

A GEOMETRIC PERSPECTIVE ON THE METHOD OF DESCENT

QIAN WANG

ABSTRACT. We derive a first order representation formula for the tensorial wave equation $\square_{\mathbf{g}}\phi^I = F^I$ in globally hyperbolic Lorentzian spacetimes $(\mathcal{M}^{2+1}, \mathbf{g})$ by giving a geometric formulation of the method of descent which is applicable for any dimension.

1. Introduction

We consider the wave equation on $(d+1)$ globally hyperbolic, smooth Lorentzian spacetimes $(\mathcal{M}, \mathbf{g})$. In the case that $(\mathcal{M}, \mathbf{g})$ is the Minkowski spacetime $(\mathbb{R}^{d+1}, \mathbf{m})$, a representation formula for the solutions of wave equation can be obtained by the classical theory ([7]). In fact, when d is odd, one can derive the formula by reducing the problem to a wave equation in $(\mathbb{R}^{1+1}, \mathbf{m})$ via the spherical mean and then apply the d'Alembert's formula; when d is even, one can derive the formula by considering a wave equation in $(\mathbb{R}^{(d+1)+1}, \mathbf{m})$ and then apply the method of descent due to Hadamard. In particular, for $d = 3$ and $d = 2$, the corresponding formulae are called the Kirchhoff formula and the Poisson formula respectively.

When $(\mathcal{M}, \mathbf{g})$ is a globally hyperbolic $(3+1)$ Lorentzian spacetime, a geometric Kirchhoff formula is provided in [11] for the tensorial wave equations $\square_{\mathbf{g}}\phi^I = F^I$. In this paper, we give the geometric formulation of the method of descent in $(2+1)$ Lorentzian spacetimes. By using this formulation, we obtain a first order, intrinsic, representation formula in physical space for the solutions of tensorial wave equations in $(2+1)$ spacetimes. Our construction is purely geometric, which potentially can be used in quasi-linear problems, such as $(2+1)$ Einstein gravity, when the geometric quantities appeared in the formula have better structures due to the curvature properties of the background geometry. One can see such topic in [16] and [3].

There are various types of parametrix for wave equations in the curved spacetime. When establishing the Strichartz estimates or the bilinear estimates for solving well-posedness problem with large rough data, one may use the Fourier type parametrix, see [13] and [21] for examples. The Kirchhoff parametrix constructed in [11] is used in [14] and [23] to provide geometric breakdown criteria for the solutions of $(3+1)$ Einstein vacuum equations with large data. The application can be traced back to the work of [6], where the authors prove the global existence result for Yang-Mills-Higgs equations. A crucial step of the proof is to use the representation formula for the wave equation in Minkowski space to represent and control the curvature. This strategy is later used in [5] to prove the same result in globally hyperbolic spacetimes, where they employed the Hadamard parametrix in [8] in the curved spacetimes.

There is a significant difference between the Hadamard parametrix for the solutions of linear tensorial wave equations in [8] with the first order formulas such as in [11] and the one presented in this paper. The former expresses the solution to the associated Cauchy problem at any given point p in the spacetime purely in terms of the Cauchy data specified on the intersection of the causal past of p with the Cauchy hypersurface. The formula relies merely on the data and the inhomogeneous term. The construction requires the causal geodesic convexity in the causal past of p . In $(3+1)$ spacetime, the parametrix, which is an integral within the domain of dependence, is not purely supported on the boundary of the causal past of the point unless the spacetime is flat. This is due to the tail term, which is defined by a series of corrections by using transport equations in the causal past of p . Even for linear problems, it is difficult to obtain estimates on this implicitly determined tail term.

In many situations, it is unnecessary to construct a precise representation of solution. First order parametrices based on the flat Kirchhoff formula are used to show the well-posedness for general second order linear wave equations with variable coefficients in [19] and the local-well-posedness for Einstein equation in [1]. Such representations provide integral equations satisfied by the corresponding solutions instead of expressions for the solutions themselves. Their importance is that they can be used to derive estimates satisfied by the solutions of curved space wave equations. It is particularly useful for nonlinear equations for which an actual, explicit solution to the Cauchy problem would not be attainable. This idea was emphasized and developed in [15], [11], [14], [16] and [17].

In particular, the Kirchhoff formula established in [11] is supported only on the null boundary of the causal past,¹ i.e. the backward lightcone of the point p , which coincides with the Huygens principle. The quantities involved in the formula are purely the geometric quantities on the null boundary, which can be controlled by a standard method, see [4]. The null boundary is actually more regular than the full causal past. For the $(3+1)$ Einstein spacetime, instead of requiring pointwise control on curvature, the null boundary can be controlled in terms of the Bel-Robinson energy flux on the null boundary by using a series of sharp trace estimates, see [10] and [22]. This advantage is very crucial for the applications in [14] and [23].

In a $(2+1)$ curved spacetime, there is no such first order formula available. To implement the method of descent based on the Kirchhoff formula in a $(3+1)$ Lorentzian spacetime, we need to establish the geometric correspondence between the geometry of the lightcone of a point in the $(3+1)$ spacetime with the causal past of the same point in the $(2+1)$ spacetime.

With ρ the Lorentzian distance inside the backward lightcone in $(d+1)$ spacetime, we observe that the vector field $-\mathbf{D}\rho$ in \mathcal{TM} corresponds to the null geodesic generator \tilde{L} in the corresponding backward light cone in $((d+1)+1)$ spacetime. This allows us to express the geometric quantities and the null frames on the light cone in $((d+1)+1)$ spacetime in terms of the hyperboloidal frames in $(d+1)$ spacetime. These quantities include the null expansion of the light rays in $((d+1)+1)$ spacetime and the area expansion of the timelike geodesic congruence in $(d+1)$ and other connection coefficients of these frames. Based on this observation, we can uncover the relation between the radius of injectivity of the corresponding

¹See in [15] a parametrix, which is modified from the conventional Hadamard/Frieland formula, has the same feature.

geodesic congruences in two spacetimes of different dimensions. This relation is in particular important since the Kirchhoff formula in [11] holds within the null radius of injectivity.

In this paper, we focus on the case that $d = 2$, while our method applies to higher dimensions. As long as the analogous representation formulae of [11] in other odd dimensions are available, we can similarly obtain the formulae in the even dimensions. This paper is organized as follows. In Section 2.1, we give the geometric set-up and the main theorem of the paper. In Section 2.2, we give the relation between the geometry of the null cones with the vertex p in $(3+1)$ spacetime and the causal past of the same vertex in $(2+1)$ spacetime. By uncovering the quantitative correspondence between connection coefficients of the null frames in $(3+1)$ spacetime and those of the triads in $(2+1)$ spacetime, we control the null radius of injectivity in terms of the causal radius of injectivity in $(2+1)$ spacetime. In Section 2.3 we then complete the proof of the main result. This result, in the flat case, coincides with the Poisson formula in Minkowski space. In Appendix, we give a proof of the Kirchhoff formula used in Section 2.

Acknowledgments. The author sincerely thanks the anonymous referee, who provided a detailed review and comparison on various physical space parametrices and recommended very useful references. The author is also grateful to the enlightening comment by the referee on the potential application of the main result of this paper.

2. A geometric method of descent

Let $(\mathcal{M}, \mathbf{g})$ be a $(2+1)$ -globally hyperbolic smooth Lorentzian spacetime. We assume that \mathcal{M} is foliated by a time function t and the metric \mathbf{g} takes the form²

$$\mathbf{g} = \mathbf{g}_{\alpha\beta} dx^\alpha dx^\beta = -n^2 dt^2 + g_{ij} dx^i dx^j,$$

where n is the lapse function and $g = g_{ij} dx^i dx^j$ are Riemannian metrics on Σ_t , the level sets of the time function t .

Let \mathbf{D} denote the covariant differentiation on $(\mathcal{M}, \mathbf{g})$ and let $\square_{\mathbf{g}} := \mathbf{g}^{\alpha\beta} \mathbf{D}_\alpha \mathbf{D}_\beta$ denote the induced d'Alembertian. Consider the tensorial wave equation $\square_{\mathbf{g}} \phi_I = F_I$ in $(\mathcal{M}, \mathbf{g})$. In this paper we will develop a geometric formulation of the method of descent to derive a representation formula for ϕ_I which can be viewed as an extension of the classical Poisson formula for the scalar wave equation in the Minkowski spacetime $(\mathbb{R}^{2+1}, \mathbf{m})$.

2.1. Set-up and main result. Let \mathbf{T} be the future directed time-like unit normal of Σ_t . Any point in $(\mathcal{M}, \mathbf{g})$ can be written as (t, x) where $x \in \Sigma_t$. Given $p \in \mathcal{M}$, we denote by $\mathcal{I}^-(p)$, $\mathcal{J}^-(p)$ and $\mathcal{N}^-(p)$ the chronological past, the causal past and the backward light-cone in $(\mathcal{M}, \mathbf{g})$ initiating from p . Note that $\mathcal{N}^-(p)$ is a surface ruled by the backward null geodesics from p . In the sequel, by Σ_t we mean $\Sigma_t \cap \mathcal{I}^-(p)$.

For a fixed point $p \in \mathcal{M}$, we consider

$$\mathbb{H}^2 := \{V \in \mathcal{T}_p \mathcal{M} : \|V\|_{\mathbf{g}(p)} = -1, V^0 = \mathbf{g}(V, \mathbf{T}) > 0\}.$$

²Throughout the paper we use the Einstein summation convention. We set $x^0 = t$. A little Greek letter is used to denote an index from $\{0, 1, 2\}$ and a little Latin letter is used to denote an index from $\{1, 2\}$, e.g. $\alpha = 0, 1, 2$ and $i = 1, 2$.

Relative to a geodesic normal coordinate at p , we can regard

$$\mathbb{H}^2 = \left\{ V = (V^0, V^1, V^2) : (V^0)^2 - \sum_{i=1}^2 (V^i)^2 = 1 \text{ and } V^0 > 0 \right\}$$

which is the canonical hyperboloid in \mathbb{R}^{2+1} . For each $V \in \mathbb{H}^2$ let $\Upsilon_V(\rho)$ be the time-like geodesic with $\Upsilon_V(0) = p$ and $\Upsilon'_V(0) = V$, and let $\rho(t)$ denote the Lorentzian distance from p to the intersection point of $\Upsilon_V(\rho)$ with Σ_t . Note that $\rho(t)$ is a function not only depending on t but also on V ; we suppress V for simplicity. We then define the past time-like radius of injectivity δ_* at p in $(\mathcal{M}, \mathbf{g})$ to be the supremum over all the values $\tau > 0$ for which the exponential map

$$\exp_p : (t, V) \rightarrow \Upsilon_V(\rho(t)) \quad (2.1)$$

is a global diffeomorphism from $(t_p - \tau, t_p) \times \mathbb{H}^2$ to its image in $\mathcal{I}^-(p)$. In this paper, we only consider the part of $\mathcal{I}^-(p)$ within the time-like radius of injectivity, which will be still denoted as $\mathcal{I}^-(p)$ by abuse of notation.

For $(t, x) \in \mathcal{J}^-(p)$ let $\rho(t, x)$ be the Lorentzian distance to p in $\mathcal{J}^-(p)$. Clearly, $\rho(t, x) = 0$ iff $(t, x) \in \mathcal{N}^-(p)$. Moreover, within $\mathcal{J}^-(p)$ with $0 < t_p - t < \delta_*$ this function is smooth and verifies

$$\mathbf{g}^{\alpha\beta} \partial_\alpha \rho \partial_\beta \rho = -1, \quad \rho(p) = 0. \quad (2.2)$$

In $\mathcal{I}^-(p) \subset \mathcal{M}$ we define the vector field \mathfrak{B} by

$$\mathfrak{B} := -\mathbf{D}\rho = -\mathbf{g}^{\alpha\beta} \partial_\alpha \rho \partial_\beta.$$

Then \mathfrak{B} is geodesic, i.e. $\mathbf{D}_{\mathfrak{B}}\mathfrak{B} = 0$ and satisfies $\mathbf{g}(\mathfrak{B}, \mathfrak{B}) = -1$. Moreover,

$$\mathfrak{B} = (d\exp_p)_{\rho V}(\partial_\rho). \quad (2.3)$$

Let H_ρ denote the level sets of ρ . Then \mathfrak{B} is the past directed unit normal of H_ρ and is the generator of the timelike geodesic $\Upsilon_V(\rho)$.

We define the frame lapse \mathbf{b} by

$$\mathbf{g}(\mathfrak{B}, \mathbf{T}) = \mathbf{b}^{-1} \frac{t_p - t}{\rho}. \quad (2.4)$$

Let $\tau := t_p - t$. Then by noting that $\mathbf{T} = n^{-1} \partial_t$, we have from (2.4) that

$$\mathfrak{B}(\tau) = n^{-1} \mathbf{b}^{-1} \frac{\tau}{\rho}. \quad (2.5)$$

Let g be the induced metric on Σ_t and let ∇ be the corresponding covariant derivative on Σ_t . We consider the lapse function $a^{-1} := |\nabla \rho|_g$. By using (2.4) we have

$$-\mathbf{T}(\rho) = \frac{\mathbf{b}^{-1} \tau}{\rho}. \quad (2.6)$$

This together with (2.2) then implies that

$$-1 = \mathbf{g}^{\alpha\beta} \partial_\alpha \rho \partial_\beta \rho = -(\mathbf{T}(\rho))^2 + g^{ij} \partial_i \rho \partial_j \rho = -\frac{\mathbf{b}^{-2} \tau^2}{\rho^2} + |\nabla \rho|_g^2.$$

Hence the lapse a can be written as

$$a^{-2} = |\nabla \rho|_g^2 = \frac{\mathbf{b}^{-2} \tau^2}{\rho^2} - 1,$$

which also implies $\mathbf{b}^{-1}\tau \geq \rho$ in $\mathcal{I}^-(p)$. By setting $\tilde{r} = \sqrt{\mathbf{b}^{-2}\tau^2 - \rho^2}$, we have

$$a^{-1} = \frac{\tilde{r}}{\rho}. \quad (2.7)$$

Let $S_{t,\rho} := H_\rho \cap \Sigma_t$. Then for each fixed t , $\{S_{t,\rho}\}_\rho$ is a family of 1-dimensional curves diffeomorphic to circles and forms the radial foliation of Σ_t . Let \mathbf{N} be the radial normal of $S_{t,\rho}$ in Σ_t . Then

$$\mathbf{N} = -\frac{\nabla\rho}{|\nabla\rho|_g} = -a\nabla\rho. \quad (2.8)$$

In view of (2.6) and (2.8), we can decompose \mathfrak{B} in terms of \mathbf{T} and \mathbf{N} as

$$\mathfrak{B} = -\frac{\mathbf{b}^{-1}\tau}{\rho}\mathbf{T} + a^{-1}\mathbf{N}. \quad (2.9)$$

We set

$$\underline{\mathfrak{B}} = -\frac{\mathbf{b}^{-1}\tau}{\rho}\mathbf{T} - a^{-1}\mathbf{N}. \quad (2.10)$$

Clearly

$$\mathbf{g}(\mathfrak{B}, \mathfrak{B}) = \mathbf{g}(\underline{\mathfrak{B}}, \underline{\mathfrak{B}}) = -1. \quad (2.11)$$

Let \underline{g} be the induced metric of \mathbf{g} on H_ρ and $\underline{\nabla}$ be the Levi-civita connection of \underline{g} . By introducing the projection tensor

$$\check{\Pi}_{\alpha\beta} = \mathbf{g}_{\alpha\beta} + \mathfrak{B}_\alpha \mathfrak{B}_\beta,$$

we have

$$\underline{\nabla}^\alpha = \check{\Pi}_{\beta\gamma} \mathbf{g}^{\alpha\beta} \mathbf{D}^\gamma \quad \text{and} \quad |\underline{\nabla}\tau|_{\underline{g}} = (an)^{-1}.$$

Let $\underline{\mathcal{N}}$ be the radial normal of $\{S_{\tau,\rho}\}_\tau \subset H_\rho$. Then we have

$$\underline{\mathcal{N}} = \frac{\underline{\nabla}\tau}{|\underline{\nabla}\tau|_{\underline{g}}} = an\underline{\nabla}\tau.$$

Similar to [25, Page 13], $\underline{\mathcal{N}}$ can be decomposed as

$$\underline{\mathcal{N}} = -\frac{\tilde{r}}{\rho}\mathbf{T} + \frac{\mathbf{b}^{-1}\tau}{\rho}\mathbf{N}. \quad (2.12)$$

We will use $\check{\nabla}$ to denote the Levi-civita connection of the induced metric on $S_{\tau,\rho}$ and use $e_{\mathcal{A}}$ to denote a unit tangent vector field on $S_{\tau,\rho}$.

Definition 2.1. (1) We denote by $\boldsymbol{\pi}$ the second fundamental form of $(\Sigma_t, g) \subset (\mathcal{M}, \mathbf{g})$, i.e.

$$\boldsymbol{\pi}(X, Y) = -\mathbf{g}(\mathbf{D}_X \mathbf{T}, Y)$$

for $X, Y \in \mathcal{T}\Sigma_t$. The trace of $\boldsymbol{\pi}$ is $\text{Tr}\boldsymbol{\pi} = g^{ij}\pi_{ij}$.

(2) We denote by k the second fundamental form of $H_\rho \subset (\mathcal{M}, \mathbf{g})$, i.e.

$$k(X, Y) = \mathbf{g}(\mathbf{D}_X \mathfrak{B}, Y) \quad (2.13)$$

for $X, Y \in \mathcal{T}H_\rho$ in $(\mathcal{M}, \mathbf{g})$. We denote the trace and traceless part of k by $\text{tr}k$ and \hat{k} respectively. Note that in Minkowski space $(\mathbb{R}^{2+1}, \mathbf{m})$ we have $\text{tr}k = \frac{2}{\rho}$.

(3) We introduce the connection coefficients

$$\omega := -\mathfrak{B}\left(\frac{\rho}{\mathbf{b}^{-1}\tau}\right) \quad \text{and} \quad \zeta_{\mathcal{A}} := \langle \mathbf{D}_{\mathfrak{B}} \underline{\mathcal{N}}, e_{\mathcal{A}} \rangle.$$

We first give some preliminary results on the geometric quantities defined above.

Lemma 2.2. *For the frame lapse \mathbf{b} and a and the connection coefficients ω and $\underline{\zeta}$, there hold*

$$\underline{\zeta}_{\mathcal{A}} = \frac{\rho}{\tilde{r}} \langle \mathbf{D}_{\mathfrak{B}} \mathbf{T}, e_{\mathcal{A}} \rangle, \quad (2.14)$$

$$\omega = \frac{\rho \tilde{r}}{\mathbf{b}^{-2} \tau^2} \langle \mathbf{D}_{\mathfrak{B}} \mathbf{T}, \mathbf{N} \rangle, \quad (2.15)$$

$$\mathfrak{B}(\mathbf{b}^{-1}) = \frac{\mathbf{b}^{-1}}{\rho} (1 - \mathbf{b}^{-1} n^{-1}) + \frac{\mathbf{b}^{-2} \tau}{\rho} \omega, \quad (2.16)$$

$$\nabla_{\mathcal{A}} \log a = \frac{\mathbf{b}^{-1} \tau}{\tilde{r}} (\pi_{\mathbf{N}\mathcal{A}} - k_{\mathcal{A}\mathcal{N}}), \quad (2.17)$$

$$\langle \mathbf{D}_{\mathbf{T}} \mathbf{N}, e_{\mathcal{A}} \rangle = k_{\mathcal{N}\mathcal{A}} + \frac{\mathbf{b}^{-1} \tau}{\tilde{r}} \langle \mathbf{D}_{\mathbf{T}} \mathbf{T}, e_{\mathcal{A}} \rangle. \quad (2.18)$$

Proof. We first derive (2.14). From (2.9) and (2.12) it follows that

$$\mathbf{T} = -\frac{\mathbf{b}^{-1} \tau}{\rho} \mathfrak{B} + \frac{\tilde{r}}{\rho} \mathcal{N} \quad (2.19)$$

and hence $\mathcal{N} = \frac{\rho}{\tilde{r}} (\mathbf{T} + \frac{\mathbf{b}^{-1} \tau}{\rho} \mathfrak{B})$. Consequently, by using $\mathbf{D}_{\mathfrak{B}} \mathfrak{B} = 0$ we have

$$\underline{\zeta}_{\mathcal{A}} = \langle \mathbf{D}_{\mathfrak{B}} \mathcal{N}, e_{\mathcal{A}} \rangle = \frac{\rho}{\tilde{r}} \langle \mathbf{D}_{\mathfrak{B}} \mathbf{T}, e_{\mathcal{A}} \rangle + \frac{\mathbf{b}^{-1} \tau}{\tilde{r}} \langle \mathbf{D}_{\mathfrak{B}} \mathfrak{B}, e_{\mathcal{A}} \rangle = \frac{\rho}{\tilde{r}} \langle \mathbf{D}_{\mathfrak{B}} \mathbf{T}, e_{\mathcal{A}} \rangle. \quad (2.20)$$

To see (2.15), we use (2.4), (2.9) and $\mathbf{D}_{\mathfrak{B}} \mathfrak{B} = 0$ to obtain

$$\begin{aligned} \omega &= \frac{\rho^2}{\mathbf{b}^{-2} \tau^2} \mathfrak{B} \left(\frac{\mathbf{b}^{-1} \tau}{\rho} \right) = \frac{\rho^2}{\mathbf{b}^{-2} \tau^2} \mathfrak{B}(\langle \mathfrak{B}, \mathbf{T} \rangle) = \frac{\rho^2}{\mathbf{b}^{-2} \tau^2} \langle \mathfrak{B}, \mathbf{D}_{\mathfrak{B}} \mathbf{T} \rangle \\ &= \frac{\rho^2}{\mathbf{b}^{-2} \tau^2} a^{-1} \mathbf{g}(\mathbf{D}_{\mathfrak{B}} \mathbf{T}, \mathbf{N}) = \frac{\rho \tilde{r}}{\mathbf{b}^{-2} \tau^2} \mathbf{g}(\mathbf{D}_{\mathfrak{B}} \mathbf{T}, \mathbf{N}). \end{aligned}$$

To obtain (2.16), we use $\mathbf{b}^{-1} = \frac{\rho}{\tau} \langle \mathfrak{B}, \mathbf{T} \rangle$. By using (2.5) we have

$$\mathfrak{B} \left(\frac{\rho}{\tau} \right) = \frac{1}{\tau} - \frac{\rho}{\tau^2} \mathfrak{B}(\tau) = \frac{1}{\tau} (1 - n^{-1} \mathbf{b}^{-1}).$$

Therefore, in view of (2.4), $\mathbf{D}_{\mathfrak{B}} \mathfrak{B} = 0$ and (2.9), it follows that

$$\begin{aligned} \mathfrak{B}(\mathbf{b}^{-1}) &= \mathfrak{B} \left(\frac{\rho}{\tau} \langle \mathfrak{B}, \mathbf{T} \rangle \right) = \mathfrak{B} \left(\frac{\rho}{\tau} \right) \langle \mathfrak{B}, \mathbf{T} \rangle + \frac{\rho}{\tau} \langle \mathfrak{B}, \mathbf{D}_{\mathfrak{B}} \mathbf{T} \rangle \\ &= \frac{\mathbf{b}^{-1}}{\rho} (1 - \mathbf{b}^{-1} n^{-1}) + \frac{\tilde{r}}{\tau} \langle \mathbf{N}, \mathbf{D}_{\mathfrak{B}} \mathbf{T} \rangle. \end{aligned}$$

In view of (2.15), we therefore obtain (2.16).

To obtain (2.17), we first use $\tilde{r}^2 = \mathbf{b}^{-2} \tau^2 - \rho^2$ to derive that $\nabla_{\mathcal{A}} \tilde{r} = \frac{\mathbf{b}^{-1} \tau}{\tilde{r}} \nabla_{\mathcal{A}} (\mathbf{b}^{-1} \tau)$. Thus, in view of (2.4) we have

$$\begin{aligned} \nabla_{\mathcal{A}} \log \tilde{r} &= \frac{\mathbf{b}^{-1} \tau}{\tilde{r}^2} \nabla_{\mathcal{A}} (\mathbf{b}^{-1} \tau) = \frac{\mathbf{b}^{-1} \tau \rho}{\tilde{r}^2} \nabla_{\mathcal{A}} \left(\frac{\mathbf{b}^{-1} \tau}{\rho} \right) = \frac{\mathbf{b}^{-1} \tau \rho}{\tilde{r}^2} \nabla_{\mathcal{A}} \langle \mathfrak{B}, \mathbf{T} \rangle \\ &= \frac{\mathbf{b}^{-1} \tau \rho}{\tilde{r}^2} (\langle \mathbf{D}_{\mathcal{A}} \mathfrak{B}, \mathbf{T} \rangle + \langle \mathfrak{B}, \mathbf{D}_{\mathcal{A}} \mathbf{T} \rangle). \end{aligned}$$

By using (2.19) and (2.9) we can further obtain

$$\nabla_{\mathcal{A}} \log \tilde{r} = \frac{\mathbf{b}^{-1} \tau}{\tilde{r}} (\langle \mathbf{D}_{\mathcal{A}} \mathfrak{B}, \mathcal{N} \rangle + \langle \mathbf{N}, \mathbf{D}_{\mathcal{A}} \mathbf{T} \rangle) = \frac{\mathbf{b}^{-1} \tau}{\tilde{r}} (k_{\mathcal{A}\mathcal{N}} - \pi_{\mathcal{A}\mathbf{N}}).$$

(2.17) then follows by using (2.7) and the above identity.

Finally, we prove (2.18). From (2.9) we have $\mathbf{N} = \frac{\rho}{\tilde{r}}\mathfrak{B} + \frac{\mathbf{b}^{-1}\tau}{\tilde{r}}\mathbf{T}$. Thus, by using $\langle \mathfrak{B}, e_{\mathcal{A}} \rangle = \langle \mathbf{T}, e_{\mathcal{A}} \rangle = 0$ we have

$$\langle \mathbf{D}_{\mathbf{T}}\mathbf{N}, e_{\mathcal{A}} \rangle = \frac{\rho}{\tilde{r}}\langle \mathbf{D}_{\mathbf{T}}\mathfrak{B}, e_{\mathcal{A}} \rangle + \frac{\mathbf{b}^{-1}\tau}{\tilde{r}}\langle \mathbf{D}_{\mathbf{T}}\mathbf{T}, e_{\mathcal{A}} \rangle.$$

By using (2.19) and $\mathbf{D}_{\mathfrak{B}}\mathfrak{B} = 0$ we obtain

$$\langle \mathbf{D}_{\mathbf{T}}\mathbf{N}, e_{\mathcal{A}} \rangle = \langle \mathbf{D}_{\underline{\mathcal{N}}}\mathfrak{B}, e_{\mathcal{A}} \rangle + \frac{\mathbf{b}^{-1}\tau}{\tilde{r}}\langle \mathbf{D}_{\mathbf{T}}\mathbf{T}, e_{\mathcal{A}} \rangle = k_{\underline{\mathcal{N}}\mathcal{A}} + \frac{\mathbf{b}^{-1}\tau}{\tilde{r}}\langle \mathbf{D}_{\mathbf{T}}\mathbf{T}, e_{\mathcal{A}} \rangle.$$

The proof is therefore complete. \square

Now we are ready to state the main result of this paper.

Theorem 2.3 (Main theorem). *Consider a tensorial wave equation*

$$\square_{\mathbf{g}}\phi_I = F_I \quad (2.21)$$

on $(\mathcal{M}, \mathbf{g})$. Let p be any point in $(\mathcal{M}, \mathbf{g})$ and t_0 verify $0 < t_p - t_0 < c_*(p, t)$.³ Denote by $\mathcal{I}_*^-(p)$ the interior of the backward lightcone from p with $t \in [t_0, t_p]$. Given a tensor J at p of the same type as ϕ_I , let A_I be a tensor field on $\mathcal{I}_*^-(p)$ satisfying

$$\mathbf{D}_{\mathfrak{B}}A_I + \left(\frac{1}{2}\text{tr}k + \frac{n^{-1}\mathbf{b}^{-1} - 1}{\rho} \right) A_I = 0, \quad \lim_{t \rightarrow t_p} \tau A_I = J. \quad (2.22)$$

Then there holds

$$2\pi(n\mathbf{g}(\phi, J))(p) = - \int_{\mathcal{I}_*^-(p)} F_I A^I \frac{\tau}{\rho} n d\mu_{\Sigma_t} dt + \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3, \quad (2.23)$$

with

$$\begin{aligned} \mathcal{I}_1 &= \int_{\Sigma_{t_0} \cap \mathcal{I}_*^-(p)} \left[-\mathbf{D}_{\mathfrak{B}}\phi_I + \frac{1}{2}\phi_I \left(\text{tr}k - 2\frac{\tau}{\mathbf{b}\rho}(\text{Tr}\boldsymbol{\pi} - \frac{\mathbf{b}^2\tilde{r}^2}{\tau^2}\boldsymbol{\pi}_{\mathbf{N}\mathbf{N}}) \right) \right] A^I d\mu_{\Sigma_{t_0}}, \\ \mathcal{I}_2 &= -2 \int_{\mathcal{I}_*^-(p)} \left[\frac{\tilde{r}}{\rho}\zeta^{\mathcal{A}}\mathbf{D}_{\mathcal{A}}\phi_I + \frac{\rho}{\tilde{r}}\omega\mathbf{D}_{\mathbf{N}}\phi_I \right] A^I \mathbf{b}n d\mu_{\Sigma_t} dt \\ &\quad - \int_{\mathcal{I}_*^-(p)} \left[\nabla(\mathbf{b}A^I)\nabla\phi_I + \frac{\rho^2}{\mathbf{b}^{-2}\tau^2}\mathbf{D}_{\mathbf{N}}(\mathbf{b}A^I)\mathbf{D}_{\mathbf{N}}\phi_I \right] \frac{\mathbf{b}^{-1}\tau n}{\rho} d\mu_{\Sigma_t} dt, \\ \mathcal{I}_3 &= \int_{\mathcal{I}_*^-(p)} \left[\frac{\tilde{r}}{\rho}(\mathbf{R} * \phi)_I + \omega\text{tr}k\phi_I + \frac{1}{2}\frac{\rho}{\mathbf{b}^{-1}\tau}(\mathbf{R}_{\mathfrak{B}\mathfrak{B}} + |\hat{k}|^2)\phi_I \right] A^I \mathbf{b}n d\mu_{\Sigma_t} dt \\ &\quad + \int_{\mathcal{I}_*^-(p)} \left(\mathfrak{B} + \frac{\text{tr}k}{2} - \frac{\mathbf{b}^{-1}\tau}{\rho}\omega \right) \left(\text{Tr}\boldsymbol{\pi} - \frac{\tilde{r}^2}{\mathbf{b}^{-2}\tau^2}\boldsymbol{\pi}_{\mathbf{N}\mathbf{N}} \right) \phi_I A^I \mathbf{b}n d\mu_{\Sigma_t} dt, \end{aligned}$$

where, for $I = \{\mu_1, \dots, \mu_l\}$,

$$(\mathbf{R} * \phi)_I = \sum_{i=1}^l \mathbf{R}_{\mu_i}^{\alpha} \mathbf{T}\mathbf{N}\phi_{\mu_1 \dots \mu_{i-1} \alpha \mu_{i+1} \dots \mu_l}.$$

As a simple application, we will use the representation formula in Theorem 2.3 to recover the Poisson formula for the scalar linear wave equation $\square_{\mathbf{m}}\phi = F$ in the (2+1)-Minkowski space-time $(\mathbb{R}^{2+1}, \mathbf{m})$, with Cauchy data given at $t = 0$. Let p be a point in $(\mathbb{R}^{2+1}, \mathbf{m})$ with coordinates (x_p, t_p) and $t_p > 0$. Note that $n = 1$,

³The definition of the causal radius of injectivity $c_*(p, t)$ is given in Theorem 2.9.

$\mathcal{I}_*^-(p) = \{(x, t) \in \mathbb{R}^{2+1} : 0 \leq t < t_p - |x - x_p|\}$ and $\rho(x, t) = \sqrt{\tau^2 - r^2}$ in $\mathcal{I}_*^-(p)$ with $\tau = t_p - t$ and $r = |x - x_p|$. We can derive that

$$\begin{aligned} \mathfrak{B} &= -\frac{\tau}{\rho}\partial_t + \frac{r}{\rho}\partial_r, \quad \underline{\mathfrak{B}} = -\frac{\tau}{\rho}\partial_t - \frac{r}{\rho}\partial_r, \quad \mathbf{b} = 1, \quad \tilde{r} = r, \quad \omega = 0, \\ \boldsymbol{\pi} &= 0, \quad \underline{\zeta} = 0, \quad \mathbf{R} = 0, \quad \text{tr}k(x, t) = \frac{2}{\rho(x, t)}, \quad \hat{k} = 0 \end{aligned}$$

For $J = 1$ we can see that $A = \tau^{-1}$. Consequently $\mathcal{I}_2 = \mathcal{I}_3 = 0$ and it follows from Theorem 2.3 that

$$\begin{aligned} 2\pi\phi(p) &= -\int_0^{t_p} \int_{|x-x_p| < t_p-t} \frac{F(x, t)}{\sqrt{(t_p-t)^2 - |x-x_p|^2}} dx dt \\ &\quad + \frac{1}{t_p} \int_{|x-x_p| < t_p} \frac{t_p \partial_t \phi(x, 0) + r \partial_r \phi(x, 0) + \phi(x, 0)}{\sqrt{t_p^2 - |x-x_p|^2}} dx. \end{aligned}$$

Hence in the Minkowski space-time $(\mathbb{R}^{2+1}, \mathbf{m})$, Theorem 2.3 gives the classical Poisson formula.

2.2. A Kirchhoff formula in 3-dimensional space-time. We will give the proof of Theorem 2.3 by a geometric method of descent. To this end, we use $(\mathcal{M}, \mathbf{g})$ to introduce the manifold $\widetilde{\mathcal{M}} = \mathcal{M} \times \mathbb{R}$ and a Lorentzian metric $\tilde{\mathbf{g}}$ on $\widetilde{\mathcal{M}}$ by⁴

$$\tilde{\mathbf{g}} = \tilde{\mathbf{g}}_{\tilde{\alpha}\tilde{\beta}} dx^{\tilde{\alpha}} dx^{\tilde{\beta}} := \mathbf{g}_{\alpha\beta} dx^\alpha dx^\beta + dz^2. \quad (2.24)$$

We use $\tilde{\mathbf{D}}$ to denote the Levi-Civita connection of $\tilde{\mathbf{g}}$ on $\widetilde{\mathcal{M}}$.

We may identify \mathcal{M} with $\mathcal{M} \times \{0\}$ as a submanifold of $\widetilde{\mathcal{M}}$. For a function or tensor on $(\mathcal{M}, \mathbf{g})$ we may use a standard procedure to extend it to a function or tensor on $(\widetilde{\mathcal{M}}, \tilde{\mathbf{g}})$ such that it is independent of z with vanishing ∂_z -components; such extensions are called \mathcal{M} -tangent extensions and are denoted by the same notation. Let $\square_{\tilde{\mathbf{g}}} := \tilde{\mathbf{g}}^{\tilde{\alpha}\tilde{\beta}} \tilde{\mathbf{D}}_{\tilde{\alpha}} \tilde{\mathbf{D}}_{\tilde{\beta}}$ be the d'Alembertian with respect to $(\widetilde{\mathcal{M}}, \tilde{\mathbf{g}})$. Then for the tensor fields ϕ and F satisfying $\square_{\mathbf{g}} \phi = F$ in \mathcal{M} , we have $\square_{\tilde{\mathbf{g}}} \phi = F$ in $\widetilde{\mathcal{M}}$. Therefore, to derive a representation formula of ϕ in $(\mathcal{M}, \mathbf{g})$, we will use a Kirchhoff formula in $(\widetilde{\mathcal{M}}, \tilde{\mathbf{g}})$. We start with some preparation.

Lemma 2.4. *Let $\tilde{\Gamma}$ and Γ denote the Christoffel symbols of $\tilde{\mathbf{g}}$ and \mathbf{g} respectively. There hold*

$$\tilde{\Gamma}_{z\gamma}^\alpha = \tilde{\Gamma}_{\alpha\gamma}^z = \tilde{\Gamma}_{\alpha z}^z = \tilde{\Gamma}_{zz}^z = \tilde{\Gamma}_{zz}^\alpha = 0; \quad \tilde{\Gamma}_{\alpha\beta}^\gamma = \Gamma_{\alpha\beta}^\gamma, \quad (2.25)$$

$$\tilde{\mathbf{D}}_{\mathfrak{B}} \mathfrak{B} = \tilde{\mathbf{D}}_{\partial_z} \mathfrak{B} = \tilde{\mathbf{D}}_{\mathfrak{B}} \partial_z = \tilde{\mathbf{D}}_{\partial_z} \partial_z = 0. \quad (2.26)$$

Proof. Note that $\tilde{\mathbf{g}}_{z\alpha} = 0$, $\tilde{\mathbf{g}}_{zz} = 1$ and $\tilde{\mathbf{g}}_{\tilde{\alpha}\tilde{\beta}}$ is independent of z . We can obtain (2.25) directly from the formula

$$\tilde{\Gamma}_{\tilde{\beta}\tilde{\gamma}}^{\tilde{\alpha}} = \frac{1}{2} \tilde{\mathbf{g}}^{\tilde{\alpha}\tilde{\eta}} \left(\partial_{\tilde{\beta}} \tilde{\mathbf{g}}_{\tilde{\eta}\tilde{\gamma}} + \partial_{\tilde{\gamma}} \tilde{\mathbf{g}}_{\tilde{\beta}\tilde{\eta}} - \partial_{\tilde{\eta}} \tilde{\mathbf{g}}_{\tilde{\beta}\tilde{\gamma}} \right).$$

Next we show (2.26). Note that

$$(\tilde{\mathbf{D}}_{\mathfrak{B}} \mathfrak{B})^{\tilde{\beta}} = \mathfrak{B}^{\tilde{\alpha}} \tilde{\mathbf{D}}_{\tilde{\alpha}} \mathfrak{B}^{\tilde{\beta}} = \mathfrak{B}^{\tilde{\alpha}} \left(\partial_{\tilde{\alpha}} \mathfrak{B}^{\tilde{\beta}} + \tilde{\Gamma}_{\tilde{\alpha}\tilde{\eta}}^{\tilde{\beta}} \mathfrak{B}^{\tilde{\eta}} \right).$$

⁴We will identify z with x^3 . Besides the convention on page 3, a Greek letter with tilde is used to denote an index from $\{0, 1, 2, 3\}$, e.g. $\tilde{\alpha} = 0, 1, 2, 3$.

Since $\tilde{\mathbf{g}}(\mathfrak{B}, \partial_z) = 0$ and \mathfrak{B} is independent of z , we may use (2.25) to obtain $(\tilde{\mathbf{D}}_{\mathfrak{B}}\mathfrak{B})^z = 0$ and

$$(\tilde{\mathbf{D}}_{\mathfrak{B}}\mathfrak{B})^\beta = \mathfrak{B}^\alpha (\partial_\alpha \mathfrak{B}^\beta + \Gamma_{\alpha\eta}^\beta \mathfrak{B}^\eta) = (\mathbf{D}_{\mathfrak{B}}\mathfrak{B})^\beta = 0,$$

where for the last equality we used the fact that \mathfrak{B} is geodesic. The remaining three equalities in (2.26) can be proved similarly. \square

For $p \in \mathcal{M}$, let $\tilde{\mathcal{N}}^-(p)$ denote the backward light cone with vertex p in $(\tilde{\mathcal{M}}, \tilde{\mathbf{g}})$. Then $\mathcal{N}^-(p)$ can be identified as a subset of $\tilde{\mathcal{N}}^-(p)$ that is ruled by null geodesics in $(\tilde{\mathcal{M}}, \tilde{\mathbf{g}})$ with vanishing z -coordinate. For $(t, x, z) \in \tilde{\mathcal{M}}$, let

$$u = u(t, x, z) := |z| - \rho(x, t).$$

Lemma 2.5. *Within $0 < \tau := t_p - t < \delta_*$, the level set $\{u = 0\}$ of u in $(\tilde{\mathcal{M}}, \tilde{\mathbf{g}})$ coincides with the backward null cone $\tilde{\mathcal{N}}^-(p)$ with vertex p .*

Proof. By the geodesic equation and (2.25) it is easy to see that $\tilde{\Upsilon}(s) := (\Upsilon(s), z(s))$ is a geodesic in $(\tilde{\mathcal{M}}, \tilde{\mathbf{g}})$ with $\tilde{\Upsilon}(0) = p$ if and only if $\Upsilon(s)$ is a geodesic in $(\mathcal{M}, \mathbf{g})$ with $\Upsilon(0) = p$ and $z(s) = cs$ for some constant c .

If $\tilde{\Upsilon}(s) := (\Upsilon(s), z(s))$ is a null geodesic in $(\tilde{\mathcal{M}}, \tilde{\mathbf{g}})$ initiating from p , then

$$0 = \tilde{\mathbf{g}}(\tilde{\Upsilon}'(s), \tilde{\Upsilon}'(s)) = \mathbf{g}(\Upsilon'(s), \Upsilon'(s)) + |z'(s)|^2$$

which implies that

$$|z'(s)| = (-\mathbf{g}(\Upsilon'(s), \Upsilon'(s)))^{1/2}.$$

Integrating this equation with respect to s and using the definition of the Lorentzian distance ρ , we can obtain

$$0 = \int_0^s \left(|z'(\tilde{s})| - (-\mathbf{g}(\Upsilon'(\tilde{s}), \Upsilon'(\tilde{s})))^{1/2} \right) d\tilde{s} = |c|s - \rho(\Upsilon(s)) = |z(s)| - \rho(\Upsilon(s)).$$

This shows that, within $\tau < \delta_*$, $\tilde{\Upsilon} \subset \{u = 0\}$ and hence $\tilde{\mathcal{N}}^-(p) \subset \{u = 0\}$.

Conversely, let (t, x, z) be any point on $\{u = 0\}$ with $\tau = t_p - t < \delta_*$. If $z = 0$ then $\rho(t, x) = 0$ and hence $(t, x, z) \in \mathcal{N}^-(p) \subset \tilde{\mathcal{N}}^-(p)$. Thus we may assume $z \neq 0$. We can find a time-like geodesic $\Upsilon(s)$ in $(\mathcal{M}, \mathbf{g})$ initiating from p , with $\Upsilon(s_0) = (t, x)$ for some $s_0 > 0$. Set

$$\tilde{\Upsilon}(s) = (\Upsilon(s), \frac{z}{s_0}s).$$

Then $\tilde{\Upsilon}(s_0) = (t, x, z)$ and $\tilde{\Upsilon}(s)$ is a geodesic in $(\tilde{\mathcal{M}}, \tilde{\mathbf{g}})$. Moreover

$$\tilde{\mathbf{g}}(\tilde{\Upsilon}'(s), \tilde{\Upsilon}'(s)) = \mathbf{g}(\Upsilon'(s), \Upsilon'(s)) + \frac{z^2}{s_0^2}.$$

Since $\Upsilon(s)$ is a geodesic in $(\mathcal{M}, \mathbf{g})$, $\mathbf{g}(\Upsilon'(s), \Upsilon'(s))$ is a constant and thus

$$\rho(t, x) = \int_0^{s_0} (-\mathbf{g}(\Upsilon'(s), \Upsilon'(s)))^{1/2} ds = s_0 (-\mathbf{g}(\Upsilon'(s), \Upsilon'(s)))^{1/2}.$$

Consequently

$$\tilde{\mathbf{g}}(\tilde{\Upsilon}'(s), \tilde{\Upsilon}'(s)) = \frac{z^2 - \rho(t, x)^2}{s_0^2} = 0.$$

This shows that $\tilde{\Upsilon}(s)$ is a null geodesic in $(\tilde{\mathcal{M}}, \tilde{\mathbf{g}})$ and hence $(t, x, z) \in \tilde{\mathcal{N}}^-(p)$. Therefore, within $\tau < \delta_*$ we have $\{u = 0\} \subset \tilde{\mathcal{N}}^-(p)$. \square

According to Lemma 2.5, within $\tau < \delta_*$, the null cone $\tilde{\mathcal{N}}^-(p)$ is the union of three parts: $\mathcal{N}^-(p)$, \mathcal{H}^+ and \mathcal{H}^- , where

$$\mathcal{H}^+ := \{z = \rho(t, x), \tau < \delta_*\} \quad \text{and} \quad \mathcal{H}^- := \{z = -\rho(t, x), \tau < \delta_*\}.$$

Note that both \mathcal{H}^+ and \mathcal{H}^- can be regarded as graphs in $(\tilde{\mathcal{M}}, \tilde{\mathbf{g}})$ over $\mathcal{I}^-(p)$.

Lemma 2.6. *Within $\tau < \delta_*$, any null geodesic in $(\tilde{\mathcal{M}}, \tilde{\mathbf{g}})$ initiating from p lies completely in either $\mathcal{N}^-(p)$, or \mathcal{H}^+ , or \mathcal{H}^- .*

Proof. Since the z -component of a geodesic in $(\tilde{\mathcal{M}}, \tilde{\mathbf{g}})$ has the form $z(s) = cs$ for some constant c , the result then follows according to the sign of c . \square

Let $\tilde{\Sigma}_t$ denote the level set of t in $(\tilde{\mathcal{M}}, \tilde{\mathbf{g}})$, let \tilde{g} be the induced metric of $\tilde{\mathbf{g}}$ on $\tilde{\Sigma}_t$, and let $\tilde{\nabla}$ denote the Levi-Civita connection of \tilde{g} on $\tilde{\Sigma}_t$. We set $S_t := \tilde{\mathcal{N}}^-(p) \cap \tilde{\Sigma}_t$. For $\tau_0 < \delta_*$, we have

$$\tilde{\mathcal{N}}^-(p) \cap \{t_p - t \leq \tau_0\} = \bigcup_{\tau \leq \tau_0} S_{t_p - \tau}$$

and thus $\{S_{t_p - \tau}\}_\tau$ forms a time foliation of $\tilde{\mathcal{N}}^-(p)$. Let \tilde{N} be the radial normal of S_t in $\tilde{\Sigma}_t$. We will derive the formula for \tilde{N} . We will only consider the half cone \mathcal{H}^+ , since \mathcal{H}^- can be treated in the same way. We first have

$$\tilde{\nabla}u = \partial_z - \nabla\rho.$$

Let $\tilde{a}^{-1} = |\tilde{\nabla}u|_{\tilde{g}}$. We have $\tilde{a}^{-2} = 1 + |\nabla\rho|_g^2$. It then follows from (2.7) that

$$\tilde{a}^{-1} = \frac{\mathbf{b}^{-1}\tau}{\rho}. \quad (2.27)$$

Consequently

$$\tilde{N} = \frac{\tilde{\nabla}u}{|\tilde{\nabla}u|_{\tilde{g}}} = \tilde{a}(\partial_z - \nabla\rho). \quad (2.28)$$

Let $\tilde{\gamma}$ denote the induced metric on S_t . We use $\tilde{\Psi}$ to denote the Levi-Civita connection of $\tilde{\gamma}$ and use $\tilde{\Delta}$ to denote the corresponding Laplace-Beltrami operator. By setting $v_t = \sqrt{|\tilde{\gamma}|}/\sqrt{|\gamma_{\mathbb{S}^2}|}$, we have $d\mu_{S_t} = v_t d\mu_{\mathbb{S}^2}$. Since S_t can be viewed as a graph over Σ_t locally, we have

$$d\mu_{S_t} = \sqrt{1 + |\nabla\rho|_g^2} d\mu_{\Sigma_t} = \tilde{a}^{-1} d\mu_{\Sigma_t}. \quad (2.29)$$

Now we introduce the null frame

$$\tilde{L} = -\mathbf{T} + \tilde{N}, \quad \underline{\tilde{L}} = -\mathbf{T} - \tilde{N}$$

on $\tilde{\mathcal{N}}^-(p) \cap \{\tau < \delta_*\}$. Clearly, $\langle \tilde{L}, \underline{\tilde{L}} \rangle = -2$. In view of (2.28), we have on \mathcal{H}^+ that

$$\begin{aligned} \tilde{L} &= \frac{\rho}{\mathbf{b}^{-1}\tau} \left(-\frac{\mathbf{b}^{-1}\tau}{\rho} \mathbf{T} + a^{-1} \mathbf{N} \right) + \frac{\rho}{\mathbf{b}^{-1}\tau} \partial_z, \\ \underline{\tilde{L}} &= \frac{\rho}{\mathbf{b}^{-1}\tau} \left(-\frac{\mathbf{b}^{-1}\tau}{\rho} \mathbf{T} - a^{-1} \mathbf{N} \right) - \frac{\rho}{\mathbf{b}^{-1}\tau} \partial_z. \end{aligned} \quad (2.30)$$

By using (2.9) and (2.10) we can write

$$\tilde{L} = \frac{\rho}{\mathbf{b}^{-1}\tau} (\mathfrak{B} + \partial_z), \quad \underline{\tilde{L}} = \frac{\rho}{\mathbf{b}^{-1}\tau} (\mathfrak{B} - \partial_z). \quad (2.31)$$

In $(\widetilde{\mathcal{M}}, \widetilde{\mathbf{g}})$ we define the following projection tensors

$$\widetilde{\Pi}^{\tilde{\alpha}\tilde{\beta}} = \widetilde{\mathbf{g}}^{\tilde{\alpha}\tilde{\beta}} - \delta_z^{\tilde{\alpha}} \delta_z^{\tilde{\beta}}, \quad \Pi^{\tilde{\alpha}\tilde{\beta}} = \widetilde{\mathbf{g}}^{\tilde{\alpha}\tilde{\beta}} + \frac{1}{2}(\tilde{L}^{\tilde{\alpha}} \tilde{L}^{\tilde{\beta}} + \tilde{L}^{\tilde{\alpha}} \tilde{L}^{\tilde{\beta}}).$$

For the induced metric $\tilde{\gamma}$ on S_t , we have $\tilde{\gamma}^{\tilde{\alpha}\tilde{\beta}} = \Pi^{\tilde{\alpha}\tilde{\beta}}$ if $\tilde{\gamma}$ is regarded as an S_t -tangent tensor in $(\widetilde{\mathcal{M}}, \widetilde{\mathbf{g}})$. We can project $\Pi^{\tilde{\alpha}\tilde{\beta}}$ to $(\mathcal{M}, \mathbf{g})$ by $\mathbb{I}^{\tilde{\alpha}\tilde{\beta}} = \Pi_{\tilde{\alpha}'\tilde{\beta}'}^{\tilde{\alpha}\tilde{\beta}} \tilde{\Pi}^{\tilde{\alpha}'\tilde{\alpha}} \tilde{\Pi}^{\tilde{\beta}'\tilde{\beta}}$. Noting that $\tilde{\Pi}^{z\tilde{\beta}} = 0$ and $\tilde{\Pi}^{\alpha\beta} = \mathbf{g}^{\alpha\beta}$, we have $\mathbb{I}^{z\tilde{\beta}} = 0$ and

$$\mathbb{I}^{\alpha\beta} = \mathbf{g}^{\alpha\alpha'} \mathbf{g}^{\beta\beta'} \Pi_{\alpha'\beta'}^{\alpha\beta} = \Pi^{\alpha\beta} = \mathbf{g}^{\alpha\beta} + \frac{1}{2}(\tilde{L}^{\alpha} \tilde{L}^{\beta} + \tilde{L}^{\beta} \tilde{L}^{\alpha}). \quad (2.32)$$

In view of (2.31) we have $\tilde{L}^{\alpha} = \frac{\rho}{\mathbf{b}^{-1}\tau} \mathfrak{B}^{\alpha}$ and $\tilde{L}^{\alpha} = \frac{\rho}{\mathbf{b}^{-1}\tau} \mathfrak{B}^{\alpha}$. Combining this with the above equation shows that

$$\mathbb{I}^{\alpha\beta} = \mathbf{g}^{\alpha\beta} + \frac{1}{2} \frac{\rho^2}{\mathbf{b}^{-2}\tau^2} (\mathfrak{B}^{\alpha} \mathfrak{B}^{\beta} + \mathfrak{B}^{\beta} \mathfrak{B}^{\alpha}). \quad (2.33)$$

By using (2.9) and (2.10), we can derive from (2.33) that

$$\mathbb{I}^{\alpha\beta} = \mathbf{g}^{\alpha\beta} + \mathbf{T}^{\alpha} \mathbf{T}^{\beta} - \frac{\tilde{r}^2}{\mathbf{b}^{-2}\tau^2} \mathbf{N}^{\alpha} \mathbf{N}^{\beta}. \quad (2.34)$$

Let ${}^{(3)}\Pi^{\alpha\beta} = \mathbf{g}^{\alpha\beta} + \mathbf{T}^{\alpha} \mathbf{T}^{\beta} - \mathbf{N}^{\alpha} \mathbf{N}^{\beta}$, the above identity can be recast as

$$\mathbb{I}^{\alpha\beta} = {}^{(3)}\Pi^{\alpha\beta} + \frac{\rho^2}{\mathbf{b}^{-2}\tau^2} \mathbf{N}^{\alpha} \mathbf{N}^{\beta}. \quad (2.35)$$

We now introduce a set of geometric notion on $(\widetilde{\mathcal{M}}, \widetilde{\mathbf{g}})$ which will be used in the Kirchhoff formula.

Definition 2.7. (1) We denote by $\tilde{\pi}$ the second fundamental form of $(\tilde{\Sigma}_t, \tilde{g}) \subset (\widetilde{\mathcal{M}}, \widetilde{\mathbf{g}})$, i.e.

$$\tilde{\pi}(X, Y) = -\tilde{\mathbf{g}}(\tilde{\mathbf{D}}_X \mathbf{T}, Y)$$

for $X, Y \in \mathcal{T}\tilde{\Sigma}_t$ and \tilde{g} is the induced metric of $\tilde{\mathbf{g}}$ on $\tilde{\Sigma}_t$. We denote by $\text{Tr}\tilde{\pi}$ the trace part of $\tilde{\pi}$.

(2) We define the null second fundamental forms on $\tilde{\mathcal{N}}^-(p)$ in the extended space-time $(\widetilde{\mathcal{M}}, \widetilde{\mathbf{g}})$ by

$$\tilde{\chi}(X, Y) = \tilde{\mathbf{g}}(\tilde{\mathbf{D}}_X \tilde{L}, Y), \quad \tilde{\underline{\chi}}(X, Y) = \tilde{\mathbf{g}}(\tilde{\mathbf{D}}_X \tilde{L}, Y), \quad (2.36)$$

where X, Y are in $\mathcal{T}S_t$. The trace parts of the above symmetric S_t -tangent tensor fields are denoted by $\text{tr}\tilde{\chi}$ and $\text{tr}\tilde{\underline{\chi}}$.

(3) We introduce the connection coefficients

$$\tilde{\omega} = -\frac{1}{2} \langle \tilde{\mathbf{D}}_{\tilde{L}} \tilde{L}, \tilde{L} \rangle, \quad \underline{\mu} = \tilde{L} \text{tr}\tilde{\underline{\chi}} + \frac{1}{2} \text{tr}\tilde{\chi} \text{tr}\tilde{\underline{\chi}}, \quad \tilde{\zeta}(X) = \frac{1}{2} \tilde{\mathbf{g}}(\tilde{\mathbf{D}}_{\tilde{L}} \tilde{L}, X), \quad (2.37)$$

where $X \in \mathcal{T}S_t$.

The following lemma gives some preliminary results on how to represent geometric quantities in $(\widetilde{\mathcal{M}}, \widetilde{\mathbf{g}})$ in terms of geometric quantities in $(\mathcal{M}, \mathbf{g})$.

Lemma 2.8. *There hold*

$$\mathrm{tr}\tilde{\chi} = \frac{\rho}{\mathbf{b}^{-1}\tau} \mathrm{tr}k, \quad (2.38)$$

$$\mathrm{tr}\tilde{\pi} = \mathrm{Tr}\pi - \frac{\mathbf{b}^2\tilde{r}^2}{\tau^2}\pi_{\mathbf{N}\mathbf{N}}, \quad (2.39)$$

$$\mathrm{tr}\tilde{\chi} + \mathrm{tr}\tilde{\chi} = 2\mathrm{tr}\tilde{\pi}, \quad (2.40)$$

$$\tilde{\pi}_{\tilde{N}\tilde{N}} = \frac{\tilde{r}^2}{\mathbf{b}^{-2}\tau^2}\pi_{\mathbf{N}\mathbf{N}}, \quad (2.41)$$

$$\tilde{\omega} = \omega, \quad (2.42)$$

$$\tilde{L} \log \mathbf{b} = \frac{n^{-1} - \mathbf{b}}{\tau} - \omega. \quad (2.43)$$

where $\tilde{\pi}$ is defined in Definition 2.7 and $\mathrm{tr}\tilde{\pi} = -\tilde{\mathbf{D}}_{\tilde{\mu}}\mathbf{T}_{\tilde{\nu}}\Pi^{\tilde{\mu}\tilde{\nu}}$ is the trace of $\tilde{\pi}$ restricted to S_t .

Proof. To obtain (2.38), we note that $\mathrm{tr}\tilde{\chi} = \Pi^{\tilde{\alpha}\tilde{\beta}}\tilde{\chi}_{\tilde{\alpha}\tilde{\beta}} = \tilde{\mathbf{D}}_{\tilde{\alpha}}\tilde{L}^{\tilde{\beta}}\Pi_{\tilde{\beta}}^{\tilde{\alpha}}$. In view of (2.31) we have

$$\mathrm{tr}\tilde{\chi} = \partial_{\tilde{\alpha}}\left(\frac{\rho}{\mathbf{b}^{-1}\tau}\right)(\mathfrak{B}^{\tilde{\beta}} + \partial_z^{\tilde{\beta}})\Pi_{\tilde{\beta}}^{\tilde{\alpha}} + \frac{\rho}{\mathbf{b}^{-1}\tau}\tilde{\mathbf{D}}_{\tilde{\alpha}}(\mathfrak{B}^{\tilde{\beta}} + \partial_z^{\tilde{\beta}})\Pi_{\tilde{\beta}}^{\tilde{\alpha}}.$$

From (2.25) we have $\tilde{\mathbf{D}}_{\tilde{\alpha}}\partial_z^{\tilde{\beta}} = 0$. By the definition of $\Pi_{\tilde{\beta}}^{\tilde{\alpha}}$ it is straightforward to check that

$$(\mathfrak{B}^{\tilde{\beta}} + \partial_z^{\tilde{\beta}})\Pi_{\tilde{\beta}}^{\tilde{\alpha}} = \frac{\mathbf{b}^{-1}\tau}{\rho}\tilde{L}^{\tilde{\beta}}\Pi_{\tilde{\beta}}^{\tilde{\alpha}} = 0.$$

Therefore

$$\mathrm{tr}\tilde{\chi} = \frac{\rho}{\mathbf{b}^{-1}\tau}\tilde{\mathbf{D}}_{\tilde{\alpha}}\mathfrak{B}^{\tilde{\beta}}\Pi_{\tilde{\beta}}^{\tilde{\alpha}} = \frac{\rho}{\mathbf{b}^{-1}\tau}\left(\tilde{\mathbf{D}}_{\tilde{\alpha}}\mathfrak{B}^{\tilde{\alpha}} + \frac{1}{2}\tilde{\mathbf{D}}_{\tilde{\alpha}}\mathfrak{B}^{\tilde{\beta}}(\tilde{L}^{\tilde{\alpha}}\tilde{L}_{\tilde{\beta}} + \tilde{L}^{\tilde{\alpha}}\tilde{L}_{\tilde{\beta}})\right).$$

By using (2.25), $\tilde{\mathbf{g}}(\mathfrak{B}, \partial_z) = 0$ and the fact that \mathfrak{B} is independent of z , we can derive that

$$\tilde{\mathbf{D}}_{\tilde{\alpha}}\mathfrak{B}^{\tilde{\alpha}} = \mathbf{D}_{\alpha}\mathfrak{B}^{\alpha}, \quad \tilde{\mathbf{D}}_{\tilde{\alpha}}\mathfrak{B}^z = 0, \quad \tilde{\mathbf{D}}_{\tilde{\alpha}}\mathfrak{B} = \mathbf{D}_{\alpha}\mathfrak{B}. \quad (2.44)$$

We claim

$$\tilde{\mathbf{D}}_{\tilde{\alpha}}\mathfrak{B}^{\tilde{\beta}}(\tilde{L}^{\tilde{\alpha}}\tilde{L}_{\tilde{\beta}} + \tilde{L}^{\tilde{\alpha}}\tilde{L}_{\tilde{\beta}}) = 0. \quad (2.45)$$

Combining the first identity in (2.44) with (2.45) implies

$$\mathrm{tr}\tilde{\chi} = \frac{\rho}{\mathbf{b}^{-1}\tau}\mathbf{D}_{\alpha}\mathfrak{B}^{\alpha} = \frac{\rho}{\mathbf{b}^{-1}\tau}\mathrm{tr}k.$$

To see (2.45), we first can obtain $\tilde{\mathbf{D}}_{\tilde{L}}\mathfrak{B}^{\tilde{\beta}} = 0$ from (2.31) and (2.26). It follows by using the second and the third identities in (2.44), the second identity in (2.26) and (2.31) that

$$\tilde{\mathbf{D}}_{\tilde{\alpha}}\mathfrak{B}^{\tilde{\beta}}\tilde{L}_{\tilde{\beta}} = \tilde{\mathbf{D}}_{\tilde{\alpha}}\mathfrak{B}^z\tilde{L}_z + \tilde{\mathbf{D}}_{\tilde{\alpha}}\mathfrak{B}^{\beta}\tilde{L}_{\beta} = \frac{\rho}{\mathbf{b}^{-1}\tau}\tilde{\mathbf{D}}_{\tilde{\alpha}}\mathfrak{B}^{\beta}\mathfrak{B}_{\beta} = 0$$

where we used $\mathbf{g}(\mathfrak{B}, \mathfrak{B}) = -1$. Hence (2.45) is proved and the proof of (2.38) is thus complete.

To show (2.39), we note that $\tilde{\mathbf{g}}(\mathbf{T}, \partial_z) = 0$ and \mathbf{T} is independent of z . Thus, by using (2.25) we can derive that $\tilde{\mathbf{D}}_{\tilde{\alpha}}\mathbf{T}_{\tilde{\beta}}\Pi^{\tilde{\alpha}\tilde{\beta}} = \mathbf{D}_{\alpha}\mathbf{T}_{\beta}\Pi^{\alpha\beta}$. Therefore, by using (2.32)

and (2.34) we deduce that

$$\begin{aligned}\mathrm{tr}\tilde{\boldsymbol{\pi}} &= -\mathbf{D}_\alpha \mathbf{T}_\beta \mathbb{I}^{\alpha\beta} = -\mathbf{D}_\alpha \mathbf{T}_\beta \left(\mathbf{g}^{\alpha\beta} + \mathbf{T}^\alpha \mathbf{T}^\beta - \frac{\tilde{r}^2}{\mathbf{b}^{-2}\tau^2} \mathbf{N}^\alpha \mathbf{N}^\beta \right) \\ &= \mathrm{Tr}\boldsymbol{\pi} + \frac{\tilde{r}^2}{\mathbf{b}^{-2}\tau^2} \langle \mathbf{D}_\mathbf{N} \mathbf{T}, \mathbf{N} \rangle = \mathrm{Tr}\boldsymbol{\pi} - \frac{\tilde{r}^2}{\mathbf{b}^{-2}\tau^2} \boldsymbol{\pi}_{\mathbf{N}\mathbf{N}},\end{aligned}$$

where we used the fact that $\mathbf{g}^{\alpha\beta} + \mathbf{T}^\alpha \mathbf{T}^\beta$ is the standard projection to $\Sigma_t \subset \mathcal{M}$. Hence (2.39) is proved.

From the definition of $\tilde{\chi}$, $\tilde{\underline{\chi}}$ and $\tilde{\boldsymbol{\pi}}$, it is straightforward to derive (2.40).

Next we derive (2.41). Note that $\mathbf{T} = n^{-1}\partial_t$ and $\tilde{\mathbf{g}}(\partial_t, \tilde{N}) = 0$, we have $\tilde{\boldsymbol{\pi}}_{\tilde{N}\tilde{N}} = -n^{-1}\tilde{\mathbf{g}}(\tilde{\mathbf{D}}_{\tilde{N}}\partial_t, \tilde{N})$. In view of (2.28), (2.8) and (2.25) we can further obtain

$$\begin{aligned}\tilde{\boldsymbol{\pi}}_{\tilde{N}\tilde{N}} &= -n^{-1}\tilde{a}^2 a^{-1} \left[\tilde{\mathbf{g}}(\tilde{\mathbf{D}}_{\mathbf{N}}\partial_t, \partial_z) + a^{-1}\tilde{\mathbf{g}}(\tilde{\mathbf{D}}_{\mathbf{N}}\partial_t, \mathbf{N}) \right] = -n^{-1}\tilde{a}^2 a^{-2} \mathbf{g}(\mathbf{D}_{\mathbf{N}}\partial_t, \mathbf{N}) \\ &= -\tilde{a}^2 a^{-2} \mathbf{g}(\mathbf{D}_{\mathbf{N}} \mathbf{T}, \mathbf{N}) = \tilde{a}^2 a^{-2} \boldsymbol{\pi}_{\mathbf{N}\mathbf{N}} = \frac{\tilde{r}^2}{\mathbf{b}^{-2}\tau^2} \boldsymbol{\pi}_{\mathbf{N}\mathbf{N}}.\end{aligned}$$

To show (2.42), from (2.31), (2.26) and $\langle \tilde{L}, \tilde{\underline{L}} \rangle = -2$ it follows that

$$\tilde{\omega} = \frac{1}{2} \left\langle (\mathfrak{B} + \partial_z) \left(\frac{\rho}{\mathbf{b}^{-1}\tau} \right) \tilde{L} + \frac{\rho^2}{\mathbf{b}^{-2}\tau^2} \tilde{\mathbf{D}}_{\mathfrak{B}+\partial_z} (\mathfrak{B} + \partial_z), \tilde{\underline{L}} \right\rangle = -(\mathfrak{B} + \partial_z) \left(\frac{\rho}{\mathbf{b}^{-1}\tau} \right).$$

Since $\frac{\rho}{\mathbf{b}^{-1}\tau}$ is independent of z , we therefore obtain $\tilde{\omega} = -\mathfrak{B}(\frac{\rho}{\mathbf{b}^{-1}\tau}) = \omega$ which is (2.42).

To obtain (2.43), by using (2.31) and the fact that \mathbf{b} is independent of z , we can conclude from (2.16) that

$$\tilde{L} \log \mathbf{b} = -\mathbf{b} \tilde{L}(\mathbf{b}^{-1}) = -\frac{\rho}{\mathbf{b}^{-2}\tau} \mathfrak{B}(\mathbf{b}^{-1}) = \frac{n^{-1} - \mathbf{b}}{\tau} - \omega.$$

The proof is therefore complete. \square

Recall that the Kirchhoff representation formula in [11] only holds on the regular part of $\tilde{\mathcal{N}}^-(p)$. To define the regular part, we need the notion of null radius of injectivity. Since we mainly rely on the time foliation to analyze $\tilde{\mathcal{N}}^-(p)$, we only need to introduce the past null radius of injectivity at p with respect to the global time function t .

Let us briefly recall the definition of past null radius of injectivity; one may consult ([11, 12, 14, 23]) for more details. We parametrize the set of past null vectors in $\mathcal{T}_p \tilde{\mathcal{M}}$ in terms of $\omega \in \mathbb{S}^2$, the standard sphere in \mathbb{R}^3 . Then, for each $\omega \in \mathbb{S}^2$, let ℓ_ω be the null vector in $\mathcal{T}_p \tilde{\mathcal{M}}$ normalized with respect to the future, unit, timelike vector \mathbf{T}_p by

$$\tilde{\mathbf{g}}(\ell_\omega, \mathbf{T}_p) = 1$$

and let $\tilde{\Gamma}_\omega(s)$ be the past null geodesic satisfying $\tilde{\Gamma}_\omega(0) = p$ and $\frac{d\tilde{\Gamma}_\omega}{ds}(0) = \ell_\omega$. We define the null vector field \tilde{L}' on $\tilde{\mathcal{N}}^-(p)$ by

$$\tilde{L}'(\tilde{\Gamma}_\omega(s)) = \frac{d}{ds} \tilde{\Gamma}_\omega(s)$$

which may only be smooth almost everywhere on $\tilde{\mathcal{N}}^-(p)$ and can be multivalued on a set of exceptional points. We can choose the parameter s with $s(p) = 0$ so that $\tilde{\mathbf{D}}_{\tilde{L}'} \tilde{L}' = 0$ and $\tilde{L}'(s) = 1$. This s is called the affine parameter.

We define the past null radius of injectivity $\tilde{i}_*(p, t)$ at p to be the supremum over all the values $\tau > 0$ for which the exponential map

$$\mathcal{G}_p : (t, \omega) \rightarrow \tilde{\Gamma}_\omega(s(t)) \quad (2.46)$$

is a global diffeomorphism from $(t(p) - \tau, t(p)) \times \mathbb{S}^2$ to its image in $\tilde{\mathcal{N}}^-(p)$. We remark that s is a function not only depending on t but also on ω . We suppress ω just for convenience. It is known that

$$\tilde{i}_*(p, t) = \min\{\tilde{s}_*(p, t), \tilde{\ell}_*(p, t)\},$$

where $\tilde{s}_*(p, t)$ is defined to be the supremum over all values $\tau > 0$ such that the map \mathcal{G}_p is a local diffeomorphism from $(t(p) - \tau, t(p)) \times \mathbb{S}^2$ to its image, and $\tilde{\ell}_*(p, t)$ is defined to be the smallest value of $\tau > 0$ for which there exist two distinct null geodesics $\tilde{\Gamma}_{\omega_1}(s(t))$ and $\tilde{\Gamma}_{\omega_2}(s(t))$ from p which intersect at a point with $t = t_p - \tau$.

We can similarly define in $(\mathcal{M}, \mathbf{g})$ the past null radius of injectivity $i_*(p, t)$ with respect to the time foliation.

The following result gives the relation between the causal radius of injectivity in $(\mathcal{M}, \mathbf{g})$ and the null radius of injectivity in $(\tilde{\mathcal{M}}, \tilde{\mathbf{g}})$.

Theorem 2.9. *For a point p in $(\mathcal{M}, \mathbf{g})$, let $c_*(p, t)$ denote the backward causal radius of injectivity at p in $(\mathcal{M}, \mathbf{g})$, which is defined by*

$$c_*(p, t) := \min(\delta_*, i_*(p, t)),$$

where δ_* and $i_*(p, t)$ are the past timelike and null radius of injectivity at p in $(\mathcal{M}, \mathbf{g})$. Then there holds

$$c_*(p, t) \leq \tilde{i}_*(p, t). \quad (2.47)$$

Proof. Recall that $\tilde{\mathcal{N}}^-(p)$ is composed by the three parts: \mathcal{H}^+ , \mathcal{H}^- and $\mathcal{N}^-(p)$, and, according to Lemma 2.6, every $\tilde{\Gamma}_\omega(s(\tau))$ lies completely in \mathcal{H}^+ , \mathcal{H}^- or $\mathcal{N}^-(p)$. Moreover, the proof of Lemma 2.5 shows that the two sets \mathcal{H}^\pm are ruled by the family of curves

$$(\Upsilon_V(\rho(\tau)), \pm\rho(\tau)), \quad V \in \mathbb{H}^2, \quad (2.48)$$

where Υ_V denotes the time-like geodesic in $(\mathcal{M}, \mathbf{g})$ with $\Upsilon_V(0) = p$ and $\Upsilon'(0) = V$. Here the sign of the z coordinate is determined by the fact whether the curves lie in \mathcal{H}^+ or \mathcal{H}^- .

(i) We first show that for any null geodesic $\tilde{\Gamma}_\omega(s)$ lying in \mathcal{H}^\pm there is a unique $V \in \mathbb{H}^2$ such that

$$\tilde{\Gamma}_\omega(s(\tau)) = \tilde{\Upsilon}_V(\tau) := (\Upsilon_V(\rho(\tau)), \pm\rho(\tau)) \quad (2.49)$$

for $\tau < \min\{\tilde{i}_*(p, t), c_*(p, t)\}$.

Since the spacetime $(\tilde{\mathcal{M}}, \tilde{\mathbf{g}})$ is symmetric about the z coordinate, it suffices to consider the case that $\tilde{\Gamma}_\omega$ lies in \mathcal{H}^+ . We can write

$$\tilde{\Gamma}_\omega(s(\tau)) := (\Gamma_\omega(s(\tau)), z(s(\tau))).$$

According to Lemma 2.5, $z(s(\tau)) = \rho(\tau)$ is the Lorentzian distance of $\Gamma_\omega(s(\tau))$ to p in $(\mathcal{M}, \mathbf{g})$ and Γ_ω is a timelike geodesic in $(\mathcal{M}, \mathbf{g})$ initiating from p . Thus, we can find a $V := (V^0, V^1, V^2) \in \mathbb{H}^2$ such that the geodesic Γ_ω can be represented by Υ_V if parametrized by ρ , i.e. $\Gamma_\omega(s(\tau)) = \Upsilon_V(\rho(\tau))$. This shows (2.49). The choice of V is unique, since otherwise there would exist two distance maximizing time-like geodesics forming a loop at certain $\tau < \delta_*$ in $(\mathcal{M}, \mathbf{g})$, which is impossible by the definition of δ_* .

(ii) We now derive the relation between the null geodesic generator of $\tilde{\Gamma}_\omega(s)$ and the geodesic generator of $\tilde{\Upsilon}_V(\rho)$. By using (2.5), (2.3) and (2.31) we have

$$\frac{d}{d\tau} \tilde{\Upsilon}_V(\tau) = \frac{d}{d\rho} \tilde{\Upsilon}_V \cdot \frac{d\rho}{d\tau} = \frac{n\rho}{\mathbf{b}^{-1}\tau} (\mathfrak{B} + \partial_z) \Big|_{\tilde{\Upsilon}_V(\tau)} = n\tilde{L} \Big|_{\tilde{\Upsilon}_V(\tau)} = n\tilde{L} \Big|_{\tilde{\Gamma}_\omega(s(\tau))}.$$

On the other hand, since $\tilde{L}'(s) = 1$ implies $\frac{ds}{d\tau} = \mathbf{a}n$ along $\tilde{\Gamma}_\omega(s(\tau))$ with $\mathbf{a}^{-1} = \langle \tilde{L}', \mathbf{T} \rangle$, we may use $\tilde{\Upsilon}_V(\tau) = \tilde{\Gamma}_\omega(s(\tau))$ to derive that

$$\frac{d}{d\tau} \tilde{\Upsilon}_V(\tau) = \frac{d}{d\tau} \tilde{\Gamma}_\omega(s(\tau)) = \frac{d}{ds} \tilde{\Gamma}_\omega(s(\tau)) \cdot \frac{ds}{d\tau} = \mathbf{a}n\tilde{L}' \Big|_{\tilde{\Gamma}_\omega(s(\tau))}$$

Combining the above two equations we can obtain $\tilde{L}' = \mathbf{a}^{-1}\tilde{L}$. Using $\langle \ell_\omega, \mathbf{T}_p \rangle = 1$ we can see that $\mathbf{a} \rightarrow 1$ as q approaches p along $\tilde{\Gamma}_\omega$.

By the definition of V^0 and (2.4), along the timelike geodesic Υ_V there holds

$$\lim_{t \rightarrow t_p} \frac{\rho}{\mathbf{b}^{-1}(t_p - t)} = \frac{1}{V^0}.$$

This together with $\tilde{\mathbf{D}}_{\tilde{L}'} \tilde{L}' = 0$, $\langle \ell_\omega, \mathbf{T}_p \rangle = 1$ and (2.4) imply that

$$\tilde{L}' = \frac{1}{V^0} (\mathfrak{B} + \partial_z), \quad \mathbf{a} = \frac{\rho}{\mathbf{b}^{-1}\tau} V^0. \quad (2.50)$$

(iii) We now show that if $\tilde{\ell}_*(p, t) \geq \tilde{s}_*(p, t)$ then $\tilde{s}_*(p, t) \geq c_*(p, t)$. To this end, consider an arbitrary null geodesic $\tilde{\Gamma}_\omega$. If $\tilde{\Gamma}_\omega(s(\tau))$ is contained in $\mathcal{N}^-(p)$, by the definition of $c_*(p, t)$, $\tilde{\Gamma}_\omega(s(\tau))$ does not contain any null conjugate point on $(0, c_*(p, t))$. So it needs only to consider the case that $\tilde{\Gamma}_\omega$ is contained in \mathcal{H}^+ or \mathcal{H}^- . By symmetry, it suffices to consider the case that $\tilde{\Gamma}_\omega$ is contained in \mathcal{H}^+ . According to [2, 9], it suffices to show that $\text{tr}\tilde{\chi}' > -\infty$ for $\tau := t_p - t < c_*(p, t)$, where $\tilde{\chi}'$ denotes the null second fundamental form defined by (2.36) with \tilde{L} replaced by \tilde{L}' and $\text{tr}\tilde{\chi}'$ denotes its trace. By using (2.31) and (2.50), it is straightforward to obtain

$$\text{tr}\tilde{\chi} = \frac{\rho}{\mathbf{b}^{-1}\tau} V^0 \text{tr}\tilde{\chi}'.$$

This together with (2.38) shows that

$$V^0 \text{tr}\tilde{\chi}' = \text{tr}k. \quad (2.51)$$

Note that the timelike geodesic Υ_V in $(\mathcal{M}, \mathbf{g})$ from p reaches a conjugate point q iff $\text{tr}k \rightarrow -\infty$ as points approach q along this geodesic. Since $\tilde{\ell}_*(p, t) \geq \tilde{s}_*(p, t)$, we can conclude from (2.51) that, $\text{tr}\tilde{\chi}'$ does not diverge to $-\infty$ along $\tilde{\Gamma}_\omega(s(\tau))$ iff $\text{tr}k$ does not along $\Upsilon_V(\rho(\tau))$. Since $\text{tr}k > -\infty$ along $\Upsilon_V(\rho(\tau))$ on $(0, c_*(p, t))$, we must have $\text{tr}\tilde{\chi}' > -\infty$ along $\tilde{\Gamma}_\omega(s(\tau))$ on $(0, c_*(p, t))$. Therefore $\tilde{s}_*(p, t) \geq c_*(p, t)$.

(iv) Finally we show that $\tilde{i}_*(p, t) \geq c_*(p, t)$. Suppose this is not true, i.e. $\tilde{i}_*(p, t) < c_*(p, t)$, we will derive a contradiction. By using the claim in (iii), we must have $\tilde{\ell}_*(p, t) < \tilde{s}_*(p, t)$. Thus there exist two distinct null geodesics $\tilde{\Gamma}_{\omega_1}(s(\tau))$ and $\tilde{\Gamma}_{\omega_2}(s(\tau))$ intersecting at some point q with $\tau = \tilde{\ell}_*(p, t)$. If $q \in \mathcal{N}^-(p)$, then Lemma 2.6 implies that $\tilde{\Gamma}_{\omega_1}(s(\tau))$ and $\tilde{\Gamma}_{\omega_2}(s(\tau))$ are both contained in $\mathcal{N}^-(p)$ and intersect at q ; this can not happen since $\tau < c_*(p, t)$. We may assume $q \in \mathcal{H}^+ \cup \mathcal{H}^-$. By symmetry we only need to consider the case that $q \in \mathcal{H}^+$. Now by Lemma 2.6

both $\tilde{\Gamma}_{\omega_1}(s(\tau))$ and $\tilde{\Gamma}_{\omega_2}(s(\tau))$ are contained in \mathcal{H}^+ . According to (i), we can find two distinct vectors $V_1, V_2 \in \mathbb{H}^2$ such that

$$\tilde{\Gamma}_{\omega_i}(s(\tau)) = \tilde{\Upsilon}_{V_i}(\tau) := (\Upsilon_{V_i}(\rho(\tau)), \rho(\tau)), \quad i = 1, 2$$

for $\tau < \tilde{\ell}_*(p, t)$. Thus there exists a point $q' \in \mathcal{I}^-(p)$ such that $\Upsilon_{V_1}(\rho(\tau)) = \Upsilon_{V_2}(\rho(\tau)) = q'$ when $\tau = \tilde{\ell}_*(p, t) < c_*(p, t)$. This is impossible by the definition of $c_*(p, t)$ and in particular the definition of δ_* . Therefore $\tilde{i}_*(p, t) \geq c_*(p, t)$ and the proof is complete. \square

According to Theorem 2.9, for any $\tau_0 < c_*(p, t)$, $\tilde{\mathcal{N}}^-(p) \cap \{t_p - t < \tau_0\}$ is a regular part of $\tilde{\mathcal{N}}^-(p)$ on which the Kirchhoff formula in [11, 20, 23] holds. By adapting the version of the Kirchhoff formula in [20, 23] with the null frame $\{\tilde{L}, \tilde{\underline{L}}, e_C, C = 1, 2\}$, where $\{e_C\}_{C=1,2}$ is an orthonormal frame on S_t , we have the following result which will be used to prove Theorem 2.3.

Proposition 2.10. *Let p be any point in \mathcal{M} , let t_0 verify $0 < \tau_0 := t_p - t_0 < c_*(p, t)$, and set $\mathcal{H}_* := \tilde{\mathcal{N}}^-(p) \cap \{t_p - t \leq \tau_0\}$. Let ϕ_I be an \mathcal{M} -tangent tensor on $(\tilde{\mathcal{M}}, \tilde{\mathbf{g}})$. For any \mathcal{M} -tangent tensor J_I at p of the same type as ϕ_I , let A^I be an \mathcal{M} -tangent tensor field satisfying*

$$\tilde{\mathbf{D}}_{\tilde{L}} A_I + \left(\frac{1}{2} \text{tr} \tilde{\chi} + \frac{n^{-1} - \mathbf{b}}{\tau} \right) A_I = 0, \quad \lim_{t \rightarrow t_p} \tau A_I = J_I. \quad (2.52)$$

along \mathcal{H}_* . Then there holds the Kirchhoff formula

$$\begin{aligned} 4\pi(n\langle \phi, J \rangle)(p) &= - \int_{S_{t_0}} \left(\tilde{\mathbf{D}}_{\tilde{\underline{L}}} \phi_I + \frac{1}{2} \text{tr} \tilde{\chi} \phi_I \right) A^I d\mu_{S_{t_0}} \\ &\quad - \int_{\mathcal{H}_*} \square_{\tilde{\mathbf{g}}} \phi_I A^I \mathbf{b} d\mu + Er[\phi] \end{aligned} \quad (2.53)$$

with

$$\begin{aligned} Er[\phi] &= \int_{\mathcal{H}_*} \left(2\tilde{\zeta}^C \tilde{\mathbf{D}}_C \phi_I + \tilde{\Delta} \phi_I \right) A^I \mathbf{b} d\mu \\ &\quad + \frac{1}{2} \int_{\mathcal{H}_*} \left(-(\tilde{\mathbf{R}} * \phi)_I + (\underline{\mu} - \omega \text{tr} \tilde{\chi}) \phi_I \right) A^I \mathbf{b} d\mu, \end{aligned} \quad (2.54)$$

where $d\mu = nd\mu_{S_t} dt$ on \mathcal{H}_* , $\underline{\mu}$ and $\tilde{\zeta}$ are defined in (2.37), and for $I = \{\mu_1, \dots, \mu_l\}$,

$$(\tilde{\mathbf{R}} * \phi)_I = \sum_{i=1}^l \tilde{\mathbf{R}}_{\mu_i}^{\alpha} \tilde{\underline{L}}_{\tilde{L}} \phi_{\mu_1 \dots \mu_{i-1} \alpha \mu_{i+1} \dots \mu_l}. \quad (2.55)$$

In Section 3, we will give an alternative proof for the Kirchhoff formula in Proposition 2.10. In the sequel we will prove Theorem 2.3 by using Proposition 2.10.

2.3. Proof of Theorem 2.3. We first give the following result which represents $\text{tr} \tilde{\chi}$, $\tilde{\zeta}$, $\tilde{\mathbf{R}}$ and $\underline{\mu} - \omega \text{tr} \tilde{\chi}$ in terms of the quantities in $(\mathcal{M}, \mathbf{g})$.

Lemma 2.11.

$$\underline{\zeta}^\beta = -\frac{\tilde{r}}{\mathbf{b}^{-1}\tau}\underline{\zeta}^\mathcal{A}e_\mathcal{A}^\beta - \frac{\rho^2}{\mathbf{b}^{-1}\tau\tilde{r}}\omega\mathbf{N}^\beta, \quad (2.56)$$

$$\mathrm{tr}\tilde{\chi} = -\frac{\rho}{\mathbf{b}^{-1}\tau}\mathrm{tr}k + 2\left(\mathrm{Tr}\boldsymbol{\pi} - \frac{\tilde{r}^2}{\mathbf{b}^{-2}\tau^2}\boldsymbol{\pi}\mathbf{N}\mathbf{N}\right), \quad (2.57)$$

$$\begin{aligned} \underline{\mu} - \omega\mathrm{tr}\tilde{\chi} &= 2\frac{\rho}{\mathbf{b}^{-1}\tau}\omega\mathrm{tr}k + \frac{\rho^2}{\mathbf{b}^{-2}\tau^2}(\mathbf{R}_{\mathfrak{B}\mathfrak{B}} + |\hat{k}|^2) \\ &\quad + 2\frac{\rho}{\mathbf{b}^{-1}\tau}\left(\mathfrak{B} + \frac{\mathrm{tr}k}{2} - \frac{\mathbf{b}^{-1}\tau\omega}{\rho}\right)\left(\mathrm{Tr}\boldsymbol{\pi} - \frac{\tilde{r}^2}{\mathbf{b}^{-2}\tau^2}\boldsymbol{\pi}\mathbf{N}\mathbf{N}\right). \end{aligned} \quad (2.58)$$

$$\frac{1}{2}\tilde{\mathbf{R}}_{\beta}^{\alpha}{}_{\tilde{L}\tilde{L}} = \frac{\tilde{r}}{\mathbf{b}^{-1}\tau}\mathbf{R}_{\beta}^{\alpha}{}_{\mathbf{T}\mathbf{N}}. \quad (2.59)$$

Proof. We first derive (2.56). Note that $\underline{\zeta}^\beta = \frac{1}{2}(\tilde{\mathbf{D}}_{\tilde{L}}\tilde{\underline{L}})^{\tilde{\alpha}}\Pi_{\tilde{\alpha}}^\beta$. By using (2.31) and $\tilde{\underline{L}}^{\tilde{\alpha}}\Pi_{\tilde{\alpha}}^\beta = 0$ we have

$$\underline{\zeta}^\beta = \frac{1}{2}\frac{\rho}{\mathbf{b}^{-1}\tau}\tilde{\mathbf{D}}_{\tilde{L}}(\mathfrak{B} - \partial_z)^{\tilde{\alpha}}\Pi_{\tilde{\alpha}}^\beta = \frac{1}{2}\frac{\rho^2}{\mathbf{b}^{-2}\tau^2}\tilde{\mathbf{D}}_{\mathfrak{B}+\partial_z}(\mathfrak{B} - \partial_z)^{\tilde{\alpha}}\Pi_{\tilde{\alpha}}^\beta.$$

In view of (2.25), (2.26) and the fact that \mathfrak{B} is independent of z , we then obtain

$$\underline{\zeta}^\beta = \frac{1}{2}\frac{\rho^2}{\mathbf{b}^{-2}\tau^2}\tilde{\mathbf{D}}_{\mathfrak{B}}\mathfrak{B}^{\tilde{\alpha}}\Pi_{\tilde{\alpha}}^\beta = \frac{1}{2}\frac{\rho^2}{\mathbf{b}^{-2}\tau^2}\mathbf{D}_{\mathfrak{B}}\mathfrak{B}^{\alpha}\mathbb{I}_{\alpha}^\beta$$

where we also used $\Pi_{\alpha}^\beta = \mathbb{I}_{\alpha}^\beta$ due to (2.32). In view of the above identity and (2.35) we can see that

$$\underline{\zeta}^\beta = \frac{1}{2}\frac{\rho^2}{\mathbf{b}^{-2}\tau^2}\left(\langle\mathbf{D}_{\mathfrak{B}}\mathfrak{B}, e_{\mathcal{A}}\rangle e_{\mathcal{A}}^\beta + \frac{\rho^2}{\mathbf{b}^{-2}\tau^2}\langle\mathbf{D}_{\mathfrak{B}}\mathfrak{B}, \mathbf{N}\rangle\mathbf{N}^\beta\right). \quad (2.60)$$

By using (2.10) we have

$$\begin{aligned} \mathbf{D}_{\mathfrak{B}}\mathfrak{B} &= -\mathbf{D}_{\mathfrak{B}}\left(\frac{\mathbf{b}^{-1}\tau}{\rho}\mathbf{T} + \frac{\tilde{r}}{\rho}\mathbf{N}\right) \\ &= -\mathfrak{B}\left(\frac{\mathbf{b}^{-1}\tau}{\rho}\right)\mathbf{T} - \mathfrak{B}\left(\frac{\tilde{r}}{\rho}\right)\mathbf{N} - \frac{\mathbf{b}^{-1}\tau}{\rho}\mathbf{D}_{\mathfrak{B}}\mathbf{T} - \frac{\tilde{r}}{\rho}\mathbf{D}_{\mathfrak{B}}\mathbf{N}. \end{aligned} \quad (2.61)$$

Since $\langle\mathbf{T}, e_{\mathcal{A}}\rangle = \langle\mathbf{N}, e_{\mathcal{A}}\rangle = 0$, it follows that

$$\langle\mathbf{D}_{\mathfrak{B}}\mathfrak{B}, e_{\mathcal{A}}\rangle = -\frac{\mathbf{b}^{-1}\tau}{\rho}\langle\mathbf{D}_{\mathfrak{B}}\mathbf{T}, e_{\mathcal{A}}\rangle - \frac{\tilde{r}}{\rho}\langle\mathbf{D}_{\mathfrak{B}}\mathbf{N}, e_{\mathcal{A}}\rangle. \quad (2.62)$$

From (2.14) we have $\langle\mathbf{D}_{\mathfrak{B}}\mathbf{T}, e_{\mathcal{A}}\rangle = \frac{\tilde{r}}{\rho}\underline{\zeta}_{\mathcal{A}}^\beta$. Recall from [25, page 17] that $\nabla_{\mathcal{A}}\log a = -\langle\mathbf{D}_{\mathbf{N}}\mathbf{N}, e_{\mathcal{A}}\rangle$. This together with (2.9) and (2.18) gives

$$\begin{aligned} \langle\mathbf{D}_{\mathfrak{B}}\mathbf{N}, e_{\mathcal{A}}\rangle &= -\frac{\mathbf{b}^{-1}\tau}{\rho}\langle\mathbf{D}_{\mathbf{T}}\mathbf{N}, e_{\mathcal{A}}\rangle + \frac{\tilde{r}}{\rho}\langle\mathbf{D}_{\mathbf{N}}\mathbf{N}, e_{\mathcal{A}}\rangle \\ &= -\frac{\mathbf{b}^{-1}\tau}{\rho}\left(k_{\mathcal{N}\mathcal{A}} + \frac{\mathbf{b}^{-1}\tau}{\tilde{r}}\langle\mathbf{D}_{\mathbf{T}}\mathbf{T}, e_{\mathcal{A}}\rangle\right) - \frac{\tilde{r}}{\rho}\nabla_{\mathcal{A}}\log a. \end{aligned}$$

By using (2.17) and (2.14), we then obtain

$$\begin{aligned}\langle \mathbf{D}_{\mathfrak{B}} \mathbf{N}, e_{\mathcal{A}} \rangle &= -\frac{\mathbf{b}^{-1}\tau}{\rho} \left(k_{\mathcal{N}\mathcal{A}} + \frac{\mathbf{b}^{-1}\tau}{\tilde{r}} \langle \mathbf{D}_{\mathbf{T}} \mathbf{T}, e_{\mathcal{A}} \rangle \right) - \frac{\tilde{r}}{\rho} \frac{\mathbf{b}^{-1}\tau}{\tilde{r}} (\pi_{\mathbf{N}\mathcal{A}} - k_{\mathcal{N}\mathcal{A}}) \\ &= \frac{\mathbf{b}^{-1}\tau}{\rho} \left(-\frac{\mathbf{b}^{-1}\tau}{\tilde{r}} \langle \mathbf{D}_{\mathbf{T}} \mathbf{T}, e_{\mathcal{A}} \rangle + \langle \mathbf{D}_{\mathbf{N}} \mathbf{T}, e_{\mathcal{A}} \rangle \right) \\ &= \frac{\mathbf{b}^{-1}\tau}{\tilde{r}} \langle \mathbf{D}_{\mathfrak{B}} \mathbf{T}, e_{\mathcal{A}} \rangle = \frac{\mathbf{b}^{-1}\tau}{\rho} \zeta_{\mathcal{A}}.\end{aligned}$$

Combining the above results with (2.62), we obtain

$$\langle \mathbf{D}_{\mathfrak{B}} \mathfrak{B}, e_{\mathcal{A}} \rangle = -2 \frac{\mathbf{b}^{-1}\tau \tilde{r}}{\rho^2} \zeta_{\mathcal{A}}. \quad (2.63)$$

Next we calculate $\langle \mathbf{D}_{\mathfrak{B}} \mathfrak{B}, \mathbf{N} \rangle$. It follows from (2.61) that

$$\langle \mathbf{D}_{\mathfrak{B}} \mathfrak{B}, \mathbf{N} \rangle = -\mathfrak{B} \left(\frac{\tilde{r}}{\rho} \right) - \frac{\mathbf{b}^{-1}\tau}{\rho} \langle \mathbf{D}_{\mathfrak{B}} \mathbf{T}, \mathbf{N} \rangle. \quad (2.64)$$

We have $\mathfrak{B}(\tilde{r}/\rho) = \mathfrak{B}(\tilde{r})/\rho - \tilde{r}/\rho^2$. From $\tilde{r}^2 = \mathbf{b}^{-2}\tau^2 - \rho^2$ it follows that $2\tilde{r}\mathfrak{B}(\tilde{r}) = 2\mathbf{b}^{-1}\tau\mathfrak{B}(\mathbf{b}^{-1}\tau) - 2\rho$. Thus

$$\mathfrak{B}(\tilde{r}) = \frac{\mathbf{b}^{-1}\tau}{\tilde{r}} \mathfrak{B}(\mathbf{b}^{-1}\tau) - \frac{\rho}{\tilde{r}}.$$

Consequently

$$\mathfrak{B} \left(\frac{\tilde{r}}{\rho} \right) = \frac{\mathbf{b}^{-1}\tau}{\rho \tilde{r}} \mathfrak{B}(\mathbf{b}^{-1}\tau) - \frac{1}{\tilde{r}} - \frac{\tilde{r}}{\rho^2} = \frac{\mathbf{b}^{-1}\tau}{\rho \tilde{r}} \mathfrak{B}(\mathbf{b}^{-1}\tau) - \frac{\mathbf{b}^{-2}\tau^2}{\rho^2 \tilde{r}}.$$

In view of (2.5), we have $\mathfrak{B}(\tau) = \frac{n^{-1}\mathbf{b}^{-1}\tau}{\rho}$. This together with (2.16) shows that

$$\mathfrak{B}(\mathbf{b}^{-1}\tau) = \mathbf{b}^{-1}\mathfrak{B}(\tau) + \tau\mathfrak{B}(\mathbf{b}^{-1}) = \frac{\mathbf{b}^{-1}\tau}{\rho} + \frac{\mathbf{b}^{-2}\tau^2}{\rho}\omega.$$

Therefore

$$\mathfrak{B} \left(\frac{\tilde{r}}{\rho} \right) = \frac{\mathbf{b}^{-3}\tau^3}{\rho^2 \tilde{r}} \omega.$$

Combining this with (2.64) and using (2.42) and (2.15) we obtain

$$\langle \mathbf{D}_{\mathfrak{B}} \mathfrak{B}, \mathbf{N} \rangle = -\frac{\mathbf{b}^{-3}\tau^3}{\rho^2 \tilde{r}} \omega - \frac{\mathbf{b}^{-1}\tau}{\rho} \langle \mathbf{D}_{\mathfrak{B}} \mathbf{T}, \mathbf{N} \rangle = -2 \frac{\mathbf{b}^{-3}\tau^3}{\rho^2 \tilde{r}} \omega. \quad (2.65)$$

The combination of (2.60), (2.63) and (2.65) then shows (2.56).

The equation (2.57) follows directly from (2.38), (2.39) and (2.40) in Lemma 2.8.

Next we derive (2.58). Note that $\text{tr} \tilde{\chi}$ is independent of z . It follows from (2.31), (2.38), the definition of $\underline{\mu}$ from (2.37) and (2.57),

$$\begin{aligned}\underline{\mu} &= \frac{\rho}{\mathbf{b}^{-1}\tau} \left(\mathfrak{B} + \frac{1}{2} \text{tr} k \right) \text{tr} \tilde{\chi} \\ &= \frac{\rho}{\mathbf{b}^{-1}\tau} \left(\mathfrak{B} + \frac{1}{2} \text{tr} k \right) \left(-\frac{\rho}{\mathbf{b}^{-1}\tau} \text{tr} k + 2 \left(\text{Tr} \pi - \frac{\tilde{r}^2}{\mathbf{b}^{-2}\tau^2} \pi_{\mathbf{N}\mathbf{N}} \right) \right).\end{aligned} \quad (2.66)$$

In view of Definition 2.1 (3), we have

$$\left(\mathfrak{B} + \frac{1}{2} \text{tr} k \right) \left(-\frac{\rho}{\mathbf{b}^{-1}\tau} \text{tr} k \right) = \omega \text{tr} k - \frac{\rho}{\mathbf{b}^{-1}\tau} \left(\mathfrak{B} \text{tr} k + \frac{1}{2} (\text{tr} k)^2 \right).$$

By using the Raychaudhuri equation [25, Section 3]

$$\mathfrak{B} \operatorname{tr} k + \frac{1}{2} (\operatorname{tr} k)^2 = -\mathbf{R}_{\mathfrak{B}\mathfrak{B}} - |\hat{k}|^2, \quad (2.67)$$

we further obtain

$$\left(\mathfrak{B} + \frac{1}{2} \operatorname{tr} k \right) \left(-\frac{\rho}{\mathbf{b}^{-1}\tau} \operatorname{tr} k \right) = \operatorname{tr} k \omega + \frac{\rho}{\mathbf{b}^{-1}\tau} (\mathbf{R}_{\mathfrak{B}\mathfrak{B}} + |\hat{k}|^2).$$

Therefore, it follows from (2.66) that

$$\underline{\mu} = \frac{\rho}{\mathbf{b}^{-1}\tau} \left[\omega \operatorname{tr} k + \frac{\rho}{\mathbf{b}^{-1}\tau} (\mathbf{R}_{\mathfrak{B}\mathfrak{B}} + |\hat{k}|^2) + 2 \left(\mathfrak{B} + \frac{1}{2} \operatorname{tr} k \right) \left(\operatorname{Tr} \boldsymbol{\pi} - \frac{\tilde{r}^2}{\mathbf{b}^{-2}\tau^2} \boldsymbol{\pi}_{\mathbf{N}\mathbf{N}} \right) \right].$$

This together with (2.57) shows (2.58).

Finally we prove (2.59). From the identity

$$\tilde{\mathbf{R}}_{\beta}^{\alpha}{}_{\gamma\delta} = \partial_{\delta} \tilde{\Gamma}_{\beta\gamma}^{\alpha} - \partial_{\gamma} \tilde{\Gamma}_{\beta\delta}^{\alpha} + \tilde{\Gamma}_{\delta\eta}^{\alpha} \tilde{\Gamma}_{\beta\gamma}^{\eta} - \tilde{\Gamma}_{\gamma\eta}^{\alpha} \tilde{\Gamma}_{\beta\delta}^{\eta}$$

and (2.25) it follows that

$$\tilde{\mathbf{R}}_{\beta}^{\alpha}{}_{z\gamma} = 0 \quad \text{and} \quad \tilde{\mathbf{R}}_{\beta}^{\alpha}{}_{\gamma\delta} = \mathbf{R}_{\beta}^{\alpha}{}_{\gamma\delta}.$$

Therefore, by using (2.9), (2.10), (2.31) we can obtain (2.59). \square

Proof of Theorem 2.3. For A_I satisfying (2.22), by viewing it as an \mathcal{M} -tangent tensor in $\tilde{\mathcal{M}}$, we may use (2.31), (2.38) and (2.25) to verify that A_I satisfies (2.52) along \mathcal{H}_* . Thus, we may represent ϕ_I by the formula (2.53) in Proposition 2.10. Since ϕ_I is \mathcal{M} -tangent, we have from (2.29), (2.31) and (2.57) that

$$\begin{aligned} \int_{\mathcal{H}_*} \square_{\mathbf{g}} \phi_I A^I \mathbf{b} d\mu &= \int_{t_p-t_0}^{t_p} \int_{S_t} \square_{\mathbf{g}} \phi_I A^I \mathbf{b} n d\mu_{S_t} dt = 2 \int_{t_p-t_0}^{t_p} \int_{\Sigma_t} \square_{\mathbf{g}} \phi_I A^I \frac{\tau}{\rho} n d\mu_{\Sigma_t} dt \\ &= 2 \int_{\mathcal{I}_*^-(p)} F_I A^I \frac{\tau}{\rho} n d\mu_{\Sigma_t} dt \end{aligned}$$

and

$$\begin{aligned} &\int_{S_{t_0}} \left(\tilde{\mathbf{D}}_{\underline{L}} \phi_I + \frac{1}{2} \operatorname{tr} \tilde{\chi} \phi_I \right) A^I d\mu_{S_{t_0}} \\ &= \int_{S_{t_0}} \left[\frac{\rho}{\mathbf{b}^{-1}\tau} \mathbf{D}_{\mathfrak{B}} \phi_I - \frac{1}{2} \frac{\rho}{\mathbf{b}^{-1}\tau} \operatorname{tr} k \phi_I + \left(\operatorname{Tr} \boldsymbol{\pi} - \frac{\tilde{r}^2}{\mathbf{b}^{-2}\tau^2} \boldsymbol{\pi}_{\mathbf{N}\mathbf{N}} \right) \phi_I \right] A^I d\mu_{S_{t_0}} \\ &= 2 \int_{\Sigma_{t_0}} \left[\mathbf{D}_{\mathfrak{B}} \phi_I - \frac{1}{2} \operatorname{tr} k \phi_I + \frac{\mathbf{b}^{-1}\tau}{\rho} \left(\operatorname{Tr} \boldsymbol{\pi} - \frac{\tilde{r}^2}{\mathbf{b}^{-2}\tau^2} \boldsymbol{\pi}_{\mathbf{N}\mathbf{N}} \right) \phi_I \right] A^I d\mu_{\Sigma_{t_0}}. \end{aligned}$$

Therefore

$$4\pi(n\mathbf{g}(\phi, J))(p) = -2 \int_{\mathcal{I}_*^-(p)} F_I A^I \frac{\tau}{\rho} n d\mu_{\Sigma_t} dt + 2\mathcal{I}_1 + Er[\phi], \quad (2.68)$$

where \mathcal{I}_1 is given in Theorem 2.3 and $Er[\phi]$ is given by (2.54) in Proposition 2.10.

We need to consider $Er[\phi]$. By the divergence theorem, (2.29) and (2.35), we have

$$\begin{aligned} \int_{S_t} \tilde{\Delta} \phi_I A^I \mathbf{b} d\mu_{S_t} &= - \int_{S_t} \tilde{\nabla} \phi_I \tilde{\nabla} (A^I \mathbf{b}) d\mu_{S_t} \\ &= -2 \int_{\Sigma_t} \left[\nabla \phi_I \nabla (A^I \mathbf{b}) + \frac{\rho^2}{\mathbf{b}^{-2}\tau^2} \mathbf{D}_{\mathbf{N}} (\mathbf{b} A^I) \mathbf{D}_{\mathbf{N}} \phi_I \right] \frac{\mathbf{b}^{-1}\tau}{\rho} n d\mu_{\Sigma_t}. \end{aligned}$$

In view of (2.56) we then obtain

$$\int_{\mathcal{H}_*} \left(2\tilde{\zeta}^C \tilde{D}_C \phi_I + \tilde{\Delta} \phi_I \right) A^I \mathbf{b} d\mu = 2\mathcal{I}_2$$

with \mathcal{I}_2 defined in Theorem 2.3. In view of (2.58), (2.11) and (2.29) we can see that

$$\frac{1}{2} \int_{\mathcal{H}_*} \left(-(\tilde{\mathbf{R}} * \phi)_I + (\underline{\mu} - \omega \text{tr} \tilde{\chi}) \phi_I \right) A^I \mathbf{b} d\mu = 2\mathcal{I}_3,$$

where \mathcal{I}_3 is defined in Theorem 2.3. Consequently $Er[\phi] = 2\mathcal{I}_2 + 2\mathcal{I}_3$. Combining this with (2.68) we therefore complete the proof. \square

3. Appendix. Proof of Proposition 2.10

In this section we give an alternative proof of the Kirchhoff formula in Proposition 2.10 by using a multiplier type approach. As in [23], we can decompose $\square_{\tilde{\mathbf{g}}} \phi_I$ as

$$\begin{aligned} \square_{\tilde{\mathbf{g}}} \phi_I &= -\tilde{\mathbf{D}}_{\tilde{L}}(\tilde{\mathbf{D}}_{\tilde{L}} \phi)_I + 2\tilde{\zeta}^C \tilde{\mathbf{D}}_C \phi_I - \frac{1}{2} \text{tr} \tilde{\chi} \tilde{\mathbf{D}}_{\tilde{L}} \phi_I - \frac{1}{2} (\text{tr} \tilde{\chi} - 2\omega) \tilde{\mathbf{D}}_{\tilde{L}} \phi_I \\ &\quad + \delta^{AB} \tilde{\nabla}_A \tilde{\nabla}_B \phi_I - \frac{1}{2} (\tilde{\mathbf{R}} * \phi)_I, \end{aligned} \quad (3.1)$$

where $(\tilde{\mathbf{R}} * \phi)_I$ is defined by (2.55).

Let $v = v_t$ and $\psi_I = v^{\frac{1}{2}} \phi_I$. Note that $\tilde{L}v = \text{tr} \tilde{\chi} v$ and $\tilde{L}v = \text{tr} \tilde{\chi} v$. We can derive that

$$\begin{aligned} \tilde{\mathbf{D}}_{\tilde{L}}(\tilde{\mathbf{D}}_{\tilde{L}} \psi)_I &= \tilde{\mathbf{D}}_{\tilde{L}} \left(v^{\frac{1}{2}} \tilde{\mathbf{D}}_{\tilde{L}} \phi + \frac{1}{2} v^{\frac{1}{2}} \text{tr} \tilde{\chi} \phi \right)_I \\ &= v^{\frac{1}{2}} \tilde{\mathbf{D}}_{\tilde{L}}(\tilde{\mathbf{D}}_{\tilde{L}} \phi)_I + \frac{1}{2} v^{\frac{1}{2}} \left(\text{tr} \tilde{\chi} \tilde{\mathbf{D}}_{\tilde{L}} \phi_I + \text{tr} \tilde{\chi} \tilde{\mathbf{D}}_{\tilde{L}} \phi_I \right) + \frac{1}{2} \mu \psi_I. \end{aligned}$$

By contracting (3.1) with $\mathbf{b}vA$ and using the above identity it follows that

$$\begin{aligned} \square_{\tilde{\mathbf{g}}} \phi_I A^I \mathbf{b} v &= - \left(\tilde{\mathbf{D}}_{\tilde{L}}(\tilde{\mathbf{D}}_{\tilde{L}} \psi)_I - \frac{1}{2} \mu \psi_I \right) A^I v^{\frac{1}{2}} \mathbf{b} + \left(2\tilde{\zeta}^C \tilde{\mathbf{D}}_C \phi_I + \omega \tilde{\mathbf{D}}_{\tilde{L}} \phi_I \right) A^I v \mathbf{b} \\ &\quad + \tilde{\Delta} \phi_I A^I v \mathbf{b} - \frac{1}{2} (\tilde{\mathbf{R}} * \phi)_I A^I v \mathbf{b}. \end{aligned} \quad (3.2)$$

Note that the first term on the right hand side can be recast as

$$\begin{aligned} -\tilde{\mathbf{D}}_{\tilde{L}}(\tilde{\mathbf{D}}_{\tilde{L}} \psi)_I A^I v^{\frac{1}{2}} \mathbf{b} &= -\tilde{L} \left(\mathbf{b} \tilde{\mathbf{D}}_{\tilde{L}} \psi_I A^I v^{\frac{1}{2}} \right) + \tilde{\mathbf{D}}_{\tilde{L}}(v^{\frac{1}{2}} \mathbf{b} A)^I \tilde{\mathbf{D}}_{\tilde{L}} \psi_I \\ &= -\tilde{L} \left(\mathbf{b} \tilde{\mathbf{D}}_{\tilde{L}} \psi_I A^I v^{\frac{1}{2}} \right) + \mathbf{b} \left(A^I v^{\frac{1}{2}} \tilde{L} \log \mathbf{b} + \tilde{\mathbf{D}}_{\tilde{L}}(v^{\frac{1}{2}} A)^I \right) \tilde{\mathbf{D}}_{\tilde{L}} \psi_I \\ &= -\tilde{L} \left(\mathbf{b} \tilde{\mathbf{D}}_{\tilde{L}} \psi_I A^I v^{\frac{1}{2}} \right) \\ &\quad + \mathbf{b} \left(\left(\frac{n^{-1} - \mathbf{b}}{\tau} - \omega \right) A^I + \left(\tilde{\mathbf{D}}_{\tilde{L}} A^I + \frac{1}{2} \text{tr} \tilde{\chi} A^I \right) \right) v^{\frac{1}{2}} \tilde{\mathbf{D}}_{\tilde{L}} \psi_I, \end{aligned}$$

where for the last equality we employed (2.43). By using (2.52) and $\tilde{L}v = \text{tr} \tilde{\chi} v$, we have

$$-\tilde{\mathbf{D}}_{\tilde{L}}(\tilde{\mathbf{D}}_{\tilde{L}} \psi)_I A^I v^{\frac{1}{2}} \mathbf{b} = -\tilde{L} \left(\mathbf{b} \tilde{\mathbf{D}}_{\tilde{L}} \psi_I A^I v^{\frac{1}{2}} \right) - \mathbf{b} \omega \left(\tilde{\mathbf{D}}_{\tilde{L}} \phi_I + \frac{1}{2} \text{tr} \tilde{\chi} \phi_I \right) A^I v. \quad (3.3)$$

Substituting (3.3) into (3.2) and integrating along \mathcal{H}_* , we have, in view of $\frac{d}{d\tau} = n\tilde{L}$, that

$$\int_{\mathcal{H}_*} \square_{\tilde{\mathbf{g}}} \phi_I A^I \mathbf{b} v d\mu = - \int_{S_{t_0}} \mathbf{b} \tilde{\mathbf{D}}_{\underline{L}} \psi_I A^I v^{-\frac{1}{2}} d\mu_{S_{t_0}} + \lim_{t \rightarrow t_p} \int_{S_t} \mathbf{b} \tilde{\mathbf{D}}_{\underline{L}} \psi_I A^I v^{-\frac{1}{2}} d\mu_{S_t} + \text{Er}[\phi],$$

where $\text{Er}[\phi]$ is defined in Proposition 2.10 and $d\mu = nd\mu_{S_t} dt$. We claim that

$$\lim_{t \rightarrow t_p} \int_{S_t} \tilde{\mathbf{D}}_{\underline{L}} \psi_I A^I v^{-\frac{1}{2}} d\mu_{S_t} = -4\pi(n\langle\phi, J\rangle)(p). \quad (3.4)$$

Using this claim we can conclude that

$$\int_{\mathcal{H}_*} \square_{\tilde{\mathbf{g}}} \phi_I A^I d\mu = - \int_{S_{t_0}} \mathbf{b} \tilde{\mathbf{D}}_{\underline{L}} \psi_I A^I v^{-\frac{1}{2}} d\mu_{S_{t_0}} - 4\pi(n\langle\phi, J\rangle)(p) + \text{Er}[\phi]$$

which is the desired formula.

To obtain (3.4), we note that

$$\tilde{\mathbf{D}}_{\underline{L}} \psi_I A^I v^{-\frac{1}{2}} = \left(\frac{1}{2} \text{tr} \tilde{\chi} \phi_I + \tilde{\mathbf{D}}_{\underline{L}} \phi_I \right) A^I. \quad (3.5)$$

Recall that $\tau = t_p - t$. By using (2.57) and the boundedness of $\text{Tr} \pi$ and $\pi_{\mathbf{NN}}$ near p we have $\tau \text{tr} \tilde{\chi} + \mathbf{b} \rho \text{tr} k \rightarrow 0$ as $t \rightarrow t_p$. Recall also that

$$\rho \text{tr} k \rightarrow 2, \quad \mathbf{b} \rightarrow \frac{1}{n(p)} \quad \text{and} \quad \frac{|S_t|}{4\pi n^2 \tau^2} \rightarrow 1 \quad \text{as } t \rightarrow t_p$$

which have been proved in [18, 25, 24]. We therefore obtain $\tau \text{tr} \tilde{\chi} \rightarrow -2/n(p)$ and hence, by using (3.5) and $\lim_{t \rightarrow t_p} \tau A_I = J_I$ in (2.52), we can obtain (3.4).

REFERENCES

- [1] Choquet-Bruhat, Y. *Théorème d'existence pour certains systèmes d'équations aux dérivées partielles non linéaires*. Acta Math. 88 (1952), 141-225.
- [2] Choquet-Bruhat, Y. *General relativity and the Einstein equations*, Oxford Mathematical Monographs. Oxford University Press, Oxford, 2009.
- [3] Choquet-Bruhat, Y. and Moncrief, V. *Future global in time Einsteinian spacetimes with $U(1)$ isometry group*. Ann. Henri Poincaré 2 (2001), no. 6, 1007-1064.
- [4] Christodoulou, D. and Klainerman, S. *The Global Nonlinear Stability of Minkowski Space*, Princeton Mathematical Series 41, 1993.
- [5] Chrusciel, P. and Shatah, J., *Global existence of solutions of the Yang-Mills equations on globally hyperbolic four-dimensional Lorentzian manifolds*, Asian J. Math. 1 (1997), no. 3, 530-548.
- [6] Eardley, D., Moncrief, V., *The global existence of Yang-Mills-Higgs fields in 4-dimensional Minkowski space II. Completion of proof*, Comm. Math. Phys. 83 (1982), no. 2, 193-212.
- [7] Evans, L. *Partial differential equations*. Second edition. Graduate Studies in Mathematics, 19. American Mathematical Society, Providence, RI, 2010. xxii+749 pp.
- [8] Friedlander, H. G., *The Wave Equation on a Curved Space-time*, Cambridge University Press, 1976.
- [9] Hawking, S. W. and Ellis, G. F. R., *The large scale structure of space-time*, Cambridge Monographs on Mathematical Physics, 1. Cambridge University Press, London New York, 1973.
- [10] Klainerman, S. and Rodnianski, I., *Causal geometry of Einstein-vacuum spacetimes with finite curvature flux*. Invent. Math. 159 (2005), no. 3, 437-529.
- [11] Klainerman, S. and Rodnianski, I., *A Kirchhoff-Sobolev parametrization for the wave equation and applications*, J. Hyperbolic Differ. Equ. 4 (2007), no. 3, 401-433.
- [12] Klainerman, S. and Rodnianski, I., *On the radius of injectivity of null hypersurfaces*, J. Amer. Math. Soc. 21 (2008), no. 3, 775-795.

- [13] Klainerman, S. and Rodnianski, I., *Bilinear estimates on curved space-times*, J. Hyperbolic Differ. Equ. 2(2), 279-291 (2005)
- [14] Klainerman, S. and Rodnianski, I., *On the breakdown criterion in general relativity*, J. Amer. Math. Soc., 23 (2010), no. 2, 345–382.
- [15] Moncrief, V. *An integral equation for spacetime curvature in general relativity*. Surveys in differential geometry. Vol. X, 109146, Surv. Differ. Geom., 10, Int. Press, Somerville, MA, 2006.
- [16] Moncrief, V. *Reflections on the $U(1)$ problem in general relativity*. J. Fixed Point Theory Appl. 14 (2013), no. 2, 397-418.
- [17] Moncrief, V. *Convergence and stability issues in mathematical cosmology*. General relativity and gravitation, 480-498, Cambridge Univ. Press, Cambridge, 2015.
- [18] Poisson, E., Pound A., and Vega, I., *The Motion of Point Particles in Curved Spacetime*, Living Rev. Relativity 14, (2011), 7. <http://www.livingreviews.org/lrr-2011-7>
- [19] S. Sobolev, *Methodes nouvelle a resoudre le probleme de Cauchy pour les equations lineaires hyperboliques normales*, Matematicheskii Sbornik, vol 1 (43) 1936, 31–79.
- [20] Shao, A., *Breakdown Criteria for Einstein Equations with Matters*, PhD thesis, Princeton University, 2010.
- [21] Smith, H. F., *A parametrix construction for wave equations with $C^{1,1}$ coefficients*, Ann. Inst. Fourier (Grenoble), 48 (1998), no. 3, 797–835.
- [22] Wang, Q., *Causal Geometry of Einstein-Vacuum Spacetimes*, PhD thesis, Princeton University, 2006.
- [23] Wang, Q., *Improved breakdown criterion for Einstein vacuum equations in CMC gauge*, Comm. Pure Appl. Math., Vol. LXV, 21–76 (2012).
- [24] Wang, Q., *Rough solutions of Einstein vacuum equations in CMCSH gauges*, Comm. Math. Phys., 328 (2014), no. 3, 1275–1340.
- [25] Wang, Q., *An intrinsic hyperboloid approach for Einstein Klein-Gordon equations*, preprint 2016, Arxiv:1607.01466

OXFORD PDE CENTER, MATHEMATICAL INSTITUTE, UNIVERSITY OF OXFORD, OXFORD, OX2 6GG, UK

E-mail address: `qian.wang@maths.ox.ac.uk`