variety.*

### 3.2. (Weak) stability conditions on $\mathcal{A}$

Next we discuss material in [53] on *stability conditions*.

**DEFINITION 3.5.** Let  $\mathcal{A}$  be an abelian category,  $K(\mathcal{A})$  be the quotient of  $K_0(\mathcal{A})$  by some fixed subgroup, and  $C(\mathcal{A})$  as in (3.2). Suppose  $(T, \leq)$  is a totally ordered set, and  $\tau : C(\mathcal{A}) \rightarrow T$  a map. We call  $(\tau, T, \leq)$  a *stability condition* on  $\mathcal{A}$  if whenever  $\alpha, \beta, \gamma \in C(\mathcal{A})$  with  $\beta = \alpha + \gamma$  then either  $\tau(\alpha) < \tau(\beta) < \tau(\gamma)$ , or  $\tau(\alpha) > \tau(\beta) > \tau(\gamma)$ , or  $\tau(\alpha) = \tau(\beta) = \tau(\gamma)$ . We call  $(\tau, T, \leq)$  a *weak stability condition* on  $\mathcal{A}$  if whenever  $\alpha, \beta, \gamma \in C(\mathcal{A})$  with  $\beta = \alpha + \gamma$  then either  $\tau(\alpha) \leq \tau(\beta) \leq \tau(\gamma)$ , or  $\tau(\alpha) \geq \tau(\beta) \geq \tau(\gamma)$ .

For such  $(\tau, T, \leq)$ , we say that a nonzero object  $E$  in  $\mathcal{A}$  is

- (i)  $\tau$ -semistable if for all  $S \subset E$  with  $S \not\cong 0, E$  we have  $\tau([S]) \leq \tau([E/S])$ ;
- (ii)  $\tau$ -stable if for all  $S \subset E$  with  $S \not\cong 0, E$  we have  $\tau([S]) < \tau([E/S])$ ; and
- (iii)  $\tau$ -unstable if it is not  $\tau$ -semistable.

Given a weak stability condition  $(\tau, T, \leq)$  on  $\mathcal{A}$ , we say that  $\mathcal{A}$  is  $\tau$ -artinian if there exist no infinite chains of subobjects  $\cdots \subset A_2 \subset A_1 \subset X$  in  $\mathcal{A}$  with  $A_{n+1} \neq A_n$  and  $\tau([A_{n+1}]) \geq \tau([A_n/A_{n+1}])$  for all  $n$ .

In [53, Th. 4.4] we prove the existence of *Harder–Narasimhan filtrations*.

**PROPOSITION 3.6.** *Let  $(\tau, T, \leq)$  be a weak stability condition on an abelian category  $\mathcal{A}$ . Suppose  $\mathcal{A}$  is noetherian and  $\tau$ -artinian. Then each  $E \in \mathcal{A}$  admits a unique filtration  $0 = E_0 \subset \cdots \subset E_n = E$  for  $n \geq 0$ , such that  $S_k = E_k/E_{k-1}$  is  $\tau$ -semistable for  $k = 1, \dots, n$ , and  $\tau([S_1]) > \tau([S_2]) > \cdots > \tau([S_n])$ .*

We define *permissible* (weak) stability conditions, a condition needed to get well-behaved invariants ‘counting’  $\tau$ -(semi)stable objects in [54].

DEFINITION 3.7. Suppose Assumption 3.2 holds for  $\mathbb{K}, \mathcal{A}, K(\mathcal{A})$ , so that the moduli stack  $\mathfrak{M}_{\mathcal{A}}$  of objects in  $\mathcal{A}$  is an Artin  $\mathbb{K}$ -stack, with substacks  $\mathfrak{M}_{\mathcal{A}}^{\alpha}$  for  $\alpha \in C(\mathcal{A})$ . Suppose too that  $\mathcal{A}$  is *noetherian*. Let  $(\tau, T, \leq)$  be a weak stability condition on  $\mathcal{A}$ . For  $\alpha \in C(\mathcal{A})$ , write  $\mathfrak{M}_{\text{ss}}^{\alpha}(\tau), \mathfrak{M}_{\text{st}}^{\alpha}(\tau)$  for the moduli substacks of  $\tau$ -(semi)stable  $E \in \mathcal{A}$  with class  $[E] = \alpha$  in  $K(\mathcal{A})$ . As in [53, §4.2],  $\mathfrak{M}_{\text{ss}}^{\alpha}(\tau), \mathfrak{M}_{\text{st}}^{\alpha}(\tau)$  are *open*  $\mathbb{K}$ -substacks of  $\mathfrak{M}_{\mathcal{A}}^{\alpha}$ . We call  $(\tau, T, \leq)$  *permissible* if:

- (a)  $\mathcal{A}$  is  $\tau$ -artinian, in the sense of Definition 3.5, and
- (b)  $\mathfrak{M}_{\text{ss}}^{\alpha}(\tau)$  is a *finite type* substack of  $\mathfrak{M}_{\mathcal{A}}^{\alpha}$  for all  $\alpha \in C(\mathcal{A})$ .

Here (b) is necessary if ‘counting’  $\tau$ -(semi)stables in class  $\alpha$  is to yield a finite answer. We will be interested in two classes of examples of permissible (weak) stability conditions on coherent sheaves, Gieseker stability and  $\mu$ -stability.

EXAMPLE 3.8. Let  $\mathbb{K}$  be an algebraically closed field,  $X$  a smooth projective  $\mathbb{K}$ -scheme of dimension  $m$ , and  $\mathcal{A} = \text{coh}(X)$  the coherent sheaves on  $X$ . Then [51, §9] defines data satisfying Assumption 3.2, with  $K(\mathcal{A}) = K^{\text{num}}(\text{coh}(X))$ . It is a finite rank lattice, that is,  $K(\mathcal{A}) \cong \mathbb{Z}^l$ .

Define  $G$  to be the set of monic rational polynomials in  $t$ :

$$G = \{p(t) = t^d + a_{d-1}t^{d-1} + \cdots + a_0 : d = 0, 1, \dots, a_0, \dots, a_{d-1} \in \mathbb{Q}\}.$$

Define a total order ‘ $\leq$ ’ on  $G$  by  $p \leq p'$  for  $p, p' \in G$  if either

- (a)  $\deg p > \deg p'$ , or
- (b)  $\deg p = \deg p'$  and  $p(t) \leq p'(t)$  for all  $t \gg 0$ .

We write  $p < q$  if  $p \leq q$  and  $p \neq q$ . Note that  $\deg p > \deg p'$  in (a) implies that  $p(t) > p'(t)$  for all  $t \gg 0$ , which is the opposite to  $p(t) \leq p'(t)$  for  $t \gg 0$  in (b), and not what you might expect, but it is necessary for Definition 3.5 to hold. The effect of (a) is that  $\tau$ -semistable sheaves are automatically *pure*, because if  $0 \neq S \subset E$  with  $\dim S < \dim E$  then  $S$  destabilizes  $E$ .

Fix a very ample line bundle  $\mathcal{O}_X(1)$  on  $X$ . For  $E \in \text{coh}(X)$ , the *Hilbert polynomial*  $P_E$  is the unique polynomial in  $\mathbb{Q}[t]$  such that  $P_E(n) = \dim H^0(E(n))$  for all  $n \gg 0$ . Equivalently,  $P_E(n) = \bar{\chi}([\mathcal{O}_X(-n)], [E])$  for all  $n \in \mathbb{Z}$ . Thus,  $P_E$  depends only on the class  $\alpha \in K^{\text{num}}(\text{coh}(X))$  of  $E$ , and we may write  $P_{\alpha}$  instead of  $P_E$ . Define  $\tau : C(\text{coh}(X)) \rightarrow G$  by  $\tau(\alpha) = P_{\alpha}/r_{\alpha}$ , where  $P_{\alpha}$  is the Hilbert polynomial of  $\alpha$ , and  $r_{\alpha}$  is the leading coefficient of  $P_{\alpha}$ , which must be positive. Then as in [53, Ex. 4.16],  $(\tau, G, \leq)$  is a *permissible stability condition* on  $\text{coh}(X)$ . It is called *Gieseker stability*, and  $\tau$ -(semi)stable sheaves are called *Gieseker (semi)stable*. Gieseker stability is studied in [44, §1.2].

For the case of Gieseker stability, as well as the moduli stacks  $\mathfrak{M}_{\text{ss}}^{\alpha}(\tau), \mathfrak{M}_{\text{st}}^{\alpha}(\tau)$  of  $\tau$ -(semi)stable sheaves  $E$  with class  $[E] = \alpha$ , later we will also use the notation  $\mathcal{M}_{\text{ss}}^{\alpha}(\tau), \mathcal{M}_{\text{st}}^{\alpha}(\tau)$  for the *coarse moduli schemes* of  $\tau$ -(semi)stable sheaves  $E$  with class  $[E] = \alpha$  in  $K^{\text{num}}(\text{coh}(X))$ . By [44, Th. 4.3.4],  $\mathcal{M}_{\text{ss}}^{\alpha}(\tau)$  is a projective  $\mathbb{K}$ -scheme whose  $\mathbb{K}$ -points correspond to S-equivalence classes of Gieseker semistable sheaves in class  $\alpha$ , and  $\mathcal{M}_{\text{st}}^{\alpha}(\tau)$  is an open  $\mathbb{K}$ -subscheme whose  $\mathbb{K}$ -points correspond to isomorphism classes of Gieseker stable sheaves.

EXAMPLE 3.9. In the situation of Example 3.8, define

$$M = \{p(t) = t^d + a_{d-1}t^{d-1} : d = 0, 1, \dots, a_{d-1} \in \mathbb{Q}, a_{-1} = 0\} \subset G$$

and restrict the total order  $\leq$  on  $G$  to  $M$ . Define  $\mu : C(\text{coh}(X)) \rightarrow M$  by  $\mu(\alpha) = t^d + a_{d-1}t^{d-1}$  when  $\tau(\alpha) = P_{\alpha}/r_{\alpha} = t^d + a_{d-1}t^{d-1} + \cdots + a_0$ , that is,  $\mu(\alpha)$  is the

truncation of the polynomial  $\tau(\alpha)$  in Example 3.8 at its second term. Then as in [53, Ex. 4.17],  $(\mu, M, \leq)$  is a *permissible weak stability condition* on  $\text{coh}(X)$ . It is called  $\mu$ -*stability*, and is studied in [44, §1.6].

In [53, §8] we define interesting stack functions  $\bar{\delta}_{\text{ss}}^\alpha(\tau), \bar{\epsilon}^\alpha(\tau)$  in  $\text{SF}_{\text{al}}(\mathfrak{M}_{\mathcal{A}})$ .

DEFINITION 3.10. Let  $\mathbb{K}, \mathcal{A}, K(\mathcal{A})$  satisfy Assumption 3.2, and  $(\tau, T, \leq)$  be a permissible weak stability condition on  $\mathcal{A}$ . Define stack functions  $\bar{\delta}_{\text{ss}}^\alpha(\tau) = \bar{\delta}_{\mathfrak{M}_{\text{ss}}^\alpha(\tau)}^\alpha$  in  $\text{SF}_{\text{al}}(\mathfrak{M}_{\mathcal{A}})$  for  $\alpha \in C(\mathcal{A})$ . That is,  $\bar{\delta}_{\text{ss}}^\alpha(\tau)$  is the characteristic function, in the sense of Definition 2.6, of the moduli substack  $\mathfrak{M}_{\text{ss}}^\alpha(\tau)$  of  $\tau$ -semistable sheaves in  $\mathfrak{M}_{\mathcal{A}}$ . In [53, Def. 8.1] we define elements  $\bar{\epsilon}^\alpha(\tau)$  in  $\text{SF}_{\text{al}}(\mathfrak{M}_{\mathcal{A}})$  by

$$\bar{\epsilon}^\alpha(\tau) = \sum_{\substack{n \geq 1, \alpha_1, \dots, \alpha_n \in C(\mathcal{A}): \\ \alpha_1 + \dots + \alpha_n = \alpha, \tau(\alpha_i) = \tau(\alpha), \text{ all } i}} \frac{(-1)^{n-1}}{n} \bar{\delta}_{\text{ss}}^{\alpha_1}(\tau) * \bar{\delta}_{\text{ss}}^{\alpha_2}(\tau) * \dots * \bar{\delta}_{\text{ss}}^{\alpha_n}(\tau), \quad (3.4)$$

where  $*$  is the Ringel–Hall multiplication in  $\text{SF}_{\text{al}}(\mathfrak{M}_{\mathcal{A}})$ . Then [53, Th. 8.2] proves

$$\bar{\delta}_{\text{ss}}^\alpha(\tau) = \sum_{\substack{n \geq 1, \alpha_1, \dots, \alpha_n \in C(\mathcal{A}): \\ \alpha_1 + \dots + \alpha_n = \alpha, \tau(\alpha_i) = \tau(\alpha), \text{ all } i}} \frac{1}{n!} \bar{\epsilon}^{\alpha_1}(\tau) * \bar{\epsilon}^{\alpha_2}(\tau) * \dots * \bar{\epsilon}^{\alpha_n}(\tau). \quad (3.5)$$

There are only finitely many nonzero terms in (3.4)–(3.5), because as the family of  $\tau$ -semistable sheaves in class  $\alpha$  is bounded, there are only finitely ways to write  $\alpha = \alpha_1 + \dots + \alpha_n$  with  $\tau$ -semistable sheaves in class  $\alpha_i$  for all  $i$ .

Here is a way to interpret (3.4) and (3.5) informally in terms of log and exp: working in a completed version  $\widehat{\text{SF}}_{\text{al}}(\mathfrak{M}_{\mathcal{A}})$  of the algebra  $\text{SF}_{\text{al}}(\mathfrak{M}_{\mathcal{A}})$ , so that appropriate classes of infinite sums make sense, for fixed  $t \in T$  we have

$$\sum_{\alpha \in C(\mathcal{A}): \tau(\alpha) = t} \bar{\epsilon}^\alpha(\tau) = \log \left[ \bar{\delta}_0 + \sum_{\alpha \in C(\mathcal{A}): \tau(\alpha) = t} \bar{\delta}_{\text{ss}}^\alpha(\tau) \right], \quad (3.6)$$

$$\bar{\delta}_0 + \sum_{\alpha \in C(\mathcal{A}): \tau(\alpha) = t} \bar{\delta}_{\text{ss}}^\alpha(\tau) = \exp \left[ \sum_{\alpha \in C(\mathcal{A}): \tau(\alpha) = t} \bar{\epsilon}^\alpha(\tau) \right], \quad (3.7)$$

where  $\bar{\delta}_0$  is the identity 1 in  $\widehat{\text{SF}}_{\text{al}}(\mathfrak{M}_{\mathcal{A}})$ . For  $\alpha \in C(\mathcal{A})$  and  $t = \tau(\alpha)$ , using the power series  $\log(1+x) = \sum_{n \geq 1} \frac{(-1)^{n-1}}{n} x^n$  and  $\exp(x) = 1 + \sum_{n \geq 1} \frac{1}{n!} x^n$  we see that (3.4)–(3.5) are the restrictions of (3.6)–(3.7) to  $\mathfrak{M}_{\mathcal{A}}^\alpha$ . This makes clear why (3.4) and (3.5) are inverse, since log and exp are inverse. Thus, knowing the  $\bar{\epsilon}^\alpha(\tau)$  is equivalent to knowing the  $\bar{\delta}_{\text{ss}}^\alpha(\tau)$ .

If  $\mathfrak{M}_{\text{ss}}^\alpha(\tau) = \mathfrak{M}_{\text{st}}^\alpha(\tau)$  then  $\bar{\epsilon}^\alpha(\tau) = \bar{\delta}_{\text{ss}}^\alpha(\tau)$ . The difference between  $\bar{\epsilon}^\alpha(\tau)$  and  $\bar{\delta}_{\text{ss}}^\alpha(\tau)$  is that  $\bar{\epsilon}^\alpha(\tau)$  ‘counts’ strictly semistable sheaves in a special, complicated way. Here [53, Th. 8.7] is an important property of the  $\bar{\epsilon}^\alpha(\tau)$ , which does not hold for the  $\bar{\delta}_{\text{ss}}^\alpha(\tau)$ . The proof is highly nontrivial, using the full power of the configurations formalism of [51–54].

THEOREM 3.11.  $\bar{\epsilon}^\alpha(\tau)$  lies in the Lie subalgebra  $\text{SF}_{\text{al}}^{\text{ind}}(\mathfrak{M}_{\mathcal{A}})$  in  $\text{SF}_{\text{al}}(\mathfrak{M}_{\mathcal{A}})$ .

### 3.3. Changing stability conditions and algebra identities

In [54] we prove transformation laws for the  $\bar{\delta}_{\text{ss}}^\alpha(\tau), \bar{\epsilon}^\alpha(\tau)$  under change of stability condition. These involve combinatorial coefficients  $S(*; \tau, \tilde{\tau}) \in \mathbb{Z}$  and  $U(*; \tau, \tilde{\tau}) \in \mathbb{Q}$  defined in [54, §4.1]. We have changed some notation from [54].

DEFINITION 3.12. Let  $\mathcal{A}, K(\mathcal{A})$  satisfy Assumption 3.2, and  $(\tau, T, \leq), (\tilde{\tau}, \tilde{T}, \leq)$  be weak stability conditions on  $\mathcal{A}$ . We say that  $(\tilde{\tau}, \tilde{T}, \leq)$  *dominates*  $(\tau, T, \leq)$  if  $\tau(\alpha) \leq \tau(\beta)$  implies  $\tilde{\tau}(\alpha) \leq \tilde{\tau}(\beta)$  for all  $\alpha, \beta \in C(\mathcal{A})$ .

Let  $n \geq 1$  and  $\alpha_1, \dots, \alpha_n \in C(\mathcal{A})$ . If for all  $i = 1, \dots, n-1$  we have either

- (a)  $\tau(\alpha_i) \leq \tau(\alpha_{i+1})$  and  $\tilde{\tau}(\alpha_1 + \dots + \alpha_i) > \tilde{\tau}(\alpha_{i+1} + \dots + \alpha_n)$  or
- (b)  $\tau(\alpha_i) > \tau(\alpha_{i+1})$  and  $\tilde{\tau}(\alpha_1 + \dots + \alpha_i) \leq \tilde{\tau}(\alpha_{i+1} + \dots + \alpha_n)$ ,

then define  $S(\alpha_1, \dots, \alpha_n; \tau, \tilde{\tau}) = (-1)^r$ , where  $r$  is the number of  $i = 1, \dots, n-1$  satisfying (a). Otherwise define  $S(\alpha_1, \dots, \alpha_n; \tau, \tilde{\tau}) = 0$ . Now define

$$U(\alpha_1, \dots, \alpha_n; \tau, \tilde{\tau}) = \sum_{\substack{1 \leq l \leq m \leq n, \ 0 = a_0 < a_1 < \dots < a_m = n, \ 0 = b_0 < b_1 < \dots < b_l = m: \\ \text{Define } \beta_1, \dots, \beta_m \in C(\mathcal{A}) \text{ by } \beta_i = \alpha_{a_{i-1}+1} + \dots + \alpha_{a_i}. \\ \text{Define } \gamma_1, \dots, \gamma_l \in C(\mathcal{A}) \text{ by } \gamma_i = \beta_{b_{i-1}+1} + \dots + \beta_{b_i}. \\ \text{Then } \tau(\beta_i) = \tau(\alpha_j), \ i = 1, \dots, m, \ a_{i-1} < j \leq a_i, \\ \text{and } \tilde{\tau}(\gamma_i) = \tilde{\tau}(\alpha_1 + \dots + \alpha_n), \ i = 1, \dots, l}} \frac{(-1)^{l-1}}{l} \cdot \prod_{i=1}^l S(\beta_{b_{i-1}+1}, \beta_{b_{i-1}+2}, \dots, \beta_{b_i}; \tau, \tilde{\tau}) \cdot \prod_{i=1}^m \frac{1}{(a_i - a_{i-1})!}. \quad (3.8)$$

Then in [54, §5] we derive wall-crossing formulae for the  $\bar{\delta}_{ss}^\alpha(\tau), \bar{\epsilon}^\alpha(\tau)$  under change of stability condition from  $(\tau, T, \leq)$  to  $(\tilde{\tau}, \tilde{T}, \leq)$ :

THEOREM 3.13. *Let Assumption 3.2 hold, and  $(\tau, T, \leq), (\tilde{\tau}, \tilde{T}, \leq), (\hat{\tau}, \hat{T}, \leq)$  be permissible weak stability conditions on  $\mathcal{A}$  with  $(\hat{\tau}, \hat{T}, \leq)$  dominating  $(\tau, T, \leq)$  and  $(\tilde{\tau}, \tilde{T}, \leq)$ . Then for all  $\alpha \in C(\mathcal{A})$  we have*

$$\bar{\delta}_{ss}^\alpha(\tilde{\tau}) = \sum_{\substack{n \geq 1, \ \alpha_1, \dots, \alpha_n \in C(\mathcal{A}): \\ \alpha_1 + \dots + \alpha_n = \alpha}} S(\alpha_1, \dots, \alpha_n; \tau, \tilde{\tau}) \cdot \bar{\delta}_{ss}^{\alpha_1}(\tau) * \bar{\delta}_{ss}^{\alpha_2}(\tau) * \dots * \bar{\delta}_{ss}^{\alpha_n}(\tau), \quad (3.9)$$

$$\bar{\epsilon}^\alpha(\tilde{\tau}) = \sum_{\substack{n \geq 1, \ \alpha_1, \dots, \alpha_n \in C(\mathcal{A}): \\ \alpha_1 + \dots + \alpha_n = \alpha}} U(\alpha_1, \dots, \alpha_n; \tau, \tilde{\tau}) \cdot \bar{\epsilon}^{\alpha_1}(\tau) * \bar{\epsilon}^{\alpha_2}(\tau) * \dots * \bar{\epsilon}^{\alpha_n}(\tau), \quad (3.10)$$

where there are only finitely many nonzero terms in (3.9)–(3.10).

Here the third stability condition  $(\hat{\tau}, \hat{T}, \leq)$  may be thought of as lying on a ‘wall’ separating  $(\tau, T, \leq)$  and  $(\tilde{\tau}, \tilde{T}, \leq)$  in the space of stability conditions. Here is how to prove (3.9)–(3.10). If  $E \in \mathcal{A}$  then by Proposition 3.6 there is a unique Harder–Narasimhan filtration  $0 = E_0 \subset \dots \subset E_n = E$  with  $S_k = E_k/E_{k-1}$   $\tau$ -semistable and  $\tau([S_1]) > \dots > \tau([S_n])$ . As  $\hat{\tau}$  dominates  $\tau$ , one can show  $E$  is  $\hat{\tau}$ -semistable if and only if  $\hat{\tau}([S_1]) = \dots = \hat{\tau}([S_n])$ . It easily follows that

$$\bar{\delta}_{ss}^\alpha(\hat{\tau}) = \sum_{\substack{n \geq 1, \ \alpha_1, \dots, \alpha_n \in C(\mathcal{A}): \alpha_1 + \dots + \alpha_n = \alpha, \\ \tau(\alpha_1) > \dots > \tau(\alpha_n), \ \hat{\tau}(\alpha_1) = \dots = \hat{\tau}(\alpha_n)}} \bar{\delta}_{ss}^{\alpha_1}(\tau) * \bar{\delta}_{ss}^{\alpha_2}(\tau) * \dots * \bar{\delta}_{ss}^{\alpha_n}(\tau). \quad (3.11)$$

By a similar argument to (3.4)–(3.7) but using the inverse functions  $x \mapsto x/(1-x)$  and  $x \mapsto x/(1+x)$  rather than  $\log, \exp$ , we find the inverse of (3.11) is

$$\bar{\delta}_{ss}^\alpha(\tau) = \sum_{\substack{n \geq 1, \ \alpha_1, \dots, \alpha_n \in C(\mathcal{A}): \alpha_1 + \dots + \alpha_n = \alpha, \\ \tau(\alpha_1) > \dots > \tau(\alpha_n), \ \hat{\tau}(\alpha_1) = \dots = \hat{\tau}(\alpha_n)}} (-1)^{n-1} \bar{\delta}_{ss}^{\alpha_1}(\hat{\tau}) * \bar{\delta}_{ss}^{\alpha_2}(\hat{\tau}) * \dots * \bar{\delta}_{ss}^{\alpha_n}(\hat{\tau}). \quad (3.12)$$

Substituting (3.11) into (3.12) with  $\tilde{\tau}$  in place of  $\tau$  gives (3.9), and (3.10) then follows from (3.4), (3.5) and (3.9). From this proof we can see that over each point

of  $\mathfrak{M}_{\mathcal{A}}$  there are only finitely many nonzero terms in (3.9)–(3.10), and also that every term in (3.9)–(3.10) is supported on the open substack  $\mathfrak{M}_{\text{ss}}^{\alpha}(\hat{\tau})$  in  $\mathfrak{M}_{\mathcal{A}}$ . Since  $\hat{\tau}$  is assumed to be permissible,  $\mathfrak{M}_{\text{ss}}^{\alpha}(\hat{\tau})$  is of finite type, and therefore there are only finitely many nonzero terms in (3.9)–(3.10). In [54, Th. 5.4] we prove:

**THEOREM 3.14.** *Equation (3.10) may be rewritten as an equation in  $\text{SF}_{\text{al}}^{\text{ind}}(\mathfrak{M}_{\mathcal{A}})$  using the Lie bracket  $[\cdot, \cdot]$  on  $\text{SF}_{\text{al}}^{\text{ind}}(\mathfrak{M}_{\mathcal{A}})$ , rather than as an equation in  $\text{SF}_{\text{al}}(\mathfrak{M}_{\mathcal{A}})$  using the Ringel–Hall product  $*$ . That is, we may rewrite (3.10) in the form*

$$\bar{\epsilon}^{\alpha}(\tilde{\tau}) = \sum_{\substack{n \geq 1, \alpha_1, \dots, \alpha_n \in C(\mathcal{A}): \\ \alpha_1 + \dots + \alpha_n = \alpha}} \tilde{U}(\alpha_1, \dots, \alpha_n; \tau, \tilde{\tau}) \cdot [[\dots [[\bar{\epsilon}^{\alpha_1}(\tau), \bar{\epsilon}^{\alpha_2}(\tau)], \bar{\epsilon}^{\alpha_3}(\tau)], \dots], \bar{\epsilon}^{\alpha_n}(\tau)], \quad (3.13)$$

for some system of combinatorial coefficients  $\tilde{U}(\alpha_1, \dots, \alpha_n; \tau, \tilde{\tau}) \in \mathbb{Q}$ , with only finitely many nonzero terms.

There is an irritating technical problem in [54, §5] in changing between stability conditions on  $\text{coh}(X)$  when  $\dim X \geq 3$ . Suppose  $(\tau, T, \leq), (\tilde{\tau}, \tilde{T}, \leq)$  are two (weak) stability conditions on  $\text{coh}(X)$  of Gieseker or  $\mu$ -stability type, as in Examples 3.8 and 3.9, defined using different ample line bundles  $\mathcal{O}_X(1), \tilde{\mathcal{O}}_X(1)$ . Then the first author was not able to show that the changes between  $(\tau, T, \leq)$  and  $(\tilde{\tau}, \tilde{T}, \leq)$  are *globally finite*. That is, we prove (3.9)–(3.10) hold in the local stack function spaces  $\text{LSF}(\mathfrak{M}_{\text{coh}(X)})$ , but we do not know there are only finitely many nonzero terms in (3.9)–(3.10), although the first author believes this is true. Instead, as in [54, §5.1], we can show that we can interpolate between any two stability conditions on  $X$  of Gieseker or  $\mu$ -stability type by a finite sequence of stability conditions, such that between successive stability conditions in the sequence the changes are globally finite, and Theorem 3.13 applies.

### 3.4. Calabi–Yau 3-folds and Lie algebra morphisms

We now specialize to the case when  $\mathcal{A} = \text{coh}(X)$  for  $X$  a Calabi–Yau 3-fold, and explain some results of [52, §6.6] and [54, §6.5]. We restrict to  $\mathbb{K}$  of characteristic zero so that Euler characteristics over  $\mathbb{K}$  are well-behaved.

**DEFINITION 3.15.** Let  $\mathbb{K}$  be an algebraically closed field of characteristic zero. A *Calabi–Yau 3-fold* is a smooth projective 3-fold  $X$  over  $\mathbb{K}$ , with trivial canonical bundle  $K_X$ . From §5 onwards we will also assume that  $H^1(\mathcal{O}_X) = 0$ , but this is not needed for the results of [51–54]. Take  $\mathcal{A}$  to be  $\text{coh}(X)$  and  $K(\text{coh}(X))$  to be  $K^{\text{num}}(\text{coh}(X))$ . As in Definition 3.1 we have the Euler form  $\bar{\chi} : K(\text{coh}(X)) \times K(\text{coh}(X)) \rightarrow \mathbb{Z}$  in (3.1). As  $X$  is a Calabi–Yau 3-fold, Serre duality gives  $\text{Ext}^i(F, E) \cong \text{Ext}^{3-i}(E, F)^*$ , so  $\dim \text{Ext}^i(F, E) = \dim \text{Ext}^{3-i}(E, F)$  for all  $E, F \in \text{coh}(X)$ . Therefore  $\bar{\chi}$  is also given by

$$\bar{\chi}([E], [F]) = (\dim \text{Hom}(E, F) - \dim \text{Ext}^1(E, F)) - (\dim \text{Hom}(F, E) - \dim \text{Ext}^1(F, E)). \quad (3.14)$$

Thus the Euler form  $\bar{\chi}$  on  $K(\text{coh}(X))$  is *antisymmetric*.

In [52, §6.5] we define an explicit Lie algebra  $L(X)$  as follows:  $L(X)$  is the  $\mathbb{Q}$ -vector space with basis of symbols  $\lambda^{\alpha}$  for  $\alpha \in K(\text{coh}(X))$ , with Lie bracket

$$[\lambda^{\alpha}, \lambda^{\beta}] = \bar{\chi}(\alpha, \beta) \lambda^{\alpha+\beta}, \quad (3.15)$$



for  $\alpha, \beta \in K(\mathrm{coh}(X))$ . As  $\bar{\chi}$  is antisymmetric, (3.15) satisfies the Jacobi identity and makes  $L(X)$  into an infinite-dimensional Lie algebra over  $\mathbb{Q}$ . (We have changed notation: in [52],  $L(X), \lambda^\alpha$  are written  $C^{\mathrm{ind}}(\mathrm{coh}(X), \mathbb{Q}, \frac{1}{2}\bar{\chi}), c^\alpha$ .)

Define a  $\mathbb{Q}$ -linear map  $\Psi^{\chi, \mathbb{Q}} : \mathrm{SF}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{M}_{\mathrm{coh}(X)}, \chi, \mathbb{Q}) \rightarrow L(X)$  by

$$\Psi^{\chi, \mathbb{Q}}(f) = \sum_{\alpha \in K(\mathrm{coh}(X))} \gamma^\alpha \lambda^\alpha, \quad (3.16)$$

where  $\gamma^\alpha \in \mathbb{Q}$  is defined as follows. Proposition 3.4 says  $\mathrm{SF}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{M}_{\mathrm{coh}(X)}, \chi, \mathbb{Q})$  is spanned by elements  $[(U \times [\mathrm{Spec} \mathbb{K}/\mathbb{G}_m], \rho)]$ . We may write

$$f|_{\mathfrak{M}_{\mathrm{coh}(X)}} = \sum_{i=1}^n \delta_i [(U_i \times [\mathrm{Spec} \mathbb{K}/\mathbb{G}_m], \rho_i)], \quad (3.17)$$

where  $\delta_i \in \mathbb{Q}$  and  $U_i$  is a quasiprojective  $\mathbb{K}$ -variety. We set

$$\gamma^\alpha = \sum_{i=1}^n \delta_i \chi(U_i). \quad (3.18)$$

This is independent of the choices in (3.17). Now define  $\Psi : \mathrm{SF}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{M}_{\mathrm{coh}(X)}) \rightarrow L(X)$  by  $\Psi = \Psi^{\chi, \mathbb{Q}} \circ \bar{\Pi}_{\mathfrak{M}_{\mathrm{coh}(X)}}^{\chi, \mathbb{Q}}$ .

In [52, Th. 6.12], using equation (3.14), we prove:

**THEOREM 3.16.**  $\Psi : \mathrm{SF}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{M}_{\mathrm{coh}(X)}) \rightarrow L(X)$  and  $\Psi^{\chi, \mathbb{Q}} : \mathrm{SF}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{M}_{\mathrm{coh}(X)}, \chi, \mathbb{Q}) \rightarrow L(X)$  are Lie algebra morphisms.

Our next example may help readers to understand why this is true.

**EXAMPLE 3.17.** Suppose  $E, F$  are simple sheaves on  $X$ , with  $[E] = \alpha$  and  $[F] = \beta$  in  $K(\mathrm{coh}(X))$ . Consider the stack functions  $\bar{\delta}_E, \bar{\delta}_F$  in  $\mathrm{SF}_{\mathrm{al}}(\mathfrak{M}_{\mathrm{coh}(X)}, \chi, \mathbb{Q})$ , the characteristic functions of the points  $E, F$  in  $\mathfrak{M}_{\mathrm{coh}(X)}$ . Since  $\mathrm{Aut}(E) = \mathrm{Aut}(F) = \mathbb{G}_m$  as  $E, F$  are simple, we may write

$$\bar{\delta}_E = [([\mathrm{Spec} \mathbb{K}/\mathbb{G}_m], e)] \quad \text{and} \quad \bar{\delta}_F = [([\mathrm{Spec} \mathbb{K}/\mathbb{G}_m], f)], \quad (3.19)$$

where the 1-morphisms  $e, f : [\mathrm{Spec} \mathbb{K}/\mathbb{G}_m] \rightarrow \mathfrak{M}_{\mathrm{coh}(X)}$  correspond to  $E, F$ . Thus  $\bar{\delta}_E, \bar{\delta}_F$  have virtual rank 1, and lie in  $\mathrm{SF}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{M}_{\mathrm{coh}(X)}, \chi, \mathbb{Q})$ . We will prove explicitly that  $\Psi^{\chi, \mathbb{Q}}([\bar{\delta}_E, \bar{\delta}_F]) = [\Psi^{\chi, \mathbb{Q}}(\bar{\delta}_E), \Psi^{\chi, \mathbb{Q}}(\bar{\delta}_F)]$ , as we need for  $\Psi^{\chi, \mathbb{Q}}$  to be a Lie algebra morphism. From (3.19) and  $[E] = \alpha, [F] = \beta$  we have  $\Psi^{\chi, \mathbb{Q}}(\bar{\delta}_E) = \lambda^\alpha$  and  $\Psi^{\chi, \mathbb{Q}}(\bar{\delta}_F) = \lambda^\beta$ , so that  $[\Psi^{\chi, \mathbb{Q}}(\bar{\delta}_E), \Psi^{\chi, \mathbb{Q}}(\bar{\delta}_F)] = \bar{\chi}(\alpha, \beta) \lambda^{\alpha+\beta}$  by (3.15).

Now in  $\mathrm{SF}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{M}_{\mathrm{coh}(X)}, \chi, \mathbb{Q})$  we have

$$\begin{aligned} \bar{\delta}_E * \bar{\delta}_F &= [([\mathrm{Spec} \mathbb{K}/\mathbb{G}_m], e)] * [([\mathrm{Spec} \mathbb{K}/\mathbb{G}_m], f)] \\ &= [([\mathrm{Spec} \mathbb{K}/\mathbb{G}_m^2] \times_{e \times f, \mathfrak{M}_{\mathrm{coh}(X)} \times \mathfrak{M}_{\mathrm{coh}(X)}, \pi_1 \times \pi_3} \mathfrak{E}^{\mathrm{r} \mathrm{act}}_{\mathrm{coh}(X)}, \pi_2 \circ \pi_{\mathfrak{E}^{\mathrm{r} \mathrm{act}}_{\mathrm{coh}(X)}})] \\ &= [([\mathrm{Ext}^1(F, E)/(\mathbb{G}_m^2 \rtimes \mathrm{Hom}(F, E))], \rho_1)] \\ &= [([\mathrm{Spec} \mathbb{K}/(\mathbb{G}_m^2 \rtimes \mathrm{Hom}(F, E))], \rho_2)] \\ &\quad + [(\mathbb{P}(\mathrm{Ext}^1(F, E)) \times [\mathrm{Spec} \mathbb{K}/(\mathbb{G}_m \times \mathrm{Hom}(F, E))], \rho_3)] \\ &= [([\mathrm{Spec} \mathbb{K}/\mathbb{G}_m^2], \rho_4)] - \dim \mathrm{Hom}(F, E) [([\mathrm{Spec} \mathbb{K}/\mathbb{G}_m], \rho_5)] \\ &\quad + [(\mathbb{P}(\mathrm{Ext}^1(F, E)) \times [\mathrm{Spec} \mathbb{K}/\mathbb{G}_m], \rho_6)], \end{aligned} \quad (3.20)$$

where  $\rho_i$  are 1-morphisms to  $\mathfrak{M}_{\mathrm{coh}(X)}^{\alpha+\beta}$ , and the group law on  $\mathbb{G}_m^2 \rtimes \mathrm{Hom}(F, E)$  is  $(\lambda, \mu, \phi) \cdot (\lambda', \mu', \phi') = (\lambda\lambda', \mu\mu', \lambda\phi' + \mu'\phi)$  for  $\lambda, \lambda', \mu, \mu'$  in  $\mathbb{G}_m$  and  $\phi, \phi'$  in  $\mathrm{Hom}(F, E)$ , and  $\mathbb{G}_m^2 \rtimes \mathrm{Hom}(F, E)$  acts on  $\mathrm{Ext}^1(F, E)$  by  $(\lambda, \mu, \phi) : \epsilon \mapsto \lambda\mu^{-1}\epsilon$ .

Here in the first step of (3.20) we use (3.19), in the second (3.3), and in the third that the fibre of  $\pi_1 \times \pi_3$  over  $E, F$  is  $[\mathrm{Ext}^1(F, E)/\mathrm{Hom}(F, E)]$ . In the

fourth step of (3.20) we use relation Definition 2.16(i) in  $\bar{\mathbf{S}}_{\text{al}}^{\text{ind}}(\mathfrak{M}_{\text{coh}(X)}, \chi, \mathbb{Q})$  to cut  $[\text{Ext}^1(F, E)/\mathbb{G}_m^2 \ltimes \text{Hom}(F, E)]$  into two pieces  $[\{0\}/(\mathbb{G}_m^2 \ltimes \text{Hom}(F, E))]$  and  $[(\text{Ext}^1(F, E) \setminus \{0\})/(\mathbb{G}_m^2 \ltimes \text{Hom}(F, E))]$ , where for the second  $\mathbb{G}_m^2 \ltimes \text{Hom}(F, E)$  acts by dilation on  $\text{Ext}^1(F, E) \setminus \{0\}$ , turning it into  $\mathbb{P}(\text{Ext}^1(F, E))$ , and the stabilizer of each point is  $\mathbb{G}_m \ltimes \text{Hom}(F, E)$ . In the fifth step of (3.20) we use relation Definition 2.16(iii) to rewrite in terms of quotients by tori  $\mathbb{G}_m, \mathbb{G}_m^2$ ; the term in  $\dim \text{Hom}(F, E)$  is there as the coefficient  $F(\mathbb{G}_m^2 \ltimes \text{Hom}(F, E), \mathbb{G}_m^2, \mathbb{G}_m)$  in (2.10) is  $-\dim \text{Hom}(F, E)$  (see equation (11.13) in §11 for this computation).

In the same way we show that

$$\begin{aligned} \bar{\delta}_F * \bar{\delta}_E = & [([\text{Spec } \mathbb{K}/\mathbb{G}_m^2], \rho_4)] - \dim \text{Hom}(E, F) [([\text{Spec } \mathbb{K}/\mathbb{G}_m], \rho_5)] \\ & + [(\mathbb{P}(\text{Ext}^1(E, F)) \times [\text{Spec } \mathbb{K}/\mathbb{G}_m], \rho_7)], \end{aligned} \quad (3.21)$$

where the terms  $[[[\text{Spec } \mathbb{K}/\mathbb{G}_m^2], \rho_4]]$  and  $[[[\text{Spec } \mathbb{K}/\mathbb{G}_m], \rho_5]]$  in (3.20)–(3.21) are the same, mapping to  $E \oplus F$ . So subtracting (3.21) from (3.20) yields

$$\begin{aligned} [\bar{\delta}_E, \bar{\delta}_F] = & (\dim \text{Hom}(E, F) - \dim \text{Hom}(F, E)) [([\text{Spec } \mathbb{K}/\mathbb{G}_m], \rho_5)] \\ & + [(\mathbb{P}(\text{Ext}^1(F, E)) \times [\text{Spec } \mathbb{K}/\mathbb{G}_m], \rho_6)] - [(\mathbb{P}(\text{Ext}^1(E, F)) \times [\text{Spec } \mathbb{K}/\mathbb{G}_m], \rho_7)]. \end{aligned}$$

Applying  $\Psi^{\chi, \mathbb{Q}}$  thus yields

$$\begin{aligned} \Psi^{\chi, \mathbb{Q}}([\bar{\delta}_E, \bar{\delta}_F]) &= (\dim \text{Hom}(E, F) - \dim \text{Hom}(F, E) + \dim \text{Ext}^1(F, E) - \dim \text{Ext}^1(E, F)) \lambda^{\alpha+\beta} \\ &= \bar{\chi}([E], [F]) \lambda^{\alpha+\beta} = \bar{\chi}(\alpha, \beta) \lambda^{\alpha+\beta} = [\Psi^{\chi, \mathbb{Q}}(\bar{\delta}_E), \Psi^{\chi, \mathbb{Q}}(\bar{\delta}_F)], \end{aligned}$$

by equation (3.14) and  $\chi(\mathbb{P}(\text{Ext}^1(E, F))) = \dim \text{Ext}^1(E, F)$ .

### 3.5. Invariants $J^\alpha(\tau)$ and transformation laws

We continue in the situation of §3.4, with  $\mathbb{K}$  of characteristic zero and  $X$  a Calabi–Yau 3-fold over  $\mathbb{K}$ . Let  $(\tau, T, \leq)$  be a permissible weak stability condition on  $\text{coh}(X)$ , for instance, Gieseker stability or  $\mu$ -stability w.r.t. some ample line bundle  $\mathcal{O}_X(1)$  on  $X$ , as in Example 3.8 or 3.9. In [54, §6.6] we define invariants  $J^\alpha(\tau) \in \mathbb{Q}$  for all  $\alpha \in C(\text{coh}(X))$  by

$$\Psi(\bar{\epsilon}^\alpha(\tau)) = J^\alpha(\tau) \lambda^\alpha. \quad (3.22)$$

This is valid by Theorem 3.11. These  $J^\alpha(\tau)$  are rational numbers ‘counting’  $\tau$ -semistable sheaves  $E$  in class  $\alpha$ . When  $\mathcal{M}_{\text{ss}}^\alpha(\tau) = \mathcal{M}_{\text{st}}^\alpha(\tau)$  we have  $J^\alpha(\tau) = \chi(\mathcal{M}_{\text{st}}^\alpha(\tau))$ , that is,  $J^\alpha(\tau)$  is the Euler characteristic of the moduli space  $\mathcal{M}_{\text{st}}^\alpha(\tau)$ . As we explain in §4, this is *not* weighted by the Behrend function  $\nu_{\mathcal{M}_{\text{st}}^\alpha(\tau)}$ , and is not the Donaldson–Thomas invariant  $DT^\alpha(\tau)$ . Also, the  $J^\alpha(\tau)$  are in general *not* unchanged under deformations of  $X$ , as we show in Example 6.9 below.

Now suppose  $(\tau, T, \leq), (\tilde{\tau}, \tilde{T}, \leq), (\hat{\tau}, \hat{T}, \leq)$  are as in Theorem 3.13, so that equation (3.10) holds, and is equivalent to a Lie algebra equation (3.13) as in Theorem 3.14. Therefore we may apply the Lie algebra morphism  $\Psi$  to equation (3.13). In fact we prefer to work with equation (3.10), since the coefficients  $\tilde{U}(\alpha_1, \dots, \alpha_n; \tau, \tilde{\tau})$  in (3.13) are difficult to write down. So we express it as an equation in the *universal enveloping algebra*  $U(L(X))$ . This gives

$$J^\alpha(\tilde{\tau}) \lambda^\alpha = \sum_{\substack{n \geq 1, \alpha_1, \dots, \alpha_n \in C(\text{coh}(X)): \\ \alpha_1 + \dots + \alpha_n = \alpha}} U(\alpha_1, \dots, \alpha_n; \tau, \tilde{\tau}) \cdot \prod_{i=1}^n J^{\alpha_i}(\tau) \cdot \lambda^{\alpha_1} \star \lambda^{\alpha_2} \star \dots \star \lambda^{\alpha_n}, \quad (3.23)$$

where  $\star$  is the product in  $U(L(X))$ .

Now in [52, §6.5], an explicit description is given of the universal enveloping algebra  $U(L(X))$  (the notation used for  $U(L(X))$  in [52] is  $C(\text{coh}(X), \mathbb{Q}, \frac{1}{2}\bar{\chi})$ ). There is an explicit basis given for  $U(L(X))$  in terms of symbols  $\lambda_{[I, \kappa]}$ , and multiplication  $\star$  in  $U(L(X))$  is given in terms of the  $\lambda_{[I, \kappa]}$  as a sum over graphs. Here  $I$  is a finite set,  $\kappa$  maps  $I \rightarrow C(\text{coh}(X))$ , and when  $|I| = 1$ , so that  $I = \{i\}$ , we have  $\lambda_{[I, \kappa]} = \lambda^{\kappa(i)}$ . Then [54, eq. (127)] gives an expression for  $\lambda^{\alpha_1} \star \dots \star \lambda^{\alpha_n}$  in  $U(L(X))$ , in terms of sums over *directed graphs* (*digraphs*):

$$\lambda^{\alpha_1} \star \dots \star \lambda^{\alpha_n} = \text{terms in } \lambda_{[I, \kappa]}, |I| > 1, \quad (3.24)$$

$$+ \left[ \frac{1}{2^{n-1}} \sum_{\substack{\text{connected, simply-connected digraphs } \Gamma: \\ \text{vertices } \{1, \dots, n\}, \text{ edge } \overset{i}{\bullet} \rightarrow \overset{j}{\bullet} \text{ implies } i < j}} \prod_{\substack{\text{edges} \\ \overset{i}{\bullet} \rightarrow \overset{j}{\bullet} \\ \text{in } \Gamma}} \bar{\chi}(\alpha_i, \alpha_j) \right] \lambda^{\alpha_1 + \dots + \alpha_n}.$$

Substitute (3.24) into (3.23). The terms in  $\lambda_{[I, \kappa]}$  for  $|I| > 1$  all cancel, as (3.23) lies in  $L(X) \subset U(L(X))$ . So equating coefficients of  $\lambda^\alpha$  yields

$$J^\alpha(\tilde{\tau}) = \sum_{\substack{n \geq 1, \alpha_1, \dots, \alpha_n \in C(\text{coh}(X)): \\ \alpha_1 + \dots + \alpha_n = \alpha}} \sum_{\substack{\text{connected, simply-connected digraphs } \Gamma: \\ \text{vertices } \{1, \dots, n\}, \text{ edge } \overset{i}{\bullet} \rightarrow \overset{j}{\bullet} \text{ implies } i < j}} \quad (3.25)$$

$$\frac{1}{2^{n-1}} U(\alpha_1, \dots, \alpha_n; \tau, \tilde{\tau}) \prod_{\substack{\text{edges } \overset{i}{\bullet} \rightarrow \overset{j}{\bullet} \\ \text{in } \Gamma}} \bar{\chi}(\alpha_i, \alpha_j) \prod_{i=1}^n J^{\alpha_i}(\tau).$$

Following [54, Def. 6.27], we define combinatorial coefficients  $V(I, \Gamma, \kappa; \tau, \tilde{\tau})$ :

DEFINITION 3.18. In the situation above, suppose  $\Gamma$  is a connected, simply-connected digraph with finite vertex set  $I$ , where  $|I| = n$ , and  $\kappa : I \rightarrow C(\text{coh}(X))$  is a map. Define  $V(I, \Gamma, \kappa; \tau, \tilde{\tau}) \in \mathbb{Q}$  by

$$V(I, \Gamma, \kappa; \tau, \tilde{\tau}) = \frac{1}{2^{n-1} n!} \sum_{\substack{\text{orderings } i_1, \dots, i_n \text{ of } I: \\ \text{edge } \overset{i_a}{\bullet} \rightarrow \overset{i_b}{\bullet} \text{ in } \Gamma \text{ implies } a < b}} U(\kappa(i_1), \kappa(i_2), \dots, \kappa(i_n); \tau, \tilde{\tau}). \quad (3.26)$$

Then as in [54, Th. 6.28], using (3.26) to rewrite (3.25) yields a transformation law for the  $J^\alpha(\tau)$  under change of stability condition:

$$J^\alpha(\tilde{\tau}) = \sum_{\substack{\text{iso.} \\ \text{classes} \\ \text{of finite} \\ \text{sets } I}} \sum_{\substack{\kappa: I \rightarrow C(\text{coh}(X)): \\ \sum_{i \in I} \kappa(i) = \alpha}} \sum_{\substack{\text{connected,} \\ \text{simply-connected} \\ \text{digraphs } \Gamma, \\ \text{vertices } I}} V(I, \Gamma, \kappa; \tau, \tilde{\tau}) \cdot \prod_{\substack{\text{edges } \overset{i}{\bullet} \rightarrow \overset{j}{\bullet} \\ \text{in } \Gamma}} \bar{\chi}(\kappa(i), \kappa(j)) \cdot \prod_{i \in I} J^{\kappa(i)}(\tau). \quad (3.27)$$

As in [54, Rem. 6.29],  $V(I, \Gamma, \kappa; \tau, \tilde{\tau})$  depends on the orientation of  $\Gamma$  only up to sign: changing the directions of  $k$  edges multiplies  $V(I, \Gamma, \kappa; \tau, \tilde{\tau})$  by  $(-1)^k$ . Since  $\bar{\chi}$  is antisymmetric, it follows that  $V(I, \Gamma, \kappa; \tau, \tilde{\tau}) \cdot \prod_{\overset{i}{\bullet} \rightarrow \overset{j}{\bullet}} \bar{\chi}(\kappa(i), \kappa(j))$  in (3.27) is independent of the orientation of  $\Gamma$ .

## CHAPTER 4

### Behrend functions and Donaldson–Thomas theory

We now discuss *Behrend functions* of schemes and stacks, and their application to Donaldson–Thomas invariants. Our primary source is Behrend’s paper [3]. But Behrend considers only  $\mathbb{C}$ -schemes and Deligne–Mumford  $\mathbb{C}$ -stacks, whereas we treat Artin stacks, and discuss which parts of the theory work over other algebraically closed fields  $\mathbb{K}$ . Some of our results, such as Theorem 4.11 below, appear to be new. Also, in §4.5 we give an exact cohomological description of the numerical Grothendieck group  $K^{\text{num}}(\text{coh}(X))$  of a Calabi–Yau 3-fold  $X$ .

We have not tried to be brief; instead, we have tried to make §4.1–§4.4 a helpful reference on Behrend functions, by collecting ideas and material which may be useful in the future. Section 4.4, and most of §4.2, will not be used in this book. We include in §4.2 a discussion of *perverse sheaves* and *vanishing cycles*, since they seem to be connected to Behrend functions at a deep level, but we expect many of our readers may not be familiar with them.

#### 4.1. The definition of Behrend functions

**DEFINITION 4.1.** Let  $\mathbb{K}$  be an algebraically closed field of characteristic zero, and  $X$  a finite type  $\mathbb{K}$ -scheme. Write  $Z_*(X)$  for the group of *algebraic cycles* on  $X$ , as in Fulton [28]. Suppose  $X \hookrightarrow M$  is an embedding of  $X$  as a closed subscheme of a smooth  $\mathbb{K}$ -scheme  $M$ . Let  $C_X M$  be the *normal cone* of  $X$  in  $M$ , as in [28, p. 73], and  $\pi : C_X M \rightarrow X$  the projection. As in [3, §1.1], define a cycle  $\mathbf{c}_{X/M} \in Z_*(X)$  by

$$\mathbf{c}_{X/M} = \sum_{C'} (-1)^{\dim \pi(C')} \text{mult}(C') \pi(C'),$$

where the sum is over all irreducible components  $C'$  of  $C_X M$ .

It turns out that  $\mathbf{c}_{X/M}$  depends only on  $X$ , and not on the embedding  $X \hookrightarrow M$ . Behrend [3, Prop. 1.1] proves that given a finite type  $\mathbb{K}$ -scheme  $X$ , there exists a unique cycle  $\mathbf{c}_X \in Z_*(X)$ , such that for any étale map  $\varphi : U \rightarrow X$  for a  $\mathbb{K}$ -scheme  $U$  and any closed embedding  $U \hookrightarrow M$  into a smooth  $\mathbb{K}$ -scheme  $M$ , we have  $\varphi^*(\mathbf{c}_X) = \mathbf{c}_{U/M}$  in  $Z_*(U)$ . If  $X$  is a subscheme of a smooth  $M$  we take  $U = X$  and get  $\mathbf{c}_X = \mathbf{c}_{X/M}$ . Behrend calls  $\mathbf{c}_X$  the *signed support of the intrinsic normal cone*, or the *distinguished cycle* of  $X$ .

Write  $\text{CF}_{\mathbb{Z}}(X)$  for the group of  $\mathbb{Z}$ -valued constructible functions on  $X$ . The *local Euler obstruction* is a group isomorphism  $\text{Eu} : Z_*(X) \rightarrow \text{CF}_{\mathbb{Z}}(X)$ . It was first defined by MacPherson [75] when  $\mathbb{K} = \mathbb{C}$ , using complex analysis, but Kennedy [60] provides an alternative algebraic definition which works over any algebraically closed field  $\mathbb{K}$  of characteristic zero. If  $V$  is a prime cycle on  $X$ , the constructible function  $\text{Eu}(V)$  is given by

$$\text{Eu}(V) : x \longmapsto \int_{\mu^{-1}(x)} c(\tilde{T}) \cap s(\mu^{-1}(x), \tilde{V}),$$

where  $\mu : \tilde{V} \rightarrow V$  is the Nash blowup of  $V$ ,  $\tilde{T}$  the dual of the universal quotient bundle,  $c$  the total Chern class and  $s$  the Segre class of the normal cone to a closed immersion. Kennedy [60, Lem. 4] proves that  $\text{Eu}(V)$  is constructible. For each finite type  $\mathbb{K}$ -scheme  $X$ , define the *Behrend function*  $\nu_X$  in  $\text{CF}(X)$  by  $\nu_X = \text{Eu}(\mathbf{c}_X)$ , as in Behrend [3, §1.2].

In the case  $\mathbb{K} = \mathbb{C}$ , using MacPherson’s complex analytic definition of the local Euler obstruction [75], the definition of  $\nu_X$  makes sense in the framework of complex analytic geometry, and so Behrend functions can be defined for *complex analytic spaces*  $X_{\text{an}}$ . Informally, we have a commutative diagram:

$$\begin{array}{ccccc}
 \text{\mathbb{C}-subvariety } X \text{ in} & \longrightarrow & \text{Algebraic cycle} & \xrightarrow{\text{Eu}} & \text{Algebraic Behrend function} \\
 \text{smooth } \mathbb{C}\text{-variety } M & & \mathbf{c}_{X/M} \text{ in } Z_*(X) & & \nu_X = \text{Eu}(\mathbf{c}_{X/M}) \text{ in } \text{CF}_{\mathbb{Z}}(X) \\
 \downarrow & & \downarrow & & \downarrow \\
 \text{complex analytic space } \tilde{X} & \longrightarrow & \text{Analytic cycle} & \xrightarrow{\text{Eu}} & \text{Analytic Behrend function} \\
 \text{in complex manifold } \tilde{M} & & \mathbf{c}_{\tilde{X}/\tilde{M}} \text{ in } Z_*^{\text{an}}(\tilde{X}) & & \nu_{\tilde{X}} = \text{Eu}(\mathbf{c}_{\tilde{X}/\tilde{M}}) \text{ in } \text{CF}_{\mathbb{Z}}^{\text{an}}(\tilde{X}),
 \end{array}$$

where the columns pass from  $\mathbb{C}$ -algebraic varieties/cycles/constructible functions to the underlying complex analytic spaces/cycles/constructible functions. Thus we deduce:

**PROPOSITION 4.2.** (a) *If  $\mathbb{K}$  is an algebraically closed field of characteristic zero, and  $X$  is a finite type  $\mathbb{K}$ -scheme, then the Behrend function  $\nu_X$  is a well-defined  $\mathbb{Z}$ -valued constructible function on  $X$ , in the Zariski topology.*  
(b) *If  $Y$  is a complex analytic space then the Behrend function  $\nu_Y$  is a well-defined  $\mathbb{Z}$ -valued locally constructible function on  $Y$ , in the analytic topology.*  
(c) *If  $X$  is a finite type  $\mathbb{C}$ -scheme, with underlying complex analytic space  $X_{\text{an}}$ , then the algebraic Behrend function  $\nu_X$  in (a) and the analytic Behrend function  $\nu_{X_{\text{an}}}$  in (b) coincide. In particular,  $\nu_X$  depends only on the complex analytic space  $X_{\text{an}}$  underlying  $X$ , locally in the analytic topology.*

Here are some important properties of Behrend functions. They are proved by Behrend [3, §1.2 & Prop. 1.5] when  $\mathbb{K} = \mathbb{C}$ , but his proof is valid for general  $\mathbb{K}$ .

**THEOREM 4.3.** *Let  $\mathbb{K}$  be an algebraically closed field of characteristic zero, and  $X, Y$  be finite type  $\mathbb{K}$ -schemes. Then:*

- (i) *If  $X$  is smooth of dimension  $n$  then  $\nu_X \equiv (-1)^n$ .*
- (ii) *If  $\varphi : X \rightarrow Y$  is smooth with relative dimension  $n$  then  $\nu_X \equiv (-1)^n \varphi^*(\nu_Y)$ .*
- (iii)  *$\nu_{X \times Y} \equiv \nu_X \square \nu_Y$ , where  $(\nu_X \square \nu_Y)(x, y) = \nu_X(x) \nu_Y(y)$ .*

We can extend the definition of Behrend functions to  $\mathbb{K}$ -schemes, algebraic  $\mathbb{K}$ -spaces, and Artin  $\mathbb{K}$ -stacks, locally of finite type.

**PROPOSITION 4.4.** *Let  $\mathbb{K}$  be an algebraically closed field of characteristic zero, and  $X$  be a  $\mathbb{K}$ -scheme, algebraic  $\mathbb{K}$ -space, or Artin  $\mathbb{K}$ -stack, locally of finite type. Then there is a well-defined **Behrend function**  $\nu_X$ , a  $\mathbb{Z}$ -valued locally constructible function on  $X$ , which is characterized uniquely by the property that if  $W$  is a finite type  $\mathbb{K}$ -scheme and  $\varphi : W \rightarrow X$  is a 1-morphism of Artin stacks that is smooth of relative dimension  $n$  then  $\varphi^*(\nu_X) = (-1)^n \nu_W$  in  $\text{CF}(W)$ .*

**PROOF.** As Artin  $\mathbb{K}$ -stacks include  $\mathbb{K}$ -schemes and algebraic  $\mathbb{K}$ -spaces, it is enough to do the Artin stack case. Suppose  $X$  is an Artin  $\mathbb{K}$ -stack, locally of finite type. Let  $x \in X(\mathbb{K})$ . Then by the existence of atlases for  $X$ , and as  $X$  is locally of

finite type, there exists a finite type  $\mathbb{K}$ -scheme  $W$  and a 1-morphism  $\varphi : W \rightarrow X$  smooth of relative dimension  $n$ , with  $x = \varphi_*(w)$  for some  $w \in W(\mathbb{K})$ . We wish to define  $\nu_X(x) = (-1)^n \nu_W(w)$ .

To show this is well-defined, suppose  $W', \varphi', n', w'$  are alternative choices for  $W, \varphi, n, w$ . Consider the fibre product  $Y = W \times_{\varphi, X, \varphi'} W'$ . This is a finite type  $\mathbb{K}$ -scheme, as  $W, W'$  are. Let  $\pi_1 : Y \rightarrow W$  and  $\pi_2 : Y \rightarrow W'$  be the projections to the factors of the fibre product. Then  $\pi_1, \pi_2$  are morphisms of  $\mathbb{K}$ -schemes, and  $\pi_1$  is smooth of relative dimension  $n'$  as  $\varphi'$  is, and  $\pi_2$  is smooth of relative dimension  $n$  as  $\varphi$  is. Hence Theorem 4.3(ii) gives

$$(-1)^{n'} \pi_1^*(\nu_W) \equiv \nu_Y \equiv (-1)^n \pi_2^*(\nu_{W'}). \quad (4.1)$$

Since  $\varphi_*(w) = x = \varphi'_*(w')$ , the fibre of  $\pi_1 \times \pi_2 : Y \rightarrow W \times W'$  over  $(w, w')$  is isomorphic as a  $\mathbb{K}$ -scheme to the stabilizer group  $\text{Iso}_X(x)$ , and so is nonempty. Thus there exists  $y \in Y(\mathbb{K})$  with  $(\pi_1)_*(y) = w$  and  $(\pi_2)_*(y) = w'$ . Equation (4.1) thus gives  $(-1)^{n'} \nu_W(w) = \nu_Y(y) = (-1)^n \nu_{W'}(w')$ , so that  $(-1)^n \nu_W(w) = (-1)^{n'} \nu_{W'}(w')$ . Hence  $\nu_X(x)$  is well-defined.

Therefore there exists a unique function  $\nu_X : X(\mathbb{K}) \rightarrow \mathbb{Z}$  with the property in the proposition. It remains only to show that  $\nu_X$  is locally constructible. For  $\varphi, W, n$  as above,  $\varphi^*(\nu_X) = (-1)^n \nu_W$  and  $\nu_W$  constructible imply that  $\nu_X$  is constructible on the constructible set  $\varphi_*(W(\mathbb{K})) \subseteq X(\mathbb{K})$ . But any constructible subset  $S$  of  $X(\mathbb{K})$  can be covered by finitely many such subsets  $\varphi_*(W(\mathbb{K}))$ , so  $\nu_X|_S$  is constructible, and thus  $\nu_X$  is locally constructible.  $\square$

It is then easy to deduce:

**COROLLARY 4.5.** *Theorem 4.3 also holds for Artin  $\mathbb{K}$ -stacks  $X, Y$  locally of finite type.*

## 4.2. Milnor fibres and vanishing cycles

We define *Milnor fibres* for holomorphic functions on complex analytic spaces.

**DEFINITION 4.6.** Let  $U$  be a complex analytic space,  $f : U \rightarrow \mathbb{C}$  a holomorphic function, and  $x \in U$ . Let  $d(\cdot, \cdot)$  be a metric on  $U$  near  $x$  induced by a local embedding of  $U$  in some  $\mathbb{C}^N$ . For  $\delta, \epsilon > 0$ , consider the holomorphic map

$$\Phi_{f,x} : \{y \in U : d(x, y) < \delta, 0 < |f(y) - f(x)| < \epsilon\} \longrightarrow \{z \in \mathbb{C} : 0 < |z| < \epsilon\}$$

given by  $\Phi_{f,x}(y) = f(y) - f(x)$ . Milnor [78], extended by Lê [68], shows that  $\Phi_{f,x}$  is a locally trivial topological fibration provided  $0 < \epsilon \ll \delta \ll 1$ . The *Milnor fibre*  $MF_f(x)$  is the fibre of  $\Phi_{f,x}$ . It is independent of the choice of  $0 < \epsilon \ll \delta \ll 1$  up to homeomorphism, or up to diffeomorphism for smooth  $U$ .

The next theorem is due to Parusiński and Pragacz [87], as in [3, §1.2].

**THEOREM 4.7.** *Let  $U$  be a complex manifold and  $f : U \rightarrow \mathbb{C}$  a holomorphic function, and define  $X$  to be the complex analytic space  $\text{Crit}(f) \subseteq U$ . Then the Behrend function  $\nu_X$  of  $X$  is given by*

$$\nu_X(x) = (-1)^{\dim U} (1 - \chi(MF_f(x))) \quad \text{for } x \in X. \quad (4.2)$$

These ideas on Milnor fibres have a deep and powerful generalization in the theory of *perverse sheaves* and *vanishing cycles*. We now sketch a few of the basics of the theory. It works both in the algebraic and complex analytic contexts, but we will explain only the complex analytic setting. A survey paper on the subject is Massey [76], and three books are Kashiwara and Schapira [57], Dimca [18], and Schürmann [93]. Over the field  $\mathbb{C}$ , Saito’s theory of *mixed Hodge modules* [92] provides a generalization of the theory of perverse sheaves with more structure, which may also be a context in which to generalize Donaldson–Thomas theory, but we will not discuss this.

What follows will not be needed to understand the rest of the book — the only result in this discussion we will use later is Theorem 4.11, which makes sense using only the definitions of §4.1. We include this material both for completeness, as it underlies the theory of Behrend functions, and also to point out to readers in Donaldson–Thomas theory that future developments in the subject, particularly in the direction of motivic Donaldson–Thomas invariants and motivic Milnor fibres envisaged by Kontsevich and Soibelman [63], will probably be framed in terms of perverse sheaves and vanishing cycles.

**DEFINITION 4.8.** Let  $X$  be a complex analytic space. Consider sheaves of  $\mathbb{Q}$ -modules  $\mathcal{C}$  on  $X$ . Note that these are *not* coherent sheaves, which are sheaves of  $\mathcal{O}_X$ -modules. A sheaf  $\mathcal{C}$  is called *constructible* if there is a locally finite stratification  $X = \bigcup_{j \in J} X_j$  of  $X$  in the complex analytic topology, such that  $\mathcal{C}|_{X_j}$  is a  $\mathbb{Q}$ -local system for all  $j \in J$ , and all the stalks  $\mathcal{C}_x$  for  $x \in X$  are finite-dimensional  $\mathbb{Q}$ -vector spaces. A complex  $\mathcal{C}^\bullet$  of sheaves of  $\mathbb{Q}$ -modules on  $X$  is called *constructible* if all its cohomology sheaves  $H^i(\mathcal{C}^\bullet)$  for  $i \in \mathbb{Z}$  are constructible.

Write  $D_{\text{con}}^b(X)$  for the bounded derived category of constructible complexes on  $X$ . It is a triangulated category. By [18, Th. 4.1.5],  $D_{\text{con}}^b(X)$  is closed under Grothendieck’s “six operations on sheaves”  $R\varphi_*$ ,  $R\varphi^*$ ,  $\varphi^*$ ,  $\varphi^!$ ,  $\mathcal{R}Hom$ ,  $\otimes$ . The *perverse sheaves* on  $X$  are a particular abelian subcategory  $\text{Per}(X)$  in  $D_{\text{con}}^b(X)$ , which is the heart of a t-structure on  $D_{\text{con}}^b(X)$ . So perverse sheaves are actually complexes of sheaves, not sheaves, on  $X$ . The category  $\text{Per}(X)$  is noetherian and locally artinian, and is artinian if  $X$  is of finite type, so every perverse sheaf has (locally) a unique filtration whose quotients are simple perverse sheaves; and the simple perverse sheaves can be described completely in terms of irreducible local systems on irreducible subvarieties in  $X$ .

Next we explain *nearby cycles* and *vanishing cycles*. Let  $X$  be a complex analytic space, and  $f : X \rightarrow \mathbb{C}$  a holomorphic function. Define  $X_0 = f^{-1}(0)$ , as a complex analytic space, and  $X^* = X \setminus X_0$ . Consider the commutative diagram

$$\begin{array}{ccccccc} X_0 & \xrightarrow{i} & X & \xleftarrow{j} & X^* & \xleftarrow{p} & \widetilde{X}^* \\ \downarrow f & & \downarrow f & & \downarrow f & & \downarrow \tilde{f} \\ \{0\} & \longrightarrow & \mathbb{C} & \longleftarrow & \mathbb{C}^* & \xleftarrow{\rho} & \widetilde{\mathbb{C}^*} \end{array}$$

Here  $i : X_0 \rightarrow X$ ,  $j : X^* \rightarrow X$  are the inclusions,  $\rho : \widetilde{\mathbb{C}^*} \rightarrow \mathbb{C}^*$  is the universal cover of  $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ , and  $\widetilde{X}^* = X^* \times_{f, \mathbb{C}^*, \rho} \widetilde{\mathbb{C}^*}$  the corresponding cover of  $X^*$ , with covering map  $p : \widetilde{X}^* \rightarrow X^*$ , and  $\pi = j \circ p$ . The *nearby cycle functor*  $\psi_f : D_{\text{con}}^b(X) \rightarrow D_{\text{con}}^b(X_0)$  is  $\psi_f = i^* R\pi_* \pi^*$ .

There is a natural transformation  $\Xi : i^* \Rightarrow \psi_f$  between the functors  $i^*, \psi_f : D_{\text{Con}}^b(X) \rightarrow D_{\text{Con}}^b(X_0)$ . The *vanishing cycle functor*  $\phi_f : D_{\text{Con}}^b(X) \rightarrow D_{\text{Con}}^b(X_0)$  is a functor such that for every  $\mathcal{C}^\bullet$  in  $D_{\text{Con}}^b(X)$  we have a distinguished triangle

$$i^*(\mathcal{C}^\bullet) \xrightarrow{\Xi(\mathcal{C}^\bullet)} \psi_f(\mathcal{C}^\bullet) \longrightarrow \phi_f(\mathcal{C}^\bullet) \xrightarrow{[+1]} i^*(\mathcal{C}^\bullet) \quad (4.3)$$

in  $D_{\text{Con}}^b(X_0)$ . So roughly speaking  $\phi_f$  is the cone on  $\Xi$ , but this is not a good definition as cones are not unique up to canonical isomorphism. The shifted functors  $\psi_f[-1], \phi_f[-1]$  take perverse sheaves to perverse sheaves.

As  $i^*, \psi_f, \phi_f$  are exact, they induce morphisms on the Grothendieck groups

$$(i^*)_*, (\psi_f)_*, (\phi_f)_* : K_0(D_{\text{Con}}^b(X)) \longrightarrow K_0(D_{\text{Con}}^b(X_0)),$$

with  $(\psi_f)_* = (i^*)_* + (\phi_f)_*$  by (4.3). Note that  $K_0(D_{\text{Con}}^b(X)) = K_0(\text{Per}(X))$  and  $K_0(D_{\text{Con}}^b(X_0)) = K_0(\text{Per}(X_0))$ , and for  $X$  of finite type  $K_0(\text{Per}(X))$  is spanned by isomorphism classes of simple perverse sheaves, which have a nice description [18, Th. 5.2.12].

Write  $\text{CF}_{\mathbb{Z}}^{\text{an}}(X)$  for the group of  $\mathbb{Z}$ -valued analytically constructible functions on  $X$ . Define a map  $\chi_X : \text{Obj}(D_{\text{Con}}^b(X)) \rightarrow \text{CF}_{\mathbb{Z}}^{\text{an}}(X)$  by taking Euler characteristics of the cohomology of stalks of complexes, given by

$$\chi_X(\mathcal{C}^\bullet) : x \mapsto \sum_{k \in \mathbb{Z}} (-1)^k \dim \mathcal{H}^k(\mathcal{C}^\bullet)_x.$$

Since distinguished triangles in  $D_{\text{Con}}^b(X)$  give long exact sequences on cohomology of stalks  $\mathcal{H}^k(-)_x$ , this  $\chi_X$  is additive over distinguished triangles, and so descends to a group morphism  $\chi_X : K_0(D_{\text{Con}}^b(X)) \rightarrow \text{CF}_{\mathbb{Z}}^{\text{an}}(X)$ .

These maps  $\chi_X : \text{Obj}(D_{\text{Con}}^b(X)) \rightarrow \text{CF}_{\mathbb{Z}}^{\text{an}}(X)$  and  $\chi_X : K_0(D_{\text{Con}}^b(X)) \rightarrow \text{CF}_{\mathbb{Z}}^{\text{an}}(X)$  are surjective, since  $\text{CF}_{\mathbb{Z}}^{\text{an}}(X)$  is spanned by the characteristic functions of closed analytic cycles  $Y$  in  $X$ , and each such  $Y$  lifts to a perverse sheaf in  $D_{\text{Con}}^b(X)$ . In category-theoretic terms,  $X \mapsto D_{\text{Con}}^b(X)$  is a functor  $D_{\text{Con}}^b$  from complex analytic spaces to triangulated categories, and  $X \mapsto \text{CF}_{\mathbb{Z}}^{\text{an}}(X)$  is a functor  $\text{CF}_{\mathbb{Z}}^{\text{an}}$  from complex analytic spaces to abelian groups, and  $X \mapsto \chi_X$  is a natural transformation  $\chi$  from  $D_{\text{Con}}^b$  to  $\text{CF}_{\mathbb{Z}}^{\text{an}}$ .

As in Schürmann [93, §2.3], the operations  $R\varphi_*, R\varphi_!, \varphi^*, \varphi^!, \mathcal{R}Hom$ , and  $\overset{L}{\otimes}$  on  $D_{\text{Con}}^b(X)$  all have analogues on constructible functions, which commute with the maps  $\chi_X$ . So, for example, if  $\varphi : X \rightarrow Y$  is a morphism of complex analytic spaces, pullback of complexes  $\varphi^*$  corresponds to pullback of constructible functions in §2.1, that is, we have a commutative diagram

$$\begin{array}{ccc} D_{\text{Con}}^b(Y) & \xrightarrow{\varphi^*} & D_{\text{Con}}^b(X) \\ \downarrow \chi_Y & & \downarrow \chi_X \\ \text{CF}_{\mathbb{Z}}^{\text{an}}(Y) & \xrightarrow{\varphi^*} & \text{CF}_{\mathbb{Z}}^{\text{an}}(X). \end{array}$$

Similarly, if  $\varphi$  is *proper* then  $R\varphi_*$  on complexes corresponds to pushforward of constructible functions  $\text{CF}(\varphi)$  in §2.1, that is, we have a commutative diagram

$$\begin{array}{ccc} D_{\text{Con}}^b(X) & \xrightarrow{R\varphi_*} & D_{\text{Con}}^b(Y) \\ \downarrow \chi_X & & \downarrow \chi_Y \\ \text{CF}_{\mathbb{Z}}^{\text{an}}(X) & \xrightarrow{\text{CF}(\varphi)} & \text{CF}_{\mathbb{Z}}^{\text{an}}(Y). \end{array} \quad (4.4)$$

Also  $\overset{L}{\otimes}$  corresponds to multiplication of constructible functions.



The functors  $\psi_f, \phi_f$  above have analogues  $\Psi_f, \Phi_f$  on constructible functions defined by Verdier [102, Prop.s 3.4 & 4.1]. For  $X, f, X_0$  as above, there is a unique morphism  $\Psi_f : \text{CF}_{\mathbb{Z}}^{\text{an}}(X) \rightarrow \text{CF}_{\mathbb{Z}}^{\text{an}}(X_0)$  such that

$$\Psi_f(1_Z) : x \mapsto \begin{cases} \chi(MF_f|_Z(x)), & x \in X_0 \cap Z, \\ 0, & x \in X_0 \setminus Z, \end{cases} \quad (4.5)$$

whenever  $Z$  is a closed complex analytic subspace of  $X$ , and  $1_Z \in \text{CF}_{\mathbb{Z}}^{\text{an}}(X)$  is given by  $1_Z(x) = 1$  if  $x \in Z$  and  $1_Z(x) = 0$  if  $x \notin Z$ . We set  $\Phi_f = \Psi_f - i^*$ , where  $i : X_0 \rightarrow X$  is the inclusion. Then we have commutative diagrams

$$\begin{array}{ccc} D_{\text{Con}}^b(X) & \xrightarrow{\psi_f} & D_{\text{Con}}^b(X_0) \\ \downarrow \chi_X & & \downarrow \chi_{X_0} \\ \text{CF}_{\mathbb{Z}}^{\text{an}}(X) & \xrightarrow{\Psi_f} & \text{CF}_{\mathbb{Z}}^{\text{an}}(X_0), \end{array} \quad \begin{array}{ccc} D_{\text{Con}}^b(X) & \xrightarrow{\phi_f} & D_{\text{Con}}^b(X_0) \\ \downarrow \chi_X & & \downarrow \chi_{X_0} \\ \text{CF}_{\mathbb{Z}}^{\text{an}}(X) & \xrightarrow{\Phi_f} & \text{CF}_{\mathbb{Z}}^{\text{an}}(X_0). \end{array} \quad (4.6)$$

Now let  $U$  be a complex manifold of dimension  $n$ , and  $f : U \rightarrow \mathbb{C}$  a holomorphic function. The critical locus  $X = \text{Crit}(f)$  is a complex analytic subspace of  $U$ , and  $f$  is locally constant on  $X$ , so locally  $X \subseteq f^{-1}(c)$  for some  $c \in \mathbb{C}$ . Suppose  $X$  is contained in  $f^{-1}(0) = U_0$ . Write  $\underline{\mathbb{Q}}$  for the constant sheaf with fibre  $\mathbb{Q}$  on  $U$ , regarded as an element of  $D_{\text{Con}}^b(U)$ . As  $U$  is smooth of dimension  $n$ , the shift  $\underline{\mathbb{Q}}[n]$  is a simple perverse sheaf on  $U$ . Since  $\psi_f[-1], \phi_f[-1]$  take perverse sheaves to perverse sheaves, it follows that  $\psi_f[-1](\underline{\mathbb{Q}}[n]) = \psi_f(\underline{\mathbb{Q}}[n-1])$  and  $\phi_f[-1](\underline{\mathbb{Q}}[n]) = \phi_f(\underline{\mathbb{Q}}[n-1])$  are perverse sheaves on  $U_0$ . We call these the *perverse sheaves of nearby cycles and vanishing cycles*, respectively.

We will compute  $\chi_{U_0}(\phi_f(\underline{\mathbb{Q}}[n-1]))$ . We have

$$\begin{aligned} \chi_{U_0}(\phi_f(\underline{\mathbb{Q}}[n-1])) &\equiv (\Phi_f \circ \chi_U(\underline{\mathbb{Q}}[n-1])) \equiv (-1)^{n-1}(\Phi_f \circ \chi_U(\underline{\mathbb{Q}})) \\ &\equiv (-1)^{n-1}(\Phi_f(1_U)) \equiv (-1)^{n-1}(\Psi_f(1_U) - i^*(1_U)) \\ &\equiv (-1)^{n-1}(\Psi_f(1_U) - 1_{U_0}) \equiv (-1)^n(1_{U_0} - \Psi_f(1_U)), \end{aligned}$$

using (4.6) commutative in the first step,  $\chi_U \circ [+1] = -\chi_U$  in the second,  $\chi_U(\underline{\mathbb{Q}}) = 1_U$  in the third and  $\Phi_f = \Psi_f - i^*$  in the fourth. So (4.5) gives

$$\chi_{U_0}(\phi_f(\underline{\mathbb{Q}}[n-1])) : x \mapsto (-1)^n(1 - \chi(MF_f(x))) \quad \text{for } x \in U_0. \quad (4.7)$$

If  $x \in U_0 \setminus X$  then  $MF_f(x)$  is an open ball, so  $\chi_{U_0}(\phi_f(\underline{\mathbb{Q}}[n-1]))(x) = 0$  by (4.7), and if  $x \in X$  then  $\chi_{U_0}(\phi_f(\underline{\mathbb{Q}}[n-1]))(x) = \nu_X(x)$  by (4.7) and Theorem 4.7. Thus we have proved:

**THEOREM 4.9.** *Let  $U$  be a complex manifold of dimension  $n$ , and  $f : U \rightarrow \mathbb{C}$  a holomorphic function with  $X = \text{Crit}(f)$  contained in  $U_0 = f^{-1}(\{0\})$ . Then the perverse sheaf of vanishing cycles  $\phi_f(\underline{\mathbb{Q}}[n-1])$  on  $U_0$  is supported on  $X$ , and*

$$\chi_{U_0}(\phi_f(\underline{\mathbb{Q}}[n-1]))(x) = \begin{cases} \nu_X(x), & x \in X, \\ 0, & x \in U_0 \setminus X, \end{cases} \quad (4.8)$$

where  $\nu_X$  is the Behrend function of the complex analytic space  $X$ .

Behrend [3, eq. (5)] gives equation (4.8) with an extra sign  $(-1)^{n-1}$ , since he omits the shift  $[n-1]$  in  $\underline{\mathbb{Q}}[n-1]$ , which makes  $\phi_f(\underline{\mathbb{Q}}[n-1])$  a perverse sheaf. Theorem 4.9 may be important for future work in Donaldson–Thomas theory, as it suggests that we should try to lift from constructible functions to perverse sheaves, or mixed Hodge modules [92], or some similar setting.

This bridge between perverse sheaves and vanishing cycles on one hand, and Milnor fibres and Behrend functions on the other, is also useful because we can take known results on the perverse sheaf side, and translate them into properties of Milnor fibres by applying the surjective functors  $\chi_X$ . Here is one such result. For constructible complexes, the functors  $\psi_f, \phi_f$  commute with proper pushdowns [18, Prop. 4.2.11]. Applying  $\chi_X$  yields:

PROPOSITION 4.10. *Let  $X, Y$  be complex analytic spaces,  $\varphi : Y \rightarrow X$  a proper morphism, and  $f : X \rightarrow \mathbb{C}$  a holomorphic function. Set  $g = f \circ \varphi$ , and write  $X_0 = f^{-1}(0)$  and  $Y_0 = g^{-1}(0)$ . Then the following diagrams commute:*

$$\begin{array}{ccc} \mathrm{CF}_{\mathbb{Z}}^{\mathrm{an}}(Y) & \xrightarrow{\mathrm{CF}(\varphi)} & \mathrm{CF}_{\mathbb{Z}}^{\mathrm{an}}(X) \\ \downarrow \Psi_g & & \downarrow \Psi_f \\ \mathrm{CF}_{\mathbb{Z}}^{\mathrm{an}}(Y_0) & \xrightarrow{\mathrm{CF}(\varphi)} & \mathrm{CF}_{\mathbb{Z}}^{\mathrm{an}}(X_0) \end{array} \quad \begin{array}{ccc} \mathrm{CF}_{\mathbb{Z}}^{\mathrm{an}}(Y) & \xrightarrow{\mathrm{CF}(\varphi)} & \mathrm{CF}_{\mathbb{Z}}^{\mathrm{an}}(X) \\ \downarrow \Phi_g & & \downarrow \Phi_f \\ \mathrm{CF}_{\mathbb{Z}}^{\mathrm{an}}(Y_0) & \xrightarrow{\mathrm{CF}(\varphi)} & \mathrm{CF}_{\mathbb{Z}}^{\mathrm{an}}(X_0) \end{array} \quad (4.9)$$

We use this to prove a property of Milnor fibres that we will need later. The authors would like to thank Jörg Schürmann for suggesting the simple proof of Theorem 4.11 below using Proposition 4.10, which replaces a longer proof using Lagrangian cycles in an earlier version of this book.

THEOREM 4.11. *Let  $U$  be a complex manifold,  $f : U \rightarrow \mathbb{C}$  a holomorphic function,  $V$  a closed, embedded complex submanifold of  $U$ , and  $v \in V \cap \mathrm{Crit}(f)$ . Define  $\tilde{U}$  to be the blowup of  $U$  along  $V$ , with blow-up map  $\pi : \tilde{U} \rightarrow U$ , and set  $\tilde{f} = f \circ \pi : \tilde{U} \rightarrow \mathbb{C}$ . Then  $\pi^{-1}(v) = \mathbb{P}(T_v U / T_v V)$  is contained in  $\mathrm{Crit}(\tilde{f})$ , and*

$$\begin{aligned} \chi(MF_f(v)) &= \int_{w \in \mathbb{P}(T_v U / T_v V)} \chi(MF_{\tilde{f}}(w)) d\chi \\ &\quad + (1 - \dim U + \dim V) \chi(MF_{f|_V}(v)). \end{aligned} \quad (4.10)$$

Here  $w \mapsto \chi(MF_{\tilde{f}}(w))$  is a constructible function on  $\mathbb{P}(T_v U / T_v V)$ , and the integral in (4.10) is the Euler characteristic of  $\mathbb{P}(T_v U / T_v V)$  weighted by this.

PROOF. Let  $U, V, \tilde{U}, v$  be as in the theorem. It is immediate that  $\pi^{-1}(v) = \mathbb{P}(T_v U / T_v V) \subseteq \mathrm{Crit}(\tilde{f})$ . Replacing  $f$  by  $f - f(v)$  if necessary, we can suppose  $f(v) = 0$ . Applying Proposition 4.10 with  $U, \tilde{U}, \pi, f, \tilde{f}$  in place of  $X, Y, \varphi, f, g$  to the function  $1_{\tilde{U}}$  on  $\tilde{U}$  shows that

$$\mathrm{CF}(\pi) \circ \Psi_{\tilde{f}}(1_{\tilde{U}}) = \Psi_f \circ \mathrm{CF}(\pi) 1_{\tilde{U}}. \quad (4.11)$$

We evaluate (4.11) at  $v \in V$ . Since  $\pi^{-1}(v) = \mathbb{P}(T_v U / T_v V) \subset \tilde{V}$ , we have

$$(\mathrm{CF}(\pi) \circ \Psi_{\tilde{f}}(1_{\tilde{U}}))(v) = \int_{w \in \mathbb{P}(T_v U / T_v V)} \Psi_{\tilde{f}}(w) d\chi = \int_{w \in \mathbb{P}(T_v U / T_v V)} \chi(MF_{\tilde{f}}(w)) d\chi, \quad (4.12)$$

by (4.5). The fibre  $\pi^{-1}(u)$  of  $\pi : \tilde{U} \rightarrow U$  is one point over  $u \in U \setminus V$ , with  $\chi(\pi^{-1}(u)) = 1$ , and a projective space  $\mathbb{P}(T_u U / T_u V)$  for  $u \in V$ , with  $\chi(\pi^{-1}(u)) = \dim U - \dim V$ . It follows that  $\mathrm{CF}(\pi) 1_{\tilde{U}}$  is 1 at  $u \in U \setminus V$  and  $\dim U - \dim V$  at  $u \in V$ , giving

$$\mathrm{CF}(\pi) 1_{\tilde{U}} = 1_U + (\dim U - \dim V - 1) 1_V. \quad (4.13)$$

Applying  $\Psi_f$  to (4.13) and using (4.5) to evaluate it at  $v$  gives

$$(\Psi_f \circ \mathrm{CF}(\pi) 1_{\tilde{U}})(v) = \chi(MF_f(v)) + (\dim U - \dim V - 1) \chi(MF_{f|_V}(v)). \quad (4.14)$$

Equation (4.10) now follows from (4.11), (4.12) and (4.14).  $\square$

### 4.3. Donaldson–Thomas invariants of Calabi–Yau 3-folds

Donaldson–Thomas invariants  $DT^\alpha(\tau)$  were defined by Richard Thomas [100], following a proposal of Donaldson and Thomas [20, §3].

DEFINITION 4.12. Let  $\mathbb{K}$  be an algebraically closed field of characteristic zero. As in §3.4, a *Calabi–Yau 3-fold* is a smooth projective 3-fold  $X$  over  $\mathbb{K}$ , with trivial canonical bundle  $K_X$ . Fix a very ample line bundle  $\mathcal{O}_X(1)$  on  $X$ , and let  $(\tau, G, \leq)$  be Gieseker stability on  $\text{coh}(X)$  w.r.t.  $\mathcal{O}_X(1)$ , as in Example 3.8. For  $\alpha \in K^{\text{num}}(\text{coh}(X))$ , write  $\mathcal{M}_{\text{ss}}^\alpha(\tau), \mathcal{M}_{\text{st}}^\alpha(\tau)$  for the coarse moduli schemes of  $\tau$ -(semi)stable sheaves  $E$  with class  $[E] = \alpha$ . Then  $\mathcal{M}_{\text{ss}}^\alpha(\tau)$  is a projective  $\mathbb{K}$ -scheme, and  $\mathcal{M}_{\text{st}}^\alpha(\tau)$  an open subscheme.

Thomas [100] constructs a symmetric obstruction theory on  $\mathcal{M}_{\text{st}}^\alpha(\tau)$ . Suppose that  $\mathcal{M}_{\text{ss}}^\alpha(\tau) = \mathcal{M}_{\text{st}}^\alpha(\tau)$ . Then  $\mathcal{M}_{\text{st}}^\alpha(\tau)$  is proper, so using the obstruction theory Behrend and Fantechi [5] define a virtual class  $[\mathcal{M}_{\text{st}}^\alpha(\tau)]^{\text{vir}} \in A_0(\mathcal{M}_{\text{st}}^\alpha(\tau))$ . The Donaldson–Thomas invariant [100] is defined to be

$$DT^\alpha(\tau) = \int_{[\mathcal{M}_{\text{st}}^\alpha(\tau)]^{\text{vir}}} 1. \quad (4.15)$$

Note that  $DT^\alpha(\tau)$  is defined only when  $\mathcal{M}_{\text{ss}}^\alpha(\tau) = \mathcal{M}_{\text{st}}^\alpha(\tau)$ , that is, there are no strictly semistable sheaves  $E$  in class  $\alpha$ . One of our main goals is to extend the definition to all  $\alpha \in K^{\text{num}}(\text{coh}(X))$ .

In fact Thomas did not define invariants  $DT^\alpha(\tau)$  counting sheaves with fixed class  $\alpha \in K^{\text{num}}(\text{coh}(X))$ , but coarser invariants  $DT^P(\tau)$  counting sheaves with fixed Hilbert polynomial  $P(t) \in \mathbb{Q}[t]$ . Since  $\mathcal{M}_{\text{ss}}^P(\tau) = \coprod_{\alpha: P_\alpha = P} \mathcal{M}_{\text{ss}}^\alpha(\tau)$ , the relationship with our version  $DT^\alpha(\tau)$  is

$$DT^P(\tau) = \sum_{\alpha \in K^{\text{num}}(\text{coh}(X)): P_\alpha = P} DT^\alpha(\tau),$$

with only finitely many nonzero terms in the sum. Thomas’ main result [100, §3], which works over an arbitrary algebraically closed base field  $\mathbb{K}$ , is that

THEOREM 4.13. *For each Hilbert polynomial  $P(t)$ , the invariant  $DT^P(\tau)$  is unchanged by continuous deformations of the underlying Calabi–Yau 3-fold  $X$ .*

The same proof shows that  $DT^\alpha(\tau)$  for  $\alpha \in K^{\text{num}}(\text{coh}(X))$  is deformation-invariant, *provided* we know that the group  $K^{\text{num}}(\text{coh}(X))$  is deformation-invariant, so that this statement makes sense. This issue will be discussed in §4.5. We show that when  $\mathbb{K} = \mathbb{C}$  we can describe  $K^{\text{num}}(\text{coh}(X))$  in terms of cohomology groups  $H^*(X; \mathbb{Z}), H^*(X; \mathbb{Q})$ , so that  $K^{\text{num}}(\text{coh}(X))$  is manifestly deformation-invariant, and therefore  $DT^\alpha(\tau)$  is also deformation-invariant.

Here is a property of Behrend functions which is crucial for Donaldson–Thomas theory. It is proved by Behrend [3, Th. 4.18] when  $\mathbb{K} = \mathbb{C}$ , but his proof is valid for general  $\mathbb{K}$ .

THEOREM 4.14. *Let  $\mathbb{K}$  be an algebraically closed field of characteristic zero,  $X$  a proper  $\mathbb{K}$ -scheme with a symmetric obstruction theory, and  $[X]^{\text{vir}} \in A_0(X)$  the corresponding virtual class from Behrend and Fantechi [5]. Then*

$$\int_{[X]^{\text{vir}}} 1 = \chi(X, \nu_X) \in \mathbb{Z},$$

where  $\chi(X, \nu_X) = \int_{X(\mathbb{K})} \nu_X d\chi$  is the Euler characteristic of  $X$  weighted by the Behrend function  $\nu_X$  of  $X$ . In particular,  $\int_{[X]^{\text{vir}}} 1$  depends only on the  $\mathbb{K}$ -scheme structure of  $X$ , not on the choice of symmetric obstruction theory.

Theorem 4.14 implies that  $DT^\alpha(\tau)$  in (4.15) is given by

$$DT^\alpha(\tau) = \chi(\mathcal{M}_{\text{st}}^\alpha(\tau), \nu_{\mathcal{M}_{\text{st}}^\alpha(\tau)}). \quad (4.16)$$

There is a big difference between the two equations (4.15) and (4.16) defining Donaldson–Thomas invariants. Equation (4.15) is non-local, and non-motivic, and makes sense only if  $\mathcal{M}_{\text{st}}^\alpha(\tau)$  is a proper  $\mathbb{K}$ -scheme. But (4.16) is local, and (in a sense) motivic, and makes sense for arbitrary finite type  $\mathbb{K}$ -schemes  $\mathcal{M}_{\text{st}}^\alpha(\tau)$ . In fact, one could take (4.16) to be the definition of Donaldson–Thomas invariants even when  $\mathcal{M}_{\text{ss}}^\alpha(\tau) \neq \mathcal{M}_{\text{st}}^\alpha(\tau)$ , but we will argue in §6.5 that this is not a good idea, as then  $DT^\alpha(\tau)$  would not be unchanged under deformations of  $X$ .

Equation (4.16) was the inspiration for this book. It shows that Donaldson–Thomas invariants  $DT^\alpha(\tau)$  can be written as *motivic* invariants, like those studied in [51–55], and so it raises the possibility that we can extend the results of [51–55] to Donaldson–Thomas invariants by including Behrend functions as weights.

#### 4.4. Behrend functions and almost closed 1-forms

The material of §4.2–§4.3 raises an obvious question. Given a proper moduli space  $\mathcal{M}$  with a symmetric obstruction theory, such as a moduli space of sheaves  $\mathcal{M}_{\text{st}}^\alpha(\tau)$  on a Calabi–Yau 3-fold when  $\mathcal{M}_{\text{st}}^\alpha(\tau) = \mathcal{M}_{\text{ss}}^\alpha(\tau)$ , we have  $\int_{[\mathcal{M}]^{\text{vir}}} 1 = \chi(\mathcal{M}, \nu_{\mathcal{M}})$  by Theorem 4.14. If we could write  $\mathcal{M}$  as  $\text{Crit}(f)$  for  $f : U \rightarrow \mathbb{C}$  a holomorphic function on a complex manifold  $U$ , we could use the results of §4.2 to study the Behrend function  $\nu_{\mathcal{M}}$ . However, as Behrend says [3, p. 5]:

‘We do not know if every scheme admitting a symmetric obstruction theory can locally be written as the critical locus of a regular function on a smooth scheme. This limits the usefulness of the above formula for  $\nu_X(x)$  in terms of the Milnor fibre.’

Later we will prove using transcendental complex analytic methods that when  $\mathbb{K} = \mathbb{C}$ , moduli spaces  $\mathcal{M}_{\text{st}}^\alpha(\tau)$  on a Calabi–Yau 3-fold can indeed be written as  $\text{Crit}(f)$  for  $f$  holomorphic on a complex manifold  $U$ , and so we can apply §4.2 to prove identities on Behrend functions (5.2)–(5.3). But here we sketch an alternative approach due to Behrend [3], which could perhaps be used to give a strictly algebraic proof of the same identities.

**DEFINITION 4.15.** Let  $\mathbb{K}$  be an algebraically closed field, and  $M$  a smooth  $\mathbb{K}$ -scheme. Let  $\omega$  be a 1-form on  $M$ , that is,  $\omega \in H^0(T^*M)$ . We call  $\omega$  *almost closed* if  $d\omega$  is a section of  $I_\omega \cdot \Lambda^2 T^*M$ , where  $I_\omega$  is the ideal sheaf of the zero locus  $\omega^{-1}(0)$  of  $\omega$ . Equivalently,  $d\omega|_{\omega^{-1}(0)}$  is zero as a section of  $\Lambda^2 T^*M|_{\omega^{-1}(0)}$ . In (étale) local coordinates  $(z_1, \dots, z_n)$  on  $M$ , if  $\omega = f_1 dz_1 + \dots + f_n dz_n$ , then  $\omega$  is almost closed provided

$$\frac{\partial f_j}{\partial z_k} \equiv \frac{\partial f_k}{\partial z_j} \pmod{(f_1, \dots, f_n)}.$$

Behrend [3, Prop. 3.14] proves the following, by a proof valid for general  $\mathbb{K}$ :

**PROPOSITION 4.16.** *Let  $\mathbb{K}$  be an algebraically closed field, and  $X$  a  $\mathbb{K}$ -scheme with a symmetric obstruction theory. Then  $X$  may be covered by Zariski open sets  $Y \subseteq X$  such that there exists a smooth  $\mathbb{K}$ -scheme  $M$ , an almost closed 1-form  $\omega$  on  $M$ , and an isomorphism of  $\mathbb{K}$ -schemes  $Y \cong \omega^{-1}(0)$ .*

If we knew the almost closed 1-form  $\omega$  was closed, then locally  $\omega = df$  for  $f : M \rightarrow \mathbb{K}$  regular, and  $X \cong \text{Crit}(f)$  as we want. Restricting to  $\mathbb{K} = \mathbb{C}$ , Behrend [3, Prop. 4.22] gives an expression for the Behrend function of the zero locus of an almost closed 1-form as a linking number. He states it in the complex algebraic case, but his proof is also valid in the complex analytic case.

**PROPOSITION 4.17.** *Let  $M$  be a complex manifold and  $\omega$  an almost closed holomorphic  $(1,0)$ -form on  $M$ , and let  $X = \omega^{-1}(0)$  as a complex analytic subspace of  $M$ . Fix  $x \in X$ , choose holomorphic coordinates  $(z_1, \dots, z_n)$  on  $X$  near  $x$  with  $z_1(x) = \dots = z_n(x) = 0$ , and let  $(z_1, \dots, z_n, w_1, \dots, w_n)$  be the induced coordinates on  $T^*M$ , with  $(z_1, \dots, w_n)$  representing the 1-form  $w_1 dz_1 + \dots + w_n dz_n$  at  $(z_1, \dots, z_n)$ , so that we identify  $T^*M$  near  $x$  with  $\mathbb{C}^{2n}$ .*

*Then for all  $\eta \in \mathbb{C}$  and  $\epsilon \in \mathbb{R}$  with  $0 < |\eta| \ll \epsilon \ll 1$  we have*

$$\nu_X(x) = L_{\mathcal{S}_\epsilon}(\Gamma_{\eta^{-1}\omega} \cap \mathcal{S}_\epsilon, \Delta \cap \mathcal{S}_\epsilon), \quad (4.17)$$

*where  $\mathcal{S}_\epsilon = \{(z_1, \dots, w_n) \in \mathbb{C}^{2n} : |z_1|^2 + \dots + |w_n|^2 = \epsilon^2\}$  is the sphere of radius  $\epsilon$  in  $\mathbb{C}^{2n}$ , and  $\Gamma_{\eta^{-1}\omega}$  the graph of  $\eta^{-1}\omega$  regarded locally as a complex submanifold of  $\mathbb{C}^{2n}$ , and  $\Delta = \{(z_1, \dots, w_n) \in \mathbb{C}^{2n} : w_j = \bar{z}_j, j=1, \dots, n\}$ , and  $L_{\mathcal{S}_\epsilon}(\cdot, \cdot)$  the linking number of two disjoint, closed, oriented  $(n-1)$ -submanifolds in  $\mathcal{S}_\epsilon$ .*

Here are some questions which seem interesting. If the answer to (a) is yes, it suggests the possibility of an alternative proof of our Behrend function identities (5.2)–(5.3) using algebraic almost closed 1-forms as in Proposition 4.16, rather than using transcendental complex analytic methods.

**QUESTION 4.18.** *Let  $M$  be a complex manifold,  $\omega$  an almost closed holomorphic  $(1,0)$ -form on  $M$ , and  $X = \omega^{-1}(0)$  as a complex analytic subspace of  $M$ .*

**(a)** *Can one prove results for Behrend functions  $\nu_X$  analogous to those one can prove for Behrend functions of  $\text{Crit}(f)$  for  $f : M \rightarrow \mathbb{C}$  holomorphic, using Proposition 4.17? For instance, is the analogue of Theorem 4.11 true with  $df$  replaced by an almost closed 1-form  $\omega$ , and  $d\tilde{f}$  replaced by  $\pi^*(\omega)$ ?*

**(b)** *Can one define a natural perverse sheaf  $\mathcal{P}$  supported on  $X$ , with  $\chi_X(\mathcal{P}) = \nu_X$ , such that  $\mathcal{P} \cong \phi_f(\mathbb{Q}[n-1])$  when  $\omega = df$  for  $f : M \rightarrow \mathbb{C}$  holomorphic?*

**(c)** *If the answer to (a) or (b) is yes, are there generalizations to the algebraic setting, which work say over  $\mathbb{K}$  algebraically closed of characteristic zero?*

One can also ask Question 4.18(b) for Saito’s mixed Hodge modules [92].

#### 4.5. Characterizing $K^{\text{num}}(\text{coh}(X))$ for Calabi–Yau 3-folds

Let  $X$  be a Calabi–Yau 3-fold over  $\mathbb{C}$ , with  $H^1(\mathcal{O}_X) = 0$ . We will now give an exact description of the numerical Grothendieck group  $K^{\text{num}}(\text{coh}(X))$  in terms of the cohomology  $H^{\text{even}}(X, \mathbb{Q})$ . A corollary of this is that  $K^{\text{num}}(\text{coh}(X))$  is unchanged by small deformations of the complex structure of  $X$ . This is necessary for our claim in §5.4 that the  $\bar{D}T^\alpha(\tau)$  for  $\alpha \in K^{\text{num}}(\text{coh}(X))$  are deformation-invariant to make sense.

To do this we will use the *Chern character*, as in Hartshorne [40, App. A] or Fulton [28]. For each  $E \in \text{coh}(X)$  we have the rank  $r(E) \in H^0(X; \mathbb{Z})$  and the Chern classes  $c_i(E) \in H^{2i}(X; \mathbb{Z})$  for  $i = 1, 2, 3$ . It is useful to organize these into the Chern

character  $\text{ch}(E)$  in  $H^{\text{even}}(X, \mathbb{Q})$ , where  $\text{ch}(E) = \text{ch}_0(E) + \text{ch}_1(E) + \text{ch}_2(E) + \text{ch}_3(E)$  with  $\text{ch}_i(E) \in H^{2i}(X; \mathbb{Q})$ , with

$$\begin{aligned} \text{ch}_0(E) &= r(E), \quad \text{ch}_1(E) = c_1(E), \quad \text{ch}_2(E) = \frac{1}{2}(c_1(E)^2 - 2c_2(E)), \\ \text{ch}_3(E) &= \frac{1}{6}(c_1(E)^3 - 3c_1(E)c_2(E) + 3c_3(E)). \end{aligned} \quad (4.18)$$

Here we use the natural morphism  $H^{\text{even}}(X; \mathbb{Z}) \rightarrow H^{\text{even}}(X; \mathbb{Q})$  to make  $r(E)$ ,  $c_i(E)$  into elements of  $H^{\text{even}}(X; \mathbb{Q})$ . The kernel of this morphism is the *torsion* of  $H^{\text{even}}(X; \mathbb{Z})$ , the subgroup of elements of finite order. From now on we will neglect torsion in  $H^{\text{even}}(X; \mathbb{Z})$ , so by an abuse of notation, when we say that an element  $\lambda_i$  of  $H^{2i}(X; \mathbb{Q})$  lies in  $H^{2i}(X; \mathbb{Z})$ , we really mean that  $\lambda_i$  lies in the image of  $H^{2i}(X; \mathbb{Z})$  in  $H^{2i}(X; \mathbb{Q})$ .

By the Hirzebruch–Riemann–Roch Theorem [40, Th. A.4.1], the Euler form on coherent sheaves  $E, F$  is given in terms of their Chern characters by

$$\bar{\chi}([E], [F]) = \deg(\text{ch}(E)^\vee \cdot \text{ch}(F) \cdot \text{td}(TX))_3, \quad (4.19)$$

where  $\text{td}(TX)$  is the *Todd class* of  $TX$ , which is  $1 + \frac{1}{12}c_2(TX)$  as  $X$  is a Calabi–Yau 3-fold, and  $(\lambda_0, \lambda_1, \lambda_2, \lambda_3)^\vee = (\lambda_0, -\lambda_1, \lambda_2, -\lambda_3)$ , writing elements of  $H^{\text{even}}(X; \mathbb{Q})$  as  $(\lambda_0, \dots, \lambda_3)$  with  $\lambda_i \in H^{2i}(X; \mathbb{Q})$ .

The Chern character is additive over short exact sequences. That is, if  $0 \rightarrow E \rightarrow F \rightarrow G \rightarrow 0$  is exact in  $\text{coh}(X)$  then  $\text{ch}(F) = \text{ch}(E) + \text{ch}(G)$ . Hence  $\text{ch}$  induces a group morphism  $\text{ch} : K_0(\text{coh}(X)) \rightarrow H^{\text{even}}(X; \mathbb{Q})$ . We have  $K^{\text{num}}(\text{coh}(X)) = K_0(\text{coh}(X))/I$ , where  $I$  is the kernel of  $\bar{\chi}$  on  $K_0(\text{coh}(X))$ . Equation (4.19) implies that  $\text{Ker ch} \subseteq I$ . Theorem 4.19 will identify the image of  $\text{ch}$  in  $H^{\text{even}}(X; \mathbb{Q})$ . This image spans  $H^{\text{even}}(X; \mathbb{Q})$  over  $\mathbb{Q}$ , so as the pairing  $(\alpha, \beta) \mapsto \deg(\alpha^\vee \cdot \beta \cdot \text{td}(TX))_3$  is nondegenerate on  $H^{\text{even}}(X; \mathbb{Q})$ , it is nondegenerate on the image of  $\text{ch}$ , so  $\text{Ker ch} = I$ .

Hence  $\text{ch}$  induces an *injective* morphism  $\text{ch} : K^{\text{num}}(\text{coh}(X)) \hookrightarrow H^{\text{even}}(X; \mathbb{Q})$ , and we may regard  $K^{\text{num}}(\text{coh}(X))$  as a subgroup of  $H^{\text{even}}(X; \mathbb{Q})$ . (Actually, this is true for any smooth projective  $\mathbb{C}$ -scheme  $X$ .) Our next theorem identifies the image of  $\text{ch}$ .

**THEOREM 4.19.** *Let  $X$  be a Calabi–Yau 3-fold over  $\mathbb{C}$  with  $H^1(\mathcal{O}_X) = 0$ . Define*

$$\begin{aligned} \Lambda_X = \{ & (\lambda_0, \lambda_1, \lambda_2, \lambda_3) \in H^{\text{even}}(X; \mathbb{Q}) : \lambda_0 \in H^0(X; \mathbb{Z}), \lambda_1 \in H^2(X; \mathbb{Z}), \\ & \lambda_2 - \frac{1}{2}\lambda_1^2 \in H^4(X; \mathbb{Z}), \lambda_3 + \frac{1}{12}\lambda_1 c_2(TX) \in H^6(X; \mathbb{Z}) \}. \end{aligned} \quad (4.20)$$

*Then  $\Lambda_X$  is a subgroup of  $H^{\text{even}}(X; \mathbb{Q})$ , a lattice of rank  $\sum_{i=0}^3 b^{2i}(X)$ , and the Chern character gives a group isomorphism  $\text{ch} : K^{\text{num}}(\text{coh}(X)) \rightarrow \Lambda_X$ .*

*Therefore the numerical Grothendieck group  $K^{\text{num}}(\text{coh}(X))$  depends only on the underlying topological space of  $X$  up to homotopy, and so  $K^{\text{num}}(\text{coh}(X))$  is unchanged by deformations of the complex structure of  $X$ .*

**PROOF.** Suppose for simplicity that  $X$  is connected, so that we have natural isomorphisms  $H^6(X; \mathbb{Q}) \cong \mathbb{Q}$  and  $H^6(X; \mathbb{Z}) \cong \mathbb{Z}$ . If it is not connected, we can run the argument below for each connected component of  $X$ .

To show  $\Lambda_X$  is a subgroup of  $H^{\text{even}}(X; \mathbb{Q})$ , we must check it is closed under addition and inverses. The only issue is that the condition  $\lambda_2 - \frac{1}{2}\lambda_1^2 \in H^4(X; \mathbb{Z})$  is not linear in  $\lambda_1$ . If  $(\lambda_0, \dots, \lambda_3), (\lambda'_0, \dots, \lambda'_3) \in \Lambda_X$  then

$$(\lambda_2 + \lambda'_2) - \frac{1}{2}(\lambda_1 + \lambda'_1)^2 = [\lambda_2 - \frac{1}{2}\lambda_1^2] + [\lambda'_2 - \frac{1}{2}(\lambda'_1)^2] + [\lambda_1 \lambda'_1],$$

and the right hand side is the sum of three terms in  $H^4(X; \mathbb{Z})$ . So  $(\lambda_0 + \lambda'_0, \dots, \lambda_3 + \lambda'_3) \in \Lambda_X$ . Also

$$(-\lambda_2) - \frac{1}{2}(-\lambda_1)^2 = -[\lambda_2 - \frac{1}{2}\lambda_1^2] - [\lambda_1^2],$$

with the right hand the sum of two terms in  $H^4(X; \mathbb{Z})$ . So  $(-\lambda_0, \dots, -\lambda_3) \in \Lambda_X$ , and  $\Lambda_X$  is a subgroup of  $H^{\text{even}}(X; \mathbb{Q})$ . We have

$$\frac{1}{6} H^{\text{even}}(X; \mathbb{Z})/\text{torsion} \subseteq \Lambda_X \subseteq H^{\text{even}}(X; \mathbb{Z})/\text{torsion} \subseteq H^{\text{even}}(X; \mathbb{Q}),$$

so  $\Lambda_X$  is a lattice of rank  $\sum_{i=0}^3 b^{2i}(X)$  as  $H^{\text{even}}(X; \mathbb{Z})/\text{torsion}$  is.

Next we show that  $\text{ch}(K^{\text{num}}(\text{coh}(X))) \subseteq \Lambda_X$ . As  $\Lambda_X$  is a subgroup, it is enough to show that  $\text{ch}(E) \in \Lambda_X$  for any  $E \in \text{coh}(X)$ . Set  $\text{ch}(E) = (\lambda_0, \dots, \lambda_3)$ . Then (4.18) gives  $\lambda_0 = r(E)$ ,  $\lambda_1 = c_1(E)$ ,  $\lambda_2 = \frac{1}{2}(c_1(E)^2 - 2c_2(E))$  and  $\lambda_3 = \frac{1}{6}(c_1(E)^3 - 3c_1(E)c_2(E) + 3c_3(E))$ , with  $r(E) \in H^0(X; \mathbb{Z})$  and  $c_i(E) \in H^{2i}(X; \mathbb{Z})$ . The conditions  $\lambda_0 \in H^0(X; \mathbb{Z})$  and  $\lambda_1 \in H^2(X; \mathbb{Z})$  are immediate, and  $\lambda_2 - \frac{1}{2}\lambda_1^2 = -c_2(E)$  which lies in  $H^4(X; \mathbb{Z})$ . For the final condition,

$$\begin{aligned} & \deg(\lambda_3 + \frac{1}{12}\lambda_1 c_2(TX)) \\ &= \deg(\frac{1}{6}(c_1(E)^3 - 3c_1(E)c_2(E) + 3c_3(E)) + \frac{1}{12}c_1(E)c_2(TX)) \\ &= \deg((1, 0, 0, 0) \cdot (r(E), c_1(E), \frac{1}{2}(c_1(E)^2 - 2c_2(E)), \\ & \quad \frac{1}{6}(c_1(E)^3 - 3c_1(E)c_2(E) + 3c_3(E))) \cdot (1, 0, \frac{1}{12}c_2(TX), 0))_3 \\ &= \deg(\text{ch}(\mathcal{O})^\vee \cdot \text{ch}(E) \cdot \text{td}(TX))_3 = \bar{\chi}([\mathcal{O}_X], [F]) \in \mathbb{Z}, \end{aligned}$$

using (4.19) in the last line. Then  $\deg(\lambda_3 + \frac{1}{12}\lambda_1 c_2(TX)) \in \mathbb{Z}$  implies  $\lambda_3 + \frac{1}{12}\lambda_1 c_2(TX) \in H^6(X; \mathbb{Z}) \cong \mathbb{Z}$ . Hence  $\text{ch}(E) = (\lambda_0, \dots, \lambda_3) \in \Lambda_X$ , as we want.

As  $X$  is a Calabi–Yau 3-fold over  $\mathbb{C}$  with  $H^1(\mathcal{O}_X) = 0$  we have  $H^{2,0}(X) = H^{0,2}(X) = 0$ , so  $H^{1,1}(X) = H^2(X; \mathbb{C})$ . Therefore every  $\beta \in H^2(X; \mathbb{Z})$  is  $c_1(L_\beta)$  for some holomorphic line bundle  $L_\beta$ , with

$$\text{ch}(L_\beta) = (1, \beta, \frac{1}{2}\beta^2, \frac{1}{6}\beta^3), \quad \text{for any } \beta \in H^2(X; \mathbb{Z}). \quad (4.21)$$

Pick  $x \in X$ , and let  $\mathcal{O}_x$  be the skyscraper sheaf at  $x$ . Then

$$\text{ch}(\mathcal{O}_x) = (0, 0, 0, 1), \quad (4.22)$$

identifying  $H^6(X; \mathbb{Q}) \cong \mathbb{Q}$  and  $H^6(X; \mathbb{Z}) \cong \mathbb{Z}$  in the natural way. Suppose  $\Sigma$  is an reduced algebraic curve in  $X$ , with homology class  $[\Sigma] \in H_2(X; \mathbb{Z}) \cong H^4(X; \mathbb{Z})$ . Then the structure sheaf  $\mathcal{O}_\Sigma$  in  $\text{coh}(X)$  has

$$\text{ch}(\mathcal{O}_\Sigma) = (0, 0, [\Sigma], k) \quad \text{for some } k \in \mathbb{Z}. \quad (4.23)$$

Now  $\text{ch}(K^{\text{num}}(\text{coh}(X)))$  is a subgroup of  $\Lambda_X$  which contains (4.21)–(4.23). We claim that the elements (4.21)–(4.23) over all  $\beta, \Sigma$  generate  $\Lambda_X$ , which forces  $\text{ch}(K^{\text{num}}(\text{coh}(X))) = \Lambda_X$  and proves the theorem. This depends on a deep fact: Voisin [103, Th. 2] proves the Hodge Conjecture over  $\mathbb{Z}$  for Calabi–Yau 3-folds  $X$  over  $\mathbb{C}$  with  $H^1(\mathcal{O}_X) = 0$ . In this case, the Hodge Conjecture over  $\mathbb{Z}$  is equivalent to the statement that  $H_2(X; \mathbb{Z}) \cong H^4(X; \mathbb{Z})$  is generated as a group by classes  $[\Sigma]$  of algebraic curves  $\Sigma$  in  $X$ . It follows that (4.22) and (4.23) taken over all  $\Sigma$  generate the subgroup of  $(0, 0, \lambda_2, \lambda_3) \in \Lambda_X$  with  $\lambda_2 \in H^4(X; \mathbb{Z})$  and  $\lambda_3 \in H^6(X; \mathbb{Z})$ . Together with (4.21) for all  $\beta \in H^2(X; \mathbb{Z})$ , these generate  $\Lambda_X$ .  $\square$

**REMARK 4.20. (a)** Our proof used the Hodge Conjecture over  $\mathbb{Z}$  for Calabi–Yau 3-folds, proved by Voisin [103]. But the Hodge Conjecture over  $\mathbb{Z}$  is false in general, so the theorem may not generalize to other classes of varieties.

(b) In fact  $\Lambda_X$  is a subring of  $H^{\text{even}}(X; \mathbb{Q})$ . Also,  $K_0(\text{coh}(X)), K^{\text{num}}(\text{coh}(X))$  naturally have the structure of rings, with multiplication ‘ $\cdot$ ’ characterized by  $[E] \cdot [F] = [E \otimes F]$  for  $E, F$  locally free. As  $\text{ch}(E \otimes F) = \text{ch}(E) \text{ch}(F)$  for  $E, F$  locally free, it follows that  $\text{ch} : K^{\text{num}}(\text{coh}(X)) \rightarrow \Lambda_X$  is a ring isomorphism. But we will make no use of these ring structures.

(c) If  $X$  is a Calabi–Yau 3-fold over  $\mathbb{C}$  but  $H^1(\mathcal{O}_X) \neq 0$  then  $\text{ch}(K^{\text{num}}(\text{coh}(X)))$  can be a proper subgroup of  $\Lambda_X$ , and this subgroup can change under deformations of  $X$ . Thus  $K^{\text{num}}(\text{coh}(X))$  need not be deformation-invariant up to isomorphism, as its rank can jump under deformation.

To see this, note that if  $H^1(\mathcal{O}_X) \neq 0$  then  $H^{2,0}(X) \neq 0$ , so  $H^{1,1}(X)$  is a proper subspace of  $H^2(X; \mathbb{C})$ , which can vary as we deform  $X$ , and the intersection  $H^{1,1}(X) \cap H^2(X; \mathbb{Z})$  can change under deformation. Let  $\beta \in H^2(X; \mathbb{Z})$ . Then  $\beta = c_1(L)$  for some holomorphic line bundle  $L$  on  $X$  if and only if  $\beta \in H^{1,1}(X)$ , and then  $\text{ch}(L) = (1, \beta, \frac{1}{2}\beta^2, \frac{1}{6}\beta^3)$ . One can show that  $(1, \beta, \frac{1}{2}\beta^2, \frac{1}{6}\beta^3)$  lies in  $\text{ch}(K^{\text{num}}(\text{coh}(X)))$  if and only if  $\beta \in H^{1,1}(X)$ .

(d) Let  $B$  be a  $\mathbb{C}$ -scheme, and  $X_b$  for  $b \in B(\mathbb{C})$  be a family of Calabi–Yau 3-folds with  $H^1(\mathcal{O}_{X_b}) = 0$ . That is, we have a smooth  $\mathbb{C}$ -scheme morphism  $\pi : X \rightarrow B$ , with fibres  $X_b$  for  $b \in B(\mathbb{C})$ . Then we can form  $K^{\text{num}}(\text{coh}(X_b))$  for each  $b \in B(\mathbb{C})$ . If  $B$  is connected, then for  $b_0, b_1 \in B(\mathbb{C})$ , we can choose a continuous path  $\gamma : [0, 1] \rightarrow B(\mathbb{C})$  joining  $b_0$  and  $b_1$ . The family  $X_{\gamma(t)}$  for  $t \in [0, 1]$  defines a homotopy  $X_{b_0} \xrightarrow{\sim} X_{b_1}$ , and so induces isomorphisms  $H^{\text{even}}(X_{b_0}; \mathbb{Q}) \cong H^{\text{even}}(X_{b_1}; \mathbb{Q})$ ,  $\Lambda_{X_{b_0}} \cong \Lambda_{X_{b_1}}$ , and  $K^{\text{num}}(\text{coh}(X_{b_0})) \cong K^{\text{num}}(\text{coh}(X_{b_1}))$ .

However, if  $B$  is not *simply-connected* this isomorphism  $K^{\text{num}}(\text{coh}(X_{b_0})) \cong K^{\text{num}}(\text{coh}(X_{b_1}))$  can depend on the homotopy class of the path  $\gamma$ . The groups  $K^{\text{num}}(\text{coh}(X_b))$  for  $b \in B(\mathbb{C})$  form a *local system* on  $B(\mathbb{C})$ , so that the fibres are all isomorphic, but going round loops in  $B(\mathbb{C})$  can induce nontrivial automorphisms of  $K^{\text{num}}(\text{coh}(X_b))$ , through an action of  $\pi_1(B(\mathbb{C}))$  on  $K^{\text{num}}(\text{coh}(X_b))$ . This phenomenon is called *monodromy*. We study it in Theorem 4.21 below.

Thus, the statement in Theorem 4.19 that  $K^{\text{num}}(\text{coh}(X))$  is unchanged by deformations of the complex structure of  $X$  should be treated with caution: it is true up to isomorphism, but in general it is only true up to canonical isomorphism if we restrict to a simply-connected family of deformations.

(e) It may be possible to extend Theorem 4.19 to work over an algebraically closed base field  $\mathbb{K}$  of characteristic zero by replacing  $H^*(X; \mathbb{Q})$  by the *algebraic de Rham cohomology*  $H_{\text{dR}}^*(X)$  of Hartshorne [39]. For  $X$  a smooth projective  $\mathbb{K}$ -scheme,  $H_{\text{dR}}^*(X)$  is a finite-dimensional vector space over  $\mathbb{K}$ . There is a Chern character map  $\text{ch} : K^{\text{num}}(\text{coh}(X)) \hookrightarrow H_{\text{dR}}^{\text{even}}(X)$ . In [39, §4], Hartshorne considers how  $H_{\text{dR}}^*(X_t)$  varies in families  $X_t : t \in T$ , and defines a Gauss–Manin connection, which makes sense of  $H_{\text{dR}}^*(X_t)$  being locally constant in  $t$ .

We now study monodromy phenomena for  $K^{\text{num}}(\text{coh}(X_u))$  in families of smooth  $\mathbb{K}$ -schemes  $X \rightarrow U$ , as in Remark 4.20(d). We find that we can always eliminate such monodromy by passing to a finite cover  $\tilde{U}$  of  $U$ . This will be used in §5.4 and §12 to prove deformation-invariance of the  $\bar{D}T^\alpha(\tau), PI^{\alpha,n}(\tau')$ .

**THEOREM 4.21.** *Let  $\mathbb{K}$  be an algebraically closed field,  $\varphi : X \rightarrow U$  a smooth projective morphism of  $\mathbb{K}$ -schemes with  $U$  connected, and  $\mathcal{O}_X(1)$  a relative very ample line on  $X$ , so that for each  $u \in U(\mathbb{K})$ , the fibre  $X_u$  of  $\varphi$  is a smooth projective  $\mathbb{K}$ -scheme with very ample line bundle  $\mathcal{O}_{X_u}(1)$ . Suppose the numerical Grothendieck*



groups  $K^{\text{num}}(\text{coh}(X_u))$  are locally constant in  $U(\mathbb{K})$ , so that  $u \mapsto K^{\text{num}}(\text{coh}(X_u))$  is a local system of abelian groups on  $U$ .

Fix a base point  $v \in U(\mathbb{K})$ , and let  $\Gamma$  be the group of automorphisms of  $K^{\text{num}}(\text{coh}(X_v))$  generated by monodromy round loops in  $U$ . Then  $\Gamma$  is a finite group. There exists a finite étale cover  $\pi : \tilde{U} \rightarrow U$  of degree  $|\Gamma|$ , with  $\tilde{U}$  a connected  $\mathbb{K}$ -scheme, such that writing  $\tilde{X} = X \times_U \tilde{U}$  and  $\tilde{\varphi} : \tilde{X} \rightarrow \tilde{U}$  for the natural projection, with fibre  $\tilde{X}_{\tilde{u}}$  at  $\tilde{u} \in \tilde{U}(\mathbb{K})$ , then  $K^{\text{num}}(\text{coh}(\tilde{X}_{\tilde{u}}))$  for all  $\tilde{u} \in \tilde{U}(\mathbb{K})$  are all globally canonically isomorphic to  $K^{\text{num}}(\text{coh}(X_v))$ . That is, the local system  $\tilde{u} \mapsto K^{\text{num}}(\text{coh}(\tilde{X}_{\tilde{u}}))$  on  $\tilde{U}$  is trivial.

PROOF. As  $K^{\text{num}}(\text{coh}(X_v))$  is a finite rank lattice, we can choose  $E_1, \dots, E_n \in \text{coh}(X_v)$  such that  $[E_1], \dots, [E_n]$  generate  $K^{\text{num}}(\text{coh}(X_v))$ . Write  $\tau$  for Gieseker stability on  $\text{coh}(X_v)$  with respect to  $\mathcal{O}_{X_v}(1)$ . Then as in §3.2 each  $E_i$  has a Harder–Narasimhan filtration with  $\tau$ -semistable factors  $S_{ij}$ . Let  $\alpha_1, \dots, \alpha_k \in K^{\text{num}}(\text{coh}(X_v))$  be the classes of the  $S_{ij}$  for all  $i, j$ . Then  $\alpha_1, \dots, \alpha_k$  generate  $K^{\text{num}}(\text{coh}(X_v))$  as an abelian group, and the coarse moduli scheme  $\mathcal{M}_{\text{ss}}^{\alpha_i}(\tau)_v$  of  $\tau$ -semistable sheaves on  $X_v$  in class  $\alpha_i$  is nonempty for  $i = 1, \dots, k$ . Let  $P_i$  be the Hilbert polynomial of  $\alpha_i$  for  $i = 1, \dots, k$ .

Let  $\gamma \in \Gamma$ , and consider the images  $\gamma \cdot \alpha_i \in K^{\text{num}}(\text{coh}(X_v))$  for  $i = 1, \dots, k$ . As we assume  $\mathcal{O}_X(1)$  is globally defined on  $U$  and does not change under monodromy, it follows that the Hilbert polynomials of classes  $\alpha \in K^{\text{num}}(\text{coh}(X_v))$  do not change under monodromy. Hence  $\gamma \cdot \alpha_i$  has Hilbert polynomial  $P_i$ . Also, the condition that  $\mathcal{M}_{\text{ss}}^{\alpha}(\tau)_u \neq \emptyset$  for  $u \in U(\mathbb{K})$  and  $\alpha \in K^{\text{num}}(\text{coh}(X_u))$  is an open and closed condition in  $(u, \alpha)$ , so as  $\mathcal{M}_{\text{ss}}^{\alpha_i}(\tau)_v \neq \emptyset$  we have  $\mathcal{M}_{\text{ss}}^{\gamma \cdot \alpha_i}(\tau)_v \neq \emptyset$ .

For each  $i = 1, \dots, k$ , the family of  $\tau$ -semistable sheaves on  $X_v$  with Hilbert polynomial  $P_i$  is bounded, and thus realizes only finitely many classes  $\beta_i^1, \dots, \beta_i^{n_i}$  in  $K^{\text{num}}(\text{coh}(X_v))$ . It follows that for each  $\gamma \in \Gamma$  we have  $\gamma \cdot \alpha_i \in \{\beta_i^1, \dots, \beta_i^{n_i}\}$ . So there are at most  $n_1 \cdots n_k$  possibilities for  $(\gamma \cdot \alpha_1, \dots, \gamma \cdot \alpha_k)$ . But  $(\gamma \cdot \alpha_1, \dots, \gamma \cdot \alpha_k)$  determines  $\gamma$  as  $\alpha_1, \dots, \alpha_k$  generate  $K^{\text{num}}(\text{coh}(X_v))$ . Hence  $|\Gamma| \leq n_1 \cdots n_k$ , and  $\Gamma$  is finite.

We can now construct an étale cover  $\pi : \tilde{U} \rightarrow U$  which is a principal  $\Gamma$ -bundle, and so has degree  $|\Gamma|$ , such that the  $\mathbb{K}$ -points of  $\tilde{U}$  are pairs  $(u, \iota)$  where  $u \in U(\mathbb{K})$  and  $\iota : K^{\text{num}}(\text{coh}(X_u)) \rightarrow K^{\text{num}}(\text{coh}(X_v))$  is an isomorphism induced by parallel transport along some path from  $u$  to  $v$ , which is possible as  $U$  is connected, and  $\Gamma$  acts freely on  $\tilde{U}(\mathbb{K})$  by  $\gamma : (u, \iota) \mapsto (u, \gamma \circ \iota)$ , so that the  $\Gamma$ -orbits correspond to points  $u \in U(\mathbb{K})$ . Then for  $\tilde{u} = (u, \iota)$  we have  $\tilde{X}_{\tilde{u}} = X_u$ , with canonical isomorphism  $\iota : K^{\text{num}}(\text{coh}(\tilde{X}_{\tilde{u}})) \rightarrow K^{\text{num}}(\text{coh}(X_v))$ .  $\square$

## CHAPTER 5

### Statements of main results

Let  $X$  be a Calabi–Yau 3-fold over the complex numbers  $\mathbb{C}$ , and  $\mathcal{O}_X(1)$  a very ample line bundle over  $X$ . For the rest of the book, our definition of Calabi–Yau 3-fold includes the assumption that  $H^1(\mathcal{O}_X) = 0$ , except where we explicitly allow otherwise. Remarks 5.1 and 5.2 discuss the reasons for assuming  $\mathbb{K} = \mathbb{C}$  and  $H^1(\mathcal{O}_X) = 0$ . Write  $\mathrm{coh}(X)$  for the abelian category of coherent sheaves on  $X$ , and  $K(\mathrm{coh}(X))$  for the numerical Grothendieck group of  $\mathrm{coh}(X)$ . Let  $(\tau, G, \leq)$  be the stability condition on  $\mathrm{coh}(X)$  of Gieseker stability with respect to  $\mathcal{O}_X(1)$ , as in Example 3.8. If  $E$  is a coherent sheaf on  $X$  then the class  $[E] \in K(\mathrm{coh}(X))$  is in effect the Chern character  $\mathrm{ch}(E)$  of  $E$ .

Write  $\mathfrak{M}$  for the moduli stack of coherent sheaves  $E$  on  $X$ . It is an Artin  $\mathbb{C}$ -stack, locally of finite type. For  $\alpha \in K(\mathrm{coh}(X))$ , write  $\mathfrak{M}^\alpha$  for the open and closed substack of  $E$  with  $[E] = \alpha$  in  $K(\mathrm{coh}(X))$ . (In §3 we used the notation  $\mathfrak{M}_{\mathrm{coh}(X)}, \mathfrak{M}_{\mathrm{coh}(X)}^\alpha$  for  $\mathfrak{M}, \mathfrak{M}^\alpha$ , but we now drop the subscript  $\mathrm{coh}(X)$  for brevity). Write  $\mathfrak{M}_{\mathrm{ss}}^\alpha(\tau), \mathfrak{M}_{\mathrm{st}}^\alpha(\tau)$  for the substacks of  $\tau$ -(semi)stable sheaves  $E$  in class  $[E] = \alpha$ , which are finite type open substacks of  $\mathfrak{M}^\alpha$ . Write  $\mathcal{M}_{\mathrm{ss}}^\alpha(\tau), \mathcal{M}_{\mathrm{st}}^\alpha(\tau)$  for the coarse moduli schemes of  $\tau$ -(semi)stable sheaves  $E$  with  $[E] = \alpha$ . Then  $\mathcal{M}_{\mathrm{ss}}^\alpha(\tau)$  is a projective  $\mathbb{C}$ -scheme whose points correspond to S-equivalence classes of  $\tau$ -semistable sheaves, and  $\mathcal{M}_{\mathrm{st}}^\alpha(\tau)$  is an open subscheme of  $\mathcal{M}_{\mathrm{ss}}^\alpha(\tau)$  whose points correspond to isomorphism classes of  $\tau$ -stable sheaves.

We divide our main results into four sections §5.1–§5.4. Section 5.1 studies local properties of the moduli stack  $\mathfrak{M}$  of coherent sheaves on  $X$ . We first show that  $\mathfrak{M}$  is Zariski locally isomorphic to the moduli stack  $\mathfrak{Vect}$  of algebraic vector bundles on  $X$ . Then we use gauge theory on complex vector bundles and transcendental complex analytic methods to show that an atlas for  $\mathfrak{M}$  may be written locally in the complex analytic topology as  $\mathrm{Crit}(f)$  for  $f : U \rightarrow \mathbb{C}$  a holomorphic function on a complex manifold  $U$ . The proofs of Theorems 5.3, 5.4, and 5.5 in §5.1 are postponed to §8–§9.

Section 5.2 uses the results of §5.1 and the Milnor fibre description of Behrend functions in §4.2 to prove two identities (5.2)–(5.3) for the Behrend function  $\nu_{\mathfrak{M}}$  of the moduli stack  $\mathfrak{M}$ . The proof of Theorem 5.11 in §5.2 is given in §10. Section 5.3, the central part of our book, constructs a Lie algebra morphism  $\tilde{\Psi} : \mathrm{SF}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{M}) \rightarrow \tilde{L}(X)$ , which modifies  $\Psi$  in §3.4 by inserting the Behrend function  $\nu_{\mathfrak{M}}$  as a weight. Then we use  $\tilde{\Psi}$  to define generalized Donaldson–Thomas invariants  $\tilde{DT}^\alpha(\tau)$ , and show they satisfy a transformation law under change of stability condition  $\tau$ . Theorem 5.14 in §5.3 is proved in §11.

Section 5.4 shows that our new invariants  $\tilde{DT}^\alpha(\tau)$  are unchanged under deformations of the underlying Calabi–Yau 3-fold  $X$ . We do this by first defining auxiliary invariants  $PI^{\alpha,n}(\tau')$  counting ‘stable pairs’  $s : \mathcal{O}_X(-n) \rightarrow E$  for  $E \in \mathrm{coh}(X)$  and  $n \gg 0$ , similar to Pandharipande–Thomas invariants [86]. We show the moduli

space of stable pairs  $\mathcal{M}_{\text{stp}}^{\alpha,n}(\tau')$  is a projective scheme with a symmetric obstruction theory, and deduce that  $PI^{\alpha,n}(\tau')$  is unchanged under deformations of  $X$ . We prove a formula for  $PI^{\alpha,n}(\tau')$  in terms of the  $\bar{DT}^{\beta}(\tau)$ , and use this to deduce that  $\bar{DT}^{\alpha}(\tau)$  is deformation-invariant. The proofs of Theorems 5.22, 5.23, 5.25, and 5.27 in §5.4 are postponed to §12–§13.

REMARK 5.1. We will use the assumption that the base field  $\mathbb{K} = \mathbb{C}$  for the Calabi–Yau 3-fold  $X$  in three main ways in the rest of the book:

- (a) Theorems 5.4 and 5.5 in §5.1 are proved using gauge theory and transcendental complex analytic methods, and work only over  $\mathbb{K} = \mathbb{C}$ . These are used to prove the Behrend function identities (5.2)–(5.3) in §5.2, which are vital for much of §5.3–§7, including the wall crossing formula (5.14) for the  $\bar{DT}^{\alpha}(\tau)$ , and the relation (5.17) between  $PI^{\alpha,n}(\tau')$ ,  $\bar{DT}^{\alpha}(\tau)$ . In examples we often compute  $PI^{\alpha,n}(\tau')$  and then use (5.17) to find  $\bar{DT}^{\alpha}(\tau)$ .
- (b) As in §4.5, when  $\mathbb{K} = \mathbb{C}$  the Chern character embeds  $K^{\text{num}}(\text{coh}(X))$  in  $H^{\text{even}}(X; \mathbb{Q})$ , and we use this to show  $K^{\text{num}}(\text{coh}(X))$  is unchanged under deformations of  $X$ . This is important for the results in §5.4 that  $\bar{DT}^{\alpha}(\tau)$  and  $PI^{\alpha,n}(\tau')$  for  $\alpha \in K^{\text{num}}(\text{coh}(X))$  are invariant under deformations of  $X$  even to make sense. Also, in §6 we use this embedding in  $H^{\text{even}}(X; \mathbb{Q})$  as a convenient way of describing classes in  $K^{\text{num}}(\text{coh}(X))$ .
- (c) Our notion of *compactly embeddable* in §6.7 is complex analytic and does not make sense for general  $\mathbb{K}$ .

We now discuss the extent to which the results can be extended to other fields  $\mathbb{K}$ . Thomas’ original definition (4.15) of  $DT^{\alpha}(\tau)$ , and our definition (5.15) of the pair invariants  $PI^{\alpha,n}(\tau')$ , are both valid over general algebraically closed fields  $\mathbb{K}$ . Apart from problem (b) with  $K^{\text{num}}(\text{coh}(X))$  above, the proofs of deformation-invariance of  $DT^{\alpha}(\tau)$ ,  $PI^{\alpha,n}(\tau')$  are also valid over general  $\mathbb{K}$ .

As noted after Theorem 2.4, constructible functions methods fail for  $\mathbb{K}$  of positive characteristic. Because of this, the alternative descriptions (4.16), (5.16) for  $DT^{\alpha}(\tau)$ ,  $PI^{\alpha,n}(\tau')$  as weighted Euler characteristics, and the definition of  $\bar{DT}^{\alpha}(\tau)$  in §5.3, are valid for algebraically closed fields  $\mathbb{K}$  of characteristic zero.

The authors believe that the Behrend function identities (5.2)–(5.3) should hold over algebraically closed fields  $\mathbb{K}$  of characteristic zero; Question 5.12(a) suggests a starting point for a purely algebraic proof of (5.2)–(5.3). This would resolve (a) above, and probably also (c), because we only need the notion of ‘compactly embeddable’ as our complex analytic proof of (5.2)–(5.3) requires  $X$  compact; an algebraic proof of (5.2)–(5.3) would presumably also work for compactly supported sheaves on a noncompact  $X$ .

For (b), one approach valid over general  $\mathbb{K}$  which is deformation-invariant is to count sheaves with fixed Hilbert polynomial, as in Thomas [100], rather than with fixed class in  $K^{\text{num}}(\text{coh}(X))$ . It seems likely that  $K^{\text{num}}(\text{coh}(X))$  is deformation-invariant for more general  $\mathbb{K}$ , so there may not be a problem.

REMARK 5.2. We will use the assumption  $H^1(\mathcal{O}_X) = 0$  for the Calabi–Yau 3-fold  $X$  in four different ways in the rest of the book:

- (i) Theorem 5.3 shows that the moduli stack  $\mathfrak{M}$  of coherent sheaves is locally isomorphic to the moduli stack  $\mathfrak{Vect}$  of vector bundles. The proof uses Seidel–Thomas twists by  $\mathcal{O}_X(-n)$ , and is only valid if  $\mathcal{O}_X(-n)$  is a spherical object in  $D^b(\text{coh}(X))$ , that is, if  $H^1(\mathcal{O}_X) = 0$ . Theorem 5.3 is needed to show that

the Behrend function identities (5.2)–(5.3) hold on  $\mathfrak{M}$  as well as on  $\mathfrak{Vect}$ , and (5.2)–(5.3) are essential for most of §5.3–§5.4.

- (ii) If  $H^1(\mathcal{O}_X) = 0$  then as in §4.5  $K^{\text{num}}(\text{coh}(X))$  is unchanged under deformations of  $X$ , which makes sense of the idea that  $\bar{DT}^\alpha(\tau)$  is deformation-invariant for  $\alpha \in K^{\text{num}}(\text{coh}(X))$ .
- (iii) In §6.3 we use that if  $H^1(\mathcal{O}_X) = 0$  then Hilbert schemes  $\text{Hilb}^d(X)$  are open subschemes of moduli schemes of sheaves on  $X$ .
- (iv) In §6.4 and §6.6 we use that if  $H^1(\mathcal{O}_X) = 0$  then every  $\beta \in H^2(X; \mathbb{Z})$  is  $c_1(L)$  for some line bundle  $L$  and the map  $E \mapsto E \otimes L$  for  $E \in \text{coh}(X)$  to deduce symmetries of the  $\bar{DT}^\alpha(\tau)$ .

Of these, in (i) Theorem 5.3 is false if  $H^1(\mathcal{O}_X) \neq 0$ , but nonetheless the authors expect (5.2)–(5.3) will be true if  $H^1(\mathcal{O}_X) \neq 0$ , and this is the important thing for most of our theory. Question 5.12(a) suggests a route towards a purely algebraic proof of (5.2)–(5.3), which is likely not to require  $H^1(\mathcal{O}_X) = 0$ .

For (ii), if  $H^1(\mathcal{O}_X) \neq 0$  then in §4.5 the Chern character induces a map  $\text{ch} : K^{\text{num}}(\text{coh}(X)) \rightarrow \Lambda_X$  which is injective but may not be surjective, where  $\Lambda_X \subset H^{\text{even}}(X; \mathbb{Q})$  is the lattice in (4.20). Let us identify  $K^{\text{num}}(\text{coh}(X))$  with its image under  $\text{ch}$  in  $\Lambda_X$ , and then extend the definition of  $\bar{DT}^\alpha(\tau)$  to all  $\alpha \in \Lambda_X$  by setting  $\bar{DT}^\alpha(\tau) = 0$  for  $\alpha \in \Lambda_X \setminus \text{ch}(K^{\text{num}}(\text{coh}(X)))$ . Then  $\bar{DT}^\alpha(\tau)$  is defined for  $\alpha \in \Lambda_X$ , where  $\Lambda_X$  is deformation-invariant, and we claim that  $\bar{DT}^\alpha(\tau)$  will then be deformation-invariant. Part (iii) is false if  $H^1(\mathcal{O}_X) \neq 0$ .

Generalizing our theory to the case  $H^1(\mathcal{O}_X) \neq 0$  is not very interesting anyway, as most invariants  $\bar{DT}^\alpha(\tau)$  (as we have defined them) will be automatically zero. If  $\dim H^1(\mathcal{O}_X) = g > 0$  then there is a  $T^{2g}$  family of flat line bundles on  $X$  up to isomorphism, which is a group under  $\otimes$ , with identity  $\mathcal{O}_X$ . Suppose  $\alpha \in K^{\text{num}}(\text{coh}(X))$  with  $\text{rank } \alpha > 0$ , so that  $\tau$ -semistable sheaves in class  $\alpha$  are torsion-free. Then  $E \mapsto E \otimes L$  for  $E \in \mathcal{M}_{\text{ss}}^\alpha(\tau)$  and  $L \in T^{2g}$  defines an action of  $T^{2g}$  on  $\mathcal{M}_{\text{ss}}^\alpha(\tau)$  which is essentially free. As  $\bar{DT}^\alpha(\tau)$  is a weighted Euler characteristic of  $\mathcal{M}_{\text{ss}}^\alpha(\tau)$ , and each orbit of  $T^{2g}$  in  $\mathcal{M}_{\text{ss}}^\alpha(\tau)$  is a copy of  $T^{2g}$  with Euler characteristic zero, it follows that  $\bar{DT}^\alpha(\tau) = 0$  when  $\text{rank } \alpha > 0$ .

One solution to this is to consider sheaves with *fixed determinant*, as in Thomas [100]. But this is not nicely compatible with viewing  $\text{coh}(X)$  as an abelian category, or with our treatment of wall-crossing formulae. In (iv) above, if  $H^1(\mathcal{O}_X) \neq 0$  and  $\beta \in H^2(X; \mathbb{Z})$  is not of the form  $c_1(L)$  for a holomorphic line bundle  $L$ , then for any  $\alpha \in K^{\text{num}}(\text{coh}(X))$  which would be moved by the symmetry of  $H^{\text{even}}(X; \mathbb{Q})$  corresponding to  $\beta$  we must have  $\bar{DT}^\alpha(\tau) = 0$  as above. Thus (iv) should still hold when  $H^1(\mathcal{O}_X) \neq 0$ , but for trivial reasons.

### 5.1. Local description of the moduli of coherent sheaves

We begin by recalling some facts about moduli spaces and moduli stacks of coherent sheaves and vector bundles over smooth projective  $\mathbb{K}$ -schemes, to establish notation. Let  $\mathbb{K}$  be an algebraically closed field, and  $X$  a smooth projective  $\mathbb{K}$ -scheme of dimension  $m$ . In Theorem 5.3 we will take  $X$  to be a Calabi–Yau  $m$ -fold over general  $\mathbb{K}$ , and from Theorem 5.4 onwards we will restrict to  $\mathbb{K} = \mathbb{C}$ ,  $m = 3$  and  $X$  a Calabi–Yau 3-fold, except for Theorems 5.22, 5.23 and 5.25, which work over general  $\mathbb{K}$ .

Some good references are Hartshorne [40, §II.5] on coherent sheaves, Huybrechts and Lehn [44] on coherent sheaves and moduli schemes, and Laumon and

Moret-Bailly [67] on algebraic spaces and stacks. When we say a coherent sheaf  $E$  is *simple* we mean that  $\mathrm{Hom}(E, E) \cong \mathbb{C}$ . (Beware that some authors use ‘simple’ with the different meaning ‘has no nontrivial subobjects’. An alternative word for ‘simple’ in our sense would be *Schurian*.) By an *algebraic vector bundle* we mean a locally free coherent sheaf on  $X$  of rank  $l \geq 0$ . (See Hartshorne [40, Ex. II.5.18] for an alternative definition of vector bundles  $E$  as a morphism of schemes  $\pi : E \rightarrow X$  with fibre  $\mathbb{K}^l$  and with extra structure, and an explanation of why these are in 1–1 correspondence with locally free sheaves.)

Write  $\mathfrak{M}$  and  $\mathfrak{Vect}$  for the moduli stacks of coherent sheaves and algebraic vector bundles on  $X$ , respectively. By Laumon and Moret-Bailly [67, §§2.4.4, 3.4.4 & 4.6.2] using results of Grothendieck [34, 35], they are Artin  $\mathbb{K}$ -stacks, locally of finite type, and  $\mathfrak{Vect}$  is an open  $\mathbb{K}$ -substack of  $\mathfrak{M}$ . Write  $\mathcal{M}_{\mathrm{si}}$  and  $\mathcal{Vect}_{\mathrm{si}}$  for the coarse moduli spaces of simple coherent sheaves and simple algebraic vector bundles. By Altman and Kleiman [1, Th. 7.4] they are algebraic  $\mathbb{K}$ -spaces, locally of finite type. Also  $\mathcal{Vect}_{\mathrm{si}}$  is an open subspace of  $\mathcal{M}_{\mathrm{si}}$ .

Here  $\mathfrak{M}, \mathfrak{Vect}, \mathcal{M}_{\mathrm{si}}, \mathcal{Vect}_{\mathrm{si}}$  being *locally of finite type* means roughly only that they are locally finite-dimensional; in general  $\mathfrak{M}, \dots, \mathcal{Vect}_{\mathrm{si}}$  are neither proper (essentially, compact), nor separated (essentially, Hausdorff), nor of finite type (finite type is necessary for invariants such as Euler characteristics to be well-defined; for the purposes of this book, ‘finite type’ means something like ‘measurable’). These results tell us little about the global geometry of  $\mathfrak{M}, \dots, \mathcal{Vect}_{\mathrm{si}}$ . But we do have some understanding of their local geometry.

For the moduli stack  $\mathfrak{M}$  of coherent sheaves on  $X$ , writing  $\mathfrak{M}(\mathbb{K})$  for the set of geometric points of  $\mathfrak{M}$ , as in Definition 2.1, elements of  $\mathfrak{M}(\mathbb{K})$  are just isomorphism classes  $[E]$  of coherent sheaves  $E$  on  $X$ . Fix some such  $E$ . Then the stabilizer group  $\mathrm{Iso}_{\mathfrak{M}}([E])$  in  $\mathfrak{M}$  is isomorphic as an algebraic  $\mathbb{K}$ -group to the automorphism group  $\mathrm{Aut}(E)$ , and the Zariski tangent space  $T_{[E]}\mathfrak{M}$  to  $\mathfrak{M}$  at  $[E]$  is isomorphic to  $\mathrm{Ext}^1(E, E)$ , and the action of  $\mathrm{Iso}_{\mathfrak{M}}([E])$  on  $T_{[E]}\mathfrak{M}$  corresponds to the action of  $\mathrm{Aut}(E)$  on  $\mathrm{Ext}^1(E, E)$  by  $\gamma : \epsilon \mapsto \gamma \circ \epsilon \circ \gamma^{-1}$ .

Since  $\mathrm{Iso}_{\mathfrak{M}}([E])$  is the group of invertible elements in the finite-dimensional  $\mathbb{K}$ -algebra  $\mathrm{Aut}(E)$ , it is an affine  $\mathbb{K}$ -group. Hence the Artin  $\mathbb{K}$ -stack  $\mathfrak{M}$  has *affine geometric stabilizers*, in the sense of Definition 2.1. If  $S$  is a  $\mathbb{K}$ -scheme, then 1-morphisms  $\phi : S \rightarrow \mathfrak{M}$  are just families of coherent sheaves on  $X$  parametrized by  $S$ , that is, they are coherent sheaves  $E_S$  on  $X \times S$  flat over  $S$ . A 1-morphism  $\phi : S \rightarrow \mathfrak{M}$  is an *atlas* for some open substack  $\mathfrak{U} \subset \mathfrak{M}$ , if and only if  $E_S$  is a *versal family* of sheaves such that  $\{[E_s] : s \in S(\mathbb{K})\} = \mathfrak{U}(\mathbb{K}) \subseteq \mathfrak{M}(\mathbb{K})$ .

For the algebraic  $\mathbb{K}$ -spaces  $\mathcal{M}_{\mathrm{si}}, \mathcal{Vect}_{\mathrm{si}}$ , elements of  $\mathcal{M}_{\mathrm{si}}(\mathbb{K}), \mathcal{Vect}_{\mathrm{si}}(\mathbb{K})$  are isomorphism classes  $[E]$  of simple coherent sheaves or vector bundles  $E$ . When  $\mathbb{K} = \mathbb{C}$ ,  $\mathcal{M}_{\mathrm{si}}, \mathcal{Vect}_{\mathrm{si}}$  are *complex algebraic spaces*, and so  $\mathcal{M}_{\mathrm{si}}(\mathbb{C}), \mathcal{Vect}_{\mathrm{si}}(\mathbb{C})$  have the induced structure of *complex analytic spaces*. Direct constructions of  $\mathcal{Vect}_{\mathrm{si}}(\mathbb{C})$  as a complex analytic space parametrizing complex analytic holomorphic vector bundles are given by Lübke and Okonek [72] and Kosarew and Okonek [65]. Miyajima [79, Th. 3] shows that these complex analytic space structures on  $\mathcal{Vect}_{\mathrm{si}}(\mathbb{C})$  coming from the algebraic side [1] and the analytic side [65, 72] are equivalent.

Our first result works for Calabi–Yau  $m$ -folds  $X$  of any dimension  $m \geq 1$ , and over any algebraically closed field  $\mathbb{K}$ . It is proved in §8. The authors are grateful to Tom Bridgeland for suggesting the approach used to prove Theorem 5.3.

**THEOREM 5.3.** *Let  $\mathbb{K}$  be an algebraically closed field, and  $X$  a projective Calabi–Yau  $m$ -fold over  $\mathbb{K}$  for  $m \geq 1$ , with  $H^i(\mathcal{O}_X) = 0$  for  $0 < i < m$ , and  $\mathfrak{M}, \mathfrak{Vect}, \mathcal{M}_{\text{si}}, \text{Vect}_{\text{si}}$  be as above. Let  $\mathfrak{U}$  be an open, finite type substack of  $\mathfrak{M}$ . Then there exists an open substack  $\mathfrak{V}$  in  $\mathfrak{Vect}$ , and a 1-isomorphism  $\varphi : \mathfrak{U} \rightarrow \mathfrak{V}$  of Artin  $\mathbb{K}$ -stacks. Similarly, let  $U$  be an open, finite type subspace of  $\mathcal{M}_{\text{si}}$ . Then there exists an open subspace  $V$  in  $\text{Vect}_{\text{si}}$  and an isomorphism  $\psi : U \rightarrow V$  of algebraic  $\mathbb{K}$ -spaces. That is,  $\mathfrak{M}, \mathcal{M}_{\text{si}}$  are locally isomorphic to  $\mathfrak{Vect}, \text{Vect}_{\text{si}}$ , in the Zariski topology. The isomorphisms  $\varphi, \psi$  are constructed as the composition of  $m$  Seidel–Thomas twists by  $\mathcal{O}_X(-n)$  for  $n \gg 0$  and a shift  $[-m]$ .*

We now restrict to Calabi–Yau 3-folds over  $\mathbb{C}$ , which includes the assumption  $H^1(\mathcal{O}_X) = 0$ . Our next two results, Theorems 5.4 and 5.5, are proved in §9. Roughly, they say that moduli spaces of coherent sheaves on Calabi–Yau 3-folds over  $\mathbb{C}$  can be written locally in the form  $\text{Crit}(f)$ , for  $f$  a holomorphic function on a complex manifold. This is a partial answer to the question of Behrend quoted at the beginning of §4.4. Because of Theorems 5.4 and 5.5, we can use the Milnor fibre formula for the Behrend function of  $\text{Crit}(f)$  in §4.2 to study the Behrend function  $\nu_{\mathfrak{M}}$ , and this will be vital in proving Theorem 5.11.

**THEOREM 5.4.** *Let  $X$  be a Calabi–Yau 3-fold over  $\mathbb{C}$ , and  $\mathcal{M}_{\text{si}}$  the coarse moduli space of simple coherent sheaves on  $X$ , so that  $\mathcal{M}_{\text{si}}(\mathbb{C})$  is the set of isomorphism classes  $[E]$  of simple coherent sheaves  $E$  on  $X$ , and is a complex analytic space. Then for each  $[E] \in \mathcal{M}_{\text{si}}(\mathbb{C})$  there exists a finite-dimensional complex manifold  $U$ , a holomorphic function  $f : U \rightarrow \mathbb{C}$ , and a point  $u \in U$  with  $f(u) = \text{df}|_u = 0$ , such that  $\mathcal{M}_{\text{si}}(\mathbb{C})$  near  $[E]$  is locally isomorphic as a complex analytic space to  $\text{Crit}(f)$  near  $u$ . We can take  $U$  to be an open neighbourhood of  $u = 0$  in the finite-dimensional complex vector space  $\text{Ext}^1(E, E)$ .*

Our next result generalizes Theorem 5.4 from simple to arbitrary coherent sheaves, and from algebraic spaces to Artin stacks.

**THEOREM 5.5.** *Let  $X$  be a Calabi–Yau 3-fold over  $\mathbb{C}$ , and  $\mathfrak{M}$  the moduli stack of coherent sheaves on  $X$ . Suppose  $E$  is a coherent sheaf on  $X$ , so that  $[E] \in \mathfrak{M}(\mathbb{C})$ . Let  $G$  be a maximal compact subgroup in  $\text{Aut}(E)$ , and  $G^{\mathbb{C}}$  its complexification. Then  $G^{\mathbb{C}}$  is an algebraic  $\mathbb{C}$ -subgroup of  $\text{Aut}(E)$ , a maximal reductive subgroup, and  $G^{\mathbb{C}} = \text{Aut}(E)$  if and only if  $\text{Aut}(E)$  is reductive.*

*There exists a quasiprojective  $\mathbb{C}$ -scheme  $S$ , an action of  $G^{\mathbb{C}}$  on  $S$ , a point  $s \in S(\mathbb{C})$  fixed by  $G^{\mathbb{C}}$ , and a 1-morphism of Artin  $\mathbb{C}$ -stacks  $\Phi : [S/G^{\mathbb{C}}] \rightarrow \mathfrak{M}$ , which is smooth of relative dimension  $\dim \text{Aut}(E) - \dim G^{\mathbb{C}}$ , where  $[S/G^{\mathbb{C}}]$  is the quotient stack, such that  $\Phi(sG^{\mathbb{C}}) = [E]$ , the induced morphism on stabilizer groups  $\Phi_* : \text{Iso}_{[S/G^{\mathbb{C}}]}(sG^{\mathbb{C}}) \rightarrow \text{Iso}_{\mathfrak{M}}([E])$  is the natural morphism  $G^{\mathbb{C}} \hookrightarrow \text{Aut}(E) \cong \text{Iso}_{\mathfrak{M}}([E])$ , and  $\text{d}\Phi|_{sG^{\mathbb{C}}} : T_s S \cong T_{sG^{\mathbb{C}}} [S/G^{\mathbb{C}}] \rightarrow T_{[E]} \mathfrak{M} \cong \text{Ext}^1(E, E)$  is an isomorphism. Furthermore,  $S$  parametrizes a formally versal family  $(S, \mathcal{D})$  of coherent sheaves on  $X$ , equivariant under the action of  $G^{\mathbb{C}}$  on  $S$ , with fibre  $\mathcal{D}_s \cong E$  at  $s$ . If  $\text{Aut}(E)$  is reductive then  $\Phi$  is étale.*

*Write  $S_{\text{an}}$  for the complex analytic space underlying the  $\mathbb{C}$ -scheme  $S$ . Then there exists an open neighbourhood  $U$  of 0 in  $\text{Ext}^1(E, E)$  in the analytic topology, a holomorphic function  $f : U \rightarrow \mathbb{C}$  with  $f(0) = \text{df}|_0 = 0$ , an open neighbourhood  $V$  of  $s$  in  $S_{\text{an}}$ , and an isomorphism of complex analytic spaces  $\Xi : \text{Crit}(f) \rightarrow V$ , such that  $\Xi(0) = s$  and  $\text{d}\Xi|_0 : T_0 \text{Crit}(f) \rightarrow T_s V$  is the inverse of  $\text{d}\Phi|_{sG^{\mathbb{C}}} : T_s S \rightarrow$*

$\text{Ext}^1(E, E)$ . Moreover we can choose  $U, f, V$  to be  $G^c$ -invariant, and  $\Xi$  to be  $G^c$ -equivariant.

Here the first paragraph is immediate, and the second has a straightforward proof in §9.3, similar to parts of the proof of Luna's Etale Slice Theorem [74]; the case in which  $\text{Aut}(E)$  is reductive, so that  $G^c = \text{Aut}(E)$  and  $\Phi$  is étale, is a fairly direct consequence of the Etale Slice Theorem. The third paragraph is what takes the hard work in the proof. Composing the projection  $\pi : S \rightarrow [S/G^c]$  with  $\Phi$  gives a smooth 1-morphism  $\Phi \circ \pi : S \rightarrow \mathfrak{M}$ , which is locally an atlas for  $\mathfrak{M}$  near  $[E]$ . Thus, Theorem 5.5 says that we can write an atlas for  $\mathfrak{M}$  in the form  $\text{Crit}(f)$ , locally in the analytic topology, where  $f : U \rightarrow \mathbb{C}$  is a holomorphic function on a complex manifold.

By Theorem 5.3, it suffices to prove Theorems 5.4 and 5.5 with  $\mathcal{V}ect_{\text{si}}, \mathfrak{Vect}$  in place of  $\mathcal{M}_{\text{si}}, \mathfrak{M}$ . We do this using gauge theory, motivated by an idea of Donaldson and Thomas [20, §3], [100, §2]. Let  $E \rightarrow X$  be a fixed complex (not holomorphic) vector bundle over  $X$ . Write  $\mathcal{A}$  for the infinite-dimensional affine space of smooth semiconnections ( $\bar{\partial}$ -operators) on  $E$ , and  $\mathcal{A}_{\text{si}}$  for the open subset of simple semiconnections, and  $\mathcal{G}$  for the infinite-dimensional Lie group of smooth gauge transformations of  $E$ . Note that we do not assume semiconnections are integrable. Then  $\mathcal{G}$  acts on  $\mathcal{A}$  and  $\mathcal{A}_{\text{si}}$ , and  $\mathcal{B} = \mathcal{A}/\mathcal{G}$  is the space of gauge-equivalence classes of semiconnections on  $E$ .

The subspace  $\mathcal{B}_{\text{si}} = \mathcal{A}_{\text{si}}/\mathcal{G}$  of simple semiconnections should be an infinite-dimensional complex manifold. Inside  $\mathcal{B}_{\text{si}}$  is the subspace  $\mathcal{V}_{\text{si}}$  of integrable simple semiconnections, which should be a finite-dimensional complex analytic space. Now the moduli scheme  $\mathcal{V}ect_{\text{si}}$  of simple complex algebraic vector bundles has an underlying complex analytic space  $\mathcal{V}ect_{\text{si}}(\mathbb{C})$ ; the idea is that  $\mathcal{V}_{\text{si}}$  is naturally isomorphic as a complex analytic space to the open subset of  $\mathcal{V}ect_{\text{si}}(\mathbb{C})$  of algebraic vector bundles with underlying complex vector bundle  $E$ .

We fix  $\bar{\partial}_E$  in  $\mathcal{A}$  coming from a holomorphic vector bundle structure on  $E$ . Then points in  $\mathcal{A}$  are of the form  $\bar{\partial}_E + A$  for  $A \in C^\infty(\text{End}(E) \otimes_{\mathbb{C}} \Lambda^{0,1} T^*X)$ , and  $\bar{\partial}_E + A$  makes  $E$  into a holomorphic vector bundle if  $F_A^{0,2} = \bar{\partial}_E A + A \wedge A$  is zero in  $C^\infty(\text{End}(E) \otimes_{\mathbb{C}} \Lambda^{0,2} T^*X)$ . Thus, the moduli space of holomorphic vector bundle structures on  $E$  is isomorphic to  $\{\bar{\partial}_E + A \in \mathcal{A} : F_A^{0,2} = 0\}/\mathcal{G}$ . Thomas observes that when  $X$  is a Calabi–Yau 3-fold, there is a natural holomorphic function  $CS : \mathcal{A} \rightarrow \mathbb{C}$  called the *holomorphic Chern–Simons functional*, invariant under  $\mathcal{G}$  up to addition of constants, such that  $\{\bar{\partial}_E + A \in \mathcal{A} : F_A^{0,2} = 0\}$  is the critical locus of  $CS$ . Thus,  $\mathcal{V}_{\text{si}}$ , and hence  $(\mathcal{V}ect_{\text{si}})(\mathbb{C})$ , is (informally) locally the set of critical points of a holomorphic function  $CS$  on an infinite-dimensional complex manifold  $\mathcal{B}_{\text{si}}$ .

In the proof of Theorem 5.4 in §9, when  $\bar{\partial}_E$  is simple, we show using results of Miyajima [79] that there is a finite-dimensional complex submanifold  $Q_\epsilon$  of  $\mathcal{A}$  containing  $\bar{\partial}_E$ , such that  $\mathcal{V}ect_{\text{si}}(\mathbb{C})$  near  $[(E, \bar{\partial}_E)]$  is isomorphic as a complex analytic space to  $\text{Crit}(CS|_{Q_\epsilon})$  near  $\bar{\partial}_E$ , where  $CS|_{Q_\epsilon} : Q_\epsilon \rightarrow \mathbb{C}$  is a holomorphic function on the finite-dimensional complex manifold  $Q_\epsilon$ . We also show  $Q_\epsilon$  is biholomorphic to an open neighbourhood  $U$  of 0 in  $\text{Ext}^1(E, E)$ .

In the proof of Theorem 5.5 in §9, without assuming  $\bar{\partial}_E$  simple, we show that a local atlas  $S$  for  $\mathfrak{Vect}$  near  $[(E, \bar{\partial}_E)]$  is isomorphic as a complex analytic space to  $\text{Crit}(CS|_{Q_\epsilon})$  near  $\bar{\partial}_E$ . The new issues in Theorem 5.5 concern to what extent we can take  $S, Q_\epsilon$  and  $CS|_{Q_\epsilon} : Q_\epsilon \rightarrow \mathbb{C}$  to be invariant under  $\text{Aut}(E, \bar{\partial}_E)$ . In fact, in Theorem 5.5 we would have preferred to take  $S, U, V, f$  invariant under the full

group  $\text{Aut}(E)$ , rather than just under the maximal reductive subgroup  $G^c$ . But we expect this is not possible.

On the algebraic geometry side, the choice of  $S, \Phi, \mathcal{D}$  in the second paragraph of Theorem 5.5, to construct  $S$  we use ideas from the proof of Luna's Etale Slice Theorem [74], which works only for reductive groups, so we can make  $S$  invariant under at most a maximal reductive subgroup  $G^c$  in  $\text{Aut}(E)$ . On the gauge theory side, constructing  $Q_\epsilon$  involves a *slice*  $\mathcal{S}_E = \{\bar{\partial}_E + A : \bar{\partial}_E^* A = 0\}$  to the action of  $\mathcal{G}$  in  $\mathcal{A}$  at  $\bar{\partial}_E \in \mathcal{A}$ , where  $\bar{\partial}_E^*$  is defined using choices of Hermitian metrics  $h_X, h_E$  on  $X$  and  $E$ . In general we cannot make  $\mathcal{S}_E$  invariant under  $\text{Aut}(E, \bar{\partial}_E)$ . The best we can do is to choose  $h_E$  invariant under a maximal compact subgroup  $G$  of  $\text{Aut}(E, \bar{\partial}_E)$ . Then  $\mathcal{S}_E$  is invariant under  $G$ , and hence under  $G^c$  as  $\mathcal{S}_E$  is a closed complex submanifold.

We can improve the group-invariance in Theorem 5.5 if we restrict to moduli stacks of *semistable* sheaves.

**COROLLARY 5.6.** *Let  $X$  be a Calabi–Yau 3-fold over  $\mathbb{C}$ . Write  $\tau$  for Gieseker stability of coherent sheaves on  $X$  w.r.t. some ample line bundle  $\mathcal{O}_X(1)$ , and  $\mathfrak{M}_{\text{ss}}^\alpha(\tau)$  for the moduli stack of  $\tau$ -semistable sheaves with Chern character  $\alpha$ . It is an open Artin  $\mathbb{C}$ -substack of  $\mathfrak{M}$ .*

*Then for each  $[E] \in \mathfrak{M}_{\text{ss}}^\alpha(\tau)(\mathbb{C})$ , there exists an affine  $\mathbb{C}$ -scheme  $S$  with associated complex analytic space  $S_{\text{an}}$ , a point  $s \in S_{\text{an}}$ , a reductive affine algebraic  $\mathbb{C}$ -group  $H$  acting on  $S$ , an étale morphism  $\Phi : [S/H] \rightarrow \mathfrak{M}_{\text{ss}}^\alpha(\tau)$  mapping  $H \cdot s \mapsto [E]$ , a finite-dimensional complex manifold  $U$  with a holomorphic action of  $H$ , an  $H$ -invariant holomorphic function  $f : U \rightarrow \mathbb{C}$ , an  $H$ -invariant open neighbourhood  $V$  of  $s$  in  $S_{\text{an}}$  in the analytic topology, and an  $H$ -equivariant isomorphism of complex analytic spaces  $\Xi : \text{Crit}(f) \rightarrow V$ .*

**PROOF.** Let  $[E] \in \mathfrak{M}_{\text{ss}}^\alpha(\tau)(\mathbb{C})$ . Then by properties of Gieseker stability,  $E$  has a Jordan–Hölder decomposition into pairwise non-isomorphic stable factors  $E_1, \dots, E_k$  with multiplicities  $m_1, \dots, m_k$  respectively, and  $E$  is an arbitrarily small deformation of  $E' = m_1 E_1 \oplus \dots \oplus m_k E_k$ . We have  $\text{Hom}(E_i, E_j) = 0$  if  $i \neq j$  and  $\text{Hom}(E_i, E_i) = \mathbb{C}$ . Thus  $\text{Aut}(E') \cong \prod_{i=1}^k \text{GL}(m_i, \mathbb{C})$ , which is the complexification of its maximal compact subgroup  $\prod_{i=1}^k \text{U}(m_i)$ . Applying Theorem 5.5 to  $E'$  with  $G = \prod_{i=1}^k \text{U}(m_i)$  and  $G^c = \text{Aut}(E')$  gives  $S, H = G^c, \Phi, U, f, V, \Xi$ . Since  $E$  is an arbitrarily small deformation of  $E'$  and  $\Phi$  is étale with  $\Phi_* : [H \cdot 0] \mapsto [E']$ ,  $[E]$  lies in the image under  $\Phi_*$  of any open neighbourhood of  $[H \cdot 0]$  in  $[S/H](\mathbb{C})$ , and thus  $[E]$  lies in the image of any  $H$ -invariant open neighbourhood  $V$  of 0 in  $S_{\text{an}}$ , in the analytic topology. Hence there exists  $s \in V \subseteq S_{\text{an}}$  with  $\Phi(H \cdot s) = [E]$ . The corollary follows.  $\square$

We can connect the last three results with the ideas on *perverse sheaves and vanishing cycles* sketched in §4.2. The first author would like to thank Kai Behrend, Jim Bryan and Balázs Szendrői for explaining the following ideas. Theorem 5.4 proves that the complex algebraic space  $\mathcal{M}_{\text{si}}$  may be written locally in the complex analytic topology as  $\text{Crit}(f)$ , for  $f : U \rightarrow \mathbb{C}$  holomorphic and  $U$  a complex manifold. Therefore Theorem 4.9 shows that locally in the complex analytic topology, there is a *perverse sheaf of vanishing cycles*  $\phi_f(\mathbb{Q}[\dim U - 1])$  supported on  $\text{Crit}(f) \cong \mathcal{M}_{\text{si}}$ , which projects to  $\nu_{\mathcal{M}_{\text{si}}}$  under  $\chi_{U_0}$ . So it is natural to ask whether we can glue these to get a global perverse sheaf on  $\mathcal{M}_{\text{si}}$ :



**QUESTION 5.7. (a)** Let  $X$  be a Calabi–Yau 3-fold over  $\mathbb{C}$ , and write  $\mathcal{M}_{\text{si}}$  for the coarse moduli space of simple coherent sheaves on  $X$ . Does there exist a natural perverse sheaf  $\mathcal{P}$  on  $\mathcal{M}_{\text{si}}$ , with  $\chi_{\mathcal{M}_{\text{si}}}(\mathcal{P}) = \nu_{\mathcal{M}_{\text{si}}}$ , which is locally isomorphic to  $\phi_f(\mathbb{Q}[\dim U - 1])$  for  $f, U$  as in Theorem 5.4?

**(b)** Is there also some Artin stack version of  $\mathcal{P}$  in **(a)** for the moduli stack  $\mathfrak{M}$ , locally isomorphic to  $\phi_f(\mathbb{Q}[\dim U - 1])$  for  $f, U$  as in Theorem 5.5?

The authors have no particular view on whether the answer is yes or no. One can also ask Question 5.7 for Saito’s mixed Hodge modules [92].

**REMARK 5.8. (i)** Question 5.7(a) could be tested by calculation in examples, such as the Hilbert scheme of  $n$  points on  $X$ . Partial results in this case can be found in Dimca and Szendrői [19] and Behrend, Bryan and Szendrői [4], see in particular [4, Rem. 3.2].

**(ii)** If the answer to Question 5.7(a) is yes, it would provide a way of *categorifying* (conventional) Donaldson–Thomas invariants  $DT^\alpha(\tau)$ . That is, if  $\alpha \in K(\text{coh}(X))$  with  $\mathcal{M}_{\text{ss}}^\alpha(\tau) = \mathcal{M}_{\text{st}}^\alpha(\tau)$ , as in §4.3, then we can restrict  $\mathcal{P}$  in Question 5.7(a) to a perverse sheaf on the open, proper subscheme  $\mathcal{M}_{\text{st}}^\alpha(\tau)$  in  $\mathcal{M}_{\text{si}}$ , and form the *hypercohomology*  $\mathbb{H}^*(\mathcal{M}_{\text{st}}^\alpha(\tau); \mathcal{P}|_{\mathcal{M}_{\text{st}}^\alpha(\tau)})$ , which is a finite-dimensional graded  $\mathbb{Q}$ -vector space. Then

$$\begin{aligned} \sum_{k \in \mathbb{Z}} (-1)^k \dim \mathbb{H}^k(\mathcal{M}_{\text{st}}^\alpha(\tau); \mathcal{P}|_{\mathcal{M}_{\text{st}}^\alpha(\tau)}) &= \chi(\mathcal{M}_{\text{st}}^\alpha(\tau), \chi_{\mathcal{M}_{\text{si}}}(\mathcal{P})|_{\mathcal{M}_{\text{st}}^\alpha(\tau)}) \\ &= \chi(\mathcal{M}_{\text{st}}^\alpha(\tau), \nu_{\mathcal{M}_{\text{si}}}|_{\mathcal{M}_{\text{st}}^\alpha(\tau)}) = \chi(\mathcal{M}_{\text{st}}^\alpha(\tau), \nu_{\mathcal{M}_{\text{st}}^\alpha(\tau)}) = DT^\alpha(\tau), \end{aligned} \quad (5.1)$$

where the first equality in (5.1) holds because we have a commutative diagram

$$\begin{array}{ccc} D_{\text{Con}}^b(\mathcal{M}_{\text{st}}^\alpha(\tau)) & \xrightarrow{R\pi_*} & D_{\text{Con}}^b(\text{Spec } \mathbb{C}) \\ \downarrow \chi_{\mathcal{M}_{\text{st}}^\alpha(\tau)} & & \downarrow \chi_{\text{Spec } \mathbb{C}} \\ \text{CF}_{\mathbb{Z}}^{\text{an}}(\mathcal{M}_{\text{st}}^\alpha(\tau)) & \xrightarrow{\text{CF}(\pi)} & \text{CF}_{\mathbb{Z}}^{\text{an}}(\text{Spec } \mathbb{C}). \end{array}$$

by (4.4), where  $\pi : \mathcal{M}_{\text{st}}^\alpha(\tau) \rightarrow \text{Spec } \mathbb{C}$  is the projection, which is proper as  $\mathcal{M}_{\text{st}}^\alpha(\tau)$  is proper, and the last equality in (5.1) holds by (4.16).

Thus,  $\mathbb{H}^*(\mathcal{M}_{\text{st}}^\alpha(\tau); \mathcal{P}|_{\mathcal{M}_{\text{st}}^\alpha(\tau)})$  would be a natural cohomology group of  $\mathcal{M}_{\text{st}}^\alpha(\tau)$  whose Euler characteristic is the Donaldson–Thomas invariant by (5.1); the Poincaré polynomial of  $\mathbb{H}^*(\mathcal{M}_{\text{st}}^\alpha(\tau); \mathcal{P}|_{\mathcal{M}_{\text{st}}^\alpha(\tau)})$  would be a lift of  $DT^\alpha(\tau)$  to  $\mathbb{Z}[t, t^{-1}]$ , which might also be interesting.

**(iii)** If the answers to Question 5.7(a),(b) are no, at least locally in the Zariski topology, this might be bad news for the programme of Kontsevich–Soibelman [63] to extend Donaldson–Thomas invariants of Calabi–Yau 3-folds to other motivic invariants. Kontsevich and Soibelman wish to associate a ‘motivic Milnor fibre’ to each point of  $\mathfrak{M}$ . The question of how these vary in families over the base  $\mathfrak{M}$  is important, but not really addressed in [63]. It appears to the authors to be a similar issue to whether one can glue perverse sheaves above; indeed,  $\mathcal{P}$  in Question 5.7 may be some kind of cohomology pushforward of the Kontsevich–Soibelman family of motivic Milnor fibres, if this exists.

The last three results use transcendental complex analysis, and so work only over  $\mathbb{C}$ . It is an important question whether analogous results can be proved using strictly algebraic methods, and over fields  $\mathbb{K}$  other than  $\mathbb{C}$ . Observe that above we locally write  $\mathcal{M}_{\text{si}}$  as  $\text{Crit}(f)$  for  $f : U \rightarrow \mathbb{C}$ , that is, we write  $\mathcal{M}_{\text{si}}$  as the zeroes  $(df)^{-1}(0)$  of a closed 1-form  $df$  on a smooth complex manifold  $U$ . A promising

way to generalize Theorems 5.4–5.5 to the algebraic context is to replace  $df$  by an *almost closed 1-form*  $\omega$ , in the sense of §4.4.

Results of Thomas [100] imply that the coarse moduli space of simple coherent sheaves  $\mathcal{M}_{\text{si}}$  on  $X$  carries a symmetric obstruction theory, and thus Proposition 4.16 shows that  $\mathcal{M}_{\text{si}}$  is locally isomorphic to the zeroes of an almost closed 1-form  $\omega$  on a smooth variety  $U$ . Étale locally near  $[E] \in \mathcal{M}_{\text{si}}(\mathbb{K})$  we can take  $U$  to be  $\text{Ext}^1(E, E)$ . Thus we deduce:

**PROPOSITION 5.9.** *Let  $\mathbb{K}$  be an algebraically closed field and  $X$  a Calabi–Yau 3-fold over  $\mathbb{K}$ , and write  $\mathcal{M}_{\text{si}}$  for the coarse moduli space of simple coherent sheaves on  $X$ , which is an algebraic  $\mathbb{K}$ -space. Then for each point  $[E] \in \mathcal{M}_{\text{si}}(\mathbb{K})$  there exists a Zariski open subset  $U$  in the affine  $\mathbb{K}$ -space  $\text{Ext}^1(E, E)$  with  $0 \in U(\mathbb{K})$ , an algebraic almost closed 1-form  $\omega$  on  $U$  with  $\omega|_0 = \partial\omega|_0 = 0$ , and an étale morphism  $\xi : \omega^{-1}(0) \rightarrow \mathcal{M}_{\text{si}}$  with  $\xi(0) = [E] \in \mathcal{M}_{\text{si}}(\mathbb{K})$  and  $d\xi|_0 : T_0(\omega^{-1}(0)) = \text{Ext}^1(E, E) \rightarrow T_{[E]}\mathcal{M}_{\text{si}}$  the natural isomorphism, where  $\omega^{-1}(0)$  is the  $\mathbb{K}$ -subscheme of  $U$  on which  $\omega \equiv 0$ .*

This is an analogue of Theorem 5.4, with  $\mathbb{C}$  replaced by any algebraically closed  $\mathbb{K}$ , the complex analytic topology replaced by the étale topology, and the closed 1-form  $df$  replaced by the almost closed 1-form  $\omega$ . We can ask whether there is a corresponding algebraic analogue of Theorem 5.5.

**QUESTION 5.10.** *Let  $\mathbb{K}$  be an algebraically closed field and  $X$  a Calabi–Yau 3-fold over  $\mathbb{K}$ , and write  $\mathfrak{M}$  for the moduli stack of coherent sheaves on  $X$ .*

- (a) *For each  $[E] \in \mathfrak{M}(\mathbb{K})$ , does there exist a Zariski open subset  $U$  in the affine  $\mathbb{K}$ -space  $\text{Ext}^1(E, E)$  with  $0 \in U(\mathbb{K})$ , an algebraic almost closed 1-form  $\omega$  on  $U$  with  $\omega|_0 = \partial\omega|_0 = 0$ , and a 1-morphism  $\xi : \omega^{-1}(0) \rightarrow \mathfrak{M}$  smooth of relative dimension  $\dim \text{Aut}(E)$ , with  $\xi(0) = [E] \in \mathfrak{M}(\mathbb{K})$  and  $d\xi|_0 : T_0(\omega^{-1}(0)) = \text{Ext}^1(E, E) \rightarrow T_{[E]}\mathfrak{M}$  the natural isomorphism?*
- (b) *In (a), let  $G$  be a maximal reductive subgroup of  $\text{Aut}(E)$ , acting on  $\text{Ext}^1(E, E)$  by  $\gamma : \epsilon \mapsto \gamma \circ \epsilon \circ \gamma^{-1}$ . Can we take  $U, \omega, \xi$  to be  $G$ -invariant?*

## 5.2. Identities on Behrend functions of moduli stacks

We use the results of §5.1 to study the *Behrend function*  $\nu_{\mathfrak{M}}$  of the moduli stack  $\mathfrak{M}$  of coherent sheaves on  $X$ , as in §4. Our next theorem is proved in §10.

**THEOREM 5.11.** *Let  $X$  be a Calabi–Yau 3-fold over  $\mathbb{C}$ , and  $\mathfrak{M}$  the moduli stack of coherent sheaves on  $X$ . The **Behrend function**  $\nu_{\mathfrak{M}} : \mathfrak{M}(\mathbb{C}) \rightarrow \mathbb{Z}$  is a natural locally constructible function on  $\mathfrak{M}$ . For all  $E_1, E_2 \in \text{coh}(X)$ , it satisfies:*

$$\nu_{\mathfrak{M}}(E_1 \oplus E_2) = (-1)^{\bar{\chi}([E_1], [E_2])} \nu_{\mathfrak{M}}(E_1) \nu_{\mathfrak{M}}(E_2), \quad (5.2)$$

$$\begin{aligned} & \int_{[\lambda] \in \mathbb{P}(\text{Ext}^1(E_2, E_1)) : \lambda \Leftrightarrow 0 \rightarrow E_1 \rightarrow F \rightarrow E_2 \rightarrow 0} \nu_{\mathfrak{M}}(F) d\chi - \int_{[\tilde{\lambda}] \in \mathbb{P}(\text{Ext}^1(E_1, E_2)) : \tilde{\lambda} \Leftrightarrow 0 \rightarrow E_2 \rightarrow \tilde{F} \rightarrow E_1 \rightarrow 0} \nu_{\mathfrak{M}}(\tilde{F}) d\chi \\ &= (\dim \text{Ext}^1(E_2, E_1) - \dim \text{Ext}^1(E_1, E_2)) \nu_{\mathfrak{M}}(E_1 \oplus E_2). \end{aligned} \quad (5.3)$$

Here  $\bar{\chi}([E_1], [E_2])$  in (5.2) is defined in (3.1), and in (5.3) the correspondence between  $[\lambda] \in \mathbb{P}(\text{Ext}^1(E_2, E_1))$  and  $F \in \text{coh}(X)$  is that  $[\lambda] \in \mathbb{P}(\text{Ext}^1(E_2, E_1))$  lifts to some  $0 \neq \lambda \in \text{Ext}^1(E_2, E_1)$ , which corresponds to a short exact sequence  $0 \rightarrow E_1 \rightarrow F \rightarrow E_2 \rightarrow 0$  in  $\text{coh}(X)$  in the usual way. The function  $[\lambda] \mapsto \nu_{\mathfrak{M}}(F)$

is a constructible function  $\mathbb{P}(\mathrm{Ext}^1(E_2, E_1)) \rightarrow \mathbb{Z}$ , and the integrals in (5.3) are integrals of constructible functions using the Euler characteristic as measure.

We will prove Theorem 5.11 using Theorem 5.5 and the Milnor fibre description of Behrend functions from §4.2. We apply Theorem 5.5 to  $E = E_1 \oplus E_2$ , and we take the maximal compact subgroup  $G$  of  $\mathrm{Aut}(E)$  to contain the subgroup  $\{\mathrm{id}_{E_1} + \lambda \mathrm{id}_{E_2} : \lambda \in \mathrm{U}(1)\}$ , so that  $G^c$  contains  $\{\mathrm{id}_{E_1} + \lambda \mathrm{id}_{E_2} : \lambda \in \mathbb{G}_m\}$ . Equations (5.2) and (5.3) are proved by a kind of localization using this  $\mathbb{G}_m$ -action on  $\mathrm{Ext}^1(E_1 \oplus E_2, E_1 \oplus E_2)$ .

Equations (5.2)–(5.3) are related to a conjecture of Kontsevich and Soibelman [63, Conj. 4] and its application in [63, §6.3], and could probably be deduced from it. The authors got the idea of proving (5.2)–(5.3) by localization using the  $\mathbb{G}_m$ -action on  $\mathrm{Ext}^1(E_1 \oplus E_2, E_1 \oplus E_2)$  from [63]. However, Kontsevich and Soibelman approach [63, Conj. 4] via formal power series and non-Archimedean geometry. Instead, in Theorem 5.5 we in effect first prove that we can choose the formal power series to be convergent, and then use ordinary differential geometry and Milnor fibres.

Note that our proof of Theorem 5.11 is *not wholly algebro-geometric* — it uses gauge theory, and transcendental complex analytic geometry methods. Therefore this method will not suffice to prove the parallel conjectures in Kontsevich and Soibelman [63, Conj. 4], which are supposed to hold for general fields  $\mathbb{K}$  as well as  $\mathbb{C}$ , and for general motivic invariants of algebraic  $\mathbb{K}$ -schemes as well as for the topological Euler characteristic.

QUESTION 5.12. (a) Suppose the answers to Questions 4.18(a) and 5.10 are both yes. Can one use these to give an alternative, strictly algebraic proof of Theorem 5.11 using **almost closed 1-forms** as in §4.4, either over  $\mathbb{K} = \mathbb{C}$  using the linking number expression for Behrend functions in (4.17), or over general algebraically closed  $\mathbb{K}$  of characteristic zero by some other means?

(b) Might the ideas of (a) provide an approach to proving [63, Conj. 4] without using formal power series methods?

(c) Can one extend Theorem 5.11 from the abelian category  $\mathrm{coh}(X)$  to the derived category  $D^b(X)$ , say to all objects  $E_1 \oplus E_2$  in  $D(X)$  with  $\mathrm{Ext}^{<0}(E_1 \oplus E_2, E_1 \oplus E_2) = 0$ ?

### 5.3. A Lie algebra morphism $\tilde{\Psi} : \mathrm{SF}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{M}) \rightarrow \tilde{L}(X)$ , and generalized Donaldson–Thomas invariants $DT^\alpha(\tau)$

In §3.4 we defined an explicit Lie algebra  $L(X)$  and Lie algebra morphisms  $\Psi : \mathrm{SF}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{M}) \rightarrow L(X)$  and  $\Psi^{\chi, \mathbb{Q}} : \mathrm{SF}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{M}, \chi, \mathbb{Q}) \rightarrow L(X)$ . We now define modified versions  $\tilde{L}(X)$ ,  $\tilde{\Psi}$ ,  $\tilde{\Psi}^{\chi, \mathbb{Q}}$ , with  $\tilde{\Psi}$ ,  $\tilde{\Psi}^{\chi, \mathbb{Q}}$  weighted by the Behrend function  $\nu_{\mathfrak{M}}$  of  $\mathfrak{M}$ . We continue to use the notation of §2–§4.

DEFINITION 5.13. Define a Lie algebra  $\tilde{L}(X)$  to be the  $\mathbb{Q}$ -vector space with basis of symbols  $\tilde{\lambda}^\alpha$  for  $\alpha \in K^{\mathrm{num}}(\mathrm{coh}(X))$ , with Lie bracket

$$[\tilde{\lambda}^\alpha, \tilde{\lambda}^\beta] = (-1)^{\bar{\chi}(\alpha, \beta)} \bar{\chi}(\alpha, \beta) \tilde{\lambda}^{\alpha+\beta}, \quad (5.4)$$

which is (3.15) with a sign change. As  $\bar{\chi}$  is antisymmetric, (5.4) satisfies the Jacobi identity, and makes  $\tilde{L}(X)$  into an infinite-dimensional Lie algebra over  $\mathbb{Q}$ .

Define a  $\mathbb{Q}$ -linear map  $\tilde{\Psi}^{\chi, \mathbb{Q}} : \bar{\mathrm{SF}}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{M}, \chi, \mathbb{Q}) \rightarrow \tilde{L}(X)$  by

$$\tilde{\Psi}^{\chi, \mathbb{Q}}(f) = \sum_{\alpha \in K^{\mathrm{num}}(\mathrm{coh}(X))} \gamma^\alpha \tilde{\lambda}^\alpha,$$

as in (3.16), where  $\gamma^\alpha \in \mathbb{Q}$  is defined as follows. Write  $f|_{\mathfrak{M}^\alpha}$  in terms of  $\delta_i, U_i, \rho_i$  as in (3.17), and set

$$\gamma^\alpha = \sum_{i=1}^n \delta_i \chi(U_i, \rho_i^*(\nu_{\mathfrak{M}})), \quad (5.5)$$

where  $\rho_i^*(\nu_{\mathfrak{M}})$  is the pullback of the Behrend function  $\nu_{\mathfrak{M}}$  to a constructible function on  $U_i \times [\mathrm{Spec} \mathbb{C}/\mathbb{G}_m]$ , or equivalently on  $U_i$ , and  $\chi(U_i, \rho_i^*(\nu_{\mathfrak{M}}))$  is the Euler characteristic of  $U_i$  weighted by  $\rho_i^*(\nu_{\mathfrak{M}})$ . One can show that the map from (3.17) to (5.5) is compatible with the relations in  $\bar{\mathrm{SF}}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{M}^\alpha, \chi, \mathbb{Q})$ , and so  $\tilde{\Psi}^{\chi, \mathbb{Q}}$  is well-defined. Define  $\tilde{\Psi} : \mathrm{SF}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{M}) \rightarrow \tilde{L}(X)$  by  $\tilde{\Psi} = \tilde{\Psi}^{\chi, \mathbb{Q}} \circ \bar{\Pi}_{\mathfrak{M}}^{\chi, \mathbb{Q}}$ .

Here is an alternative way to write  $\tilde{\Psi}^{\chi, \mathbb{Q}}, \tilde{\Psi}$  using constructible functions. Define a  $\mathbb{Q}$ -linear map  $\Pi_{\mathrm{CF}} : \bar{\mathrm{SF}}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{M}, \chi, \mathbb{Q}) \rightarrow \mathrm{CF}(\mathfrak{M})$  by

$$\Pi_{\mathrm{CF}} : \sum_{i=1}^n \delta_i [(U_i \times [\mathrm{Spec} \mathbb{C}/\mathbb{G}_m], \rho_i)] \mapsto \sum_{i=1}^n \delta_i \mathrm{CF}^{\mathrm{na}}(\rho_i)(1_{U_i}),$$

where by Proposition 3.4 elements of  $\bar{\mathrm{SF}}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{M}, \chi, \mathbb{Q})$  may be written  $\sum_{i=1}^n \delta_i [(U_i \times [\mathrm{Spec} \mathbb{C}/\mathbb{G}_m], \rho_i)]$  for  $\delta_i \in \mathbb{Q}$ ,  $U_i$  a quasiprojective  $\mathbb{C}$ -variety, and  $[(U_i \times [\mathrm{Spec} \mathbb{C}/\mathbb{G}_m], \rho_i)]$  with algebra stabilizers, and  $1_{U_i} \in \mathrm{CF}(U_i)$  is the function 1, and  $\mathrm{CF}^{\mathrm{na}}(\rho_i)$  is as in Definition 2.3. Then we have

$$\begin{aligned} \tilde{\Psi}^{\chi, \mathbb{Q}}(f) &= \sum_{\alpha \in K^{\mathrm{num}}(\mathrm{coh}(X))} \chi^{\mathrm{na}}(\mathfrak{M}^\alpha, (\Pi_{\mathrm{CF}}(f) \cdot \nu_{\mathfrak{M}})|_{\mathfrak{M}^\alpha}) \tilde{\lambda}^\alpha, \\ \tilde{\Psi}(f) &= \sum_{\alpha \in K^{\mathrm{num}}(\mathrm{coh}(X))} \chi^{\mathrm{na}}(\mathfrak{M}^\alpha, (\Pi_{\mathrm{CF}} \circ \bar{\Pi}_{\mathfrak{M}}^{\chi, \mathbb{Q}}(f) \cdot \nu_{\mathfrak{M}})|_{\mathfrak{M}^\alpha}) \tilde{\lambda}^\alpha. \end{aligned} \quad (5.6)$$

Our Lie algebra  $\tilde{L}(X)$  is essentially the same as the Lie algebra  $\mathfrak{g}_\Gamma$  of Kontsevich and Soibelman [63, §1.4]. They also define a Lie algebra  $\mathfrak{g}_V$  which is a completion of a Lie subalgebra of  $\mathfrak{g}_\Gamma$ , and a pro-nilpotent Lie group  $G_V$  with Lie algebra  $\mathfrak{g}_V$ . Kontsevich and Soibelman express wall-crossing for Donaldson–Thomas type invariants in terms of multiplication in the Lie group  $G_V$ , whereas we do it in terms of Lie brackets in the Lie algebra  $\tilde{L}(X)$ . Applying  $\exp : \mathfrak{g}_V \rightarrow G_V$  should transform our approach to that of Kontsevich and Soibelman.

The reason for the sign change between (3.15) and (5.4) is the signs involved in Behrend functions, in particular, the  $(-1)^n$  in Theorem 4.3(ii), which is responsible for the factor  $(-1)^{\bar{\chi}([E_1], [E_2])}$  in (5.2).

Here is the analogue of Theorem 3.16. It is proved in §11.

**THEOREM 5.14.**  $\tilde{\Psi} : \mathrm{SF}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{M}) \rightarrow \tilde{L}(X)$  and  $\tilde{\Psi}^{\chi, \mathbb{Q}} : \bar{\mathrm{SF}}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{M}, \chi, \mathbb{Q}) \rightarrow \tilde{L}(X)$  are Lie algebra morphisms.

Theorem 5.14 should be compared with Kontsevich and Soibelman [63, §6.3], which gives a conjectural construction of an *algebra morphism*  $\Phi : \mathrm{SF}(\mathfrak{M}) \rightarrow \mathcal{R}_{K(\mathrm{coh}(X))}$ , where  $\mathcal{R}_{K(\mathrm{coh}(X))}$  is a certain explicit algebra. We expect our  $\tilde{\Psi}$  should be obtained from their  $\Phi$  by restricting to  $\mathrm{SF}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{M})$ , and obtaining  $\tilde{L}(X)$  from a Lie subalgebra of  $\mathcal{R}_{K(\mathrm{coh}(X))}$  by taking a limit, the limit corresponding to specializing from virtual Poincaré polynomials or more general motivic invariants of  $\mathbb{C}$ -varieties to Euler characteristics.

We can now define generalized Donaldson–Thomas invariants.

**DEFINITION 5.15.** Let  $X$  be a projective Calabi–Yau 3-fold over  $\mathbb{C}$ , let  $\mathcal{O}_X(1)$  be a very ample line bundle on  $X$ , and let  $(\tau, G, \leq)$  be Gieseker stability and  $(\mu, M, \leq)$  be  $\mu$ -stability on  $\mathrm{coh}(X)$  w.r.t.  $\mathcal{O}_X(1)$ , as in Examples 3.8 and 3.9. As in (3.22),

define *generalized Donaldson–Thomas invariants*  $\bar{DT}^\alpha(\tau) \in \mathbb{Q}$  and  $\bar{DT}^\alpha(\mu) \in \mathbb{Q}$  for all  $\alpha \in C(\text{coh}(X))$  by

$$\tilde{\Psi}(\bar{\epsilon}^\alpha(\tau)) = -\bar{DT}^\alpha(\tau)\tilde{\lambda}^\alpha \quad \text{and} \quad \tilde{\Psi}(\bar{\epsilon}^\alpha(\mu)) = -\bar{DT}^\alpha(\mu)\tilde{\lambda}^\alpha. \quad (5.7)$$

Here  $\bar{\epsilon}^\alpha(\tau), \bar{\epsilon}^\alpha(\mu)$  are defined in (3.4), and lie in  $\text{SF}_{\text{al}}^{\text{ind}}(\mathfrak{M})$  by Theorem 3.11, so  $\bar{DT}^\alpha(\tau), \bar{DT}^\alpha(\mu)$  are well-defined. The signs in (5.7) will be explained after Proposition 5.17. Equation (5.6) implies that an alternative expression is

$$\begin{aligned} \bar{DT}^\alpha(\tau) &= -\chi^{\text{na}}(\mathfrak{M}_{\text{ss}}^\alpha(\tau), \Pi_{\text{CF}} \circ \bar{\Pi}_{\mathfrak{M}}^{\chi, \mathbb{Q}}(\bar{\epsilon}^\alpha(\tau)) \cdot \nu_{\mathfrak{M}}), \\ \bar{DT}^\alpha(\mu) &= -\chi^{\text{na}}(\mathfrak{M}_{\text{ss}}^\alpha(\mu), \Pi_{\text{CF}} \circ \bar{\Pi}_{\mathfrak{M}}^{\chi, \mathbb{Q}}(\bar{\epsilon}^\alpha(\mu)) \cdot \nu_{\mathfrak{M}}). \end{aligned} \quad (5.8)$$

For the case of Gieseker stability  $(\tau, G, \leq)$ , we have a projective coarse moduli scheme  $\mathcal{M}_{\text{ss}}^\alpha(\tau)$ . Write  $\pi : \mathfrak{M}_{\text{ss}}^\alpha(\tau) \rightarrow \mathcal{M}_{\text{ss}}^\alpha(\tau)$  for the natural projection. Then by functoriality of the naïve pushforward (2.1), we can rewrite the first line of (5.8) as a weighted Euler characteristic of  $\mathcal{M}_{\text{ss}}^\alpha(\tau)$ :

$$\bar{DT}^\alpha(\tau) = -\chi(\mathcal{M}_{\text{ss}}^\alpha(\tau), \text{CF}^{\text{na}}(\pi)[\Pi_{\text{CF}} \circ \bar{\Pi}_{\mathfrak{M}}^{\chi, \mathbb{Q}}(\bar{\epsilon}^\alpha(\tau)) \cdot \nu_{\mathfrak{M}}]). \quad (5.9)$$

The constructible functions  $-\Pi_{\text{CF}} \circ \bar{\Pi}_{\mathfrak{M}}^{\chi, \mathbb{Q}}(\bar{\epsilon}^\alpha(\tau)) \cdot \nu_{\mathfrak{M}}$  on  $\mathfrak{M}_{\text{ss}}^\alpha(\tau)$  in (5.8), and  $-\text{CF}^{\text{na}}(\pi)[\Pi_{\text{CF}} \circ \bar{\Pi}_{\mathfrak{M}}^{\chi, \mathbb{Q}}(\bar{\epsilon}^\alpha(\tau)) \cdot \nu_{\mathfrak{M}}]$  on  $\mathcal{M}_{\text{ss}}^\alpha(\tau)$  in (5.9), are the contributions to  $\bar{DT}^\alpha(\tau)$  from each  $\tau$ -semistable, and each S-equivalence class of  $\tau$ -semistables (or  $\tau$ -polystable), respectively. We will return to (5.9) in §6.2.

REMARK 5.16. We show in Corollary 5.28 below that  $\bar{DT}^\alpha(\tau)$  is *unchanged under deformations of  $X$* . Our definition of  $\bar{DT}^\alpha(\tau)$  is very complicated. It counts sheaves using two kinds of weights: firstly, we define  $\bar{\epsilon}^\alpha(\tau)$  from the  $\bar{\delta}_{\text{ss}}^\beta(\tau)$  by (3.4), with  $\mathbb{Q}$ -valued weights  $(-1)^{n-1}/n$ , and then we apply the Lie algebra morphism  $\tilde{\Psi}$ , which takes Euler characteristics weighted by the  $\mathbb{Z}$ -valued Behrend function  $\nu_{\mathfrak{M}}$ . Furthermore, to compute  $\tilde{\Psi}(\bar{\epsilon}^\alpha(\tau))$  we must first write  $\bar{\epsilon}^\alpha(\tau)$  in the form (3.17) using Proposition 3.4, and this uses relation Definition 2.16(iii) involving coefficients  $F(G, T^G, Q) \in \mathbb{Q}$ .

In §6.5 we will show in an example that all this complexity is *really necessary* to make  $\bar{DT}^\alpha(\tau)$  deformation-invariant. In particular, we will show that strictly  $\tau$ -semistable sheaves must be counted with *non-integral* weights, and also that the obvious definition  $DT^\alpha(\tau) = \chi(\mathcal{M}_{\text{st}}^\alpha(\tau), \nu_{\mathcal{M}_{\text{st}}^\alpha(\tau)})$  from (4.16) is not deformation-invariant when  $\mathcal{M}_{\text{ss}}^\alpha(\tau) \neq \mathcal{M}_{\text{st}}^\alpha(\tau)$ .

Suppose that  $\mathcal{M}_{\text{ss}}^\alpha(\tau) = \mathcal{M}_{\text{st}}^\alpha(\tau)$ , that is, there are no strictly  $\tau$ -semistable sheaves in class  $\alpha$ . Then the only nonzero term in (3.4) is  $n = 1$  and  $\alpha_1 = \alpha$ , so

$$\bar{\epsilon}^\alpha(\tau) = \bar{\delta}_{\text{ss}}^\alpha(\tau) = \bar{\delta}_{\mathfrak{M}_{\text{st}}^\alpha(\tau)} = [(\mathfrak{M}_{\text{st}}^\alpha(\tau), \iota)], \quad (5.10)$$

where  $\iota : \mathfrak{M}_{\text{st}}^\alpha(\tau) \rightarrow \mathfrak{M}$  is the inclusion 1-morphism. Write  $\pi : \mathfrak{M}_{\text{st}}^\alpha(\tau) \rightarrow \mathcal{M}_{\text{st}}^\alpha(\tau)$  for the projection from  $\mathfrak{M}_{\text{st}}^\alpha(\tau)$  to its coarse moduli scheme  $\mathcal{M}_{\text{st}}^\alpha(\tau)$ . Then

$$\begin{aligned} \bar{DT}^\alpha(\tau) &= -\chi^{\text{na}}(\mathfrak{M}_{\text{st}}^\alpha(\tau), \iota^*(\nu_{\mathfrak{M}})) = -\chi^{\text{na}}(\mathfrak{M}_{\text{st}}^\alpha(\tau), \nu_{\mathfrak{M}_{\text{st}}^\alpha(\tau)}) \\ &= \chi^{\text{na}}(\mathfrak{M}_{\text{st}}^\alpha(\tau), \pi^*(\nu_{\mathcal{M}_{\text{st}}^\alpha(\tau)})) = \chi(\mathcal{M}_{\text{st}}^\alpha(\tau), \nu_{\mathcal{M}_{\text{st}}^\alpha(\tau)}) = DT^\alpha(\tau), \end{aligned}$$

using Definition 5.13 and (5.7) in the first step,  $\mathfrak{M}_{\text{st}}^\alpha(\tau)$  open in  $\mathfrak{M}$  in the second,  $\pi$  smooth of relative dimension  $-1$  and Corollary 4.5 to deduce  $\pi^*(\nu_{\mathcal{M}_{\text{st}}^\alpha(\tau)}) \equiv -\nu_{\mathfrak{M}_{\text{st}}^\alpha(\tau)}$  in the third,  $\pi_* : \mathfrak{M}_{\text{st}}^\alpha(\tau)(\mathbb{C}) \rightarrow \mathcal{M}_{\text{st}}^\alpha(\tau)(\mathbb{C})$  an isomorphism of constructible sets in the fourth, and (4.16) in the fifth. Thus we have proved:

PROPOSITION 5.17. *If  $\mathcal{M}_{\text{ss}}^\alpha(\tau) = \mathcal{M}_{\text{st}}^\alpha(\tau)$  then  $\bar{D}T^\alpha(\tau) = DT^\alpha(\tau)$ . That is, our new generalized Donaldson–Thomas invariants  $\bar{D}T^\alpha(\tau)$  are equal to the original Donaldson–Thomas invariants  $DT^\alpha(\tau)$  whenever the  $DT^\alpha(\tau)$  are defined.*

We include the minus signs in (5.7) to cancel that in  $\pi^*(\nu_{\mathcal{M}_{\text{st}}^\alpha(\tau)}) = -\nu_{\mathcal{M}_{\text{st}}^\alpha(\tau)}$ . Omitting the signs in (5.7) would have given  $\bar{D}T^\alpha(\tau) = -DT^\alpha(\tau)$  above.

We can now repeat the argument of §3.5 to deduce transformation laws for generalized Donaldson–Thomas invariants under change of stability condition. Suppose  $(\tau, T, \leq), (\tilde{\tau}, \tilde{T}, \leq), (\hat{\tau}, \hat{T}, \leq)$  are as in Theorem 3.13 for  $\mathcal{A} = \text{coh}(X)$ . Then as in §3.2 equation (3.10) holds, and by Theorem 3.14 it is equivalent to a Lie algebra equation (3.13) in  $\text{SF}_{\text{al}}^{\text{ind}}(\mathfrak{M})$ . Thus we may apply the Lie algebra morphism  $\tilde{\Psi}$  to transform (3.13) (or equivalently (3.10)) into an identity in the Lie algebra  $\tilde{L}(X)$ , and use (5.7) to write this in terms of generalized Donaldson–Thomas invariants. As for (3.23), this gives an equation in the universal enveloping algebra  $U(\tilde{L}(X))$ :

$$\bar{D}T^\alpha(\tilde{\tau})\tilde{\lambda}^\alpha = \sum_{\substack{n \geq 1, \alpha_1, \dots, \alpha_n \in C(\text{coh}(X)): \\ \alpha_1 + \dots + \alpha_n = \alpha}} U(\alpha_1, \dots, \alpha_n; \tau, \tilde{\tau}) \cdot (-1)^{n-1} \prod_{i=1}^n \bar{D}T^{\alpha_i}(\tau) \cdot \tilde{\lambda}^{\alpha_1} \star \tilde{\lambda}^{\alpha_2} \star \dots \star \tilde{\lambda}^{\alpha_n}. \quad (5.11)$$

As in [52, §6.5], we describe  $U(\tilde{L}(X))$  explicitly, and the analogue of (3.24) is

$$\begin{aligned} \tilde{\lambda}^{\alpha_1} \star \dots \star \tilde{\lambda}^{\alpha_n} &= \text{terms in } \tilde{\lambda}_{[I, \kappa]}, |I| > 1, \\ &+ \left[ \frac{(-1)^{\sum_{1 \leq i < j \leq n} \bar{\chi}(\alpha_i, \alpha_j)}}{2^{n-1}} \sum_{\substack{\text{connected, simply-connected} \\ \text{digraphs } \Gamma: \text{ vertices } \{1, \dots, n\}, \\ \text{edge } \overset{i}{\bullet} \rightarrow \overset{j}{\bullet} \text{ implies } i < j}} \prod_{\substack{\text{edges} \\ \overset{i}{\bullet} \rightarrow \overset{j}{\bullet} \\ \text{in } \Gamma}} \bar{\chi}(\alpha_i, \alpha_j) \right] \tilde{\lambda}^{\alpha_1 + \dots + \alpha_n}. \end{aligned} \quad (5.12)$$

Substitute (5.12) into (5.11). As for (3.25), equating coefficients of  $\tilde{\lambda}^\alpha$  yields

$$\begin{aligned} \bar{D}T^\alpha(\tilde{\tau}) &= \sum_{\substack{\alpha_1, \dots, \alpha_n \in C(\text{coh}(X)): \\ n \geq 1, \alpha_1 + \dots + \alpha_n = \alpha}} \sum_{\substack{\text{connected, simply-connected digraphs } \Gamma: \\ \text{vertices } \{1, \dots, n\}, \text{ edge } \overset{i}{\bullet} \rightarrow \overset{j}{\bullet} \text{ implies } i < j}} \quad (5.13) \\ &\frac{(-1)^{n-1 + \sum_{1 \leq i < j \leq n} \bar{\chi}(\alpha_i, \alpha_j)}}{2^{n-1}} U(\alpha_1, \dots, \alpha_n; \tau, \tilde{\tau}) \prod_{\substack{\text{edges } \overset{i}{\bullet} \rightarrow \overset{j}{\bullet} \text{ in } \Gamma}} \bar{\chi}(\alpha_i, \alpha_j) \prod_{i=1}^n \bar{D}T^{\alpha_i}(\tau). \end{aligned}$$

Using the coefficients  $V(I, \Gamma, \kappa; \tau, \tilde{\tau})$  of Definition 3.18 to rewrite (5.13), we obtain an analogue of (3.27), as in [54, Th. 6.28]:

THEOREM 5.18. *Under the assumptions above, for all  $\alpha \in C(\text{coh}(X))$  we have*

$$\begin{aligned} \bar{D}T^\alpha(\tilde{\tau}) &= \sum_{\substack{\text{iso.} \\ \text{classes} \\ \text{of finite} \\ \text{sets } I}} \sum_{\substack{\kappa: I \rightarrow C(\text{coh}(X)): \\ \sum_{i \in I} \kappa(i) = \alpha}} \sum_{\substack{\text{connected,} \\ \text{simply-} \\ \text{connected} \\ \text{digraphs } \Gamma, \\ \text{vertices } I}} (-1)^{|I|-1} V(I, \Gamma, \kappa; \tau, \tilde{\tau}) \cdot \prod_{i \in I} \bar{D}T^{\kappa(i)}(\tau) \\ &\quad \cdot (-1)^{\frac{1}{2} \sum_{i, j \in I} |\bar{\chi}(\kappa(i), \kappa(j))|} \cdot \prod_{\substack{\text{edges } \overset{i}{\bullet} \rightarrow \overset{j}{\bullet} \\ \text{in } \Gamma}} \bar{\chi}(\kappa(i), \kappa(j)), \end{aligned} \quad (5.14)$$

with only finitely many nonzero terms.

As we explained at the end of §3.3, for technical reasons the authors do not know whether the changes between every two weak stability conditions of Gieseker or  $\mu$ -stability type on  $\text{coh}(X)$  are globally finite, so we cannot apply Theorem 5.18 directly. But as in [54, §5.1], we can interpolate between any two such stability

conditions on  $X$  of Gieseker stability or  $\mu$ -stability type by a finite sequence of stability conditions, such that between successive stability conditions in the sequence the changes are globally finite. Thus we deduce:

**COROLLARY 5.19.** *Let  $(\tau, T, \leq), (\tilde{\tau}, \tilde{T}, \leq)$  be two permissible weak stability conditions on  $\text{coh}(X)$  of Gieseker or  $\mu$ -stability type, as in Examples 3.8 and 3.9. Then the  $\bar{D}T^\alpha(\tau)$  for all  $\alpha \in C(\text{coh}(X))$  completely determine the  $\bar{D}T^\alpha(\tilde{\tau})$  for all  $\alpha \in C(\text{coh}(X))$ , and vice versa, through finitely many applications of the transformation law (5.14).*

#### 5.4. Invariants $PI^{\alpha,n}(\tau')$ counting stable pairs, and deformation-invariance of the $\bar{D}T^\alpha(\tau)$

Next we define *stable pairs* on  $X$ . Our next three results, Theorems 5.22, 5.23 and 5.25, will be proved in §12. They work for  $X$  a Calabi–Yau 3-fold over a general algebraically closed field  $\mathbb{K}$ , without assuming  $H^1(\mathcal{O}_X) = 0$ .

**DEFINITION 5.20.** Let  $\mathbb{K}$  be an algebraically closed field, and  $X$  a Calabi–Yau 3-fold over  $\mathbb{K}$ , which may have  $H^1(\mathcal{O}_X) \neq 0$ . Choose a very ample line bundle  $\mathcal{O}_X(1)$  on  $X$ , and write  $(\tau, G, \leq)$  for Gieseker stability w.r.t.  $\mathcal{O}_X(1)$ , as in Example 3.8.

Fix  $n \gg 0$  in  $\mathbb{Z}$ . A *pair* is a nonzero morphism of sheaves  $s : \mathcal{O}_X(-n) \rightarrow E$ , where  $E$  is a nonzero sheaf. A *morphism* between two pairs  $s : \mathcal{O}_X(-n) \rightarrow E$  and  $t : \mathcal{O}_X(-n) \rightarrow F$  is a morphism of  $\mathcal{O}_X$ -modules  $f : E \rightarrow F$ , with  $f \circ s = t$ . A pair  $s : \mathcal{O}_X(-n) \rightarrow E$  is called *stable* if:

- (i)  $\tau([E']) \leq \tau([E])$  for all subsheaves  $E'$  of  $E$  with  $0 \neq E' \neq E$ ; and
- (ii) If also  $s$  factors through  $E'$ , then  $\tau([E']) < \tau([E])$ .

Note that (i) implies that if  $s : \mathcal{O}_X(-n) \rightarrow E$  is stable then  $E$  is  $\tau$ -semistable. The class of a pair  $s : \mathcal{O}_X(-n) \rightarrow E$  is the numerical class  $[E]$  in  $K^{\text{num}}(\text{coh}(X))$ .

We have no notion of semistable pairs. We will use  $\tau'$  to denote *stability of pairs*, defined using  $\mathcal{O}_X(1)$ . Note that pairs do not form an abelian category, so  $\tau'$  is not a (weak) stability condition on an abelian category in the sense of §3.2. However, in §13.1 we will define an auxiliary abelian category  $\mathcal{B}_p$  and relate stability of pairs  $\tau'$  to a weak stability condition  $(\tilde{\tau}, \tilde{T}, \leq)$  on  $\mathcal{B}_p$ .

**DEFINITION 5.21.** Use the notation of Definition 5.20. Let  $T$  be a  $\mathbb{K}$ -scheme, and write  $\pi_X : X \times T \rightarrow X$  for the projection. A  *$T$ -family of stable pairs with class  $\alpha$*  in  $K^{\text{num}}(\text{coh}(X))$  is a morphism of  $\mathcal{O}_{X \times T}$ -modules  $s : \pi_X^*(\mathcal{O}_X(-n)) \rightarrow E$ , where  $E$  is flat over  $T$ , and when restricting to  $\mathbb{K}$ -points  $t \in T(\mathbb{K})$ ,  $s_t : \mathcal{O}_X(-n) \rightarrow E_t$  is a stable pair, with  $[E_t] = \alpha$ . Note that since  $E$  is flat over  $T$ , the class  $[E_t]$  in  $K^{\text{num}}(\text{coh}(X))$  is locally constant on  $T$ , so requiring  $[E_t] = \alpha$  for all  $t \in T(\mathbb{K})$  is an open condition on such families.

Two  $T$ -families of stable pairs  $s_1 : \pi_X^*(\mathcal{O}_X(-n)) \rightarrow E_1, s_2 : \pi_X^*(\mathcal{O}_X(-n)) \rightarrow E_2$  are called *isomorphic* if there exists an isomorphism  $f : E_1 \rightarrow E_2$ , such that the following diagram commutes:

$$\begin{array}{ccc} \pi_X^*(\mathcal{O}_X(-n)) & \xrightarrow{s_1} & E_1 \\ \parallel & & \downarrow f \\ \pi_X^*(\mathcal{O}_X(-n)) & \xrightarrow{s_2} & E_2. \end{array}$$

The *moduli functor of stable pairs with class  $\alpha$* :

$$\mathbb{M}_{\text{stp}}^{\alpha,n}(\tau') : \text{Sch}_{\mathbb{K}} \longrightarrow \mathbf{Sets}$$

is defined to be the functor that takes a  $\mathbb{K}$ -scheme  $T$  to the set of isomorphism classes of  $T$ -families of stable pairs with class  $\alpha$ .

In §12.1 we will use results of Le Potier to prove:

**THEOREM 5.22.** *The moduli functor  $\mathbb{M}_{\text{stp}}^{\alpha,n}(\tau')$  is represented by a projective  $\mathbb{K}$ -scheme  $\mathcal{M}_{\text{stp}}^{\alpha,n}(\tau')$ .*

Broadly following similar proofs by Pandharipande and Thomas [86, §2] and Huybrechts and Thomas [45, §4], in §12.5–§12.7 we prove:

**THEOREM 5.23.** *If  $n$  is sufficiently large then the projective  $\mathbb{K}$ -scheme  $\mathcal{M}_{\text{stp}}^{\alpha,n}(\tau')$  has a symmetric obstruction theory.*

Here  $n$  is sufficiently large if all  $\tau$ -semistable sheaves  $E$  in class  $\alpha$  are  $n$ -regular. Using this symmetric obstruction theory, Behrend and Fantechi [5] construct a canonical Chow class  $[\mathcal{M}_{\text{stp}}^{\alpha,n}(\tau')]^{\text{vir}} \in A_*(\mathcal{M}_{\text{stp}}^{\alpha,n}(\tau'))$ . It lies in degree zero since the obstruction theory is symmetric. Since  $\mathcal{M}_{\text{stp}}^{\alpha,n}(\tau')$  is proper, there is a degree map on  $A_0(\mathcal{M}_{\text{stp}}^{\alpha,n}(\tau'))$ . We define an invariant counting stable pairs of class  $(\alpha, n)$  to be the degree of this virtual fundamental class.

**DEFINITION 5.24.** In the situation above, if  $\alpha \in K^{\text{num}}(\text{coh}(X))$  and  $n \gg 0$  is sufficiently large, define *stable pair invariants*  $PI^{\alpha,n}(\tau')$  in  $\mathbb{Z}$  by

$$PI^{\alpha,n}(\tau') = \int_{[\mathcal{M}_{\text{stp}}^{\alpha,n}(\tau')]^{\text{vir}}} 1. \quad (5.15)$$

Theorem 4.14 implies that when  $\mathbb{K}$  has characteristic zero, the stable pair invariants may also be written

$$PI^{\alpha,n}(\tau') = \chi(\mathcal{M}_{\text{stp}}^{\alpha,n}(\tau'), \nu_{\mathcal{M}_{\text{stp}}^{\alpha,n}(\tau')}). \quad (5.16)$$

Our invariants  $PI^{\alpha,n}(\tau')$  were inspired by Pandharipande and Thomas [86], who use invariants counting pairs to study curve counting in Calabi–Yau 3-folds. Observe an important difference between Donaldson–Thomas and stable pair invariants:  $DT^\alpha(\tau)$  is defined only for classes  $\alpha \in K^{\text{num}}(\text{coh}(X))$  with  $\mathcal{M}_{\text{ss}}^\alpha(\tau) = \mathcal{M}_{\text{st}}^\alpha(\tau)$ , but  $PI^{\alpha,n}(\tau')$  is defined for all  $\alpha \in K^{\text{num}}(\text{coh}(X))$  and all  $n \gg 0$ . We wish to show the  $PI^{\alpha,n}(\tau')$  for  $\alpha \in K^{\text{num}}(\text{coh}(X))$  are unchanged under deformations of the underlying Calabi–Yau 3-fold  $X$ . For this to make sense,  $K^{\text{num}}(\text{coh}(X))$  should also be unchanged under deformations of  $X$ , so we put this in as an assumption. Our next theorem will be proved in §12.8.

**THEOREM 5.25.** *Let  $\mathbb{K}$  be an algebraically closed field,  $U$  a connected algebraic  $\mathbb{K}$ -variety, and  $X \xrightarrow{\varphi} U$  be a family of Calabi–Yau 3-folds, so that for each  $u \in U(\mathbb{K})$  the fibre  $X_u = X \times_{\varphi, U, u} \text{Spec } \mathbb{K}$  of  $\varphi$  over  $u$  is a Calabi–Yau 3-fold over  $\mathbb{K}$ , which may have  $H^1(\mathcal{O}_{X_u}) \neq 0$ . Let  $\mathcal{O}_X(1)$  be a relative very ample line bundle for  $X \xrightarrow{\varphi} U$ , and write  $\mathcal{O}_{X_u}(1)$  for  $\mathcal{O}_X(1)|_{X_u}$ . Suppose that the numerical Grothendieck groups  $K^{\text{num}}(\text{coh}(X_u))$  for  $u \in U(\mathbb{K})$  are canonically isomorphic locally in  $U(\mathbb{K})$ , and write  $K(\text{coh}(X))$  for this group  $K^{\text{num}}(\text{coh}(X_u))$  up to canonical isomorphism. Then the stable pair invariants  $PI^{\alpha,n}(\tau')_u$  of  $X_u, \mathcal{O}_{X_u}(1)$  for  $\alpha \in K(\text{coh}(X))$  and  $n \gg 0$  are independent of  $u \in U(\mathbb{K})$ .*

Here is what we mean by saying that the numerical Grothendieck groups  $K^{\text{num}}(\text{coh}(X_u))$  for  $u \in U(\mathbb{K})$  are canonically isomorphic locally in  $U(\mathbb{K})$ : we do



not require canonical isomorphisms  $K^{\text{num}}(\text{coh}(X_u)) \cong K(\text{coh}(X))$  for all  $u \in U(\mathbb{K})$  (this would be canonically isomorphic *globally* in  $U(\mathbb{K})$ ). Instead, we mean that the groups  $K^{\text{num}}(\text{coh}(X_u))$  for  $u \in U(\mathbb{K})$  form a *local system of abelian groups* over  $U(\mathbb{K})$ , with fibre  $K(\text{coh}(X))$ .

When  $\mathbb{K} = \mathbb{C}$ , this means that in simply-connected regions of  $U(\mathbb{C})$  in the complex analytic topology the  $K^{\text{num}}(\text{coh}(X_u))$  are all canonically isomorphic, and isomorphic to  $K(\text{coh}(X))$ . But around loops in  $U(\mathbb{C})$ , this isomorphism with  $K(\text{coh}(X))$  can change by *monodromy*, by an automorphism  $\mu : K(\text{coh}(X)) \rightarrow K(\text{coh}(X))$  of  $K(\text{coh}(X))$ , as in Remark 4.20(d). In Theorem 4.21 we showed that the group of such monodromies  $\mu$  is finite, and we can make it trivial by passing to a finite cover  $\tilde{U}$  of  $U$ . If we worked instead with invariants  $PI^{P,n}(\tau')$  counting pairs  $s : \mathcal{O}_X(-n) \rightarrow E$  in which  $E$  has fixed Hilbert polynomial  $P$ , rather than fixed class  $\alpha \in K^{\text{num}}(\text{coh}(X))$ , as in Thomas' original definition of Donaldson–Thomas invariants [100], then we could drop the assumption on  $K^{\text{num}}(\text{coh}(X_u))$  in Theorem 5.25.

In Theorem 4.19 we showed that when  $\mathbb{K} = \mathbb{C}$  and  $H^1(\mathcal{O}_X) = 0$  the numerical Grothendieck group  $K^{\text{num}}(\text{coh}(X))$  is unchanged under small deformations of  $X$  up to canonical isomorphism. So Theorem 5.25 yields:

**COROLLARY 5.26.** *Let  $X$  be a Calabi–Yau 3-fold over  $\mathbb{K} = \mathbb{C}$ , with  $H^1(\mathcal{O}_X) = 0$ . Then the pair invariants  $PI^{\alpha,n}(\tau')$  are unchanged by continuous deformations of the complex structure of  $X$ .*

The following result, proved in §13, expresses the pair invariants  $PI^{\alpha,n}(\tau')$  above in terms of the generalized Donaldson–Thomas invariants  $\bar{DT}^\beta(\tau)$  of §5.3. Although the theorem makes sense for general algebraically closed fields  $\mathbb{K}$ , our proof works only for  $\mathbb{K} = \mathbb{C}$ , since it involves a version of the Behrend function identities (5.2)–(5.3), which are proved using complex analytic methods.

**THEOREM 5.27.** *Let  $\mathbb{K} = \mathbb{C}$ . Then for  $\alpha \in C(\text{coh}(X))$  and  $n \gg 0$  we have*

$$PI^{\alpha,n}(\tau') = \sum_{\substack{\alpha_1, \dots, \alpha_l \in C(\text{coh}(X)), \\ l \geq 1: \alpha_1 + \dots + \alpha_l = \alpha, \\ \tau(\alpha_i) = \tau(\alpha), \text{ all } i}} \frac{(-1)^l}{l!} \prod_{i=1}^l [(-1)^{\bar{\chi}([\mathcal{O}_X(-n)] - \alpha_1 - \dots - \alpha_{i-1}, \alpha_i)} \bar{\chi}([\mathcal{O}_X(-n)] - \alpha_1 - \dots - \alpha_{i-1}, \alpha_i) \bar{DT}^{\alpha_i}(\tau)], \quad (5.17)$$

where there are only finitely many nonzero terms in the sum.

As we will see in §6, equation (5.17) is useful for computing invariants  $\bar{DT}^\alpha(\tau)$  in examples. We also use it to deduce the  $\bar{DT}^\alpha(\tau)$  are deformation-invariant.

**COROLLARY 5.28.** *The generalized Donaldson–Thomas invariants  $\bar{DT}^\alpha(\tau)$  of §5.3 are unchanged under continuous deformations of the underlying Calabi–Yau 3-fold  $X$ .*

**PROOF.** Let  $\alpha \in C(\text{coh}(X))$  have dimension  $\dim \alpha = d = 0, 1, 2$  or  $3$ . Then the Hilbert polynomial  $P_\alpha$  is of the form  $P_\alpha(t) = \frac{k}{d!} t^d + a_{d-1} t^{d-1} + \dots + a_0$  for  $k$  a positive integer and  $a_{d-1}, \dots, a_0 \in \mathbb{Q}$ . Fix  $d$ , and suppose by induction on  $K \geq 0$  that  $\bar{DT}^\alpha(\tau)$  is deformation-invariant for all  $\alpha \in C(\text{coh}(X))$  with  $\dim \alpha = d$  and  $P_\alpha(t) = \frac{k}{d!} t^d + \dots + a_0$  for  $k \leq K$ . This is vacuous for  $K = 0$ .

Let  $\alpha \in C(\text{coh}(X))$  with  $\dim \alpha = d$  and  $P_\alpha(t) = \frac{K+1}{d!}t^d + \dots + a_0$ . We rewrite (5.17) by splitting into terms  $l = 1$  and  $l \geq 2$  as

$$\begin{aligned} & (-1)^{\bar{\chi}([\mathcal{O}_X(-n)], \alpha)} \bar{\chi}([\mathcal{O}_X(-n)], \alpha) \bar{D}T^\alpha(\tau) = -PI^{\alpha, n}(\tau') \\ & + \sum_{\substack{\alpha_1, \dots, \alpha_l \in C(\text{coh}(X)), \\ l \geq 2: \alpha_1 + \dots + \alpha_l = \alpha, \\ \tau(\alpha_i) = \tau(\alpha), \text{ all } i}} \frac{(-1)^l}{l!} \prod_{i=1}^l [(-1)^{\bar{\chi}([\mathcal{O}_X(-n)] - \alpha_1 - \dots - \alpha_{i-1}, \alpha_i)} \\ & \bar{\chi}([\mathcal{O}_X(-n)] - \alpha_1 - \dots - \alpha_{i-1}, \alpha_i) \bar{D}T^{\alpha_i}(\tau)]. \end{aligned} \quad (5.18)$$

Here  $\bar{\chi}([\mathcal{O}_X(-n)], \alpha) > 0$  for  $n \gg 0$ , so the coefficient of  $\bar{D}T^\alpha(\tau)$  on the left hand side of (5.18) is nonzero. On the right hand side,  $PI^{\alpha, n}(\tau')$  is unchanged under deformations of  $X$  by Corollary 5.26.

For terms  $l \geq 2$ ,  $\alpha_1, \dots, \alpha_l \in C(\text{coh}(X))$  with  $\alpha_1 + \dots + \alpha_l = \alpha$  and  $\tau(\alpha_i) = \tau(\alpha)$  in (5.18), we have  $\dim \alpha_i = d$  and  $P_{\alpha_i}(t) = \frac{k_i}{d!}t^d + \dots + a_0$ , where  $k_1, \dots, k_l$  are positive integers with  $k_1 + \dots + k_l = K + 1$ . Thus  $k_i \leq K$  for each  $i$ , and  $\bar{D}T^{\alpha_i}(\tau)$  is deformation-invariant by the inductive hypothesis. Therefore everything on the right hand side of (5.18) is deformation-invariant, so  $\bar{D}T^\alpha(\tau)$  is deformation-invariant. This proves the inductive step.  $\square$

In many interesting cases the terms  $\bar{\chi}(\alpha_i, \alpha_j)$  in (5.17) are automatically zero. Then (5.17) simplifies, and we can encode it in a generating function equation. The proof of the next proposition is immediate. Note that there is a problem with choosing  $n$  in (5.20), as (5.19) only holds for  $n \gg 0$  depending on  $\alpha$ , but (5.20) involves one fixed  $n$  but infinitely many  $\alpha$ . We can regard the initial term 1 in (5.20) as  $PI^{\alpha, n}(\tau') q^\alpha$  for  $\alpha = 0$ . In Conjecture 6.12 we will call  $(\tau, T, \leq)$  *generic* if  $\bar{\chi}(\beta, \gamma) = 0$  for all  $\beta, \gamma$  with  $\tau(\beta) = \tau(\gamma)$ .

**PROPOSITION 5.29.** *In the situation above, with  $(\tau, T, \leq)$  a weak stability condition on  $\text{coh}(X)$ , suppose  $t \in T$  is such that  $\bar{\chi}(\beta, \gamma) = 0$  for all  $\beta, \gamma \in C(\text{coh}(X))$  with  $\tau(\beta) = \tau(\gamma) = t$ . Then for all  $\alpha \in C(\text{coh}(X))$  with  $\tau(\alpha) = t$  and  $n \gg 0$  depending on  $\alpha$ , equation (5.17) becomes*

$$PI^{\alpha, n}(\tau') = \sum_{\substack{\alpha_1, \dots, \alpha_l \in C(\text{coh}(X)), \\ l \geq 1: \alpha_1 + \dots + \alpha_l = \alpha, \\ \tau(\alpha_i) = t, \text{ all } i}} \frac{(-1)^l}{l!} \prod_{i=1}^l [(-1)^{\bar{\chi}([\mathcal{O}_X(-n)], \alpha_i)} \bar{\chi}([\mathcal{O}_X(-n)], \alpha_i) \bar{D}T^{\alpha_i}(\tau)]. \quad (5.19)$$

Ignore for the moment the fact that (5.19) only holds for  $n \gg 0$  depending on  $\alpha$ . Then (5.19) can be encoded as the  $q^\alpha$  term in the formal power series

$$\begin{aligned} & 1 + \sum_{\alpha \in C(\text{coh}(X)): \tau(\alpha) = t} PI^{\alpha, n}(\tau') q^\alpha = \\ & \exp \left[ - \sum_{\alpha \in C(\text{coh}(X)): \tau(\alpha) = t} (-1)^{\bar{\chi}([\mathcal{O}_X(-n)], \alpha)} \bar{\chi}([\mathcal{O}_X(-n)], \alpha) \bar{D}T^\alpha(\tau) q^\alpha \right], \end{aligned} \quad (5.20)$$

where  $q^\alpha$  for  $\alpha \in C(\text{coh}(X))$  are formal symbols satisfying  $q^\alpha \cdot q^\beta = q^{\alpha + \beta}$ .

Now Theorem 5.27 relates the invariants  $PI^{\alpha, n}(\tau')$  and  $\bar{D}T^\beta(\tau)$ , which can both be written in terms of Euler characteristics weighted by Behrend functions. There is an analogue in which we simply omit the Behrend functions. Omitting the Behrend function  $\nu_{\mathcal{M}_{\text{stp}}^{\alpha, n}(\tau')}$  in the expression (5.16) for  $PI^{\alpha, n}(\tau')$  shows that the unweighted analogue of  $PI^{\alpha, n}(\tau')$  is  $\chi(\mathcal{M}_{\text{stp}}^{\alpha, n}(\tau'))$ . Comparing (3.22) and (5.7)

shows that (up to sign) the unweighted analogue of  $\bar{D}T^\beta(\tau)$  is the invariant  $J^\beta(\tau)$  of §3.5. The proof of Theorem 5.27 in §13 involves a Lie algebra morphism  $\tilde{\Psi}^{\mathcal{B}_p}$  in §13.4; for the unweighted case we must replace this by a Lie algebra morphism  $\Psi^{\mathcal{B}_p}$  which is related to  $\tilde{\Psi}^{\mathcal{B}_p}$  in the same way that  $\Psi$  in §3.4 is related to  $\tilde{\Psi}$  in §5.3, and maps to a Lie algebra  $L(\mathcal{B}_p)$  with the sign omitted in (13.29). In this way we obtain the following unweighted version of Theorem 5.27:

THEOREM 5.30. *For  $\alpha \in C(\text{coh}(X))$  and  $n \gg 0$  we have*

$$\chi(\mathcal{M}_{\text{stp}}^{\alpha,n}(\tau')) = \sum_{\substack{\alpha_1, \dots, \alpha_l \in C(\text{coh}(X)), \\ l \geq 1: \alpha_1 + \dots + \alpha_l = \alpha, \\ \tau(\alpha_i) = \tau(\alpha), \text{ all } i}} \frac{1}{l!} \prod_{i=1}^l [\tilde{\chi}([\mathcal{O}_X(-n)] - \alpha_1 - \dots - \alpha_{i-1}, \alpha_i) \cdot J^{\alpha_i}(\tau)], \quad (5.21)$$

for  $J^{\alpha_i}(\tau)$  as in §3.5, with only finitely many nonzero terms in the sum.

## CHAPTER 6

### Examples, applications, and generalizations

We now give many worked examples of the theory of §5, and some consequences and further developments. This section considers Donaldson–Thomas theory in  $\text{coh}(X)$ , for  $X$  a Calabi–Yau 3-fold over  $\mathbb{C}$ . As in §5, our definition of Calabi–Yau 3-fold includes the assumption that  $H^1(\mathcal{O}_X) = 0$ , as discussed in Remark 5.2.

Section 7 will discuss Donaldson–Thomas theory in categories of quiver representations  $\text{mod-}\mathbb{C}Q/I$  coming from a superpotential  $W$  on  $Q$ , which is a fertile source of easily computable examples.

#### 6.1. Computing $PI^{\alpha,n}(\tau')$ , $\bar{DT}^{\alpha}(\tau)$ and $J^{\alpha}(\tau)$ in examples

Here are a series of simple situations in which we can calculate contributions to the invariants  $PI^{\alpha,n}(\tau')$  and  $\bar{DT}^{\alpha}(\tau)$  of §5, and  $J^{\alpha}(\tau)$  of §3.5.

**EXAMPLE 6.1.** Let  $X$  be a Calabi–Yau 3-fold over  $\mathbb{C}$  equipped with a very ample line bundle  $\mathcal{O}_X(1)$ . Suppose  $\alpha \in C(\text{coh}(X))$ , and that  $E \in \text{coh}(X)$  with  $[E] = \alpha$  is  $\tau$ -stable and rigid, so that  $\text{Ext}^1(E, E) = 0$ . Then  $mE = E \oplus \cdots \oplus E$  ( $m$  copies) for  $m \geq 2$  is a strictly  $\tau$ -semistable sheaf of class  $m\alpha$ , which is also rigid. Hence  $\{[mE]\}$  is a connected component of  $\mathcal{M}_{\text{ss}}^{m\alpha}(\tau)$ , and  $\pi^{-1}([mE])$  is a connected component of  $\mathcal{M}_{\text{stp}}^{m\alpha,n}(\tau')$  for  $m \geq 1$ , where  $\pi : \mathcal{M}_{\text{stp}}^{m\alpha,n}(\tau') \rightarrow \mathcal{M}_{\text{ss}}^{m\alpha}(\tau)$  is the projection from a stable pair  $s : \mathcal{O}(-n) \rightarrow E'$  to the S-equivalence class  $[E']$  of the underlying  $\tau$ -semistable sheaf  $E'$ . Suppose for simplicity that  $mE$  is the only  $\tau$ -semistable sheaf of class  $m\alpha$ ; alternatively, we can consider the following as computing the contribution to  $PI^{m\alpha,n}(\tau')$  from stable pairs  $s : \mathcal{O}(-n) \rightarrow mE$ .

A pair  $s : \mathcal{O}(-n) \rightarrow mE$  may be regarded as  $m$  elements  $s^1, \dots, s^m$  of  $H^0(E(n)) \cong \mathbb{C}^{P_{\alpha}(n)}$  for  $n \gg 0$ , where  $P_{\alpha}$  is the Hilbert polynomial of  $E$ . Such a pair turns out to be stable if and only if  $s^1, \dots, s^m$  are linearly independent in  $H^0(E(n))$ . Two such pairs are equivalent if they are identified under the action of  $\text{Aut}(mE) \cong \text{GL}(m, \mathbb{C})$ , acting in the obvious way on  $(s^1, \dots, s^m)$ . Thus, equivalence classes of stable pairs correspond to linear subspaces of dimension  $m$  in  $H^0(E(n))$ , so the moduli space  $\mathcal{M}_{\text{stp}}^{m\alpha,n}(\tau')$  is isomorphic as a  $\mathbb{C}$ -scheme to the Grassmannian  $\text{Gr}(\mathbb{C}^m, \mathbb{C}^{P_{\alpha}(n)})$ . This is smooth of dimension  $m(P_{\alpha}(n) - m)$ , so that  $\nu_{\mathcal{M}_{\text{stp}}^{m\alpha,n}(\tau')} \equiv (-1)^{m(P_{\alpha}(n) - m)}$  by Theorem 4.3(i). Also  $\text{Gr}(\mathbb{C}^m, \mathbb{C}^{P_{\alpha}(n)})$  has Euler characteristic the binomial coefficient  $\binom{P_{\alpha}(n)}{m}$ . Therefore (5.16) gives

$$PI^{m\alpha,n}(\tau') = (-1)^{m(P_{\alpha}(n) - m)} \binom{P_{\alpha}(n)}{m}. \quad (6.1)$$

We can use equations (5.17) and (5.21) to compute the generalized Donaldson–Thomas invariants  $\bar{DT}^{m\alpha}(\tau)$  and invariants  $J^{m\alpha}(\tau)$  in Example 6.1.

EXAMPLE 6.2. Work in the situation of Example 6.1, and assume that  $mE$  is the only  $\tau$ -semistable sheaf of class  $m\alpha$  for all  $m \geq 1$ , up to isomorphism. Consider (5.17) with  $m\alpha$  in place of  $\alpha$ . If  $\alpha_1, \dots, \alpha_l$  give a nonzero term on the right hand side of (5.17) then  $m\alpha = \alpha_1 + \dots + \alpha_l$ , and  $\bar{D}T^{\alpha_i}(\tau) \neq 0$ , so there exists a  $\tau$ -semistable  $E_i$  in class  $\alpha_i$ . Thus  $E_1 \oplus \dots \oplus E_l$  lies in class  $m\alpha$ , and is  $\tau$ -semistable as  $\tau(\alpha_i) = \tau(\alpha)$  for all  $i$ . Hence  $E_1 \oplus \dots \oplus E_l \cong mE$ , which implies that  $E_i \cong k_i E$  for some  $k_1, \dots, k_l \geq 1$  with  $k_1 + \dots + k_l = m$ , and  $\alpha_i = k_i \alpha$ .

Setting  $\alpha_i = k_i \alpha$ , we see that  $\bar{\chi}(\alpha_j, \alpha_i) = 0$  and  $\bar{\chi}([\mathcal{O}_X(-n)], \alpha_i) = k_i P_\alpha(n)$ , where  $P_\alpha$  is the Hilbert polynomial of  $E$ . Thus in (5.17) we have  $\bar{\chi}([\mathcal{O}_X(-n)] - \alpha_1 - \dots - \alpha_{i-1}, \alpha_i) = k_i P_\alpha(n)$ . Combining (6.1), and (5.17) with these substitutions, and cancelling a factor of  $(-1)^{mP_\alpha(n)}$  on both sides, yields

$$(-1)^m \binom{P_\alpha(n)}{m} = \sum_{\substack{l, k_1, \dots, k_l \geq 1: \\ k_1 + \dots + k_l = m}} \frac{(-1)^l}{l!} \prod_{i=1}^l k_i P_\alpha(n) \bar{D}T^{k_i \alpha}(\tau). \quad (6.2)$$

Regarding each side as a polynomial in  $P_\alpha(n)$  and taking the linear term in  $P_\alpha(n)$  we see that

$$\bar{D}T^{m\alpha}(\tau) = \frac{1}{m^2} \quad \text{for all } m \geq 1. \quad (6.3)$$

Setting  $\bar{D}T^{k_i \alpha}(\tau) = 1/k_i^2$ , we see that (6.2) is the  $x^m$  term in the power series expansion of the identity

$$(1-x)^{P_\alpha(n)} = \exp[-P_\alpha(n) \sum_{k=1}^{\infty} x^k/k].$$

This provides a consistency check for (5.17) in this example: there exist unique values for  $\bar{D}T^{k\alpha}(\tau)$  for  $k = 1, 2, \dots$  such that (6.2) holds for all  $n, m$ .

In the same way, by (5.21) the analogue of (6.2) is

$$\binom{P_\alpha(n)}{m} = \sum_{\substack{l, k_1, \dots, k_l \geq 1: \\ k_1 + \dots + k_l = m}} \frac{1}{l!} \prod_{i=1}^l k_i P_\alpha(n) J^{k_i \alpha}(\tau).$$

Taking the linear term in  $P_\alpha(n)$  on both sides gives

$$J^{m\alpha}(\tau) = \frac{(-1)^{m-1}}{m^2} \quad \text{for all } m \geq 1. \quad (6.4)$$

From (6.3)–(6.4) we see that

COROLLARY 6.3. *The invariants  $\bar{D}T^\alpha(\tau), J^\alpha(\tau) \in \mathbb{Q}$  need not be integers.*

EXAMPLE 6.4. Work in the situation of Example 6.1, but suppose now that  $E_1, \dots, E_l$  are rigid, pairwise non-isomorphic  $\tau$ -stable coherent sheaves with  $[E_i] = \alpha_i \in C(\text{coh}(X))$ , where  $\alpha_1, \dots, \alpha_l$  are distinct with  $\tau(\alpha_1) = \dots = \tau(\alpha_l) = \tau(\alpha)$  for  $\alpha = \alpha_1 + \dots + \alpha_l$ , and suppose  $E = E_1 \oplus \dots \oplus E_l$  is the only  $\tau$ -semistable sheaf in class  $\alpha \in C(\text{coh}(X))$ , up to isomorphism. Then by properties of (semi)stable sheaves we have  $\text{Hom}(E_i, E_j) = 0$  for all  $i \neq j$ , so  $\text{Hom}(E, E) = \bigoplus_{i=1}^l \text{Hom}(E_i, E_i)$ , and  $\text{Aut}(E) = \prod_{i=1}^l \text{Aut}(E_i) \cong \mathbb{G}_m^l$ . A pair  $s : \mathcal{O}(-n) \rightarrow E$  is an  $l$ -tuple  $(s_1, \dots, s_l)$  with  $s_i \in H^0(E_i(n)) \cong \mathbb{C}^{P_{\alpha_i}(n)}$ .

The condition for  $s : \mathcal{O}(-n) \rightarrow E$  to be stable is  $s_i \neq 0$  for  $i = 1, \dots, l$ . Thus  $\mathcal{M}_{\text{stp}}^{\alpha, n}(\tau')$  is the quotient of  $\prod_{i=1}^l (H^0(E_i(n)) \setminus \{0\})$  by  $\text{Aut}(E) \cong \mathbb{G}_m^l$ , so that

$\mathcal{M}_{\text{stp}}^{\alpha,n}(\tau') \cong \prod_{i=1}^l \mathbb{CP}^{P_{\alpha_i}(n)-1}$  as a smooth  $\mathbb{C}$ -scheme, where  $E_i$  has Hilbert polynomial  $P_{\alpha_i}$ . This has Euler characteristic  $\prod_{i=1}^l P_{\alpha_i}(n)$  and dimension  $\sum_{i=1}^l (P_{\alpha_i}(n) - 1)$ , so that  $\nu_{\mathcal{M}_{\text{stp}}^{\alpha,n}(\tau')} \equiv (-1)^{\sum_{i=1}^l (P_{\alpha_i}(n)-1)}$ . So (5.16) gives

$$PI^{\alpha,n}(\tau') = (-1)^{\sum_{i=1}^l (P_{\alpha_i}(n)-1)} \prod_{i=1}^l P_{\alpha_i}(n). \quad (6.5)$$

EXAMPLE 6.5. We work in the situation of Example 6.4. Let  $i, j = 1, \dots, l$  with  $i \neq j$ . Since  $E_i, E_j$  are nonisomorphic  $\tau$ -stable sheaves with  $\tau([E_i]) = \tau([E_j])$  we have  $\text{Hom}(E_i, E_j) = \text{Hom}(E_j, E_i) = 0$ . As by assumption  $E = E_1 \oplus \dots \oplus E_l$  is the only  $\tau$ -semistable sheaf in class  $\alpha$ , we also have  $\text{Ext}^1(E_i, E_j) = \text{Ext}^1(E_j, E_i) = 0$ , since if  $\text{Ext}^1(E_i, E_j) \neq 0$  we would have a nontrivial extension  $0 \rightarrow E_j \rightarrow F \rightarrow E_i \rightarrow 0$ , and then  $F \oplus \bigoplus_{k \neq i,j} E_k$  would be a  $\tau$ -semistable sheaf in class  $\alpha$  not isomorphic to  $E$ . So by (3.14) we have  $\bar{\chi}([E_i], [E_j]) = \bar{\chi}(\alpha_i, \alpha_j) = 0$ .

If  $\emptyset \neq I \subseteq \{1, \dots, l\}$ , using (3.4) and  $\text{Hom}(E_i, E_j) = 0 = \text{Ext}^1(E_i, E_j)$  gives

$$\begin{aligned} \bar{\epsilon}^{\sum_{i \in I} \alpha_i}(\tau) &= \sum_{\substack{I=I_1 \amalg \dots \amalg I_n: \\ n \geq 1, I_j \neq \emptyset}} \frac{(-1)^{n-1}}{n} \bar{\delta}_{\text{ss}}^{\sum_{i \in I_1} \alpha_i}(\tau) * \dots * \bar{\delta}_{\text{ss}}^{\sum_{i \in I_n} \alpha_i}(\tau) \\ &= \sum_{\substack{I=I_1 \amalg \dots \amalg I_n: \\ n \geq 1, I_j \neq \emptyset}} \frac{(-1)^{n-1}}{n} \bar{\delta}_{[\bigoplus_{i \in I_1} E_i]} * \dots * \bar{\delta}_{[\bigoplus_{i \in I_n} E_i]} \\ &= \left[ \sum_{\substack{I=I_1 \amalg \dots \amalg I_n: \\ n \geq 1, I_j \neq \emptyset}} \frac{(-1)^{n-1}}{n} \right] \bar{\delta}_{[\bigoplus_{i \in I} E_i]} = \begin{cases} \bar{\delta}_{[E_i]}, & I = \{i\}, \\ 0, & |I| \geq 2, \end{cases} \end{aligned} \quad (6.6)$$

where the combinatorial sum  $[\dots]$  in the last line is evaluated as in the proof of [53, Th. 7.8]. It follows from (5.7) that  $\bar{D}T^{\alpha_i}(\tau) = 1$  for all  $i = 1, \dots, l$ , and  $\bar{D}T^{\sum_{i \in I} \alpha_i}(\tau) = 0$  for all subsets  $I \subseteq \{1, \dots, l\}$  with  $|I| \geq 2$ . Substituting these values into (5.17), the only nonzero terms come from splitting  $\alpha = \alpha_{\sigma(1)} + \alpha_{\sigma(2)} + \dots + \alpha_{\sigma(l)}$ , where  $\sigma : \{1, \dots, l\} \rightarrow \{1, \dots, l\}$  is a permutation of  $\{1, \dots, l\}$ . This term contributes

$$\begin{aligned} &\frac{(-1)^l}{l!} \prod_{i=1}^l [(-1)^{\bar{\chi}([\mathcal{O}_X(-n)] - \alpha_{\sigma(1)} - \dots - \alpha_{\sigma(i-1)}, \alpha_{\sigma(i)})} \\ &\quad \bar{\chi}([\mathcal{O}_X(-n)] - \alpha_{\sigma(1)} - \dots - \alpha_{\sigma(i-1)}, \alpha_{\sigma(i)}) \cdot 1] \\ &= \frac{(-1)^l}{l!} \prod_{i=1}^l [(-1)^{P_{\alpha_{\sigma(i)}}(n)} P_{\alpha_{\sigma(i)}}(n)] = \frac{1}{l!} \cdot (-1)^{\sum_{i=1}^l (P_{\alpha_i}(n)-1)} \prod_{i=1}^l P_{\alpha_i}(n) \end{aligned}$$

to the r.h.s. of (5.17). As there are  $l!$  permutations  $\sigma$ , summing these contributions in (5.17) gives (6.5). A similar computation with (5.21) shows that  $J^{\alpha_i}(\tau) = 1$  and  $J^{\sum_{i \in I} \alpha_i}(\tau) = 0$  when  $|I| \geq 2$ . Thus we see that if  $E_1, \dots, E_l$  are pairwise nonisomorphic  $\tau$ -stable sheaves for  $l \geq 2$  with  $[E_i] = \alpha_i$ , and  $\tau(\alpha_i) = \tau(\alpha_j)$  and  $\text{Ext}^1(E_i, E_j) = 0$ , then the  $\tau$ -semistable sheaf  $E_1 \oplus \dots \oplus E_l$  contributes zero to  $\bar{D}T^{\alpha_1 + \dots + \alpha_l}(\tau)$  and  $J^{\alpha_1 + \dots + \alpha_l}(\tau)$ .

EXAMPLE 6.6. We combine Examples 6.1 and 6.4. Suppose  $E_1, \dots, E_l$  are rigid, pairwise non-isomorphic stable coherent sheaves, where  $E_i$  has Hilbert polynomial  $P_{\alpha_i}$ , that  $m_1, \dots, m_l \geq 1$ , and that  $E = m_1 E_1 \oplus \dots \oplus m_l E_l$  is the only semistable sheaf in class  $\alpha \in C(\text{coh}(X))$ , up to isomorphism.

Then a pair  $s : \mathcal{O}(-n) \rightarrow E$  is a collection of  $s_i^j \in H^0(E_i(n)) \cong \mathbb{C}^{P_{\alpha_i}(n)}$  for  $i = 1, \dots, l$  and  $j = 1, \dots, m_i$ , and is stable if and only if  $s_i^1, \dots, s_i^{m_i}$  are linearly independent in  $H^0(E_i(n))$  for all  $i = 1, \dots, l$ . The automorphism group  $\text{Aut}(E) \cong \prod_{i=1}^l \text{GL}(m_i, \mathbb{C})$  acts upon the set of such stable pairs, and taking the quotient shows that the moduli space  $\mathcal{M}_{\text{stp}}^{\alpha, n}(\tau')$  is isomorphic to the product of Grassmannians  $\prod_{i=1}^l \text{Gr}(\mathbb{C}^{m_i}, \mathbb{C}^{P_{\alpha_i}(n)})$ . Hence

$$PI^{\alpha, n}(\tau') = \prod_{i=1}^l (-1)^{m_i(P_{\alpha_i}(n) - m_i)} \binom{P_{\alpha_i}(n)}{m_i}. \quad (6.7)$$

Equation (6.7) includes both (6.1) as the case  $l = 1$  with  $P_1 = P$ ,  $m_1 = m$  and  $\alpha$  in place of  $m\alpha$ , and (6.5) as the case  $m_1 = \dots = m_l = 1$ .

EXAMPLE 6.7. We combine Examples 6.2 and 6.5. Work in the situation of Example 6.6. Then  $\bar{\chi}(\alpha_i, \alpha_j) = 0$  for  $i \neq j$ . Suppose for simplicity that  $\alpha_1, \dots, \alpha_l$  are linearly independent over  $\mathbb{Z}$  in  $K^{\text{num}}(\text{coh}(X))$ , and that  $m_1 E_1 \oplus \dots \oplus m_l E_l$  is the only  $\tau$ -semistable sheaf in class  $m_1 \alpha_1 + \dots + m_l \alpha_l$  for all  $m_1, \dots, m_l \geq 0$ . We claim that  $\bar{D}T^{m_i \alpha_i}(\tau) = 1/m_i^2$  and  $J^{m_i \alpha_i}(\tau) = (-1)^{m_i-1}/m_i^2$  for all  $i = 1, \dots, l$  and  $m_i = 1, 2, \dots$ , and  $\bar{D}T^{m_1 \alpha_1 + \dots + m_l \alpha_l}(\tau) = J^{m_1 \alpha_1 + \dots + m_l \alpha_l}(\tau) = 0$  whenever at least two  $m_i$  are positive. The latter holds as  $\bar{\epsilon}^{m_1 \alpha_1 + \dots + m_l \alpha_l}(\tau) = 0$  whenever at least two  $m_i$  are positive, as in (6.6). It is not difficult to show, as in Examples 6.2 and 6.5, that substituting these values into the r.h.s. of (5.17) gives (6.7).

EXAMPLE 6.8. Suppose now that  $E_1, E_2$  are rigid  $\tau$ -stable sheaves in classes  $\alpha_1, \alpha_2$  in  $C(\text{coh}(X))$  with  $\alpha_1 \neq \alpha_2$  and  $\tau(\alpha_1) = \tau(\alpha_2) = \tau(\alpha)$ , where  $\alpha = \alpha_1 + \alpha_2$ . Suppose too that  $\text{Ext}^1(E_1, E_2) = 0$  and  $\text{Ext}^1(E_2, E_1) \cong \mathbb{C}^d$ . We have  $\text{Hom}(E_1, E_2) = \text{Hom}(E_2, E_1) = 0$ , as  $E_1, E_2$  are nonisomorphic  $\tau$ -stable sheaves with  $\tau([E_1]) = \tau([E_2])$ . So by (3.14) we have  $\bar{\chi}(\alpha_1, \alpha_2) = d$ .

As  $E_1, E_2$  are rigid we have  $\text{Ext}^1(E_1, E_1) = \text{Ext}^1(E_2, E_2) = 0$ . Hence  $\text{Ext}^1(E_1 \oplus E_2, E_1 \oplus E_2) = \text{Ext}^1(E_2, E_1) \cong \mathbb{C}^d$ . Now  $\text{Ext}^1(E_1 \oplus E_2, E_1 \oplus E_2)$  parametrizes infinitesimal deformations of  $E_1 \oplus E_2$ . All deformations in  $\text{Ext}^1(E_2, E_1)$  are realized by sheaves  $F$  in exact sequences  $0 \rightarrow E_1 \rightarrow F \rightarrow E_2 \rightarrow 0$ . Therefore as  $\text{Ext}^1(E_1 \oplus E_2, E_1 \oplus E_2) = \text{Ext}^1(E_2, E_1)$ , all deformations of  $E_1 \oplus E_2$  are unobstructed, and the moduli stack of deformations of  $E_1 \oplus E_2$  is the quotient stack  $[\text{Ext}^1(E_1 \oplus E_2, E_1 \oplus E_2) / \text{Aut}(E_1 \oplus E_2)] \cong [\mathbb{C}^d / \mathbb{G}_m^2]$ , where  $\mathbb{G}_m^2$  acts on  $\mathbb{C}^d$  by  $(\lambda, \mu) : v \mapsto \lambda \mu^{-1} v$ .

Suppose now that the only  $\tau$ -semistable sheaf up to isomorphism in class  $\alpha_1$  is  $E_1$ , and the only in class  $\alpha_2$  is  $E_2$ , and the only in class  $\alpha_1 + \alpha_2$  are extensions  $F$  in  $0 \rightarrow E_1 \rightarrow F \rightarrow E_2 \rightarrow 0$ . Then we have  $\mathfrak{M}_{\text{ss}}^{\alpha_1}(\tau) \cong [\text{Spec } \mathbb{C} / \mathbb{G}_m] \cong \mathfrak{M}_{\text{ss}}^{\alpha_2}(\tau)$ , and  $\mathfrak{M}_{\text{ss}}^{\alpha_1 + \alpha_2}(\tau) \cong [\mathbb{C}^d / \mathbb{G}_m^2]$ . These are smooth of dimensions  $-1, -1, d-2$  respectively, and  $\mathfrak{M}_{\text{ss}}^{\alpha_1 + \alpha_2}(\tau)$  is the non-separated disjoint union of a projective space  $\mathbb{CP}^{d-1}$  with stabilizer groups  $\mathbb{G}_m$ , and a point with stabilizer group  $\mathbb{G}_m^2$ .

The moduli space  $\mathcal{M}_{\text{stp}}^{\alpha_1 + \alpha_2, n}(\tau')$  has points  $s : \mathcal{O}_X(-n) \rightarrow F_\epsilon$ , for  $0 \rightarrow E_1 \rightarrow F_\epsilon \rightarrow E_2 \rightarrow 0$  exact. Here  $F_\epsilon$  corresponds to some  $\epsilon \in \text{Ext}^1(E_2, E_1)$ , and  $s \in H^0(F_\epsilon(n))$ , where the exact sequence  $0 \rightarrow E_1 \rightarrow F_\epsilon \rightarrow E_2 \rightarrow 0$  and  $E_1, F_\epsilon, E_2$   $n$ -regular give an exact sequence

$$0 \longrightarrow H^0(E_1(n)) \longrightarrow H^0(F_\epsilon(n)) \longrightarrow H^0(E_2(n)) \longrightarrow 0.$$

Globally over  $\epsilon \in \text{Ext}^1(E_2, E_1)$  we can (noncanonically) split this short exact sequence and identify  $H^0(F_\epsilon(n)) \cong H^0(E_1(n)) \oplus H^0(E_2(n))$ , so  $s \in H^0(F_\epsilon(n))$  is identified with  $(s_1, s_2) \in H^0(E_1(n)) \oplus H^0(E_2(n)) \cong \mathbb{C}^{P_{\alpha_1}(n)} \oplus \mathbb{C}^{P_{\alpha_2}(n)}$ .

The condition that  $s : \mathcal{O}_X(-n) \rightarrow F_\epsilon$  is a stable pair turns out to be that either  $\epsilon \neq 0$  and  $s_2 \neq 0$ , or  $\epsilon = 0$  and  $s_1, s_2 \neq 0$ . The equivalence relation on triples  $(s_1, s_2, \epsilon)$  is that  $(s_1, s_2, \epsilon) \sim (\lambda s_1, \mu s_2, \lambda \mu^{-1} \epsilon)$ , for  $\lambda \in \text{Aut}(E_1) \cong \mathbb{G}_m$  and  $\mu \in \text{Aut}(E_2) \cong \mathbb{G}_m$ . This proves that

$$\mathcal{M}_{\text{stp}}^{\alpha_1+\alpha_2,n}(\tau') \cong \{(s_1, s_2, \epsilon) \in \mathbb{C}^{P_{\alpha_1}(n)} \oplus \mathbb{C}^{P_{\alpha_2}(n)} \oplus \mathbb{C}^d : \epsilon \neq 0 \text{ and } s_2 \neq 0, \\ \text{or } \epsilon = 0 \text{ and } s_1, s_2 \neq 0\} / \mathbb{G}_m^2.$$

Therefore  $\mathcal{M}_{\text{stp}}^{\alpha_1+\alpha_2,n}(\tau')$  is a smooth projective variety of dimension  $P_{\alpha_1}(n) + P_{\alpha_2}(n) + d - 2$ , so  $\nu_{\mathcal{M}_{\text{stp}}^{\alpha_1+\alpha_2,n}(\tau')} = (-1)^{P_{\alpha_1}(n)+P_{\alpha_2}(n)+d-2}$ . We cut  $\mathcal{M}_{\text{stp}}^{\alpha_1+\alpha_2,n}(\tau')$  into the disjoint union of two pieces (a) points with  $\epsilon = 0$ , and (b) points with  $\epsilon \neq 0$ . Piece (a) is isomorphic to  $\mathbb{CP}^{P_{\alpha_1}(n)-1} \times \mathbb{CP}^{P_{\alpha_2}(n)-1}$ , and has Euler characteristic  $P_{\alpha_1}(n)P_{\alpha_2}(n)$ . Piece (b) is a vector bundle over  $\mathbb{CP}^{P_{\alpha_2}(n)-1} \times \mathbb{CP}^{d-1}$  with fibre  $\mathbb{C}^{P_{\alpha_1}(n)}$ , and has Euler characteristic  $P_{\alpha_2}(n)d$ . Hence  $\mathcal{M}_{\text{stp}}^{\alpha_1+\alpha_2,n}(\tau')$  has Euler characteristic  $(P_{\alpha_1}(n) + d)P_{\alpha_2}(n)$ , and (5.16) yields

$$PI^{\alpha_1+\alpha_2,n}(\tau') = (-1)^{P_{\alpha_1}(n)+P_{\alpha_2}(n)+d-2} (P_{\alpha_1}(n) + d)P_{\alpha_2}(n). \quad (6.8)$$

The expression (5.17) for  $PI^{\alpha_1+\alpha_2,n}(\tau')$  yields

$$PI^{\alpha_1+\alpha_2,n}(\tau') = -(-1)^{P_{\alpha_1}(n)+P_{\alpha_2}(n)} (P_{\alpha_1}(n) + P_{\alpha_2}(n)) \bar{D}T^{\alpha_1+\alpha_2}(\tau) \\ + \frac{1}{2}(-1)^{P_{\alpha_1}(n)} P_{\alpha_1}(n) (-1)^{P_{\alpha_2}(n)-d} (P_{\alpha_2}(n) - d) \bar{D}T^{\alpha_1}(\tau) \bar{D}T^{\alpha_2}(\tau) \\ + \frac{1}{2}(-1)^{P_{\alpha_2}(n)} P_{\alpha_2}(n) (-1)^{P_{\alpha_1}(n)+d} (P_{\alpha_1}(n) + d) \bar{D}T^{\alpha_2}(\tau) \bar{D}T^{\alpha_1}(\tau), \quad (6.9)$$

where the three terms on the right correspond to splitting  $\alpha$  into  $\alpha = \alpha$  with  $l = 1$ , into  $\alpha = \alpha_1 + \alpha_2$  with  $l = 2$ , and into  $\alpha = \alpha_2 + \alpha_1$  with  $l = 2$  respectively. We have  $\bar{D}T^{\alpha_i}(\tau) = 1$  by Example 6.2. So comparing (6.8) and (6.9) shows that  $\bar{D}T^{\alpha_1+\alpha_2}(\tau) = (-1)^{d-1}d/2$ , and similarly  $J^{\alpha_1+\alpha_2}(\tau) = d/2$ .

Here is a more complicated example illustrating non-smooth moduli spaces, nontrivial Behrend functions, and failure of deformation-invariance of the  $J^\alpha(\tau)$ .

EXAMPLE 6.9. Let  $X_t$  for  $t \in \mathbb{C}$  be a smooth family of Calabi–Yau 3-folds over  $\mathbb{C}$ , equipped with a smooth family of very ample line bundles  $\mathcal{O}_{X_t}(1)$ ; note that our definition of Calabi–Yau 3-fold requires that  $H^1(\mathcal{O}_{X_t}) = 0$ . Then by Theorem 4.19 the numerical Grothendieck groups  $K^{\text{num}}(\text{coh}(X_t))$  for  $t \in \mathbb{C}$  are all canonically isomorphic, so we identify them with  $K^{\text{num}}(\text{coh}(X_0))$ . Suppose  $\alpha \in K^{\text{num}}(\text{coh}(X_0))$ , and that

$$\mathfrak{M}_{\text{st}}^\alpha(\tau)_t = \mathfrak{M}_{\text{ss}}^\alpha(\tau)_t \cong \text{Spec}(\mathbb{C}[z]/(z^2 - t^2)) \times [\text{Spec } \mathbb{C}/\mathbb{G}_m]$$

for all  $t \in \mathbb{C}$ , where the subscript  $t$  means the moduli space for  $X_t$ . That is,  $\mathfrak{M}_{\text{ss}}^\alpha(\tau)_t$  for  $t \neq 0$  is the disjoint union of two points  $[\text{Spec } \mathbb{C}/\mathbb{G}_m]$  at  $z = t$  and  $z = -t$ , which correspond to rigid, stable sheaves  $E_+, E_-$  with  $[E_\pm] = \alpha$ . But  $\mathfrak{M}_{\text{ss}}^\alpha(\tau)_0$  is  $\text{Spec}(\mathbb{C}[z]/(z^2)) \times [\text{Spec } \mathbb{C}/\mathbb{G}_m]$ . This contains only one stable sheaf  $E_0$ , whose moduli space is a double point. That is,  $E_0$  has one infinitesimal deformation, so that  $\text{Ext}^1(E_0, E_0) = \mathbb{C}$ , but this deformation is obstructed to second order. So the picture is that as  $t \rightarrow 0$ , the two distinct rigid stable sheaves  $E_+, E_-$  come together, and at  $t = 0$  they are replaced by one stable, non-rigid sheaf  $E_0$  with an infinitesimal deformation.

First consider the invariants  $\bar{D}T^\alpha(\tau)_t$  and  $J^\alpha(\tau)_t$ . Since  $\mathfrak{M}_{\text{st}}^\alpha(\tau)_t = \mathfrak{M}_{\text{ss}}^\alpha(\tau)_t$  we have  $\bar{e}^\alpha(\tau)_t = \bar{\delta}_{\mathfrak{M}_{\text{ss}}^\alpha(\tau)_t}$ . When  $t \neq 0$ ,  $\mathfrak{M}_{\text{ss}}^\alpha(\tau)_t \cong [\text{Spec } \mathbb{C}/\mathbb{G}_m] \amalg [\text{Spec } \mathbb{C}/\mathbb{G}_m]$  is



smooth of dimension  $-1$ , so  $\nu_{\mathfrak{M}_{\text{ss}}^\alpha(\tau)_t} \equiv -1$ . It follows that  $\tilde{\Psi}(\bar{\epsilon}^\alpha(\tau)_t) = -2\tilde{\lambda}^\alpha$  in the notation of §5.3, so  $\bar{D}T^\alpha(\tau)_t = 2$  by (5.7). Similarly,  $\Psi(\bar{\epsilon}^\alpha(\tau)_t) = 2\lambda^\alpha$  in the notation of §3.4, so  $J^\alpha(\tau)_t = 2$  by (3.22).

When  $t = 0$ ,  $\mathfrak{M}_{\text{ss}}^\alpha(\tau)_0$  is not smooth. As  $\text{Spec}(\mathbb{C}[z]/(z^2)) = \text{Crit}(\frac{1}{3}z^3)$ , the Milnor fibre of  $\frac{1}{3}z^3$  is 3 points, and  $\dim \mathbb{C} = 1$ , we have  $\nu_{\text{Spec}(\mathbb{C}[z]/(z^2))} \equiv 2$  by Theorem 4.7, so  $\nu_{\mathfrak{M}_{\text{ss}}^\alpha(\tau)_0} = -2$  by Theorem 4.3(i) and Corollary 4.5. Thus, as  $\mathfrak{M}_{\text{ss}}^\alpha(\tau)_0$  is a single point with Behrend function  $-2$  we have  $\tilde{\Psi}(\bar{\epsilon}^\alpha(\tau)_0) = -2\tilde{\lambda}^\alpha$ , so  $\bar{D}T^\alpha(\tau)_0 = 2$ , but  $\Psi(\bar{\epsilon}^\alpha(\tau)_0) = \lambda^\alpha$ , so  $J^\alpha(\tau)_0 = 1$ . To summarize,

$$\bar{D}T^\alpha(\tau)_t = 2, \quad \text{all } t, \quad \text{and} \quad J^\alpha(\tau)_t = \begin{cases} 2, & t \neq 0, \\ 1, & t = 0. \end{cases} \quad (6.10)$$

Now let us assume that the only  $\tau$ -semistable sheaves in class  $2\alpha$  are those with stable factors in class  $\alpha$ . Thus, when  $t \neq 0$  the  $\tau$ -semistable sheaves in class  $\alpha$  are  $E_+ \oplus E_+$ , and  $E_- \oplus E_-$ , and  $E_+ \oplus E_-$ . Example 6.2 when  $m = 2$  implies that  $E_+ \oplus E_+$ , and  $E_- \oplus E_-$  each contribute  $\frac{1}{4}$  to  $\bar{D}T^{2\alpha}(\tau)_t$  and  $-\frac{1}{4}$  to  $J^{2\alpha}(\tau)_t$ , and Example 6.5 shows that  $E_+ \oplus E_-$  contributes 0 to both. Therefore  $\bar{D}T^{2\alpha}(\tau)_t = \frac{1}{2}$  and  $J^{2\alpha}(\tau)_t = -\frac{1}{2}$ .

When  $t = 0$ , as  $\text{Ext}^1(E_0, E_0) = \mathbb{C}$ , there is one nontrivial extension  $F$  in  $0 \rightarrow E_0 \rightarrow F \rightarrow E_0 \rightarrow 0$ . Hence  $\mathfrak{M}_{\text{ss}}^{2\alpha}(\tau)_0(\mathbb{C})$  consists of two points  $[E_0 \oplus E_0]$  and  $[F]$ . Since  $\text{Aut}(E_0 \oplus E_0) \cong \text{GL}(2, \mathbb{C})$  is the complexification of its maximal subgroup  $\text{U}(2)$ , Theorem 5.5 implies that we may write  $\mathfrak{M}_{\text{ss}}^{2\alpha}(\tau)_0$  locally near  $[E_0 \oplus E_0]$  in the complex analytic topology as  $\text{Crit}(f)/\text{Aut}(E_0 \oplus E_0)$ , where  $U \subseteq \text{Ext}^1(E_0 \oplus E_0, E_0 \oplus E_0)$  is an  $\text{Aut}(E_0 \oplus E_0)$ -invariant open set, and  $f : U \rightarrow \mathbb{C}$  is an  $\text{Aut}(E_0 \oplus E_0)$ -invariant holomorphic function. As  $\text{Ext}^1(E_0, E_0) \cong \mathbb{C}$ , we may identify  $\text{Ext}^1(E_0 \oplus E_0, E_0 \oplus E_0)$  with  $2 \times 2$  complex matrices  $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ , with  $\text{Aut}(E_0 \oplus E_0) \cong \text{GL}(2, \mathbb{C})$  acting by conjugation.

Since  $f$  is a conjugation-invariant holomorphic function, it must be a function of  $\text{Tr}(A)$  and  $\det(A)$ . But when we restrict to diagonal matrices  $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ ,  $f$  must reduce to the potential defining  $\mathfrak{M}_0^\alpha \times \mathfrak{M}_0^\alpha$  at  $(E_0, E_0)$ . As  $\text{Spec}(\mathbb{C}[z]/(z^2)) = \text{Crit}(\frac{1}{3}z^3)$ , we want  $f\left(\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}\right) = \frac{1}{3}a^3 + \frac{1}{3}d^3$ . But  $f$  is a function of  $\text{Tr}(A)$  and  $\det(A)$ , so we see that

$$f\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \frac{1}{3}(\text{Tr } A)^3 - \text{Tr } A \det A = \frac{1}{3}(a^3 + d^3) + (a + d)bc.$$

We can then take  $U = \text{Ext}^1(E_0 \oplus E_0, E_0 \oplus E_0)$ , and we see that  $\mathfrak{M}_{\text{ss}}^{2\alpha}(\tau)_0 \cong [\text{Crit}(f)/\text{GL}(2, \mathbb{C})]$  as an Artin stack.

Now  $\text{Crit}(f)$  consists of two  $\text{GL}(2, \mathbb{C})$ -orbits, the point 0 which corresponds to  $E_0 \oplus E_0$ , and the orbit of  $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$  which corresponds to  $F$ . As  $f$  is a homogeneous polynomial, the Milnor fibre  $MF_f(0)$  is diffeomorphic to  $f^{-1}(1)$ . One can show that  $\chi(f^{-1}(1)) = -3$ , so  $\nu_{\text{Crit}(f)}(0) = 4$  by (4.2), and as the projection  $\text{Crit}(f) \rightarrow [\text{Crit } f/\text{GL}(2, \mathbb{C})]$  is smooth of relative dimension 4, we deduce that  $\nu_{\mathfrak{M}_{\text{ss}}^{2\alpha}(\tau)_0}(E_0 \oplus E_0) = 4$  by Corollary 4.5. This also follows from  $\nu_{\mathfrak{M}_{\text{ss}}^\alpha(\tau)_0}(E_0) = -2$  and equation (5.2). The orbit of  $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$  in  $\text{Crit}(f)$  is smooth of dimension 2, so Theorem 4.3(i) gives  $\nu_{\text{Crit}(f)}\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) = 1$ , and  $\nu_{\mathfrak{M}_{\text{ss}}^{2\alpha}(\tau)_0}(F) = 1$ .

Using the definition (3.4) of  $\bar{\epsilon}^{2\alpha}(\tau)$  and the relations in  $\text{SF}_{\text{al}}^{\text{ind}}(\mathfrak{M}, \chi, \mathbb{Q})$  in §2.4, reasoning as in the proof of Theorem 5.14 in §11 we can show that

$$\bar{\Pi}_{\mathfrak{M}}^{\chi, \mathbb{Q}}(\bar{\epsilon}^{2\alpha}(\tau)_0) = -\frac{1}{4}[(\text{Spec } \mathbb{C}/\mathbb{G}_m], \rho_{E_0 \oplus E_0})] + \frac{1}{2}[(\text{Spec } \mathbb{C}/\mathbb{G}_m], \rho_F)], \quad (6.11)$$

where  $\rho_{E_0 \oplus E_0}, \rho_F$  map  $[\mathrm{Spec} \mathbb{C}/\mathbb{G}_m]$  to  $E_0 \oplus E_0$  and  $F$  respectively. So Definition 5.15 gives

$$\bar{D}T^{2\alpha}(\tau)_0 = -\left(-\frac{1}{4}\nu_{\mathfrak{M}_{ss}^{2\alpha}(\tau)_0}(E_0 \oplus E_0) + \frac{1}{2}\nu_{\mathfrak{M}_{ss}^{2\alpha}(\tau)_0}(F)\right) = \frac{1}{4} \cdot 4 - \frac{1}{2} \cdot 1 = \frac{1}{2}. \quad (6.12)$$

Similarly  $J^{2\alpha}(\tau)_0 = -\frac{1}{4} + \frac{1}{2} = \frac{1}{4}$ . To summarize,

$$\bar{D}T^{2\alpha}(\tau)_t = \frac{1}{2}, \quad \text{all } t, \quad \text{and} \quad J^{2\alpha}(\tau)_t = \begin{cases} -\frac{1}{2}, & t \neq 0, \\ \frac{1}{4}, & t = 0. \end{cases} \quad (6.13)$$

Equations (6.10), (6.13) illustrate the fact that the  $\bar{D}T^\alpha(\tau)$  are deformation-invariant, as in Corollary 5.28, but the  $J^\alpha(\tau)$  of §3.5 are not.

### 6.2. Integrality properties of the $\bar{D}T^\alpha(\tau)$

This subsection is based on ideas in Kontsevich and Soibelman [63, §2.5 & §7.1]. Example 6.2 shows that given a rigid  $\tau$ -stable sheaf  $E$  in class  $\alpha$ , the sheaves  $mE$  contribute  $1/m^2$  to  $\bar{D}T^{m\alpha}(\tau)$  for all  $m \geq 1$ . We can regard this as a kind of ‘multiple cover formula’, analogous to the well known Aspinwall–Morrison computation for a Calabi–Yau 3-fold  $X$  that a rigid embedded  $\mathbb{CP}^1$  in class  $\alpha \in H_2(X; \mathbb{Z})$  contributes  $1/m^3$  to the genus zero Gromov–Witten invariant of  $X$  in class  $m\alpha$  for all  $m \geq 1$ . So we can define new invariants  $\hat{D}T^\alpha(\tau)$  which subtract out these contributions from  $mE$  for  $m > 1$ .

**DEFINITION 6.10.** Let  $X$  be a projective Calabi–Yau 3-fold over  $\mathbb{C}$ , let  $\mathcal{O}_X(1)$  be a very ample line bundle on  $X$ , and let  $(\tau, T, \leq)$  be a weak stability condition on  $\mathrm{coh}(X)$  of Gieseker or  $\mu$ -stability type. Then Definition 5.15 defines generalized Donaldson–Thomas invariants  $\bar{D}T^\alpha(\tau) \in \mathbb{Q}$  for  $\alpha \in C(\mathrm{coh}(X))$ .

Let us define new invariants  $\hat{D}T^\alpha(\tau)$  for  $\alpha \in C(\mathrm{coh}(X))$  to satisfy

$$\bar{D}T^\alpha(\tau) = \sum_{m \geq 1, m|\alpha} \frac{1}{m^2} \hat{D}T^{\alpha/m}(\tau). \quad (6.14)$$

We can invert (6.14) explicitly to write  $\hat{D}T^\alpha(\tau)$  in terms of the  $\bar{D}T^{\alpha/m}(\tau)$ . The *Möbius function*  $\mathrm{Mö} : \mathbb{N} \rightarrow \{-1, 0, 1\}$  in elementary number theory and combinatorics is given by  $\mathrm{Mö}(n) = (-1)^d$  if  $n = 1, 2, \dots$  is square-free and has  $d$  prime factors, and  $\mathrm{Mö}(n) = 0$  if  $n$  is not square-free. Then the *Möbius inversion formula* says that if  $f, g : \mathbb{N} \rightarrow \mathbb{Q}$  are functions with  $g(n) = \sum_{m|n} f(n/m)$  for  $n = 1, 2, \dots$  then  $f(n) = \sum_{m|n} \mathrm{Mö}(m)g(n/m)$  for  $n = 1, 2, \dots$ . Suppose  $\beta \in C(\mathrm{coh}(X))$  is primitive. Applying the Möbius inversion formula with  $f(n) = n^2 \hat{D}T^{n\beta}(\tau)$  and  $g(n) = n^2 \bar{D}T^{n\beta}(\tau)$ , we find the inverse of (6.14) is

$$\hat{D}T^\alpha(\tau) = \sum_{m \geq 1, m|\alpha} \frac{\mathrm{Mö}(m)}{m^2} \bar{D}T^{\alpha/m}(\tau). \quad (6.15)$$

We take (6.15) to be the *definition* of  $\hat{D}T^\alpha(\tau)$ , and then reversing the argument shows that (6.14) holds. The  $\hat{D}T^\alpha(\tau)$  are our analogues of invariants  $\Omega(\alpha)$  discussed in [63, §2.5 & §7.1]. We call  $\hat{D}T^\alpha(\tau)$  the *BPS invariants* of  $X$ , since Kontsevich and Soibelman suggest that their  $\Omega(\alpha)$  count BPS states. The coefficients  $1/m^2$  in (6.14) are related to the appearance of *dilogarithms* in Kontsevich and Soibelman [63, §2.5]. The dilogarithm is  $Li_2(t) = \sum_{m \geq 1} t^m/m^2$  for  $|t| < 1$ , and the inverse function

for  $Li_2$  near  $t = 0$  is  $Li_2^{-1}(t) = \sum_{m \geq 1} \text{Mö}(m)t^m/m^2$  for  $|t| < 1$ , with power series coefficients  $\text{Mö}(m)/m^2$  as in (6.15).

If  $\mathcal{M}_{\text{ss}}^\alpha(\tau) = \mathcal{M}_{\text{st}}^\alpha(\tau)$  then  $\mathcal{M}_{\text{ss}}^{\alpha/m}(\tau) = \emptyset$  for all  $m \geq 2$  dividing  $\alpha$ , since if  $[E] \in \mathcal{M}_{\text{ss}}^{\alpha/m}(\tau)$  then  $[mE] \in \mathcal{M}_{\text{ss}}^\alpha(\tau) \setminus \mathcal{M}_{\text{st}}^\alpha(\tau)$ . So  $\bar{D}T^{\alpha/m}(\tau) = 0$ , and hence (6.15) and Proposition 5.17 give:

PROPOSITION 6.11. *If  $\mathcal{M}_{\text{ss}}^\alpha(\tau) = \mathcal{M}_{\text{st}}^\alpha(\tau)$  then  $\hat{D}T^\alpha(\tau) = DT^\alpha(\tau)$ .*

Thus the  $\hat{D}T^\alpha(\tau)$  are also generalizations of Donaldson–Thomas invariants  $DT^\alpha(\tau)$ . Using (6.14) we evaluate the  $\hat{D}T^\alpha(\tau)$  in each of the examples of §6.1:

- In Examples 6.1–6.2 we have  $\hat{D}T^\alpha(\tau) = 1$  and  $\hat{D}T^{m\alpha}(\tau) = 0$  for all  $m > 1$ . Thus, a rigid stable sheaf  $E$  and its ‘multiple covers’  $mE$  for  $m \geq 2$  contribute 1 to  $\hat{D}T^\alpha(\tau)$  and 0 to  $\hat{D}T^{m\alpha}(\tau)$  for  $m \geq 2$ . The point of (6.14) was to achieve this, as it suggests that the  $\hat{D}T^\alpha(\tau)$  are a more meaningful way to ‘count’ stable sheaves.
- In Examples 6.4–6.5  $\hat{D}T^{\alpha_i}(\tau) = 1$  for all  $i = 1, \dots, l$ , and  $\hat{D}T^{\sum_{i \in I} \alpha_i}(\tau) = 0$  for all subsets  $I \subseteq \{1, \dots, l\}$  with  $|I| \geq 2$ .
- In Examples 6.6–6.7  $\hat{D}T^{m_1\alpha_1 + \dots + m_l\alpha_l}(\tau) = 1$  if  $m_i = 1$  for some  $i = 1, \dots, l$  and  $m_j = 0$  for  $i \neq j$ , and  $\hat{D}T^{m_1\alpha_1 + \dots + m_l\alpha_l}(\tau) = 0$  otherwise.
- In Example 6.8  $\hat{D}T^{\alpha_1 + \alpha_2}(\tau) = (-1)^{d-1}d/2$ , where  $\bar{\chi}(\alpha_1, \alpha_2) = d$ . Note that  $\hat{D}T^{\alpha_1 + \alpha_2}(\tau) \notin \mathbb{Z}$  when  $d$  is odd.
- In Example 6.9  $\hat{D}T^\alpha(\tau)_t = 2$  and  $\hat{D}T^{2\alpha}(\tau)_t = 0$ .

Here is our version of a conjecture by Kontsevich and Soibelman [63, Conj. 6].

CONJECTURE 6.12. *Let  $X$  be a Calabi–Yau 3-fold over  $\mathbb{C}$ , and  $(\tau, T, \leq)$  a weak stability condition on  $\text{coh}(X)$  of Gieseker or  $\mu$ -stability type. Call  $(\tau, T, \leq)$  **generic** if for all  $\alpha, \beta \in C(\text{coh}(X))$  with  $\tau(\alpha) = \tau(\beta)$  we have  $\bar{\chi}(\alpha, \beta) = 0$ .*

*If  $(\tau, T, \leq)$  is generic, then  $\hat{D}T^\alpha(\tau) \in \mathbb{Z}$  for all  $\alpha \in C(\text{coh}(X))$ .*

Kontsevich and Soibelman deal with Bridgeland stability conditions on derived categories, and their notion of *generic* stability condition is stronger than ours: they require that  $\tau(\alpha) = \tau(\beta)$  implies  $\alpha, \beta$  are linearly dependent in  $\mathbb{Z}$ . But we believe  $\bar{\chi}(\alpha, \beta) = 0$  is sufficient. Note that Conjecture 6.12 holds in the examples above: the only case in which  $\hat{D}T^\alpha(\tau) \notin \mathbb{Z}$  is Example 6.8 when  $d$  is odd, and then  $(\tau, T, \leq)$  is not generic, as  $\tau(\alpha_1) = \tau(\alpha_2)$  but  $\bar{\chi}(\alpha_1, \alpha_2) = d \neq 0$ .

Suppose now that  $(\tau, T, \leq)$  is a *stability condition*, such as Gieseker stability, rather than a weak stability condition. This is necessary for decomposition of  $\tau$ -semistables into  $\tau$ -stables to be well-behaved, as in [53, Th. 4.5]. Then as in (5.9) we can write  $\bar{D}T^\alpha(\tau)$  as the Euler characteristic of the *coarse moduli scheme*  $\mathcal{M}_{\text{ss}}^\alpha(\tau)$  weighted by a constructible function. (The existence of coarse moduli schemes  $\mathcal{M}_{\text{ss}}^\alpha(\tau)$  is known in the two cases we consider in this book, Gieseker stability for coherent sheaves and slope stability for quiver representations. For the argument below to work, we do not need  $\mathcal{M}_{\text{ss}}^\alpha(\tau)$  to be a scheme, but only a constructible set, the quotient of  $\mathfrak{M}_{\text{ss}}^\alpha(\tau)(\mathbb{C})$  by a constructible equivalence relation, and this should always be true.)

We will write  $\hat{D}T^\alpha(\tau)$  as a weighted Euler characteristic of  $\mathcal{M}_{\text{ss}}^\alpha(\tau)$  in the same way. For  $m \geq 1$ , define a 1-morphism  $P_m : \mathfrak{M} \rightarrow \mathfrak{M}$  by  $P_m : [E] \mapsto [mE]$  for  $E \in \text{coh}(X)$ , where  $mE = \overset{\text{“} m \text{ copies “}}{E \oplus \dots \oplus E}$ . Then from equations (2.1), (5.9) and

(6.15), for  $\alpha \in C(\text{coh}(X))$  we deduce that

$$\begin{aligned} \hat{D}T^\alpha(\tau) &= \chi(\mathcal{M}_{\text{ss}}^\alpha(\tau), F^\alpha(\tau)), \quad \text{where} \\ F^\alpha(\tau) &= - \sum_{m \geq 1, m|\alpha} \frac{\text{Mö}(m)}{m^2} \text{CF}^{\text{na}}(\pi) [\text{CF}^{\text{na}}(P_m) \circ \Pi_{\text{CF}} \circ \bar{\Pi}_{\mathfrak{M}}^{\chi, \mathbb{Q}}(\bar{\epsilon}^{\alpha/m}(\tau)) \cdot \nu_{\mathfrak{M}}], \end{aligned} \quad (6.16)$$

and  $\pi : \mathfrak{M}_{\text{ss}}^\alpha(\tau) \rightarrow \mathcal{M}_{\text{ss}}^\alpha(\tau)$  is the projection to the coarse moduli scheme. The following conjecture implies Conjecture 6.12, at least for stability conditions rather than weak stability conditions.

**CONJECTURE 6.13.** *Let  $X$  be a Calabi–Yau 3-fold over  $\mathbb{C}$ , and  $(\tau, G, \leq)$  a generic Gieseker stability condition. Then the functions  $F^\alpha(\tau) \in \text{CF}(\mathcal{M}_{\text{ss}}^\alpha(\tau))$  of (6.16) are  $\mathbb{Z}$ -valued for all  $\alpha \in C(\text{coh}(X))$ .*

That is, the contributions to  $\hat{D}T^\alpha(\tau)$  from each S-equivalence class of  $\tau$ -semi-stables (or each  $\tau$ -polystable) are integral. In §7.6 we will prove versions of Conjectures 6.12 and 6.13 for Donaldson–Thomas type invariants  $\hat{D}T_Q^d(\mu)$  for quivers without relations. By analogy with Question 5.7(a), we can ask:

**QUESTION 6.14.** *Suppose Conjecture 6.13 is true. For generic  $(\tau, G, \leq)$ , does there exist a natural perverse sheaf  $\mathcal{Q}$  on  $\mathcal{M}_{\text{ss}}^\alpha(\tau)$  with  $\chi_{\mathcal{M}_{\text{ss}}^\alpha(\tau)}(\mathcal{Q}) \equiv F^\alpha(\tau)$ ?*

Such a perverse sheaf  $\mathcal{Q}$  would be interesting as it would provide a ‘categorification’ of the BPS invariants  $\hat{D}T^\alpha(\tau)$ , and help explain their integrality.

We can also ask whether the unweighted invariants  $J^\alpha(\tau)$  of §3.5 also have similar integrality properties to those suggested in Conjectures 6.12 and 6.13. The answer is no. Following the argument above but using (6.4) rather than (6.3), one would expect that the correct analogue of (6.14) is

$$J^\alpha(\tau) = \sum_{m \geq 1, m|\alpha} \frac{(-1)^{m-1}}{m^2} \hat{J}^{\alpha/m}(\tau).$$

But then in Example 6.9, from (6.10) and (6.13) we see that  $\hat{J}^{2\alpha}(\tau)_0 = \frac{1}{2}$ , so the  $\hat{J}^\alpha(\tau)$  need not be integers even for a generic stability condition. In fact, using (6.4) in Example 6.2 and (6.10) and (6.13) when  $t = 0$  in Example 6.9, one can show that there is no universal formula with  $c_1, c_2, \dots \in \mathbb{Q}$  and  $c_1 = 1$  defining

$$\hat{J}^\alpha(\tau) = \sum_{m \geq 1, m|\alpha} c_m J^{\alpha/m}(\tau),$$

such that  $\hat{J}^\alpha(\tau) \in \mathbb{Z}$  for all generic  $(\tau, T, \leq)$  and  $\alpha \in C(\text{coh}(X))$ . One conclusion (at least if you believe Conjecture 6.12) is that counting sheaves weighted by the Behrend function is essential to ensure good integrality properties.

### 6.3. Counting dimension zero sheaves

Let  $X$  be a Calabi–Yau 3-fold over  $\mathbb{C}$  with  $H^1(\mathcal{O}_X) = 0$ , let  $\mathcal{O}_X(1)$  be a very ample line bundle on  $X$ , and  $(\tau, G, \leq)$  the associated Gieseker stability condition on  $\text{coh}(X)$ , as in Example 3.8. For  $x \in X(\mathbb{C})$ , write  $\mathcal{O}_x$  for the skyscraper sheaf at  $x$ , and define  $p = [\mathcal{O}_x]$  in  $K^{\text{num}}(\text{coh}(X))$ , the ‘point class’ on  $X$ . Then  $p$  is independent of the choice of  $x$  in  $X(\mathbb{C})$ , as  $X$  is connected.

For  $d \geq 0$ , the Hilbert scheme  $\text{Hilb}^d X$  of  $d$  points on  $X$  parametrizes 0-dimensional subschemes  $S$  of  $X$  of length  $d$ . It is a projective  $\mathbb{C}$ -scheme, which is singular for  $d \geq 4$ , and for  $d \gg 0$  has many irreducible components. The virtual count

of  $\text{Hilb}^d X$  may be written as a weighted Euler characteristic  $\chi(\text{Hilb}^d X, \nu_{\text{Hilb}^d X})$  as in §4.3. Values for these virtual counts were conjectured by Maulik et al. [80, Conj. 1], and different proofs are given by Behrend and Fantechi [6, Th. 4.12], Li [71, Th. 0.2], and Levine and Pandharipande [70, §14.1].

**THEOREM 6.15.**  $\sum_{d=0}^{\infty} \chi(\text{Hilb}^d X, \nu_{\text{Hilb}^d X}) s^d = M(-s)^{\chi(X)}$ , where  $\chi(X)$  is the Euler characteristic of  $X$ , and  $M(q) = \prod_{k \geq 1} (1 - q^k)^{-k}$  the MacMahon function.

We will compute the generalized Donaldson–Thomas invariants  $\bar{DT}^{dp}(\tau)$  counting dimension 0 sheaves in class  $dp$  for  $d \geq 1$ . Our calculation is parallel to Kontsevich and Soibelman [63, §6.5]. First consider the pair invariants  $PI^{dp,n}(\tau')$ . These count stable pairs  $s : \mathcal{O}_X(-n) \rightarrow E$  for  $E \in \text{coh}(X)$  with  $[E] = dp$ . The condition for  $s : \mathcal{O}_X(-n) \rightarrow E$  to be a stable pair is simply that  $s$  is surjective. Tensoring by  $\mathcal{O}_X(n)$  gives a morphism  $s(n) : \mathcal{O}_X \rightarrow E(n)$ . But  $E(n) \cong E$  as  $E$  has dimension 0. Thus, tensoring by  $\mathcal{O}_X(n)$  gives an isomorphism  $\mathcal{M}_{\text{stp}}^{dp,n}(\tau') \cong \mathcal{M}_{\text{stp}}^{dp,0}(\tau')$ , so that  $\mathcal{M}_{\text{stp}}^{dp,n}(\tau')$  and  $PI^{dp,n}(\tau')$  are independent of  $n$ . Furthermore,  $\mathcal{M}_{\text{stp}}^{dp,0}(\tau')$  parametrizes surjective  $s : \mathcal{O}_X \rightarrow E$  for  $E \in \text{coh}(X)$  with  $[E] = dp$ , which are just points of  $\text{Hilb}^d X$ . Therefore  $\mathcal{M}_{\text{stp}}^{dp,n}(\tau') \cong \text{Hilb}^d X$ , and

$$PI^{dp,n}(\tau') = \chi(\text{Hilb}^d X, \nu_{\text{Hilb}^d X}), \quad \text{for all } n \in \mathbb{Z} \text{ and } d \geq 0. \quad (6.17)$$

We have  $\tau(dp) = 1$  in  $G$ , and any  $\beta, \gamma \in C(\text{coh}(X))$  with  $\tau(\beta) = \tau(\gamma) = 1$  are of the form  $\beta = dp$ ,  $\gamma = ep$ , so that  $\bar{\chi}(\beta, \gamma) = 0$ . Therefore Proposition 5.29 applies with  $t = 1$  in  $G$ . So from Theorem 6.15 and (6.17) we see that

$$\begin{aligned} M(-s)^{\chi(X)} &= 1 + \sum_{d \geq 1} PI^{dp,n}(\tau') s^d = \\ &\exp \left[ - \sum_{d \geq 1} (-1)^{\bar{\chi}([\mathcal{O}_X(-n)], dp)} \bar{\chi}([\mathcal{O}_X(-n)], dp) \bar{DT}^{dp}(\tau) s^d \right]. \end{aligned} \quad (6.18)$$

Here we have replaced the sums over  $\alpha \in C(\text{coh}(X))$  with  $\tau(\alpha) = 1$  by a sum over  $dp$  for  $d \geq 1$ , and used the formal variable  $s$  in place of  $q^p$  in (5.20), so that  $q^{dp}$  is replaced by  $s^d$ .

Now  $\bar{\chi}([\mathcal{O}_X(-n)], p) = \sum_i (-1)^i \dim H^i(\mathcal{O}_x(n)) = 1$ , so  $\bar{\chi}([\mathcal{O}_X(-n)], dp) = d$ . Substituting this into (6.18), taking logs, and using  $M(q) = \prod_{k \geq 1} (1 - q^k)^{-k}$  yields

$$- \sum_{d \geq 1} (-1)^d d \bar{DT}^{dp}(\tau) s^d = \chi(X) \sum_{k \geq 1} (-k) \log(1 - (-s)^k) = \chi(X) \sum_{k, l \geq 1} \frac{k}{l} (-s)^{kl}.$$

Equating coefficients of  $s^d$  yields after a short calculation

$$\bar{DT}^{dp}(\tau) = -\chi(X) \sum_{l \geq 1, l|d} \frac{1}{l^2}. \quad (6.19)$$

So from (6.14) we deduce that

$$\hat{DT}^{dp}(\tau) = -\chi(X), \quad \text{all } d \geq 1. \quad (6.20)$$

This is a satisfying result, and confirms Conjecture 6.12 for dimension 0 sheaves.

### 6.4. Counting dimension one sheaves

Let  $X$  be a Calabi–Yau 3-fold over  $\mathbb{C}$ , and  $\mathcal{O}_X(1)$  a very ample line bundle on  $X$ . From §4.5 the Chern character  $\text{ch}$  identifies  $K^{\text{num}}(\text{coh}(X))$  with a particular lattice  $\Lambda_X$  in  $H^{\text{even}}(X; \mathbb{Q})$ , so we may write  $\alpha \in K^{\text{num}}(\text{coh}(X))$  as  $(\alpha_0, \alpha_2, \alpha_4, \alpha_6)$  with  $\alpha_{2j} \in H^{2j}(X; \mathbb{Q})$ . If  $E \rightarrow X$  is a vector bundle with  $[E] = \alpha$  then  $\alpha_0 = \text{rank } E \in \mathbb{Z}$ .

Let us consider invariants  $\bar{D}T^\alpha(\tau), \hat{D}T^\alpha(\tau)$  counting pure sheaves  $E$  of dimension 1 on  $X$ , that is, sheaves  $E$  supported on curves  $C$  in  $X$ . If  $[E] = \alpha$  then  $\alpha_0 = \alpha_2 = 0$  for dimensional reasons, so we may write  $\alpha = (0, 0, \beta, k)$ . By [40, §A.4] we have  $\beta = -c_2(E)$  and  $k = \frac{1}{2}c_3(E)$ , where  $c_2(E) \in H^4(X; \mathbb{Z}) \cong H_2(X; \mathbb{Z})$  and  $c_3(E) \in H^6(X; \mathbb{Z}) \cong \mathbb{Z}$  are Chern classes of  $E$ . Write  $\delta = c_1(\mathcal{O}_X(1))$ . If  $[E] = \alpha$  then  $[E(n)] = \exp(n\delta)\alpha = (0, 0, \beta, k + n\beta \cup \delta)$ . Hence by the Hirzebruch–Riemann–Roch Theorem [40, Th. A.4.1] we have

$$\begin{aligned} \chi(E(n)) &= \deg(\text{ch}(E(n)) \cdot \text{td}(TX))_3 \\ &= \deg((0, 0, \beta, k + n\beta \cup \delta) \cdot (1, 0, *, *))_3 = k + n\beta \cup \delta, \end{aligned} \quad (6.21)$$

using  $c_1(X) = 0$  to simplify  $\text{td}(TX)$ . So the Hilbert polynomial of  $E$  is

$$P_{(0,0,\beta,k)}(t) = (\beta \cup \delta)t + k. \quad (6.22)$$

Note that  $\beta \cup \delta, k \in \mathbb{Z}$  as  $P_{(0,0,\beta,k)}$  maps  $\mathbb{Z} \rightarrow \mathbb{Z}$ . Note too that for dimension 1 sheaves, Gieseker stability in Example 3.8 and  $\mu$ -stability in Example 3.9 coincide, since truncating a degree 1 polynomial at its second term has no effect.

Here are some properties of the  $\bar{D}T^\alpha(\tau), \hat{D}T^\alpha(\tau)$  in dimension 1. Part (a) may be new, and answers a question of Sheldon Katz in [58, §3.2]. The proof of (c) uses  $H^1(\mathcal{O}_X) = 0$ , which we assume for Calabi–Yau 3-folds in this section.

**THEOREM 6.16.** *Let  $X$  be a Calabi–Yau 3-fold over  $\mathbb{C}$ , and  $(\tau, T, \leq)$  a weak stability condition on  $\text{coh}(X)$  of Gieseker or  $\mu$ -stability type. Consider invariants  $\bar{D}T^{(0,0,\beta,k)}(\tau), \hat{D}T^{(0,0,\beta,k)}(\tau)$  for  $0 \neq \beta \in H^4(X; \mathbb{Z})$  and  $k \in \mathbb{Z}$  counting  $\tau$ -semistable dimension 1 sheaves in  $X$ . Then*

- (a)  $\bar{D}T^{(0,0,\beta,k)}(\tau), \hat{D}T^{(0,0,\beta,k)}(\tau)$  are independent of the choice of  $(\tau, T, \leq)$ .
- (b) Assume Conjecture 6.12 holds. Then  $\hat{D}T^{(0,0,\beta,k)}(\tau) \in \mathbb{Z}$ .
- (c) For any  $l \in \beta \cup H^2(X; \mathbb{Z}) \subseteq \mathbb{Z}$  we have  $\bar{D}T^{(0,0,\beta,k)}(\tau) = \bar{D}T^{(0,0,\beta,k+l)}(\tau)$  and  $\hat{D}T^{(0,0,\beta,k)}(\tau) = \hat{D}T^{(0,0,\beta,k+l)}(\tau)$ .

**PROOF.** For (a), let  $(\tau, T, \leq), (\tilde{\tau}, \tilde{T}, \leq)$  be two weak stability conditions on  $\text{coh}(X)$  of Gieseker or  $\mu$ -stability type. Then Corollary 5.19 shows that we may write  $\bar{D}T^{(0,0,\beta,k)}(\tilde{\tau})$  in terms of the  $\bar{D}T^{(0,0,\beta',k')}(\tau)$  by finitely many applications of the transformation law (5.14). Now each of these changes of stability condition involves only sheaves in the abelian subcategory  $\text{coh}_{\leq 1}(X)$  of sheaves in  $\text{coh}(X)$  with dimension  $\leq 1$ . However, the Euler form  $\bar{\chi}$  vanishes on  $K_0(\text{coh}_{\leq 1}(X))$  for dimensional reasons. Each term  $I, \kappa, \Gamma$  in (5.14) has  $|I| - 1$  factors  $\bar{\chi}(\kappa(i), \kappa(j))$  so all terms with  $|I| > 1$  are zero as  $\bar{\chi} \equiv 0$  on this part of  $\text{coh}(X)$ . So (5.14) reduces to  $\bar{D}T^\alpha(\tilde{\tau}) = \bar{D}T^\alpha(\tau)$ . Therefore each of the finitely many applications of (5.14) leaves the  $\bar{D}T^{(0,0,\beta',k')}(\tau)$  unchanged, proving (a).

For (b), note that any stability condition  $(\tau, T, \leq)$  on  $\text{coh}(X)$  is generic on  $\text{coh}_{\leq 1}(X)$ , since  $\bar{\chi} = 0$  on  $K_0(\text{coh}_{\leq 1}(X))$ . Alternatively, one can show that if  $\tilde{\mathcal{O}}_X(1)$  is a sufficiently general very ample line bundle on  $X$  the resulting Gieseker

stability condition  $(\tilde{\tau}, G, \leq)$  is generic on all of  $\text{coh}(X)$ , and then apply (a). Either way, Conjecture 6.12 implies that  $\hat{DT}^{(0,0,\beta,k)}(\tau) \in \mathbb{Z}$ .

For (c), let  $L \rightarrow X$  be a line bundle, let  $(\tau, T, \leq)$  be a weak stability condition on  $\text{coh}(X)$  of Gieseker or  $\mu$ -stability type, and define another weak stability condition  $(\tilde{\tau}, T, \leq)$  on  $\text{coh}(X)$  by  $\tilde{\tau}([E]) = \tau([E \otimes L^{-1}])$ . There is an automorphism  $F^L : \text{coh}(X) \rightarrow \text{coh}(X)$  acting as  $F^L : E \mapsto E \otimes L$  on objects. It induces a 1-isomorphism  $F_*^L : \mathfrak{M} \rightarrow \mathfrak{M}$ . Also  $E$  is  $\tau$ -semistable if and only if  $E \otimes L$  is  $\tilde{\tau}$ -semistable, so  $F_*^L$  maps  $\mathfrak{M}_{\text{ss}}^\alpha(\tau) \rightarrow \mathfrak{M}_{\text{ss}}^{F_*^L(\alpha)}(\tilde{\tau})$ , where  $F_*^L(\alpha) = \exp(c_1(L)) \cdot \alpha$  in  $H^{\text{even}}(X; \mathbb{Q})$ .

Clearly we have  $\hat{DT}^\alpha(\tau) = \hat{DT}^{F_*^L(\alpha)}(\tilde{\tau})$  for all  $\alpha \in C(\text{coh}(X))$ . When  $\alpha = (0, 0, \beta, k)$  we have  $F_*^L(\alpha) = (0, 0, \beta, k + \beta \cup c_1(L))$ . Thus

$$\bar{DT}^{(0,0,\beta,k)}(\tau) = \bar{DT}^{(0,0,\beta,k+\beta \cup c_1(L))}(\tilde{\tau}) = \bar{DT}^{(0,0,\beta,k+\beta \cup c_1(L))}(\tau),$$

by (a). Since  $H^1(\mathcal{O}_X) = 0$ ,  $c_1(L)$  can take any value in  $H^2(X; \mathbb{Z})$ , and so  $\beta \cup c_1(L)$  can take any value  $l \in \beta \cup H^2(X; \mathbb{Z})$ , proving the first part of (c). The second part follows by (6.15).  $\square$

We will compute contributions to  $\bar{DT}^{(0,0,\beta,k)}(\tau)$ ,  $\hat{DT}^{(0,0,\beta,k)}(\tau)$  from sheaves supported on nice curves  $C$  in  $X$ . We begin with a rigid rational curve. The proof of the next proposition is based on Hosono et al. [41, Prop. 4.3].

**PROPOSITION 6.17.** *Let  $X$  be a Calabi–Yau 3-fold over  $\mathbb{C}$ , and  $(\tau, T, \leq)$  a weak stability condition on  $\text{coh}(X)$  of Gieseker or  $\mu$ -stability type. Suppose  $i : \mathbb{CP}^1 \rightarrow X$  is an embedding, and  $i(\mathbb{CP}^1)$  has normal bundle  $\mathcal{O}_{\mathbb{CP}^1}(-1) \oplus \mathcal{O}_{\mathbb{CP}^1}(-1)$ . Then the only  $\tau$ -semistable dimension 1 sheaves supported set-theoretically on  $i(\mathbb{CP}^1)$  in  $X$  are  $i_*(m\mathcal{O}_{\mathbb{CP}^1}(k))$  for  $m \geq 1$  and  $k \in \mathbb{Z}$ , and these sheaves are rigid in  $\text{coh}(X)$ .*

**PROOF.** Let  $\beta \in H^4(X; \mathbb{Z})$  be Poincaré dual to  $[i(\mathbb{CP}^1)] \in H_2(X; \mathbb{Z})$ . Suppose  $E$  is a pure dimension 1 sheaf supported on  $i(\mathbb{CP}^1)$  in  $X$ . Then  $[E] = (0, 0, m\beta, k)$  in  $K^{\text{num}}(\text{coh}(X)) \subset H^{\text{even}}(X; \mathbb{Q})$ , where  $m \geq 1$  is the multiplicity of  $E$  at a generic point of  $i(\mathbb{CP}^1)$ . Any subsheaf  $0 \neq E' \subset E$  has  $[E'] = (0, 0, m'\beta, k')$  for  $1 \leq m' \leq m$  and  $k' \in \mathbb{Z}$ . Let  $(\tau, T, \leq)$  be defined using an ample line bundle  $\mathcal{O}_X(1)$  with  $c_1(L) = \delta$ . Then by (6.22), the Hilbert polynomials of  $E$  and  $E'$  are  $m(\beta \cup \delta)t + k$  and  $m'(\beta \cup \delta)t + k'$ , where  $\beta \cup \delta > 0$ .

By Example 3.8 or 3.9,  $E$  is  $\tau$ -semistable if and only if, for all  $0 \neq E' \subset E$ , when  $[E'] = (0, 0, m'\beta, k')$ , we have  $k'/m'(\beta \cup \delta) \leq k/m(\beta \cup \delta)$ , that is,  $k'/m \leq k/m$ . Note that this condition is *independent of the choice of stability condition*  $(\tau, T, \leq)$ . This is a stronger analogue of Theorem 6.16(a): if  $\Sigma \subset X$  is an irreducible curve in  $X$ , then the moduli stacks  $\mathfrak{M}_{\text{ss}}^\alpha(\tau)_\Sigma$  of  $\tau$ -semistable sheaves supported on  $\Sigma$  are independent of  $(\tau, T, \leq)$ .

Suppose  $E$  is  $\tau$ -semistable and dimension 1 with  $[E] = (0, 0, m\beta, k)$ . Locally in the étale or complex analytic topology near  $i(\mathbb{CP}^1)$  in  $X$  we can find a line bundle  $L$  such that  $i^*(L) \cong \mathcal{O}_{\mathbb{CP}^1}(1)$  (this holds as the obstructions to finding such an  $L$  lie in  $\text{Ext}^2(\mathcal{O}_{\mathbb{CP}^1}(1), \mathcal{O}_{\mathbb{CP}^1}(1))$ , which is zero as  $2 > \dim \mathbb{CP}^1$ ). Then  $[E \otimes L^n] = (0, 0, m\beta, k + mn)$  for  $n \in \mathbb{Z}$ , and  $E \otimes L^n$  is  $\tau$ -semistable by the proof of Theorem 6.16(c) and  $\mathfrak{M}_{\text{ss}}^\alpha(\tau)_{i(\mathbb{CP}^1)}$  independent of  $\tau$  above. Let  $n \in \mathbb{Z}$  be unique such that  $d = k + mn$  lies in  $\{1, 2, \dots, m\}$ . Then  $[E \otimes L^n] = (0, 0, m\beta, d)$ , so  $\chi(E \otimes L^n) = P_{[E \otimes L^n]}(0) = d$  by (6.22). But  $H^i(E \otimes L^n) = 0$  for  $i > 1$  as  $E \otimes L^n$  is supported in dimension 1, so  $\dim H^0(E \otimes L^n) \geq \dim H^0(E \otimes L^n) - \dim H^1(E \otimes L^n) = d > 0$ , and we can choose  $0 \neq s \in H^0(E \otimes L^n)$ .

Thus we have a nonzero morphism  $s : \mathcal{O}_X \rightarrow E \otimes L^n$  in  $\text{coh}(X)$ . Write  $J$  for the image and  $K$  for the kernel of  $s$ . Then  $0 \neq J \subset E \otimes L^n$  and  $K \subset \mathcal{O}_X$ . As  $E \otimes L^n$  is pure of dimension 1,  $J$  is pure of dimension 1. Let  $I$  be the ideal sheaf of  $i(\mathbb{CP}^1)$ . Since  $\text{supp}(J) = i(\mathbb{CP}^1)$  which is reducible and reduced we see that  $K \subset I \subset \mathcal{O}_X$ . Consider the two cases (a)  $K = I$  and (b)  $K \neq I$ . In case (a) we have  $J = \mathcal{O}_{i(\mathbb{CP}^1)} = i_*(\mathcal{O}_{\mathbb{CP}^1}(0))$ , which has class  $[J] = (0, 0, \beta, 1)$ . Since  $E \otimes L^n$  is  $\tau$ -semistable with  $[E \otimes L^n] = (0, 0, m\beta, d)$  and  $0 \neq J \subset E \otimes L^n$ , this implies that  $1 \leq d/m$ . But  $d = 1, 2, \dots, m$  by choice of  $n$ , so this forces  $d = m$ .

In case (b), there is a unique  $l \geq 1$  such that  $I^{l+1} \subset K$  and  $I^l \not\subset K$ . Then  $K + I^l/K$  is a nontrivial subsheaf of  $\mathcal{O}_X/L \cong J$ , and so  $K + I^l/K \subset E \otimes L^n$ . But  $I^l/I^{l+1}$  is the  $l^{\text{th}}$  symmetric power of the conormal bundle of  $i(\mathbb{CP}^1)$  in  $X$ , so that  $I^l/I^{l+1} \cong i_*((l+1)\mathcal{O}_{\mathbb{CP}^1}(l))$ . As  $I^{l+1} \subset K$  there is a surjection  $I^l/I^{l+1} \rightarrow K + I^l/K$ . Let  $[K + I^l/K] = (0, 0, m'\beta, k')$ . Since  $[i_*((l+1)\mathcal{O}_{\mathbb{CP}^1}(l))] = (0, 0, (l+1)\beta, (l+1)^2)$  and  $K + I^l/K$  is a quotient sheaf of  $i_*((l+1)\mathcal{O}_{\mathbb{CP}^1}(l))$  which is  $\tau$ -semistable, we deduce that  $l+1 \leq k'/m'$ . But  $K + I^l/K \subset E \otimes L^n$ , so  $E \otimes L^n$   $\tau$ -semistable implies  $k'/m' \leq d/m$ . Hence  $l+1 \leq k'/m' \leq d/m$ , a contradiction as  $l \geq 1$  and  $d \leq m$ .

Thus, case (b) does not happen, and in case (a) we must have  $d = m$ , and  $E \otimes L^n$  has a subsheaf  $J \cong i_*(\mathcal{O}_{\mathbb{CP}^1}(0))$ . As  $\tau(J) = \tau(E \otimes L^n) = t + 1/(\beta \cup \delta)$ , the quotient  $(E \otimes L^n)/J$  is also  $\tau$ -semistable with  $[(E \otimes L^n)/J] = (0, 0, (m-1)\beta, m-1)$ . By induction on  $m$  we now see that  $E \otimes L^n$  has a filtration  $0 = F_0 \subset F_1 \subset \dots \subset F_m = E \otimes L^n$  with  $F_i/F_{i-1} \cong i_*(\mathcal{O}_{\mathbb{CP}^1}(0))$  for  $i = 1, \dots, m$ . As the normal bundle is  $\mathcal{O}_{\mathbb{CP}^1}(-1) \oplus \mathcal{O}_{\mathbb{CP}^1}(-1)$  the curve  $i(\mathbb{CP}^1)$  is rigid in  $X$ , which implies that  $\text{Ext}^1(i_*(\mathcal{O}_{\mathbb{CP}^1}(0)), i_*(\mathcal{O}_{\mathbb{CP}^1}(0))) = 0$ . It follows by induction on  $m$  that  $E \otimes L^n \cong i_*(m\mathcal{O}_{\mathbb{CP}^1}(0))$ , and also that  $E \otimes L^n$  is rigid. Tensoring by  $L^{-n}$  shows that  $E \cong i_*(m\mathcal{O}_{\mathbb{CP}^1}(-n))$  and  $E$  is rigid. This completes the proof.  $\square$

Combining Proposition 6.17 with Examples 6.1 and 6.2, and taking  $E$  in Example 6.1 to be  $i_*(\mathcal{O}_{\mathbb{CP}^1}(k))$  for  $k \in \mathbb{Z}$ , we deduce:

**PROPOSITION 6.18.** *Let  $X$  be a Calabi–Yau 3-fold over  $\mathbb{C}$ , and  $(\tau, T, \leq)$  a weak stability condition on  $\text{coh}(X)$  of Gieseker or  $\mu$ -stability type. Suppose  $i : \mathbb{CP}^1 \rightarrow X$  is an embedding, and  $i(\mathbb{CP}^1)$  has normal bundle  $\mathcal{O}_{\mathbb{CP}^1}(-1) \oplus \mathcal{O}_{\mathbb{CP}^1}(-1)$ . Let  $\beta \in H^4(X; \mathbb{Z})$  be Poincaré dual to  $[i(\mathbb{CP}^1)] \in H_2(X; \mathbb{Z})$ .*

*If  $m \geq 1$  and  $m \mid k$  then  $\tau$ -semistable sheaves supported on  $i(\mathbb{CP}^1)$  contribute  $1/m^2$  to  $\bar{DT}^{(0,0,m\beta,k)}(\tau)$ , and contribute 1 if  $m = 1$  and 0 if  $m > 1$  to  $\hat{DT}^{(0,0,m\beta,k)}(\tau)$ . If  $m \geq 1$  and  $m \nmid k$  there are no  $\tau$ -semistable sheaves in class  $(0, 0, m\beta, k)$  supported on  $i(\mathbb{CP}^1)$ , so no contribution to  $\bar{DT}, \hat{DT}^{(0,0,m\beta,k)}(\tau)$ .*

For higher genus curves the contributions are zero. Note that we do not need  $i(\Sigma)$  to be rigid, the contributions are local via weighted Euler characteristics.

**PROPOSITION 6.19.** *Let  $X$  be a Calabi–Yau 3-fold over  $\mathbb{C}$ , and  $(\tau, T, \leq)$  a weak stability condition on  $\text{coh}(X)$  of Gieseker or  $\mu$ -stability type. Suppose  $\Sigma$  is a connected, nonsingular Riemann surface of genus  $g \geq 1$  and  $i : \Sigma \rightarrow X$  is an embedding. Let  $\beta \in H^4(X; \mathbb{Z})$  be Poincaré dual to  $[i(\Sigma)] \in H_2(X; \mathbb{Z})$ . Then  $\tau$ -semistable dimension 1 sheaves supported on  $i(\Sigma)$  contribute 0 to both  $\bar{DT}^{(0,0,m\beta,k)}(\tau)$  and  $\hat{DT}^{(0,0,m\beta,k)}(\tau)$  for all  $m \geq 1$  and  $k \in \mathbb{Z}$ .*

**PROOF.** The family of line bundles  $L_t \rightarrow \Sigma$  with  $c_1(L) = 0$  form a group  $T^{2g}$  under  $\otimes$ . As  $i$  is an embedding, locally near  $i(\Sigma)$  in  $X$  we can find a family



of line bundles  $\tilde{L}_t$  for  $t \in T^{2g}$  which form a group under  $\otimes$ , with  $i^*(\tilde{L}_t) = L_t$ . Write  $\mathfrak{M}_{\text{ss}}^{(0,0,m\beta,k)}(\tau)_{i(\Sigma)}$  for the substack of  $\mathfrak{M}_{\text{ss}}^{(0,0,m\beta,k)}(\tau)$  supported on  $i(\Sigma)$ . Then  $t : E \mapsto E \otimes \tilde{L}_t$  defines an action of  $T^{2g}$  on  $\mathfrak{M}_{\text{ss}}^{(0,0,m\beta,k)}(\tau)_{i(\Sigma)}$ . For  $m \geq 1$ , the stabilizer groups of this action are finite. So  $\mathfrak{M}_{\text{ss}}^{(0,0,m\beta,k)}(\tau)_{i(\Sigma)}(\mathbb{C})$  is fibred by orbits of  $T^{2g}$  isomorphic to  $T^{2g}/K \cong T^{2g}$  for  $K$  finite.

Since the  $T^{2g}$ -action extends to an open neighbourhood of  $\mathfrak{M}_{\text{ss}}^{(0,0,m\beta,k)}(\tau)_{i(\Sigma)}$  in  $\mathfrak{M}_{\text{ss}}^{(0,0,m\beta,k)}(\tau)$ , the restriction to  $\mathfrak{M}_{\text{ss}}^{(0,0,m\beta,k)}(\tau)_{i(\Sigma)}$  of the Behrend function of  $\mathfrak{M}_{\text{ss}}^{(0,0,m\beta,k)}(\tau)$  is  $T^{2g}$ -invariant. Now the contribution to  $\bar{DT}^{(0,0,m\beta,k)}(\tau)$  from sheaves supported on  $i(\Sigma)$  is the Euler characteristic of  $\mathfrak{M}_{\text{ss}}^{(0,0,m\beta,k)}(\tau)_{i(\Sigma)}$  weighted by a constructible function built from  $\bar{e}^{(0,0,m\beta,k)}(\tau)$  and  $\nu_{\mathfrak{M}_{\text{ss}}^{(0,0,m\beta,k)}(\tau)}$ , as in §5.3. This constructible function is  $T^{2g}$ -invariant, as  $\bar{e}^{(0,0,m\beta,k)}(\tau)$  and  $\nu_{\mathfrak{M}_{\text{ss}}^{(0,0,m\beta,k)}(\tau)}$  are. But  $\chi(T^{2g}) = 0$  as  $g \geq 1$ , so each  $T^{2g}$ -orbit  $T^{2g}/K \cong T^{2g}$  contributes zero to the weighted Euler characteristic. Thus sheaves supported on  $i(\Sigma)$  contribute 0 to  $\bar{DT}^{(0,0,m\beta,k)}(\tau)$  for all  $m, k$ , and so contribute 0 to  $\hat{DT}^{(0,0,m\beta,k)}(\tau)$  by (6.15).  $\square$

Let  $X$  be a Calabi–Yau 3-fold over  $\mathbb{C}$ , and for  $\gamma \in H_2(X; \mathbb{Z})$  write  $GW_0(\gamma) \in \mathbb{Q}$  for the genus zero Gromov–Witten invariants of  $X$ . The *genus zero Gopakumar–Vafa invariants*  $GV_0(\gamma)$  may be defined by the formula

$$GW_0(\gamma) = \sum_{m|\gamma} \frac{1}{m^3} GV_0(\gamma/m).$$

A priori we have  $GV_0(\gamma) \in \mathbb{Q}$ , but Gopakumar and Vafa [32] conjecture that the  $GV_0(\gamma)$  are integers, and count something meaningful in String Theory.

Katz [58] considers the moduli spaces  $\mathcal{M}_{\text{ss}}^{(0,0,\beta,1)}(\tau)$  when  $k = 1$ , where  $\beta \in H^4(X; \mathbb{Z})$  is Poincaré dual to  $\gamma$ . Then  $\mathcal{M}_{\text{ss}}^{(0,0,\beta,1)}(\tau) = \mathcal{M}_{\text{st}}^{(0,0,\beta,1)}(\tau)$  as  $(\beta, 1)$  is primitive, so  $\bar{DT}^{(0,0,\beta,1)}(\tau) = \hat{DT}^{(0,0,\beta,1)}(\tau) = DT^{(0,0,\beta,1)}(\tau)$  by Propositions 5.17 and 6.11. Katz [58, Conj. 2.3] conjectures that  $GV_0(\gamma) = DT^{(0,0,\beta,1)}(\tau)$ ; this had also earlier been conjectured by Hosono, Saito and Takahashi [41, Conj. 3.2]. We can now extend their conjecture to all  $k \in \mathbb{Z}$ .

**CONJECTURE 6.20.** *Let  $X$  be a Calabi–Yau 3-fold over  $\mathbb{C}$ , and  $(\tau, T, \leq)$  a weak stability condition on  $\text{coh}(X)$  of Gieseker or  $\mu$ -stability type. Then for  $\gamma \in H_2(X; \mathbb{Z})$  with  $\beta \in H^4(X; \mathbb{Z})$  Poincaré dual to  $\gamma$  and all  $k \in \mathbb{Z}$  we have  $\hat{DT}^{(0,0,\beta,k)}(\tau) = GV_0(\gamma)$ . In particular,  $\hat{DT}^{(0,0,\beta,k)}(\tau)$  is independent of  $k, \tau$ .*

For evidence for this, see [58] for the case  $k = 1$ , and note also that Theorem 6.16(a) shows  $\hat{DT}^{(0,0,\beta,k)}(\tau)$  is independent of  $\tau$ , Theorem 6.16(b) suggests  $\hat{DT}^{(0,0,\beta,k)}(\tau) \in \mathbb{Z}$ , and Propositions 6.18 and 6.19 show that the contributions to  $\hat{DT}^{(0,0,\beta,k)}(\tau)$  from rigid rational curves and embedded higher genus curves are as expected, and independent of  $k$ . Also Theorem 6.16(c) implies that  $\hat{DT}^{(0,0,\beta,k)}(\tau)$  is periodic in  $k$ , which supports the idea that it is independent of  $k$ . The first author would like to thank Sheldon Katz and Daves Maulik for conversations about Conjecture 6.20.

**REMARK 6.21.** There are other ways to count curves using Donaldson–Thomas theory than counting dimension 1 sheaves. The (ordinary) Donaldson–Thomas invariants  $DT^{(1,0,\beta,k)}(\tau)$  for  $\beta \in H^4(X; \mathbb{Z})$  and  $k \in \mathbb{Z}$  ‘count’ ideal sheaves of subschemes  $S$  of  $X$  with  $\dim S \leq 1$ , and the celebrated *MNOP Conjecture* [80, 81]

expresses  $DT^{(1,0,\beta,k)}(\tau)$  in terms of the Gromov–Witten invariants  $GW_g(\gamma)$  of  $X$  for all genera  $g \geq 0$ , or equivalently in terms of the Gopakumar–Vafa invariants  $GV_g(\gamma)$  of  $X$  for all  $g \geq 0$ . Pandharipande–Thomas invariants  $PT_{n,\beta}$  in [86] count pairs  $s : \mathcal{O}_X \rightarrow E$  for  $E$  a pure dimension 1 sheaf, like our  $PI^{\alpha,n}(\tau)$  but with a different stability condition, and these also have conjectural equivalences [86, §3] with  $DT^{(1,0,\beta,k)}(\tau)$ ,  $GW_g(\gamma)$  and  $GV_g(\gamma)$ .

We will not discuss these further in this book. However, we note that the results of this book should lead to advances in the theory of these curve counting invariants, and the relations between them. In particular, our wall-crossing formula Theorem 5.18 should be used to prove the correspondence between Donaldson–Thomas invariants  $DT^{(1,0,\beta,k)}(\tau)$  and Pandharipande–Thomas invariants  $PT_{n,\beta}$ . Recent papers by Toda [101] and Stoppa and Thomas [98] prove a version of this for invariants without Behrend functions as weights, and using our methods to include Behrend functions should yield the proof. Bridgeland [12] also proves the correspondence assuming conjectures in [63].

### 6.5. Why it all has to be so complicated: an example

Our definitions of  $\bar{DT}^\alpha(\tau)$  and  $\hat{DT}^\alpha(\tau)$  are very complicated. They count sheaves using two kinds of weights: firstly, we define  $\bar{\epsilon}^\alpha(\tau)$  from the  $\bar{\delta}_{ss}^\beta(\tau)$  by (3.4), with rational weights  $(-1)^{n-1}/n$ , and then we apply the Lie algebra morphism  $\tilde{\Psi}$  of §5.3, which takes Euler characteristics weighted by the  $\mathbb{Z}$ -valued Behrend function  $\nu_{\mathfrak{M}}$ . Some readers may have wondered whether all this complexity is really necessary. For instance, following (4.16) when  $\mathcal{M}_{ss}^\alpha(\tau) = \mathcal{M}_{st}^\alpha(\tau)$ , we could simply have defined  $DT^\alpha(\tau)$  for all  $\alpha \in C(\text{coh}(X))$  by

$$DT^\alpha(\tau) = \chi(\mathcal{M}_{st}^\alpha(\tau), \nu_{\mathcal{M}_{st}^\alpha(\tau)}). \quad (6.23)$$

We will now show, by carefully studying an example of dimension 1 sheaves supported on two rigid  $\mathbb{CP}^1$ 's in  $X$  which cross under deformation, that to get invariants unchanged under deformations of  $X$ , the extra layer of complexity with the  $\bar{\epsilon}^\alpha(\tau)$  and rational weights really is necessary. Our example will show that (6.23) is *not* deformation-invariant when  $\mathcal{M}_{ss}^\alpha(\tau) \neq \mathcal{M}_{st}^\alpha(\tau)$ , and the same holds if we replace  $\mathcal{M}_{st}^\alpha(\tau)$  by  $\mathcal{M}_{ss}^\alpha(\tau)$  or  $\mathfrak{M}_{ss}^\alpha(\tau)$ ; also, we will see that to get a deformation-invariant answer, it can be necessary to count strictly  $\tau$ -semistable sheaves with *rational, non-integral* weights, so we do need the  $\bar{\epsilon}^\alpha(\tau)$ .

For  $\epsilon > 0$  write  $\Delta_\epsilon = \{t \in \mathbb{C} : |t| < \epsilon\}$ . Let  $X_t$  for  $t \in \Delta_\epsilon$  be a smooth family of Calabi–Yau 3-folds over  $\mathbb{C}$ , equipped with a family of very ample line bundles  $\mathcal{O}_{X_t}(1)$ . Identify  $H^*(X_t; \mathbb{Q}) \cong H^*(X_0; \mathbb{Q})$ ,  $H_*(X_t; \mathbb{Z}) \cong H_*(X_0; \mathbb{Z})$  for all  $t$ . Suppose  $i_t : \mathbb{CP}^1 \rightarrow X_t$  and  $j_t : \mathbb{CP}^1 \rightarrow X_t$  are two smooth families of embeddings for  $t \in \Delta_\epsilon$ , and  $i_t(\mathbb{CP}^1), j_t(\mathbb{CP}^1)$  have normal bundle  $\mathcal{O}_{\mathbb{CP}^1}(-1) \oplus \mathcal{O}_{\mathbb{CP}^1}(-1)$  for all  $t$ . Suppose that  $i_t(\mathbb{CP}^1) \cap j_t(\mathbb{CP}^1) = \emptyset$  for  $t \neq 0$ , and that  $i_0(\mathbb{CP}^1), j_0(\mathbb{CP}^1)$  intersect in a single point  $x \in X_0$ , with  $T_x(i_0(\mathbb{CP}^1)) \cap T_x(j_0(\mathbb{CP}^1)) = 0$  in  $T_x X_0$ .

Now  $i_0(\mathbb{CP}^1) \cup j_0(\mathbb{CP}^1)$  is a nodal  $\mathbb{CP}^1$  in  $X_0$ , so we can regard it as the image of a genus 0 stable map  $k_0 : \Sigma_0 \rightarrow X_0$  from a prestable curve  $\Sigma_0 = \mathbb{CP}^1 \cup_x \mathbb{CP}^1$ , in the sense of Gromov–Witten theory. As we have prescribed the normal bundles and intersection of  $i_0(\mathbb{CP}^1), j_0(\mathbb{CP}^1)$ , we can show that  $k_0 : \Sigma_0 \rightarrow X_0$  is a *rigid* stable map, and so it persists as a stable map under small deformations of  $X_0$ . Thus, making  $\epsilon > 0$  smaller if necessary, for  $t \in \Delta_\epsilon$  there is a continuous family of genus 0 stable maps  $k_t : \Sigma_t \rightarrow X_t$ . Now  $k_t(\Sigma_t)$  cannot be reducible for small  $t \neq 0$ ,

since the irreducible components would have to be  $i_t(\mathbb{CP}^1), j_t(\mathbb{CP}^1)$ , but these do not intersect. So, making  $\epsilon > 0$  smaller if necessary, we can suppose  $\Sigma_t \cong \mathbb{CP}^1$ , and  $k_t$  is an embedding, and  $k_t(\mathbb{CP}^1)$  has normal bundle  $\mathcal{O}_{\mathbb{CP}^1}(-1) \oplus \mathcal{O}_{\mathbb{CP}^1}(-1)$ , for all  $0 \neq t \in \Delta_\epsilon$ .

Let  $\beta, \gamma \in H^4(X_0; \mathbb{Z})$  be Poincaré dual to  $[i_0(\mathbb{CP}^1)], [j_0(\mathbb{CP}^1)]$  in  $H_2(X_0; \mathbb{Z})$ . Suppose  $\beta, \gamma$  are linearly independent over  $\mathbb{Z}$ . Let  $\delta = c_1(\mathcal{O}_{X_0}(1))$  in  $H^2(X_0; \mathbb{Z})$ . Set  $c_\beta = \beta \cup \delta$  and  $c_\gamma = \gamma \cup \delta$  and  $c_{\beta+\gamma} = c_\beta + c_\gamma$ , so that  $c_\beta, c_\gamma, c_{\beta+\gamma} \in \mathbb{N}$ . Write classes  $\alpha \in K^{\text{num}}(\text{coh}(X_0))$  as  $(\alpha_0, \alpha_2, \alpha_4, \alpha_6)$  as in §6.4. We will consider  $\tau$ -semistable sheaves  $E$  on  $X_t$  in classes  $(0, 0, \beta, k)$ ,  $(0, 0, \gamma, l)$  and  $(0, 0, \beta + \gamma, m)$  for  $k, l, m \in \mathbb{Z}$  and  $t \in \Delta_\epsilon$ . Suppose for simplicity that all such sheaves are supported on  $i_t(\mathbb{CP}^1) \cup j_t(\mathbb{CP}^1) \cup k_t(\Sigma_t)$ ; alternatively, we can consider the following as computing the contributions to  $\bar{D}T^{(0,0,\beta,k)}(\tau)_t, \dots, \bar{D}T^{(0,0,\beta+\gamma,m)}(\tau)_t$  from sheaves supported on  $i_t(\mathbb{CP}^1) \cup j_t(\mathbb{CP}^1) \cup k_t(\Sigma_t)$ .

Here is a way to model all this explicitly in a family of compact Calabi–Yau 3-folds. Let  $\mathbb{CP}^2 \times \mathbb{CP}^2$  have homogeneous coordinates  $([x_0, x_1, x_2], [y_0, y_1, y_2])$ , write  $\mathbf{x} = (x_0, x_1, x_2)$ ,  $\mathbf{y} = (y_0, y_1, y_2)$ , and let  $X_t$  be the bicubic  $F_t(\mathbf{x}, \mathbf{y}) = 0$  in  $\mathbb{CP}^2 \times \mathbb{CP}^2$ , with very ample line bundle  $\mathcal{O}_{X_t}(1) = \mathcal{O}_{\mathbb{CP}^1 \times \mathbb{CP}^1}(1, 1)|_{X_t}$ , where

$$F_t(\mathbf{x}, \mathbf{y}) = x_0^2 x_1 y_0^2 y_1 + x_0^3 y_0^2 y_2 + x_0^2 x_2 y_0^3 - t x_0^3 y_0^3 \\ + x_1 x_2 P_{1,3}(\mathbf{x}, \mathbf{y}) + x_2 y_2 P'_{2,2}(\mathbf{x}, \mathbf{y}) + y_1 y_2 P''_{3,1}(\mathbf{x}, \mathbf{y}),$$

with  $P_{1,3}, P'_{2,2}, P''_{3,1}$  homogeneous polynomials of the given bidegrees.

Define  $i_t, j_t, k_t : \mathbb{CP}^1 \rightarrow X_t$  by  $i_t : [u, v] \mapsto ([u, v, 0], [1, 0, t])$  and  $j_t : [u, v] \mapsto ([1, 0, t], [u, v, 0])$  for all  $t$ , and  $k_t : [u, v] \mapsto ([u, v, 0], [v, tu, 0])$  for  $t \neq 0$ . Then the conditions above hold for  $P_{1,3}, P'_{2,2}, P''_{3,1}$  generic and  $\epsilon > 0$  small. Consider first the moduli spaces  $\mathfrak{M}_{\text{ss}}^{(0,0,\beta,k)}(\tau)_t, \mathfrak{M}_{\text{ss}}^{(0,0,\gamma,l)}(\tau)_t$  over  $X_t$ . These are  $\tau$ -semistable sheaves supported on  $i_t(\mathbb{CP}^1), j_t(\mathbb{CP}^1)$ , so by Proposition 6.17 we see that the only  $\tau$ -semistable sheaves in classes  $(0, 0, \beta, k)$  and  $(0, 0, \gamma, l)$  are  $E_t(k) = (i_t)_*(\mathcal{O}_{\mathbb{CP}^1}(k-1))$  and  $F_t(l) = (j_t)_*(\mathcal{O}_{\mathbb{CP}^1}(l-1))$  respectively, and both are  $\tau$ -stable and rigid.

The Hilbert polynomials of  $E_t(k)$  and  $F_t(l)$  are  $P_{(0,0,\beta,k)}(t) = c_\beta t + k$  and  $P_{(0,0,\gamma,l)}(t) = c_\gamma t + l$  by (6.22), so we have

$$\tau([E_t(k)]) = t + k/c_\beta, \quad \tau([F_t(l)]) = t + l/c_\gamma. \quad (6.24)$$

Therefore the sheaf  $E_t(k) \oplus F_t(l)$  in class  $(0, 0, \beta + \gamma, k + l)$  is  $\tau$ -semistable if and only if  $k c_\gamma = l c_\beta$ .

We can now describe  $\mathfrak{M}_{\text{ss}}^{(0,0,\beta+\gamma,m)}(\tau)_t$  for  $t \neq 0$ . For all  $m \in \mathbb{Z}$ , we have a rigid  $\tau$ -stable sheaf  $G_t(m) = (k_t)_*(\mathcal{O}_{\mathbb{CP}^1}(m-1))$  in class  $(0, 0, \beta + \gamma, m)$  supported on  $k_t(\Sigma_t)$ , which contributes  $[\text{Spec } \mathbb{C}/\mathbb{G}_m]$  to  $\mathfrak{M}_{\text{ss}}^{(0,0,\beta+\gamma,m)}(\tau)_t$ . In addition, if there exist  $k, l \in \mathbb{Z}$  with  $k + l = m$  and  $k c_\gamma = l c_\beta$ , then  $E_t(k) \oplus F_t(l)$  is a rigid, strictly  $\tau$ -semistable sheaf in class  $(0, 0, \beta + \gamma, m)$  supported on  $i_t(\mathbb{CP}^1) \amalg j_t(\mathbb{CP}^1)$ , which contributes  $[\text{Spec } \mathbb{C}/\mathbb{G}_m^2]$  to  $\mathfrak{M}_{\text{ss}}^{(0,0,\beta+\gamma,m)}(\tau)_t$ . We have  $k = m c_\beta / c_{\beta+\gamma}$ ,  $l = m c_\gamma / c_{\beta+\gamma}$ , which lie in  $\mathbb{Z}$  if and only if  $c_{\beta+\gamma} \mid m c_\beta$ . These are all the  $\tau$ -semistable

sheaves in class  $(0, 0, \beta + \gamma, m)$ . Thus we see that

$$\begin{aligned} t \neq 0, c_{\beta+\gamma} \nmid m c_{\beta} &\text{ imply } \mathfrak{M}_{\text{ss}}^{(0,0,\beta+\gamma,m)}(\tau)_t \cong [\text{Spec } \mathbb{C}/\mathbb{G}_m] \\ \text{and } \mathcal{M}_{\text{ss}}^{(0,0,\beta+\gamma,m)}(\tau)_t &= \mathcal{M}_{\text{st}}^{(0,0,\beta+\gamma,m)}(\tau)_t \cong \text{Spec } \mathbb{C}, \end{aligned} \quad (6.25)$$

$$\begin{aligned} t \neq 0, c_{\beta+\gamma} \mid m c_{\beta} &\text{ imply } \mathfrak{M}_{\text{ss}}^{(0,0,\beta+\gamma,m)}(\tau)_t \cong [\text{Spec } \mathbb{C}/\mathbb{G}_m] \amalg [\text{Spec } \mathbb{C}/\mathbb{G}_m^2], \\ \mathcal{M}_{\text{ss}}^{(0,0,\beta+\gamma,m)}(\tau)_t &\cong \text{Spec } \mathbb{C} \amalg \text{Spec } \mathbb{C}, \text{ and } \mathcal{M}_{\text{st}}^{(0,0,\beta+\gamma,m)}(\tau)_t \cong \text{Spec } \mathbb{C}. \end{aligned} \quad (6.26)$$

Now consider  $\mathfrak{M}_{\text{ss}}^{(0,0,\beta+\gamma,m)}(\tau)_0$  when  $t = 0$ . Writing  $\mathcal{O}_x$  for the structure sheaf of intersection point of  $i_0(\mathbb{CP}^1)$  and  $j_0(\mathbb{CP}^1)$ , we have exact sequences

$$\begin{aligned} 0 \longrightarrow E_0(k) \longrightarrow E_0(k+1) \xrightarrow{\pi_x} \mathcal{O}_x \longrightarrow 0, \\ 0 \longrightarrow F_0(l) \longrightarrow F_0(l+1) \xrightarrow{\pi_x} \mathcal{O}_x \longrightarrow 0. \end{aligned} \quad (6.27)$$

Define  $G_0(k, l)$  to be the kernel of the morphism in  $\text{coh}(X_0)$

$$\pi_x \oplus \pi_x : E_0(k+1) \oplus F_0(l+1) \longrightarrow \mathcal{O}_x.$$

Since  $[E_0(k+1)] = (0, 0, \beta, k+1)$ ,  $[F_0(l+1)] = (0, 0, \beta, l+1)$  and  $[\mathcal{O}_x] = (0, 0, 0, 1)$  and each  $\pi_x$  is surjective we have  $[G_0(k, l)] = (0, 0, \beta + \gamma, k + l + 1)$ .

From (6.27) we see that we have non-split exact sequences

$$\begin{aligned} 0 \longrightarrow E_0(k) \longrightarrow G_0(k, l) \longrightarrow F_0(l+1) \longrightarrow 0, \\ 0 \longrightarrow F_0(l) \longrightarrow G_0(k, l) \longrightarrow E_0(k+1) \longrightarrow 0. \end{aligned} \quad (6.28)$$

By (6.24), the first sequence of (6.28) destabilizes  $G_0(k, l)$  if  $k/c_{\beta} > (l+1)/c_{\gamma}$ , and the second sequence destabilizes  $G_0(k, l)$  if  $l/c_{\gamma} > (k+1)/c_{\beta}$ . The sequences (6.28) are sufficient to test the  $\tau$ -(semi)stability of  $G_0(k, l)$ . It follows that  $G_0(k, l)$  is  $\tau$ -semistable if  $k/c_{\beta} \leq (l+1)/c_{\gamma}$  and  $l/c_{\gamma} \leq (k+1)/c_{\beta}$ , and  $G_0(k, l)$  is  $\tau$ -stable if  $k/c_{\beta} < (l+1)/c_{\gamma}$  and  $l/c_{\gamma} < (k+1)/c_{\beta}$ .

Now fix  $m \in \mathbb{Z}$ . It is easy to show from these inequalities that if  $c_{\beta+\gamma} \nmid m c_{\beta}$  there is exactly one choice of  $k, l \in \mathbb{Z}$  with  $k + l + 1 = m$  and  $G_0(k, l)$   $\tau$ -semistable in class  $(0, 0, \beta + \gamma, m)$ , and in fact this  $G_0(k, l)$  is  $\tau$ -stable. And if  $c_{\beta+\gamma} \mid m c_{\beta}$  then setting  $k = m c_{\beta}/c_{\beta+\gamma}$ ,  $l = m c_{\gamma}/c_{\beta+\gamma}$  in  $\mathbb{Z}$  we find that  $G_0(k-1, l)$  and  $G_0(k, l-1)$  are both strictly  $\tau$ -semistable in class  $(0, 0, \beta + \gamma, m)$ , and in addition  $E_0(k) \oplus F_0(l)$  is strictly  $\tau$ -semistable in class  $(0, 0, \beta + \gamma, m)$ . These are all the  $\tau$ -semistables in class  $(0, 0, \beta + \gamma, m)$ . So

$$\begin{aligned} c_{\beta+\gamma} \nmid m c_{\beta} &\text{ implies } \mathfrak{M}_{\text{ss}}^{(0,0,\beta+\gamma,m)}(\tau)_0 \cong [\text{Spec } \mathbb{C}/\mathbb{G}_m] \\ \text{and } \mathcal{M}_{\text{ss}}^{(0,0,\beta+\gamma,m)}(\tau)_0 &= \mathcal{M}_{\text{st}}^{(0,0,\beta+\gamma,m)}(\tau)_0 \cong \text{Spec } \mathbb{C}, \end{aligned} \quad (6.29)$$

$$\begin{aligned} c_{\beta+\gamma} \mid m c_{\beta} &\text{ implies } \mathcal{M}_{\text{st}}^{(0,0,\beta+\gamma,m)}(\tau)_0 = \emptyset \text{ and} \\ \mathfrak{M}_{\text{ss}}^{(0,0,\beta+\gamma,m)}(\tau)_0(\mathbb{C}) &= \{G_0(k-1, l), G_0(k, l-1), E_0(k) \oplus F_0(l)\}. \end{aligned} \quad (6.30)$$

Next we describe the stack structure on  $\mathfrak{M}_{\text{ss}}^{(0,0,\beta+\gamma,m)}(\tau)_0$  when  $c_{\beta+\gamma} \mid m c_{\beta}$ . As  $E_0(k), F_0(l)$  are rigid, we have

$$\begin{aligned} \text{Ext}^1(E_0(k) \oplus F_0(l), E_0(k) \oplus F_0(l)) \\ = \text{Ext}^1(E_0(k), F_0(l)) \oplus \text{Ext}^1(F_0(l), E_0(k)) \cong \mathbb{C} \oplus \mathbb{C}, \end{aligned} \quad (6.31)$$

where a nonzero element of  $\text{Ext}^1(E_0(k), F_0(l))$  corresponds to  $G_0(k-1, l)$ , and a nonzero element of  $\text{Ext}^1(F_0(l), E_0(k))$  corresponds to  $G_0(k, l-1)$ , by (6.28). Let  $(y, z)$  be coordinates on  $\mathbb{C} \oplus \mathbb{C}$  in (6.31). Then  $\text{Aut}(E_0(k) \oplus F_0(l)) \cong \mathbb{G}_m^2$  acts on  $\mathbb{C} \oplus \mathbb{C}$  by  $(\lambda, \mu) : (y, z) \mapsto (\lambda\mu^{-1}y, \lambda^{-1}\mu z)$ .

By Theorem 5.5,  $\mathfrak{M}_{\text{ss}}^{(0,0,\beta+\gamma,m)}(\tau)_0$  is locally isomorphic as an Artin stack near  $E_0(k) \oplus F_0(l)$  to  $[\text{Crit}(f)/\mathbb{G}_m^2]$ , where  $U \subseteq \mathbb{C} \oplus \mathbb{C}$  is a  $\mathbb{G}_m^2$ -invariant analytic open neighbourhood of 0, and  $f : U \rightarrow \mathbb{C}$  is a  $\mathbb{G}_m^2$ -invariant holomorphic function. Since  $f$  is  $\mathbb{G}_m^2$ -invariant, it must be a function of  $yz$ . Now  $(y, 0)$  for  $y \neq 0$  in  $\mathbb{C} \oplus \mathbb{C}$  represents  $G_0(k-1, l)$ , which is rigid; also  $(0, z)$  for  $z \neq 0$  represents  $G_0(k, l-1)$ , which is rigid. Therefore  $\{(y, 0) : 0 \neq y \in \mathbb{C}\}$  and  $\{(0, z) : 0 \neq z \in \mathbb{C}\}$  must be smooth open sets in  $\text{Crit}(f)$ , so that  $f$  is nondegenerate quadratic normal to them, to leading order.

It follows that we may take  $U = \mathbb{C} \oplus \mathbb{C}$  and  $f(y, z) = y^2 z^2$ , giving

$$\mathfrak{M}_{\text{ss}}^{(0,0,\beta+\gamma,m)}(\tau)_0 \cong [\text{Crit}(y^2 z^2)/\mathbb{G}_m^2]. \quad (6.32)$$

There are three  $\mathbb{G}_m^2$  orbits in  $\text{Crit}(y^2 z^2)$ :  $\{(y, 0) : 0 \neq y \in \mathbb{C}\}$  corresponding to  $G_0(k-1, l)$ , and  $\{(0, z) : 0 \neq z \in \mathbb{C}\}$  corresponding to  $G_0(k, l-1)$ , and  $(0, 0)$  corresponding to  $E_0(k) \oplus F_0(l)$ . The Milnor fibres of  $y^2 z^2$  at  $(1, 0)$  and  $(0, 1)$  are both two discs, with Euler characteristic 2. Since  $y^2 z^2$  is homogeneous, the Milnor fibre of  $y^2 z^2$  at  $(0, 0)$  is diffeomorphic to  $\{(y, z) \in \mathbb{C}^2 : y^2 z^2 = 1\}$ , which is the disjoint union of two copies of  $\mathbb{C} \setminus \{0\}$ , with Euler characteristic zero. By results in §4 we deduce that

$$\begin{aligned} \nu_{\mathfrak{M}_{\text{ss}}^{(0,0,\beta+\gamma,m)}(\tau)_0}(G_0(k-1, l)) &= \nu_{\mathfrak{M}_{\text{ss}}^{(0,0,\beta+\gamma,m)}(\tau)_0}(G_0(k, l-1)) = -1 \\ \text{and} \quad \nu_{\mathfrak{M}_{\text{ss}}^{(0,0,\beta+\gamma,m)}(\tau)_0}(E_0(k) \oplus F_0(l)) &= 1. \end{aligned} \quad (6.33)$$

From (6.32) we find that the coarse moduli space  $\mathcal{M}_{\text{ss}}^{(0,0,\beta+\gamma,m)}(\tau)_0$  is

$$\mathcal{M}_{\text{ss}}^{(0,0,\beta+\gamma,m)}(\tau)_0 \cong \text{Spec } \mathbb{C}. \quad (6.34)$$

As in equation (6.11) of Example 6.9 we find that

$$\begin{aligned} \bar{\Pi}_{\mathfrak{M}}^{\chi, \mathbb{Q}}(\bar{\epsilon}^{(0,0,\beta+\gamma,m)}(\tau)_0) &= \frac{1}{2} [([\text{Spec } \mathbb{C}/\mathbb{G}_m], \rho_{G_0(k-1, l)})] \\ &\quad + \frac{1}{2} [([\text{Spec } \mathbb{C}/\mathbb{G}_m], \rho_{G_0(k, l-1)})], \end{aligned}$$

where  $\rho_{G_0(k-1, l)}, \rho_{G_0(k, l-1)}$  map  $[\text{Spec } \mathbb{C}/\mathbb{G}_m]$  to  $G_0(k-1, l), G_0(k, l-1)$ . Note that  $\bar{\epsilon}^{(0,0,\beta+\gamma,m)}(\tau)_0$  is zero over  $E_0(k) \oplus F_0(l)$ . As for (6.12) we have

$$\begin{aligned} \bar{D}T^{(0,0,\beta+\gamma,m)}(\tau)_0 &= \frac{1}{2} \nu_{\mathfrak{M}_{\text{ss}}^{(0,0,\beta+\gamma,m)}(\tau)_0}(G_0(k-1, l)) \\ &\quad + \frac{1}{2} \nu_{\mathfrak{M}_{\text{ss}}^{(0,0,\beta+\gamma,m)}(\tau)_0}(G_0(k, l-1)) = \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot 1 = 1. \end{aligned} \quad (6.35)$$

Thus  $G_0(k-1, l)$  and  $G_0(k, l-1)$  each contribute  $\frac{1}{2}$  to  $\bar{D}T^{(0,0,\beta+\gamma,m)}(\tau)_0$ .

From equations (6.25),(6.26),(6.29),(6.30),(6.33),(6.34) and (6.35) we deduce:

$$\bar{D}T^{(0,0,\beta+\gamma,m)}(\tau)_t = 1, \quad \text{all } t \in \Delta_\epsilon \text{ and } m \in \mathbb{Z},$$

$$\chi(\mathcal{M}_{\text{st}}^{(0,0,\beta+\gamma,m)}(\tau)_t, \nu_{\mathcal{M}_{\text{st}}^{(0,0,\beta+\gamma,m)}(\tau)_t}) = \begin{cases} 1, & t \neq 0 \text{ or } c_{\beta+\gamma} \nmid m c_\beta, \\ 0, & t = 0 \text{ and } c_{\beta+\gamma} \mid m c_\beta, \end{cases} \quad (6.36)$$

$$\chi(\mathcal{M}_{\text{ss}}^{(0,0,\beta+\gamma,m)}(\tau)_t, \nu_{\mathcal{M}_{\text{ss}}^{(0,0,\beta+\gamma,m)}(\tau)_t}) = \begin{cases} 1, & t \neq 0 \text{ or } c_{\beta+\gamma} \nmid m c_\beta, \\ 2, & t = 0 \text{ and } c_{\beta+\gamma} \mid m c_\beta, \end{cases} \quad (6.37)$$

$$\chi^{\text{na}}(\mathfrak{M}_{\text{ss}}^{(0,0,\beta+\gamma,m)}(\tau)_t, \nu_{\mathfrak{M}_{\text{ss}}^{(0,0,\beta+\gamma,m)}(\tau)_t}) = \begin{cases} -1, & t = 0 \text{ or } c_{\beta+\gamma} \nmid m c_\beta, \\ 0, & t \neq 0 \text{ and } c_{\beta+\gamma} \mid m c_\beta. \end{cases} \quad (6.38)$$

Equations (6.36)–(6.38) imply:

**COROLLARY 6.22.** *Let  $X$  be a Calabi–Yau 3-fold over  $\mathbb{C}$  and  $\alpha \in K^{\text{num}}(\text{coh}(X))$  with  $\mathcal{M}_{\text{ss}}^\alpha(\tau) \neq \mathcal{M}_{\text{st}}^\alpha(\tau)$ . Then none of  $\chi(\mathcal{M}_{\text{st}}^\alpha(\tau), \nu_{\mathcal{M}_{\text{st}}^\alpha(\tau)})$ ,  $\chi(\mathcal{M}_{\text{ss}}^\alpha(\tau), \nu_{\mathcal{M}_{\text{ss}}^\alpha(\tau)})$  or  $\chi^{\text{na}}(\mathfrak{M}_{\text{ss}}^\alpha(\tau), \nu_{\mathfrak{M}_{\text{ss}}^\alpha(\tau)})$  need be unchanged under deformations of  $X$ .*

We can also use these calculations to justify the necessity of rational weights in the  $\bar{\epsilon}^\alpha(\tau)$  in our definition of  $\bar{D}T^\alpha(\tau)$ . Let  $c_{\beta+\gamma} \mid m c_\beta$ . Then when  $t \neq 0$ , we have one stable, rigid sheaf  $G_t(m)$  in class  $(0,0,\beta+\gamma,m)$ , which is counted with weight 1 in  $\bar{D}T^{(0,0,\beta+\gamma,m)}(\tau)_t$ . But when  $t = 0$ ,  $G_t(m)$  is replaced by two strictly  $\tau$ -semistable sheaves  $G_0(k-1,l)$  and  $G_0(k,l-1)$ , which are counted with weight  $\frac{1}{2}$  in  $\bar{D}T^{(0,0,\beta+\gamma,m)}(\tau)_0$ . By symmetry between  $G_0(k-1,l), G_0(k,l-1)$ , to get deformation-invariance it is necessary that they are each counted with weight  $\frac{1}{2}$ , which means that *we must allow non-integral weights for strictly  $\tau$ -semistables in our counting scheme to get a deformation-invariant answer.*

Also, we cannot tell that  $G_0(k-1,l), G_0(k,l-1)$  should have weight  $\frac{1}{2}$  just from the stack  $\mathfrak{M}_{\text{ss}}^{(0,0,\beta+\gamma,m)}(\tau)_0$ , as they are rigid with stabilizer group  $\mathbb{G}_m$ , and look just like  $\tau$ -stables. The strict  $\tau$ -semistability of  $G_0(k-1,l), G_0(k,l-1)$  is measured by the fact that  $\bar{\delta}_{\text{ss}}^{(0,0,\gamma,l)}(\tau) * \bar{\delta}_{\text{ss}}^{(0,0,\beta,k)}(\tau), \bar{\delta}_{\text{ss}}^{(0,0,\beta,k)}(\tau) * \bar{\delta}_{\text{ss}}^{(0,0,\gamma,l)}(\tau)$  are nonzero over  $G_0(k-1,l), G_0(k,l-1)$  respectively. But  $\bar{\delta}_{\text{ss}}^{(0,0,\gamma,l)}(\tau) * \bar{\delta}_{\text{ss}}^{(0,0,\beta,k)}(\tau)$  and  $\bar{\delta}_{\text{ss}}^{(0,0,\beta,k)}(\tau) * \bar{\delta}_{\text{ss}}^{(0,0,\gamma,l)}(\tau)$  occur with coefficient  $-\frac{1}{2}$  in the expression (3.4) for  $\bar{\epsilon}^{(0,0,\beta+\gamma,m)}(\tau)$ . This suggests that *using  $\bar{\epsilon}^\alpha(\tau)$  or something like it is necessary to make  $\bar{D}T^\alpha(\tau)$  deformation-invariant.*

## 6.6. $\mu$ -stability and invariants $\bar{D}T^\alpha(\mu)$

So far we have mostly discussed invariants  $\bar{D}T^\alpha(\tau)$ , where  $(\tau, G, \leq)$  is Gieseker stability w.r.t. a very ample line bundle  $\mathcal{O}_X(1)$ , as in Example 3.8. We can also consider  $\bar{D}T^\alpha(\mu)$ , where  $(\mu, M, \leq)$  is  $\mu$ -stability w.r.t.  $\mathcal{O}_X(1)$ , as in Example 3.9. We now prove some simple but nontrivial facts about the  $\bar{D}T^\alpha(\mu)$ .

First note that as  $(\mu, M, \leq)$  is a truncation of  $(\tau, G, \leq)$ , we have  $\tau(\beta) \leq \tau(\gamma)$  implies  $\mu(\beta) \leq \mu(\gamma)$  for  $\beta, \gamma \in C(\text{coh}(X))$ , and so  $(\mu, M, \leq)$  dominates  $(\tau, G, \leq)$  in the sense of Definition 3.12. In Theorem 3.13 we can use  $(\hat{\tau}, \hat{T}, \leq) = (\mu, M, \leq)$  as the dominating weak stability condition to write  $\bar{\epsilon}^\alpha(\mu)$  in terms of the  $\bar{\epsilon}^\beta(\tau)$  and vice versa, and then Theorem 5.18 writes  $\bar{D}T^\alpha(\mu)$  in terms of the  $\bar{D}T^\beta(\tau)$  and vice versa. Since the Gieseker stability invariants  $\bar{D}T^\beta(\tau)$  are deformation-invariant by Corollary 5.28, we deduce:

**COROLLARY 6.23.** *The  $\mu$ -stability invariants  $\bar{D}T^\alpha(\mu)$  are unchanged under continuous deformations of the underlying Calabi–Yau 3-fold  $X$ .*

The next well-known lemma says that for torsion-free sheaves,  $\mu$ -stability is unchanged by tensoring by a line bundle. It holds on any smooth projective scheme  $X$ , not just Calabi–Yau 3-folds, and works because  $\mu([E \otimes L]) - \mu([E])$  is independent of  $E$  when  $\dim E = \dim X$ . The corresponding results are not true for Gieseker stability, nor for  $\mu$ -stability for non-torsion-free sheaves.

**LEMMA 6.24.** *Let  $E$  be a nonzero torsion-free sheaf on  $X$ , and  $L$  a line bundle. Then  $E \otimes L$  is  $\mu$ -semistable if and only if  $E$  is  $\mu$ -semistable.*

Now  $E \mapsto E \otimes L$  induces an automorphism of the abelian category  $\text{coh}(X)$ , which acts on  $K^{\text{num}}(\text{coh}(X)) \subset H^{\text{even}}(X; \mathbb{Q})$  by  $\alpha \mapsto \alpha \exp(\gamma)$ . For torsion-free sheaves, this automorphism takes  $\mu$ -semistables to  $\mu$ -semistables, and so maps  $\bar{\delta}_{\text{ss}}^\alpha(\mu)$  to  $\bar{\delta}_{\text{ss}}^{\alpha \exp(\gamma)}(\mu)$  and  $\bar{\epsilon}^\alpha(\mu)$  to  $\bar{\epsilon}^{\alpha \exp(\gamma)}(\mu)$  for  $\text{rank}(\alpha) > 0$ . Applying  $\tilde{\Psi}$  as in §5.3, we see that  $\bar{D}T^{\alpha \exp(\gamma)}(\mu) = \bar{D}T^\alpha(\mu)$ . Since we assume  $H^1(\mathcal{O}_X) = 0$ , every  $\gamma \in H^2(X; \mathbb{Z})$  is  $c_1(L)$  for some line bundle  $L$ . Thus we deduce:

**THEOREM 6.25.** *Let  $X$  be a Calabi–Yau 3-fold over  $\mathbb{C}$  and  $(\mu, M, \leq)$  be  $\mu$ -stability with respect to a very ample line bundle  $\mathcal{O}_X(1)$  on  $X$ , as in Example 3.9. Write elements  $\alpha$  of  $K^{\text{num}}(\text{coh}(X)) \subset H^{\text{even}}(X; \mathbb{Q})$  as  $(\alpha_0, \alpha_2, \alpha_4, \alpha_6)$ , as in §6.4. Then for all  $\alpha \in C(\text{coh}(X))$  with  $\alpha_0 > 0$  and all  $\gamma \in H^2(X; \mathbb{Z})$  we have  $\bar{D}T^{\alpha \exp(\gamma)}(\mu) = \bar{D}T^\alpha(\mu)$ .*

Theorem 6.25 encodes a big symmetry group of generalized Donaldson–Thomas invariants  $\bar{D}T^\alpha(\mu)$  in positive rank, which would be much more complicated to write down for Gieseker stability.

### 6.7. Extension to noncompact Calabi–Yau 3-folds

So far we have considered only *compact* Calabi–Yau 3-folds, and indeed our convention is that Calabi–Yau 3-folds are by definition compact, unless we explicitly say that they are noncompact. Suppose  $X$  is a *noncompact Calabi–Yau 3-fold* over  $\mathbb{C}$ , by which we mean a smooth quasiprojective 3-fold over  $\mathbb{C}$ , with trivial canonical bundle  $K_X$ . (We will impose further conditions on  $X$  shortly.) Then the abelian category  $\text{coh}(X)$  of coherent sheaves on  $X$  is badly behaved, from our point of view – for instance, groups  $\text{Hom}(E, F)$  for  $E, F \in \text{coh}(X)$  may be infinite-dimensional, so the Euler form  $\bar{\chi}$  on  $\text{coh}(X)$  may not be defined.

However, the abelian category  $\text{coh}_{\text{cs}}(X)$  of *compactly-supported* coherent sheaves on  $X$  is well-behaved:  $\text{Ext}^i(E, F)$  is finite-dimensional for  $E, F \in \text{coh}_{\text{cs}}(X)$  and satisfies Serre duality  $\text{Ext}^i(F, E) \cong \text{Ext}^{3-i}(E, F)^*$ , so  $\text{coh}_{\text{cs}}(X)$  has a well-defined Euler form. If  $X$  has no compact connected components then  $\text{coh}_{\text{cs}}(X)$  consists of torsion sheaves, supported in dimension 0, 1 or 2.

We propose that a good generalization of Donaldson–Thomas theory to noncompact Calabi–Yau 3-folds is to define invariants counting sheaves in  $\text{coh}_{\text{cs}}(X)$ . Note that this is *not* the route that has been taken by other authors such as Szendrői [99, §2.8], who instead consider invariants counting *ideal sheaves*  $I$  of compact subschemes of  $X$ . Such  $I$  are not compactly-supported, but are isomorphic to  $\mathcal{O}_X$  outside a compact subset of  $X$ .

Going through the theory of §4–§5, we find that the assumption that  $X$  is compact (proper, or projective) is used in three important ways:

- (a) The Euler form  $\bar{\chi}$  on  $K_0(\text{coh}(X))$  is undefined for noncompact  $X$  as  $\text{Ext}^i(E, F)$  may be infinite-dimensional, so  $K^{\text{num}}(\text{coh}(X))$  is undefined. For  $\text{coh}_{\text{cs}}(X)$ ,  $K^{\text{num}}(\text{coh}_{\text{cs}}(X))$  is well defined, but may not be isomorphic to the image of  $\text{ch} : K_0(\text{coh}_{\text{cs}}(X)) \rightarrow H_{\text{cs}}^{\text{even}}(X; \mathbb{Q})$ . Hilbert polynomials  $P_E$  of  $E \in \text{coh}_{\text{cs}}(X)$  need not factor through the class  $[E]$  in  $K^{\text{num}}(\text{coh}_{\text{cs}}(X))$  for  $X$  noncompact.
- (b) For noncompact  $X$ , Theorem 5.3 in §5.1 fails because nonzero vector bundles on  $X$  are not compactly-supported. But Theorems 5.4 and 5.5 depend on Theorem 5.3, and Theorem 5.11 in §5.2 depends on Theorem 5.5, and most of the rest of §5–§6 depends on Theorem 5.11.
- (c) For noncompact  $X$ , the moduli schemes  $\mathcal{M}_{\text{ss}}^{\alpha}(\tau)$  and  $\mathcal{M}_{\text{stp}}^{\alpha, n}(\tau')$  of §4.3 and §5.4 need not be proper. This means that the virtual cycle definitions of  $DT^{\alpha}(\tau)$  in (4.15) when  $\mathcal{M}_{\text{ss}}^{\alpha}(\tau) = \mathcal{M}_{\text{st}}^{\alpha}(\tau)$ , and of  $PI^{\alpha, n}(\tau')$  in (5.15), are not valid. The weighted Euler characteristic expressions (4.16), (5.16) for  $DT^{\alpha}(\tau)$  and  $PI^{\alpha, n}(\tau')$  still make sense. But the proofs that  $DT^{\alpha}(\tau), \bar{DT}^{\alpha}(\tau), PI^{\alpha, n}(\tau')$  are unchanged by deformations of  $X$  no longer work, as they are based on the virtual cycle definitions (4.15), (5.15).

Here is how we deal with these issues. For (a), with  $X$  noncompact, note that although  $\text{coh}(X)$  may not have a well-defined Euler form, there is an Euler pairing  $\bar{\chi} : K_0(\text{coh}(X)) \times K_0(\text{coh}_{\text{cs}}(X)) \rightarrow \mathbb{Z}$ . Under the Chern character maps  $\text{ch} : K_0(\text{coh}(X)) \rightarrow H^{\text{even}}(X; \mathbb{Q})$  and  $\text{ch}_{\text{cs}} : K_0(\text{coh}_{\text{cs}}(X)) \rightarrow H_{\text{cs}}^{\text{even}}(X; \mathbb{Q})$ , this  $\bar{\chi}$  is mapped to the pairing  $H^{\text{even}}(X; \mathbb{Q}) \times H_{\text{cs}}^{\text{even}}(X; \mathbb{Q}) \rightarrow \mathbb{Q}$  given by  $(\alpha, \beta) \mapsto \deg(\alpha^{\vee} \cdot \beta \cdot \text{td}(TX))_3$ , which is nondegenerate by the invertibility of  $\text{td}(TX) = 1 + \frac{1}{12} c_2(X)$  and Poincaré duality  $H^{2k}(X; \mathbb{Q}) \cong H_{\text{cs}}^{6-2k}(X; \mathbb{Q})^*$ .

In Assumption 3.2, with  $\mathcal{A} = \text{coh}_{\text{cs}}(X)$ , we should take  $K(\text{coh}_{\text{cs}}(X))$  to be the quotient of  $K_0(\text{coh}_{\text{cs}}(X))$  by the kernel in  $K_0(\text{coh}_{\text{cs}}(X))$  of the Euler pairing  $\bar{\chi} : K_0(\text{coh}(X)) \times K_0(\text{coh}_{\text{cs}}(X)) \rightarrow \mathbb{Z}$ . This is *not* the same as the numerical Grothendieck group  $K^{\text{num}}(\text{coh}_{\text{cs}}(X))$ , which is the quotient of  $K_0(\text{coh}_{\text{cs}}(X))$  by the kernel of  $\bar{\chi} : K_0(\text{coh}_{\text{cs}}(X)) \times K_0(\text{coh}_{\text{cs}}(X)) \rightarrow \mathbb{Z}$ ; in general  $K^{\text{num}}(\text{coh}_{\text{cs}}(X))$  is a quotient of  $K(\text{coh}_{\text{cs}}(X))$ , but they may not be equal. In Example 6.30 below we will have  $K(\text{coh}_{\text{cs}}(X)) \cong \mathbb{Z}^2$  but  $K^{\text{num}}(\text{coh}_{\text{cs}}(X)) = 0$ . As the pairing above is nondegenerate, this  $K(\text{coh}_{\text{cs}}(X))$  is naturally identified with the image of the *compactly-supported Chern character*  $\text{ch}_{\text{cs}} : K_0(\text{coh}_{\text{cs}}(X)) \rightarrow H_{\text{cs}}^{\text{even}}(X; \mathbb{Q})$ . This can be defined by combining the ‘localized Chern character’  $\text{ch}_Z^X$  of Fulton [28, §18.1 & p. 368] with the compactly supported Chow groups and homology in [28, Ex.s 10.2.8 & 19.1.12], using the limiting process over all compact subschemes  $Z \subset X$  in [28, Ex. 10.2.8].

Then if  $E \in \text{coh}(X)$  and  $F \in \text{coh}_{\text{cs}}(X)$ , the Euler form  $\bar{\chi}(E, F)$  depends only on  $E$  and  $[F]$  in  $K(\text{coh}_{\text{cs}}(X))$ . In particular, given a very ample line bundle  $\mathcal{O}_X(1)$  on  $X$ , the Hilbert polynomial  $P_F(n) = \bar{\chi}(\mathcal{O}_X(-n), F)$  of  $F$  depends only on the class  $[F]$  in  $K(\text{coh}_{\text{cs}}(X))$ . Since  $\mathcal{O}_X(-n)$  is not compactly-supported, in general  $P_F$  does not depend only on  $[F]$  in  $K^{\text{num}}(\text{coh}_{\text{cs}}(X))$ , as in Example 6.30.

This is important for two reasons. Firstly, equation (5.17) in §5.4 involves  $\bar{\chi}([\mathcal{O}_X(-n)], \alpha)$  for  $\alpha \in K(\text{coh}(X))$ , and if  $K(\text{coh}_{\text{cs}}(X)) = K^{\text{num}}(\text{coh}_{\text{cs}}(X))$  then  $\bar{\chi}([\mathcal{O}_X(-n)], \alpha)$  would not be well-defined for  $\alpha \in K(\text{coh}_{\text{cs}}(X))$ , and Theorem 5.27 would fail. Secondly, the proof that moduli spaces  $\mathcal{M}_{\text{st}}^{\alpha}(\tau), \mathcal{M}_{\text{ss}}^{\alpha}(\tau), \mathfrak{M}_{\text{st}}^{\alpha}(\tau), \mathfrak{M}_{\text{ss}}^{\alpha}(\tau)$  of  $\tau$ -(semi)stable sheaves  $E$  in class  $\alpha$  in  $K(\text{coh}(X))$  are of *finite type* depends on the fact that  $\alpha$  determines the Hilbert polynomial of  $E$ . If we took  $K(\text{coh}_{\text{cs}}(X)) = K^{\text{num}}(\text{coh}_{\text{cs}}(X))$ , this would not be true, the moduli spaces might not be of finite



type, and then weighted Euler characteristic expressions such as (4.16), (5.16) would not make sense.

For (b), we will show in Theorem 6.28 that under extra assumptions on  $X$  we can deduce Theorems 5.4 and 5.5 for noncompact  $X$  from the compact case. This is enough to generalize §5.2–§5.3 and parts of §5.4 to the noncompact case. For (c), we should accept that for noncompact  $X$ , the invariants  $\bar{D}T^\alpha(\tau)$ ,  $\hat{D}T^\alpha(\tau)$ ,  $PI^{\alpha,n}(\tau')$  may not be deformation-invariant, as they can change when the (compact) support of a sheaf ‘goes to infinity’ in  $X$  as we deform the complex structure of  $X$ . Here is an example.

EXAMPLE 6.26. For  $t \in \mathbb{C}$ , define

$$Y_t = \{(z_1, z_2, z_3, z_4) \in \mathbb{C}^4 : (1 - t^2 z_1)^2 + z_2^2 + z_3^2 + z_4^2 = 0\}. \quad (6.39)$$

Then  $Y_0$  is nonsingular, and  $Y_t$  for  $t \neq 0$  has one singular point at  $(t^{-2}, 0, 0, 0)$ , an ordinary double point, which has two small resolutions. Because (6.39) involves  $t^2$  rather than  $t$  we may choose one of these small resolutions continuously in  $t$  for all  $t \neq 0$  to get a smooth family of noncompact Calabi–Yau 3-folds  $X_t$  for  $t \in \mathbb{C}$ , which is also smooth over  $t = 0$ .

As a complex 3-fold,  $X_t$  for  $t \neq 0$  is the total space of  $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \rightarrow \mathbb{CP}^1$ , and  $X_0$  is  $\mathbb{C} \times Q$ , where  $Q$  is a smooth quadric in  $\mathbb{C}^3$ . All the  $X_t$  are diffeomorphic to  $\mathbb{R}^2 \times T^*\mathcal{S}^2$ , and we can regard them as a smooth family of complex structures  $J_t$  for  $t \in \mathbb{C}$  on the fixed 6-manifold  $\mathbb{R}^2 \times T^*\mathcal{S}^2$ . Then  $X_t$  for  $t \neq 0$  contains a curve  $\Sigma_t \cong \mathbb{CP}^1$ , the fibre over  $(t^{-2}, 0, 0, 0)$  in  $Y_t$ . As  $t \rightarrow 0$ , this  $\Sigma_t$  goes to infinity in  $X_t$ , and  $X_0$  contains no compact curves, as it is affine.

From Example 6.30 below it follows that for  $t \neq 0$  there are nonzero invariants  $\bar{D}T^\alpha(\tau)_t$  with  $\dim \alpha = 1$  counting dimension 1 sheaves supported on  $\Sigma_t$  in  $X_t$ . But when  $t = 0$  there are no compactly-supported dimension 1 sheaves on  $X_0$ , as there are no curves on which they could be supported, so  $\bar{D}T^\alpha(\tau)_0 = 0$ , and  $\bar{D}T^\alpha(\tau)$  is not deformation invariant.

Here is the extra condition we need to extend Theorems 5.4–5.5 to  $\text{coh}_{\text{cs}}(X)$ .

DEFINITION 6.27. Let  $X$  be a noncompact Calabi–Yau 3-fold over  $\mathbb{C}$ . We call  $X$  *compactly embeddable* if whenever  $K \subset X$  is a compact subset, in the analytic topology, there exists an open neighbourhood  $U$  of  $K$  in  $X$  in the analytic topology, a compact Calabi–Yau 3-fold  $Y$  over  $\mathbb{C}$  with  $H^1(\mathcal{O}_Y) = 0$ , an open subset  $V$  of  $Y$  in the analytic topology, and an isomorphism of complex manifolds  $\phi : U \rightarrow V$ .

THEOREM 6.28. *Let  $X$  be a noncompact Calabi–Yau 3-fold over  $\mathbb{C}$ , and suppose  $X$  is compactly embeddable. Then Theorems 5.4 and 5.5 hold in  $\text{coh}_{\text{cs}}(X)$ .*

PROOF. Write  $\mathfrak{M}^X$  for the moduli stack of compactly-supported coherent sheaves on  $X$  and  $\mathcal{M}_{\text{si}}^X$  for the complex algebraic space of simple compactly-supported coherent sheaves on  $X$ . For each compactly-supported (algebraic) coherent sheaf  $E$  on  $X$  there is an underlying compactly-supported complex analytic coherent sheaf  $E_{\text{an}}$ , and by Serre [97] this map  $E \mapsto E_{\text{an}}$  is an equivalence of categories.

Let  $E \in \text{coh}_{\text{cs}}(X)$ , so that  $[E] \in \mathfrak{M}^X(\mathbb{C})$ , or  $[E] \in \mathcal{M}_{\text{si}}^X(\mathbb{C})$  if  $E$  is simple. Then  $K = \text{supp } E$  is a compact subset of  $X$ . Let  $U, Y, V$  be as in Definition 6.27 for this  $K$ , and write  $\mathfrak{M}^Y$  for the moduli stack of coherent sheaves on  $Y$ , and  $\mathcal{M}_{\text{si}}^Y$  for the complex algebraic space of simple coherent sheaves on  $Y$ . Then  $E_{\text{an}}|_U$  is a complex analytic coherent sheaf on  $U \subset X$ , so  $\phi_*(E_{\text{an}})$  is a complex analytic coherent sheaf

on  $V \subset Y$ , which we extend by zero to get a complex analytic coherent sheaf  $F_{\text{an}}$  on  $Y$ , and this is associated to a unique (algebraic) coherent sheaf  $F$  on  $Y$  by [97], with  $[F] \in \mathfrak{M}^Y(\mathbb{C})$ , and  $[F] \in \mathcal{M}_{\text{si}}^Y(\mathbb{C})$  if  $F$  (or equivalently  $E$ ) is simple.

For Theorem 5.4, let  $E$  be simple, and write  $W^X$  for the subset of  $[E'] \in \mathcal{M}_{\text{si}}^X(\mathbb{C})$  with  $E'$  supported on  $U$ , and  $W^Y$  for the subset of  $[F'] \in \mathcal{M}_{\text{si}}^Y(\mathbb{C})$  with  $F'$  supported on  $V$ . Then  $W^X, W^Y$  are open neighbourhoods of  $[E], [F]$  in  $\mathcal{M}_{\text{si}}^X(\mathbb{C}), \mathcal{M}_{\text{si}}^Y(\mathbb{C})$  in the complex analytic topology, and there is a unique map  $\phi_* : W^X \rightarrow W^Y$  with  $\phi_*([E']) = [F']$  if  $\phi_*(E'_{\text{an}}) \cong F'_{\text{an}}$ . Since  $\phi$  is an isomorphism of complex manifolds, it is easy to see that  $\phi_*$  is an isomorphism of complex analytic spaces.

By Theorem 5.4,  $W^Y$  near  $[F]$  is locally isomorphic to  $\text{Crit}(f)$  as a complex analytic space, for  $f : U \rightarrow \mathbb{C}$  holomorphic and  $U \subset \text{Ext}^1(F, F)$  open. Since  $W^X \cong W^Y$  as complex analytic spaces and  $\text{Ext}^1(E, E) \cong \text{Ext}^1(E_{\text{an}}, E_{\text{an}}) \cong \text{Ext}^1(F_{\text{an}}, F_{\text{an}}) \cong \text{Ext}^1(F, F)$  by [97], Theorem 5.4 for  $\text{coh}_{\text{cs}}(X)$  follows.

For Theorem 5.5, let  $S^X, \Phi^X$  be as in the second paragraph of Theorem 5.5 for  $\mathfrak{M}^X, E$  in  $\text{coh}_{\text{cs}}(X)$ , and  $S^Y, \Phi^Y$  for  $\mathfrak{M}^Y, F$  on  $\text{coh}(Y)$ . Then as in Proposition 9.10(b) in the sheaf case there are formally versal families  $(S^X, \mathcal{D}^X)$  of compactly-supported coherent sheaves on  $X$  with  $\mathcal{D}_0^X \cong E$ , and  $(S^Y, \mathcal{D}^Y)$  of coherent sheaves on  $Y$  with  $\mathcal{D}_0^Y \cong F$ . The corresponding families  $(S^X(\mathbb{C}), \mathcal{D}_{\text{an}}^X)$ ,  $(S^Y(\mathbb{C}), \mathcal{D}_{\text{an}}^Y)$  of complex analytic coherent sheaves are versal. Let  $W^X, W^Y$  be the subsets of  $S^X(\mathbb{C}), S^Y(\mathbb{C})$  representing sheaves supported on  $U, V$ . Then  $W^X, W^Y$  are open neighbourhoods of 0 in  $S^X(\mathbb{C}), S^Y(\mathbb{C})$ , in the analytic topology.

Since  $\phi : U \rightarrow V$  is an isomorphism of complex manifolds,  $\phi_*$  takes versal families of complex analytic sheaves on  $U$  to versal families of complex analytic sheaves on  $V$ . Therefore  $(W^X, \phi_*(\mathcal{D}_{\text{an}}^X|_{W^X}))$  and  $(W^Y, \mathcal{D}_{\text{an}}^Y|_{W^Y})$  are both versal families of complex analytic coherent sheaves on  $V$  with  $\phi_*(\mathcal{D}_{\text{an}}^X|_{W^X})_0 \cong F_{\text{an}} \cong (\mathcal{D}_{\text{an}}^Y|_{W^Y})_0$ . We can now argue as in Proposition 9.11 using the fact that  $T_0 W^X \cong \text{Ext}^1(E_{\text{an}}, E_{\text{an}}) \cong \text{Ext}^1(F_{\text{an}}, F_{\text{an}}) \cong T_0 W^Y$  that  $W^X$  near 0 is isomorphic as a complex analytic space to  $W^Y$  near 0. Theorem 5.5 for  $X$  then follows from Theorem 5.5 for  $Y$ .  $\square$

QUESTION 6.29. *Let  $X$  be a noncompact Calabi–Yau 3-fold over  $\mathbb{C}$ . Can you prove Theorems 5.4 and 5.5 hold in  $\text{coh}_{\text{cs}}(X)$  without assuming  $X$  is compactly embeddable?*

All of §5.2–§5.3 now extends immediately to  $\text{coh}_{\text{cs}}(X)$  for  $X$  a compactly embeddable noncompact Calabi–Yau 3-fold: the Behrend function identities (5.2)–(5.3), the Lie algebra morphisms  $\tilde{\Psi}, \tilde{\Psi}^{\chi, \mathbb{Q}}$ , the definition of generalized Donaldson–Thomas invariants  $\bar{D}T^\alpha(\tau)$  for  $\alpha \in K(\text{coh}_{\text{cs}}(X))$ , and the transformation law (5.14) under change of stability condition.

In §5.4 the definition of stable pairs still works, and the moduli scheme  $\mathcal{M}_{\text{stp}}^{\alpha, n}(\tau')$  is well-defined, but may not be proper. So (5.15) does not make sense, and we take the weighted Euler characteristic (5.16) to be the *definition* of the pair invariants  $PI^{\alpha, n}(\tau')$ . The deformation-invariance of  $\bar{D}T^\alpha(\tau), PI^{\alpha, n}(\tau')$  in Corollaries 5.26 and 5.28 will not hold for  $\text{coh}_{\text{cs}}(X)$  in general, as Example 6.26 shows. But Theorem 5.27, expressing the  $PI^{\alpha, n}(\tau')$  in terms of the  $\bar{D}T^{\beta}(\tau)$ , is still valid, with proof essentially unchanged; it does not matter that  $\mathcal{O}_X(-n)$  lies in  $\text{coh}(X)$  rather than  $\text{coh}_{\text{cs}}(X)$ .

As in §6.2 we define BPS invariants  $\hat{D}T^\alpha(\tau)$  for  $\text{coh}_{\text{cs}}(X)$  from the  $\bar{D}T^\alpha(\tau)$ , and conjecture they are integers for generic  $(\tau, T, \leq)$ . The results of §6.3 computing

invariants counting dimension zero sheaves also hold in the noncompact case, as the proof of Theorem 6.15 in [6] does not need  $X$  compact.

EXAMPLE 6.30. Let  $X$  be the noncompact Calabi–Yau 3-fold  $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \rightarrow \mathbb{CP}^1$ , that is, the total space of the rank 2 vector bundle  $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$  over  $\mathbb{CP}^1$ . This is a very familiar example from the Mathematics and String Theory literature; it is a crepant resolution of the *conifold*  $z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0$  in  $\mathbb{C}^4$ , so it is often known as the *resolved conifold*.

Let  $Y$  be any compact Calabi–Yau 3-fold containing a rational curve  $C \cong \mathbb{CP}^1$  with normal bundle  $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ ; explicit examples such as quintics are easy to find. Then  $Y$  near  $C$  is isomorphic as a complex manifold to  $X$  near the zero section. Since any compact subset  $K$  in  $X$  can be mapped into any open neighbourhood of the zero section in  $X$  by a sufficiently small dilation, it follows that  $X$  is *compactly embeddable*, and our theory applies for  $\text{coh}_{\text{cs}}(X)$ .

We have  $H_{\text{cs}}^{2j}(X; \mathbb{Q}) = \mathbb{Q}$  for  $j = 2, 3$  and  $H_{\text{cs}}^{2j}(X; \mathbb{Q}) = 0$  otherwise. For  $E \in \text{coh}_{\text{cs}}(X)$  we have  $\text{ch}_{\text{cs}}(E) = (0, 0, \text{ch}_2(E), \text{ch}_3(E))$ , where  $\text{ch}_j(E) \in \mathbb{Z} \subset \mathbb{Q} = H_{\text{cs}}^{2j}(X; \mathbb{Q})$  for  $j = 2, 3$ . Thus we can identify  $K(\text{coh}_{\text{cs}}(X))$  with  $\mathbb{Z}^2$  with coordinates  $(a_2, a_3)$ , where  $[E] = (a_2, a_3)$  if  $\text{ch}_j(E) = a_j$  for  $j = 2, 3$ . The class of a point sheaf  $\mathcal{O}_x$  for  $x \in X$  is  $(0, 1)$ , and if  $i : \mathbb{CP}^1 \rightarrow X$  is the zero section, the class of  $i_*(\mathcal{O}_{\mathbb{CP}^1}(k))$  is  $(1, 1 + k)$ . The positive cone  $C(\text{coh}_{\text{cs}}(X))$  is

$$C(\text{coh}_{\text{cs}}(X)) = \{(a_2, a_3) \in \mathbb{Z}^2 : a_2 = 0 \text{ and } a_3 > 0, \text{ or } a_2 > 0\}. \quad (6.40)$$

The Euler form  $\bar{\chi}$  on  $\text{coh}_{\text{cs}}(X)$  is zero, so  $K^{\text{num}}(\text{coh}_{\text{cs}}(X)) = 0$ .

Let  $(\tau, G, \leq)$  be Gieseker stability on  $X$  with respect to the ample line bundle  $\pi^*(\mathcal{O}_{\mathbb{CP}^1}(1))$ . We can write down the full Donaldson–Thomas and BPS invariants  $\bar{DT}^\alpha(\tau), \hat{DT}^\alpha(\tau)$  using the work of §6.3–§6.4. We have

$$\bar{DT}^{(a_2, a_3)}(\tau) = \begin{cases} -2 \sum_{m \geq 1, m|a_3} \frac{1}{m^2}, & a_2 = 0, a_3 \geq 1, \\ \frac{1}{a_2^2}, & a_2 > 0, a_2 \mid a_3, \\ 0, & \text{otherwise,} \end{cases} \quad (6.41)$$

$$\hat{DT}^{(a_2, a_3)}(\tau) = \begin{cases} -2, & a_2 = 0, a_3 \geq 1, \\ 1, & a_2 = 1, \\ 0, & \text{otherwise.} \end{cases} \quad (6.42)$$

Here the first lines of (6.41)–(6.42) count dimension 0 sheaves and are taken from (6.19)–(6.20), noting that  $\chi(X) = 2$ , and the rest which count dimension 1 sheaves follow from Proposition 6.18. We will return to this example in §7.5.2.

It is easy to show that other important examples of noncompact Calabi–Yau 3-folds such as  $K_{\mathbb{CP}^2}$  and  $K_{\mathbb{CP}^1 \times \mathbb{CP}^1}$  are also compactly embeddable.

### 6.8. Configuration operations and extended Donaldson–Thomas invariants

Let  $X$  be a Calabi–Yau 3-fold over  $\mathbb{C}$ , and  $(\tau, T, \leq)$  a weak stability condition on  $\text{coh}(X)$  of Gieseker or  $\mu$ -stability type. In §3.2 we explained how to construct elements  $\bar{\delta}_{\text{ss}}^\alpha(\tau)$  in the algebra  $\text{SF}_{\text{al}}(\mathfrak{M})$  and  $\bar{\epsilon}^\alpha(\tau)$  in the Lie algebra  $\text{SF}_{\text{al}}^{\text{ind}}(\mathfrak{M})$  for

$\alpha \in C(\text{coh}(X))$ . Then in §5.3 we defined a Lie algebra morphism  $\tilde{\Psi} : \text{SF}_{\text{al}}^{\text{ind}}(\mathfrak{M}) \rightarrow \tilde{L}(X)$ , and applied  $\tilde{\Psi}$  to  $\bar{\epsilon}^\alpha(\tau)$  to define  $\bar{D}T^\alpha(\tau)$ .

Now the theory of [51–54] is even more complicated than was explained in §3. As well as the Ringel–Hall product  $*$  on  $\text{SF}_{\text{al}}(\mathfrak{M})$  and the Lie bracket  $[\cdot, \cdot]$  on  $\text{SF}_{\text{al}}^{\text{ind}}(\mathfrak{M})$ , in [52, Def. 5.3] using the idea of ‘configurations’ we define an infinite family of multilinear operations  $P_{(I, \preceq)}$  on  $\text{SF}_{\text{al}}(\mathfrak{M})$  depending on a finite partially ordered set (poset)  $(I, \preceq)$ , with  $*$  =  $P_{(\{1,2\}, \leq)}$ . It follows from [52, Th. 5.17] that certain linear combinations of the  $P_{(I, \preceq)}$  are multilinear operations on  $\text{SF}_{\text{al}}^{\text{ind}}(\mathfrak{M})$ , with  $[\cdot, \cdot]$  being the simplest of these.

Also, in [53, §8], given  $(\tau, T, \leq)$  on  $\text{coh}(X)$  we construct much larger families of interesting elements of  $\text{SF}_{\text{al}}(\mathfrak{M})$  and  $\text{SF}_{\text{al}}^{\text{ind}}(\mathfrak{M})$  than just the  $\bar{\delta}_{\text{ss}}^\alpha(\tau)$  and  $\bar{\epsilon}^\alpha(\tau)$ . In [53, Def. 8.9] we define a Lie subalgebra  $\tilde{\mathcal{L}}_\tau^{\text{pa}}$  of  $\text{SF}_{\text{al}}^{\text{ind}}(\mathfrak{M})$  which is spanned by certain elements  $\sigma_*(I)\bar{\delta}_{\text{si}}^{\text{b}}(I, \preceq, \kappa, \tau)$  of  $\text{SF}_{\text{al}}^{\text{ind}}(\mathfrak{M})$ , where  $(I, \preceq)$  is a finite, connected poset and  $\kappa : I \rightarrow C(\text{coh}(X))$  is a map. The  $\bar{\epsilon}^\alpha(\tau)$  lie in  $\tilde{\mathcal{L}}_\tau^{\text{pa}}$ , and may be written as finite  $\mathbb{Q}$ -linear combinations of  $\sigma_*(I)\bar{\delta}_{\text{si}}^{\text{b}}(I, \preceq, \kappa, \tau)$ , but the  $\bar{\epsilon}^\alpha(\tau)$  do not generate  $\tilde{\mathcal{L}}_\tau^{\text{pa}}$  as a Lie algebra, they only generate a smaller Lie algebra  $\tilde{\mathcal{L}}_\tau^{\text{to}}$ . The  $\bar{\epsilon}^\alpha(\tau)$  do generate  $\tilde{\mathcal{L}}_\tau^{\text{pa}}$  over the infinite family of multilinear operations on  $\text{SF}_{\text{al}}^{\text{ind}}(\mathfrak{M})$  defined from the  $P_{(I, \preceq)}$ . In [54, §6.5] we apply the Lie algebra morphism  $\Psi : \text{SF}_{\text{al}}^{\text{ind}}(\mathfrak{M}) \rightarrow L(X)$  of §3.4 to the  $\sigma_*(I)\bar{\delta}_{\text{si}}^{\text{b}}(I, \preceq, \kappa, \tau)$  to define invariants  $J_{\text{si}}^{\text{b}}(I, \preceq, \kappa, \tau) \in \mathbb{Q}$ , and prove they satisfy a transformation law under change of stability condition. So replacing  $\Psi$  by  $\tilde{\Psi}$  we define:

DEFINITION 6.31. In the situation above, define *extended Donaldson–Thomas invariants*  $\tilde{J}_{\text{si}}^{\text{b}}(I, \preceq, \kappa, \tau) \in \mathbb{Q}$ , where  $(I, \preceq)$  is a finite, connected poset and  $\kappa : I \rightarrow C(\text{coh}(X))$  is a map, by

$$\tilde{\Psi}(\sigma_*(I)\bar{\delta}_{\text{si}}^{\text{b}}(I, \preceq, \kappa, \tau)) = \tilde{J}_{\text{si}}^{\text{b}}(I, \preceq, \kappa, \tau) \tilde{\lambda}^{\kappa(I)}, \quad (6.43)$$

where  $\sigma_*(I)\bar{\delta}_{\text{si}}^{\text{b}}(I, \preceq, \kappa, \tau) \in \text{SF}_{\text{al}}^{\text{ind}}(\mathfrak{M})$  is as in [53, Def. 8.1].

Here are some good properties of the  $\tilde{J}_{\text{si}}^{\text{b}}(I, \preceq, \kappa, \tau)$ :

- $\bar{\epsilon}^\alpha(\tau)$  may be written as a  $\mathbb{Q}$ -linear combination of the  $\sigma_*(I)\bar{\delta}_{\text{si}}^{\text{b}}(I, \preceq, \kappa, \tau)$ . Thus comparing (5.7) and (6.43) shows that  $\bar{D}T^\alpha(\tau)$  is a  $\mathbb{Q}$ -linear combination of the  $\tilde{J}_{\text{si}}^{\text{b}}(I, \preceq, \kappa, \tau)$ .
- $\tilde{\mathcal{L}}_\tau^{\text{pa}}$  is a Lie algebra spanned by the  $\sigma_*(I)\bar{\delta}_{\text{si}}^{\text{b}}(I, \preceq, \kappa, \tau)$ , and the Lie bracket of two generators  $\sigma_*(I)\bar{\delta}_{\text{si}}^{\text{b}}(I, \preceq, \kappa, \tau)$  may be written as an explicit  $\mathbb{Q}$ -linear combination of other generators. So since  $\tilde{\Psi}$  is a Lie algebra morphism, we can deduce many *multiplicative relations* between the  $\tilde{J}_{\text{si}}^{\text{b}}(I, \preceq, \kappa, \tau)$ .
- As for the  $J_{\text{si}}^{\text{b}}(I, \preceq, \kappa, \tau)$  in [54], there is a known *wall-crossing formula* for the  $\tilde{J}_{\text{si}}^{\text{b}}(I, \preceq, \kappa, \tau)$  under change of stability condition.

Since the  $\bar{D}T^\alpha(\tau)$  are deformation-invariant by Corollary 5.28, we can ask whether the  $\tilde{J}_{\text{si}}^{\text{b}}(I, \preceq, \kappa, \tau)$  are deformation-invariant. Also, we can ask whether the multilinear operations on  $\text{SF}_{\text{al}}^{\text{ind}}(\mathfrak{M})$  above are taken by  $\tilde{\Psi}$  to multilinear operations on  $\tilde{L}(X)$  by  $\tilde{\Psi}$ . The answer to both is no, as we show by an example.

EXAMPLE 6.32. Define a 1-morphism  $\phi : \mathfrak{M} \times \mathfrak{M} \rightarrow \mathfrak{M}$  by  $\phi(E, F) = E \oplus F$  on objects. In a similar way to the Ringel–Hall product  $*$  in §3.1, define a bilinear operation  $\bullet$  on  $\text{SF}_{\text{al}}(\mathfrak{M})$  by  $f \bullet g = \phi_*(f \otimes g)$ . Then  $\bullet$  is commutative and associative; in the notation of [52, Def. 5.3] we have  $\bullet = P_{(\{1,2\}, \triangleleft)}$ , where  $i \triangleleft j$  if  $i = j$ . Define a bilinear operation  $\diamond$  on  $\text{SF}_{\text{al}}(\mathfrak{M})$  by  $f \diamond g = f * g - f \bullet g$ . Then [52, Th. 5.17]

implies that if  $f, g \in \mathrm{SF}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{M})$  then  $f \diamond g \in \mathrm{SF}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{M})$ , so  $\diamond$  restricts to a bilinear operation on  $\mathrm{SF}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{M})$ . We have  $[f, g] = f \diamond g - g \diamond f$ , since  $\bullet$  is commutative. The Lie algebra  $\tilde{\mathcal{L}}_{\tau}^{\mathrm{pa}}$  above is closed under  $\diamond$ .

Now let us work in the situation of §6.5. Consider the elements  $\bar{\epsilon}^{(0,0,\beta,k)}(\tau)_t$ ,  $\bar{\epsilon}^{(0,0,\gamma,l)}(\tau)_t$  and  $\bar{\epsilon}^{(0,0,\beta,k)}(\tau)_t \diamond \bar{\epsilon}^{(0,0,\gamma,l)}(\tau)_t$  in  $\mathrm{SF}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{M})_t$ , for  $t \in \Delta_{\epsilon}$ , and their images under  $\tilde{\Psi}$ . We find that  $\bar{\epsilon}^{(0,0,\beta,k)}(\tau)_t = \bar{\delta}_{E_t(k)}$  and  $\bar{\epsilon}^{(0,0,\gamma,l)}(\tau)_t = \bar{\delta}_{F_t(l)}$ , so  $\bar{\epsilon}^{(0,0,\beta,k)}(\tau)_t \bullet \bar{\epsilon}^{(0,0,\gamma,l)}(\tau)_t = \bar{\delta}_{E_t(k) \oplus F_t(l)}$ . But  $\bar{\epsilon}^{(0,0,\beta,k)}(\tau)_t * \bar{\epsilon}^{(0,0,\gamma,l)}(\tau)_t$  is  $\bar{\delta}_{E_t(k) \oplus F_t(l)}$  when  $t \neq 0$ , and  $\bar{\delta}_{E_0(k) \oplus F_t(l)} + \bar{\delta}_{G_0(k,l-1)}$  when  $t = 0$ . Hence

$$\bar{\epsilon}^{(0,0,\beta,k)}(\tau)_t \diamond \bar{\epsilon}^{(0,0,\gamma,l)}(\tau)_t = \begin{cases} 0, & t \neq 0, \\ \bar{\delta}_{G_0(k,l-1)}, & t = 0. \end{cases}$$

Since each of  $E_t(k)$ ,  $F_t(l)$  and  $G_0(k, l-1)$  are simple and rigid, we see that

$$\begin{aligned} \tilde{\Psi}(\bar{\epsilon}^{(0,0,\beta,k)}(\tau)_t) &= -\tilde{\lambda}^{(0,0,\beta,k)}, & \tilde{\Psi}(\bar{\epsilon}^{(0,0,\gamma,l)}(\tau)_t) &= -\tilde{\lambda}^{(0,0,\gamma,l)}, \\ \text{and} \quad \tilde{\Psi}(\bar{\epsilon}^{(0,0,\beta,k)}(\tau)_t \diamond \bar{\epsilon}^{(0,0,\gamma,l)}(\tau)_t) &= \begin{cases} 0, & t \neq 0, \\ -\tilde{\lambda}^{(0,0,\beta+\gamma,k+l)}, & t = 0. \end{cases} \end{aligned} \quad (6.44)$$

Equation (6.44) tells us three important things. Firstly, there cannot exist a deformation-invariant bilinear operation  $\blacklozenge$  on  $\tilde{L}(X)$  with  $\tilde{\Psi}(f \diamond g) = \tilde{\Psi}(f) \blacklozenge \tilde{\Psi}(g)$  for all  $f, g \in \mathrm{SF}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{M})_t$ . Thus, although  $\tilde{\Psi}$  is compatible with the Lie bracket  $[\cdot, \cdot]$  on  $\mathrm{SF}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{M})$ , it will not be nicely compatible with the more general multilinear operations on  $\mathrm{SF}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{M})$  defined using the  $P_{(I, \preceq)}$ .

Secondly,  $\bar{\epsilon}^{(0,0,\beta,k)}(\tau)_t \diamond \bar{\epsilon}^{(0,0,\gamma,l)}(\tau)_t$  is an element of  $\tilde{\mathcal{L}}_{\tau}^{\mathrm{pa}}$ , a  $\mathbb{Q}$ -linear combination of elements  $\sigma_*(I) \bar{\delta}_{\mathrm{si}}^b(I, \preceq, \kappa, \tau)$ , and its image under  $\tilde{\Psi}$  is a  $\mathbb{Q}$ -linear combination of  $\tilde{J}_{\mathrm{si}}^b(I, \preceq, \kappa, \tau)$ , multiplied by  $\tilde{\lambda}^{(0,0,\beta+\gamma,k+l)}$ . Equation (6.44) shows that this  $\mathbb{Q}$ -linear combination of  $\tilde{J}_{\mathrm{si}}^b(I, \preceq, \kappa, \tau)$  is not deformation-invariant, so some at least of the extended Donaldson–Thomas invariants in Definition 6.31 are not deformation-invariant. Thirdly, the  $\tilde{J}_{\mathrm{si}}^b(I, \preceq, \kappa, \tau)$  do in general include extra information not encoded in the  $\bar{DT}^{\alpha}(\tau)$ , as if they did not they would have to be deformation-invariant.

Here and in §6.5 we have considered several ways of defining invariants by counting sheaves weighted by the Behrend function  $\nu_{\mathfrak{M}}$ , but which turn out not to be deformation-invariant. It seems to the authors that deformation-invariance arises in situations where you have proper moduli schemes with obstruction theories, such as  $\mathcal{M}_{\mathrm{stp}}^{\alpha,n}(\tau')$  in §5.4, and that you should not expect deformation-invariance if you cannot find such proper moduli schemes in the problem.

**QUESTION 6.33.** *Let  $X$  be a Calabi–Yau 3-fold over  $\mathbb{C}$ . Are there any  $\mathbb{Q}$ -linear combinations of extended Donaldson–Thomas invariants  $\tilde{J}_{\mathrm{si}}^b(I, \preceq, \kappa, \tau)$  of  $X$ , which are unchanged by deformations of  $X$  for all  $X$ , but which cannot be written in terms of the  $\bar{DT}^{\alpha}(\tau)$ ?*

## CHAPTER 7

### Donaldson–Thomas theory for quivers with superpotentials

The theory of §5–§6 relied on three properties of the abelian category  $\text{coh}(X)$  of coherent sheaves on a compact Calabi–Yau 3-fold  $X$ :

- (a) The moduli stack  $\mathfrak{M}$  of objects in  $\text{coh}(X)$  can locally be written in terms of  $\text{Crit}(f)$  for  $f : U \rightarrow \mathbb{C}$  holomorphic and  $U$  smooth, as in Theorem 5.5;
- (b) For all  $D, E \in \text{coh}(X)$  we have

$$\bar{\chi}([D], [E]) = (\dim \text{Hom}(D, E) - \dim \text{Ext}^1(D, E)) - (\dim \text{Hom}(E, D) - \dim \text{Ext}^1(E, D)),$$

where  $\bar{\chi} : K(\text{coh}(X)) \times K(\text{coh}(X)) \rightarrow \mathbb{Z}$  is biadditive and antisymmetric. This is a consequence of Serre duality in dimension 3, that is,  $\text{Ext}^i(D, E) \cong \text{Ext}^{3-i}(E, D)^*$ , but we do not actually need Serre duality to hold; and

- (c) We can form *proper* moduli schemes  $\mathcal{M}_{\text{st}}^\alpha(\tau)$  when  $\mathcal{M}_{\text{ss}}^\alpha(\tau) = \mathcal{M}_{\text{st}}^\alpha(\tau)$ , and  $\mathcal{M}_{\text{stp}}^{\alpha,n}(\tau')$  in general, which have symmetric obstruction theories.

As in §6.7, for a noncompact Calabi–Yau 3-fold  $X$ , properties (a),(b) hold for compactly-supported sheaves  $\text{coh}_{\text{cs}}(X)$ , but the properness in (c) fails. Properness is essential in proving  $\bar{DT}^\alpha(\tau)$ ,  $PI^{\alpha,n}(\tau')$  are *deformation-invariant* in §5.4.

We will show that properties (a) and (b) also hold for  $\mathbb{C}$ -linear abelian categories of representations  $\text{mod-}\mathbb{C}Q/I$  of a quiver  $Q$  with relations  $I$  coming from a superpotential  $W$ . So we can extend much of §5–§6 to these categories. As property (c) does not hold, the Donaldson–Thomas type invariants we define may not be unchanged under deformations of the underlying geometry or algebra. Much work has already been done in this area, and we will explain as we go along how our results relate to those in the literature.

#### 7.1. Introduction to quivers

Let  $\mathbb{K}$  be an algebraically closed field of characteristic zero. Here are the basic definitions in quiver theory. Benson [7, §4.1] is a good reference.

**DEFINITION 7.1.** A *quiver*  $Q$  is a finite directed graph. That is,  $Q$  is a quadruple  $(Q_0, Q_1, h, t)$ , where  $Q_0$  is a finite set of *vertices*,  $Q_1$  is a finite set of *edges*, and  $h, t : Q_1 \rightarrow Q_0$  are maps giving the *head* and *tail* of each edge.

The *path algebra*  $\mathbb{K}Q$  is an associative algebra over  $\mathbb{K}$  with basis all *paths* of length  $k \geq 0$ , that is, sequences of the form

$$v_0 \xrightarrow{e_1} v_1 \rightarrow \cdots \rightarrow v_{k-1} \xrightarrow{e_k} v_k, \tag{7.1}$$

where  $v_0, \dots, v_k \in Q_0$ ,  $e_1, \dots, e_k \in Q_1$ ,  $t(a_i) = v_{i-1}$  and  $h(a_i) = v_i$ . Multiplication is given by composition of paths in reverse order.

For  $n \geq 0$ , write  $\mathbb{K}Q_{(n)}$  for the vector subspace of  $\mathbb{K}Q$  with basis all paths of length  $k \geq n$ . It is an ideal in  $\mathbb{K}Q$ . A *quiver with relations*  $(Q, I)$  is defined to be a quiver  $Q$  together with a two-sided ideal  $I$  in  $\mathbb{K}Q$  with  $I \subseteq \mathbb{K}Q_{(2)}$ . Then  $\mathbb{K}Q/I$  is an associative  $\mathbb{K}$ -algebra.

We define *representations* of quivers, and of quivers with relations.

DEFINITION 7.2. Let  $Q = (Q_0, Q_1, h, t)$  be a quiver. A *representation* of  $Q$  consists of finite-dimensional  $\mathbb{K}$ -vector spaces  $X_v$  for each  $v \in Q_0$ , and linear maps  $\rho_e : X_{t(e)} \rightarrow X_{h(e)}$  for each  $e \in Q_1$ . Representations of  $Q$  are in 1-1 correspondence with *finite-dimensional left  $\mathbb{K}Q$ -modules*  $(X, \rho)$ , as follows.

Given  $X_v, \rho_e$ , define  $X = \bigoplus_{v \in Q_0} X_v$ , and a linear  $\rho : \mathbb{K}Q \rightarrow \text{End}(X)$  taking (7.1) to the linear map  $X \rightarrow X$  acting as  $\rho_{e_k} \circ \rho_{e_{k-1}} \circ \cdots \circ \rho_{e_1}$  on  $X_{v_0}$ , and 0 on  $X_v$  for  $v \neq v_0$ . Then  $(X, \rho)$  is a left  $\mathbb{K}Q$ -module. Conversely, any such  $(X, \rho)$  comes from a unique representation of  $Q$ . If  $(Q, I)$  is a quiver with relations, a *representation of  $(Q, I)$*  is a representation of  $Q$  such that the corresponding left  $\mathbb{K}Q$ -module  $(X, \rho)$  has  $\rho(I) = 0$ .

A *morphism of representations*  $\phi : (X, \rho) \rightarrow (Y, \sigma)$  is a linear map  $\phi : X \rightarrow Y$  with  $\phi \circ \rho(\gamma) = \sigma(\gamma) \circ \phi$  for all  $\gamma \in \mathbb{K}Q$ . Equivalently,  $\phi$  defines linear maps  $\phi_v : X_v \rightarrow Y_v$  for all  $v \in Q_0$  with  $\phi_{h(e)} \circ \rho_e = \sigma_e \circ \phi_{t(e)}$  for all  $e \in Q_1$ . Write  $\text{mod-}\mathbb{K}Q, \text{mod-}\mathbb{K}Q/I$  for the categories of representations of  $Q$  and  $(Q, I)$ . They are  $\mathbb{K}$ -linear abelian categories, of finite length.

If  $(X, \rho)$  is a representation of  $Q$  or  $(Q, I)$ , the *dimension vector*  $\mathbf{dim}(X, \rho)$  of  $(X, \rho)$  in  $\mathbb{Z}_{\geq 0}^{Q_0} \subset \mathbb{Z}^{Q_0}$  is  $\mathbf{dim}(X, \rho) : v \mapsto \dim_{\mathbb{K}} X_v$ . This induces surjective morphisms  $\mathbf{dim} : K_0(\text{mod-}\mathbb{K}Q)$  or  $K_0(\text{mod-}\mathbb{K}Q/I) \rightarrow \mathbb{Z}^{Q_0}$ .

In [51, §10] we show that  $\text{mod-}\mathbb{K}Q$  and  $\text{mod-}\mathbb{K}Q/I$  satisfy Assumption 3.2, where we choose the quotient group  $K(\text{mod-}\mathbb{K}Q)$  or  $K(\text{mod-}\mathbb{K}Q/I)$  to be  $\mathbb{Z}^{Q_0}$ , using this morphism  $\mathbf{dim}$ . For quivers we will always take  $K(\text{mod-}\mathbb{K}Q/I)$  to be  $\mathbb{Z}^{Q_0}$  rather than the numerical Grothendieck group  $K^{\text{num}}(\text{mod-}\mathbb{K}Q/I)$ ; one reason is that in some interesting cases the Euler form  $\bar{\chi}$  on  $K_0(\text{mod-}\mathbb{K}Q/I)$  is zero, so that  $K^{\text{num}}(\text{mod-}\mathbb{K}Q/I) = 0$ , but  $\mathbb{Z}^{Q_0}$  is nonzero.

REMARK 7.3. There is an analogy between quiver representations and sheaves on noncompact schemes, as in §6.7. For  $\mathbb{K}, Q$  or  $(Q, I)$  as above, we define  $\text{proj-}\mathbb{K}Q$  and  $\text{proj-}\mathbb{K}Q/I$  to be the exact categories of finitely generated, projective, but not necessarily finite-dimensional modules over  $\mathbb{K}Q$  and  $\mathbb{K}Q/I$ . Here a representation is *projective* if it is a direct summand of a free module.

If  $E \in \text{proj-}\mathbb{K}Q/I$  and  $F \in \text{mod-}\mathbb{K}Q/I$  then  $\text{Ext}^*(E, F)$  is finite-dimensional, so we have a biadditive pairing  $\bar{\chi} : K_0(\text{proj-}\mathbb{K}Q/I) \times K_0(\text{mod-}\mathbb{K}Q/I) \rightarrow \mathbb{Z}$  given by  $\bar{\chi}([E], [F]) = \sum_{i \geq 0} (-1)^i \dim \text{Ext}^i(E, F)$ . The quotient of  $K_0(\text{proj-}\mathbb{K}Q/I)$  by the left kernel of  $\bar{\chi}$ , and the quotient of  $K_0(\text{mod-}\mathbb{K}Q/I)$  by the right kernel of  $\bar{\chi}$ , are both naturally isomorphic to the dimension vectors  $\mathbb{Z}^{Q_0}$ .

We can think of  $\text{mod-}\mathbb{K}Q, \text{mod-}\mathbb{K}Q/I$  as like the category of compactly supported sheaves  $\text{coh}_{\text{cs}}(X)$  for some smooth noncompact  $\mathbb{K}$ -scheme  $X$ , and  $\text{proj-}\mathbb{K}Q, \text{proj-}\mathbb{K}Q/I$  as like the category  $\text{coh}(X)$  of all coherent sheaves. The pairing  $\bar{\chi} : K_0(\text{proj-}\mathbb{K}Q/I) \times K_0(\text{mod-}\mathbb{K}Q/I) \rightarrow \mathbb{Z}$  is like the pairing  $\bar{\chi} : K_0(\text{coh}(X)) \times K_0(\text{coh}_{\text{cs}}(X)) \rightarrow \mathbb{Z}$  in §6.7. Thus, our choice of  $K(\text{mod-}\mathbb{K}Q) = \mathbb{Z}^{Q_0}$  is directly analogous to our definition of  $K(\text{coh}_{\text{cs}}(X))$  in §6.7.

As we will discuss briefly in §7.5, there are examples known in which there are equivalences of derived categories  $D^b(\text{mod-}\mathbb{K}Q/I) \sim D^b(\text{coh}_{\text{cs}}(X))$  for  $X$  a

noncompact Calabi–Yau 3-fold over  $\mathbb{K}$ . Then we also expect equivalences of derived categories  $D^b(\text{proj-}\mathbb{K}Q/I) \sim D^b(\text{coh}(X))$ .

If  $Q$  is a quiver, the moduli stack  $\mathfrak{M}_Q$  of objects  $(X, \rho)$  in  $\text{mod-}\mathbb{K}Q$  is an Artin  $\mathbb{K}$ -stack. For  $\mathbf{d} \in \mathbb{Z}_{\geq 0}^{Q_0}$ , the open substack  $\mathfrak{M}_Q^{\mathbf{d}}$  of  $(X, \rho)$  with  $\mathbf{dim}(X, \rho) = \mathbf{d}$  has a very explicit description: as a quotient  $\mathbb{K}$ -stack we have

$$\mathfrak{M}_Q^{\mathbf{d}} \cong [\prod_{e \in Q_1} \text{Hom}(\mathbb{K}^{\mathbf{d}(t(e))}, \mathbb{K}^{\mathbf{d}(h(e))}) / \prod_{v \in Q_0} \text{GL}(\mathbf{d}(v))]. \quad (7.2)$$

If  $(Q, I)$  is a quiver with relations, the moduli stack  $\mathfrak{M}_{Q,I}$  of objects  $(X, \rho)$  in  $\text{mod-}\mathbb{K}Q/I$  is a substack of  $\mathfrak{M}_Q$ , and for  $\mathbf{d} \in \mathbb{Z}_{\geq 0}^{Q_0}$  we may write

$$\mathfrak{M}_{Q,I}^{\mathbf{d}} \cong [V_{Q,I}^{\mathbf{d}} / \prod_{v \in Q_0} \text{GL}(\mathbf{d}(v))], \quad (7.3)$$

with  $V_{Q,I}^{\mathbf{d}}$  a closed  $\prod_{v \in Q_0} \text{GL}(\mathbf{d}(v))$ -invariant  $\mathbb{K}$ -subscheme of  $\prod_{e \in Q_1} \text{Hom}(\mathbb{K}^{\mathbf{d}(t(e))}, \mathbb{K}^{\mathbf{d}(h(e))})$  defined using the relations  $I$ .

Let  $Q = (Q_0, Q_1, h, t)$  be a quiver, without relations. It is well known that  $\text{Ext}^i(D, E) = 0$  for all  $D, E \in \text{mod-}\mathbb{K}Q$  and  $i > 1$ , and

$$\dim_{\mathbb{K}} \text{Hom}(D, E) - \dim_{\mathbb{K}} \text{Ext}^1(D, E) = \hat{\chi}(\mathbf{dim} D, \mathbf{dim} E), \quad (7.4)$$

where  $\hat{\chi} : \mathbb{Z}^{Q_0} \times \mathbb{Z}^{Q_0} \rightarrow \mathbb{Z}$  is the Euler form of  $\text{mod-}\mathbb{K}Q$ , given by

$$\hat{\chi}(\mathbf{d}, \mathbf{e}) = \sum_{v \in Q_0} \mathbf{d}(v) \mathbf{e}(v) - \sum_{e \in Q_1} \mathbf{d}(t(e)) \mathbf{e}(h(e)). \quad (7.5)$$

Note that  $\hat{\chi}$  need not be antisymmetric. Define  $\bar{\chi} : \mathbb{Z}^{Q_0} \times \mathbb{Z}^{Q_0} \rightarrow \mathbb{Z}$  by

$$\bar{\chi}(\mathbf{d}, \mathbf{e}) = \hat{\chi}(\mathbf{d}, \mathbf{e}) - \hat{\chi}(\mathbf{e}, \mathbf{d}) = \sum_{e \in Q_1} (\mathbf{d}(h(e)) \mathbf{e}(t(e)) - \mathbf{d}(t(e)) \mathbf{e}(h(e))). \quad (7.6)$$

Then  $\bar{\chi}$  is antisymmetric, and as in (b) above, for all  $D, E \in \text{mod-}\mathbb{K}Q$  we have

$$\begin{aligned} \bar{\chi}(\mathbf{dim} D, \mathbf{dim} E) &= (\dim \text{Hom}(D, E) - \dim \text{Ext}^1(D, E)) - \\ &\quad (\dim \text{Hom}(E, D) - \dim \text{Ext}^1(E, D)). \end{aligned} \quad (7.7)$$

This is the analogue of (3.14) for Calabi–Yau 3-folds, property (b) at the beginning of §7. Theorem 7.6 generalizes (7.7) to quivers with a superpotential.

We define a class of *stability conditions* on  $\text{mod-}\mathbb{K}Q/I$ , [55, Ex. 4.14].

EXAMPLE 7.4. Let  $(Q, I)$  be a quiver with relations, and take  $K(\text{mod-}\mathbb{K}Q/I)$  to be  $\mathbb{Z}^{Q_0}$ , as above. Then  $C(\text{mod-}\mathbb{K}Q/I) = \mathbb{Z}_{\geq 0}^{Q_0} \setminus \{0\}$ . Let  $c : Q_0 \rightarrow \mathbb{R}$  and  $r : Q_0 \rightarrow (0, \infty)$  be maps. Define  $\mu : C(\text{mod-}\mathbb{K}Q/I) \rightarrow \mathbb{R}$  by

$$\mu(\mathbf{d}) = \frac{\sum_{v \in Q_0} c(v) \mathbf{d}(v)}{\sum_{v \in Q_0} r(v) \mathbf{d}(v)}.$$

Note that  $\sum_{v \in Q_0} r(v) \mathbf{d}(v) > 0$  as  $r(v) > 0$  for all  $v \in Q_0$ , and  $\mathbf{d}(v) \geq 0$  for all  $v$  with  $\mathbf{d}(v) > 0$  for some  $v$ . Then [55, Ex. 4.14] shows that  $(\mu, \mathbb{R}, \leq)$  is a *permissible stability condition* on  $\text{mod-}\mathbb{K}Q/I$  which we call *slope stability*. Write  $\mathfrak{M}_{\text{ss}}^{\mathbf{d}}(\mu)$  for the open  $\mathbb{K}$ -substack of  $\mu$ -semistable objects in class  $\mathbf{d}$  in  $\mathfrak{M}_{Q,I}^{\mathbf{d}}$ .

A simple case is to take  $c \equiv 0$  and  $r \equiv 1$ , so that  $\mu \equiv 0$ . Then  $(0, \mathbb{R}, \leq)$  is a trivial stability condition on  $\text{mod-}\mathbb{K}Q$  or  $\text{mod-}\mathbb{K}Q/I$ , and every nonzero object in  $\text{mod-}\mathbb{K}Q$  or  $\text{mod-}\mathbb{K}Q/I$  is 0-semistable, so that  $\mathfrak{M}_{\text{ss}}^{\mathbf{d}}(0) = \mathfrak{M}_{Q,I}^{\mathbf{d}}$ .



## 7.2. Quivers with superpotentials, and 3-Calabi–Yau categories

We shall be interested in quivers with relations coming from a *superpotential*. This is an idea which originated in Physics. Two foundational mathematical papers on them are Ginzburg [30] and Derksen, Weyman and Zelevinsky [17]. Again,  $\mathbb{K}$  is an algebraically closed field of characteristic zero throughout.

DEFINITION 7.5. Let  $Q$  be a quiver. A *superpotential*  $W$  for  $Q$  over  $\mathbb{K}$  is an element of  $\mathbb{K}Q/[\mathbb{K}Q, \mathbb{K}Q]$ . The cycles in  $Q$  up to cyclic permutation form a basis for  $\mathbb{K}Q/[\mathbb{K}Q, \mathbb{K}Q]$  over  $\mathbb{K}$ , so we can think of  $W$  as a finite  $\mathbb{K}$ -linear combination of cycles up to cyclic permutation. Following [63], we call  $W$  *minimal* if all cycles in  $W$  have length at least 3. We will consider only minimal superpotentials  $W$ .

Define  $I$  to be the two-sided ideal in  $\mathbb{K}Q$  generated by  $\partial_e W$  for all edges  $e \in Q_1$ , where if  $C$  is a cycle in  $Q$ , we define  $\partial_e C$  to be the sum over all occurrences of the edge  $e$  in  $C$  of the path obtained by cyclically permuting  $C$  until  $e$  is in first position, and then deleting it. Since  $W$  is minimal,  $I$  lies in  $\mathbb{K}Q_{(2)}$ , so that  $(Q, I)$  is a quiver with relations.

We allow  $W \equiv 0$ , so that  $I = 0$ , and  $\text{mod-}\mathbb{K}Q/I = \text{mod-}\mathbb{K}Q$ .

When  $I$  comes from a superpotential  $W$ , we can improve the description (7.3) of the moduli stacks  $\mathfrak{M}_{Q,I}^d$ . Define a  $\prod_{v \in Q_0} \text{GL}(d(v))$ -invariant polynomial

$$W^d : \prod_{e \in Q_1} \text{Hom}(\mathbb{K}^{d(t(e))}, \mathbb{K}^{d(h(e))}) \longrightarrow \mathbb{K}$$

as follows. Write  $W$  as a finite sum  $\sum_i \gamma^i C^i$ , where  $\gamma^i \in \mathbb{K}$  and  $C^i$  is a cycle  $v_0 \xrightarrow{e_1^i} v_1 \rightarrow \cdots \rightarrow v_{k^i-1}^i \xrightarrow{e_{k^i}^i} v_{k^i}^i = v_0$  in  $Q$ . Set

$$W^d(A_e : e \in Q_1) = \sum_i \gamma^i \text{Tr}(A_{e_{k^i}^i} \circ A_{e_{k^i-1}^i} \circ \cdots \circ A_{e_1^i}).$$

Then  $V_{Q,I}^d = \text{Crit}(W^d)$  in (7.3), so that

$$\mathfrak{M}_{Q,I}^d \cong [\text{Crit}(W^d) / \prod_{v \in Q_0} \text{GL}(d(v))]. \quad (7.8)$$

Equation (7.8) is an analogue of Theorem 5.5 for categories  $\text{mod-}\mathbb{K}Q/I$  coming from a superpotential  $W$  on  $Q$ , and gives property (a) at the beginning of §7.

We now show that property (b) at the beginning of §7 holds for quivers with relations  $(Q, I)$  coming from a minimal superpotential  $W$ . Note that we do not impose any other condition on  $W$ , and in particular, we do not require  $\text{mod-}\mathbb{K}Q/I$  to be 3-Calabi–Yau. Also,  $\bar{\chi}$  is in general not the Euler form of the abelian category  $\text{mod-}\mathbb{K}Q/I$ . When  $W \equiv 0$ , so that  $\text{mod-}\mathbb{K}Q/I = \text{mod-}\mathbb{K}Q$ , Theorem 7.6 reduces to equations (7.6)–(7.7). We have not been able to find a reference for Theorem 7.6 and it may be new, though it is probably obvious to experts in the context of Remark 7.10 below.

THEOREM 7.6. Let  $Q = (Q_0, Q_1, h, t)$  be a quiver with relations  $I$  coming from a minimal superpotential  $W$  on  $Q$  over  $\mathbb{K}$ . Define  $\bar{\chi} : \mathbb{Z}^{Q_0} \times \mathbb{Z}^{Q_0} \rightarrow \mathbb{Z}$  by

$$\bar{\chi}(\mathbf{d}, \mathbf{e}) = \sum_{e \in Q_1} (\mathbf{d}(h(e))\mathbf{e}(t(e)) - \mathbf{d}(t(e))\mathbf{e}(h(e))). \quad (7.9)$$

Then for any  $D, E \in \text{mod-}\mathbb{K}Q/I$  we have

$$\begin{aligned} \bar{\chi}(\mathbf{dim} D, \mathbf{dim} E) &= (\dim \text{Hom}(D, E) - \dim \text{Ext}^1(D, E)) - \\ &\quad (\dim \text{Hom}(E, D) - \dim \text{Ext}^1(E, D)). \end{aligned} \quad (7.10)$$

PROOF. Write  $D = (X_v : v \in Q_0, \rho_e : e \in Q_1)$  and  $E = (Y_v : v \in Q_0, \sigma_e : e \in Q_1)$ . Define a sequence of  $\mathbb{K}$ -vector spaces and linear maps

$$\begin{aligned} 0 \longrightarrow \bigoplus_{v \in Q_0} X_v^* \otimes Y_v &\xrightarrow{d_1} \bigoplus_{e \in Q_1} X_{t(e)}^* \otimes Y_{h(e)} \xrightarrow{d_2} \\ \bigoplus_{e \in Q_1} X_{h(e)}^* \otimes Y_{t(e)} &\xrightarrow{d_3} \bigoplus_{v \in Q_0} X_v^* \otimes Y_v \longrightarrow 0, \end{aligned} \quad (7.11)$$

where  $d_1, d_2, d_3$  are given by

$$d_1 : (\phi_v)_{v \in Q_0} \mapsto (\phi_{h(e)} \circ \rho_e - \sigma_e \circ \phi_{t(v)})_{e \in Q_1}, \quad (7.12)$$

$$d_2 : (\psi_e)_{e \in Q_1} \mapsto (\sum_{e \in Q_1} L_{e,f}^{W,D,E}(\psi_e))_{f \in Q_1}, \quad \text{where}$$

$$L_{e,f}^{W,D,E}(\psi_e) = \sum_{\substack{\text{terms } c \left( \overset{t(f)}{\bullet} \xrightarrow{f} \overset{h(f)}{\bullet} \xrightarrow{g_1} \bullet \cdots \bullet \xrightarrow{g_k} \overset{t(e)}{\bullet} \xrightarrow{e} \overset{h(e)}{\bullet} \xrightarrow{h_1} \bullet \cdots \bullet \xrightarrow{h_l} \overset{t(f)}{\bullet} \right) \\ \text{in } W \text{ up to cyclic permutation, } c \in \mathbb{K}}} c \sigma_{h_l} \circ \cdots \circ \sigma_{h_1} \circ \psi_e \circ \rho_{g_k} \circ \cdots \circ \rho_{g_1}, \quad (7.13)$$

$$d_3 : (\xi_e)_{e \in Q_1} \mapsto (\sum_{e \in Q_1: t(e)=v} \xi_e \circ \rho_e - \sum_{e \in Q_1: h(e)=v} \sigma_e \circ \xi_e)_{v \in Q_0}. \quad (7.14)$$

Observe that the dual sequence of (7.11), namely

$$\begin{aligned} 0 \longrightarrow \bigoplus_{v \in Q_0} Y_v^* \otimes X_v &\xrightarrow{d_3^*} \bigoplus_{e \in Q_1} Y_{t(e)}^* \otimes X_{h(e)} \xrightarrow{d_2^*} \\ \bigoplus_{e \in Q_1} Y_{h(e)}^* \otimes X_{t(e)} &\xrightarrow{d_1^*} \bigoplus_{v \in Q_0} Y_v^* \otimes X_v \longrightarrow 0, \end{aligned} \quad (7.15)$$

is (7.11) with  $D$  and  $E$  exchanged. That  $d_3^*, d_1^*$  correspond to  $d_1, d_3$  with  $D, E$  exchanged is immediate from (7.12) and (7.14); for  $d_2^*$ , we find from (7.13) that  $(L_{e,f}^{W,D,E})^* = L_{f,e}^{W,E,D}$ , by cyclically permuting the term  $\overset{t(f)}{\bullet} \xrightarrow{f} \overset{h(f)}{\bullet} \cdots \bullet \xrightarrow{h_l} \overset{t(f)}{\bullet}$  in (7.13) to obtain  $\overset{t(e)}{\bullet} \xrightarrow{e} \overset{h(e)}{\bullet} \cdots \bullet \xrightarrow{g_k} \overset{t(e)}{\bullet}$ .

We claim that (7.11), and hence (7.15), are *complexes*, that is,  $d_2 \circ d_1 = 0$  and  $d_3 \circ d_2 = 0$ . To show  $d_2 \circ d_1 = 0$ , for  $(\phi_v)_{v \in Q_0}$  in  $\bigoplus_{v \in Q_0} X_v^* \otimes Y_v$  we have

$$\begin{aligned} &d_2 \circ d_1((\phi_v)_{v \in Q_0}) \\ &= \left( \sum_{c \left( \overset{t(f)}{\bullet} \xrightarrow{f} \overset{h(f)}{\bullet} \xrightarrow{g_1} \bullet \cdots \bullet \xrightarrow{g_k} \overset{v}{\bullet} \xrightarrow{h_1} \bullet \cdots \bullet \xrightarrow{h_l} \overset{t(f)}{\bullet} \right) \text{ in } W: k \geq 1, l \geq 0} c \sigma_{h_l} \circ \cdots \circ \sigma_{h_1} \circ \phi_v \circ \rho_{g_k} \circ \cdots \circ \rho_{g_1} \right)_{f \in Q_1} \\ &\quad - \left( \sum_{c \left( \overset{t(f)}{\bullet} \xrightarrow{f} \overset{h(f)}{\bullet} \xrightarrow{g_1} \bullet \cdots \bullet \xrightarrow{g_k} \overset{v}{\bullet} \xrightarrow{h_1} \bullet \cdots \bullet \xrightarrow{h_l} \overset{t(f)}{\bullet} \right) \text{ in } W: k \geq 0, l \geq 1} c \sigma_{h_l} \circ \cdots \circ \sigma_{h_1} \circ \phi_v \circ \rho_{g_k} \circ \cdots \circ \rho_{g_1} \right)_{f \in Q_1} \\ &= \left( \sum_{c \left( \overset{t(f)}{\bullet} \xrightarrow{f} \overset{h(f)}{\bullet} \xrightarrow{g_1} \bullet \cdots \bullet \xrightarrow{g_k} \overset{t(f)}{\bullet} \right) \text{ in } W} c \phi_{t(f)} \circ \rho_{g_k} \circ \cdots \circ \rho_{g_1} \right)_{f \in Q_1} \\ &\quad - \left( \sum_{c \left( \overset{t(f)}{\bullet} \xrightarrow{f} \overset{h(f)}{\bullet} \xrightarrow{h_1} \bullet \cdots \bullet \xrightarrow{h_l} \overset{t(f)}{\bullet} \right) \text{ in } W} c \sigma_{h_l} \circ \cdots \circ \sigma_{h_1} \circ \phi_{h(f)} \right)_{f \in Q_1} = 0. \end{aligned} \quad (7.16)$$

Here the second line of (7.16) comes from the first term  $\phi_{h(e)} \circ \rho_e$  on the r.h.s. of (7.12), and we have included  $\rho_e$  as  $\rho_{g_k}$  in  $\rho_{g_k} \circ \cdots \circ \rho_{g_1}$  by replacing  $k$  by  $k+1$ , which is why we have the condition  $k \geq 1$ . The third line of (7.16) comes from the second term  $-\sigma_e \circ \phi_{t(v)}$  on the r.h.s. of (7.12), and we have included  $\sigma_e$  as  $\sigma_{h_1}$  in  $\sigma_{h_l} \circ \cdots \circ \sigma_{h_1}$  replacing  $l$  by  $l+1$ , which is why we have  $l \geq 1$ . The fourth and fifth lines of (7.16) cancel the terms  $k \geq 1, l \geq 1$  in the second and third lines. Finally, we note that the sums on the fourth and fifth lines vanish as they are the

compositions of  $\phi_{t(f)}, \phi_{h(f)}$  with the relations satisfied by  $(\rho_e)_{e \in Q_1}$  and  $(\sigma_e)_{e \in Q_1}$  coming from the cyclic derivative  $\partial_f W$ . Thus  $d_2 \circ d_1 = 0$ . Since (7.15) is (7.11) with  $D$  and  $E$  exchanged, the same proof shows that  $d_2^* \circ d_3^* = 0$ , and hence  $d_3 \circ d_2 = 0$ . Therefore (7.11), (7.15) are complexes.

Thus we can form the cohomology of (7.11). We will show that it satisfies

$$\text{Ker } d_1 \cong \text{Hom}(D, E), \quad \text{Ker } d_2 / \text{Im } d_1 \cong \text{Ext}^1(D, E), \quad (7.17)$$

$$\text{Ker } d_3 / \text{Im } d_2 \cong \text{Ext}^1(E, D)^*, \quad (\bigoplus_{v \in Q_0} X_v^* \otimes Y_v) / \text{Im } d_3 \cong \text{Hom}(E, D)^*. \quad (7.18)$$

For the first equation of (7.17), observe that  $d_1((\phi_v)_{v \in Q_0}) = 0$  is equivalent to  $\phi_{h(e)} \circ \rho_e = \sigma_e \circ \phi_{t(e)}$  for all  $e \in Q_1$ , which is the condition for  $(\phi_v)_{v \in Q_0}$  to define a morphism of representations  $\phi : (X, \rho) \rightarrow (Y, \sigma)$  in Definition 7.2.

For the second equation of (7.17), note that elements of  $\text{Ext}^1(D, E)$  correspond to isomorphism classes of exact sequences  $0 \rightarrow E \xrightarrow{\alpha} F \xrightarrow{\beta} D \rightarrow 0$  in  $\text{mod-}\mathbb{K}Q/I$ . Write  $F = (Z_v : v \in Q_0, \tau_e : e \in Q_1)$ . Then for all  $v \in Q_0$  we have exact sequences of  $\mathbb{K}$ -vector spaces

$$0 \longrightarrow Y_v \xrightarrow{\alpha_v} Z_v \xrightarrow{\beta_v} X_v \longrightarrow 0. \quad (7.19)$$

Choose isomorphisms  $Z_v \cong Y_v \oplus X_v$  for all  $v \in Q_0$  compatible with (7.19). Then for each  $e \in Q_1$ , we have linear maps  $\tau_e : Y_{t(e)} \oplus X_{t(e)} \rightarrow Y_{h(e)} \oplus X_{h(e)}$ . As  $\alpha, \beta$  are morphisms of representations, we see that in matrix notation

$$\tau_e = \begin{pmatrix} \rho_e & \psi_e \\ 0 & \sigma_e \end{pmatrix}. \quad (7.20)$$

Thus  $(\psi_e)_{e \in Q_1}$  lies in  $\bigoplus_{e \in Q_1} X_{t(e)}^* \otimes Y_{h(e)}$ , the second space in (7.11).

Given that  $(\rho_e)_{e \in Q_1}$  and  $(\sigma_e)_{e \in Q_1}$  satisfy the relations in  $\text{mod-}\mathbb{K}Q/I$ , which come from the cyclic derivatives  $\partial_f W$  for  $f \in Q_1$ , it is not difficult to show that  $(\tau_e)_{e \in Q_1}$  of the form (7.20) satisfy the relations in  $\text{mod-}\mathbb{K}Q/I$  if and only if  $d_2(\psi_e)_{e \in Q_1} = 0$ . Therefore exact sequences  $0 \rightarrow E \rightarrow F \rightarrow D \rightarrow 0$  in  $\text{mod-}\mathbb{K}Q/I$  together with choices of isomorphisms  $Z_v \cong Y_v \oplus X_v$  for  $v \in Q_0$  splitting (7.19) correspond to elements  $(\psi_e)_{e \in Q_1}$  in  $\text{Ker } d_2$ . The freedom to choose splittings of (7.19) is  $X_v^* \otimes Y_v$ . Summing this over all  $v \in Q_0$  gives the first space in (7.11), and quotienting by this freedom corresponds to quotienting  $\text{Ker } d_2$  by  $\text{Im } d_1$ . This proves the second equation of (7.17).

Equation (7.18) follows from (7.17) and the fact that the dual complex (7.15) of (7.11) is (7.11) with  $D, E$  exchanged, so that the dual of the cohomology of (7.11) is the cohomology of (7.11) with  $D, E$  exchanged. Taking the Euler characteristic of (7.11) and using (7.9) and (7.17)–(7.18) then yields (7.10).  $\square$

Equation (7.8) and Theorem 7.6 are analogues for categories  $\text{mod-}\mathbb{K}Q/I$  coming from quivers with superpotentials of (a),(b) at the beginning of §7. Now (a),(b) for  $\text{coh}(X)$  depend crucially on  $X$  being a *Calabi–Yau 3-fold*. We now discuss two senses in which  $\text{mod-}\mathbb{K}Q/I$  can be like a Calabi–Yau 3-fold.

**DEFINITION 7.7.** A  $\mathbb{K}$ -linear abelian category  $\mathcal{A}$  is called *3-Calabi–Yau* if for all  $D, E \in \mathcal{A}$  we have  $\text{Ext}^i(D, E) = 0$  for  $i > 3$ , and there are choices of isomorphisms  $\text{Ext}^i(D, E) \cong \text{Ext}^{3-i}(E, D)^*$  for  $i = 0, \dots, 3$ , which are functorial in an appropriate way. That is,  $\mathcal{A}$  has Serre duality in dimension 3. When  $X$  is a Calabi–Yau 3-fold over  $\mathbb{K}$ , the coherent sheaves  $\text{coh}(X)$  are 3-Calabi–Yau. For more details, see Ginzburg [30], Bocklandt [8], and Segal [95].

An interesting problem in this field is to find examples of 3-Calabi–Yau abelian categories. A lot of work has been done on this. It has become clear that categories  $\text{mod-}\mathbb{K}Q/I$  coming from a superpotential  $W$  on  $Q$  are often, but not always, 3-Calabi–Yau. Here are two classes of examples.

EXAMPLE 7.8. Let  $G$  be a finite subgroup of  $\text{SL}(3, \mathbb{C})$ . The *McKay quiver*  $Q_G$  of  $G$  is defined as follows. Let the vertex set of  $Q_G$  be the set of isomorphism classes of irreducible representations of  $G$ . If vertices  $i, j$  correspond to  $G$ -representations  $V_i, V_j$ , let the number of edges  $\bullet^i \rightarrow \bullet^j$  be  $\dim \text{Hom}_G(V_i, V_j \otimes \mathbb{C}^3)$ , where  $\mathbb{C}^3$  has the natural representation of  $G \subset \text{SL}(3, \mathbb{C})$ . Identify these edges with a basis for  $\text{Hom}_G(V_i, V_j \otimes \mathbb{C}^3)$ .

Following Ginzburg [30, §4.4], define a cubic superpotential  $W_G$  for  $Q_G$  by

$$W_G = \sum_{\text{triangles } \bullet^i \xrightarrow{e} \bullet^j \xrightarrow{f} \bullet^k \xrightarrow{g} \bullet^i \text{ in } Q_G} \text{Tr}(V_i \xrightarrow{e} V_j \otimes \mathbb{C}^3 \xrightarrow{f \otimes \text{id}} V_k \otimes (\mathbb{C}^3)^{\otimes 2} \xrightarrow{g \otimes \text{id}} V_i \otimes (\mathbb{C}^3)^{\otimes 3} \xrightarrow{\text{id} \otimes \Omega} V_i) gfe,$$

where  $\Omega : (\mathbb{C}^3)^{\otimes 3} \rightarrow \mathbb{C}$  is induced by the holomorphic volume form  $dz_1 \wedge dz_2 \wedge dz_3$  on  $\mathbb{C}^3$ . Let  $I_G$  be the relations on  $Q_G$  defined using  $W_G$ . Then Ginzburg [30, Th. 4.4.6] shows that  $\text{mod-}\mathbb{C}Q_G/I_G$  is a 3-Calabi–Yau category, which is equivalent to the abelian category of  $G$ -equivariant compactly-supported coherent sheaves on  $\mathbb{C}^3$ . Using Bridgeland, King and Reid [13], he deduces [30, Cor. 4.4.8] that if  $X$  is any crepant resolution of  $\mathbb{C}^3/G$ , then the derived categories  $D^b(\text{coh}_{\text{cs}}(X))$  and  $D^b(\text{mod-}\mathbb{C}Q_G/I_G)$  are equivalent, where  $\text{coh}_{\text{cs}}(X)$  is the abelian category of compactly-supported coherent sheaves on  $X$ .

EXAMPLE 7.9. A *brane tiling* is a bipartite graph drawn on the 2-torus  $T^2$ , dividing  $T^2$  into simply-connected polygons. From such a graph one can write down a quiver  $Q$  and superpotential  $W$ , yielding a quiver with relations  $(Q, I)$ . If the brane tiling satisfies certain *consistency conditions*,  $\text{mod-}\mathbb{C}Q/I$  is a 3-Calabi–Yau category. For some noncompact toric Calabi–Yau 3-fold  $X$  constructed from the brane tiling, the derived categories  $D^b(\text{mod-}\mathbb{C}Q_G/I_G)$  and  $D^b(\text{coh}_{\text{cs}}(X))$  are equivalent. This class of examples arose in String Theory, where they are known as ‘quiver gauge theories’ or ‘dimer models’, and appear in the work of Hanany and others, see for instance [26, 36–38]. Some mathematical references are Ishii and Ueda [48, §2] and Mozgovoy and Reineke [82, §3].

The abelian categories  $\text{mod-}\mathbb{K}Q/I$  are only 3-Calabi–Yau for some special quivers  $Q$  and superpotentials  $W$ . For instance, if  $Q \neq \emptyset$  and  $W \equiv 0$ , so that  $\text{mod-}\mathbb{K}Q/I = \text{mod-}\mathbb{K}Q$ , then  $\text{mod-}\mathbb{K}Q$  is never 3-Calabi–Yau, since  $\text{Hom}(*, *)$ ,  $\text{Ext}^1(*, *)$  in  $\text{mod-}\mathbb{K}Q$  are nonzero but  $\text{Ext}^2(*, *)$ ,  $\text{Ext}^3(*, *)$  are zero. We now describe a way to embed any  $\text{mod-}\mathbb{K}Q/I$  coming from a minimal superpotential  $W$  in a 3-Calabi–Yau *triangulated* category. The first author is grateful to Alastair King and Bernhard Keller for explaining this to him.

REMARK 7.10. By analogy with Definition 7.7, there is also a notion of when a  $\mathbb{K}$ -linear triangulated category  $\mathcal{T}$  is 3-Calabi–Yau, discussed in Keller [59]. Let  $Q$  be a quiver with relations  $I$  coming from a minimal superpotential  $W$  for  $Q$  over  $\mathbb{K}$ . Then there is a natural way to construct a  $\mathbb{K}$ -linear, 3-Calabi–Yau triangulated category  $\mathcal{T}$ , and a t-structure  $\mathcal{F}$  on  $\mathcal{T}$  whose heart  $\mathcal{A} = \mathcal{F} \cap \mathcal{F}^\perp[1]$  is equivalent to  $\text{mod-}\mathbb{K}Q/I$ . This is briefly discussed in Keller [59, §5].

Given  $Q, W$ , Ginzburg [30] constructs a DG-algebra  $\mathcal{D}(\mathbb{K}Q, W)$  (we want the non-complete version). Then  $\mathcal{T}$  is the full triangulated subcategory of the derived category of DG-modules of  $\mathcal{D}(\mathbb{K}Q, W)$  whose objects are DG-modules with homology of finite total dimension. The standard t-structure on  $\mathcal{T}$  has heart  $\mathcal{A}$  the DG-modules  $M^\bullet$  with  $H^0(M^\bullet)$  finite-dimensional and  $H^i(M^\bullet) = 0$  for  $i \neq 0$ . Here  $H^0(M^\bullet)$  is a representation of  $H^0(\mathcal{D}(\mathbb{K}Q, W)) = \mathbb{K}Q/I$ . Thus  $M^\bullet \mapsto H^0(M^\bullet)$  induces a functor  $\mathcal{A} \mapsto \text{mod-}\mathbb{K}Q/I$ , which is an equivalence. Inverting this induces a functor  $D^b(\text{mod-}\mathbb{K}Q/I) \rightarrow \mathcal{T}$ . If this is an equivalence then  $\text{mod-}\mathbb{K}Q/I$  is 3-Calabi–Yau.

Kontsevich and Soibelman [63, Th. 9, §8.1] prove a related result, giving a 1-1 correspondence between  $\mathbb{K}$ -linear 3-Calabi–Yau triangulated categories  $\hat{\mathcal{T}}$  satisfying certain conditions, and quivers  $Q$  with minimal superpotential  $W$  over  $\mathbb{K}$ . But their set-up is slightly different: in effect they use Ginzburg’s completed DG-algebra  $\hat{\mathcal{D}}(\mathbb{K}Q, W)$  instead of  $\mathcal{D}(\mathbb{K}Q, W)$ , they allow  $W$  to be a formal power series rather than just a finite sum, and the heart  $\hat{\mathcal{A}}$  of the t-structure on  $\hat{\mathcal{T}}$  is  $\text{nil-}\mathbb{K}Q/I$ , the abelian category of *nilpotent* representations of  $(Q, I)$ .

Identify  $\text{mod-}\mathbb{K}Q/I$  with the heart  $\mathcal{A}$  in  $\mathcal{T}$ . Then for  $E, F \in \text{mod-}\mathbb{K}Q/I$ , we can compute the Ext groups  $\text{Ext}^i(E, F)$  in either  $\text{mod-}\mathbb{K}Q/I$  or  $\mathcal{T}$ . We have  $\text{Ext}_{\text{mod-}\mathbb{K}Q/I}^i(E, F) \cong \text{Ext}_{\mathcal{T}}^i(E, F)$  for  $i = 0, 1$ , as  $\text{mod-}\mathbb{K}Q/I$  is the heart of a t-structure, but if  $\text{mod-}\mathbb{K}Q/I$  is not 3-Calabi–Yau then in general  $\text{Ext}_{\text{mod-}\mathbb{K}Q/I}^i(E, F) \not\cong \text{Ext}_{\mathcal{T}}^i(E, F)$  for  $i > 1$ . The cohomology of the complex (7.11) is  $\text{Ext}_{\mathcal{T}}^*(E, F)$ , and  $\bar{\chi}$  in (7.9) is the Euler form of  $\mathcal{T}$ , which may not be the same as the Euler form of  $\text{mod-}\mathbb{K}Q/I$ , if this exists.

In the style of Kontsevich and Soibelman [63], we can regard the Donaldson–Thomas type invariants  $\bar{DT}_{Q,I}^d(\mu), \bar{DT}_Q^d(\mu), \hat{DT}_{Q,I}^d(\mu), \hat{DT}_Q^d(\mu)$  of §7.3 below as counting  $Z$ -semistable objects in the 3-Calabi–Yau category  $\mathcal{T}$ , where  $(Z, \mathcal{P})$  is the Bridgeland stability condition [11] on  $\mathcal{T}$  constructed from the t-structure  $\mathcal{F}$  on  $\mathcal{T}$  and the slope stability condition  $(\mu, \mathbb{R}, \leq)$  on the heart of  $\mathcal{F}$ .

From this point of view, the question of whether or not  $\text{mod-}\mathbb{K}Q/I$  is 3-Calabi–Yau seems less important, as we always have a natural 3-Calabi–Yau triangulated category  $\mathcal{T}$  containing  $\text{mod-}\mathbb{K}Q/I$  to work in.

### 7.3. Behrend function identities, Lie algebra morphisms, and Donaldson–Thomas type invariants

We now develop analogues of §5.2, §5.3 and §6.2 for quivers. Let  $Q$  be a quiver with relations  $I$  coming from a minimal superpotential  $W$  on  $Q$  over  $\mathbb{C}$ . Write  $\mathfrak{M}_{Q,I}$  for the moduli stack of objects in  $\text{mod-}\mathbb{C}Q/I$ , an Artin  $\mathbb{C}$ -stack locally of finite type, and  $\mathfrak{M}_{Q,I}^d$  for the open substack of objects with dimension vector  $d$ , which is of finite type.

The proof of Theorem 5.11 in §10 depends on two things: the description of  $\mathfrak{M}$  in terms of  $\text{Crit}(f)$  in Theorem 5.5, and equation (3.14). For  $\text{mod-}\mathbb{C}Q/I$  equation (7.8) provides an analogue of Theorem 5.5, and Theorem 7.6 an analogue of (3.14). Thus, the proof of Theorem 5.11 also yields:

**THEOREM 7.11.** *In the situation above, with  $\mathfrak{M}_{Q,I}$  the moduli stack of objects in a category  $\text{mod-}\mathbb{C}Q/I$  coming from a quiver  $Q$  with minimal superpotential  $W$ , and  $\bar{\chi}$  defined in (7.9), the Behrend function  $\nu_{\mathfrak{M}_{Q,I}}$  of  $\mathfrak{M}_{Q,I}$  satisfies the identities (5.2)–(5.3) for all  $E_1, E_2 \in \text{mod-}\mathbb{C}Q/I$ .*

Since the description of  $\mathfrak{M}_{Q,I}$  in terms of  $\text{Crit}(W^d)$  in (7.8) is algebraic rather than complex analytic, and holds over any field  $\mathbb{K}$ , we ask:

QUESTION 7.12. *Can you prove Theorem 7.11 over an arbitrary algebraically closed field  $\mathbb{K}$  of characteristic zero, using the ideas of §4.2?*

Here is the analogue of Definition 5.13.

DEFINITION 7.13. Define a Lie algebra  $\tilde{L}(Q)$  to be the  $\mathbb{Q}$ -vector space with basis of symbols  $\tilde{\lambda}^d$  for  $d \in \mathbb{Z}^{Q_0}$ , with Lie bracket

$$[\tilde{\lambda}^d, \tilde{\lambda}^e] = (-1)^{\bar{\chi}(d,e)} \tilde{\chi}(d,e) \tilde{\lambda}^{d+e},$$

as for (5.4). This makes  $\tilde{L}(Q)$  into an infinite-dimensional Lie algebra over  $\mathbb{Q}$ . Define  $\mathbb{Q}$ -linear maps  $\tilde{\Psi}_{Q,I}^{\chi,\mathbb{Q}} : \text{SF}_{\text{al}}^{\text{ind}}(\mathfrak{M}_{Q,I}, \chi, \mathbb{Q}) \rightarrow \tilde{L}(Q)$  and  $\tilde{\Psi}_{Q,I} : \text{SF}_{\text{al}}^{\text{ind}}(\mathfrak{M}_{Q,I}) \rightarrow \tilde{L}(Q)$  exactly as for  $\tilde{\Psi}^{\chi,\mathbb{Q}}, \tilde{\Psi}$  in Definition 5.13.

The proof of Theorem 5.14 in §11 has two ingredients: equation (3.14) and Theorem 5.11. Theorems 7.6 and 7.11 are analogues of these in the quiver case. So the proof of Theorem 5.14 also yields:

THEOREM 7.14.  $\tilde{\Psi}_{Q,I} : \text{SF}_{\text{al}}^{\text{ind}}(\mathfrak{M}_{Q,I}) \rightarrow \tilde{L}(Q)$  and  $\tilde{\Psi}_{Q,I}^{\chi,\mathbb{Q}} : \text{SF}_{\text{al}}^{\text{ind}}(\mathfrak{M}_{Q,I}, \chi, \mathbb{Q}) \rightarrow \tilde{L}(Q)$  are Lie algebra morphisms.

Here is the analogue of Definitions 5.15 and 6.10.

DEFINITION 7.15. Let  $(\mu, \mathbb{R}, \leq)$  be a slope stability condition on  $\text{mod-}\mathbb{C}Q/I$  as in Example 7.4. It is permissible, as in [55, Ex. 4.14]. So as in §3.2 we have elements  $\bar{\delta}_{\text{ss}}^d(\mu) \in \text{SF}_{\text{al}}(\mathfrak{M}_{Q,I})$  and  $\bar{\epsilon}^d(\mu) \in \text{SF}_{\text{al}}^{\text{ind}}(\mathfrak{M}_{Q,I})$  for all  $d \in C(\text{mod-}\mathbb{C}Q/I) = \mathbb{Z}_{\geq 0}^{Q_0} \setminus \{0\} \subset \mathbb{Z}^{Q_0}$ . As in (5.7), define *quiver generalized Donaldson–Thomas invariants*  $\bar{D}T_{Q,I}^d(\mu) \in \mathbb{Q}$  for all  $d \in C(\text{mod-}\mathbb{C}Q/I)$  by

$$\tilde{\Psi}_{Q,I}(\bar{\epsilon}^d(\mu)) = -\bar{D}T_{Q,I}^d(\mu) \tilde{\lambda}^d.$$

As in (6.15), define *quiver BPS invariants*  $\hat{D}T_{Q,I}^d(\mu) \in \mathbb{Q}$  by

$$\hat{D}T_{Q,I}^d(\mu) = \sum_{m \geq 1, m|d} \frac{\text{Mö}(m)}{m^2} \bar{D}T_{Q,I}^{d/m}(\mu), \quad (7.21)$$

where  $\text{Mö} : \mathbb{N} \rightarrow \mathbb{Q}$  is the Möbius function. As for (6.14), the inverse of (7.21) is

$$\bar{D}T_{Q,I}^d(\mu) = \sum_{m \geq 1, m|d} \frac{1}{m^2} \hat{D}T_{Q,I}^{d/m}(\mu). \quad (7.22)$$

If  $W \equiv 0$ , so that  $\text{mod-}\mathbb{C}Q/I = \text{mod-}\mathbb{C}Q$ , we write  $\bar{D}T_Q^d(\mu), \hat{D}T_Q^d(\mu)$  for  $\bar{D}T_{Q,I}^d(\mu), \hat{D}T_{Q,I}^d(\mu)$ . Note that  $\mu \equiv 0$  is allowed as a slope stability condition, with every object in  $\text{mod-}\mathbb{C}Q/I$  0-semistable, and this is in many ways the most natural choice. So we have invariants  $\bar{D}T_{Q,I}^d(0), \hat{D}T_{Q,I}^d(0)$  and  $\bar{D}T_Q^d(0), \hat{D}T_Q^d(0)$ . We cannot do this in the coherent sheaf case; the difference is that for quivers  $\mathfrak{M}_{Q,I}^d$  is of finite type for all  $d \in C(\text{mod-}\mathbb{C}Q/I)$ , so  $(0, \mathbb{R}, \leq)$  is permissible on  $\text{mod-}\mathbb{C}Q/I$ , but for coherent sheaves  $\mathfrak{M}^\alpha$  is generally not of finite type for  $\alpha \in C(\text{coh}(X))$  with  $\dim \alpha > 0$ , so  $(0, \mathbb{R}, \leq)$  is not permissible.

Here is the analogue of the integrality conjecture, Conjecture 6.12. We will prove the conjecture in §7.6 for the invariants  $\hat{D}T_Q^d(\mu)$ , that is, the case  $W \equiv 0$ .

CONJECTURE 7.16. Call  $(\mu, \mathbb{R}, \leq)$  **generic** if for all  $\mathbf{d}, \mathbf{e} \in C(\text{mod-}\mathbb{C}Q/I)$  with  $\mu(\mathbf{d}) = \mu(\mathbf{e})$  we have  $\bar{\chi}(\mathbf{d}, \mathbf{e}) = 0$ . If  $(\mu, \mathbb{R}, \leq)$  is generic, then  $\hat{D}T_{Q,I}^{\mathbf{d}}(\mu) \in \mathbb{Z}$  for all  $\mathbf{d} \in C(\text{mod-}\mathbb{C}Q/I)$ .

If the maps  $c : Q_0 \rightarrow \mathbb{R}$  and  $r : Q_0 \rightarrow (0, \infty)$  defining  $\mu$  in Example 7.4 are generic, it is easy to see that  $\mu(\mathbf{d}) = \mu(\mathbf{e})$  only if  $\mathbf{d}, \mathbf{e}$  are linearly dependent over  $\mathbb{Q}$  in  $\mathbb{Z}^{Q_0}$ , so that  $\bar{\chi}(\mathbf{d}, \mathbf{e}) = 0$  by antisymmetry of  $\bar{\chi}$ , and  $(\mu, \mathbb{R}, \leq)$  is generic in the sense of Conjecture 7.16. Thus, there exist generic stability conditions  $(\mu, \mathbb{R}, \leq)$  on any  $\text{mod-}\mathbb{C}Q/I$ .

Let  $(\mu, \mathbb{R}, \leq)$ ,  $(\tilde{\mu}, \mathbb{R}, \leq)$  be slope stability conditions on  $\text{mod-}\mathbb{C}Q/I$ . Then  $(0, \mathbb{R}, \leq)$  dominates both, so Theorem 3.13 with  $(\mu, \mathbb{R}, \leq)$ ,  $(\tilde{\mu}, \mathbb{R}, \leq)$ ,  $(0, \mathbb{R}, \leq)$  in place of  $(\tau, T, \leq)$ ,  $(\tilde{\tau}, \tilde{T}, \leq)$ ,  $(\hat{\tau}, \hat{T}, \leq)$  writes  $\bar{\epsilon}^{\mathbf{d}}(\tilde{\mu})$  in terms of the  $\bar{\epsilon}^{\mathbf{e}}(\mu)$  in (3.10). Applying  $\tilde{\Psi}_{Q,I}$ , which is a Lie algebra morphism by Theorem 7.14, to this identity gives an analogue of Theorem 5.18:

THEOREM 7.17. Let  $(\mu, \mathbb{R}, \leq)$  and  $(\tilde{\mu}, \mathbb{R}, \leq)$  be any two slope stability conditions on  $\text{mod-}\mathbb{C}Q/I$ , and  $\bar{\chi}$  be as in (7.9). Then for all  $\mathbf{d} \in C(\text{mod-}\mathbb{C}Q/I)$  we have

$$\begin{aligned} \bar{D}T_{Q,I}^{\mathbf{d}}(\tilde{\mu}) = & \sum_{\substack{\text{iso.} \\ \text{classes} \\ \text{of finite} \\ \text{sets } I}} \sum_{\substack{\kappa: I \rightarrow C(\text{mod-}\mathbb{C}Q/I): \\ \sum_{i \in I} \kappa(i) = \mathbf{d}}} \sum_{\substack{\text{connected,} \\ \text{simply-} \\ \text{connected} \\ \text{digraphs } \Gamma, \\ \text{vertices } I}} (-1)^{|I|-1} V(I, \Gamma, \kappa; \mu, \tilde{\mu}) \cdot \prod_{i \in I} \bar{D}T_{Q,I}^{\kappa(i)}(\mu) \\ & \cdot (-1)^{\frac{1}{2} \sum_{i,j \in I} |\bar{\chi}(\kappa(i), \kappa(j))|} \cdot \prod_{\substack{\text{edges } i \rightarrow j \text{ in } \Gamma}} \bar{\chi}(\kappa(i), \kappa(j)), \end{aligned} \quad (7.23)$$

with only finitely many nonzero terms.

The form  $\bar{\chi}$  in (7.9) is zero if and only if for all vertices  $i, j$  in  $Q$ , there are the same number of edges  $i \rightarrow j$  and  $j \rightarrow i$  in  $Q$ . Then (7.23) gives:

COROLLARY 7.18. Suppose that  $\bar{\chi}$  in (7.9) is zero. Then for any slope stability conditions  $(\mu, \mathbb{R}, \leq)$  and  $(\tilde{\mu}, \mathbb{R}, \leq)$  on  $\text{mod-}\mathbb{C}Q/I$  and all  $\mathbf{d}$  in  $C(\text{mod-}\mathbb{C}Q/I)$  we have  $\bar{D}T_{Q,I}^{\mathbf{d}}(\tilde{\mu}) = \bar{D}T_{Q,I}^{\mathbf{d}}(\mu)$  and  $\hat{D}T_{Q,I}^{\mathbf{d}}(\tilde{\mu}) = \hat{D}T_{Q,I}^{\mathbf{d}}(\mu)$ .

Here is a case in which we can evaluate the invariants very easily.

EXAMPLE 7.19. Let  $Q$  be a quiver without oriented cycles. Choose a slope stability condition  $(\mu, \mathbb{R}, \leq)$  on  $\text{mod-}\mathbb{C}Q$  such that  $\mu(\delta_v) > \mu(\delta_w)$  for all edges  $v \rightarrow w$  in  $Q$ . This is possible as  $Q$  has no oriented cycles. Then up to isomorphism the only  $\mu$ -stable objects in  $\text{mod-}\mathbb{C}Q$  are the simple representations  $S^v$  for  $v \in Q_0$  and the only  $\mu$ -semistables are  $kS^v$  for  $v \in Q_0$  and  $k \geq 1$ . Here  $S^v = (X^v, \rho^v)$ , where  $X_w^v = \mathbb{C}$  if  $v = w$  and  $X_w^v = 0$  if  $v \neq w \in Q_0$ , and  $\rho_e^v = 0$  for  $e \in Q_1$ . Examples 6.1–6.2 and equations (7.21)–(7.22) now imply that

$$\bar{D}T_Q^{\mathbf{d}}(\mu) = \begin{cases} \frac{1}{l^2}, & \mathbf{d} = l\delta_v, l \geq 1, v \in Q_0, \\ 0, & \text{otherwise,} \end{cases} \quad \hat{D}T_Q^{\mathbf{d}}(\mu) = \begin{cases} 1, & \mathbf{d} = \delta_v, v \in Q_0, \\ 0, & \text{otherwise.} \end{cases}$$

#### 7.4. Pair invariants for quivers

We now discuss analogues for quivers of the moduli spaces of stable pairs  $\mathcal{M}_{\text{stp}}^{\alpha,n}(\tau')$  and stable pair invariants  $PI^{\alpha,n}(\tau')$  in §5.4, and the identity (5.17) in Theorem 5.27 relating  $PI^{\alpha,n}(\tau')$  and the  $\bar{D}T^{\beta}(\tau)$ . Here are the basic definitions.

These quiver analogues of  $\mathcal{M}_{\text{stp}}^{\alpha,n}(\tau')$ ,  $PI^{\alpha,n}(\tau')$  are not new, similar things have been studied in quiver theory by Nakajima, Reineke, Szendrői and other authors for some years [24, 82–85, 88, 89, 99]. We explain the relations between our definitions and the literature after Definition 7.21.

**DEFINITION 7.20.** Let  $Q$  be a quiver with relations  $I$  coming from a superpotential  $W$  on  $Q$  over an algebraically closed field  $\mathbb{K}$  of characteristic zero. Suppose  $(\mu, \mathbb{R}, \leq)$  is a slope stability condition on  $\text{mod-}\mathbb{K}Q/I$ , as in Example 7.4.

Let  $\mathbf{d}, \mathbf{e} \in \mathbb{Z}_{\geq 0}^{Q_0}$  be dimension vectors. A *framed representation*  $(X, \rho, \sigma)$  of  $(Q, I)$  of *type*  $(\mathbf{d}, \mathbf{e})$  consists of a representation  $(X, \rho) = (X_v : v \in Q_0, \rho_e : e \in Q_1)$  of  $(Q, I)$  over  $\mathbb{K}$  with  $\dim X_v = \mathbf{d}(v)$  for all  $v \in Q_0$ , together with linear maps  $\sigma_v : \mathbb{K}^{e(v)} \rightarrow X_v$  for all  $v \in Q_0$ . An *isomorphism* between framed representations  $(X, \rho, \sigma), (X', \rho', \sigma')$  consists of isomorphisms  $i_v : X_v \rightarrow X'_v$  for all  $v \in Q_0$  such that  $i_{h(e)} \circ \rho_e = \rho'_e \circ i_{t(e)}$  for all  $e \in Q_1$  and  $i_v \circ \sigma_v = \sigma'_v$  for all  $v \in Q_0$ . We call a framed representation  $(X, \rho, \sigma)$  *stable* if

- (i)  $\mu([(X', \rho')]) \leq \mu([(X, \rho)])$  for all subobjects  $0 \neq (X', \rho') \subset (X, \rho)$  in  $\text{mod-}\mathbb{K}Q/I$  or  $\text{mod-}\mathbb{K}Q$ ; and
- (ii) If also  $\sigma$  factors through  $(X', \rho')$ , that is,  $\sigma_v(\mathbb{C}^{e(v)}) \subseteq X'_v \subseteq X_v$  for all  $v \in Q_0$ , then  $\mu([(X', \rho')]) < \mu([(X, \rho)])$ .

We will use  $\mu'$  to denote *stability of framed representations*, defined using  $\mu$ .

Following Engel and Reineke [24, §3] or Szendrői [99, §1.2], we can in a standard way define moduli problems for all framed representations, and for stable framed representations. When  $W \equiv 0$ , so that  $\text{mod-}\mathbb{K}Q/I = \text{mod-}\mathbb{K}Q$ , the moduli space of all framed representations of type  $(\mathbf{d}, \mathbf{e})$  is an Artin  $\mathbb{K}$ -stack  $\mathfrak{M}_{\text{fr } Q}^{\mathbf{d}, \mathbf{e}}$ . By analogy with (7.2) we have

$$\mathfrak{M}_{\text{fr } Q}^{\mathbf{d}, \mathbf{e}} \cong \left[ \frac{\prod_{e \in Q_1} \text{Hom}(\mathbb{K}^{\mathbf{d}(t(e))}, \mathbb{K}^{\mathbf{d}(h(e))}) \times \prod_{v \in Q_0} \text{Hom}(\mathbb{K}^{e(v)}, \mathbb{K}^{\mathbf{d}(v)})}{\prod_{v \in Q_0} \text{GL}(\mathbf{d}(v))} \right]. \quad (7.24)$$

The moduli space of stable framed representations of type  $(\mathbf{d}, \mathbf{e})$  is a fine moduli  $\mathbb{K}$ -scheme  $\mathcal{M}_{\text{stf } Q}^{\mathbf{d}, \mathbf{e}}(\mu')$ , an open  $\mathbb{K}$ -substack of  $\mathfrak{M}_{\text{fr } Q}^{\mathbf{d}, \mathbf{e}}$ , with

$$\mathcal{M}_{\text{stf } Q}^{\mathbf{d}, \mathbf{e}}(\mu') \cong U_{\text{stf}, Q}^{\mathbf{d}, \mathbf{e}}(\mu') / \prod_{v \in Q_0} \text{GL}(\mathbf{d}(v)), \quad (7.25)$$

where  $U_{\text{stf}, Q}^{\mathbf{d}, \mathbf{e}}(\mu')$  is open in  $\prod_e \text{Hom}(\mathbb{K}^{\mathbf{d}(t(e))}, \mathbb{K}^{\mathbf{d}(h(e))}) \times \prod_v \text{Hom}(\mathbb{K}^{e(v)}, \mathbb{K}^{\mathbf{d}(v)})$ , and  $\prod_v \text{GL}(\mathbf{d}(v))$  acts freely on  $U_{\text{stf}, Q}^{\mathbf{d}, \mathbf{e}}(\mu')$ , and (7.25) may be written as a GIT quotient for an appropriate linearization. From (7.24)–(7.25) we see that  $\mathfrak{M}_{\text{fr } Q}^{\mathbf{d}, \mathbf{e}}, \mathcal{M}_{\text{stf } Q}^{\mathbf{d}, \mathbf{e}}(\mu')$  are both smooth with dimension

$$\dim \mathfrak{M}_{\text{fr } Q}^{\mathbf{d}, \mathbf{e}} = \mathcal{M}_{\text{stf } Q}^{\mathbf{d}, \mathbf{e}}(\mu') = \hat{\chi}(\mathbf{d}, \mathbf{d}) + \sum_{v \in Q_0} \mathbf{e}(v) \mathbf{d}(v). \quad (7.26)$$

Similarly, for general  $W$ , the moduli space of all framed representations of type  $(\mathbf{d}, \mathbf{e})$  is an Artin  $\mathbb{K}$ -stack  $\mathfrak{M}_{\text{fr } Q, I}^{\mathbf{d}, \mathbf{e}}$ . By analogy with (7.8) we have

$$\mathfrak{M}_{\text{fr } Q, I}^{\mathbf{d}, \mathbf{e}} \cong [\text{Crit}(W^{\mathbf{d}}) \times \prod_{v \in Q_0} \text{Hom}(\mathbb{K}^{e(v)}, \mathbb{K}^{\mathbf{d}(v)}) / \prod_{v \in Q_0} \text{GL}(\mathbf{d}(v))],$$

where  $\text{Crit}(W^{\mathbf{d}}) \subseteq \prod_{e \in Q_1} \text{Hom}(\mathbb{K}^{\mathbf{d}(t(e))}, \mathbb{K}^{\mathbf{d}(h(e))})$  is as in (7.8), and the moduli space of stable framed representations of type  $(\mathbf{d}, \mathbf{e})$  is a fine moduli  $\mathbb{K}$ -scheme



$\mathcal{M}_{\text{stf } Q, I}^{\mathbf{d}, \mathbf{e}}(\mu')$ , an open  $\mathbb{K}$ -substack of  $\mathfrak{M}_{\text{fr } Q, I}^{\mathbf{d}, \mathbf{e}}$ , with

$$\mathcal{M}_{\text{stf } Q, I}^{\mathbf{d}, \mathbf{e}}(\mu') \cong \frac{(\text{Crit}(W^{\mathbf{d}}) \times \prod_{v \in Q_0} \text{Hom}(\mathbb{K}^{e(v)}, \mathbb{K}^{\mathbf{d}(v)})) \cap U_{\text{stf}, Q}^{\mathbf{d}, \mathbf{e}}(\mu')}{\prod_{v \in Q_0} \text{GL}(\mathbf{d}(v))}.$$

We can now define our analogues of invariants  $PI^{\alpha, n}(\tau')$  for quivers, which following Szendrői [99] we call *noncommutative Donaldson–Thomas invariants*.

DEFINITION 7.21. In the situation above, define

$$NDT_{Q, I}^{\mathbf{d}, \mathbf{e}}(\mu') = \chi(\mathcal{M}_{\text{stf } Q, I}^{\mathbf{d}, \mathbf{e}}(\mu'), \nu_{\mathcal{M}_{\text{stf } Q, I}^{\mathbf{d}, \mathbf{e}}(\mu')}), \quad (7.27)$$

$$\begin{aligned} NDT_Q^{\mathbf{d}, \mathbf{e}}(\mu') &= \chi(\mathcal{M}_{\text{stf } Q}^{\mathbf{d}, \mathbf{e}}(\mu'), \nu_{\mathcal{M}_{\text{stf } Q}^{\mathbf{d}, \mathbf{e}}(\mu')}) \\ &= (-1)^{\hat{\chi}(\mathbf{d}, \mathbf{d}) + \sum_{v \in Q_0} e(v)\mathbf{d}(v)} \chi(\mathcal{M}_{\text{stf } Q}^{\mathbf{d}, \mathbf{e}}(\mu')), \end{aligned} \quad (7.28)$$

where the second line in (7.28) holds as  $\mathcal{M}_{\text{stf } Q}^{\mathbf{d}, \mathbf{e}}(\mu')$  is smooth of dimension (7.26), so  $\nu_{\mathcal{M}_{\text{stf } Q}^{\mathbf{d}, \mathbf{e}}(\mu')} \equiv (-1)^{\hat{\chi}(\mathbf{d}, \mathbf{d}) + \sum_{v \in Q_0} e(v)\mathbf{d}(v)}$  by Theorem 4.3(i).

Here is how Definitions 7.20 and 7.21 relate to the literature. We first discuss the case of quivers without relations.

- ‘Framed’ moduli spaces of quivers appear in the work of Nakajima, see for instance [85, §3]. His framed moduli schemes  $\mathfrak{R}_\theta(\mathbf{d}, \mathbf{e})$  are similar to our moduli schemes  $\mathcal{M}_{\text{stf } Q}^{\mathbf{d}, \mathbf{e}}(\mu')$ , with one difference: rather than framing  $(X_v : v \in Q_0, \rho_e : e \in Q_1)$  using linear maps  $\sigma_v : \mathbb{K}^{e(v)} \rightarrow X_v$  for  $v \in Q_0$ , as we do, he uses linear maps  $\sigma_v : X_v \rightarrow \mathbb{K}^{e(v)}$  going the other way.

Here is a natural way to relate framings of his type to framings of our type. Given a quiver  $Q = (Q_0, Q_1, h, t)$ , let  $Q^{\text{op}}$  be  $Q$  with directions of edges reversed, that is,  $Q^{\text{op}} = (Q_0, Q_1, t, h)$ . If  $(X_v : v \in Q_0, \rho_e : e \in Q_1)$  is a representation of  $Q$  then  $(X_v^* : v \in Q_0, \rho_e^* : e \in Q_1)$  is a representation of  $Q^{\text{op}}$ , and this identifies  $\text{mod-}\mathbb{K}Q^{\text{op}}$  with the opposite category of  $\text{mod-}\mathbb{K}Q$ . Then Nakajima-style framings in  $\text{mod-}\mathbb{K}Q$  correspond to our framings in  $\text{mod-}\mathbb{K}Q^{\text{op}}$ , and vice versa.

- Let  $Q_m$  be the quiver  $Q$  with one vertex  $v$  and  $m$  edges  $v \rightarrow v$ , and consider the trivial stability condition  $(0, \mathbb{R}, \leq)$  on  $\text{mod-}\mathbb{K}Q_m$ . Reineke [88] studied ‘noncommutative Hilbert schemes’  $H_{\mathbf{d}, \mathbf{e}}^{(m)}$  for  $\mathbf{d}, \mathbf{e} \in \mathbb{N}$ , and determines their Poincaré polynomials. In our notation we have  $H_{\mathbf{d}, \mathbf{e}}^{(m)} = \mathcal{M}_{\text{stf } Q_m}^{\mathbf{d}, \mathbf{e}}(0')$ , and Reineke’s calculations and (7.28) yield a formula for  $NDT_{Q_m}^{\mathbf{d}, \mathbf{e}}(0')$ . In [88] Reineke uses framings as in Definition 7.20, not following Nakajima.
- Let  $Q$  be a quiver, and  $\mathbf{d}, \mathbf{e}$  be dimension vectors. Reineke [89] defined ‘framed quiver moduli’  $\mathcal{M}_{\mathbf{d}, \mathbf{e}}(Q)$ . These are the same as Nakajima’s moduli spaces  $\mathfrak{R}_0(\mathbf{d}, \mathbf{e})$  with trivial stability condition  $\theta = 0$ , and correspond to our moduli spaces  $\mathcal{M}_{\text{stf } Q}^{\mathbf{d}, \mathbf{e}}(0')$ , except that the framing uses maps  $\sigma_v : X_v \rightarrow \mathbb{K}^{e(v)}$ .

Reineke studies  $\mathcal{M}_{\mathbf{d}, \mathbf{e}}(Q)$  for  $Q$  without oriented cycles. This yields the Euler characteristic of  $\mathcal{M}_{\text{stf } Q}^{\mathbf{d}, \mathbf{e}}(0')$ , and so gives  $NDT_Q^{\mathbf{d}, \mathbf{e}}(0')$  in (7.28).

- Engel and Reineke [24] study ‘smooth models of quiver moduli’  $M_{\mathbf{d}, \mathbf{e}}^\Theta(Q)$ , which agree with our  $\mathcal{M}_{\text{stf } Q}^{\mathbf{d}, \mathbf{e}}(\mu')$  for a slope stability condition  $(\mu, \mathbb{R}, \leq)$  on  $\text{mod-}\mathbb{K}Q$  defined using a map  $\Theta : Q_0 \rightarrow \mathbb{Q}$ , with framing as in Definition 7.20. They give combinatorial formulae for the Poincaré polynomials of  $M_{\mathbf{d}, \mathbf{e}}^\Theta(Q)$ , allowing us to compute  $NDT_Q^{\mathbf{d}, \mathbf{e}}(\mu')$  in (7.28).

Next we consider quivers with relations coming from a superpotential:

- Let  $\text{mod-}\mathbb{C}Q/I$  come from a minimal superpotential  $W$  over  $\mathbb{C}$  on a quiver  $Q$ . Fix a vertex  $v \in Q_0$  of  $Q$ . Let  $(X, \rho) \in \text{mod-}\mathbb{C}Q/I$ . We say that  $(X, \rho)$  is *cyclic*, and *generated by a vector*  $x \in X_v$  if  $X = \mathbb{C}Q/I \cdot x$ . That is, there is no subobject  $(X', \rho') \subset (X, \rho)$  in  $\text{mod-}\mathbb{C}Q/I$  with  $(X', \rho') \neq (X, \rho)$  and  $x \in X'_v \subseteq X_v$ .

Szendrői [99, §1.2] calls  $((X, \rho), x)$  a *framed cyclic module* for  $(Q, I)$ , and defines a moduli space  $\mathcal{M}_{v, \mathbf{d}}$  of framed cyclic modules with  $\dim(X, \rho) = \mathbf{d}$ . Szendrői defines the *noncommutative Donaldson–Thomas invariant*  $Z_{v, \mathbf{d}}$  to be  $\chi(\mathcal{M}_{v, \mathbf{d}}, \nu_{\mathcal{M}_{v, \mathbf{d}}})$ . He computes the  $Z_{v, \mathbf{d}}$  in an example, the ‘noncommutative conifold’, and shows the generating function of the  $Z_{v, \mathbf{d}}$  may be written explicitly as an infinite product. In our notation  $\mathcal{M}_{v, \mathbf{d}}$  is  $\mathcal{M}_{\text{stf } Q, I}^{\mathbf{d}, \delta_v}(0')$ , where the framing dimension vector  $\mathbf{e}$  is  $\delta_v$ , that is,  $\delta_v(w) = 1$  for  $v = w$  and 0 for  $v \neq w \in Q_0$ , and the stability condition  $(\mu, \mathbb{R}, \leq)$  on  $\text{mod-}\mathbb{C}Q/I$  is zero. Thus by (7.27), Szendrői’s invariants are our  $NDT_{Q, I}^{\mathbf{d}, \delta_v}(0')$ .

For the conifold, Nagao and Nakajima [84] prove relationships between Szendrői’s invariants, Donaldson–Thomas invariants, and Pandharipande–Thomas invariants, via wall-crossing for stability conditions on the derived category. Nagao [83] generalizes this to other toric Calabi–Yau 3-folds.

- Let  $G$  be a finite subgroup of  $\text{SL}(3, \mathbb{C})$ . Young and Bryan [105, §A] discuss *Donaldson–Thomas invariants*  $N^{\mathbf{d}}(\mathbb{C}^3/G)$  of the orbifold  $[\mathbb{C}^3/G]$ . By this they mean invariants counting ideal sheaves of compactly-supported  $G$ -equivariant sheaves on  $\mathbb{C}^3$ . In two cases  $G = \mathbb{Z}_2 \times \mathbb{Z}_2$  and  $G = \mathbb{Z}_n$ , they show that the generating function of  $N^{\mathbf{d}}(\mathbb{C}^3/G)$  can be written explicitly as an infinite product, in a similar way to the conifold case [99].

As in Example 7.8, Ginzburg defines a quiver  $Q_G$  with superpotential  $W_G$  such that  $\text{mod-}\mathbb{C}Q_G/I_G$  is 3-Calabi–Yau and equivalent to the category of  $G$ -equivariant compactly-supported coherent sheaves on  $\mathbb{C}^3$ . The definitions imply that Bryan and Young’s  $N^{\mathbf{d}}(\mathbb{C}^3/G)$  is Szendrői’s  $Z_{v, \mathbf{d}}$  for  $(Q_G, I_G)$ , where the vertex  $v$  in  $Q_G$  corresponds to the trivial representation  $\mathbb{C}$  of  $G$ . Thus in our notation,  $N^{\mathbf{d}}(\mathbb{C}^3/G) = NDT_{Q_G, I_G}^{\mathbf{d}, \delta_v}(0')$ .

- Let  $Q, W, I$  come from a consistent brane tiling, as in Example 7.9. Then Mozgovoy and Reineke [82] write Szendrői’s invariants  $Z_{v, \mathbf{d}}$  for  $Q, I$  as combinatorial sums, allowing evaluation of them on a computer.

In §7.5–§7.6 we will use the results of [88, 99, 105] to write down the values of  $NDT_{Q, I}^{\mathbf{d}, \mathbf{e}}(\mu')$  and  $NDT_Q^{\mathbf{d}, \mathbf{e}}(\mu')$  in some of these examples. Then we will use Theorem 7.23 below to compute  $\bar{DT}_{Q, I}^{\mathbf{d}}(\mu)$  and  $\bar{DT}_Q^{\mathbf{d}}(\mu)$ , and equation (7.21) to find  $\hat{DT}_{Q, I}^{\mathbf{d}}(\mu)$  and  $\hat{DT}_Q^{\mathbf{d}}(\mu)$ .

**REMARK 7.22. (a)** Definitions 7.20 and 7.21 are fairly direct analogues of Definitions 5.20 and 5.24, with  $\text{coh}(X)$  and  $(\tau, T, \leq)$  replaced by  $\text{mod-}\mathbb{K}Q/I$  and  $(\mu, \mathbb{R}, \leq)$ . Note that the moduli spaces  $\mathcal{M}_{\text{stf } Q, I}^{\mathbf{d}, \mathbf{e}}(\mu')$ ,  $\mathcal{M}_{\text{stf } Q}^{\mathbf{d}, \mathbf{e}}(\mu')$  will in general not be proper. So we cannot define virtual classes for  $\mathcal{M}_{\text{stf } Q, I}^{\mathbf{d}, \mathbf{e}}(\mu')$ ,  $\mathcal{M}_{\text{stf } Q}^{\mathbf{d}, \mathbf{e}}(\mu')$ , and we have no analogue of (5.15); we are forced to define the invariants as weighted Euler characteristics, following (5.16).

**(b)** Here is why the framing data  $\sigma$  for  $(X, \rho) \in \text{mod-}\mathbb{K}Q/I$  or  $\text{mod-}\mathbb{K}Q$  in Definition 7.20 is a good analogue of the framing  $s : \mathcal{O}(-n) \rightarrow E$  for  $E \in \text{coh}(X)$  when  $n \gg 0$  in Definition 5.20.

In a well-behaved abelian category  $\mathcal{A}$ , an object  $P \in \mathcal{A}$  is called *projective* if  $\text{Ext}^i(P, E) = 0$  for all  $E \in \mathcal{A}$  and  $i > 0$ . Therefore  $\dim \text{Hom}(P, E) = \bar{\chi}([P], [E])$ , where  $\bar{\chi}$  is the Euler form of  $\mathcal{A}$ . If  $X$  is a Calabi–Yau 3-fold, there will generally be no nonzero projectives in  $\text{coh}(X)$ . However, for any bounded family  $\mathcal{F}$  of sheaves in  $\text{coh}(X)$ , for  $n \gg 0$  we have  $\text{Ext}^i(\mathcal{O}(-n), E) = 0$  for all  $E$  in  $\mathcal{F}$  and  $i > 0$ . Thus  $\mathcal{O}(-n)$  for  $n \gg 0$  *acts like* a projective object in  $\text{coh}(X)$ , and this is what is important in §5.4. Thus, a good generalization of stable pairs in  $\text{coh}(X)$  to an abelian category  $\mathcal{A}$  is to consider morphisms  $s : P \rightarrow E$  in  $\mathcal{A}$ , where  $P$  is some fixed projective object in  $\mathcal{A}$ , and  $E \in \mathcal{A}$ .

Now when  $Q$  has oriented cycles,  $\text{mod-}\mathbb{K}Q/I$  or  $\text{mod-}\mathbb{K}Q$  (which consist of *finite-dimensional* representations) generally do not contain enough projective objects for this to be a good definition. However, if we allow *infinite-dimensional* representations  $P$  of  $\mathbb{K}Q/I$  or  $\mathbb{K}Q$ , then we can define projective representations. Let  $\mathbf{e}$  be a dimension vector, and define

$$P^{\mathbf{e}} = \bigoplus_{v \in Q_0} ((\mathbb{K}Q/I) \cdot i_v) \otimes \mathbb{K}^{\mathbf{e}(v)} \quad \text{or} \quad P^{\mathbf{e}} = \bigoplus_{v \in Q_0} (\mathbb{K}Q \cdot i_v) \otimes \mathbb{K}^{\mathbf{e}(v)},$$

where the idempotent  $i_v$  in the algebra  $\mathbb{K}Q/I$  or  $\mathbb{K}Q$  is the path of length zero at  $v$ , so that  $\mathbb{K}Q \cdot i_v$  has basis the set of oriented paths in  $Q$  starting at  $v$ .

Then  $P^{\mathbf{e}}$  is a left representation of  $\mathbb{K}Q/I$  or  $\mathbb{K}Q$ , which may be infinite-dimensional if  $Q$  has oriented cycles. In the abelian category of possibly infinite-dimensional representations of  $\mathbb{K}Q/I$  or  $\mathbb{K}Q$ , it is projective. If  $(X, \rho)$  lies in  $\text{mod-}\mathbb{K}Q/I$  or  $\text{mod-}\mathbb{K}Q$  with  $\mathbf{dim}(X, \rho) = \mathbf{d}$  then

$$\text{Hom}(P^{\mathbf{e}}, (X, \rho)) \cong \bigoplus_{v \in Q_0} \text{Hom}(\mathbb{K}^{\mathbf{e}(v)}, X_v), \quad (7.29)$$

so that  $\dim \text{Hom}(P^{\mathbf{e}}, (X, \rho)) = \sum_{v \in Q_0} \mathbf{e}(v) \mathbf{d}(v)$ . (Note that this is not  $\bar{\chi}(\mathbf{e}, \mathbf{d})$ .) Equation (7.29) implies that morphisms of representations  $P^{\mathbf{e}} \rightarrow (X, \rho)$  are the same as choices of  $\sigma$  in Definition 7.20. Thus, framed representations  $(X, \rho, \sigma)$  in Definition 7.20 are equivalent to morphisms  $\sigma : P^{\mathbf{e}} \rightarrow (X, \rho)$ , where  $P^{\mathbf{e}}$  is a fixed projective. The comparison with  $s : \mathcal{O}(-n) \rightarrow E$  in §5.4 is clear.

(c) Here is another interpretation of framed representations, following Reineke [89, §3.1]. Given  $(Q, I)$  or  $Q$ ,  $\mathbf{d}, \mathbf{e}$  as above, define another quiver  $\tilde{Q}$  to be  $Q$  together with an extra vertex  $\infty$ , so that  $\tilde{Q}_0 = Q_0 \amalg \{\infty\}$ , and with  $\mathbf{e}(v)$  extra edges  $\infty \rightarrow v$  for each  $v \in Q_0$ . Let the relations  $\tilde{I}$  for  $\tilde{Q}$  be the lift of  $I$  to  $\mathbb{K}\tilde{Q}$ , with no extra relations. Define  $\tilde{\mathbf{d}} : \tilde{Q}_0 \rightarrow \mathbb{Z}_{\geq 0}$  by  $\tilde{\mathbf{d}}(v) = \mathbf{d}(v)$  for  $v \in Q_0$  and  $\tilde{\mathbf{d}}(\infty) = 1$ . It is then easy to show that framed representations of  $(Q, I)$  or  $Q$  of type  $(\mathbf{d}, \mathbf{e})$  correspond naturally to representations of  $(\tilde{Q}, \tilde{I})$  or  $\tilde{Q}$  of type  $\tilde{\mathbf{d}}$ , and one can define a stability condition  $(\tilde{\mu}, \mathbb{R}, \leq)$  on  $\text{mod-}\mathbb{K}\tilde{Q}/\tilde{I}$  or  $\text{mod-}\mathbb{K}\tilde{Q}$  such that  $\mu'$ -stable framed representations correspond to  $\tilde{\mu}$ -stable representations.

We now prove the analogue of Theorem 5.27 for quivers.

**THEOREM 7.23.** *Suppose  $Q$  is a quiver with relations  $I$  coming from a minimal superpotential  $W$  on  $Q$  over  $\mathbb{C}$ . Let  $(\mu, \mathbb{R}, \leq)$  be a slope stability condition on  $\text{mod-}\mathbb{C}Q/I$ , as in Example 7.4, and  $\bar{\chi}$  be as in (7.9). Then for all  $\mathbf{d}, \mathbf{e}$  in  $C(\text{mod-}\mathbb{C}Q/I) = \mathbb{Z}_{\geq 0}^{Q_0} \setminus \{0\} \subset \mathbb{Z}^{Q_0}$ , we have*

$$NDT_{Q,I}^{\mathbf{d}, \mathbf{e}}(\mu') = \sum_{\substack{\mathbf{d}_1, \dots, \mathbf{d}_l \in C(\text{mod-}\mathbb{C}Q/I), \\ l \geq 1: \mathbf{d}_1 + \dots + \mathbf{d}_l = \mathbf{d}, \\ \mu(\mathbf{d}_i) = \mu(\mathbf{d}), \text{ all } i}} \frac{(-1)^l}{l!} \prod_{i=1}^l [(-1)^{\mathbf{e} \cdot \mathbf{d}_i - \bar{\chi}(\mathbf{d}_1 + \dots + \mathbf{d}_{i-1}, \mathbf{d}_i)} (\mathbf{e} \cdot \mathbf{d}_i - \bar{\chi}(\mathbf{d}_1 + \dots + \mathbf{d}_{i-1}, \mathbf{d}_i)) \bar{DT}_{Q,I}^{\mathbf{d}_i}(\mu)], \quad (7.30)$$

with  $\mathbf{e} \cdot \mathbf{d}_i = \sum_{v \in Q_0} \mathbf{e}(v) \mathbf{d}_i(v)$ , and  $\bar{D}T_{Q,I}^{\mathbf{d}_i}(\mu)$ ,  $NDT_{Q,I}^{\mathbf{d},\mathbf{e}}(\mu')$  as in Definitions 7.15, 7.21. When  $W \equiv 0$ , the same equation holds for  $NDT_{Q,I}^{\mathbf{d},\mathbf{e}}(\mu')$ ,  $\bar{D}T_{Q,I}^{\mathbf{d}}(\mu)$ .

PROOF. The proof follows that of Theorem 5.27 in §13 closely. We need to explain the analogues of the abelian categories  $\mathcal{A}_p, \mathcal{B}_p$  in §13.1. When  $\mu \equiv 0$ , we have  $\mathcal{A}_p = \text{mod-}\mathbb{C}Q/I$  and  $\mathcal{B}_p = \text{mod-}\mathbb{C}\tilde{Q}/\tilde{I}$ , where  $(\tilde{Q}, \tilde{I})$  is as in Remark 7.22(c). For general  $\mu$ , with  $\mathbf{d}$  fixed, we take  $\mathcal{A}_p$  to be the abelian subcategory of objects  $(X, \rho)$  in  $\text{mod-}\mathbb{C}Q/I$  with  $\mu([X, \rho]) = \mu(\mathbf{d})$ , together with 0, and  $\mathcal{B}_p$  to be the abelian subcategory of objects  $(\tilde{X}, \tilde{\rho})$  in  $\text{mod-}\mathbb{C}\tilde{Q}/\tilde{I}$  with  $\tilde{\mu}([\tilde{X}, \tilde{\rho}]) = \tilde{\mu}(\tilde{\mathbf{d}})$ , together with 0, for  $\tilde{\mathbf{d}}, (\tilde{\mu}, \mathbb{R}, \leq)$  as in Remark 7.22(c).

Then we have  $K(\mathcal{A}_p) \subseteq K(\text{mod-}\mathbb{C}Q/I) = \mathbb{Z}^{Q_0}$ , and  $K(\mathcal{B}_p) = K(\mathcal{A}_p) \oplus \mathbb{Z}$ , as in §13.1, and  $\bar{\chi}^{\mathcal{A}_p} = \bar{\chi}|_{K(\mathcal{A}_p)}$ . The analogue of (13.5) giving the ‘Euler form’  $\bar{\chi}^{\mathcal{B}_p}$  on  $K(\mathcal{B}_p)$  is

$$\bar{\chi}^{\mathcal{B}_p}((\mathbf{d}, k), (\mathbf{d}', k')) = \bar{\chi}(\mathbf{d}, \mathbf{d}') - k \mathbf{e} \cdot \mathbf{d}' + k' \mathbf{e} \cdot \mathbf{d}. \quad (7.31)$$

The analogue of Proposition 13.4 then holds for all pairs of elements in  $\mathcal{B}_p$ , without the restrictions that  $\dim V + \dim W \leq 1$  and  $k, l \leq N$ . The point here is that  $\mathcal{O}_X(-n)$  is not actually a projective object in  $\text{coh}(X)$  for fixed  $n \gg 0$ , so we have to restrict to a bounded part of the category  $\mathcal{A}_p$  in which it acts as a projective. But as in Remark 7.22(b), in the quiver case we are in effect dealing with genuine projectives, so no boundedness assumptions are necessary.

The rest of the proof in §13 goes through without significant changes. Using (7.31) rather than (13.5) eventually yields equation (7.30).  $\square$

The proof of Proposition 5.29 now yields:

COROLLARY 7.24. *In the situation above, suppose  $c \in \mathbb{R}$  with  $\bar{\chi}(\mathbf{d}, \mathbf{d}') = 0$  for all  $\mathbf{d}, \mathbf{d}'$  in  $C(\text{mod-}\mathbb{C}Q/I)$  with  $\mu(\mathbf{d}) = \mu(\mathbf{d}') = c$ . Then for any  $\mathbf{e}$  in  $C(\text{mod-}\mathbb{C}Q/I)$ , in formal power series we have*

$$1 + \sum_{\mathbf{d} \in C(\text{mod-}\mathbb{C}Q/I): \mu(\mathbf{d})=c} NDT_{Q,I}^{\mathbf{d},\mathbf{e}}(\mu') q^{\mathbf{d}} = \exp \left[ - \sum_{\mathbf{d} \in C(\text{mod-}\mathbb{C}Q/I): \mu(\mathbf{d})=c} (-1)^{\mathbf{e} \cdot \mathbf{d}} (\mathbf{e} \cdot \mathbf{d}) \bar{D}T_{Q,I}^{\mathbf{d}}(\mu) q^{\mathbf{d}} \right], \quad (7.32)$$

where  $q^{\mathbf{d}}$  for  $\mathbf{d} \in C(\text{mod-}\mathbb{C}Q/I)$  are formal symbols satisfying  $q^{\mathbf{d}} \cdot q^{\mathbf{d}'} = q^{\mathbf{d}+\mathbf{d}'}$ . When  $W \equiv 0$ , the same equation holds for  $NDT_{Q,I}^{\mathbf{d},\mathbf{e}}(\mu')$ ,  $\bar{D}T_{Q,I}^{\mathbf{d}}(\mu)$ .

REMARK 7.25. In the coherent sheaf case of §5–§6, we regarded the generalized Donaldson–Thomas invariants  $\bar{D}T^\alpha(\tau)$ , or equivalently the BPS invariants  $\hat{D}T^\alpha(\tau)$ , as being the central objects of interest. The pair invariants  $PI^{\alpha,n}(\tau')$  appeared as auxiliary invariants, not of that much interest in themselves, but useful for computing the  $\bar{D}T^\alpha(\tau)$ ,  $\hat{D}T^\alpha(\tau)$  and proving their deformation-invariance.

In contrast, in the quiver literature to date, the invariants  $\bar{D}T_{Q,I}^{\mathbf{d}}(\mu)$ ,  $\bar{D}T_{Q,I}^{\mathbf{d}}(\mu)$  and  $\hat{D}T_{Q,I}^{\mathbf{d}}(\mu)$ ,  $\hat{D}T_{Q,I}^{\mathbf{d}}(\mu)$  have not been seriously considered even in the stable = semistable case, and the analogues  $NDT_{Q,I}^{\mathbf{d},\mathbf{e}}(\mu')$ ,  $NDT_{Q,I}^{\mathbf{d},\mathbf{e}}(\mu')$  of pair invariants  $PI^{\alpha,n}(\tau')$  have been the central object of study.

We argue that the  $\bar{D}T_{Q,I}^{\mathbf{d}}(\mu), \dots, \hat{D}T_{Q,I}^{\mathbf{d}}(\mu)$  should be regarded as more fundamental and more interesting than the  $NDT_{Q,I}^{\mathbf{d},\mathbf{e}}(\mu')$ ,  $NDT_{Q,I}^{\mathbf{d},\mathbf{e}}(\mu')$ . We offer two reasons for this. Firstly, as Theorem 7.23 shows, the  $NDT_{Q,I}^{\mathbf{d},\mathbf{e}}(\mu')$ ,  $NDT_{Q,I}^{\mathbf{d},\mathbf{e}}(\mu')$  can be written in terms of the  $\bar{D}T_{Q,I}^{\mathbf{d}}(\mu)$ ,  $\bar{D}T_{Q,I}^{\mathbf{d}}(\mu)$ , and hence by (7.21) in terms of the  $\hat{D}T_{Q,I}^{\mathbf{d}}(\mu)$ ,  $\hat{D}T_{Q,I}^{\mathbf{d}}(\mu)$ , so the pair invariants contain no more information. The  $\bar{D}T_{Q,I}^{\mathbf{d}}(\mu)$ ,  $\bar{D}T_{Q,I}^{\mathbf{d}}(\mu)$  are simpler than the  $NDT_{Q,I}^{\mathbf{d},\mathbf{e}}(\mu')$ ,  $NDT_{Q,I}^{\mathbf{d},\mathbf{e}}(\mu')$  as they depend

only on  $\mathbf{d}$  rather than on  $\mathbf{d}, \mathbf{e}$ , and in examples in §7.5–§7.6 we will see that the values of  $\bar{DT}_{Q,I}^{\mathbf{d}}(\mu), \hat{DT}_{Q,I}^{\mathbf{d}}(\mu)$  and especially of  $\bar{DT}_{Q,I}^{\mathbf{d}}(\mu), \hat{DT}_{Q,I}^{\mathbf{d}}(\mu)$  may be much simpler and more illuminating than the values of the  $NDT_{Q,I}^{\mathbf{d},\mathbf{e}}(\mu'), NDT_Q^{\mathbf{d},\mathbf{e}}(\mu')$ .

Secondly, the case in [88, 99, 105] for regarding  $NDT_{Q,I}^{\mathbf{d},\mathbf{e}}(\mu'), NDT_Q^{\mathbf{d},\mathbf{e}}(\mu')$  as analogues of rank 1 Donaldson–Thomas invariants counting ideal sheaves, that is, of counting surjective morphisms  $s : \mathcal{O}_X \rightarrow E$ , is in some ways misleading. The  $NDT_{Q,I}^{\mathbf{d},\mathbf{e}}(\mu'), NDT_Q^{\mathbf{d},\mathbf{e}}(\mu')$  are closer to our invariants  $PI^{\alpha,n}(\tau')$  counting  $s : \mathcal{O}_X(-n) \rightarrow E$  for  $n \gg 0$  than they are to counting morphisms  $s : \mathcal{O}_X \rightarrow E$ . The difference is that  $\mathcal{O}_X$  is *not a projective object* in  $\text{coh}(X)$ , but  $\mathcal{O}_X(-n)$  for  $n \gg 0$  is effectively a projective object in  $\text{coh}(X)$ , as in (b) above.

To see the difference between counting morphisms  $s : \mathcal{O}_X \rightarrow E$  and counting morphisms  $s : \mathcal{O}_X(-n) \rightarrow E$  for  $n \gg 0$ , consider the case where  $E$  is a dimension 1 sheaf on a Calabi–Yau 3-fold  $X$ . Then the MNOP Conjecture [80, 81] predicts that invariants  $DT^{(1,0,\beta,m)}(\tau)$  counting morphisms  $s : \mathcal{O}_X \rightarrow E$  encode the Gopakumar–Vafa invariants  $GV_g(\beta)$  of  $X$  for all genera  $g \geq 0$ . But Theorem 5.27 and Conjecture 6.20 in §6.4 imply that invariants  $PI^{(0,0,\beta,m),n}(\tau)$  counting morphisms  $s : \mathcal{O}_X(-n) \rightarrow E$  for  $n \gg 0$  encode only the Gopakumar–Vafa invariants  $GV_0(\beta)$  of  $X$  for genus  $g = 0$ .

The point is that since  $\mathcal{O}_X$  is not a projective, counting morphisms  $s : \mathcal{O}_X \rightarrow E$  gives you information not just about counting sheaves  $E$ , but also extra information about how  $\mathcal{O}_X$  and  $E$  interact. But as  $\mathcal{O}_X(-n)$  for  $n \gg 0$  is effectively a projective, counting morphisms  $s : \mathcal{O}_X(-n) \rightarrow E$  gives you information only about counting sheaves  $E$ , so we might as well just count sheaves  $E$  directly using (generalized) Donaldson–Thomas invariants.

### 7.5. Computing $\bar{DT}_{Q,I}^{\mathbf{d}}(\mu), \hat{DT}_{Q,I}^{\mathbf{d}}(\mu)$ in examples

We now use calculations of noncommutative Donaldson–Thomas invariants in examples by Szendrői [99] and Young and Bryan [105] to write down generating functions for  $NDT_{Q,I}^{\mathbf{d},\mathbf{e}}(\mu')$ , and then apply (7.32) to deduce values of  $\bar{DT}_{Q,I}^{\mathbf{d}}(\mu)$ , and (7.22) to deduce values of  $\hat{DT}_{Q,I}^{\mathbf{d}}(\mu)$ . These values of  $\bar{DT}_{Q,I}^{\mathbf{d}}(\mu)$  turn out to be much simpler than those of the  $NDT_{Q,I}^{\mathbf{d},\mathbf{e}}(\mu')$ , and explain the MacMahon function product form of the generating functions in [99, 105]. The translation between the notation of [99, 105] and our notation was explained after Definition 7.21, and we assume it below.

**7.5.1. Coherent sheaves on  $\mathbb{C}^3$ .** As in Szendrői [99, §1.5], let  $Q = (Q_0, Q_1, h, t)$  have one vertex  $Q_0 = \{v\}$  and three edges  $Q_1 = \{e_1, e_2, e_3\}$ , so that  $h(e_j) = t(e_j) = v$  for  $j = 1, 2, 3$ . Define a superpotential  $W$  on  $Q$  by  $W = e_1 e_2 e_3 - e_1 e_3 e_2$ . Then the ideal  $I$  in  $\mathbb{C}Q$  is generated by  $e_2 e_3 - e_3 e_2, e_3 e_1 - e_1 e_3, e_1 e_2 - e_2 e_1$ , and is  $[\mathbb{C}Q, \mathbb{C}Q]$ , so  $\mathbb{C}Q/I$  is the commutative polynomial algebra  $\mathbb{C}[e_1, e_2, e_3]$ , the coordinate ring of the noncompact Calabi–Yau 3-fold  $\mathbb{C}^3$ , and  $\text{mod-}\mathbb{C}Q/I$  is isomorphic to the abelian category  $\text{coh}_{\text{cs}}(\mathbb{C}^3)$ .

We have  $C(\text{mod-}\mathbb{C}Q/I) = \mathbb{N}$ , so taking  $\mathbf{d} = d \in \mathbb{N}$ ,  $\mathbf{e} = 1$ , and  $(\mu, \mathbb{R}, \leq)$  to be the trivial stability condition  $(0, \mathbb{R}, \leq)$  on  $\text{mod-}\mathbb{C}Q/I$ , we form invariants  $NDT_{Q,I}^{d,1}(0') \in \mathbb{Z}$ . Then as in [99, §1.5], by torus localization one can show that

$$1 + \sum_{d \geq 1} NDT_{Q,I}^{d,1}(0') q^d = \prod_{k \geq 1} (1 - (-q)^k)^{-k}, \quad (7.33)$$

which is Theorem 6.15 for the noncompact Calabi–Yau 3-fold  $X = \mathbb{C}^3$ . Taking logs of (7.33) and using (7.32), which holds as  $\bar{\chi} \equiv 0$ , gives

$$-\sum_{d \geq 1} (-1)^d d \bar{D}T_{Q,I}^d(0) q^d = \sum_{k \geq 1} (-k) \log(1 - (-q)^k) = \sum_{k, l \geq 1} \frac{k}{l} (-q)^{kl}.$$

Equating coefficients of  $q^d$  yields

$$\bar{D}T_{Q,I}^d(0) = - \sum_{l \geq 1, l|d} \frac{1}{l^2}.$$

So from (7.22) we deduce that

$$\hat{D}T_{Q,I}^d(0) = -1, \quad \text{all } d \geq 1. \quad (7.34)$$

This is (6.20) for the noncompact Calabi–Yau 3-fold  $X = \mathbb{C}^3$ , as in §6.7.

**7.5.2. The noncommutative conifold, following Szendrői.** As in Szendrői [99, §2.1], let  $Q = (Q_0, Q_1, h, t)$  have two vertices  $Q_0 = \{v_0, v_1\}$  and edges  $e_1, e_2 : v_0 \rightarrow v_1$  and  $f_1, f_2 : v_1 \rightarrow v_0$ , as below:

$$\begin{array}{ccc} & \xrightarrow{e_1} & \\ & \xrightarrow{e_2} & \\ \bullet & & \bullet \\ v_0 & & v_1 \\ & \xleftarrow{f_1} & \\ & \xleftarrow{f_2} & \end{array} \quad (7.35)$$

Define a superpotential  $W$  on  $Q$  by  $W = e_1 f_1 e_2 f_2 - e_1 f_2 e_2 f_1$ , and let  $I$  be the associated relations. Then  $\text{mod-}\mathbb{C}Q/I$  is a 3-Calabi–Yau category. Theorem 7.6 shows that the Euler form  $\bar{\chi}$  on  $\text{mod-}\mathbb{C}Q/I$  is zero.

We have equivalences of derived categories

$$D^b(\text{mod-}\mathbb{C}Q/I) \sim D^b(\text{coh}_{\text{cs}}(X)) \sim D^b(\text{coh}_{\text{cs}}(X_+)), \quad (7.36)$$

where  $\pi : X \rightarrow Y$  and  $\pi_+ : X_+ \rightarrow Y$  are the two crepant resolutions of the conifold  $Y = \{(z_1, z_2, z_3, z_4) \in \mathbb{C}^4 : z_1^2 + \dots + z_4^2 = 0\}$ , and  $X, X_+$  are related by a flop. Here  $X, X_+$  are regarded as ‘commutative’ crepant resolutions of  $Y$ , and  $\text{mod-}\mathbb{C}Q/I$  as a ‘noncommutative’ resolution of  $Y$ , in the sense that  $\text{mod-}\mathbb{C}Q/I$  can be regarded as the coherent sheaves on the ‘noncommutative scheme’  $\text{Spec}(\mathbb{C}Q/I)$  constructed from the noncommutative  $\mathbb{C}$ -algebra  $\mathbb{C}Q/I$ .

Szendrői [99, Th. 2.7.1] computed the noncommutative Donaldson–Thomas invariants  $NDT_{Q,I}^{d, \delta_{v_0}}(0')$  for  $\text{mod-}\mathbb{C}Q/I$  with  $e = \delta_{v_0}$ , as combinatorial sums, and using work of Young [104] wrote the generating function of the  $NDT_{Q,I}^{d, \delta_{v_0}}(0')$  as a product [99, Th. 2.7.2], giving

$$\begin{aligned} 1 + \sum_{d \in C(\text{mod-}\mathbb{C}Q/I)} NDT_{Q,I}^{d, \delta_{v_0}}(0') q_0^{d(v_0)} q_1^{d(v_1)} \\ = \prod_{k \geq 1} (1 - (-q_0 q_1)^k)^{-2k} (1 - (-q_0)^k q_1^{k-1})^k (1 - (-q_0)^k q_1^{k+1})^k. \end{aligned} \quad (7.37)$$

Taking logs of (7.37) and using (7.32) gives

$$\begin{aligned}
& - \sum_{\mathbf{d} \in C(\text{mod-}\mathbb{C}Q/I)} (-1)^{\mathbf{d}(v_0)} \mathbf{d}(v_0) \bar{DT}_{Q,I}^{\mathbf{d}}(0) q_0^{\mathbf{d}(v_0)} q_1^{\mathbf{d}(v_1)} \\
& = \sum_{k \geq 1} [-2k \log(1 - (-q_0 q_1)^k) + k \log(1 - (-q_0)^k q_1^{k-1}) \\
& \quad + k \log(1 - (-q_0)^k q_1^{k+1})] \\
& = \sum_{k, l \geq 1} \left[ \frac{2k}{l} (-q_0 q_1)^{kl} - \frac{k}{l} (-q_0)^{kl} q_1^{(k-1)l} - \frac{k}{l} (-q_0)^{kl} q_1^{(k+1)l} \right] \\
& = - \sum_{k, l \geq 1} (-1)^{kl} kl \cdot \left[ -\frac{2}{l^2} q_0^{kl} q_1^{kl} + \frac{1}{l^2} q_0^{kl} q_1^{(k-1)l} + \frac{1}{l^2} q_0^{kl} q_1^{(k+1)l} \right].
\end{aligned} \tag{7.38}$$

Writing  $\mathbf{d} = (d_0, d_1)$  with  $d_j = \mathbf{d}(v_j)$  and equating coefficients of  $q_0^{d_0} q_1^{d_1}$  yields

$$\bar{DT}_{Q,I}^{(d_0, d_1)}(0) = \begin{cases} -2 \sum_{l \geq 1, l|d} \frac{1}{l^2}, & d_0 = d_1 = d \geq 1, \\ \frac{1}{l^2}, & d_0 = kl, d_1 = (k-1)l, k, l \geq 1, \\ \frac{1}{l^2}, & d_0 = kl, d_1 = (k+1)l, k \geq 0, l \geq 1, \\ 0, & \text{otherwise.} \end{cases} \tag{7.39}$$

Actually we have cheated a bit here: because of the factor  $\mathbf{d}(v_0)$  on the first line, equation (7.38) only determines  $\bar{DT}_{Q,I}^{(d_0, d_1)}(0)$  when  $d_0 > 0$ . But by symmetry between  $v_0$  and  $v_1$  in (7.35) we have  $\bar{DT}_{Q,I}^{(d_0, d_1)}(0) = \bar{DT}_{Q,I}^{(d_1, d_0)}(0)$ , so we can deduce the answer for  $d_0 = 0, d_1 > 0$  from that for  $d_0 > 0, d_1 = 0$ . This is why we included the case  $k = 0, l \geq 1$  on the third line of (7.39).

Combining (7.22) and (7.39) we see that

$$\hat{DT}_{Q,I}^{(d_0, d_1)}(0) = \begin{cases} -2, & (d_0, d_1) = (k, k), k \geq 1, \\ 1, & (d_0, d_1) = (k, k-1), k \geq 1, \\ 1, & (d_0, d_1) = (k-1, k), k \geq 1, \\ 0, & \text{otherwise.} \end{cases} \tag{7.40}$$

Note that the values of the  $\hat{DT}_{Q,I}^{(d_0, d_1)}(0)$  in (7.40) lie in  $\mathbb{Z}$ , as in Conjecture 7.16, and are far simpler than those of the  $NDT_{Q,I}^{\mathbf{d}, \delta_{v_0}}(0')$  in (7.37). Also, (7.40) restores the symmetry between  $v_0, v_1$  in (7.35), which is broken in (7.37) by choosing the vertex  $v_0$  in  $\mathbf{e} = \delta_{v_0}$ .

Since  $\bar{\chi} \equiv 0$  on  $\text{mod-}\mathbb{C}Q/I$ , by Corollary 7.18 equations (7.39)–(7.40) also give  $\bar{DT}_{Q,I}^{(d_0, d_1)}(\mu), \hat{DT}_{Q,I}^{(d_0, d_1)}(\mu)$  for any stability condition  $(\mu, \mathbb{R}, \leq)$  on  $\text{mod-}\mathbb{C}Q/I$ . It should not be difficult to prove (7.39)–(7.40) directly, without going via pair invariants. If  $(\mu, \mathbb{R}, \leq)$  is a nontrivial slope stability condition on  $\text{mod-}\mathbb{C}Q/I$ , then Nagao and Nakajima [84, §3.2] prove that every  $\mu$ -stable object in  $\text{mod-}\mathbb{C}Q/I$  lies in class  $(k, k)$  or  $(k, k-1)$  or  $(k-1, k)$  in  $K(\text{mod-}\mathbb{C}Q/I)$  for  $k \geq 1$ , and the  $\mu$ -stable objects in classes  $(k, k-1)$  and  $(k-1, k)$  are unique up to isomorphism. The bottom three lines of (7.40) can be deduced from this.

Szendrői [99, §2.9] relates noncommutative Donaldson–Thomas invariants counting (framed) objects in  $\text{mod-}\mathbb{C}Q/I$  with Donaldson–Thomas invariants counting

(ideal sheaves of) objects in  $\text{coh}_{\text{cs}}(X)$ ,  $\text{coh}_{\text{cs}}(X_+)$ , under the equivalences (7.36). He ends up with the generating functions

$$Z_{\text{mod-}\mathbb{C}Q/I}(q, z) = \prod_{k \geq 1} (1 - (-q)^k)^{-2k} (1 + (-q)^k z)^k (1 + (-q)^k z^{-1})^k, \quad (7.41)$$

$$Z_{\text{coh}_{\text{cs}}(X)}(q, z) = \prod_{k \geq 1} (1 - (-q)^k)^{-2k} (1 + (-q)^k z)^k, \quad (7.42)$$

$$Z_{\text{coh}_{\text{cs}}(X_+)}(q, z) = \prod_{k \geq 1} (1 - (-q)^k)^{-2k} (1 + (-q)^k z^{-1})^k, \quad (7.43)$$

where (7.41) is (7.37) with the variable change  $q = q_0 q_1$ ,  $z = q_1$ , and (7.42)–(7.43) encode counting invariants in  $\text{coh}_{\text{cs}}(X)$  and  $\text{coh}_{\text{cs}}(X_+)$  in a similar way. Nagao and Nakajima [84] explain the relationship between (7.41)–(7.43) in terms of stability conditions and wall-crossing on the triangulated category (7.36).

We can offer a much simpler explanation for the relationship between our invariants  $\hat{DT}_{Q,I}^{(d_0, d_1)}(\mu)$  counting (unframed) objects in  $\text{mod-}\mathbb{C}Q/I$ , and the analogous invariants counting objects (not ideal sheaves) in  $\text{coh}_{\text{cs}}(X), \text{coh}_{\text{cs}}(X_+)$ . We base it on the following conjecture:

**CONJECTURE 7.26.** *Let  $\mathcal{T}$  be a  $\mathbb{C}$ -linear 3-Calabi–Yau triangulated category, and abelian categories  $\mathcal{A}, \mathcal{B} \subset \mathcal{T}$  be the hearts of t-structures on  $\mathcal{T}$ . Suppose the Euler form  $\bar{\chi}$  of  $\mathcal{T}$  is zero. Let  $K(\mathcal{T})$  be a quotient of  $K_0(\mathcal{T})$ , and  $K(\mathcal{A}), K(\mathcal{B})$  the corresponding quotients of  $K_0(\mathcal{A}), K_0(\mathcal{B})$  under  $K_0(\mathcal{A}) \cong K_0(\mathcal{T}) \cong K_0(\mathcal{B})$ .*

*Suppose we can define Donaldson–Thomas type invariants  $\bar{DT}_{\mathcal{A}}^{\alpha}(\tau), \hat{DT}_{\mathcal{A}}^{\alpha}(\tau)$  counting objects in  $\mathcal{A}$ , for  $\alpha \in K(\mathcal{A})$  and  $(\tau, T, \leq)$  a stability condition on  $\mathcal{A}$ , and  $\bar{DT}_{\mathcal{B}}^{\beta}(\tilde{\tau}), \hat{DT}_{\mathcal{B}}^{\beta}(\tilde{\tau})$  counting objects in  $\mathcal{B}$ , for  $\beta \in K(\mathcal{B})$  and  $(\tilde{\tau}, \tilde{T}, \leq)$  a stability condition on  $\mathcal{B}$ , as for  $\mathcal{A} = \text{coh}(X)$  in §5–§6 and  $\mathcal{A} = \text{mod-}\mathbb{C}Q/I$  in §7.3.*

*Define  $\bar{DT}_{\mathcal{A}}, \hat{DT}_{\mathcal{A}} : K(\mathcal{A}) \rightarrow \mathbb{Q}$  and  $\bar{DT}_{\mathcal{B}}, \hat{DT}_{\mathcal{B}} : K(\mathcal{B}) \rightarrow \mathbb{Q}$  by*

$$\bar{DT}_{\mathcal{A}}(\alpha) = \begin{cases} \bar{DT}_{\mathcal{A}}^{\alpha}(\tau), & \alpha \in C(\mathcal{A}), \\ \bar{DT}_{\mathcal{A}}^{-\alpha}(\tau), & -\alpha \in C(\mathcal{A}), \\ 0, & \text{otherwise,} \end{cases} \quad \hat{DT}_{\mathcal{A}}(\alpha) = \begin{cases} \hat{DT}_{\mathcal{A}}^{\alpha}(\tau), & \alpha \in C(\mathcal{A}), \\ \hat{DT}_{\mathcal{A}}^{-\alpha}(\tau), & -\alpha \in C(\mathcal{A}), \\ 0, & \text{otherwise,} \end{cases}$$

*and similarly for  $\bar{DT}_{\mathcal{B}}, \hat{DT}_{\mathcal{B}}$ . Then (possibly under some extra conditions), under  $K(\mathcal{A}) \cong K(\mathcal{B})$  we have  $\bar{DT}_{\mathcal{A}} \equiv \bar{DT}_{\mathcal{B}}$ , or equivalently  $\hat{DT}_{\mathcal{A}} \equiv \hat{DT}_{\mathcal{B}}$ .*

Here is why we believe this. We expect that there should be some extension of Donaldson–Thomas theory from abelian categories to 3-Calabi–Yau triangulated categories  $\mathcal{T}$ , in the style of Kontsevich–Soibelman [63], using Bridgeland stability conditions on triangulated categories [11]. Invariants  $\bar{DT}_{\mathcal{A}}^{\alpha}(\tau)$  for an abelian category  $\mathcal{A}$  embedded as the heart of a t-structure in  $\mathcal{T}$  should be a special case of triangulated category invariants on  $\mathcal{T}$ , in which the Bridgeland stability condition  $(Z, \mathcal{P})$  on  $D^b(\mathcal{A})$  is constructed from  $(\tau, T, \leq)$  on  $\mathcal{A}$ . If  $\mathcal{A}$  is a 3-Calabi–Yau abelian category then we take  $\mathcal{T} = D^b(\mathcal{A})$ .

Now the  $Z$ -(semi)stable objects in  $\mathcal{T}$  should be shifts  $E[k]$  for  $k \in \mathbb{Z}$  and  $E \in \mathcal{A}$   $\tau$ -(semistable). The class  $[E[k]]$  of  $E[k]$  in  $K(\mathcal{T}) \cong K(\mathcal{A})$  is  $(-1)^k[E]$ . Thus, invariants  $\bar{DT}_{\mathcal{T}}^{\alpha}(Z)$  for  $\alpha \in K(\mathcal{A})$  should have contributions  $\bar{DT}_{\mathcal{A}}^{\alpha}(\tau)$  for  $\alpha \in C(\mathcal{A})$  counting  $E[2k]$  for  $E \in \mathcal{A}$   $\tau$ -(semi)stable and  $k \in \mathbb{Z}$ , and  $\bar{DT}_{\mathcal{A}}^{-\alpha}(\tau)$  for  $\alpha \in C(\mathcal{A})$  counting  $E[2k+1]$  for  $E \in \mathcal{A}$   $\tau$ -(semi)stable and  $k \in \mathbb{Z}$ . This explains the definitions of  $\bar{DT}_{\mathcal{A}}, \hat{DT}_{\mathcal{A}} : K(\mathcal{A}) \rightarrow \mathbb{Q}$  above.



As in Corollary 7.18, if the form  $\bar{\chi}$  on  $\mathcal{A}$  (which is the Euler form of  $\mathcal{T}$ ) is zero then (5.14), (7.23) imply that invariants  $\bar{DT}_{\mathcal{A}}^{\alpha}(\tau), \hat{DT}_{\mathcal{A}}^{\alpha}(\tau)$  are independent of the choice of stability condition  $(\tau, T, \leq)$  on  $\mathcal{A}$ , since the changes when we cross a wall always include factors  $\bar{\chi}(\beta, \gamma)$ . The point of Conjecture 7.26 is that we expect this to be true for triangulated categories too, so computing invariants in  $\mathcal{T}$  either in  $\mathcal{A}$  or  $\mathcal{B}$  should give the same answers, i.e.  $\bar{DT}_{\mathcal{A}} \equiv \bar{DT}_{\mathcal{B}}$ .

In the noncommutative conifold example above, from (7.40) we have

$$\hat{DT}_{Q,I}(d_0, d_1) = \begin{cases} -2, & (d_0, d_1) = (k, k), 0 \neq k \in \mathbb{Z}, \\ 1, & (d_0, d_1) = (k, k-1), k \in \mathbb{Z}, \\ 1, & (d_0, d_1) = (k-1, k), k \in \mathbb{Z}, \\ 0, & \text{otherwise.} \end{cases} \quad (7.44)$$

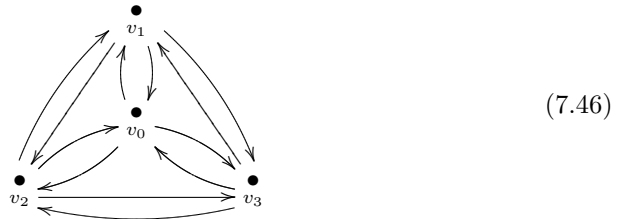
The Donaldson–Thomas invariants for  $\text{coh}_{\text{cs}}(X) \cong \text{coh}_{\text{cs}}(X_+)$  were computed in Example 6.30, and from (6.40)–(6.42) we have

$$\hat{DT}_{\text{coh}_{\text{cs}}(X)}(a_2, a_3) = \hat{DT}_{\text{coh}_{\text{cs}}(X_+)}(a_2, a_3) = \begin{cases} -2, & a_2 = 0, 0 \neq a_3 \in \mathbb{Z}, \\ 1, & a_2 = \pm 1, \\ 0, & \text{otherwise.} \end{cases} \quad (7.45)$$

As in Szendrői [99, §2.8–§2.9], the identification  $K(\text{mod-}\mathbb{C}Q/I) \rightarrow K(\text{coh}_{\text{cs}}(X))$  induced by  $D^b(\text{mod-}\mathbb{C}Q/I) \sim D^b(\text{coh}_{\text{cs}}(X))$  in (7.36) is  $(d_0, d_1) \mapsto (-d_0 + d_1, d_0) = (a_2, a_3)$ , and under this identification we have  $\hat{DT}_{Q,I} \equiv \hat{DT}_{\text{coh}_{\text{cs}}(X)}$  by (7.44)–(7.45). Similarly, the identification  $K(\text{mod-}\mathbb{C}Q/I) \rightarrow K(\text{coh}_{\text{cs}}(X_+))$  induced by  $D^b(\text{mod-}\mathbb{C}Q/I) \sim D^b(\text{coh}_{\text{cs}}(X_+))$  in (7.36) is  $(d_0, d_1) \mapsto (d_0 - d_1, d_0) = (a_2, a_3)$ , and again we have  $\hat{DT}_{Q,I} \equiv \hat{DT}_{\text{coh}_{\text{cs}}(X_+)}$ .

Thus  $\hat{DT}_{Q,I} \equiv \hat{DT}_{\text{coh}_{\text{cs}}(X)} \equiv \hat{DT}_{\text{coh}_{\text{cs}}(X_+)}$ , verifying Conjecture 7.26 for the equivalences (7.36). This seems a much simpler way of relating enumerative invariants in  $\text{mod-}\mathbb{C}Q/I, \text{coh}_{\text{cs}}(X)$  and  $\text{coh}_{\text{cs}}(X_+)$  than those in [84, 99].

**7.5.3. Coherent sheaves on  $\mathbb{C}^3/\mathbb{Z}_2^2$ , following Young.** Let  $G$  be the subgroup  $\mathbb{Z}_2^2$  in  $\text{SL}(3, \mathbb{C})$  generated by  $(z_1, z_2, z_3) \mapsto (-z_1, -z_2, z_3)$  and  $(z_1, z_2, z_3) \mapsto (z_1, -z_2, -z_3)$ . Then the Ginzburg construction in Example 7.8 yields a quiver  $Q_G$  and a cubic superpotential  $W_G$  giving relations  $I_G$  such that  $\text{mod-}\mathbb{C}Q_G/I_G$  is 3-Calabi–Yau and equivalent to the abelian category of  $G$ -equivariant compactly-supported coherent sheaves on  $\mathbb{C}^3$ . Write  $(Q, I)$  for  $(Q_G, I_G)$ . Then  $Q$  has 4 vertices  $v_0, \dots, v_3$  corresponding to the irreducible representations of  $\mathbb{Z}_2^2$ , with  $v_0$  the trivial representation, and 12 edges, as below:



Theorem 7.6 implies that the Euler form  $\bar{\chi}$  of  $\text{mod-}\mathbb{C}Q/I$  is zero.

As for (7.37), Young and Bryan [105, Th.s 1.5 & 1.6] prove that

$$\begin{aligned}
& 1 + \sum_{\mathbf{d} \in C(\text{mod-}\mathbb{C}Q/I)} NDT_{Q,I}^{\mathbf{d}, \delta_{v_0}}(0') q_0^{\mathbf{d}(v_0)} q_1^{\mathbf{d}(v_1)} q_2^{\mathbf{d}(v_2)} q_3^{\mathbf{d}(v_3)} \\
&= \prod_{k \geq 1} (1 - (-q_0 q_1 q_2 q_3)^k)^{-4k} \\
&\quad (1 - (-q_0 q_1)^k (q_2 q_3)^{k+1})^{-k} (1 - (-q_0 q_1)^k (q_2 q_3)^{k-1})^{-k} \\
&\quad (1 - (-q_0 q_2)^k (q_3 q_1)^{k+1})^{-k} (1 - (-q_0 q_2)^k (q_3 q_1)^{k-1})^{-k} \\
&\quad (1 - (-q_0 q_3)^k (q_1 q_2)^{k+1})^{-k} (1 - (-q_0 q_3)^k (q_1 q_2)^{k-1})^{-k} \\
&\quad (1 - (-q_0 q_2 q_3)^k q_1^{k+1})^k (1 - (-q_0 q_2 q_3)^k q_1^{k-1})^k \\
&\quad (1 - (-q_0 q_3 q_1)^k q_2^{k+1})^k (1 - (-q_0 q_3 q_1)^k q_2^{k-1})^k \\
&\quad (1 - (-q_0 q_1 q_2)^k q_3^{k+1})^k (1 - (-q_0 q_1 q_2)^k q_3^{k-1})^k \\
&\quad (1 - (-q_0)^k (q_1 q_2 q_3)^{k+1})^k (1 - (-q_0)^k (q_1 q_2 q_3)^{k-1})^k.
\end{aligned} \tag{7.47}$$

Arguing as for (7.38)–(7.40) and writing  $\mathbf{d} = (d_0, \dots, d_3)$  with  $d_j = \mathbf{d}(v_j)$  yields

$$\hat{DT}_{Q,I}^{(d_0, \dots, d_3)}(0) = \begin{cases} -4, & d_j = k \text{ for all } j, k \geq 1, \\ -1, & d_j = k \text{ for two } j, d_j = k-1 \text{ for two } j, k \geq 1, \\ 1, & d_j = k \text{ for three } j, d_j = k-1 \text{ for one } j, k \geq 1, \\ 0, & \text{otherwise.} \end{cases} \tag{7.48}$$

This is clearly much simpler than (7.47), and restores the symmetry between  $v_0, \dots, v_3$  in (7.46) which is lost in (7.47) by selecting the vertex  $v_0$ .

If  $X$  is any crepant resolution of  $\mathbb{C}^3/G$  then by Ginzburg [30, Cor. 4.4.8] we have  $D^b(\text{mod-}\mathbb{C}Q/I) \sim D^b(\text{coh}_{\text{cs}}(X))$ . As the Euler forms  $\bar{\chi}$  are zero on  $\text{mod-}\mathbb{C}Q/I, \text{coh}_{\text{cs}}(X)$ , using Conjecture 7.26 we can read off a prediction for the invariants  $\hat{DT}_{\text{coh}_{\text{cs}}(X)}^{\alpha}(\tau)$ . The first line of (7.48) corresponds to (6.20) for  $X$ , as one can show that  $\chi(X) = 4$ .

**7.5.4. Coherent sheaves on  $\mathbb{C}^3/\mathbb{Z}_n$ , following Young.** Let  $G$  be the subgroup  $\mathbb{Z}_n$  in  $\text{SL}(3, \mathbb{C})$  generated by  $(z_1, z_2, z_3) \mapsto (e^{2\pi i/n} z_1, z_2, e^{-2\pi i/n} z_3)$ . Then the Ginzburg construction in Example 7.8 gives a quiver  $Q$  and a cubic superpotential  $W$  giving relations  $I$  such that  $\text{mod-}\mathbb{C}Q/I$  is 3-Calabi–Yau and equivalent to the abelian category of  $G$ -equivariant compactly-supported coherent sheaves on  $\mathbb{C}^3$ . Then  $Q$  has vertices  $v_0, \dots, v_{n-1}$ , with  $v_0$  the trivial representation. We take  $v_i$  to be indexed by  $i \in \mathbb{Z}_n$ , so that  $v_i = v_j$  if  $i \equiv j \pmod n$ . With this convention,  $Q$  has edges  $v_i \rightarrow v_{i+1}, v_i \rightarrow v_i, v_i \rightarrow v_{i-1}$  for  $i = 0, \dots, n-1$ . The case  $n = 3$  is shown below:



Theorem 7.6 implies that the Euler form  $\bar{\chi}$  of  $\text{mod-}\mathbb{C}Q/I$  is zero.

As for (7.37) and (7.47), Young and Bryan [105, Th.s 1.4 & 1.6] prove that

$$1 + \sum_{\mathbf{d} \in C(\text{mod-}\mathbb{C}Q/I)} NDT_{Q,I}^{\mathbf{d}, \delta_{v_0}}(0') q_0^{\mathbf{d}(v_0)} q_1^{\mathbf{d}(v_1)} \cdots q_{n-1}^{\mathbf{d}(v_{n-1})} = \prod_{k \geq 1} (1 - (-q_0 \cdots q_{n-1})^k)^{-n}. \quad (7.50)$$

$$\prod_{0 < a < b \leq n} \prod_{k \geq 1} (1 - (-q_0 \cdots q_{n-1})^k (q_a q_{a+1} \cdots q_{b-1}))^{-k} \\ (1 - (-q_0 \cdots q_{n-1})^k (q_a q_{a+1} \cdots q_{b-1})^{-1})^{-k}.$$

Arguing as for (7.38)–(7.40) and writing  $\mathbf{d} = (d_0, \dots, d_{n-1})$  with  $d_j = \mathbf{d}(v_j)$  yields

$$\hat{DT}_{Q,I}^{(d_0, \dots, d_{n-1})}(0) = \begin{cases} -n, & d_i = k \text{ for all } i, k \geq 1, \\ -1, & \begin{cases} d_i = k \text{ for } i = a, \dots, b-1 \text{ and} \\ d_i = k-1 \text{ for } i = b, \dots, a+n-1, \\ 0 \leq a < n, a < b < a+n, k \geq 1, \end{cases} \\ 0, & \text{otherwise.} \end{cases} \quad (7.51)$$

This is simpler than (7.50), and restores the dihedral symmetry group of (7.49), which is lost in (7.50) by selecting the vertex  $v_0$ .

**7.5.5. Conclusions.** In each of our four examples, the noncommutative Donaldson–Thomas invariants  $NDT_{Q,I}^{\mathbf{d}, \delta_v}(0')$  can be written in a generating function as an explicit infinite product involving MacMahon type factors (7.33), (7.37), (7.47), (7.50). In each case, this product form held because the Euler form  $\bar{\chi}$  of  $\text{mod-}\mathbb{C}Q/I$  was zero, so that the generating function for  $NDT_{Q,I}^{\mathbf{d}, \delta_v}(0')$  has an exponential expression (7.32) in terms of the  $\bar{DT}_{Q,I}^{\mathbf{d}}(0)$ , and because of simple explicit formulae (7.34), (7.40), (7.48), (7.51) for the BPS invariants  $\hat{DT}_{Q,I}^{\mathbf{d}}(0)$ .

In these examples, the BPS invariants  $\hat{DT}_{Q,I}^{\mathbf{d}}(\mu)$  seem to be a simpler and more illuminating invariant than the noncommutative Donaldson–Thomas invariants  $NDT_{Q,I}^{\mathbf{d}, \delta_v}(\mu')$ . That  $\hat{DT}_{Q,I}^{\mathbf{d}}(\mu)$  has such a simple form probably says something interesting about the representation theory of  $\mathbb{C}Q/I$ , which may be worth pursuing. Also, when we pass from the invariants  $\hat{DT}_{Q,I}^{\mathbf{d}}(\mu)$  for the abelian category  $\text{mod-}\mathbb{C}Q/I$  to  $\hat{DT}_{Q,I} : K(\text{mod-}\mathbb{C}Q/I) \rightarrow \mathbb{Z}$  for the derived category  $D^b(\text{mod-}\mathbb{C}Q/I)$  as in Conjecture 7.26, in §7.5.2–§7.5.4 things actually become simpler, in that pairs of entries parametrized by  $k \geq 1$  combine to give one entry parametrized by  $k \in \mathbb{Z}$ . So maybe these phenomena will be best understood in the derived category.

We can also ask whether there are other categories  $\text{mod-}\mathbb{C}Q/I$  which admit the same kind of explicit computation of invariants. For the programme above to work we need the Euler form  $\bar{\chi}$  to be zero, which by Theorem 7.6 means that for all vertices  $i, j$  in  $Q$  there must be the same number of edges  $i \rightarrow j$  as edges  $j \rightarrow i$ . Suppose  $\text{mod-}\mathbb{C}Q/I$  comes from a finite subgroup  $G \subset \text{SL}(3, \mathbb{C})$  as in Example 7.8, and let  $\pi : X \rightarrow \mathbb{C}^3/G$  be a crepant resolution. Then as  $D^b(\text{mod-}\mathbb{C}Q/I) \sim D^b(\text{coh}_{\text{cs}}(X))$ , the Euler form of  $\text{mod-}\mathbb{C}Q/I$  is zero if and only if that of  $\text{coh}_{\text{cs}}(X)$  is zero.

The Euler form of  $\text{coh}_{\text{cs}}(X)$  is zero if and only if  $\pi : X \rightarrow \mathbb{C}^3/G$  is *semismall*, that is, no divisors in  $X$  lie over points in  $\mathbb{C}^3/G$ . This is equivalent to the ‘hard Lefschetz condition’ for  $\mathbb{C}^3/G$ , and by Bryan and Gholampour [14, Lem. 3.4.1] holds if and only if  $G$  is conjugate to a subgroup of either  $\text{SO}(3) \subset \text{SL}(3, \mathbb{C})$  or  $\text{SU}(2) \subset \text{SL}(3, \mathbb{C})$ ; in §7.5.3 we have  $\mathbb{Z}_2^2 \subset \text{SO}(3) \subset \text{SL}(3, \mathbb{C})$ , and in §7.5.4 we have  $\mathbb{Z}_n \subset \text{SU}(2) \subset \text{SL}(3, \mathbb{C})$ . Following discussion in Bryan and Gholampour [14, §1.2.1], Young and Bryan [105, Conj. A.6 & Rem. A.9], and Szendrői [99, §2.12],

it seems likely that formulae similar to (7.47) and (7.50) hold for all finite  $G$  in  $\mathrm{SO}(3) \subset \mathrm{SL}(3, \mathbb{C})$  or  $\mathrm{SU}(2) \subset \mathrm{SL}(3, \mathbb{C})$ , so that the  $\hat{DT}_{Q,I}^d(0)$  have a simple form. But note as in [105, Rem. A.10] that the Gromov–Witten invariants of  $X$  computed in [14] are not always the right ones for computing Donaldson–Thomas invariants, because of the way they count curves going to infinity.

### 7.6. Integrality of $\hat{DT}_Q^d(\mu)$ for generic $(\mu, \mathbb{R}, \leq)$

We now prove Conjecture 7.16 when  $W \equiv 0$ , that for  $Q$  a quiver and  $(\mu, \mathbb{R}, \leq)$  generic we have  $\hat{DT}_Q^d(\mu) \in \mathbb{Z}$ . We first compute the invariants when  $Q$  has only one vertex and verify their integrality, using Reineke [88, 90]. This example is also discussed by Kontsevich and Soibelman [63, §7.5].

EXAMPLE 7.27. Let  $Q_m$  be the quiver with one vertex  $v$  and  $m$  edges  $v \rightarrow v$ , for  $m \geq 0$ . Then  $K(\mathrm{mod}\text{-}\mathbb{C}Q_m) = \mathbb{Z}$  and  $C(\mathrm{mod}\text{-}\mathbb{C}Q_m) = \mathbb{N}$ . Consider the trivial stability condition  $(0, \mathbb{R}, \leq)$  on  $\mathrm{mod}\text{-}\mathbb{C}Q_m$ . Then our framed moduli space  $\mathcal{M}_{\mathrm{stf} Q_m}^{d,e}(0')$  is  $H_{d,e}^{(m)}$  in Reineke’s notation [88]. Reineke [88, Th. 1.4] proves that

$$\chi(\mathcal{M}_{\mathrm{stf} Q_m}^{d,e}(0')) = \frac{e}{(m-1)d+1} \binom{md+e-1}{d},$$

so by (7.28) we have

$$NDT_{Q_m}^{d,e}(0') = (-1)^{d(1-m)+ed} \frac{e}{(m-1)d+1} \binom{md+e-1}{d}.$$

Fixing  $e = 1$ , we see as in [63, §7.5] that

$$\begin{aligned} 1 + \sum_{d \geq 1} NDT_{Q_m}^{d,1}(0') q^d &= \sum_{d \geq 0} \frac{(-1)^{md}}{(m-1)d+1} \binom{md}{d} q^d \\ &= \exp \left[ \sum_{d \geq 1} \frac{(-1)^{md}}{md} \binom{md}{d} q^d \right]. \end{aligned} \tag{7.52}$$

Taking logs of (7.52) and using (7.32) yields

$$\bar{DT}_{Q_m}^d(0) = \frac{(-1)^{(m+1)d+1}}{md^2} \binom{md}{d},$$

so by (7.21) we have

$$\hat{DT}_{Q_m}^d(0) = \frac{1}{md^2} \sum_{e \geq 1, e|d} \mathrm{M\ddot{o}}(d/e) (-1)^{(m+1)e+1} \binom{me}{e}. \tag{7.53}$$

By Reineke [90, Th. 5.9] applied with  $N = m$ ,  $b_1 = 1$  and  $b_i = 0$  for  $i > 1$  as in [90, Ex., §5], so that the r.h.s. of (7.53) is  $-a_d$  in Reineke’s notation, we have  $\hat{DT}_{Q_m}^d(0) \in \mathbb{Z}$  for  $d \geq 1$ . This will be important in the proof of Theorem 7.29.

Now let  $Q$  be an arbitrary quiver without relations, and  $(\mu, \mathbb{R}, \leq)$  a slope stability condition on  $\mathrm{mod}\text{-}\mathbb{C}Q$  which is generic in the sense of Conjecture 7.16. As in §6.2, define a 1-morphism  $P_m : \mathfrak{M}_Q \rightarrow \mathfrak{M}_Q$  for  $m \geq 1$  by  $P_m : [E] \mapsto [mE]$  for

$E \in \text{mod-}\mathbb{C}Q$ . Then as for (5.9) and (6.16), for all  $\mathbf{d} \in C(\text{mod-}\mathbb{C}Q)$  we have

$$\begin{aligned} \hat{DT}_Q^{\mathbf{d}}(\mu) &= \chi(\mathcal{M}_{\text{ss}}^{\mathbf{d}}(\mu), F_Q^{\mathbf{d}}(\mu)), \quad \text{where} \\ F_Q^{\mathbf{d}}(\mu) &= - \sum_{m \geq 1, m|\mathbf{d}} \frac{\text{Mö}(m)}{m^2} \text{CF}^{\text{na}}(\pi) [\text{CF}^{\text{na}}(P_m) \circ \Pi_{\text{CF}} \circ \\ &\quad \bar{\Pi}_{\mathfrak{M}_Q}^{\chi, \mathbb{Q}}(\bar{\epsilon}^{\mathbf{d}/m}(\mu)) \cdot \nu_{\mathfrak{M}_Q}]. \end{aligned} \quad (7.54)$$

Here  $\mathfrak{M}_{\text{ss}}^{\mathbf{d}}(\mu)$  is the moduli stack of  $\mu$ -semistable objects of class  $\mathbf{d}$  in  $\text{mod-}\mathbb{C}Q$ , an open substack of  $\mathfrak{M}_Q$ , and  $\mathcal{M}_{\text{ss}}^{\mathbf{d}}(\mu)$  is the quasiprojective coarse moduli scheme of  $\mu$ -semistable objects of class  $\mathbf{d}$  in  $\text{mod-}\mathbb{C}Q$ , and  $\pi : \mathfrak{M}_{\text{ss}}^{\mathbf{d}}(\mu) \rightarrow \mathcal{M}_{\text{ss}}^{\mathbf{d}}(\mu)$  is the natural projection 1-morphism.

An object  $E$  in  $\text{mod-}\mathbb{C}Q$  is called  $\mu$ -polystable if it is  $\mu$ -semistable and a direct sum of  $\mu$ -stable objects. That is,  $E$  is  $\mu$ -polystable if and only if  $E \cong a_1 E_1 \oplus \cdots \oplus a_k E_k$ , where  $E_1, \dots, E_k$  are pairwise nonisomorphic  $\mu$ -stables in  $\text{mod-}\mathbb{C}Q$  with  $\mu([E_1]) = \cdots = \mu([E_k])$  and  $a_1, \dots, a_k \geq 1$ , and  $E$  determines  $E_1, \dots, E_k$  and  $a_1, \dots, a_k$  up to order and isomorphism. Since  $\mu$  is a stability condition, each  $\mathbb{C}$ -point of  $\mathcal{M}_{\text{ss}}^{\mathbf{d}}(\mu)$  is represented uniquely up to isomorphism by a  $\mu$ -polystable. That is, if  $E'$  is  $\mu$ -semistable then  $E'$  admits a Jordan–Hölder filtration with  $\mu$ -stable factors  $E_1, \dots, E_k$  of multiplicities  $a_1, \dots, a_k$ , and  $E = a_1 E_1 \oplus \cdots \oplus a_k E_k$  is the  $\mu$ -polystable representing  $[E'] \in \mathcal{M}_{\text{ss}}^{\mathbf{d}}(\mu)(\mathbb{C})$ .

Here is a useful expression for  $F_Q^{\mathbf{d}}(\mu)$  in (7.54) at a  $\mu$ -polystable  $E$ :

**PROPOSITION 7.28.** *Let  $Q$  be a quiver,  $(\mu, \mathbb{R}, \leq)$  a slope stability condition on  $\text{mod-}\mathbb{C}Q$ , and  $E = a_1 E_1 \oplus \cdots \oplus a_k E_k$  a  $\mu$ -polystable representing a  $\mathbb{C}$ -point  $[E]$  in  $\mathcal{M}_{\text{ss}}^{\mathbf{d}}(\mu)$  for  $\mathbf{d} \in C(\text{mod-}\mathbb{C}Q)$ , where  $E_1, \dots, E_k$  are pairwise nonisomorphic  $\mu$ -stables in  $\text{mod-}\mathbb{C}Q$  with  $\mu([E_1]) = \cdots = \mu([E_k])$  and  $a_1, \dots, a_k \geq 1$ .*

*Define the **Ext quiver**  $Q_E$  of  $E$  to have vertices  $\{1, 2, \dots, k\}$  and  $d_{ij} = \dim \text{Ext}^1(E_i, E_j)$  edges  $i \rightarrow j$  for all  $i, j = 1, \dots, k$ , and define a dimension vector  $\mathbf{a}$  in  $C(\text{mod-}\mathbb{C}Q_E)$  by  $\mathbf{a}(i) = a_i$  for  $i = 1, \dots, k$ . For  $i = 1, \dots, k$ , define  $\hat{E}_i \in \text{mod-}\mathbb{C}Q_E$  to have vector spaces  $X_v = \mathbb{C}$  for vertex  $v = i$  and  $X_v = 0$  for vertices  $v \neq i$  in  $Q_E$ , and linear maps  $\rho_e = 0$  for all edges  $e$  in  $Q_E$ . Set  $\hat{E} = a_1 \hat{E}_1 \oplus \cdots \oplus a_k \hat{E}_k$  in  $\text{mod-}\mathbb{C}Q_E$ . Then for  $F_Q^{\mathbf{d}}(\mu)$  as in (7.54), we have*

$$F_Q^{\mathbf{d}}(\mu)([E]) = F_{Q_E}^{\mathbf{a}}(0)([\hat{E}]) = \hat{DT}_{Q_E}^{\mathbf{a}}(0). \quad (7.55)$$

**PROOF.** Write  $(\text{mod-}\mathbb{C}Q_E)_{\hat{E}_1, \dots, \hat{E}_k}$  for the full subcategory of  $F$  in  $\text{mod-}\mathbb{C}Q_E$  generated by  $\hat{E}_1, \dots, \hat{E}_k$  by repeated extensions. Then  $(X, \rho)$  in  $\text{mod-}\mathbb{C}Q_E$  lies in  $(\text{mod-}\mathbb{C}Q_E)_{\hat{E}_1, \dots, \hat{E}_k}$  if and only if it is *nilpotent*, that is,  $\rho(\mathbb{C}Q_{E(n)}) = 0$  for some  $n \geq 0$ , where the ideal  $\mathbb{C}Q_{E(n)}$  of paths of length at least  $n$  in  $\mathbb{C}Q_E$  is as in Definition 7.1. Similarly, write  $(\text{mod-}\mathbb{C}Q)_{E_1, \dots, E_k}$  for the full subcategory of objects in  $\text{mod-}\mathbb{C}Q$  generated by  $E_1, \dots, E_k$  by repeated extensions. Both are  $\mathbb{C}$ -linear abelian subcategories.

In  $\text{mod-}\mathbb{C}Q_E$  we have  $\text{Hom}(\hat{E}_i, \hat{E}_j) = \mathbb{C}$  for  $i = j$  and  $\text{Hom}(\hat{E}_i, \hat{E}_j) = 0$  for  $i \neq j$ , and  $\text{Ext}^1(\hat{E}_i, \hat{E}_j) \cong \mathbb{C}^{d_{ij}}$  for all  $i, j$ . In  $\text{mod-}\mathbb{C}Q$  we have  $\text{Hom}(E_i, E_j) = \mathbb{C}$  for  $i = j$  and  $\text{Hom}(E_i, E_j) = 0$  for  $i \neq j$ , and  $\text{Ext}^1(E_i, E_j) \cong \mathbb{C}^{d_{ij}}$  for all  $i, j$ . Choose isomorphisms  $\text{Ext}^1(\hat{E}_i, \hat{E}_j) \cong \text{Ext}^1(E_i, E_j)$  for all  $i, j$ . It is then easy to construct an equivalence of  $\mathbb{C}$ -linear abelian categories

$$G : (\text{mod-}\mathbb{C}Q_E)_{\hat{E}_1, \dots, \hat{E}_k} \longrightarrow (\text{mod-}\mathbb{C}Q)_{E_1, \dots, E_k} \quad (7.56)$$

using linear algebra, such that  $G(\hat{E}_i) = E_i$  for  $i = 1, \dots, k$ , and  $G$  induces the chosen isomorphisms  $\text{Ext}^1(\hat{E}_i, \hat{E}_j) \rightarrow \text{Ext}^1(E_i, E_j)$ .

Write  $(\mathfrak{M}_{Q_E})_{\hat{E}_1, \dots, \hat{E}_k}, (\mathfrak{M}_Q)_{E_1, \dots, E_k}$  for the locally closed  $\mathbb{C}$ -substacks of objects in  $(\text{mod-}\mathbb{C}Q_E)_{\hat{E}_1, \dots, \hat{E}_k}, (\text{mod-}\mathbb{C}Q)_{E_1, \dots, E_k}$  in the moduli stacks  $\mathfrak{M}_{Q_E}, \mathfrak{M}_Q$  of  $\text{mod-}\mathbb{C}Q_E, \text{mod-}\mathbb{C}Q$ . Then  $G$  induces a 1-isomorphism of Artin  $\mathbb{C}$ -stacks

$$\dot{G} : (\mathfrak{M}_{Q_E})_{\hat{E}_1, \dots, \hat{E}_k} \longrightarrow (\mathfrak{M}_Q)_{E_1, \dots, E_k}.$$

As  $G$  identifies  $\text{Hom}(\hat{E}_i, \hat{E}_j), \text{Ext}^1(\hat{E}_i, \hat{E}_j)$  with  $\text{Hom}(E_i, E_j), \text{Ext}^1(E_i, E_j)$ , it follows that  $G$  takes the restriction to  $(\text{mod-}\mathbb{C}Q_E)_{\hat{E}_1, \dots, \hat{E}_k}$  of the Euler form  $\hat{\chi}_{Q_E}$  on  $\text{mod-}\mathbb{C}Q_E$  to the restriction to  $(\text{mod-}\mathbb{C}Q)_{E_1, \dots, E_k}$  of the Euler form  $\hat{\chi}_Q$  on  $\text{mod-}\mathbb{C}Q$ . By (7.2)  $\mathfrak{M}_Q^d$  is smooth of dimension  $-\hat{\chi}_Q(d, d)$ , so the Behrend function  $\nu_{\mathfrak{M}_Q^d} \equiv (-1)^{-\hat{\chi}_Q(d, d)}$  by Corollary 4.5, and similarly  $\nu_{\mathfrak{M}_{Q_E}^a} \equiv (-1)^{-\hat{\chi}_{Q_E}(a, a)}$ . As  $\dot{G}$  takes  $\hat{\chi}_{Q_E}$  to  $\hat{\chi}_Q$ , it follows that

$$\dot{G}_*(\nu_{\mathfrak{M}_{Q_E}}|_{(\mathfrak{M}_{Q_E})_{\hat{E}_1, \dots, \hat{E}_k}}) = \nu_{\mathfrak{M}_Q}|_{(\mathfrak{M}_Q)_{E_1, \dots, E_k}}. \quad (7.57)$$

Since all objects in  $(\text{mod-}\mathbb{C}Q_E)_{\hat{E}_1, \dots, \hat{E}_k}$  are 0-semistable, and all objects in  $(\text{mod-}\mathbb{C}Q)_{E_1, \dots, E_k}$  are  $\mu$ -semistable, and  $G$  is a 1-isomorphism, we see that

$$\dot{G}_*(\bar{\delta}_{\text{ss } Q_E}^{a/m}(0)|_{(\mathfrak{M}_{Q_E})_{\hat{E}_1, \dots, \hat{E}_k}}) = \bar{\delta}_{\text{ss } Q}^{d/m}(\mu)|_{(\mathfrak{M}_Q)_{E_1, \dots, E_k}}$$

for  $m \geq 1$  with  $m \mid d$ . So from (3.4) we deduce that

$$\dot{G}_*(\bar{\epsilon}_{Q_E}^{a/m}(0)|_{(\mathfrak{M}_{Q_E})_{\hat{E}_1, \dots, \hat{E}_k}}) = \bar{\epsilon}_Q^{d/m}(\mu)|_{(\mathfrak{M}_Q)_{E_1, \dots, E_k}}. \quad (7.58)$$

Equations (7.57) and (7.58) imply that

$$\dot{G}_*(F_{Q_E}^a(0)|_{(\mathfrak{M}_{Q_E})_{\hat{E}_1, \dots, \hat{E}_k}(\mathbb{C})}) = F_Q^d(\mu)|_{(\mathfrak{M}_Q)_{E_1, \dots, E_k}(\mathbb{C})},$$

since  $\dot{G}_*$  identifies (7.54) for  $\text{mod-}\mathbb{C}Q_E$  on  $(\mathfrak{M}_{Q_E})_{\hat{E}_1, \dots, \hat{E}_k}$  term-by-term with (7.54) for  $\text{mod-}\mathbb{C}Q$  on  $(\mathfrak{M}_Q)_{E_1, \dots, E_k}$ . As  $\dot{G}([\hat{E}]) = [E]$ , this implies the first equality of (7.55).

Now consider the  $\mathbb{G}_m$ -action on  $\text{mod-}\mathbb{C}Q_E$  acting by  $\lambda : (X, \rho) \mapsto (X, \lambda\rho)$  for  $\lambda \in \mathbb{G}_m$  and  $(X, \rho) \in \text{mod-}\mathbb{C}Q_E$ . This induces  $\mathbb{G}_m$ -actions on the moduli stack  $\mathfrak{M}_{Q_E}^a$  and the coarse moduli space  $\mathcal{M}_{\text{ss } Q_E}^a(0)$ . By (7.2) we have  $\mathfrak{M}_{Q_E}^a \cong [V/H]$  where  $V$  is a vector space and  $H = \prod_{i=1}^k \text{GL}(a_i, \mathbb{C})$ , and the  $\mathbb{G}_m$ -action on  $\mathfrak{M}_{Q_E}^a$  is induced by multiplication by  $\mathbb{G}_m$  in  $V$ . Let  $A_E$  be the  $\mathbb{C}$ -algebra of  $H$ -invariant polynomials on  $V$ . Then  $\mathcal{M}_{\text{ss } Q_E}^a(0) = \text{Spec } A_E$  by GIT.

This  $A_E$  is graded by homogeneous polynomials of degree  $d = 0, 1, \dots$  on  $V$ , and  $\mathbb{G}_m$  acts on homogeneous polynomial  $f$  of degree  $d$  by  $\lambda : f \mapsto \lambda^d f$ . Thus there is exactly one point in  $\mathcal{M}_{\text{ss } Q_E}^a(0)$  fixed by the  $\mathbb{G}_m$ -action, the ideal of polynomials in  $A_E$  which vanish at  $0 \in V$ . Since  $0 \in V$  corresponds to  $[\hat{E}]$  in  $\mathfrak{M}_{Q_E}^a \cong [V/H]$ , we see that there is a  $\mathbb{G}_m$ -action on  $\mathcal{M}_{\text{ss } Q_E}^a(0)$  with unique fixed point  $[\hat{E}]$ . By (7.54) we have  $\hat{DT}_{Q_E}^a(0) = \chi(\mathcal{M}_{\text{ss } Q_E}^a(0), F_{Q_E}^a(0))$ . The  $\mathbb{G}_m$ -action on  $\mathcal{M}_{\text{ss } Q_E}^a(0)$  preserves  $F_{Q_E}^a(0)$ . The second equality of (7.55) follows by the usual torus localization argument, as all  $\mathbb{G}_m$ -orbits other than  $[\hat{E}]$  are copies of  $\mathbb{G}_m$ , and contribute 0 to the weighted Euler characteristic.  $\square$

Here is our integrality result, which proves Conjecture 7.16 for  $\text{mod-}\mathbb{C}Q$ . We give two different proofs of it. The first proof works by first proving the analogue of Conjecture 6.13 in our quiver context, and so is evidence for Conjecture 6.13. It

relies on the integrality of the invariants  $\hat{DT}_{Q_m}^d(0)$  in equation (7.53) of Example 7.27, which we proved using Reineke [90, Th. 5.9].

Reineke [90] proves an integrality conjecture of Kontsevich and Soibelman [63, Conj. 1]. The authors believe [63, Conj. 1] concerns integrality of *transformation laws*, rather than of invariants themselves. That is, if  $\mu, \tilde{\mu}$  are generic stability conditions on  $\text{mod-}\mathbb{C}Q/I$ , then translated into our framework [63, Conj. 1] should imply that  $\hat{DT}_{Q,I}^d(\tilde{\mu}) \in \mathbb{Z}$  for all  $d$  if and only if  $\hat{DT}_{Q,I}^{d'}(\mu) \in \mathbb{Z}$  for all  $d'$ , where  $\hat{DT}_{Q,I}^d(\tilde{\mu}), \hat{DT}_{Q,I}^{d'}(\mu)$  are related using (7.21)–(7.23).

Despite this, in our second proof of Theorem 7.29 we prove integrality of the  $\hat{DT}_Q^d(\mu)$  using Reineke [90] in a more direct way than the first proof. We include this second proof to try to clarify the relationship between our work and Reineke’s. Also, the second proof implicitly expresses the  $\hat{DT}_Q^d(\mu)$  in terms of Euler characteristics  $\chi(\mathcal{M}_{\text{st } Q}^{d'}(\mu))$  of  $\mu$ -stable moduli schemes  $\mathcal{M}_{\text{st } Q}^{d'}(\mu)$ , such that integrality of the  $\hat{DT}_Q^d(\mu)$  follows from integrality of the  $\chi(\mathcal{M}_{\text{st } Q}^{d'}(\mu))$ . Since in our set up we never use stable moduli schemes, and one of our major themes is counting strictly semistables correctly, to involve invariants counting only stables and ignoring strictly semistables seems curious.

**THEOREM 7.29.** *Let  $Q$  be a quiver, and write  $\hat{\chi}_Q : K(\text{mod-}\mathbb{C}Q) \times K(\text{mod-}\mathbb{C}Q) \rightarrow \mathbb{Z}$  for the Euler form of  $Q$  and  $\bar{\chi}_Q : K(\text{mod-}\mathbb{C}Q) \times K(\text{mod-}\mathbb{C}Q) \rightarrow \mathbb{Z}$  for its antisymmetrization, as in (7.4)–(7.6). Let  $(\mu, \mathbb{R}, \leq)$  be a **generic** slope stability condition on  $\text{mod-}\mathbb{C}Q$ , that is, for all  $d, e \in C(\text{mod-}\mathbb{C}Q)$  with  $\mu(d) = \mu(e)$  we have  $\bar{\chi}_Q(d, e) = 0$ . Then for all  $d \in C(\text{mod-}\mathbb{C}Q)$  the constructible function  $F_Q^d(\mu)$  on  $\mathcal{M}_{\text{ss}}^d(\mu)$  in (7.54) is  $\mathbb{Z}$ -valued, so that  $\hat{DT}_Q^d(\mu) \in \mathbb{Z}$ .*

**FIRST PROOF.** For  $Q, (\mu, \mathbb{R}, \leq), d$  as in the theorem, let a  $\mathbb{C}$ -point in  $\mathcal{M}_{\text{ss}}^d(\mu)$  be represented by a  $\mu$ -polystable  $E = a_1 E_1 \oplus \cdots \oplus a_k E_k$ , where  $E_1, \dots, E_k$  are pairwise nonisomorphic  $\mu$ -stables in  $\text{mod-}\mathbb{C}Q$  with  $\mu([E_1]) = \cdots = \mu([E_k])$ , and  $a_1, \dots, a_k \geq 1$ . Use the notation of Proposition 7.28. As  $(\mu, \mathbb{R}, \leq)$  is generic and  $\mu([E_i]) = \mu([E_j])$  we have  $\bar{\chi}_Q([E_i], [E_j]) = 0$  for all  $i, j$ . But  $G$  in (7.56) takes  $\hat{\chi}_{Q_E}$  to  $\hat{\chi}_Q$  and  $\bar{\chi}_{Q_E}$  to  $\bar{\chi}_Q$ , so  $\bar{\chi}_{Q_E}([\hat{E}_i], [\hat{E}_j]) = 0$  for all  $i, j$ . Since the  $[\hat{E}_i]$  for  $i = 1, \dots, k$  span  $K(\text{mod-}\mathbb{C}Q_E)$ , this implies that  $\bar{\chi}_{Q_E} \equiv 0$ . We must show that  $F_Q^d(\mu)([E]) \in \mathbb{Z}$ , which by Proposition 7.28 is equivalent to  $\hat{DT}_{Q_E}^a(0) \in \mathbb{Z}$ .

Thus, replacing  $Q_E, a$  by  $Q, d$ , it is enough to show that for all quivers  $Q$  with  $\bar{\chi}_Q \equiv 0$  and all  $d \in C(\text{mod-}\mathbb{C}Q)$  we have  $\hat{DT}_Q^d(0) \in \mathbb{Z}$ . Note that as in Corollary 7.18,  $\bar{\chi}_Q \equiv 0$  implies that  $\hat{DT}_Q^d(\mu)$  is independent of the choice of stability condition  $(\mu, \mathbb{R}, \leq)$ , so  $\hat{DT}_Q^d(0) \in \mathbb{Z}$  is equivalent to  $\hat{DT}_Q^d(\mu) \in \mathbb{Z}$  for any  $(\mu, \mathbb{R}, \leq)$  on  $\text{mod-}\mathbb{C}Q$ . Write  $|d|$  for the *total dimension*  $\sum_{v \in Q_0} d(v)$  of  $d$ . We will prove the theorem by induction on  $|d|$ .

Let  $N \geq 0$ . Suppose by induction that for all quivers  $Q$  with  $\bar{\chi}_Q \equiv 0$  and all  $d \in C(\text{mod-}\mathbb{C}Q)$  with  $|d| \leq N$  we have  $\hat{DT}_Q^d(0) \in \mathbb{Z}$ . (The first step  $N = 0$  is vacuous.) Let  $Q$  be a quiver with  $\bar{\chi}_Q \equiv 0$  and  $d \in C(\text{mod-}\mathbb{C}Q)$  with  $|d| = N + 1$ . We divide into two cases:

- (a)  $d(v) = N + 1$  for some  $v \in Q_0$ , and  $d(w) = 0$  for  $v \neq w \in Q_0$ ; and
- (b) there are  $v \neq w$  in  $Q_0$  with  $d(v), d(w) > 0$ .

In case (a), the vertices  $w$  in  $Q$  with  $w \neq v$ , and the edges joined to them make no difference to  $\hat{DT}_Q^d(0)$ , as in  $(X, \rho)$  with  $[(X, \rho)] = d$  in  $C(\text{mod-}\mathbb{C}Q)$  the vector

spaces  $X_w$  are zero for  $w \neq v$ . Thus  $\hat{DT}_Q^{\mathbf{d}}(0) = \hat{DT}_{Q_m}^{N+1}(0)$ , where  $m$  is the number of edges  $v \rightarrow v$  in  $Q$ , and  $Q_m$  is the quiver with one vertex  $v$  and  $m$  edges  $v \rightarrow v$ . Example 7.27 then shows that  $\hat{DT}_Q^{\mathbf{d}}(0) \in \mathbb{Z}$ , as we want.

In case (b), choose a stability condition  $(\mu, \mathbb{R}, \leq)$  on  $\text{mod-}\mathbb{C}Q$  with  $\mu(\delta_v) \neq \mu(\delta_w)$ . Then  $\hat{DT}_Q^{\mathbf{d}}(0) = \hat{DT}_Q^{\mathbf{d}}(\mu)$  by Corollary 7.18. So (7.54)–(7.55) give

$$\hat{DT}_Q^{\mathbf{d}}(0) = \hat{DT}_Q^{\mathbf{d}}(\mu) = \chi(\mathcal{M}_{\text{ss}}^{\mathbf{d}}(\mu), F_Q^{\mathbf{d}}(\mu)) = \int_{\substack{[E] \in \mathcal{M}_{\text{ss}}^{\mathbf{d}}(\mu): \\ E = a_1 E_1 \oplus \dots \oplus a_k E_k, \\ E \text{ } \mu\text{-polystable}}} \hat{DT}_{Q_E}^{\mathbf{a}}(0) d\chi. \quad (7.59)$$

Let  $E = a_1 E_1 \oplus \dots \oplus a_k E_k$  be as in (7.59). Then  $\sum_{i=1}^k a_i [E_i] = \mathbf{d}$ , so  $\sum_{i=1}^k a_i |[E_i]| = |\mathbf{d}| = N + 1$ . Suppose for a contradiction that  $|[E_i]| = 1$  for all  $i = 1, \dots, k$ . Then each  $E_i$  is 1-dimensional, and located at some vertex  $u \in Q_0$ , so  $[E_i] = \delta_u$  in  $C(\text{mod-}\mathbb{C}Q)$ , and  $\mu([E_i]) = \mu(u)$ . For each  $u \in Q_0$ , we have  $\sum_{i: [E_i] = \delta_u} a_i = \mathbf{d}(u)$ . As  $\mathbf{d}(v), \mathbf{d}(w) > 0$ , this implies there exist  $i, j = 1, \dots, k$  with  $[E_i] = \delta_v$  and  $[E_j] = \delta_w$ . But then  $\mu([E_i]) = \mu(\delta_v) \neq \mu(\delta_w) = \mu([E_j])$ , which contradicts  $\mu([E_1]) = \dots = \mu([E_k])$  as  $E$  is  $\mu$ -polystable.

Therefore  $|[E_i]| \geq 1$  for all  $i = 1, \dots, k$ , and  $|[E_i]| > 1$  for some  $i$ . As  $\sum_{i=1}^k a_i |[E_i]| = N + 1$  we see that  $|\mathbf{a}| = \sum_{i=1}^k a_i \leq N$ , so  $\hat{DT}_{Q_E}^{\mathbf{a}}(0) \in \mathbb{Z}$  by the inductive hypothesis. As this holds for all  $E$  in (7.59),  $\hat{DT}_Q^{\mathbf{d}}(0)$  is the Euler characteristic integral of a  $\mathbb{Z}$ -valued constructible function, so  $\hat{DT}_Q^{\mathbf{d}}(0) \in \mathbb{Z}$ . This completes the inductive step, and the first proof of Theorem 7.29.  $\square$

SECOND PROOF OF THEOREM 7.29. Let  $Q, \bar{\chi}_Q, \mu$  be as in the theorem. Then by perturbing  $\mu$  slightly we can find a second slope stability condition  $(\tilde{\mu}, \mathbb{R}, \leq)$  on  $\text{mod-}\mathbb{C}Q$  such that  $\mu$  dominates  $\tilde{\mu}$  in the sense of Definition 3.12, and if  $\mathbf{d}, \mathbf{e} \in C(\text{mod-}\mathbb{C}Q)$  then  $\tilde{\mu}(\mathbf{d}) = \tilde{\mu}(\mathbf{e})$  if and only if  $\mathbf{d}, \mathbf{e}$  are proportional.

Let  $\mathbf{d} \in C(\text{mod-}\mathbb{C}Q)$  be primitive, and fix  $\mathbf{e} \in C(\text{mod-}\mathbb{C}Q)$  with  $\mathbf{e} \cdot \mathbf{d} = p > 0$ . Write  $N = \hat{\chi}_Q(\mathbf{d}, \mathbf{d})$ . Then we have

$$\begin{aligned} F^{\mathbf{d}, \mathbf{e}}(t) &:= 1 + \sum_{n \geq 1} \chi(\mathcal{M}_{\text{stf } Q}^{\mathbf{d}, \mathbf{e}}(\tilde{\mu}')) t^n = 1 + \sum_{n \geq 1} ((-1)^{n^2 p + nN} N \hat{DT}_Q^{n\mathbf{d}, \mathbf{e}}(\tilde{\mu}')) t^n \\ &= \exp \left[ - \sum_{n \geq 1} (-1)^{np} np \cdot \bar{DT}_Q^{n\mathbf{d}}(\tilde{\mu}) \cdot (-1)^{np + nN} t^n \right] \\ &= \exp \left[ - \sum_{n \geq 1} np \sum_{m \geq 1, m|n} \frac{1}{m^2} \hat{DT}_Q^{n\mathbf{d}/m}(\tilde{\mu}) ((-1)^N t)^n \right] \quad (7.60) \\ &= \exp \left[ - \sum_{i \geq 1} ip \cdot \hat{DT}_Q^{i\mathbf{d}}(\tilde{\mu}) \sum_{m \geq 1} \frac{((-1)^N t)^{im}}{m} \right] \\ &= \prod_{i \geq 1} (1 - ((-1)^N t)^i)^{ip \cdot \bar{DT}_Q^{i\mathbf{d}}(\tilde{\mu})}, \end{aligned}$$

in formal power series in  $t$ , where the first step uses (7.28), the second step is Corollary 7.24 with  $W \equiv 0$ ,  $c = \tilde{\mu}(\mathbf{d})$  and  $(-1)^{p+N} t$  in place of  $q^{\mathbf{d}}$ , the third substitutes in (7.22), the fourth sets  $i = n/m$ , and the fifth uses

$$1 - x = \exp[\log(1 - x)] = \exp \left[ - \sum_{m \geq 1} \frac{x^m}{m} \right].$$



Set  $S^{\mathbf{d}}(t) = F^{\mathbf{d}, \mathbf{e}}(t)^{1/p}$ , so that in the notation of Reineke [90, p. 10] we have  $S^{\mathbf{d}}_\mu(t) = S^{\mathbf{d}}(t)^N$ . Then (7.60) gives

$$S^{\mathbf{d}}(t) = \prod_{i \geq 1} (1 - ((-1)^N t)^i)^{-ia_i}, \quad \text{with } a_i = -\hat{D}T_Q^{i\mathbf{d}}(\tilde{\mu}). \quad (7.61)$$

Note that  $S^{\mathbf{d}}(t)$  is independent of the choice of  $\mathbf{e}$ , which also follows from [90, Th. 4.2]. By Reineke [90, Cor. 4.3],  $S^{\mathbf{d}}(t)$  satisfies the functional equation

$$S^{\mathbf{d}}(t)^N = \prod_{i \geq 1} (1 - t^i S^{\mathbf{d}}(t)^{iN})^{-iN \cdot \chi(\mathcal{M}_{\text{st}Q}^{i\mathbf{d}}(\tilde{\mu}))},$$

where  $\chi(\mathcal{M}_{\text{st}Q}^{i\mathbf{d}}(\tilde{\mu}))$  is the Euler characteristic of the moduli scheme  $\mathcal{M}_{\text{st}Q}^{i\mathbf{d}}(\tilde{\mu})$  of  $\tilde{\mu}$ -stable  $E$  in  $\text{mod-}\mathbb{C}Q$  with  $\dim E = i\mathbf{d}$ . If  $N \neq 0$ , taking  $N^{\text{th}}$  roots gives

$$S^{\mathbf{d}}(t) = \prod_{i \geq 1} (1 - t^i S^{\mathbf{d}}(t)^{iN})^{-i \cdot \chi(\mathcal{M}_{\text{st}Q}^{i\mathbf{d}}(\tilde{\mu}))}, \quad (7.62)$$

so that

$$S^{\mathbf{d}}(t) = \prod_{i \geq 1} (1 - (t S^{\mathbf{d}}(t)^N)^i)^{ib_i}, \quad \text{where } b_i = -\chi(\mathcal{M}_{\text{st}Q}^{i\mathbf{d}}(\tilde{\mu})). \quad (7.63)$$

When  $N = 0$ , equations (7.62)–(7.63) follow directly from [90, Th. 6.2].

Now Reineke [90, Th. 5.9] shows that if a formal power series  $S^{\mathbf{d}}(t)$  satisfies equations (7.61) and (7.63) for  $N \in \mathbb{Z}$  and  $a_i, b_i \in \mathbb{Q}$ , then  $a_i \in \mathbb{Z}$  for all  $i \geq 1$  if and only if  $b_i \in \mathbb{Z}$  for all  $i \geq 1$ . In our case  $b_i \in \mathbb{Z}$  is immediate, so  $a_i \in \mathbb{Z}$ , and thus  $\hat{D}T_Q^{i\mathbf{d}}(\tilde{\mu}) \in \mathbb{Z}$ . As this holds for all primitive  $\mathbf{d} \in C(\text{mod-}\mathbb{C}Q)$ , we have  $\hat{D}T_Q^{\mathbf{d}}(\tilde{\mu}) \in \mathbb{Z}$  for all  $\mathbf{d} \in C(\text{mod-}\mathbb{C}Q)$ . Also, comparing (7.61) and (7.63) shows that we could compute the  $\hat{D}T_Q^{i\mathbf{d}}(\tilde{\mu})$  from the  $\chi(\mathcal{M}_{\text{st}Q}^{j\mathbf{d}}(\tilde{\mu}))$  for  $j = 1, \dots, i$ .

Theorem 7.17 now writes  $\bar{D}T_Q^{\mathbf{d}}(\mu)$  in terms of the  $\bar{D}T_Q^{\mathbf{e}}(\tilde{\mu})$ , in the form

$$\bar{D}T_Q^{\mathbf{d}}(\mu) = \bar{D}T_Q^{\mathbf{d}}(\tilde{\mu}) + \text{higher order terms.}$$

Each higher order term involves a finite set  $I$  with  $|I| > 1$ , a splitting  $\mathbf{d} = \sum_{i \in I} \kappa(i)$  for  $\kappa(i) \in C(\text{mod-}\mathbb{C}Q)$ , a combinatorial coefficient  $V(I, \Gamma, \kappa; \tilde{\mu}, \mu) \in \mathbb{Q}$ , a product of  $|I| - 1$  terms  $\bar{\chi}(\kappa(i), \kappa(j))$ , and the product of the  $\bar{D}T_Q^{ka(i)}(\tilde{\mu})$ . Now as  $\mu$  dominates  $\tilde{\mu}$  it follows that  $V(I, \Gamma, \kappa; \tilde{\mu}, \mu) = 0$  unless  $\mu(\kappa(i)) = \mu(\mathbf{d})$  for all  $i \in I$ , as in [54]. But then  $\mu$  generic implies that  $\bar{\chi}(\kappa(i), \kappa(j)) = 0$ . Hence all the higher order terms are zero, and  $\bar{D}T_Q^{\mathbf{d}}(\mu) = \bar{D}T_Q^{\mathbf{d}}(\tilde{\mu})$  for all  $\mathbf{d} \in C(\text{mod-}\mathbb{C}Q)$ . Therefore  $\hat{D}T_Q^{\mathbf{d}}(\mu) = \hat{D}T_Q^{\mathbf{d}}(\tilde{\mu})$  for all  $\mathbf{d} \in C(\text{mod-}\mathbb{C}Q)$ , so  $\hat{D}T_Q^{\mathbf{d}}(\mu) \in \mathbb{Z}$ , as we have to prove.  $\square$

As for Question 6.14, we can ask:

QUESTION 7.30. *In the situation of Theorem 7.29, does there exist a natural perverse sheaf  $\mathcal{Q}$  on  $\mathcal{M}_{\text{ss}}^{\mathbf{d}}(\mu)$  with  $\chi_{\mathcal{M}_{\text{ss}}^{\mathbf{d}}(\mu)}(\mathcal{Q}) \equiv F_Q^{\mathbf{d}}(\mu)$ ?*

One can ask the same question about Saito's mixed Hodge modules [92]. These questions should be amenable to study in explicit examples.

REMARK 7.31. The proof of Theorem 7.29 also holds without change for arbitrary generic stability conditions  $(\tau, T, \leq)$  on  $\text{mod-}\mathbb{C}Q$  in the sense of §3.2 with  $K(\text{mod-}\mathbb{C}Q) = \mathbb{Z}^{Q_0}$ , not just for slope stability conditions  $(\mu, \mathbb{R}, \leq)$ .

## CHAPTER 8

### The proof of Theorem 5.3

Let  $X$  be a projective Calabi–Yau  $m$ -fold over an algebraically closed field  $\mathbb{K}$  with a very ample line bundle  $\mathcal{O}_X(1)$ . Our definition of Calabi–Yau  $m$ -fold requires that  $X$  should be smooth, the canonical bundle  $K_X$  should be trivial, and that  $H^i(\mathcal{O}_X) = 0$  for  $0 < i < m$ . Let  $\mathfrak{M}$  and  $\mathfrak{Vect}$  be the moduli stacks of coherent sheaves and algebraic vector bundles on  $X$ , respectively. Then  $\mathfrak{M}, \mathfrak{Vect}$  are both Artin  $\mathbb{K}$ -stacks, locally of finite type. This section proves Theorem 5.3, which says that  $\mathfrak{M}$  is locally isomorphic to  $\mathfrak{Vect}$ , in the Zariski topology.

Recall the following definition of *Seidel–Thomas twist*, [96, Ex. 3.3]:

DEFINITION 8.1. For each  $n \in \mathbb{Z}$ , the *Seidel–Thomas twist*  $T_{\mathcal{O}_X(-n)}$  by  $\mathcal{O}_X(-n)$  is the Fourier–Mukai transform from  $D(X)$  to  $D(X)$  with kernel

$$K = \text{cone}(\mathcal{O}_X(n) \boxtimes \mathcal{O}_X(-n) \longrightarrow \mathcal{O}_\Delta).$$

A good book on derived categories and Fourier–Mukai transforms is Huybrechts [42]. Since  $X$  is Calabi–Yau, which includes the assumption that  $H^i(\mathcal{O}_X) = 0$  for  $0 < i < m$ , we see that  $\text{Hom}_{D(X)}^i(\mathcal{O}_X(-n), \mathcal{O}_X(-n))$  is  $\mathbb{C}$  for  $i = 0, m$  and zero otherwise, so  $\mathcal{O}_X(-n)$  is a *spherical object* in the sense of [96, Def. 1.1], and by [96, Th. 1.2] the Seidel–Thomas twist  $T_{\mathcal{O}_X(-n)}$  is an autoequivalence of  $D(X)$ . Define  $T_n = T_{\mathcal{O}_X(-n)}[-1]$ , the composition of  $T_{\mathcal{O}_X(-n)}$  and the shift  $[-1]$ . Then  $T_n$  is also an autoequivalence of  $D(X)$ .

The functors  $T_n$  do not preserve the subcategory  $\text{coh}(X)$  in  $D(X)$ , that is, in general they take sheaves to complexes of sheaves. However, given any bounded family of sheaves  $E_U$  on  $X$  we can choose  $n \gg 0$  such that  $T_n$  takes the sheaves in  $E_U$  to sheaves, rather than complexes.

LEMMA 8.2. *Let  $U$  be a finite type  $\mathbb{K}$ -scheme and  $E_U$  a coherent sheaf on  $X \times U$  flat over  $U$ , that is, a  $U$ -family of coherent sheaves on  $X$ . Then for  $n \gg 0$ ,  $F_U = T_n(E_U)$  is also a  $U$ -family of coherent sheaves on  $X$ .*

PROOF. Since  $U$  is of finite type, the family of coherent sheaves  $E_U$  is bounded, so there exists  $n \gg 0$  such that  $H^i(E_u(n)) = 0$  vanishes for all  $u \in U$  and  $i > 0$ , and  $E_u(n)$  is globally generated. Then we have

$$\begin{aligned} T_n(E_u) &= \text{cone}\left(\bigoplus_{i \geq 0} \text{Ext}^i(\mathcal{O}_X(-n), E_u) \otimes \mathcal{O}_X(-n)[-i] \longrightarrow E_u\right)[-1] \\ &= \text{cone}\left(\text{Hom}(\mathcal{O}_X(-n), E_u) \otimes \mathcal{O}_X(-n) \longrightarrow E_u\right)[-1] \\ &= \text{Ker}\left(\text{Hom}(\mathcal{O}_X(-n), E_u) \otimes \mathcal{O}_X(-n) \longrightarrow E_u\right), \end{aligned} \tag{8.1}$$

where the first line is from the definition, the second as  $H^i(E_u(n)) = 0$  for  $i > 0$ , and the third as  $\text{Hom}(\mathcal{O}_X(-n), E_u) \otimes \mathcal{O}_X(-n) \rightarrow E_u$  is surjective in  $\text{coh}(X)$  as  $E_u(n)$  is globally generated. Thus  $F_u = T_n(E_u)$  is a sheaf, rather than a complex,

for all  $u \in U$ . In sheaves on  $X \times U$ , we have an exact sequence:

$$0 \longrightarrow F_U \longrightarrow p_{X,*}^*(E_U(n)) \otimes p_X^*(\mathcal{O}_X(-n)) \longrightarrow E_U \longrightarrow 0,$$

and  $F_U$  is flat over  $U$  as  $E_U$  and  $p_{X,*}^*(E_U(n)) \otimes p_X^*(\mathcal{O}_X(-n))$  are.  $\square$

We recall the notion of *homological dimension* of a sheaf, as in Hartshorne [40, p. 238] and Huybrechts and Lehn [44, p. 4]:

DEFINITION 8.3. For a nonzero sheaf  $E$  in  $\text{coh}(X)$ , the *homological dimension*  $\text{hd}(E)$  is the smallest  $n \geq 0$  for which there exists an exact sequence in  $\text{coh}(X)$

$$0 \rightarrow V_n \rightarrow V_{n-1} \rightarrow \cdots \rightarrow V_0 \rightarrow E \rightarrow 0,$$

with  $V_0, \dots, V_n$  vector bundles (locally free sheaves). Clearly  $\text{hd}(E) = 0$  if and only if  $E$  is a vector bundle. Equivalently,  $\text{hd}(E)$  is the largest  $n \geq 0$  such that  $\text{Ext}^n(E, \mathcal{O}_x) = 0$  for some  $x \in X$ , where  $\mathcal{O}_x$  is the skyscraper sheaf at  $x$ . Since  $X$  is smooth of dimension  $m$  we have  $\text{hd}(E) \leq m$  for all  $E \in \text{coh}(X)$ .

The operators  $T_n$  above decrease  $\text{hd}(E)$  by 1 when  $n \gg 0$ , unless  $\text{hd}(E) = 0$  when they fix  $\text{hd}(E)$ .

LEMMA 8.4. Let  $U, E_U$  and  $n \gg 0$  be as in Lemma 8.2. Then for all  $u \in U$  we have  $\text{hd}(T_n(E_u)) = \max(\text{hd}(E_u) - 1, 0)$ .

PROOF. As in (8.1) we have an exact sequence

$$0 \longrightarrow T_n(E_u) \rightarrow H^0(E_u(n)) \otimes \mathcal{O}_X(-n) \longrightarrow E_u \longrightarrow 0.$$

Applying  $\text{Ext}(-, \mathcal{O}_x)$  to this sequence for  $x \in X$  gives a long exact sequence

$$\begin{aligned} \cdots \longrightarrow H^0(E_u(n)) \otimes \text{Ext}^i(\mathcal{O}_X(-n), \mathcal{O}_x) &\longrightarrow \text{Ext}^i(T_n(E_u), \mathcal{O}_x) \\ &\longrightarrow \text{Ext}^{i+1}(E_u, \mathcal{O}_x) \longrightarrow H^0(E_u(n)) \otimes \text{Ext}^{i+1}(\mathcal{O}_X(-n), \mathcal{O}_x) \longrightarrow \cdots \end{aligned}$$

Thus  $\text{Ext}^i(T_n(E_u), \mathcal{O}_x) \cong \text{Ext}^{i+1}(E_u, \mathcal{O}_x)$  for  $i > 0$ , as  $\text{Ext}^i(\mathcal{O}_X(-n), \mathcal{O}_x) = 0$  for  $i > 0$ . Since  $\text{hd}(E_u)$  is the largest  $n \geq 0$  such that  $\text{Ext}^n(E, \mathcal{O}_x) = 0$  for some  $x \in X$ , this implies that  $\text{hd}(T_n(E_u)) = \text{hd}(E_u) - 1$  if  $\text{hd}(E_u) > 0$ , and  $\text{hd}(T_n(E_u)) = 0$  if  $\text{hd}(E_u) = 0$ .  $\square$

COROLLARY 8.5. Let  $U$  be a finite type  $\mathbb{K}$ -scheme and  $E_U$  a  $U$ -family of coherent sheaves on  $X$ . Then there exist  $n_1, \dots, n_m \gg 0$  such that  $T_{n_m} \circ T_{n_{m-1}} \circ \cdots \circ T_{n_1}(E_U)$  is a  $U$ -family of vector bundles on  $X$ .

PROOF. Apply Lemma 8.2  $m$  times to  $E_U$ , where by induction on  $i = 1, \dots, m$ ,  $n_i$  is the  $n$  in Lemma 8.2 applied to the  $U$ -family of sheaves  $T_{n_{i-1}} \circ \cdots \circ T_{n_1}(E_U)$ . For each  $u \in U$  we have  $\text{hd}(E_u) \leq m$ , since  $X$  is smooth of dimension  $m$ . So Lemma 8.4 implies that  $\text{hd}(T_{n_1}(E_u)) \leq m-1$ , and by induction  $\text{hd}(T_{n_i} \circ \cdots \circ T_{n_1}(E_u)) \leq m-i$  for  $i = 1, \dots, m$ . Hence  $\text{hd}(T_{n_m} \circ \cdots \circ T_{n_1}(E_u)) = 0$ , so that  $T_{n_m} \circ \cdots \circ T_{n_1}(E_u)$  is a vector bundle on  $X$  for all  $u \in U$ .  $\square$

We can now prove Theorem 5.3. Let  $\mathfrak{U}$  be an open, finite type substack of  $\mathfrak{M}$ . Then  $\mathfrak{U}$  admits an atlas  $\pi : U \rightarrow \mathfrak{U}$ , with  $U$  a finite type  $K$ -scheme. Let  $E_U$  be the corresponding  $U$ -family of coherent sheaves on  $X$ . Then  $E_U$  is a versal family of coherent sheaves on  $X$  which parametrizes all the sheaves represented by points in  $\mathfrak{U}$ , up to isomorphism. Let  $n_1, \dots, n_m$  be as in Corollary 8.5 for these  $U, E_U$ . Then by Lemma 8.2 applied  $m$  times and Corollary 8.5,  $F_U = T_{n_m} \circ \cdots \circ T_{n_1}(E_U)$  is a  $U$ -family of vector bundles. Thus  $F_U$  corresponds to a 1-morphism  $\pi' : U \rightarrow \mathfrak{Vect}$ .

Since Fourier–Mukai transforms and shifts preserve moduli families of (complexes of) sheaves, and  $E_U$  is a versal family,  $F_U$  is a versal family. Hence  $\pi'$  is an atlas for an open substack  $\mathfrak{V}$  in  $\mathfrak{Vect}$ . If  $S$  is a  $\mathbb{K}$ -scheme then  $\mathrm{Hom}(S, \mathfrak{U})$  is the category of  $S$ -families  $E_S$  of coherent sheaves on  $X$  which lie in  $\mathfrak{U}$  for all  $s \in S$ , and  $\mathrm{Hom}(S, \mathfrak{V})$  the category of  $S$ -families  $F_S$  of vector bundles on  $X$  which lie in  $\mathfrak{V}$  for all  $s \in S$ . Then  $E_S \mapsto T_{n_m} \circ \cdots \circ T_{n_1}(E_S)$  defines an equivalence of categories  $\mathrm{Hom}(S, \mathfrak{U}) \rightarrow \mathrm{Hom}(S, \mathfrak{V})$ . Hence  $T_{n_m} \circ \cdots \circ T_{n_1}$  induces a 1-isomorphism  $\varphi : \mathfrak{U} \rightarrow \mathfrak{V}$ , proving the first part of Theorem 5.3. The second part follows by passing to coarse moduli spaces.



## The proofs of Theorems 5.4 and 5.5

To prove Theorem 5.4 we will need a local description of the complex analytic space  $\mathcal{Vect}_{\text{si}}(\mathbb{C})$  underlying the coarse moduli space  $\mathcal{Vect}_{\text{si}}$  of simple algebraic vector bundles on a projective Calabi–Yau 3-fold  $X$ , in terms of gauge theory on a complex vector bundle  $E \rightarrow X$ , and infinite-dimensional Sobolev spaces of sections of  $\text{End}(E) \otimes \Lambda^{0,q} T^*X$ . For Theorem 5.5 we will need a similar local description for the moduli stack  $\mathfrak{Vect}$  of algebraic vector bundles on  $X$ . Fortunately, there is already a substantial literature on this subject, mostly aimed at proving the *Hitchin–Kobayashi correspondence*, so we will be able to quote many of the results we need.

Some background references are Hartshorne [40, App. B] on complex analytic spaces (in finite dimensions) and the functor to them from  $\mathbb{C}$ -schemes, Laumon and Moret-Bailly [67] on Artin stacks, and Lang [66] on Banach manifolds. The general theory of analytic functions on infinite-dimensional spaces, and (possibly infinite-dimensional) complex analytic spaces is developed in Douady [21, 22], and summarized in [27, §4.1.3] and [73, §7.5]. Some books covering much of §9.1–§9.5 are Kobayashi [62, §VII.3], Lübke and Teleman [73, §4.1 & §4.3], and Friedman and Morgan [27, §4.1–§4.2]. Our main reference is Miyajima [79], who proves that the complex-algebraic and gauge-theoretic descriptions of  $\mathcal{Vect}_{\text{si}}(\mathbb{C})$  are isomorphic as complex analytic spaces.

Let  $X$  be a projective complex algebraic manifold of dimension  $m$ . Then Miyajima considers three different moduli problems:

- The moduli of *holomorphic structures* on a fixed  $C^\infty$  complex vector bundle  $E \rightarrow X$ . For simple holomorphic structures we form the coarse moduli space  $\mathcal{Hol}_{\text{si}}(E) = \{\bar{\partial}_E \in \mathcal{A}_{\text{si}} : \bar{\partial}_E^2 = 0\}/\mathcal{G}$ , a complex analytic space.
- The moduli of *complex analytic vector bundles* over  $X$ . For simple vector bundles we form a coarse moduli space  $\mathcal{Vect}_{\text{si}}^{\text{an}}$ , a complex analytic space.
- The moduli of *complex algebraic vector bundles* over  $X$ . For simple vector bundles we form a coarse moduli space  $\mathcal{Vect}_{\text{si}}$ , a complex algebraic space. For all vector bundles we form a moduli stack  $\mathfrak{Vect}$ , an Artin  $\mathbb{C}$ -stack.

Miyajima [79, §3] proves that  $\mathcal{Hol}_{\text{si}}(E) \cong \mathcal{Vect}_{\text{si}}^{\text{an}} \cong \mathcal{Vect}_{\text{si}}(\mathbb{C})$  locally as complex analytic spaces. Presumably one can also prove analogous results for moduli stacks of all vector bundles, working in some class of analytic  $\mathbb{C}$ -stacks, but the authors have not found references on this in the literature. Instead, to prove what we need about the moduli stack  $\mathfrak{Vect}$ , we will express our results in terms of *versal families* of objects.

Sections 9.1, 9.2, 9.4 and 9.5 explain moduli spaces of holomorphic structures, of analytic vector bundles, and of algebraic vector bundles, respectively, and the isomorphisms between them. Section 9.3 is an aside on existence of local atlases for

$\mathfrak{M}$  with group-invariance properties. All of §§9.1, 9.2 and 9.4 is from Miyajima [79] and other sources, or is easily deduced from them. Sections 9.6–9.8 prove Theorems 5.4 and 5.5.

### 9.1. Holomorphic structures on a complex vector bundle

Let  $X$  be a compact complex manifold of complex dimension  $m$ . Fix a nonzero  $C^\infty$  complex vector bundle  $E \rightarrow X$  of rank  $l > 0$ . That is,  $E$  is a smooth vector bundle whose fibres have the structure of complex vector spaces isomorphic to  $\mathbb{C}^l$ , but  $E$  does not (yet) have the structure of a holomorphic vector bundle. Here are some basic definitions.

DEFINITION 9.1. A (smooth) *semiconnection* (or  $\bar{\partial}$ -operator) is a first order differential operator  $\bar{\partial}_E : C^\infty(E) \rightarrow C^\infty(E \otimes_{\mathbb{C}} \Lambda^{0,1} T^* X)$  satisfying the Leibnitz rule  $\bar{\partial}_E(f \cdot e) = e \otimes (\bar{\partial} f) + f \cdot \bar{\partial}_E e$  for all smooth  $f : X \rightarrow \mathbb{C}$  and  $e \in C^\infty(E)$ , where  $\bar{\partial}$  is the usual operator on complex functions. They are called semiconnections since they arise as the projections to the  $(0, 1)$ -forms  $\Lambda^{0,1} T^* X$  of connections  $\nabla : C^\infty(E) \rightarrow C^\infty(E \otimes_{\mathbb{C}} (T^* X \otimes_{\mathbb{R}} \mathbb{C}))$ , so they are half of an ordinary connection.

Any semiconnection  $\bar{\partial}_E : C^\infty(E) \rightarrow C^\infty(E \otimes_{\mathbb{C}} \Lambda^{0,1} T^* X)$  extends uniquely to operators  $\bar{\partial}_E : C^\infty(E \otimes_{\mathbb{C}} \Lambda^{p,q} T^* X) \rightarrow C^\infty(E \otimes_{\mathbb{C}} \Lambda^{p,q+1} T^* X)$  for all  $0 \leq p \leq m$  and  $0 \leq q < m$  satisfying  $\bar{\partial}_E(e \wedge \alpha) = (-1)^{r+s} e \otimes \bar{\partial} \alpha + (\bar{\partial}_E e) \wedge \alpha$  for all smooth  $e \in C^\infty(E \otimes_{\mathbb{C}} \Lambda^{r,s} T^* X)$  and  $\alpha \in C^\infty(\Lambda^{p-r,q-s} T^* X)$  with  $0 \leq r \leq p$ ,  $0 \leq s \leq q$ . In particular we can consider the composition

$$C^\infty(E) \xrightarrow{\bar{\partial}_E} C^\infty(E \otimes_{\mathbb{C}} \Lambda^{0,1} T^* X) \xrightarrow{\bar{\partial}_E} C^\infty(E \otimes_{\mathbb{C}} \Lambda^{0,2} T^* X).$$

The composition  $\bar{\partial}_E^2$  can be regarded as a section of  $C^\infty(\text{End}(E) \otimes_{\mathbb{C}} \Lambda^{0,2} T^* X)$  called the  $(0, 2)$ -curvature, analogous to the curvature of a connection.

The semiconnection  $\bar{\partial}_E$  defines a *holomorphic structure* on  $E$  if  $\bar{\partial}_E^2 = 0$ . That is, if  $U$  is an open set in  $X$  (in the complex analytic topology) we can define  $\mathcal{E}(U) = \{e \in C^\infty(E|_U) : \bar{\partial}_E e = 0\}$ , and if  $V \subseteq U \subseteq X$  are open we have a restriction map  $\rho_{UV} : \mathcal{E}(U) \rightarrow \mathcal{E}(V)$ . Then  $\mathcal{E}$  is a *complex analytic coherent sheaf* on  $X$ . Since  $\bar{\partial}_E^2 = 0$  we can use the Newlander–Nirenberg Theorem to show that near each  $x \in X$  there exists a basis of holomorphic sections for  $E$ , and thus  $\mathcal{E}$  is locally free, that is, it is an *analytic vector bundle*. Conversely, given a locally free complex analytic coherent sheaf  $\mathcal{E}$  on  $X$ , we can write it as a subsheaf of the sheaf of smooth sections of a complex vector bundle  $E \rightarrow X$ , and then there is a unique semiconnection  $\bar{\partial}_E$  on  $E$  with  $\bar{\partial}_E^2 = 0$  such that if  $U \subseteq X$  is open and  $e \in C^\infty(U)$  then  $e \in \mathcal{E}(U)$  if and only if  $\bar{\partial}_E e = 0$ .

Fix a semiconnection  $\bar{\partial}_E$  with  $\bar{\partial}_E^2 = 0$ . Then any other semiconnection  $\bar{\partial}'_E$  may be written uniquely as  $\bar{\partial}_E + A$  for  $A \in C^\infty(\text{End}(E) \otimes_{\mathbb{C}} \Lambda^{0,1} T^* X)$ . Thus the set  $\mathcal{A}$  of smooth semiconnections on  $E$  is an infinite-dimensional affine space. The  $(0, 2)$ -curvature of  $\bar{\partial}'_E = \bar{\partial}_E + A$  is

$$F_A^{0,2} = \bar{\partial}_E A + A \wedge A.$$

Here to form  $\bar{\partial}_E A$  we extend the action of  $\bar{\partial}_E$  on  $E$  to  $\text{End}(E) \otimes_{\mathbb{C}} \Lambda^{0,1} T^* X = E \otimes_{\mathbb{C}} E^* \otimes_{\mathbb{C}} \Lambda^{0,1} T^* X$  in the natural way, and  $A \wedge A$  combines the Lie bracket on  $\text{End}(E)$  with the wedge product  $\wedge : \Lambda^{0,1} T^* X \times \Lambda^{0,1} T^* X \rightarrow \Lambda^{0,2} T^* X$ .

Write  $\text{Aut}(E)$  for the subbundle of invertible elements in  $\text{End}(E)$ . It is a smooth bundle of complex Lie groups over  $X$ , with fibre  $\text{GL}(l, \mathbb{C})$ . Define the *gauge group*

$\mathcal{G} = C^\infty(\text{Aut}(E))$  to be the space of smooth sections of  $\text{Aut}(E)$ . It is an infinite-dimensional Lie group, with Lie algebra  $\mathfrak{g} = C^\infty(\text{End}(E))$ . It acts on the right on  $\mathcal{A}$  by  $\gamma : \bar{\partial}'_E \mapsto \bar{\partial}'_E{}^\gamma = \gamma^{-1} \circ \bar{\partial}'_E \circ \gamma$ . That is,  $\bar{\partial}'_E{}^\gamma$  is the first order differential operator  $C^\infty(E) \rightarrow C^\infty(E \otimes_{\mathbb{C}} \Lambda^{0,1} T^*X)$  acting by  $e \mapsto \gamma^{-1} \cdot (\bar{\partial}'_E(\gamma \cdot e))$ . One can show that  $\bar{\partial}'_E{}^\gamma$  satisfies the Leibnitz rule, so that  $\bar{\partial}'_E{}^\gamma \in \mathcal{A}$ , and this defines an action of  $\mathcal{G}$  on  $\mathcal{A}$ . Writing  $\bar{\partial}'_E = \bar{\partial}_E + A$  we have

$$(\bar{\partial}_E + A)^\gamma = \bar{\partial}_E + (\gamma^{-1} \circ A \circ \gamma + \gamma^{-1} \bar{\partial} \gamma). \quad (9.1)$$

Write  $\text{Stab}_{\mathcal{G}}(\bar{\partial}'_E)$  for the stabilizer group of  $\bar{\partial}'_E \in \mathcal{A}$  in  $\mathcal{G}$ . It is a complex Lie group with Lie algebra

$$\begin{aligned} \mathfrak{stab}_{\mathcal{G}}(\bar{\partial}'_E) &= \text{Ker}(\bar{\partial}'_E : C^\infty(\text{End}(E)) \rightarrow C^\infty(\text{End}(E) \otimes_{\mathbb{C}} \Lambda^{0,1})) \\ &= \text{Ker}((\bar{\partial}'_E)^* \bar{\partial}'_E : C^\infty(\text{End}(E)) \rightarrow C^\infty(\text{End}(E))), \end{aligned}$$

which is the kernel of an elliptic operator on a compact manifold, and so is finite-dimensional. In fact  $\mathfrak{stab}_{\mathcal{G}}(\bar{\partial}'_E)$  is a finite-dimensional  $\mathbb{C}$ -algebra, and  $\text{Stab}_{\mathcal{G}}(\bar{\partial}'_E)$  is the group of invertible elements in  $\mathfrak{stab}_{\mathcal{G}}(\bar{\partial}'_E)$ . If  $\bar{\partial}'_E$  is a holomorphic structure then  $\mathfrak{stab}_{\mathcal{G}}(\bar{\partial}'_E)$  is the sheaf cohomology group  $H^0(\text{End}(E, \bar{\partial}'_E))$ .

The multiples of the identity  $\mathbb{G}_m \cdot \text{id}_E$  in  $\mathcal{G}$  act trivially on  $\mathcal{A}$ , so  $\mathbb{G}_m \cdot \text{id}_E \subseteq \text{Stab}_{\mathcal{G}}(\bar{\partial}'_E)$  for all  $\bar{\partial}'_E \in \mathcal{A}$ . Call a semiconnection  $\bar{\partial}'_E$  *simple* if  $\text{Stab}_{\mathcal{G}}(\bar{\partial}'_E) = \mathbb{G}_m \cdot \text{id}_E$ . Write  $\mathcal{A}_{\text{si}}$  for the subset of simple  $\bar{\partial}'_E$  in  $\mathcal{A}$ . It is a  $\mathcal{G}$ -invariant open subset of  $\mathcal{A}$ , in the natural topology.

Now  $\mathcal{A}, \mathcal{A}_{\text{si}}, \mathcal{G}$  have the disadvantage that they are not Banach manifolds. Choose Hermitian metrics  $h_X$  on  $X$  and  $h_E$  on the fibres of  $E$ . As in Miyajima [79, §1], fix an integer  $k > 2m + 1$ , and write  $\mathcal{A}^{2,k}, \mathcal{A}_{\text{si}}^{2,k}$  for the completions of  $\mathcal{A}, \mathcal{A}_{\text{si}}$  in the Sobolev norm  $L^2_k$ , and  $\mathcal{G}^{2,k+1}$  for the completion of  $\mathcal{G}$  in the Sobolev norm  $L^2_{k+1}$ , defining norms using  $h_X, h_E$ . Then

$$\mathcal{A}^{2,k} = \{\bar{\partial}_E + A : A \in L^2_k(\text{End}(E) \otimes_{\mathbb{C}} \Lambda^{0,1} T^*X)\}, \quad (9.2)$$

Also  $\mathcal{A}^{2,k}, \mathcal{A}_{\text{si}}^{2,k}$  are complex Banach manifolds, and  $\mathcal{G}^{2,k+1}$  is a complex Banach Lie group acting holomorphically on  $\mathcal{A}^{2,k}, \mathcal{A}_{\text{si}}^{2,k}$  by (9.1).

Define  $P_k : \mathcal{A}^{2,k} \rightarrow L^2_{k-1}(\text{End}(E) \otimes_{\mathbb{C}} \Lambda^{0,1} T^*X)$  by

$$P_k : \bar{\partial}_E + A \mapsto F_A^{0,2} = \bar{\partial}_E A + A \wedge A. \quad (9.3)$$

Using the Sobolev Embedding Theorem we see that  $P_k$  is a well-defined, holomorphic map between complex Banach manifolds.

**DEFINITION 9.2.** A *family of holomorphic structures*  $(T, \tau)$  on  $E$  is a (finite-dimensional) complex analytic space  $T$  and a complex analytic map of complex analytic spaces  $\tau : T \rightarrow P_k^{-1}(0)$ , where  $P_k^{-1}(0) \subset \mathcal{A}^{2,k}$  as above. Two families  $(T, \tau), (T, \tau')$  with the same base  $T$  are *equivalent* if there exists a complex analytic map  $\sigma : T \rightarrow \mathcal{G}^{2,k+1}$  such that  $\tau' \equiv \sigma \cdot \tau$ , using the product  $\cdot : \mathcal{G}^{2,k+1} \times \mathcal{A}^{2,k} \rightarrow \mathcal{A}^{2,k}$  which restricts to  $\cdot : \mathcal{G}^{2,k+1} \times P_k^{-1}(0) \rightarrow P_k^{-1}(0)$ .

A family  $(T, \tau)$  is called *versal at*  $t \in T$  if whenever  $(T', \tau')$  is a family of holomorphic structures on  $E$  and  $t' \in T'$  with  $\tau'(t') = \tau(t)$ , there exists an open neighbourhood  $U'$  of  $t'$  in  $T'$  and complex analytic maps  $v : U' \rightarrow T$  and  $\sigma : U' \rightarrow \mathcal{G}^{2,k+1}$  such that  $v(t') = t$ ,  $\sigma(t') = \text{id}_E$ , and  $\tau \circ v \equiv \sigma \cdot \tau'|_{U'}$  as complex analytic maps  $U' \rightarrow P^{-1}(0)$ . We call  $(T, \tau)$  *universal at*  $t \in T$  if in addition the map  $v : U' \rightarrow T$  is unique, provided the neighbourhood  $U'$  is sufficiently small.



(Note that we do not require  $\sigma$  to be unique. Thus, this notion of universal is appropriate for defining a coarse moduli space, not a fine moduli space or moduli stack.) The family  $(T, \tau)$  is called *versal* (or *universal*) if it is versal (or universal) at every  $t \in T$ .

Fix a smooth holomorphic structure  $\bar{\partial}_E$  on  $E$ , as above. In [79, Th. 1], Miyajima constructs a versal family of holomorphic structures  $(T, \tau)$  containing  $\bar{\partial}_E$ . We now explain his construction. Write  $\bar{\partial}_E^*$  for the formal adjoint of  $\bar{\partial}_E$  computed using the Hermitian metrics  $h_X$  on  $X$  and  $h_E$  on the fibres of  $E$ . Then  $\bar{\partial}_E^* : C^\infty(E \otimes_{\mathbb{C}} \Lambda^{p,q+1} T^* X) \rightarrow C^\infty(E \otimes_{\mathbb{C}} \Lambda^{p,q} T^* X)$  for all  $p, q$  is a first order differential operator such that  $\langle \bar{\partial}_E e, e' \rangle_{L^2} = \langle e, \bar{\partial}_E^* e' \rangle_{L^2}$  for all  $e \in C^\infty(E \otimes_{\mathbb{C}} \Lambda^{p,q} T^* X)$  and  $e' \in C^\infty(E \otimes_{\mathbb{C}} \Lambda^{p,q+1} T^* X)$ , where  $\langle \cdot, \cdot \rangle_{L^2}$  is the  $L^2$  inner product defined using  $h_X, h_E$ . Also  $\bar{\partial}_E^*$  extends to Sobolev spaces  $L_k^2$ .

Using Hodge theory for  $(C^\infty(\text{End}(E) \otimes_{\mathbb{C}} \Lambda^{0,*} T^* X), \bar{\partial}_E)$ , we give expressions for the Ext groups of the holomorphic vector bundle  $(E, \bar{\partial}_E)$  with itself:

$$\begin{aligned} & \text{Ext}^q((E, \bar{\partial}_E), (E, \bar{\partial}_E)) \\ & \cong \frac{\text{Ker}(\bar{\partial}_E : C^\infty(\text{End}(E) \otimes_{\mathbb{C}} \Lambda^{0,q} T^* X) \rightarrow C^\infty(\text{End}(E) \otimes_{\mathbb{C}} \Lambda^{0,q+1} T^* X))}{\text{Im}(\bar{\partial}_E : C^\infty(\text{End}(E) \otimes_{\mathbb{C}} \Lambda^{0,q-1} T^* X) \rightarrow C^\infty(\text{End}(E) \otimes_{\mathbb{C}} \Lambda^{0,q} T^* X))} \\ & \cong \{e \in C^\infty(\text{End}(E) \otimes_{\mathbb{C}} \Lambda^{0,q} T^* X) : \bar{\partial}_E e = \bar{\partial}_E^* e = 0\} \\ & = \{e \in C^\infty(\text{End}(E) \otimes_{\mathbb{C}} \Lambda^{0,q} T^* X) : (\bar{\partial}_E \bar{\partial}_E^* + \bar{\partial}_E^* \bar{\partial}_E)e = 0\}. \end{aligned}$$

Hence the finite-dimensional complex vector space

$$\mathcal{E}^q = \{e \in C^\infty(\text{End}(E) \otimes_{\mathbb{C}} \Lambda^{0,q} T^* X) : (\bar{\partial}_E \bar{\partial}_E^* + \bar{\partial}_E^* \bar{\partial}_E)e = 0\}$$

is isomorphic to  $\text{Ext}^q((E, \bar{\partial}_E), (E, \bar{\partial}_E))$ . Miyajima [79, §1] proves:

**PROPOSITION 9.3. (a)** *In the situation above, for sufficiently small  $\epsilon > 0$ ,*

$$\begin{aligned} Q_\epsilon &= \{\bar{\partial}_E + A : A \in L_k^2(\text{End}(E) \otimes_{\mathbb{C}} \Lambda^{0,1} T^* X), \quad \|A\|_{L_k^2} < \epsilon, \\ &\quad \bar{\partial}_E^* A = 0, \quad \bar{\partial}_E^*(\bar{\partial}_E A + A \wedge A) = 0\} \end{aligned} \quad (9.4)$$

*is a finite-dimensional complex submanifold of  $\mathcal{A}^{2,k}$ , of complex dimension  $\dim \text{Ext}^1((E, \bar{\partial}_E), (E, \bar{\partial}_E))$ , such that  $\bar{\partial}_E \in Q_\epsilon$  and  $T_{\bar{\partial}_E} Q_\epsilon = \mathcal{E}^1$ . Furthermore,  $Q_\epsilon \subset \mathcal{A} \subset \mathcal{A}^{2,k}$ , that is, if  $\bar{\partial}_E + A \in Q_\epsilon$  then  $A$  is smooth.*

**(b)** *Now define  $\pi : Q_\epsilon \rightarrow \mathcal{E}^2$  by  $\pi : \bar{\partial}_E + A \mapsto \pi_{\mathcal{E}^2}(\bar{\partial}_E A + A \wedge A)$ , where  $\pi_{\mathcal{E}^2} : L_{k-1}^2(\text{End}(E) \otimes_{\mathbb{C}} \Lambda^{0,2} T^* X) \rightarrow \mathcal{E}^2$  is orthogonal projection using the  $L^2$  inner product. Then  $\pi$  is a holomorphic map of finite-dimensional complex manifolds. Let  $T = \pi^{-1}(0)$ , as a complex analytic subspace of  $Q_\epsilon$ . Then  $T = Q_\epsilon \cap P_k^{-1}(0)$ , as an intersection of complex analytic subspaces in  $\mathcal{A}^{2,k}$ , so  $T$  is a complex analytic subspace of  $P_k^{-1}(0)$ . Also  $t = \bar{\partial}_E \in T$ , with  $\tau(t) = \bar{\partial}_E$ , and the Zariski tangent space  $T_t T$  is  $\mathcal{E}^1 \cong \text{Ext}^1((E, \bar{\partial}_E), (E, \bar{\partial}_E))$ .*

**(c)** *Making  $\epsilon$  smaller if necessary,  $(T, \tau)$  is a **versal** family of smooth holomorphic structures on  $E$ , which includes  $\bar{\partial}_E$ . If  $\bar{\partial}_E$  is simple, then  $(T, \tau)$  is a **universal** family of smooth, simple holomorphic structures on  $E$ .*

This gives the standard Kuranishi picture: there exists a versal family of deformations of  $\bar{\partial}_E$ , with base space the zeroes of a holomorphic map from  $\text{Ext}^1((E, \bar{\partial}_E), (E, \bar{\partial}_E))$  to  $\text{Ext}^2((E, \bar{\partial}_E), (E, \bar{\partial}_E))$ . Here is a sketch of the proof.

For (a), we consider the nonlinear elliptic operator  $F : L_k^2(\text{End}(E) \otimes_{\mathbb{C}} \Lambda^{0,1} T^* X) \rightarrow L_{k-2}^2(\text{End}(E) \otimes_{\mathbb{C}} \Lambda^{0,1} T^* X)$  mapping  $F : A \mapsto (\bar{\partial}_E \bar{\partial}_E^* + \bar{\partial}_E^* \bar{\partial}_E)A + \bar{\partial}_E^*(A \wedge A)$ . The image of  $F$  lies in the orthogonal subspace  $(\mathcal{E}^1)^\perp$  to  $\mathcal{E}^1$  in  $L_{k-2}^2(\text{End}(E) \otimes_{\mathbb{C}} \Lambda^{0,1} T^* X)$ , using the  $L^2$  inner product. So we can consider  $F$  as mapping  $F : L_k^2(\text{End}(E) \otimes_{\mathbb{C}} \Lambda^{0,1} T^* X) \rightarrow (\mathcal{E}^1)^\perp$ . The linearization of  $F$  at  $A = 0$  is then surjective, with kernel  $\mathcal{E}^1$ . Part (a) then follows from the Implicit Function Theorem for Banach spaces, together with elliptic regularity for  $F$  to deduce smoothness in the last part.

For (b), one must show that  $(P_k|_{Q_\epsilon})^{-1}(0)$  and  $\pi^{-1}(0)$  coincide as complex analytic subspaces of  $Q_\epsilon$ . Since  $\pi$  factors through  $P_k$  we have  $(P_k|_{Q_\epsilon})^{-1}(0) \subseteq \pi^{-1}(0)$  as complex analytic subspaces. It is enough to show that any local holomorphic function  $Q_\epsilon \rightarrow \mathbb{C}$  of the form  $f \circ P_k$  for a local holomorphic function  $f : L_{k-1}^2(\text{End}(E) \otimes_{\mathbb{C}} \Lambda^{0,2} T^* X) \rightarrow \mathbb{C}$  may also be written in the form  $\tilde{f} \circ \pi$  for a local holomorphic function  $\tilde{f} : \mathcal{E}^2 \rightarrow \mathbb{C}$ .

For (c), the main point is that the condition  $\bar{\partial}_E^* A = 0$  is a ‘slice’ to the action of  $\mathcal{G}^{2,k+1}$  on  $\mathcal{A}^{2,k}$  at  $\bar{\partial}_E$ . That is, the Hilbert submanifold  $\{\bar{\partial}_E + A : \bar{\partial}_E^* A = 0\}$  in  $\mathcal{A}^{2,k}$  intersects the orbit  $\mathcal{G}^{2,k+1} \cdot \bar{\partial}_E$  transversely, and it also intersects every nearby orbit of  $\mathcal{G}^{2,k+1}$  in  $\mathcal{A}^{2,k}$ . The complex analytic space  $T$  is exactly the intersection (as Douady complex analytic subspaces of  $\mathcal{A}^{2,k}$ ) of  $P_k^{-1}(0)$ , the slice  $\{\bar{\partial}_E + A : \bar{\partial}_E^* A = 0\}$ , and the ball of radius  $\epsilon$  around  $\bar{\partial}_E$  in  $\mathcal{A}^{2,k}$ . The point of introducing  $Q_\epsilon, \mathcal{E}^1, \mathcal{E}^2, \pi$  is to describe this complex analytic space  $T$  in strictly finite-dimensional terms.

## 9.2. Moduli spaces of analytic vector bundles on $X$

Let  $X$  be a compact complex manifold. Here is the analogue of Definition 9.2 for analytic vector bundles.

**DEFINITION 9.4.** A *family of analytic vector bundles*  $(T, \mathcal{F})$  on  $X$  is a (finite-dimensional) complex analytic space  $T$  and a complex analytic vector bundle  $\mathcal{F}$  over  $X \times T$  which is flat over  $T$ . For each  $t \in T$ , the *fibre*  $\mathcal{F}_t$  of the family is  $\mathcal{F}|_{X \times \{t\}}$ , regarded as a complex analytic vector bundle over  $X \cong X \times \{t\}$ .

A family  $(T, \mathcal{F})$  is called *versal* at  $t \in T$  if whenever  $(T', \mathcal{F}')$  is a family of analytic vector bundles on  $X$  and  $t' \in T'$  with  $\mathcal{F}_t \cong \mathcal{F}'_{t'}$  as analytic vector bundles on  $X$ , there exists an open neighbourhood  $U'$  of  $t'$  in  $T'$ , a complex analytic map  $v : U' \rightarrow T$  with  $v(t') = t$  and an isomorphism  $v^*(\mathcal{F}) \cong \mathcal{F}'|_{X \times U'}$  as vector bundles over  $X \times U'$ .

It is called *universal* at  $t \in T$  if in addition the map  $v : U' \rightarrow T$  is unique, provided the neighbourhood  $U'$  is sufficiently small. (Note that we do not require the isomorphism  $v^*(\mathcal{F}) \cong \mathcal{F}'|_{X \times U'}$  to be unique.) The family  $(T, \mathcal{F})$  is called *versal* (or *universal*) if it is versal (or universal) at every  $t \in T$ .

In a parallel result to Proposition 9.3(c), Forster and Knorr [25] prove that any analytic vector bundle on  $X$  can be extended to a versal family of analytic vector bundles. Then Miyajima [79, §2] proves:

**PROPOSITION 9.5.** *Let  $X$  be a compact complex manifold,  $E \rightarrow X$  a  $C^\infty$  complex vector bundle, and  $\bar{\partial}_E$  a holomorphic structure on  $E$ , so that  $(E, \bar{\partial}_E)$  is an analytic vector bundle over  $X$ . Let  $(T, \tau)$  be the versal family of holomorphic structures on  $E$  containing  $\bar{\partial}_E$  constructed in Proposition 9.3.*

*Then there exists a versal family of analytic vector bundles  $(T, \mathcal{F})$  over  $X$ , and an isomorphism  $\mathcal{F} \rightarrow E \times T$  of  $C^\infty$  complex vector bundles over  $X \times T$  which induces the family of holomorphic structures  $(T, \tau)$ . If  $(E, \bar{\partial}_E)$  is simple then  $(T, \mathcal{F})$  is a universal family of simple analytic vector bundles.*

Here is an idea of the proof. Let  $(T, \mathcal{F})$  be a family of analytic vector bundles over  $X$ , let  $t \in T$ , and let  $E \rightarrow X$  be the complex vector bundle underlying the analytic vector bundle  $\mathcal{F}_t \rightarrow X$ . Then for some small open neighbourhood  $U$  of  $t$  in  $T$ , we can identify  $\mathcal{F}|_{X \times U}$  with  $(E \times U) \rightarrow (X \times U)$  as complex vector bundles, where  $(E \times U) \rightarrow (X \times U)$  is the pullback of  $E$  from  $X$  to  $X \times U$ .

Thus, the analytic vector bundle structure on  $\mathcal{F}|_{X \times U}$  induces an analytic vector bundle structure on  $(E \times U) \rightarrow (X \times U)$ . We can regard this as a first order differential operator  $\bar{\partial}_{E,U} : C^\infty(E) \rightarrow C^\infty(E \otimes \Lambda^{0,1} T^* X \oplus E \otimes \Lambda^{0,1} T^* U)$  on bundles over  $X \times U$ . Thus,  $\bar{\partial}_{E,U}$  has two components, a  $\bar{\partial}$ -operator in the  $X$  directions and a  $\bar{\partial}$ -operator in the  $U$  directions in  $X \times U$ . The first of these components is a *family of holomorphic structures*  $(U, \tau)$  on  $E$ .

Therefore, by choosing a (local) trivialization in the  $T$ -directions, a family  $(T, \mathcal{F})$  of analytic vector bundles induces a family  $(T, \tau)$  of holomorphic structures on  $E$ , by forgetting part of the structure. Conversely, given a family  $(T, \tau)$  of holomorphic structures on  $E$ , we can try to add extra structure, a  $\bar{\partial}$ -operator in the  $T$  directions in  $X \times T$ , to make  $(T, \tau)$  into a family of analytic vector bundles  $(T, \mathcal{F})$ . Miyajima proves that this can be done, and that the local deformation functors are isomorphic. Hence the (uni)versal family in Proposition 9.3 lifts to a (uni)versal family of analytic vector bundles.

### 9.3. Constructing a good local atlas $S$ for $\mathfrak{M}$ near $[E]$

We divert briefly from our main argument to prove the first part of the second paragraph of Theorem 5.5, the existence of a 1-morphism  $\Phi : [S/G^c] \rightarrow \mathfrak{M}$  for  $\mathfrak{M}$  satisfying various conditions. Let  $X, \mathfrak{M}, E, \text{Aut}(E)$  and  $G^c$  be as in Theorem 5.5.

**PROPOSITION 9.6.** *There exists a finite type open  $\mathbb{C}$ -substack  $\Omega$  in  $\mathfrak{M}$  with  $[E] \in \Omega(\mathbb{C}) \subset \mathfrak{M}(\mathbb{C})$  and a 1-isomorphism  $\Omega \cong [Q/H]$ , where  $Q$  is a finite type  $\mathbb{C}$ -scheme and  $H$  an algebraic  $\mathbb{C}$ -group acting on  $Q$ . Furthermore we may realize  $Q$  as a quasiprojective  $\mathbb{C}$ -scheme, a locally closed  $\mathbb{C}$ -subscheme of a complex projective space  $\mathbb{P}(W)$ , such that the action of  $H$  on  $Q$  is the restriction of an action of  $H$  on  $\mathbb{P}(W)$  coming from a representation of  $H$  on  $W$ .*

**PROOF.** We follow the standard method for constructing coarse moduli schemes of semistable coherent sheaves in Huybrechts and Lehn [44], adapting it for Artin stacks. Choose an ample line bundle  $\mathcal{O}_X(1)$  on  $X$ , let  $P$  be the Hilbert polynomial of  $E$  and fix  $n \in \mathbb{Z}$ . Consider Grothendieck's Quot Scheme  $\text{Quot}_X(\mathbb{C}^{P(n)} \otimes \mathcal{O}_X(-n), P)$ , explained in [44, §2.2], which parametrizes quotients  $\mathbb{C}^{P(n)} \otimes \mathcal{O}_X(-n) \twoheadrightarrow E'$ , where  $E'$  has Hilbert polynomial  $P$ . Then the  $\mathbb{C}$ -group  $H = \text{GL}(P(n), \mathbb{C})$  acts on  $\text{Quot}_X(\mathbb{C}^{P(n)} \otimes \mathcal{O}_X(-n), P)$ .

Let  $Q$  be the  $H$ -invariant open  $\mathbb{C}$ -subscheme of  $\text{Quot}_X(\mathbb{C}^{P(n)} \otimes \mathcal{O}_X(-n), P)$  parametrizing  $\phi : \mathbb{C}^{P(n)} \otimes \mathcal{O}_X(-n) \twoheadrightarrow E'$  for which  $H^i(E'(n)) = 0$  for  $i > 0$  and  $\phi_* : \mathbb{C}^{P(n)} \rightarrow H^0(E'(n))$  is an isomorphism. Then the projection  $[Q/H] \rightarrow \mathfrak{M}$  taking the  $\text{GL}(P(n), \mathbb{C})$ -orbit of  $\phi : \mathbb{C}^{P(n)} \otimes \mathcal{O}_X(-n) \twoheadrightarrow E'$  to  $E'$  is a 1-isomorphism with an open  $\mathbb{C}$ -substack  $\Omega$  of  $\mathfrak{M}$ . If  $n \gg 0$  then  $[E] \in \Omega(\mathbb{C})$ . For the last

part,  $\text{Quot}_X(\mathbb{C}^{P(n)} \otimes \mathcal{O}_X(-n), P)$  is projective as in [44, Th. 2.2.4], so it comes embedded as a closed  $\mathbb{C}$ -subscheme in some  $\mathbb{P}(W)$ , and by construction the  $H$ -action on  $\text{Quot}_X(\mathbb{C}^{P(n)} \otimes \mathcal{O}_X(-n), P)$  is the restriction of an  $H$ -action on  $\mathbb{P}(W)$  from a representation of  $H$  on  $W$ .  $\square$

The proof of the next proposition is similar to parts of that of Luna's Etale Slice Theorem [74, §III].

**PROPOSITION 9.7.** *In the situation of Proposition 9.6, let  $s \in Q(\mathbb{C})$  project to the point  $sH$  in  $\mathfrak{Q}(\mathbb{C})$  identified with  $[E] \in \mathfrak{M}(\mathbb{C})$  under the 1-isomorphism  $\mathfrak{Q} \cong [Q/H]$ . This 1-isomorphism identifies the stabilizer groups  $\text{Iso}_{\mathfrak{M}}([E]) = \text{Aut}(E)$  and  $\text{Iso}_{[Q/H]}(sH) = \text{Stab}_H(s)$ , and the Zariski tangent spaces  $T_{[E]}\mathfrak{M} \cong \text{Ext}^1(E, E)$  and  $T_{sH}[Q/H] \cong T_s Q / T_s(sH)$ , so we have natural isomorphisms  $\text{Aut}(E) \cong \text{Stab}_H(s)$  and  $\text{Ext}^1(E, E) \cong T_s Q / T_s(sH)$ .*

*Let  $K$  be the  $\mathbb{C}$ -subgroup of  $\text{Stab}_H(s) \subseteq H$  identified with  $G^c$  in  $\text{Aut}(E)$ . Then there exists a  $K$ -invariant, locally closed  $\mathbb{C}$ -subscheme  $S$  in  $Q$  with  $s \in S(\mathbb{C})$ , such that  $T_s Q = T_s S \oplus T_s(sH)$ , and the morphism  $\mu : S \times H \rightarrow Q$  induced by the inclusion  $S \hookrightarrow Q$  and the  $H$ -action on  $Q$  is smooth of relative dimension  $\dim \text{Aut}(E)$ .*

**PROOF.** We have  $s \in Q(\mathbb{C}) \subset \mathbb{P}(W)$ , so  $s$  corresponds to a 1-dimensional vector subspace  $L$  of  $W$ . As  $K \subseteq \text{Stab}_H(s) \subseteq H$ , this  $L$  is preserved by  $K$ , that is,  $L$  is a  $K$ -subrepresentation of  $W$ . Since  $K \cong G^c$  is reductive, we can decompose  $W$  into a direct sum of  $K$ -representations  $W = L \oplus W'$ . Then there is a natural identification of  $K$ -representations  $T_s \mathbb{P}(W) = W' \otimes L^*$ .

We have  $s \in (sH)(\mathbb{C}) \subseteq Q(\mathbb{C}) \subseteq \mathbb{P}(W)$ , where  $sH$  and  $Q$  are both  $K$ -invariant  $\mathbb{C}$ -subschemes of  $\mathbb{P}(W)$ . Hence we have inclusions of Zariski tangent spaces, which are  $K$ -representations

$$T_s(sH) \subseteq T_s Q \subseteq T_s \mathbb{P}(W) = W' \otimes L^*.$$

As  $K$  is reductive, we may choose  $K$ -subrepresentations  $W'', W'''$  of  $T_s \mathbb{P}(W)$  such that  $T_s Q = T_s(sH) \oplus W''$  and  $T_s \mathbb{P}(W) = T_s Q \oplus W'''$ . Then  $W' \otimes L^* = T_s(sH) \oplus W'' \oplus W'''$ . Tensoring by  $L$  and using  $W = L \oplus W'$  gives a direct sum of  $K$ -representations

$$W = L \oplus (T_s(sH) \otimes L) \oplus (W'' \otimes L) \oplus (W''' \otimes L). \quad (9.5)$$

Define  $S' = Q \cap \mathbb{P}(L \oplus (W'' \otimes L) \oplus (W''' \otimes L))$ , where we have omitted the factor  $T_s(sH) \otimes L$  in (9.5) in the projective space. Then  $S'$  is a locally closed  $\mathbb{C}$ -subscheme of  $\mathbb{P}(W)$ , and is  $K$ -invariant as it is the intersection of two  $K$ -invariant subschemes, with  $s \in S'(\mathbb{C})$ . In the decomposition  $T_s \mathbb{P}(W) = T_s(sH) \oplus W'' \oplus W'''$  we have  $T_s Q = T_s(sH) \oplus W''$  and  $T_s \mathbb{P}(L \oplus (W'' \otimes L) \oplus (W''' \otimes L)) = W'' \oplus W'''$ , which intersect transversely in  $W''$ . Hence  $Q$  and  $\mathbb{P}(L \oplus (W'' \otimes L) \oplus (W''' \otimes L))$  intersect transversely at  $s \in S'(\mathbb{C})$ , and  $T_s S' = W''$ , so that  $T_s Q = T_s S' \oplus T_s(sH)$ .

Since  $T_s Q = T_s S' \oplus T_s(sH)$ , we see that  $S'$  intersects the  $H$ -orbit  $sH$  transversely at  $s$ . It follows that the morphism  $\mu' : S' \times H \rightarrow Q$  induced by the inclusion  $S' \hookrightarrow Q$  and the  $H$ -action on  $Q$  is smooth at  $(s, 1)$ , of relative dimension  $\dim \text{Stab}_H(s) = \dim \text{Aut}(E)$ . Let  $S$  be the  $\mathbb{C}$ -subscheme of points  $s \in S'(\mathbb{C})$  such that  $\mu'$  is smooth of relative dimension  $\dim \text{Aut}(E)$  at  $(s, 1)$ . This is an open condition, so  $S$  is open in  $S'$ , and contains  $s$ . Therefore  $T_s Q = T_s S \oplus T_s(sH)$ , as we want. Write  $\mu = \mu'|_{S \times H}$ . Then  $\mu : S \times H \rightarrow Q$  is smooth of relative dimension  $\dim \text{Aut}(E)$  along  $S \times \{1\}$ , and so is smooth of relative dimension  $\dim \text{Aut}(E)$  on all of

$S \times H$ , since  $\mu$  is equivariant under the action of  $H$  on  $S \times H$  acting trivially on  $S$  and on the right on  $H$ , and the right action of  $H$  on  $Q$ .  $\square$

Since  $S$  is invariant under the  $\mathbb{C}$ -subgroup  $K$  of the  $\mathbb{C}$ -group  $H$  acting on  $Q$ , the inclusion  $i : S \hookrightarrow Q$  induces a representable 1-morphism of quotient stacks  $i_* : [S/K] \rightarrow [Q/H]$ . We claim that  $i_*$  is smooth of relative dimension  $\dim \operatorname{Aut}(E) - \dim G^c$ . Since the projection  $\pi : Q \rightarrow [Q/H]$  is an atlas for  $[Q/H]$ , this is true if and only if the projection from the fibre product  $\pi_2 : [S/K] \times_{i_*, [Q/H], \pi} Q \rightarrow Q$  is a smooth morphism of  $\mathbb{C}$ -schemes of relative dimension  $\dim \operatorname{Aut}(E) - \dim G^c$ . But  $[S/K] \times_{i_*, [Q/H], \pi} Q \cong (S \times H)/K$ , where  $K$  acts in the given way on  $S$  and on the left on  $H$ . The projection  $\pi_2 : (S \times H)/K \rightarrow Q$  fits into the commutative diagram

$$\begin{array}{ccc} S \times H & \xrightarrow{\pi} & (S \times H)/K \\ & \searrow \mu & \swarrow \pi_2 \\ & & Q, \end{array}$$

where  $\pi : S \times H \rightarrow (S \times H)/K$  is the projection to the quotient. As  $\pi$  is smooth of relative dimension  $\dim K = \dim G^c$  and surjective, and  $\mu$  is smooth of relative dimension  $\dim \operatorname{Aut}(E)$  by Proposition 9.7, it follows that  $\pi_2$  and hence  $i_*$  are smooth of relative dimension  $\dim \operatorname{Aut}(E) - \dim G^c$ .

Combining the isomorphism  $K \cong G^c$ , the 1-morphism  $i_* : [S/K] \rightarrow [Q/H]$ , the 1-isomorphism  $\Omega \cong [Q/H]$ , and the open inclusion  $\Omega \hookrightarrow \mathfrak{M}$ , yields a 1-morphism  $\Phi : [S/G^c] \rightarrow \mathfrak{M}$ , as in Theorem 5.5. This  $\Phi$  is smooth of relative dimension  $\dim \operatorname{Aut}(E) - \dim G^c$ , as  $i_*$  is. If  $\operatorname{Aut}(E)$  is reductive, so that  $G^c = \operatorname{Aut}(E)$ , then  $\Phi$  is smooth of dimension 0, that is,  $\Phi$  is étale.

The conditions  $\Phi(sG^c) = [E]$  and  $\Phi_* : \operatorname{Iso}_{[S/G^c]}(sG^c) \rightarrow \operatorname{Iso}_{\mathfrak{M}}([E])$  is the natural  $G^c \hookrightarrow \operatorname{Aut}(E) \cong \operatorname{Iso}_{\mathfrak{M}}([E])$  in Theorem 5.5 are immediate from the construction. That  $d\Phi|_{sG^c} : T_s S \cong T_{sG^c}[S/G^c] \rightarrow T_{[E]}\mathfrak{M} \cong \operatorname{Ext}^1(E, E)$  is an isomorphism follows from  $T_{[E]}\mathfrak{M} \cong T_{sH}[Q/H] \cong T_s Q/T_s(sH)$  and  $T_s Q = T_s S \oplus T_s(sH)$  in Proposition 9.7.

#### 9.4. Moduli spaces of algebraic vector bundles on $X$

We can now discuss results in algebraic geometry corresponding to §9.1–§9.2. Let  $X$  be a projective complex algebraic manifold.

**DEFINITION 9.8.** A family of algebraic vector bundles  $(T, \mathcal{F})$  on  $X$  is a  $\mathbb{C}$ -scheme  $T$ , locally of finite type, and an algebraic vector bundle  $\mathcal{F}$  over  $X \times T$ . For each  $t \in T(\mathbb{C})$ , the fibre  $\mathcal{F}_t$  of the family is  $\mathcal{F}|_{X \times \{t\}}$ , regarded as an algebraic vector bundle over  $X \cong X \times \{t\}$ .

A family  $(T, \mathcal{F})$  is called *formally versal* at  $t \in T$  if whenever  $T'$  is a  $\mathbb{C}$ -scheme of finite length with exactly one  $\mathbb{C}$ -point  $t'$ , and  $(T', \mathcal{F}')$  is a family of algebraic vector bundles on  $X$  with  $\mathcal{F}_t \cong \mathcal{F}'_{t'}$  as algebraic vector bundles on  $X$ , there exists a morphism  $v : T' \rightarrow T$  with  $v(t') = t$ , and an isomorphism  $v^*(\mathcal{F}) \cong \mathcal{F}'$  as vector bundles over  $X \times T'$ . It is called *formally universal* at  $t \in T$  if in addition the morphism  $v : T' \rightarrow T$  is unique. The family  $(T, \mathcal{F})$  is called *formally versal* (or *formally universal*) if it is formally versal (or formally universal) at every  $t \in T$ .

By work of Grothendieck and others, as in Laumon and Moret-Bailly [67, Th. 4.6.2.1] for instance, we have:

**PROPOSITION 9.9.** The moduli functor  $\mathbb{V}\mathbb{B}_{\text{si}}^{\text{ét}} : (\mathbb{C}\text{-schemes}) \rightarrow (\text{sets})$  of isomorphism classes of families of simple algebraic vector bundles on  $X$ , sheafified in the

étale topology, is represented by a complex algebraic space  $\mathcal{Vect}_{\text{si}}$  locally of finite type, the moduli space of simple algebraic vector bundles on  $X$ .

The moduli functor  $\mathbb{V}\mathcal{B} : (\mathbb{C}\text{-schemes}) \rightarrow (\text{groupoids})$  of families of algebraic vector bundles on  $X$  is represented by an Artin  $\mathbb{C}$ -stack  $\mathfrak{Vect}$  locally of finite type, the moduli stack of algebraic vector bundles on  $X$ .

As in Miyajima [79, §3], the existence of  $\mathcal{Vect}_{\text{si}}$  as a complex algebraic space implies the existence étale locally of formally universal families of simple vector bundles on  $X$ . Similarly, the existence of  $\mathfrak{Vect}$  as an Artin  $\mathbb{C}$ -stack implies that a smooth 1-morphism  $\phi : S \rightarrow \mathfrak{Vect}$  from a scheme  $S$  corresponds naturally to a formally versal family  $(S, \mathcal{D})$  of vector bundles on  $X$ .

**PROPOSITION 9.10. (a)** *Let  $\mathcal{E}$  be a simple algebraic vector bundle on  $X$ . Then there exists an affine  $\mathbb{C}$ -scheme  $S$ , a  $\mathbb{C}$ -point  $s \in S$ , and a formally universal family of simple algebraic vector bundles  $(S, \mathcal{D})$  on  $X$  with  $\mathcal{D}_s \cong \mathcal{E}$ . This family  $(S, \mathcal{D})$  induces an étale map of complex algebraic spaces  $\pi : S \rightarrow \mathcal{Vect}_{\text{si}}$  with  $\pi(s) = [\mathcal{E}]$ . There is a natural isomorphism between the Zariski tangent space  $T_s S$  and  $\text{Ext}^1(\mathcal{E}, \mathcal{E})$ .*

**(b)** *Let  $\mathcal{E}$  be an algebraic vector bundle on  $X$ , and  $G^c$  a maximal reductive subgroup of  $\text{Aut}(\mathcal{E})$ . Then §9.3 constructs a quasiprojective  $\mathbb{C}$ -scheme  $S$ , an action of  $G^c$  on  $S$ , a  $G^c$ -invariant point  $s \in S(\mathbb{C})$ , and a 1-morphism  $\Phi : [S/G^c] \rightarrow \mathfrak{Vect}$  smooth of relative dimension  $\dim \text{Aut}(\mathcal{E}) - \dim G^c$ , such that  $\Phi(s G^c) = [\mathcal{E}]$ ,  $\Phi_* : \text{Iso}_{[S/G^c]}(s G^c) \rightarrow \text{Iso}_{\mathfrak{Vect}}([\mathcal{E}])$  is the inclusion  $G^c \hookrightarrow \text{Aut}(\mathcal{E})$ , and  $d\Phi|_{s G^c} : T_s S \cong T_{s G^c} [S/G^c] \rightarrow T_{[\mathcal{E}]} \mathfrak{Vect} \cong \text{Ext}^1(\mathcal{E}, \mathcal{E})$  is an isomorphism.*

*Let  $(S, \mathcal{D})$  be the family of algebraic vector bundles on  $X$  corresponding to the 1-morphism  $\Phi \circ \pi : S \rightarrow \mathfrak{Vect}$ , where  $\pi : S \rightarrow [S/G]$  is the projection. Then  $(S, \mathcal{D})$  is formally versal and  $G^c$  equivariant, with  $\mathcal{D}_s \cong \mathcal{E}$ .*

Together with §9.3, part (b) completes the proof of the second paragraph of Theorem 5.5 for  $\mathfrak{Vect}$ . The proof for  $\mathfrak{M}$  follows from Theorem 5.3.

### 9.5. Identifying versal families of holomorphic structures and algebraic vector bundles

Let  $\mathcal{E}$  be an algebraic vector bundle on  $X$ . Write  $E \rightarrow X$  for the underlying  $C^\infty$  complex vector bundle, and  $\bar{\partial}_E$  for the induced holomorphic structure on  $E$ . Then  $(E, \bar{\partial}_E)$  is the analytic vector bundle associated to  $\mathcal{E}$ . By Serre [97] we have  $\text{Ext}^1(\mathcal{E}, \mathcal{E}) \cong \text{Ext}^1((E, \bar{\partial}_E), (E, \bar{\partial}_E))$ , that is, Ext groups computed in the complex algebraic or complex analytic categories are the same.

Then Proposition 9.3 constructs a *versal family*  $(T, \tau)$  of *holomorphic structures* on  $E$ , with  $\tau(t) = \bar{\partial}_E$  for  $t \in T$  and  $T_t T \cong \text{Ext}^1((E, \bar{\partial}_E), (E, \bar{\partial}_E))$ . If  $(E, \bar{\partial}_E)$  is simple then  $(T, \tau)$  is a *universal* family of simple holomorphic structures. Proposition 9.5 shows that we may lift  $(T, \tau)$  to a versal family  $(T, \mathcal{F})$  of analytic vector bundles over  $X$ , with isomorphism  $(\mathcal{F} \rightarrow (X \times T)) \cong ((E \times T) \rightarrow (X \times T))$  as  $C^\infty$  complex vector bundles inducing  $(T, \tau)$ . If  $(E, \bar{\partial}_E)$  is simple then  $(T, \mathcal{F})$  is a universal family of simple analytic vector bundles.

On the other hand, using algebraic geometry, Proposition 9.10 gives a formally versal family of algebraic vector bundles  $(S, \mathcal{D})$  on  $X$  with  $\mathcal{D}_s \cong \mathcal{E}$  and  $T_s S \cong \text{Ext}^1(\mathcal{E}, \mathcal{E})$ , and if  $\mathcal{E}$  is simple then  $(S, \mathcal{D})$  is a formally universal family of simple algebraic vector bundles. Now Miyajima [79, §3] quotes Serre [97] and Schuster [94]

to say that if  $(S, \mathcal{D})$  is a formally versal (or formally universal) family of algebraic vector bundles on  $X$ , then the induced family of complex analytic vector bundles  $(S_{\text{an}}, \mathcal{D}_{\text{an}})$  is versal (or universal) in the sense of Definition 9.4.

Hence we have two versal families of complex analytic vector bundles:  $(T, \mathcal{F})$  from Propositions 9.3 and 9.5, with  $\mathcal{F}_t \cong (E, \bar{\partial}_E)$ , and  $(S_{\text{an}}, \mathcal{D}_{\text{an}})$  from Proposition 9.10, with  $\mathcal{D}_s \cong (E, \bar{\partial}_E)$ . We will prove these two families are locally isomorphic near  $s, t$ . In the universal case this is obvious, as in Miyajima [79, §3]. In the versal case we use the isomorphisms  $T_t T \cong \text{Ext}^1((E, \bar{\partial}_E), (E, \bar{\partial}_E)) \cong T_s S$ .

**PROPOSITION 9.11.** *Let  $\mathcal{E}$  be an algebraic vector bundle on  $X$ , with underlying complex vector bundle  $E$  and holomorphic structure  $\bar{\partial}_E$ . Let  $(T, \tau), (T, \mathcal{F}), (S, \mathcal{D})$  be the versal families of holomorphic structures, analytic vector bundles, and algebraic vector bundles from Propositions 9.3, 9.5, 9.10, so  $t \in T, s \in S(\mathbb{C})$  with  $\tau(t) = \bar{\partial}_E$ ,  $\mathcal{F}_t \cong (E, \bar{\partial}_E)$ ,  $\mathcal{D}_s \cong \mathcal{E}$  and  $T_t T \cong \text{Ext}^1((E, \bar{\partial}_E), (E, \bar{\partial}_E)) \cong T_s S$ . Write  $(S_{\text{an}}, \mathcal{D}_{\text{an}})$  for the family of analytic vector bundles underlying  $(S, \mathcal{D})$ .*

*Then there exist open neighbourhoods  $T'$  of  $t$  in  $T$  and  $S'_{\text{an}}$  of  $s$  in  $S_{\text{an}}$  and an isomorphism of complex analytic spaces  $\varphi : T' \rightarrow S'_{\text{an}}$  such that  $\varphi(t) = s$  and  $\varphi^*(\mathcal{D}_{\text{an}}) \cong \mathcal{F}|_{X \times T'}$  as analytic vector bundles over  $X \times T'$ .*

**PROOF.** From above,  $(T, \mathcal{F})$  and  $(S_{\text{an}}, \mathcal{D}_{\text{an}})$  are both versal families of analytic vector bundles on  $X$  with  $\mathcal{F}_t \cong (E, \bar{\partial}_E) \cong (\mathcal{D}_{\text{an}})_s$ . By Definition 9.4, since  $(S_{\text{an}}, \mathcal{D}_{\text{an}})$  is versal, there exists an open neighbourhood  $\tilde{T}$  of  $t$  in  $T$  and a morphism of complex analytic spaces  $\tilde{\varphi} : \tilde{T} \rightarrow S_{\text{an}}$  such that  $\tilde{\varphi}(t) = s$  and  $\tilde{\varphi}^*(\mathcal{D}_{\text{an}}) \cong \mathcal{F}|_{\tilde{T}}$ . Similarly, since  $(T, \mathcal{F})$  is versal, there exists an open neighbourhood  $\tilde{S}_{\text{an}}$  of  $s$  in  $S_{\text{an}}$  and a morphism of complex analytic spaces  $\tilde{\psi} : \tilde{S}_{\text{an}} \rightarrow T$  such that  $\tilde{\psi}(s) = t$  and  $\tilde{\psi}^*(\mathcal{F}) \cong \mathcal{D}_{\text{an}}|_{\tilde{S}_{\text{an}}}$ .

Restricting the isomorphism  $\tilde{\varphi}^*(\mathcal{D}_{\text{an}}) \cong \mathcal{F}|_{\tilde{T}}$  to the fibres at  $t$  gives an isomorphism  $\mathcal{D}_s \cong \mathcal{F}_t$ . We are also given isomorphisms  $\mathcal{F}_t \cong (E, \bar{\partial}_E)$  and  $(E, \bar{\partial}_E) \cong \mathcal{D}_s$ . Composing these three  $(E, \bar{\partial}_E) \cong \mathcal{D}_s \cong \mathcal{F}_t \cong (E, \bar{\partial}_E)$  gives an automorphism  $\gamma$  of  $(E, \bar{\partial}_E)$ . Differentiating  $\tilde{\varphi}$  at  $t$  gives a  $\mathbb{C}$ -linear map  $d\tilde{\varphi}|_t : T_t \tilde{T} \rightarrow T_s \tilde{S}_{\text{an}}$ . We also have isomorphisms  $T_t T \cong \text{Ext}^1((E, \bar{\partial}_E), (E, \bar{\partial}_E)) \cong T_s S$ . Using the interpretation of  $\text{Ext}^1((E, \bar{\partial}_E), (E, \bar{\partial}_E))$  as infinitesimal deformations of  $(E, \bar{\partial}_E)$ , one can show that under these identifications  $T_t T \cong \text{Ext}^1((E, \bar{\partial}_E), (E, \bar{\partial}_E)) \cong T_s S$ , the map  $d\tilde{\varphi}|_t : T_t \tilde{T} \rightarrow T_s \tilde{S}_{\text{an}}$  corresponds to conjugation by  $\gamma \in \text{Aut}(E, \bar{\partial}_E)$  in  $\text{Ext}^1((E, \bar{\partial}_E), (E, \bar{\partial}_E))$ . This implies that  $d\tilde{\varphi}|_t : T_t \tilde{T} \rightarrow T_s \tilde{S}_{\text{an}}$  is an isomorphism. Similarly,  $d\tilde{\psi}|_s : T_s \tilde{S}_{\text{an}} \rightarrow T_t \tilde{T}$  is an isomorphism.

Suppose first that  $\mathcal{E}$  is simple. Then  $(T, \mathcal{F}), (S_{\text{an}}, \mathcal{D}_{\text{an}})$  are universal families, so  $\tilde{\varphi}, \tilde{\psi}$  above are unique. Also by universality of  $(T, \mathcal{F})$  we see that  $\tilde{\psi} \circ \tilde{\varphi} \cong \text{id}_T$  on  $\tilde{T} \cap \tilde{\varphi}^{-1}(\tilde{S}_{\text{an}})$ , and similarly  $\tilde{\varphi} \circ \tilde{\psi} \cong \text{id}_{S_{\text{an}}}$  on  $\tilde{S}_{\text{an}} \cap \tilde{\psi}^{-1}(\tilde{T})$ . Hence the restrictions of  $\tilde{\varphi}$  to  $\tilde{T} \cap \tilde{\varphi}^{-1}(\tilde{S}_{\text{an}})$  and  $\tilde{\psi}$  to  $\tilde{S}_{\text{an}} \cap \tilde{\psi}^{-1}(\tilde{T})$  are inverse, and setting  $T' = \tilde{T} \cap \tilde{\varphi}^{-1}(\tilde{S}_{\text{an}})$  and  $\varphi' = \tilde{\varphi}|_{T'}$  gives the result. This argument was used by Miyajima [79, §3].

For the general case,  $\tilde{\psi} \circ \tilde{\varphi} : \tilde{T} \cap \tilde{\varphi}^{-1}(\tilde{S}_{\text{an}}) \rightarrow T$  is a morphism of complex analytic spaces with  $\tilde{\psi} \circ \tilde{\varphi}(t) = t$  and  $d(\tilde{\psi} \circ \tilde{\varphi})|_t : T_t T \rightarrow T_t T$  an isomorphism. We will show that this implies  $\tilde{\psi} \circ \tilde{\varphi}$  is an isomorphism of complex analytic spaces near  $t$ . A similar result in algebraic geometry is Eisenbud [23, Cor. 7.17]. Write  $\mathcal{O}_{t,T}$  for the algebra of germs of analytic functions on  $T$  defined near  $t$ . Write  $\mathfrak{m}_{t,T}$  for the maximal ideal of  $\mathcal{O}_{t,T}$  with  $f(t) = 0$ .

In complex analytic geometry, the operations on  $\mathcal{O}_{t,T}$  are not just addition and multiplication. We can also apply holomorphic functions of several variables: if  $W$  is an open neighbourhood of 0 in  $\mathbb{C}^l$  and  $F : W \rightarrow \mathbb{C}$  is holomorphic, then there is an operation  $F_* : \mathfrak{m}_{t,T}^{\oplus l} \rightarrow \mathcal{O}_{t,T}$  mapping  $F_* : (f_1, \dots, f_l) \mapsto F(f_1, \dots, f_l)$ . Let  $N = \dim T_t T$ , and choose  $g_1, \dots, g_N \in \mathfrak{m}_{t,T}$  such that  $g_1 + \mathfrak{m}_{t,T}^2, \dots, g_N + \mathfrak{m}_{t,T}^2$  are a basis for  $\mathfrak{m}_{t,T}/\mathfrak{m}_{t,T}^2 \cong T_t^* T$ . Then  $g_1, \dots, g_N$  generate  $\mathcal{O}_{t,T}$  over such operations  $F_*$ .

Let  $f \in \mathcal{O}_{t,T}$ . Since  $(\tilde{\psi} \circ \tilde{\varphi})^* : \mathfrak{m}_{t,T}/\mathfrak{m}_{t,T}^2 \rightarrow \mathfrak{m}_{t,T}/\mathfrak{m}_{t,T}^2$  is an isomorphism, we see that  $(\tilde{\psi} \circ \tilde{\varphi})^*(g_1), \dots, (\tilde{\psi} \circ \tilde{\varphi})^*(g_N)$  project to a basis for  $\mathfrak{m}_{t,T}/\mathfrak{m}_{t,T}^2$ , so they generate  $\mathcal{O}_{t,T}$  over operations  $F_*$ . Thus there exists a holomorphic function  $F$  defined near 0 in  $\mathbb{C}^N$  with  $f = F((\tilde{\psi} \circ \tilde{\varphi})^*(g_1), \dots, (\tilde{\psi} \circ \tilde{\varphi})^*(g_N))$ . Hence  $f = (\tilde{\psi} \circ \tilde{\varphi})^*(F(g_1, \dots, g_N))$ . Thus  $(\tilde{\psi} \circ \tilde{\varphi})^* : \mathcal{O}_{t,T} \rightarrow \mathcal{O}_{t,T}$  is surjective. But  $\mathcal{O}_{t,T}$  is noetherian as in Griffiths and Harris [33, p. 679], and a surjective endomorphism of a noetherian ring is an isomorphism. Therefore  $(\tilde{\psi} \circ \tilde{\varphi})^* : \mathcal{O}_{t,T} \rightarrow \mathcal{O}_{t,T}$  is an isomorphism of local algebras.

Since  $\mathcal{O}_{t,T}$  determines  $(T, t)$  as a germ of complex analytic spaces, this implies  $\tilde{\psi} \circ \tilde{\varphi}$  is an isomorphism of complex analytic spaces near  $t$ , as we claimed above. Similarly,  $\tilde{\varphi} \circ \tilde{\psi}$  is an isomorphism of complex analytic spaces near  $s$ . Thus  $\tilde{\varphi}$  and  $\tilde{\psi}$  are isomorphisms of complex analytic spaces near  $s, t$ . So we can choose an open neighbourhood  $T'$  of  $t$  in  $\tilde{T} \cap \tilde{\varphi}^{-1}(\tilde{S}_{\text{an}})$  such that  $\varphi = \tilde{\varphi}|_{T'} : T' \rightarrow S'_{\text{an}} = \tilde{\varphi}(T')$  is an isomorphism of complex analytic spaces. The conditions  $\varphi(t) = s$  and  $\varphi^*(\mathcal{D}_{\text{an}}) \cong \mathcal{F}|_{X \times T'}$  are immediate.  $\square$

### 9.6. Writing the moduli space as $\text{Crit}(f)$

We now return to the situation of §9.1, and suppose  $X$  is a Calabi–Yau 3-fold. Let  $X$  be a compact complex 3-manifold with trivial canonical bundle  $K_X$ , and pick a nonzero section of  $K_X$ , that is, a nonvanishing closed  $(3, 0)$ -form  $\Omega$  on  $X$ .

Fix a  $C^\infty$  complex vector bundle  $E \rightarrow X$  on  $X$ , and choose a holomorphic structure  $\bar{\partial}_E$  on  $E$ . Then  $\mathcal{A}^{2,k}$  is given by (9.2) as in §9.1. Following Thomas [100, §3], define the *holomorphic Chern–Simons functional*  $CS : \mathcal{A}^{2,k} \rightarrow \mathbb{C}$  by

$$CS : \bar{\partial}_E + A \mapsto \frac{1}{4\pi^2} \int_X \text{Tr} \left( \frac{1}{2} (\bar{\partial}_E A) \wedge A + \frac{1}{3} A \wedge A \wedge A \right) \wedge \Omega. \quad (9.6)$$

Here  $A \in L_k^2(\text{End}(E) \otimes_{\mathbb{C}} \Lambda^{0,1} T^* X)$  and  $\bar{\partial}_E A \in L_{k-1}^2(\text{End}(E) \otimes_{\mathbb{C}} \Lambda^{0,2} T^* X)$ . To form  $(\bar{\partial}_E A) \wedge A$  and  $A \wedge A \wedge A$  we take the exterior product of the  $\Lambda^{0,q} T^* X$  factors, and multiply the  $\text{End}(E)$  factors. So  $\frac{1}{2} (\bar{\partial}_E A) \wedge A + \frac{1}{3} A \wedge A \wedge A$  is a section of  $\text{End}(E) \otimes_{\mathbb{C}} \Lambda^{0,3} T^* X$ . We then apply the trace  $\text{Tr} : \text{End}(E) \rightarrow \mathbb{C}$  to get a  $(0, 3)$ -form, wedge with  $\Omega$ , and integrate over  $X$ . As  $k \geq 1$ , Sobolev Embedding and  $X$  compact with  $\dim X = 6$  imply  $A$  is  $L^3$  and  $\bar{\partial}_E A$  is  $L^{3/2}$ , so the integrand in (9.6) is  $L^1$  by Hölder’s inequality, and  $CS$  is well-defined.

This  $CS$  is a cubic polynomial on the infinite-dimensional affine space  $\mathcal{A}^{2,k}$ . It is a well-defined analytic function on  $\mathcal{A}^{2,k}$  in the sense of Douady [21, 22]. An easy calculation shows that for all  $A, a \in L_k^2(\text{End}(E) \otimes_{\mathbb{C}} \Lambda^{0,1} T^* X)$  we have

$$\frac{d}{dt} [CS(\bar{\partial}_E + A + ta)]|_{t=0} = \frac{1}{4\pi^2} \int_X \text{Tr} (a \wedge (\bar{\partial}_E A + A \wedge A)) \wedge \Omega, \quad (9.7)$$



where  $\bar{\partial}_E A + A \wedge A = F_A^{0,2} = P_k(\bar{\partial}_E + A)$  as in (9.3). Essentially, equation (9.7) says that the 1-form  $dCS$  on the affine space  $\mathcal{A}^{2,k}$  is given at  $\bar{\partial}_E + A$  by the  $(0, 2)$ -curvature  $F_A^{0,2}$  of  $\bar{\partial}_E + A$ .

**PROPOSITION 9.12.** *Suppose  $X$  is a compact complex 3-manifold with trivial canonical bundle,  $E \rightarrow X$  a  $C^\infty$  complex vector bundle on  $X$ , and  $\bar{\partial}_E$  a holomorphic structure on  $E$ . Define  $CS : \mathcal{A}^{2,k} \rightarrow \mathbb{C}$  by (9.6). Let  $Q_\epsilon, T$  be as in Proposition 9.3. Then for sufficiently small  $\epsilon > 0$ , as a complex analytic subspace of the finite-dimensional complex submanifold  $Q_\epsilon$ ,  $T$  is the critical locus of the holomorphic function  $CS|_{Q_\epsilon} : Q_\epsilon \rightarrow \mathbb{C}$ .*

**PROOF.** Following [79, §1], define  $R_\epsilon \subset Q_\epsilon \times L_{k-1}^2(\text{End}(E) \otimes_{\mathbb{C}} \Lambda^{0,2} T^* X)$  by

$$R_\epsilon = \{(\bar{\partial}_E + A, B) \in Q_\epsilon \times L_{k-1}^2(\text{End}(E) \otimes_{\mathbb{C}} \Lambda^{0,2} T^* X) : \bar{\partial}_E^* B = 0, \bar{\partial}_E^*(\bar{\partial}_E B - B \wedge A + A \wedge B) = 0\}. \quad (9.8)$$

Then Miyajima [79, Lem. 1.5] shows that for sufficiently small  $\epsilon > 0$ ,  $R_\epsilon$  is a complex submanifold of  $Q_\epsilon \times L_{k-1}^2(\text{End}(E) \otimes_{\mathbb{C}} \Lambda^{0,2} T^* X)$ , and in the notation of §9.1, the projection  $\text{id} \times \pi_{\mathcal{E}^2} : R_\epsilon \rightarrow Q_\epsilon \times \mathcal{E}^2$  is a biholomorphism. Thus the projection  $\pi_{Q_\epsilon} : R_\epsilon \rightarrow Q_\epsilon$  makes  $R_\epsilon$  into a holomorphic vector bundle over  $Q_\epsilon$ , with fibre  $\mathcal{E}^2 \cong \text{Ext}^2((E, \bar{\partial}_E), (E, \bar{\partial}_E))$ . Note from (9.8) that the fibres of  $\pi_{Q_\epsilon}$  are vector subspaces of  $L_{k-1}^2(\text{End}(E) \otimes_{\mathbb{C}} \Lambda^{0,2} T^* X)$ , so  $R_\epsilon$  is a vector subbundle of the infinite-dimensional vector bundle  $Q_\epsilon \times L_{k-1}^2(\text{End}(E) \otimes_{\mathbb{C}} \Lambda^{0,2} T^* X) \rightarrow Q_\epsilon$ .

Let  $\bar{\partial}_E + A \in Q_\epsilon$ , and set  $B = P_k(\bar{\partial}_E + A) = F_A^{0,2} = \bar{\partial}_E A + A \wedge A$ . Then  $\bar{\partial}_E^* B = 0$  by the definition (9.4) of  $Q_\epsilon$ , and  $\bar{\partial}_E B - B \wedge A + A \wedge B = 0$  by the Bianchi identity. So  $(\bar{\partial}_E + A, P_k(\bar{\partial}_E + A)) \in R_\epsilon$ . Thus  $P_k|_{Q_\epsilon}$  is actually a holomorphic section of the holomorphic vector bundle  $R_\epsilon \rightarrow Q_\epsilon$ . The complex analytic subspace  $T$  in  $Q_\epsilon$  is  $T = (P_k|_{Q_\epsilon})^{-1}(0)$ . So we can regard  $T$  as the zeroes of the holomorphic section  $P_k|_{Q_\epsilon}$  of the holomorphic vector bundle  $R_\epsilon \rightarrow Q_\epsilon$ .

Define a holomorphic map  $\Xi : R_\epsilon \rightarrow T^* Q_\epsilon$  by  $\Xi : (\bar{\partial}_E + A, B) \mapsto (\bar{\partial}_E + A, \alpha_B)$ , where  $\alpha_B \in T_{\bar{\partial}_E + A}^* Q_\epsilon$  is defined by

$$\alpha_B(a) = \frac{1}{4\pi^2} \int_X \text{Tr}(a \wedge B \wedge \Omega) \quad (9.9)$$

for all  $a \in T_{\bar{\partial}_E + A} Q_\epsilon \subset L_k^2(\text{End}(E) \otimes_{\mathbb{C}} \Lambda^{0,1} T^* X)$ . Then  $\Xi$  is linear between the fibres of  $R_\epsilon, T^* Q_\epsilon$ , so it is a morphism of holomorphic vector bundles over  $Q_\epsilon$ . Comparing (9.7) and (9.9) we see that when  $B = P_k(\bar{\partial}_E + A) = \bar{\partial}_E A + A \wedge A$  we have  $\alpha_B = d(CS|_{Q_\epsilon})|_{\bar{\partial}_E + A}$ . Hence  $\Xi \circ P_k|_{Q_\epsilon} \equiv d(CS|_{Q_\epsilon})$ , that is,  $\Xi$  takes the holomorphic section  $P_k$  of  $R_\epsilon$  to the holomorphic section  $d(CS|_{Q_\epsilon})$  of  $T^* Q_\epsilon$ .

Now consider the fibres of  $R_\epsilon$  and  $T^* Q_\epsilon$  at  $\bar{\partial}_E \in Q_\epsilon$ . As in [79, §1] we have  $T_{\bar{\partial}_E} Q_\epsilon = \mathcal{E}^1 \cong \text{Ext}^1((E, \bar{\partial}_E), (E, \bar{\partial}_E))$  and  $R_\epsilon|_{\bar{\partial}_E} = \mathcal{E}^2 \cong \text{Ext}^2((E, \bar{\partial}_E), (E, \bar{\partial}_E))$ . But  $X$  is a Calabi–Yau 3-fold, so by Serre duality we have an isomorphism  $\text{Ext}^2((E, \bar{\partial}_E), (E, \bar{\partial}_E)) \cong \text{Ext}^1((E, \bar{\partial}_E), (E, \bar{\partial}_E))^*$ . The linear map  $\Xi|_{\bar{\partial}_E} : R_\epsilon|_{\bar{\partial}_E} \rightarrow T_{\bar{\partial}_E}^* Q_\epsilon$  is a multiple of this isomorphism, so  $\Xi|_{\bar{\partial}_E}$  is an isomorphism. This is an open condition, so by making  $\epsilon > 0$  smaller if necessary we can suppose that  $\Xi : R_\epsilon \rightarrow T^* Q_\epsilon$  is an isomorphism of holomorphic bundles. Since  $\Xi \circ P_k|_{Q_\epsilon} \equiv d(CS|_{Q_\epsilon})$ , it follows that  $T = (P_k|_{Q_\epsilon})^{-1}(0)$  coincides with  $(d(CS|_{Q_\epsilon}))^{-1}(0)$  as a complex analytic subspace of  $Q_\epsilon$ , as we have to prove.  $\square$

### 9.7. The proof of Theorem 5.4

We can now prove Theorem 5.4. The second part of Theorem 5.3 shows that it is enough to prove Theorem 5.4 with  $\mathcal{Vect}_{\text{si}}$  in place of  $\mathcal{M}_{\text{si}}$ . Let  $X$  be a projective Calabi–Yau 3-fold over  $\mathbb{C}$ , and  $\mathcal{E}$  a simple algebraic vector bundle on  $X$ , with underlying  $C^\infty$  complex vector bundle  $E \rightarrow X$  and holomorphic structure  $\bar{\partial}_E$ . Then Proposition 9.3 gives a complex analytic space  $T$ , a point  $t \in T$  with  $T_t T \cong \text{Ext}^1(\mathcal{E}, \mathcal{E})$ , and a universal family  $(T, \tau)$  of simple holomorphic structures on  $E$  with  $\tau(t) = \bar{\partial}_E$ .

Proposition 9.5 shows that  $(T, \tau)$  extends to a universal family  $(T, \mathcal{F})$  of simple analytic vector bundles. Then Proposition 9.10(a) gives an affine  $\mathbb{C}$ -scheme  $S$ , a point  $s \in S_{\text{an}}$ , a formally universal family of simple algebraic vector bundles  $(S, \mathcal{D})$  on  $X$  with  $\mathcal{D}_s \cong \mathcal{E}$ , and an étale map of complex algebraic spaces  $\pi : S \rightarrow \mathcal{Vect}_{\text{si}}$  with  $\pi(s) = [\mathcal{E}]$ . Write  $(S_{\text{an}}, \mathcal{D}_{\text{an}})$  for the underlying family of simple analytic vector bundles. Proposition 9.11 gives an isomorphism of complex analytic spaces  $\varphi : T' \rightarrow S'_{\text{an}}$  between open neighbourhoods  $T'$  of  $t$  in  $T$  and  $S'_{\text{an}}$  of  $s$  in  $S_{\text{an}}$ , with  $\varphi(t) = s$  and  $\varphi^*(\mathcal{D}_{\text{an}}) \cong \mathcal{F}|_{X \times T'}$ . Proposition 9.12 shows that we may write  $T$  as the critical locus of  $CS|_{Q_\epsilon} : Q_\epsilon \rightarrow \mathbb{C}$ , where  $Q_\epsilon$  is a complex manifold with  $T_t Q_\epsilon \cong \text{Ext}^1(\mathcal{E}, \mathcal{E})$ .

Since  $Q_\epsilon$  is a complex manifold with  $T_t Q_\epsilon \cong \text{Ext}^1(\mathcal{E}, \mathcal{E})$ , we may identify  $Q_\epsilon$  near  $t$  with an open neighbourhood  $U$  of  $u = 0$  in  $\text{Ext}^1(\mathcal{E}, \mathcal{E})$ . A natural way to do this is to map  $Q_\epsilon \rightarrow \mathcal{E}^1$  by  $\bar{\partial}_E + A \mapsto \pi_{\mathcal{E}^1}(A)$ , and then use the isomorphism  $\mathcal{E}^1 \cong \text{Ext}^1(\mathcal{E}, \mathcal{E})$ . Let  $f : U \rightarrow \mathbb{C}$  be the holomorphic function identified with  $CS|_{Q_\epsilon} : Q_\epsilon \rightarrow \mathbb{C}$ . Since étale maps of complex algebraic spaces induce local isomorphisms of the underlying complex analytic spaces, putting all this together yields an isomorphism of complex analytic spaces between  $\mathcal{M}_{\text{si}}(\mathbb{C})$  near  $[E]$  and  $\text{Crit}(f)$  near 0, as we want.

### 9.8. The proof of Theorem 5.5

The first part of Theorem 5.3 shows that it is enough to prove Theorem 5.5 with  $\mathfrak{Vect}$  in place of  $\mathfrak{M}$ . Let  $X$  be a projective Calabi–Yau 3-fold over  $\mathbb{C}$ , and  $\mathcal{E}$  an algebraic vector bundle on  $X$ , with underlying  $C^\infty$  complex vector bundle  $E \rightarrow X$  and holomorphic structure  $\bar{\partial}_E$ . Let  $G$  be a maximal compact subgroup in  $\text{Aut}(E)$ , and its complexification  $G^c$  a maximal reductive subgroup in  $\text{Aut}(E)$ . Then Propositions 9.3 and 9.5 give a complex analytic space  $T$ , a point  $t \in T$  with  $T_t T \cong \text{Ext}^1(\mathcal{E}, \mathcal{E})$ , and versal families  $(T, \tau)$  of holomorphic structures on  $E$  and  $(T, \mathcal{F})$  of analytic vector bundles on  $E$ , with  $\tau(t) = \bar{\partial}_E$  and  $\mathcal{F}_t \cong (E, \bar{\partial}_E)$ .

Proposition 9.10(b) gives a quasiprojective  $\mathbb{C}$ -scheme  $S$ , an action of  $G^c$  on  $S$ , a 1-morphism  $\Phi : [S/G^c] \rightarrow \mathfrak{Vect}$  smooth of relative dimension  $\dim \text{Aut}(\mathcal{E}) - \dim G^c$ , a  $G^c$ -invariant point  $s \in S(\mathbb{C})$  with  $\Phi(s^{G^c}) = [\mathcal{E}]$  and  $T_s S \cong \text{Ext}^1(\mathcal{E}, \mathcal{E})$ , and a  $G^c$ -invariant, formally versal family of algebraic vector bundles  $(S, \mathcal{D})$  on  $X$  with  $\mathcal{D}_s \cong \mathcal{E}$ . By Serre [97] we have  $\text{Aut}(\mathcal{E}) = \text{Aut}(E, \bar{\partial}_E)$ , that is, the automorphisms of  $\mathcal{E}$  as an algebraic vector bundle coincide with the automorphisms of  $(E, \bar{\partial}_E)$  as an analytic vector bundle.

Proposition 9.11 gives a local isomorphism of complex analytic spaces between  $T$  near  $t$  and  $S_{\text{an}}$  near  $s$ , and Proposition 9.12 gives an open neighbourhood  $U$  of 0 in  $\text{Ext}^1(\mathcal{E}, \mathcal{E})$  and a holomorphic function  $f : U \rightarrow \mathbb{C}$ , where  $U \cong Q_\epsilon$  and  $f \cong CS|_{Q_\epsilon}$ , and an isomorphism of complex analytic spaces between  $T$  and  $\text{Crit}(f)$  identifying

$t$  with 0. Putting these two isomorphisms together yields an open neighbourhood  $V$  of  $s$  in  $S_{\text{an}}$ , and an isomorphism of complex analytic spaces  $\Xi : \text{Crit}(f) \rightarrow V$  with  $\Xi(0) = s$ .

Consider  $d\Xi|_0 : T_0 \text{Crit}(f) \rightarrow T_s V$ . We have  $T_0 \text{Crit}(f) \cong \text{Ext}^1(\mathcal{E}, \mathcal{E}) \cong T_s V$  by Propositions 9.3 and 9.10(b). The isomorphism  $T_0 \text{Crit}(f) \cong \text{Ext}^1(\mathcal{E}, \mathcal{E})$  is determined by a choice of isomorphism of analytic vector bundles  $\eta_1 : (E, \bar{\partial}_E) \rightarrow \mathcal{F}_t$ . The isomorphism  $\text{Ext}^1(\mathcal{E}, \mathcal{E}) \cong T_s V$  is determined by a choice of isomorphism of analytic vector bundles  $\eta_2 : (\mathcal{D}_{\text{an}})_s \rightarrow (E, \bar{\partial}_E)$ . The map  $\Xi$  is determined by a choice of local isomorphism of versal families of analytic vector bundles  $\eta_3$  from  $(T, \mathcal{F})$  near  $t$  to  $(S_{\text{an}}, \mathcal{D}_{\text{an}})$  near  $s$ . Composing gives an isomorphism  $\eta_2 \circ \eta_3|_t \circ \eta_1 : (E, \bar{\partial}_E) \rightarrow (E, \bar{\partial}_E)$ , so that  $\eta_2 \circ \eta_3|_t \circ \eta_1$  lies in  $\text{Aut}(E, \bar{\partial}_E)$ .

Following the definitions through we find that  $d\Xi|_0 : \text{Ext}^1(\mathcal{E}, \mathcal{E}) \rightarrow \text{Ext}^1(\mathcal{E}, \mathcal{E})$  is conjugation by  $\gamma = \eta_2 \circ \eta_3|_t \circ \eta_1$  in  $\text{Aut}(E, \bar{\partial}_E) = \text{Aut}(\mathcal{E})$ . But in constructing  $\eta_3$  we were free to choose the isomorphism  $\eta_3|_t : \mathcal{F}_t \rightarrow (\mathcal{D}_{\text{an}})_s$ , and we choose it to make  $\gamma = \text{id}_{\mathcal{E}}$ , so that  $d\Xi|_0$  is the identity on  $\text{Ext}^1(\mathcal{E}, \mathcal{E})$ . This proves the first part of the third paragraph of Theorem 5.5. It remains to prove the final part, that if  $G$  is a maximal compact subgroup of  $\text{Aut}(\mathcal{E})$  then we can take  $U, f$  to be  $G^c$ -invariant, and  $\Xi$  to be  $G^c$ -equivariant.

First we show that we can take  $U, f$  to be  $G$ -invariant. Now  $\text{Aut}(E, \bar{\partial}_E)$  acts on  $\mathcal{A}^{2,k}$  fixing  $\bar{\partial}_E$  by  $\gamma : \bar{\partial}_E + A \mapsto \bar{\partial}_E + \gamma^{-1} \circ A \circ \gamma$ , as in (9.1), since  $\bar{\partial}_E \gamma = 0$  for  $\gamma \in \text{Aut}(E, \bar{\partial}_E)$ . However, the construction of  $(T, \tau)$  in §9.1 involves a choice of metric  $h_E$  on the fibres of  $E$ , which is used to define  $\bar{\partial}_E^*$ , and the norm in the condition  $\|A\|_{L_k^2} < \epsilon$  in (9.4). By averaging  $h_E$  over the action of  $G$ , which is compact, we can choose  $h_E$  to be  $G$ -invariant. Then  $\bar{\partial}_E^*$  is  $G$ -equivariant, and  $\|\cdot\|_{L_k^2}$  is  $G$ -invariant, so  $Q_\epsilon$  in (9.4) is  $G$ -invariant, and as  $P_k$  is  $G$ -equivariant the analytic subspace  $T = (P_k|_{Q_\epsilon})^{-1}(0)$  in  $Q_\epsilon$  is also  $G$ -invariant.

In §9.6, since  $dCS$  maps  $\bar{\partial}_E'$  to its  $(0, 2)$ -curvature by (9.7), it is equivariant under the gauge group  $\mathcal{G}$ , so its first integral  $CS : \mathcal{A}^{2,k} \rightarrow \mathbb{C}$  is invariant under the subgroup  $\text{Aut}(E, \bar{\partial}_E)$  of  $\mathcal{G}$  fixing the point  $\bar{\partial}_E$  in  $\mathcal{A}^{2,k}$ , and  $CS|_{Q_\epsilon}$  is invariant under the maximal compact subgroup  $G$  of  $\text{Aut}(E, \bar{\partial}_E)$ . We choose the identification of  $Q_\epsilon$  with an open subset  $U$  of  $\text{Ext}^1(\mathcal{E}, \mathcal{E})$  to be the composition of the map  $Q_\epsilon \rightarrow \mathcal{E}^1$  taking  $\bar{\partial}_E + A \mapsto \pi_{\mathcal{E}^1}(A)$  with the isomorphism  $\mathcal{E}^1 \cong \text{Ext}^1(\mathcal{E}, \mathcal{E})$ . As both of these are  $G$ -equivariant, we see that  $U \subset \text{Ext}^1(\mathcal{E}, \mathcal{E})$  and  $f : U \rightarrow \mathbb{C}$  are both  $G$ -invariant.

Then in Proposition 9.11, each of  $(T, \tau), (T, \mathcal{F}), (S, \mathcal{D})$  is equivariant under an action of  $G$ , which fixes  $t, 0$  and acts on  $T_t T \cong \text{Ext}^1(\mathcal{E}, \mathcal{E}) \cong T_0 S$  through the action of  $\text{Aut}(\mathcal{E})$  on  $\text{Ext}^1(\mathcal{E}, \mathcal{E})$ . We can choose the isomorphism of versal families of analytic vector bundles in Proposition 9.11 to be  $G$ -equivariant, since the proofs of the versality property extend readily to equivariant versality under a compact Lie group. This then implies that  $\Xi : \text{Crit}(f) \rightarrow V$  is  $G$ -equivariant.

Next we modify  $U, f, \Xi$  to make them  $G^c$ -invariant or  $G^c$ -equivariant. Let  $U'$  be a  $G$ -invariant connected open neighbourhood of 0 in  $U \subseteq \text{Ext}^1(\mathcal{E}, \mathcal{E})$ . Define  $V' = \Xi(U') \subset S_{\text{an}}$ . Define  $U^c = G^c \cdot U'$  in  $\text{Ext}^1(\mathcal{E}, \mathcal{E})$  and  $V^c = G^c \cdot V'$  in  $S_{\text{an}}$ . Then  $U^c, V^c$  are  $G^c$ -invariant, and are open in  $\text{Ext}^1(\mathcal{E}, \mathcal{E}), S_{\text{an}}$ , as they are unions of open sets  $\gamma \cdot U, \gamma \cdot V$  over all  $\gamma \in G^c$ .

We wish to define  $f^c : U^c \rightarrow \mathbb{C}$  by  $f^c(\gamma \cdot u) = f(u)$  for  $\gamma \in G^c$  and  $u \in U'$ , and  $\Xi^c : \text{Crit}(f^c) \rightarrow V^c$  by  $\Xi^c(\gamma \cdot u) = \gamma \cdot \Xi(u)$  for  $\gamma \in G^c$  and  $u \in \text{Crit}(f|_{U'})$ . Clearly  $f^c$  is  $G^c$ -invariant, and  $\Xi^c$  is  $G^c$ -equivariant, provided they are well-defined. To show

they are, we must prove that if  $\gamma_1, \gamma_2 \in G^c$  and  $u_1, u_2 \in U'$  with  $\gamma_1 \cdot u_1 = \gamma_2 \cdot u_2$  then  $f(u_1) = f(u_2)$ , and  $\gamma_1 \cdot \Xi(u_1) = \gamma_2 \cdot \Xi(u_2)$ .

The  $G^c$ -orbit  $G^c \cdot u_1 = G^c \cdot u_2$  is a  $G$ -invariant complex submanifold of  $\text{Ext}^1(\mathcal{E}, \mathcal{E})$ , so  $(G^c \cdot u_1) \cap U$  is a  $G$ -invariant complex submanifold of  $U$ . Since  $f$  is  $G$ -invariant, it is constant on each  $G$ -orbit in  $(G^c \cdot u_1) \cap U$ , so as  $f$  is holomorphic it is constant on each connected component of  $(G^c \cdot u_1) \cap U$ . We require that the  $G$ -invariant open neighbourhood  $U'$  of 0 in  $U$  should satisfy the following condition: whenever  $u_1, u_2 \in U'$  with  $G^c \cdot u_1 = G^c \cdot u_2$ , then the connected component of  $(G^c \cdot u_1) \cap U$  containing  $u_1$  should intersect  $G \cdot u_2$ . This is true provided  $U'$  is sufficiently small.

Suppose this condition holds. Then  $f$  is constant on the connected component of  $(G^c \cdot u_1) \cap U$  containing  $u_1$ , with value  $f(u_1)$ . This component intersects  $G \cdot u_2$ , so it contains  $\gamma \cdot u_2$  for  $\gamma \in G$ . Hence  $f(u_1) = f(\gamma \cdot u_2) = f(u_2)$  by  $G$ -invariance of  $f$ , and  $f^c$  is well-defined. To show  $\Xi^c$  is well-defined we use a similar argument, based on the fact that if  $\gamma \in G^c$  and  $u, \gamma \cdot u$  lie in the same connected component of  $(G^c \cdot u) \cap U$  then  $\Xi(\gamma \cdot u) = \gamma \cdot \Xi(u)$ , since this holds for  $\gamma \in G$  and  $\Xi$  is holomorphic. Then  $U^c, f^c, V^c, \Xi^c$  satisfy the last part of Theorem 5.5, completing the proof.



## The proof of Theorem 5.11

Next we prove Theorem 5.11. Sections 10.1 and 10.2 prove equations (5.2) and (5.3). The authors got an important idea in the proof, that of proving (5.2)–(5.3) by localizing at the fixed points of the action of  $\{\mathrm{id}_{E_1} + \lambda \mathrm{id}_{E_2} : \lambda \in \mathrm{U}(1)\}$  on  $\mathrm{Ext}^1(E_1 \oplus E_2, E_1 \oplus E_2)$ , from Kontsevich and Soibelman [63, §4.4 & §6.3].

### 10.1. Proof of equation (5.2)

We now prove equation (5.2) of Theorem 5.11. Let  $X$  be a Calabi–Yau 3-fold over  $\mathbb{C}$ ,  $\mathfrak{M}$  the moduli stack of coherent sheaves on  $X$ , and  $E_1, E_2$  be coherent sheaves on  $X$ . Set  $E = E_1 \oplus E_2$ . Choose a maximal compact subgroup  $G$  of  $\mathrm{Aut}(E)$  which contains the  $\mathrm{U}(1)$ -subgroup  $T = \{\mathrm{id}_{E_1} + \lambda \mathrm{id}_{E_2} : \lambda \in \mathrm{U}(1)\}$ . Apply Theorem 5.5 with these  $E$  and  $G$ . This gives an  $\mathrm{Aut}(E)$ -invariant  $\mathbb{C}$ -subscheme  $S$  in  $\mathrm{Ext}^1(E, E)$  with  $0 \in S$  and  $T_0 S = \mathrm{Ext}^1(E, E)$ , an étale 1-morphism  $\Phi : [S / \mathrm{Aut}(E)] \rightarrow \mathfrak{M}$  with  $\Phi([0]) = [E]$ , a  $G^c$ -invariant open neighbourhood  $U$  of 0 in  $\mathrm{Ext}^1(E, E)$  in the analytic topology, a  $G^c$ -invariant holomorphic function  $f : U \rightarrow \mathbb{C}$  with  $f(0) = \mathrm{d}f|_0 = 0$ , a  $G^c$ -invariant open neighbourhood  $V$  of 0 in  $S_{\mathrm{an}}$ , and a  $G^c$ -equivariant isomorphism of complex analytic spaces  $\Xi : \mathrm{Crit}(f) \rightarrow V$  with  $\Xi(0) = 0$  and  $\mathrm{d}\Xi|_0$  the identity map on  $\mathrm{Ext}^1(E, E)$ .

Then the Behrend function  $\nu_{\mathfrak{M}}$  at  $[E] = [E_1 \oplus E_2]$  satisfies

$$\begin{aligned} \nu_{\mathfrak{M}}(E_1 \oplus E_2) &= \nu_{[S / \mathrm{Aut}(E)]}(0) = (-1)^{\dim \mathrm{Aut}(E)} \nu_S(0) \\ &= (-1)^{\dim \mathrm{Aut}(E) + \dim \mathrm{Ext}^1(E, E)} (1 - \chi(MF_f(0))), \end{aligned} \quad (10.1)$$

where in the first step we use that as  $\Phi$  is étale it is smooth of relative dimension 0, Theorem 4.3(ii), and Corollary 4.5, in the second step Proposition 4.4, and in the third Theorem 4.7.

To define the Milnor fibre  $MF_f(0)$  of  $f$  we use a Hermitian metric on  $\mathrm{Ext}^1(E, E)$  invariant under the action of the compact Lie group  $G$ . Since  $U, f$  are  $G$ -invariant, it follows that  $\Phi_{f,0}$  and its domain is  $G$ -invariant, so each fibre  $\Phi_{f,0}^{-1}(z)$  for  $0 < |z| < \epsilon$  is  $G$ -invariant. Thus  $G$ , and its  $\mathrm{U}(1)$ -subgroup  $T$ , acts on the Milnor fibre  $MF_f(0)$ . Now  $MF_f(0)$  is a manifold, the interior of a compact manifold with boundary  $\overline{MF_f(0)}$ , and  $T$  acts smoothly on  $MF_f(0)$  and  $\overline{MF_f(0)}$ . Each orbit of  $T$  on  $MF_f(0)$  is either a single point, a fixed point of  $T$ , or a circle  $S^1$ . The circle orbits contribute zero to  $\chi(MF_f(0))$ , as  $\chi(S^1) = 0$ , so

$$\chi(MF_f(0)) = \chi(MF_f(0)^T), \quad (10.2)$$

where  $MF_f(0)^T$  is the fixed point set of  $T$  in  $MF_f(0)$ .

Consider how  $T = \{\mathrm{id}_{E_1} + \lambda \mathrm{id}_{E_2} : \lambda \in \mathrm{U}(1)\}$  acts on

$$\mathrm{Ext}^1(E, E) = \mathrm{Ext}^1(E_1, E_1) \times \mathrm{Ext}^1(E_2, E_2) \times \mathrm{Ext}^1(E_1, E_2) \times \mathrm{Ext}^1(E_2, E_1). \quad (10.3)$$

As in Theorem 5.5,  $\gamma \in T$  acts on  $\epsilon \in \text{Ext}^1(E, E)$  by  $\gamma : \epsilon \mapsto \gamma \circ \epsilon \circ \gamma^{-1}$ . So  $\text{id}_{E_1} + \lambda \text{id}_{E_2}$  fixes the first two factors on the r.h.s. of (10.3), multiplies the third by  $\lambda^{-1}$  and the fourth by  $\lambda$ . Therefore

$$\text{Ext}^1(E, E)^T = \text{Ext}^1(E_1, E_1) \times \text{Ext}^1(E_2, E_2) \times \{0\} \times \{0\}. \quad (10.4)$$

Now  $MF_f(0)^T = MF_f(0) \cap \text{Ext}^1(E, E)^T = MF_f|_{\text{Ext}^1(E, E)^T}(0)$ . But  $\text{Crit}(f)^T = \text{Crit}(f|_{\text{Ext}^1(E, E)^T})$ . Also as  $\Xi$  is  $T$ -equivariant, it induces a local isomorphism of complex analytic spaces between  $S_{\text{an}}^T$  near 0 and  $\text{Crit}(f)^T$  near 0. Hence

$$\begin{aligned} \nu_{S^T}(0) &= (-1)^{\dim \text{Ext}^1(E_1, E_1) + \dim \text{Ext}^1(E_2, E_2)} (1 - \chi(MF_f|_{\text{Ext}^1(E, E)^T}(0))) \\ &= (-1)^{\dim \text{Ext}^1(E_1, E_1) + \dim \text{Ext}^1(E_2, E_2)} (1 - \chi(MF_f(0)^T)) \\ &= (-1)^{\dim \text{Ext}^1(E_1, E_1) + \dim \text{Ext}^1(E_2, E_2)} (1 - \chi(MF_f(0))), \end{aligned} \quad (10.5)$$

using Theorem 4.7 and equations (10.2) and (10.4).

Let  $s' \in S^T(\mathbb{C}) \subseteq S(\mathbb{C})$ , and set  $[E'] = \Phi_*(s')$  in  $\mathfrak{M}(\mathbb{C})$ , so that  $E' \in \text{coh}(X)$ . As  $\Phi$  is étale, it induces isomorphisms of stabilizer groups. But  $\text{Iso}_{[S/\text{Aut}(E)]}(s') = \text{Stab}_{\text{Aut}(E)}(s')$ , and  $\text{Iso}_{\mathfrak{M}}([E']) = \text{Aut}(E')$ , so we have an isomorphism of complex Lie groups  $\Phi_* : \text{Stab}_{\text{Aut}(E)}(s') \rightarrow \text{Aut}(E')$ . As  $s' \in S^T(\mathbb{C})$  we have  $T \subset \text{Stab}_{\text{Aut}(E)}(s')$ , so  $\Phi_*|_T : T \rightarrow \text{Aut}(E')$  is an injective morphism of Lie groups. Let  $R$  be the  $\mathbb{C}$ -subscheme of points  $s'$  in  $S^T$  for which  $\Phi_*|_T(\text{id}_{E_1} + \lambda \text{id}_{E_2}) = \text{id}_{E'_1} + \lambda \text{id}_{E'_2}$  for some splitting  $E' \cong E'_1 \oplus E'_2$  and all  $\lambda \in \text{U}(1)$ . Taking  $E'_1 = E_1$ ,  $E'_2 = E_2$  shows that  $0 \in R(\mathbb{C})$ .

We claim  $R$  is open and closed in  $S^T$ . To see this, note that  $\Phi_T$  is of the form  $\Phi_*|_T(\text{id}_{E_1} + \lambda \text{id}_{E_2}) = \lambda^{a_1} \text{id}_{F_1} + \dots + \lambda^{a_k} \text{id}_{F_k}$ , for some splitting  $E' = F_1 \oplus \dots \oplus F_k$  with  $F_1, \dots, F_k$  indecomposable and  $a_1, \dots, a_k \in \mathbb{Z}$ . Then  $R$  is the subset of  $s'$  with  $\{a_1, \dots, a_k\} = \{0, 1\}$ . Therefore we see that

$$E' \cong \bigoplus_{a \in \mathbb{Z}} \text{Ker}(\lambda^a \text{id}_{E'} - \Phi_*|_T(\text{id}_{E_1} + \lambda \text{id}_{E_2})) \quad (10.6)$$

for  $\lambda \in \text{U}(1)$  not of finite order, with only finitely many nonzero terms. Now the Hilbert polynomial at  $n \gg 0$  of each term on the r.h.s. of (10.6) is upper semicontinuous in  $S^T$ , and of the l.h.s. is locally constant in  $S^T$ . Hence the Hilbert polynomials of each term in (10.6) are locally constant in  $S^T$ , and in particular, whether  $\text{Ker}(\lambda^a \text{id}_{E'} - \Phi_*|_T(\text{id}_{E_1} + \lambda \text{id}_{E_2})) \neq 0$  is locally constant in  $S^T$ . As  $R$  is the subset of  $s'$  with  $\text{Ker}(\lambda^a \text{id}_{E'} - \Phi_*|_T(\text{id}_{E_1} + \lambda \text{id}_{E_2})) \neq 0$  if and only if  $a = 0, 1$ , we see  $R$  is open and closed in  $S^T$ .

The subgroup  $\text{Aut}(E_1) \times \text{Aut}(E_2)$  of  $\text{Aut}(E)$  commutes with  $T$ . Hence the action of  $\text{Aut}(E_1) \times \text{Aut}(E_2)$  on  $S$  induced by the action of  $\text{Aut}(E)$  on  $S$  preserves  $S^T$ . The action of  $\text{Aut}(E_1) \times \text{Aut}(E_2)$  on  $s' \in S^T(\mathbb{C})$  does not change  $E'$  or  $\Phi_*|_T : T \rightarrow \text{Aut}(E')$  above up to isomorphism, so  $\text{Aut}(E_1) \times \text{Aut}(E_2)$  also preserves  $R$ . Hence we can form the quotient stack  $[R/\text{Aut}(E_1) \times \text{Aut}(E_2)]$ . The inclusions  $R \hookrightarrow S$ ,  $\text{Aut}(E_1) \times \text{Aut}(E_2) \hookrightarrow \text{Aut}(E)$  induce a 1-morphism of quotient stacks  $\iota : [R/\text{Aut}(E_1) \times \text{Aut}(E_2)] \rightarrow [S/\text{Aut}(E)]$ . The family of coherent sheaves parametrized by  $S$ ,  $E_S$ , pulls back to a family of coherent sheaves,  $E_R$ , parametrized by  $R$ . By definition of  $R$  we have a global splitting  $E_R \cong E_{R,1} \oplus E_{R,2}$ , where  $E_{R,1}, E_{R,2}$  are the eigensubsheaves of  $\Phi_*|_T(\text{id}_{E_1} + \lambda \text{id}_{E_2})$  in  $E_R$  with eigenvalues 1,  $\lambda$ . These  $E_{R,1}, E_{R,2}$  induce a 1-morphism  $\Psi$  from  $[R/\text{Aut}(E_1) \times \text{Aut}(E_2)]$  to  $\mathfrak{M} \times \mathfrak{M}$ .

Then we have a commutative diagram of 1-morphisms of Artin  $\mathbb{C}$ -stacks

$$\begin{array}{ccc}
 [R/\mathrm{Aut}(E_1) \times \mathrm{Aut}(E_2)] & \xrightarrow{\quad \iota \quad} & [S/\mathrm{Aut}(E)] \\
 \downarrow \Psi & & \downarrow \Phi \\
 \mathfrak{M} \times \mathfrak{M} & \xrightarrow{\quad \Lambda \quad} & \mathfrak{M},
 \end{array} \tag{10.7}$$

where  $\Lambda : \mathfrak{M} \times \mathfrak{M} \rightarrow \mathfrak{M}$  is the 1-morphism acting on points as  $\Lambda : (E'_1, E'_2) \mapsto E'_1 \oplus E'_2$ , such that  $\Psi$  maps  $[0]$  to  $[(E_1, E_2)]$ , with  $\Psi_* : \mathrm{Iso}_{[R/\mathrm{Aut}(E_1) \times \mathrm{Aut}(E_2)]}(0) \rightarrow \mathrm{Iso}_{\mathfrak{M} \times \mathfrak{M}}(E_1, E_2)$  the identity map on  $\mathrm{Aut}(E_1) \times \mathrm{Aut}(E_2)$ . Furthermore, we will show that (10.7) is *locally 2-Cartesian*, in the sense that  $[R/\mathrm{Aut}(E_1) \times \mathrm{Aut}(E_2)]$  is 1-isomorphic to an open substack  $\mathfrak{N}$  of the fibre product  $(\mathfrak{M} \times \mathfrak{M}) \times_{\Lambda, \mathfrak{M}, \Phi} [S/\mathrm{Aut}(E)]$ . Since the diagram (10.7) commutes, there exists a 1-morphism  $\chi : [R/\mathrm{Aut}(E_1) \times \mathrm{Aut}(E_2)] \rightarrow (\mathfrak{M} \times \mathfrak{M}) \times_{\Lambda, \mathfrak{M}, \Phi} [S/\mathrm{Aut}(E)]$ . It is sufficient to construct a local inverse for  $\chi$ .

The reason it may not be globally 2-Cartesian is that there might be points  $s' \in S$  with  $\Phi_*([s']) = [E'_1 \oplus E'_2]$ , so that  $\Phi_* : \mathrm{Stab}_{\mathrm{Aut}(E)}(s') \rightarrow \mathrm{Aut}(E'_1 \oplus E'_2)$  is an isomorphism, but such that the  $\mathrm{U}(1)$ -subgroup  $\Phi_*^{-1}(\{\mathrm{id}_{E'_1} + \lambda \mathrm{id}_{E'_2} : \lambda \in \mathrm{U}(1)\})$  in  $\mathrm{Aut}(E)$  is not conjugate to  $T$  in  $\mathrm{Aut}(E)$ . Then  $s', E'_1, E'_2$  would yield a point in  $(\mathfrak{M} \times \mathfrak{M}) \times_{\Lambda, \mathfrak{M}, \Phi} [S/\mathrm{Aut}(E)]$  not corresponding to a point of  $[R/\mathrm{Aut}(E_1) \times \mathrm{Aut}(E_2)]$ . However, since  $\mathrm{U}(1)$ -subgroups of  $\mathrm{Aut}(E)$  up to conjugation are discrete objects, the condition that  $\Phi_*^{-1}(\{\mathrm{id}_{E'_1} + \lambda \mathrm{id}_{E'_2} : \lambda \in \mathrm{U}(1)\})$  is conjugate to  $T$  in  $\mathrm{Aut}(E)$  is open in  $(\mathfrak{M} \times \mathfrak{M}) \times_{\Lambda, \mathfrak{M}, \Phi} [S/\mathrm{Aut}(E)]$ . Write  $\mathfrak{N}$  for this open substack of  $(\mathfrak{M} \times \mathfrak{M}) \times_{\Lambda, \mathfrak{M}, \Phi} [S/\mathrm{Aut}(E)]$ . Then  $\chi$  maps  $[R/\mathrm{Aut}(E_1) \times \mathrm{Aut}(E_2)] \rightarrow \mathfrak{N}$ .

Let  $B$  be a base  $\mathbb{C}$ -scheme and  $\theta : B \rightarrow \mathfrak{N}$  a 1-morphism. Then  $(B, \theta)$  parametrizes the following objects: a principal  $\mathrm{Aut}(E)$ -torsor  $\eta : P \rightarrow B$ ; an  $\mathrm{Aut}(E)$ -equivariant morphism  $\zeta : P \rightarrow S$ ; a  $B$ -family of coherent sheaves  $E_B \cong E_{B,1} \oplus E_{B,2}$ ; and an isomorphism  $\zeta^*(E_S) \cong \eta^*(E_B)$ , where  $E_S$  is the family of coherent sheaves parametrized by  $S$ . The open condition on  $\mathfrak{N}$  implies that  $\zeta$  maps  $P$  into  $R \subset S^T$ . The isomorphism between  $\zeta^*(E_S)$  and  $\eta^*(E_B)$  implies there exists an  $(\mathrm{Aut}(E_1) \times \mathrm{Aut}(E_2))$ -subtorsor  $Q$  of  $P$  over  $B$  and the restriction of  $\zeta$  to  $Q$  is  $(\mathrm{Aut}(E_1) \times \mathrm{Aut}(E_2))$ -equivariant. Therefore  $\theta$  induces a 1-morphism  $\kappa : B \rightarrow [R/\mathrm{Aut}(E_1) \times \mathrm{Aut}(E_2)]$ . As this holds functorially for all  $B, \theta$  there is a 1-morphism  $\xi : \mathfrak{N} \rightarrow [R/\mathrm{Aut}(E_1) \times \mathrm{Aut}(E_2)]$  with  $\kappa$  2-isomorphic to  $\xi \circ \theta$  for all such  $B, \theta$ , and  $\xi$  is the required inverse for  $\chi$ .

Since (10.7) is locally 2-Cartesian and  $\Phi$  is étale,  $\Psi$  is étale. Thus  $\Psi$  is smooth of relative dimension 0, and Corollary 4.5 and Theorem 4.3(ii) imply that  $\nu_{[R/\mathrm{Aut}(E_1) \times \mathrm{Aut}(E_2)]} = \Psi^*(\nu_{\mathfrak{M} \times \mathfrak{M}})$ . Hence

$$\begin{aligned}
 \nu_{\mathfrak{M}}(E_1) \nu_{\mathfrak{M}}(E_2) &= \nu_{\mathfrak{M} \times \mathfrak{M}}(E_1, E_2) = \nu_{[R/\mathrm{Aut}(E_1) \times \mathrm{Aut}(E_2)]}(0) \\
 &= (-1)^{\dim \mathrm{Aut}(E_1) + \dim \mathrm{Aut}(E_2)} \nu_R(0) \\
 &= (-1)^{\dim \mathrm{Aut}(E_1) + \dim \mathrm{Aut}(E_2)} \nu_{S^T}(0),
 \end{aligned} \tag{10.8}$$

using Theorem 4.3(iii) and Corollary 4.5 in the first step,  $\nu_{[R/\mathrm{Aut}(E_1) \times \mathrm{Aut}(E_2)]} = \Psi^*(\nu_{\mathfrak{M} \times \mathfrak{M}})$  and  $\Psi_*([0]) = [(E_1, E_2)]$  in the second, Proposition 4.4 in the third, and  $R$  open in  $S^T$  in the fourth.



Combining equations (10.1), (10.5) and (10.8) yields

$$\begin{aligned} \nu_{\mathfrak{M}}(E_1 \oplus E_2) &= (-1)^{\dim \operatorname{Aut}(E) + \dim \operatorname{Ext}^1(E, E)} \\ &\quad (-1)^{\dim \operatorname{Ext}^1(E_1, E_1) + \dim \operatorname{Ext}^1(E_2, E_2)} \\ &\quad (-1)^{\dim \operatorname{Aut}(E_1) + \dim \operatorname{Aut}(E_2)} \nu_{\mathfrak{M}}(E_1) \nu_{\mathfrak{M}}(E_2). \end{aligned} \quad (10.9)$$

To sort out the signs, note that  $\operatorname{Aut}(E)$  is open in

$$\operatorname{Hom}(E, E) = \operatorname{Hom}(E_1, E_1) \oplus \operatorname{Hom}(E_2, E_2) \oplus \operatorname{Hom}(E_1, E_2) \oplus \operatorname{Hom}(E_2, E_1).$$

Cancelling  $(-1)^{\dim \operatorname{Hom}(E_i, E_i)}$ ,  $(-1)^{\dim \operatorname{Ext}^1(E_i, E_i)}$  for  $i = 1, 2$ , the sign in (10.9) becomes  $(-1)^{\dim \operatorname{Hom}(E_1, E_2) + \dim \operatorname{Hom}(E_2, E_1) + \dim \operatorname{Ext}^1(E_1, E_2) + \dim \operatorname{Ext}^1(E_2, E_1)}$ . As  $X$  is a Calabi–Yau 3-fold, Serre duality gives  $\dim \operatorname{Hom}(E_2, E_1) = \dim \operatorname{Ext}^3(E_1, E_2)$  and  $\dim \operatorname{Ext}^1(E_2, E_1) = \dim \operatorname{Ext}^2(E_1, E_2)$ . Hence the overall sign in (10.9) is

$$(-1)^{\dim \operatorname{Hom}(E_1, E_2) - \dim \operatorname{Ext}^1(E_1, E_2) + \dim \operatorname{Ext}^2(E_1, E_2) - \dim \operatorname{Ext}^3(E_1, E_2)},$$

which is  $(-1)^{\bar{\chi}([E_1], [E_2])}$ , proving (5.2).

## 10.2. Proof of equation (5.3)

We continue to use the notation of §10.1. Using the splitting (10.3), write elements of  $\operatorname{Ext}^1(E, E)$  as  $(\epsilon_{11}, \epsilon_{22}, \epsilon_{12}, \epsilon_{21})$  with  $\epsilon_{ij} \in \operatorname{Ext}^1(E_i, E_j)$ .

PROPOSITION 10.1. *Let  $\epsilon_{12} \in \operatorname{Ext}^1(E_1, E_2)$  and  $\epsilon_{21} \in \operatorname{Ext}^1(E_2, E_1)$ . Then*

- (i)  $(0, 0, \epsilon_{12}, 0), (0, 0, 0, \epsilon_{21}) \in \operatorname{Crit}(f) \subseteq U \subseteq \operatorname{Ext}^1(E, E)$ , and  $(0, 0, \epsilon_{12}, 0), (0, 0, 0, \epsilon_{21}) \in V \subseteq S(\mathbb{C}) \subseteq \operatorname{Ext}^1(E, E)$ ;
- (ii)  $\Xi$  maps  $(0, 0, \epsilon_{12}, 0) \mapsto (0, 0, \epsilon_{12}, 0)$  and  $(0, 0, 0, \epsilon_{21}) \mapsto (0, 0, 0, \epsilon_{21})$ ; and
- (iii)  $\Phi_* : [S/\operatorname{Aut}(E)](\mathbb{C}) \rightarrow \mathfrak{M}(\mathbb{C})$ , the induced morphism on closed points, maps  $[(0, 0, 0, \epsilon_{21})] \mapsto [F]$  and  $[(0, 0, \epsilon_{12}, 0)] \mapsto [F']$ , where the exact sequences  $0 \rightarrow E_1 \rightarrow F \rightarrow E_2 \rightarrow 0$  and  $0 \rightarrow E_2 \rightarrow F' \rightarrow E_1 \rightarrow 0$  in  $\operatorname{coh}(X)$  correspond to  $\epsilon_{21} \in \operatorname{Ext}^1(E_2, E_1)$  and  $\epsilon_{12} \in \operatorname{Ext}^1(E_1, E_2)$ , respectively.

PROOF. For (i)  $T^c = \{\operatorname{id}_{E_1} + \lambda \operatorname{id}_{E_2} : \lambda \in \mathbb{G}_m\}$ , which acts on  $\operatorname{Ext}^1(E, E)$  by

$$\lambda : (\epsilon_{11}, \epsilon_{22}, \epsilon_{12}, \epsilon_{21}) \mapsto (\epsilon_{11}, \epsilon_{22}, \lambda^{-1} \epsilon_{12}, \lambda \epsilon_{21}). \quad (10.10)$$

Since  $U$  is an open neighbourhood of 0 in  $\operatorname{Ext}^1(E, E)$  in the analytic topology, we see that  $(0, 0, \lambda^{-1} \epsilon_{12}, 0) \in U$  for  $|\lambda| \gg 1$  and  $(0, 0, 0, \lambda \epsilon_{21}) \in U$  for  $0 < |\lambda| \ll 1$ . Hence  $(0, 0, \epsilon_{12}, 0), (0, 0, 0, \epsilon_{21}) \in U$  as  $U$  is  $G^c$ -invariant, and so  $T^c$ -invariant.

As  $f$  is  $T^c$ -invariant we have  $f(\epsilon_{11}, \epsilon_{22}, \epsilon_{12}, 0) = f(\epsilon_{11}, \epsilon_{22}, \lambda^{-1} \epsilon_{12}, 0)$ , so taking the limit  $\lambda \rightarrow \infty$  and using continuity of  $f$  gives  $f(\epsilon_{11}, \epsilon_{22}, \epsilon_{12}, 0) = f(\epsilon_{11}, \epsilon_{22}, 0, 0)$ . Similarly  $f(\epsilon_{11}, \epsilon_{22}, 0, \epsilon_{21}) = f(\epsilon_{11}, \epsilon_{22}, 0, 0)$ . But  $f(0, 0, 0, 0) = \operatorname{df}|_0 = 0$ , so we see that  $f(0, 0, \epsilon_{12}, 0) = f(0, 0, 0, \epsilon_{21}) = 0$ , and

$$\operatorname{df}|_{(0, 0, \epsilon_{12}, 0)} \cdot (\epsilon'_{11}, \epsilon'_{22}, \epsilon'_{12}, 0) = 0, \quad \operatorname{df}|_{(0, 0, 0, \epsilon_{21})} \cdot (\epsilon'_{11}, \epsilon'_{22}, 0, \epsilon'_{21}) = 0. \quad (10.11)$$

Now by (10.10),  $T^c$ -invariance of  $f$  and linearity in  $\epsilon'_{12}$  we see that

$$\operatorname{df}|_{(0, 0, 0, \epsilon_{21})} \cdot (0, 0, \epsilon'_{12}, 0) = \lambda^{-1} \operatorname{df}|_{(0, 0, 0, \lambda \epsilon_{21})} \cdot (0, 0, \epsilon'_{12}, 0).$$

Using this and  $df|_0 = 0$  to differentiate  $df \cdot (0, 0, \epsilon'_{12}, 0)$  at 0, we find that

$$\begin{aligned} & (\partial^2 f)|_0 \cdot (\epsilon_{21} \otimes \epsilon'_{12}) \\ &= \lim_{\lambda \rightarrow 0} \lambda^{-1} (df|_{(0,0,0,\lambda\epsilon_{21})} \cdot (0, 0, \epsilon'_{12}, 0) - df|_{(0,0,0,0)} \cdot (0, 0, \epsilon'_{12}, 0)) \\ &= \lim_{\lambda \rightarrow 0} (df|_{(0,0,0,\epsilon_{21})} \cdot (0, 0, \epsilon'_{12}, 0) - 0) = df|_{(0,0,0,\epsilon_{21})} \cdot (0, 0, \epsilon'_{12}, 0). \end{aligned}$$

But  $T_0 \text{Crit}(f) = \text{Ext}^1(E, E)$ , which implies that  $(\partial^2 f)|_0 = 0$ , so  $df|_{(0,0,0,\epsilon_{21})} \cdot (0, 0, \epsilon'_{12}, 0) = 0$ . Together with (10.11) this gives  $df|_{(0,0,0,\epsilon_{21})} = 0$ , and similarly  $df|_{(0,0,\epsilon_{12},0)} = 0$ . Therefore  $(0, 0, \epsilon_{12}, 0), (0, 0, 0, \epsilon_{21}) \in \text{Crit}(f) \subseteq U \subseteq \text{Ext}^1(E, E)$ , as we have to prove.

For (ii), let  $\Xi(0, 0, 0, \epsilon_{21}) = (\epsilon'_{11}, \epsilon'_{22}, \epsilon'_{12}, \epsilon'_{21})$ . As  $\Xi$  is  $T^c$ -equivariant, this gives  $\Xi(0, 0, 0, \lambda\epsilon_{21}) = (\epsilon'_{11}, \epsilon'_{22}, \lambda^{-1}\epsilon'_{12}, \lambda\epsilon'_{21})$ . But  $\Xi(0) = 0$  and  $\Xi$  is continuous, so taking the limit  $\lambda \rightarrow 0$  gives  $\Xi(0, 0, 0, \epsilon_{21}) = (0, 0, 0, \epsilon'_{21})$ . Thus  $\Xi(0, 0, 0, \lambda\epsilon_{21}) = (0, 0, 0, \lambda\epsilon'_{21})$ . But  $d\Xi|_0$  is the identity on  $\text{Ext}^1(E, E)$ , which forces  $\epsilon'_{21} = \epsilon_{21}$ . Hence  $\Xi(0, 0, 0, \epsilon_{21}) = (0, 0, 0, \epsilon_{21})$ , so that  $(0, 0, 0, \epsilon_{21}) \in V$ , and similarly  $\Xi(0, 0, \epsilon_{12}, 0) = (0, 0, \epsilon_{12}, 0)$  with  $(0, 0, \epsilon_{12}, 0) \in V$ , as we want.

Part (iii) is trivial when  $\epsilon_{21} = \epsilon_{12} = 0$  and  $F = F' = E$ , so suppose  $\epsilon_{21}, \epsilon_{12} \neq 0$ . Then  $[F]$  is the unique point in  $\mathfrak{M}(\mathbb{C})$ , with its nonseparated topology, which is distinct from  $[E]$  but infinitesimally close to  $[E]$  in direction  $(0, 0, 0, \epsilon_{21})$  in  $T_{[E]}\mathfrak{M} = \text{Ext}^1(E, E)$ . Similarly,  $[(0, 0, \epsilon_{12}, 0)]$  is the unique point in  $[S/\text{Aut}(E)]$ , with its nonseparated topology, which is distinct from  $[0]$  but infinitesimally close to  $[0]$  in direction  $(0, 0, \epsilon_{12}, 0)$  in  $T_{[0]}[S/\text{Aut}(E)] = \text{Ext}^1(E, E)$ . But  $\Phi_*$  maps  $[0] \mapsto [E]$ , and  $d\Phi_* : T_{[0]}[S/\text{Aut}(E)] \rightarrow T_{[E]}\mathfrak{M}$  is the identity on  $\text{Ext}^1(E, E)$ . It follows that  $\Phi_*$  maps  $[(0, 0, 0, \epsilon_{21})] \mapsto [F]$ , and similarly  $\Phi_*$  maps  $[(0, 0, \epsilon_{12}, 0)] \mapsto [F']$ .  $\square$

Let  $0 \neq \epsilon_{21} \in \text{Ext}^1(E_2, E_1)$  correspond to the short exact sequence  $0 \rightarrow E_1 \rightarrow F \rightarrow E_2 \rightarrow 0$  in  $\text{coh}(X)$ . Then

$$\begin{aligned} \nu_{\mathfrak{M}}(F) &= \nu_{[S/\text{Aut}(E)]}(0, 0, 0, \epsilon_{21}) = (-1)^{\dim \text{Aut}(E)} \nu_S(0, 0, 0, \epsilon_{21}) \\ &= (-1)^{\dim \text{Aut}(E) + \dim \text{Ext}^1(E, E)} (1 - \chi(MF_f(0, 0, 0, \epsilon_{21}))), \end{aligned} \quad (10.12)$$

using  $\Phi_* : [(0, 0, 0, \epsilon_{21})] \mapsto [F]$  from Proposition 10.1,  $\Phi$  smooth of relative dimension 0, Corollary 4.5 and Theorem 4.3(ii) in the first step, Proposition 4.4 in the second, and  $\Xi : (0, 0, 0, \epsilon_{21}) \mapsto (0, 0, 0, \epsilon_{21})$  from Proposition 10.1 and Theorem 4.7 in the last step.

Substituting (10.12) and its analogue for  $F'$  into (5.3), using equation (10.1) and  $\chi(MF_f(0)) = \chi(MF_f|_{\text{Ext}^1(E, E)^T}(0))$  from §10.1 to substitute for  $\nu_{\mathfrak{M}}(E_1 \oplus E_2)$ , and cancelling factors of  $(-1)^{\dim \text{Aut}(E) + \dim \text{Ext}^1(E, E)}$ , we see that (5.3) is equivalent to

$$\begin{aligned} & \int_{[\epsilon_{21}] \in \mathbb{P}(\text{Ext}^1(E_2, E_1))} (1 - \chi(MF_f(0, 0, 0, \epsilon_{21}))) d\chi - \int_{[\epsilon_{12}] \in \mathbb{P}(\text{Ext}^1(E_1, E_2))} (1 - \chi(MF_f(0, 0, \epsilon_{12}, 0))) d\chi \\ &= (\dim \text{Ext}^1(E_2, E_1) - \dim \text{Ext}^1(E_1, E_2)) (1 - \chi(MF_f|_{\text{Ext}^1(E, E)^T}(0))). \end{aligned} \quad (10.13)$$

Here  $\chi(MF_f(0, 0, 0, \epsilon_{21}))$  is independent of the choice of  $\epsilon_{21}$  representing the point  $[\epsilon_{21}] \in \mathbb{P}(\text{Ext}^1(E_2, E_1))$ , and is a constructible function of  $[\epsilon_{21}]$ , so the integrals in (10.13) are well-defined.

Set  $U' = \{(\epsilon_{11}, \epsilon_{22}, \epsilon_{12}, \epsilon_{21}) \in U : \epsilon_{21} \neq 0\}$ , an open set in  $U$ , and write  $V'$  for the submanifold of  $(\epsilon_{11}, \epsilon_{22}, \epsilon_{12}, \epsilon_{21}) \in U'$  with  $\epsilon_{12} = 0$ . Let  $\tilde{U}'$  be the blowup of  $U'$  along  $V'$ , with projection  $\pi' : \tilde{U}' \rightarrow U'$ . Points of  $\tilde{U}'$  may be written

$(\epsilon_{11}, \epsilon_{22}, [\epsilon_{12}], \lambda \epsilon_{12}, \epsilon_{21})$ , where  $[\epsilon_{12}] \in \mathbb{P}(\text{Ext}^1(E_1, E_2))$ , and  $\lambda \in \mathbb{C}$ , and  $\epsilon_{21} \neq 0$ . Write  $f' = f|_{U'}$  and  $\tilde{f}' = f' \circ \pi'$ . Then applying Theorem 4.11 to  $U', V', f', \tilde{U}', \pi', \tilde{f}'$  at the point  $(0, 0, 0, \epsilon_{21}) \in U'$  gives

$$\begin{aligned} \chi(MF_f(0, 0, 0, \epsilon_{21})) &= \int_{[\epsilon_{12}] \in \mathbb{P}(\text{Ext}^1(E_1, E_2))} \chi(MF_{\tilde{f}'}(0, 0, [\epsilon_{12}], 0, \epsilon_{21})) d\chi \\ &\quad + (1 - \dim \text{Ext}^1(E_1, E_2)) \chi(MF_{f|_{V'}}(0, 0, 0, \epsilon_{21})). \end{aligned} \quad (10.14)$$

Let  $L_{12} \rightarrow \mathbb{P}(\text{Ext}^1(E_1, E_2))$  and  $L_{21} \rightarrow \mathbb{P}(\text{Ext}^1(E_2, E_1))$  be the tautological line bundles, so that the fibre of  $L_{12}$  over a point  $[\epsilon_{12}]$  in  $\mathbb{P}(\text{Ext}^1(E_1, E_2))$  is the 1-dimensional subspace  $\{\lambda \epsilon_{12} : \lambda \in \mathbb{C}\}$  in  $\text{Ext}^1(E_1, E_2)$ . Consider the line bundle  $L_{12} \otimes L_{21} \rightarrow \mathbb{P}(\text{Ext}^1(E_1, E_2)) \times \mathbb{P}(\text{Ext}^1(E_2, E_1))$ . The fibre of  $L_{12} \otimes L_{21}$  over  $([\epsilon_{12}], [\epsilon_{21}])$  is  $\{\lambda \epsilon_{12} \otimes \epsilon_{21} : \lambda \in \mathbb{C}\}$ .

Write points of the total space of  $L_{12} \otimes L_{21}$  as  $([\epsilon_{12}], [\epsilon_{21}], \lambda \epsilon_{12} \otimes \epsilon_{21})$ . Define  $W \subseteq \text{Ext}^1(E_1, E_1) \times \text{Ext}^1(E_2, E_2) \times (L_{12} \otimes L_{21})$  to be the open subset of points  $(\epsilon_{11}, \epsilon_{22}, [\epsilon_{12}], [\epsilon_{21}], \lambda \epsilon_{12} \otimes \epsilon_{21})$  for which  $(\epsilon_{21}, \epsilon_{22}, \lambda \epsilon_{12}, \epsilon_{21})$  lies in  $U$ . Since  $U$  is  $T^c$ -invariant, this definition is independent of the choice of representatives  $\epsilon_{12}, \epsilon_{21}$  for  $[\epsilon_{12}], [\epsilon_{21}]$ , since any other choice would replace  $(\epsilon_{11}, \epsilon_{22}, \lambda \epsilon_{12}, \epsilon_{21})$  by  $(\epsilon_{11}, \epsilon_{22}, \lambda \mu \epsilon_{12}, \mu^{-1} \epsilon_{21})$  for some  $\mu \in \mathbb{G}_m$ . Define a holomorphic function  $h : W \rightarrow \mathbb{C}$  by  $h(\epsilon_{11}, \epsilon_{22}, [\epsilon_{12}], [\epsilon_{21}], \lambda \epsilon_{12} \otimes \epsilon_{21}) = f(\epsilon_{11}, \epsilon_{22}, \lambda \epsilon_{12}, \epsilon_{21})$ . As  $f$  is  $T^c$ -invariant, the same argument shows  $h$  is well-defined.

Define a projection  $\Pi : \tilde{U}' \rightarrow W$  by  $\Pi : (\epsilon_{11}, \epsilon_{22}, [\epsilon_{12}], \lambda \epsilon_{12}, \epsilon_{21}) \mapsto (\epsilon_{11}, \epsilon_{22}, [\epsilon_{12}], [\epsilon_{21}], \lambda \epsilon_{12} \otimes \epsilon_{21})$ . Then  $\Pi$  is a smooth holomorphic submersion, with fibre  $\mathbb{G}_m$ . Furthermore, we have  $\tilde{f}' \equiv h \circ \Pi$ . Thus the Milnor fibre of  $\tilde{f}'$  at  $(\epsilon_{11}, \epsilon_{22}, [\epsilon_{12}], \lambda \epsilon_{12}, \epsilon_{21})$  is the product of the Milnor fibre of  $h$  at  $(\epsilon_{11}, \epsilon_{22}, [\epsilon_{12}], [\epsilon_{21}], \lambda \epsilon_{12} \otimes \epsilon_{21})$  with a small ball in  $\mathbb{C}$ , so they have the same Euler characteristic. That is,

$$\chi(MF_{\tilde{f}'}(0, 0, [\epsilon_{12}], 0, \epsilon_{21})) = \chi(MF_h(0, 0, [\epsilon_{12}], [\epsilon_{21}], 0)). \quad (10.15)$$

Also, we have  $f(\epsilon_{11}, \epsilon_{22}, 0, \epsilon_{21}) = f(\epsilon_{11}, \epsilon_{22}, 0, 0)$  as in the proof of Proposition 10.1, so the Milnor fibre of  $f|_{V'}$  at  $(0, 0, 0, \epsilon_{21})$  is the product of the Milnor fibre of  $f|_{\text{Ext}^1(E, E)^T}$  at 0 with a small ball in  $\text{Ext}^1(E_2, E_1)$ , and they have the same Euler characteristic. That is,

$$\chi(MF_{f|_{V'}}(0, 0, 0, \epsilon_{21})) = \chi(MF_{f|_{\text{Ext}^1(E, E)^T}}(0)). \quad (10.16)$$

Substituting (10.15) and (10.16) into (10.14) gives

$$\begin{aligned} \chi(MF_f(0, 0, 0, \epsilon_{21})) &= \int_{[\epsilon_{12}] \in \mathbb{P}(\text{Ext}^1(E_1, E_2))} \chi(MF_h(0, 0, [\epsilon_{12}], [\epsilon_{21}], 0)) d\chi \\ &\quad + (1 - \dim \text{Ext}^1(E_1, E_2)) \chi(MF_{f|_{\text{Ext}^1(E, E)^T}}(0)). \end{aligned}$$

Integrating this over  $[\epsilon_{21}] \in \mathbb{P}(\text{Ext}^1(E_2, E_1))$  yields

$$\begin{aligned} \int_{[\epsilon_{21}] \in \mathbb{P}(\text{Ext}^1(E_2, E_1))} \chi(MF_f(0, 0, 0, \epsilon_{21})) d\chi &= \int_{([\epsilon_{12}], [\epsilon_{21}]) \in \mathbb{P}(\text{Ext}^1(E_1, E_2)) \times \mathbb{P}(\text{Ext}^1(E_2, E_1))} \chi(MF_h(0, 0, [\epsilon_{12}], [\epsilon_{21}], 0)) d\chi \\ &\quad + (1 - \dim \text{Ext}^1(E_1, E_2)) \dim \text{Ext}^1(E_2, E_1) \cdot \chi(MF_{f|_{\text{Ext}^1(E, E)^T}}(0)), \end{aligned} \quad (10.17)$$

since  $\chi(\mathbb{P}(\text{Ext}^1(E_2, E_1))) = \dim \text{Ext}^1(E_2, E_1)$ . Similarly we have

$$\begin{aligned} \int_{[\epsilon_{12}] \in \mathbb{P}(\text{Ext}^1(E_1, E_2))} \chi(MF_f(0, 0, \epsilon_{12}, 0)) \, d\chi &= \int_{([\epsilon_{12}], [\epsilon_{21}]) \in \mathbb{P}(\text{Ext}^1(E_1, E_2)) \times \mathbb{P}(\text{Ext}^1(E_2, E_1))} \chi(MF_h(0, 0, [\epsilon_{12}], [\epsilon_{21}], 0)) \, d\chi \\ &\quad + (1 - \dim \text{Ext}^1(E_2, E_1)) \dim \text{Ext}^1(E_1, E_2) \cdot \chi(MF_{f|_{\text{Ext}^1(E, E)^T}}(0)). \end{aligned} \quad (10.18)$$

Equation (10.13) now follows from (10.18) minus (10.17). This completes the proof of (5.3).



## The proof of Theorem 5.14

We use the notation of §2–§4 and Theorem 5.14. It is sufficient to prove that  $\tilde{\Psi}^{\chi, \mathbb{Q}}$  is a Lie algebra morphism, as  $\tilde{\Psi} = \tilde{\Psi}^{\chi, \mathbb{Q}} \circ \bar{\Pi}_{\mathfrak{M}}^{\chi, \mathbb{Q}}$  and  $\bar{\Pi}_{\mathfrak{M}}^{\chi, \mathbb{Q}} : \mathrm{SF}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{M}) \rightarrow \mathrm{SF}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{M}, \chi, \mathbb{Q})$  is a Lie algebra morphism as in §3.1. The rough idea is to insert Behrend functions  $\nu_{\mathfrak{M}}$  as weights in the proof of Theorem 3.16 in [52, §6.4], and use the identities (5.2)–(5.3). However, [52, §6.4] involved lifting from Euler characteristics to virtual Poincaré polynomials; here we give an alternative proof involving only Euler characteristics, and also change some methods in the proof.

We must show  $\tilde{\Psi}^{\chi, \mathbb{Q}}([f, g]) = [\tilde{\Psi}^{\chi, \mathbb{Q}}(f), \tilde{\Psi}^{\chi, \mathbb{Q}}(g)]$  for  $f, g \in \mathrm{SF}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{M}, \chi, \mathbb{Q})$ . It is enough to do this for  $f, g$  supported on  $\mathfrak{M}^{\alpha}, \mathfrak{M}^{\beta}$  respectively, for  $\alpha, \beta \in C(\mathrm{coh}(X)) \cup \{0\}$ . Choose finite type, open  $\mathbb{C}$ -substacks  $\mathfrak{U}$  in  $\mathfrak{M}^{\alpha}$  and  $\mathfrak{V}$  in  $\mathfrak{M}^{\beta}$  such that  $f, g$  are supported on  $\mathfrak{U}, \mathfrak{V}$ . This is possible as  $f, g$  are supported on constructible sets and  $\mathfrak{M}^{\alpha}, \mathfrak{M}^{\beta}$  are locally of finite type. As  $\mathfrak{U}, \mathfrak{V}$  are of finite type the families of sheaves they parametrize are bounded, so by Serre vanishing [44, Lem. 1.7.6] we can choose  $n \gg 0$  such that for all  $[E_1] \in \mathfrak{U}(\mathbb{C})$  and  $[E_2] \in \mathfrak{V}(\mathbb{C})$  we have  $H^i(E_j(n)) = 0$  for all  $i > 0$  and  $j = 1, 2$ . Hence  $\dim H^0(E_1(n)) = \bar{\chi}([\mathcal{O}_X(-n)], \alpha) = P_{\alpha}(n)$  and  $\dim H^0(E_2(n)) = \bar{\chi}([\mathcal{O}_X(-n)], \beta) = P_{\beta}(n)$ , where  $P_{\alpha}, P_{\beta}$  are the Hilbert polynomials of  $\alpha, \beta$ .

Consider Grothendieck's Quot Scheme  $\mathrm{Quot}_X(U \otimes \mathcal{O}_X(-n), P_{\alpha})$ , explained in [44, §2.2], which parametrizes quotients  $U \otimes \mathcal{O}_X(-n) \twoheadrightarrow E$  of the fixed coherent sheaf  $U \otimes \mathcal{O}_X(-n)$  over  $X$ , such that  $E$  has fixed Hilbert polynomial  $P_{\alpha}$ . By [44, Th. 2.2.4],  $\mathrm{Quot}_X(U \otimes \mathcal{O}_X(-n), P_{\alpha})$  is a projective  $\mathbb{C}$ -scheme representing the moduli functor  $\underline{\mathrm{Quot}}_X(U \otimes \mathcal{O}_X(-n), P_{\alpha})$  of such quotients.

Define  $Q_{\mathfrak{U}, n}$  to be the subscheme of  $\mathrm{Quot}_X(U \otimes \mathcal{O}_X(-n), P_{\alpha})$  representing quotients  $U \otimes \mathcal{O}_X(-n) \twoheadrightarrow E_1$  such that  $[E_1] \in \mathfrak{U}(\mathbb{C})$ , and the morphism  $U \otimes \mathcal{O}_X(-n) \twoheadrightarrow E_1$  is induced by an isomorphism  $\phi : U \rightarrow H^0(E_1(n))$ , noting that  $[E_1] \in \mathfrak{U}(\mathbb{C})$  implies that  $\dim H^0(E_1(n)) = P_{\alpha}(n) = \dim U$ . This is an open condition on  $U \otimes \mathcal{O}_X(-n) \twoheadrightarrow E_1$ , as  $\mathfrak{U}$  is open in  $\mathfrak{M}^{\alpha}$ , so  $Q_{\mathfrak{U}, n}$  is open in  $\mathrm{Quot}_X(U \otimes \mathcal{O}_X(-n), P_{\alpha})$ , and is a quasiprojective  $\mathbb{C}$ -scheme, with

$$Q_{\mathfrak{U}, n}(\mathbb{C}) \cong \left\{ \text{isomorphism classes } [(E_1, \phi_1)] \text{ of pairs } (E_1, \phi_1) : \right. \\ \left. [E_1] \in \mathfrak{U}(\mathbb{C}), \phi_1 : U \rightarrow H^0(E_1(n)) \text{ is an isomorphism} \right\}. \quad (11.1)$$

The algebraic  $\mathbb{C}$ -group  $\mathrm{GL}(U) \cong \mathrm{GL}(P_{\alpha}(n), \mathbb{C})$  acts on the right on  $Q_{\mathfrak{U}, n}$ , on points as  $\gamma : [(E_1, \phi_1)] \mapsto [(E_1, \phi_1 \circ \gamma)]$  in the representation (11.1). Similarly, we define an open subscheme  $Q_{\mathfrak{V}, n}$  in  $\mathrm{Quot}_X(V \otimes \mathcal{O}_X(-n), P_{\beta})$  with a right action of  $\mathrm{GL}(V)$ . In the usual way we have 1-isomorphisms of Artin  $\mathbb{C}$ -stacks

$$\mathfrak{U} \cong [Q_{\mathfrak{U}, n} / \mathrm{GL}(U)], \quad \mathfrak{V} \cong [Q_{\mathfrak{V}, n} / \mathrm{GL}(V)], \quad (11.2)$$

which write  $\mathfrak{U}, \mathfrak{V}$  as global quotient stacks.

The definition of the Ringel–Hall multiplication  $*$  on  $\mathrm{SF}_{\mathrm{al}}(\mathfrak{M})$  in §3.1 involves the moduli stack  $\mathfrak{E}\mathfrak{r}\mathfrak{a}\mathfrak{c}\mathfrak{t}$  of short exact sequences  $0 \rightarrow E_1 \rightarrow F \rightarrow E_2 \rightarrow 0$  in  $\mathrm{coh}(X)$ , and 1-morphisms  $\pi_1, \pi_2, \pi_3 : \mathfrak{E}\mathfrak{r}\mathfrak{a}\mathfrak{c}\mathfrak{t} \rightarrow \mathfrak{M}$  mapping  $0 \rightarrow E_1 \rightarrow F \rightarrow E_2 \rightarrow 0$  to  $E_1, F, E_2$  respectively. Thus we have a 1-morphism  $\pi_1 \times \pi_3 : \mathfrak{E}\mathfrak{r}\mathfrak{a}\mathfrak{c}\mathfrak{t} \rightarrow \mathfrak{M} \times \mathfrak{M}$ . We wish to describe  $\mathfrak{E}\mathfrak{r}\mathfrak{a}\mathfrak{c}\mathfrak{t}$  and  $\pi_1 \times \pi_3$  over  $\mathfrak{U} \times \mathfrak{V}$  in  $\mathfrak{M} \times \mathfrak{M}$ . Suppose  $[0 \rightarrow E_1 \rightarrow F \rightarrow E_2 \rightarrow 0]$  is a point in  $\mathfrak{E}\mathfrak{r}\mathfrak{a}\mathfrak{c}\mathfrak{t}(\mathbb{C})$  which is mapped to  $(\mathfrak{U} \times \mathfrak{V})(\mathbb{C})$  by  $\pi_1 \times \pi_3$ . Then  $[E_1] \in \mathfrak{U}(\mathbb{C})$  and  $[E_2] \in \mathfrak{V}(\mathbb{C})$ , so  $E_1, E_2$  have Hilbert polynomials  $P_\alpha, P_\beta$ , and thus  $F$  has Hilbert polynomial  $P_{\alpha+\beta}$ . Also  $H^i(E_j(n)) = 0$  for all  $i > 0$  and  $j = 1, 2$  and  $\dim H^0(E_1(n)) = P_\alpha(n)$ ,  $\dim H^0(E_2(n)) = P_\beta(n)$ . Applying  $\mathrm{Hom}(\mathcal{O}_X(-n), *)$  to  $0 \rightarrow E_1 \rightarrow F \rightarrow E_2 \rightarrow 0$  shows that

$$0 \longrightarrow H^0(E_1(n)) \longrightarrow H^0(F(n)) \longrightarrow H^0(E_2(n)) \longrightarrow 0$$

is exact, so that  $\dim H^0(F(n)) = P_{\alpha+\beta}(n)$ , and  $H^i(F(n)) = 0$  for  $i > 0$ .

By a similar argument to the construction of the Quot scheme in [44, §2.2], one can construct a ‘Quot scheme for exact sequences’  $0 \rightarrow E_1 \rightarrow F \rightarrow E_2 \rightarrow 0$ , which are quotients of the natural split short exact sequence of coherent sheaves  $0 \rightarrow U \otimes \mathcal{O}_X(-n) \rightarrow (U \oplus V) \otimes \mathcal{O}_X(-n) \rightarrow V \otimes \mathcal{O}_X(-n) \rightarrow 0$ . There is an open subscheme  $Q_{\mathfrak{U}, \mathfrak{V}, n}$  of this Quot scheme for exact sequences such that, in a similar way to (11.1), there is a natural identification between  $Q_{\mathfrak{U}, \mathfrak{V}, n}(\mathbb{C})$  and the set of isomorphism classes of data  $(0 \rightarrow E_1 \rightarrow F \rightarrow E_2 \rightarrow 0, \phi_1, \phi, \phi_2)$  where  $\phi_1 : U \rightarrow H^0(E_1(n))$ ,  $\phi : U \oplus V \rightarrow H^0(F(n))$  and  $\phi_2 : V \rightarrow H^0(E_2(n))$  are isomorphisms, and the following diagram commutes:

$$\begin{array}{ccccccc} 0 & \longrightarrow & U & \longrightarrow & U \oplus V & \longrightarrow & V \longrightarrow 0 \\ & & \cong \downarrow \phi_1 & & \cong \downarrow \phi & & \cong \downarrow \phi_2 \\ 0 & \longrightarrow & H^0(E_1(n)) & \longrightarrow & H^0(F(n)) & \longrightarrow & H^0(E_2(n)) \longrightarrow 0. \end{array}$$

The automorphism group of the sequence  $0 \rightarrow U \rightarrow U \oplus V \rightarrow V \rightarrow 0$  is the algebraic  $\mathbb{C}$ -group  $(\mathrm{GL}(U) \times \mathrm{GL}(V)) \ltimes \mathrm{Hom}(V, U)$ , with multiplication

$$(\gamma, \delta, \epsilon) \cdot (\gamma', \delta', \epsilon') = (\gamma \circ \gamma', \delta \circ \delta', \gamma \circ \epsilon' + \epsilon \circ \delta')$$

for  $\gamma, \gamma' \in \mathrm{GL}(U)$ ,  $\delta, \delta' \in \mathrm{GL}(V)$ ,  $\epsilon, \epsilon' \in \mathrm{Hom}(V, U)$ . It is the subgroup of elements  $\begin{pmatrix} \gamma & \epsilon \\ 0 & \delta \end{pmatrix}$  in  $\mathrm{GL}(U \oplus V)$ . Then  $(\mathrm{GL}(U) \times \mathrm{GL}(V)) \ltimes \mathrm{Hom}(V, U)$  acts naturally on the right on  $Q_{\mathfrak{U}, \mathfrak{V}, n}$ . On points in the representation above it acts by

$$\begin{aligned} (\gamma, \delta, \epsilon) : (0 \rightarrow E_1 \rightarrow F \rightarrow E_2 \rightarrow 0, \phi_1, \phi, \phi_2) &\longmapsto \\ (0 \rightarrow E_1 \rightarrow F \rightarrow E_2 \rightarrow 0, \phi_1 \circ \gamma, \phi \circ \begin{pmatrix} \gamma & \epsilon \\ 0 & \delta \end{pmatrix}, \phi_2 \circ \delta). \end{aligned}$$

As for (11.2), we have a 1-isomorphism

$$(\mathfrak{U} \times \mathfrak{V})_{\iota_{\mathfrak{U}} \times \iota_{\mathfrak{V}}, \mathfrak{M} \times \mathfrak{M}, \pi_1 \times \pi_3} \mathfrak{E}\mathfrak{r}\mathfrak{a}\mathfrak{c}\mathfrak{t} \cong [Q_{\mathfrak{U}, \mathfrak{V}, n} / ((\mathrm{GL}(U) \times \mathrm{GL}(V)) \ltimes \mathrm{Hom}(V, U))], \quad (11.3)$$

where  $\iota_{\mathfrak{U}} : \mathfrak{U} \rightarrow \mathfrak{M}$ ,  $\iota_{\mathfrak{V}} : \mathfrak{V} \rightarrow \mathfrak{M}$  are the inclusions, and the l.h.s. of (11.3) is the open  $\mathbb{C}$ -substack of  $\mathfrak{E}\mathfrak{r}\mathfrak{a}\mathfrak{c}\mathfrak{t}$  taken to  $\mathfrak{U} \times \mathfrak{V}$  in  $\mathfrak{M} \times \mathfrak{M}$  by  $\pi_1 \times \pi_3$ .

There are projections  $\Pi_{\mathfrak{U}} : Q_{\mathfrak{U}, \mathfrak{V}, n} \rightarrow Q_{\mathfrak{U}, n}$ ,  $\Pi_{\mathfrak{V}} : Q_{\mathfrak{U}, \mathfrak{V}, n} \rightarrow Q_{\mathfrak{V}, n}$  acting by

$$\Pi_{\mathfrak{U}}, \Pi_{\mathfrak{V}} : [(0 \rightarrow E_1 \rightarrow F \rightarrow E_2 \rightarrow 0, \phi_1, \phi, \phi_2)] \longmapsto [(E_1, \phi_1)], [(E_2, \phi_2)].$$

Combining  $\Pi_{\mathfrak{U}}, \Pi_{\mathfrak{V}}$  with the natural projections of algebraic  $\mathbb{C}$ -groups  $(\mathrm{GL}(U) \times \mathrm{GL}(V)) \ltimes \mathrm{Hom}(V, U) \rightarrow \mathrm{GL}(U)$ ,  $\mathrm{GL}(V)$  gives 1-morphisms

$$\begin{aligned} \Pi'_{\mathfrak{U}} : [Q_{\mathfrak{U}, \mathfrak{V}, n} / ((\mathrm{GL}(U) \times \mathrm{GL}(V)) \ltimes \mathrm{Hom}(V, U))] &\longrightarrow [Q_{\mathfrak{U}, n} / \mathrm{GL}(U)], \\ \Pi'_{\mathfrak{V}} : [Q_{\mathfrak{U}, \mathfrak{V}, n} / ((\mathrm{GL}(U) \times \mathrm{GL}(V)) \ltimes \mathrm{Hom}(V, U))] &\longrightarrow [Q_{\mathfrak{V}, n} / \mathrm{GL}(V)], \end{aligned} \quad (11.4)$$

which are 2-isomorphic to  $\pi_1, \pi_3$  under the 1-isomorphisms (11.2), (11.3). There is a morphism  $z : Q_{\mathfrak{U},n} \times Q_{\mathfrak{V},n} \rightarrow Q_{\mathfrak{U},\mathfrak{V},n}$  which embeds  $Q_{\mathfrak{U},n} \times Q_{\mathfrak{V},n}$  as a closed subscheme of  $Q_{\mathfrak{U},\mathfrak{V},n}$ , given on points by

$$z : [(E_1, \phi_1), (E_2, \phi_2)] \mapsto [(0 \rightarrow E_1 \rightarrow E_1 \oplus E_2 \rightarrow E_2 \rightarrow 0, \phi_1, \phi_1 \oplus \phi_2, \phi_2)].$$

Write  $Q'_{\mathfrak{U},\mathfrak{V},n} = Q_{\mathfrak{U},\mathfrak{V},n} \setminus z(Q_{\mathfrak{U},n} \times Q_{\mathfrak{V},n})$ , an open subscheme of  $Q_{\mathfrak{U},\mathfrak{V},n}$ .

Let  $q_1 \in Q_{\mathfrak{U},n}(\mathbb{C})$  correspond to  $[(E_1, \phi_1)]$  under (11.1), and  $q_2 \in Q_{\mathfrak{V},n}(\mathbb{C})$  correspond to  $[(E_2, \phi_2)]$ . Then the fibre  $(\Pi_{\mathfrak{U}} \times \Pi_{\mathfrak{V}})^*(q_1, q_2)$  of  $\Pi_{\mathfrak{U}} \times \Pi_{\mathfrak{V}}$  over  $(q_1, q_2)$  is a subscheme of  $Q_{\mathfrak{U},\mathfrak{V},n}$  of points  $[(0 \rightarrow E_1 \rightarrow F \rightarrow E_2 \rightarrow 0, \phi_1, \phi, \phi_2)]$  with  $E_1, \phi_1, E_2, \phi_2$  fixed. By the usual correspondence between extensions and vector spaces  $\text{Ext}^1(\cdot, \cdot)$  we find  $(\Pi_{\mathfrak{U}} \times \Pi_{\mathfrak{V}})^*(q_1, q_2)$  is a  $\mathbb{C}$ -vector space, which we write as  $W_{\mathfrak{U},\mathfrak{V},n}^{q_1, q_2}$ , where  $0 \in W_{\mathfrak{U},\mathfrak{V},n}^{q_1, q_2}$  is  $z(q_1, q_2)$ . The subgroup  $\text{Hom}(V, U)$  of  $(\text{GL}(U) \times \text{GL}(V)) \ltimes \text{Hom}(V, U)$  acts on  $(\Pi_{\mathfrak{U}} \times \Pi_{\mathfrak{V}})^*(q_1, q_2) \cong W_{\mathfrak{U},\mathfrak{V},n}^{q_1, q_2}$  by translations. Write this action as a linear map  $L_{\mathfrak{U},\mathfrak{V},n}^{q_1, q_2} : \text{Hom}(V, U) \rightarrow W_{\mathfrak{U},\mathfrak{V},n}^{q_1, q_2}$ .

We claim this fits into an exact sequence

$$0 \gg \text{Hom}(E_2, E_1) \gg \text{Hom}(V, U) \xrightarrow{L_{\mathfrak{U},\mathfrak{V},n}^{q_1, q_2}} W_{\mathfrak{U},\mathfrak{V},n}^{q_1, q_2} \xrightarrow{\pi_{E_2, E_1}} \text{Ext}^1(E_2, E_1) \gg 0. \quad (11.5)$$

To see this, note that the fibre of  $\Pi'_{\mathfrak{U}} \times \Pi'_{\mathfrak{V}}$  over  $(q_1, q_2)$  is the quotient stack  $[W_{\mathfrak{U},\mathfrak{V},n}^{q_1, q_2} / \text{Hom}(V, U)]$ , where  $\text{Hom}(V, U)$  acts on  $W_{\mathfrak{U},\mathfrak{V},n}^{q_1, q_2}$  by  $\epsilon : w \mapsto w + L_{\mathfrak{U},\mathfrak{V},n}^{q_1, q_2}(\epsilon)$ , whereas the fibre of  $\pi_1 \times \pi_3 : \mathfrak{E}\mathfrak{r}\mathfrak{a}\mathfrak{c}\mathfrak{t} \rightarrow \mathfrak{M} \times \mathfrak{M}$  over  $(E_1, E_2)$  is the quotient stack  $[\text{Ext}^1(E_2, E_1) / \text{Hom}(E_2, E_1)]$ , where  $\text{Hom}(E_2, E_1)$  acts trivially on  $\text{Ext}^1(E_2, E_1)$ . The 1-isomorphisms (11.2) and (11.3) induce a 1-isomorphism  $[W_{\mathfrak{U},\mathfrak{V},n}^{q_1, q_2} / \text{Hom}(V, U)] \cong [\text{Ext}^1(E_2, E_1) / \text{Hom}(E_2, E_1)]$ , which gives (11.5).

We can repeat all the above material on  $Q_{\mathfrak{U},\mathfrak{V},n}$  with  $\mathfrak{U}, \mathfrak{V}$  exchanged. We use the corresponding notation with accents  $\sim$ . We obtain a quasiprojective  $\mathbb{C}$ -scheme  $\tilde{Q}_{\mathfrak{V},\mathfrak{U},n}$  whose  $\mathbb{C}$ -points are isomorphism classes of data  $(0 \rightarrow E_2 \rightarrow \tilde{F} \rightarrow E_1 \rightarrow 0, \phi_2, \tilde{\phi}, \phi_1)$  where  $[E_2] \in \mathfrak{V}(\mathbb{C})$ ,  $[E_1] \in \mathfrak{U}(\mathbb{C})$ ,  $\phi_2 : V \rightarrow H^0(E_2(n))$ ,  $\tilde{\phi} : V \oplus U \rightarrow H^0(\tilde{F}(n))$  and  $\phi_1 : U \rightarrow H^0(E_1(n))$  are isomorphisms, and the following diagram commutes:

$$\begin{array}{ccccccc} 0 & \longrightarrow & V & \longrightarrow & V \oplus U & \longrightarrow & U \longrightarrow 0 \\ & & \cong \downarrow \phi_2 & & \cong \downarrow \tilde{\phi} & & \cong \downarrow \phi_1 \\ 0 & \longrightarrow & H^0(E_2(n)) & \longrightarrow & H^0(\tilde{F}(n)) & \longrightarrow & H^0(E_1(n)) \longrightarrow 0. \end{array}$$

There is a closed embedding  $\tilde{z} : Q_{\mathfrak{V},n} \times Q_{\mathfrak{U},n} \rightarrow \tilde{Q}_{\mathfrak{V},\mathfrak{U},n}$ , and we write  $\tilde{Q}'_{\mathfrak{V},\mathfrak{U},n} = \tilde{Q}_{\mathfrak{V},\mathfrak{U},n} \setminus \tilde{z}(Q_{\mathfrak{V},n} \times Q_{\mathfrak{U},n})$ .

The algebraic  $\mathbb{C}$ -group  $(\text{GL}(V) \times \text{GL}(U)) \ltimes \text{Hom}(U, V)$  acts on  $\tilde{Q}_{\mathfrak{V},\mathfrak{U},n}$  with

$$(\mathfrak{V} \times \mathfrak{U})_{\iota_{\mathfrak{V}} \times \iota_{\mathfrak{U}}, \mathfrak{M} \times \mathfrak{M}, \pi_1 \times \pi_3} \mathfrak{E}\mathfrak{r}\mathfrak{a}\mathfrak{c}\mathfrak{t} \cong [\tilde{Q}_{\mathfrak{V},\mathfrak{U},n} / (\text{GL}(V) \times \text{GL}(U)) \ltimes \text{Hom}(U, V)].$$

There are natural projections  $\tilde{\Pi}_{\mathfrak{V}}, \tilde{\Pi}_{\mathfrak{U}} : \tilde{Q}_{\mathfrak{V},\mathfrak{U},n} \rightarrow Q_{\mathfrak{V},n}, Q_{\mathfrak{U},n}$  and  $\tilde{\Pi}'_{\mathfrak{V}}, \tilde{\Pi}'_{\mathfrak{U}}$  from  $[\tilde{Q}_{\mathfrak{V},\mathfrak{U},n} / (\text{GL}(V) \times \text{GL}(U)) \ltimes \text{Hom}(U, V)]$  to  $[Q_{\mathfrak{V},n} / \text{GL}(V)], [Q_{\mathfrak{U},n} / \text{GL}(U)]$ . If  $q_1 \in Q_{\mathfrak{U},n}(\mathbb{C})$  and  $q_2 \in Q_{\mathfrak{V},n}(\mathbb{C})$  correspond to  $[(E_1, \phi_1)]$  and  $[(E_2, \phi_2)]$  then  $(\tilde{\Pi}_{\mathfrak{V}} \times \tilde{\Pi}_{\mathfrak{U}})^*(q_2, q_1)$  in  $\tilde{Q}_{\mathfrak{V},\mathfrak{U},n}$  is a  $\mathbb{C}$ -vector space  $\tilde{W}_{\mathfrak{V},\mathfrak{U},n}^{q_2, q_1}$  with an exact sequence

$$0 \gg \text{Hom}(E_1, E_2) \gg \text{Hom}(U, V) \xrightarrow{\tilde{L}_{\mathfrak{V},\mathfrak{U},n}^{q_2, q_1}} \tilde{W}_{\mathfrak{V},\mathfrak{U},n}^{q_2, q_1} \xrightarrow{\tilde{\pi}_{E_1, E_2}} \text{Ext}^1(E_1, E_2) \gg 0. \quad (11.6)$$

Now consider the stack function  $f \in \text{SF}_{\text{al}}^{\text{ind}}(\mathfrak{M}, \chi, \mathbb{Q})$ . Since  $f$  is supported on  $\mathfrak{U}$ , by Proposition 3.4 we may write  $f$  in the form

$$f = \sum_{i=1}^n \delta_i[(Z_i \times [\text{Spec } \mathbb{C}/\mathbb{G}_m], \iota_{\mathfrak{U}} \circ \rho_i)], \quad (11.7)$$



where  $\delta_i \in \mathbb{Q}$ ,  $Z_i$  is a quasiprojective  $\mathbb{C}$ -variety, and  $\rho_i : Z_i \times [\text{Spec } \mathbb{C}/\mathbb{G}_m] \rightarrow \mathfrak{U}$  is representable for  $i = 1, \dots, n$ , and  $\iota_{\mathfrak{U}} : \mathfrak{U} \rightarrow \mathfrak{M}$  is the inclusion, and each term in (11.7) has algebra stabilizers. Consider the fibre product  $P_i = Z_i \times_{\rho_i, \mathfrak{U}, \pi_{\mathfrak{U}}} Q_{\mathfrak{U}, n}$ , where  $\pi_{\mathfrak{U}} : Q_{\mathfrak{U}, n} \rightarrow \mathfrak{U}$  is the projection induced by (11.2). As  $\pi_{\mathfrak{U}}$  is a principal  $\text{GL}(U)$ -bundle of Artin  $\mathbb{C}$ -stacks,  $\pi_1 : P_i \rightarrow Z_i$  is a principal  $\text{GL}(U)$ -bundle of  $\mathbb{C}$ -schemes, and so is Zariski locally trivial as  $\text{GL}(U)$  is special. Thus by cutting the  $Z_i$  into smaller pieces using relation Definition 2.16(i), we can suppose the fibrations  $\pi_1 : P_i \rightarrow Z_i$  are trivial, with trivializations  $P_i \cong Z_i \times \text{GL}(U)$ . Composing the morphisms  $Z_i \hookrightarrow Z_i \times \{1\} \subset P_i \xrightarrow{\pi_2} Q_{\mathfrak{U}, n}$  gives a morphism  $\xi_i : Z_i \rightarrow Q_{\mathfrak{U}, n}$ .

The algebra stabilizers condition implies that if  $z \in Z_i(\mathbb{C})$  and  $(\iota_{\mathfrak{U}} \circ \rho_i)_*(z)$  is a point  $[E] \in \mathfrak{M}(\mathbb{C})$  then on stabilizer groups  $(\iota_{\mathfrak{U}} \circ \rho_i)_* : \mathbb{G}_m \rightarrow \text{Aut}(E)$  must map  $\lambda \mapsto \lambda \text{id}_E$ . If  $q \in Q_{\mathfrak{U}, n}(\mathbb{C})$  with  $(\pi_{\mathfrak{U}})_*(q) = [E]$  then  $(\pi_{\mathfrak{U}})_* : \text{Stab}_{\text{GL}(U)}(q) \rightarrow \text{Aut}(E)$  is an isomorphism, and from the construction it follows that  $(\pi_{\mathfrak{U}})_*$  maps  $\lambda \text{id}_U \mapsto \lambda \text{id}_E$  for  $\lambda \in \mathbb{G}_m$ . Hence the 1-morphism  $\rho_i : Z_i \times [\text{Spec } \mathbb{C}/\mathbb{G}_m] \rightarrow [Q_{\mathfrak{U}, n}/\text{GL}(U)] \cong \mathfrak{U}$  acts on stabilizer groups as  $(\rho_i)_* : \lambda \mapsto \lambda \text{id}_U$  for  $\lambda \in \mathbb{G}_m$ , for all  $z \in Z_i(\mathbb{C})$ . It is now easy to see that the 1-morphism  $\rho_i : Z_i \times [\text{Spec } \mathbb{C}/\mathbb{G}_m] \rightarrow \mathfrak{U}$ , regarded as a morphism of global quotient stacks  $\rho_i : [Z_i/\mathbb{G}_m] \rightarrow [Q_{\mathfrak{U}, n}/\text{GL}(U)]$  where  $\mathbb{G}_m$  acts trivially on  $Z_i$ , is induced by the morphisms  $\xi_i : Z_i \rightarrow Q_{\mathfrak{U}, n}$  of  $\mathbb{C}$ -schemes and  $I_U : \mathbb{G}_m \rightarrow \text{GL}(U)$  of algebraic  $\mathbb{C}$ -groups mapping  $I_U : \lambda \mapsto \lambda \text{id}_U$ .

Thus we may write  $f$  in the form (11.7), where each  $Z_i$  is a quasiprojective  $\mathbb{C}$ -variety and each  $\rho_i : Z_i \times [\text{Spec } \mathbb{C}/\mathbb{G}_m] \rightarrow [Q_{\mathfrak{U}, n}/\text{GL}(U)] \cong \mathfrak{U}$  is induced by  $\xi_i : Z_i \rightarrow Q_{\mathfrak{U}, n}$  and  $I_U : \mathbb{G}_m \rightarrow \text{GL}(U)$ ,  $I_U : \lambda \mapsto \lambda \text{id}_U$ . Similarly, we may write

$$g = \sum_{j=1}^{\hat{n}} \hat{\delta}_j [(\hat{Z}_j \times [\text{Spec } \mathbb{C}/\mathbb{G}_m], \iota_{\mathfrak{V}} \circ \hat{\rho}_j)], \quad (11.8)$$

where  $\hat{Z}_j$  is quasiprojective and  $\hat{\rho}_j : \hat{Z}_j \times [\text{Spec } \mathbb{C}/\mathbb{G}_m] \rightarrow [Q_{\mathfrak{V}, n}/\text{GL}(V)] \cong \mathfrak{V}$  is induced by  $\hat{\xi}_j : \hat{Z}_j \rightarrow Q_{\mathfrak{V}, n}$  and  $I_V : \mathbb{G}_m \rightarrow \text{GL}(V)$ ,  $I_V : \lambda \mapsto \lambda \text{id}_V$ .

Combining (11.7)–(11.8) gives an expression for  $f \otimes g$  in  $\text{SF}(\mathfrak{M} \times \mathfrak{M}, \chi, \mathbb{Q})$ :

$$f \otimes g = \sum_{i=1}^n \sum_{j=1}^{\hat{n}} \delta_i \hat{\delta}_j [(Z_i \times \hat{Z}_j \times [\text{Spec } \mathbb{C}/\mathbb{G}_m^2], (\iota_{\mathfrak{U}} \times \iota_{\mathfrak{V}}) \circ (\rho_i \times \hat{\rho}_j))]. \quad (11.9)$$

Using the 1-isomorphisms (11.2), (11.3) and the correspondence between the 1-morphisms  $\pi_1, \pi_3$  and  $\Pi'_{\mathfrak{U}}, \Pi'_{\mathfrak{V}}$  in (11.4) and  $\Pi_{\mathfrak{U}}, \Pi_{\mathfrak{V}}$ , we obtain 1-isomorphisms

$$\begin{aligned} & (Z_i \times \hat{Z}_j \times [\text{Spec } \mathbb{C}/\mathbb{G}_m^2]) \times_{(\iota_{\mathfrak{U}} \times \iota_{\mathfrak{V}}) \circ (\rho_i \times \hat{\rho}_j), \mathfrak{M} \times \mathfrak{M}, \pi_1 \times \pi_3} \mathbf{E}^{\text{fact}} \\ & \cong (Z_i \times \hat{Z}_j \times [\text{Spec } \mathbb{C}/\mathbb{G}_m^2]) \times_{\substack{\rho_i \times \hat{\rho}_j, [Q_{\mathfrak{U}, n}/\text{GL}(U)] \times [Q_{\mathfrak{V}, n}/\text{GL}(V)], \Pi'_{\mathfrak{U}} \times \Pi'_{\mathfrak{V}}}} [Q_{\mathfrak{U}, \mathfrak{V}, n}/(\text{GL}(U) \times \text{GL}(V)) \ltimes \text{Hom}(V, U)] \\ & \cong [(Z_i \times \hat{Z}_j) \times_{\xi_i \times \hat{\xi}_j, Q_{\mathfrak{U}, n} \times Q_{\mathfrak{V}, n}, \Pi_{\mathfrak{U}} \times \Pi_{\mathfrak{V}}} Q_{\mathfrak{U}, \mathfrak{V}, n}/\mathbb{G}_m^2 \ltimes \text{Hom}(V, U)], \end{aligned} \quad (11.10)$$

where in the last line, the multiplication in  $\mathbb{G}_m^2 \ltimes \text{Hom}(V, U)$  is  $(\lambda, \mu, \epsilon) \cdot (\lambda', \mu', \epsilon') = (\lambda\lambda', \mu\mu', \lambda\epsilon' + \mu'\epsilon)$  for  $\lambda, \lambda', \mu, \mu' \in \mathbb{G}_m$  and  $\epsilon, \epsilon' \in \text{Hom}(V, U)$ , and  $\mathbb{G}_m^2 \ltimes \text{Hom}(V, U)$  acts on  $(Z_i \times \hat{Z}_j) \times \dots Q_{\mathfrak{U}, \mathfrak{V}, n}$  by the composition of the morphism  $\mathbb{G}_m^2 \ltimes \text{Hom}(V, U) \rightarrow (\text{GL}(U) \times \text{GL}(V)) \ltimes \text{Hom}(V, U)$  mapping  $(\lambda, \mu, \epsilon) \mapsto (\lambda \text{id}_U, \mu \text{id}_V, \epsilon)$  and the action of  $(\text{GL}(U) \times \text{GL}(V)) \ltimes \text{Hom}(V, U)$  on  $Q_{\mathfrak{U}, \mathfrak{V}, n}$ , with the trivial action on  $Z_i \times \hat{Z}_j$ .

Now  $f * g = (\pi_2)_*((\pi_1 \times \pi_3)^*(f \otimes g))$  by (3.3). Applying  $(\pi_1 \times \pi_3)^*$  to each term in (11.9) involves the fibre product in the first line of (11.10). So from (3.3), (11.9) and (11.10) we see that

$$f * g = \sum_{i=1}^n \sum_{j=1}^{\hat{n}} \delta_i \hat{\delta}_j [((Z_i \times \hat{Z}_j) \times_{Q_{\mathfrak{U}, n} \times Q_{\mathfrak{V}, n}} Q_{\mathfrak{U}, \mathfrak{V}, n}/\mathbb{G}_m^2 \ltimes \text{Hom}(V, U)), \psi_{ij}], \quad (11.11)$$

for 1-morphisms  $\psi_{ij} : [(Z_i \times \hat{Z}_j) \times \dots Q_{\mathfrak{U}, \mathfrak{V}, n} / \mathbb{G}_m^2 \ltimes \text{Hom}(V, U)] \rightarrow \mathfrak{M}^{\alpha+\beta}$ . Similarly

$$g * f = \sum_{i=1}^n \sum_{j=1}^{\hat{n}} \delta_i \hat{\delta}_j [((\hat{Z}_j \times Z_i) \times_{Q_{\mathfrak{V}, n} \times Q_{\mathfrak{U}, n}} \tilde{Q}_{\mathfrak{V}, \mathfrak{U}, n} / \mathbb{G}_m^2 \ltimes \text{Hom}(U, V)), \tilde{\psi}_{ji}]. \quad (11.12)$$

Next we use relations Definition 2.16(i)–(iii) in  $\bar{\text{SF}}(\mathfrak{M}, \chi, \mathbb{Q})$  to write (11.11)–(11.12) in a more useful form. When  $G = \mathbb{G}_m^2 \ltimes \text{Hom}(V, U)$  and  $T^G = \mathbb{G}_m^2 \times \{0\} \subset \mathbb{G}_m^2 \ltimes \text{Hom}(V, U)$ , we find that  $\mathcal{Q}(G, T^G) = \{T^G, \{(\lambda, \lambda) : \lambda \in \mathbb{G}_m\}\} = \{\mathbb{G}_m^2, \mathbb{G}_m\}$ , in the notation of Definition 2.15. We need to compute the coefficients  $F(G, T^G, \mathcal{Q})$  in Definition 2.16(iii) for  $\mathcal{Q} = \mathbb{G}_m^2, \mathbb{G}_m$ . Let  $X$  be the homogeneous space  $\mathbb{G}_m^2 \backslash G \cong \text{Hom}(V, U)$ , considered as a  $\mathbb{C}$ -scheme, with a right action of  $G$ . Then  $\mathbb{G}_m^2 \subset G$  acts on  $X \cong \text{Hom}(V, U)$  by  $(\lambda, \mu) : \epsilon \mapsto \lambda\mu^{-1}\epsilon$  and  $\mathbb{G}_m \subset G$  acts trivially on  $X$ . Then in  $\bar{\text{SF}}(\text{Spec } \mathbb{C}, \chi, \mathbb{Q})$  we have

$$\begin{aligned} [[\text{Spec } \mathbb{C} / \mathbb{G}_m^2]] &= [[X/G]] = F(G, T^G, \mathbb{G}_m^2)[[X/\mathbb{G}_m^2]] + F(G, T^G, \mathbb{G}_m)[[X/\mathbb{G}_m]] \\ &= F(G, T^G, \mathbb{G}_m^2)([[\text{Spec } \mathbb{C} / \mathbb{G}_m^2]] + [P(\text{Hom}(V, U)) \times [\text{Spec } \mathbb{C} / \mathbb{G}_m]]) \\ &\quad + F(G, T^G, \mathbb{G}_m)[\text{Hom}(V, U) \times [\text{Spec } \mathbb{C} / \mathbb{G}_m]] \\ &= F(G, T^G, \mathbb{G}_m^2)[[\text{Spec } \mathbb{C} / \mathbb{G}_m^2]] \\ &\quad + (F(G, T^G, \mathbb{G}_m^2) \dim \text{Hom}(V, U) + F(G, T^G, \mathbb{G}_m)) [[\text{Spec } \mathbb{C} / \mathbb{G}_m]], \end{aligned} \quad (11.13)$$

where in the second step we use Definition 2.16(iii), in the third we divide  $[X/\mathbb{G}_m^2]$  into  $[(\text{Hom}(V, U) \setminus \{0\})/\mathbb{G}_m^2] \cong P(\text{Hom}(V, U)) \times [\text{Spec } \mathbb{C} / \mathbb{G}_m]$  and  $[\{0\}/\mathbb{G}_m^2] \cong [\text{Spec } \mathbb{C} / \mathbb{G}_m^2]$  and use Definition 2.16(i), and in the fourth we use Definition 2.16(ii) and  $\chi(P(\text{Hom}(V, U))) = \dim \text{Hom}(V, U)$ ,  $\chi(\text{Hom}(V, U)) = 1$ .

As  $[[\text{Spec } \mathbb{C} / \mathbb{G}_m]]$ ,  $[[\text{Spec } \mathbb{C} / \mathbb{G}_m^2]]$  are independent in  $\bar{\text{SF}}(\text{Spec } \mathbb{C}, \chi, \mathbb{Q})$  by Proposition 2.18, equating coefficients in (11.13) gives  $F(G, T^G, \mathbb{G}_m^2) = 1$  and  $F(G, T^G, \mathbb{G}_m) = -\dim U \dim V$ . Therefore Definition 2.16(iii) gives

$$\begin{aligned} &[[[(Z_i \times \hat{Z}_j) \times_{Q_{\mathfrak{U}, n} \times Q_{\mathfrak{V}, n}} Q_{\mathfrak{U}, \mathfrak{V}, n} / \mathbb{G}_m^2 \ltimes \text{Hom}(V, U)], \psi_{ij}]] = \\ &[[[(Z_i \times \hat{Z}_j) \times_{Q_{\mathfrak{U}, n} \times Q_{\mathfrak{V}, n}} Q_{\mathfrak{U}, \mathfrak{V}, n} / \mathbb{G}_m^2], \psi_{ij} \circ \iota^{\mathbb{G}_m^2}]] \\ &\quad - \dim U \dim V [[[(Z_i \times \hat{Z}_j) \times_{Q_{\mathfrak{U}, n} \times Q_{\mathfrak{V}, n}} Q_{\mathfrak{U}, \mathfrak{V}, n} / \mathbb{G}_m], \psi_{ij} \circ \iota^{\mathbb{G}_m}]]. \end{aligned} \quad (11.14)$$

Split  $Q_{\mathfrak{U}, \mathfrak{V}, n}$  into  $z(Q_{\mathfrak{U}, n} \times Q_{\mathfrak{V}, n}) \cong Q_{\mathfrak{U}, n} \times Q_{\mathfrak{V}, n}$  and  $Q'_{\mathfrak{U}, \mathfrak{V}, n}$ . In the second line of (11.14), the action of  $\mathbb{G}_m^2$  is trivial on  $Z_i \times \hat{Z}_j$  and on  $z(Q_{\mathfrak{U}, n} \times Q_{\mathfrak{V}, n})$ . On  $Q'_{\mathfrak{U}, \mathfrak{V}, n}$ , the element  $(\lambda, \mu)$  in  $\mathbb{G}_m^2$  acts by dilation by  $\lambda\mu^{-1}$  in the fibres  $W_{\mathfrak{U}, \mathfrak{V}, n}^{q_1, q_2} \setminus \{0\}$ . Thus we can write  $\mathbb{G}_m^2$  as a product of the diagonal  $\mathbb{G}_m$  factor  $\{(\lambda, \lambda) : \lambda \in \mathbb{G}_m\}$  which acts trivially, and a  $\mathbb{G}_m$  factor  $\{(\lambda, 1) : \lambda \in \mathbb{G}_m\}$  which acts freely on  $Q'_{\mathfrak{U}, \mathfrak{V}, n}$ . Hence Definition 2.16(i) gives

$$\begin{aligned} &[[[(Z_i \times \hat{Z}_j) \times_{Q_{\mathfrak{U}, n} \times Q_{\mathfrak{V}, n}} Q_{\mathfrak{U}, \mathfrak{V}, n} / \mathbb{G}_m^2], \psi_{ij} \circ \iota^{\mathbb{G}_m^2}]] = \\ &[[[Z_i \times \hat{Z}_j \times [\text{Spec } \mathbb{C} / \mathbb{G}_m^2], \psi_{ij} \circ \iota^{\mathbb{G}_m^2} \circ z]] \end{aligned} \quad (11.15)$$

$$\begin{aligned} &\quad + [[[(Z_i \times \hat{Z}_j) \times_{Q_{\mathfrak{U}, n} \times Q_{\mathfrak{V}, n}} (Q'_{\mathfrak{U}, \mathfrak{V}, n} / \mathbb{G}_m) \times [\text{Spec } \mathbb{C} / \mathbb{G}_m], \psi_{ij} \circ \iota^{\mathbb{G}_m^2}]], \\ &[[[(Z_i \times \hat{Z}_j) \times_{Q_{\mathfrak{U}, n} \times Q_{\mathfrak{V}, n}} Q_{\mathfrak{U}, \mathfrak{V}, n} / \mathbb{G}_m], \psi_{ij} \circ \iota^{\mathbb{G}_m}]] = \\ &[[[Z_i \times \hat{Z}_j \times [\text{Spec } \mathbb{C} / \mathbb{G}_m], \psi_{ij} \circ \iota^{\mathbb{G}_m} \circ z]] \\ &\quad + [[[(Z_i \times \hat{Z}_j) \times_{Q_{\mathfrak{U}, n} \times Q_{\mathfrak{V}, n}} Q'_{\mathfrak{U}, \mathfrak{V}, n} \times [\text{Spec } \mathbb{C} / \mathbb{G}_m], \psi_{ij} \circ \iota^{\mathbb{G}_m}]], \end{aligned} \quad (11.16)$$

since  $(Z_i \times \hat{Z}_j) \times_{Q_{\mathfrak{U},n} \times Q_{\mathfrak{V},n}} z(Q_{\mathfrak{U},n} \times Q_{\mathfrak{V},n}) \cong Z_i \times \hat{Z}_j$ . Here  $Q'_{\mathfrak{U},\mathfrak{V},n}/\mathbb{G}_m$  is a quasiprojective  $\mathbb{C}$ -variety, with projection  $\Pi_{\mathfrak{U}} \times \Pi_{\mathfrak{V}} : Q'_{\mathfrak{U},\mathfrak{V},n} \rightarrow Q_{\mathfrak{U},n} \times Q_{\mathfrak{V},n}$  with fibre  $\mathbb{P}(W_{\mathfrak{U},\mathfrak{V},n}^{q_1, q_2})$  over  $(q_1, q_2) \in (Q_{\mathfrak{U},n} \times Q_{\mathfrak{V},n})(\mathbb{C})$ . The action of  $\mathbb{G}_m$  on  $Q'_{\mathfrak{U},\mathfrak{V},n}$  is given on points by  $\lambda : [(0 \rightarrow E_1 \rightarrow F \rightarrow E_2 \rightarrow 0, \phi_1, \phi, \phi_2)] \mapsto [(0 \rightarrow E_1 \rightarrow F \rightarrow E_2 \rightarrow 0, \lambda\phi_1, \phi \circ \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}, \phi_2)]$ , for  $\lambda \in \mathbb{G}_m$ .

In the final term in (11.16), the 1-morphism  $\psi_{ij} \circ \iota^{\mathbb{G}_m}$  factors via the projection  $Q'_{\mathfrak{U},\mathfrak{V},n} \rightarrow Q'_{\mathfrak{U},\mathfrak{V},n}/\mathbb{G}_m$ , since  $[(0 \rightarrow E_1 \rightarrow F \rightarrow E_2 \rightarrow 0, \lambda\phi_1, \phi \circ \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}, \phi_2)]$  maps to  $[F]$  for all  $\lambda \in \mathbb{G}_m$ . The projection  $(Z_i \times \hat{Z}_j) \times_{Q_{\mathfrak{U},n} \times Q_{\mathfrak{V},n}} Q'_{\mathfrak{U},\mathfrak{V},n} \rightarrow (Z_i \times \hat{Z}_j) \times_{Q_{\mathfrak{U},n} \times Q_{\mathfrak{V},n}} (Q'_{\mathfrak{U},\mathfrak{V},n}/\mathbb{G}_m)$  is a principal bundle with fibre  $\mathbb{G}_m$ , and so is Zariski locally trivial as  $\mathbb{G}_m$  is special. Therefore cutting  $(Z_i \times \hat{Z}_j) \times_{Q_{\mathfrak{U},n} \times Q_{\mathfrak{V},n}} (Q'_{\mathfrak{U},\mathfrak{V},n}/\mathbb{G}_m)$  into disjoint pieces over which the fibration is trivial and using relations Definition 2.16(i),(ii) and  $\chi(\mathbb{G}_m) = 0$  shows that

$$[(Z_i \times \hat{Z}_j) \times_{Q_{\mathfrak{U},n} \times Q_{\mathfrak{V},n}} Q'_{\mathfrak{U},\mathfrak{V},n} \times [\text{Spec } \mathbb{C}/\mathbb{G}_m], \psi_{ij} \circ \iota^{\mathbb{G}_m}] = 0. \quad (11.17)$$

Combining equations (11.11) and (11.14)–(11.17) now gives

$$\begin{aligned} f * g &= \sum_{i=1}^n \sum_{j=1}^{\hat{n}} \delta_i \hat{\delta}_j [(Z_i \times \hat{Z}_j \times [\text{Spec } \mathbb{C}/\mathbb{G}_m^2], \psi_{ij} \circ \iota^{\mathbb{G}_m^2} \circ z)] \\ &\quad + \sum_{i=1}^n \sum_{j=1}^{\hat{n}} \delta_i \hat{\delta}_j [(Z_i \times \hat{Z}_j) \times_{Q_{\mathfrak{U},n} \times Q_{\mathfrak{V},n}} (Q'_{\mathfrak{U},\mathfrak{V},n}/\mathbb{G}_m) \\ &\quad \times [\text{Spec } \mathbb{C}/\mathbb{G}_m], \psi_{ij} \circ \iota^{\mathbb{G}_m^2}]] \\ &\quad - \dim U \dim V \sum_{i=1}^n \sum_{j=1}^{\hat{n}} \delta_i \hat{\delta}_j [(Z_i \times \hat{Z}_j \times [\text{Spec } \mathbb{C}/\mathbb{G}_m], \psi_{ij} \circ \iota^{\mathbb{G}_m} \circ z)]. \end{aligned} \quad (11.18)$$

Similarly, from equation (11.12) we deduce that

$$\begin{aligned} g * f &= \sum_{i=1}^n \sum_{j=1}^{\hat{n}} \delta_i \hat{\delta}_j [(\hat{Z}_j \times Z_i \times [\text{Spec } \mathbb{C}/\mathbb{G}_m^2], \tilde{\psi}_{ji} \circ \iota^{\mathbb{G}_m^2} \circ \tilde{z})] \\ &\quad + \sum_{i=1}^n \sum_{j=1}^{\hat{n}} \delta_i \hat{\delta}_j [(\hat{Z}_j \times Z_i) \times_{Q_{\mathfrak{V},n} \times Q_{\mathfrak{U},n}} (\tilde{Q}'_{\mathfrak{V},\mathfrak{U},n}/\mathbb{G}_m) \\ &\quad \times [\text{Spec } \mathbb{C}/\mathbb{G}_m], \tilde{\psi}_{ji} \circ \iota^{\mathbb{G}_m^2}]] \\ &\quad - \dim U \dim V \sum_{i=1}^n \sum_{j=1}^{\hat{n}} \delta_i \hat{\delta}_j [(\hat{Z}_j \times \hat{Z}_i \times [\text{Spec } \mathbb{C}/\mathbb{G}_m], \tilde{\psi}_{ji} \circ \iota^{\mathbb{G}_m} \circ \tilde{z})]. \end{aligned} \quad (11.19)$$

Subtracting (11.19) from (11.18) gives an expression for the Lie bracket  $[f, g]$ . Now the first terms on the right hand sides of (11.18) and (11.19) are equal, as over points  $z_1 \in Z_i(\mathbb{C})$  and  $\hat{z}_2 \in \hat{Z}_j(\mathbb{C})$  projecting to  $[E_1] \in \mathfrak{U}(\mathbb{C})$  and  $[E_2] \in \mathfrak{V}(\mathbb{C})$  they correspond to exact sequences  $[0 \rightarrow E_1 \rightarrow E_1 \oplus E_2 \rightarrow E_2 \rightarrow 0]$  and  $[0 \rightarrow E_2 \rightarrow E_2 \oplus E_1 \rightarrow E_1 \rightarrow 0]$  respectively, and so project to the same point  $[E_1 \oplus E_2]$  in  $\mathfrak{M}$ . Similarly, the final terms on the right hand sides of (11.18) and (11.19) are equal. Hence

$$\begin{aligned} [f, g] &= \sum_{i=1}^n \sum_{j=1}^{\hat{n}} \delta_i \hat{\delta}_j \cdot \\ &\quad \{ [((Z_i \times \hat{Z}_j) \times_{Q_{\mathfrak{U},n} \times Q_{\mathfrak{V},n}} (Q'_{\mathfrak{U},\mathfrak{V},n}/\mathbb{G}_m) \times [\text{Spec } \mathbb{C}/\mathbb{G}_m], \psi_{ij} \circ \iota^{\mathbb{G}_m^2})] \\ &\quad - [((\hat{Z}_j \times Z_i) \times_{Q_{\mathfrak{V},n} \times Q_{\mathfrak{U},n}} (\tilde{Q}'_{\mathfrak{V},\mathfrak{U},n}/\mathbb{G}_m) \times [\text{Spec } \mathbb{C}/\mathbb{G}_m], \tilde{\psi}_{ji} \circ \iota^{\mathbb{G}_m^2})] \}. \end{aligned} \quad (11.20)$$

Note that (11.20) writes  $[f, g] \in \text{SF}_{\text{al}}^{\text{ind}}(\mathfrak{M}, \chi, \mathbb{Q})$  as a  $\mathbb{Q}$ -linear combination of  $[(U \times [\text{Spec } \mathbb{C}/\mathbb{G}_m], \rho)]$  for  $U$  a quasiprojective  $\mathbb{C}$ -variety, as in Proposition 3.4.

We now apply the  $\mathbb{Q}$ -linear map  $\tilde{\Psi}^{\chi, \mathbb{Q}}$  to  $f, g$  and  $[f, g]$ . Since  $f, g$  are supported on  $\mathfrak{M}^\alpha, \mathfrak{M}^\beta$ , Definition 5.13 and equations (11.7) and (11.8) yield

$$\tilde{\Psi}^{\chi, \mathbb{Q}}(f) = \gamma \tilde{\lambda}^\alpha \quad \text{and} \quad \tilde{\Psi}^{\chi, \mathbb{Q}}(g) = \hat{\gamma} \tilde{\lambda}^\beta, \quad (11.21)$$

where  $\gamma, \hat{\gamma} \in \mathbb{Q}$  are given by

$$\gamma = \sum_{i=1}^n \delta_i \chi(Z_i, (\iota_{\mathfrak{U}} \circ \rho_i)^*(\nu_{\mathfrak{M}})), \quad \hat{\gamma} = \sum_{j=1}^{\hat{n}} \hat{\delta}_j \chi(\hat{Z}_j, (\iota_{\mathfrak{V}} \circ \hat{\rho}_j)^*(\nu_{\mathfrak{M}})). \quad (11.22)$$

Using Theorem 4.3(iii) and Corollary 4.5 we have

$$\begin{aligned} & \chi(Z_i, (\iota_{\mathfrak{U}} \circ \rho_i)^*(\nu_{\mathfrak{M}})) \chi(\hat{Z}_j, (\iota_{\mathfrak{V}} \circ \hat{\rho}_j)^*(\nu_{\mathfrak{M}})) \\ &= \chi(Z_i \times \hat{Z}_j, (\iota_{\mathfrak{U}} \circ \rho_i)^*(\nu_{\mathfrak{M}}) \square (\iota_{\mathfrak{V}} \circ \hat{\rho}_j)^*(\nu_{\mathfrak{M}})) \\ &= \chi(Z_i \times \hat{Z}_j, (\iota_{\mathfrak{U}} \circ \rho_i \times \iota_{\mathfrak{V}} \circ \hat{\rho}_j)^*(\nu_{\mathfrak{M} \times \mathfrak{M}})). \end{aligned}$$

Thus multiplying the two equations of (11.22) together gives

$$\gamma \hat{\gamma} = \sum_{i=1}^n \sum_{j=1}^{\hat{n}} \delta_i \hat{\delta}_j \chi(Z_i \times \hat{Z}_j, (\iota_{\mathfrak{U}} \circ \rho_i \times \iota_{\mathfrak{V}} \circ \hat{\rho}_j)^*(\nu_{\mathfrak{M} \times \mathfrak{M}})). \quad (11.23)$$

In the same way, since  $[f, g]$  is supported on  $\mathfrak{M}^{\alpha+\beta}$ , using (11.20) we have

$$\begin{aligned} & \tilde{\Psi}^{\chi, \mathbb{Q}}([f, g]) = \zeta \tilde{\lambda}^{\alpha+\beta}, \quad \text{where} \\ & \zeta = \sum_{i=1}^n \sum_{j=1}^{\hat{n}} \delta_i \hat{\delta}_j \chi((Z_i \times \hat{Z}_j) \times_{Q_{\mathfrak{U}, n} \times Q_{\mathfrak{V}, n}} (Q'_{\mathfrak{U}, \mathfrak{V}, n}/\mathbb{G}_m), \psi_{ij}^*(\nu_{\mathfrak{M}})) \\ & \quad - \sum_{i=1}^n \sum_{j=1}^{\hat{n}} \delta_i \hat{\delta}_j \chi((\hat{Z}_j \times Z_i) \times_{Q_{\mathfrak{V}, n} \times Q_{\mathfrak{U}, n}} (\tilde{Q}'_{\mathfrak{V}, \mathfrak{U}, n}/\mathbb{G}_m), \tilde{\psi}_{ji}^*(\nu_{\mathfrak{M}})). \end{aligned} \quad (11.24)$$

Write  $\pi_{ij} : (Z_i \times \hat{Z}_j) \times_{Q_{\mathfrak{U}, n} \times Q_{\mathfrak{V}, n}} (Q'_{\mathfrak{U}, \mathfrak{V}, n}/\mathbb{G}_m) \rightarrow Z_i \times \hat{Z}_j$  for the projection, and  $\tilde{\pi}_{ji}$  for its analogue with  $\mathfrak{U}, \mathfrak{V}$  exchanged. Then from [49], we have

$$\chi((Z_i \times \hat{Z}_j) \times_{Q_{\mathfrak{U}, n} \times Q_{\mathfrak{V}, n}} (Q'_{\mathfrak{U}, \mathfrak{V}, n}/\mathbb{G}_m), \psi_{ij}^*(\nu_{\mathfrak{M}})) = \chi(Z_i \times \hat{Z}_j, \text{CF}(\pi_{ij})(\psi_{ij}^*(\nu_{\mathfrak{M}}))),$$

where  $\text{CF}(\pi_{ij})$  is the pushforward of constructible functions. Substituting this and its analogue for  $\tilde{\pi}_{ji}$  into (11.24) and identifying  $Z_i \times \hat{Z}_j \cong \hat{Z}_j \times Z_i$  yields

$$\begin{aligned} & \zeta = \sum_{i=1}^n \sum_{j=1}^{\hat{n}} \delta_i \hat{\delta}_j \chi(Z_i \times \hat{Z}_j, F_{ij}), \quad \text{where} \\ & F_{ij} = \text{CF}(\pi_{ij})(\psi_{ij}^*(\nu_{\mathfrak{M}})) - \text{CF}(\tilde{\pi}_{ji})(\tilde{\psi}_{ji}^*(\nu_{\mathfrak{M}})) \quad \text{in } \text{CF}(Z_i \times \hat{Z}_j). \end{aligned} \quad (11.25)$$

Let  $z_1 \in Z_i(\mathbb{C})$  for some  $i = 1, \dots, n$ , and  $\hat{z}_2 \in \hat{Z}_j(\mathbb{C})$  for some  $j = 1, \dots, \hat{n}$ . Set  $q_1 = (\xi_i)_*(z_1)$  in  $Q_{\mathfrak{U}, n}(\mathbb{C})$  and  $q_2 = (\hat{\xi}_j)_*(\hat{z}_2)$  in  $Q_{\mathfrak{V}, n}(\mathbb{C})$ , and let  $q_1, q_2$  correspond to isomorphism classes  $[(E_1, \phi_1)], [(E_2, \phi_2)]$  with  $[E_1] \in \mathfrak{U}(\mathbb{C})$  and  $[E_2] \in \mathfrak{V}(\mathbb{C})$ . We will compute an expression for  $F_{ij}(z_1, \hat{z}_2)$  in terms of  $E_1, E_2$ . The fibre of  $\pi_{ij} : (Z_i \times \hat{Z}_j) \times_{Q_{\mathfrak{U}, n} \times Q_{\mathfrak{V}, n}} (Q'_{\mathfrak{U}, \mathfrak{V}, n}/\mathbb{G}_m) \rightarrow Z_i \times \hat{Z}_j$  over  $(z_1, \hat{z}_2)$  is the fibre of  $\Pi_{\mathfrak{U}} \times \Pi_{\mathfrak{V}} : Q'_{\mathfrak{U}, \mathfrak{V}, n}/\mathbb{G}_m \rightarrow Q_{\mathfrak{U}, n} \times Q_{\mathfrak{V}, n}$  over  $(q_1, q_2)$ , which is the projective space  $\mathbb{P}(W_{\mathfrak{U}, \mathfrak{V}, n}^{q_1, q_2})$ . Thus the definition of  $\text{CF}(\pi_{ij})$  in §2.1 implies that

$$(\text{CF}(\pi_{ij})(\psi_{ij}^*(\nu_{\mathfrak{M}})))(z_1, \hat{z}_2) = \chi(\mathbb{P}(W_{\mathfrak{U}, \mathfrak{V}, n}^{q_1, q_2}), \psi_{ij}^*(\nu_{\mathfrak{M}})). \quad (11.26)$$

To understand the constructible function  $\psi_{ij}^*(\nu_{\mathfrak{M}})$  on  $\mathbb{P}(W_{\mathfrak{U}, \mathfrak{V}, n}^{q_1, q_2})$ , consider the linear map  $\pi_{E_2, E_1} : W_{\mathfrak{U}, \mathfrak{V}, n}^{q_1, q_2} \rightarrow \text{Ext}^1(E_2, E_1)$  in (11.5). The kernel  $\text{Ker } \pi_{E_2, E_1}$  is a subspace of  $W_{\mathfrak{U}, \mathfrak{V}, n}^{q_1, q_2}$ , so  $\mathbb{P}(\text{Ker } \pi_{E_2, E_1}) \subseteq \mathbb{P}(W_{\mathfrak{U}, \mathfrak{V}, n}^{q_1, q_2})$ . The induced map

$$(\pi_{E_2, E_1})_* : \mathbb{P}(W_{\mathfrak{U}, \mathfrak{V}, n}^{q_1, q_2}) \setminus \mathbb{P}(\text{Ker } \pi_{E_2, E_1}) \longrightarrow \mathbb{P}(\text{Ext}^1(E_2, E_1)) \quad (11.27)$$

is surjective as  $\pi_{E_2, E_1}$  is, and has fibre  $\text{Ker } \pi_{E_2, E_1}$ . Let  $[w] \in \mathbb{P}(W_{\mathfrak{U}, \mathfrak{V}, n}^{q_1, q_2})$ . If  $[w] \notin \mathbb{P}(\text{Ker } \pi_{E_2, E_1})$ , write  $(\pi_{E_2, E_1})_*([w]) = [\lambda]$  for  $0 \neq \lambda \in \text{Ext}^1(E_2, E_1)$ , and then  $(\psi_{ij})_*([w]) = [F]$  in  $\mathfrak{M}(\mathbb{C})$  where the exact sequence  $0 \rightarrow E_1 \rightarrow F \rightarrow E_2 \rightarrow 0$  corresponds to  $\lambda \in \text{Ext}^1(E_2, E_1)$ , and  $(\psi_{ij}^*(\nu_{\mathfrak{M}}))([w]) = \nu_{\mathfrak{M}}(F)$ . If  $[w] \in \mathbb{P}(\text{Ker } \pi_{E_2, E_1})$

then  $(\psi_{ij})_*([w]) = [E_1 \oplus E_2]$  in  $\mathfrak{M}(\mathbb{C})$ , so  $(\psi_{ij}^*(\nu_{\mathfrak{M}}))([w]) = \nu_{\mathfrak{M}}(E_1 \oplus E_2)$ . Therefore

$$\begin{aligned} \chi(\mathbb{P}(W_{\mathfrak{U}, \mathfrak{V}, n}^{q_1, q_2}), \psi_{ij}^*(\nu_{\mathfrak{M}})) &= \int_{\substack{[\lambda] \in \mathbb{P}(\text{Ext}^1(E_2, E_1)): \\ \lambda \Leftrightarrow 0 \rightarrow E_1 \rightarrow F \rightarrow E_2 \rightarrow 0}} \nu_{\mathfrak{M}}(F) d\chi \\ &\quad + \dim \text{Ker } \pi_{E_2, E_1} \cdot \nu_{\mathfrak{M}}(E_1 \oplus E_2), \end{aligned} \quad (11.28)$$

since the fibres  $\text{Ker } \pi_{E_2, E_1}$  of  $(\pi_{E_2, E_1})_*$  in (11.27) have Euler characteristic 1, and  $\chi(\mathbb{P}(\text{Ker } \pi_{E_2, E_1})) = \dim \text{Ker } \pi_{E_2, E_1}$ .

Combining (11.26) and (11.28) with their analogues with  $\mathfrak{U}, \mathfrak{V}$  exchanged and substituting into (11.25) yields

$$\begin{aligned} F_{ij}(z_1, \hat{z}_2) &= \int_{\substack{[\lambda] \in \mathbb{P}(\text{Ext}^1(E_2, E_1)): \\ \lambda \Leftrightarrow 0 \rightarrow E_1 \rightarrow F \rightarrow E_2 \rightarrow 0}} \nu_{\mathfrak{M}}(F) d\chi - \int_{\substack{[\tilde{\lambda}] \in \mathbb{P}(\text{Ext}^1(E_1, E_2)): \\ \tilde{\lambda} \Leftrightarrow 0 \rightarrow E_2 \rightarrow \tilde{F} \rightarrow E_1 \rightarrow 0}} \nu_{\mathfrak{M}}(\tilde{F}) d\chi \\ &\quad + (\dim \text{Ker } \pi_{E_2, E_1} - \dim \text{Ker } \tilde{\pi}_{E_1, E_2}) \nu_{\mathfrak{M}}(E_1 \oplus E_2). \end{aligned} \quad (11.29)$$

From the exact sequences (11.5)–(11.6) we see that

$$\begin{aligned} &\dim \text{Ker } \pi_{E_2, E_1} - \dim \text{Ker } \tilde{\pi}_{E_1, E_2} \\ &= (\dim \text{Hom}(V, U) - \dim \text{Hom}(E_2, E_1)) - (\dim \text{Hom}(U, V) - \dim \text{Hom}(E_1, E_2)) \\ &= \dim \text{Hom}(E_1, E_2) - \dim \text{Hom}(E_2, E_1). \end{aligned}$$

Substituting this into (11.29) and using (3.14), (5.2) and (5.3) gives

$$\begin{aligned} F_{ij}(z_1, \hat{z}_2) &= (\dim \text{Ext}^1(E_2, E_1) - \dim \text{Ext}^1(E_1, E_2)) \\ &\quad + \dim \text{Hom}(E_1, E_2) - \dim \text{Hom}(E_2, E_1) \nu_{\mathfrak{M}}(E_1 \oplus E_2) \\ &= (-1)^{\bar{\chi}(\alpha, \beta)} \bar{\chi}(\alpha, \beta) \nu_{\mathfrak{M} \times \mathfrak{M}}(E_1, E_2) \\ &= (-1)^{\bar{\chi}(\alpha, \beta)} \bar{\chi}(\alpha, \beta) (\iota_{\mathfrak{U}} \circ \rho_i \times \iota_{\mathfrak{V}} \circ \hat{\rho}_j)^* (\nu_{\mathfrak{M} \times \mathfrak{M}})(z_1, \hat{z}_2). \end{aligned}$$

Hence  $F_{ij} \equiv (-1)^{\bar{\chi}(\alpha, \beta)} \bar{\chi}(\alpha, \beta) (\iota_{\mathfrak{U}} \circ \rho_i \times \iota_{\mathfrak{V}} \circ \hat{\rho}_j)^* (\nu_{\mathfrak{M} \times \mathfrak{M}})$ . So (11.23), (11.25) give

$$\begin{aligned} \zeta &= \sum_{i=1}^n \sum_{j=1}^{\hat{n}} \delta_i \hat{\delta}_j \chi(Z_i \times \hat{Z}_j, (-1)^{\bar{\chi}(\alpha, \beta)} \bar{\chi}(\alpha, \beta) (\iota_{\mathfrak{U}} \circ \rho_i \times \iota_{\mathfrak{V}} \circ \hat{\rho}_j)^* (\nu_{\mathfrak{M} \times \mathfrak{M}})) \\ &= (-1)^{\bar{\chi}(\alpha, \beta)} \bar{\chi}(\alpha, \beta) \gamma \hat{\gamma}. \end{aligned}$$

From equations (11.21) and (11.24) we now have

$$\tilde{\Psi}^{\chi, \mathbb{Q}}(f) = \gamma \tilde{\lambda}^\alpha, \quad \tilde{\Psi}^{\chi, \mathbb{Q}}(g) = \hat{\gamma} \tilde{\lambda}^\beta, \quad \tilde{\Psi}^{\chi, \mathbb{Q}}([f, g]) = (-1)^{\bar{\chi}(\alpha, \beta)} \bar{\chi}(\alpha, \beta) \gamma \hat{\gamma} \tilde{\lambda}^{\alpha+\beta},$$

so  $\tilde{\Psi}^{\chi, \mathbb{Q}}([f, g]) = [\tilde{\Psi}^{\chi, \mathbb{Q}}(f), \tilde{\Psi}^{\chi, \mathbb{Q}}(g)]$  by (5.4), and  $\tilde{\Psi}^{\chi, \mathbb{Q}}$  is a Lie algebra morphism. This completes the proof of Theorem 5.14.

## CHAPTER 12

### The proofs of Theorems 5.22, 5.23 and 5.25

This section will prove Theorems 5.22, 5.23 and 5.25, which say that the moduli space of *stable pairs* introduced in §5.4 is a projective  $\mathbb{K}$ -scheme  $\mathcal{M}_{\text{stp}}^{\alpha,n}(\tau')$  with a symmetric obstruction theory, and the corresponding invariants  $PI^{\alpha,n}(\tau') = \int_{[\mathcal{M}_{\text{stp}}^{\alpha,n}(\tau')]^{\text{vir}}} 1$  are unchanged under deformations of  $X$ . Throughout  $\mathbb{K}$  is an arbitrary algebraically closed field, and when we consider Calabi–Yau 3-folds  $X$  over  $\mathbb{K}$ , we do not assume  $H^1(\mathcal{O}_X) = 0$ . A good reference for the material we use on derived categories of coherent sheaves is Huybrechts [42].

#### 12.1. The moduli scheme of stable pairs $\mathcal{M}_{\text{stp}}^{\alpha,n}(\tau')$

To prove deformation-invariance in Theorem 5.25 we will need to work not with a single Calabi–Yau 3-fold  $X$  over  $\mathbb{K}$ , but with a *family* of Calabi–Yau 3-folds  $X \xrightarrow{\varphi} U$  over a base  $\mathbb{K}$ -scheme  $U$ . Taking  $U = \text{Spec } \mathbb{K}$  recovers the case of one Calabi–Yau 3-fold. Here are our assumptions and notation for such families.

**DEFINITION 12.1.** Let  $\mathbb{K}$  be an algebraically closed field, and  $X \xrightarrow{\varphi} U$  be a smooth projective morphism of algebraic  $\mathbb{K}$ -varieties  $X, U$ , with  $U$  connected. Let  $\mathcal{O}_X(1)$  be a relative very ample line bundle for  $X \xrightarrow{\varphi} U$ . For each  $u \in U(\mathbb{K})$ , write  $X_u$  for the fibre  $X \times_{\varphi, U, u} \text{Spec } \mathbb{K}$  of  $\varphi$  over  $u$ , and  $\mathcal{O}_{X_u}(1)$  for  $\mathcal{O}_X(1)|_{X_u}$ . Suppose that  $X_u$  is a smooth Calabi–Yau 3-fold over  $\mathbb{K}$  for all  $u \in U(\mathbb{K})$ , which may have  $H^1(\mathcal{O}_{X_u}) \neq 0$ . The Calabi–Yau condition implies that the dualizing complex  $\omega_{\varphi}$  of  $\varphi$  is a line bundle trivial on the fibres of  $\varphi$ .

The hypotheses of Theorem 5.25 require the  $K^{\text{num}}(\text{coh}(X_u))$  to be canonically isomorphic *locally in*  $U(\mathbb{K})$ . But by Theorem 4.21, we can pass to a finite cover  $\tilde{U}$  of  $U$ , so that the  $K^{\text{num}}(\text{coh}(\tilde{X}_{\tilde{u}}))$  are canonically isomorphic *globally in*  $\tilde{U}(\mathbb{K})$ . So, replacing  $X, U$  by  $\tilde{X}, \tilde{U}$ , we will assume from here until Theorem 12.21 that the numerical Grothendieck groups  $K^{\text{num}}(\text{coh}(X_u))$  for  $u \in U(\mathbb{K})$  are all canonically isomorphic *globally in*  $U(\mathbb{K})$ , and we write  $K(\text{coh}(X))$  for this group  $K^{\text{num}}(\text{coh}(X_u))$  up to canonical isomorphism. We return to the locally isomorphic case after Theorem 12.21.

Let  $E$  be a coherent sheaf on  $X$  which is flat over  $U$ . Then the fibre  $E_u$  over  $u \in U(\mathbb{K})$  is a coherent sheaf on  $X_u$ , and as  $E$  is flat over  $U$  and  $U(\mathbb{K})$  is connected, the class  $[E_u] \in K^{\text{num}}(\text{coh}(X_u)) \cong K(\text{coh}(X))$  is independent of  $u \in U(\mathbb{K})$ . We will write  $[E] \in K(\text{coh}(X))$  for this class  $[E_u]$ .

For any  $\alpha \in K(\text{coh}(X))$ , write  $P_{\alpha}$  for the *Hilbert polynomial* of  $\alpha$  with respect to  $\mathcal{O}_X$ . Then for any  $u \in U(\mathbb{K})$ , if  $E_u \in \text{coh}(X_u)$  with  $[E_u] = \alpha$  in  $K^{\text{num}}(\text{coh}(X_u)) \cong K(\text{coh}(X))$ , the Hilbert polynomial  $P_{E_u}$  of  $E_u$  w.r.t.  $\mathcal{O}_{X_u}(1)$  is  $P_{\alpha}$ . Define  $\tau : C(\text{coh}(X)) \rightarrow G$  by  $\tau(\alpha) = P_{\alpha}/r_{\alpha}$  as in Example 3.8, where  $r_{\alpha}$  is the leading coefficient of  $P_{\alpha}$ . Then  $(\tau, G, \leq)$  is Gieseker stability on  $\text{coh}(X_u)$ , for each  $u \in U(\mathbb{K})$ .

Later, we will fix  $\alpha \in K(\mathrm{coh}(X))$ , and we will fix an integer  $n \gg 0$ , such that every Gieseker semistable coherent sheaf  $E$  over any fibre  $X_u$  of  $X \rightarrow U$  with  $[E] = \alpha \in K^{\mathrm{num}}(\mathrm{coh}(X_u)) \cong K(\mathrm{coh}(X))$  is  $n$ -regular. This is possible as  $U$  is of finite type. We follow the convention in [5] of taking  $D(X)$  to be the derived category of complexes of *quasi-coherent* sheaves on  $X$ , even though complexes in this book will always have coherent cohomology.

We generalize Definitions 5.20 and 5.21 to the families case:

DEFINITION 12.2. Let  $\mathbb{K}, X \xrightarrow{\varphi} U, \mathcal{O}_X(1)$  be as above. Fix  $n \gg 0$  in  $\mathbb{Z}$ . A *pair* is a nonzero morphism of sheaves  $s : \mathcal{O}_X(-n) \rightarrow E$ , where  $E$  is a nonzero sheaf on  $X$ , flat over  $U$ . A *morphism* between two pairs  $s : \mathcal{O}_X(-n) \rightarrow E$  and  $t : \mathcal{O}_X(-n) \rightarrow F$  is a morphism of  $\mathcal{O}_X$ -modules  $f : E \rightarrow F$ , with  $f \circ s = t$ . A pair  $s : \mathcal{O}_X(-n) \rightarrow E$  is called *stable* if:

- (i)  $\tau([E']) \leq \tau([E])$  for all subsheaves  $E'$  of  $E$  with  $0 \neq E' \neq E$ ; and
- (ii) if also  $s$  factors through  $E'$ , then  $\tau([E']) < \tau([E])$ .

The *class* of a pair  $s : \mathcal{O}_X(-n) \rightarrow E$  is the numerical class  $[E]$  in  $K(\mathrm{coh}(X))$ . We will use  $\tau'$  to denote *stability of pairs*, defined using  $\mathcal{O}_X(1)$ .

Let  $T$  be a  $U$ -scheme, that is, a morphism of  $\mathbb{K}$ -schemes  $\psi : T \rightarrow U$ . Let  $\pi : X_T \rightarrow T$  be the pullback of  $X$  to  $T$ , that is,  $X_T = X \times_{\varphi, U, \psi} T$ . A  $T$ -family of *stable pairs with class  $\alpha$*  is a morphism of  $\mathcal{O}_{X_T}$ -modules  $s : \mathcal{O}_{X_T}(-n) \rightarrow E$ , where  $E$  is flat over  $T$ , and when restricting to  $U$ -points  $t$  in  $T$ ,  $s_t : \mathcal{O}_{X_t}(-n) \rightarrow E_t$  is a stable pair, and  $[E_t] = \alpha$  in  $K(\mathrm{coh}(X))$ . As in Definition 5.21 we define the *moduli functor of stable pairs with class  $\alpha$* :

$$\mathbb{M}_{\mathrm{stp}}^{\alpha, n}(\tau') : \mathrm{Sch}_U \longrightarrow \mathbf{Sets}.$$

Pairs, or framed modules, have been studied extensively for the last twenty years, especially on curves. Some references are Bradlow et al. [15], Huybrechts and Lehn [43] and Le Potier [69]. Note that stable pairs can have no automorphisms. Le Potier gives the construction of the moduli spaces in our generality in [69, Th. 4.11]. It follows directly from his construction that in our case of stable pairs, with no strictly semistables, we always get a fine moduli scheme. Theorem 5.22 follows when  $U = \mathrm{Spec} \mathbb{K}$ . Later in the section we will abbreviate  $\mathcal{M}_{\mathrm{stp}}^{\alpha, n}(\tau')$  to  $\mathcal{M}$ , especially in subscripts  $X_{\mathcal{M}}$ .

THEOREM 12.3. *Let  $\alpha \in K(\mathrm{coh}(X))$  and  $n \in \mathbb{Z}$ . Then the moduli functor  $\mathbb{M}_{\mathrm{stp}}^{\alpha, n}(\tau')$  is represented by a projective  $U$ -scheme  $\mathcal{M}_{\mathrm{stp}}^{\alpha, n}(\tau')$ .*

PROOF. This follows from a more general result of Le Potier [69, Th. 4.11], which we now explain. Let  $X$  be a smooth projective  $U$ -scheme of dimension  $m$ , with very ample line bundle  $\mathcal{O}_X(1)$ . In [69, §4], Le Potier defines a *coherent system* to be a pair  $(\Gamma, E)$ , where  $E \in \mathrm{coh}(X)$  and  $\Gamma \subseteq H^0(E)$  is a vector subspace. A *morphism of coherent systems*  $f : (\Gamma, E) \rightarrow (\Gamma', E')$  is a morphism  $f : E \rightarrow E'$  in  $\mathrm{coh}(X)$  with  $f(\Gamma) \subseteq \Gamma'$ .

Let  $q \in \mathbb{Q}[t]$  be a polynomial with positive leading coefficient. For a coherent system  $(\Gamma, E)$  with  $E \neq 0$ , define a polynomial  $p_{(\Gamma, E)}$  in  $\mathbb{Q}[t]$  by

$$p_{(\Gamma, E)}(t) = \frac{P_E(t) + \dim \Gamma \cdot q(t)}{r_E},$$

where  $P_E$  is the Hilbert polynomial of  $E$  and  $r_E > 0$  its leading coefficient. Then we call  $(\Gamma, E)$  *q-semistable* (respectively *q-stable*) if  $E$  is pure and whenever  $E' \subset E$  is a subsheaf with  $E' \neq 0, E$  and  $\Gamma' = \Gamma \cap H^0(E') \subset H^0(E)$  we have  $p_{(\Gamma', E')} \leq p_{(\Gamma, E)}$  (respectively  $p_{(\Gamma', E')} < p_{(\Gamma, E)}$ ), using the total order  $\leq$  on polynomials  $\mathbb{Q}[t]$  in Example 3.8.

Then Le Potier [69, Th. 4.11] shows that if  $\alpha \in K^{\text{num}}(\text{coh}(X))$ , then the moduli functor  $\text{Sys}^\alpha(q)$  of  $q$ -semistable coherent systems  $(\Gamma, E)$  with  $[E] = \alpha$  is represented by a projective moduli  $U$ -scheme  $\text{Sys}^\alpha(q)$ , such that  $U$ -points of  $\text{Sys}^\alpha(q)$  correspond to  $S$ -equivalence classes of coherent systems  $(\Gamma, E)$ . The method is to fix  $N \gg 0$  and a vector space  $V$  of dimension  $P_\alpha(N)$ , and to define a projective ‘Quot scheme’  $\text{Quot}^{\alpha, N}(q)$  of pairs  $((\Gamma, E), \varphi)$ , where  $(\Gamma, E)$  is a coherent system with  $[E] = \alpha$  and  $\varphi : V \rightarrow H^0(E(N))$  is an isomorphism, and  $\text{GL}(V)$  acts on  $\text{Quot}^{\alpha, N}(q)$ . Le Potier shows that there exists a linearization  $\mathcal{L}$  for this action of  $\text{GL}(V)$ , depending on  $q$ , such that GIT (semi)stability of  $((\Gamma, E), \varphi)$  coincides with  $q$ -(semi)stability of  $(\Gamma, E)$ . Then  $\text{Sys}^\alpha(q)$  is the GIT quotient  $\text{Quot}^{\alpha, N}(q) //_{\mathcal{L}} \text{GL}(V)$ .

Here is how to relate this to our situation. Fix  $\alpha \in K(\text{coh}(X))$  and  $n \in \mathbb{Z}$ . To a pair  $s : \mathcal{O}_X(-n) \rightarrow E$  in the sense of Definition 12.2 we associate the coherent system  $(\langle s \rangle, E(n))$ , with sheaf  $E(n) = E \otimes \mathcal{O}_X(n)$  and 1-dimensional subspace  $\Gamma \subset H^0(E(n))$  spanned by  $0 \neq s \in H^0(E(n))$ . We take  $q \in \mathbb{Q}[t]$  to have degree 0, so that  $q \in \mathbb{Q}_{>0}$ , and to be sufficiently small (in fact  $0 < q \leq 1/d! r_\alpha$  is enough, where  $d = \dim \alpha$  and  $r_\alpha$  is the leading coefficient of the Hilbert polynomial  $P_\alpha$ ). Then it is easy to show that  $s : \mathcal{O}_X(-n) \rightarrow E$  is stable if and only if  $(\langle s \rangle, E(n))$  is  $q$ -stable if and only if  $(\langle s \rangle, E(n))$  is  $q$ -semistable.

Hence [69, Th. 4.11] gives a projective coarse moduli  $U$ -scheme  $\mathcal{M}_{\text{stp}}^{\alpha, n}(\tau')$ . Since there are no strictly  $q$ -semistable  $(\langle s \rangle, E(n))$  in this moduli space,  $U$ -points of  $\mathcal{M}_{\text{stp}}^{\alpha, n}(\tau')$  correspond to isomorphism classes of pairs  $s : \mathcal{O}_X(-n) \rightarrow E$ , not just  $S$ -equivalence classes. Also, as stable pairs  $s : \mathcal{O}_X(-n) \rightarrow E$  have no automorphisms,  $\mathcal{M}_{\text{stp}}^{\alpha, n}(\tau')$  is actually a fine moduli scheme.  $\square$

## 12.2. Pairs as objects of the derived category

We can consider a stable pair  $s : \mathcal{O}_X(-n) \rightarrow E$  on  $X$  as a complex  $I$  in the derived category  $D(X)$ , with  $\mathcal{O}_X(-n)$  in degree  $-1$  and  $E$  in degree  $0$ . We evaluate some Ext groups for such a complex  $I$ .

**PROPOSITION 12.4.** *Let  $s : \mathcal{O}_X(-n) \rightarrow E$  be a stable pair, and suppose  $n$  is large enough that  $H^i(E(n)) = 0$  for  $i > 0$ . Write  $I$  for  $\mathcal{O}_X(-n) \xrightarrow{s} E$  considered as an object of  $D(X)$ , with  $E$  in degree  $0$ . Then:*

- (a)  $\text{Ext}_{D(X)}^i(I, E) = 0$  for  $i < 1, i > 3$ .
- (b)  $\text{Ext}_{D(X)}^i(I, \mathcal{O}_X(-n)) = 0$  for  $i < 1, i > 3$ , and  $\text{Ext}_{D(X)}^1(I, \mathcal{O}_X(-n)) \cong \mathbb{K}$ .
- (c)  $\text{Ext}_{D(X)}^i(I, I) = 0$  for  $i < 0, i > 3$ , and  $\text{Ext}_{D(X)}^i(I, I) \cong \mathbb{K}$  for  $i = 0, 3$ .

**PROOF.** We have a distinguished triangle  $I[-1] \rightarrow \mathcal{O}_X(-n) \xrightarrow{s} E \rightarrow I$  in  $D(X)$ . Taking  $\text{Ext}_{D(X)}^*$  of this with  $E, \mathcal{O}_X(-n)$  and  $I$  gives long exact sequences:

$$\begin{aligned} \text{Ext}^{i-1}(\mathcal{O}(-n), E) &\longrightarrow \text{Ext}^i(I, E) \longrightarrow \text{Ext}^i(E, E) \xrightarrow{\circ s} \text{Ext}^i(\mathcal{O}(-n), E), \\ \text{Ext}^{i-1}(\mathcal{O}(-n), \mathcal{O}(-n)) &\longrightarrow \text{Ext}^i(I, \mathcal{O}(-n)) \longrightarrow \text{Ext}^i(E, \mathcal{O}(-n)) \xrightarrow{\circ s} \text{Ext}^i(\mathcal{O}(-n), \mathcal{O}(-n)), \quad (12.1) \\ \text{Ext}^{i-1}(\mathcal{O}(-n), I) &\longrightarrow \text{Ext}^i(I, I) \longrightarrow \text{Ext}^i(E, I) \xrightarrow{\circ s} \text{Ext}^i(\mathcal{O}(-n), I). \end{aligned}$$



In the first row of (12.1), since  $\text{Ext}_{D(X)}^i(\mathcal{O}_X(-n), E) = H^i(E(n)) = 0$  for  $i \neq 0$  and  $\text{Ext}_{D(X)}^i(E, E) = 0$  for  $i < 0$  and  $i > 3$ , this gives  $\text{Ext}_{D(X)}^i(I, E) = 0$  for  $i < 0$  and  $i > 3$ , and

$$\text{Hom}_{D(X)}(I, E) \cong \text{Ker}(\circ s : \text{Hom}(E, E) \longrightarrow \text{Hom}(\mathcal{O}_X(-n), E)). \quad (12.2)$$

Write  $\pi : E \rightarrow F$  for the cokernel of  $s : \mathcal{O}_X(-n) \rightarrow E$ . Suppose  $0 \neq \beta \in \text{Hom}(E, E)$  with  $\beta \circ s = 0$  in  $\text{Hom}(\mathcal{O}_X(-n), E)$ . Both  $\text{Ker}(\beta)$  and  $\text{Im}(\beta)$  are in fact subsheaves of  $E$  with  $[E] = [\text{Ker} \beta] + [\text{Im} \beta]$ , and  $\text{Ker} \beta \neq 0$  as  $\beta \circ s = 0$  with  $s \neq 0$ , and  $\text{Im} \beta \neq 0$  as  $\beta \neq 0$ . Since  $E$  is  $\tau$ -semistable, the seesaw inequalities imply that  $\tau([\text{Ker} \beta]) = \tau([\text{Im} \beta]) = \tau([E])$ . But as  $s$  factors through  $\text{Ker} \beta$ , stability of the pair implies that  $\tau([\text{Ker} \beta]) < \tau([E])$ , a contradiction. So  $\text{Hom}_{D(X)}(I, E) = 0$  by (12.2), proving (a).

We have  $\text{Ext}_{D(X)}^i(\mathcal{O}_X(-n), \mathcal{O}_X(-n)) = H^i(\mathcal{O}_X) = 0$  for  $i < 0$  or  $i > 3$  and is  $\mathbb{K}$  for  $i = 0$ , and  $\text{Ext}_{D(X)}^i(E, \mathcal{O}_X(-n)) \cong \text{Ext}_{D(X)}^{3-i}(\mathcal{O}_X(-n), E)^* \cong H^{3-i}(E(n))^*$  by Serre duality, which is zero unless  $i = 3$ . Part (b) follows from the second row of (12.1) and the fact that  $\circ s : \text{Ext}^3(E, \mathcal{O}_X(-n)) \rightarrow \text{Ext}^3(\mathcal{O}_X(-n), \mathcal{O}_X(-n)) \cong \mathbb{K}$  is nonzero, as this is Serre dual to the morphism  $\text{Hom}(\mathcal{O}_X(-n), \mathcal{O}_X(-n)) \rightarrow \text{Hom}(\mathcal{O}_X(-n), E)$  taking  $1 \mapsto s \neq 0$ .

For (c), Serre duality and part (a) gives  $\text{Ext}_{D(X)}^i(E, I) = 0$  for  $i < 0$  and  $i > 2$ . Thus the third row of (12.1) yields  $\text{Ext}_{D(X)}^i(I, I) \cong \text{Ext}_{D(X)}^{i-1}(\mathcal{O}_X(-n), I) \cong \text{Ext}_{D(X)}^{4-i}(I, \mathcal{O}_X(-n))^*$  for  $i < 0$  and  $i > 3$ . So  $\text{Ext}_{D(X)}^i(I, I) = 0$  for  $i < 0$  and  $i > 3$  by (b). Also  $\mathbb{K} \cong \text{Ext}_{D(X)}^2(\mathcal{O}_X(-n), I) \rightarrow \text{Ext}_{D(X)}^3(I, I) \rightarrow 0$  is exact, and  $\text{Ext}_{D(X)}^3(I, I) \cong \text{Hom}_{D(X)}(I, I)^*$  cannot be zero as  $I$  is nonzero in  $D(X)$ , so  $\text{Ext}_{D(X)}^3(I, I) \cong \mathbb{K}$  and hence  $\text{Ext}_{D(X)}^0(I, I) \cong \mathbb{K}$  by Serre duality, giving (c).  $\square$

**COROLLARY 12.5.** *In the situation of Proposition 12.4, the object  $I$  in  $D(X)$  up to quasi-isomorphism and the integer  $n$  determine the stable pair  $s : \mathcal{O}_X(-n) \rightarrow E$  up to isomorphism.*

**PROOF.** In the distinguished triangle  $I[-1] \xrightarrow{\pi} \mathcal{O}_X(-n) \xrightarrow{s} E \rightarrow I$ , the morphism  $\pi$  is nonzero since otherwise  $E \cong \mathcal{O}_X(-n) \oplus I$  which is not a sheaf. But  $\text{Ext}_{D(X)}^1(I, \mathcal{O}_X(-n)) \cong \mathbb{K}$  by Proposition 12.4(b), so  $\text{Ext}_{D(X)}^1(I, \mathcal{O}_X(-n)) = \mathbb{K} \cdot \pi$ . Thus the morphism  $\pi$  is determined by  $I, n$  up to  $\mathbb{G}_m$  rescalings, so  $E \cong \text{cone}(\pi)$  and  $s$  are determined by  $I, n$  up to isomorphism.  $\square$

We have  $U$ -schemes  $X$  and  $\mathcal{M}_{\text{stp}}^{\alpha, n}(\tau')$ , by Theorem 12.3, so we can form the fibre product  $X_{\mathcal{M}} = X \times_U \mathcal{M}_{\text{stp}}^{\alpha, n}(\tau')$ , which we regard as a family of Calabi–Yau 3-folds  $X_{\mathcal{M}} \xrightarrow{\pi} \mathcal{M}_{\text{stp}}^{\alpha, n}(\tau')$  over the base  $U$ -scheme  $\mathcal{M}_{\text{stp}}^{\alpha, n}(\tau')$ . Since  $\mathcal{M}_{\text{stp}}^{\alpha, n}(\tau')$  is a fine moduli scheme for pairs on  $X$ , on  $X_{\mathcal{M}}$  we have a universal pair, which we denote by  $\mathbb{S} : \mathcal{O}_{X_{\mathcal{M}}}(-n) \rightarrow \mathbb{E}$ . We can regard this as an object in the derived category  $D(X_{\mathcal{M}_{\text{stp}}^{\alpha, n}(\tau')})$ , with  $\mathcal{O}_{X_{\mathcal{M}_{\text{stp}}^{\alpha, n}(\tau')}}(-n)$  in degree  $-1$  and  $\mathbb{E}$  in degree  $0$ , and we will denote this object by  $\mathbb{I} = \text{cone}(\mathbb{S})$ .

### 12.3. Cotangent complexes and obstruction theories

Suppose  $X, Y$  are schemes over some base  $\mathbb{K}$ -scheme  $U$ , and  $X \xrightarrow{\phi} Y$  is a morphism of  $U$ -schemes. Then one can define the *cotangent sheaf* (or *sheaf of relative differentials*)  $\Omega_{X/Y}$  in  $\text{coh}(X)$ , as in Hartshorne [40, §II.8]. This generalizes cotangent bundles of smooth schemes: if  $X$  is a smooth  $\mathbb{K}$ -scheme then  $\Omega_{X/\text{Spec } \mathbb{K}}$  is the

cotangent bundle  $T^*X$ , a locally free sheaf of rank  $\dim X$  on  $X$ . If  $X \xrightarrow{\phi} Y \xrightarrow{\psi} Z$  are morphisms of  $U$ -schemes then there is an exact sequence

$$\phi^*(\Omega_{Y/Z}) \longrightarrow \Omega_{X/Z} \longrightarrow \Omega_{X/Y} \longrightarrow 0 \quad (12.3)$$

in  $\text{coh}(X)$ . Note that the morphism  $\phi^*(\Omega_{Y/Z}) \rightarrow \Omega_{X/Z}$  need not be injective, that is, (12.3) may not be a short exact sequence. Morally speaking, this says that  $\phi \mapsto \Omega_{X/Y}$  is a right exact functor, but may not be left exact.

Cotangent complexes are derived versions of cotangent sheaves, for which (12.3) is replaced by a distinguished triangle (12.4), making it fully exact. The *cotangent complex*  $L_{X/Y}$  of a morphism  $X \xrightarrow{\phi} Y$  is an object in the derived category  $D(X)$ , constructed by Illusie [46]; a helpful review is given in Illusie [47, §1]. It has  $h^0(L_{X/Y}) \cong \Omega_{X/Y}$ . If  $\phi$  is smooth then  $L_{X/Y} = \Omega_{X/Y}$ . Here are some properties of cotangent complexes:

- (a) Suppose  $X \xrightarrow{\phi} Y \xrightarrow{\psi} Z$  are morphisms of  $U$ -schemes. Then there is a distinguished triangle in  $D(X)$ , [46, §2.1], [47, §1.2]:

$$L\phi^*(L_{Y/Z}) \longrightarrow L_{X/Z} \longrightarrow L_{X/Y} \longrightarrow L\phi^*(L_{Y/Z})[1]. \quad (12.4)$$

This is called the *distinguished triangle of transitivity*.

- (b) Suppose we have a commutative diagram of morphisms of  $U$ -schemes:

$$\begin{array}{ccccc} R & \xrightarrow{\rho} & S & \xrightarrow{\sigma} & T \\ \downarrow \epsilon & & \downarrow & & \downarrow \\ X & \xrightarrow{\phi} & Y & \xrightarrow{\psi} & Z. \end{array}$$

Then we get a commutative diagram in  $D(R)$ , [46, §2.1]:

$$\begin{array}{ccccccc} L\rho^*(L_{S/T}) & \longrightarrow & L_{R/T} & \longrightarrow & L_{R/S} & \longrightarrow & L\phi^*(L_{S/T})[1] \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ L\epsilon^*(L\phi^*(L_{Y/Z})) & \rightarrow & L\epsilon^*(L_{X/Z}) & \rightarrow & L\epsilon^*(L_{X/Z}) & \rightarrow & L\epsilon^*(L\phi^*(L_{Y/Z})[1]), \end{array}$$

where the rows come from the distinguished triangles of transitivity for  $R \rightarrow S \rightarrow T$  and  $X \rightarrow Y \rightarrow Z$ .

- (c) Suppose we have a Cartesian diagram of  $U$ -schemes:

$$\begin{array}{ccc} X \times_Z Y & \xrightarrow{\pi_Y} & Y \\ \pi_X \downarrow & & \downarrow \psi \\ X & \xrightarrow{\phi} & Z. \end{array}$$

If  $\phi$  or  $\psi$  is flat then we have *base change isomorphisms* [46, §2.2], [47, §1.3]:

$$\begin{aligned} L_{X \times_Z Y/Y} &\cong L\pi_X^*(L_{X/Z}), & L_{X \times_Z Y/X} &\cong L\pi_Y^*(L_{Y/Z}), \\ \text{and} & & L_{X \times_Z Y/Z} &\cong L\pi_X^*(L_{X/Z}) \oplus L\pi_Y^*(L_{Y/Z}). \end{aligned} \quad (12.5)$$

Recall the following definitions from Behrend and Fantechi [3, 5, 6]:

**DEFINITION 12.6.** Let  $Y$  be a  $\mathbb{K}$ -scheme, and  $D(Y)$  the derived category of quasicoherent sheaves on  $Y$ .

- (a) A complex  $E^\bullet \in D(Y)$  is *perfect of perfect amplitude contained in  $[a, b]$* , if étale locally on  $Y$ ,  $E^\bullet$  is quasi-isomorphic to a complex of locally free sheaves of finite rank in degrees  $a, a+1, \dots, b$ .
- (b) We say that a complex  $E^\bullet \in D(Y)$  *satisfies condition (\*)* if
- (i)  $h^i(E^\bullet) = 0$  for all  $i > 0$ ,

- (ii)  $h^i(E^\bullet)$  is coherent for  $i = 0, -1$ .
- (c) An *obstruction theory* for  $Y$  is a morphism  $\phi : E^\bullet \rightarrow L_Y$  in  $D(Y)$ , where  $L_Y = L_{Y/\mathrm{Spec} \mathbb{K}}$  is the cotangent complex of  $Y$ , and  $E$  satisfies condition (\*), and  $h^0(\phi)$  is an isomorphism, and  $h^{-1}(\phi)$  is an epimorphism.
- (d) An obstruction theory  $\phi : E^\bullet \rightarrow L_Y$  is called *perfect* if  $E^\bullet$  is perfect of perfect amplitude contained in  $[-1, 0]$ .
- (e) A perfect obstruction theory  $\phi : E^\bullet \rightarrow L_Y$  on  $Y$  is called *symmetric* if there exists an isomorphism  $\theta : E^\bullet \rightarrow E^{\bullet\vee}[1]$ , such that  $\theta^\vee[1] = \theta$ . Here  $E^{\bullet\vee} = R\mathcal{H}om(E^\bullet, \mathcal{O}_Y)$  is the *dual* of  $E^\bullet$ , and  $\theta^\vee$  the dual morphism of  $\theta$ .

If instead  $Y \xrightarrow{\psi} U$  is a morphism of  $\mathbb{K}$ -schemes, so  $Y$  is a  $U$ -scheme, we define *relative perfect obstruction theories*  $\phi : E^\bullet \rightarrow L_{Y/U}$  in the obvious way.

A closed immersion of  $\mathbb{K}$ -schemes  $j : T \rightarrow \bar{T}$  is called a *square zero extension* with *ideal sheaf*  $J$  if  $J$  is the ideal sheaf of  $T$  in  $\bar{T}$  and  $J^2 = 0$ , so that we have an exact sequence in  $\mathrm{coh}(\bar{T})$ :

$$0 \longrightarrow J \longrightarrow \mathcal{O}_{\bar{T}} \longrightarrow \mathcal{O}_T \longrightarrow 0. \quad (12.6)$$

We will always take  $T, \bar{T}$  to be *affine* schemes.

The deformation theory of a  $\mathbb{K}$ -scheme  $Y$  is largely governed by its cotangent complex  $L_Y \in D(Y)$ , in the following sense. Suppose that we are given a square-zero extension  $\bar{T}$  of  $T$  with ideal sheaf  $J$ , with  $T, \bar{T}$  affine, and a morphism  $g : T \rightarrow Y$ . Then the theory of cotangent complexes gives a canonical morphism

$$g^*(L_Y) \longrightarrow L_T \longrightarrow J[1]$$

in  $D(T)$ . This morphism,  $\omega(g) \in \mathrm{Ext}^1(g^*L_Y, J)$ , is equal to zero if and only if there exists an extension  $\bar{g} : \bar{T} \rightarrow Y$  of  $g$ . Moreover, when  $\omega(g) = 0$ , the set of isomorphism extensions form a torsor under  $\mathrm{Hom}(g^*L_Y, J)$ .

Behrend and Fantechi prove the following theorem, which both explains the term obstruction theory and provides a criterion for verification in practice:

**THEOREM 12.7** (Behrend and Fantechi [5, Th. 4.5]). *The following two conditions are equivalent for  $E^\bullet \in D(Y)$  satisfying condition (\*).*

- (a) *The morphism  $\phi : E^\bullet \rightarrow L_Y$  is an obstruction theory.*
- (b) *Suppose we are given a setup  $(T, \bar{T}, J, g)$  as above. The morphism  $\phi$  induces an element  $\phi^*(\omega(g)) \in \mathrm{Ext}^1(g^*E^\bullet, J)$  from  $\omega(g) \in \mathrm{Ext}^1(g^*L_Y, J)$  by composition. Then  $\phi^*(\omega(g))$  vanishes if and only if there exists an extension  $\bar{g}$  of  $g$ . If it vanishes, then the set of extensions form a torsor under  $\mathrm{Hom}(g^*E^\bullet, J)$ .*

*The analogue also holds for relative obstruction theories.*

#### 12.4. Deformation theory for pairs

Let  $X, T, \bar{T}$  be  $U$ -schemes with  $T, \bar{T}$  affine, and  $T \rightarrow \bar{T}$  a square zero extension. Write  $X_T = X \times_U T$  and  $X_{\bar{T}} = X \times_U \bar{T}$ , which are  $U$ -schemes with projections

$X_T \rightarrow T$ ,  $X_{\bar{T}} \rightarrow \bar{T}$  and  $X_T \rightarrow X_{\bar{T}}$ . We have a Cartesian diagram:

$$\begin{array}{ccccc} & & X_{\bar{T}} & & \\ & j_* \nearrow & \downarrow \pi & \searrow & \\ X_T & \xrightarrow{\tau} & X & & \\ \downarrow \pi & j \nearrow & \downarrow \varphi & & \\ T & \xrightarrow{\quad} & U, & & \end{array}$$

and an exact sequence in  $\text{coh}(X_{\bar{T}})$ :

$$0 \longrightarrow \pi^* J \longrightarrow \mathcal{O}_{X_{\bar{T}}} \longrightarrow \mathcal{O}_{X_T} \longrightarrow 0. \quad (12.7)$$

Let  $s : \mathcal{O}_{X_T}(-n) \rightarrow E$  be a  $T$ -family of stable pairs. The deformation theory of stable pairs involves studying  $\bar{T}$ -families of stable pairs  $\bar{s} : \mathcal{O}_{X_{\bar{T}}}(-n) \rightarrow \bar{E}$  extending  $s : \mathcal{O}_{X_T}(-n) \rightarrow E$ , that is, with  $(j_*)^*((\bar{E}, \bar{s})) \cong (E, s)$ . Tensoring (12.7) with  $\bar{s} : \mathcal{O}_{X_{\bar{T}}}(-n) \rightarrow \bar{E}$  gives a commutative diagram in  $\text{coh}(X_{\bar{T}})$ , with exact rows:

$$\begin{array}{ccccccc} 0 \rightarrow \pi^*(J) \otimes_{\mathcal{O}_{X_{\bar{T}}}} \mathcal{O}_{X_{\bar{T}}}(-n) & \rightarrow & \mathcal{O}_{X_{\bar{T}}}(-n) & \rightarrow & \mathcal{O}_{X_{\bar{T}}}(-n) \otimes_{\mathcal{O}_{X_{\bar{T}}}} \mathcal{O}_{X_T} & \rightarrow & 0 \\ & \downarrow & \downarrow \bar{s} & & \downarrow s \otimes \text{id}_{\mathcal{O}_{X_T}} & & \\ 0 \longrightarrow \pi^*(J) \otimes_{\mathcal{O}_{X_{\bar{T}}}} \bar{E} & \longrightarrow & \bar{E} & \longrightarrow & \bar{E} \otimes_{\mathcal{O}_{X_{\bar{T}}}} \mathcal{O}_{X_T} & \longrightarrow & 0. \end{array} \quad (12.8)$$

But  $\bar{E} \otimes_{\mathcal{O}_{X_{\bar{T}}}} \mathcal{O}_{X_T} \cong (j_*)^*(\bar{E}) \cong E$ , and  $\mathcal{O}_{X_{\bar{T}}}(-n) \otimes_{\mathcal{O}_{X_{\bar{T}}}} \mathcal{O}_{X_T} \cong \mathcal{O}_{X_T}(-n)$ , and as  $J^2 = 0$  we have  $J \otimes_{\mathcal{O}_{\bar{T}}} \mathcal{O}_T \cong J$ , so  $\pi^*(J) \otimes_{\mathcal{O}_{X_{\bar{T}}}} \mathcal{O}_{X_T} \cong \pi^*(J)$ , and thus

$$\pi^*(J) \otimes_{\mathcal{O}_{X_{\bar{T}}}} \bar{E} \cong \pi^*(J) \otimes_{\mathcal{O}_{X_{\bar{T}}}} \mathcal{O}_{X_T} \otimes_{\mathcal{O}_{X_{\bar{T}}}} \bar{E} \cong \pi^*(J) \otimes_{\mathcal{O}_{X_{\bar{T}}}} E \cong \pi^*(J) \otimes_{\mathcal{O}_{X_T}} E.$$

Hence (12.8) is equivalent to the commutative diagram in  $\text{coh}(X_{\bar{T}})$ :

$$\begin{array}{ccccccc} 0 \longrightarrow \pi^* J \otimes_{\mathcal{O}_{X_T}} \mathcal{O}_{X_T}(-n) & \longrightarrow & \mathcal{O}_{X_{\bar{T}}}(-n) & \longrightarrow & \mathcal{O}_{X_T}(-n) & \longrightarrow & 0 \\ & \downarrow & \downarrow \bar{s} & & \downarrow s & & \\ 0 \longrightarrow \pi^* J \otimes_{\mathcal{O}_{X_T}} E & \longrightarrow & \bar{E} & \longrightarrow & E & \longrightarrow & 0. \end{array} \quad (12.9)$$

Both rows are exact sequences of  $\mathcal{O}_{X_{\bar{T}}}$ -modules. Since  $E$  is flat over  $T$ , such  $\bar{E}$ , if it exists, is necessarily flat over  $\bar{T}$ . Now Illusie [46, §IV.3] studies the problem of completing a diagram of the form (12.9), and proves:

**THEOREM 12.8** (Illusie [46, Prop. IV.3.2.12]). *There exists an element  $ob$  in  $\text{Ext}_{D(X_T)}^2(\text{cone}(s), \pi^* J \otimes E)$ , whose vanishing is necessary and sufficient to complete the diagram (12.9). If  $ob = 0$  then the set of isomorphism classes of deformations forms a torsor under  $\text{Ext}_{D(X_T)}^1(\text{cone}(s), \pi^* J \otimes E)$ .*

Illusie also shows in [46, §IV.3.2.14] that this element  $ob$  can be written as the composition of three morphisms in  $D(X_T)$ , in the commutative diagram:

$$\begin{array}{ccc} \text{cone}(s) & \xrightarrow{\text{At}_{\mathcal{O}_{X_T}/\mathcal{O}_X}(s)} & k^1(L_{(\mathcal{O}_{X_T} \oplus \mathcal{O}_{X_T}(-n))/\mathcal{O}_X}^{gr} \otimes (\mathcal{O}_{X_T} \oplus E))[1] \\ \downarrow ob & & \downarrow k^1(e(\mathcal{O}_{X_{\bar{T}}} \oplus \mathcal{O}_{X_{\bar{T}}}(-n)) \otimes \text{id}_{\mathcal{O}_{X_T} \oplus E}) \\ \pi^* J \otimes E[2] & \xleftarrow{\Pi_1} & k^1(\pi^* J \otimes (\mathcal{O}_{X_T} \oplus \mathcal{O}_{X_T}(-n)) \otimes (\mathcal{O}_{X_T} \oplus E))[2]. \end{array} \quad (12.10)$$

Here in Illusie's set up, we regard  $\mathcal{O}_{X_T} \oplus \mathcal{O}_{X_T}(-n)$  and  $\mathcal{O}_{X_T} \oplus E$  as sheaves of graded algebras on  $X_T$ , with  $\mathcal{O}_{X_T}$  in degree 0, and  $\mathcal{O}_{X_T}(-n), E$  in degree 1; we also regard  $\mathcal{O}_X$  as a sheaf of graded algebras concentrated in degree 0. Then

$L_{(\mathcal{O}_{X_T} \oplus \mathcal{O}_{X_T}(-n))/\mathcal{O}_X}^{gr}$  is the cotangent complex for sheaves of graded algebras. The notation  $k^1(\dots)$  means take the degree 1 part of ‘ $\dots$ ’ in the grading.

The morphism  $\text{At}_{\mathcal{O}_{X_T}/\mathcal{O}_X}(s)$  in (12.10) is called the *Atiyah class* of  $s$ , as in Illusie [46, §IV.2.3], and the morphism

$$e(\mathcal{O}_{X_T} \oplus \mathcal{O}_{X_T}(-n)) : L_{(\mathcal{O}_{X_T} \oplus \mathcal{O}_{X_T}(-n))/\mathcal{O}_X}^{gr} \rightarrow \pi^* J \otimes (\mathcal{O}_{X_T} \oplus \mathcal{O}_{X_T}(-n))[1] \quad (12.11)$$

corresponds as in [46, §IV.2.4] to the following extension of graded  $\mathcal{O}_X$ -algebras:

$$0 \rightarrow \pi^* J \otimes (\mathcal{O}_{X_T} \oplus \mathcal{O}_{X_T}(-n)) \rightarrow \mathcal{O}_{X_T} \oplus \mathcal{O}_{X_T}(-n) \rightarrow \mathcal{O}_{X_T} \oplus \mathcal{O}_{X_T}(-n) \rightarrow 0, \quad (12.12)$$

and the morphism  $\Pi_1$  in (12.10) is projection to the first factor on the right in

$$k^1(\pi^* J \otimes (\mathcal{O}_{X_T} \oplus \mathcal{O}_{X_T}(-n)) \otimes (\mathcal{O}_{X_T} \oplus E)) = (\pi^* J \otimes E) \oplus (\pi^* J \otimes \mathcal{O}_{X_T}(-n)).$$

We will factorize (12.10) further. We have a cocartesian diagram

$$\begin{array}{ccc} \mathcal{O}_{X_T} & \longrightarrow & \mathcal{O}_{X_T} \oplus \mathcal{O}_{X_T}(-n) \\ \uparrow & & \uparrow \\ \mathcal{O}_X & \longrightarrow & \mathcal{O}_X \oplus \mathcal{O}_X(-n) \end{array}$$

of sheaves of graded algebras. Since  $\mathcal{O}_X(-n)$  is a flat  $\mathcal{O}_X$ -module,  $\mathcal{O}_X \oplus \mathcal{O}_X(-n)$  is a flat graded  $\mathcal{O}_X$ -algebra. Therefore, by (12.5) we have an isomorphism:

$$\begin{aligned} L_{(\mathcal{O}_{X_T}/\mathcal{O}_X)} \otimes (\mathcal{O}_{X_T} \oplus \mathcal{O}_{X_T}(-n)) \oplus L_{(\mathcal{O}_X \oplus \mathcal{O}_X(-n))/\mathcal{O}_X}^{gr} \otimes (\mathcal{O}_{X_T} \oplus \mathcal{O}_{X_T}(-n)) \\ \xrightarrow{\cong} L_{(\mathcal{O}_{X_T} \oplus \mathcal{O}_{X_T}(-n))/\mathcal{O}_X}^{gr}. \end{aligned} \quad (12.13)$$

Since  $\varphi$  is flat in the following Cartesian diagram:

$$\begin{array}{ccc} X_T & \longrightarrow & X \\ \pi \downarrow & & \downarrow \varphi \\ T & \longrightarrow & U, \end{array}$$

equation (12.5) gives

$$L_{\mathcal{O}_{X_T}/\mathcal{O}_X} \cong \pi^* L_{\mathcal{O}_T/\mathcal{O}_U}. \quad (12.14)$$

Let  $e(\mathcal{O}_{\overline{T}}) \in \text{Ext}^1(L_{\mathcal{O}_T/\mathcal{O}_U}, J)$  and  $e(\mathcal{O}_{X_{\overline{T}}}) \in \text{Ext}^1(L_{\mathcal{O}_{X_T}/\mathcal{O}_X}, \pi^* J)$  correspond to algebra extensions (12.6) and (12.7) as in [46, §IV.2.4]. Since (12.7) is the pullback of (12.6) by  $\pi$  we have

$$e(\mathcal{O}_{X_{\overline{T}}}) = \pi^* e(\mathcal{O}_{\overline{T}}) \in \text{Ext}^1(L_{\mathcal{O}_{X_T}/\mathcal{O}_X}, \pi^* J) \cong \text{Ext}^1(\pi^* L_{\mathcal{O}_T/\mathcal{O}_U}, \pi^* J), \quad (12.15)$$

using (12.14). The extension (12.12) is  $\otimes_{\mathcal{O}_X}(\mathcal{O}_X \oplus \mathcal{O}_X(-n))$  applied to (12.7). Thus the following diagram commutes:

$$\begin{array}{ccc} L_{\mathcal{O}_{X_T} \oplus \mathcal{O}_{X_T}(-n)/\mathcal{O}_X}^{gr} & \xrightarrow{e(\mathcal{O}_{X_{\overline{T}} \oplus \mathcal{O}_{X_{\overline{T}}}(-n))}} & \pi^* J \otimes (\mathcal{O}_{X_T} \oplus \mathcal{O}_{X_T}(-n))[1], \\ \downarrow pr_1 & \searrow & \\ L_{\mathcal{O}_{X_T}/\mathcal{O}_X} \otimes (\mathcal{O}_{X_T} \oplus \mathcal{O}_{X_T}(-n)) & \xrightarrow{e(\mathcal{O}_{X_{\overline{T}}}) \otimes \text{id}_{\mathcal{O}_{X_T} \oplus \mathcal{O}_{X_T}(-n)}} & \pi^* J \otimes (\mathcal{O}_{X_T} \oplus \mathcal{O}_{X_T}(-n))[1], \end{array} \quad (12.16)$$

where  $pr_1$  is projection to the first factor in (12.13).

Combining equations (12.10)–(12.16) gives a commutative diagram:

$$\begin{array}{ccc}
 \text{cone}(s) & & \\
 \downarrow \text{At}_{\mathcal{O}_{X_T}/\mathcal{O}_X}(s) & & \\
 k^1(L_{(\mathcal{O}_{X_T} \oplus \mathcal{O}_{X_T}(-n))/\mathcal{O}_X} \otimes (\mathcal{O}_{X_T} \oplus E))[1] & \xrightarrow{k^1(e(\mathcal{O}_{X_T} \oplus \mathcal{O}_{X_T}(-n)) \otimes \text{id}_{\mathcal{O}_{X_T} \oplus E})} & \\
 \downarrow \text{pr}_1 & & \downarrow k^1(e(\mathcal{O}_{X_T} \oplus \mathcal{O}_{X_T}(-n)) \otimes \text{id}_{\mathcal{O}_{X_T} \oplus E}) \\
 k^1(L_{(\mathcal{O}_{X_T}/\mathcal{O}_X) \otimes (\mathcal{O}_{X_T} \oplus \mathcal{O}_{X_T}(-n))} \otimes (\mathcal{O}_{X_T} \oplus E))[1] & \xrightarrow{k^1(e(\mathcal{O}_{X_T} \oplus \mathcal{O}_{X_T}(-n)) \otimes \text{id}_{\mathcal{O}_{X_T} \oplus E})} & k^1(\pi^* J \otimes (\mathcal{O}_{X_T} \oplus \mathcal{O}_{X_T}(-n)) \otimes (\mathcal{O}_{X_T} \oplus E))[2] \\
 \downarrow \text{project} & \text{ob} & \downarrow \text{project} \\
 k^1(L_{(\mathcal{O}_{X_T}/\mathcal{O}_X)} \otimes (\mathcal{O}_{X_T} \oplus E))[1] & \xrightarrow{k^1(e(\mathcal{O}_{X_T} \oplus \mathcal{O}_{X_T}(-n)) \otimes \text{id}_{\mathcal{O}_{X_T} \oplus E})} & k^1(\pi^* J \otimes (\mathcal{O}_{X_T} \oplus E))[2] \\
 \downarrow = & & \downarrow = \\
 L_{\mathcal{O}_{X_T}/\mathcal{O}_X} \otimes E[1] & \xrightarrow{e(\mathcal{O}_{X_T}) \otimes \text{id}_E} & \pi^* J \otimes E[2] \\
 \downarrow \cong \text{ by (12.14)} & \pi^* e(\mathcal{O}_{\overline{T}}) \otimes \text{id}_E & \downarrow \\
 \pi^* L_{\mathcal{O}_T/\mathcal{O}_U} \otimes E[1] & \xrightarrow{\pi^* e(\mathcal{O}_{\overline{T}}) \otimes \text{id}_E} & \pi^* J \otimes E[2].
 \end{array} \tag{12.17}$$

Let  $\text{At}(E, s)$  denote the composition of all the morphisms in the left column of diagram (12.17). We call this the *Atiyah class* of the family of pairs  $s : \mathcal{O}_X(-n) \rightarrow E$  over  $X_T$ . Thus, we may restate Illusie's results as:

**THEOREM 12.9.** *In Theorem 12.8, the obstruction morphism  $ob$  factorizes as*

$$\text{cone}(s) \xrightarrow{\text{At}(E, s)} \pi^* L_{\mathcal{O}_T/\mathcal{O}_U} \otimes E[1] \xrightarrow{\pi^* e(\mathcal{O}_{\overline{T}}) \otimes \text{id}_E} \pi^* J \otimes E[2]. \tag{12.18}$$

Note that in (12.18),  $\text{At}(E, s)$  is independent of the choice of  $J, \overline{T}$ , and  $\pi^* e(\mathcal{O}_{\overline{T}})$  depends on the square zero extension  $J, \overline{T}$  but is independent of the choice of pair  $E, s$ . A very similar picture is explained by Huybrechts and Thomas [45], when they show that obstruction class  $ob$  for deforming a complex  $E^\bullet$  in  $D^b(\text{coh}(X))$  is the composition of an *Atiyah class* depending on  $E^\bullet$ , and a *Kodaira–Spencer class* depending on the square-zero extension.

### 12.5. A non-perfect obstruction theory for $\mathcal{M}_{\text{stp}}^{\alpha,n}(\tau')/U$

Now suppose  $\alpha \in K(\text{coh}(X))$ , and  $n \gg 0$  is large enough that  $H^i(E(n)) = 0$  for all  $i > 0$  and all  $\tau$ -semistable sheaves  $E$  of class  $\alpha$ . We will construct the natural relative obstruction theory  $\phi : B^\bullet \rightarrow L_{\mathcal{M}_{\text{stp}}^{\alpha,n}(\tau')/U}$  for  $\mathcal{M}_{\text{stp}}^{\alpha,n}(\tau')$ , which unfortunately is neither perfect nor symmetric. Sections 12.6 and 12.7 explain how to modify  $\phi$  to a perfect, symmetric obstruction theory, firstly in the case  $\text{rank } \mathbb{I} \neq 0$ , and then a more complicated construction for the general case.

**REMARK 12.10.** Here is an informal sketch of what is going on in §12.5–§12.7. Consider a single stable pair  $s : \mathcal{O}_X(-n) \rightarrow E$  on  $X$ , and let  $I$  be the pair considered as an object of  $D(X)$ . Think of  $(E, s)$  as a point of the moduli scheme  $\mathcal{M}_{\text{stp}}^{\alpha,n}(\tau')$ . It turns out that *deformations* of  $(E, s)$  are given by  $\text{Ext}_{D(X)}^1(I, E)$ , so that the tangent space  $T_{(E,s)} \mathcal{M}_{\text{stp}}^{\alpha,n}(\tau') \cong \text{Ext}_{D(X)}^1(I, E)$ , and the *obstruction space* to deforming  $(E, s)$  is  $\text{Ext}_{D(X)}^2(I, E)$ , so we may informally write  $O_{(E,s)} \mathcal{M}_{\text{stp}}^{\alpha,n}(\tau') \cong \text{Ext}_{D(X)}^2(I, E)$ .

Now obstruction theories concern cotangent not tangent spaces, as they map to the cotangent complex  $L_{\mathcal{M}_{\text{stp}}^{\alpha,n}(\tau')}$ , so we are interested in the dual spaces

$$\begin{aligned} T_{(E,s)}^* \mathcal{M}_{\text{stp}}^{\alpha,n}(\tau') &\cong \text{Ext}_{D(X)}^1(I, E)^* \cong \text{Ext}_{D(X)}^2(E, I \otimes K_X) \cong H^0(B^\bullet), \\ O_{(E,s)}^* \mathcal{M}_{\text{stp}}^{\alpha,n}(\tau') &\cong \text{Ext}_{D(X)}^2(I, E)^* \cong \text{Ext}_{D(X)}^1(E, I \otimes K_X) \cong H^{-1}(B^\bullet), \end{aligned}$$

using Serre duality in the second steps, and where  $B^\bullet$  in the third steps is defined in (12.21) below, and  $\omega_\pi$  in (12.21) plays the rôle of  $K_X$ .

Thus, the complex  $B^\bullet$  in  $D(X)$  encodes the (dual of the) deformations of  $s : \mathcal{O}_X(-n) \rightarrow E$  in its degree 0 cohomology, and the (dual of the) obstructions in its degree  $-1$  cohomology. This is what we want from an obstruction theory, and we will show in Proposition 12.12 that  $B^\bullet$  can indeed be made into an obstruction theory for  $\mathcal{M}_{\text{stp}}^{\alpha,n}(\tau')$ . However, there are two problems with it. Firstly,  $H^{-2}(B^\bullet)$  may be nonzero, so  $B^\bullet$  is not concentrated in degrees  $[-1, 0]$ , that is, it is not a *perfect* obstruction theory. Secondly, as  $\text{Ext}_{D(X)}^1(I, E)$  and  $\text{Ext}_{D(X)}^2(I, E)$  are not dual spaces,  $B^\bullet$  is not *symmetric*.

Here is how we fix these problems. There are natural *identity* and *trace morphisms*  $\text{id}_I : H^i(\mathcal{O}_X) \rightarrow \text{Ext}_{D(X)}^i(I, I)$  and  $\text{tr}_I : \text{Ext}_{D(X)}^i(I, I) \rightarrow H^i(\mathcal{O}_X)$ , with  $\text{tr}_I \circ \text{id}_I = \text{rank } I \cdot 1_{H^i(\mathcal{O}_X)}$ . Suppose for the moment that  $\text{rank } I \neq 0$ . Then  $\text{Ext}_{D(X)}^i(I, I) \cong \text{Ext}_{D(X)}^i(I, I)_0 \oplus H^i(\mathcal{O}_X)$ , where  $\text{Ext}_{D(X)}^i(I, I)_0 = \text{Ker } \text{tr}_I$  is the trace-free part of  $\text{Ext}^i(I, I)$ . We have natural morphisms

$$\text{Ext}_{D(X)}^i(I, E) \xrightarrow{T \circ} \text{Ext}_{D(X)}^i(I, I) \longrightarrow \text{Ext}_{D(X)}^i(I, I)/H^i(\mathcal{O}_X) \cong \text{Ext}_{D(X)}^i(I, I)_0,$$

where  $T : E \rightarrow I$  is the natural morphism in  $D(X)$ . Now  $\text{Ext}_{D(X)}^1(I, E) \rightarrow \text{Ext}_{D(X)}^1(I, I)_0$  is an isomorphism, and  $\text{Ext}_{D(X)}^2(I, E) \rightarrow \text{Ext}_{D(X)}^2(I, I)_0$  is injective. The idea of §12.6 is to replace  $\text{Ext}_{D(X)}^\bullet(I, E)$  by  $\text{Ext}_{D(X)}^\bullet(I, I)_0$ . We construct a complex  $G^\bullet$  with  $H^i(G^\bullet) \cong \text{Ext}_{D(X)}^{1-i}(I, I)_0^*$ , and this is our symmetric perfect obstruction theory for  $\mathcal{M}_{\text{stp}}^{\alpha,n}(\tau')$ .

If  $\text{rank } I = 0$  then this construction fails, since  $\text{tr}_I \circ \text{id}_I = 0$ , and we no longer have a canonical splitting  $\text{Ext}_{D(X)}^i(I, I) \cong \text{Ext}_{D(X)}^i(I, I)_0 \oplus H^i(\mathcal{O}_X)$ . We deal with this in §12.7 in a peculiar way. The basic idea is to replace  $\text{Ext}_{D(X)}^i(I, E)$  by  $\text{Ext}_{D(X)}^i(I, I)/\text{id}_I(H^i(\mathcal{O}_X))$  when  $i = 0, 1$ , and by  $\text{Ker}(\text{tr } I : \text{Ext}_{D(X)}^i(I, I) \rightarrow H^i(\mathcal{O}_X))$  when  $i = 2, 3$ . This yields groups which are zero in degrees 0 and 3 and dual in degrees 1 and 2, so again they give us a symmetric perfect obstruction theory for  $\mathcal{M}_{\text{stp}}^{\alpha,n}(\tau')$ .

As in §12.2, write  $X_{\mathcal{M}} = X \times_U \mathcal{M}_{\text{stp}}^{\alpha,n}(\tau')$  with projection  $\pi : X_{\mathcal{M}} \rightarrow \mathcal{M}_{\text{stp}}^{\alpha,n}(\tau')$ , and write  $\mathbb{S} : \mathcal{O}_{X_{\mathcal{M}}}(-n) \rightarrow \mathbb{E}$  for the universal stable pair on  $X_{\mathcal{M}}$ . We will also regard this as an object  $\mathbb{I} = \text{cone}(\mathbb{S})$  in  $D(X_{\mathcal{M}})$  with  $\mathcal{O}_{X_{\mathcal{M}}}(-n)$  in degree  $-1$  and  $\mathbb{E}$  in degree 0. Thus we have a distinguished triangle in  $D(X_{\mathcal{M}})$ :

$$\mathbb{I}[-1] \longrightarrow \mathcal{O}_{X_{\mathcal{M}}}(-n) \xrightarrow{\mathbb{S}} \mathbb{E} \xrightarrow{\mathbb{T}} \mathbb{I}. \quad (12.19)$$

Define objects in  $D(\mathcal{M}_{\text{stp}}^{\alpha,n}(\tau'))$ :

$$A^\bullet = R\pi_*(R\mathcal{H}om(\mathbb{I}, \mathbb{I}) \otimes \omega_\pi)[2], \quad (12.20)$$

$$B^\bullet = R\pi_*(R\mathcal{H}om(\mathbb{E}, \mathbb{I}) \otimes \omega_\pi)[2], \quad (12.21)$$

$$\check{B}^\bullet = R\pi_*(R\mathcal{H}om(\mathbb{I}, \mathbb{E}) \otimes \omega_\pi)[2], \quad (12.22)$$

$$C^\bullet = R\pi_*(R\mathcal{H}om(\mathcal{O}_{X_{\mathcal{M}}}(-n), \mathbb{I}) \otimes \omega_\pi)[2], \quad (12.23)$$

$$\check{C}^\bullet = R\pi_*(R\mathcal{H}om(\mathbb{I}, \mathcal{O}_{X_{\mathcal{M}}}(-n)) \otimes \omega_\pi)[2], \quad (12.24)$$

$$D^\bullet = R\pi_*(R\mathcal{H}om(\mathbb{E}, \mathbb{E}) \otimes \omega_\pi)[2], \quad (12.25)$$

$$E^\bullet = R\pi_*(R\mathcal{H}om(\mathcal{O}_{X_{\mathcal{M}}}(-n), \mathbb{E}) \otimes \omega_\pi)[2], \quad (12.26)$$

$$\check{E}^\bullet = R\pi_*(R\mathcal{H}om(\mathbb{E}, \mathcal{O}_{X_{\mathcal{M}}}(-n)) \otimes \omega_\pi)[2], \quad (12.27)$$

$$F^\bullet = R\pi_*(\omega_\pi)[2] \cong R\pi_*(R\mathcal{H}om(\mathcal{O}_{X_{\mathcal{M}}}(-n), \mathcal{O}_{X_{\mathcal{M}}}(-n)) \otimes \omega_\pi)[2]. \quad (12.28)$$

Applying  $R\pi_*(R\mathcal{H}om(-, *) \otimes \omega_\pi)[2]$ ,  $R\pi_*(R\mathcal{H}om(*, -) \otimes \omega_\pi)[2]$  to (12.19) for  $*$  =  $\mathbb{I}, \mathbb{E}, \mathcal{O}_{X_{\mathcal{M}}}$  gives six distinguished triangles in  $D(\mathcal{M}_{\text{stp}}^{\alpha,n}(\tau'))$ , which we write as a commutative diagram with rows and columns distinguished triangles:

$$\begin{array}{ccccccc} A^\bullet & \xrightarrow{\beta} & B^\bullet & \xrightarrow{\gamma} & C^\bullet & \xrightarrow{\delta} & A^\bullet[1] \\ \beta^\flat \uparrow & & \epsilon^\flat \uparrow & & \tilde{\epsilon}^\flat \uparrow & & \beta^\flat \uparrow \\ \check{B}^\bullet & \xrightarrow{\epsilon} & D^\bullet & \xrightarrow{\zeta} & E^\bullet & \xrightarrow{\eta} & \check{B}^\bullet[1] \\ \tilde{\gamma}^\flat \uparrow & & \zeta^\flat \uparrow & & \tilde{\kappa}^\flat \uparrow & & \tilde{\gamma}^\flat \uparrow \\ \check{C}^\bullet & \xrightarrow{\iota} & \check{E}^\bullet & \xrightarrow{\kappa} & F^\bullet & \xrightarrow{\lambda} & \check{C}^\bullet[1] \\ \delta^\flat \uparrow & & \tilde{\eta}^\flat \uparrow & & \tilde{\lambda}^\flat \uparrow & & \delta^\flat \uparrow \\ A^\bullet[-1] & \xrightarrow{\beta} & B^\bullet[-1] & \xrightarrow{\gamma} & C^\bullet[-1] & \xrightarrow{\delta} & A^\bullet. \end{array} \quad (12.29)$$

We take (12.29) to be the definition of the morphisms  $\beta, \dots, \tilde{\lambda}$ .

Here is the point of the notation with accents ‘ $\sim$ ’: the Calabi–Yau condition on  $\varphi : X \rightarrow U$  implies that the dualizing complex  $\omega_\varphi$  is a line bundle on  $X$  trivial on the fibres of  $\varphi$ , so it is  $\varphi^*(L)$  for some line bundle  $L$  on  $U$ . Suppose  $U$  is affine. Then any line bundle on  $U$  is trivial, so we may choose an isomorphism  $L \cong \mathcal{O}_U$ , and then  $\omega_\varphi \cong \mathcal{O}_X$ , and on  $X_{\mathcal{M}}$  we have  $\omega_\pi \cong \pi_X^*(\omega_\varphi) \cong \pi_X^*(\mathcal{O}_X) \cong \mathcal{O}_{X_{\mathcal{M}}}$ . Using this isomorphism  $\omega_\pi \cong \mathcal{O}_{X_{\mathcal{M}}}$  we get isomorphisms

$$\begin{aligned} A^{\bullet\vee}[1] &\cong A^\bullet, \quad B^{\bullet\vee}[1] \cong \check{B}^\bullet, \quad \check{B}^{\bullet\vee}[1] \cong B^\bullet, \quad C^{\bullet\vee}[1] \cong \check{C}^\bullet, \quad \check{C}^{\bullet\vee}[1] \cong C^\bullet, \\ D^{\bullet\vee}[1] &\cong D^\bullet, \quad E^{\bullet\vee}[1] \cong \check{E}^\bullet, \quad \check{E}^{\bullet\vee}[1] \cong E^\bullet, \quad F^{\bullet\vee}[1] \cong F^\bullet, \end{aligned} \quad (12.30)$$

where  $A^{\bullet\vee}, \dots, F^{\bullet\vee}$  are the *duals* of  $A^\bullet, \dots, F^\bullet$ , dualizing in (12.29) corresponds to taking the transpose along the diagonal, and  $\check{\beta}, \dots, \check{\lambda}$  would be the dual morphisms of  $\beta, \dots, \lambda$ , respectively.

LEMMA 12.11. *The complex  $B^\bullet$  in (12.21) satisfies condition (\*).*

PROOF. As  $B^\bullet$  is obtained by applying standard derived functors to a complex of quasi-coherent sheaves with coherent cohomology, the general theory guarantees that  $B^\bullet$  is also a complex of quasi-coherent sheaves with coherent cohomology. Thus we only have to check that  $h^i(B^\bullet) = 0$  for  $i > 0$ . But the fibre of  $h^i(B^\bullet)$  at a  $U$ -point  $p$  of  $\mathcal{M}_{\text{stp}}^{\alpha,n}(\tau')$  corresponding to a stable pair  $I_p = \mathcal{O}_X(-n) \xrightarrow{s_p} E_p$



is  $\text{Ext}_{D(X)}^{2+i}(E_p, I_p \otimes \omega_\pi)$ , which is dual to  $\text{Ext}_{D(X)}^{1-i}(I_p, E_p)$  by Serre duality, and so vanishes when  $i > 0$  by Proposition 12.4(a).  $\square$

As in §12.4, the *Atiyah class* of the universal pair  $\mathbb{S} : \mathcal{O}_{X_M}(-n) \rightarrow \mathbb{E}$  is

$$\text{At}(\mathbb{E}, \mathbb{S}) : \mathbb{I} \longrightarrow \pi^*(L_{\mathcal{M}_{\text{stp}}^{\alpha,n}(\tau')/U}) \otimes \mathbb{E}[1]. \quad (12.31)$$

We have isomorphisms

$$\begin{aligned} & \text{Ext}^1(\mathbb{I}, \pi^*(L_{\mathcal{M}_{\text{stp}}^{\alpha,n}(\tau')/U}) \otimes \mathbb{E}) \\ & \cong \text{Ext}^1(R\mathcal{H}om(\mathbb{E}, \mathbb{I}), \pi^*(L_{\mathcal{M}_{\text{stp}}^{\alpha,n}(\tau')/U})) \\ & \cong \text{Ext}^1(R\mathcal{H}om(\mathbb{E}, \mathbb{I}) \otimes \omega_\pi[3], \pi^*(L_{\mathcal{M}_{\text{stp}}^{\alpha,n}(\tau')/U}) \otimes \omega_\pi[3]) \\ & \cong \text{Ext}^1(R\mathcal{H}om(\mathbb{E}, \mathbb{I}) \otimes \omega_\pi[3], \pi^!(L_{\mathcal{M}_{\text{stp}}^{\alpha,n}(\tau')/U})) \\ & \cong \text{Ext}^1(R\pi_*(R\mathcal{H}om(\mathbb{E}, \mathbb{I}) \otimes \omega_\pi[3]), L_{\mathcal{M}_{\text{stp}}^{\alpha,n}(\tau')/U}) \\ & \cong \text{Hom}(B^\bullet, L_{\mathcal{M}_{\text{stp}}^{\alpha,n}(\tau')/U}), \end{aligned} \quad (12.32)$$

using  $\pi^!(K^\bullet) = \pi^*(K^\bullet) \otimes \omega_\pi[3]$  for all  $K^\bullet \in D(\mathcal{M}_{\text{stp}}^{\alpha,n}(\tau'))$  as  $\pi$  is smooth of dimension 3 in the third isomorphism, the adjoint pair  $(R\pi_*, \pi^!)$  in the fourth, and (12.21) in the fifth. Define

$$\phi : B^\bullet \longrightarrow L_{\mathcal{M}_{\text{stp}}^{\alpha,n}(\tau')/U} \quad (12.33)$$

to correspond to  $\text{At}(\mathbb{E}, \mathbb{S})$  in (12.31) under the isomorphisms (12.32).

**PROPOSITION 12.12.** *The morphism  $\phi$  in (12.33) makes  $B^\bullet$  into a (generally not perfect) relative obstruction theory for  $\mathcal{M}_{\text{stp}}^{\alpha,n}(\tau')/U$ .*

**PROOF.** Suppose we are given an affine  $U$ -scheme  $T$ , a square-zero extension  $\overline{T}$  of  $T$  with ideal sheaf  $J$  as in §12.3, and a morphism of  $U$ -schemes  $g : T \rightarrow \mathcal{M}_{\text{stp}}^{\alpha,n}(\tau')$ . Set  $X_T = X \times_U T$ . Then we have a Cartesian diagram:

$$\begin{array}{ccc} X_T & \xrightarrow{f} & X_{\mathcal{M}} \\ \downarrow \xi & \searrow \pi & \downarrow \\ T & \xrightarrow{g} & \mathcal{M}_{\text{stp}}^{\alpha,n}(\tau') \\ & \searrow & \downarrow \\ & & X \\ & & \downarrow \\ & & U. \end{array}$$

The universal stable pair  $\mathbb{S} : \mathcal{O}_{X_{\mathcal{M}}}(-n) \rightarrow \mathbb{E}$  on  $X_{\mathcal{M}}$  pulls back under  $f$  to a  $T$ -family  $\mathbb{S}_T : \mathcal{O}_{X_T}(-n) \rightarrow \mathbb{E}_T$  of stable pairs, which we write as  $\mathbb{I}_T$  when considered as an object of  $D(X_T)$ . Since  $\mathcal{M}_{\text{stp}}^{\alpha,n}(\tau')$  is a fine moduli space, extensions  $\overline{g} : \overline{T} \rightarrow \mathcal{M}_{\text{stp}}^{\alpha,n}(\tau')$  of  $g$  to  $\overline{T}$  are equivalent to extensions  $\overline{\mathbb{S}}_T : \mathcal{O}_{X_{\overline{T}}}(-n) \rightarrow \overline{\mathbb{E}}_T$  of  $\mathbb{S}_T : \mathcal{O}_{X_T}(-n) \rightarrow \mathbb{E}_T$   $s : \mathcal{O}_{X_T}(-n) \rightarrow E$  to  $\overline{T}$ , which is exactly the problem

considered in §12.4. For  $i \in \mathbb{Z}$  we have the following isomorphisms:

$$\begin{aligned}
\text{Ext}^i(g^*B^\bullet, J) &\cong \text{Ext}^i(g^*(R\pi_*(R\mathcal{H}om(\mathbb{E}, \mathbb{I}) \otimes \omega_\pi[2])), J) \\
&\cong \text{Ext}^i(R\pi_*(R\mathcal{H}om(\mathbb{E}, \mathbb{I}) \otimes \omega_\pi[2]), Rg_*J) \\
&\cong \text{Ext}^i(R\mathcal{H}om(\mathbb{E}, \mathbb{I}) \otimes \omega_\pi[2], \pi^*(Rg_*J) \otimes \omega_\pi[3]) \\
&\cong \text{Ext}^{i+1}(R\mathcal{H}om(\mathbb{E}, \mathbb{I}), \pi^*(Rg_*J)) \cong \text{Ext}^{i+1}(R\mathcal{H}om(\mathbb{E}, \mathbb{I}), Rf_*(\xi^*J)) \\
&\cong \text{Ext}^{i+1}(Lf^*(R\mathcal{H}om(\mathbb{E}, \mathbb{I})), \xi^*J) \cong \text{Ext}^{i+1}(R\mathcal{H}om(Lf^*\mathbb{E}, Lf^*\mathbb{I}), \xi^*J) \\
&\cong \text{Ext}^{i+1}(Lf^*\mathbb{I}, \xi^*J \otimes Lf^*\mathbb{E}) \cong \text{Ext}^{i+1}(f^*\mathbb{I}, \xi^*J \otimes f^*\mathbb{E}),
\end{aligned} \tag{12.34}$$

using (12.21) in the first step, the adjoint pair  $(g^*, Rg_*)$  in the second, the adjoint pair  $(R\pi_*, \pi^!)$  and  $\pi^!(A) = \pi^*(A) \otimes \omega_\pi[3]$  in the third, base change for the flat morphism  $\pi$  in the fifth, and the adjoint pair  $(Lf^*, Rf_*)$  in the sixth. In the final step, as  $\mathcal{O}_{X_{\mathcal{M}}}(-n)$  and  $\mathbb{E}$  are flat over  $\mathcal{M}_{\text{stp}}^{\alpha, n}(\tau')$  we have  $Lf^*\mathbb{E} \cong f^*\mathbb{E}$ , and  $Lf^*(\mathbb{I})$  is quasi-isomorphic to  $\mathcal{O}_{X_T}(-n) \rightarrow \mathbb{E}_T$ , which we denote as  $f^*\mathbb{I} = \mathbb{I}_T$ .

In a similar way to isomorphisms (12.32), the composition

$$g^*(B^\bullet) \xrightarrow{g^*\phi} g^*(L\mathcal{M}_{\text{stp}}^{\alpha, n}(\tau')/U) \longrightarrow L_{T/U}$$

lifts to

$$\text{At}(\mathbb{E}_T, \mathbb{S}_T) : \mathbb{I}_T \longrightarrow \xi^*(L_{T/U}) \otimes \mathbb{E}_T[1],$$

the Atiyah class of the family  $\mathbb{S}_T : \mathcal{O}_{X_T}(-n) \rightarrow \mathbb{E}_T$ . Thus the composition

$$\mathbb{I}_T \xrightarrow{\text{At}(\mathbb{E}_T, \mathbb{S}_T)} \xi^*L_{T/U} \otimes \mathbb{E}_T[1] \xrightarrow{\xi^*e(\mathcal{O}_{\overline{T}}) \otimes \text{id}_{\mathbb{E}_T}} \xi^*J \otimes \mathbb{E}_T[2]$$

is the element  $\phi^*(\omega(g))$  under the isomorphism (12.34) for  $i = 1$ , by Theorems 12.8 and 12.9. The morphism  $g : T \rightarrow \mathcal{M}_{\text{stp}}^{\alpha, n}(\tau')$  extends to  $\bar{g} : \bar{T} \rightarrow \mathcal{M}_{\text{stp}}^{\alpha, n}(\tau')$  if and only if the family of pairs extend from  $T$  to  $\bar{T}$ . Therefore, by Theorems 12.7–12.9, Lemma 12.11, and equation (12.34) for  $i = 0$  we conclude that  $\phi$  is an obstruction theory for  $\mathcal{M}_{\text{stp}}^{\alpha, n}(\tau')$ .  $\square$

As in the proof of Lemma 12.11, the fibre of  $h^i(B^\bullet)$  at a  $U$ -point  $p$  is  $\text{Ext}^{1-i}(I_p, E_p)^*$ , so that  $h^i(B^\bullet) = 0$  unless  $i = -2, -1, 0$  by Proposition 12.4(a). The first row of (12.1) then shows the fibre of  $h^{-2}(B^\bullet)$  is  $\text{Ext}^3(E_p, E_p)^*$ , which is isomorphic to  $\text{Hom}(E_p, E_p)$  if  $\omega_\varphi \cong \mathcal{O}_X$ , and so is never zero as  $\alpha = [E_p] \neq 0$ . Thus,  $B^\bullet$  is perfect of perfect amplitude contained in  $[-2, 0]$  but not in  $[-1, 0]$ , and  $\phi$  is not a perfect obstruction theory.

### 12.6. A perfect obstruction theory when $\text{rank } \alpha \neq 1$

We now modify  $\phi$  to get a perfect obstruction theory  $\psi : G^\bullet \rightarrow L\mathcal{M}_{\text{stp}}^{\alpha, n}(\tau')/U$  which is symmetric when  $U$  is affine. Parts of the construction fail when  $\text{rank } \alpha = 1$ , and we explain how to fix this in §12.7. The identity and trace morphisms  $\mathcal{O}_{X_{\mathcal{M}}} \rightarrow R\mathcal{H}om(\mathbb{I}, \mathbb{I})$  and  $R\mathcal{H}om(\mathbb{I}, \mathbb{I}) \rightarrow \mathcal{O}_{X_{\mathcal{M}}}$  induce morphisms  $\text{id}_{\mathbb{I}} : F^\bullet \rightarrow A^\bullet$  and  $\text{tr}_{\mathbb{I}} : A^\bullet \rightarrow F^\bullet$  in  $D(\mathcal{M}_{\text{stp}}^{\alpha, n}(\tau'))$ . Since  $\text{rank } \mathbb{E} = \text{rank } \alpha$  and  $\text{rank } \mathbb{I} = \text{rank } \alpha - 1$ , as in [45, §4] we see that

$$\text{tr}_{\mathbb{I}} \circ \text{id}_{\mathbb{I}} = (\text{rank } \alpha - 1) 1_{F^\bullet}, \tag{12.35}$$

where  $1_{F^\bullet} : F^\bullet \rightarrow F^\bullet$  is the identity map. Because of (12.35) we must treat the rank  $\alpha \neq 1$  and rank  $\alpha = 1$  cases differently. For  $\delta, \check{\delta}, \lambda, \check{\lambda}$  as in (12.29) we have natural identities

$$\check{\delta} \circ \text{id}_{\mathbb{I}} = \lambda : F^\bullet \rightarrow \check{C}^\bullet[1] \quad \text{and} \quad \text{tr}_{\mathbb{I}} \circ \delta = \check{\lambda} : C^\bullet[-1] \rightarrow F^\bullet. \quad (12.36)$$

Define objects  $G^\bullet, \check{G}^\bullet$  in  $D(\mathcal{M}_{\text{stp}}^{\alpha,n}(\tau'))$  by

$$G^\bullet = \text{cone}(\text{tr}_{\mathbb{I}})[-1] \quad \text{and} \quad \check{G}^\bullet = \text{cone}(\text{id}_{\mathbb{I}}), \quad (12.37)$$

so that we have distinguished triangles:

$$\begin{aligned} G^\bullet &\xrightarrow{\nu} A^\bullet \xrightarrow{\text{tr}_{\mathbb{I}}} F^\bullet \xrightarrow{\mu} G^\bullet[1], \\ \check{G}^\bullet[-1] &\xrightarrow{\check{\mu}} F^\bullet \xrightarrow{\text{id}_{\mathbb{I}}} A^\bullet \xrightarrow{\check{\nu}} \check{G}^\bullet. \end{aligned} \quad (12.38)$$

We take (12.38) to be the definition of  $\mu, \nu, \check{\mu}, \check{\nu}$ . Again, if we were given an isomorphism  $\omega_\pi \cong \mathcal{O}_{X_{\mathcal{M}}}$  then we would have isomorphisms  $G^{\bullet\vee}[1] \cong \check{G}^\bullet$  and  $\check{G}^{\bullet\vee}[1] \cong G^\bullet$  as in (12.30), and  $\check{\mu}, \check{\nu}$  would be the dual morphisms of  $\mu, \nu$ , and dualizing would exchange the two lines of (12.38).

Applying the *octahedral axiom* in the triangulated category  $D(\mathcal{M}_{\text{stp}}^{\alpha,n}(\tau'))$ , as in Gelfand and Manin [29, §IV.1], gives diagrams:

where ‘ $\star$ ’ indicates a distinguished triangle, and ‘ $\circlearrowright$ ’ a commutative triangle, and we have used the first row and third column of (12.29), equation  $\text{tr}_{\mathbb{I}} \circ \delta = \check{\lambda}$  in (12.36), and the first row of (12.38). Thus, the octahedral axiom gives morphisms  $\rho, \sigma$  in a distinguished triangle:

$$G^\bullet \xrightarrow{\beta \circ \nu} B^\bullet \xrightarrow{\rho} E^\bullet \xrightarrow{\sigma} G^\bullet[1]. \quad (12.39)$$

The complex  $E^\bullet$  in  $D(\mathcal{M}_{\text{stp}}^{\alpha,n}(\tau'))$  has cohomology  $h^i(E^\bullet) \in \text{coh}(\mathcal{M}_{\text{stp}}^{\alpha,n}(\tau'))$  for  $i \in \mathbb{Z}$ . At a  $U$ -point  $p$  of  $\mathcal{M}_{\text{stp}}^{\alpha,n}(\tau')$  corresponding to a stable pair  $s_p : \mathcal{O}_X(-n) \rightarrow E_p$ , the fibre of  $h^i(E^\bullet)$  is  $\text{Ext}^{i-2}(\mathcal{O}_X(-n), E_p)$  by (12.26), and this is zero for  $i \neq -2$  by choice of  $n \gg 0$ . Therefore we have:

LEMMA 12.13. *The cohomology sheaves  $h^i(E^\bullet)$  satisfy  $h^i(E^\bullet) = 0$  for  $i \neq -2$ .*

Now define a morphism

$$\psi = \phi \circ \beta \circ \nu : G^\bullet \longrightarrow L_{\mathcal{M}_{\text{stp}}^{\alpha,n}(\tau')/U}. \quad (12.40)$$

PROPOSITION 12.14. *The morphism  $\psi : G^\bullet \longrightarrow L_{\mathcal{M}_{\text{stp}}^{\alpha,n}(\tau')/U}$  in (12.40) makes  $G^\bullet$  into a relative obstruction theory for  $\mathcal{M}_{\text{stp}}^{\alpha,n}(\tau')/U$ .*

PROOF. Taking cohomology sheaves of the distinguished triangle (12.39) gives a long exact sequence in  $\text{coh}(\mathcal{M}_{\text{stp}}^{\alpha,n}(\tau'))$ :

$$\cdots \longrightarrow h^i(G^\bullet) \xrightarrow{h^i(\beta \circ \nu)} h^i(B^\bullet) \xrightarrow{h^i(\rho)} h^i(E^\bullet) \xrightarrow{h^i(\sigma)} h^{i+1}(G^\bullet) \longrightarrow \cdots \quad (12.41)$$

Lemma 12.13 then implies that  $h^i(\beta \circ \nu) : h^i(G^\bullet) \rightarrow h^i(B^\bullet)$  is an isomorphism for  $i = 0$  and an epimorphism for  $i = -1$ . Also  $h^0(\phi)$  is an isomorphism and  $h^{-1}(\phi)$  is an epimorphism, since  $\phi$  is an obstruction theory by Proposition 12.12. Thus  $h^0(\psi) = h^0(\phi) \circ h^0(\beta \circ \nu)$  is an isomorphism and  $h^{-1}(\psi) = h^{-1}(\phi) \circ h^{-1}(\beta \circ \nu)$  is an epimorphism. We can also see from Lemma 12.11 and (12.41) that  $G^\bullet$  satisfies condition (\*). Hence  $\psi$  is an obstruction theory by Definition 12.6(c).  $\square$

Suppose now that  $\text{rank } \alpha \neq 1$ , and also if  $\mathbb{K}$  has positive characteristic that  $\text{char } \mathbb{K}$  does not divide  $\text{rank } \alpha - 1$ . This implies that the morphism  $\text{tr}_{\mathbb{I}} \circ \text{id}_{\mathbb{I}}$  in (12.35) is invertible.

LEMMA 12.15. *Suppose  $\text{rank } \alpha \not\equiv 1 \pmod{\text{char } \mathbb{K}}$ . Then  $\theta = \check{\nu} \circ \nu : G^\bullet \rightarrow \check{G}^\bullet$  is an isomorphism. Also  $A^\bullet \cong F^\bullet \oplus G^\bullet \cong F^\bullet \oplus \check{G}^\bullet$ .*

PROOF. Use the octahedral axiom [29, §IV.1] and (12.35), (12.38) to form diagrams:

$$\begin{array}{ccc} G^\bullet[1] & \xleftarrow{\mu} & F^\bullet \\ \downarrow [1] \theta & \swarrow \nu & \uparrow \text{tr}_{\mathbb{I}} \\ & A^\bullet & \\ \downarrow [1] \check{\nu} & \nwarrow \check{\mu} & \uparrow \text{id}_{\mathbb{I}} \\ \check{G}^\bullet & \xrightarrow{[1]} & F^\bullet \end{array} \quad \begin{array}{ccc} G^\bullet[1] & \xleftarrow{\mu} & F^\bullet \\ \downarrow [1] \theta & \swarrow \star & \uparrow (\text{rank } \alpha - 1) 1_{F^\bullet} \\ & 0 & \\ \downarrow [1] \check{\mu} & \nwarrow \star & \uparrow [1] \\ \check{G}^\bullet & \xrightarrow{[1]} & F^\bullet \end{array} \quad (12.42)$$

Since  $(\text{rank } \alpha - 1)1_{F^\bullet}$  is an isomorphism, the cone on it is zero, so the central object in the right hand square is zero. This in turn implies that the cone on  $\theta$  is zero, so  $\theta$  is an isomorphism. Also in the second square the morphisms  $\mu, \check{\mu}$  factor through zero, so  $\mu = \check{\mu} = 0$ , and as  $A^\bullet$  is the cone on  $\mu, \check{\mu}$  this gives  $A^\bullet \cong F^\bullet \oplus G^\bullet \cong F^\bullet \oplus \check{G}^\bullet$ .  $\square$

The splitting  $A^\bullet \cong F^\bullet \oplus G^\bullet$  corresponds to

$$R\pi_*(R\mathcal{H}om(\mathbb{I}, \mathbb{I}) \otimes \omega_\pi)[2] \cong R\pi_*(\omega_\pi)[2] \oplus R\pi_*(R\mathcal{H}om(\mathbb{I}, \mathbb{I})_0 \otimes \omega_\pi)[2],$$

where  $R\mathcal{H}om(\mathbb{I}, \mathbb{I})_0$  are the trace-free automorphisms of  $\mathbb{I}$ .

LEMMA 12.16. *If  $\text{rank } \alpha \not\equiv 1 \pmod{\text{char } \mathbb{K}}$  then  $h^i(G^\bullet) = 0$  for  $i \neq 0, -1$ .*

PROOF. Let  $p$  be a  $U$ -point of  $\mathcal{M}_{\text{stp}}^{\alpha,n}(\tau')$  corresponding to a stable pair  $s_p : \mathcal{O}_X(-n) \rightarrow E_p$ . Specializing the first row of (12.38) and (12.39) at  $p$  to get distinguished triangles in  $D^b(U)$ , taking long exact sequences in cohomology, and using (12.20)–(12.26) to substitute for  $A_p^\bullet, B_p^\bullet, E_p^\bullet, F_p^\bullet$  gives long exact sequences:

$$\begin{aligned} \cdots \rightarrow \text{Ext}^{i+1}(I_p, I_p) &\rightarrow H^{i+1}(\mathcal{O}_X) \rightarrow H^i(G_p^\bullet) \rightarrow \text{Ext}^{i+2}(I_p, I_p) \rightarrow \cdots, \\ \cdots \rightarrow H^i(G_p^\bullet) &\rightarrow \text{Ext}^{i+3}(E_p, I_p) \rightarrow \text{Ext}^{i+2}(\mathcal{O}_X(-n), E_p) \rightarrow H^{i+1}(G_p^\bullet) \rightarrow \cdots. \end{aligned}$$

In the first line, by Proposition 12.4(c) we have  $\text{Ext}^i(I_p, I_p), H^i(\mathcal{O}_X) = 0$  for  $i < 0, i > 3$  and  $\text{Ext}^i(I_p, I_p) \rightarrow H^i(\mathcal{O}_X)$  is a morphism  $\mathbb{K} \rightarrow \mathbb{K}$  for  $i = 0, 3$ . The morphism  $\text{Ext}^3(I_p, I_p) \rightarrow H^3(\mathcal{O}_X)$  is dual to the identity morphism  $H^0(\mathcal{O}_X) \rightarrow$

$\text{Ext}^0(I_p, I_p)$ , and so is an isomorphism without conditions on  $\text{rank } \alpha$ . The composition  $H^0(\mathcal{O}_X) \rightarrow \text{Ext}^0(I_p, I_p) \rightarrow H^0(\mathcal{O}_X)$  is multiplication by  $\text{rank } \alpha - 1$  by (12.35), which is nonzero by assumption, so  $\text{Ext}^0(I_p, I_p) \rightarrow H^0(\mathcal{O}_X)$  is also an isomorphism. Hence so  $H^i(G_p^\bullet) = 0$  for  $i < -1$  and  $i > 1$ . In the second line,  $\text{Ext}^i(E_p, I_p) = 0$  for  $i < 1$  and  $i > 3$  by Proposition 12.4(a) and Serre duality, and  $\text{Ext}^i(\mathcal{O}_X(-n), E_p) = 0$  for  $i \neq 0$  by choice of  $n \gg 0$ , so  $H^i(G_p^\bullet) = 0$  for  $i < -2$  and  $i > 0$ . The lemma follows.  $\square$

Lemma 12.16 and the proofs of [86, Lem. 2.10] or [45, Lem. 4.2] imply:

LEMMA 12.17. *If  $\text{rank } \alpha \not\equiv 1 \pmod{\text{char } \mathbb{K}}$  then  $G^\bullet$  in (12.37) is perfect of perfect amplitude contained in  $[-1, 0]$ .*

Putting all this together gives:

THEOREM 12.18. *If  $\text{rank } \alpha \not\equiv 1 \pmod{\text{char } \mathbb{K}}$  then  $\psi$  is a perfect relative obstruction theory for  $\mathcal{M}_{\text{stp}}^{\alpha, n}(\tau')/U$ . If  $U$  is affine it is symmetric.*

PROOF. By Proposition 12.14  $\psi$  is a relative obstruction theory, which is perfect by Lemma 12.17. When  $U$  is affine, as in §12.5 we may choose an isomorphism  $\omega_\pi \cong \mathcal{O}_{X, \mathcal{M}}$ , and this induces isomorphisms (12.30) and  $\check{G}^\bullet \cong G^{\bullet \vee}[1]$ . Thus Lemma 12.15 gives an isomorphism  $\theta : G^\bullet \rightarrow G^{\bullet \vee}[1]$ . Since  $\theta = \check{\nu} \circ \nu$  and  $\check{\nu} = \nu^\vee$  we see that  $\theta^\vee[1] = \theta$ . Hence  $\psi$  is a symmetric obstruction theory.  $\square$

This proves Theorem 5.23 in the case that  $\text{rank } \alpha \not\equiv 1 \pmod{\text{char } \mathbb{K}}$ .

### 12.7. An alternative construction for all $\text{rank } \alpha$

If either  $\text{rank } \alpha = 1$ , or  $\text{char } \mathbb{K} > 0$  divides  $\text{rank } \alpha - 1$ , then  $\text{tr}_\mathbb{I} \circ \text{id}_\mathbb{I} = 0$  in (12.35). The proofs in §12.6 then fail in two ways:

- In Lemma 12.15, equation (12.42) must be replaced by the diagrams:

$$\begin{array}{ccc}
 G^\bullet[1] & \xleftarrow{\quad \mu \quad} & F^\bullet \\
 \downarrow \theta & \searrow \nu & \uparrow \text{tr}_\mathbb{I} \\
 & A^\bullet & \\
 \downarrow \check{\nu} & \swarrow \check{\mu} & \downarrow 0 \\
 \check{G}^\bullet & \xrightarrow{\quad [1] \quad} & F^\bullet
 \end{array}
 \quad
 \begin{array}{ccc}
 G^\bullet[1] & \xleftarrow{\quad \mu \quad} & F^\bullet \\
 \downarrow \theta & \searrow \star & \uparrow \star \\
 & F^\bullet \oplus F^\bullet[1] & \\
 \downarrow \check{\nu} & \swarrow \check{\mu} & \downarrow 0 \\
 \check{G}^\bullet & \xrightarrow{\quad [1] \quad} & F^\bullet
 \end{array}$$

As the central object of the right hand square is no longer zero,  $\theta : G^\bullet \rightarrow \check{G}^\bullet$  is not an isomorphism, so we cannot show  $\psi$  is symmetric for  $U = \text{Spec } \mathbb{K}$  in Theorem 12.18. Also we cannot conclude that  $\mu = \check{\mu} = 0$ , so we do not have  $A^\bullet \cong F^\bullet \oplus G^\bullet \cong F^\bullet \oplus \check{G}^\bullet$ .

- In the proof of Lemma 12.16 the morphism  $\text{Ext}^0(I_p, I_p) \rightarrow H^0(\mathcal{O}_X)$  is zero. This implies that  $H^{-2}(G_p^\bullet) \cong \mathbb{K}$ , so  $G^\bullet$  is perfect of amplitude contained in  $[-2, 0]$  rather than  $[-1, 0]$ , and  $\psi$  is not perfect in Theorem 12.18.

The same problem occurs the construction of obstruction theories for moduli schemes of simple complexes  $\mathbb{I}$  in  $D^b(X)$  in Huybrechts and Thomas [45, §4]. When  $\text{rank } \mathbb{I} \not\equiv 0 \pmod{\text{char } \mathbb{K}}$ , so that  $\text{tr}_\mathbb{I} \circ \text{id}_\mathbb{I} \neq 0$ , they consider complexes  $\mathbb{I}$  with fixed determinant [45, §4.2], and obtain a perfect obstruction theory similar to our  $\psi$  in §12.6. When  $\text{rank } \mathbb{I} \equiv 0 \pmod{\text{char } \mathbb{K}}$ , so that  $\text{tr}_\mathbb{I} \circ \text{id}_\mathbb{I} = 0$ , they instead consider

complexes  $\mathbb{I}$  without fixed determinant [45, §4.4], and they modify their obstruction theory using truncation functors.

We now present an alternative, more complex construction of a perfect obstruction theory for  $L_{\mathcal{M}_{\text{stp}}^{\alpha,n}(\tau')/U}$  which works for all  $\alpha$ , and in fact is isomorphic to  $\psi$  in §12.6 when  $\text{rank } \alpha \not\equiv 1 \pmod{\text{char } \mathbb{K}}$ . Applying the truncation functors  $\tau^{\leq -1}, \tau^{\geq 0}$  to  $F^\bullet$  gives a distinguished triangle

$$\tau^{\leq -1} F^\bullet \xrightarrow{\tau^{\leq -1}} F^\bullet \xrightarrow{\tau^{\geq 0}} \tau^{\geq 0} F^\bullet \longrightarrow (\tau^{\leq -1} F^\bullet)[1]. \quad (12.43)$$

PROPOSITION 12.19. *The following composition of morphisms in  $D(\mathcal{M}_{\text{stp}}^{\alpha,n}(\tau'))$  is zero:*

$$\tau^{\leq -1} F^\bullet \xrightarrow{\tau^{\leq -1}} F^\bullet \xrightarrow{\text{id}_{\mathbb{I}}} A^\bullet \xrightarrow{\beta} B^\bullet \xrightarrow{\phi} L_{\mathcal{M}_{\text{stp}}^{\alpha,n}(\tau')/U}. \quad (12.44)$$

PROOF. Let  $u \in U(\mathbb{K})$ , with Calabi–Yau 3-fold  $X_u$ , and let  $s_u : \mathcal{O}_{X_u}(-n) \rightarrow E_u$  be a stable pair on  $X_u$ , written as  $I_u$  when considered as an object in  $D^b(X_u)$ . The *determinant*  $\det I_u$  of  $I_u$  is a line bundle  $L$  over  $X_u$ , with first Chern class  $c_1(L) = \text{ch}_1(\alpha - [\mathcal{O}_X(-n)])$ .

Write  $\mathfrak{J}^\alpha$  for the relative moduli  $U$ -stack of line bundles  $L$  over  $X_u$  for  $u \in U(\mathbb{K})$  with first Chern class  $\text{ch}_1(\alpha - [\mathcal{O}_X(-n)])$ . Then  $\mathfrak{J}^\alpha$  is an Artin  $\mathbb{K}$ -stack with a 1-morphism  $\mathfrak{J}^\alpha \rightarrow U$ , whose fibre  $\mathfrak{J}_u^\alpha$  over  $u \in U(\mathbb{K})$  is the moduli stack of line bundles over  $X_u$  with first Chern class  $\text{ch}_1(\alpha - [\mathcal{O}_X(-n)])$ . Each  $\mathbb{K}$ -point in  $\mathfrak{J}_u^\alpha$  has stabilizer group  $\mathbb{K}^\times$ , since line bundles are simple sheaves, and the coarse moduli scheme of  $\mathfrak{J}_u^\alpha$  is the usual Jacobian of line bundles on  $X_u$ .

There is a natural 1-morphism  $\Pi_{\mathfrak{J}} : \mathcal{M}_{\text{stp}}^{\alpha,n}(\tau') \rightarrow \mathfrak{J}^\alpha$  taking  $s_u : \mathcal{O}_{X_u}(-n) \rightarrow E_u$  to  $\det I_u$ . Thus, from the sequence of 1-morphisms of  $\mathbb{K}$ -stacks  $\mathcal{M}_{\text{stp}}^{\alpha,n}(\tau') \rightarrow \mathfrak{J}^\alpha \rightarrow U$ , by (12.4) we get a distinguished triangle in  $D(\mathcal{M}_{\text{stp}}^{\alpha,n}(\tau'))$ :

$$L\Pi_{\mathfrak{J}}^*(L_{\mathfrak{J}^\alpha/U}) \xrightarrow{\text{d}\Pi_{\mathfrak{J}}} L_{\mathcal{M}_{\text{stp}}^{\alpha,n}(\tau')/U} \longrightarrow L_{\mathcal{M}_{\text{stp}}^{\alpha,n}(\tau')/\mathfrak{J}^\alpha} \longrightarrow L\Pi_{\mathfrak{J}}^*(L_{\mathfrak{J}^\alpha/U}).$$

Now in (12.44), we may think of  $B^\bullet$  as the obstruction theory of pairs  $s : \mathcal{O}_X(-n) \rightarrow E$ , the morphism  $\beta : A^\bullet \rightarrow B^\bullet$  as the dual of the morphism taking a stable pair  $s_u : \mathcal{O}_{X_u}(-n) \rightarrow E_u$  to the associated complex  $I_u$ , and the morphism  $\text{id}_{\mathbb{I}} : F^\bullet \rightarrow A^\bullet$  as the dual of the morphism taking  $I_u$  to  $\det I_u$  in  $\mathfrak{J}_u^\alpha$ . So  $\beta \circ \text{id}_{\mathbb{I}} : F^\bullet \rightarrow B^\bullet$  is, on the level of obstruction theories, the dual of  $\Pi_{\mathfrak{J}}$  taking  $s_u : \mathcal{O}_{X_u}(-n) \rightarrow E_u$  to  $\det I_u$ . Therefore there exists a morphism  $v$  in a commutative diagram:

$$\begin{array}{ccc} F^\bullet & \xrightarrow{\beta \circ \text{id}_{\mathbb{I}}} & B^\bullet \\ v \downarrow \vdots & & \downarrow \phi \\ L_{\Pi_{\mathfrak{J}}}^*(L_{\mathfrak{J}^\alpha/U}) & \xrightarrow{\text{d}\Pi_{\mathfrak{J}}} & L_{\mathcal{M}_{\text{stp}}^{\alpha,n}(\tau')/U}. \end{array} \quad (12.45)$$

By assumption the numerical Grothendieck groups  $K^{\text{num}}(\text{coh}(X_u))$  are all canonically isomorphic for  $u \in U(\mathbb{K})$ . If  $\mathcal{M}_{\text{stp}}^{\alpha,n}(\tau') \neq \emptyset$ , this implies that line bundles with first Chern class  $\text{ch}_1(\alpha - [\mathcal{O}_X(-n)])$  exist for all  $u \in U(\mathbb{K})$ , since otherwise  $K^{\text{num}}(\text{coh}(X_u))$  would depend on  $u$ . As  $U$  is an algebraic  $\mathbb{K}$ -variety, and so reduced, it follows that deformations of line bundles with first Chern class  $\text{ch}_1(\alpha - [\mathcal{O}_X(-n)])$  on  $X_u$  are unobstructed in the family of Calabi–Yau 3-folds  $\varphi : X \rightarrow U$ . Therefore  $\mathfrak{J}^\alpha \rightarrow U$  is a *smooth* 1-morphism of Artin  $\mathbb{K}$ -stacks. It is clear that  $\mathfrak{J}_u^\alpha$  is a smooth Artin  $\mathbb{K}$ -stack for each  $u \in U(\mathbb{K})$ , since Jacobians

are smooth abelian varieties. We are here saying a bit more, that the 1-morphism  $\mathfrak{J}^\alpha \rightarrow U$  is smooth.

For a smooth morphism of  $\mathbb{K}$ -schemes  $\phi : X \rightarrow Y$  the cotangent complex  $L_{X/Y}$  is concentrated in degree 0. But for the smooth 1-morphism  $\mathfrak{J}^\alpha \rightarrow U$ , as  $\mathfrak{J}^\alpha$  is an Artin  $\mathbb{K}$ -stack rather than a  $\mathbb{K}$ -scheme, the cotangent complex  $L_{\mathfrak{J}^\alpha/U}$  is concentrated in degrees 0 and 1. It follows that the morphism  $v \circ \tau^{\leq -1} : \tau^{\leq -1} F^\bullet \rightarrow L_{\Pi_{\mathfrak{J}}}^*(L_{\mathfrak{J}^\alpha/U})$  is zero, since  $\tau^{\leq -1} F^\bullet$  lives in degree  $\leq -1$  and  $L_{\Pi_{\mathfrak{J}}}^*(L_{\mathfrak{J}^\alpha/U})$  in degree  $\geq 0$ . Hence in (12.44) we have  $\phi \circ \beta \circ \text{id}_{\mathbb{I}} \circ \tau^{\leq -1} = d\Pi_{\mathfrak{J}} \circ v \circ \tau^{\leq -1} = 0$ , by (12.45), which proves the proposition.

In fact in (12.45) we have an isomorphism  $L_{\Pi_{\mathfrak{J}}}^*(L_{\mathfrak{J}^\alpha/U}) \cong \tau^{\geq 0} F^\bullet$ , which identifies  $v$  with the projection  $\tau^{\geq 0} : F^\bullet \rightarrow \tau^{\geq 0} F^\bullet$  in (12.43).  $\square$

Set  $H^\bullet = \text{cone}(\tau^{\geq 0} \circ \text{tr}_{\mathbb{I}})[-1]$ ,  $\check{H}^\bullet = \text{cone}(\text{id}_{\mathbb{I}} \circ \tau^{\leq -1})$ , giving triangles

$$\begin{aligned} H^\bullet &\xrightarrow{a} A^\bullet \xrightarrow{\tau^{\geq 0} \circ \text{tr}_{\mathbb{I}}} \tau^{\geq 0} F^\bullet \xrightarrow{b} H^\bullet[1], \\ \check{H}^\bullet[-1] &\xrightarrow{\check{b}} \tau^{\leq -1} F^\bullet \xrightarrow{\text{id}_{\mathbb{I}} \circ \tau^{\leq -1}} A^\bullet \xrightarrow{\check{a}} \check{H}^\bullet. \end{aligned} \quad (12.46)$$

We have  $(\tau^{\geq 0} \circ \text{tr}_{\mathbb{I}}) \circ (\text{id}_{\mathbb{I}} \circ \tau^{\leq -1}) = (\text{rank } \alpha - 1) \tau^{\geq 0} \circ \tau^{\leq -1} = 0$  by (12.35), so by the first line of (12.46) there exists  $c : \tau^{\leq -1} F^\bullet \rightarrow H^\bullet$  with  $a \circ c = \text{id}_{\mathbb{I}} \circ \tau^{\leq -1}$ . Define  $K^\bullet = \text{cone}(c)$ , in a distinguished triangle

$$\tau^{\leq -1} F^\bullet \xrightarrow{c} H^\bullet \xrightarrow{d} K^\bullet \xrightarrow{e} (\tau^{\leq -1} F^\bullet)[1]. \quad (12.47)$$

Then by the octahedral axiom we have diagrams

$$\begin{array}{ccc} \tau^{\geq 0} F^\bullet & \xleftarrow{\tau^{\geq 0} \circ \text{tr}_{\mathbb{I}}} & A^\bullet \\ \downarrow d \circ b \stackrel{=}{=} \check{e} & \begin{array}{c} \swarrow [1] \star \searrow a \\ \downarrow [1] \circlearrowleft \star \uparrow [1] \circlearrowright \\ \downarrow d \star \uparrow c \end{array} & \uparrow \text{id}_{\mathbb{I}} \circ \tau^{\leq -1} \\ K^\bullet & \xrightarrow[e]{[1]} & \tau^{\leq -1} F^\bullet \end{array} \quad \begin{array}{ccc} \tau^{\geq 0} F^\bullet & \xleftarrow{\tau^{\geq 0} \circ \text{tr}_{\mathbb{I}}} & A^\bullet \\ \downarrow d \circ b \stackrel{=}{=} \check{e} & \begin{array}{c} \swarrow \check{c} \searrow \check{a} \\ \downarrow [1] \star \uparrow [1] \circlearrowright \\ \downarrow \check{d} \star \uparrow \check{b} \end{array} & \uparrow \text{id}_{\mathbb{I}} \circ \tau^{\leq -1} \\ K^\bullet & \xrightarrow[e]{[1]} & \tau^{\leq -1} F^\bullet \end{array}$$

Thus we get a morphism  $\check{c} : \check{H}^\bullet \rightarrow \tau^{\geq 0} F^\bullet$  with  $\check{c} \circ \check{a} = \tau^{\geq 0} \circ \text{tr}_{\mathbb{I}}$  and  $K^\bullet \cong \text{cone}(\check{c})[-1]$ , and a distinguished triangle with  $\check{e} = d \circ b$ :

$$(\tau^{\geq 0} F^\bullet)[-1] \xrightarrow{\check{e}} K^\bullet \xrightarrow{\check{d}} \check{H}^\bullet \xrightarrow{\check{c}} \tau^{\geq 0} F^\bullet. \quad (12.48)$$

In morphisms  $\tau^{\leq -1} F^\bullet \rightarrow L_{\mathcal{M}_{\text{stp}}^{\alpha,n}(\tau')/U}$  we have

$$\phi \circ \beta \circ a \circ c = \phi \circ \beta \circ \text{id}_{\mathbb{I}} \circ \tau^{\leq -1} = 0,$$

by  $a \circ c = \text{id}_{\mathbb{I}} \circ \tau^{\leq -1}$  and Proposition 12.19. Thus as (12.47) is distinguished there exists a morphism  $\xi : K^\bullet \rightarrow L_{\mathcal{M}_{\text{stp}}^{\alpha,n}(\tau')/U}$  with  $\xi \circ d = \phi \circ \beta \circ a$ .

**THEOREM 12.20.** *This  $\xi : K^\bullet \rightarrow L_{\mathcal{M}_{\text{stp}}^{\alpha,n}(\tau')/U}$  is a perfect relative obstruction theory for  $\mathcal{M}_{\text{stp}}^{\alpha,n}(\tau')/U$ . If  $U$  is affine it is symmetric.*

This proves Theorem 5.23. We will sketch the proof of Theorem 12.20 in four steps, leaving the details to the reader:

- (a) Show that  $h^i(K^\bullet) = 0$  for  $i \neq 0, -1$ .
- (b) Show that  $K^\bullet$  is perfect of perfect amplitude contained in  $[-1, 0]$ .

(c) When  $U$  is affine, construct  $\theta : K^\bullet \xrightarrow{\cong} K^{\bullet\vee}[1]$  with  $\theta^\vee[1] = \theta$ .

(d) Show that  $h^0(\xi)$  is an isomorphism and  $h^{-1}(\xi)$  an epimorphism.

For (a), we have  $h^i(A^\bullet) = h^i(F^\bullet) = 0$  for  $i > 1$  and  $h^1(A^\bullet) \cong h^1(F^\bullet) \cong \mathcal{O}_{\mathcal{M}}$ , where  $h^1(A^\bullet) \cong \mathcal{O}_{\mathcal{M}}$  follows from  $\text{Ext}^3(I, I) \cong \mathbb{K}$  in Proposition 12.4(c). Since  $h^i(E^\bullet) = 0$  for  $i \neq -2$  by Serre vanishing, taking the long exact sequence  $h^*(-)$  in the third column of (12.29) implies that  $h^i(\check{\lambda}) : h^{i-1}(C^\bullet) \rightarrow h^i(F^\bullet)$  is an isomorphism for  $i = 0, 1$ . But (12.36) yields  $h^i(\check{\lambda}) = h^i(\text{tr}_{\mathbb{I}}) \circ h^i(\delta)$ . Therefore  $h^i(\text{tr}_{\mathbb{I}}) : h^i(A^\bullet) \rightarrow h^i(F^\bullet)$  is surjective for  $i = 0, 1$ . As  $h^1(A^\bullet) \cong h^1(F^\bullet) \cong \mathcal{O}_{\mathcal{M}}$  we see that  $h^1(\text{tr}_{\mathbb{I}})$  is an isomorphism.

Taking the long exact sequence  $h^*(-)$  in the first line of (12.46) and using  $h^1(\text{tr}_{\mathbb{I}})$  an isomorphism and  $h^0(\text{tr}_{\mathbb{I}})$  surjective then gives  $h^i(H^\bullet) = 0$  for  $i > 0$ . Then  $h^i(\tau^{\leq -1}F^\bullet) = 0$  for  $i \geq 0$  and (12.47) imply that  $h^i(K^\bullet) = 0$  for  $i > 0$ . Similarly, from the third row of (12.29), the equation  $h^i(\lambda) = h^i(\check{\delta}) \circ h^i(\text{id}_{\mathbb{I}})$  from (12.36) and the second line of (12.46) we get  $h^i(\check{H}^\bullet) = 0$  for  $i < -1$ , and then  $h^i(K^\bullet) = 0$  for  $i < -1$  by (12.48). Step (b) then follows as for Lemma 12.17.

For (c), if  $U$  is affine then choosing an isomorphism  $\omega_\pi \cong \mathcal{O}_{X_{\mathcal{M}}}$  induces isomorphisms  $A^{\bullet\vee}[1] \cong A^\bullet$  and  $F^{\bullet\vee}[1] \cong F^\bullet$  as in (12.30). We then have  $(\tau^{\geq 0}F^\bullet)^\vee[1] \cong \tau^{\leq -1}F^\bullet$  and  $(\tau^{\leq -1}F^\bullet)^\vee[1] \cong \tau^{\geq 0}F^\bullet$ . Under these identifications  $\tau^{\geq 0} \circ \text{tr}_{\mathbb{I}}$  and  $\text{id}_{\mathbb{I}} \circ \tau^{\leq -1}$  are dual morphisms. Hence the two distinguished triangles (12.46) are dual, and we get isomorphisms  $H^{\bullet\vee}[1] \cong \check{H}^\bullet$ ,  $\check{H}^{\bullet\vee}[1] \cong H^\bullet$ . In (12.46)–(12.48)  $\check{a}, \dots, \check{e}$  are dual to  $a, \dots, e$ , and  $K^{\bullet\vee}[1] \cong K^\bullet$  as we want.

For (d), as  $\xi \circ d = \phi \circ \beta \circ a$  we have commutative diagrams in  $\text{coh}(\mathcal{M}_{\text{stp}}^{\alpha,n}(\tau'))$

$$\begin{array}{ccccc} h^i(H^\bullet) & \xrightarrow{h^i(\beta \circ a)} & h^i(B^\bullet) & \xrightarrow{h^i(\phi)} & h^i(L_{\mathcal{M}_{\text{stp}}^{\alpha,n}(\tau')/U}) \\ h^i(d) \downarrow & & & \nearrow h^i(\xi) & \\ h^i(K^\bullet) & & & & \end{array}$$

for each  $i \in \mathbb{Z}$ . We know  $h^0(\phi)$  is an isomorphism and  $h^{-1}(\phi)$  an epimorphism by Proposition 12.12. By similar arguments to (a), we show that  $h^0(\beta \circ a)$  is an isomorphism and  $h^{-1}(\beta \circ a)$  an epimorphism, and from the distinguished triangle (12.47) and  $h^i(\tau^{\leq -1}F^\bullet) = 0$  for  $i > -1$  we see that  $h^0(d)$  is an isomorphism and  $h^{-1}(d)$  an epimorphism. Step (d) follows.

## 12.8. Deformation-invariance of the $PI^{\alpha,n}(\tau')$

We now prove Theorem 5.25. In §12.1–§12.7 we assumed that the numerical Grothendieck groups  $K^{\text{num}}(\text{coh}(X_u))$  for  $u \in U(\mathbb{K})$  are all canonically isomorphic *globally in*  $U(\mathbb{K})$ , and we wrote  $K(\text{coh}(X))$  for  $K^{\text{num}}(\text{coh}(X_u))$  up to canonical isomorphism. We first prove Theorem 5.25 under this assumption.

As in Definition 12.1 we have a family of Calabi–Yau 3-folds  $X \xrightarrow{\varphi} U$  with  $X, U$  algebraic  $\mathbb{K}$ -varieties and  $U$  connected, and a relative very ample line bundle  $\mathcal{O}_X(1)$  for  $X \xrightarrow{\varphi} U$ , and we suppose  $K^{\text{num}}(\text{coh}(X_u))$  for  $u \in U(\mathbb{K})$  are all globally canonically isomorphic to  $K(\text{coh}(X))$ . Then for  $\alpha \in K(\text{coh}(X))$ , as in §12.5 we choose  $n \gg 0$  large enough that  $H^i(E_u(n)) = 0$  for all  $i > 0$  and all  $\tau$ -semistable sheaves  $E_u$  on  $X_u$  of class  $\alpha \in K^{\text{num}}(\text{coh}(X_u))$  for any  $u \in U(\mathbb{K})$ .

Then §12.1 constructs a projective  $U$ -scheme  $\mathcal{M}_{\text{stp}}^{\alpha,n}(\tau')$ , and §12.5–§12.7 construct perfect relative obstruction theories  $\psi : G^\bullet \rightarrow L_{\mathcal{M}_{\text{stp}}^{\alpha,n}(\tau')/U}$  when  $\text{rank } \alpha \neq 1 \pmod{\text{char } \mathbb{K}}$  and  $\xi : K^\bullet \rightarrow L_{\mathcal{M}_{\text{stp}}^{\alpha,n}(\tau')/U}$  for all  $\alpha$ . For each  $u \in U(\mathbb{K})$  the fibre



of  $\mathcal{M}_{\text{stp}}^{\alpha,n}(\tau') \rightarrow U$  at  $u$  is a projective  $\mathbb{K}$ -scheme  $\mathcal{M}_{\text{stp}}^{\alpha,n}(\tau')_u$ , and  $\psi, \xi$  specialize to perfect obstruction theories  $\psi_u, \xi_u$  for  $\mathcal{M}_{\text{stp}}^{\alpha,n}(\tau')_u$ , which are symmetric by the case  $U = \text{Spec } \mathbb{K}$  in Theorems 12.18 and 12.20.

Using these perfect obstruction theories, Behrend and Fantechi [5] construct virtual classes  $[\mathcal{M}_{\text{stp}}^{\alpha,n}(\tau')]^{\text{vir}}$  in the relative Chow homology  $A_0(\mathcal{M}_{\text{stp}}^{\alpha,n}(\tau') \rightarrow U)$ , and  $[\mathcal{M}_{\text{stp}}^{\alpha,n}(\tau')_u]^{\text{vir}}$  in the absolute Chow homology  $A_0(\mathcal{M}_{\text{stp}}^{\alpha,n}(\tau')_u)$ . In (5.15) we define  $PI^{\alpha,n}(\tau')_u = \int_{[\mathcal{M}_{\text{stp}}^{\alpha,n}(\tau')_u]^{\text{vir}}} 1$ . Since  $\psi_u, \xi_u$  are the specializations of  $\psi, \xi$  at  $u \in U(\mathbb{K})$  we have

$$[\mathcal{M}_{\text{stp}}^{\alpha,n}(\tau')_u]^{\text{vir}} = u^*([\mathcal{M}_{\text{stp}}^{\alpha,n}(\tau')]^{\text{vir}}).$$

By ‘conservation of number’, as in [28, Prop. 10.2] for instance,  $[\mathcal{M}_{\text{stp}}^{\alpha,n}(\tau')_u]^{\text{vir}}$  has the same degree for all  $u \in U(\mathbb{K})$ , as  $U$  is connected, which proves:

**THEOREM 12.21.** *Let  $\mathbb{K}$  be an algebraically closed field,  $U$  a connected algebraic  $\mathbb{K}$ -variety,  $X \rightarrow U$  a family of Calabi–Yau 3-folds  $X_u$  over  $\mathbb{K}$ , which may have  $H^1(\mathcal{O}_{X_u}) \neq 0$ , and  $\mathcal{O}_X(1)$  a relative very ample line on  $X$ . Suppose  $K^{\text{num}}(\text{coh}(X_u))$  is globally canonically isomorphic to  $K(\text{coh}(X))$  for  $u \in U(\mathbb{K})$ . Then for all  $\alpha \in K(\text{coh}(X))$  and  $n \gg 0$  the invariants  $PI^{\alpha,n}(\tau')_u$  of stable pairs on each fibre  $X_u$  of  $X \rightarrow U$ , computed using the ample line bundle  $\mathcal{O}_{X_u}(1)$  on  $X_u$ , are independent of  $u \in U(\mathbb{K})$ .*

This is the first part of Theorem 5.25. If instead the  $K^{\text{num}}(\text{coh}(X_u))$  are only canonically isomorphic *locally* in  $U(\mathbb{K})$ , then by Theorem 4.21 we can pass to a finite étale cover  $\pi : \tilde{U} \rightarrow U$ , such that the induced family of Calabi–Yau 3-folds  $\tilde{\varphi} : \tilde{X} \rightarrow \tilde{U}$  has  $K^{\text{num}}(\text{coh}(\tilde{X}_{\tilde{u}}))$  globally canonically isomorphic for  $\tilde{u} \in \tilde{U}(\mathbb{K})$ , where  $\tilde{u}$  is of the form  $(u, \iota)$  for  $u \in U(\mathbb{K})$  and  $\iota : K^{\text{num}}(\text{coh}(X_u)) \xrightarrow{\cong} K(\text{coh}(X))$ , and  $\tilde{X}_{\tilde{u}} = X_u$ . Theorem 12.21 for this family shows that  $PI^{\alpha,n}(\tau')_{(u,\iota)}$  is independent of  $(u, \iota) \in \tilde{U}(\mathbb{K})$ . But  $PI^{\alpha,n}(\tau')_{(u,\iota)} = PI^{\alpha,n}(\tau')_u$  as  $\tilde{X}_{\tilde{u}} = X_u$ , and the proof of Theorem 5.25 is complete.

## CHAPTER 13

### The proof of Theorem 5.27

In this section we will prove Theorem 5.27, which says that the invariants  $PI^{\alpha,n}(\tau')$  counting stable pairs, defined in §5.4, can be written in terms of the generalized Donaldson–Thomas invariants  $\bar{DT}^\beta(\tau)$  in §5.3 by

$$PI^{\alpha,n}(\tau') = \sum_{\substack{\alpha_1, \dots, \alpha_l \in C(\text{coh}(X)), \\ l \geq 1: \alpha_1 + \dots + \alpha_l = \alpha, \\ \tau(\alpha_i) = \tau(\alpha), \text{ all } i}} \frac{(-1)^l}{l!} \prod_{i=1}^l [(-1)^{\bar{\chi}([\mathcal{O}_X(-n)] - \alpha_1 - \dots - \alpha_{i-1}, \alpha_i)} \bar{\chi}([\mathcal{O}_X(-n)] - \alpha_1 - \dots - \alpha_{i-1}, \alpha_i) \bar{DT}^{\alpha_i}(\tau)], \quad (13.1)$$

for  $n \gg 0$ . As the  $PI^{\alpha,n}(\tau')$  are deformation-invariant by Theorem 12.21, it follows by induction in Corollary 5.28 that the  $\bar{DT}^\alpha(\tau)$  are deformation-invariant. Equation (13.1) is also useful for computing the  $\bar{DT}^\alpha(\tau)$  in examples.

#### 13.1. Auxiliary abelian categories $\mathcal{A}_p, \mathcal{B}_p$

In order to relate the invariants of stable pairs and the generalized Donaldson–Thomas invariants, we will introduce auxiliary abelian categories  $\mathcal{A}_p, \mathcal{B}_p$  and apply wall-crossing formulae in  $\mathcal{B}_p$  to obtain equation (13.1).

**DEFINITION 13.1.** We continue to use the notation of §3–§5, so that  $X$  is a Calabi–Yau 3-fold with ample line bundle  $\mathcal{O}_X(1)$ ,  $\tau$  is Gieseker stability on the abelian category  $\text{coh}(X)$  of coherent sheaves on  $X$ , and so on.

Fix some nonzero  $\alpha \in K(\text{coh}(X))$  with  $\mathcal{M}_{\text{ss}}^\alpha(\tau) \neq 0$ , for which we will prove (13.1). Then  $\alpha$  has Hilbert polynomial  $P_\alpha(t)$  with leading coefficient  $r_\alpha$ . Write  $p(t) = P_\alpha(t)/r_\alpha$  for the reduced Hilbert polynomial of  $\alpha$ . Let  $d = \dim \alpha$ . Then  $d = 1, 2$  or  $3$ , and  $p(t) = t^d + a_{d-1}t^{d-1} + \dots + a_0$ , for  $a_0, \dots, a_{d-1} \in \mathbb{Q}$ .

Define  $\mathcal{A}_p$  to be the subcategory of  $\text{coh}(X)$  whose objects are zero sheaves and nonzero  $\tau$ -semistable sheaves  $E \in \text{coh}(X)$  with  $\tau([E]) = p$ , that is,  $E$  has reduced Hilbert polynomial  $p$ , and such that  $\text{Hom}_{\mathcal{A}_p}(E, F) = \text{Hom}(E, F)$  for all  $E, F \in \mathcal{A}_p$ . Then  $\mathcal{A}_p$  is a full and faithful abelian subcategory of  $\text{coh}(X)$ .

If  $E \in \mathcal{A}_p$  then the Hilbert polynomial  $P_E$  of  $E$  is a rational multiple of  $p(t)$ . Since  $P_E : \mathbb{Z} \rightarrow \mathbb{Z}$  and  $P_E(l) \geq 0$  for  $l \gg 0$ , we see that  $P_E(t) \equiv \frac{k}{d!}p(t)$  for some  $k \in \mathbb{Z}_{\geq 0}$ . Let  $P_\alpha(t) = \frac{N}{d!}p(t)$  for some  $N > 0$ . It will turn out that to prove (13.1), we need only consider sheaves  $E \in \mathcal{A}_p$  with  $P_E(t) \equiv \frac{k}{d!}p(t)$  for  $k = 0, 1, \dots, N$ , that is, we need consider only  $\tau$ -semistable sheaves with *finitely many different Hilbert polynomials*.

By Huybrechts and Lehn [44, Th. 3.37], the family of  $\tau$ -semistable sheaves  $E$  on  $X$  with a fixed Hilbert polynomial is bounded, so the family of  $\tau$ -semistable sheaves  $E$  on  $X$  with Hilbert polynomial  $P_E(t) \equiv \frac{k}{d!}p(t)$  for any  $k = 0, 1, \dots, N$  is also bounded. Hence by Serre vanishing [44, Lem. 1.7.6] we can choose  $n \gg 0$  such that every  $\tau$ -semistable sheaf  $E$  on  $X$  with Hilbert polynomial  $P_E(t) \equiv \frac{k}{d!}p(t)$  for

some  $k = 0, 1, \dots, N$  has  $H^i(E(n)) = 0$  for all  $i > 0$ . That is,  $\text{Ext}^i(\mathcal{O}_X(-n), E) = 0$  for  $i > 0$ , so equation (3.1) implies that

$$\dim \text{Hom}(\mathcal{O}_X(-n), E) = \frac{k}{d!} p(n) = \bar{\chi}([\mathcal{O}_X(-n)], [E]). \quad (13.2)$$

We use this  $n$  to define  $\mathcal{M}_{\text{stp}}^{\alpha, n}(\tau')$  and  $PI^{\alpha, n}(\tau')$  in §5.4, and  $\mathcal{B}_p$  below.

Now define a category  $\mathcal{B}_p$  to have objects triples  $(E, V, s)$ , where  $E$  lies in  $\mathcal{A}_p$ ,  $V$  is a finite-dimensional  $\mathbb{C}$ -vector space, and  $s : V \rightarrow \text{Hom}(\mathcal{O}_X(-n), E)$  is a  $\mathbb{C}$ -linear map. Given objects  $(E, V, s), (E', V', s')$  in  $\mathcal{B}_p$ , define morphisms  $(f, g) : (E, V, s) \rightarrow (E', V', s')$  in  $\mathcal{B}_p$  to be pairs  $(f, g)$ , where  $f : E \rightarrow E'$  is a morphism in  $\mathcal{A}_p$  and  $g : V \rightarrow V'$  is a  $\mathbb{C}$ -linear map, such that the following diagram commutes:

$$\begin{array}{ccc} V & \xrightarrow{s} & \text{Hom}(\mathcal{O}_X(-n), E) \\ \downarrow g & & \downarrow f \circ \\ V' & \xrightarrow{s'} & \text{Hom}(\mathcal{O}_X(-n), E'), \end{array}$$

where ' $f \circ$ ' maps  $t \mapsto f \circ t$ .

Define  $K(\mathcal{A}_p)$  to be the image of  $K_0(\mathcal{A}_p)$  in  $K(\text{coh}(X)) = K^{\text{num}}(\text{coh}(X))$ . Then each  $E \in \mathcal{A}_p \subset \text{coh}(X)$  has numerical class  $[E] \in K(\mathcal{A}_p) \subset K(\text{coh}(X))$ . Define  $K(\mathcal{B}_p) = K(\mathcal{A}_p) \oplus \mathbb{Z}$ , and for  $(E, V, s)$  in  $\mathcal{B}_p$  define the numerical class  $[(E, V, s)]$  in  $K(\mathcal{B}_p)$  to be  $([E], \dim V)$ .

For coherent sheaves, the auxiliary category  $\mathcal{B}_p$  is a generalization of the coherent systems introduced by Le Potier [69]. A version of the category  $\mathcal{B}_p$  for representations of quivers was discussed in §7.4. It is now straightforward using the methods of [51] to prove:

LEMMA 13.2. *The category  $\mathcal{B}_p$  is abelian and  $\mathcal{B}_p, K(\mathcal{B}_p)$  satisfy Assumption 3.2 over  $\mathbb{K} = \mathbb{C}$ . Also  $\mathcal{B}_p$  is noetherian and artinian, and the moduli stacks  $\mathfrak{M}_{\mathcal{B}_p}^{(\beta, d)}$  are of **finite type** for all  $(\beta, d) \in C(\mathcal{B}_p)$ .*

Here  $\mathfrak{M}_{\mathcal{B}_p}^{(\beta, d)}$  is of finite type as it is built out of  $\tau$ -semistable sheaves  $E$  in class  $\beta$  in  $K(\text{coh}(X))$ , which form a bounded family by [44, Th. 3.37]. Lemma 13.2 means that we can apply the results of [51–54] to  $\mathcal{B}_p$ . Note that  $\mathcal{A}_p$  embeds as a full and faithful subcategory in  $\mathcal{B}_p$  by  $E \mapsto (E, 0, 0)$ . Every object  $(E, V, s)$  in  $\mathcal{B}_p$  fits into a short exact sequence

$$0 \longrightarrow (E, 0, 0) \longrightarrow (E, V, s) \longrightarrow (0, V, 0) \longrightarrow 0 \quad (13.3)$$

in  $\mathcal{B}_p$ , and  $(0, V, 0)$  is isomorphic to the direct sum of  $\dim V$  copies of the object  $(0, \mathbb{C}, 0)$  in  $\mathcal{B}_p$ . Thus, regarding  $\mathcal{A}_p$  as a subcategory of  $\mathcal{B}_p$ , we see that  $\mathcal{B}_p$  is generated over extensions by  $\mathcal{A}_p$  and one extra object  $(0, \mathbb{C}, 0)$ .

By considering short exact sequences (13.3) with  $V = \mathbb{C}$  we see that

$$\begin{aligned} \text{Ext}_{\mathcal{B}_p}^1((0, \mathbb{C}, 0), (E, 0, 0)) &= H^0(E(n)) \cong \text{Hom}(\mathcal{O}_X(-n), E) \\ &\cong \text{Ext}_{D(X)}^1(\mathcal{O}_X(-n)[-1], E), \end{aligned} \quad (13.4)$$

where  $\mathcal{O}_X(-n)[-1]$  is the shift of the sheaf  $\mathcal{O}_X(-n)$  in the derived category  $D(X)$ . Thus the extra element  $(0, \mathbb{C}, 0)$  in  $\mathcal{B}_p$  behaves like  $\mathcal{O}_X(-n)[-1]$  in  $D(X)$ . In fact there is a natural embedding functor  $F : \mathcal{B}_p \rightarrow D(X)$  which takes  $(E, V, s)$  in  $\mathcal{B}_p$  to the complex  $\cdots \rightarrow 0 \rightarrow V \otimes \mathcal{O}_X(-n) \xrightarrow{s} E \rightarrow 0 \rightarrow \cdots$  in  $D(X)$ , where  $V \otimes \mathcal{O}_X(-n), E$  appear in positions  $-1, 0$  respectively. Then  $F$  takes  $\mathcal{A}_p$  to  $\mathcal{A}_p \subset \text{coh}(X) \subset D(X)$ , and  $(0, \mathbb{C}, 0)$  to  $\mathcal{O}_X(-n)[-1]$  in  $D(X)$ .

Therefore we can think of  $\mathcal{B}_p$  as the abelian subcategory of  $D(X)$  generated by  $\mathcal{A}_p$  and  $\mathcal{O}_X(-n)[-1]$ . But working in the derived category would lead to complications about forming moduli stacks of objects in  $D(X)$ , classifying objects up to quasi-isomorphism, and so on, so we prefer just to use the explicit description of  $\mathcal{B}_p$  in Definition 13.1.

Although  $D(X)$  is a 3-Calabi–Yau triangulated category, and  $\mathcal{B}_p$  is embedded in  $D(X)$ , it does not follow that  $\mathcal{B}_p$  is a 3-Calabi–Yau abelian category, and we do not claim this. In §3.2 we defined the Euler form  $\bar{\chi}$  of  $\text{coh}(X)$ , and used the Calabi–Yau 3-fold property to prove (3.14), which was the crucial equation in proving the wall-crossing formulae (3.27), (5.14) for the invariants  $J^\alpha(\tau), \bar{DT}^\alpha(\tau)$ . We will show that even though  $\mathcal{B}_p$  may not be a 3-Calabi–Yau abelian category, a weakened version of (3.14) still holds in  $\mathcal{B}_p$ , which will be enough to prove wall-crossing formulae for invariants in  $\mathcal{B}_p$ .

DEFINITION 13.3. Define  $\bar{\chi}^{\mathcal{B}_p} : K(\mathcal{B}_p) \times K(\mathcal{B}_p) \rightarrow \mathbb{Z}$  by

$$\begin{aligned} \bar{\chi}^{\mathcal{B}_p}((\beta, d), (\gamma, e)) &= \bar{\chi}(\beta - d[\mathcal{O}_X(-n)], \gamma - e[\mathcal{O}_X(-n)]) \\ &= \bar{\chi}(\beta, \gamma) - d\bar{\chi}([\mathcal{O}_X(-n)], \gamma) + e\bar{\chi}([\mathcal{O}_X(-n)], \beta). \end{aligned} \quad (13.5)$$

This is the natural Euler form on  $K(\mathcal{B}_p)$  induced by the functor  $F : \mathcal{B}_p \rightarrow D(X)$ , since  $K^{\text{num}}(D(X)) = K^{\text{num}}(\text{coh}(X))$ , and

$$\begin{aligned} [F(E, V, s)] &= [V \otimes \mathcal{O}_X(-n) \xrightarrow{s} E] = \dim V[\mathcal{O}_X(-n)[-1]] + [E] \\ &= [E] - \dim V[\mathcal{O}_X(-n)] \end{aligned}$$

in  $K^{\text{num}}(D(X))$ , and  $\text{coh}(X), D(X)$  have the same Euler form  $\bar{\chi}$ .

PROPOSITION 13.4. Suppose  $(E, V, s), (F, W, t)$  lie in  $\mathcal{B}_p$  with  $\dim V + \dim W \leq 1$  and  $P_E(t) \equiv \frac{k}{d!} p(t), P_F(t) \equiv \frac{l}{d!} p(t)$  for some  $k, l = 0, 1, \dots, N$ . Then

$$\begin{aligned} \bar{\chi}^{\mathcal{B}_p}([(E, V, s)], [(F, W, t)]) &= \\ &(\dim \text{Hom}_{\mathcal{B}_p}((E, V, s), (F, W, t)) - \dim \text{Ext}_{\mathcal{B}_p}^1((E, V, s), (F, W, t))) - \\ &(\dim \text{Hom}_{\mathcal{B}_p}((F, W, t), (E, V, s)) - \dim \text{Ext}_{\mathcal{B}_p}^1((F, W, t), (E, V, s))). \end{aligned} \quad (13.6)$$

PROOF. The possibilities for  $(\dim V, \dim W)$  are  $(0, 0), (1, 0)$  or  $(0, 1)$ . For  $(0, 0)$  we have  $V = W = s = t = 0$ , and then  $\bar{\chi}^{\mathcal{B}_p}([(E, 0, 0)], [(F, 0, 0)]) = \bar{\chi}([E], [F])$ ,  $\text{Hom}_{\mathcal{B}_p}((E, 0, 0), (F, 0, 0)) = \text{Hom}(E, F)$ , and so on, so (13.6) follows from (3.14). The cases  $(1, 0), (0, 1)$  are equivalent after exchanging  $(E, V, s), (F, W, t)$ , so it is enough to do the  $(0, 1)$  case. Thus we must verify (13.6) for  $(E, 0, 0)$  and  $(F, \mathbb{C}, t)$ .

By Definition 13.1,  $\text{Hom}_{\mathcal{B}_p}((E, 0, 0), (F, \mathbb{C}, t))$  is the vector space of  $(f, 0)$  for  $f \in \text{Hom}(E, F)$  such that the following diagram commutes:

$$\begin{array}{ccc} 0 & \longrightarrow & \mathcal{O}_X(-n) \\ \downarrow & & \downarrow t \\ E & \xrightarrow{f} & F. \end{array}$$

This is no restriction on  $f$ , so

$$\text{Hom}_{\mathcal{B}_p}((E, 0, 0), (F, \mathbb{C}, t)) \cong \text{Hom}(E, F). \quad (13.7)$$

Also  $\text{Ext}_{\mathcal{B}_p}^1((E, 0, 0), (F, \mathbb{C}, t))$  corresponds to the set of isomorphism classes of commutative diagrams with exact rows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{C} \otimes \mathcal{O}_X(-n) & \xrightarrow{g \otimes \text{id}_{\mathcal{O}_X(-n)}} & Y \otimes \mathcal{O}_X(-n) & \longrightarrow & 0 \longrightarrow 0 \\ & & \downarrow t & & \downarrow u & & \downarrow \\ 0 & \longrightarrow & F & \xrightarrow{f} & G & \xrightarrow{f'} & E \longrightarrow 0. \end{array} \quad (13.8)$$

Here  $Y$  is a  $\mathbb{C}$ -vector space,  $g : \mathbb{C} \rightarrow Y$  is linear,  $G \in \mathcal{A}_p$ , and  $f, f', u$  are morphisms in  $\text{coh}(X)$ . By exactness of the top row,  $g$  is an isomorphism, so we can identify  $Y = \mathbb{C}$  and  $g = \text{id}_{\mathbb{C}}$ . Then for any exact  $0 \rightarrow F \xrightarrow{f} G \xrightarrow{f'} E \rightarrow 0$  in  $\mathcal{A}_p$  we define  $u = f \circ t$  to complete (13.8). Hence diagrams (13.8) correspond up to isomorphisms with exact  $0 \rightarrow F \rightarrow G \rightarrow E \rightarrow 0$  in  $\mathcal{A}_p$ , giving

$$\text{Ext}_{\mathcal{B}_p}^1((E, 0, 0), (F, \mathbb{C}, t)) \cong \text{Ext}^1(E, F). \quad (13.9)$$

Similarly,  $\text{Hom}_{\mathcal{B}_p}((F, \mathbb{C}, t), (E, 0, 0))$  is the set of  $(f, 0)$  for  $f \in \text{Hom}(F, E)$  such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{O}_X(-n) & \longrightarrow & 0 \\ \downarrow t & & \downarrow \\ F & \xrightarrow{f} & E. \end{array}$$

That is, we need  $f \circ t = 0$ . So

$$\begin{aligned} \text{Hom}_{\mathcal{B}_p}((F, \mathbb{C}, t), (E, 0, 0)) \\ \cong \text{Ker}(\text{Hom}(F, E) \xrightarrow{\circ t} \text{Hom}(\mathcal{O}_X(-n), E)). \end{aligned} \quad (13.10)$$

And  $\text{Ext}_{\mathcal{B}_p}^1((F, \mathbb{C}, t), (E, 0, 0))$  corresponds to the set of isomorphism classes of commutative diagrams with exact rows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & 0 & \longrightarrow & Y \otimes \mathcal{O}_X(-n) & \xrightarrow{g \otimes \text{id}_{\mathcal{O}_X(-n)}} & \mathbb{C} \otimes \mathcal{O}_X(-n) \longrightarrow 0 \\ & & \downarrow & & \downarrow u & & \downarrow t \\ 0 & \longrightarrow & E & \xrightarrow{f} & G & \xrightarrow{f'} & F \longrightarrow 0. \end{array} \quad (13.11)$$

Again, we identify  $Y = \mathbb{C}$  and  $g = \text{id}_{\mathbb{C}}$ . Then for a given exact sequence  $0 \rightarrow E \xrightarrow{f} G \xrightarrow{f'} F \rightarrow 0$  in  $\mathcal{A}_p$ , we want to know what are the possibilities for  $u$  to complete (13.11). Applying  $\text{Hom}(\mathcal{O}_X(-n), -)$  to  $0 \rightarrow E \xrightarrow{f} G \xrightarrow{f'} F \rightarrow 0$  yields an exact sequence

$$\begin{aligned} 0 & \longrightarrow \text{Hom}(\mathcal{O}_X(-n), E) \xrightarrow{f \circ} \text{Hom}(\mathcal{O}_X(-n), G) \xrightarrow{f' \circ} \\ & \text{Hom}(\mathcal{O}_X(-n), F) \longrightarrow \text{Ext}^1(\mathcal{O}_X(-n), E) \longrightarrow \dots \end{aligned} \quad (13.12)$$

But as  $P_E(t) \equiv \frac{k}{dt} p(t)$ , for  $k \leq N$ , by choice of  $n$  in Definition 13.1 we have  $\text{Ext}^1(\mathcal{O}_X(-n), E) = 0$ , so ' $f' \circ$ ' in (13.12) is surjective, and there exists at least one  $u \in \text{Hom}(\mathcal{O}_X(-n), G)$  with  $t = f' \circ u$ . If  $u, \tilde{u}$  are possible choices for  $u$  then  $f' \circ (u - \tilde{u}) = 0$ , so  $u - \tilde{u}$  lies in the kernel of ' $f' \circ$ ' in (13.12), which is the image of ' $f \circ$ ' by exactness, and is isomorphic to  $\text{Hom}(\mathcal{O}_X(-n), E)$ .

Naïvely this appears to show that  $\text{Ext}_{\mathcal{B}_p}^1((F, \mathbb{C}, t), (E, 0, 0))$  is the direct sum of  $\text{Ext}^1(F, E)$ , which represents the freedom to choose  $G, f, f'$  in (13.11) up to isomorphism, and  $\text{Hom}(\mathcal{O}_X(-n), E)$ , which parametrizes the additional freedom to choose  $u$  in (13.11). However, this is not quite true.  $\text{Ext}_{\mathcal{B}_p}^1((F, \mathbb{C}, t), (E, 0, 0))$

parametrizes isomorphism classes of diagrams (13.11), up to isomorphisms which are the identity on the second and fourth columns. Two different choices  $u, u'$  for  $u$  in (13.11) might still be isomorphic in this sense, through an isomorphism  $g$  in the following commutative diagram:

$$\begin{array}{ccccc}
 & & \mathbb{C} \otimes \mathcal{O}_X(-n) & \xrightarrow{\quad} & \mathbb{C} \otimes \mathcal{O}_X(-n) \\
 & & \downarrow u & & \downarrow t \\
 E & \xrightarrow{f} & G & \xrightarrow{f'} & F \\
 & \searrow \text{id}_E & \downarrow f & \searrow f' & \downarrow \text{id}_F \\
 & & E & \xrightarrow{g} & G & \xrightarrow{f'} & F
 \end{array} \quad (13.13)$$

Reasoning in the abelian category  $\text{coh}(X)$ , as  $f' \circ g = \text{id}_F \circ f'$  we have  $f' \circ (g - \text{id}_G) = 0$ , so  $g - \text{id}_G$  factorizes through the kernel  $f$  of  $f'$ , that is,  $g - \text{id}_G = f \circ h$ , where  $h : G \rightarrow E$ . Also  $g \circ f = f \circ \text{id}_E = f$ , so  $(g - \text{id}_G) \circ f = 0$ , and  $f \circ h \circ f = 0$ . As  $f$  is injective this gives  $h \circ f = 0$ . So  $h$  factorizes via the cokernel  $f'$  of  $f$ , and  $h = k \circ f'$  for  $k : F \rightarrow E$ . Therefore in (13.13) we may write  $g = \text{id}_G + f \circ k \circ f'$  for  $k \in \text{Hom}(F, E)$ . Hence, for any given choice  $u$  in (13.13), the equivalent choices  $u'$  are of the form  $u' = u + (f \circ k \circ f') \circ u = u + f \circ k \circ t$ . Thus we must quotient by the vector space of morphisms  $f \circ k \circ t$ , for  $k \in \text{Hom}(F, E)$ . As  $f$  is injective, this is isomorphic to the vector space of morphisms  $k \circ t$  in  $\text{Hom}(\mathcal{O}_X(-n), E)$ . This proves that there is an exact sequence

$$\begin{aligned}
 0 &\rightarrow \text{Coker}(\text{Hom}(F, E) \xrightarrow{\circ t} \text{Hom}(\mathcal{O}_X(-n), E)) \\
 &\rightarrow \text{Ext}_{\mathcal{B}_p}^1((F, \mathbb{C}, t), (E, 0, 0)) \rightarrow \text{Ext}^1(F, E) \rightarrow 0.
 \end{aligned} \quad (13.14)$$

Now taking dimensions in equations (13.7), (13.9), (13.10) and (13.14), and noting in (13.10) and (13.14) that if  $F : U \rightarrow V$  is a linear map of finite-dimensional vector spaces then  $\dim \text{Ker } F - \dim \text{Coker } F = \dim U - \dim V$ , we see that

$$\begin{aligned}
 &(\dim \text{Hom}_{\mathcal{B}_p}((E, 0, 0), (F, \mathbb{C}, t)) - \dim \text{Ext}_{\mathcal{B}_p}^1((E, 0, 0), (F, \mathbb{C}, t))) - \\
 &(\dim \text{Hom}_{\mathcal{B}_p}((F, \mathbb{C}, t), (E, 0, 0)) - \dim \text{Ext}_{\mathcal{B}_p}^1((F, \mathbb{C}, t), (E, 0, 0))) \\
 &= \dim \text{Hom}(E, F) - \dim \text{Ext}^1(E, F) - \dim \text{Hom}(F, E) + \dim \text{Ext}^1(F, E) \\
 &\quad + \dim \text{Hom}(\mathcal{O}_X(-n), E) \\
 &= \bar{\chi}([E], [F]) + \bar{\chi}([\mathcal{O}_X(-n)], [E]) = \bar{\chi}^{\mathcal{B}_p}([(E, 0, 0)], [(F, \mathbb{C}, t)]),
 \end{aligned}$$

using equations (3.14), (13.2) which holds as  $P_E(t) \equiv \frac{k}{d!} p(t)$  for  $k \leq N$ , and (13.5). This completes the proof of Proposition 13.4.  $\square$

### 13.2. Three weak stability conditions on $\mathcal{B}_p$

DEFINITION 13.5. It is easy to see that the positive cone  $C(\mathcal{B}_p)$  of  $\mathcal{B}_p$  is

$$C(\mathcal{B}_p) = \{(\beta, d) : \beta \in C(\mathcal{A}) \text{ and } d \geq 0 \text{ or } \beta = 0 \text{ and } d > 0\}.$$

Define weak stability conditions  $(\dot{\tau}, \dot{T}, \leq), (\tilde{\tau}, \tilde{T}, \leq), (\hat{\tau}, \hat{T}, \leq)$  on  $\mathcal{B}_p$  by:

- $\dot{T} = \{-1, 0\}$  with the natural order  $-1 < 0$ , and  $\dot{\tau}(\beta, d) = 0$  if  $d = 0$ , and  $\dot{\tau}(\beta, d) = -1$  if  $d > 0$ ; and
- $\tilde{T} = \{0, 1\}$  with the natural order  $0 < 1$ , and  $\tilde{\tau}(\beta, d) = 0$  if  $d = 0$ , and  $\tilde{\tau}(\beta, d) = 1$  if  $d > 0$ ;
- $\hat{T} = \{0\}$ , and  $\hat{\tau}(\beta, d) = 0$  for all  $(\beta, d)$ .

Since  $\mathcal{B}_p$  is artinian by Lemma 13.2, it is  $\dot{\tau}$ -artinian, and as  $\mathfrak{M}_{ss}^{(\beta,d)}(\dot{\tau})$  is a substack of  $\mathfrak{M}_{\mathcal{B}_p}^{(\beta,d)}$  which is of finite type by Lemma 13.2,  $\mathfrak{M}_{ss}^{(\beta,d)}(\dot{\tau})$  is of finite type for all  $(\beta, d) \in C(\mathcal{B}_p)$ . Therefore  $(\dot{\tau}, \dot{T}, \leq)$  is *permissible* by Definition 3.7, and similarly so are  $(\tilde{\tau}, \tilde{T}, \leq)$ ,  $(\hat{\tau}, \hat{T}, \leq)$ . Note too that  $(\hat{\tau}, \hat{T}, \leq)$  *dominates*  $(\dot{\tau}, \dot{T}, \leq)$ ,  $(\tilde{\tau}, \tilde{T}, \leq)$ , in the sense of Definition 3.12.

We can describe some of the moduli spaces  $\mathfrak{M}_{ss}^{(\beta,d)}(\dot{\tau})$ ,  $\mathfrak{M}_{ss}^{(\beta,d)}(\tilde{\tau})$ .

**PROPOSITION 13.6. (a)** *For all  $\beta \in C(\mathcal{A}_p)$  we have natural stack isomorphisms  $\mathfrak{M}_{ss}^{(\beta,0)}(\dot{\tau}) \cong \mathfrak{M}_{ss}^\beta(\tau)$  identifying  $(E, 0, 0)$  with  $E$ , where  $\mathfrak{M}_{ss}^\beta(\tau)$  is as in §3.2. Also  $\mathfrak{M}_{ss}^{(0,1)}(\dot{\tau}) \cong [\text{Spec } \mathbb{C}/\mathbb{G}_m]$  is the point  $(0, \mathbb{C}, 0)$ , and  $\mathfrak{M}_{ss}^{(\beta,1)}(\dot{\tau}) = \emptyset$  for  $\beta \neq 0$ .*

**(b)** *Let  $\alpha, n$  be as in Definition 13.1,  $\mathcal{M}_{stp}^{\alpha,n}(\tau')$  the moduli scheme of stable pairs  $s : \mathcal{O}_X(-n) \rightarrow E$  from §12, and  $\mathfrak{M}_{ss}^{(\alpha,1)}(\tilde{\tau})$  the moduli stack of  $\tilde{\tau}$ -semistable objects in class  $(\alpha, 1)$  in  $\mathcal{B}_p$ . Then  $\mathfrak{M}_{ss}^{(\alpha,1)}(\tilde{\tau}) \cong \mathcal{M}_{stp}^{\alpha,n}(\tau') \times [\text{Spec } \mathbb{C}/\mathbb{G}_m]$ .*

**PROOF.** For (a), all objects  $(E, 0, 0)$  in class  $(\beta, 0)$  are  $\dot{\tau}$ -semistable, so  $\mathfrak{M}_{ss}^{(\beta,0)}(\dot{\tau}) = \mathfrak{M}_{\mathcal{B}_p}^{(\beta,0)} \cong \mathfrak{M}_{\mathcal{A}_p}^\beta \cong \mathfrak{M}_{ss}^\beta(\tau)$ . The unique object in class  $(0, 1)$  in  $\mathcal{B}_p$  up to isomorphism is  $(0, \mathbb{C}, 0)$ , and it has no nontrivial subobjects, so it is  $\dot{\tau}$ -semistable. The automorphism group of  $(0, \mathbb{C}, 0)$  in  $\mathcal{B}_p$  is  $\mathbb{G}_m$ . Therefore  $\mathfrak{M}_{ss}^{(0,1)}(\dot{\tau}) \cong [\text{Spec } \mathbb{C}/\mathbb{G}_m]$  is the point  $(0, \mathbb{C}, 0)$ . Suppose  $(E, V, s)$  lies in class  $(\beta, 1)$  in  $\mathcal{B}_p$  for  $\beta \neq 0$  in  $C(\mathcal{A}_p)$ . Consider the short exact sequence in  $\mathcal{B}_p$ :

$$\begin{array}{ccccccc} 0 & \longrightarrow & 0 & \longrightarrow & V \otimes \mathcal{O}_X(-n) & \xrightarrow{\text{id}} & V \otimes \mathcal{O}_X(-n) \longrightarrow 0 \\ & & \downarrow & & \downarrow^s & & \downarrow \\ 0 & \longrightarrow & E & \longrightarrow & E & \longrightarrow & 0 \longrightarrow 0, \end{array} \quad (13.15)$$

that is,  $0 \rightarrow (E, 0, 0) \rightarrow (E, V, s) \rightarrow (0, V, 0) \rightarrow 0$ . We have  $[(E, 0, 0)] = (\beta, 0)$  and  $[(0, V, 0)] = (0, 1)$  in  $K(\mathcal{B}_p)$ , and  $\dot{\tau}(\beta, 0) = 0 > -1 = \dot{\tau}(0, 1)$ , so (13.15)  $\dot{\tau}$ -destabilizes  $(E, V, s)$ . Thus any object  $(E, V, s)$  in class  $(\beta, 1)$  in  $\mathcal{B}_p$  is  $\dot{\tau}$ -unstable, and  $\mathfrak{M}_{ss}^{(\beta,1)}(\dot{\tau}) = \emptyset$ , proving (a).

For (b), points of  $\mathcal{M}_{stp}^{\alpha,n}(\tau')$  are morphisms  $s : \mathcal{O}_X(-n) \rightarrow E$  with  $[E] = \alpha$ , and points of  $\mathfrak{M}_{ss}^{(\alpha,1)}(\tilde{\tau})$  are triples  $(E, V, s)$  with  $[E] = \alpha$ ,  $\dim V = 1$  and  $s : V \otimes \mathcal{O}_X(-n) \rightarrow E$  a morphism. Define a 1-morphism  $\pi_1 : \mathcal{M}_{stp}^{\alpha,n}(\tau') \rightarrow \mathfrak{M}_{ss}^{(\alpha,1)}(\tilde{\tau})$  by  $\pi_1 : (s : \mathcal{O}_X(-n) \rightarrow E) \mapsto (E, \mathbb{C}, s)$ . It is straightforward to check that  $s : \mathcal{O}_X(-n) \rightarrow E$  is a  $\tau'$ -stable pair if and only if  $(E, \mathbb{C}, s)$  is  $\tilde{\tau}$ -semistable in  $\mathcal{B}_p$ .

Define another 1-morphism  $\pi_2 : \mathfrak{M}_{ss}^{(\alpha,1)}(\tilde{\tau}) \rightarrow \mathcal{M}_{stp}^{\alpha,n}(\tau')$  by  $\pi_2 : (E, V, s) \mapsto (s(v) : \mathcal{O}_X(-n) \rightarrow E)$ , for some choice of  $0 \neq v \in V$ . If  $v, v'$  are possible choices then  $v' = \lambda v$  for some  $\lambda \in \mathbb{G}_m$ , since  $\dim V = 1$ . The isomorphism  $\lambda \text{id}_E : E \rightarrow E$  is an isomorphism between the stable pairs  $s(v) : \mathcal{O}_X(-n) \rightarrow E$  and  $s(v') : \mathcal{O}_X(-n) \rightarrow E$ , so they have the same isomorphism class, and define the same point in  $\mathcal{M}_{stp}^{\alpha,n}(\tau')$ . Thus  $\pi_2$  is well-defined.

On  $\mathbb{C}$ -points,  $\pi_1, \pi_2$  define inverse maps. The scheme  $\mathcal{M}_{stp}^{\alpha,n}(\tau')$  parametrizes isomorphism classes of objects parametrized by  $\mathfrak{M}_{ss}^{(\alpha,1)}(\tilde{\tau})$ . Therefore, by [67, Rem. 3.19],  $\pi_2 : \mathfrak{M}_{ss}^{(\alpha,1)}(\tilde{\tau}) \rightarrow \mathcal{M}_{stp}^{\alpha,n}(\tau')$  is a gerbe, which has fibre  $[\text{Spec } \mathbb{C}/\mathbb{G}_m]$ . Also  $\pi_1$  is a trivializing section of  $\pi_2$ , so by [67, Lem. 3.21],  $\mathfrak{M}_{ss}^{(\alpha,1)}(\tilde{\tau})$  is a trivial  $\mathbb{G}_m$ -gerbe over  $\mathcal{M}_{stp}^{\alpha,n}(\tau')$ , that is,  $\mathfrak{M}_{ss}^{(\alpha,1)}(\tilde{\tau}) \cong \mathcal{M}_{stp}^{\alpha,n}(\tau') \times [\text{Spec } \mathbb{C}/\mathbb{G}_m]$ .  $\square$

**13.3. Stack function identities in  $\text{SF}_{\text{al}}(\mathfrak{M}_{\mathcal{B}_p})$** 

As in §3.1 we have a Ringel–Hall algebra  $\text{SF}_{\text{al}}(\mathfrak{M}_{\mathcal{B}_p})$  with multiplication  $*$ , and a Lie subalgebra  $\text{SF}_{\text{al}}^{\text{ind}}(\mathfrak{M}_{\mathcal{B}_p})$ . As in §3.2, since  $(\tilde{\tau}, \tilde{T}, \leq)$  and  $(\dot{\tau}, \dot{T}, \leq)$  are permissible we have elements  $\bar{\delta}_{\text{ss}}^{(\beta, d)}(\tilde{\tau}), \bar{\delta}_{\text{ss}}^{(\beta, d)}(\dot{\tau})$  in  $\text{SF}_{\text{al}}(\mathfrak{M}_{\mathcal{B}_p})$  for  $(\beta, d) \in C(\mathcal{B}_p)$ , and we define  $\bar{\epsilon}^{(\beta, d)}(\tilde{\tau}), \bar{\epsilon}^{(\beta, d)}(\dot{\tau})$  by (3.4), which lie in  $\text{SF}_{\text{al}}^{\text{ind}}(\mathfrak{M}_{\mathcal{B}_p})$  by Theorem 3.11. Applying Theorem 3.13 with dominating permissible stability condition  $(\hat{\tau}, \hat{T}, \leq)$  yields:

PROPOSITION 13.7. *For all  $(\beta, d)$  in  $C(\mathcal{B}_p)$  we have the identity in  $\text{SF}_{\text{al}}(\mathcal{B}_p)$ :*

$$\bar{\epsilon}^{(\beta, d)}(\tilde{\tau}) = \sum_{\substack{n \geq 1, (\beta_1, d_1), \dots, (\beta_n, d_n) \in C(\mathcal{B}_p): \\ (\beta_1, d_1) + \dots + (\beta_n, d_n) = (\beta, d)}} U((\beta_1, d_1), \dots, (\beta_n, d_n); \tilde{\tau}, \tilde{\tau}) \cdot \bar{\epsilon}^{(\beta_1, d_1)}(\dot{\tau}) * \dots * \bar{\epsilon}^{(\beta_n, d_n)}(\dot{\tau}). \quad (13.16)$$

There are only finitely many nonzero terms in (13.16).

We now take  $(\beta, d) = (\alpha, 1)$  in (13.16), where  $\alpha$  is as fixed in Definition 13.1. Then each term has  $d_1 + \dots + d_n = 1$  with  $d_i \geq 0$ , so we have  $d_k = 1$  for some  $k = 1, \dots, n$  and  $d_i = 0$  for  $i \neq k$ . But  $\bar{\epsilon}^{(\beta_k, 1)}(\dot{\tau})$  is supported on  $\mathfrak{M}_{\text{ss}}^{(\beta_k, 1)}(\dot{\tau})$  which is empty for  $\beta_k \neq 0$  by Proposition 13.6(a). Thus the only nonzero terms in (13.16) have  $(\beta_i, d_i) = (\beta_i, 0)$  for  $i \neq k$  and  $\beta_i \in C(\mathcal{A}_p)$  and  $(\beta_k, d_k) = (0, 1)$ . Changing notation to  $\alpha_i = \beta_i$  for  $i < k$  and  $\alpha_i = \beta_{i+1}$  for  $i \geq k$  gives:

$$\bar{\epsilon}^{(\alpha, 1)}(\tilde{\tau}) = \sum_{\substack{1 \leq k \leq n, \\ \alpha_1, \dots, \alpha_{n-1} \in C(\mathcal{A}_p): \\ \alpha_1 + \dots + \alpha_{n-1} = \alpha}} U((\alpha_1, 0), \dots, (\alpha_{k-1}, 0), (0, 1), (\alpha_k, 0), \dots, (\alpha_{n-1}, 0); \tilde{\tau}, \tilde{\tau}) \cdot \bar{\epsilon}^{(\alpha_1, 0)}(\dot{\tau}) * \dots * \bar{\epsilon}^{(\alpha_{k-1}, 0)}(\dot{\tau}) * \bar{\epsilon}^{(0, 1)}(\dot{\tau}) * \bar{\epsilon}^{(\alpha_k, 0)}(\dot{\tau}) * \dots * \bar{\epsilon}^{(\alpha_{n-1}, 0)}(\dot{\tau}). \quad (13.17)$$

PROPOSITION 13.8. *In equation (13.17) we have*

$$U((\alpha_1, 0), \dots, (\alpha_{k-1}, 0), (0, 1), (\alpha_k, 0), \dots, (\alpha_{n-1}, 0); \tilde{\tau}, \tilde{\tau}) = \frac{(-1)^{n-k}}{(k-1)!(n-k)!}. \quad (13.18)$$

PROOF. The coefficient  $U(\dots; \tilde{\tau}, \tilde{\tau})$  is defined in equation (3.8). Consider some choices  $l, m, a_i, b_i, \beta_i, \gamma_i$  in this sum. There are two conditions in (3.8). The first, that  $\dot{\tau}(\beta_i) = \dot{\tau}(\alpha_j)$ ,  $i = 1, \dots, m$ ,  $a_{i-1} < j \leq a_i$ , holds if and only if we have  $a_{p-1} = k-1$  and  $a_p = k$  for some  $p = 1, \dots, m$ . The second, that  $\tilde{\tau}(\gamma_i) = \tilde{\tau}((\alpha, 1))$ ,  $i = 1, \dots, l$ , is equivalent to  $l = 1$ , since if  $l > 1$  then one  $\gamma_i$  is of the form  $(\beta, 1)$ , with  $\tilde{\tau}(\gamma_i) = 1$  and the other  $\gamma_j$  are of the form  $(\beta, 0)$ , with  $\tilde{\tau}(\gamma_j) = 0$ . Thus we may rewrite (3.8) as

$$U((\alpha_1, 0), \dots, (\alpha_{k-1}, 0), (0, 1), (\alpha_k, 0), \dots, (\alpha_{n-1}, 0); \tilde{\tau}, \tilde{\tau}) = \quad (13.19)$$

$$\sum S(\beta_1, \beta_2, \dots, \beta_m; \dot{\tau}, \tilde{\tau}) \cdot \prod_{i=1}^m \frac{1}{(a_i - a_{i-1})!}.$$

$1 \leq p \leq m \leq n$ ,  $0 = a_0 < a_1 < \dots < a_{p-1} = k-1$ ,  $k = a_p < a_{p+1} < \dots < a_m = n$ .  
Define  $\beta_1, \dots, \beta_m \in C(\mathcal{B}_p)$  by  $\beta_i = (\alpha_{a_{i-1}+1} + \dots + \alpha_{a_i}, 0)$ ,  $i < p$ ,  
 $\beta_p = (0, 1)$ ,  $\beta_i = (\alpha_{a_{i-1}} + \dots + \alpha_{a_i-1}, 0)$ ,  $i > p$ .



In (13.19) we have  $\beta_p = (0, 1)$  and  $\beta_i = (\alpha'_i, 0)$  for  $\alpha'_i \in C(\mathcal{A}_p)$ ,  $i \neq p$ . Using Definition 13.5, in Definition 3.12 we see that  $i = 1, \dots, m-1$  satisfies neither (a) nor (b) if  $i < p-1$ , satisfies (b) when  $i = p-1$ , and satisfies (a) for  $i \geq p$ . Therefore

$$S(\beta_1, \beta_2, \dots, \beta_m; \dot{\tau}, \tilde{\tau}) = \begin{cases} (-1)^{m-1}, & p = 1, \\ (-1)^{m-2}, & p = 2, \\ 0, & p > 2. \end{cases} \quad (13.20)$$

Since  $0 = a_0 < \dots < a_{p-1} = k-1$ , we see that  $p = 1$  if  $k = 1$ , and  $p > 1$  if  $k > 1$ . So we divide into two cases  $k = 1$  in (13.21) and  $k > 1$  in (13.22), and rewrite (13.19) using (13.20) in each case:

$$U((0, 1), (\alpha_1, 0), \dots, (\alpha_{n-1}, 0); \dot{\tau}, \tilde{\tau}) = \sum_{1 \leq m \leq n, \ 1=a_1 < a_2 < \dots < a_m=n} (-1)^{m-1} \cdot \prod_{i=2}^m \frac{1}{(a_i - a_{i-1})!}, \quad (13.21)$$

$$U((\alpha_1, 0), \dots, (\alpha_{k-1}, 0), (0, 1), (\alpha_k, 0), \dots, (\alpha_{n-1}, 0); \dot{\tau}, \tilde{\tau}) = \frac{1}{(k-1)!} \cdot \sum_{2 \leq m \leq n, \ k=a_2 < a_3 < \dots < a_m=n} (-1)^{m-2} \cdot \prod_{i=3}^m \frac{1}{(a_i - a_{i-1})!}. \quad (13.22)$$

Here the factor  $1/(k-1)!$  in (13.22) is  $1/(a_1 - a_0)!$  in (13.19), since  $a_0 = 0$ ,  $a_1 = k-1$ , and  $a_2 = k$ . We evaluate a rewritten version of the sums in (13.21) and (13.22):

LEMMA 13.9. *For all  $l \geq 1$  we have*

$$\sum_{1 \leq m \leq l, \ 0=a_0 < a_1 < \dots < a_m=l} (-1)^m \cdot \prod_{i=1}^m \frac{1}{(a_i - a_{i-1})!} = \frac{(-1)^l}{l!}. \quad (13.23)$$

PROOF. Write  $T_l$  for the l.h.s. of (13.23). Then in formal power series we have:

$$\begin{aligned} \sum_{l=1}^{\infty} T_l t^l &= \sum_{l=1}^{\infty} \sum_{1 \leq m \leq l, \ 0=a_0 < a_1 < \dots < a_m=l} (-1)^m \cdot \prod_{i=1}^m \frac{t^{a_i - a_{i-1}}}{(a_i - a_{i-1})!} = \sum_{l=1}^{\infty} \sum_{1 \leq m \leq l, \ b_1, \dots, b_m \geq 1, \ b_1 + \dots + b_m = l} (-1)^m \cdot \prod_{i=1}^m \frac{t^{b_i}}{(b_i)!} \\ &= \sum_{m=1}^{\infty} (-1)^m \left[ \sum_{j=1}^{\infty} \frac{t^j}{j!} \right]^m = \sum_{m=1}^{\infty} (-1)^m (e^t - 1)^m = \frac{-(e^t - 1)}{1 + (e^t - 1)} = e^{-t} - 1, \end{aligned} \quad (13.24)$$

where in the second step we set  $b_i = a_i - a_{i-1}$ , and in the third we regard  $l$  as defined by  $b_1 + \dots + b_m = l$  and drop the sum over  $l$ , and then replace the sum over  $b_1, \dots, b_m$  by an  $m^{\text{th}}$  power of a sum over  $j$ . Equating coefficients of  $t^l$  in (13.24) gives (13.23).  $\square$

Now the r.h.s. of (13.21) agrees with the l.h.s. of (13.23) with  $l = n-1$ , replacing  $n, m, a_1, \dots, a_m$  in (13.21) by  $l+1, m+1, a_0, \dots, a_m$  respectively. Thus (13.21) and Lemma 13.9 prove the case  $k = 1$  of (13.18). Similarly, apart from the factor  $1/(k-1)!$ , the r.h.s. of (13.22) agrees with the l.h.s. of (13.23) with  $l = n-k$ , replacing  $n, m, a_2, \dots, a_m$  in (13.21) by  $l+k, m+2, a_0, \dots, a_m$  respectively. This gives the case  $k > 1$  of (13.18), and completes the proof of Proposition 13.8.  $\square$

Substituting (13.18) into (13.17) and replacing  $n$  by  $l+1$  and  $k$  by  $k-1$  gives

$$\bar{\epsilon}^{(\alpha,1)}(\dot{\tau}) = \sum_{\substack{0 \leq k \leq l, \\ \alpha_1 + \dots + \alpha_l = \alpha}} \frac{(-1)^{l-k}}{k!(l-k)!} \cdot \bar{\epsilon}^{(\alpha_1,0)}(\dot{\tau}) * \dots * \bar{\epsilon}^{(\alpha_k,0)}(\dot{\tau}) * \bar{\epsilon}^{(0,1)}(\dot{\tau}) * \bar{\epsilon}^{(\alpha_{k+1},0)}(\dot{\tau}) * \dots * \bar{\epsilon}^{(\alpha_l,0)}(\dot{\tau}). \quad (13.25)$$

As in Theorem 3.14, by [54, Th. 5.4] we can rewrite the wall crossing formula (13.25) in terms of the Lie bracket  $[\cdot, \cdot]$  on  $\text{SF}_{\text{al}}^{\text{ind}}(\mathfrak{M}_{\mathcal{B}_p})$ , rather than the Ringel–Hall multiplication  $*$  on  $\text{SF}_{\text{al}}(\mathfrak{M}_{\mathcal{B}_p})$ . In this case, we can do it explicitly.

PROPOSITION 13.10. *In the situation above we have*

$$\bar{\epsilon}^{(\alpha,1)}(\dot{\tau}) = \sum_{l \geq 1, \alpha_1, \dots, \alpha_l \in C(\mathcal{A}_p): \alpha_1 + \dots + \alpha_l = \alpha} \frac{(-1)^l}{l!} [[\dots [[\bar{\epsilon}^{(0,1)}(\dot{\tau}), \bar{\epsilon}^{(\alpha_1,0)}(\dot{\tau})], \bar{\epsilon}^{(\alpha_2,0)}(\dot{\tau})], \dots], \bar{\epsilon}^{(\alpha_l,0)}(\dot{\tau})]. \quad (13.26)$$

PROOF. The term  $[[\dots [\bar{\epsilon}^{(0,1)}(\dot{\tau}), \bar{\epsilon}^{(\alpha_1,0)}(\dot{\tau})], \dots], \bar{\epsilon}^{(\alpha_l,0)}(\dot{\tau})]$  in (13.26) has  $l$  nested commutators  $[\cdot, \cdot]$ , and so consists of  $2^l$  terms. For each of these  $2^l$  terms, let  $k$  be the number of the  $l$  commutators in which we reverse the order of multiplication. Then the sign of this term is  $(-1)^k$ , and  $k$   $\bar{\epsilon}^{(\alpha_i,0)}(\dot{\tau})$ 's appear before  $\bar{\epsilon}^{(0,1)}(\dot{\tau})$  in the product. There are  $\binom{l}{k}$  such terms for fixed  $k$ . Thus we have

$$[[\dots [[\bar{\epsilon}^{(0,1)}(\dot{\tau}), \bar{\epsilon}^{(\alpha_1,0)}(\dot{\tau})], \bar{\epsilon}^{(\alpha_2,0)}(\dot{\tau})], \dots], \bar{\epsilon}^{(\alpha_l,0)}(\dot{\tau})] = \sum_{k=0}^l \binom{l}{k} \text{ terms of the form } (-1)^k \bar{\epsilon}^{(\alpha_{i_1},0)}(\dot{\tau}) * \dots * \bar{\epsilon}^{(\alpha_{i_k},0)}(\dot{\tau}) * \bar{\epsilon}^{(0,1)}(\dot{\tau}) * \bar{\epsilon}^{(\alpha_{i_{k+1}},0)}(\dot{\tau}) * \dots * \bar{\epsilon}^{(\alpha_{i_l},0)}(\dot{\tau}), \quad (13.27)$$

where  $\{i_1, \dots, i_l\}$  is some permutation of  $\{1, \dots, l\}$ .

Let us now sum (13.27) over all permutations of  $\{1, \dots, l\}$ , acting by permuting  $\alpha_1, \dots, \alpha_l$ . The permutations  $\{i_1, \dots, i_l\}$  are then also summed over all permutations of  $\{1, \dots, l\}$ , giving

$$\sum_{\sigma \in S_l} [[\dots [[\bar{\epsilon}^{(0,1)}(\dot{\tau}), \bar{\epsilon}^{(\alpha_{\sigma(1)},0)}(\dot{\tau})], \bar{\epsilon}^{(\alpha_{\sigma(2)},0)}(\dot{\tau})], \dots], \bar{\epsilon}^{(\alpha_{\sigma(l)},0)}(\dot{\tau})] = \sum_{\sigma \in S_l} \sum_{k=0}^l \binom{l}{k} (-1)^k \bar{\epsilon}^{(\alpha_{\sigma(1)},0)}(\dot{\tau}) * \dots * \bar{\epsilon}^{(\alpha_{\sigma(k)},0)}(\dot{\tau}) * \bar{\epsilon}^{(0,1)}(\dot{\tau}) * \bar{\epsilon}^{(\alpha_{\sigma(k+1)},0)}(\dot{\tau}) * \dots * \bar{\epsilon}^{(\alpha_{\sigma(l)},0)}(\dot{\tau}), \quad (13.28)$$

where  $S_l$  is the symmetric group of permutations  $\sigma : \{1, \dots, l\} \rightarrow \{1, \dots, l\}$ .

We now have

$$\begin{aligned}
& \sum_{l \geq 1, \alpha_1, \dots, \alpha_l \in C(\mathcal{A}_p): \alpha_1 + \dots + \alpha_l = \alpha} \frac{(-1)^l}{l!} [[\dots [\bar{\epsilon}^{(0,1)}(\dot{\tau}), \bar{\epsilon}^{(\alpha_1,0)}(\dot{\tau})], \bar{\epsilon}^{(\alpha_2,0)}(\dot{\tau})], \dots], \bar{\epsilon}^{(\alpha_l,0)}(\dot{\tau})] = \\
& \sum_{l \geq 1, \alpha_1, \dots, \alpha_l \in C(\mathcal{A}_p): \alpha_1 + \dots + \alpha_l = \alpha} \frac{(-1)^l}{(l!)^2} \sum_{\sigma \in S_l} [[\dots [\bar{\epsilon}^{(0,1)}(\dot{\tau}), \bar{\epsilon}^{(\alpha_{\sigma(1)},0)}(\dot{\tau})], \bar{\epsilon}^{(\alpha_{\sigma(2)},0)}(\dot{\tau})], \dots], \\
& \hspace{15em} \bar{\epsilon}^{(\alpha_{\sigma(l)},0)}(\dot{\tau})] = \\
& \sum_{l \geq 1, \alpha_1, \dots, \alpha_l \in C(\mathcal{A}_p): \alpha_1 + \dots + \alpha_l = \alpha} \frac{(-1)^l}{(l!)^2} \sum_{\sigma \in S_l} \sum_{k=0}^l \binom{l}{k} (-1)^k \bar{\epsilon}^{(\alpha_{\sigma(1)},0)}(\dot{\tau}) * \dots * \bar{\epsilon}^{(\alpha_{\sigma(k)},0)}(\dot{\tau}) * \bar{\epsilon}^{(0,1)}(\dot{\tau}) \\
& \hspace{15em} * \bar{\epsilon}^{(\alpha_{\sigma(k+1)},0)}(\dot{\tau}) * \dots * \bar{\epsilon}^{(\alpha_{\sigma(l)},0)}(\dot{\tau}) = \\
& \sum_{l \geq 1, \alpha_1, \dots, \alpha_l \in C(\mathcal{A}_p): \alpha_1 + \dots + \alpha_l = \alpha} \frac{(-1)^l}{l!} \sum_{k=0}^l \binom{l}{k} (-1)^k \bar{\epsilon}^{(\alpha_1,0)}(\dot{\tau}) * \dots * \bar{\epsilon}^{(\alpha_k,0)}(\dot{\tau}) * \bar{\epsilon}^{(0,1)}(\dot{\tau}) \\
& \hspace{15em} * \bar{\epsilon}^{(\alpha_{k+1},0)}(\dot{\tau}) * \dots * \bar{\epsilon}^{(\alpha_l,0)}(\dot{\tau}) = \\
& \sum_{0 \leq k \leq l, \alpha_1, \dots, \alpha_l \in C(\mathcal{A}_p): \alpha_1 + \dots + \alpha_l = \alpha} \frac{(-1)^{l-k}}{k!(l-k)!} \cdot \bar{\epsilon}^{(\alpha_1,0)}(\dot{\tau}) * \dots * \bar{\epsilon}^{(\alpha_k,0)}(\dot{\tau}) * \bar{\epsilon}^{(0,1)}(\dot{\tau}) \\
& \hspace{15em} * \bar{\epsilon}^{(\alpha_{k+1},0)}(\dot{\tau}) * \dots * \bar{\epsilon}^{(\alpha_l,0)}(\dot{\tau}) = \bar{\epsilon}^{(\alpha,1)}(\tilde{\tau}),
\end{aligned}$$

using the fact that the sums over  $\alpha_1, \dots, \alpha_l \in C(\mathcal{A}_p)$  with  $\alpha_1 + \dots + \alpha_l = \alpha$  are symmetric in permutations of  $\{1, \dots, l\}$  in the first and third steps, (13.28) in the second, and (13.25) in the fifth. This proves equation (13.26).  $\square$

#### 13.4. A Lie algebra morphism $\tilde{\Psi}^{\mathcal{B}_p} : \mathrm{SF}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{M}_{\mathcal{B}_p}) \rightarrow \tilde{L}(\mathcal{B}_p)$

We now define a Lie algebra morphism  $\tilde{\Psi}^{\mathcal{B}_p} : \mathrm{SF}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{M}_{\mathcal{B}_p}) \rightarrow \tilde{L}(\mathcal{B}_p)$ , which is a version of  $\tilde{\Psi} : \mathrm{SF}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{M}) \rightarrow \tilde{L}(X)$  in §5.3 for our auxiliary abelian category  $\mathcal{B}_p$ . Since as in §13.1 we do not know  $\mathcal{B}_p$  is 3-Calabi–Yau, and also as we will see below we only have good control of the Behrend function  $\nu_{\mathfrak{M}_{\mathcal{B}_p}}$  on a bounded part of  $\mathfrak{M}_{\mathcal{B}_p}$ , we will choose the Lie algebra  $\tilde{L}(\mathcal{B}_p)$  to be small, a finite-dimensional, nilpotent Lie algebra, and define  $\tilde{\Psi}^{\mathcal{B}_p}$  to be supported on  $\mathfrak{M}_{\mathcal{B}_p}^{(\beta,d)}$  for only finitely many  $(\beta, d) \in K(\mathcal{B}_p)$ .

**DEFINITION 13.11.** Define  $\mathcal{S}$  to be the subset of  $(\beta, d)$  in  $C(\mathcal{B}_p) \subset K(\mathcal{B}_p)$  such that  $P_\beta(t) = \frac{k}{d!} p(t)$  for  $k = 0, \dots, N$  and  $d = 0$  or  $1$ . (These were the conditions on numerical classes in Proposition 13.4.) Then  $\mathcal{S}$  is a finite set, as [44, Th. 3.37] implies that  $\tau$ -semistable sheaves  $E$  on  $X$  with Hilbert polynomials  $\frac{k}{d!} p(t)$  for  $k = 0, \dots, N$  can realize only finite many numerical classes  $\beta \in K(\mathcal{A}_p) \subset K(\mathrm{coh}(X))$ . Define a Lie algebra  $\tilde{L}(\mathcal{B}_p)$  to be the  $\mathbb{Q}$ -vector space with basis of symbols  $\tilde{\lambda}^{(\beta,d)}$  for  $(\beta, d) \in \mathcal{S}$ , with Lie bracket

$$\begin{aligned}
& [\tilde{\lambda}^{(\beta,d)}, \tilde{\lambda}^{(\gamma,e)}] = \\
& \begin{cases} (-1)^{\bar{\chi}^{\mathcal{B}_p}((\beta,d),(\gamma,e))} \bar{\chi}^{\mathcal{B}_p}((\beta,d),(\gamma,e)) \tilde{\lambda}^{(\beta+\gamma,d+e)}, & (\beta+\gamma, d+e) \in \mathcal{S}, \\ 0, & \text{otherwise,} \end{cases} \quad (13.29)
\end{aligned}$$

as in (5.4). As  $\bar{\chi}^{\mathcal{B}_p}$  is antisymmetric, and  $\mathcal{S} \subset K(\mathcal{B}_p)$  has the property that if  $\epsilon, \zeta, \eta \in \mathcal{S}$  and  $\epsilon + \zeta + \eta \in \mathcal{S}$  then  $\epsilon + \zeta, \epsilon + \eta, \zeta + \eta \in \mathcal{S}$ , equation (13.29) satisfies the Jacobi identity, and makes  $\tilde{L}(\mathcal{B}_p)$  into a finite-dimensional, nilpotent Lie algebra

over  $\mathbb{Q}$ . Now define a  $\mathbb{Q}$ -linear map  $\tilde{\Psi}^{\mathcal{B}_p} : \mathrm{SF}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{M}_{\mathcal{B}_p}) \rightarrow \tilde{L}(\mathcal{B}_p)$  exactly as for  $\tilde{\Psi} : \mathrm{SF}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{M}) \rightarrow \tilde{L}(X)$  in Definition 5.13.

We shall show that  $\tilde{\Psi}^{\mathcal{B}_p}$  is a *Lie algebra morphism*, by modifying the proof for  $\tilde{\Psi}$  in Theorem 5.14. The two key ingredients in the proof of Theorem 5.14 were, firstly, equation (3.14) writing the Euler form  $\bar{\chi}$  of  $\mathrm{coh}(X)$  in terms of  $\dim \mathrm{Hom}$  and  $\dim \mathrm{Ext}^1$  in  $\mathrm{coh}(X)$ , and secondly, the identities (5.2)–(5.3) for the Behrend function  $\nu_{\mathfrak{M}}$  in Theorem 5.11. Proposition 13.4 proves the analogue of (3.14) in the bounded part of  $\mathcal{B}_p$  we need it for. Here is an analogue of Theorem 5.11.

**PROPOSITION 13.12. (a)** *If  $(\beta, 0) \in \mathcal{S}$  then  $\pi : \mathfrak{M}_{\mathcal{B}_p}^{(\beta, 0)} \rightarrow \mathfrak{M}^\beta$  mapping  $(E, 0, 0) \mapsto E$  is a 1-isomorphism, and the Behrend functions satisfy  $\nu_{\mathfrak{M}_{\mathcal{B}_p}^{(\beta, 0)}} \equiv \pi^*(\nu_{\mathfrak{M}}^\beta)$ .*

*If  $(\beta, 1) \in \mathcal{S}$  then  $\pi : \mathfrak{M}_{\mathcal{B}_p}^{(\beta, 1)} \rightarrow \mathfrak{M}^\beta$  mapping  $(E, V, s) \mapsto E$  is smooth of relative dimension  $\bar{\chi}([\mathcal{O}_X(-n)], \beta) - 1$ , and  $\nu_{\mathfrak{M}_{\mathcal{B}_p}^{(\beta, 1)}} \equiv (-1)^{\bar{\chi}([\mathcal{O}_X(-n)], \beta) - 1} \pi^*(\nu_{\mathfrak{M}}^\beta)$ .*

**(b)** *An analogue of Theorem 5.11 holds in  $\mathcal{B}_p$ , with  $E_1, E_2 \in \mathrm{coh}(X)$  replaced by  $(E_1, V_1, s_1), (E_2, V_2, s_2) \in \mathcal{B}_p$  such that  $[(E_1 \oplus E_2, V_1 \oplus V_2, s_1 \oplus s_2)] \in \mathcal{S}$ , and  $\mathrm{Ext}^1$  replaced by  $\mathrm{Ext}_{\mathcal{B}_p}^1$ , and  $\bar{\chi}$  replaced by  $\bar{\chi}^{\mathcal{B}_p}$ .*

**PROOF.** The first part of (a) is immediate. For the second, note that if  $(\beta, 1) \in \mathcal{S}$  and  $(E, V, s)$  is a point in  $\mathfrak{M}_{\mathcal{B}_p}^{(\beta, 1)}$  then  $[E] = \beta$  in  $K(\mathcal{A}_p)$ , we may identify  $V \cong \mathbb{C}$ , and then  $s : \mathcal{O}_X(-n) \rightarrow E$ , that is,  $s \in H^0(E(n))$ . But by choice of  $E$  and of  $n$  in Definition 13.1, we have  $H^i(E(n)) = 0$  for  $i > 0$ , so  $H^0(E(n))$  is a vector space of fixed dimension  $P_\beta(n) = \bar{\chi}([\mathcal{O}_X(-n)], \beta)$ . Furthermore,  $E \mapsto H^0(E(n))$  is a *vector bundle* (in the Artin stack sense) over the stack  $\mathfrak{M}^\beta$ , with fibre  $H^0(E(n)) \cong \mathbb{C}^{\bar{\chi}([\mathcal{O}_X(-n)], \beta)}$  over  $E$ .

Now consider the fibre of  $\pi : \mathfrak{M}_{\mathcal{B}_p}^{(\beta, 1)} \rightarrow \mathfrak{M}^\beta$  over  $E$ . It is a set of pairs  $(V, s)$  with  $V \cong \mathbb{C}$  and  $s : V \rightarrow H^0(E(n))$  linear, satisfying a stability condition. This stability condition requires  $s \neq 0$ , and selects an open set of such  $s$ . Dividing out by automorphisms of  $V$  turns  $H^0(E(n)) \setminus 0$  into the projective space  $\mathbb{P}(H^0(E(n)))$ . Hence the fibre of  $\pi$  over  $E$  is an open subset of the projective space  $\mathbb{P}(H^0(E(n)))$ . Since  $E \mapsto H^0(E(n))$  is a vector bundle over  $\mathfrak{M}^\beta$ ,  $E \mapsto \mathbb{P}(H^0(E(n)))$  is a projective space bundle over  $\mathfrak{M}^\beta$ . Therefore  $\mathfrak{M}_{\mathcal{B}_p}^{(\beta, 1)}$  is an open subset of a smooth fibration over  $\mathfrak{M}^\beta$  with fibre  $\mathbb{C}^{\bar{\chi}([\mathcal{O}_X(-n)], \beta) - 1}$ . So  $\pi$  is smooth of relative dimension  $\bar{\chi}([\mathcal{O}_X(-n)], \beta) - 1$ . The Behrend function equation follows from Theorem 4.3(ii) and Corollary 4.5.

For (b), we can now follow the proof of Theorem 5.11, using facts from (a) above. In Theorem 5.5, we proved that an atlas for  $\mathfrak{M}^\beta$  near  $E$  may be written locally in the complex analytic topology as  $\mathrm{Crit}(f)$  for holomorphic  $f : U \rightarrow \mathbb{C}$ , where  $U$  is an open neighbourhood of 0 in  $\mathrm{Ext}^1(E, E)$ , and  $U, f$  are invariant under the complexification  $G^c$  of a maximal compact subgroup  $G$  of  $\mathrm{Aut}(E)$ . From the second part of (a), it follows that an atlas for  $\mathfrak{M}_{\mathcal{B}_p}^{(\beta, 1)}$  near  $E, V, s$  may be written locally in the complex analytic topology as  $\mathrm{Crit}(f) \times W$ , where  $W$  is an open set in  $H^0(E(n))$ . But  $\mathrm{Crit}(f) \times W = \mathrm{Crit}(f \circ \pi_U)$ , where  $f \circ \pi_U : U \times W \rightarrow \mathbb{C}$  is a holomorphic function on a smooth complex manifold.

Therefore, just as we can write the moduli stack  $\mathfrak{M}$  locally as  $\mathrm{Crit}(f)$ , and so use differential-geometric reasoning with the Milnor fibres of  $f$  to prove (5.2)–(5.3)

in Theorem 5.11, so we can write the moduli stacks  $\mathfrak{M}_{\mathcal{B}_p}^{(\beta,d)}$  for  $(\beta, d) \in \mathcal{S}$  locally as  $\text{Crit}(f \circ \pi_U)$ , and the proof of Theorem 5.11 extends to give (b).  $\square$

We can now follow the proof of Theorem 5.14 using Proposition 13.4 in place of (3.14) and Proposition 13.12(b) in place of Theorem 5.11 to prove:

PROPOSITION 13.13.  $\tilde{\Psi}^{\mathcal{B}_p} : \text{SF}_{\text{al}}^{\text{ind}}(\mathfrak{M}_{\mathcal{B}_p}) \rightarrow \tilde{L}(\mathcal{B}_p)$  is a Lie algebra morphism.

### 13.5. Proof of Theorem 5.27

Finally we prove Theorem 5.27. We will apply the Lie algebra morphism  $\tilde{\Psi}^{\mathcal{B}_p}$  to the Lie algebra equation (13.26). Observe that the terms  $(\alpha, 1)$ ,  $(1, 0)$  and  $(\alpha_i, 0)$  occurring in (13.26) all lie in  $\mathcal{S}$ . We will prove that

$$\begin{aligned} \tilde{\Psi}^{\mathcal{B}_p}(\bar{\epsilon}^{(\alpha,1)}(\tilde{\tau})) &= -PI^{\alpha,n}(\tau')\tilde{\lambda}^{(\alpha,1)}, & \tilde{\Psi}^{\mathcal{B}_p}(\bar{\epsilon}^{(0,1)}(\dot{\tau})) &= -\tilde{\lambda}^{(0,1)}, \\ \text{and} \quad \tilde{\Psi}^{\mathcal{B}_p}(\bar{\epsilon}^{(\alpha_i,0)}(\dot{\tau})) &= -\bar{D}T^{\alpha_i}(\tau)\tilde{\lambda}^{(\alpha_i,0)}. \end{aligned} \quad (13.30)$$

For the first equation, there are no strictly  $\tilde{\tau}$ -semistables in  $\mathfrak{M}_{\text{ss}}^{(\alpha,1)}(\tilde{\tau})$ , so  $\bar{\epsilon}^{(\alpha,1)}(\tilde{\tau}) = \bar{\delta}_{\text{ss}}^{(\alpha,1)}(\tilde{\tau})$ , and  $\tilde{\Psi}^{\mathcal{B}_p}(\bar{\epsilon}^{(\alpha,1)}(\tilde{\tau})) = \chi^{\text{na}}(\mathfrak{M}_{\text{ss}}^{(\alpha,1)}(\tilde{\tau}), \nu_{\mathfrak{M}_{\text{ss}}^{(\alpha,1)}(\tilde{\tau})})\tilde{\lambda}^{(\alpha,1)}$  in the notation of Definition 2.3. But  $\mathfrak{M}_{\text{ss}}^{(\alpha,1)}(\tilde{\tau}) \cong \mathcal{M}_{\text{stp}}^{\alpha,n}(\tau') \times [\text{Spec } \mathbb{C}/\mathbb{G}_m]$  by Proposition 13.6(b), so the projection  $\pi : \mathfrak{M}_{\text{ss}}^{(\alpha,1)}(\tilde{\tau}) \rightarrow \mathcal{M}_{\text{stp}}^{\alpha,n}(\tau')$  is smooth of relative dimension  $-1$ , and  $\nu_{\mathfrak{M}_{\text{ss}}^{(\alpha,1)}(\tilde{\tau})} = -\pi^*(\nu_{\mathcal{M}_{\text{stp}}^{\alpha,n}(\tau')})$  by Theorem 4.3(ii) and Corollary 4.5. Hence

$$\chi^{\text{na}}(\mathfrak{M}_{\text{ss}}^{(\alpha,1)}(\tilde{\tau}), \nu_{\mathfrak{M}_{\text{ss}}^{(\alpha,1)}(\tilde{\tau})}) = -\chi(\mathcal{M}_{\text{stp}}^{\alpha,n}(\tau'), \nu_{\mathcal{M}_{\text{stp}}^{\alpha,n}(\tau')}) = -PI^{\alpha,n}(\tau')$$

by (5.16), proving the first equation of (13.30). We have  $\mathfrak{M}_{\text{ss}}^{(0,1)}(\dot{\tau}) \cong [\text{Spec } \mathbb{C}/\mathbb{G}_m]$  by Proposition 13.6(a), so  $\bar{\epsilon}^{(0,1)}(\dot{\tau})$  is the stack characteristic function of  $[\text{Spec } \mathbb{C}/\mathbb{G}_m]$ . But  $[\text{Spec } \mathbb{C}/\mathbb{G}_m]$  is a single point with Behrend function  $-1$ , so the second equation follows. And the isomorphism  $\mathfrak{M}_{\text{ss}}^{(\alpha_i,0)}(\dot{\tau}) \cong \mathfrak{M}_{\text{ss}}^{\alpha_i}(\tau) \subset \mathfrak{M}$  identifies  $\bar{\epsilon}^{(\alpha_i,0)}(\dot{\tau})$  with  $\bar{\epsilon}^{\alpha_i}(\tau)$ , so the third equation of (13.30) follows from (5.7).

Hence, applying  $\tilde{\Psi}^{\mathcal{B}_p}$  (which is a Lie algebra morphism by Proposition 13.13) to (13.26) and substituting in (13.30) gives an equation in the Lie algebra  $\tilde{L}(\mathcal{B}_p)$ :

$$\begin{aligned} -PI^{\alpha,n}(\tau')\tilde{\lambda}^{(\alpha,1)} &= \\ \sum_{l \geq 1, \alpha_1, \dots, \alpha_l \in C(\mathcal{A}_p): \alpha_1 + \dots + \alpha_l = \alpha} \frac{(-1)^l}{l!} &[[\dots [[-\tilde{\lambda}^{(0,1)}, -\bar{D}T^{\alpha_1}(\tau)\tilde{\lambda}^{(\alpha_1,0)}], -\bar{D}T^{\alpha_2}(\tau)\tilde{\lambda}^{(\alpha_2,0)}], \dots], \\ &- \bar{D}T^{\alpha_l}(\tau)\tilde{\lambda}^{(\alpha_l,0)}]. \end{aligned} \quad (13.31)$$

Using the definitions (13.5) of  $\tilde{\chi}^{\mathcal{B}_p}$  and (13.29) of the Lie bracket in  $\tilde{L}(\mathcal{B}_p)$ , and noting that the condition  $\alpha_i \in C(\text{coh}(X))$  with  $\tau(\alpha_i) = \tau(\alpha)$  in (5.17) corresponds to  $\alpha_i \in C(\mathcal{A}_p)$  in (13.31), we see that (13.31) reduces to (5.17). There are only finitely many nonzero terms in each of these equations, as in Proposition 13.7. This completes the proof of Theorem 5.27.

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## Glossary of Notation

$\Lambda_X$	sublattice of $H^{\text{even}}(X, \mathbb{Q})$ for Calabi–Yau 3-fold $X$ , 41
$\Phi_f$	vanishing cycle functor on constructible functions, 36
$\Pi_{\text{CF}}$	projection $\text{SF}_{\text{al}}^{\text{ind}}(\mathfrak{M}, \chi, \mathbb{Q}) \rightarrow \text{CF}(\mathfrak{M})$ , 55
$\Pi_n^{\text{vi}}$	projection to stack functions with ‘virtual rank $n$ ’, 18
$\Psi^{\chi, \mathbb{Q}}$	Lie algebra morphism $\text{SF}_{\text{al}}^{\text{ind}}(\mathfrak{M}, \chi, \mathbb{Q}) \rightarrow L(X)$ , 28
$\tilde{\Psi}$	Lie algebra morphism $\text{SF}_{\text{al}}^{\text{ind}}(\mathfrak{M}) \rightarrow \tilde{L}(X)$ , 55
$\tilde{\Psi}^{\chi, \mathbb{Q}}$	Lie algebra morphism $\text{SF}_{\text{al}}^{\text{ind}}(\mathfrak{M}, \chi, \mathbb{Q}) \rightarrow \tilde{L}(X)$ , 55
$\tilde{\Psi}_{Q, I}$	Lie algebra morphism $\text{SF}_{\text{al}}^{\text{ind}}(\mathfrak{M}_{Q, I}) \rightarrow \tilde{L}(Q)$ , 97
$\tilde{\Psi}_{Q, I}^{\chi, \mathbb{Q}}$	Lie algebra morphism $\text{SF}_{\text{al}}^{\text{ind}}(\mathfrak{M}_{Q, I}, \chi, \mathbb{Q}) \rightarrow \tilde{L}(Q)$ , 97
$\Psi_f$	nearby cycle functor on constructible functions, 36
$\Psi$	Lie algebra morphism $\text{SF}_{\text{al}}^{\text{ind}}(\mathfrak{M}) \rightarrow L(X)$ , 28
$\chi^{\text{na}}(C)$	naïve Euler characteristic of a constructible set $C$ in a stack, 14
$\chi^{\text{na}}(\mathfrak{F}, f)$	naïve Euler characteristic of an Artin stack $\mathfrak{F}$ weighted by a constructible function $f$ , 14
$\bar{\chi}$	Euler form of an abelian category, 21
$\bar{\delta}_{\text{ss}}^{\alpha}(\tau)$	element of the Ringel–Hall algebra $\text{SF}_{\text{al}}(\mathfrak{M})$ that ‘counts’ $\tau$ -semistable objects in class $\alpha$ , 25
$\bar{\epsilon}^{\alpha}(\tau)$	element of the Ringel–Hall Lie algebra $\text{SF}_{\text{al}}^{\text{ind}}(\mathfrak{M})$ that ‘counts’ $\tau$ -semistable objects in class $\alpha$ , 25
$\tilde{\lambda}^{\alpha}$	basis element of Lie algebra $\tilde{L}(X)$ , 54
$\lambda^{\alpha}$	basis element of Lie algebra $L(X)$ , 27
$(\mu, \mathbb{R}, \leq)$	slope stability condition on $\text{mod-}\mathbb{K}Q$ or $\text{mod-}\mathbb{K}Q/I$ , 91
$(\mu, M, \leq)$	the weak stability condition of $\mu$ -stability on $\text{coh}(X)$ , 25
$\nu_X$	Behrend function of a scheme or stack $X$ , 32
$\phi_f$	vanishing cycle functor on derived category of constructible sheaves, 35
$\psi_f$	nearby cycle functor on derived category of constructible sheaves, 34
$(\tau, G, \leq)$	the stability condition of Gieseker stability on $\text{coh}(X)$ , 24
$(\tau, T, \leq)$	stability condition on an abelian category, 23
$A^{\bullet}, B^{\bullet}, \dots$	complexes in the derived category $D(\mathcal{M}_{\text{stp}}^{\alpha, n}(\tau'))$ , 163
$\mathcal{A}$	affine space of smooth semiconnections on a vector bundle, 122
$\mathcal{A}_{\text{si}}$	open subset of simple semiconnections in $\mathcal{A}$ , 123
$\mathcal{A}^{2, k}$	affine space of $L_k^2$ semiconnections on a vector bundle, 123
$\mathcal{A}_{\text{si}}^{2, k}$	open subset of simple semiconnections in $\mathcal{A}^{2, k}$ , 123
$\mathcal{A}_p$	an abelian subcategory of $\tau$ -semistable sheaves in $\text{coh}(X)$ , 173
$\text{At}(E, s)$	Atiyah class of a family of stable pairs $s : \mathcal{O}_{X_T}(-n) \rightarrow E$ , 161
$\mathcal{B}_p$	abelian category of coherent sheaves extended by vector spaces, 174
$C(\mathcal{A})$	the positive cone in $K(\mathcal{A})$ , 22
$C(G)$	centre of an algebraic $\mathbb{K}$ -group $G$ , 17

- $\mathrm{CF}(\mathfrak{F})$   $\mathbb{Q}$ -vector space of constructible functions on a stack  $\mathfrak{F}$ , 13
- $\mathrm{CF}^{\mathrm{na}}(\phi)$  naïve pushforward of constructible functions along 1-morphism  $\phi$ , 14
- $\mathrm{CF}^{\mathrm{stk}}(\phi)$  stack pushforward of constructible functions along representable  $\phi$ , 14
- $\mathrm{CF}_{\mathbb{Z}}^{\mathrm{an}}(X)$  group of  $\mathbb{Z}$ -valued analytically constructible functions on  $X$ , 35
- $C_G(S)$  centralizer of a subset  $S$  in a group  $G$ , 17
- $\mathrm{ch}(E)$  Chern character of a coherent sheaf  $E$  in  $H^{\mathrm{even}}(X, \mathbb{Q})$ , 41
- $\mathrm{ch}_{\mathrm{cs}}(E)$  Chern character of a compactly-supported coherent sheaf  $E$ , 83
- $\mathrm{ch}_i(E)$   $i^{\mathrm{th}}$  component of Chern character of  $E$  in  $H^{2i}(X, \mathbb{Q})$ , 41
- $\mathrm{coh}(X)$  abelian category of coherent sheaves on a scheme  $X$
- $\mathrm{coh}_{\mathrm{cs}}(X)$  abelian category of compactly supported coherent sheaves on  $X$ , 82
- $\mathrm{Crit}(f)$  critical locus of a holomorphic function  $f$ , as a complex analytic space
- $CS$  holomorphic Chern–Simons functional, 131
- $D(X)$  derived category of quasi-coherent sheaves on  $X$ , 154
- $D^b(\mathrm{coh}(X))$  or  $D^b(X)$  bounded derived category of coherent sheaves on  $X$
- $D_{\mathrm{con}}^b(X)$  bounded derived category of constructible complexes on  $X$ , 34
- $\bar{\partial}_E$  semiconnection on complex vector bundle  $E$ , 122
- $DT^{\alpha}(\tau)$  original Donaldson–Thomas invariants defined in [100], 38
- $\bar{DT}^{\alpha}(\tau)$  generalized Donaldson–Thomas invariants, 56
- $\hat{DT}^{\alpha}(\tau)$  BPS invariants of a Calabi–Yau 3-fold, 69
- $\bar{DT}^{\alpha}(\mu)$  generalized Donaldson–Thomas invariants for  $\mu$ -stability, 56
- $\bar{DT}_Q^d(\mu)$  Donaldson–Thomas type invariants for a quiver  $Q$ , 97
- $\hat{DT}_Q^d(\mu)$  BPS invariants for a quiver  $Q$ , 97
- $\bar{DT}_{Q,I}^d(\mu)$  Donaldson–Thomas type invariants for  $(Q, I)$ , 97
- $\hat{DT}_{Q,I}^d(\mu)$  BPS invariants for  $(Q, I)$ , 97
- $\mathfrak{E}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H}, \dots$  Artin  $\mathbb{K}$ -stacks
- $\mathrm{Eu}$  the ‘local Euler obstruction’, an isomorphism  $Z_*(X) \rightarrow \mathrm{CF}_{\mathbb{Z}}(X)$ , 31
- $\mathfrak{Exact}_{\mathcal{A}}$  moduli stack of short exact sequences in an abelian category  $\mathcal{A}$ , 22
- $F(G, T^G, Q)$  rational coefficients used in the definition of stack function spaces
- $\underline{\mathrm{SF}}, \bar{\mathrm{SF}}(\mathfrak{F}, \chi, \mathbb{Q})$ , 19
- $\mathfrak{F}(\mathbb{K})$  set of  $\mathbb{K}$ -points of an Artin  $\mathbb{K}$ -stack  $\mathfrak{F}$ , 13
- $F^{\alpha}(\tau)$  function in  $\mathrm{CF}(\mathcal{M}_{\mathrm{ss}}^{\alpha}(\tau))$  with  $\chi(\mathcal{M}_{\mathrm{ss}}^{\alpha}(\tau), F^{\alpha}(\tau)) = \hat{DT}^{\alpha}(\tau)$ , 71
- $\mathcal{G}$  gauge group of smooth gauge transformations of a vector bundle, 123
- $\mathcal{G}^{2,k+1}$  gauge group of  $L_{k+1}^2$  gauge transformations of a vector bundle, 123
- $\mathbb{G}_m$  the algebraic  $\mathbb{K}$ -group  $\mathbb{K} \setminus \{0\}$ , 17
- $GV_g(\alpha)$  Gopakumar–Vafa invariants, 76
- $GW_{g,m}(\alpha)$  Gromov–Witten invariants, 76
- $\mathrm{hd}(E)$  homological dimension of a coherent sheaf  $E$ , 118
- $H^{\mathrm{even}}(X; \mathbb{Q})$  even cohomology of a complex manifold  $X$ , 41
- $H_{\mathrm{cs}}^{\mathrm{even}}(X; \mathbb{Q})$  compactly-supported even cohomology of  $X$ , 83
- $\mathrm{Hilb}^d(X)$  Hilbert scheme of  $d$  points on  $X$ , 71
- $H^{p,q}(X)$  Dolbeault cohomology groups of a Kähler manifold  $X$
- $\mathbb{H}^*(\mathcal{M}; \mathcal{P})$  hypercohomology of a perverse sheaf  $\mathcal{P}$  on a scheme  $\mathcal{M}$ , 52
- $\mathrm{Iso}_{\mathfrak{F}}(x)$  stabilizer group of an Artin stack  $\mathfrak{F}$  at the point  $x$ , 13
- $J^{\alpha}(\tau)$  invariant counting  $\tau$ -semistable sheaves in class  $\alpha$  on a Calabi–Yau 3-fold, introduced in [54], 29
- $\tilde{J}_{\mathrm{si}}^b(I, \preceq, \kappa, \tau)$  extended Donaldson–Thomas invariants, 87
- $\mathbb{K}$  base field, usually algebraically closed

- $K(\mathcal{A})$  a quotient of the Grothendieck group  $K_0(\mathcal{A})$  of an abelian category  $\mathcal{A}$ . Often  $K(\mathcal{A}) = K^{\text{num}}(\mathcal{A})$ , 21  
 $K_0(\mathcal{A})$  Grothendieck group of an abelian category  $\mathcal{A}$ , 21  
 $K^{\text{num}}(\mathcal{A})$  numerical Grothendieck group of an abelian category  $\mathcal{A}$ , 21  
 $\mathbb{K}Q$  path algebra of a quiver, 89  
 $\mathbb{K}Q/I$  algebra of a quiver with relations  $(Q, I)$ , 89  
 $K_X$  canonical bundle of a smooth scheme  $X$ , 27  
 $k^1(\cdots)$  degree 1 part of graded object ‘ $\cdots$ ’, 160  
 $\tilde{L}(Q)$  Lie algebra depending on a quiver  $Q$ , 97  
 $L(X)$  Lie algebra depending on a Calabi–Yau 3-fold  $X$ , 27  
 $\tilde{L}(X)$  Lie algebra depending on a Calabi–Yau 3-fold  $X$ , variant of  $L(X)$ , 54  
 $\pi^!$  functor  $D(Y) \rightarrow D(X)$  mapping  $\mathcal{E} \mapsto Lf^*(\mathcal{E}) \otimes \omega_\pi[\dim \pi]$  for a smooth scheme morphism  $\pi : X \rightarrow Y$ . See Huybrechts [42, §3.3].  
 $L\pi^*$  left derived pullback functor  $D(Y) \rightarrow D(X)$  of a scheme morphism  $\pi : X \rightarrow Y$ . See Huybrechts [42, §3.3].  
 $\text{LCF}(\mathfrak{F})$   $\mathbb{Q}$ -vector space of locally constructible functions on a stack  $\mathfrak{F}$ , 13  
 $Li_2(t)$  dilogarithm function, 69  
 $\tilde{\mathcal{L}}_\tau^{\text{pa}}, \tilde{\mathcal{L}}_\tau^{\text{to}}$  Lie subalgebras of Ringel–Hall Lie algebra  $\text{SF}_{\text{al}}^{\text{ind}}(\mathfrak{M})$ , 87  
 $\underline{\text{LSF}}(\mathfrak{F})$  vector space of ‘local stack functions’ on an Artin stack  $\mathfrak{F}$ , 16  
 $\text{LSF}(\mathfrak{F})$  vector space of ‘local stack functions’ on an Artin stack  $\mathfrak{F}$ , defined using representable 1-morphisms, 16  
 $\mathfrak{M}_{\mathcal{A}}$  or  $\mathfrak{M}$  moduli stack of objects in an abelian category  $\mathcal{A}$ , 22  
 $\mathfrak{M}_{\mathcal{A}}^\alpha$  or  $\mathfrak{M}^\alpha$  moduli stack of objects in  $\mathcal{A}$  with class  $\alpha \in K(\mathcal{A})$ , 22  
 $\mathfrak{M}_{\text{ss}}^\alpha(\tau)$  moduli stack of  $\tau$ -semistable objects in class  $\alpha$ , 24  
 $\mathcal{M}_{\text{ss}}^\alpha(\tau)$  coarse moduli scheme of  $\tau$ -semistable objects in class  $\alpha$ , 24  
 $\mathfrak{M}_{\text{st}}^\alpha(\tau)$  moduli stack of  $\tau$ -stable objects in class  $\alpha$ , 24  
 $\mathcal{M}_{\text{st}}^\alpha(\tau)$  coarse moduli scheme of  $\tau$ -stable objects in class  $\alpha$ , 24  
 $\mathcal{M}_{\text{si}}$  coarse moduli space of simple coherent sheaves, 48  
 $\mathfrak{M}_Q$  moduli stack of representations of a quiver  $Q$ , 91  
 $\mathfrak{M}_{Q,I}$  moduli stack of representations of a quiver with relations, 91  
 $\mathfrak{M}_{\text{fr } Q}^{d,e}$  moduli stack of framed representations of a quiver  $Q$ , 99  
 $\mathfrak{M}_{\text{fr } Q,I}^{d,e}$  moduli stack of framed representations of  $(Q, I)$ , 99  
 $\mathcal{M}_{\text{stf } Q}^{d,e}(\mu')$  fine moduli scheme of stable framed representations of  $Q$ , 99  
 $\mathcal{M}_{\text{stf } Q,I}^{d,e}(\mu')$  fine moduli scheme of stable framed representations of  $(Q, I)$ , 100  
 $MF_f(x)$  Milnor fibre of a holomorphic function  $f$  at point  $x$ , 33  
 $\text{Mö}(m)$  Möbius function, 69  
 $\text{mod-}\mathbb{K}Q$  abelian category of representations of a quiver  $Q$ , 90  
 $\text{mod-}\mathbb{K}Q/I$  abelian category of representations of a quiver with relations, 90  
 $[\mathcal{M}]^{\text{vir}}$  virtual cycle of a proper moduli scheme  $\mathcal{M}$ , defined using an obstruction theory on  $\mathcal{M}$ , 38  
 $NDT_Q^{d,e}(\mu')$  noncommutative Donaldson–Thomas invariants for  $Q$ , 100  
 $NDT_{Q,I}^{d,e}(\mu')$  noncommutative Donaldson–Thomas invariants for  $(Q, I)$ , 100  
 $N_G(S)$  normalizer of a subset  $S$  in a group  $G$ , 17  
 $\text{nil-}\mathbb{K}Q/I$  abelian category of nilpotent representations of  $(Q, I)$ , 96  
 $\mathcal{O}_X$  structure sheaf of a scheme  $X$   
 $\mathcal{O}_X(1)$  very ample line bundle on a scheme  $X$ , 24  
 $\mathbb{P}(V)$  projective space of a vector space  $V$

- $\mathrm{Per}(X)$  abelian category of perverse sheaves on  $X$ , 34  
 $P_{(I, \preceq)}$  multilinear operation on  $\mathrm{SF}_{\mathrm{al}}(\mathfrak{M})$  depending on a poset  $(I, \preceq)$ , 87  
 $PI^{\alpha, n}(\tau')$  invariants counting ‘stable pairs’  $s : \mathcal{O}_X(-n) \rightarrow E$ , 59  
 $P_m$  1-morphism  $\mathfrak{M} \rightarrow \mathfrak{M}$  taking  $E \mapsto mE = E \oplus \cdots \oplus E$ , 70  
 $\mathrm{proj}\text{-}\mathbb{K}Q$  exact category of projective representations of a quiver  $Q$ , 90  
 $\mathrm{proj}\text{-}\mathbb{K}Q/I$  exact category of projective representations of  $(Q, I)$ , 90  
 $PT_{n, \beta}$  Pandharipande–Thomas invariants, 77  
 $\mathcal{Q}(G, T^G)$  a set of subtori of the maximal torus  $T^G$  of a  $\mathbb{K}$ -group  $G$ , 19  
 $(Q, I)$  quiver with relations, 89  
 $R\pi_*$  right derived pushforward functor  $D(X) \rightarrow D(Y)$  of a scheme morphism  $\pi : X \rightarrow Y$ . See Huybrechts [42, §3.3].  
 $R\mathcal{H}om$  derived sheaf Hom functor. If  $\mathcal{E}, \mathcal{F} \in D(X)$  then  $R\mathcal{H}om(\mathcal{E}, \mathcal{F}) \in D(X)$ . See Huybrechts [42, §3.3].  
 $S(\alpha_1, \dots, \alpha_n; \tau, \tilde{\tau})$  combinatorial coefficient used in wall-crossing formulae, 26  
 $\mathrm{SF}(\mathfrak{F})$  vector space of ‘stack functions’ on an Artin stack  $\mathfrak{F}$ , 15  
 $\mathrm{SF}(\mathfrak{F})$  vector space of ‘stack functions’ on an Artin stack  $\mathfrak{F}$ , defined using representable 1-morphisms, 15  
 $\mathrm{SF}(\mathfrak{F}, \chi, \mathbb{Q})$  vector space of ‘stack functions’ on an Artin stack  $\mathfrak{F}$  with extra relations involving the Euler characteristic, 19  
 $\mathrm{SF}(\mathfrak{F}, \chi, \mathbb{Q})$  vector space of ‘stack functions’ on an Artin stack  $\mathfrak{F}$  with extra relations, defined using representable 1-morphisms, 19  
 $\mathrm{td}(TX)$  Todd class of  $TX$  in  $H^{\mathrm{even}}(X, \mathbb{Q})$ , 41  
 $T^G$  maximal torus in an algebraic  $\mathbb{K}$ -group  $G$ , 17  
 $U(\alpha_1, \dots, \alpha_n; \tau, \tilde{\tau})$  combinatorial coefficient used in wall-crossing formulae, 26  
 $U(L(X))$  universal enveloping algebra of Lie algebra  $L(X)$ , 29  
 $\mathfrak{Vect}$  moduli stack of algebraic vector bundles, 48  
 $\mathcal{Vect}_{\mathrm{si}}$  coarse moduli space of simple algebraic vector bundles, 48  
 $V(I, \Gamma, \kappa; \tau, \tilde{\tau})$  combinatorial coefficient used in wall-crossing formulae, 30  
 $W(G, T^G)$  Weyl group of algebraic  $\mathbb{K}$ -group  $G$ , 17  
 $Z_*(X)$  group of algebraic cycles on a scheme  $X$ , 31

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