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Regularity Theory for Variational Problems on BD

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Declaration of Authorship

I hereby declare that this thesis is entirely the work of my own except where otherwise indicated. All direct or indirect sources are acknowledged as references. It has not been submitted, either wholly or substantially, for another course of this Department or University, or for a course at any other institution.

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Franz Gmeineder

To my parents
for never asking why
and to Tabea
for telling me why

Abstract

The aim of this thesis is to contribute to the regularity of minima in the linear growth context. More precisely, we study (semi–)convex variational integrals of the form

$$\mathfrak{F}_1[u] := \int_\Omega f(\mathrm{D}u)\,\mathrm{d}x \quad \text{and} \quad \mathfrak{F}_2[u] := \int_\Omega f(\boldsymbol{\varepsilon}(u))\,\mathrm{d}x,$$

where Ω is an open and bounded Lipschitz subset of $\mathbb{R}^n$ and $f\colon \mathbb{R}^{n\times n} \to \mathbb{R}$ satisfies

$$c_1|z| \leqslant f(z) \leqslant c_2(1+|z|)$$

for all $z \in \mathbb{R}^{n\times n}$ or $z \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$, respectively, within the framework of Dirichlet or Neumann problems. Such problems are typically relaxed to and studied on spaces of weakly differentiable functions for which $\mathrm{D}u$ or the symmetric gradient $\boldsymbol{\varepsilon}(u) := \frac{1}{2}(\mathrm{D}u + \mathrm{D}^{\mathsf{T}}u)$ are finite matrix–valued Radon measures. This leads to the space of functions of bounded variation BV in the gradient case, and to the space of functions of bounded deformation BD in the symmetric gradient case. Most notably, by Ornstein's Non–Inequality, $\mathrm{BV} \subsetneq \mathrm{BD}$, and in general, elements of BD need not have distributional gradients which are measures. In a natural way, this raises the question of which conditions need to be imposed on f to obtain minima whose gradients or symmetric gradients belong to L^1 instead of merely qualifying as Radon measures. Moreover, in the symmetric gradient case, we also wish to find conditions on the integrands to produce minima in BV. Apart from carefully examining inequalities of Korn–type and non–inequalities of Ornstein–type, we establish the following regularity results:

In the convex linear growth situation, we prove that for a certain range of strong ellipticity (and under additional assumptions), solutions of the Dirichlet problem for $\mathfrak{F}_2$ indeed belong to some $\mathrm{W}^{1,p}_{\mathrm{loc}}$ for some $1 < p < \infty$. This makes use of the scale of μ–elliptic integrands, and a similar result was previously only known for functionals of the type $\mathfrak{F}_1$. In employing a difference quotient–type approach, we need to deal with the non–availability of full derivatives in L^1. In doing so, a crucial tool is provided by convex analysis and the regularity of the dual solution. This comprises a joint work with J. Kristensen (Oxford).

For a convex Neumann problem in the full gradient case related to $\mathfrak{F}_1$, we subsequently establish a $\mathrm{W}^{1,1}$–regularity result. This is the first $\mathrm{W}^{1,1}$–regularity result for vectorial problems of linear growth without requiring a strong convexity (such as μ–ellipticity) condition. This result has been obtained in collaboration with L. Beck (Augsburg) and M. Bulicek (Prague).

The third main regularity result concerns the partial regularity of generalised minima in the convex setting. Partial regularity, i.e., Hölder continuity apart from a relatively closed set of Lebesgue measure zero, is the standard notion for Hölder regularity in the vectorial setting. Here, adapting a strategy which was employed by Anzellotti & Giaquinta [22] first, we show how partial regularity can be established in the symmetric–convex case, too. We provide a novel Poincaré–type inequality to overcome the problems arising from the aforementioned approach. In particular, the result is 'local–in–phase–space' and does not require any strong ellipticity condition on the integrands.

In the final part of this thesis, we examine the regularity theory for symmetric–quasiconvex and rank–one convex functionals of linear growth. Here, on the one hand, we give a partial regularity result based on the recent work [134] of Kristensen and the author in the BV–case. This, in turn, contributes to the partial regularity program for quasiconvex variational integrals as initiated by Evans [99] and Acerbi & Fusco [3]. We supply this result with an ε–regularity result which yields full Hölder continuity under suitable smallness and smoothness assumptions on the Dirichlet data.

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Contents

CHAPTER 1

Introduction

1.1. The Direct Method of the Calculus of Variations

The results presented in this thesis are linked to the variational principle

$$\text{to minimise } \mathfrak{F}[u] := \int_\Omega f(\boldsymbol{\varepsilon}(u))\,\mathrm{d}x \text{ over a class } \mathcal{X}, \tag{1.1}$$

where Ω is an open, bounded Lipschitz subset of $\mathbb{R}^n$ and $f\colon \mathbb{R}^{n\times n}_{\mathrm{sym}} \to \mathbb{R}$ is a continuous integrand satisfying for some fixed $1 \leqslant p < \infty$ the *p–growth and coercivity condition*

$$c_1|\xi|^p - c_2 \leqslant f(\xi) \leqslant c_3|\xi|^p + c_4 \qquad \text{for all } \xi \in \mathbb{R}^{n\times n}_{\mathrm{sym}}, \tag{1.2}$$

$c_1, ..., c_4 > 0$ being constants. Regarding the class $\mathcal{X}$ of admissible maps, we mostly consider a suitable Dirichlet class. Thus we fix a given map $u_0 \in \mathrm{W}^{1,p}(\Omega;\mathbb{R}^n)$ and put $\mathcal{X} := \mathcal{D}_{u_0} := u_0 + \mathrm{W}^{1,p}_0(\Omega;\mathbb{R}^n)$. Based on (1.2), it is natural to consider (1.1) on the spaces

$$\mathrm{LD}^p(\Omega) := \{v \in \mathrm{L}^p(\Omega;\mathbb{R}^n)\colon\ \boldsymbol{\varepsilon}(v) \in \mathrm{L}^p(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})\}, \tag{1.3}$$

where $\boldsymbol{\varepsilon}(v) \in \mathrm{L}^p(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})$ needs to be understood in the sense that the distributional symmetric gradient $\boldsymbol{\varepsilon}(v)$ can be represented by an $\mathrm{L}^p(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})$–map. The space $\mathrm{LD}^p(\Omega)$ is endowed with norm

$$\|v\|_{\mathrm{LD}^p(\Omega)} := \left(\|v\|^p_{\mathrm{L}^p(\Omega;\mathbb{R}^n)} + \|\boldsymbol{\varepsilon}(v)\|^p_{\mathrm{L}^p(\Omega;\mathbb{R}^{n\times n})}\right)^{\frac{1}{p}}, \qquad v \in \mathrm{LD}^p(\Omega),$$

and it is customary to define the space $\mathrm{LD}^p_0(\Omega)$ as the closure of $\mathrm{C}^\infty_c(\Omega;\mathbb{R}^n)$ with respect to this norm.

In studying the variational principle (1.1), it turns out to be convenient to distinguish the cases $1 < p < \infty$ and $p = 1$. So assume $1 < p < \infty$ first. Since f is bounded below and Ω is bounded, too, we have $m := \inf_{\mathcal{D}_{u_0}} \mathfrak{F} > -\infty$. Hence we find $(u_k) \subset \mathcal{D}_{u_0}$ such that $\mathfrak{F}[u_k] \to m$ as $k \to \infty$. Let us note that $\mathfrak{F}$ is coercive on $\mathcal{D}_{u_0}$ with respect to the $\mathrm{W}^{1,p}$–norm by (1.2): Indeed, for $k \in \mathbb{N}$, we write $u_k = u_0 + v_k$ with some $v_k \in \mathrm{W}^{1,p}_0(\Omega;\mathbb{R}^n)$. We find (by Poincaré's inequality for symmetric gradients)

$$\begin{aligned}\int_\Omega |u_k|^p + |\boldsymbol{\varepsilon}(u_k)|^p\,\mathrm{d}x &\leqslant c\left(\int_\Omega |u_0|^p + |\boldsymbol{\varepsilon}(u_0)|^p\,\mathrm{d}x + \int_\Omega |v_k|^p + |\boldsymbol{\varepsilon}(v_k)|^p\,\mathrm{d}x\right)\\ &\leqslant c\left(\int_\Omega |u_0|^p + |\boldsymbol{\varepsilon}(u_0)|^p\,\mathrm{d}x + \int_\Omega |\boldsymbol{\varepsilon}(v_k)|^p\,\mathrm{d}x\right)\\ &\leqslant c + c\left(\int_\Omega |u_0|^p + |\boldsymbol{\varepsilon}(u_0)|^p\,\mathrm{d}x + \int_\Omega f(\boldsymbol{\varepsilon}(u_k))\,\mathrm{d}x\right).\end{aligned} \tag{1.4}$$

The crucial difference between $1 < p < \infty$ and $p = 1$ now lies in the fact that in the former case, a *Korn inequality* is available: Given $1 < p < \infty$, there exists a constant $C > 0$ such that for all $\varphi \in \mathrm{C}_c^\infty(\Omega;\mathbb{R}^n)$ there holds

$$\|\,\mathrm{D}\varphi\|_{\mathrm{L}^p(\Omega;\mathbb{R}^{n\times n})} \leqslant C\|\boldsymbol{\varepsilon}(\varphi)\|_{\mathrm{L}^p(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})}. \tag{1.5}$$

This is a remarkable inequality indeed: Note that the L^p–norm of the *full* gradient can be bounded by that of its *symmetric* part. As a consequence, we obtain $\mathrm{W}_0^{1,p}(\Omega;\mathbb{R}^n) = \mathrm{LD}_0^p(\Omega)$ for $1 < p < \infty$ and, even more generally, we have $\mathrm{W}^{1,p}(\Omega;\mathbb{R}^n) = \mathrm{LD}^p(\Omega)$ provided $\partial\Omega$ is Lipschitz. We will discuss such inequalities at length as the thesis evolves, but accepting them for the time being, we record that in conjunction with (1.4), (1.5) implies coerciveness of $\mathfrak{F}$ on $\mathcal{D}_{u_0}$ with respect to the $\mathrm{W}^{1,p}$–norm. In fact, by the Poincaré inequality involving symmetric gradients,

$$\int_\Omega |u_k|^p + |\,\mathrm{D}u_k|^p \,\mathrm{d}x \overset{(1.5)+\text{Poincaré}}{\leqslant} c\Big(\int_\Omega |u_0|^p + |\,\mathrm{D}u_0|^p\,\mathrm{d}x + \int_\Omega |\boldsymbol{\varepsilon}(u_k)|^p\,\mathrm{d}x\Big),$$

the coerciveness follows by (1.4). Since $1 < p < \infty$, the Sobolev spaces $\mathrm{W}^{1,p}(\Omega;\mathbb{R}^n)$ are reflexive and hence the Banach–Alaoglu Theorem allows to extract a subsequence $(u_{k(j)})$ of (u_k) such that for some $u \in \mathrm{W}^{1,p}(\Omega;\mathbb{R}^n)$ there holds $u_{k(j)} \rightharpoonup u$ as $j \to \infty$. Now assume that $\mathfrak{F}$ is sequentially weakly lower semicontinuous in $\mathrm{W}^{1,p}(\Omega;\mathbb{R}^n)$, that is, if $(v_k) \subset \mathrm{W}^{1,p}(\Omega;\mathbb{R}^n)$ and $v \in \mathrm{W}^{1,p}(\Omega;\mathbb{R}^n)$ are such that $v_k \rightharpoonup v$ in $\mathrm{W}^{1,p}(\Omega;\mathbb{R}^n)$ as $k \to \infty$, then $\mathfrak{F}[v] \leqslant \liminf_{k\to\infty}\mathfrak{F}[v_k]$. In this situation, we obtain by $u_{k(j)} \rightharpoonup u$ as $j \to \infty$ that

$$m \leqslant \mathfrak{F}[u] \leqslant \liminf_{j\to\infty}\mathfrak{F}[u_{k(j)}] = m,$$

and hence $\mathfrak{F}[u] = m$, showing that $u \in \mathrm{W}^{1,p}(\Omega;\mathbb{R}^n)$ is a minimiser indeed. Finally, since the trace operator on $\mathrm{W}^{1,p}(\Omega;\mathbb{R}^n)$ is continuous with respect to weak convergence, we conclude $u \in \mathcal{D}_{u_0}$ and hence $u \in \mathcal{D}_{u_0}$ is a solution of the variational problem (1.1).

We now turn to the linear growth case and hereafter assume (1.2) with $p = 1$. In this situation, there is no Korn–type inequality, a fundamental result which is referred to as Ornstein's *Non–Inequality*:

Theorem 1.1 (Ornstein, [194])**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded and open Lipschitz domain. Then there exists a sequence $(u_k) \subset \mathrm{C}_c^\infty(\Omega;\mathbb{R}^n)$ such that $\sup_{k\in\mathbb{N}}\|\boldsymbol{\varepsilon}(u_k)\|_{\mathrm{L}^1(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})} < \infty$ but $\limsup_{k\to\infty}\|\,\mathrm{D}u_k\|_{\mathrm{L}^1(\Omega;\mathbb{R}^{n\times n})} = \infty$.*

Imitating the approach pursued above, we immediately record three main difficulties:

(i) Considering the variational principle (1.1) on Dirichlet classes $u_0 + \mathrm{W}_0^{1,1}(\Omega;\mathbb{R}^n)$, the estimate (1.4) remains valid but does not imply coerciveness on $\mathrm{W}^{1,1}(\Omega;\mathbb{R}^n)$. This fundamental obstruction is a direct consequence of Ornstein's Non–Inequality. In view of a fruitful existence theory, we therefore must firstly change the function space framework, and at this level, the correct substitute is given by the space

$$\mathrm{LD}(\Omega) := \mathrm{LD}^1(\Omega) = \{v \in \mathrm{L}^1(\Omega;\mathbb{R}^n)\colon\ \boldsymbol{\varepsilon}(v) \in \mathrm{L}^1(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})\} \tag{LD}$$

endowed with norm $\|v\|_{\mathrm{LD}(\Omega)} := \|v\|_{\mathrm{L}^1(\Omega;\mathbb{R}^n)} + \|\boldsymbol{\varepsilon}(v)\|_{\mathrm{L}^1(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})}$. This is a symmetric–gradient variant of the usual Sobolev spaces $\mathrm{W}^{1,1}$ satisfying $\mathrm{W}^{1,1}(\Omega;\mathbb{R}^n) \subsetneq \mathrm{LD}(\Omega)$. To define the adequate Dirichlet classes, we put

$$\mathrm{LD}_0(\Omega) := \overline{\mathrm{C}_c^1(\Omega;\mathbb{R}^n)}^{\|\cdot\|_{\mathrm{LD}(\Omega)}}$$

and hereafter define $\widetilde{\mathcal{D}}_{u_0} := u_0 + \mathrm{LD}_0(\Omega)$, where now $u_0 \in \mathrm{LD}(\Omega)$ is a fixed map.

(ii) Since $p = 1$, neither $\mathrm{W}^{1,1}(\Omega;\mathbb{R}^n)$ nor $\mathrm{LD}(\Omega)$ are reflexive spaces and so we lack an appropriate version of the Banach–Alaoglu Theorem. Most importantly, by possible concentration effects, boundedness of minimising sequences with respect to the LD–norm does not imply weak convergence in $\mathrm{LD}(\Omega)$. The second objective therefore is to overcome the lack of sequential weak compactness and leads us to the space $\mathrm{BD}(\Omega)$ of *functions of bounded deformation* given by

$$\mathrm{BD}(\Omega) := \{v \in \mathrm{L}^1(\Omega;\mathbb{R}^n)\colon\ \boldsymbol{\varepsilon}(v) \in \mathcal{M}_{<\infty}(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})\}, \tag{BD}$$

where the defining property needs to be read as that the distributional symmetric gradients of BD–functions can be represented by bounded $\mathbb{R}^{n\times n}_{\mathrm{sym}}$–valued Radon measures (see Chapter 4.2 for more detail). The overarching philosophy here is that by passing from functions to measures, we may use the fact that the Radon measures arise as the dual of continuous functions vanishing at the boundary. Thus they allow the application of the Banach–Alaoglu Theorem (with respect to convergence in a suitable weak*–sense). From an application perspective, the space BD arises naturally in the study of perfect plasticity, cp. [23, 115, 227], for instance.

(iii) The drawback of (ii) is that the symmetric gradients of weak*–limits of minimising sequences are measures rather than functions, however, $\mathfrak{F}$ is a priori not defined when $\boldsymbol{\varepsilon}(v)$ is a Radon measure. Therefore the functional $\mathfrak{F}$ must be relaxed to the space $\mathrm{BD}(\Omega)$. In passing to a suitable relaxation, we also need to extend the concept of minima. Moreover, we note in advance that such *generalised minimisers* might not attain the prescribed boundary values. This is discussed in detail in Chapter 4.

As we have seen above, the decisive feature for functionals $\mathfrak{F}$ to ensure the existence of minima are suitable lower semicontinuity properties. These, in turn, depend on the (semi–)convexity properties of the integrands. A first and very strong assumption on f to ensure a variety of useful lower semicontinuity results is *symmetric convexity*:

Definition 1.2 (Symmetric Convexity). *A convex function $f\colon \mathbb{R}^{n\times n}_{\mathrm{sym}} \to \mathbb{R}$ is called* symmetric–convex. *More generally, a function $f\colon \mathbb{R}^{n\times n} \to \mathbb{R}$ is called* symmetric–convex *if its restriction to $\mathbb{R}^{n\times n}_{\mathrm{sym}}$ is a convex function.*

It is clear that if $f\colon \mathbb{R}^{n\times n} \to \mathbb{R}$ is convex, then $f|_{\mathbb{R}^{n\times n}_{\mathrm{sym}}}\colon \mathbb{R}^{n\times n}_{\mathrm{sym}} \to \mathbb{R}$ is symmetric–convex, and if $f\colon \mathbb{R}^{n\times n}_{\mathrm{sym}} \to \mathbb{R}$ is convex, then $\widetilde{f} := f \circ \pi_{\mathbb{R}^{n\times n}_{\mathrm{sym}}}$ (where $\pi\colon \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}_{\mathrm{sym}}$ is the orthogonal projection onto $\mathbb{R}^{n\times n}_{\mathrm{sym}}$) is a convex function with $\widetilde{f}|_{\mathbb{R}^{n\times n}_{\mathrm{sym}}} = f$. In particular, every symmetric–convex function arises as the restriction to $\mathbb{R}^{n\times n}_{\mathrm{sym}}$ of a convex function on $\mathbb{R}^{n\times n}$. Coming back to the lower semicontinuity properties of (symmetric–)convex functionals in slightly more detail in Section 1.1.2, we pause to discuss the full gradient case (i.e., where $\boldsymbol{\varepsilon}$ in (1.1) is replaced by the full gradient D).

1.1.1. Full Gradient Functionals Throughout both the introduction and the main part of the text, it is instructive to keep in mind the related class of variational principles

$$\text{to minimise } \ \mathfrak{F}[u] := \int_\Omega \widetilde{f}(\mathrm{D}u)\,\mathrm{d}x \ \text{ over the Dirichlet class } u_0 + \mathrm{W}^{1,1}_0(\Omega;\mathbb{R}^N), \tag{1.6}$$

where $\widetilde{f}\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ is a continuous function satisfying (1.2) with $p = 1$ and the obvious modifications. Here, condition (1.2) with $p = 1$ directly yields boundedness of minimising sequences in $\mathrm{W}^{1,1}(\Omega;\mathbb{R}^N)$ (clearly, there is no need for Korn's inequality here) so that we only need to keep track of the respective versions of (ii) and (iii). This gives rise to the space $\mathrm{BV}(\Omega;\mathbb{R}^N)$ of *functions of bounded variation* which precisely consists of all those $v \in \mathrm{L}^1(\Omega;\mathbb{R}^N)$ for which the distributional gradient can be represented by a bounded $\mathbb{R}^{N\times n}$–valued Radon measure (see Chapter 4.2 for more detail). The study of the variational principle (1.6) arises for instance in the scalar case $N = 1$ in the study of non–parametric minimal surface–type functionals (the latter corresponding to the area–integrand $\langle z\rangle := \sqrt{1+|z|^2}$) which, among others, has been extensively dealt with in [126, 46, 179, 180, 181, 195, 196, 222]. If $f = |\cdot|$, variational problems such as (1.6) often arise in imaging problems (see, e.g., [205, 62]). Let us remark in advance that these examples are particular borderline cases of the scale of μ–elliptic integrands to be recalled and discussed in Chapter 5.2.

1.1.2. Semiconvexity Notions Beyond the usual convexity, there are various generalisations which are strictly weaker and hereafter shall be referred to as *semiconvexity notions*. We recall from MORREY's fundamental work [184, 185] that a continuous function $f\colon \mathbb{R}^{N\times n}\to\mathbb{R}$ is called *quasiconvex* if and only if it satisfies

$$f(A) \leqslant \fint_{\Omega} f(A+\mathrm{D}\varphi)\,\mathrm{d}x \tag{QC}$$

for all open and bounded Lipschitz sets $\Omega\subset\mathbb{R}^n$, all $A\in\mathbb{R}^{N\times n}$ and $\varphi\in\mathrm{W}_0^{1,\infty}(\Omega;\mathbb{R}^N)$. The importance of this condition stems from the fact that in a plenty of settings, it is a necessary and sufficient condition for variational integrals of the form (1.6) to be sequentially weakly lower semicontinuous in spaces of weakly differentiable functions. For completeness, we also recall that due to [27], $f\colon\mathbb{R}^{N\times n}\to\mathbb{R}$ is called *polyconvex* provided it can be written as a convex function of the minors of its argument, and *rank–one convex* if it is convex with respect to directions in the rank–one cone $\mathbb{R}^N\otimes\mathbb{R}^n$, i.e., for all $A\in\mathbb{R}^{N\times n}$, $a\in\mathbb{R}^N$ and $b\in\mathbb{R}^n$, $t\mapsto f(A+ta\otimes b)$ is a convex map $[0,1]\to\mathbb{R}$. We shall come back to this fact in more detail in Chapters 3 and 4.

Quasiconvexity, however, is still far from being well–understood. So, for instance, in the case $n=N=2$, the ALIBERT–DACOROGNA–MARCELLINI example (cp. [11, 68]) is given by the family of functions $f_\gamma\colon\mathbb{R}^{2\times2}\to\mathbb{R}$ ($\gamma\in\mathbb{R}$), where $f_\gamma(A) := |A|^2(|A|^2 - 2\gamma\det(A))$ for $A\in\mathbb{R}^{2\times 2}$. As shown in Table 1.1, it is well–known what is the precise range of γ for f_γ to be convex, polyconvex or rank–1–convex, respectively. However, such a characterisation is still not available in the quasiconvex case. The semiconvexity notions gathered so far are linked by the diagram

$$\text{convexity} \Rightarrow \text{polyconvexity} \Rightarrow \text{quasiconvexity} \Rightarrow \text{rank–one convexity},$$

and in general, none of the corresponding reverse implications holds true. Note, however, that it is still not clear whether rank–1–convexity implies quasiconvexity for $n\geqslant N=2$ whereas for the remaining constellations of N and n this has been settled in the negative by ŠVERÁK [226].

All of these notions are easily seen to be not adequate if f is an integrand being defined on the symmetric matrices, say. Having briefly encountered the notion of *symmetric convexity*, we shall now proceed with the corresponding generalisations of quasiconvexity and rank–one convexity (also see Chapter 3 for a unifying treatment).

Definition 1.3. *A continuous function $f\colon\mathbb{R}^{n\times n}_{\mathrm{sym}}\to\mathbb{R}$ is called*

(a) symmetric–quasiconvex *if and only if for all open and bounded Lipschitz sets $\Omega\subset\mathbb{R}^n$, all $\varphi\in\mathrm{W}_0^{1,\infty}(\Omega;\mathbb{R}^n)$ and all $A\in\mathbb{R}^{n\times n}_{\mathrm{sym}}$ there holds*

$$f(A) \leqslant \fint_{\Omega} f(A+\boldsymbol{\varepsilon}(\varphi))\,\mathrm{d}x. \tag{SQC}$$

(b) symmetric–rank–one–convex *if and only if for all $A\in\mathbb{R}^{n\times n}_{\mathrm{sym}}$ and all $a,b\in\mathbb{R}^n$ the function $t\mapsto f(A+ta\odot b)$ is convex on $[0,1]$. Here, for given $a,b\in\mathbb{R}^n$, $a\odot b := \frac{1}{2}(ab^{\mathsf{T}}+ba^{\mathsf{T}})$ denotes the symmetric tensor product of a and b.*

f_γ is	convex	polyconvex	quasiconvex	rank–1–convex
$\Leftrightarrow$	$\lvert\gamma\rvert \leqslant \gamma_c := 2\sqrt{2}/3$	$\lvert\gamma\rvert \leqslant \gamma_{pc} := 1$	$\lvert\gamma\rvert \leqslant \gamma_{qc}$	$\lvert\gamma\rvert \leqslant \gamma_{rc} := 2/\sqrt{3}$

Table 1.1: Characterisation of semiconvexity properties of the Alibert–Dacorogna–Marcellini family f_γ. It is known that $\gamma_{qc} \in (1, 2/\sqrt{3}]$, but the precise value is unknown.

It was expected for a long while that symmetric quasiconvexity for linear growth functionals $f\colon \mathbb{R}^{n\times n}_{\mathrm{sym}} \to \mathbb{R}$ is a necessary and sufficient condition for functionals of the type (1.1) to be sequentially weak*–lower semicontinuous on $\mathrm{BD}(\Omega)$ (see Chapter 4.2 for this notion), a problem which has been partially resolved in the space $\mathrm{SBD}(\Omega)$ of *functions of special*[1] *bounded deformation* by Barroso, Fonseca & Toader in [30] and in full generality by Rindler [202]. The chief obstruction in establishing such a result was the non–availability of a symmetric version of the Alberti–rank–one theorem [8]. The main novelty of Rindler's work consequently was to come up with a proof that does not use the symmetric–rank–one structure of the singular part of the symmetric gradients of BD–maps. By now, this issue has been completely resolved by the seminal work of De Philippis & Rindler [76] (also see (4.12)ff. below).

1.2. Sketches of Regularity Theory

With the existence of minima being established among weakly differentiable functions, the objective of *regularity theory* is to study the extent to which minima of (elliptic) variational integrals share better smoothness properties than generic competitor maps. In this sense, it crucially depends on the underlying space scale being used to measure smoothness, prominent examples of which are given by the Hölder space scales $(\mathrm{C}^{m,\alpha})_{m\in\mathbb{N}_0, 0<\alpha\leqslant 1}$ or the (fractional) Sobolev space scales $(\mathrm{W}^{\theta,p})_{\theta>0, 1\leqslant p\leqslant\infty}$, for instance.

One of the most crucial features of the variational principle (1.1) is that it represents a so–called *vectorial problem*, meaning that the target space of the competitor maps is $\mathbb{R}^N$ for some $N > 1$ (here, $N = n > 1$). As to the regularity of minima, it was independently discovered by De Giorgi [73] and Maz'ya [169] first that the classical statements for *scalar problems* do not extend to the vectorial case. More precisely, following the previous work of De Giorgi, Nash and Moser (see [72, 191, 187]) in the 50s and 60s of the last century on the full $\mathrm{C}^{1,\alpha}$–regularity for quadratic functionals under suitable measurability, boundedness and ellipticity assumptions imposed on the coefficients defining the integrands, this was a surprising novelty: It showed that Hilbert's 19th problem which had been posed at the ICM in Paris in 1900,

> **Hilbert's 19th Problem:** "Sind die Lösungen regulärer Variationsprobleme stets notwendig analytisch?" (Are the solutions of regular variational problems necessarily analytic throughout?)

was only resolved in the scalar situation (for quadratic functionals). Two of the initial counterexamples to full regularity (subsequently extended and sharpened by many others, see [110, 111, 144, 183, 193, 228, 229] for a non–exhaustive list) read as follows:

Example 1.4 (De Giorgi, Giusti & Miranda, [72, 129], [33, Chpt. 4.1])**.** *Let* $n \geqslant 3$.

[1] This space is defined to consist of all BD–maps for which the Cantor part of the symmetric gradient vanishes; compare Chapter 4.2 for the requisite terminology.

We define bilinear forms on $\mathbb{R}^{n\times n}$ *by*

$$
\begin{aligned}
A_{ij}^{\kappa,\lambda}(n-2,n,x) &:= \delta_{\kappa\lambda}\delta_{ij} + \Big((n-2)\delta_{i\kappa} + n\frac{x_i x_\kappa}{|x|^2}\Big)\Big((n-2)\delta_{j\lambda} + n\frac{x_j x_\lambda}{|x|^2}\Big), \qquad x \neq 0,\\
\mathcal{A}_{n-2,n,x}[y,z] &:= \sum_{\kappa,\lambda,i,j=1}^{n} A_{ij}^{\kappa\lambda}(n-2,n,x) y_i^\kappa z_j^\lambda, \qquad y = (y_i^\kappa),\ z = (z_j^\lambda),\\
\widetilde{A}_{ij}^{\kappa,\lambda}(u) &:= \delta_{\kappa\lambda}\delta_{ij} + \Big(\delta_{i\kappa} + \frac{4}{n-2}\frac{u_i u_\kappa}{1+|u|^2}\Big)\Big(\delta_{j\lambda} + \frac{4}{n-2}\frac{u_j u_\lambda}{1+|u|^2}\Big), \qquad u \in \mathbb{R}^n,\\
\widetilde{\mathcal{A}}(u)[y,z] &:= \sum_{i,j,\kappa,\lambda=1}^{n} \widetilde{A}_{ij}^{\kappa,\lambda}(u) y_i^\kappa z_j^\lambda, \qquad y = (y_i^\kappa),\ z = (z_j^\lambda).
\end{aligned}
$$

(a) (De Giorgi, [72]). *The function* $u\colon \mathrm{B}(0,1) \to \mathbb{R}^n$ *given by* $u(x) = u_\alpha(x) := x/|x|^\alpha$ *for* $\alpha := (n/2)(1-((2n-2)^2+1)^{-1/2})$ *belongs to* $\mathrm{W}^{1,2}(\mathrm{B}(0,1);\mathbb{R}^n)$ *and is an unbounded weak solution of the elliptic system* $\operatorname{div}(\mathcal{A}_{n-2,n,x}(\mathrm{D}u)) = 0$ *in* $\mathrm{B}(0,1)$*.*

(b) (Giusti & Miranda, [129]). *The function* $u\colon \mathrm{B}(0,1) \to \mathbb{R}^n$ *given by* $u(x) = x/|x|$ *belongs to* $(\mathrm{W}^{1,2} \cap \mathrm{L}^\infty)(\mathrm{B}(0,1);\mathbb{R}^n)$ *and is a discontinuous weak solution of the elliptic system* $\operatorname{div}(\widetilde{\mathcal{A}}(u)\,\mathrm{D}u) = 0$ *in* $\mathrm{B}(0,1)$*.*

These results show that in general, the full (Hölder–)regularity of solutions is not to be expected in the vectorial case. Instead, it motivates to study as to whether the set of points in whose neighbourhood weak solutions or minima are *not* suitably Hölder continuous, is $\mathscr{L}^n$–negligible or even has Hausdorff dimension strictly smaller than the ambient space dimension. Following the terminology of MORREY [186], such regularity properties of minima are usually referred to as *partial regularity*, and have been a constant source of research interest; see, among others, the monographs [128, 121, 33] and MINGIONE's survey papers [176, 177] for an extensive list of relevant contributions and techniques to arrive at partial regularity results in both the convex and quasiconvex setup.

For the purpose of the present thesis the distinction between scalar and vectorial problems is of special importance as the variational principle (1.1) is genuinely vectorial. In particular, even though the integrands are assumed to be of *nice* structure, so for instance of radial structure [237, 238], $f = g(|\cdot|)$, the dependence on the symmetric gradients seems to make this special structure inapplicable to retrieve the strong results which apply to such problems in the full gradient setting:

Remark 1.5 (Radial Structure and Symmetric Gradients)**.** *Based on the fundamental work [237] of* UHLENBECK *and [238] of* URAL'CEVA*, it is a general paradigm that minimisers of vectorial problems of the form* (1.6) *(also for the superlinear growth regime) can be proven to be* fully $\mathrm{C}^{1,\alpha}$*–regular provided* $\widetilde{f}$ *exhibits* radial structure*, i.e.,* $\widetilde{f}(z) = g(|z|)$ *for* $z \in \mathbb{R}^{N\times n}$ *and* $g\colon \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$*. In this situation, scalar methods reminiscent of the classical De Giorgi–Nash–Moser theory can be employed. In the* BV*–case, this has been investigated in [34, 35]. However, if we pass from* (1.6) *to functionals of the type* (1.1)*, then the appearance of the symmetric gradient 'destroys' the scalar structure of the resulting problem. This relies on the fact that there is no scalar equivalent of the symmetric gradient, whereas the gradient in the scalar situation trivially makes sense. In particular, in Chapters 5 and 6 we therefore will not be able to recover the regularity results available for the* BV*–case in the symmetric gradient case in full generality.*

We note that this phenomenon is at variance with the gradient case, where the radial structure indeed can be used to establish full (hence local) L^∞–regularity of minima of variational integrals (1.6), see [34, Sec. 4].

Even though partial regularity is a very weak notion of regularity and, starting with EVANS' seminal paper [99] in the superquadratic, quasiconvex case, there are still various open questions. So, for instance, unlike for convex integrands, to the best of our knowledge there is no unconditional[2] Hausdorff dimension bound for the singular set of minima of quasiconvex functionals. Moreover, a large variety of results has been obtained for quasiconvex variational integrals in the superlinear growth regime (cp. [122, 22, 4, 209, 89, 90, 82, 121, 33, 32, 31, 128] for a non–exhaustive list), but the linear growth case has been settled only recently by KRISTENSEN and the author [134]. We shall discuss its generalisation to the symmetric gradient case in detail below.

Depending on the underlying space scale invoked to measure regularity, however, still various results survive when passing from the scalar to the vectorial situation. For example, the so–called *Nikolskii regularity* to be discussed below can be read as an L^p–version of Hölder continuity and still can be established for solutions of vectorial problems, see [175, 177]. By suitable embedding theorems (see Section 2.5.1), such Nikolskii–type estimates can be transferred to fractional Sobolev estimates and hence yield higher integrability of the (symmetric) gradients. This is in line with the results of BILDHAUER [40, 42] who, based on previous work by SEREGIN [212, 213, 214] initiated the first systematic Sobolev regularity program for vectorial linear growth problems of the form (1.6).

A common gateway to both the aforementioned partial (Hölder–)regularity and higher Sobolev regularity of minima is given by use of smoothness spaces defined in terms of maximal functions. Referring the reader to Chapter 2.6 for the requisite definition of the fractional sharp maximal operator $\mathcal{M}_\alpha^{\#}$, the key estimate is given by

$$|u(x)-u(y)|^p \leqslant c|x-y|^{\alpha p}\big(\mathcal{M}_\alpha^{\#}u(x)+\mathcal{M}_\alpha^{\#}u(y)\big)^p \qquad \text{for all } u\in\mathrm{L}^p_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^m), \tag{1.7}$$

where $0<\alpha\leqslant 1$, $c=c(n,\alpha)$ is a positive constant and $1\leqslant p<\infty$. From (1.7), it is immediate that if $\mathcal{M}_\alpha^{\#}u\in\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^n)$, then $u\in\mathrm{C}^{0,\alpha}_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^m)$. This is a convenient maximal function version of the usual CAMPANATO–type characterisation of Hölder continuity, and partial regularity proofs are indeed centered around establishing the validity of $\mathcal{M}_\alpha^{\#}(\mathrm{D}u)\in\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^n)$ by use of certain *excess decay estimates*. These control the mean deviation of the minimisers' gradients from their mean values on small balls. The latter crucially relies on ellipticity assumptions imposed on the variational integrands f. On the other hand, letting $1<p<\infty$ and considering the condition $\mathcal{M}_\alpha^{\#}u\in\mathrm{L}^p_{\mathrm{loc}}(\mathbb{R}^n)$ yields the local Nikolskii–type condition (where, without loss of generality, $h\in\mathbb{R}^n$ with $|h|<R$ for some suitable $R>0$)

$$\int_{\mathrm{B}(x_0,R)}|u(x+h)-u(x)|^p\,\mathrm{d}x\leqslant CR^{\alpha p}\int_{\mathrm{B}(x_0,2R)}|\mathcal{M}_\alpha^{\#}u(x)|^p\,\mathrm{d}x.$$

The previous estimate entails $u\in\mathcal{N}^{\alpha,p}_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^m)\hookrightarrow\mathrm{W}^{\alpha-\varepsilon,p}_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^m)$ (see Chapters 2.5.1 and 2.5.5 for the requisite definitions) for $0<\varepsilon<\alpha$, which gives a direct regularity improvement. Here it depends in an essential way on which ellipticity condition is being used, and the conditions we shall encounter in this thesis are the so–called *strong* versions of convexity, quasiconvexity and rank–one convexity.

A different perspective on the regularity of minima is given by interpreting the higher Nikolskii regularity for the gradients as an L^p–modulus of continuity. By the Riesz–Kolmogorov criterion, this in turn amounts to gaining compactness in L^p for the gradients (of minimising sequences) and, following the analysis of FARACO & KRISTENSEN [103], regularity and compactness in fact are the same qualitatively. Coming back to the case $p=1$, it is therefore clear that the main difficulties in proving regularity stem from the comparatively weak compactness properties of L^1–based spaces. For the purposes of the present thesis, the non–availability of L^1–estimates for the gradients of members of minimising sequences to variational principles such as (1.1) even deteriorates coerciveness and hence a careful approach to the regularity is required. This, in turn, shall be described in the following section.

[2] KRISTENSEN & MINGIONE [159] established that for strongly quasiconvex integrals of p–growth with $p\geqslant 2$, minima of class $\mathrm{W}^{1,\infty}$ possess a singular set of Hausdorff dimension strictly smaller than the ambient space dimension.

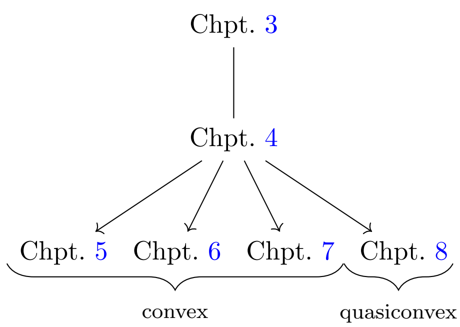

Figure 1.1: Dependencies of the single chapters of this thesis.

1.3. Outline of the Thesis and Description of Main Results

The main objective of this thesis is to extend the regularity theory for variational problems of the form (1.6) to the symmetric gradient setting (1.1). In doing so, the foremost difficulty lies in the non–availability of suitable a–priori estimates for the *full gradients* of the single members of minimising sequences. Depending on the specific scale of regularity, this will be achieved in different manners. In this respect, a common principle which we will employ is to *switch to scales of fractional differentiability* by use of Sobolev–type embeddings, and thereby to give some differentiability away but to gain higher integrability for fractional derivatives.

In what follows, we give a brief description of the structure of the thesis and, for each chapter, single out the rough contents of the main results. For the precise statements, the reader is referred to the single chapters. Moreover, each chapter contains a variety of further results, but for the sake of exposition, their description is deferred to the beginning of each chapter.

Publications and preprints which are part of this thesis

(P0) F. Gmeineder: Symmetric-Convex Functionals of Linear Growth. J. Elliptic Parabol. Equ. 2 (2016), no. 1-2, 59–71.

(P1) F. Gmeineder: Partial regularity for symmetric–convex functionals with linear growth. Preprint.

(P2) F. Gmeineder: Update on Korn's Inequality and Ornstein's Non–Inequality. Preprint.

(P3) F. Gmeineder, J. Kristensen: Sobolev Regularity for symmetric–convex functionals of linear growth. Preprint.

(P4) L. Beck, M. Bulicek, F. Gmeineder: On the Neumann Problem on the space of functions of bounded variation. Preprint.

(P5) F. Gmeineder: Partial Regularity for symmetric–quasiconvex functionals of linear growth. Preprint.

Chapter 3 further contains some minor auxiliary results which appeared firstly in

- D. Breit, L. Diening, F. Gmeineder: Traces of functions of bounded $\mathbb{A}$–variation and variational problems of linear growth. Preprint.

Moreover, Chapter 8 is the adaption to the BD–case of the joint work

- F. Gmeineder, J. Kristensen: Partial Regularity for BV–minimisers. Preprint.

We now give a systematic outline of the structure of the thesis.

Chapter 2: Preliminaries. The purpose of this chapter is to fix notation and to introduce auxiliary function spaces, maximal operators and measure theory to the extent as is needed in the main text.

Chapter 3: Korn's Inequality and Ornstein's Non–Inequality. As the main obstruction to establish both existence and regularity results for minima of symmetric–convex or –quasiconvex problems, Ornstein's Non–Inequality plays a substantial rôle in all of what follows. In this chapter we present a very general form of such a non–inequality, a proof of which is based on the semiconvexity approach of KIRCHHEIM & KRISTENSEN [151]. We set up a very general framework of elliptic first order operators and study the geometry which they induce in matrix space. This chapter contains material from (P2).
Chapter 4: The Dirichlet Problem on BV and BD. With Ornstein's Non–Inequality at our disposal, the objective of this chapter is to introduce and study the space BD of functions of bounded deformation together with the Dirichlet problem on this function space. As the main concepts already appear in the classical BV–context, we briefly review the BV–theory and point out similarities and differences to the BD–setup. Within the convex setting, we introduce functionals of measures and give the corresponding RESHETNYAK (lower semi–)continuity theorems for such functionals.
Chapter 5: Sobolev Regularity for Symmetric–Convex Problems. Using the scale of μ–ellipticity, this chapter is dedicated to the proof of higher integrability of the symmetric gradients of generalised minima for μ–elliptic variational integrals. Here we employ a novel embedding for $\mathrm{BD}\cap\mathrm{BMO}$ and combine it with an extensive study of the dual problem (in the sense of convex duality) along the lines of SEREGIN [212]–[215] to prove higher integrability of minima. This is accomplished subject to a condition which generalises BILDHAUER's *local boundedness* assumption, cp. [40]. To the best of our knowledge, this is the first higher integrability result for variational problems on BD.

We further discuss the (direct) regularity impact of Caccioppoli–type inequalities in the linear growth regime, uniqueness assertions for generalised minima and how the techniques employed here can be used in different contexts. This chapter contains material from (P3).
Chapter 6: Interlude: $\mathrm{W}^{1,1}$–Regularity for the Neumann Problem on BV. Before we return to functionals depending on the symmetric gradient in the subsequent chapters, we study here the Neumann problem on the space of functions of bounded variation. Assuming radial symmetry of the integrands at our disposal, we establish $\mathrm{W}^{1,1}$–regularity of minima provided the integrand is of class C^2 and strictly convex. This seems to constitute the first unconditional (i.e., without imposing the condition of μ–ellipticity) $\mathrm{W}^{1,1}$–regularity result for convex linear growth problems.

To clarify terminology, strict convexity here is understood in the sense that

$$f'' \text{ is positive definite everywhere,}$$

and we hereafter also refer to this notion sometimes as C^2–*strict convexity*. When speaking of *strong* convexity, however, we think of integrands whose Hessians satisfy a prescribed growth bound from below (which, e.g., is the case for μ–elliptic integrands).

We further compare the result with the known results on $\mathrm{W}^{1,1}$–regularity for Dirichlet problems and discuss the difficulties which come up for the Neumann problem on BD (cp. Remark 1.5). This chapter comprises the material of (P4).
Chapter 7: Partial Regularity for Symmetric–Convex Problems.
In this chapter we turn to the partial regularity for strictly symmetric–convex variational integrals of linear growth and provide the first partial regularity result for such problems on BD. In a similar vein as above, by *strict symmetric convexity* we understand that in (1.1) we consider C^2–integrands $f\colon \mathbb{R}^{n\times n}_{\mathrm{sym}} \to \mathbb{R}$ whose Hessian $f''(\xi)$ is positive definite at every $\xi \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$. Similarly as in Chapter 6 and hereafter by the lack of a strong ellipticity assumption, we cannot expect higher integrability of the symmetric gradients of minima to hold true. Instead, we show that partial regularity survives, and in doing so, we use a direct approach that was employed by ANZELLOTTI & GIAQUINTA [22] in the BV–setting first and later on refined by SCHMIDT [210]. As the main idea here is to compare the minimisers with suitable mollifications, we need to establish appropriate

Poincaré–type inequalities which help to estimate the L^1–distance between BD–maps and their mollifications. Finally, we discuss implications for the dual solution and further give Hausdorff dimension estimates on the singular set. This chapter comprises the material of (P1).

Chapter 8: Partial Regularity for Symmetric–Quasiconvex Problems.
In the final chapter of this thesis we deal with symmetric–quasiconvex and –rank one convex functionals in the sense of Definition 1.3. For strongly symmetric–rank one convex functionals of linear growth, we give a *full* $C^{1,\alpha}$–regularity result subject to a smoothness and smallness assumption on the given Dirichlet data. In the second part of this chapter, we extend the results of the previous chapter to the symmetric quasiconvex setting[3]. In doing so, we derive a Fubini–type theorem for fractional Sobolev spaces and the space BD that might be of independent interest, leading to the first partial regularity result for symmetric quasiconvex integrals of linear growth. The chapter comprises the content from (P5).

The appendix contains various auxiliary results which are used in the main text, and in particular gives an overview of Euler–Lagrange equations for measures, an approach to the approximate differentiability of LD–maps via Calderón–Zygmund–type decompositions for measures and auxiliary estimates.

We conclude with some general remarks.

- Sometimes it is necessary to distinguish with which sort of *linear growth* assumption we work throughout a chapter (e.g., in the sense of signed integrands). Therefore, the linear growth assumption in action will be fixed at the beginning of each of Chapters 5–8.
- Unless otherwise stated, all vector spaces are assumed to be over the real numbers.
- Apart from the preliminary second chapter, each of the chapters contains a very rough outline of the contents at the very beginning. In Chapters 3, 5–8, the main results and a contextualised description of the structure of each chapter is consequently given in the section *Main Results* or *Description of Results*, respectively.
- Most of the notation used throughout will be explained carefully in Chapter 2 and especially Section 2.1 therein. A comprehensive list of symbols is given at the end of this thesis on page 284.

[3]Note that the notion of *extension* here merely applies to a weakening of the convexity condition but not its strength in the sense of strong convexity or quasiconvexity hypotheses. In particular, there are convex integrands to which the results from Chapter 7 apply, whereas those of Chapter 8 do not.

CHAPTER 2

Preliminaries

The purpose of this chapter is to fix notation and to record basic facts regarding measure theory, Fourier analysis and function spaces as will be required in the main part of the thesis. As such, Section 2.2 is along the lines of [16, Chpts. 1.1, 1.3] and [97, Chpts. 1, 2], where also a more detailled treatment of the measure theoretic concepts described here can be found. The remaining sections of this chapter settle the function space framework which we shall use in the subsequent chapters. In this respect, a concise overall treatment is given in [6, 170, 168] where most of the material as presented here is taken from.

2.1. General Notation

Unless otherwise stated, Ω denotes an open and bounded Lipschitz subset of $\mathbb{R}^n$ and $\mathscr{B}(\Omega)$ its Borel σ–algebra. All norms on $\mathbb{R}^m$ or $\mathbb{R}^{m\times n}$ are the usual euclidean or Frobenius norms, respectively. In this respect, given $x \in \mathbb{R}^n$ and $r > 0$, we denote $\mathrm{B}(x,r) := \{y \in \mathbb{R}^n\colon\ |x-y| < r\}$ the open ball centered at x with radius r and further put $\mathbb{S}^{n-1} := \{x \in \mathbb{R}^n\colon\ |x| = 1\}$. Sometimes it is useful to distinguish between the unit ball in $\mathbb{R}^n$ and the unit ball in the symmetric matrices $\mathbb{R}^{n\times n}_{\mathrm{sym}}$, and so we further define for $A \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$ and $R > 0$

$$\mathbb{B}(A,R) := \big\{C \in \mathbb{R}^{n\times n}_{\mathrm{sym}}\colon\ |A-C| < R\big\}.$$

Lastly, we denote the euclidean inner product by $\langle\cdot,\cdot\rangle$, and the underlying space will be clear from the context. Note that pairings of distributions with test functions or of measures with continuous functions are always denoted with subscript, e.g., $\langle\cdot,\cdot\rangle_{\mathscr{D}'\times\mathscr{D}}$ or $\langle\cdot,\cdot\rangle_{\mathcal{M}\times\mathrm{C}}$. Moreover, $\mathcal{H}^{n-1}$ denotes the $(n-1)$–dimensional Hausdorff measure and $\mathscr{L}^n$ the n–dimensional Lebesgue measure (sometimes we use the shorthand $|U| := \mathscr{L}^n(U)$ for a measurable set $U \subset \mathbb{R}^n$, cp. Section 2.2). Given a measure μ on $\mathscr{B}(\Omega)$ and $A \in \mathscr{B}(\Omega)$, we use the standard shorthand $(\mu \llcorner A)(B) := \mu(A\cap B)$ for the measure μ restricted to A. Given an open set $U \subset \mathbb{R}^n$ and $u \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^m)$, we use the equivalent notations

$$(u)_U := \langle u\rangle_U := \fint_U u\,\mathrm{d}x := \frac{1}{|U|}\int_U u\,\mathrm{d}x.$$

The shorthand $a \lesssim b$ for $a, b \in \mathbb{R}$ asserts that there exists a constant $C > 0$ such that $a \leqslant Cb$, where $C > 0$ does not depend on a, b or any essential quantities implicit in a or b. If $a \lesssim b$ and $b \lesssim a$, then we write $a \approx b$. When we write $a \ll b$, then a is understood to be essentially smaller variable than the fixed value b in the sense that the difference $b-a$ cannot be made arbitrarily small. This is not to be confused with the sign '$\ll$' in the context of absolute continuity of measures, but no ambiguities will arise from this. As is customary, we use $c, C > 0$ for two generic constants whose value might change from line to line unless they are specified explicitely.

It is important to note that some of the function spaces to be dealt with throughout are purely vectorial in the sense that they have no scalar analogue. On the other hand, if $(X(\Omega), \|\cdot\|_{X(\Omega)})$ is a function space consisting of scalar functions defined on Ω, then we denote (for given $m \in \mathbb{N}$) $X(\Omega;\mathbb{R}^m)$ the space of mappings $v = (v^1, ..., v^m)$ such that for each $j = 1, ..., m$ there holds $v^j \in X(\Omega)$ and equip it with its canonical norm $\|v\|_{X(\Omega;\mathbb{R}^m)} := \sum_{j=1}^m \|v^j\|_{X(\Omega)}$. For two function spaces $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$, we use $X \simeq Y$ to indicate $X = Y$ and that the norms $\|\cdot\|_X$ and $\|\cdot\|_Y$ are equivalent. In this context, the symbol '$\approx$' is strictly reserved for isometrically isomorphic spaces. We denote $\mathscr{L}(X, Y)$ the space of bounded linear operators between the two normed spaces X and Y and consider it endowed with the usual operator norm throughout.

Lastly, as it is at variance with standard usage, we write $(u_k) \subset X$ to indicate that the sequence (u_k) is contained in the set X and $(u_{k(j)}) \subset (u_k)$ to express that $(u_{k(j)})$ is a subsequence of (u_k).

2.2. Measure Theory

In the following, let (Ω, Σ) be a measurable space and let $m \in \mathbb{N}$. A mapping $\mu\colon \Sigma \to \mathbb{R}^m$ is called a *(vector) measure* provided $\mu(\emptyset) = 0$ and whenever (E_k) is a sequence of pairwise disjoint elements of Σ, then $\mu(\bigcup_k E_k) = \sum_k \mu(E_k)$, the latter property being referred to as σ–additivity. In this situation, the **total variation measure** (associated with μ) is given by

$$|\mu|(E) := \sup\left\{\sum_{j=1}^{\infty} |\mu(E_j)|\colon\ E_j \in \Sigma \text{ pairwise disjoint with } E = \bigcup_{j=1}^{\infty} E_j\right\}$$

for $E \in \Sigma$. Let us also say that μ is a *positive measure* if $\mu\colon \Sigma \to [0, \infty](\subset \mathbb{R})$ is a measure in the above sense. Throuhout, we shall be particularly interested in the situation where Σ arises as the Borel σ–algebra induced by the natural topology on a metric space (X, d) which is here assumed to be locally compact and separable. Supressing d for notational simplicity, we write $\mathscr{B}(X)$ for the Borel σ–algebra on (X, d).

A positive measure on $(X, \mathscr{B}(X))$ is called a *Borel measure*, and if it is finite on compact subsets of X, then it is called a *positive Radon measure*. Similarly, if μ is an $\mathbb{R}^m$–valued set function defined on the relatively compact Borel subsets of X which is a measure on $(K, \mathscr{B}(K))$ for any compact subset K of X, then we say that μ is *Radon measure* (on X), and if it additionally is a measure in the sense described above, then we say that it is a *finite Radon measure*. We denote the set of all (finite) $\mathbb{R}^m$–Radon measures on (X, d) by[1] $\mathcal{M}_{<\infty}(X;\mathbb{R}^m)$ or $\mathcal{M}(X;\mathbb{R}^m)$, respectively. Let us note that $\mathcal{M}_{<\infty}(X;\mathbb{R}^m)$ is a normed space when being equipped with the total variation $\|\mu\| := |\mu|(X)$ as norm.

To set up the duality framework for Radon measures within which convenient compactness results can be obtained by the topological vector space variant of the Banach–Alaoglu–Bourbaki Theorem, we denote $\mathrm{C}_c(X;\mathbb{R}^m)$ the compactly supported, $\mathbb{R}^m$–valued continuous functions on X, and $\mathrm{C}_0(X;\mathbb{R}^m)$ the closure of $\mathrm{C}_c(X;\mathbb{R}^m)$ with respect to the sup–norm. Now, if $\mu \in \mathcal{M}_{<\infty}(X;\mathbb{R}^m)$, then for every open $A \subset X$ there holds

$$|\mu|(A) = \sup\left\{\int_X \langle\varphi, \mathrm{d}\mu\rangle\colon\ \varphi \in \mathrm{C}_c(X;\mathbb{R}^m),\ |\varphi| \leqslant 1\right\},$$

where $\langle\varphi, \mathrm{d}\mu\rangle := \sum_{j=1}^m \varphi^j\,\mathrm{d}\mu^j$ for $\varphi = (\varphi^1, ..., \varphi^m)$ and $\mu = (\mu^1, ..., \mu^m)$.

In the specific case where Ω is a locally compact, separable metric space, we have the isometric isomorphy

$$\mathcal{M}_{<\infty}(X;\mathbb{R}^m) \cong (\mathrm{C}_0(X;\mathbb{R}^m))^*, \tag{2.1}$$

[1] This notation is at variance with that used in [16], where the authors use the notation $\mathcal{M}_{\mathrm{loc}}(X;\mathbb{R}^m)$ for the $\mathbb{R}^m$–valued Radon measures on X.

the duality pairing given by $\langle \mu, \varphi \rangle_{\mathcal{M}\times \mathrm{C}} := \int_\Omega \varphi \,\mathrm{d}\mu$. This is a very general form of the *Riesz representation theorem*, a more specific form of which we record explicitely:

Theorem 2.1 (Riesz, [16, Thm. 1.54]). *Let (X, d) be a locally compact and separable metric space and let $\mathcal{L}\colon \mathrm{C}_0(X;\mathbb{R}^m) \to \mathbb{R}$ be an additive and bounded functional, that is, $\mathcal{L}$ satisfies the following two conditions:*

(a) *For all $u, v \in \mathrm{C}_0(X;\mathbb{R}^m)$ there holds $\mathcal{L}(u+v) = \mathcal{L}(u) + \mathcal{L}(v)$.*

(b) $\|\mathcal{L}\| := \sup\{\mathcal{L}(u)\colon\ u \in \mathrm{C}_0(X;\mathbb{R}^m),\ |u| \leqslant 1\} < \infty$.

Then there exists a unique $\mathbb{R}^m$–valued finite Radon measure $\mu = (\mu^1, ..., \mu^m)$ on X such that for all $u = (u^1, ..., u^m) \in \mathrm{C}_0(X;\mathbb{R}^m)$ there holds

$$\mathcal{L}(u) = \sum_{j=1}^{m} \int_X u^j \,\mathrm{d}\mu^j.$$

Locally, we can identify $\mathcal{M}(X;\mathbb{R}^m)$ with $\mathrm{C}_c(X;\mathbb{R}^m)^*$ in the sense of duality of locally convex topological vector spaces (briefly LCTVS), but we will not need this in the sequel. The duality (2.1), however, is of special interest to us: As a consequence of the Banach–Alaoglu–Bourbaki Theorem on weak*–compactness in duals of LCTVS [102, Thm. 4.31], we immediately obtain the following compactness principle [16, Thm. 1.59]: If $(\mu_k) \subset \mathcal{M}_{<\infty}(X;\mathbb{R}^m)$ satisfies $\sup_k \|\mu_k\| < \infty$, then it has a weakly convergent subsequence. That is, there exists a subsequence $(\mu_{k(j)}) \subset (\mu_k)$ and $\mu \in \mathcal{M}_{<\infty}(X;\mathbb{R}^m)$ such that $\mu_{k(j)} \overset{*}{\rightharpoonup} \mu$ as $j \to \infty$, i.e., for all $\varphi \in \mathrm{C}_0(X;\mathbb{R}^m)$ we have $\langle \mu_{k(j)}, \varphi \rangle_{\mathcal{M}\times\mathrm{C}} \to \langle \mu, \varphi \rangle_{\mathcal{M}\times\mathrm{C}}$ for all $\varphi \in \mathrm{C}_0(X;\mathbb{R}^m)$ as $j \to \infty$.

In an intermediate step, let us recall the definition of both the Hausdorff measures and dimensions. Let $0 \leqslant s < \infty$ and $0 < \delta \leqslant \infty$. For a set $A \subset \mathbb{R}^n$, we define

$$\mathcal{H}^s_\delta(A) := \inf\Big\{ \sum_{j=1}^{\infty} \alpha(s) \Big(\frac{\mathrm{diam}(C_j)}{2}\Big)^s\colon\ A \subset \bigcup_{j=1}^{\infty} C_j,\ \mathrm{diam}(C_j) \leqslant \delta \Big\},$$

where $\alpha(s) := \pi^{s/2}/\Gamma(\frac{s}{2}+1)$ with the Eulerian gamma function Γ, and $\mathcal{H}^s(A) := \sup_{\delta>0} \mathcal{H}^s_\delta(A)$. As shown in [97, Chpt. 2.1], $\mathcal{H}^s$ is a well–defined Borel regular measure on $\mathbb{R}^n$, and we refer to $\mathcal{H}^s$ as the s–dimensional *Hausdorff measure* . Let $0 \leqslant s < t < \infty$. Given a set $A \subset \mathbb{R}^n$, we have the implication $(\mathcal{H}^s(A) < \infty \Rightarrow \mathcal{H}^t(A) = 0)$, and thus we may define the *Hausdorff dimension* $\dim_{\mathcal{H}}(A)$ of A by

$$\dim_{\mathcal{H}}(A) := \inf\{s \geqslant 0\colon\ \mathcal{H}^s(A) = 0\}. \tag{2.2}$$

For our purposes, it is useful to record the so–called *measure density lemma* which can often be used to derive Hausdorff dimension bounds.

Lemma 2.2 (Measure Density Lemma, [128, Prop. 2.7]). *Let $A \subset \mathbb{R}^n$ be open and let μ be a positive Radon measure on A with $\mu(A) < \infty$. Define, for $0 < \alpha < n$,*

$$E^\alpha := \Big\{ x \in A\colon\ \limsup_{r \searrow 0} \frac{\mu(\mathrm{B}(x,r))}{r^\alpha} > 0 \Big\}.$$

Then we have $\dim_{\mathcal{H}}(E^\alpha) \leqslant \alpha$.

We conclude our exposition of measure theory with elementary density results and a vectorial version of the Radon–Nikodym theorem. Let (Ω, Σ) be a measurable space, μ a positive measure and ν a vector measure on (Ω, Σ). The measure ν is *absolutely continuous* with respect to μ (in formulas $\nu \ll \mu$) if for any $B \in \Sigma$ with $\mu(B) = 0$ there holds $|\nu|(\mathrm{B}) = 0$. In turn, if μ and ν are positive measures, we call μ and ν

mutually singular (in formulas $\mu\perp\nu$) provided there exists $E\in\Sigma$ such that $\mu(E)=0$ and $\nu(X\setminus E)=0$. Finally, if μ and ν are vector–valued, then we call them mutually singular provided $|\mu|$ and $|\nu|$ are mutually singular in the above sense.

Letting ν be a σ–finite vector measure and μ a positive measure on (Ω,Σ), the *Radon–Nikodym Theorem* asserts that there exists a unique pair of $\mathbb{R}^m$–valued measures ν^{ac},ν^s with the properties

$$\nu^{ac}\ll\mu,\quad \nu^{ac}\perp\mu,\quad \nu=\nu^{ac}+\nu^s.$$

Moreover, there exists a unique function $f\in\mathrm{L}^1(\Omega,\mu;\mathbb{R}^m)$ such that $\nu^{ac}=f\mu$, and in this situation f is called the *density* of ν with respect to μ. We also use the notation $\frac{\mathrm{d}\nu}{\mathrm{d}\mu}:=f$ for the density. The explicit computation of the density of ν with respect to μ is accomplished by the Lebesgue differentiation theorem (also often referred to as *Besicovitch derivation theorem*) for Radon measures which we state for the euclidean setup only (cp. [97, Secs. 1.6, 1,7]):

Theorem 2.3 (Lebesgue)**.** *Let ν be a $\mathbb{R}^m$–valued Radon measure on $\mathbb{R}^n$ and μ a positive Radon measure on an open set $\Omega\subset\mathbb{R}^n$. Then the density $\frac{\mathrm{d}\nu}{\mathrm{d}\mu}$ exists and is finite μ–a.e., μ–measurable and is for μ–a.e. $x\in\mathbb{R}^n$ given by*

$$f(x):=\frac{\mathrm{d}\nu}{\mathrm{d}\mu}(x)=\begin{cases}\lim_{r\searrow0}\frac{\nu(\mathrm{B}(x,r))}{\mu(\mathrm{B}(x,r))} & \textit{if } \mu(\mathrm{B}(x,r))>0 \textit{ for all } r>0,\\ +\infty & \textit{otherwise.}\end{cases}\tag{2.3}$$

The Radon–Nikodym decomposition of ν then is given by $\nu=f\mu+\nu^s$, where $\nu^s=\nu\llcorner E$ with the μ–negligible set

$$E:=(\Omega\setminus\mu)\cup\Big\{x\in\operatorname{spt}(\mu)\colon\ \lim_{r\searrow0}\frac{|\nu|(\mathrm{B}(x,r))}{\mu(\mathrm{B}(x,r))}=\infty\Big\}.$$

We then have the following result on Lebesgue points of Radon measures.

Theorem 2.4. *Let μ be a positive Radon measure on an open set $\Omega\subset\mathbb{R}^n$ and let $f\in\mathrm{L}^1(\Omega;\mu)$. Then for μ–a.e. $x\in\Omega$ there exists $\widetilde{f}(x)\in\mathbb{R}$*

$$\lim_{r\searrow0}\frac{1}{\mu(\mathrm{B}(x,r))}\int_{\mathrm{B}(x,r)}|f(y)-\widetilde{f}(x)|\,\mathrm{d}\mu(y)=0,$$

and any point x with this property shall be referred to as Lebesgue point *of μ.*

Let us further recall the notion of *approximate differentiability*. Let $f\colon\mathbb{R}^n\to\mathbb{R}^m$ be a map. We say that $a\in\mathbb{R}^m$ is the *approximate limit* of $f(y)$ as $y\to x$ (in symbols $a=\operatorname{aplim}_{y\to x}f(y)$) if for each $\varepsilon>0$,

$$\lim_{r\searrow0}\frac{\mathscr{L}^n(\mathrm{B}(x,r)\cap\{|f-a|\geqslant\varepsilon\})}{\mathscr{L}^n(\mathrm{B}(x,r))}=0,$$

compare with [97, Chpt. 1.7.2]. If existent, approximate limits are unique. Now, we say that f is *approximately differentiable* at $x\in\mathbb{R}^n$ if and only if there exists a linear mapping $L\colon\mathbb{R}^n\to\mathbb{R}^m$ such that

$$\operatorname{aplim}_{\substack{y\to x\\ y\neq x}}\frac{|f(y)-f(x)-L(y-x)|}{|y-x|}=0.$$

By Rademacher's Theorem [97, Chpt. 3.1.2, Thm. 2], every locally Lipschitz continuous map $g\colon\mathbb{R}^n\to\mathbb{R}^m$ is differentiable (and hence in particular approximately differentiable) $\mathscr{L}^n$–a.e.. As a consequence, if for a given measurable map $f\colon\mathbb{R}^n\to\mathbb{R}^m$ there exists a sequence (g_k) of locally Lipschitz maps with $\mathscr{L}^n(\{f\neq g_k\})\to0$ as $k\to\infty$, then f is approximately differentiable $\mathscr{L}^n$–a.e.. This in turn is the easy direction of the following result.

Theorem 2.5 (Whitney, [244]). *The following conditions are equivalent for a measurable map $f\colon \mathbb{R}^n \to \mathbb{R}^m$:*

(a) *f is approximately differentiable $\mathscr{L}^n$–a.e. in $\mathbb{R}^n$.*

(b) *For each $\varepsilon > 0$, there exists a locally Lipschitz continuous function $g\colon \mathbb{R}^n \to \mathbb{R}^m$ such that $\mathscr{L}^n(\{f \neq g\}) < \varepsilon$.*

2.3. Fourier Analysis

Many properties of locally integrable mappings $v\colon \mathbb{R}^n \to \mathbb{R}^N$ for which we only have control over certain combinations of their weak derivatives can be extracted by means of Fourier analysis and accordingly so–called singular integral operators. In this section we aim to give a concise overview of the related concepts, with particular emphasis on Mihlin's theorem on the boundedness of Fourier multiplier operators with symbol homogeneous of degree zero, see Theorem 2.6 below. Let $v \in \mathscr{S}(\mathbb{R}^n)$ with the Schwartz class $\mathscr{S}(\mathbb{R}^n)$ of rapidly decreasing functions. We define the *Fourier transform* of v by

$$(\mathscr{F}v)(\xi) := \widehat{v}(\xi) := \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} v(x)\, \mathrm{e}^{-i\langle \xi, x\rangle}\, \mathrm{d}x \tag{2.4}$$

Sometimes, when we want to stress that the Fourier transform takes a function in x and yields a function in ξ, we also write $\mathscr{F}_{x\mapsto\xi}v$ and similarly for the inverse Fourier transform $\mathscr{F}^{-1}_{\xi\mapsto x}$. As one of its main applications, the Fourier transform allows to set up a functional calculus for (pseudo)differential operators. More precisely, it helps to assign a (pseudo)differential operator to a given function. This is known as *quantisation*, and the following result due to Mihlin shows that for a large class of multipliers m the corresponding operators are bounded on L^p for $1 < p < \infty$.

Theorem 2.6 (Mihlin, [1, Thm. 4.23]). *Let $m\colon \mathbb{R}^n \setminus \{0\} \to \mathbb{C}$ be an $(n+2)$–times continuously differentiable function such that $|\partial_\xi^\alpha m(\xi)| \leqslant A|\xi|^{-|\alpha|}$ holds for all $\xi \neq 0$ and all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leqslant n+2$. Then for every $1 < p < \infty$ the operator $m(\mathrm{D}_x)$ defined by*

$$m(\mathrm{D}_x)f := \mathscr{F}^{-1}_{\xi\mapsto x}[m(\xi)\widehat{f}(\xi)] \qquad \textit{for } f \in \mathscr{S}(\mathbb{R}^n)$$

extends to a bounded linear operator $m(\mathrm{D}_x)\colon\ \mathrm{L}^p(\mathbb{R}^n) \to \mathrm{L}^p(\mathbb{R}^n)$.

We refer to the function m as *multiplier*. In Chapter 3, we shall give an equivalent of this theorem on the torus $\mathbb{T}^n$.

2.3.1. Symbol Classes and Pseudodifferential Operators At several stages of the main text it is useful to have the rudiments of the general theory of pseudodifferential operators at our disposal. Our exposition here is based on Abels' monograph [1], also see [221, 234].

Let $m \in \mathbb{R}$, $n, N \in \mathbb{N}$. We define the (Hörmander) *symbol class* $S^m_{1,0}(\mathbb{R}^N \times \mathbb{R}^n)$ as the space of functions $p \in \mathrm{C}^\infty(\mathbb{R}^N \times \mathbb{R}^n; \mathbb{C})$ such that for every $(\alpha, \beta) \in \mathbb{N}_0^n \times \mathbb{N}_0^N$ there exists a constant $c_{\alpha,\beta} > 0$ such that

$$|\partial_\xi^\alpha \partial_x^\beta p(x,\xi)| \leqslant c_{\alpha,\beta}(1 + |\xi|)^{m-|\alpha|}.$$

For $p \in S^m_{1,0}(\mathbb{R}^N \times \mathbb{R}^n)$, we define an operator $p(x, \mathrm{D}_x)\colon \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$ by

$$p(x,\mathrm{D}_x)f(x) := \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} p(x,\xi)\widehat{f}(\xi)\exp(\mathrm{i}\langle x,\xi\rangle)\,\mathrm{d}\xi, \qquad x \in \mathbb{R}^n.$$

A particular class we shall deal with frequently are the (translation invariant) *singular integral operators*, cp. [1, Chpt. 4]. Given $f \in \mathscr{S}'(\mathbb{R}^n)$, we call the operator $T\colon f \mapsto \mathscr{F}^{-1}(\widehat{K}\widehat{f})$ a *translation invariant singular integral operator* provided $K\colon \mathbb{R}^n \to \mathbb{R}$ is such that

(S1) $\widehat{K} \in \mathrm{L}^\infty(\mathbb{R}^n)$,

(S2) there exists $k \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\})$ such that for all $f \in \mathrm{C}^\infty_c(\mathbb{R}^n)$ there holds

$$(Tf)(x) = \int_{\mathbb{R}^n} k(x-y)f(y)\,\mathrm{d}y \qquad \text{for almost all } x \notin \mathrm{spt}(f),$$

(S3) and the kernel function k from (S2) satisfies the *Hörmander condition*

$$\sup_{y\in\mathbb{R}^n} \int_{\{|x|>2y\}} |k(x-y)-k(x)|\,\mathrm{d}x < \infty.$$

The singular integrals as considered here can in turn be written as the Cauchy principal value integrals of convolutions against the kernel k, and qualify as pseudodifferential operators of order zero. It is possible to give a more general approach to such operators, and we refer the reader to [232, 221, 138] for more background information.

2.4. Convex Analysis

2.4.1. The Convex Conjugate We now introduce the relevant notions and sketch aspects of the general theory of convex duality as shall be encountered in Chapters 5 to 7. Here, a general reference are the monographs [96, 47, 204], where a much more detailled presentation of the concepts to be discussed now can be found.

Given a extended real–valued convex function $h\colon \mathbb{R}^m \to \mathbb{R}\cup\{\infty\}$, we define the *conjugate function* (or *Fenchel conjugate*) $h^*\colon \mathbb{R}^m \to \mathbb{R}\cup\{\infty\}$ by

$$h^*(z^*) := \sup_{z\in\mathbb{R}^m} \big\{\langle z^*, z\rangle - h(z)\big\}, \qquad \text{for all } z^* \in \mathbb{R}^m.$$

The *effective domain* of h^* is given by

$$\mathrm{dom}(h^*) := \big\{z^* \in \mathbb{R}^m \colon h^*(z^*) \neq +\infty\big\},$$

and in general, we define the *domain* of any function $g\colon \mathbb{R}^m \to (-\infty;\infty]$ as the set of $x \in \mathbb{R}^m$ such that $g(x) < \infty$.

Furthermore, we define the *biconjugate function* $h^{**}\colon \mathbb{R}^m \to \mathbb{R}\cup\{\infty\}$ as $h^{**} := (h^*)^*$. By definition, h^* is convex and lower semicontinuous, and if $h^* \neq \infty$ then h^{**} coincides with the lower semicontinuous, convex envelope of h (thus, it is h itself if h is already convex and lower semicontinuous). We denote

$$\partial h(z) := \{z^* \in \mathbb{R}^m \colon h(z) + \langle z^*, y-z\rangle \leqslant h(y) \text{ for all } y \in \mathbb{R}^m\}$$

the *subdifferential* of h at z. Then we have the *duality relation*

$$z^* \in \partial h(z) \qquad \text{if and only if} \qquad h(z) + h^*(z^*) = \langle z^*, z\rangle \tag{2.5}$$

for $z \in \mathbb{R}^m$, and it is clear that $\partial h(z)$ is single–valued if h is differentiable at z. Moreover, the general form of *Young's inequality* asserts that if $h\colon \mathbb{R}^m \to \mathbb{R}\cup\{\infty\}$ is convex, then for all $y \in \mathbb{R}^m$ and all $x \in \mathrm{dom}(h)$ there holds

$$\langle y, x\rangle \leqslant h(x) + h^*(y), \tag{Young}$$

and equality in (Young) occurs if and only if $y \in \partial h(x)$. In what follows, we record some examples of conjugate functions we shall refer to frequently.

Example 2.7 ([47, Chpt. 3.3]). *In what follows, we put $m = 1$.*

(a) *If $h(t) := |t|^p/p$ for $t \in \mathbb{R}$ with $1 < p < \infty$, then $h^*(t) = t^{p'}/p'$ for $t \in \mathbb{R}$ with the Hölder conjugate exponent $p' := p/(p-1)$ of p. In particular,* $\mathrm{dom}(h^*) = \mathbb{R}$.

(b) *If* $h(t) := \sqrt{1+t^2}$ *for* $t \in \mathbb{R}$, *then* $h^*(t) = -\sqrt{1-|t|^2}$ *for* $t \in [-1,1]$. *In particular,* $\mathrm{dom}(h^*) = [-1,1]$.

(c) *If* $h(t) := \exp(t)$ *for* $t \in \mathbb{R}$, *then* h^* *is given by*

$$h^*(t) = \exp^*(t) = \begin{cases} t\log(t) - t & \text{if } t > 0, \\ 0 & \text{if } t = 0, \\ +\infty & \text{if } t < 0. \end{cases}$$

We conclude our general presentation with a remark on radial symmetry.

Remark 2.8 (Radially Symmetric Integrands)**.** *If* h *is radially symmetric, i.e. it can be written in the form* $h(\cdot) = f(|\cdot|)$ *for some function* $f\colon \mathbb{R} \to \mathbb{R}$, *then we have* $h^*(\cdot) = f^*(|\cdot|)$. *To see this, note that, for each* $z^* \in \mathbb{R}^m$, *we have*

$$h^*(z^*) = \sup_{z \in \mathbb{R}^m} \big\{ \langle z^*, z\rangle - f(|z|) \big\} = \sup_{s \geqslant 0} \big\{ s|z^*| - f(s) \big\} = f^*(|z^*|).$$

2.4.2. Jensen's Inequality For expository completeness, we further record the following version of Jensen's Inequality which will be used frequently.

Lemma 2.9. *Let* $(\Omega, \mathscr{F}, \mu)$ *be a probability space,* $g\colon \Omega \to \mathbb{R}$ *a* μ*–integrable function and* $\varphi\colon \mathbb{R} \to \mathbb{R}$ *a convex function. Then we have*

$$\varphi\left(\int_\Omega g \,\mathrm{d}\mu\right) \leqslant \int_\Omega \varphi \circ g \,\mathrm{d}\mu.$$

2.5. Function Spaces

In this section we give an overview of different ways to measure the smoothness of functions. Sometimes, this can be achieved in terms of suitable maximal operators (cp. Chapter 2.6). Here, we focus on the systematic spaces scale given by that of Besov spaces which is depicted in Figure 2.1. To provide an overall approach which covers many applications, we introduce them through Littlewood–Paley theory and subsequently give more acccesible characterisations. Finally, we conclude with some remarks on Orlicz spaces.

2.5.1. Besov Spaces In this section we briefly revisit Besov spaces. For more background information, the reader is referred to [7, Chpt. 4] and [233, Chpts. 1 and 2]. Let $\varphi \in \mathscr{S}(\mathbb{R}^n)$ be such that $\mathrm{spt}(\hat{\varphi}) \subset \mathrm{B}(0,1) =: \mathrm{B}_0$ and $\hat{\varphi}(\xi) = 1$ on $\mathrm{B}(0,\frac{1}{2}) =: \mathrm{B}_1$. We put, for $k \in \mathbb{Z}$, $\varphi_k(x) := 2^{kn}\varphi(2^k x)$ and let $\Phi_k(x) := \varphi_k(x) - \varphi_{k-1}(x)$. In consequence, $\mathrm{spt}(\widehat{\Phi_k}) \subset \mathrm{B}(0,2^k) \setminus \mathrm{B}(0,2^{k-2}) =: \mathrm{B}_{-k} \setminus \mathrm{B}_{-k+2}$. Given $\alpha \in \mathbb{R}$ and $0 < p, q \leqslant \infty$, the *Besov spaces* $\mathrm{B}^{\alpha,p}_q(\mathbb{R}^n)$ then are defined by

$$\mathrm{B}^{\alpha,p}_q(\mathbb{R}^n) := \big\{ u \in \mathscr{S}'(\mathbb{R}^n)\colon\ \|\varphi * u\|_{\mathrm{L}^p} + \sum_{k=1}^{\infty} (2^{k\alpha}\|\Phi_k * u\|_{\mathrm{L}^p})^q < \infty \big\}, \qquad 0 < q < \infty,$$

$$\mathrm{B}^{\alpha,p}_\infty(\mathbb{R}^n) := \big\{ u \in \mathscr{S}'(\mathbb{R}^n)\colon\ \|\varphi * u\|_{\mathrm{L}^p} < \infty \text{ and } \sup_{k \geqslant 1} 2^{k\alpha}\|\Phi_k * u\|_{\mathrm{L}^p} < \infty \big\}, \qquad q = \infty.$$

Here, $\mathscr{S}'(\mathbb{R}^n)$ are the tempered distributions on $\mathbb{R}^n$. To define the homogeneous variants of these spaces, note that since $\sum_{k\in\mathbb{Z}} \widehat{\Phi_k}(\xi) = 1$ for any $\xi \in \mathbb{R}^n \setminus \{0\}$ and $\widehat{\Phi_k}(0) = 0$ for any $k \in \mathbb{Z}$, there holds $\widehat{\Phi_k}\,\mathrm{D}^\beta\delta_0 =$ for any $k \in \mathbb{Z}$ and $\beta \in \mathbb{N}_0^n$, where δ_0 as usual denotes the Dirac delta distribution centered at zero. Hence, $\Phi_k * \mathcal{P} = 0$ for any polynomial $\mathcal{P}\colon \mathbb{R}^n \to \mathbb{R}$. Let $\mathfrak{P}(\mathbb{R}^n)$ denote the space of $\mathbb{R}$–valued polynomials on $\mathbb{R}^n$. For α, p and

q as above, we then define

$$\dot{\mathrm{B}}_q^{\alpha,p}(\mathbb{R}^n) := \big\{u \in (\mathscr{S}'/\mathfrak{P})(\mathbb{R}^n)\colon \sum_{k\in\mathbb{Z}} \big(2^{k\alpha}\|\Phi_k * u\|_{\mathrm{L}^p}\big)^q < \infty\big\}, \qquad 0<q<\infty,$$
$$\dot{\mathrm{B}}_\infty^{\alpha,p}(\mathbb{R}^n) := \big\{u \in (\mathscr{S}'/\mathfrak{P})(\mathbb{R}^n)\colon \sup_{k\in\mathbb{Z}} 2^{k\alpha}\|\Phi_k * u\|_{\mathrm{L}^p} < \infty\big\}, \qquad q=\infty.$$

In the literature, one finds the equivalent notation $\mathrm{B}_q^{\alpha,p} = \mathrm{B}_{p,q}^\alpha$, and no confusion arises from this as long as the order of the indices is obeyed. It is also possible to define Besov spaces on domains, however, this is not needed here. We conclude by collecting the following result on embeddings of Besov spaces.

Lemma 2.10 (Embeddings, cp. [233, Thm. 2.7.1])**.** *Let $0 < p_0 \leqslant p_1 \leqslant \infty$, $0 < q \leqslant \infty$ and $0 < s_1 \leqslant s_0 < \infty$. Then we have*

$$\mathrm{B}_q^{s_0,p_0}(\mathbb{R}^n) \subset \mathrm{B}_q^{s_1,p_1}(\mathbb{R}^n) \qquad \textit{provided } s_0 - \frac{n}{p_0} = s_1 - \frac{n}{p_1}. \tag{2.6}$$

Note that the third parameter q is fixed in the preceding theorem. Finally, the local variants of these spaces are defined in the obvious manner.

2.5.2. The Hölder–Zygmund Spaces In the main part, we shall need boundedness of singular integrals on spaces of Hölder continuous functions. The singular integrals as required here are special instances of pseudodifferential operators which, in turn, have convenient boundedness properties on Besov spaces, cp. Chapter 2.5.4 below. The following result admits to reduce to the Besov space situation.

Lemma 2.11 ([1, Thm. 6.1])**.** *Let $0 < \alpha < 1$. Then the Hölder space $\mathrm{C}^{0,\alpha}(\mathbb{R}^n)$ given by*

$$\mathrm{C}^{0,\alpha}(\mathbb{R}^n) := \left\{\varphi\colon \mathbb{R}^n \to \mathbb{R}\colon\ \|\varphi\|_{\mathrm{C}^{0,\alpha}} := \|\varphi\|_{\mathrm{L}^\infty(\mathbb{R}^n)} + \sup_{x\neq y} \frac{|\varphi(x)-\varphi(y)|}{|x-y|^\alpha} < \infty\right\}$$

coincides with the inhomogeneous Besov space $\mathrm{B}_\infty^{\alpha,\infty}(\mathbb{R}^n)$, and $\|\cdot\|_{\mathrm{B}_\infty^{\alpha,\infty}}$ is an equivalent norm on $\mathrm{C}^{0,\alpha}(\mathbb{R}^n)$.

Let us note that despite the equivalence $\mathrm{B}_\infty^{\alpha,\infty} \simeq \mathrm{C}^{0,\alpha}$ for $0<\alpha<1$, the spaces $\mathrm{B}_\infty^{\alpha,\infty}$ are not directly linked to the spaces $\mathrm{C}^{k,\alpha}$ for $\alpha > 1$. More generally, we refer to $\mathrm{B}_\infty^{\alpha,\infty}$ for $\alpha > 0$ as *Hölder–Zygmund spaces*.

2.5.3. Difference Quotient Characterisation of Besov Spaces We also shall need a characterisation of the Besov space in terms of difference quotients. Let $1 \leqslant p < \infty$, $1 \leqslant q \leqslant \infty$ and $0 < \alpha < 1$. For $u \in \mathrm{L}^p(\mathbb{R}^n)$, $x \in \mathbb{R}^n$, $h \in \mathbb{R}$ and a unit vector e_s, $s \in \{1, ..., n\}$, we put

$$\tau_{s,h}u(x) := u(x+he_s) - u(x). \tag{2.7}$$

In this situation, we define for u as above

$$[u]^*_{\mathrm{B}^\alpha_{p,q}(\mathbb{R}^n)} := \sum_{s=1}^n \left(\int_0^\infty \left(\frac{\|\tau_{s,h}u\|_{\mathrm{L}^p(\mathbb{R}^n)}}{t^\alpha}\right)^q \frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}} \qquad \text{if } 1 \leqslant p,q < \infty,$$
$$[u]^*_{\mathrm{B}^\alpha_{p,\infty}(\mathbb{R}^n)} := \sup_{h\neq 0}\sup_{x\in\mathbb{R}^n} \frac{\|\tau_{s,h}u\|_{\mathrm{L}^p(\mathbb{R}^n)}}{h^\alpha} \qquad \text{if } 1 \leqslant p < \infty,\ q=\infty.$$

Then (cp. [233, Chpt. 2.5.12]) we have $u \in \mathrm{B}^\alpha_{p,q}(\mathbb{R}^n)$ if and only if $[u]^*_{\mathrm{B}^\alpha_{p,q}} < \infty$, and in this situation the norms $\|\cdot\|_{\mathrm{B}^\alpha_{p,q}}$ and the norm given by $\|u\|^*_{\mathrm{B}^\alpha_{p,q}} := \|u\|_{\mathrm{L}^p(\mathbb{R}^n)} + [u]^*_{\mathrm{B}^\alpha_{p,q}}$ are equivalent. In this context, the spaces $\mathcal{N}^{\alpha,p}(\mathbb{R}^n) := \mathrm{B}^\alpha_{p,\infty}(\mathbb{R}^n)$ are often referred to as *Nikolskii spaces* . Local versions of such spaces are obtained in the obvious manner. We now have the following embedding.

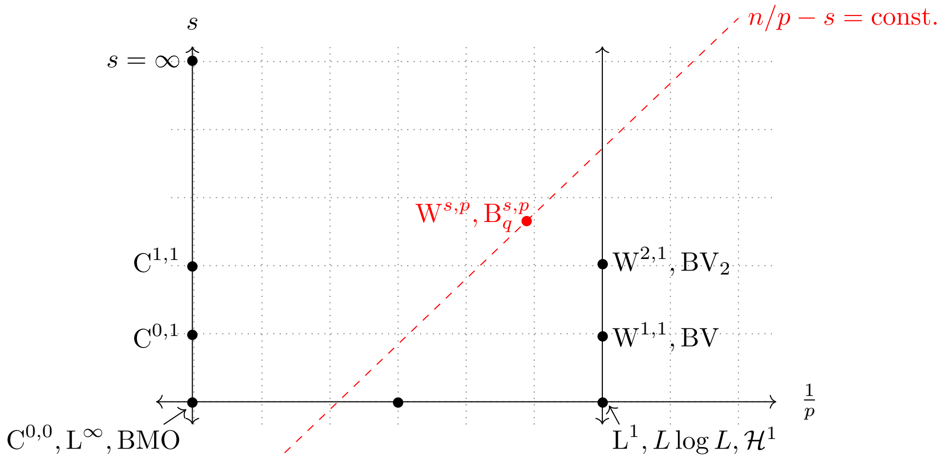

Figure 2.1: Adaptivity Diagram; schematic presentation of function spaces. The parameter p displays the integrability parameter and s displays the smoothness parameter. The spaces BV and BV_k will be introduced in Chapter 4. Note that by Lemma 2.10, if two Besov spaces with the same q–parameter lie on a fixed red line as indicated above, then the higher space embeds into the lower one.

Lemma 2.12. *Let $0 < s < 1$. Then we have $(\mathrm{B}^{s,1}_\infty)_{\mathrm{loc}}(\mathbb{R}^n) \hookrightarrow \mathrm{L}^q_{\mathrm{loc}}(\mathbb{R}^n)$ for any $1 \leqslant q < \frac{n}{n-s}$.*

Proof. Fix $1 \leqslant q < \frac{n}{n-s}$. Localising, we firstly obtain by Lemma 2.10 $(\mathrm{B}^{s,1}_\infty)_{\mathrm{loc}} \subset (\mathrm{B}^{t,p(t)}_\infty)_{\mathrm{loc}}$ for $0 < t < s$, t being sufficiently close to s, where $p(t)$ is determined via

$$s - n = t - \frac{n}{p(t)}, \quad \text{i.e.,} \quad p(t) = \frac{n}{n - s + t}.$$

Now, since $(\mathrm{B}^{\alpha,\beta}_\infty)_{\mathrm{loc}} \hookrightarrow (\mathrm{B}^{\alpha-\varepsilon,\beta}_\beta)_{\mathrm{loc}}$ for any $0 < \alpha < 1$, $\beta \geqslant 1$ and $\varepsilon > 0$ sufficiently small, we conclude that

$$(\mathrm{B}^{s,1}_\infty)_{\mathrm{loc}} \subset (\mathrm{B}^{t,p(t)}_\infty)_{\mathrm{loc}} \subset (\mathrm{B}^{t-\varepsilon,p(t)}_{p(t)})_{\mathrm{loc}} \hookrightarrow \mathrm{L}^{q(t)}, \qquad q(t) = \frac{np(t)}{n - (t-\varepsilon)p(t)}.$$

Now it suffices to note that we may choose $t < s$ arbitrary close to s and $\varepsilon > 0$ arbitrarily small to conclude. □

2.5.4. Pseudodifferential Operators on Besov Spaces In an intermediate step, we collect a useful boundedness result of the pseudodifferential operators as introduced in Section 2.3.1.

Theorem 2.13 ([1, Thm. 6.19])**.** *Let $m \in \mathbb{R}$, $s \in \mathbb{R}$ and $1 \leqslant q, r \leqslant \infty$. Let $p \in S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$. Then*

$$p(x, \mathrm{D}_x)\colon\ \mathrm{B}^{s+m,q}_r(\mathbb{R}^n) \to \mathrm{B}^{s,q}_r(\mathbb{R}^n) \tag{2.8}$$

is a bounded linear operator.

2.5.5. Sobolev–Slobodeckjii Spaces Let $\Omega \subset \mathbb{R}^n$ be a measurable set. Given $1 \leqslant p < \infty$ and $0 < s < 1$, we say that a measurable mapping $v\colon \Omega \to \mathbb{R}^m$ belongs to the *Sobolev–Slobodeckjii* or *fractional Sobolev space* $\mathrm{W}^{s,p}(\Omega;\mathbb{R}^m)$ if and only if

$$\|v\|^p_{\mathrm{W}^{s,p}(\Omega;\mathbb{R}^m)} := \|v\|^p_{\mathrm{L}^p(\Omega;\mathbb{R}^m)} + \iint_{\Omega\times\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,\mathrm{d}(x,y) < \infty. \tag{2.9}$$

It is easy to see that $(\mathrm{W}^{s,p}(\Omega;\mathbb{R}^m);\|\cdot\|_{\mathrm{W}^{s,p}(\Omega;\mathbb{R}^m)})$ is a Banach space. Moreover, for sufficiently regular sets Ω, the fractional Sobolev spaces $\mathrm{W}^{s,p}(\Omega;,\mathbb{R}^m)$ can be retrieved from the Besov spaces. To keep our presentation simple, let $\Omega=\mathbb{R}^n$ for the time being. Then there holds

$$\mathrm{W}^{s,p}(\mathbb{R}^n;\mathbb{R}^m)\simeq \mathrm{B}^{s,p}_p(\mathbb{R}^n;\mathbb{R}^m)\qquad \text{for all } 0<s<1,\ 1\leqslant p<\infty. \tag{2.10}$$

Remark 2.14 (Brezis, [54]). *Formally setting $s=1$ in (2.9) does* not *yield the usual Sobolev space $\mathrm{W}^{1,p}(\Omega;\mathbb{R}^m)$. Indeed, as* Brezis *could show in [54], a measurable mapping $v\colon\Omega\to\mathbb{R}$ with Ω open and connected satisfies*

$$\iint_{\Omega\times\Omega}\frac{|v(x)-v(y)|^p}{|x-y|^{n+p}}\,\mathrm{d}(x,y)<\infty$$

if and only if v is constant.

The observation of the preceding remark is connected to the fact that if $v\in\mathrm{W}^{1,p}(\Omega)$ is not constant, then $\limsup_{s\nearrow 1}\|v\|_{\mathrm{W}^{s,p}(\Omega)}=+\infty$. As pointed out by Bourgain, Brezis & Mironescu in [50] (also cp. [170, Chpt. 10]), this lack of 'scale continuity' can be circumvented by a suitable renormalisation. Lastly, we remind the reader of another remarkable feature of the fractional Sobolev scale:

Remark 2.15 (Mironescu & Sickel, [182]). *Let $\Omega\subset\mathbb{R}^n$ be open and bounded. Different from the usual Sobolev scale $(\mathrm{W}^{k,p}(\Omega;\mathbb{R}^m))_{1\leqslant p<\infty}$ for $k\in\mathbb{N}$ fixed, where one has the embedding $\mathrm{W}^{k,p}(\Omega;\mathbb{R}^m)\hookrightarrow\mathrm{W}^{k,q}(\Omega;\mathbb{R}^n)$ whenever $p>q$, it was proved in the recent paper [182] that whenever $0<s<1$ and $p>q$, then $\mathrm{W}^{s,p}(\Omega;\mathbb{R}^m)\not\subset\mathrm{W}^{s,q}(\Omega;\mathbb{R}^m)$.*

Finally, for later applications we record a special version of the measure density lemma, Lemma 2.2, taylored for finite difference estimates due to Mingione.

Lemma 2.16 ([175, Sec. 4]). *Let $\Omega\subset\mathbb{R}^n$ be open and bounded and let $\varphi\in\mathrm{W}^{\theta,p}(\Omega;\mathbb{R}^N)$ for some $0<\theta\leqslant 1$ and $1\leqslant p<\infty$ with $\theta p<n$. We define*

$$\Sigma:=\Big\{x_0\in\Omega\colon\ \limsup_{r\searrow 0}\fint_{\Omega\cap\mathrm{B}(x_0,r)}|f-(f)_{\mathrm{B}(x_0,r)}|^p\,\mathrm{d}x>0\Big\}. \tag{2.11}$$

Then we have $\dim_{\mathcal{H}}(\Sigma)\leqslant n-\theta p$.

2.5.6. Orlicz Spaces We next roughly sketch the fundamentals of Orlicz space theory and refer the reader to [198, 157] for more detail. Throughout, a function $\varphi\colon\mathbb{R}_{\geqslant 0}\to\mathbb{R}_{\geqslant 0}$ is called an *N–function* (or *Orlicz function*) provided $\varphi(0)=0$, φ is differentiable with right–continuous, non–decreasing derivative φ' satisfying $\varphi'(0)=0$, $\varphi'(t)>0$ for $t>0$ and $\lim_{t\nearrow\infty}\varphi'(t)=\infty$. Let us note that, since φ' is non–decreasing, any N–function is necessarily convex.

In a bit more general flavor, we say that a convex, left–continuous function $\varphi\colon\mathbb{R}_{\geqslant 0}\to[0,\infty]$ with

$$\varphi(0)=0,\quad \lim_{t\searrow 0}\varphi(t)=0,\quad\text{and}\quad \lim_{t\nearrow\infty}\varphi(t)=+\infty$$

is a *Φ–function* (cp. [51]) .

Let now $\varphi\colon\mathbb{R}_{\geqslant 0}\to\mathbb{R}_{\geqslant 0}$ be an N–function. We say that φ is

- of class Δ_2 provided there exists $K>0$ such that $\varphi(2t)\leqslant K\varphi(t)$ for all $t\geqslant 0$,
- of class ∇_2 provided the convex conjugate function φ^* (in the sense of Chapter 2.4.1) is of class Δ_2.

The Δ_2–condition rules out too fast a growth, whereas the ∇_2–condition rules out too slow a growth. When φ is both Δ_2 and ∇_2, then we say that φ is of class $\Delta_2 \cap \nabla_2$. We turn to some examples:

- The N–function $\varphi(t) := |t| \log(1 + |t|)$ for $t \in \mathbb{R}$ belongs to the Δ_2-class but not to ∇_2. In fact, we have $\varphi^*(t) \approx t(\exp(t) - 1)$ for $t \geqslant 0$. Hence φ^* has exponential growth and thus cannot satisfy the doubling criterion of the Δ_2–condition.

- The power functions $\varphi(t) := |t|^p$ for $t \in \mathbb{R}$ are N–functions if and only if $1 < p < \infty$. This is an easy consequence of Example 2.7(a).

Let φ be an N–function. The space $\mathrm{L}^{\varphi}_{\dagger}(\Omega; \mathbb{R}^m)$ consists of all measurable functions $u \colon \Omega \to \mathbb{R}^m$ for which there holds $\int_\Omega \varphi(|u|)\,\mathrm{d}x < \infty$, but in general this is no vector space. Instead, one defines the *Orlicz–Birnbaum* space $\mathrm{L}^{\varphi}(\Omega; \mathbb{R}^m)$ as the span of $\mathrm{L}^{\varphi}_{\dagger}(\Omega; \mathbb{R}^n)$ in the measurable functions from Ω to $\mathbb{R}^m$ and endows it with the Luxembourg–norm

$$\|f\|_\varphi := \inf \Big\{0 < k < \infty\colon \ \int_\Omega \varphi\Big(\frac{|u|}{k}\Big)\,\mathrm{d}x \leqslant 1\Big\}.$$

The Orlicz–Sobolev spaces $\mathrm{W}^{1,\varphi}(\Omega; \mathbb{R}^m)$ are then defined in the obvious manner, namely, as the collection of all $u \in \mathrm{L}^{\varphi}(\Omega; \mathbb{R}^m)$ for which there holds $\mathrm{D}u \in \mathrm{L}^{\varphi}(\Omega; \mathbb{R}^{m\times n})$.

In the following we adopt the notion of Chapter 2.5.4 and record the following result:

Theorem 2.17 ([198, 241, 65])**.** *Let T be a translation invariant singular integral operator of the form and let φ be an N–function. Then T extends to a bounded operator $\mathrm{L}^{\varphi}(\mathbb{R}^n) \to \mathrm{L}^{\varphi}(\mathbb{R}^n)$ if and only if $\varphi \in (\Delta_2 \cap \nabla_2)$.*

2.6. Maximal Operators

In this section we give an overview of various maximal operators which we shall encounter in different settings as the thesis evolves. Usually, maximal operators $\widetilde{\mathcal{M}}$ have the form $\widetilde{\mathcal{M}}f(x) := \sup_{r>0} \widetilde{M}_{\mathrm{B}(x,r)}(f)$, where $\widetilde{M}_{\mathrm{B}(x,r)}(f)$ is a quantity that crucially depends on the ball $\mathrm{B}(x, r)$. If f is a mapping that is not defined on the entire $\mathbb{R}^n$ but on a open proper subset $\Omega \subset \mathbb{R}^n$, we arrange that for any $x \in \Omega$ the expressions $(\widetilde{\mathcal{M}}f)(x)$ have to be interpreted as

$$(\widetilde{\mathcal{M}}f)(x) = \sup_{0<r<\mathrm{dist}(x,\partial\Omega)} \widetilde{M}_{\mathrm{B}(x,r)}(f). \tag{2.12}$$

Moreover, it is sometimes convenient to work with cubes rather than balls, in which case the corresponding *centered* and *non–centered* maximal operators read

$$\widetilde{\mathcal{M}}^c f(x) := \sup_{\substack{Q\ni x\,\text{cube} \\ Q\text{ is centered at }x}} \widetilde{M}_Q(f), \quad \widetilde{\mathcal{M}}^{nc} f(x) := \sup_{Q\ni x\,\text{cube}} \widetilde{M}_Q(f),$$

where $\widetilde{M}_Q$ are the obvious analogues of $\widetilde{M}_{\mathrm{B}(x,r)}$ with a similar general generalisation for proper subsets of $\mathbb{R}^n$ in the spirit of (2.12). For most of these maximal operators it is not crucial whether balls or cubes are involved in their definition, and we shall simultaneously use them unless otherwise stated.

To begin with, we define the *(Hardy–Littlewood) maximal operator* for a given $\mathbb{R}^m$–valued Radon measure μ on $\mathbb{R}^n$ by[2]

$$(\mathcal{M}\mu)(x) := \sup_{r>0} \fint_{\mathrm{B}(x,r)} \mathrm{d}|\mu| := \sup_{r>0} \frac{|\mu|(\mathrm{B}(x,r))}{\mathscr{L}^n(\mathrm{B}(x,r))}, \qquad x \in \mathbb{R}^n. \tag{2.13}$$

[2]Here the dash is with respect to the n–Lebesgue measure indeed.

Specifically, if μ is absolutely continuous with respect to $\mathscr{L}^n$ and hence there exists $v \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^m)$ such that $\mu = \mu_v := v\mathscr{L}^n$, we put $\mathcal{M}v := \mathcal{M}\mu_v$. As a consequence of LEBESGUE's differentiation theorem for Radon measures, it is easily seen that there holds $|v| \leqslant \mathcal{M}v$ $\mathscr{L}^n$–a.e., a fact which turns out particularly useful in the study of the mapping properties of singular integral operators and on which we shall elaborate in more detail below.

Given $1 < p \leqslant \infty$, the Hardy–Littlewood maximal operator maps $\mathcal{M}\colon \mathrm{L}^p(\mathbb{R}^n;\mathbb{R}^m) \to \mathrm{L}^p(\mathbb{R}^n)$ boundedly, meaning that there exists a constant $c = c_{n,p} > 0$ such that for all $v \in \mathrm{L}^p(\mathbb{R}^n;\mathbb{R}^m)$ there holds $\|\mathcal{M}v\|_{\mathrm{L}^p(\mathbb{R}^n)} \leqslant c\|v\|_{\mathrm{L}^p(\mathbb{R}^n;\mathbb{R}^m)}$. However, if $p = 1$, then $\mathcal{M}$ merely maps $\mathrm{L}^1(\mathbb{R}^n;\mathbb{R}^m)$ to the weak–L^1 space $\mathrm{L}^1_{\mathrm{w}}(\mathbb{R}^n)$ to be defined next:

Definition 2.18 (Weak–L^q Space)**.** *Let $1 \leqslant q < \infty$. The* weak–L^q space $\mathrm{L}^q_{\mathrm{w}}(\mathbb{R}^n;\mathbb{R}^m)$ *consists of all measurable mappings $v\colon \mathbb{R}^n \to \mathbb{R}^m$ such that*

$$\|v\|^q_{\mathrm{L}^q_{\mathrm{w}}(\mathbb{R}^n;\mathbb{R}^m)} := \sup_{t>0} t^q \mathscr{L}^n\big(\{x \in \mathbb{R}^n\colon\ |v(x)| > t\}\big) < \infty.$$

The weak–L^q spaces are special instances of Lorentz spaces (in fact, one has $\mathrm{L}^q_{\mathrm{w}} \simeq \mathrm{L}^{q,\infty}$), and hence the quantities $\|\cdot\|_{\mathrm{L}^q_{\mathrm{w}}}$ are quasinorms rather than norms. In this sense, the aforementioned boundedness of the Hardy–Littlewood maximal operators asserts that there exists a constant $c = c_n > 0$ such that for all $v \in \mathrm{L}^1(\mathbb{R}^n;\mathbb{R}^m)$ there holds $\|\mathcal{M}v\|_{\mathrm{L}^1_{\mathrm{w}}(\mathbb{R}^n)} \leqslant c\|v\|_{\mathrm{L}^1(\mathbb{R}^n;\mathbb{R}^m)}$, a fact which remains valid for measures, too:

Lemma 2.19 ([140, Lem. 4])**.** *There exists a constant $C = C(n) > 0$ with the following property: Let μ be a signed Borel measure on $\mathbb{R}^n$ with finite total variation. Then for all $t > 0$ there holds*

$$\mathscr{L}^n\big(\{x \in \mathbb{R}^n\colon\ (\mathcal{M}\mu)(x) > t\}\big) \leqslant C\frac{|\mu|(\mathbb{R}^n)}{t}.$$

A natural question is how much more integrability we have to require for a mapping such that its maximal operator is integrable indeed and not only belongs to $\mathrm{L}^1_{\mathrm{w}}$. It is an observation due to STEIN that this is achieved by passing to the space $L\log L$. More precisely, letting Ω be an open subset of $\mathbb{R}^n$, we put

$$L\log L(\Omega;\mathbb{R}^m) := \big\{v\colon \Omega \to \mathbb{R}^m \text{ meas.}\colon\quad |f|_{L\log L(\Omega;\mathbb{R}^m)} := \int_\Omega |f|\log(e+|f|)\,\mathrm{d}x < \infty\big\}.$$

If $\mathscr{L}^n(\Omega) < \infty$, then there holds

$$\bigcup_{1<p<\infty} \mathrm{L}^p(\Omega;\mathbb{R}^m) \subsetneq L\log L(\Omega;\mathbb{R}^m) \subsetneq \mathrm{L}^1(\Omega;\mathbb{R}^m).$$

STEIN's Theorem now asserts that for a mapping $f \in \mathrm{L}^1(\mathbb{R}^n;\mathbb{R}^m)$ we have $(\mathcal{M}f)|_{\mathrm{B}} \in \mathrm{L}^1(\mathrm{B})$ if and only if $f|_{\mathrm{B}} \in L\log L(\mathrm{B};\mathbb{R}^m)$. The space $L\log L$ is strictly connected to the Hardy space $\mathcal{H}^1$, the latter being defined by use of grand maximal operators, and we refer the reader to [138] for more detail.

Similarly, given a locally integrable mapping $v\colon \mathbb{R}^n \to \mathbb{R}^m$, we define the *sharp maximal operator* $\mathcal{M}^\sharp$ through its action on v by

$$(\mathcal{M}^\sharp v)(x) := \sup_{r>0} \fint_{\mathrm{B}(x,r)} |v - (v)_{x,r}|\,\mathrm{d}y, \qquad x \in \mathbb{R}^n,$$

where $(v)_{x,r} := \fint_{\mathrm{B}(x,r)} v\,\mathrm{d}y$ is the mean value of v over $\mathrm{B}(x,r)$. It is easy to see that $\mathcal{M}^\sharp$ can be pointwisely controlled by $\mathcal{M}$ and thus, defining

$$\mathrm{BMO}(\mathbb{R}^n;\mathbb{R}^m) := \big\{v \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^m)\colon\ \mathcal{M}^\sharp v \in \mathrm{L}^\infty(\mathbb{R}^n)\big\},$$

we immediately obtain $\mathrm{L}^\infty \subset \mathrm{BMO}$. This inclusion, however, is strict as can be seen by variants of the log–function on suitable subintervals of $\mathbb{R}$. Let us further note that due to the seminal work of Fefferman & Stein [104], BMO can be characterised as the dual of the Hardy space $\mathcal{H}^1$ and thus, on an informal level, differs as much from L^∞ as $\mathcal{H}^1$ from L^1. This is also reflected in the John–Nirenberg Inequality which reads as follows.

Lemma 2.20 (John–Nirenberg Lemma, [128, Thm. 2.11])**.** *Let $Q_0 \subset \mathbb{R}^n$ be a cube and $f\colon Q_0 \to \mathbb{R}^N$ be a measurable map. Then we have $f \in \mathrm{BMO}(Q_0;\mathbb{R}^N)$ if and only if there exist $c_1, c_2 > 0$ such that for all cubes $Q \subset Q_0$ there holds*

$$\fint_Q \big(\exp(c_1|f-(f)_Q|)-1\big)\,\mathrm{d}x \leqslant c_2.$$

Now, aiming at an interpolation result, recall that $\mathrm{L}^q(\mathbb{R}^n) \subset (\mathrm{L}^p \cap \mathrm{L}^\infty)(\mathbb{R}^n)$ provided $1 \leqslant p \leqslant q < \infty$. We wish to replace L^∞ on the right side of the previous embedding, and for this purpose firstly record the following lemma due to Iwaniec:

Lemma 2.21 ([147, Lem. 4])**.** *Suppose that $\mathcal{M}^\sharp_{Q_0}(f) \in \mathrm{L}^r(Q_0)$, $1 \leqslant r < \infty$. Then $\mathcal{M}_{Q_0}f \in \mathrm{L}^r(Q_0)$ with*

$$\left(\fint_{Q_0} |\mathcal{M}_{Q_0}f|^r\,\mathrm{d}x\right)^{\frac{1}{r}} \leqslant 10^{5nr}\left(\fint_{Q_0} |\mathcal{M}^\sharp_{Q_0}f|^r\,\mathrm{d}x\right)^{\frac{1}{r}} + 10^{n+1}\fint_{Q_0}|f|\,\mathrm{d}x. \tag{2.14}$$

We now have the following result:

Lemma 2.22. *Let $1 < p < \infty$. Then, for every $p \leqslant q < \infty$, $(\mathrm{L}^p \cap \mathrm{BMO})(\mathbb{R}^n) \subset \mathrm{L}^q(\mathbb{R}^n)$.*

Though this result should be fairly standard, we could find a precise reference and provide its proof in the appendix, cp. Chapter 10.6.

Both the Hardy–Littlewood and sharp maximal operators in turn are special instances of the so–called (restricted) *fractional* and *fractional sharp maximal operators*. Hence let $s > 0$, $R > 0$ and define for $x \in \mathbb{R}^n$

$$\begin{aligned}
(\mathcal{M}_{s,R}f)(x) &:= \sup_{0<r<R} r^s \fint_{\mathrm{B}(x,r)} |f(y)|\,\mathrm{d}y,\\
(\mathcal{M}^\sharp_{s,R}f)(x) &:= \sup_{0<r<R}\inf_{p\in\mathscr{P}_{\lfloor s\rfloor}(\mathrm{B}(x,r))} r^{-s}\fint_{\mathrm{B}(x,r)} |f(y)-p|\,\mathrm{d}y.
\end{aligned}$$

The *fractional flat maximal operator* is similarly given by

$$(\mathcal{M}^\flat_{s,R}f)(x) := \sup_{0<r<R}\inf_{p\in\mathscr{P}_k(\mathrm{B}(x,r))} r^{-s}\fint_{\mathrm{B}(x,r)} |f(y)-p|\,\mathrm{d}y,$$

where now $k \in \mathbb{N}_0$ is the largest integer smaller than s. Note that this only gives a difference between $\mathcal{M}^\flat_{s,R}$ and $\mathcal{M}^\sharp_{s,R}$ if and only if $s \in \mathbb{N}$.

To abbreviate notation, we also put $\mathcal{M}_s := \mathcal{M}_{s,\infty}$, $\mathcal{M}^\sharp_s := \mathcal{M}^\sharp_{s,\infty}$ and $\mathcal{M}^\flat_s := \mathcal{M}^\flat_{s,\infty}$. The importance of these operators is that they allow for convenient integral criteria for functions to be smooth, for instance, Hölder continuous (compare Table 2.1). This, in turn, is an substantial ingredient in the partial regularity proofs in the main part of this thesis. In describing these connections, we consequently introduced function spaces associated with these maximal operators and study their embedding properties. For simplicity, we start with functions defined on the entire $\mathbb{R}^n$.

Definition 2.23 (Calderón Spaces on $\mathbb{R}^n$, [77, Chpt. 6])**.** *Let $1 \leqslant p \leqslant \infty$ and $s > 0$. The* Calderón space *$\mathscr{C}^{s,p}(\mathbb{R}^n)$ and the* flat Calderón space *$\mathcal{C}^{s,p}(\mathbb{R}^n)$ are defined by*

$$\begin{aligned}
\mathscr{C}^{s,p}(\mathbb{R}^n;\mathbb{R}^m) &:= \{v \in \mathrm{L}^p(\mathbb{R}^n;\mathbb{R}^m)\colon\ \mathcal{M}^\sharp_s v \in \mathrm{L}^p(\mathbb{R}^n)\},\\
\mathcal{C}^{s,p}(\mathbb{R}^n;\mathbb{R}^m) &:= \{v \in \mathrm{L}^p(\mathbb{R}^n;\mathbb{R}^m)\colon\ \mathcal{M}^\flat_s v \in \mathrm{L}^p(\mathbb{R}^n)\}
\end{aligned}$$

If $p = \infty$, we call $\mathrm{L}^s_C := \mathscr{C}^{s,\infty}$ s–th order Campanato space*.*

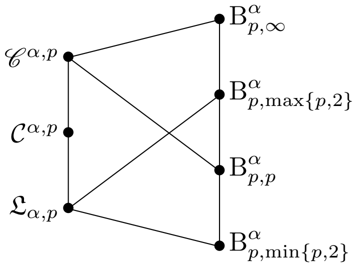

Figure 2.2: Dependencies of the Besov, Calderón and Bessel potential spaces (cp. Remark 2.31 for the latter) for $\alpha > 0$ and $1 < p < \infty$, as depicted in [77, Fig. I]. If two spaces are at the endpoints of a straight line segment, then the lower space is embedded in the higher space.

We finally link the Besov spaces to the Calderón spaces:

Lemma 2.24 ([77], Theorems 7.1 and 7.5)**.** *Let $\alpha > 0$. Then the following holds:*

(a) *If $1 \leqslant p < \infty$, then*

$$\mathrm{B}^{\alpha,p}_{p}(\mathbb{R}^n) \hookrightarrow \mathscr{C}^{\alpha,p}(\mathbb{R}^n) \hookrightarrow \mathrm{B}^{\alpha,p}_{\infty}(\mathbb{R}^n).$$

(b) *If $1 < p < \infty$, then*

$$\mathfrak{L}^{\alpha,p}(\mathbb{R}^n) \hookrightarrow \mathcal{C}^{\alpha,p}(\mathbb{R}^n) \hookrightarrow \mathscr{C}^{\alpha,p}(\mathbb{R}^n).$$

As an important consequence of the preceding lemma, we record the embeddings

$$\mathrm{W}^{\alpha,p}(\mathbb{R}^n) \hookrightarrow \mathcal{C}^{\alpha,p}(\mathbb{R}^n) \hookrightarrow \mathrm{B}^{\alpha,p}_{\infty}(\mathbb{R}^n), \qquad 1 < p < \infty,\ \alpha > 0, \tag{2.15}$$

which immediately follows from Lemma 2.24 in conjunction with the characterisation $\mathrm{W}^{\alpha,p} \simeq \mathcal{L}^{\alpha,p}$ of the fractional Sobolev spaces $\mathrm{W}^{\alpha,p}$ and the Bessel potential spaces $\mathcal{L}^{\alpha,p}$.

With a plenty of maximal operators at our disposal, we resume the theme of Section 2.5.5 and conclude this section with an alternative characterisation of the fractional Sobolev spaces and, more generally, Besov spaces involving the decay of mean deviations due to Dorronsoro and Triebel (cp. [86, 235]).

Theorem 2.25 (Dorronsoro, [86, Thm. 1])**.** *Let $0 < \alpha < 1$ and $1 \leqslant p, q < \infty$. A measurable function $f\colon \mathbb{R}^n \to \mathbb{R}$ belongs to $\dot{\mathrm{B}}^{\alpha}_{p,q}(\mathbb{R}^n)$ if and only if*

$$\|f\|^{*}_{\alpha,p,q} := \left(\int_0^\infty \left(\frac{\|\Omega_f(\cdot,t)\|_{\mathrm{L}^p(\mathbb{R}^n)}}{t^\alpha}\right)^q \frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}} < \infty.$$

Here, we have set

$$\Omega_f(x,t) := \sup\left\{ \fint_Q |f-(f)_Q|\,\mathrm{d}y\colon\ Q \ni x,\ \mathscr{L}^n(Q) = t^n \right\}.$$

In this situation, $\|\cdot\|^{*}_{\alpha,p,q}$ and $\|\cdot\|_{\dot{\mathrm{B}}^{\alpha}_{p,q}}$ are equivalent norms on $\dot{\mathrm{B}}^{\alpha}_{p,q}(\mathbb{R}^n)$.

2.6.1. Morrey and Campanato Spaces We also need versions of these spaces on domains. These are usually referred to as *Campanato spaces*. Let $\Omega \subset \mathbb{R}^n$ be open and let $1 \leqslant p < \infty$, $\lambda \geqslant 0$. We say that $f\colon \Omega \to \mathbb{R}^m$ belongs to the

	$\lambda=0$	$\lambda\in(0,n)$	$\lambda=n$	$\lambda\in(n,n+p]$	$\lambda>n+p$
$\mathrm{L}_M^{p,\lambda}$	$\cong\mathrm{L}^p$	$\cong\mathrm{L}_C^{p,\lambda}$	$\cong\mathrm{L}^\infty$	$=\{0\}$	$=\{0\}$
$\mathrm{L}_C^{p,\lambda}$	$\cong\mathrm{L}^p$	$\cong\mathrm{L}_M^{p,\lambda}$	$\subsetneq\mathrm{L}^\infty$	$\mathrm{C}^{0,\alpha}$	$=\{\mathrm{const.}\}$

Table 2.1: Relations between Morrey and Campanato Spaces assuming that $\Omega\subset\mathbb{R}^n$ is an Ahlfors regular domain, cp. [33, Rem. 1.25]. Here 'const.' describes the constant functions and we have set $\alpha:=\frac{\lambda-n}{p}$.

- *Morrey space* $\mathrm{L}_M^{p,\lambda}(\Omega;\mathbb{R}^m)$ provided $f\in\mathrm{L}^p(\Omega;\mathbb{R}^m)$ and
$$\|f\|^p_{\mathrm{L}_M^{p,\lambda}(\Omega;\mathbb{R}^m)}:=\sup_{\substack{x_0\in\overline{\Omega}\\ r>0}}\min\{r,1\}^{-\lambda}\int_{\Omega\cap\mathrm{B}(x_0,r)}|f|^p\,\mathrm{d}x<\infty.$$

- *Campanato space* $\mathrm{L}_C^{p,\lambda}(\Omega;\mathbb{R}^m)$ provided $f\in\mathrm{L}^p(\Omega;\mathbb{R}^m)$ and
$$[f]^p_{\mathrm{L}_C^{p,\lambda}(\Omega;\mathbb{R}^m)}:=\sup_{\substack{x_0\in\overline{\Omega}\\ r>0}}\min\{r,1\}^{-\lambda}\int_{\Omega\cap\mathrm{B}(x_0,r)}|f-(f)_{\mathrm{B}(x_0,r)}|^p\,\mathrm{d}x<\infty.$$

 We also write $\mathcal{L}^{p,\lambda}(\Omega;\mathbb{R}^m):=\mathrm{L}_C^{p,\lambda}(\Omega;\mathbb{R}^m)$ for the Campanato spaces.

Note that $\|\cdot\|_{\mathrm{L}_M^{p,\lambda}(\Omega;\mathbb{R}^m)}$ defines a norm on $\mathrm{L}_M^{p,\lambda}(\Omega;\mathbb{R}^m)$, whereas $[\cdot]_{\mathrm{L}_C^{p,\lambda}(\Omega;\mathbb{R}^m)}$ merely defines a seminorm. The corresponding full norm on $\mathrm{L}_C^{p,\lambda}(\Omega;\mathbb{R}^m)$ consequently is given by
$$\|f\|_{\mathrm{L}_C^{p,\lambda}(\Omega;\mathbb{R}^m)}:=\|f\|_{\mathrm{L}^p(\Omega;\mathbb{R}^m)}+[f]_{\mathrm{L}_C^{p,\lambda}(\Omega;\mathbb{R}^m)}.$$
To establish embeddings for the spaces just introduced, it is convenient with a geometric condition on Ω which is usually referred to as *Ahlfors* regularity, cp. [33, Chpt. 1.1]:

Remark 2.26 (Ahlfors regular domains)**.** *We say that $\Omega\subset\mathbb{R}^n$ is* Ahlfors regular *if there exists a constant $A>0$ such that for each $x_0\in\overline{\Omega}$ and all $0<r\leqslant\mathrm{diam}(\Omega)$ there holds $|\Omega\cap\mathrm{B}(x_0,r)|\geqslant Ar^n$. This geometrical condition for instance rules out exterior cusps and is satisfied for, e.g., Lipschitz domains. Assuming the Ahlfors regularity condition for Ω, the relations for Morrey– and Campanato spaces as exposed in Table 2.1 hold true.*

We then have the following embedding result.

Theorem 2.27 (Campanato, cp. [33, Thm. 1.27], & Meyers [173])**.** *Let $\Omega\subset\mathbb{R}^n$ be an open set which is Ahlfors regular and let $1\leqslant p<\infty$ and $0<\alpha\leqslant1$. Then there holds*
$$\mathcal{L}^{p,n+\alpha p}(\Omega;\mathbb{R}^m):=\mathrm{L}_C^{p,n+\alpha p}(\Omega;\mathbb{R}^m)\simeq\mathrm{C}^{0,\alpha}(\overline{\Omega};\mathbb{R}^m),\tag{2.16}$$
and also the seminorms $[\cdot]_{\mathrm{L}_C^{p,n+\alpha p}}$ and $[\cdot]_{\mathrm{C}^{0,\alpha}}$ are equivalent.

In Chapter 8.5 we will moreover need a higher order Campanato–type characterisation of Hölder spaces $\mathrm{C}^{m,\alpha}(\overline{\Omega})$, $m\geqslant1$. For this we denote for $k\in\mathbb{N}_0$, $\lambda>0$, a map $v\colon\Omega\to\mathbb{R}$ and $x\in\Omega$
$$\mathcal{M}^{\#}_{\lambda,k,\Omega}v(x):=\sup_{r>0}\left(\max\left\{1,\frac{1}{r}\right\}^\lambda\inf_{\pi\in\mathcal{P}_k(\Omega\cap\mathrm{B}(x,r))}\int_{\Omega\cap\mathrm{B}(x,r)}|f-\pi|^p\,\mathrm{d}y\right)^{\frac{1}{p}}.\tag{2.17}$$
Similarly to the Campanato spaces from above, we define $\mathcal{L}_k^{p,\lambda}(\Omega):=\mathrm{L}_{C,k}^{p,\lambda}(\Omega)$ as the collection of all $f\in\mathrm{L}^p(\Omega)$ such that $\mathcal{M}^{\#}_{\lambda,k,\Omega}v\in\mathrm{L}^\infty(\Omega)$. We then have the following generalisation of Theorem 2.27:

Theorem 2.28 ([197, Thm. 4.4])**.** *Let Ω be a bounded and Ahlfors regular subset of $\mathbb{R}^n$ and let $1\leqslant p<\infty$, $\lambda>0$ and $k\in\mathbb{N}_0$. For any $m\in\{0,...,k\}$ with $n+mp<\lambda<n+(m+1)p$ we then have $\mathcal{L}_k^{p,\lambda}(\Omega)\simeq\mathrm{C}^{m,\alpha}(\overline{\Omega})$ with $\alpha=(\lambda-n)/p-m$.*

2.6.2. Fractional Integrals Given $0 < s < n$, the *fractional integration operator of order s* is given by its action on $f \in \mathscr{S}(\mathbb{R}^n)$ defined as

$$(\mathcal{I}_s f)(x) := \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-s}}\,\mathrm{d}y, \qquad x \in \mathbb{R}^n,$$

and motivates the following definition of the *Riesz potential spaces*:

Definition 2.29. *Let $0 < s < n$ and $1 < p < \infty$. The* Riesz potential space $L_{s,p}(\mathbb{R}^n)$ *precisely consists of all those measurable functions $f\colon \mathbb{R}^n \to \mathbb{R}$ for which there exists $g \in \mathrm{L}^p(\mathbb{R}^n)$ such that $f = \mathcal{I}_s g$. The corresponding norm is given by $\|f\|_{\mathrm{L}_{s,p}} := \inf_g \|g\|_{\mathrm{L}^p(\mathbb{R}^n)}$, the infimum running over all $g \in \mathrm{L}^p(\mathbb{R}^n)$ with $f = \mathcal{I}_s g$.*

We pass on to a Sobolev–type theorem for the Riesz potential spaces.

Theorem 2.30 (Fractional Integration Theorem, [7, Thm. 3.1.4])**.** *Let $1 < p < n$ and $0 < s \leqslant 1$ be such that $0 < sp < n$. Then $\mathcal{I}_s$ extends to a bounded linear operator*

$$\mathcal{I}_s\colon\ \mathrm{L}^p(\mathbb{R}^n) \to \mathrm{L}^{\frac{np}{n-sp}}(\mathbb{R}^n).$$

If $p = 1$, then $\mathcal{I}_s$ extends to a bounded linear operator $\mathcal{I}_s\colon\ \mathrm{L}^1(\mathbb{R}^n) \to \mathrm{L}_{\mathrm{w}}^{\frac{n}{n-s}}(\mathbb{R}^n)$.

The Riesz potentials arise as particular Fourier multiplication operators. In fact, if $0 < s < n$, then the locally integrable function $\mathbb{R}^n \setminus \{0\} \ni \xi \mapsto |\xi|^{-s} \in \mathbb{R}$ is the Fourier multiplier of the fractional Laplacian $(-\Delta)^{\frac{s}{2}}\colon \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}'(\mathbb{R}^n)$. That is, for $f \in \mathscr{S}(\mathbb{R}^n)$ we have $(-\Delta)^{\frac{s}{2}} f = \mathscr{F}^{-1}_{\xi\mapsto x}(|\xi|^{-s}\widehat{f}(\xi))$, and the (distributional) Fourier transform of $\xi \mapsto |\xi|^{-s}$ is given by $x \mapsto \gamma_{n,s}/|x|^{n-s}$ (here, $\gamma_{n,s} > 0$ is a constant only depending on n and s). As a general principle, we have that if $m \in \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ is homogeneous of degree $(-s)$, then the Fourier multiplication operator $T_s \mathscr{S}(\mathbb{R}^n) \ni f \mapsto \mathscr{F}^{-1}(m\widehat{f})$ is a *fractional integration operator* of order $(n-s)$ (in the sense that $m(\xi) \lesssim |\xi|^{-s}$ for all $\xi \neq 0$).

Remark 2.31 (Bessel potential spaces)**.** *Systematically replacing the Fourier multiplier $\xi \mapsto |\xi|^{-s}$ by $\xi \mapsto (1+|\xi|^2)^{-\frac{s}{2}}$ and modifying Definition 2.29 in the obvious way yields the so–called* Bessel potential spaces $\mathfrak{L}_{s,p}$*; see [7, Chpt. 1] for more information.*

2.7. The Ekeland Variational Principle

The final part of this preliminary chapter is devoted to the Ekeland variational principle. The latter will frequently prove useful in obtaining *good* sequences which approximate given generalised minima and as such, we not only state but also provide its proof (compare with [128, Thm. 5.6]):

Theorem 2.32 (Ekeland Variational Principle)**.** *Let (X, d) be a complete metric space and let $\mathcal{F}\colon X \to \overline{\mathbb{R}}$ be a lower semicontinuous functional (with respect to the metric topology induced by d) which moreover is bounded from below and attains a finite value at some $x_0 \in X$. Assume that for some $u \in X$ and some $\varepsilon > 0$ there holds*

$$\mathcal{F}[u] \leqslant \inf_X \mathcal{F} + \varepsilon.$$

Then there exists $v \in X$ such that

(a) $d(u, v) \leqslant 1$,

(b) $\mathcal{F}[v] \leqslant \mathcal{F}[u]$,

(c) $\mathcal{F}[v] \leqslant \mathcal{F}[u] + \varepsilon d(v, w)$ *for all $w \in X$.*

Proof. We construct the element $v \in X$ as the limit of a particular Cauchy sequence (u_k) which is defined inductively as follows. We put $u_1 := u$. Assuming that for $k \in \mathbb{N}$ the members $u_1, ..., u_k$ have been constructed, we note that the set $S_k := \{w \in X\colon \mathcal{F}[w] \leqslant \mathcal{F}[u_k] - \varepsilon d(u_k, w)\}$ contains u_k and thus is non–empty. In conclusion, we find $u_{k+1} \in S_k$ such that

$$\mathcal{F}[u_{k+1}] \leqslant \frac{1}{2}\Big(\mathcal{F}[u_k] + \inf_{S_k} \mathcal{F}\Big). \tag{2.18}$$

We claim that (u_k) is d–Cauchy. Indeed, fix $k \in \mathbb{N}$ and note that, since $u_{k+1} \in S_k$, there holds

$$\varepsilon d(u_{k+1}, u_k) \leqslant \mathcal{F}[u_k] - \mathcal{F}[u_{k+1}]. \tag{2.19}$$

Consequently, a telescope sum argument yields for all $m \in \mathbb{N}$

$$\varepsilon d(u_{k+m}, u_k) \leqslant \varepsilon \sum_{i=1}^{m} d(u_{k+i}, u_{k+i-1}) \leqslant \mathcal{F}[u_k] - \mathcal{F}[u_{k+m}]. \tag{2.20}$$

However, since $(\mathcal{F}[u_k])$ is decreasing because of (2.19) and is bounded below by assumption, it converges to some $a \in \mathbb{R}$. Hence, (2.20) yields that (u_k) is d–Cauchy and therefore, by completeness of (X, d), converges to some $v \in X$. Lower semicontinuity of $\mathcal{F}$ consequently gives $\mathcal{F}[v] \leqslant \liminf_{k\to\infty} \mathcal{F}[u_k] = a$ and hence, sending $m \to \infty$ in (2.20), $\varepsilon d(v, u_k) \leqslant \mathcal{F}[u_k] - \mathcal{F}[v]$. Setting $k = 1$ then yields

$$0 \leqslant \varepsilon d(u, v) \leqslant \mathcal{F}[u] - \mathcal{F}[v] \leqslant \mathcal{F}[u] - \inf_X \mathcal{F} \leqslant \varepsilon$$

so that (a) and (b) follow. In view of (c), we argue by contradiction and hereafter suppose that there exists $w \in X$ such that $\mathcal{F}[w] < \mathcal{F}[v] - \varepsilon d(w, v)$. Then, since $\mathcal{F}[v] \leqslant \mathcal{F}[u_k] - \varepsilon d(u_k, v)$, we obtain by the triangle inequality

$$\mathcal{F}[w] < \mathcal{F}[v] - \varepsilon d(w, v) \leqslant \mathcal{F}[u_k] - \varepsilon(d(u_k, v) + d(v, w)) \leqslant \mathcal{F}[u_k] - \varepsilon d(u_k, w),$$

and so we conclude that for each $k \in \mathbb{N}$ we have $w \in S_k$. Therefore, we have $\inf_{S_k} \mathcal{F} \leqslant \mathcal{F}[w]$ for all $k \in \mathbb{N}$ and thus, by (2.18),

$$2\mathcal{F}[u_{k+1}] - \mathcal{F}[u_k] \leqslant \inf_{S_k} \mathcal{F} \leqslant \mathcal{F}[w] < \mathcal{F}[v] - d(v, w).$$

Passing to the limit $k \to \infty$ in the preceding inequality yields $\mathcal{F}[v] \leqslant \mathcal{F}[w] < \mathcal{F}[v] - \varepsilon d(v, w)$ which is impossible. The proof is complete. □

As a simple consequence of the preceding theorem, we obtain the following corollary.

Corollary 2.33 ([128, Rem. 5.5])**.** *Let (X, d) be a complete metric space and let $\mathcal{F}\colon X \to \overline{\mathbb{R}}$ be a lower semicontinuous functional (with respect to the metric topology induced by d) which moreover is bounded from below and attains a finite value at some $x_0 \in X$. Assume that for some $u \in X$ and some $\varepsilon > 0$ there holds*

$$\mathcal{F}[u] \leqslant \inf_X \mathcal{F} + \varepsilon.$$

Then there exists $v \in X$ such that

(a) $d(u, v) \leqslant \sqrt{\varepsilon}$,

(b) $\mathcal{F}[v] \leqslant \mathcal{F}[u]$,

(c) $\mathcal{F}[v] \leqslant \mathcal{F}[u] + \sqrt{\varepsilon} d(v, w)$ *for all* $w \in X$.

Proof. Define a new metric by $\widetilde{d} := d/\sqrt{\varepsilon}$. Then the claim follows immediately from the Ekeland variational principle. $\square$

We now record a version of this lemma suitable for our purposes in Chapter 5.4.4.

Corollary 2.34 ([36, Prop. 5.2]). *Let Ω be an open and bounded subset of $\mathbb{R}^n$. Suppose that $f\colon \mathbb{R}^m \to \mathbb{R}$ is a convex function which satisfies*

$$|f(z)| \leqslant L(1+|z|) \qquad \text{for all } z \in \mathbb{R}^m$$

with a constant $L \geqslant 0$. Moreover, consider $\varepsilon, \chi \in (0,\infty)$ and a closed subspace S of $\mathrm{L}^2(\Omega;\mathbb{R}^m)$ such that $\|\Phi\|_{\mathrm{L}^1(\Omega;R^m)} \leqslant M\|\Phi\|_{\mathrm{L}^2(\Omega;\mathbb{R}^m)}$ holds for all $\Phi \in S$ and a constant $M > 0$. Then for every $w \in \mathrm{L}^1(\Omega;\mathbb{R}^m)$ with

$$\int_\Omega f(w)\,\mathrm{d}x \leqslant \inf_{\Theta \in w+S} \int_S f(\Theta)\,\mathrm{d}x + \varepsilon$$

there exist approximate solutions $v \in w+S$ and $\tau \in \mathrm{L}^\infty(\Omega;\mathbb{R}^m)$ such that we have

$$\int_\Omega f(v)\,\mathrm{d}x \leqslant \inf_{\Theta \in w+S} \int_\Omega f(\Theta)\,\mathrm{d}x + 2\varepsilon, \tag{2.21}$$

$$\|v-w\|_{\mathrm{L}^2(\Omega;\mathbb{R}^m)} \leqslant \chi, \tag{2.22}$$

$$\tau(x) \in \partial_z f(v(x)) \quad \text{for } \mathscr{L}^n - \text{a.e. } x \in \Omega, \tag{2.23}$$

$$\int_\Omega \langle \tau, \Phi\rangle\,\mathrm{d}x \leqslant \frac{2\varepsilon}{\chi}\|\Phi\|_{\mathrm{L}^2(\Omega;\mathbb{R}^m)} \qquad \text{for all } \Phi \in S. \tag{2.24}$$

CHAPTER 3

Korn's Inequality and Ornstein's Non–Inequality

The purpose of this chapter is to give a self–contained proof of Ornstein's Non–Inequality by generalising the approach of KIRCHHEIM & KRISTENSEN [151]. We thereby set the mood for the subsequent chapters, where the non–availability of Korn–type inequalities plays an important role.

Structure of the chapter. After introducing the setup and discussing Ornstein's Non–Inequality in Section 3.1, we firstly give positive results on Korn–type Inequalities and their derivation by singular integrals in Sections 3.2–3.5 in the spirit of SMITH's representation theory [218]. In a second step we approach Ornstein's Non–Inequality through a semiconvexity approach in Sections 3.6–3.8.

3.1. Ornstein's Non–Inequality: Setup and Terminology

In [194], ORNSTEIN proved the following result which rules out the possibility of non–trivial L^1–estimates:

Theorem 3.1 (Ornstein's Non–Inequality, [194, Thm. 1]). *Let $B, D_1, ..., D_L$ be a set of linearly independent linear homogeneous differential operators in n variables of degree m. For any $K > 0$, there exists an $f \in \mathrm{C}^\infty$ vanishing outside the unit cube such that $\int |Bf|\,\mathrm{d}x > K$ and $\int |D_i f|\,\mathrm{d}x < 1$ for all $i \in \{1, ..., L\}$.*

This negative result is linked to singular integral estimates in the borderline case $p = 1$, a fact which we shall describe in Section 3.2. As explained in Chapter 1, the implications of the previous theorem encompass the need of a new function space framework when studying, e.g., perfect plasticity (and hereafter functionals that depend on the (trace–free) symmetric gradient, see [23, 115]) but also rule out the option of a L^1–potential theory: Indeed, if $f \in \mathrm{L}^1(Q; \mathbb{R}^{n\times n})$ and u solves the distributional elliptic equation $-\Delta u = \mathrm{div}(f)$ on the open unit cube $Q \subset \mathbb{R}^n$, then we *cannot* necessarily conclude $\mathrm{D}u \in \mathrm{L}^1(Q; \mathbb{R}^{n\times n})$.

Ornstein–type non–inequalities have been studied by a plenty of authors, among others [63, 151, 165, 166]. The present chapter, in turn, is devoted to an alternative proof of the following variant of an Ornstein Non–Inequality, in this form originally given by KIRCHHEIM & KRISTENSEN [151].

Theorem 3.2 ([151, Thm. 1.3]). *Let V, W, X be three finite–dimensional vector spaces, $k \in \mathbb{N}$ and consider two k–th order linear and homogeneous differential operators $\mathbb{A}_1[D]$ and $\mathbb{A}_2[D]$ of the form*

$$\mathbb{A}_1[D] = \sum_{\substack{|\alpha|=k \\ \alpha\in\mathbb{N}_0^n}} \mathbb{A}^1_\alpha(x)\partial^\alpha \quad \text{and} \quad \mathbb{A}_2[D] = \sum_{\substack{|\alpha|=k \\ \alpha\in\mathbb{N}_0^n}} \mathbb{A}^2_\alpha(x)\partial^\alpha,$$

with coefficient maps $\mathbb{A}^1_\alpha \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^n; \mathscr{L}(V;W))$ and $\mathbb{A}^2_\alpha \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^n; \mathscr{L}(V;X))$ for all $|\alpha| = k$, respectively. Then the following are equivalent:

(a) *There exists a constant $c > 0$ such that*

$$\|\mathbb{A}_2[D]\varphi\|_{\mathrm{L}^1(\mathbb{R}^n;X)} \leqslant c\|\mathbb{A}_1[D]\varphi\|_{\mathrm{L}^1(\mathbb{R}^n;W)}$$

holds for all $\varphi \in \mathrm{C}_c^\infty(\mathbb{R}^n;V)$.

(b) *There exists $T \in \mathrm{L}^\infty(\mathbb{R}^n;\mathscr{L}(W;X))$ with $\|T\|_{\mathrm{L}^\infty(\mathbb{R}^n;\mathscr{L}(W;X))} \leqslant c$ such that $\mathbb{A}_\alpha^2(x) = T(x)\mathbb{A}_\alpha^1(x)$ for $\mathscr{L}^n$–a.e. $x \in \mathbb{R}^n$ and each $\alpha \in \mathbb{N}_0^n$ with $|\alpha| = k$.*

Thus, the only L^1–estimates that are available indeed are then *trivial* in the sense that they directly follow from an application of Hölder's inequality. One aim of the present chapter is to give a self–contained proof of the preceding theorem by a positivity result for differential operators which do not admit a potential (Theorem 3.3). This is achieved by modifying and combining the arguments given in [151] with a generalisation of relaxation results from [109] to the higher order case.

For this, we introduce some terminology. Given a k–th order linear, homogeneous, constant–coefficient differential operator between the two finite–dimensional vector spaces W and X of the form

$$\mathcal{A}[D] := \sum_{|\alpha|=k} \mathcal{A}_\alpha \partial^\alpha \tag{3.1}$$

with fixed linear mappings $\mathcal{A}_\alpha \colon W \to X$ for $\alpha \in \mathbb{N}_0^n$ with $|\alpha| = k$, we say that $\mathcal{A}[D]$ has *constant rank* provided its *symbol map*

$$\mathcal{A}\colon \xi \mapsto \mathcal{A}[\xi] := \sum_{|\alpha|=k} \xi_\alpha \mathcal{A}_\alpha := \sum_{|\alpha|=k} \xi_1^{\alpha_1}\cdots\xi_n^{\alpha_n}\mathcal{A}_\alpha \qquad \text{for } \xi = (\xi_1^{\alpha_1},...,\xi_n^{\alpha_n}), \tag{3.2}$$

for $\alpha \in \mathbb{N}_0^n$ with $\alpha = (\alpha_1,...,\alpha_n)$ and $|\alpha| = |\alpha_1| + ... + |\alpha_n| = k$, has constant rank for each $\xi \neq 0$. We associate with $\mathcal{A}$ its *characteristic cone* (or *wave cone*)

$$\Lambda_{\mathcal{A}} := \bigcup_{\xi\neq 0} \ker(\mathcal{A}[\xi]), \tag{3.3}$$

which is a subset of W. Particular examples of constant rank operators are given by the elliptic operators, for which the characteristic cone is trivial, i.e., $\Lambda_{\mathcal{A}} = \{0\}$. Given a cone $\mathcal{C} \subset W$, we say that $\mathcal{C}$ is *spanning* provided its linear hull is the entire W. Using this terminology, the main result of this chapter is the following theorem, an immediate consequence of which is Theorem 3.2 (see Section 3.8 for the proof):

Theorem 3.3. *Let $f\colon \mathbb{R}^n \times W \to \mathbb{R}$ be a Carathéodory integrand satisfying*

$$f(x,t\xi) = |t|f(x,\xi) \quad \text{and } |f(x,\xi)| \leqslant a(x)|\xi| \tag{3.4}$$

for $\mathscr{L}^n$–a.e. $x \in \mathbb{R}^n$, all $t \in \mathbb{R}$ and $\xi \in W$, where $a \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^n)$ is a given function. Moreover, let $\mathcal{A}[D]$ be a constant rank differential operator of the above form, i.e., $\mathcal{A}[D]$ has symbol (3.2). Then the characteristic cone $\Lambda_{\mathcal{A}}$ is spanning if and only if the following are equivalent:

(a) *For all $w \in \mathrm{C}^\infty(\mathbb{T}^n;W)$ with $\mathcal{A}w = 0$ and $(w)_Q = 0$ there holds*

$$\int_{\mathbb{T}^n} f(x,w(x))\,\mathrm{d}x \geqslant 0. \tag{3.5}$$

(b) *There holds $f(x,\xi) \geqslant 0$ for $\mathscr{L}^n$–a.e. $x \in \mathbb{R}^n$ and for all $\xi \in W$.*

Here, $\mathbb{T}^n := \mathbb{R}^n/\mathbb{Z}^n$ denotes the n–dimensional torus.

Remark 3.4. *To fix ideas, let us argue that the property of $\Lambda_{\mathcal{A}}$ to be spanning is necessary indeed. If $\mathcal{A}$ is the gradient operator, for instance, then it is elliptic and $\Lambda_{\mathcal{A}} = \{0\}$. In particular,* $\ker(\mathcal{A}) \cap \{(w)_Q = 0\}$ *only contains the zero function, and* (3.5) *becomes vacuous due to* (3.4).

The decisive feature for the theory outlined in this chapter to depart from the setting considered in [151] is that we do *not* require $\mathcal{A}$ to have a potential: Let us say that $\mathcal{A}[D]$ of the form (3.1) *has a potential* if and only if there exists a real, finite dimensional vector space V and a homogeneous, linear, constant coefficient differential operator $\mathbb{A} = \mathbb{A}[D]$ on $\mathbb{R}^n$ from V to W such that the symbol complex

$$V \xrightarrow{\mathbb{A}[\xi]} W \xrightarrow{\mathcal{A}[\xi]} X \qquad \text{is exact for all } \xi \neq 0, \tag{3.6}$$

i.e., for any $\xi \neq 0$, $\ker(\mathcal{A}[\xi]) = \mathbb{A}[\xi](V)$. In this situation, we also call $\mathcal{A}$ an *annihilator* for $\mathbb{A}$. Examples of differential operators which admit potentials are given by curl and $\operatorname{curl} \circ \operatorname{curl}$, having as potentials the usual gradient or symmetric gradient, respectively:

Example 3.5 (The Curl–Gradient Complex, [109, Rem. 3.3(iii)])**.** *If in* (3.6), $\mathcal{A} = \operatorname{curl}$, *then a potential of $\mathcal{A}$ is given by $\mathbb{A} = \nabla$, so we let $k = 1$. More precisely, following [109, Rem. 3.3(iii)], for a given map $V\colon \mathbb{R}^n \to \mathbb{R}^{N\times m} \cong \mathbb{R}^d$ with $d := Nm$, $m = n + \rho$ for $\rho \geqslant 0$, write $V = (F|G)$ with $F \in \mathbb{R}^{N\times n}$, $G \in \mathbb{R}^{N\times \rho}$. The equation $\operatorname{curl} V = 0$ then translates to*

$$\partial_{x_i} F_{jh} - \partial_{x_h} F_{ji} = 0 \qquad \text{for all } 1 \leqslant j \leqslant N,\ 1 \leqslant i,h \leqslant n.$$

Within the framework of (3.1)*, this can be reformulated as $\mathcal{A}V = 0$ with $X = \mathbb{R}^l$ with $l = n^2 N$ and*

$$\mathcal{A}^{(r)}_{(j,h,i),(q,p)} := \delta_{ri}\delta_{qj}\delta_{ph} - \delta_{rh}\delta_{qj}\delta_{pi} \qquad \text{for } 1 \leqslant j,q \leqslant N,\ 1 \leqslant i,h,p,r \leqslant n,$$

and $\mathcal{A}^{(r)}_{(j,h,i),(q,p)} = 0$ if $p = n+1, ..., m$, where we used the obvious indexing. In particular, we notice that curl *is a constant rank operator. Moreover, we have in this case*

$$\begin{aligned} \ker(\mathcal{A}[\xi]) &= \{B \in \mathbb{R}^{N\times m}\colon\ \mathcal{A}[\xi]B = 0\} \\ &= \{B = (F|\zeta) \in \mathbb{R}^{N\times m}\colon\ \xi_i B_{jh} - \xi_h B_{ji} = 0 \text{ for all } 1 \leqslant j \leqslant N,\ 1 \leqslant i,h \leqslant n\} \\ &= \{B = (F|\zeta) \in \mathbb{R}^{N\times m}\colon\ B = a \otimes w \ \text{ for some } \ a \in \mathbb{R}^N\}, \end{aligned}$$

and as a consequence, the characteristic cone of curl *equals the usual rank–one cone. On the other hand, if $\mathbb{A}[D] = D$, then $D[\xi]w = w \otimes \xi$ (cp. Lemma 3.14).*

Example 3.6 (The Curl–Curl–Symmetric Gradient Complex)**.** *For a given map $e = (e_{ij})_{i,j=1,\dots,n} \in \mathrm{C}^2(\mathbb{R}^n;\mathbb{R}^{n\times n}_{\mathrm{sym}})$, define the differential operator* $\operatorname{curl}\circ\operatorname{curl}$ *by*

$$(\operatorname{curl}\circ\operatorname{curl} e)_{i,j,h,l} := (\partial_{x_l x_j} e_{ih} + \partial_{x_h x_i} e_{jl} - \partial_{x_l x_i} e_{jh} - \partial_{x_h x_j} e_{il})_{i,j,h,l}.$$

The Saint–Venant compatibility conditions *(cp., e.g., [17]) then assert that a map $e \in \mathrm{C}^2(\mathbb{R}^n;\mathbb{R}^{n\times n}_{\mathrm{sym}})$ is the symmetric gradient of $u \in \mathrm{C}^3(\mathbb{R}^n;\mathbb{R}^{n\times n}_{\mathrm{sym}})$, i.e., $e = \boldsymbol{\varepsilon}(u)$, if and only if $\operatorname{curl}\circ\operatorname{curl} e = 0$. Similarly as in Example 3.5, one can verify that $\Lambda_{\operatorname{curl}\circ\operatorname{curl}} = \mathbb{R}^n \odot \mathbb{R}^n := \{a \odot b\colon\ a,b \in \mathbb{R}^n\}$ is the symmetric rank–one cone.*

Before we pass to the proof of Theorem 3.2 in Section 3.8 after collecting various auxiliary results on cones, relaxation and $\mathcal{A}$–quasiconvexity in Sections 3.6 and 3.7, we firstly contextualise the conclusion of Ornstein's Non–Inequality and give a derivation of Korn–type inequalities for first order differential operators both on $\mathbb{R}^n$ and on bounded domains (cp. Sections 3.2 and 3.3). This heavily relies on singular integral estimates and indicates the failure of the corresponding estimates for $p = 1$. Working on domains,

SIs of Convolution Type bounded on	
L^1:	no
L^φ:	if and only if $\varphi \in \Delta_2 \cap \nabla_2$
L^∞:	no

Figure 3.1: Boundedness of Singular Integrals on L^p– and L^φ–spaces (cp. Chapter 2.5.6 for more information on Orlicz spaces).

we shall identify the finite dimensionality of the nullspace of a differential operator as the crucial feature to yield suitable Korn–type inequalities. Although approached in a slightly different manner, we wish to stress that large parts of this theory had already been developed in a fairly too unnoted paper by SMITH [218]. Our approach through extension operators, however, immediately can be used to treat Korn–type inequalities on very rough domains, too. Based on the finite dimensional nullspace property, we will further establish suitable Poincaré–type inequalities in Section 3.5 that shall prove useful later on.

3.2. The Singular Integral Viewpoint

Both Korn–type Inequalities and Ornstein–type Non–Inequalities are deeply linked to the (un–)boundedness of singular integral operators on Lebesgue spaces, a viewpoint we elaborate on now. Let $\mathbb{A}[D]$ be a first order[1], constant coefficient, linear and homogeneous differential operator of the form

$$\mathbb{A}[D] = \sum_{\substack{|\alpha|=1 \\ \alpha \in \mathbb{N}_0^n}} \mathbb{A}_\alpha \partial^\alpha, \tag{3.7}$$

where for each $\alpha \in \mathbb{N}_0$ with $|\alpha| = 1$, $\mathbb{A}_\alpha$ is a fixed linear map between the finite dimensional vector spaces V and W. We say that $\mathbb{A}[D]$ is ($\mathbb{R}$–)*elliptic* if and only if for each $\xi = (\xi_1, ..., \xi_n) \in \mathbb{R}^n \setminus \{0\}$ the *symbol mapping*

$$\mathbb{A}[\xi] = \sum_{j=1}^n \xi_j \mathbb{A}_j \colon V \to W$$

is injective. In particular, if $\mathbb{A}$ is elliptic, then $\mathbb{A}^*[\xi] \circ \mathbb{A}[\xi]$ (with an obvious definition of the adjoint symbol $\mathbb{A}^*[\xi]$) is an invertible mapping from V to V. In turn, a routine calculation shows that there exists a constant $c_n > 0$ such that for each $u \in \mathrm{C}_c^\infty(\mathbb{R}^n; V)$ there holds (with a suitable complex constant $c_n \in \mathbb{C} \setminus \{0\}$)

$$u(x) = c_n \mathscr{F}^{-1}_{\xi \mapsto x}((\mathbb{A}^*[\xi] \circ \mathbb{A}[\xi])^{-1} \mathbb{A}^*[\xi] \widehat{\mathbb{A}[D]u}) =: \Phi(\mathbb{A}[D]u)(x) \qquad \text{for all } x \in \mathbb{R}^n. \tag{3.8}$$

The operator Φ is also called the *Green's operator for* $\mathbb{A}[D]$ and $\Phi(\mathbb{A}[D]u)$ the *Green's function* for u. To emphasize its dependence on $\mathbb{A}$, we shall also write $\mathbf{G}_\mathbb{A} := \Phi$. We note that $(\mathbb{A}^*[\xi] \circ \mathbb{A}[\xi])^{-1} \mathbb{A}^*[\xi]$ is homogeneous of degree (-1) in ξ. As a consequence of the discussion in Chapter 2.6.2, the operator Φ therefore is a fractional integration operator of order $(n-1)$.

Remark 3.7. *Differentiating (3.8) with respect to x and keeping in mind the fact that Φ is a fractional integration operator of order $(n-1)$, it is straightforward to deduce that the mapping $\Psi\colon \mathrm{C}_c^\infty(\mathbb{R}^n; V) \to \mathrm{C}_c^\infty(\mathbb{R}^n; V \times \mathbb{R}^n)$ given by $\Psi\colon \mathbb{A}[D](u) \mapsto Du$ is given by a singular integral operator of convolution type (cp. Chapter 2.3.1). This operator arises*

[1] It is clearly possible to develop these ideas also in the context of k–th order differential operators, $k \in \mathbb{N}_{>1}$, too, but we refrain from doing so for simplicity.

as the Cauchy principal value integral of $\mathbb{A}[D]u$ with a kernel that grows like $|x|^{-n}$. In conclusion, the standard mapping properties of such operators imply that Ψ continuously extends to a mapping $\mathrm{L}^p \to \mathrm{L}^p$ if and only if $1 < p < \infty$.

Since taking the Fourier transform of a convolution product amounts to multiplying the respective Fourier transforms, implementing the Fourier inversion in (3.8) yiels the Smith representation formulas [218], here presented for the particular case of the symmetric gradient (cp. [140]): Given $u = (u^1, ..., u^n) \in \mathrm{C}_c^\infty(\mathbb{R}^n; \mathbb{R}^n)$, we may write

$$u^k = \frac{2}{n\omega_n} \sum_{i,j=1}^n \frac{\partial^2 u^k}{\partial x_j \partial x_j} * K_{ij}, \qquad \text{where } K_{ij}(x) = \frac{x_i x_j}{|x|^n} \text{ for } x \in \mathbb{R}^n \setminus \{0\}. \tag{3.9}$$

Setting $\boldsymbol{\varepsilon}(u) := (\boldsymbol{\varepsilon}(u)_{jk})_{jk}$, we observe that

$$\frac{\partial^2 u^k}{\partial x_i \partial x_j} = \frac{\partial \boldsymbol{\varepsilon}(u)_{jk}}{\partial x_i} - \frac{\partial \boldsymbol{\varepsilon}(u)_{ij}}{\partial x_k} + \frac{\partial \boldsymbol{\varepsilon}(u)_{ki}}{\partial x_j}$$

and hence, inserting this relation into (3.9), we obtain after integrating by parts

$$u^k = \frac{2}{n\omega_n} \sum_{i,j=1}^n \Big(\boldsymbol{\varepsilon}(u)_{jk} * \frac{\partial K_{ij}}{\partial x_i} - \boldsymbol{\varepsilon}(u)_{ij} * \frac{\partial K_{ij}}{\partial x_k} + \boldsymbol{\varepsilon}(u)_{ki} * \frac{\partial K_{ij}}{\partial x_j} \Big) \tag{3.10}$$

for all $k = 1, ..., n$.

3.3. Characterisation of Korn–type Operators

Throughout this section, let $\mathbb{A}[D]$ be a first order, linear, homogeneous and constant–coefficient differential operator from V to W, so that $\mathbb{A}[D]$ can be written in the form

$$\mathbb{A}[D] = \sum_{k=1}^n \mathbb{A}_k \partial_k, \tag{3.11}$$

with fixed linear mappings $\mathbb{A}_k \colon V \to W$, where V and W are two finite–dimensional $\mathbb{R}$–vector spaces. Given $1 < p < \infty$, we say that $\mathbb{A}[D]$ supports a p–Korn inequality of the *first kind* provided there exists $C > 0$ such that for all $v \in \mathrm{C}_c^\infty(\mathrm{B}; V)$ there holds

$$\|\nabla v\|_{\mathrm{L}^p(\mathbb{R}^n; V \times \mathbb{R}^n)} \leqslant C \|\mathbb{A}[D]v\|_{\mathrm{L}^p(\mathbb{R}^n; W)}. \tag{K1}$$

Let us further say that $\mathbb{A}[D]$ supports a Korn inequality of the *second kind* provided for each $1 < p < \infty$ there exists a constant $C = C(p) > 0$ such that for all $v \in \mathrm{C}^\infty(\mathrm{B}; V)$ such that $v \in \mathrm{L}^p(\mathrm{B}; V)$ and $\mathbb{A}[D]u \in \mathrm{L}^p(\mathrm{B}; W)$ there holds

$$\|v\|_{\mathrm{W}^{1,p}(\mathrm{B}; V)} \leqslant C \|v\|_{\mathrm{W}^{\mathbb{A},p}(\mathrm{B})} := C(\|v\|_{\mathrm{L}^p(\mathrm{B}; V)} + \|\mathbb{A}[D]v\|_{\mathrm{L}^p(\mathrm{B}; W)}). \tag{K2}$$

Given an open set $\Omega \subset \mathbb{R}^n$, let us define the $(\mathbb{A}, p)$–Sobolev space $\mathrm{W}^{\mathbb{A},p}(\mathrm{B})$ to consist of all those maps $v \in \mathrm{L}^p(\Omega; V)$ such that $\mathbb{A}[D]v \in \mathrm{L}^p(\Omega; W)$ (in the sense that the distributional differential expression $\mathbb{A}[D]v$ can be represented by a L^p–function). On this space, $\|\cdot\|_{\mathrm{W}^{\mathbb{A},p}(\Omega)}$ given by (K2) (with the obvious modification) is a norm which makes $\mathrm{W}^{\mathbb{A},p}(\Omega)$ a Banach space. Let us moreover define $\mathrm{W}_0^{\mathbb{A},p}(\Omega)$ to be the closure of $\mathrm{C}_c^\infty(\Omega; V)$ with respect to this norm. Consequently, (K1) or (K2) can be recast as follows: Characterise $\mathbb{A}[D]$ of the form (3.11) such that

$$\mathrm{W}_0^{1,p}(\mathrm{B}; V) \simeq \mathrm{W}_0^{\mathbb{A},p}(\mathrm{B}) \quad \text{or} \quad \mathrm{W}^{1,p}(\mathrm{B}; V) \simeq \mathrm{W}^{\mathbb{A},p}(\mathrm{B}) \tag{3.12}$$

hold, respectively (with '$X \simeq Y$' denoting that X and Y are mutually continuously embedded into each other).

In order to derive such estimates, one might be inclined to primarily work with ($\mathbb{R}$–)*elliptic operators*. We recall that these are the operators for which their *symbol map*

$$\mathbb{A}[\xi]\colon V \to W, \quad \mathbb{A}[\xi] := \sum_{k=1}^{n} \xi_k \mathbb{A}_k \tag{3.13}$$

is injective for every $\xi \in \mathbb{R}^n \setminus \{0\}$; note that, by homogeneity, it would suffice to require the latter property for all $\xi \in \mathbb{S}^{n-1} := \{x \in \mathbb{R}^n\colon\ |x| = 1\}$. As we shall see, mere ellipticity is too weak to yield (K2). Given an operator $\mathbb{A}[D]$ of the form (3.11), we shall say that it has the *finite dimensional nullspace property* (shorthand FDN) provided for some $1 \leqslant p < \infty$

$$\dim(\ker(\mathbb{A}[D]; \mathrm{B}) \cap \mathrm{L}^p(\mathrm{B}; V)) < \infty, \tag{FDN$_p$}$$

i.e., the space of L^p–functions which are in the (distributional) nullspace of $\mathbb{A}[D]$, is finite. We could drop the dependence of p in (FDN$_p$), see Lemma 3.15. Let us note in advance that the operators which have the FDN are a (strict) subset of the elliptic operators:

Lemma 3.8. *The set of FDN operators is a subset of the elliptic operators.*

The claimed ellipticity of FDN operators will be established in Section 3.4.1 but follows independently from Thm. 3.10 below. The strictness of the inclusion follows, e.g., from Example 3.13 in $n = 2$ dimensions below. In studying how much FDN differs from ellipticity, we firstly note the following.

Proposition 3.9 (Korn operators of the first kind)**.** *Let $\mathbb{A}[D]$ be a differential operator of the form* (3.11)*. Then the following are equivalent:*

(a) *For all $1 < p < \infty$, $\mathrm{W}_0^{1,p}(\mathrm{B}; V) \simeq \mathrm{W}_0^{\mathbb{A},p}(\mathrm{B})$ (i.e.,* (K1) *holds).*

(b) *For all $1 < p < \infty$, $\mathrm{W}^{1,p}(\mathbb{R}^n; V) \simeq \mathrm{W}^{\mathbb{A},p}(\mathbb{R}^n)$.*

(c) *For all $1 < p < \infty$, there exists $c > 0$ such that $\|Du\|_{\mathrm{L}^p(\mathbb{R}^n; V \times \mathbb{R}^n)} \leqslant c\|\mathbb{A}[D]u\|_{\mathrm{L}^p(\mathbb{R}^n; W)}$ holds for all $u \in \mathrm{C}_c^\infty(\mathbb{R}^n; V)$.*

(d) *$\mathbb{A}[D]$ is elliptic.*

The preceding proposition will be established in Section 3.4.2, however, follows mostly standard arguments using singular integral operators. We now turn to (K2) and therefore allow the admissible maps to have non–zero boundary values. Our second main result extends by (b) the results of Smith [218] and reads as follows.

Theorem 3.10 (Korn operators of the second kind)**.** *Let $\mathbb{A}[D]$ be a differential operator of the form* (3.11)*. Then the following are equivalent:*

(a) *$\mathbb{A}[D]$ has the FDN.*

(b) *$\mathbb{A}[D]$ is elliptic and for each $1 < p < \infty$ there exists a bounded extension operator from $\mathrm{W}^{\mathbb{A},p}(\mathrm{B})$ to $\mathrm{W}^{\mathbb{A},p}(\mathbb{R}^n; V)$ mapping continuous maps to continuous maps.*

(c) *$\mathrm{W}^{1,p}(\mathrm{B}; V) \simeq \mathrm{W}^{\mathbb{A},p}(\mathrm{B}; V)$, i.e.,* (K2) *holds for all $1 < p < \infty$.*

This theorem will be established in Section 3.4.3, and crucially relies on the equivalence of the FDN with the *type–(C) condition* introduced by Kalamajska [149]. Let us remark that in passing from zero to non–zero boundary values (i.e., from $\mathrm{W}_0^{\mathbb{A},p}(\mathrm{B})$ to $\mathrm{W}^{\mathbb{A},p}(\mathrm{B})$), the equivalence with the corresponding standard Sobolev spaces is achieved by passing from ellipticity to the stronger FDN condition. In this sense, the zero boundary value condition imposed in Proposition 3.9 compensates the possibly infinite dimensional nullspace of $\mathbb{A}[D]$.

3.3.1. Examples In what follows, we discuss three elliptic operators we shall refer to frequently. For the first two examples, assume $n = 2$ and identify $\mathbb{R}^{2\times 2} \cong \mathbb{R}^4$ by means of $(a_{ij})_{1\leqslant i,j\leqslant 2} \leftrightarrow (a_{11}, a_{12}, a_{21}, a_{22})$. Moreover, note that since all of the following examples deal with operators that fit into the above framework (cp. (3.7)) with $V = \mathbb{R}^2$ and $W = \mathbb{R}^{2\times 2} \cong \mathbb{R}^4$, the symbols $\mathbb{A}[\xi]$ take values in $V \times W \cong \mathbb{R}^{4\times 2}$.

Example 3.11 (The Gradient Operator)**.** We consider the gradient operator ∇ acting on vector functions $v\colon \mathbb{R}^2 \to \mathbb{R}^2$, clearly, $\ker(\nabla)$ are the constant maps. Its symbol then reads as

$$\nabla[\xi] = \begin{pmatrix} \xi_1 & \xi_2 & 0 & 0 \\ 0 & 0 & \xi_1 & \xi_2 \end{pmatrix}^{\mathsf{T}}, \qquad \xi = (\xi_1, \xi_2) \in \mathbb{R}^2.$$

Example 3.12 (The Symmetric Gradient Operator)**.** The *symmetric gradient operator* $\mathbb{A}[D] := \boldsymbol{\varepsilon} := \frac{1}{2}(\nabla + \nabla^{\mathsf{T}})$ is elliptic as demonstrated in [239, Sec. 6.3.2], and its kernel can be computed explicitely as the so–called *rigid deformations* $\mathcal{R} := \{x \mapsto Ax + b\colon\ A \in \mathbb{R}^{n\times n}_{\mathrm{scew}},\ b \in \mathbb{R}^n\}$. In particular, $\boldsymbol{\varepsilon}$ has the FDN. The symbol of $\boldsymbol{\varepsilon}$ reads as

$$\boldsymbol{\varepsilon}[\xi] = \frac{1}{2}\begin{pmatrix} \xi_1 & \xi_1 & \xi_1 & 0 \\ 0 & \xi_2 & \xi_2 & \xi_2 \end{pmatrix}^{\mathsf{T}}, \qquad \xi = (\xi_1, \xi_2) \in \mathbb{R}^2.$$

Example 3.13 (The Trace–Free Symmetric Gradient Operator)**.** The *trace–free symmetric gradient operator* is defined by $\boldsymbol{\varepsilon}^D := \boldsymbol{\varepsilon} - \frac{1}{n}\operatorname{div}\cdot \mathbb{1}_{n\times n}$, where $\mathbb{1}_{n\times n}$ is the $(n\times n)$–unit matrix. We note that in $n = 2$ this operator and its symbol can be written explicity as

$$\boldsymbol{\varepsilon}^D v := \frac{1}{2}\begin{pmatrix} \partial_1 v_1 - \partial_2 v_2 & \partial_1 v_2 + \partial_2 v_1 \\ \partial_1 v_2 + \partial_2 v_1 & \partial_2 v_2 - \partial_1 v_1 \end{pmatrix},$$

$$\boldsymbol{\varepsilon}^D[\xi] = \frac{1}{2}\begin{pmatrix} \xi_1 & \xi_2 & \xi_2 & -\xi_1 \\ -\xi_2 & \xi_1 & \xi_1 & \xi_2 \end{pmatrix}^{\mathsf{T}}, \qquad \xi = (\xi_1, \xi_2) \in \mathbb{R}^2.$$

Therefore, identifying $\mathbb{C} \cong \mathbb{R}^2$ and the open unit disc $\mathbb{D}$ in $\mathbb{C}$ with B, we see that $\boldsymbol{\varepsilon}^D$ corresponds to the Wirtinger differential operator $\partial_{\overline{z}}$, so $\ker(\boldsymbol{\varepsilon}^D; \mathrm{B})$ coincides with the holomorphic functions $\mathcal{O}(\mathbb{D})$ and hence $\boldsymbol{\varepsilon}^D$ does not have the FDN if $n = 2$. If $n \geqslant 3$, then the kernel of $\boldsymbol{\varepsilon}^D$ relative to B is given by

$$\ker(\boldsymbol{\varepsilon}^D; \mathrm{B}) := \left\{ x \mapsto 2(a\cdot x)x - |x|^2 a + Qx + \rho x + b\colon \begin{array}{c} a, b \in \mathbb{R}^n,\ \rho \in \mathbb{R}, \\ Q \in \mathbb{R}^{n\times n}_{\mathrm{scew}} \end{array} \right\},$$

and its elements are usually referred to as *conformal Killing vectors*. We refer the reader to [69] for more information.

3.3.2. Bilinear Pairings Every differential operator $\mathbb{A}[D]$ of the form (3.11) induces a bilinear pairing or tensor product in a canonical way (such as the gradient induces the usual tensor product $a \otimes b := ab^{\mathsf{T}}$ or the symmetric gradient induces the symmetric tensor product $a \odot b := \frac{1}{2}(a \otimes b + b \otimes a)$). In this section we shall collect the properties of this mapping and examine the geometrical consequences which it implies. Let $\varphi \in \mathrm{C}^1(\mathbb{R}^n)$ and $v \in \mathrm{C}^1(\mathbb{R}^n; V)$. Applying the operator $\mathbb{A}[D]$ to φv, we obtain

$$\mathbb{A}[D](\varphi v) = \varphi \mathbb{A}[D]v + v \otimes_{\mathbb{A}} \nabla\varphi, \tag{3.14}$$

and a straightforward computation shows that $v \otimes_{\mathbb{A}} u = \sum_{k=1}^n u_k \mathbb{A}_k v$, where $u = (u_1, ..., u_n)$ and $v \in V$. For completeness, we now collect some elementary properties of the mapping $\otimes_{\mathbb{A}}\colon V \times \mathbb{R}^n \to W$.

Lemma 3.14. *Let $\mathbb{A}[D]$ be of the form (3.11). Then the following holds:*

(a) *For all* $u \in \mathscr{S}(\mathbb{R}^n;V)$ *there holds*

$$\mathscr{F}(\mathbb{A}[D]u)(\xi) = -\,\mathrm{i}\,\mathbb{A}[\xi]\hat{u}(\xi) = -\,\mathrm{i}\,\widehat{u}(\xi)\otimes_{\mathbb{A}}\xi \qquad \text{for all } \xi\in\mathbb{R}^n.$$

(b) *If* $\mathbb{L}[D] = \sum_{|\alpha|=l}\mathbb{L}_\alpha\partial^\alpha$ *is a constant coefficient, linear and homogeneous differential operator of order* $l\in\mathbb{N}$ *from* W *to some finite dimensional real vector space* X *such that the symbol complex*

$$V \xrightarrow{\mathbb{A}[\xi]} W \xrightarrow{\mathbb{L}[\xi]} X$$

is exact at W *for all* $\xi\in\mathbb{S}^{n-1}$*, then* $\Lambda_{\mathbb{L}} = \bigcup_{\xi\in\mathbb{S}^{n-1}}\ker(\mathbb{L}[\xi]) = V\otimes_{\mathbb{A}}\mathbb{R}^n$*. In this situation, we call* $\mathbb{A}[D]$ *a* potential *for* $\mathbb{L}[D]$.

Proof. Ad (a). We have, for $u\in\mathscr{S}(\mathbb{R}^n;V)$ and all $\xi\in\mathbb{R}^n$,

$$\mathscr{F}(\mathbb{A}[D]u)(\xi) = \sum_{k=1}^n \mathbb{A}_k\mathscr{F}(\partial_k u)(\xi) = -\,\mathrm{i}\sum_{k=1}^n \xi_k\mathbb{A}_k\widehat{u}(\xi) = -\,\mathrm{i}\,\widehat{u}(\xi)\otimes_{\mathbb{A}}\xi.$$

Ad (b). Note that for all $v\in V$ and all $w\in\mathbb{R}^n$, $\mathbb{L}[w](v\otimes_{\mathbb{A}} w) = \mathbb{L}[w](\mathbb{A}[w]v) = 0$ by (a) and the given exactness of the symbol complex. On the other hand, if $\eta\in\Lambda_{\mathbb{L}}$, $\mathbb{L}[\xi]\eta = 0$ for some $\xi\in\mathbb{S}^{n-1}$ and hence, by exactness (i.e., $\ker(\mathbb{L}[\xi]) = \mathbb{A}[\xi](V)$ for all $\xi\in\mathbb{R}^n\setminus\{0\}$), we find $v\in V$ and $\xi\in\mathbb{R}^n$ with $\eta = \mathbb{A}[\xi]v = v\otimes_{\mathbb{A}}\xi\in V\otimes_{\mathbb{A}}\mathbb{R}^n$. The proof is complete. $\square$

3.3.3. Independence of (FDN$_p$) **of** p In an intermediate step, we further supply the following lemma.

Lemma 3.15. *Suppose that an elliptic differential operator* $\mathbb{A}[D]$ *of the form* (3.11) *satisfies* (FDN$_p$) *for some* $1\leqslant p<\infty$*. Then it satisfies* (FDN$_p$) *for all* $1\leqslant p\leqslant\infty$*. In particular, the following are equivalent for any* $1<p<\infty$*:* $\mathbb{A}[D]$ *has finite dimensional nullspace with respect to*

(a) $\mathrm{L}^1_{\mathrm{loc}}(\mathrm{B};V)$

(b) $\mathrm{L}^1(\mathrm{B};V)$,

(c) $\mathrm{L}^p_{\mathrm{loc}}(\mathrm{B};V)$,

(d) $\mathrm{L}^p(\mathrm{B};V)$.

Proof. Clearly, (a) implies (b) as $\mathrm{L}^1(\mathrm{B};V)\subset\mathrm{L}^1_{\mathrm{loc}}(\mathrm{B};V)$. On the other hand, if $B\subset B'$ are two concentric balls, then the map given by $I\colon\ker_{\mathrm{L}^1_{\mathrm{loc}}}(\mathbb{A}[D];B)\to\ker_{\mathrm{L}^1_{\mathrm{loc}}}(\mathbb{A}[D];B')$ given by $If(x) := f(r(B)x/r(B'))$ for $x\in B'$, where $r(B), r(B')$ are the radii of B or B', respectively, is an isomorphism between $\ker_{\mathrm{L}^1_{\mathrm{loc}}}(\mathbb{A}[D];B)$ and $\ker_{\mathrm{L}^1_{\mathrm{loc}}}(\mathbb{A}[D];B')$. So we have $\dim(\ker_{\mathrm{L}^1_{\mathrm{loc}}}(\mathbb{A}[D];B)) = \dim(\ker_{\mathrm{L}^1_{\mathrm{loc}}}(\mathbb{A}[D];B'))$. On the other hand, since for every $f\in\ker_{\mathrm{L}^1_{\mathrm{loc}}}(\mathbb{A}[D];B'$ the map $f|_B$ belongs to $\ker_{\mathrm{L}^1_{\mathrm{loc}}}(\mathbb{A}[D];B)$, we see that every $f\in\ker_{\mathrm{L}^1_{\mathrm{loc}}}(\mathbb{A}[D];B)$ arises as the restriction of some $\widetilde{f}\in\ker_{\mathrm{L}^1_{\mathrm{loc}}}(\mathbb{A}[D];B')$ to B. Now let $f\in\ker_{\mathrm{L}^1_{\mathrm{loc}}}(\mathbb{A}[D];B)$. Then in particular, f is the restriction of a locally integrable mapping $\widetilde{f}\in\ker_{\mathrm{L}^1_{\mathrm{loc}}}(\mathbb{A}[D];B')$ to $\overline{B}$ and hence belongs to $\mathrm{L}^1(B;V)$. Hence $\ker_{\mathrm{L}^1_{\mathrm{loc}}}(\mathbb{A}[D];B)\subset\ker_{\mathrm{L}^1}(\mathbb{A}[D];B)$ and so (b) implies (a). Similarly, we see that (a) and (c) are equivalent. Clearly, (a) implies (c) because of $\mathrm{L}^p_{\mathrm{loc}}(\mathrm{B};V)\subset\mathrm{L}^1_{\mathrm{loc}}(\mathrm{B};V)$. Lastly, elliptic regularity shows that if $f\in\mathrm{L}^1_{\mathrm{loc}}$ belongs to $\ker_{\mathrm{L}^1_{\mathrm{loc}}}(\mathbb{A}[D];B)$, then it belongs to $\ker_{\mathrm{L}^p_{\mathrm{loc}}}(\mathbb{A}[D];B)$, too, and so the equivalence of (c) implies (a). The proof is complete. $\square$

Throughout the remainder of the document, we shall also denote $\ker_X(\mathbb{A}[D];\Omega)$ the elements u of the function space $X(\Omega)$ (where Ω is an open subset of $\mathbb{R}^n$) for which $\mathbb{A}[D]u$ vanishes. It is not difficult to see (e.g., by passing to the corresponding Laplace–Beltrami operators $-\mathbb{A}^*\circ\mathbb{A}$ and the usual elliptic regularity theory for second order elliptic operators) that elements of $\ker_{\mathcal{D}'}(\mathbb{A}[D];\mathrm{B})$ are locally integrable, even smooth, and hence each of the conditions given in the previous lemma applies to the distributional case, too.

3.3.4. Type–(C)– and $\mathbb{C}$–elliptic operators In this section we reduce the FDN framework to a different setup which allows to infer Sobolev–Poincaré–type inequalities in the next paragraph. To do so, we borrow the terminology of *type–(C) operators* as introduced by KALAMAJSKA in [149] and study their relation to the FDN class. We begin with a strengthening of the usual ellipticity (which we shall refer to in the sequel as $\mathbb{R}$–*ellipticity*) and call an operator of the form (3.11) $\mathbb{C}$–*elliptic* if and only if

$$\mathbb{A}[\xi]\colon V + \mathrm{i}\,V \to W + \mathrm{i}\,W \tag{$\mathbb{C}$-ell}$$

is injective for all $\xi \in \mathbb{C}^n \setminus \{0\}$. The rest of this section consequently is devoted to showing equivalence of the FDN and $\mathbb{C}$–ellipticity. Let us note that the value of this characterisation is that the $\mathbb{C}$–ellipticity is relatively easy to check whereas the FDN might be not.

In what follows, let $k, N \in \mathbb{N}$ and $\mathrm{P}_j = (\mathrm{P}_{j1}, ..., \mathrm{P}_{jk})$ for $j = 1, ..., N$ be scalar differential operators which act on vector–valued functions $v = (v_1, ..., v_k)$ by

$$\mathrm{P}_j\, v = \sum_{i=1}^{k} \mathrm{P}_{ji}\, v_i.$$

Postponing the connection between P and $\mathbb{A}[D]$ to the appendix (cp. Chapter 10.1), we borrow from [149] the following definition.

Definition 3.16 (Condition (C))**.** *The family* (P_j) *is said to satisfy the condition* (C) *(or to be of* type–(C)*) if and only if*

(a) *all the* P_j *have constant coefficients,*

(b) *each* P_j *is homogeneous of order* $m_j \in \mathbb{N}$*,*

(c) *the matrix* $(\mathrm{P}_{ij}(\xi))_{i=1,\dots,k}^{j=1,\dots,N}$ *has rank* k *for any complex* $\xi \in \mathbb{C}^n \setminus \{0\}$.

As will be explained in Chapter 10.1 (also cp. [53]), the type–(C) condition equals the $\mathbb{C}$–ellipticity. The main result of this section reads as follows.

Theorem 3.17. *Let* $\mathbb{A}[D]$ *be a linear differential operator of the form* (3.11)*. If* $\mathbb{A}[D]$ *has the* FDN*, then* $\mathbb{A}[D]$ *is a* type–(C) *operator and thus* $\mathbb{C}$*–elliptic.*

Proof. For the time being, let us identify $V \cong \mathbb{R}^N$ and $W \cong \mathbb{R}^K$. Since $\mathbb{A}[D]$ has a finite dimensional nullspace, it is $\mathbb{R}$-elliptic by Lemma 3.8.

We proceed by contradiction and hence assume that $\mathbb{A}[D]$ is not $\mathbb{C}$-elliptic. Then there exists $\xi \in \mathbb{C}^n \setminus \{0\}$ and $\eta \in \mathbb{C}^N \setminus \{0\}$ with $0 = \mathbb{A}[\xi]\eta = \eta \otimes_{\mathbb{A}} \xi$. We split ξ and η into its real and imaginary parts by $\xi =: \xi_1 + i\xi_2$ and $\eta =: \eta_1 + i\eta_2$. Then $\mathbb{A}[\xi]\eta = 0$ implies

$$\mathbb{A}[\xi_1]\eta_1 - \mathbb{A}[\xi_2]\eta_2 = 0 \qquad \text{and} \qquad \mathbb{A}[\xi_1]\eta_2 + \mathbb{A}[\xi_2]\eta_1 = 0. \tag{3.15}$$

We will demonstrate that ξ_1 and x_2, respectively, η_1 and η_2, are linearly independent.

We begin with the linear independence of ξ_1 and ξ_2. If $\xi_1 = 0$, then $\xi_2 \neq 0$ and then the $\mathbb{R}$–ellipticity of $\mathbb{A}[D]$ and (3.15) implies $\eta_1 = \eta_2 = 0$, which contradicts $\eta \neq 0$. By the same argument, also $\xi_2 = 0$ is not possible. Hence, we have $\xi_1 \neq 0$ and $\xi_2 \neq 0$. We now show the linear independency of ξ_1 and ξ_2 by contradiction, so let us assume that $\xi_2 = \lambda\xi_1$ with $\lambda \neq 0$. Then it follows from (3.15) that

$$\mathbb{A}[\xi_1]\eta_1 = \mathbb{A}[\xi_2]\eta_2 = \lambda\mathbb{A}[\xi_1]\eta_2 = -\lambda\mathbb{A}[\xi_2]\eta_1 = -\lambda^2\mathbb{A}[\xi_1][\eta_1].$$

This implies $\mathbb{A}[\xi_1][\eta_1] = 0$. Hence by $\mathbb{R}$-ellipticity of $\mathbb{A}[D]$ and $\xi_1 \neq 0$, we get $\eta_1 = 0$. Now, (3.15) implies $\mathbb{A}[\xi_2][\eta_2] = 0$, so again the $\mathbb{R}$-ellipticity of $\mathbb{A}[D]$ gives $\eta_2 = 0$. Overall, $\eta = 0$, which is a contradiction. This proves that ξ_1 and ξ_2 are linearly independent.

The proof of the linear independence of η_1 and η_2 is completely analogous. Indeed, $\eta_1 = \gamma\eta_2$ implies $\mathbb{A}[\xi_1]\eta_1 = -\gamma^2\mathbb{A}[\xi_1]\eta_1$, so $\mathbb{A}[\xi_1][\eta_1] = 0$. As above this implies $\eta = 0$, which is a contradiction.

Let us define now $\tau : \mathbb{R}^n \to \mathbb{C}$ and $\sigma : \mathbb{C} \to \mathbb{R}^N$ by

$$\begin{aligned}\tau(x) &:= \langle \xi_1, x\rangle + \mathrm{i}\langle \xi_2, x\rangle,\\ \sigma(z) &:= \Re(z)\eta_1 - \Im(z)\eta_2.\end{aligned}$$

Let $\mathcal{O}(\mathbb{C})$ denote the set of holomorphic functions on $\mathbb{C}$. Then $\dim(\mathcal{O}(\mathbb{C})) = \infty$. Moreover, for $f \in \mathcal{O}(\mathbb{C})$ we have $\partial_{\bar z} f(z) = 0$ in the sense of Wirtinger derivatives (cp. [155, Chpt. 1.1]). Let us define $h_f\colon \mathbb{R}^n \to \mathbb{R}^N$ by $h_f := \sigma \circ f \circ \tau$. Our goal is to prove $\mathbb{A}[D]h_f = 0$. In the following, we identify $\mathbb{C}$ with $\mathbb{R}^2$. With the chain rule we then conclude

$$\begin{aligned}(\mathbb{A}[D]h_f)(x) = \; & \mathbb{A}[\xi_1]\eta_1(\partial_1 f_1)(\tau(x)) - \mathbb{A}[\xi_1]\eta_2(\partial_1 f_2)(\tau(x))\\ & + \mathbb{A}[\xi_2]\eta_1(\partial_2 f_1)(\tau(x)) - \mathbb{A}[\xi_2]\eta_2(\partial_2 f_2)(\tau(x)).\end{aligned} \tag{3.16}$$

Using the Cauchy-Riemann equations $\partial_1 f_1 = \partial_2 f_2$ and $\partial_1 f_2 = -\partial_2 f_1$ and (3.15) we get

$$(\mathbb{A}[D]h_f)(x) = (\mathbb{A}[\xi_1]\eta_1 - \mathbb{A}[\xi_2]\eta_2)(\partial_1 f_1)(\tau(x)) + (\mathbb{A}[\xi_1]\eta_2 + \mathbb{A}[\xi_2]\eta_1)(\partial_2 f_1)(\tau(x)) = 0.$$

So for each $f \in \mathcal{O}(\mathbb{C})$, we have constructed a function $h_f : \mathbb{R}^n \to \mathbb{R}^N$ such that $\mathbb{A}[D]h_f = 0$. We need to show that $\dim(\{h_f : f \in \mathcal{O}(\mathbb{C})\}) = \infty$. For this, it suffices to show that the linear mapping $f \mapsto h_f$ is injective. Recall that $h_f = \sigma \circ f \circ \tau$. Hence, it suffices to show that σ is injective and that τ is surjective. This, however, follows from the fact that ξ_1 and ξ_2, respectively η_1 and η_2, are linearly independent. The proof is complete. □

For type–(C) operators, Kalamajska established in [149] integral representation formulas which are in the spirit of those of Smith [218] and natural generalisations of the usual Sobolev integral representations of $\mathrm{W}^{1,p}$–functions on bounded, star–shaped domains as discussed, for instance, by Maz'ya in [170, Chapter 1.1.10]. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain which is starshaped with respect to a ball $\mathrm{B} \neq \emptyset$ with $\overline{\mathrm{B}} \subseteq \Omega$ and choose $\omega \in \mathrm{C}^1_c(\mathrm{B})$ with $\int_{\mathrm{B}} \omega \,\mathrm{d}x = 1$. We then put for $m \in \mathbb{N}$, $\alpha \in \mathbb{N}_0^n$ and a given mapping $v \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)$

$$\mathfrak{P}^{m-1}_\omega v(x) := \int_\Omega \left(\sum_{|\beta|<m} \partial_y^\beta \left(\frac{(y-x)^\beta}{\beta!}\omega(y)\right)\right) v(y)\,\mathrm{d}y, \tag{3.17}$$

$$K_\alpha(x,y) := \frac{(-1)^m m}{\alpha!} \frac{(y-x)^\alpha}{|y-x|^n} \int_{|y-x|}^\infty \omega\Big(x + t\frac{y-x}{|y-x|}\Big) t^{n-1}\,\mathrm{d}t,\; x,y \in \mathbb{R}^n. \tag{3.18}$$

We record the following lemma.

Lemma 3.18 ([149, Thm. 4])**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded, starshaped domain and let $(\mathrm{P}_j)_{j=1,\dots,N}$ be a family of differential operators acting on vector–valued functions $v = (v_1, \dots, v_k)$ and satisfying the condition* (C) *for some positive integers $m_1, \dots, m_N$. Then there exist vector–valued functions $K_j = (K_{j1}, \dots, K_{jk})\colon (\mathbb{R}^n \times \mathbb{R}^n) \setminus \Delta \to \mathbb{R}^k$ satisfying the following conditions:*

(a) $K_{ji} \in \mathrm{C}^\infty((\mathbb{R}^n \times \mathbb{R}^n) \setminus \Delta)$,

(b) $K_{ji}(x,\cdot) = 0$ *near* $\partial\Omega$ *for any* $x \in \Omega$,

(c) $|\partial_x^\alpha \partial_y^\beta K_{ji}(x,y)| \leqslant C/|x-y|^{n-m_j+|\alpha|+|\beta|}$ *for any* $x, y \in \Omega$ *and* $\alpha, \beta \in \mathbb{N}_0^n$,

(d) *there exists a positive integer l not smaller than $\max_j m_j$ and scalar differential operators $\mathrm{P}_{j,i,\alpha}$ ($j = 1, ..., N$, $i = 1, ..., k$, $|\alpha| = l$), homogeneous of order $l - m_j$, with constant coefficients such that*

$$K_{ij}(x, y) = \sum_{|\alpha|=l} (\mathrm{P}_{j,i,\alpha})_y K_\alpha(x, y),$$

where $(\mathrm{P}_{j,i,\alpha})_y$ indicates that the operator $\mathrm{P}_{j,i,\alpha}$ acts on functions of y,

(e) *if l is as in* (d)*, then for any $v = (v_1, ..., v_k) \in \mathrm{C}^\infty(\Omega; \mathbb{R}^k)$ there holds*

$$v_i(x) = \mathfrak{P}^{l-1}_\omega v_i(x) + \sum_{j=1}^{N} \int_\Omega K_{ji}(x, y)\, \mathrm{P}_j\, v(y)\, \mathrm{d}y, \tag{3.19}$$

where $\mathfrak{P}^{l-1}_\omega v_i$ is as in (3.17).

Corollary 3.19. *Let $\mathbb{A}[D]$ be a differential operator of the form* (3.11)*. Then $\mathbb{A}[D]$ has the FDN if and only if it is of type–*(C)*.*

Proof. The direction '⇒' has been established in Theorem 3.17. For the other direction '⇐', observe that by Lemma 3.18(e) elements of the nullspace of $\mathbb{A}[D]$ relative to the unit ball B are polynomials of a bounded degree, and hence $\mathbb{A}[D]$ has the FDN. By Lemma 3.15, this statement is independent of p, and the proof is complete. □

3.4. Proof of the Korn–type Inequalities

3.4.1. Proof of Lemma 3.8 In a first step, we give the

Proof of Lemma 3.8. We argue by contradiction and thus assume that $\mathbb{A}[D]$ is not elliptic. Then we find $\xi \in \mathbb{S}^{n-1}$ and $\eta \in V \setminus \{0\}$ such that $\mathbb{A}[\xi]\eta = 0$. Let (f_k) be a sequence of linearly independent smooth functions $f_k\colon (-1, 1) \to \mathbb{R}$. We define $g_k\colon \mathrm{B} \to V$ by $g_k(x) := f_k(\langle x, \xi\rangle)\eta$. Then $g_1, g_2, ...$ are linearly independent: Indeed, suppose that there exist $m \in \mathbb{N}$ and $\lambda_1, ..., \lambda_m \in \mathbb{R}$ such that

$$\lambda_1 g_1 + ... + \lambda_m g_m = g_{m+1}.$$

Then there holds for all $x \in \mathrm{B}$

$$\lambda_1 f_1(\langle x, \xi\rangle) + ... + \lambda_m f_m(\langle x, \xi\rangle) - f_{m+1}(\langle x, \xi\rangle) = 0, \tag{3.20}$$

and taking $x = a\xi$ for an arbitrary $a \in (-1, 1)$ yields by (3.20)

$$\lambda_1 f_1(a) + ... + \lambda_m f_m(a) = f_{m+1}(a) \qquad \text{for all } a \in (-1, 1), \tag{3.21}$$

which is at variance with the linear independence of the sequence (f_k). Hence the span of the sequence (f_k) is infinite–dimensional, and since $\mathbb{A}[D]f_k(x) = f_k'(\langle x, \xi\rangle)\mathbb{A}[\xi]\eta = 0$, it is contained in the nullspace of $\mathbb{A}[D]$ and so $\mathbb{A}[D]$ does not have the FDN. The proof is complete. □

3.4.2. Proof of Proposition 3.9 We now pass to the

Proof of Proposition 3.9. We prove (d)⇒(c)⇒(b)⇒(a)⇒(d). Ad (d)⇒(c). Consider the operator (with pv. denoting the Cauchy principal value)

$$T\colon \mathbb{A}[D]\varphi \mapsto \mathrm{pv.}\, D\mathbf{G}_\mathbb{A}(\mathbb{A}[D]\varphi)$$

for $\varphi \in \mathrm{C}^\infty_c(\mathbb{R}^n; V)$, where $\mathbf{G}_\mathbb{A}$ is the Green's operator of the elliptic operator $\mathbb{A}[D]$ (cp. Section 3.2). By the results of Section 3.2, this is a singular integral operator of convolution type, and since $D\varphi = \mathrm{pv.}\, D\mathbf{G}_\mathbb{A}(\mathbb{A}[D]\varphi)$ $\mathscr{L}^n$–a.e. in $\mathbb{R}^n$, we have

$\|D\varphi\|_{\mathrm{L}^p(\mathbb{R}^n;V\times\mathbb{R}^n)} = \|T(\mathbb{A}[D]\varphi)\|_{\mathrm{L}^p(\mathbb{R}^n;V\times\mathbb{R}^n)} \leqslant c\|\mathbb{A}[D]\varphi\|_{\mathrm{L}^p(\mathbb{R}^n;W)}$ which, by arbitrariness of φ, is (c) (note that $c>0$ only depends on p and $\mathbb{A}[D]$). Ad (c)⇒(b). Let $\varphi\in\mathrm{C}_c^\infty(\mathbb{R}^n;V)$. Clearly, $\|\varphi\|_{\mathrm{W}^{\mathbb{A},p}} \leqslant c\|\varphi\|_{\mathrm{W}^{1,p}}$ with some $c>0$ merely depending on the coefficients of $\mathbb{A}[D]$. On the other hand, by (c),

$$\begin{aligned}\|\varphi\|_{\mathrm{W}^{1,p}(\mathbb{R}^n;V)} &= \|\varphi\|_{\mathrm{L}^p(\mathbb{R}^n;V} + \|D\varphi\|_{\mathrm{L}^p(\mathbb{R}^n;V\times\mathbb{R}^n)} \\ &\leqslant \|\varphi\|_{\mathrm{L}^p(\mathbb{R}^n;V)} + c\|\mathbb{A}[D]\varphi\|_{\mathrm{L}^p(\mathbb{R}^n;W)} \leqslant c\|\varphi\|_{\mathrm{W}^{\mathbb{A},p}(\mathbb{R}^n)}.\end{aligned}$$

As $\mathrm{C}_c^\infty(\mathbb{R}^n;V)$ is easily seen to be dense in $\mathrm{W}^{\mathbb{A},p}(\mathbb{R}^n)$ with respect to the $\mathrm{W}^{\mathbb{A},p}(\mathbb{R}^n)$–norm, (b) follows. Ad (b)⇒(a). Given $\varphi\in\mathrm{C}_c^\infty(\mathrm{B};V)$, denote $\widetilde{\varphi}\in\mathrm{C}_c^\infty(\mathbb{R}^n;V)$ the extension of φ to $\mathbb{R}^n$ by zero. Then by (b), $\|\varphi\|_{\mathrm{W}^{1,p}(\mathrm{B};V)} \leqslant c\|\widetilde{\varphi}\|_{\mathrm{W}^{1,p}(\mathbb{R}^n;V)} \leqslant c\|\widetilde{\varphi}\|_{\mathrm{W}^{\mathbb{A},p}(\mathbb{R}^n)} = c\|\varphi\|_{\mathrm{W}^{\mathbb{A},p}(\mathbb{R}^n)}$, and since $\|\varphi\|_{\mathrm{W}^{\mathbb{A},p}(\mathrm{B})} \leqslant c\|\varphi\|_{\mathrm{W}^{1,p}(\mathrm{B};V)}$ with a constant $c>0$ depending on $\mathbb{A}[D]$ only, (a) follows. Ad (a)⇒(d). We show the contrapositive statement and so assume that (d) does not hold, i.e., assume that $\mathbb{A}[D]$ is not elliptic. Thus we find $\xi\in\mathbb{R}^n\setminus\{0\}$ and $\eta\in V\setminus\{0\}$ such that $\mathbb{A}[\xi]\eta=0$. For $0<a<1$ sufficiently small, we pick a sequence of functions $(\rho_k)\subset\mathrm{C}_c^\infty((-a,a);(0,1))$ such that $\|\rho_k'\|_{\mathrm{L}^p((-a,a))}\to\infty$ as $k\to\infty$. Similarly as in the proof of Lemma 3.8, we put $u_k(x):=\rho_k(\langle x,\xi\rangle)\eta$, $x\in\mathrm{B}$; note that diminishing $a>0$ if necessary, we may assume $\mathrm{spt}(u_k)\subset\mathrm{B}$ for all $k\in\mathbb{N}$. Since $|\rho_k|\leqslant 1$ and is compactly supported in a fixed set, (u_k) is bounded in $\mathrm{L}^p(\mathbb{R}^n;V)$. Therefore, if (a) holds, we get by $\mathbb{A}[D]u_k=0$ in B that

$$\|u_k\|_{\mathrm{W}^{A,p}(\mathrm{B})} = \big(\|u_k\|_{\mathrm{L}^p(\mathrm{B};V)} + \|\mathbb{A}[D]u_k\|_{\mathrm{L}^p(\mathrm{B};W)}\big) \leqslant c\|u_k\|_{\mathrm{L}^p(\mathrm{B};V)} \leqslant c. \tag{3.22}$$

On the other hand, since $\xi\neq 0$, we find $j\in\{1,...,n\}$ with $\xi_j\neq 0$. Therefore, we obtain by a change of coordinates

$$\begin{aligned}\int_{\mathrm{B}} |Du_k(x)|^p\,\mathrm{d}x &\geqslant \int_{\mathbb{R}^n} |\partial_j u_k(x)|^p\,\mathrm{d}x = |\eta|^p \int_{\mathbb{R}^n} |\rho_k'(\langle x,\xi\rangle)|^p\,\mathrm{d}x \\ &\geqslant c|\eta|^p \int_{\mathbb{R}^n} |\rho_k'(\langle x,\xi\rangle)|^p\,\mathrm{d}x \geqslant c|\eta|^p \int_{-1}^1 |\rho_k'(t)|^p\,\mathrm{d}t \to \infty.\end{aligned}$$

This is at variance with (3.22) and so (a) cannot hold. Hence, (a) implies (d) and the proof is complete. □

3.4.3. Proof of Theorem 3.10 Before we prove Theorem 3.10, we record the so–called *equivalence lemma* due to Peetre and Tartar, the quick proof of which we recall for the reader's convenience.

Lemma 3.20 (Equivalence Lemma, [231, Lemma 11.1])**.** *Let X_1,X_2,X_3 be three Banach spaces with corresponding norms $\|\cdot\|_j$, $j=1,2,3$, and assume $A\colon X_1\to X_3$ is a continuous linear operator, $B\colon X_1\to X_2$ is a compact operator such that $\|\cdot\|_*$ given by $\|x\|_*=\|Ax\|_3+\|Bx\|_2$ is a norm on X_1 which is equivalent to $\|\cdot\|_1$. Then* $\dim(\ker(A))<\infty$.

Proof. Let $(x_k)\subset\ker(A)$ be a bounded sequence (with respect to $\|\cdot\|_1$). Since B is a compact, there exists a subsequence $(x_{k(j)})$ of (x_k) such that $(Bx_{k(j)})$ converges in X_3. In particular, $(Bx_{k(j)})$ is Cauchy in X_3, and since $Ax_{k(j)}=0$ for all $j\in\mathbb{N}$, we conclude by the equivalence of $\|\cdot\|_1$ and $\|\cdot\|_*$ that $(x_{k(j)})$ is $\|\cdot\|_1$–Cauchy in X_1. By completeness of $(X_1,\|\cdot\|_1)$, $(x_{k(j)})$ therefore converges to some $x\in X_1$, and by continuity of A, $x\in\ker(A)$. By arbitrariness of (x_k), we conclude that the unit ball in $\ker(A)$ is relatively compact in X_1. By the well–known Riesz Lemma (cp. [102, Thm. 1.24]), this is possible if and only if $\dim(\ker(A))<\infty$, and the proof is complete. □

Recalling theorem 3.17, we can now give the

Proof of Theorem 3.10. Ad (a)⇒(b). Let $\mathbb{A}[D]$ have the FDN. Then, by Thm. 3.17, it is of type–(C) and hence elliptic (specifically, as $\mathbb{C}$– trivially implies $\mathbb{R}$–ellipticity). Moreover, by Kalamajska's result, Lemma 3.18, we can write each $u \in \mathrm{W}^{\mathbb{A},1}(\Omega;\mathbb{R}^N)$ as $u = \Pi_\ell u + (\mathrm{Id}-\Pi_\ell)u$, where $\Pi_\ell\colon \mathrm{W}^{\mathbb{A},1}(\mathrm{B}) \to \mathcal{P}_\ell(\mathrm{B};V)$ is a projection onto the V–values polynomials on B of degree at most $\ell = \ell(\mathbb{A}) \in \mathbb{N}$. As remarked in [150], this gives rise to an extension operator as follows. Since $\Pi_\ell u$ is a polynomial, it can be trivially extended to a polynomial $\overline{\Pi_\ell u}\colon \mathbb{R}^n \to V$, and since by Lemma 3.18, $(\mathrm{Id}-\Pi_\ell)$ can be represented by a suitable Riesz potential, $(\mathrm{Id}-\Pi_\ell)f = \int_{\mathrm{B}} k_{\mathbb{A}}(x-y)\mathbb{A}[D]f(y)\,\mathrm{d}y$, we may put for some C_c^∞–function $\eta\colon \mathbb{R}^n \to \mathbb{R}$ with $\eta \equiv 1$ on B,

$$Eu(x) = \eta(x)\left(\overline{\Pi_\ell u}(x) + \int_{\mathbb{R}^n} \mathbb{1}_\Omega(y)k_{\mathbb{A}}(x-y)\mathbb{A}[D]u(y)\,\mathrm{d}y\right), \qquad x \in \mathbb{R}^n,$$

for $u \in (\mathrm{C}^\infty \cap \mathrm{W}^{\mathbb{A},p})(\mathrm{B})$ and extend this operator by density of $(\mathrm{C}^\infty \cap \mathrm{W}^{\mathbb{A},p})(\mathrm{B})$ boundedly to $\mathrm{W}^{\mathbb{A},p}(\mathrm{B})$, and (b) follows. Ad (b)⇒(c). Clearly, $\mathrm{W}^{1,p}(\mathrm{B};V) \hookrightarrow \mathrm{W}^{\mathbb{A},p}(\mathrm{B})$. To prove the reverse implication, let $u \in \mathrm{W}^{\mathbb{A},p}(\mathrm{B})$ be arbitrary. Denoting the extension operator from (b) by E, we firstly note that E can always be modified such that $\mathrm{spt}(E\varphi) \subset \mathrm{B}(0,R)$ for some fixed $R > 0$ and all $\varphi \in \mathrm{W}^{\mathbb{A},p}(\overline{\mathrm{B}};V)$. Since $\mathbb{A}[D]$ is elliptic by (b) and $Eu \in \mathrm{W}^{\mathbb{A},p}(\mathbb{R}^n;V)$, Proposition 3.9 gives $\|DEu\|_{\mathrm{L}^p(\mathbb{R}^n;V\times\mathbb{R}^n)} \leqslant C\|\mathbb{A}[D]Eu\|_{\mathrm{L}^p(\mathbb{R}^n;W)}$ for some constant $C > 0$ independent of u. In conclusion, using the fact that $Eu|_{\mathrm{B}} = u$ in the first step, we thereby obtain

$$\begin{aligned}\|u\|_{\mathrm{W}^{1,p}(\mathrm{B};V)} &\leqslant \|Eu\|_{\mathrm{W}^{1,p}(\mathbb{R}^n;V)} = \|Eu\|_{\mathrm{L}^p(\mathbb{R}^n;V)} + \|DEu\|_{\mathrm{L}^p(\mathbb{R}^n;V\times\mathbb{R}^n)} \\ &\leqslant \|Eu\|_{\mathrm{L}^p(\mathbb{R}^n;V)} + C\|\mathbb{A}[D]Eu\|_{\mathrm{L}^p(\mathbb{R}^n;W)} \leqslant C\|Eu\|_{\mathrm{W}^{\mathbb{A},p}(\mathbb{R}^n)} \\ &\leqslant C\|u\|_{\mathrm{W}^{\mathbb{A},p}(\mathrm{B})},\end{aligned}$$

where we have used the boundedness of E in the last step, and so (c) follows. Ad (c)⇒(a). This is a consequence of the equivalence lemma, Lemma 3.20, which we apply to $X_1 = \mathrm{W}^{1,p}(\mathrm{B};V)$, $X_2 = \mathrm{L}^p(\mathrm{B};V)$ and $X_3 = \mathrm{L}^p(\mathrm{B};W)$ together with $A = \mathbb{A}[D]$ and B being the compact injection $X_1 = \mathrm{W}^{1,p}(\mathrm{B};V) \hookrightarrow\hookrightarrow \mathrm{L}^p(\mathrm{B};V) = X_2$. Clearly, $\mathbb{A}[D]\colon X_1 \to X_3$ is continuous and by assumption, the conditions of the equivalence lemma are satisfied. Therefore, $\ker_{\mathrm{W}^{1,p}(\mathrm{B};V)}(\mathbb{A}[D])$ is finite dimensional, and an argument similar to that invoked in the proof of Lemma 3.15 this yields that $\mathbb{A}[D]$ has the FDN, and (a) follows. The proof is complete. □

Remark 3.21 (The Wirtinger Differential)**.** Write, for $f \in \mathrm{C}^1(\overline{\mathrm{B}};\mathbb{C}) \simeq \mathrm{C}^1(\overline{\mathbb{D}};\mathbb{C})$ with the complex unit disc $\mathbb{D}$ and an arbitrary $z \in \mathrm{B}$,

$$f(z) = \frac{1}{2\pi i}\int_{\partial\mathrm{B}} \frac{f(y)}{y-z}\,\mathrm{d}\mathcal{H}^1(y) - \frac{1}{\pi}\int_{\mathrm{B}} \frac{\partial_{\overline{z}}f(y)}{y-z}\,\mathrm{d}y = F(f)(z) + G(f)(z), \tag{3.23}$$

with an obvious definition for F and G. In this situation, the extension operator as given in the proof of Thm. 3.10 cannot be constructed in the same way. This is due to the discontinuity of F which, in turn, is due to the fact that even if $\mathrm{W}^{\mathbb{A},p}(\mathrm{B})$–maps can be approximated by smooth maps up to the the boundary, $\mathrm{W}^{\mathbb{A},p}(\mathrm{B})$–maps in general do not attain traces in $\mathrm{L}^1_{\mathrm{loc}}(\partial\mathrm{B})$. Indeed, let $f\colon \mathbb{D} \to \mathbb{C}$ be the function given by $f(z) := (z-1)^{-1}$. Then $f \in \mathrm{L}^p(\mathbb{D};\mathbb{C})$ for any $1 \leqslant p < 2$, it is holomorphic, and trivially $\int_{\partial\mathrm{B}} |f|\,\mathrm{d}\mathcal{H}^1 = \infty$. Thus $|f| \notin \mathrm{L}^1_{\mathrm{loc}}(\partial\mathbb{D})$. This has been observed by Fuchs & Repin [118].

Remark 3.22 (More general domains)**.** Working from Thm. 3.10, it is possible to construct an extension operator for spaces $\mathrm{W}^{\mathbb{A},p}(\Omega)$ with $\mathbb{A}[D]$ possessing the FDN for Reifenberg or (ε,δ)–domains, cp. [136]. This can be achieved by adapting the Jones extension method [148] as done by Duran & Muschietti [88] for the symmetric gradient case which relies on essentially reflecting elements of the nullspaces on interior cubes to the complement, and it is obvious that such an extension will be bounded provided suitable comparison estimates for the reflected nullspace elements can be established. This in turn relies on inverse estimates (i.e., estimating the L^∞– against the L^p–norms) which are available for finite dimensional normed function spaces only.

3.5. Poincaré–Type Inequalities

3.5.1. General Framework for Poincaré Inequalities For later purposes, we record here some stability estimates for the projections onto the nullspaces of $\mathbb{A}[D]$ which appear in Poincaré–type inequalities. We firstly recall an abstract Poincaré inequality from [7].

Lemma 3.23 ([7, Lemma 8.3.1]). *Let X_0 be a Banach space with norm $\|\cdot\|_0$ and let $X \subset X_0$ be a Banach space with norm $\|\cdot\|_X := \|\cdot\|_0 + \|\cdot\|_1$, where $\|\cdot\|_1$ is a seminorm with nullspace $\mathcal{N} \neq \{0\}$. If the embedding $X \hookrightarrow X_0$ is compact, then there exists a constant $A > 0$ such that $\|x - Lx\|_0 \leqslant A\|L\|\,\|x\|_1$ for all $x \in X$ and all projections L of X onto $\mathcal{N}$.*

In order to fruitfully apply the previous lemma, we need to demonstrate compactness of the injection $\mathrm{W}^{\mathbb{A},p}(\mathrm{B}) \hookrightarrow \mathrm{L}^p(\mathrm{B};V)$ for all $1 \leqslant p < \infty$ first; we recall that here and in what follows $\mathrm{B} := \mathrm{B}(0,1)$ denotes the open unit ball in $\mathbb{R}^n$.

Lemma 3.24. *Let $\mathbb{A}[D]$ be a $\mathbb{C}$–elliptic operator of the form (3.11) and let $1 \leqslant p < \infty$. Then the embedding $\mathrm{W}^{\mathbb{A},p}(\mathrm{B}) \hookrightarrow \mathrm{L}^p(\mathrm{B};V)$ is compact.*

Proof. We only sketch the proof. If $1 < p < \infty$, then $\mathrm{W}^{\mathbb{A},p}(\mathrm{B}) \simeq \mathrm{W}^{1,p}(\mathrm{B};V)$ by Theorem 3.10 and so the conclusion follows from $\mathrm{W}^{1,p}(\mathrm{B};V) \hookrightarrow \mathrm{L}^p(\mathrm{B};V)$ (see, e.g., [97, Chpt. 4.6, Thm. 1]). If $p = 1$, we cannot reduce to $\mathrm{W}^{1,1}(\mathrm{B};V)$ and hence invoke the Riesz–Kolmogorov compactness criterion together with the extension part of Theorem 3.10 for $\mathrm{W}^{\mathbb{A},1}$–spaces for $\mathbb{C}$–elliptic differential operators $\mathbb{A}$, cp. [136] for more detail. This yields the claim. □

Theorem 3.25. *Let $\mathbb{A}[D]$ have the finite dimensional nullspace property or, equivalently, be a $\mathbb{C}$–elliptic operator of the form (3.11) and let $1 \leqslant p < \infty$. Let $\mathrm{B} \subset \mathbb{R}^n$ be a ball with radius $r = r(B) > 0$. Then there exists a projection $\Pi_{\mathrm{B},\mathbb{A}}\colon\ \mathrm{W}^{\mathbb{A},p}(\mathrm{B}) \to \ker_{\mathrm{W}^{\mathbb{A},p}}(\mathbb{A}[D];\mathrm{B})$ such that the following hold:*

(a) *There exists a constant $c = c(p,\mathbb{A}) > 0$ independent of B such that for all $w \in \mathrm{W}^{\mathbb{A},p}(\mathrm{B})$ there holds*

$$\fint_{\mathrm{B}} |\Pi_{B,\mathbb{A}} w|^p \,\mathrm{d}x \leqslant c \fint_{\mathrm{B}} |w|^p \,\mathrm{d}x.$$

(b) *There exists a constant $c > 0$ independent of B such that for all $u \in \mathrm{W}^{\mathbb{A},p}(\mathrm{B})$ there holds*

$$\fint_{\mathrm{B}} |u - \Pi_{\mathrm{B},\mathbb{A}} u|^p \,\mathrm{d}x \leqslant c r^p \fint_{\mathrm{B}} |\mathbb{A}[D]u|^p \,\mathrm{d}x.$$

Proof. Throughout the proof, we assume without loss of generality $r = r(\mathrm{B}) = 1$. The general case follows by a standard rescaling which we omit here. Since $\mathbb{A}$ has finite dimensional nullspace on $\mathcal{D}'(\mathrm{B};V)$ and consists of polynomials of a fixed degree, $\mathcal{K} := \ker_{\mathrm{W}^{\mathbb{A},p}}(\mathbb{A}[D];\mathrm{B}) \subset (\mathrm{L}^1 \cap \mathrm{L}^\infty)(\mathrm{B};V)$. We may thus pick an orthonormal basis $\{R_1, ..., R_\ell\}$ of $\ker_{\mathrm{W}^{\mathbb{A},p}}(\mathbb{A}[D];\mathrm{B})$ (with respect to the $\mathrm{L}^2(\mathrm{B};V)$–inner product) and define, for $w \in \mathrm{L}^2(\mathrm{B};V)$,

$$\Pi_{\mathrm{B};\mathbb{A}} w := \sum_{j=1}^{\ell} \langle R_j, w\rangle_{\mathrm{L}^2(\mathrm{B};V)} R_j. \tag{3.24}$$

Most crucially, note that since $\dim(\mathcal{K}) < \infty$, all norms are equivalent on $\mathcal{K}$ and because of $\mathcal{K} \subset (\mathrm{L}^1 \cap \mathrm{L}^\infty)(\mathrm{B}; V)$, it is easily seen that $\Pi_{\mathrm{B};\mathbb{A}} w$ is well–defined for any $w \in \mathrm{L}^1(\mathrm{B}; V)$, too. In fact,

$$\begin{aligned}\int_{\mathrm{B}} |\Pi_{\mathrm{B};\mathbb{A}} w(x)| \,\mathrm{d}x &\leqslant \sum_{j=1}^{\ell} \int_{\mathrm{B}} |R_j(x)|\,|w(x)| \,\mathrm{d}x \int_{\mathrm{B}} |R_j(x)| \,\mathrm{d}x \\ &\leqslant \sum_{j=1}^{\ell} \int_{\mathrm{B}} |w(x)| \,\mathrm{d}x \int_{\mathrm{B}} \|R_j\|^2_{\mathrm{L}^\infty(\mathrm{B};V)} \,\mathrm{d}x \leqslant C \int_{\mathrm{B}} |w(x)| \,\mathrm{d}x,\end{aligned}$$

and scaling shows that the constant $C > 0$ is independent of the ball B. This implies (a). Ad (b). Since $\Pi_{\mathrm{B};\mathbb{A}}$ is well–defined and continuous both on $\mathrm{L}^2(\mathrm{B}; V)$ and $\mathrm{L}^1(\mathrm{B}; V)$, it is continuous on each $\mathrm{L}^p(\mathrm{B}; V)$ for every $1 \leqslant p \leqslant \infty$. Now, $\Pi_{\mathrm{B};\mathbb{A}}$ gives rise to a (not renamed) bounded linear operator $\Pi_{\mathrm{B};\mathbb{A}} \colon \mathrm{W}^{\mathbb{A},p}(\mathrm{B}) \to \mathrm{W}^{\mathbb{A},p}(\mathrm{B})$. Indeed, since $\Pi_{\mathrm{B};\mathbb{A}}$ is $\mathrm{L}^p(\mathrm{B}; V)$–stable, this is immediate from $\mathbb{A}[D]\Pi_{\mathrm{B};\mathbb{A}} w \equiv 0$ for all $w \in \mathrm{W}^{\mathbb{A},p}(\mathrm{B})$. This operator is the identity on $\mathcal{K}$, is bounded as a map $\mathrm{W}^{\mathbb{A},p}(\mathrm{B}) \to \mathrm{W}^{\mathbb{A},p}(\mathrm{B})$ and since the embedding $\mathrm{W}^{\mathbb{A},p}(\mathrm{B}) \hookrightarrow \mathrm{L}^p(\mathrm{B}; V)$ is compact by Theorem 3.24, we are in the situation to apply the abstract Poincaré inequality, Lemma 3.23. From here the claim is immediate for the unit ball, and as for (a), the general case follows by rescaling. $\square$

3.5.2. Remarks on other Korn–type inequalities Let now $\mathbb{A}[D]$ be of the form (3.11). We say that $\mathbb{A}[D]$ satisfies a *Korn inequality of the third kind* provided there exists a constant $C = C(p) > 0$ such that for all $v \in \mathrm{C}^\infty(\mathrm{B}; V)$ such that $v \in \mathrm{L}^p(\mathrm{B}; V)$ and $\mathbb{A}[D]u \in \mathrm{L}^p(\mathrm{B}; W)$ there holds

$$\|\nabla(v - \pi(v))\|_{\mathrm{L}^p(\mathrm{B};V)} \leqslant C\|\mathbb{A}[D]v\|_{\mathrm{L}^p(\mathrm{B};W)}. \tag{K3}$$

Here, the mapping π is a projection onto $\ker(\mathbb{A}[D]; \mathrm{B}) := \{v \in \mathcal{D}'(\mathrm{B}; V) \colon\ \mathbb{A}[D]v = 0\}$. Subtracting an element of the nullspace of $\mathbb{A}[D]$ on the left side of (K3) is easily seen to be necessary for (K3) to hold as $\mathbb{A}[D]$ ignores adding elements of its nullspace, whereas these can be seen by the gradient operator. We wish to stress that the Korn inequalities of the second and third kind are *not* equivalent. Indeed, by Theorem 3.10, the Korn inequality of the second kind implies that the kernel of $\mathbb{A}[D]$ relative to B is finite–dimensional. This, however, need not be the case for the Korn inequality of the third kind as can be seen from the Poincaré–type inequality proven by Fuchs [114] which relies on the representation formula given in Remark 3.21[2].

Note that if (K2) holds and thus $\mathbb{A}[D]$ is $\mathbb{C}$–elliptic, we may take $\pi = \Pi$ (with Π from Theorem 3.25) and obtain

$$\begin{aligned}\|\nabla(v - \pi(v))\|_{\mathrm{L}^p(\mathrm{B};V)} &\leqslant \|v - \pi(v)\|_{\mathrm{W}^{1,p}(\mathrm{B};V)} \\ &\overset{(\mathrm{K2})}{\leqslant} C\big(\|v - \pi(v)\|_{\mathrm{L}^p(\mathrm{B};V)} + \|\mathbb{A}[D](v - \pi(v))\|_{\mathrm{L}^p(\mathrm{B};W)}\big) \\ &\overset{\text{Poincaré}}{\leqslant} C\|\mathbb{A}[D](v)\|_{\mathrm{L}^p(\mathrm{B};W)}.\end{aligned}$$

On the other hand, we have seen that (K3) does not require the kernel of $\mathbb{A}[D]$ to be finite–dimensional. However, assuming that $\mathbb{A}[D]$ has finite dimensional nullspace, we

[2] In fact, going back to (3.23), we bring $F(f)(x)$ to the left hand side of the equation and integrate a suitable power of the resulting sides with respect to x.

obtain (with $\pi = \Pi_{\mathrm{B},\mathbb{A}}$ from Theorem 3.25)

$$\begin{aligned}\|v\|_{\mathrm{W}^{1,p}(\mathrm{B};V)} &\leqslant \|v-\pi(v)\|_{\mathrm{W}^{1,p}(\mathrm{B};V)} + \|\pi(v)\|_{\mathrm{W}^{1,p}(\mathrm{B};V)} \\ &\leqslant C\big(\|\nabla(v-\pi(v))\|_{\mathrm{L}^p(\mathrm{B};V)} + \|v-\pi(v)\|_{\mathrm{L}^p(\mathrm{B};V)} + \|\pi(v)\|_{\mathrm{W}^{1,p}(\mathrm{B};V)}\big) \\ &\overset{\text{(K3), Poincaré}}{\leqslant} C\|\mathbb{A}[D]v\|_{\mathrm{L}^p(\mathrm{B};W)} + \|\pi(v)\|_{\mathrm{L}^p(\mathrm{B};V)} \\ &\overset{\mathrm{L}^p\text{-stability}}{\leqslant} C\big(\|v\|_{\mathrm{L}^p(\mathrm{B};V)} + \|\mathbb{A}[D]v\|_{\mathrm{L}^p(\mathrm{B};W)}\big).\end{aligned}$$

Note that $\|\pi(v)\|_{\mathrm{W}^{1,p}(\mathrm{B};V)} \leqslant C\|\pi(v)\|_{\mathrm{L}^p(\mathrm{B};V)}$ with a finite constant $C>0$ since the space $\ker(\mathbb{A}[D];\mathrm{B})\cap\mathrm{L}^p(\mathrm{B};V)$ is finite dimensional and all norms are equivalent on finite dimensional vector spaces. This yields the following corollary.

Corollary 3.26. *Let $\mathbb{A}[D]$ be a FDN differential operator of the form (3.11) and let $1<p<\infty$. Then $\mathbb{A}[D]$ satisfies (K2) if and only if its satisfies (K3).*

3.6. The Kirchheim–Kristensen Theorem, Revisited

Working towards the proof of Theorem 3.2, in this section we collect some preliminary results from [151] on convex cones and functions, and generalise the Helmholtz–projection estimates of [109] (therein given for first order operators) to higher order differential operators.

To do so, we firstly recall some terminology: Given a finite dimensional vector space V over $\mathbb{R}$ and D a cone of directions in V, we say that $A\subset X$ is D*–convex* provided for all $x,y\in V$ with $x-y\in D$ the the closed convex hull $[x,y]$ is contained in A too. Moreover, we say that for a subset $S\subset X$ a function $f\colon S\to\overline{\mathbb{R}}$ is D*–convex* provided the function

$$F(x) := \begin{cases} f(x) & \text{if } x\in S, \\ \infty & \text{if } x\in V\setminus S\end{cases}$$

has the property that for any $x,y\in S$ with $[x,y]\subset S$ and $x-y\in D$, the restriction $f|_{[x,y]}$ is convex. We begin with a simple lemma for (not extended) real–valued functions which are convex with respect to a cone.

Lemma 3.27. *Let W be a real vector space and $\mathcal{C}$ be a cone in W. For a function $f\colon W\to\mathbb{R}$, the following are equivalent:*

(a) *For all $A\in W$ and $B\in\mathcal{C}$, the function $[0,1]\ni t\mapsto f(A+tB)\in\mathbb{R}$ is convex.*

(b) *f is $\mathcal{C}$–convex, i.e., for all $0\leqslant\theta\leqslant 1$ and all $z,w\in W$ with $z-w\in\mathcal{C}$ there holds $f(\theta z+(1-\theta)w)\leqslant\theta f(z)+(1-\theta)f(w)$.*

Proof. Assume (a) and let $z,w\in W$ be such that $z-w\in\mathcal{C}$. Then for any $0\leqslant\theta\leqslant 1$,

$$f(\theta w+(1-\theta)z) = f(z+\theta(w-z)+(1-\theta)0)\leqslant\theta f(w)+(1-\theta)f(z),$$

where we have used $\mathcal{C}$–convexity in the ultimate step together with $z+(w-z)=w$. Now assume (b) and hence let $A\in W$ and $B\in\mathcal{C}$. Then for any $t,s\in[0,1]$, $0\leqslant\theta\leqslant 1$, there holds

$$\begin{aligned} f(A+(\theta t+(1-\theta)s)B) &= f(\theta(A+tB)+(1-\theta)(A+sB)) \\ &\leqslant \theta f(A+tB)+(1-\theta)f(A+sB) \qquad \text{(by (b))}\end{aligned}$$

and so (a) follows. The proof is complete. ☐

We now collect various auxiliary results from [151].

Lemma 3.28 ([151, Lemma 2.3])**.** *Let $(\mathcal{V}, \|\cdot\|)$ be a normed finite dimensional real vector space and $\mathcal{D}$ be a balanced cone the linear span of which is $\mathcal{V}$. If $f\colon \mathrm{B}(x_0, 2r) \to \mathbb{R}$ is $\mathcal{D}$–convex, then f is locally Lipschitz in $\mathrm{B}(x_0, 2r)$. More precisely, we have $|f(x)-f(y)| \leqslant L\|x-y\|$ for all $x, y \in \mathrm{B}(x_0, r)$, where $L = (c_0/r)\operatorname{osc}(f; \mathrm{B}(x_0, 2r))$ and the constant $c_0 > 0$ depends only on the norm $\|\cdot\|$ and the cone $\mathcal{D}$.*

Lemma 3.29 ([151, Lemma 2.2])**.** *Let V be a normed finite dimensional real vector space and D a cone of directions in V. Assume that $f\colon S \to [-\infty, \infty)$ is a D–convex function defined on an arbitrary subset S of V. If A is a connected component of the interior of S, then either $f \equiv -\infty$ on A or $f > -\infty$ on A.*

Lemma 3.30 ([151, Thm. 1.1])**.** *Let $\mathcal{C}$ be an open convex cone in a normed finite dimensional real vector space V, and D a cone of directions in V such that D spans V. If $f\colon \mathcal{C} \to \mathbb{R}$ is D–convex and positively 1–homogeneous, then f is convex at each point of $\mathcal{C} \cap D$. More precisely, for each $x_0 \in \mathcal{C} \cap D$ there exists a linear function $\ell\colon V \to \mathbb{R}$ such that $\ell(x_0) = f(x_0)$ and $f \geqslant \ell$ on $\mathcal{C}$.*

We conclude this subsection with various results from Fourier analysis on the torus $\mathbb{T}_n := \mathbb{R}^n/\mathbb{Z}^n$. To this end, we define for $f \in \mathrm{L}^1(\mathbb{T}_n; \mathbb{R}^m)$ its *Fourier coefficients* by

$$\widehat{f}(\xi) := \int_{\mathbb{T}_n} f(x) e^{-2\pi\,\mathrm{i}\langle x,\xi\rangle}\,\mathrm{d}x, \qquad \xi \in \mathbb{Z}^n, \tag{3.25}$$

and call $\Gamma_* := \mathbb{Z}^n_* := \mathbb{Z}^n \setminus \{0\}$ or $\Gamma := \mathbb{Z}^n$ the (restricted) *unit lattices*, whereas $Q := [0,1]^n$ is called the *unit cell* in this context. We then record the following variant of Mihlin's multiplier theorem.

Proposition 3.31 (Mihlin, [109, Prop. 2.13])**.** *Let $\Theta\colon \mathbb{R}^n \setminus \{0\} \to \mathbb{C}$ is homogeneous of degree zero and infinitely differentiable on $\mathbb{S}^{n-1}$ and let $1 < p < \infty$. Then the operator T_Θ given by*

$$T_\Theta f(x) := \sum_{\xi \in \mathbb{Z}^n \setminus \{0\}} \Theta(\xi) \widehat{f}(\xi) e^{2\pi\,\mathrm{i}\langle x,\xi\rangle}$$

provided $f \in \mathrm{L}^p(\mathbb{T}_n)$ is given by $f(x) = \sum_{\xi \in \mathbb{Z}^n} \widehat{f}(\xi) e^{-2\pi\,\mathrm{i}\langle x,\xi\rangle}$, extends to a bounded linear operator $T_\Theta\colon \mathrm{L}^p(\mathbb{T}_n) \to \mathrm{L}^p(\mathbb{T}_n)$.

Now, let $\mathcal{A} = \mathcal{A}[D]$ be a given differential operator of the form

$$\mathcal{A} = \sum_{|\alpha|=k} \mathcal{A}_\alpha \partial^\alpha, \tag{3.26}$$

where $\mathcal{A}_\alpha\colon W \to X$ are given fixed homomorphisms and W, X are two finite dimensional vector spaces. In the following, we shall sometimes identify $W \simeq \mathbb{R}^N$. We further assume that $\mathcal{A}$ is of *constant rank*, meaning that the symbol $\mathcal{A}[\xi]$ has rank independent of $\xi \neq 0$. For fixed $\xi \in \mathbb{R}^n \setminus \{0\}$, let $\mathbb{P}_\mathcal{A}[\xi]\colon W \to W$ be the orthogonal projection of W onto $\ker(\mathcal{A}[\xi])$. Then by homogeneity of $\xi \mapsto \mathcal{A}[\xi]$,

$$\mathbb{P}_\mathcal{A}\colon \mathbb{R}^n \setminus \{0\} \to \mathscr{L}(W; W) \text{ is smooth and homogeneous of degree zero.} \tag{3.27}$$

We moreover define $\mathbb{Q}_\mathcal{A}\colon \mathbb{R}^n \setminus \{0\} \to \mathscr{L}(Z; W)$ for $\xi \in \mathbb{R}^n \setminus \{0\}$ by

$$\mathbb{Q}_\mathcal{A}[\xi]y := \begin{cases} z - \mathbb{P}_\mathcal{A}[\xi]z & \text{if } y = \mathcal{A}[\xi]z \text{ for some } z \in W, \\ 0 & \text{if } y \in \mathcal{A}[\xi](V)^\perp. \end{cases} \tag{3.28}$$

Note that this is well–defined: If $z, z' \in W$ are such that $y = \mathcal{A}[\xi]z = \mathcal{A}[\xi]z'$, then $(z - z') \in \ker(\mathcal{A}[\xi])$ and so $z - z' = \mathbb{P}_\mathcal{A}[\xi](z - z')$, i.e., $z - \mathbb{P}_\mathcal{A}[\xi]z = z' - \mathbb{P}_\mathcal{A}[\xi]z'$. Lastly, since $\mathcal{A}[\cdot]$ is homogeneous of degree k and we have

$$\mathbb{Q}_\mathcal{A}[\xi] \circ \mathcal{A}[\xi]z = z - \mathbb{P}_\mathcal{A}[\xi]z = z - \mathbb{P}_\mathcal{A}[\lambda\xi]z = \mathbb{Q}_\mathcal{A}[\lambda\xi] \circ \mathcal{A}[\lambda\xi]z$$

for any $\lambda > 0$, we conclude that $\mathbb{Q}_{\mathcal{A}}\colon \mathbb{R}^n \setminus \{0\} \to \mathscr{L}(Z;W)$ is necessarily homogeneous of degree $(-k)$, and we note that both $\mathbb{P}_{\mathcal{A}}$ and $\mathbb{Q}_{\mathcal{A}}$ smoothly depend on their arguments.

For $i,j \in \{1,...,N\}$, we define $\Theta_{ij}\colon \mathbb{R}^n \ni \xi \mapsto \mathbb{P}_{\mathcal{A}}^{ij}(\xi)$, where $(\mathbb{P}_{\mathcal{A}}^{ij}(\xi))$ is the matrix representation of $\mathbb{P}_{\mathcal{A}}(\xi) \in \mathscr{L}(W;W)$. By the above homogeneity considerations, this defines the Fourier multiplier operators $T_{\Theta_{ij}}$ and thus allows to introduce the operator $\mathbf{T}$ acting on $u = (u_1,...,u_N) \in \mathrm{L}^p(\mathbb{T}_n;W)$ by

$$(\mathbf{T}u)_i(x) := \sum_{j=1}^{N} T_{\Theta_{ij}} u_j(x), \qquad i = 1,...,N, \tag{3.29}$$

for $x \in \mathbb{T}_n$. We proceed by examining the properties of $\mathbf{T}$ (cp. [109, Lem. 2.14] for the corresponding first order statement):

Lemma 3.32. *Let $1 < p < \infty$ and define $\mathbf{T}$ by (3.29). Then the following hold:*

(a) $\mathbf{T}\colon \mathrm{L}^p(\mathbb{T}_n;W) \to \mathrm{L}^p(\mathbb{T}_n;W)$ *is a bounded linear operator which vanishes on constant mappings.*

(b) *For any $u \in \mathrm{L}^1(\mathbb{T}_n)$ there holds $(\mathbf{T}u)_Q = 0$.*

(c) *For any $u \in \mathrm{L}^p(\mathbb{T}_n;W)$ there holds $(\mathbf{T}\circ\mathbf{T})u = \mathbf{T}u$ and $\mathcal{A}[D](\mathbf{T}u) = 0$.*

(d) (Helmholtz–Poincaré–type Inequality.) *There exists a constant $C = C(p) > 0$ such that for all $u \in \mathrm{L}^p(\mathbb{T}_n;W)$ with $(u)_Q = 0$ there holds*

$$\|u - \mathbf{T}u\|_{\mathrm{L}^p(Q;W)} \leqslant C\|\mathcal{A}[D]u\|_{\mathrm{W}^{-k,p}(Q;X)}.$$

Proof. For (a), the boundedness follows from the general properties of Fourier multipliers outlined above, and if f is constant, then we use $\int_Q \mathrm{e}^{-2\pi\,\mathrm{i}\langle x,\xi\rangle}\,\mathrm{d}x = 0$ for $\xi \neq 0$ to conclude. For (b), we obtain similarly

$$\int_Q \mathbf{T}u\,\mathrm{d}x = \sum_{\xi\in\mathbb{Z}^n_*} \mathbb{P}_{\mathcal{A}}[\xi]\widehat{u}(\xi)\int_Q \mathrm{e}^{2\pi\,\mathrm{i}\langle x,\xi\rangle}\,\mathrm{d}x \overset{\xi\neq 0}{=} 0.$$

The first part of assertion (c) can be established by similar means, now using the fact that

$$\int_Q \mathrm{e}^{2\pi\,\mathrm{i}\langle x,\xi-\xi'\rangle}\,\mathrm{d}x = 0 \quad \text{if } \xi,\xi' \in \mathbb{Z}^n,\ \xi \neq \xi'.$$

For the second part of (c), we observe that $\mathcal{A}[\xi]\circ\mathbb{P}_{\mathcal{A}}[\xi] = 0$ so that

$$\mathcal{A}[D](\mathbf{T}u) = \sum_{|\alpha|=k}\sum_{\xi\in\mathbb{Z}^n_*} \mathcal{A}_\alpha \xi_{\alpha_1}\cdots\xi_{\alpha_n}\mathbb{P}_{\mathcal{A}}[\xi]\widehat{f}(\xi)\,\mathrm{e}^{2\pi\,\mathrm{i}\langle x,\xi\rangle}\,\mathrm{d}x = 0.$$

For (d), note that by $(u)_Q = \widehat{u}(0) = 0$ and the construction of the projectors $\mathbb{P}_{\mathcal{A}}$ and $\mathbb{Q}_{\mathcal{A}}$, we obtain

$$
\begin{aligned}
\|u-\mathbf{T}u\|_{\mathrm{L}^p}^p &= \Big\| \sum_{\xi\in\mathbb{Z}^n} \widehat{u}(\xi)\,\mathrm{e}^{2\pi\,\mathrm{i}\langle x,\xi\rangle} - \sum_{\xi\in\mathbb{Z}^n_*} \mathbb{P}_{\mathcal{A}}[\xi]\widehat{u}(\xi)\,\mathrm{e}^{2\pi\,\mathrm{i}\langle x,\xi\rangle}\Big\|_{\mathrm{L}^p}^p \\
&\overset{\widehat{v}(0)=0}{=} \Big\| \sum_{\xi\in\mathbb{Z}^n_*} \mathbb{Q}_{\mathcal{A}}[\xi]\mathcal{A}[\xi]\widehat{u}(\xi)\,\mathrm{e}^{2\pi\,\mathrm{i}\langle x,\xi\rangle}\Big\|_{\mathrm{L}^p}^p \\
&= \Big\| \sum_{\xi\in\mathbb{Z}^n_*} \mathbb{Q}_{\mathcal{A}}\left[\frac{\xi}{|\xi|}\right]\mathcal{A}\left[\frac{\xi}{|\xi|}\right]\widehat{u}(\xi)\,\mathrm{e}^{2\pi\,\mathrm{i}\langle x,\xi\rangle}\Big\|_{\mathrm{L}^p}^p && \text{(by homogeneity)} \\
&\overset{(*)}{\leqslant} c\,\Big\| \sum_{\xi\in\mathbb{Z}^n_*} \mathcal{A}\left[\frac{\xi}{|\xi|}\right]\widehat{u}(\xi)\,\mathrm{e}^{2\pi\,\mathrm{i}\langle x,\xi\rangle}\Big\|_{\mathrm{L}^p}^p && \text{(by Mihlin)} \\
&\leqslant c\,\Big\| \sum_{\xi\in\mathbb{Z}^n_*} \frac{1}{|\xi|^k}\widehat{\mathcal{A}[D]u}(\xi)\,\mathrm{e}^{2\pi\,\mathrm{i}\langle x,\xi\rangle}\Big\|_{\mathrm{L}^p}^p && \text{(by homogeneity)} \\
&\leqslant c\,\|\mathcal{A}[D]u\|_{\mathrm{W}^{-k,p}}^p\,,
\end{aligned}
\tag{3.30}
$$

where $(*)$ holds as $\sup_{\mathbb{S}^{n-1}}|\mathbb{Q}_{\mathcal{A}}|<\infty$. Since the ultimate inequality is valid by definition, the proof is complete. □

Lemma 3.33. *Let $(u_j)\subset\mathrm{L}^\infty(\mathbb{T}_n;W)$ and $u\in\mathrm{L}^\infty(\mathbb{T}_n;W)$ be such that $u_j\overset{*}{\rightharpoonup}u$ as $j\to\infty$ in $\mathrm{L}^\infty(\mathbb{T}_n;W)$ and $\mathcal{A}[D]u_j\overset{*}{\rightharpoonup}0$ as $j\to\infty$ in $(\mathrm{W}^{k-1,1}(\mathbb{T}_n;X))^*$. Then there exists a sequence $(v_j)\subset(\mathrm{W}^{k-1,1}(\mathbb{T}_n;W))^*\cap\mathrm{L}^1(Q;W)\cap\ker(\mathcal{A})$ such that*

$$
\int_{\mathbb{T}_n} v_j\,\mathrm{d}x=\int_{\mathbb{T}_n}u\,\mathrm{d}x \quad\textit{and}\quad \|v_j-u_j\|_{\mathrm{L}^\infty(\mathbb{T}_n;W)}\to 0,\qquad \textit{as } j\to\infty. \tag{3.31}
$$

Proof. We only give the main arguments of the proof which, in turn, is a minor modification of [109, Lem. 2.17]. Without loss of generality, assume that $u=0$. For $j\in\mathbb{N}$, we put $v_j:=\mathbf{T}(u_j-(u_j)_Q)$. By Lemma 3.32(a) and linearity of $\mathbf{T}$, $(v_j)_Q=(u)_Q=0$ for all $j\in\mathbb{N}$. We firstly show that (u_j-v_j) is uniformly bounded in some $\mathrm{W}^{1,p}(Q;W)$ for $p>n$. Mollifying if necessary, we obtain

$$
\begin{aligned}
\|\mathrm{D}(u_j-v_j)\|_{\mathrm{L}^p}^p &\leqslant c\,\Big\|\sum_{\xi\in\mathbb{Z}^n_*}|\xi|(\widehat{u_j}(\xi)-\mathbb{P}_{\mathcal{A}}[\xi]\widehat{u_j}(\xi))\,\mathrm{e}^{2\pi\,\mathrm{i}\langle x,\xi\rangle}\Big\|_{\mathrm{L}^p}^p \\
&\leqslant c\,\Big\|\sum_{\xi\in\mathbb{Z}^n_*}\frac{1}{|\xi|^{k-1}}\mathbb{Q}_{\mathcal{A}}\left[\frac{\xi}{|\xi|}\right]\mathcal{A}[\xi]\widehat{u_j}(\xi)\,\mathrm{e}^{2\pi\,\mathrm{i}\langle x,\xi\rangle}\Big\|_{\mathrm{L}^p}^p \\
&\leqslant c\,\Big\|\sum_{\xi\in\mathbb{Z}^n_*}\frac{1}{|\xi|^{k-1}}\mathcal{A}[\xi]\widehat{u_j}(\xi)\,\mathrm{e}^{2\pi\,\mathrm{i}\langle x,\xi\rangle}\Big\|_{\mathrm{L}^p}^p && \text{(by Mihlin)} \\
&\leqslant c\,\|\mathcal{A}[D]u_j\|_{\mathrm{W}^{-k+1,p}}^p\,.
\end{aligned}
\tag{3.32}
$$

Now note that by Lemma 3.32(d), we can further estimate for $p > n$

$$\begin{aligned}\|u_j - v_j\|_{\mathrm{W}^{1,p}(Q;W)} &= \|u_j - v_j\|_{\mathrm{L}^p(Q;W)} + \|\,\mathrm{D}(u_j - v_j)\|_{\mathrm{L}^p(Q;\mathbb{R}^n\times W)} \\ &\leqslant \|u_j - (u_j)_Q - \mathbf{T}(u_j - (u_j)_Q)\|_{\mathrm{L}^p(Q;W)} + \left|\int_Q u_j\,\mathrm{d}x\right| \\ &\quad + \|\mathcal{A}[D]u_j\|_{\mathrm{W}^{-k,p}(Q;X)} \\ &\leqslant C\|\mathcal{A}[D]\big(u_j\big)\|_{\mathrm{W}^{-k,p}(Q;X)} + \left|\int_Q u_j\,\mathrm{d}x\right| \leqslant C\end{aligned}$$

by Lemma 3.32(b) and boundedness of $(\int_Q u_j\,\mathrm{d}x)$ due to the weak*–convergence of (u_j) (note that actually $\int_Q u_j\,\mathrm{d}x \to 0$). For an arbitrary subsequence of $(u_j - v_j)$ we consequently obtain that it converges in $\mathrm{L}^\infty(\mathbb{T}_n;W)$ to some element $w \in \mathrm{C}^{0,\alpha}(\mathbb{T}_n;W)$ for some $\alpha < 1-\frac{n}{p}$, and since $\|u_j - v_j\|_{\mathrm{L}^p} \leqslant C\|\mathcal{A}[D]u_j\|_{\mathrm{W}^{-k,p}}$, we obtain by use of the embeddings $(\mathrm{W}^{k-1,1}(\mathbb{T}_n;W))^* \hookrightarrow\hookrightarrow \mathrm{W}^{-k,p}(\mathbb{T}_n;W)$ that $w = 0$. The proof is complete. □

3.7. $\mathcal{A}$–Quasiconvexity

We now wish to generalise the concepts of quasiconvexity and rank–one convexity to the setting of $\mathcal{A}$–free maps. Adopting the terminology introduced by Fonseca & Müller in [109] (also see Dacorogna [66] for an earlier approach), we make the following definition.

Definition 3.34 ($\mathcal{A}$–Quasiconvexity)**.** *Let $f\colon W \to \mathbb{R}$ be a Borel measurable function. The* $\mathcal{A}$–quasiconvexification *(or $\mathcal{A}$–quasiconvex envelope) of f at $A \in W$ is given by*

$$(\widetilde{Q}_{\mathcal{A}}f)(A) := \inf\left\{\int_{\mathbb{T}_n} f(A+\varphi(x))\,\mathrm{d}x\colon\;\varphi \in \ker(\mathcal{A})\cap \mathrm{C}^\infty(\mathbb{T}_n;W)\text{ and }(\varphi)_Q = 0\right\}.$$

We moreover say that f is $\mathcal{A}$–quasiconvex *if and only if it coincides with its $\mathcal{A}$–quasiconvex envelope, i.e., $f = \widetilde{Q}_{\mathcal{A}}f$.*

In this situation, Theorem 3.35 below roughly asserts that $\mathcal{A}$–quasiconvexity implies convexity with respect to directions in the characteristic cone $\Lambda_{\mathcal{A}}$; also note that we allow for higher order differential operators $\mathcal{A}$, which is not the case in [109]. Let us now assume that $\mathcal{A}$ admits a potential $\mathbb{A}$. As shown in Lemma 3.14 (b), if a potential exists, then the characteristic cone can be written as some sort of rank one–cone, and thus in this situation, the theorem can be read as a generalisation of the fact that quasiconvexity implies rank–one convexity. To embed two perhaps more familiar settings into this framework, we wish to continue Examples 3.5 and 3.6:

(a) If $\mathcal{A} = \mathrm{curl}$, then $\mathcal{A}(v) = 0$ and $v \in \mathrm{C}^1(\mathbb{R}^n;\mathbb{R}^{N\times n})$ hold if and only if $v = \nabla w$ for some $w \in \mathrm{C}^2(\mathbb{R}^n;\mathbb{R}^N)$. Here, $\mathcal{A}$–quasiconvexity gives back the usual quasiconvexity.

(b) If $\mathcal{A} = \mathrm{curl}\circ\mathrm{curl}$, then $\mathcal{A}(v) = 0$ and $v \in \mathrm{C}^1(\mathbb{R}^n;\mathbb{R}^{n\times n}_{\mathrm{sym}})$ hold if and only if $v = \boldsymbol{\varepsilon}(w)$ for some $w \in \mathrm{C}^2(\mathbb{R}^n;\mathbb{R}^{n\times n})$. Then the $\mathcal{A}$–quasiconvexity reduces to the so–called *symmetric quasiconvexity* as introduced in [30]; also see Example 3.38 below.

Theorem 3.35. *Let $\mathcal{S}$ be an open and $\Lambda_{\mathcal{A}}$–convex subset of W and let the extended real–valued function $f\colon \mathcal{S} \to [-\infty,\infty)$ be locally bounded above and Borel measurable. Then the relaxation of f defined as*

$$\tilde{f}(\eta) := \inf\left\{\int_{\mathbb{T}_n} f(\eta + w(x))\,\mathrm{d}x\colon\;\begin{array}{c}\varphi \in \mathrm{C}^\infty_{\mathrm{per}}(Q;W)\cap\ker(\mathcal{A}),\;(w)_{\mathbb{T}^n} = 0,\\ \eta + w(x) \in \mathcal{S}\text{ for all }x\end{array}\right\} \tag{3.33}$$

for $\eta \in \mathcal{S}$ is a $\Lambda_{\mathcal{A}}$–convex function $\tilde{f}\colon \mathcal{S} \to [-\infty,\infty)$.

Before we come to the proof of the preceding theorem, we follow [109] and first give a remark.

Remark 3.36 ([109, Rem. 3.3(ii)]). *The class of functions over which the infimum is taken in (3.33) equivalently can be taken to be the subset $\mathcal{U}$ of $\mathrm{L}^\infty_{\mathrm{per}}(Q;W)\cap\ker(\mathcal{A})$ whose elements have zero mean value. In fact, given any element φ in this class, we pick a mollification $\varphi_\varepsilon := \rho_\varepsilon * \varphi$, where ρ is a radially symmetric standard mollifier. Then the map $v_\varepsilon := \varphi_\varepsilon - \fint_{\mathbb{T}_n}\varphi_\varepsilon\,\mathrm{d}x$ belongs to $\mathrm{C}^\infty_{\mathrm{per}}(Q;W)\cap\ker(\mathcal{A})$ and has zero mean value. By the assumptions on f and Fatou's lemma, we then obtain for any admissible η*

$$\int_{\mathbb{T}_n} f(\eta+\varphi)\,\mathrm{d}x \leqslant \liminf_{\varepsilon\searrow 0}\int_{\mathbb{T}_n} f(\eta+v_\varepsilon)\,\mathrm{d}x.$$

Clearly, the relaxation with respect to maps from $\mathcal{U}$ is smaller than that from (3.33), and by the foregoing remarks, it must be equal to that from (3.33).

Proof of Theorem 3.35. We follow the proof of Proposition 3.4 in [109]. Assuming that $\tilde{f}$ is lower semicontinuous, let $\lambda\in[0,1]$ and $\eta_1,\eta_2\in\mathcal{S}$ with $\eta_1-\eta_2\in\Lambda_{\mathcal{A}}$ so that by assumption $[\eta_1,\eta_2]\subset\mathcal{S}$. Hence we have to show

$$\widetilde{f}(\lambda\eta_1+(1-\lambda)\eta_2)\leqslant\lambda\tilde{f}(\eta_1)+(1-\lambda)\tilde{f}(\eta_2). \tag{3.34}$$

For $\varepsilon>0$ sufficiently small, we define $\chi_\varepsilon\colon\mathbb{R}\to\mathbb{R}$ as the 1–periodic extension of the function $\widetilde{\chi}_\varepsilon\colon(0,1)\to\mathbb{R}$ given by

$$\widetilde{\chi}_\varepsilon(t) := \begin{cases} -(1-\lambda) & \text{if } \varepsilon<t<\lambda-\varepsilon,\\ \lambda & \text{if } \lambda+\varepsilon<t<1-\varepsilon,\\ 0 & \text{otherwise.}\end{cases}$$

In a second step, we define $u_j\colon Q\to W$ by $u_j(x):=(\eta_2-\eta_1)\chi_\varepsilon(jx\cdot\xi)$ with $\varepsilon>0$ to be determined below; clearly, $u_j\overset{*}{\rightharpoonup}0$ in $\mathrm{L}^\infty(Q;W)$ as $j\to\infty$. Given $\delta>0$, let $\varphi\in\mathrm{C}^\infty_c(Q;[0,1])$ with $\mathscr{L}^n(\{\varphi=1\})=1-\delta$ and note that for any $g\in\mathrm{W}^{k-1,1}(Q;W)$ there holds (with $\langle\cdot,\cdot\rangle$ now denoting the *duality pairing*)

$$\begin{aligned}\big\langle\mathcal{A}[D]\big(\varphi u_j\big),g\big\rangle &= \sum_{|\alpha|=k}\sum_{\beta+\gamma=\alpha}\binom{\alpha}{\beta}\big\langle\partial^\gamma\varphi\partial^\beta\mathcal{A}_\alpha u_j,g\big\rangle\\ &= \Big\langle\varphi\sum_{|\alpha|=k}\mathcal{A}_\alpha\partial^\alpha u_j,g\Big\rangle+\sum_{|\alpha|=k}\sum_{\substack{\beta+\gamma=\alpha\\|\gamma|\geqslant1}}\binom{\alpha}{\beta}\big\langle\partial^\gamma\varphi\partial^\beta\mathcal{A}_\alpha u_j,g\big\rangle\\ &= \sum_{|\alpha|=k}\sum_{\substack{\beta+\gamma=\alpha\\|\gamma|\geqslant1}}\binom{\alpha}{\beta}(-1)^{|\beta|}\big\langle u_j,\mathcal{A}_\alpha^*\big(\partial^\beta\big((\partial^\gamma\varphi)g\big)\big)\big\rangle\to0,\qquad j\to\infty\end{aligned}$$

as for all $\beta,\gamma\in\mathbb{N}_0^n$ with $|\beta|\geqslant1$ and $\beta+\gamma=\alpha$ there holds $\mathcal{A}_\alpha^*\big(\partial^\beta\big((\partial^\gamma\varphi)g\big)\big)\in\mathrm{L}^1(Q;W)$. In conclusion, we have $\mathcal{A}[D]\big(\varphi u_j\big)\overset{*}{\rightharpoonup}0$ in $(\mathrm{W}^{k-1,1}(Q;X))^*$ as $j\to\infty$ and thus are in position to apply Lemma 3.33. This yields a sequence $(\mathbf{u}_j)$ such that $(\mathbf{u}_j)_Q=0$ for all $j\in\mathbb{N}$ and $\|\varphi u_j-\mathbf{u}_j\|_{\mathrm{L}^\infty(Q;W)}\to0$ as $j\to\infty$. We conclude by the definition of the relaxation (3.33)

$$
\begin{aligned}
\widetilde{f}(\lambda\eta_1+(1-\lambda)\eta_2) &\leqslant \liminf_{j\to\infty}\int_{\mathbb{T}_n} f(\lambda\eta_1+(1-\lambda)\eta_2+\mathbf{u}_j)\,\mathrm{d}x \\
&\leqslant \liminf_{j\to\infty}\int_{\mathbb{T}_n} f(\lambda\eta_1+(1-\lambda)\eta_2+\varphi u_j+(\mathbf{u}_j-\varphi u_j))\,\mathrm{d}x \\
&\leqslant (\liminf_{j\to\infty}\int_{\mathbb{T}_n} f(\lambda\eta_1+(1-\lambda)\eta_2+u_j)\,\mathrm{d}x+M\delta) \\
&+\liminf_{j\to\infty}\int_{\mathbb{T}_n} f(\lambda\eta_1+(1-\lambda)\eta_2+\varphi u_j+(\mathbf{u}_j-\varphi u_j)) \\
&\qquad\qquad -f(\lambda\eta_1+(1-\lambda)\eta_2+\varphi u_j)\,\mathrm{d}x =: \mathbf{I}+\mathbf{II},
\end{aligned}
$$

with an obvious definition for $\mathbf{I}$ and $\mathbf{II}$. Here, M is a finite constant depending on the values of $|f|$ on a sufficiently large ball. First, we have by the definition of u_j

$$
\begin{aligned}
\mathbf{I} &\leqslant \lambda f(\lambda\eta_1+(1-\lambda)\eta_2-(1-\lambda)(\eta_2-\eta_1)) \\
&\quad+(1-\lambda)f(\lambda\eta_1+(1-\lambda)\eta_2+\lambda(\eta_2-\eta_1)) = \lambda f(\eta_1)+(1-\lambda)f(\eta_2)+M\delta.
\end{aligned}
$$

Second, letting $g_j := f(\lambda\eta_1+(1-\lambda)\eta_2+u_j+(\mathbf{u}_j-u_j))-f(\lambda\eta_1+(1-\lambda)\eta_2+u_j)$, we see by uniform convergence of $(\mathbf{u}_j-\varphi u_j)$ to zero, local boundedness of f and Lebesgue's dominated convergence theorem that $\mathbf{II}\searrow 0$ as $j\to\infty$. Finally, we send $\delta\searrow 0$. By Remark 3.36, the proof is complete. □

If $\mathcal{A}[D]$ admits a potential $\mathbb{A}[D]$, then the relaxation as discussed above can be characterised by $\mathbb{A}[D]$. This is standard and appears in the same vein as when aims at characterising the usual quasiconvexity in terms of curl–free maps on the torus.

Lemma 3.37 (Coincidence). *Let $\mathbb{A}[D]=\sum_{|\alpha|=m}\mathbb{A}_\alpha\partial^\alpha$ be a m–th order, linear, homogeneous constant coefficient differential operator on $\mathbb{R}^n$ from V to W (where both $\dim(V),\dim(W)<\infty$) and let $\mathcal{A}[D]$ be an annihilator for $\mathbb{A}[D]$, so that in particular $\mathcal{A}[D]$ is a l–th order, linear, homogeneous constant coefficient differential operator on $\mathbb{R}^n$ from W to Y (where $\dim(Y)<\infty$). Let $f\colon W\to\mathbb{R}$ be Borel measurable and locally bounded. Then for any $A\in W$,*

$$
\widetilde{f}(A)=\inf\left\{\int_{(0,1)^n} f(A+\mathbb{A}[D]\varphi)\,\mathrm{d}x\colon\ \varphi\in\mathrm{C}_c^\infty((0,1)^n;V)\right\}. \tag{3.35}
$$

We omit the proof. As an important example, we note the following.

Example 3.38 (Symmetric–Quasiconvexity). *As introduced in [30], a function $f\in\mathrm{C}(\mathbb{R}^{n\times n}_{\mathrm{sym}})$ is a called* symmetric–quasiconvex *provided for all $A\in\mathbb{R}^{n\times n}_{\mathrm{sym}}$ there holds $f(A)\leqslant\int_Q f(A+\boldsymbol{\varepsilon}(\varphi))\,\mathrm{d}x$ for all $\varphi\in\mathrm{C}_c^\infty(Q;\mathbb{R}^n)$ (here, $Q:=[0,1]^n$). Since Q is simply connected, a suitable version of the classical Poincaré Lemma in conjunction with Example 3.6 yields that $w\in\mathrm{C}_c^\infty(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})$ arises as the symmetric gradient of some $\varphi\in\mathrm{C}_c^\infty(\Omega;\mathbb{R}^n)$ if and only if $\mathrm{curl}\circ\mathrm{curl}(\psi)=0$ in Ω. Hence, Theorem 3.10 and Lemma 3.37 imply for a given $f\in\mathrm{C}(\mathbb{R}^{n\times n}_{\mathrm{sym}})$ that the symmetric quasiconvexification f_{sqc}, i.e., the largest symmtric–quasiconvex function below f, can be computed by (3.33) with $\mathcal{A}=\mathrm{curl}\circ\mathrm{curl}$. Moreover, as can be easily computed, the bilinear pairing $\otimes_{\mathbb{A}}$ for $\mathbb{A}[D]=\boldsymbol{\varepsilon}$ is given by $a\odot b:=\frac{1}{2}(a\otimes b+b\otimes a)$ for $a,b\in\mathbb{R}^n$, where $a\otimes b:=ab^{\mathsf{T}}$ is the usual tensor product. Hence, Theorem 3.10 moreover implies that f_{sqc} is convex with respect to directions in the symmetric–rank–1–cone $\mathbb{R}^n\odot\mathbb{R}^n$.*

3.8. Conclusion and Proof of Theorem 3.2

In this final section, we give the proof of Theorem 3.2. To prove the main auxiliary result, Theorem 3.3, we need another simple ingredient as follows.

Lemma 3.39. *Suppose that $w \in \mathrm{C}^\infty(\mathbb{T}^n; W)$ satisfies $\mathcal{A}w \equiv 0$. Then there holds $w(x) \in \operatorname{span}(\Lambda_{\mathcal{A}})$ for $\mathscr{L}^n$–a.e. $x \in \mathbb{T}^n$.*

Proof. Let $w \in \mathrm{C}^\infty(\mathbb{T}^n; W)$ satisfy $\mathcal{A}w \equiv 0$ so that, taking the Fourier transform, $\mathcal{A}[\xi]\widehat{w}(\xi) = 0$ for every $\xi \in \mathbb{R}^n$ by the Riemann–Lebesgue Lemma. In consequence, $\widehat{w}(\xi) \in \operatorname{span}\big(\bigcup_\lambda \ker(\mathcal{A}[\lambda])\big)$ for every $\xi \in \mathbb{R}^n$ and thus, writing $\widehat{w}(\xi) = \sum_{j=1}^m a_j(\xi)v_j$ (where $v_1, ..., v_m$ is a fixed basis of $\operatorname{span}\big(\bigcup_\lambda \ker(\mathcal{A}[\lambda])\big)$, we find by applying Fourier inversion

$$w(x) = \sum_{\lambda\in\mathbb{Z}^n} \widehat{w}(\lambda)\,\mathrm{e}^{2\pi\,\mathrm{i}\,\lambda\cdot x} = \sum_{j=1}^m \Big(\sum_{\lambda\in\mathbb{Z}^n} a_j(\lambda)\,\mathrm{e}^{2\pi\,\mathrm{i}\,x\cdot\lambda} \Big) v_j \qquad \text{for all } x \in \mathbb{T}_n,$$

from where the claim is immediate. □

We now give the

Proof of Theorem 3.3. Arguing as in the proof of [151, Thm. 5.1], we may assume without loss of generality that f is autonomous, i.e., has no x–dependence.

Necessity Part. We show the contrapositive statement and hereafter assume that $\Lambda_{\mathcal{A}}$ is not spanning. Consequently, we may write

$$W = \operatorname{span}\Big(\bigcup_{\xi\neq 0}\ker(\mathcal{A}[\xi])\Big) \oplus \Big(\operatorname{span}\Big(\bigcup_{\xi\neq 0}\ker(\mathcal{A}[\xi])\Big)\Big)^\perp =: W_1 \oplus W_1^\perp$$

where $W_1^\perp \neq \{0\}$. Note that we may assume without loss of generality that $W_1 \neq \{0\}$ (cp. Remark 3.4). We define an integrand $f\colon W \to \mathbb{R}$ by

$$f(w) := |\Pi_{W_1}w| - |\Pi_{W_1^\perp}w|, \qquad x \in \mathbb{R}^n,\ w \in W,$$

where $\Pi_{W_1}w$ and $\Pi_{W_1^\perp}$ are the orthogonal projections onto W_1 or $W_1^\perp$, respectively. Since these projections are continuous it follows that f is continuous and so is Carathéodory. Moreover, it is clear that $|f(w)| \leqslant |\Pi_{W_1}w| + |\Pi_{W_1^\perp}w| \leqslant C|w|$, where $C > 0$ is a constant which is independent of w. We may thus choose $a(x) := C$ for all $x \in \mathbb{R}^n$ and hence $a \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^n)$. Since f is clearly positively 1–homogeneous, it thus satisfies the conditions of Theorem 3.3. Now let $w \in \mathrm{C}^\infty(\mathbb{T}^n; W)$ with $\mathcal{A}w = 0$ and $(w)_Q = 0$ be arbitrary. Hence, by Lemma 3.39, $w(x) \in \operatorname{span}(\Lambda_{\mathcal{A}}) = W_1$ for $\mathscr{L}^n$–a.e. $x \in \mathbb{R}^n$. Consequently, condition (i) of Theorem 3.3 is satisfied whereas (ii) is not since $f(w) < 0$ for any $w \in W_1^\perp \setminus \{0\}$. This establishes the necessity part.

Sufficiency Part. Now suppose that $\Lambda_{\mathcal{A}}$ is spanning. Assuming (a), we obtain for the relaxation $\tilde{f}$ given by (3.33) that $\tilde{f}(0) \geqslant 0$. Now, we may use Theorem 3.35 to deduce that $\tilde{f}$ is a $\Lambda_{\mathcal{A}}$–convex function which is real–valued on the entire set $\mathcal{S} = W$, and by homogeneity of $\mathcal{A}$ in conjunction with that of f, we easily see that $\tilde{f}$ satisfies $\tilde{f}(t\xi) = |t|\tilde{f}(\xi)$ for all $t \in \mathbb{R}$ and $\xi \in \mathcal{S}$. Since $\Lambda_{\mathcal{A}}$ is spanning, Lemma 3.30 implies that $\tilde{f}$ is convex at 0 together with the existence of some $\zeta \in W$ such that $\tilde{f}(\xi) \geqslant \langle\zeta,\xi\rangle$ for all $\xi \in W$. Consequently, letting $\eta \in W$ be arbitrary and inserting $\xi = \pm\eta$, we find by positive homogeneity of $\tilde{f}$ both $-\langle\zeta,\eta\rangle = \langle\zeta,-\eta\rangle \leqslant \tilde{f}(-\eta) = \tilde{f}(\eta)$ and $\langle\zeta,\eta\rangle \leqslant f(\eta)$ so that $|\langle\zeta,\eta\rangle| = \max\{-\langle\zeta,\eta\rangle, \langle\zeta,\eta\rangle\} \leqslant \tilde{f}(\eta)$ and so, by $f \geqslant \tilde{f}$, $\tilde{f}(\eta) \geqslant 0$ for all $\eta \in W$. Hence (b) follows. Since (b) trivially implies (a), the proof is complete. □

Having established Theorem 3.3, we now turn to the

Proof of Theorem 3.2. We reduce to the setting as dealt with in [151]. Let $k \in \mathbb{N}$ be given and denote for two finite–dimensional spaces X and Y by $\odot^k(X,Y)$ the Y–valued symmetric k–linear maps. As explained in [151, Sec. 4], the cone $\mathcal{D}_s(n,k;Y) := \{b\otimes a\otimes ...\otimes a = b\otimes a^{\otimes k}\colon\ a \in \mathbb{R}^n,\ b \in Y\}$ spans $\odot^k(X,Y)$ and moreover, if $\mathcal{A}$ is a differential operator annihilating precisely the full k–th order derivative (applied to functions $\mathbb{R}^n \to \mathbb{R}^N$, which we think of as a symmetric k–linear map), then the characteristic cone of $\mathcal{A}$ is exactly $\mathcal{D}_s(n,k;\mathbb{R}^N)$. This, in turn, spans $\odot^k(\mathbb{R}^n,\mathbb{R}^N)$. Localising or extending periodically, it makes not difference here whether we consider $\odot^k(\mathbb{R}^n,\mathbb{R}^N)$ or $\odot^k(\mathbb{T}_n,\mathbb{R}^N)$. By the positivity result of Theorem 3.3, we then deduce that if $F\colon \mathbb{R}^n \odot^k (\mathbb{R}^n;\mathbb{R}^N) \to \mathbb{R}$ is a Carathéodory integrand which satisfies $F(x,t\xi) = |t|F(x,\xi)$ and $|F(x,\xi)| \leqslant a(x)|\xi|$ for $\mathscr{L}^n$–a.e. $x \in \mathbb{R}^n$, all $t \in \mathbb{R}$ and all $\xi \in \odot^k(\mathbb{R}^n;\mathbb{R}^N)$ with a given $\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^n)$–map a, then we have $F(x,\xi) \geqslant 0$ for all $\xi \in \odot^k(\mathbb{R}^n;\mathbb{R}^N)$ and $\mathscr{L}^n$–a.e. $x \in \mathbb{R}^n$ if and only if

$$\int_{\mathbb{R}^n} F(x,\mathrm{D}^k\varphi(x))\,\mathrm{d}x \geqslant 0 \qquad \text{holds for all } \varphi \in \mathrm{C}^\infty_c(\mathbb{R}^n;\mathbb{R}^N).$$

This is an exact repetition of [151, Thm. 5.1]. The implication '(b)⇒(a)' is clear. To prove the other direction, we follow [151] and define $A_1(x)\colon \odot^k (\mathbb{R}^n;V) \to W$ and $A_2(x)\colon \odot^k (\mathbb{R}^n;V) \to X$ by their action on $\varphi \in \mathrm{C}^\infty_c(\mathbb{R}^n;V)$ through

$$A_1(x)\,\mathrm{D}^k\varphi(x) := \sum_{|\alpha|=k} \mathbb{A}^1_\alpha(x)\partial^\alpha\varphi(x), \quad A_2(x)\,\mathrm{D}^k\varphi(x) := \sum_{|\alpha|=k} \mathbb{A}^2_\alpha(x)\partial^\alpha\varphi(x).$$

We then define the integrand $f\colon \mathbb{R}^n \times \odot^k(\mathbb{R}^n;V) \to \mathbb{R}$ by $f(x,\eta) := c|A_1(x)\eta|_W - |A_2(x)\eta|_X$ with $c > 0$ as in (a). Since $\mathbb{A}^1_\alpha, \mathbb{A}^2_\alpha$ are locally integrable mappings, it easily follows that f is Carathéodory, and moreover satisfies (3.4). We then deduce from above that $f(x,\eta) \geqslant 0$ for all $(x,\eta) \in \mathbb{R}^n \times \odot^k(\mathbb{R}^n;V)$ and so, for $\mathscr{L}^n$–a.e. $x \in \mathbb{R}^n$, $\ker(A_1(x)) \subset \ker(A_2(x))$. Then we fix $x \in \mathbb{R}^n$ such that the partial map $f(x,\cdot)$ is non–negative and let $C(x)\colon W \to X$ be given by

$$C(x) := A_2(x) \circ \big(A_1(x)|_{\ker(A_1(x))^\perp}\big)^{-1} \circ \Pi_{A_1(x)(\odot^k(\mathbb{R}^n;V))},$$

where $\Pi_{A_1(x)(\odot^k(\mathbb{R}^n;V))}$ is the projection onto $A_1(x)(\odot^k(\mathbb{R}^n;V))$. In conclusion, $C\colon \mathbb{R}^n \to \mathscr{L}(W;X)$ is defined $\mathscr{L}^n$–a.e., measurable and further satisfies $C(x)A_1(x) = A_2(x)$ for $\mathscr{L}^n$–a.e. $x \in \mathbb{R}^n$. From this equality and the non–negativity of f it follows that $\|C\|_{\mathrm{L}^\infty(\mathbb{R}^n;\mathscr{L}(W;X))} \leqslant c$, and the proof is complete. □

CHAPTER 4

The Dirichlet Problem on BV and BD

This chapter is devoted to setting up the Dirichlet problem in the space BD of functions of bounded deformation and to establish the existence of generalised minimisers as firstly done by GIAQUINTA, MODICA & SOUČEK [126] in the BV–case. We recall the following from the introduction: Even though a suitably coercive functional of linear growth has a clear meaning when being applied to the (symmetric) gradients of the single members of a minimising sequence, the weak*–limit of the latter need not be a $\mathrm{L}^1_{\mathrm{loc}}$–mapping but rather a matrix–valued Radon measure. To make sense of a functional being applied to (symmetric) gradients which are $\mathbb{R}^{n\times n}$– or $\mathbb{R}^{n\times n}_{\mathrm{sym}}$–valued measures, it needs to be relaxed to BV or BD. Here we have to simultaneously keep track of the Dirichlet boundary data when aiming at a solution of the boundary value problem of our interest. In this regard, we shall give an overview of BV– and BD–spaces and report on the lower semicontinuity of the relaxed functionals (which, in the convex context, is known as RESHETHNYAK (lower semi–)continuity theory), finally leading to a satisfactory existence theory.

The contents of the present chapter focus on the classical convex situation, and the reader is referred to Chapter 8 for a more detailled discussion of the quasiconvex case.

Structure of the chapter. In Sections 4.1 and 4.2 we introduce the setting and the requisite function spaces BV and BD. We focus on trace, extension and smooth approximation results to finally deal with enhanced embeddings and fine properties of those functions. In Section 4.3 we introduce convex functionals of measures and explain the existence theory for convex linear growth functionals. Finally, in Section 4.4 we discuss the classical SANTI and FINN counterexamples to the attainment of the correct boundary values.

4.1. The Setting

Throughout, let Ω be an open and bounded Lipschitz subset of $\mathbb{R}^n$. As in the introduction, letting either $X = \mathbb{R}^{N\times n}$ or $X = \mathbb{R}^{n\times n}_{\mathrm{sym}}$ (which will be clear from the context), we assume that $f \in \mathrm{C}(X)$ is a convex intgrand for which there exist $0 < c_1 \leqslant c_2 < \infty$ and $\varepsilon > 0$ such that

$$c_1|z| - \varepsilon \leqslant f(z) \leqslant c_2(1 + |z|) \qquad \text{for all } z \in X. \tag{LG}$$

The convexity hypothesis can be relaxed in an essential way to weaker semiconvexity notions, a point which shall be addressed below in Chapter 8 in more detail. In what follows, we fix a Dirichlet datum[1] $u_0 \in \mathrm{W}^{1,1}(\Omega;\mathbb{R}^N)$ and shall consider the following

[1] The regularity assumption $u_0 \in \mathrm{W}^{1,1}(\Omega;\mathbb{R}^n)$ might seem artificial, however, as we shall discuss in Chapter 4.2.2, the trace space of both $\mathrm{W}^{1,1}$ and LD is $\mathrm{L}^1(\partial\Omega;\mathbb{R}^n)$. In conclusion, it does not matter whether we assume $\mathrm{W}^{1,1}$ or LD for the boundary datum.

problems:

$$\text{to minimise } \mathfrak{F}_1[v] := \int_\Omega f(\mathrm{D}v)\,\mathrm{d}x \ \text{ over } \mathcal{D}_{u_0} := u_0 + \mathrm{W}_0^{1,1}(\Omega;\mathbb{R}^N) \tag{P}$$

$$\text{to minimise } \mathfrak{F}_2[v] := \int_\Omega f(\boldsymbol{\varepsilon}(v))\,\mathrm{d}x \ \text{ over } \mathcal{D}_{u_0} := u_0 + \mathrm{LD}_0(\Omega). \tag{P$_\varepsilon$}$$

Here and in all of what follows, once it is clear whether we deal with problems of the type (P) or (P_ε), respectively, we shall write $\mathfrak{F} = \mathfrak{F}_1$ or $\mathfrak{F} = \mathfrak{F}_2$ for simplicity. Similarly, we tacitly assume $N = n$ when dealing with the symmetric gradient case.

As we indicated in the introduction, to treat minimisation problems of the type (P) or (P_ε), we need an extension of the usual setup of weakly differentiable (Sobolev) functions and consequently keep track of the way in which elements of such suitable extensions attain boundary values, too. We illustrate this by the following one–dimensional example for problems of the type (P); note that by $n = N = 1$ in this example, it also applies to those of the form (P_ε).

Example 4.1 (Strang & Temam [223]). *Consider the minimisation of the functional* $\mathfrak{F}[v] := \int_0^1 |v'|\,\mathrm{d}x$ *subject to* $\int_0^1 v\,\mathrm{d}x = 1$ *and* $v(0) = v(1) = 0$ *over* $\mathrm{W}_0^{1,1}((0,1))$. *The minimum value of the functional then is* 2, *and attained by the function* $u \equiv 1$ *on* $(0,1)$. *This function in turn violates the boundary conditions, however, this is not so if we understand them as* $u \equiv 0$ *on the exterior of* $\Omega := (0,1)$. *Doing so, we allow for jumps along* $\partial\Omega$ *so that the corresponding minimiser cannot belong to* $\mathrm{W}_0^{1,1}(\mathbb{R})$ *when being extended to* $\mathbb{R}$ *by zero.*

Based on the preceding example, we give an overview of BV–and BD–spaces with particular emphasis on embeddings and fine properties. Section 4.2 is devoted to this objective, whereas Section 4.3 is concerned with the relaxation of functionals given by (P) or (P_ε) to these spaces. This shall lead to the notion of *generalised minima* in a natural way, whose existence is eventually established in Section 4.3.2.

4.2. Functions of Bounded Variation and Bounded Deformation

4.2.1. Definition and elementary properties. We begin with the definition of the spaces BV and BD, the latter of which has been introduced and firstly studied in [64, 227] whereas the former space is much more classical and can be traced back to the works of Jordan, Tonelli and others; see [16, 61] for a historical overview.

Definition 4.2. *Let* $\Omega \subset \mathbb{R}^n$ *be open. We say that a measurable map*

(a) $u\colon \Omega \to \mathbb{R}^N$ *is of* bounded variation *if and only if* $u \in \mathrm{L}^1(\Omega;\mathbb{R}^N)$ *and* $\mathrm{D}u \in \mathcal{M}_{<\infty}(\Omega;\mathbb{R}^{N\times n})$. *In this situation,* $|\mathrm{D}u|(\Omega)$ *is called the* total variation *of* u.

(b) $u\colon \Omega \to \mathbb{R}^n$ *is of* bounded deformation *if and only if* $u \in \mathrm{L}^1(\Omega;\mathbb{R}^N)$ *and* $\boldsymbol{\varepsilon}(u) \in \mathcal{M}_{<\infty}(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})$. *In this situation,* $|\boldsymbol{\varepsilon}(u)|(\Omega)$ *is called the* total deformation *of* u.

We lastly put

$$\begin{aligned}\mathrm{BV}(\Omega;\mathbb{R}^N) &:= \big\{u \in \mathrm{L}^1(\Omega;\mathbb{R}^N)\colon\ \mathrm{D}u \in \mathcal{M}_{<\infty}(\Omega;\mathbb{R}^{N\times n})\big\},\\ \mathrm{BD}(\Omega) &:= \big\{u \in \mathrm{L}^1(\Omega;\mathbb{R}^n)\colon\ \boldsymbol{\varepsilon}(u) \in \mathcal{M}_{<\infty}(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})\big\}.\end{aligned} \tag{4.1}$$

Here, $\mathrm{D}u \in \mathcal{M}_{<\infty}(\Omega;\mathbb{R}^{N\times n})$ or $\boldsymbol{\varepsilon}(u) \in \mathcal{M}_{<\infty}(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})$ need to be understood in the sense that the distributional gradient or symmetric gradient, respectively, can be represented by an $\mathbb{R}^{N\times n}$– or $\mathbb{R}^{n\times n}_{\mathrm{sym}}$–valued Radon measure of finite (that is, bounded) total variation on Ω. Throughout the remainder of this chapter, we shall exclusively stick to the slightly more general BD–case and refer the reader to [16, 97] for a development of the corresponding notions in the BV–context. Let us note that these analogous notions in the BV–case for maps $u\colon \Omega \to \mathbb{R}^N$ can be obtained by sytematically replacing $\mathbb{R}^{n\times n}_{\mathrm{sym}}$ by $\mathbb{R}^{N\times n}$ and $\boldsymbol{\varepsilon}$ by D below.

Since the finite Radon measure arise as the dual of the continuous functions vanishing at the boundary as a consequence of the Riesz representation theorem, Theorem 2.1, it is easy to establish that

$$|\,\mathrm{D}u|(\Omega) = \sup\left\{\int_\Omega \langle u, \mathrm{div}(\varphi)\rangle\,\mathrm{d}x\colon\;\; \varphi\in \mathrm{C}^1_c(\Omega;\mathbb{R}^{N\times n}),\;\; |\varphi|\leqslant 1\right\}$$

for $u\in \mathrm{BV}(\Omega;\mathbb{R}^N)$ and

$$|\boldsymbol{\varepsilon}(u)|(\Omega) = \sup\left\{\int_\Omega \langle u, \mathrm{div}(\varphi)\rangle\,\mathrm{d}x\colon\;\; \varphi\in \mathrm{C}^1_c(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}}),\;\; |\varphi|\leqslant 1\right\},$$

for $u\in\mathrm{BD}(\Omega)$. To emphasize that $\boldsymbol{\varepsilon}(u)$ is a measure for $u\in\mathrm{BD}$, we will also use the notation $\mathrm{E}u = \boldsymbol{\varepsilon}(u)$ in this case, but no ambiguities will arise from this. We shall now introduce and study convergences on these function spaces and, in particular, end up with a crucial compactness result. Let $u, u_1, u_2, ... \in \mathrm{BD}(\Omega)$, where $\Omega\subset\mathbb{R}^n$ is open and Lipschitz. We say that

(i) (u_k) converges to u *in the weak*–sense* (in symbols $u_k \overset{*}{\rightharpoonup} u$) as $k\to\infty$ if and only if $u_k\to u$ strongly in $\mathrm{L}^1(\Omega;\mathbb{R}^n)$ and $\boldsymbol{\varepsilon}(u_k)\overset{*}{\rightharpoonup}\boldsymbol{\varepsilon}(u)$ in the sense of weak*–convergence of $\mathbb{R}^{n\times n}_{\mathrm{sym}}$–valued finite Radon measures on Ω.

(ii) (u_k) converges to u *in the strict sense* (in symbols $u_k \overset{s}{\rightharpoonup} u$) as $k\to\infty$ if and only if $u_k\to u$ strongly in $\mathrm{L}^1(\Omega;\mathbb{R}^n)$ and $|\boldsymbol{\varepsilon}(u_k)|(\Omega)\to|\boldsymbol{\varepsilon}(u)|(\Omega)$ as $k\to\infty$.

(iii) (u_k) converges to u *in the area–strict sense* (in symbols $u_k \overset{\langle\cdot\rangle}{\to} u$) as $k\to\infty$ if and only if $u_k\to u$ strongly in $\mathrm{L}^1(\Omega;\mathbb{R}^n)$ and

$$\int_\Omega\sqrt{1+\left|\frac{\mathrm{dE}u_k}{\mathrm{d}\mathscr{L}^n}\right|^2}\,\mathrm{d}x + |\,\mathrm{E}^s u_k|(\Omega) \to \int_\Omega\sqrt{1+\left|\frac{\mathrm{dE}u}{\mathrm{d}\mathscr{L}^n}\right|^2}\,\mathrm{d}x + |\,\mathrm{E}^s u|(\Omega)$$

as $k\to\infty$, where $\mathrm{E}v := \boldsymbol{\varepsilon}(v) = \frac{\mathrm{dE}u_k}{\mathrm{d}\mathscr{L}^n} + \mathrm{E}^s u_k$ are the corresponding Radon–Nikodym decompositions of the single u_k's; adopting the general theory of convex functionals of measures as exposed in Section 4.3 below, this amounts to $\sqrt{1+|\boldsymbol{\varepsilon}(u_k)|^2}(\Omega)\to\sqrt{1+|\boldsymbol{\varepsilon}(u)|^2}(\Omega)$ as $k\to\infty$.

We shall sometimes also speak of *symmetric* weak*–, strict– and area–strict convergence when there might be ambiguities with these notions for the BV–case. Lastly, we say that (u_k) *norm–converges* to u as $k\to\infty$ if and only if $\|u-u_k\|_{\mathrm{BD}(\Omega)}\to 0$ as $k\to\infty$, where $\|v\|_{\mathrm{BD}(\Omega)} := \|v\|_{\mathrm{L}^1(\Omega;\mathbb{R}^n)} + |\boldsymbol{\varepsilon}(v)|(\Omega)$ is the BD*–norm*. However, due its bad compactness properties, the BD–norm is of little use for our purposes and shall only be touched throughout the document. These notions of convergence introduced above are linked by the following diagram:

norm convergence $\Rightarrow$ area–strict convergence $\Rightarrow$ strict convergence $\Rightarrow$ weak*–convergence.

None of these implications can be read in the other direction, a fact which we discuss with some examples. Also, note that if $N=n=1$, then BV = BD, and so the upcoming examples also apply to the BD–case.

Example 4.3 (Norm versus Area–Strict)**.** *Put* $\Omega := (-1,1)$*. Given* $k\in\mathbb{N}$*, let* $u_k\colon\Omega\to(0,1)$ *be the function which equals* -1 *on* $(-1,-\frac{1}{2k})$*, is given by* $u_k(x) := 2kx$ *on* $(-\frac{1}{2k},\frac{1}{2k})$ *and equals* 1 *on* $(\frac{1}{2k},1)$*. Then we have* $\langle u_k'\rangle(\Omega) = \frac{1}{k}\sqrt{1+(2k)^2}\to 2$*,* $u_k\to\mathrm{sgn}$ *strongly* *in* $\mathrm{L}^1(\Omega)$ *and* $u_k'\mathscr{L}^1\llcorner\Omega \overset{*}{\rightharpoonup} -2\delta_0$ *as* $k\to\infty$. *In conclusion* $u_k \to u$ $\langle\cdot\rangle$*–strictly in* $\mathrm{BV}(\Omega)$, *but* $u_k \not\to \mathrm{sgn}$ *in the* BV*–norm. Indeed, assume towards a contradiction that* $u_k \to \mathrm{sgn}$ *with respect to norm convergence in* BV. *Then* $(u_k)\subset \mathrm{W}^{1,1}(\Omega)$ *is a* BV*–norm Cauchy sequence and since the* $\mathrm{W}^{1,1}$*– and* BV*–norms coincide on* $\mathrm{W}^{1,1}(\Omega)$, *we conclude that* (u_k) *is* $\mathrm{W}^{1,1}$*–Cauchy and hence converges to some* $u\in\mathrm{W}^{1,1}(\Omega)$. *But since norm convergence implies area–strict convergence, we must have* $u=\mathrm{sgn}\notin \mathrm{W}^{1,1}(\Omega)$, *a contradiction.*

Example 4.4 (Area–Strict versus Strict, [203, Ex. 14]). *Put* $\Omega := (0,1)$ *and let, for* $k\in\mathbb{N}$, $u_k(x) := x + (2\pi k)^{-1}\sin(2\pi k x)$ *for* $x\in\Omega$. *We note that for all* $k\in\mathbb{N}$ *there holds* $u_k'(x) = 1+\cos(2\pi k x)\geqslant 0$ *and hence it is easy to see that* $u_k \to \mathrm{Id}|_{(0,1)}$ *strongly in* $\mathrm{L}^1(\Omega)$, $u_k'\mathscr{L}^1\llcorner(0,1) \overset{*}{\rightharpoonup} \mathscr{L}^1\llcorner(0,1)$ *in* $\mathcal{M}(\Omega)$ *and* $|u_k'|(\Omega)\to 1 = |\mathrm{Id}'|(\Omega)$ *as* $k\to\infty$. *However, by strict concavity of* $t\mapsto\sqrt{t}$ *on* $\mathbb{R}_{\geqslant 0}$ *(or, put differently, strict convexity of* $t\mapsto\sqrt{1+|t|^2}$ *away from zero), it is easy to see that* (u_k) *cannot converge to* u *in the area–strict sense.*

Example 4.5 (Strict versus Weak*). *Put* $\Omega := (0,1)$ *and define* $u := \mathbb{1}_\Omega$. *We define, for* $k\in\mathbb{N}_{\geqslant 3}$, $u_k\colon\Omega\to[0,1]$ *by*

$$u_k(x) := \begin{cases} kx & \text{for } 0\leqslant x\leqslant \frac{1}{k},\\ 1 & \text{for } \frac{1}{k}\leqslant x\leqslant 1-\frac{1}{k},\\ -kx+k & \text{for } 1-\frac{1}{k}\leqslant x\leqslant 1.\end{cases}$$

Then clearly $u_k\to u$ *strongly in* $\mathrm{L}^1(\Omega)$ *as* $k\to\infty$. *For each* $k\in\mathbb{N}$ *we have* $u_k' = k(\mathbb{1}_{(0,\frac{1}{k})} - \mathbb{1}_{(1-\frac{1}{k},1)})$. *Let* $\varphi\in \mathrm{C}_0(\Omega)$. *We compute*

$$\langle u_k'\mathscr{L}^1\llcorner(0,1),\varphi\rangle_{\mathcal{M}\times\mathrm{C}} = \fint_0^{\frac{1}{k}} \varphi(y)\,\mathrm{d}y - \fint_{1-\frac{1}{k}}^1 \varphi(y)\,\mathrm{d}y \to \varphi(0)-\varphi(1) = 0$$

as $k\to\infty$ *and hence* $u_k \overset{*}{\rightharpoonup} u$ *in* $\mathrm{BV}(\Omega)$ *as* $k\to\infty$. *However, since* $|u_k'|(\Omega)=2$ *for all* $k\in\mathbb{N}$, *we have* $|u_k'|(\Omega) = 2\not\to 0 = |u'|(\Omega)$.

Let us now record two important properties that shall prove useful throughout (see [23] for the proofs):

- (Smooth Approximation). Let $\Omega\subset\mathbb{R}^n$ be open. Then for any $u\in\mathrm{BD}(\Omega)$ there exists a sequence $(u_k)\subset(\mathrm{C}^\infty\cap\mathrm{BD})(\Omega)$ such that $u_k\overset{s}{\to} u$ as $k\to\infty$.
- (Compactness). Let $\Omega\subset\mathbb{R}^n$ be open and bounded with Lipschitz boundary and suppose that $(u_k)\subset\mathrm{BD}(\Omega)$ is uniformly bounded with respect to the BD–norm. Then there exists $u\in\mathrm{BD}(\Omega)$ such that for a non–relabelled subsequence we have $u_k\overset{*}{\rightharpoonup} u$ as $k\to\infty$.

Remark 4.6. We wish to stress that both strict and area–strict convergence are genuinely nonlinear notions of convergence in the sense that if $u_k\to u$ strictly in BD, then $u_k-u\not\to 0$ strictly in BD in general. In fact, let $u\in(\mathrm{BD}\setminus\mathrm{LD})(\Omega)$ and choose, according to the above smooth approximation result, $(u_k)\subset \mathrm{C}^\infty(\Omega;\mathbb{R}^n)\cap\mathrm{LD}(\Omega)$ such that $u_k\to u$ strictly. Since $|\mathrm{E}^s u|(\Omega)>0$, we obtain by $\mathrm{E}^s u_k\equiv 0$ for all $k\in\mathbb{N}$ that

$$|\mathrm{E}(u-u_k)|(\Omega) = |\mathrm{E}^{ac}(u-u_k)|(\Omega) + |\mathrm{E}^s u|(\Omega)\geqslant |\mathrm{E}^s u|(\Omega)>0$$

and so $u-u_k\not\to 0$ strictly in $\mathrm{BD}(\Omega)$. The same proof also works for area–strict convergence, cp. Lemma 4.9 below.

Lastly, let us briefly comment on higher order BV–spaces which shall be encountered in Chapters 6 and 7. These are defined inductively, and for the use in this thesis, we put for $k\in\mathbb{N}$

$$\mathrm{BV}_k(\Omega;\mathbb{R}^N) := \big\{u\in\mathrm{BV}_{k-1}(\Omega;\mathbb{R}^N)\colon\ \forall\alpha\in\mathbb{N}_0^n\colon\ |\alpha|\leqslant k-1 \Rightarrow \partial^\alpha u\in\mathrm{BV}(\Omega;\mathbb{R}^N)\big\}.$$

4.2.2. Traces and Extensions As Ω is assumed to be a bounded Lipschitz domain throughout, the Korn–type inequality of Theorem 3.10 immediately imply that $\mathrm{LD}^p(\Omega)$ as defined in (1.3) equals $\mathrm{W}^{1,p}(\Omega;\mathbb{R}^n)$ provided $1<p<\infty$, and hence has the same trace space as $\mathrm{W}^{1,p}(\Omega;\mathbb{R}^n)$. If $p=1$, then the non–availability of Korn–type inequalities does not allow the conclusion that $\mathrm{LD}(\Omega)(:=\mathrm{LD}^1(\Omega))$ or $\mathrm{BD}(\Omega)$ possess $\mathrm{L}^1(\partial\Omega;\mathbb{R}^n)$ as trace space[2] (which, in turn, is the trace space of both $\mathrm{W}^{1,1}(\Omega;\mathbb{R}^n)$ and $\mathrm{BV}(\Omega;\mathbb{R}^n)$) and so the L^1–trace theory of LD– or BD–maps must be established by different means. The corresponding result, attributable to Strang & Temam [223] (also see the recent work of Babadjian [25]), then reads as follows.

Theorem 4.7 (Strang & Temam [223] + Babadjian [25])**.** *Let $\Omega\subset\mathbb{R}^n$ be an open and bounded Lipschitz set. Then there exists a bounded linear operator* $\mathrm{Tr}\colon \mathrm{BD}(\Omega)\to \mathrm{L}^1(\partial\Omega;\mathbb{R}^n)$ *such that for all $v\in \mathrm{C}^1(\Omega;\mathbb{R}^n)\cap \mathrm{C}(\overline{\Omega};\mathbb{R}^n)$ there holds $\mathrm{Tr}(u)=u|_{\partial\Omega}$ $\mathcal{H}^{n-1}$–a.e. on $\partial\Omega$. This trace operator is even surjective when restricted to* $\mathrm{LD}(\Omega)$.

Moreover,

$$\mathrm{Tr}\colon\ \mathrm{BD}(\Omega)\to \mathrm{L}^1(\partial\Omega;\mathbb{R}^n)\qquad \text{is continuous for the strict topology} \tag{4.2}$$

in the sense that if $u,u_1,u_2,...\in\mathrm{BD}(\Omega)$ are such that $u_k\to u$ strictly in $\mathrm{BD}(\Omega)$, then $\mathrm{Tr}(u_k)\to\mathrm{Tr}(u)$ strongly in $\mathrm{L}^1(\partial\Omega;\mathbb{R}^n)$ as $k\to\infty$. Morever, for all $u\in\mathrm{BD}(\Omega)$ and all $\varphi\in\mathrm{C}^1(\overline{\Omega})$ we have the Gauß-Green formula

$$\int_\Omega u\odot\nabla\varphi\,\mathrm{d}x+\int_\Omega \varphi\,\mathrm{dE}u=\int_{\partial\Omega}\varphi(x)(\mathrm{Tr}(u)\odot\nu_{\partial\Omega})(x)\,\mathrm{d}\mathcal{H}^{n-1}(x),$$

where $\nu_{\partial\Omega}$ is the outward unit normal to $\partial\Omega$.

When dealing with different domains, we sometimes use the notation $\mathrm{Tr}_{\partial\Omega}$ to express that we are concerned with the boundary trace operator for $\partial\Omega$. It is important to remark that the trace operator of Theorem 4.7 is not continuous with respect to weak*–convergence (indeed, the (symmetric) weak*–closure of $\mathrm{C}_c^\infty(\Omega;\mathbb{R}^n)$ is the entire space $\mathrm{BD}(\Omega)$). We proceed with an extension and gluing result.

Theorem 4.8 (Extensions and Gluing, cf. [23, Thm. 1.5])**.** *Let $\Omega\subset\mathbb{R}^n$ be an open and bounded Lipschitz set.*

(a) *There exists a bounded*[3] *linear extension operator $E\colon$ $\mathrm{LD}(\Omega)\to\mathrm{LD}(\mathbb{R}^n)$.*

(b) *If $v\in\mathrm{BD}(\Omega)$ and $w\in\mathrm{BD}(\overline{\Omega}^c)$, then the* glued function *$u\colon\mathbb{R}^n\to\mathbb{R}^n$ which is defined for $\mathscr{L}^n$–a.e. $x\in\mathbb{R}^n$ by*

$$u(x):=\begin{cases} v(x) & \text{for } x\in\Omega,\\ w(x) & \text{for } x\in\mathbb{R}^n\setminus\overline{\Omega}\end{cases}$$

belongs to $\mathrm{BD}(\mathbb{R}^n)$, and its symmetric gradient $\mathrm{E}u\in\mathcal{M}_{<\infty}(\mathbb{R}^n;\mathbb{R}^{n\times n}_{\mathrm{sym}})$ is given by

$$\mathrm{E}u=\mathrm{E}v\llcorner\Omega+\mathrm{E}w\llcorner\overline{\Omega}^c+(\mathrm{Tr}(v)-\mathrm{Tr}(w))\odot\nu_{\partial\Omega}\mathcal{H}^{n-1}\llcorner\partial\Omega, \tag{4.3}$$

where $\nu_{\partial\Omega}$ denotes the outward unit normal to $\partial\Omega$ and $\mathrm{Tr}(v),\mathrm{Tr}(w)$ the outer traces of v and w on $\partial\Omega$, respectively.

We finally come to a trace–preserving smooth approximation result; see [40, Chpt. A2] in the gradient case and [115, Chpt. 3] for the symmetric gradient case.

[2] Here and in what follows, $\mathrm{L}^1(\partial\Omega;\mathbb{R}^n)$ is the space of all maps $v\colon\partial\Omega\to\mathbb{R}^n$ which are integrable with respect to $\mathcal{H}^{n-1}\llcorner\partial\Omega$.

[3] With respect to the norm topology on LD.

Lemma 4.9. *Let Ω be an open and bounded Lipschitz subset of $\mathbb{R}^n$ and let $u_0 \in \mathrm{L}^1(\partial\Omega;\mathbb{R}^n)$ be a boundary datum, being identified with $u_0 \in \mathrm{LD}(\Omega)$ such that $\mathrm{Tr}_{\partial\Omega}(u_0) = u_0$ $\mathcal{H}^{n-1}$–a.e. on $\partial\Omega$. Fix an open and bounded Lipschitz superset $\widetilde{\Omega} \supset \Omega$ and extend $u_0 \in \mathrm{LD}(\Omega)$ to $\widetilde{u_0} \in \mathrm{LD}_0(\widetilde{\Omega})$. For $w \in \mathrm{BD}(\Omega)$, denote $\widetilde{w}$ its extension to $\widetilde{\Omega}$ by $\widetilde{u_0}$. Let $u \in \mathrm{BD}(\Omega)$ be arbitrary. Then there exists a sequence $(v_k) \subset u_0|_\Omega + \mathrm{C}_c^\infty(\Omega;\mathbb{R}^n)$ such that $(\widetilde{v_k})$ converges area–strictly to u in $\mathrm{BD}(\widetilde{\Omega})$ as $k \to \infty$.*

The appendix contains a variant of this approximation results, cp. Lemma 10.9.

4.2.3. Embeddings Let $n \geqslant 2$. As a consequence of the embedding $\mathrm{W}^{1,1}(\mathbb{R}^n;\mathbb{R}^N) \hookrightarrow \mathrm{L}^{\frac{n}{n-1}}(\mathbb{R}^n;\mathbb{R}^N)$, it is easy to prove by use of smooth approximation and Fatou's lemma that $\mathrm{BV}(\mathbb{R}^n;\mathbb{R}^N) \hookrightarrow \mathrm{L}^{\frac{n}{n-1}}(\mathbb{R}^n;\mathbb{R}^N)$, continuity of the embedding being understood with respect to norm convergence in both spaces. The same argument yields that $\mathrm{BD}(\mathbb{R}^n) \hookrightarrow \mathrm{L}^{\frac{n}{n-1}}(\mathbb{R}^n)$ given $\mathrm{LD}(\mathbb{R}^n) \hookrightarrow \mathrm{L}^{\frac{n}{n-1}}(\mathbb{R}^n)$. By Ornstein's Non–Inequality, the latter does not follow from the usual Gagliardo–Nirenberg–Sobolev inequality and was proved by Strauss [224] first. From a different perspective, the aforementioned embedding theorems refer to the optimal integrability of arbitrary $\mathrm{W}^{1,p}$, BV or BD–maps within the scale of L^q–spaces, and it is natural to inquire as to which Sobolev–Slobodeckjii spaces $\mathrm{W}^{\theta,q}$ they embed into (see Chapter 2.5.5 for their definition). This is of special interest to us: As a substantial tool in our proof of higher Sobolev regularity of generalised BD–minimisers, we shall prove an embedding theorem for the intersection space $\mathrm{BD} \cap \mathrm{BMO}$ (cp. Theorem 5.29), for which a crucial role is played by such fractional embedding results combined with the *Doronsorro–type characterisation* of weakly differentiable functions (cp. Theorem 2.25).

An overarching approach to limiting embeddings in the case $p = 1$ involving differential operators has been given recently by Van Schaftingen [239] which, due to its importance for our regularity investigations, we outline below in some detail. Before doing so, we wish to embed the theme of this section into the framework of Chapter 3.2.

Let us remark that none of the aforementioned embeddings follows from general mapping properties of fractional integration operators as often used in the case $1 < p < n$. Let $1 < p < n$. In order to prove that there exists a constant $C = C(p,n) > 0$ such that for all $v \in \mathrm{C}_c^\infty(\mathbb{R}^n;\mathbb{R}^n)$ there holds $\|v\|_{\mathrm{L}^{\frac{np}{n-p}}} \leqslant C\|\boldsymbol{\varepsilon}(v)\|_{\mathrm{L}^p}$, we may write $v = \Phi(\boldsymbol{\varepsilon}(v))$ with the fractional integration operator given by (3.8), the latter essentially behaving like a Riesz potential of order $(n-1)$. Since such operators map $\mathrm{L}^p(\mathbb{R}^n) \to \mathrm{L}^{\frac{np}{n-p}}(\mathbb{R}^n)$ boundedly (see Theorem 2.30), the Sobolev–type embedding theorem is immediate in this case. However, for $p = 1$, this mapping property fails and needs to be replaced by $\mathrm{L}^1(\mathbb{R}^n) \to \mathrm{L}_{\mathrm{w}}^{\frac{n}{n-1}}(\mathbb{R}^n)$ with the weak–$\mathrm{L}^{\frac{n}{n-1}}$–space $\mathrm{L}_{\mathrm{w}}^{\frac{n}{n-1}}(\mathbb{R}^n)$ as introduced in Definition 2.18, also compare with Theorem 2.30.

The general spirit behind fractional embedding theorems is that gaining higher fractional differentiability for generic weakly differentiable functions is coupled with a certain loss of integrability, and the precise balance is given by conditions on the Sobolev indices. So, for instance, it is well–known[4] that if $n \geqslant 2$, then $\mathrm{W}^{1,p}(\mathbb{R}^n) \hookrightarrow \mathrm{W}^{\theta,np/(n-(1-\theta)p)}(\mathbb{R}^n)$ if $(1-\theta)p < n$. However, in the case $p = 1$, it has been firstly established by Kolyada [154] and later on by Bourgain, Brezis & Mironescu [49] that if $n \geqslant 2$, then

$$\mathrm{W}^{1,1}(\mathbb{R}^n) \hookrightarrow \mathrm{W}^{\theta,\frac{n}{n-1+\theta}}(\mathbb{R}^n) \qquad \text{for all } 0 < \theta < 1. \tag{4.4}$$

Both approaches rely on rearrangement arguments for scalar–valued functions and carry over to the vectorial situation by componentwise application, however, thus cannot made work for the symmetric gradient operator by trivial means; in fact, to the best of our knowledge, no such proof is known. Moreover, let us note that the dimensional condition

[4] This is a consequence of the Fractional Integration Theorem 2.30, also see [168, Thm. 14.32].

$n \geqslant 2$ is necessary indeed, as for $n = 1$, the corresponding embedding of $\mathrm{W}^{1,1}$ into $\mathrm{W}^{\frac{1}{2},2}$ turns out to be false; compare with Remark 5.30.

In [239], Van Schaftingen introduced a general framework (in the spirit of chapter 3) within which sharp conditions on a standard differential operator $\mathbb{A}[D]$ (i.e., of the form (3.11)) on $\mathbb{R}^n$ from V to W (where V, W are two finite dimensional real vector spaces) can be stated for various Sobolev–type inequalities to hold. Keeping in mind the condition of ellipticity of a first order differential operator, the second crucial notion shall turn out to be that of *cancelling operators*:

Definition 4.10 (Cancellation, [239, Def. 1.2]). *Let $\mathbb{A}[D]$ be a standard operator on $\mathbb{R}^n$ from V to W. We say that $\mathbb{A}[D]$ is* cancelling *if and only if*

$$\bigcap_{\xi \neq 0} \mathbb{A}[\xi](V) = \{0\}. \tag{Canc}$$

Adopting this terminology, we have the following result.

Theorem 4.11 (Van Schaftingen [239, Thm. 1.4]). *Let $n \geqslant 2$ and let $\mathbb{A}[D]$ be an elliptic and cancelling standard operator on $\mathbb{R}^n$ from V to W. Then*

(a) *There exists a constant $C > 0$ such that $\|v\|_{\mathrm{L}^{\frac{n}{n-1}}(\mathbb{R}^n;V)} \leqslant C\|\mathbb{A}[D]v\|_{\mathrm{L}^1(\mathbb{R}^n;W)}$ holds for all $v \in \mathrm{C}_c^1(\mathbb{R}^n; V)$.*

(b) *For each $0 < \theta < 1$, there exists a constant $C > 0$ such that*

$$\|v\|_{\mathrm{W}^{\theta,p(\theta)}(\mathbb{R}^n;V)} \leqslant C\|\mathbb{A}[D]v\|_{\mathrm{L}^1(\mathbb{R}^n;W)}$$

for all $v \in \mathrm{C}_c^1(\mathbb{R}^n; V)$, where $p(\theta) := n/(n-1+\theta)$.

Conversely, if (a) *or* (b) *holds, then $\mathbb{A}[D]$ is elliptic and cancelling.*

The upshot of this theorem is as follows: By the mere representation of v in terms of $\mathbb{A}[D]v$ by use of a fractional integration operator of order $(n-1)$, Theorem 2.30 does not imply the preceding theorem. It is, however, the special differential structure of the Green's operator associated with $\mathbb{A}[D]$ that makes the conclusion of Theorem 4.11 valid.

For later purposes, we explicitely note the following.

Lemma 4.12 (Van Schaftingen, [239, Prop. 6.4]). *For $n \geqslant 2$, the symmetric gradient operator is an elliptic and cancelling standard operator on $\mathbb{R}^n$ from $\mathbb{R}^n$ to $\mathbb{R}^{n\times n}_{\mathrm{sym}}$.*

Proof. The statement can be recast as follows: Let $\mathbb{A}[D]$ be the homogeneous, linear, first order differential operator on $\mathbb{R}^n$ from $\mathbb{R}^n$ to[5] $S^2\mathbb{R}^n$ which is defined for $\xi \in \mathbb{R}^n$, $v \in \mathbb{R}^n$ and $w, z \in \mathbb{R}^n$ by $(\mathbb{A}[\xi]v)(w,z) := \frac{1}{2}(\langle \xi, w\rangle\langle v, z\rangle + \langle \xi, z\rangle\langle v, w\rangle)$. Then $\mathbb{A}[D]$ is elliptic. Indeed, if $v \in \mathbb{R}^n$ and $\xi \in \mathbb{R}^n \setminus \{0\}$ are such that $0 = \mathbb{A}[\xi]v \in S^2\mathbb{R}^n$, then $\mathbb{A}[\xi]v(w,z) = 0$ for all $w, z \in \mathbb{R}^n$ and so in particular $0 = \mathbb{A}[\xi]v(w,w) = \langle \xi, w\rangle\langle v, w\rangle$ for all $w \in \mathbb{R}^n$. Thus, if $w \in \mathbb{R}^n$ is such that $\langle \xi, w\rangle \neq 0$, then $\langle v, w\rangle = 0$. But since $\mathrm{lin}(\{w \in \mathbb{R}^n\colon\ \xi, w\rangle \neq 0\}) = \mathbb{R}^n$, we deduce $v = 0$ and so $\mathbb{A}[D]$ is elliptic. Finally, $\mathbb{A}[D]$ is cancelling if $n \geqslant 2$: Let $e \in \bigcap_{\xi \in \mathbb{R}^n\setminus\{0\}} \mathbb{A}[\xi](\mathbb{R}^n)$. For $w \in \mathbb{R}^n$ arbitrary but fixed, we pick $\xi \in \mathbb{R}^n \setminus \{0\}$ with $\xi \perp w$ and conclude for any $v \in \mathbb{R}^n$ that $\mathbb{A}[\xi]v(w,w) = 0$ so that $e(w,w) = 0$. By arbitrariness of w and symmetry of e, we must have $e = 0$ and the proof is complete. $\square$

[5] Here, $S^2\mathbb{R}^n$ denotes the space of symmetric bilinear forms on $\mathbb{R}^n$.

Van Schaftingen's Theorem, Lebesgue Scale

The rest of this section is devoted to the sufficiency part of VAN SCHAFTINGEN's original proof of Theorem 4.11(a) and as such, the material is entirely recorded from [239]. The necessity part is not needed here, and we refer the reader to [239, Sec. 5] for more detail. Before embarking on the precise arguments, we introduce a notion dual to the cancellation:

Definition 4.13 (Cocancellation, [239, Def. 1.3])**.** *Let $\mathbb{L}[D]$ be a k–th order standard operator on $\mathbb{R}^n$ between the two finite–dimensional vector spaces W and Z. We say that $\mathbb{L}[D]$ is* cocancelling *if and only if*

$$\bigcap_{\xi\neq 0}\ker\mathbb{L}[\xi]=\{0\}. \tag{4.5}$$

For each $\xi\in\mathbb{R}^n\setminus\{0\}$, consider the short sequence

$$V\xrightarrow{\mathbb{A}[\xi]}W\xrightarrow{\mathbb{L}[\xi]}Z. \tag{4.6}$$

It is clear that if this sequence is exact at W, meaning that for each $\xi\in\mathbb{R}^n\setminus\{0\}$ there holds $\mathbb{A}[\xi](V)=\ker(\mathbb{L}[\xi])$, then $\mathbb{A}[D]$ is cancelling if and only if $\mathbb{L}[D]$ is cocancelling.

Proposition 4.14 ([239, Thm. 1.4])**.** *Let $n\geqslant 2$ and assume that $\mathbb{L}[D]$ is a standard operator on $\mathbb{R}^n$ from W to Z. Then the following conditions are equivalent:*

(a) *There exists $C>0$ such that for every $f\in\mathrm{L}^1(\mathbb{R}^n;W)$ with $\mathbb{L}[D]f=0$ and $\varphi\in\mathrm{C}_c^\infty(\mathbb{R}^n;W)$ there holds*

$$\int_{\mathbb{R}^n}\langle f,\varphi\rangle\,\mathrm{d}x\leqslant C\|f\|_{\mathrm{L}^1}\|\mathrm{D}\varphi\|_{\mathrm{L}^n}. \tag{4.7}$$

(b) *$\mathbb{L}[D]$ is cocancelling.*

Assuming the previous proposition for the time being, the proof of the sufficiency part of Theorem 4.11 evolves as follows:

(i) Show that for any elliptic $\mathbb{A}[D]$ there exists a finite–dimensional real vector space X and a standard operator $\mathbb{L}[D]$ on $\mathbb{R}^n$ from W to X such that the symbol complex (4.6) is exact at W for any $\xi\in\mathbb{S}^{n-1}$. Any such $\mathbb{L}[D]$ is cocancelling provided $\mathbb{A}[D]$ is cancelling.

(ii) By Proposition 4.14(i) we then have for any $u\in\mathrm{C}_c^\infty(\mathbb{R}^n;V)$ (setting $f=\mathbb{A}[D]u$)

$$\|\mathbb{A}[D]u\|_{\dot{\mathrm{W}}^{-1,\frac{n}{n-1}}}\leqslant C\|\mathbb{A}[D]u\|_{\mathrm{L}^1},$$

which immediately follows from (4.7) by testing against all $\varphi\in\mathrm{C}_c^\infty(\mathbb{R}^n;W)$ with $\|\mathrm{D}\varphi\|_{\mathrm{L}^n}\leqslant 1$.

(iii) In view of (ii), we conclude by noting that there exists a constant $C>0$ such that for all $u\in\mathrm{C}_c^1(\mathbb{R}^n;V)$ there holds

$$\|u\|_{\mathrm{L}^{\frac{n}{n-1}}}\leqslant C\|\mathbb{A}[D]u\|_{\dot{\mathrm{W}}^{-1,\frac{n}{n-1}}}. \tag{4.8}$$

To see this, note that if $\mathbb{A}[D]$ is elliptic, then the Fourier multiplier $m\colon\xi\mapsto(\mathbb{A}^*[\xi]\mathbb{A}[\xi])^{-1}\mathbb{A}^*[\xi]$ is homogeneous of degree -1, and so the Fourier multiplier corresponding to the map $u\mapsto(-\Delta)^{-\frac{1}{2}}\mathbb{A}[D]u$ has symbol homogeneous of degree zero. Therefore, by the Mihlin multiplier theorem, the operator $u\mapsto(-\Delta)^{-\frac{1}{2}}\mathbb{A}[D]u$ is bounded on L^p for any $1<p<\infty$ and elliptic as the composition of two elliptic operators. Hence, by the Fourier characterisation of the negative Sobolev spaces $\mathrm{W}^{-1,q}$, $1<q<\infty$, we have $\|\varphi\|_{\dot{\mathrm{W}}^{-1,q}}\sim\|(-\Delta)^{-\frac{1}{2}}\varphi\|_{\mathrm{L}^q}$ for all $\varphi\in\mathrm{W}^{-1,q}$ and so

$$\|u\|_{\mathrm{L}^{\frac{n}{n-1}}}\leqslant C\|(-\Delta)^{-\frac{1}{2}}\mathbb{A}[D]u\|_{\mathrm{L}^{\frac{n}{n-1}}}\leqslant C\|\mathbb{A}[D]u\|_{\dot{\mathrm{W}}^{-1,\frac{n}{n-1}}}.$$

For the necessity part of both ellipticity and the cancellation condition, the reader is referred to [239, Sec. 5]. In conclusion, Theorem 4.11 is proven once we give the

Proof of (b)⇒(a) in Proposition 4.14. By [240, Thm. 4], if $k \in \mathbb{N}$ and $f_\alpha \in \mathrm{L}^1(\mathbb{R}^n)$ for $\alpha \in \mathbb{N}^n$ with $|\alpha| = k$ are such that $\sum_{|\alpha|=k} \partial^\alpha f_\alpha = 0$ being satisfied in the sense of distributions, then for every $\alpha \in \mathbb{N}^n$ with $|\alpha| = k$ and every $\varphi \in \mathrm{C}_c^\infty(\mathbb{R}^n)$ there holds $\int_{\mathbb{R}^n} \langle f_\alpha, \varphi\rangle \,\mathrm{d}x \leqslant C\|f\|_{\mathrm{L}^1}\|\,\mathrm{D}\varphi\|_{\mathrm{L}^n}$. On the other hand, if $\mathbb{L}[D] = \sum_{|\alpha|=k} \mathbb{L}_\alpha \partial^\alpha$ is a homogeneous differential operator of order k on $\mathbb{R}^n$ from W to X, then $\mathbb{L}[D]$ is cocancelling if and only if there exist $\mathbb{K}_\alpha \in \mathscr{L}(X;W)$ such that for every $\alpha \in \mathbb{N}^n$ with $|\alpha| = k$ there holds $\sum_{|\alpha|=k} \mathbb{K}_\alpha \circ \mathbb{L}_\alpha = \mathrm{Id}$.

Now let $f \in \mathrm{L}^1(\mathbb{R}^n;W)$ be such that $\mathbb{L}[D]f = 0$ in the sense of distributions. By the result just mentioned, we have for every $\alpha \in \mathbb{N}^n$ with $|\alpha| = k$ and $\psi \in \mathrm{C}_c^\infty(\mathbb{R}^n;V)$ that $\int_{\mathbb{R}^n} \langle \mathbb{L}_\alpha(f), \psi\rangle \,\mathrm{d}x \leqslant C\|f\|_{\mathrm{L}^1}\|\,\mathrm{D}\psi\|_{\mathrm{L}^n}$. Let $\varphi \in \mathrm{C}_c^\infty(\mathbb{R}^n;W)$ be arbitrary but fixed. We then obtain

$$\begin{aligned}\int_{\mathbb{R}^n} \langle f, \varphi\rangle \,\mathrm{d}x &= \sum_{|\alpha|=k} \int_{\mathbb{R}^n} \langle \mathbb{L}_\alpha(f), \mathbb{K}_\alpha^*(\varphi)\rangle \,\mathrm{d}x \\ &\leqslant C \sum_{|\alpha|=k} \|\mathbb{L}_\alpha(f)\|_{\mathrm{L}^1}\|\,\mathrm{D}\mathbb{K}_\alpha^*(\varphi)\|_{\mathrm{L}^n} \leqslant C\|f\|_{\mathrm{L}^1}\|\,\mathrm{D}\varphi\|_{\mathrm{L}^n}.\end{aligned}$$

The proof of the direction (b)⇒(a) is complete. □

Van Schaftingen's Theorem, Slobodeckjii Scale

To obtain an embedding result for $\mathrm{BD}(\mathbb{R}^n)$ into fractional Sobolev spaces, one extends Prop. 4.14 (see [239, Sec. 8.2]). By similar means as above we then deduce that if $\mathbb{L}[D]$ is a cocancelling homogeneous differential operator from W to Z, $0 < s < 1$ and $1 < q < \infty$ such that $sq = n$, then there exists a constant $c > 0$ such that for all $f \in \mathrm{L}^1(\mathbb{R}^n;W)$ with $\mathbb{L}[D]f = 0$ in the sense of distributions there holds $\int_{\mathbb{R}^n} \langle f, \varphi\rangle \,\mathrm{d}x \leqslant c\|f\|_{\mathrm{L}^1}\|\varphi\|_{\dot{\mathrm{W}}^{s,q}}$. Following the scheme (i)–(iii) from above, the conclusion of Theorem 4.11 is immediate once we show that for $0 < \theta < 1$, there exists a constant $C > 0$ such that for all $\varphi \in \mathrm{C}_c^1(\mathbb{R}^n;V)$ we have

$$\|\varphi\|_{\mathrm{W}^{\theta,p(\theta)}} \leqslant C\|\mathbb{A}[D]\varphi\|_{(\dot{\mathrm{W}}^{1-\theta,p(\theta)'})^*}.$$

Since $\mathbb{A}[D]$ is elliptic, it has a left inverse, and we may write $\varphi = \Phi_{\mathbb{A}}(\mathbb{A}[D]\varphi)$ (compare with (3.8)). The operator $T := (-\Delta)^{\frac{\theta}{2}} \circ \Phi$ has symbol homogeneous of degree $(\theta - 1)$. Let $s = 1 - \theta$ and $q = p(\theta) = n/(n-1+\theta)$. The Hölder exponent of q reads

$$q' = \frac{n}{n-(n-1+\theta)} = \frac{n}{1-\theta} \quad \text{and hence} \quad (1-\theta)q' = n,$$

so that the above result may be applied. In conclusion, by Mihlin's theorem, the characterisation of the dual of $\dot{\mathrm{W}}^{s,r}$ (see [7, Thm. 4.1.3]) we see that

$$\|\varphi\|_{\dot{\mathrm{W}}^{\theta,p(\theta)}} \leqslant c\|(-\Delta)^{\frac{\theta}{2}}\Phi(\mathbb{A}[D]\varphi)\|_{\mathrm{L}^{p(\theta)}} \leqslant c\|\mathbb{A}[D]\varphi\|_{(\dot{\mathrm{W}}^{1-\theta,p(\theta)'})^*},$$

and the proof of the sufficiency part of Theorem 4.11 is complete.

4.2.4. Differentiability $\mathscr{L}^n$–a.e. To round off our discussion of functions of bounded deformation, we now report on an approximate differentiability result for BD–maps as firstly stated independently by Hajlasz [140] and Ambrosio, Coscia & Dal Maso [14], where (in the terminology of [9]) the stronger notion of L^p–differentiability $\mathscr{L}^n$ was used in the latter (cp. Chapter 2.2 for the notion of approximate differentiability).

Proposition 4.15 ([140, Thm. 5], [14, Thm. 7.4]). *Let $u \in \mathrm{BD}_{\mathrm{loc}}(\mathbb{R}^n)$. Then u is approximately differentiable $\mathscr{L}^n$–a.e. in $\mathbb{R}^n$.*

The main issue in proving this theorem stems from Ornstein's Non–Inequality which excludes the availability of $\mathrm{L}^1_{\mathrm{loc}}$–estimates on the full gradients of BD–maps. Indeed, otherwise we could reduce to, e.g., [97, Chpt. 6.1.2, Thm. 2, Chpt. 6.1.3, Thm. 4]. In conclusion, we see that the main obstruction arises already when dealing with the approximate differentiability of LD–maps, and for the reader's convenience, we shall give an independent proof of this result by use of the so–called $\mathrm{W}^{1,\mathrm{BMO}}$–truncation technique in the appendix, cp. Chapter 10.2. This method is a suitable adaption of the Lipschitz truncation technique (cp. [5, 84]) to the situation at our disposal.

4.2.5. Decomposition of Derivatives and Rank One–Type Properties Working from the Radon–Nikodym decomposition of the measures $\mathrm{D}u$ or $\mathrm{E}u$ for BV– or BD–functions $u\colon \mathbb{R}^n \to \mathbb{R}^N$ or $u\colon \mathbb{R}^n \to \mathbb{R}^n$, respectively, it is possible to obtain more information on the structure of their absolutely continuous and singular parts. To do so, we recall the fundamental notions in some generality first and follow [14, Sec. 2] and [16]. Let $\Omega \subset \mathbb{R}^n$ be open and let $u \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^N)$. We say that $x_0 \in \mathbb{R}^n$ is a *Lebesgue point* for u if and only if there exists $\widetilde{u}(x_0) \in \mathbb{R}^N$ such that

$$\lim_{r\searrow 0} \fint_{\mathrm{B}(x_0,r)} |u(x) - \widetilde{u}(x_0)| \,\mathrm{d}x = 0.$$

The set of all Lebesgue points of u is denoted Ω_u, and its complement $S_u := \Omega \setminus \Omega_u$ is called the *discontinuity set* of u. This is the standard notation for these sets, and we wish to stress that in the context of regularity theory, the symbols Ω_u are often used in a different way. We shall make this precise in Chapters 7 and 8. It is clear that by Lebesgue's Differentiation Theorem (cp. Theorem 2.4), $\mathscr{L}^n$–a.e. $x_0 \in \Omega$ is a Lebesgue point for u, and we call the mapping $\widetilde{u}\colon \Omega \to \mathbb{R}^N$ given by

$$\widetilde{u}(x_0) := \begin{cases} \lim_{r\searrow 0} \fint_{\mathrm{B}(x_0,r)} u \,\mathrm{d}x & \text{if } x_0 \in \Omega_u, \\ 0 & \text{otherwise,} \end{cases}$$

the *Lebesgue representative* of u. We further say that u has *one–sided Lebesgue limits* $u^{\pm}(x_0)$ *at* $x_0 \in \Omega$ provided there exists a suitable normal direction $\nu_u(x) \in \mathbb{S}^{n-1}$ such that there holds

$$\lim_{r\searrow 0} \frac{1}{r^n} \int_{\mathrm{B}_r^{\pm}(x,\nu_u(x))} |u(y) - u^{\pm}(x)| \,\mathrm{d}y = 0. \tag{4.9}$$

Here we have set $\mathrm{B}_r^{\pm}(x,\nu_u(x)) := \{y \in \mathrm{B}(x,r)\colon\ \langle y-x, \pm\nu_u(x)\rangle > 0\}$. As a consequence of (4.9), the mappings $u_r(y) := u(x+ry)$ converge in $\mathrm{L}^1(\mathrm{B}(0,1);\mathbb{R}^N)$ as $r \searrow 0$ to $U\colon\ \mathrm{B}(0,1) \to \mathbb{R}^N$ given by

$$U(y) := \begin{cases} u^+(x) & \text{if } \langle y, \nu_u(x)\rangle > 0, \\ u^-(x) & \text{if } \langle y, \nu_u(x)\rangle < 0. \end{cases}$$

With this representation at our disposal, it is easily seen that if $u^+(x) \neq u^-(x)$, then $(u^+(x), u^-(x), \nu_u(x)) \in \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{S}^{n-1}$ is essentially uniquely determined (i.e., up to a permutation of $(u^+(x), u^-(x))$ and the sign of $\nu_u(x)$). We then define the *jump set* by

$$J_u := \big\{x_0 \in S_u\colon\ u \text{ has one–sided Lebesgue limits at } x_0\big\}.$$

Let now $u \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^N)$. With $\mathrm{D}u$ being an $\mathbb{R}^{N\times n}$–valued Radon measure, we consider its Radon–Nikodym decomposition with respect to $\mathscr{L}^n$, reading as $\mathrm{D}u = \mathrm{D}^{ac}u + \mathrm{D}^s u$, where $\mathrm{D}^{ac}u \ll \mathscr{L}^n$ and $\mathrm{D}^s u \perp \mathscr{L}^n$ (cp. Chapter 2.2 for this notation). By the Lebesgue differentiation theorem (see Section 2.2), we may further write

$$\mathrm{D}u = \mathrm{D}^{ac}u + \mathrm{D}^s u = \underbrace{\frac{\mathrm{d}\mathrm{D}^{ac}u}{\mathrm{d}\mathscr{L}^n}\mathscr{L}^n}_{=\mathrm{D}u\llcorner(\Omega\setminus S)} + \underbrace{\frac{\mathrm{d}\mathrm{D}^s u}{\mathrm{d}|\mathrm{D}^s u|}|\mathrm{D}^s u|}_{=\mathrm{D}u\llcorner S}, \tag{4.10}$$

where $S := \{x \in \mathbb{R}^n\colon\ \lim_{r\searrow 0} r^{-n}|\,\mathrm{D}u|(\mathrm{B}(x,r)) = \infty\}$. In this situation, the density $\frac{\mathrm{d}\mathrm{D}^{ac}u}{\mathrm{d}\mathscr{L}^n} \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^{N\times n})$ equals the approximate gradient of u $\mathscr{L}^n$–a.e. (cf. [16, Thm. 3.83]), a fact which is symbolically represented by $\frac{\mathrm{d}\mathrm{D}^{ac}u}{\mathrm{d}\mathscr{L}^n}\mathscr{L}^n = \nabla u\mathscr{L}^n$. However, we note carefully that in this representation, ∇u in general does *not* arise as the gradient of a $\mathrm{W}^{1,1}$–map. This is so because the approximate gradient in general is not curl–free.

On the other hand, the singular part $\mathrm{D}^s u$ can be decomposed even further into a *jump* and a *Cantor part* $\mathrm{D}^j u := \mathrm{D}u \llcorner \Theta_u$ and $\mathrm{D}^c u := \mathrm{D}u \llcorner (S\setminus\Theta_u)$, where

$$\Theta_u := \{x \in \mathbb{R}^n\colon\ \liminf_{r\searrow 0} r^{1-n}|\,\mathrm{D}u|(\mathrm{B}(x,r)) > 0\} \tag{4.11}$$

and we refer the reader to [16, Prop. 3.92] for more background information. It is comparatively easy to demonstrate that the jump part has rank one structure, meaning that $\frac{\mathrm{d}\mathrm{D}^j u}{\mathrm{d}\mathcal{H}^{n-1}}$ is a rank–1–matrix $\mathcal{H}^{n-1}$–a.e.. The corresponding result for the Cantor part is substantially harder to prove and is attributable to Alberti [8], who showed that $\frac{\mathrm{d}\mathrm{D}^c u}{\mathrm{d}|\,\mathrm{D}^c u|}$ is a rank–1–matrix $|\,\mathrm{D}^c u|$–a.e. in $\mathbb{R}^n$. In the sequel, we shall hereafter refer to this result as *Alberti's rank–1–theorem.*

With Alberti's approach not being applicable to the more general situation of the symmetric gradient, say, De Philippis & Rindler recently proved in [76] an extension of the Alberti rank–1–theorem which we shall describe now. Going back to the framework of Chapter 3, let $\mathcal{A}[D]$ be a constant rank, homogeneous, linear differential operator of the form (3.26) from W to X and call a W–valued Radon measure μ on $\mathbb{R}^n$ $\mathcal{A}$*–free* provided $\mathcal{A}[D]\mu \equiv 0$ in the sense of distributions, i.e., $\int_{\mathbb{R}^n} \mathcal{A}^*[D]\varphi\,\mathrm{d}\mu = 0$ for all $\varphi \in \mathrm{C}_c^\infty(\mathbb{R}^n;X^*)$, where $\mathcal{A}^*[D]$ is the formal adjoint of $\mathcal{A}[D]$. Recalling that the associated wave cone is given by $\Lambda_{\mathcal{A}} = \bigcup_{\xi\neq 0}\ker(\mathcal{A}[\xi])$, the main result of [76] asserts that if μ is a W–valued $\mathcal{A}$–free Radon measure on $\mathbb{R}^n$ and $\mu = \mu^{ac} + \mu^s$ is its Radon–Nikodym decomposition with respect to $\mathscr{L}^n$, then

$$\frac{\mathrm{d}\mu^s}{\mathrm{d}|\mu^s|} \in \Lambda_{\mathcal{A}} \quad |\mu^s| - \text{a.e. in } \mathbb{R}^n. \tag{4.12}$$

This result gives back the Alberti rank–one theorem: Indeed, within the framework of Example 3.5 we see that if $\mathcal{A} = \mathrm{curl}$, then $\Lambda_{\mathrm{curl}} = \mathbb{R}^N \otimes \mathbb{R}^n$ which is the usual rank–1–cone. However, since $\mathbb{R}^n$ is simply connected, the only curl–free measures are those which can be written as gradients, and the Alberti rank–one theorem follows. By the Saint Venant–compatibility conditions (cp. Example 3.6 and the definition of the operator $\mathrm{curl}\circ\mathrm{curl}$), the particular choice $\mathcal{A} = \mathrm{curl}\circ\mathrm{curl}$ yields that if μ is a $\mathcal{A}$–free measure (as is $\boldsymbol{\varepsilon}(u)$ for $u \in \mathrm{BD}_{\mathrm{loc}}(\mathbb{R}^n)$), then the singular part satisfies $\frac{\mathrm{d}\mu^s}{\mathrm{d}|\mu^s|} \in \Lambda_{\mathcal{A}}$. In this situation, however, Lemma 3.14 yields that $\Lambda_{\mathcal{A}} = \mathbb{R}^n \odot \mathbb{R}^n$, the symmetric rank–one cone, and we thereby obtain $\frac{\mathrm{d}\mathrm{E}^s u}{\mathrm{d}|\,\mathrm{E}^s u|} \in \mathbb{R}^n \odot \mathbb{R}^n$ $|\,\mathrm{E}^s u|$–a.e. in $\mathbb{R}^n$.

In the BD–case, the splitting analogous to (4.10) has been addressed in [14, 152, 153] and reads as follows. Given $u \in \mathrm{BD}_{\mathrm{loc}}(\mathbb{R}^n)$, we define

$$\widetilde{\Theta}_u := \Big\{x \in \mathbb{R}^n\colon\ \limsup_{r\searrow 0}\frac{|\,\mathrm{E}u|(\mathrm{B}(x,r))}{r^{n-1}} > 0,\Big\}$$

and introduce the discontinuity set S_u and the jump set J_u as above. We then have

$$\mathrm{E}u = \mathrm{E}^{ac}u + \mathrm{E}^s u = \mathscr{E}u\mathscr{L}^n + \underbrace{\mathrm{E}^s u \llcorner J_u}_{=:\mathrm{E}^j u} + \underbrace{\mathrm{E}^s u \llcorner (\Omega\setminus J_u)}_{=:\mathrm{E}^c u}, \tag{4.13}$$

where $\mathscr{E}u$ is the approximate symmetric gradient as in Section 4.2.4. The parts $\mathrm{E}^j u$ and $\mathrm{E}^c u$ are called the *jump* and *Cantor parts* of $\mathrm{E}u$, respectively. In this situation, it is proved in [14] that J_u is countably $(\mathcal{H}^{n-1}, n-1)$–rectifiable and equals $\widetilde{\Theta}_u$ up to a set which is $\mathcal{H}^{n-1}$–negligible. Moreover, $\mathrm{E}^c u$ is known to vanish on any Borel set $B \in \mathscr{B}(\mathbb{R}^n)$

which is σ–finite with respect to $\mathcal{H}^{n-1}$. For future reference, we record the following lemma attributable to KOHN [152], here given in the version of AMBROSIO, COSCIA & DAL MASO [14, Sec. 3(ii)].

Lemma 4.16 ([152, Chpt. II, Sec. 2.2])**.** *Let $\Omega\subset\mathbb{R}^n$ be open and let $\Sigma\subset\Omega$ be a C^1–hypersurface. For every $u\in\mathrm{BD}(\Omega)$, the one–sided Lebesgue limits $u^{\pm}(x)$ on both sides of Σ (with respect to a[6] unit normal ν_Σ to Σ) exist for $\mathcal{H}^{n-1}$–a.e. $x\in\Sigma$ and satisfy*

$$\mathrm{E}u\llcorner\Sigma=(u^+-u^-)\odot\nu_\Sigma\mathcal{H}^{n-1}\llcorner\Sigma.$$

In analogy with the BV–case, where a structure theory by means of one–dimensional sections is available, BD–functions can be characterised in a similar way, but we will not need this in the sequel and refer the reader to the extensive study [14] for more detail.

4.3. Functions of Measures and Reshetnyak–Type Theorems

After the preparations of the previous sections, we now pass on to the existence of solutions for the problems (P) and (P_ε) subject to the linear growth hypothesis (LG). We shall develop the theory for BD–maps, but the approach is completely analogous for BV–maps.

4.3.1. Functions of Measures Our first objective is to make sense of expressions $\int_\Omega f(\mu)$ when μ is a $\mathbb{R}^m$–valued Radon measure for arbitrary $m\in\mathbb{N}$ and, moreover, provide suitable lower semicontinuity results that place the definition of $\int_\Omega f(\mu)$ in relation with $\liminf_j\int_\Omega f(\mu_j)$ whenever (μ_j) is a sequence of $\mathbb{R}^m$–valued Radon measures converging to μ in a suitable sense. Finally, we shall apply this theory to the (symmetric) gradients of BV– or BD–maps. Our presentation mainly relies on a technique introduced by GIAQUINTA, MODICA & SOUCEK in [126], for a general exposition of applying functions to measures see [16, 75].

Let $f\colon\mathbb{R}^m\to\mathbb{R}$ be a convex function of linear growth, i.e., f satisfies (LG). To capture the behaviour of f at infinity, we define the *recession function* $f^\infty:\mathbb{R}^m\to\mathbb{R}_0^+$ for $\xi\in\mathbb{R}^m$ by

$$f^\infty(\xi):=\lim_{t\searrow 0}tf\left(\frac{\xi}{t}\right). \tag{4.14}$$

Owing to convexity[7] and linear growth of f this limit can be shown to exist in $\mathbb{R}_{\geqslant 0}$ for all $\xi\in\mathbb{R}^m$ (see Lemma 4.17 below). The *linear perspective function* $\widetilde{f}:\mathbb{R}_{\geqslant 0}\times\mathbb{R}^m\to\mathbb{R}_0^+$ associated with f is then given by

$$f^{\#}(t,\xi):=\begin{cases}tf\left(\frac{\xi}{t}\right) & \text{for } t>0\\ f^\infty(\xi) & \text{for } t=0.\end{cases} \tag{4.15}$$

We now collect some properties of f^∞ and $f^{\#}$ that will turn out important in the sequel:

Lemma 4.17. *Let $f\colon\mathbb{R}^m\to\mathbb{R}$ be a convex function of linear growth with* (LG)*. Then the following hold:*

(a) *f^∞ is well–defined, convex, homogeneous of degree one and satisfies $c_1|z|\leqslant f^\infty(z)\leqslant c_2|z|$ for all $z\in\mathbb{R}^m$, where $c_1,c_2>0$ are the same constants as in* (LG)*.*

[6] This is to be understood in the sense that we pick unit normals of a certain orientation and leave this orientation fixed, as this will also influence the sign of the jump u^+-u^-.

[7] For the recession function to exist, convexity it is an essential requirement. For instance, in the quasiconvex situation, one needs to work with different notions of recession functions, cp. Rem. 4.21 below.

(b) *$f^\#$ is a convex function, homogeneous of degree one and satisfies $c_1|z| \leqslant f^\#(t,z) \leqslant c_2(t+|z|)$ for all $(t,z) \in [0,\infty)\times\mathbb{R}^m$.*

(c) *If f is lower semicontinuous, then so are f^∞ and $f^\#$.*

Proof. Ad (a) and (b). Note that $t\mapsto f(tz)$ is convex on $\mathbb{R}$ so that for each $z\in\mathbb{R}^m$, the difference quotients $(f(tz)-f(0))/t$ are monotonically increasing in $t\in\mathbb{R}_{\geqslant 0}$. By (LG), we deduce for all $z\in\mathbb{R}^{N\times n}$

$$\frac{f(tz)-f(0)}{t} \leqslant \frac{c_2(1+|tz|)-f(0)}{t} \overset{t\to\infty}{\longrightarrow} c_2|z|$$

and record that the monotone increase in conjunction with boundedness given by the first of the previous two inqualities gives the existence of $f^\infty(z)(=\lim_{t\nearrow\infty}(f(tz)-f(0))/t)$. Now let $(t_1,z_1),(t_2,z_2)\in(0,\infty)\times\mathbb{R}^{N\times n}$ and $0\leqslant\lambda\leqslant 1$. Then, by convexity of f,

$$\begin{aligned}
f^\#(\lambda t_1+(1-\lambda)t_2,\lambda z_1+(1-\lambda)z_2) &\overset{\text{def}}{=} (\lambda t_1+(1-\lambda)t_2)f\left(\frac{\lambda z_1+(1-\lambda)z_2}{\lambda t_1+(1-\lambda)t_2}\right)\\
&\overset{f\text{ convex}}{\leqslant} (\lambda t_1+(1-\lambda)t_2)f\left(\frac{\lambda t_1}{\lambda t_1+(1-\lambda)t_2}\frac{z_1}{t_1}\right.\\
&\qquad\left.+\frac{(1-\lambda)t_2}{\lambda t_1+(1-\lambda)t_2}\frac{z_2}{t_2}\right)\\
&= \lambda f^\#(t_1,z_1)+(1-\lambda)f^\#(t_2,z_2).
\end{aligned}$$

Moreover, if $t_1=0$ or $t_2=0$, we may argue similarly to deduce convexity and hence continuity of f and $f^\#$ provided f is assumed continuous, and the claimed homogeneity holds trivially. This settles all assertions apart from the lower semicontinuity part (c), which we tackle now: Let $(t_k,z_k)\to(t,z)$ in $\mathbb{R}_{\geqslant 0}\times\mathbb{R}^{N\times n}$. If $t>0$, we immediately obtain by lower semicontinuity of f that $f^\#(t,z)\leqslant\liminf_{k\to\infty}f^\#(t_k,z_k)$. If $t=0$, then for given $\varepsilon>0$ we find $s>0$ satisfying $f^\infty(z)\leqslant(f(sz)-c_2)/s+\varepsilon$ and, invoking monotonicity of the difference quotients of f, we obtain (as $s\leqslant 1/t_k$ for all $k\in\mathbb{N}$)

$$\begin{aligned}
f^\#(t,z) &\leqslant \liminf_{k\to\infty}\frac{f(sz_k)-f(0)}{s}+\varepsilon \leqslant \liminf_{k\to\infty} t_k\big(f(z_k/t_k)-f(0)\big)+\varepsilon\\
&\leqslant \liminf_{k\to\infty} f^\#(t_k,z_k)+\varepsilon.
\end{aligned}$$

The proof is complete. □

To define $\mathfrak{F}$ on measures, let $\mu\in\mathcal{M}(\Omega;\mathbb{R}^m)$ and choose $\nu\in\mathcal{M}(\Omega;\mathbb{R}_{\geqslant 0})$ with $(\mathscr{L}^n,\mu)\ll\nu$ (e.g., choose $\nu:=\mathscr{L}^n+|\mu^s|$). We then define $\overline{\mathfrak{F}}:\mathcal{M}_{\text{loc}}(\Omega;\mathbb{R}^m)\to[0,\infty]$ by

$$\overline{\mathfrak{F}}(\mu) := \int_\Omega f^\#\left(\frac{\mathrm{d}\mathscr{L}^n}{\mathrm{d}\nu},\frac{\mathrm{d}\mu}{\mathrm{d}\nu}\right)\mathrm{d}\nu. \tag{4.16}$$

Homogeneity of the linear perspective function $f^\#$ yields that the preceding definition (4.16) is independent of the particular choice of ν. Consequently, $\overline{\mathfrak{F}}(\mu)$ is well–defined. Denoting the Lebesgue–Radon–Nikodym decomposition of μ with respect to $\mathscr{L}^n$ by

$$\mu=\mu^{ac}+\mu^s=\frac{\mathrm{d}\mu}{\mathrm{d}\mathscr{L}^n}\mathscr{L}^n+\frac{\mathrm{d}\mu^s}{\mathrm{d}|\mu^s|}|\mu^s|,$$

we have $\mu^{ac}\ll\mathscr{L}^n$ and $\mu^s\perp\mathscr{L}^n$. Hence there exist disjoint subsets $S,A\subset\Omega$ with $\mathscr{L}^n(S)=|\mu^s|(A)=0$. Setting $\nu:=|\mu^s|+\mathscr{L}^n$, we have $\nu\llcorner A=\mathscr{L}^n\llcorner A$ and $\nu\llcorner S=|\mu^s|\llcorner S$. We split

$$\begin{aligned}
\overline{\mathfrak{F}}(\mu) &= \int_\Omega f^\#\Big(\frac{\mathrm{d}\nu}{\mathrm{d}\mathscr{L}^n},\frac{\mathrm{d}\mu}{\mathrm{d}\nu}\Big)\,\mathrm{d}\nu\\
&= \int_A f^\#\Big(\frac{\mathrm{d}\nu}{\mathrm{d}\mathscr{L}^n},\frac{\mathrm{d}\mu}{\mathrm{d}\nu}\Big)\,\mathrm{d}\nu+\int_S f^\#\Big(\frac{\mathrm{d}\nu}{\mathrm{d}\mathscr{L}^n},\frac{\mathrm{d}\mu}{\mathrm{d}\nu}\Big)\,\mathrm{d}\nu=(*).
\end{aligned}$$

Now we note that on A there holds $|\mu^s| \equiv 0$, hence $\nu \llcorner A = \mathscr{L}^n$. Consequently, we have $\frac{\mathrm{d}\nu}{\mathrm{d}\mathscr{L}^n} = 1$ and hence $\frac{\mathrm{d}\mathscr{L}^n}{\mathrm{d}\nu} = 1$, too. Similarly, on S we have $\mathscr{L}^n \llcorner S = 0$ and hence $\frac{\mathrm{d}\mathscr{L}^n}{\mathrm{d}\nu} = \frac{\mathrm{d}\mathscr{L}^n}{\mathrm{d}|\mu^s|} = 0$. In this sense, the parameter t in the definition of the linear perspective function is introduced to distinguish conveniently between the absolutely continuous and singular parts of the reference measures. We then obtain

$$\begin{aligned}(*) &= \int_A f^\#\Big(1, \frac{\mathrm{d}\mu}{\mathrm{d}\nu}\Big)\,\mathrm{d}\nu + \int_S f^\#\Big(0, \frac{\mathrm{d}\mu}{\mathrm{d}\nu}\Big)\,\mathrm{d}\nu \\ &= \int_A f^\#\Big(1, \frac{\mathrm{d}\mu^{ac}}{\mathrm{d}\mathscr{L}^n}\Big)\,\mathrm{d}\mathscr{L}^n + \int_S f^\#\Big(0, \frac{\mathrm{d}\mu^s}{\mathrm{d}|\mu^s|}\Big)\,\mathrm{d}|\mu^s| = (**),\end{aligned}$$

where we have used that

$$\nu \llcorner A = \mathscr{L}^n \llcorner A \Rightarrow \frac{\mathrm{d}\mu}{\mathrm{d}\nu}|_A = \frac{\mathrm{d}\mu^{ac}}{\mathrm{d}\mathscr{L}^n}|_A,$$

and similarly for μ^s. In conclusion, gathering terms, we obtain by $f^\#(1,\cdot) = f(\cdot)$ and $f^\#(0,\cdot) = f^\infty(\cdot)$ that

$$\overline{\mathfrak{F}}[\mu] = \int_\Omega f\Big(\frac{\mathrm{d}\mu^{ac}}{\mathrm{d}\mathscr{L}^n}\Big)\,\mathrm{d}\mathscr{L}^n + \int_\Omega f^\infty\Big(\frac{\mathrm{d}\mu^s}{\mathrm{d}|\mu^s|}\Big)\,\mathrm{d}|\mu^s|. \tag{4.17}$$

If $f(\mathbf{A}) := \sqrt{1+|\mathbf{A}|^2}$ is the standard area integrand, then $f^\infty(\mathbf{A}) = |\mathbf{A}|$ for $\mathbf{A} \in \mathbb{R}^m$, so that (4.17) gives

$$\langle\mu\rangle := \overline{\mathfrak{F}}(\mu) = \int_\Omega \sqrt{1 + \Big|\frac{\mathrm{d}\mu}{\mathrm{d}\mathscr{L}^n}\Big|^2}\,\mathrm{d}\mathscr{L}^n + |\mu^s|(\Omega). \tag{4.18}$$

We next turn to (lower semi–)continuity result for functionals of measures with respect to (local) weak*–convergence which is often referred to as Reshetnyak–type theorems, following the seminal work [199]. The version we give here is due to Beck & Schmidt in [34, 36], thereby including the work of Kristensen & Rindler [163]; also see the work of Spector [219] for elementary proofs.

Proposition 4.18 (Reshetnyak [199])**.** *Let $m \in \mathbb{N}$, $\Omega \subset \mathbb{R}^n$ open and let (μ_k) be a sequence of $\mathbb{R}^m$–valued Radon measures of finite total variation which converges to a $\mathbb{R}^m$–valued Radon measure of finite total variation μ on Ω in the weak*–sense. Moreover, assume that all measures μ_k and μ take values in some closed convex cone $K \subset \mathbb{R}^m$. Then the following holds:*

(a) Lower Semicontinuity. *If $\widetilde{f}\colon K \to [0,\infty]$ is a lower semicontinuous function, then there holds*

$$\int_\Omega \widetilde{f}\left(\frac{\mathrm{d}\mu}{\mathrm{d}|\mu|}\right)\mathrm{d}|\mu| \leqslant \liminf_{k\to\infty} \int_\Omega \widetilde{f}\left(\frac{\mathrm{d}\mu_k}{\mathrm{d}|\mu_k|}\right)\mathrm{d}|\mu_k|.$$

(b) *If $\mu_k \to \mu$ strictly*[8] *as $k \to \infty$ and $\widetilde{f}\colon K \to [0,\infty)$ is a continuous and 1–homogeneous function, then there holds*

$$\int_\Omega \widetilde{f}\left(\frac{\mathrm{d}\mu}{\mathrm{d}|\mu|}\right)\mathrm{d}|\mu| = \lim_{k\to\infty} \int_\Omega \widetilde{f}\left(\frac{\mathrm{d}\mu_k}{\mathrm{d}|\mu_k|}\right)\mathrm{d}|\mu_k|.$$

(c) *If $f\colon \mathbb{R}^m \to \mathbb{R}$ is a convex function with (LG) and $\mu_k \to \mu$ $\langle\cdot\rangle$–strictly on Ω, i.e.,*

$$\mu_k \overset{*}{\rightharpoonup} \mu \quad \textit{and} \quad \langle\mu_j\rangle(\Omega) + \langle\mu\rangle(\Omega) \qquad \textit{as } k \to \infty,$$

then $\overline{\mathfrak{F}}$ given by (4.17) satisfies $\overline{\mathfrak{F}}(\mu_k) \to \overline{\mathfrak{F}}(\mu)$ as $k \to \infty$.

[8] In the sense that $\mu_k \overset{*}{\rightharpoonup} \mu$ and $|\mu_k|(\Omega) \to |\mu|(\Omega)$ as $k \to \infty$.

Usually, this result is applied in the setting of (4.16). Whereas parts (a) and (b) can be traced back to RESHETNYAK's classical work, part (c) of the previous proposition is a consequence of [163, Thm. 5] (therein stated in a more general context). Let us moreover note that the conditions on f to be positively homogeneous are not restrictive as $f(\mu)$ is defined by the positively homogeneous perspective function. Applying the previous proposition to the particular choice $\mu = \boldsymbol{\varepsilon}(u)$ for $u \in \mathrm{BD}$, we immediately obtain the following corollary.

Corollary 4.19. *Let Ω be an open and bounded Lipschitz subset of $\mathbb{R}^n$, $f\colon \mathbb{R}^{n\times n}_{\mathrm{sym}} \to \mathbb{R}_{\geqslant 0}$ be a convex function satisfying (LG) and let $u, u_1, u_2, ... \in \mathrm{BD}(\Omega)$. Then the following hold for the functional $\overline{\mathfrak{F}}$ given by (4.17):*

(a) *If $u_k \overset{*}{\rightharpoonup} u$ as $k \to \infty$ (in the sense of Section 4.2.1), then $\overline{\mathfrak{F}}(\boldsymbol{\varepsilon}(u)) \leqslant \liminf_{k\to\infty} \overline{\mathfrak{F}}(\boldsymbol{\varepsilon}(u_k))$.*

(b) *If $u_k \to u$ $\langle\cdot\rangle$–strictly as $k \to \infty$ (in the sense of Section 4.2.1), then $\overline{\mathfrak{F}}(\boldsymbol{\varepsilon}(u)) = \lim_{k\to\infty} \overline{\mathfrak{F}}(\boldsymbol{\varepsilon}(u_k))$.*

The corresponding statement for BV is obvious. We conclude this section with two remarks on the relaxations previously discussed.

Remark 4.20 (Lebesgue–Serrin)**.** *In light of Corollary 4.19, we obtain that if $f\colon \mathbb{R}^{n\times n}_{\mathrm{sym}} \to \mathbb{R}_{\geqslant 0}$ is a convex function with (LG), then $u \mapsto \overline{\mathfrak{F}}(\boldsymbol{\varepsilon}(u))$ with $\overline{\mathfrak{F}}$ given by (4.17), is lower semicontinuous with respect to weak*–convergence in* BD*. Moreover, following [137], we even have that $\overline{\mathfrak{F}}(\boldsymbol{\varepsilon}(u_k))$ arises as the weak*–relaxation (or weak*–Lebesgue–Serrin extension of the functional $\mathfrak{F}$ as given by $(\mathrm{P}_{\varepsilon})$ to the space $\mathrm{BD}(\Omega)$, i.e.,*

$$\begin{aligned}\overline{\mathfrak{F}}[u] &:= \int_\Omega f\left(\mathscr{E}u\right)\mathrm{d}x + \int_\Omega f^\infty\left(\frac{\mathrm{d}\mathrm{E}^s u}{\mathrm{d}|\,\mathrm{E}^s u|}\right)\mathrm{d}|\,\mathrm{E}^s u| \\ &= \inf\left\{\liminf_{j\to\infty}\int_\Omega f(\boldsymbol{\varepsilon}(u_j))\,\mathrm{d}x\colon \begin{array}{c}(u_k)\subset \mathrm{LD}(\Omega),\\ u_k \overset{*}{\rightharpoonup} u \text{ in } \mathrm{BD}(\Omega)\end{array}\right\} =: \overline{\mathfrak{F}}^{\mathrm{w}^*}_{\mathrm{LS}}[u].\end{aligned}$$

for all $u \in \mathrm{BD}(\Omega)$ (to keep notation simple, we have put $\overline{\mathfrak{F}}[u] := \overline{\mathfrak{F}}(\boldsymbol{\varepsilon}(u))$). Moreover, by convexity of f, it can be shown [137] that the $\mathrm{L}^1_{\mathrm{loc}}$–relaxation coincides with $\overline{\mathfrak{F}}^{\mathrm{w}^*}_{\mathrm{LS}}$, i.e.,

$$\overline{\mathfrak{F}}^{\mathrm{w}^*}_{\mathrm{LS}}[u] = \inf\left\{\liminf_{j\to\infty}\int_\Omega f(\boldsymbol{\varepsilon}(u_j))\,\mathrm{d}x\colon \begin{array}{c}(u_k)\subset \mathrm{LD}(\Omega),\\ u_k \to u \text{ in } \mathrm{L}^1_{\mathrm{loc}}(\Omega;\mathbb{R}^n)\end{array}\right\}.$$

See also [70, 71] for related integral representations.

Remark 4.21 (On recession functions in the quasiconvex– and rank–one cases)**.** *If the convexity notion is weakened, so for instance f is merely assumed (symmetric) quasiconvex or rank–one–convex, then the limit (4.14) does not necessarily exist and one defines the (upper) recession function $f^\infty\colon \mathbb{R}^{n\times n}_{\mathrm{sym}} \to \mathbb{R}$ by*

$$f^\infty(\mathbf{A}) := \limsup_{t\to\infty}\frac{f(t\mathbf{A})}{t}, \qquad \mathbf{A} \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$$

and we shall see in Chapter 8 how the upper recession function canonically fits into the framework of relaxations in the (symmetric) quasi– and rank–one–convex context; for more detail, see [15, 106, 107, 163] in the BV*– and [202] in the* BD*–case.*

4.3.2. Existence Theory Let $\Omega \subset \mathbb{R}^n$ be a bounded, open Lipschitz set. We recall that the trace operators $\mathrm{Tr}\colon \mathrm{BV}(\Omega;\mathbb{R}^N) \to \mathrm{L}^1(\partial\Omega;\mathbb{R}^N)$ or $\mathrm{Tr}\colon \mathrm{BD}(\Omega) \to \mathrm{L}^1(\partial\Omega;\mathbb{R}^n)$ are not weak*–continuous, and hence we do not have stability of traces with respect to this convergence. This particularly means that the weak*–limits of minimising sequences might have traces different from those of the given Dirichlet data.

To put this observation in context with the theory of convex functionals of measures, we choose an open and bounded, Lipschitz reference domain $\widetilde{\Omega} \subset \mathbb{R}^n$ with $\Omega \Subset \widetilde{\Omega}$. Since $u_0 \in \mathrm{LD}(\Omega)$, it attains traces in $\mathrm{L}^1(\partial\Omega;\mathbb{R}^n)$ and hence, surjectivity[9] of $\mathrm{Tr}\colon \mathrm{LD}(U) \to \mathrm{L}^1(\partial U;\mathbb{R}^n)$ for arbitrary Lipschitz domains $U \subset \mathbb{R}^n$ implies that there exists a map $\widetilde{u_0} \in \mathrm{LD}(\widetilde{\Omega} \setminus \overline{\Omega})$ such that $\mathrm{Tr}(u_0) = \mathrm{Tr}(\widetilde{u_0})$ $\mathcal{H}^{n-1}$–a.e. on $\partial\Omega$ and $\widetilde{u_0}|_{\partial\widetilde{\Omega}} = 0$. We then define the class of competitors $\mathcal{C}_{u_0} := \{w_v\colon\ v \in \mathrm{BD}(\Omega)\}$, where

$$w_v(x) := \begin{cases} v(x) & \text{for } x \in \Omega \\ \widetilde{u_0}(x) & \text{for } x \in \widetilde{\Omega} \setminus \Omega. \end{cases} \tag{4.19}$$

Note that by the BD–gluing principle, Theorem 4.8, $w_v \in \mathrm{BD}(\widetilde{\Omega})$. We now consider the minimisation of

$$\mathcal{F}_{u_0}[w] := \int_{\widetilde{\Omega}} f(\mathrm{E}w) \overset{(4.17)}{=} \int_{\widetilde{\Omega}} f\Big(\frac{\mathrm{dE}w}{\mathrm{d}\mathscr{L}^n}\Big)\,\mathrm{d}\mathscr{L}^n + \int_{\widetilde{\Omega}} f^{\infty}\Big(\frac{\mathrm{dE}^s w}{\mathrm{d}|\,\mathrm{E}^s w|}\Big)\,\mathrm{d}|\,\mathrm{E}^s w| \tag{4.20}$$

over all competitor maps $w = w_v \in \mathcal{C}_{u_0}$. To obtain a more precise representation of the structure of (4.20), we observe that the measures $\mathrm{E}w_v$ can be decomposed as follows due to Theorem 4.8:

$$\begin{aligned} \mathrm{E}w_v &= \mathrm{E}v\llcorner\Omega + (u_0 - \mathrm{Tr}_{\partial\Omega}(u)) \odot \nu_{\partial\Omega}\mathcal{H}^{n-1}\llcorner\partial\Omega + \mathscr{E}\widetilde{u_0}\mathscr{L}^n \llcorner(\widetilde{\Omega}\setminus\Omega) \\ &= \mathscr{E}v\mathscr{L}^n\llcorner\Omega + \mathrm{E}^s v\llcorner\Omega \\ &\quad + (u_0 - \mathrm{Tr}_{\partial\Omega}(w_v)) \odot \nu_{\partial\Omega}\mathcal{H}^{n-1}\llcorner\partial\Omega + \mathscr{E}\widetilde{u_0}\mathscr{L}^n\llcorner(\widetilde{\Omega}\setminus\Omega). \end{aligned} \tag{4.21}$$

Here, $\mathrm{Tr}_{\partial\Omega}$ denotes the boundary trace operator for $\partial\Omega$. Note that $\mathrm{E}\widetilde{u_0}\llcorner(\widetilde{\Omega}\setminus\Omega) = \mathscr{E}\widetilde{u_0}\mathscr{L}^n\llcorner(\widetilde{\Omega}\setminus\Omega)$ because $\widetilde{u_0} \in \mathrm{LD}(\widetilde{\Omega}\setminus\overline{\Omega})$. Let us now insert the expression $\mathrm{E}w_v$ obtained in (4.21) into formula (4.17) to obtain

$$\begin{aligned} \mathcal{F}_{u_0}[w_v] &= \int_{\Omega} f(\mathscr{E}v)\,\mathrm{d}\mathscr{L}^n + \int_{\Omega} f^{\infty}\big(\frac{\mathrm{dE}^s v}{\mathrm{d}|\,\mathrm{E}^s v|}\big)\,\mathrm{d}|\,\mathrm{E}^s v| \\ &\quad + \int_{\partial\Omega} f^{\infty}((\mathrm{Tr}_{\partial\Omega}(u_0 - w_v)) \odot \nu_{\partial\Omega})\,\mathrm{d}\mathcal{H}^{n-1} + \int_{\widetilde{\Omega}\setminus\Omega} f(\mathscr{E}\widetilde{u_0})\,\mathrm{d}\mathscr{L}^n. \end{aligned}$$

As to Theorem 4.23 below (i.e., the absence of gaps when passing to the relaxation) in conjunction with problem (P_ε), this is the functional we actually wish to minimise over all $v \in \mathrm{BD}(\Omega)$ when considering the Dirichlet problem

$$\text{to minimise } \mathfrak{F}[\cdot] \text{ among the Dirichlet class } u \in u_0 + \mathrm{LD}_0(\Omega). \tag{4.22}$$

Before we give the proof of this fact, let us note that since $\widetilde{u_0}$ is fixed, $w_v \in \mathrm{BD}(\Omega)$ minimises $\mathcal{F}_{u_0}$ if and only if v minimises

$$\begin{aligned} \widetilde{\mathfrak{F}}_{u_0}[u] := \int_{\Omega} f(\mathscr{E}u)\,\mathrm{d}\mathscr{L}^n + \int_{\Omega} f^{\infty}\big(\frac{\mathrm{dE}^s u}{\mathrm{d}|\,\mathrm{E}^s u|}\big)\,\mathrm{d}|\,\mathrm{E}^s u| \\ + \int_{\partial\Omega} f^{\infty}((\mathrm{Tr}_{\partial\Omega}(u_0 - u)) \odot \nu_{\partial\Omega})\,\mathrm{d}\mathcal{H}^{n-1}. \end{aligned} \tag{4.23}$$

It is worth noting that the last term takes on the role of a *penalisation term*, penalising the deviation of the boundary values of u from u_0. Another way of viewing this is to introduce the space

$$\mathrm{BD}_{u_0}(\overline{\Omega}) := \big\{v \in \mathrm{BD}(\Omega)\colon\ w_v \in \mathrm{BD}(\widetilde{\Omega})\big\}.$$

[9] By a result of Gagliardo [120], the trace operator $\mathrm{Tr}\colon \mathrm{BV}(\Omega;\mathbb{R}^n) \to \mathrm{L}^1(\partial\Omega;\mathbb{R}^n)$ is onto and, as a consequence of [223], the trace operator on BD coincides with the BV–trace operator on $\mathrm{BV}(\Omega;\mathbb{R}^n)$ and hereafter is onto itself.

The previous definition is taylored in a way such that elements $w \in \mathrm{BD}_{u_0}(\overline{\Omega})$ have symmetric gradient measures $\mathrm{E}w$ which extend to measures $\overline{\mathrm{E}w}$ on $\overline{\Omega}$. Then it follows from (4.23) that if we introduce the functional $\overline{F}_{u_0}\colon \mathrm{BD}_{u_0}(\overline{\Omega}) \to \mathbb{R}$ by

$$\overline{F}_{u_0}[w] := \int_\Omega f(\mathscr{E}w)\,\mathrm{d}x + \int_{\overline{\Omega}} f^\infty\Big(\frac{\mathrm{d}\mathrm{E}^s w}{\mathrm{d}|\,\mathrm{E}^s w|}\Big)\,\mathrm{d}|\,\mathrm{E}^s w|, \qquad w \in \mathrm{BD}_{u_0}(\overline{\Omega})$$

where $\mathrm{E}w$ is now thought as the aforementioned extension of $\mathrm{E}w\llcorner\Omega$ to $\overline{\Omega}$, we obtain that $u \in \mathrm{BD}(\Omega)$ minimises $\widetilde{\mathfrak{F}}_{u_0}$ if and only if $u \in \mathrm{BD}_{u_0}(\overline{\Omega})$ and u minimises $\overline{F}_{u_0}$ over all competitors in $\mathrm{BD}_{u_0}(\overline{\Omega})$. This formulation is a practical formalisation of '*a minimiser having jumps at the boundary*'.

For later purposes, we provide the following convenient reformulation of the Reshetnyak theorem, Proposition 4.18, for $\overline{F}_{u_0}[\cdot]$ as given above.

Lemma 4.22. *Let Ω be an open and bounded Lipschitz domain in $\mathbb{R}^n$ and let $f\colon \mathbb{R}^m \to \mathbb{R}$ be a convex function which satisfies $|f(z)| \leqslant 1 + L|z|$ for some $L > 0$ and all $z \in \mathbb{R}^m$. Assume moreover that (μ_k) is a sequence of finite Radon measures on $\overline{\Omega}$ which converges*[10] *in the weak*–sense to μ in the space of finite Radon measures on $\overline{\Omega}$. If there holds $\lim_{k\to\infty} |(\mathscr{L}^n, \mu_k)|(\overline{\Omega}) = |(\mathscr{L}^n, \mu)|(\overline{\Omega})$ for the $\mathbb{R}^{m+1}$–valued measures $(\mathscr{L}^n, \mu_k)$ and $(\mathscr{L}^n, \mu)$, then we also have*

$$\begin{aligned}\int_\Omega f\Big(\frac{\mathrm{d}\mu^{ac}}{\mathrm{d}\mathscr{L}^n}\Big)\,\mathrm{d}x &+ \int_{\overline{\Omega}} f^\infty\Big(\frac{\mathrm{d}\mu^s}{\mathrm{d}|\mu^s|}\Big)\,\mathrm{d}|\mu^s| \\ &= \lim_{k\to\infty}\Big(\int_\Omega f\Big(\frac{\mathrm{d}\mu_k^{ac}}{\mathrm{d}\mathscr{L}^n}\Big)\,\mathrm{d}x + \int_{\overline{\Omega}} f^\infty\Big(\frac{\mathrm{d}\mu_k^s}{\mathrm{d}|\mu_k^s|}\Big)\,\mathrm{d}|\mu_k^s|\Big),\end{aligned}$$

where μ^{ac}, μ^s are the absolutely or singular parts of μ with respect to $\mathscr{L}^n$ (and analogously for μ_k^{ac}, μ_k^s).

We now come to

Theorem 4.23. *Let $f\colon \mathbb{R}^{n\times n}_{\mathrm{sym}} \to \mathbb{R}$ be a convex integrand with (LG) and define $\widetilde{\mathfrak{F}}_{u_0}\colon \mathrm{BD}(\Omega) \to \mathbb{R}$ by (4.23). Then the following hold:*

(i) $\widetilde{\mathfrak{F}}_{u_0}$ *possesses a minimiser over* $\mathrm{BD}(\Omega)$.

(ii) We have

$$\min_{\mathrm{BD}(\Omega)} \widetilde{\mathfrak{F}}_{u_0} = \inf_{\mathcal{D}_{u_0}} \mathfrak{F}, \tag{4.24}$$

where $\mathcal{D}_{u_0} := u_0 + \mathrm{LD}_0(\Omega)$ is the Dirichlet class corresponding to u_0.

Proof. Ad (i). We start by recalling that the minimisation of $\widetilde{\mathfrak{F}}_{u_0}$ over $\mathrm{BD}(\Omega)$ is equivalent to the minimisation of $\mathcal{F}_{u_0}$ over $\mathcal{C}_{u_0}$. By (LG), we see that $\mathcal{F}_{u_0}$ is bounded below on $\mathcal{C}_{u_0}$ and thus we find $(w_{v_k}) \subset \mathcal{C}_{u_0}$ such that $\mathcal{F}_{u_0}[w_{v_k}] \to \inf_{\mathcal{C}_{u_0}} \mathcal{F}_{u_0}$. We note that if (LG) holds, then we find by sending $t \to \infty$ in the inequality

$$\frac{c_1|tz| - \varepsilon}{t} \leqslant \frac{f(tz)}{t} \qquad \text{for all } z \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$$

that $c_1|z| \leqslant f^\infty(z)$ for all $z \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$. Moreover, recall that there exists a constant $\widetilde{c} > 0$ such that $\widetilde{c}|a|\,|b| \leqslant f^\infty(a \odot b)$ for all $a, b \in \mathbb{R}^n$. From this we infer that, with a fixed constant $M := \max_k \mathcal{F}_{u_0}[w_{v_k}]$,

$$M \geqslant \mathcal{F}_{u_0}[w_{v_k}] \geqslant c_1|\varepsilon(v_k)|(\Omega) + \widetilde{c}\int_{\partial\Omega} |\operatorname{Tr}(u_0 - v_k)|\,\mathrm{d}\mathcal{H}^{n-1} + \int_{\widetilde{\Omega}\setminus\overline{\Omega}} f(\mathscr{E}\widetilde{u_0})\,\mathrm{d}\mathscr{L}^n - \varepsilon\mathscr{L}^n(\Omega).$$

[10] Here, weak*–convergence in the space of finite Radon measures on $\overline{\Omega}$ requires testing against all continuous functions $\varphi\colon \overline{\Omega} \to \mathbb{R}^m$ with compact support.

From this we deduce that $(\mathrm{E}v_k)$ is uniformly bounded with respect to the total variation norm, and since $\mathrm{Tr}(u_0) \in \mathrm{L}^1(\partial\Omega;\mathbb{R}^n)$, we must have uniform boundedness of $(\mathrm{Tr}(v_k)|_{\partial\Omega})$ in $\mathrm{L}^1(\partial\Omega;\mathbb{R}^n)$. In conclusion, we find that (w_{v_k}) is uniformly bounded in $\mathrm{BD}(\widetilde{\Omega})$. By the BD–weak*–compactness principle (see section 4.2.1), there exists a subsequence $(w_{v_{k(j)}}) \subset (w_{v_k})$ and an element $w \in \mathrm{BD}(\widetilde{\Omega})$ such that $w_{v_{k(j)}} \overset{*}{\rightharpoonup} w$ as $j \to \infty$. Clearly, w coincides with $\widetilde{u_0}$ on $\widetilde{\Omega}\setminus\overline{\Omega}$ and can be written in the form w_v for some $v \in \mathrm{BD}(\Omega)$. By Reshetnyak's lower semicontinuity theorem, Proposition 4.18, we find

$$\inf_{\mathcal{C}_{u_0}} \mathcal{F}_{u_0} \leqslant \mathcal{F}_{u_0}[w_v] \leqslant \liminf_{j\to\infty} \mathcal{F}_{u_0}[w_{v_{k(j)}}] = \inf_{\mathcal{C}_{u_0}} \mathcal{F}_{u_0}.$$

In conclusion, w_v is a minimiser for $\mathcal{F}_{u_0}$ over $\mathcal{C}_{u_0}$ and consequently v is a minimiser for $\widetilde{\mathfrak{F}}_{u_0}$ over $\mathrm{BD}(\Omega)$.

Ad (ii). Let us firstly note that since $\mathfrak{F}$ and $\widetilde{\mathfrak{F}}_{u_0}$ coincide on $\mathcal{D}_{u_0}$, we obviously have '$\leqslant$' in (4.24). To obtain the reverse direction, let $u \in \mathrm{BD}(\Omega)$ be a minimiser of $\widetilde{\mathfrak{F}}_{u_0}$ as shown to exist in (i). We next apply the area–strict approximation theorem, Lemma 4.9, to the extended function w_u (defined by (4.19)) and thereby obtain a sequence $(v_k) \subset u_0|_\Omega + \mathrm{C}_c^\infty(\Omega;\mathbb{R}^n)(\subset \mathcal{D}_{u_0})$ such that $w_{v_k} \to w_u$ $\langle\cdot\rangle$–strictly in $\widetilde{\Omega}$. By the continuity part of Proposition 4.18, we conclude $\mathcal{F}_{u_0}[w_{v_k}] \to \mathcal{F}_{u_0}[w_u]$ as $k \to \infty$. Hence, by minimality of u for $\widetilde{\mathfrak{F}}_{u_0}$ and the fact that $\widetilde{\mathfrak{F}}_{u_0}$ differs from $\mathcal{F}_{u_0}$ only by a constant, we see that $(v_k) \subset \mathcal{D}_{u_0}$ is a minimising sequence for $\mathfrak{F}$. Since $\mathfrak{F}$ and $\widetilde{\mathfrak{F}}_{u_0}$ coincide on $\mathcal{D}_{u_0}$, we deduce

$$\inf_{\mathcal{D}_{u_0}} \mathfrak{F} \leqslant \mathfrak{F}[v_k] = \widetilde{\mathfrak{F}}_{u_0}[v_k] \to \min_{\mathrm{BD}(\Omega)} \widetilde{\mathfrak{F}}_{u_0}, \qquad k \to \infty$$

and the proof is complete. □

Now we generalise two common concepts of minimisers from the BV–context (cp. [40]) to the BD–setting. In doing so, the above convexity and linear growth assumptions on the integrand f are tacitly assumed together with $u_0 \in \mathrm{LD}(\Omega)$ for the Dirichlet datum.

Definition 4.24 (BD–minima)**.** *A function $u \in \mathrm{BD}(\Omega)$ is called a* BD–minimiser (for $\mathfrak{F}$) *(subject to the Dirichlet class $\mathcal{D}_{u_0}$) provided u minimises the relaxed functional $\widetilde{\mathfrak{F}}_{u_0}$ given by (4.23), i.e., for all $v \in \mathrm{BD}(\Omega)$ there holds $\widetilde{\mathfrak{F}}_{u_0}[u] \leqslant \widetilde{\mathfrak{F}}_{u_0}[v]$.*

With this terminology, Theorem 4.23 asserts that a BD–minimiser always exists. Another notion of minimiser that incorporates in a more direct way the specific compactness features of the space BD is that of *generalised minima* with origins in the work of Bildhauer [40, 42]. Due to the weak*–compactness principle for BD (cp. Section 4.2.1), any sequence being uniformly bounded with respect to the BD–norm allows to extract a subsequence which strongly converges in L^1 to some function $u \in \mathrm{BD}(\Omega)$. This motivates the following definition.

Definition 4.25 (Generalised minima)**.** *The set $\mathrm{GM}(\mathfrak{F};u_0)$ of* generalised minimisers *for a given variational integral $\mathfrak{F}$ the form (P_ε) of linear growth consists of all those $u \in \mathrm{BD}(\Omega)$ for which there exists an $\mathfrak{F}$–minimising sequence $(u_k) \subset \mathcal{D}_{u_0}$ whose L^1–limit is u.*

If there is no confusion about the Dirichlet datum u_0, we simply write $\mathrm{GM}(\mathfrak{F}) := \mathrm{GM}(\mathfrak{F};u_0)$. Though defined by different means, BD– and generalised minima give rise to the same concept of minima which shall allow us to switch between the two concepts.

Proposition 4.26. *In the situation of the last definition, a map $u \in \mathrm{BD}(\Omega)$ is a BD–minimiser for $\mathfrak{F}$ (subject to the Dirichlet data u_0) if and only if $u \in \mathrm{GM}(\mathfrak{F};u_0)$.*

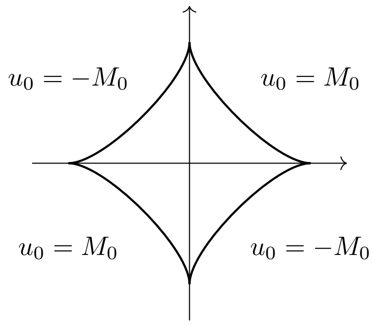

Figure 4.1: Santi's domain.

Proof. Assume $u \in \mathrm{GM}(\mathfrak{F}; u_0)$. Then we find a $\mathfrak{F}$–minimising sequence $(u_k) \subset \mathcal{D}_{u_0}$ such that $u_k \to u$ in $\mathrm{L}^1(\Omega; \mathbb{R}^n)$ as $k \to \infty$. By Remark 4.20, the above easily implies lower semicontinuity of $\overline{\mathfrak{F}}_{u_0}$ with respect to L^1–convergence and since $\mathfrak{F}[v] = \widetilde{\mathfrak{F}}_{u_0}[v]$ for all $v \in \mathcal{D}_{u_0}$,

$$\widetilde{\mathfrak{F}}_{u_0}[u] \leqslant \liminf_{k\to\infty} \widetilde{\mathfrak{F}}_{u_0}[u_k] = \liminf_{k\to\infty} \mathfrak{F}[u_k] = \inf_{\mathcal{D}_{u_0}} \mathfrak{F},$$

we see by (4.24) that u is a BD–minimiser. Now, if u is a BD–minimiser, we may proceed as in the proof of Theorem 4.23 to conclude the existence of a sequence $(u_k) \subset u_0 + \mathrm{C}_c^\infty(\Omega; \mathbb{R}^n)$ such that $u_k \to u$ $\langle \cdot \rangle$–strictly as $k \to \infty$, and by Proposition 4.18 we see that (u_k) is $\mathfrak{F}$–minimising. Because weak*–convergence implies L^1–convergence, $u \in \mathrm{GM}(\mathfrak{F}; u_0)$. The proof is complete. □

We conclude this section by remarking that BD–minimisers can be characterised by an Euler–Lagrange equation as well, but due to the possible singular part, a greater flexibility with respect to admissible test maps is required. This has been firstly accomplished by Anzellotti [20] and is explained for the reader's convenience in Chapter 10.4 in the appendix.

4.4. (Non–)Attainment of Boundary Values: The Santi and Finn Examples

We finally give two examples which demonstrate the interplay of uniqueness of minima of variational principles (1.1) subject to linear growth hypotheses and the geometry of the underlying domains. These examples are discussed in slight more detail than here in Schmidt's habilitation thesis [211, Chpt. 1] which also serves as a general reference for the present section.

As it will turn out crucial in the following chapters, let us again explicitely note the occurance of the third term on the right side of (4.23) which penalises the deviation from the given boundary values u_0. Since f^∞ is positively 1–homogeneous, it is never strictly convex and thereby gives rise to possible non–uniqueness of BD– (or BV–minima). Before we present two examples that demonstrate this phenomenon, let us briefly recall a criterion for the attainment of the correct boundary values due to Miranda [180, 181], taylored for functionals of the form (P) with general $n \in \mathbb{N}$ and $N = 1$. Assuming that Ω is an open and bounded Lipschitz domain in $\mathbb{R}^n$, $\partial\Omega$ has non–negative (generalised) mean curvature in a neighbourhood of some boundary point $x_0 \in \partial\Omega$ and the restriction of the extended boundary datum $\widetilde{u_0}$ to $\overline{\Omega}^c$ is continuous at x_0, then *every* BV–minimiser of the relaxed area integral attains the boundary value $\widetilde{u_0}(x_0)$ at x_0 continuously (cp. [211, p.10]). As a consequence, if this is the case for some x_0, then BV–minimisers of the relaxed area–functional are unique.

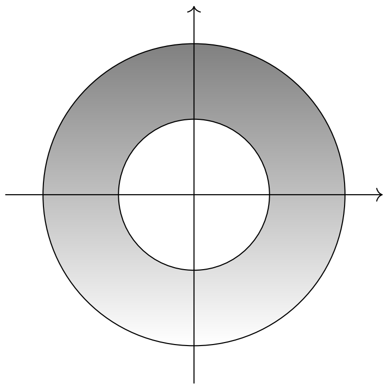

Figure 4.2: Finn's domain.

To produce non–uniqueness of minimisers, it is therefore sensible to work with domains violating the condition of non–negative mean curvature which, in turn, is the starting point in both the aforementioned examples due FINN [105] and SANTI [206] to be discussed next. Both examples deal with the relaxed version of the scalar area–integral

$$\widetilde{\mathfrak{F}}_{u_0}[u] = \int_\Omega \sqrt{1+|\nabla u|^2}\,\mathrm{d}x + |\,\mathrm{D}^s u|(\Omega) + \int_{\partial\Omega} |u-u_0|\,\mathrm{d}\mathcal{H}^{n-1} \tag{4.25}$$

for $u \in \mathrm{BV}(\Omega)$; again, $\mathrm{D}u = \nabla u \mathscr{L}^n + \mathrm{D}^s u$ is the Radon–Nikodym decomposition of $\mathrm{D}u$ for $u \in \mathrm{BV}(\Omega)$.

- The situation of Santi's example is illustrated in Figure[11] 4.1. Hence, we have $n = 2$, and the boundary values are adjusted in a way such $u_0 = M_0$ on the two opposite sides in the first and third quadrant and $u_0 = -M_0$ on the remaining opposite sides of the curvilinear rhombus as given in Figure 4.1 for some $M_0 \in \mathbb{R}$. Santi then shows that a particular minimiser of the relaxed area integral belongs to $\mathrm{C}^\infty(\overline{\Omega})$ is bounded independently of M_0. In consequence, if M_0 is chosen sufficiently large, then the boundary interior trace of $u \in \mathrm{BV}_{u_0}(\overline{\Omega})$ deviates at least by $M_0/2$ from u_0 on the boundary of the domain and so it is seen that $u + \mathbb{1}_\Omega z$ with $-M_0/2 \leqslant z \leqslant M_0/2$ is a (generalised) minimiser of the relaxed area integral.

- The situation of Finn's example is illustrated in Figure 4.2. Hence, we have again $n = 2$ together with the annulus $\Omega = \mathrm{B}(0,2) \setminus \overline{\mathrm{B}(0,1)}$ in $\mathbb{R}^2$. Here we assume that $u_0 \equiv M_0$ on $\partial\mathrm{B}(0,1)$ and $u_0 \equiv 0$ on $\partial\mathrm{B}(0,2)$, where $M_0 \in \mathbb{R}$ is a constant. The approach of Finn now relies on the fact that, by radial symmetry of the integrand and symmetry of Ω with respect to the origin, the minimiser can be computed explicitely as the solution of a suitable ordinary differential equation and therefore is unique. In a second step, it is shown that the minimiser is bounded independently of M_0 and hence cannot attain the correct boundary values provided M_0 is chosen suitably large.

We conclude this chapter with the observation that the examples just discussed above are in accordance with the Miranda criterion; in fact, both the Santi and Finn domains as illustrated in Figures 4.1 and 4.2 do not have non–negative generalised mean curvature.

[11] I am grateful to Lisa Beck for having provided me with the graphic depicted in Figure 4.1.

CHAPTER 5

Sobolev Regularity for Symmetric–Convex Problems

The aim of the present chapter is to give sufficient conditions on the ellipticity of a variational integrand $f\colon \mathbb{R}^{n\times n}_{\mathrm{sym}} \to \mathbb{R}$ to produce generalised minima (in the sense of the preceding chapter) which not only belong to $\mathrm{BD}(\Omega)$ but actually to some local Sobolev space $\mathrm{W}^{1,p}_{\mathrm{loc}}(\Omega;\mathbb{R}^n)$. We give an overview of known regularity results in the BV–setting and moreover discuss the extent to which Caccioppoli–type inequalities improve the regularity of generalised minima. Finally, we conclude with a discussion of related results in the p–growth setting, $1<p<\infty$, where a Korn–type inequality is available and compare the corresponding results.

Structure of the chapter. In Section 5.1, we give an overview of available higher integrability results in the full gradient, i.e., BV–case and indicate the difficulties for the symmetric gradient case. In Section 5.2, we carefully examine the class of μ–elliptic integrands and with this scale at our disposal, we state and discuss the main result of this chapter in Section 5.3. Section 5.4 is devoted to the dual problem and gathers regularity and uniqueness results for the dual solution. Before we deal with the Sobolev regularity, we establish in Section 5.5 two crucial interpolation results. In a next step, we discuss the impact of Caccioppoli–type inequalities in the linear growth setup as to higher integrability in Section 5.6. Finally, we prove in Section 5.7 the main result of this chapter and conclude in Section 5.8 with an outlook of other settings.

5.1. Description and Contextualisation of Results

In the present chapter we are concerned with the higher Sobolev regularity of solutions of the variational principle

$$\text{to minimise}\quad \mathfrak{F}[v] := \int_\Omega f(\boldsymbol{\varepsilon}(v))\,\mathrm{d}x \qquad \text{over } \mathcal{D}_{u_0} := u_0 + \mathrm{LD}_0(\Omega), \tag{5.1}$$

where $u_0 \in \mathrm{LD}(\Omega)$ and $f\colon \mathbb{R}^{n\times n}_{\mathrm{sym}} \to \mathbb{R}$ is a convex C^2–integrand of linear growth meaning that there exist constants $a_1, a_2, a_3 > 0$ such that

$$a_1|\xi| - a_2 \leqslant f(\xi) \leqslant a_3(1+|\xi|) \qquad \text{holds for all } \xi \in \mathbb{R}^{n\times n}_{\mathrm{sym}}. \tag{5.2}$$

From the discussion in Chapter 4.3.2 and, in particular, Ornstein's Non–Inequality, existence of generalised minima a priori can only be established in the space $\mathrm{BD}(\Omega)$. As $\mathrm{BV}(\Omega;\mathbb{R}^n) \subsetneq \mathrm{BD}(\Omega)$, it is natural to ask under which additional assumptions (particularly on f) we are able to demonstrate that the *full gradients* of minima qualify as finite $\mathbb{R}^{n\times n}$–valued Radon measures. The chief goal of this chapter is to give such a condition. As we shall see, in the framework as considered here, it always comes with some improvement of higher integrability and thus restricts the minimal space which certain minima are contained in to $\mathrm{W}^{1,p}_{\mathrm{loc}}$ for some $p>1$.

	result	additional assumptions	contribution
$1<\mu<1+\frac{2}{n}$	all GMs in $\mathrm{W}^{1,L\log L}_{\mathrm{loc}}$	–	Bildhauer [40]
$1<\mu<3$	one GM in $\mathrm{W}^{1,p}_{\mathrm{loc}}$ for some $1<p<\infty$	local boundedness	Bildhauer [43]
$\mu=3$	one GM in $\mathrm{W}^{1,L\log L}_{\mathrm{loc}}$	local boundedness	Bildhauer [43]
$\mu=3$	all GMs in $\mathrm{W}^{1,L\log L}_{\mathrm{loc}}$	local boundedness	Beck & Schmidt [34]
$\mu>3$	–	–	–

Table 5.1: Known Sobolev regularity results for generalised minima (GM) of μ–elliptic integrands in the *full* gradient case. In the borderline case $\mu=3$, $\mathrm{W}^{1,L\log L}_{\mathrm{loc}}$–regularity is known whereas for $\mu>3$ neither a positive nor a negative result is available in the *autonomous setting* (in the non–autonomous case, a counterexample is due to Bildhauer [43], also see Remark 5.1).

5.1.1. Results on the Dirichlet Problem on BV To explain our method, it is convenient to firstly report on the available higher integrability results for a particular scale of strongly convex, the *μ–elliptic* variational integrals (see Section 5.2 for the requisite definition and discusssion) in the full gradient case

$$\mathcal{F}[u]:=\int_\Omega f(\nabla v)\,\mathrm{d}x,\qquad v\in\mathcal{D}:=u_0+\mathrm{W}^{1,1}_0(\Omega;\mathbb{R}^N),\tag{5.3}$$

where $u_0\in\mathrm{W}^{1,1}(\Omega;\mathbb{R}^N)$ is a given boundary datum. Using a vanishing viscosity approach, Bildhauer [43] provided the first higher integrability results for gradients of generalised minima. Precisely, assuming $u_0\in\mathrm{W}^{1,2}(\Omega;\mathbb{R}^N)$ for the boundary data, the functional $\mathcal{F}$ is stabilised by adding small Laplacians, i.e., we consider

$$\mathcal{F}_\delta[v]:=\mathcal{F}[v]+\frac{\delta}{2}\int_\Omega|\nabla v|^2\,\mathrm{d}x\qquad\text{on }\ \mathcal{D}:=u_0+\mathrm{W}^{1,2}_0(\Omega;\mathbb{R}^N)$$

and finally aim for sending $\delta\searrow 0$. Denoting the unique minimiser of $\mathcal{F}_\delta$ over $\mathcal{D}$ by u_δ, it is easy to prove that (u_δ) is a minimising sequence for $\mathcal{F}$. Bildhauer then was able to show that if (u_δ) satisfies the *local boundedness assumption*

$$\text{for all } K\Subset\Omega \text{ there exists } C(K)>0 \text{ with } \sup_{0<\delta<1}\|u_\delta\|_{\mathrm{L}^\infty(K;\mathbb{R}^N)}\leqslant C(K),\tag{5.4}$$

then the weak*–limit u of (u_δ) belongs to $\mathrm{W}^{1,p}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ for some $p=p(\mu)>1$ provided $1<\mu<3$, and to $\mathrm{W}^{1,L\log L}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ provided $\mu=3$. Apart from the strong assumptions made on the particular minimising sequence, the boundary data and the functional itself, the result merely applies to one particular such generalised minimiser. The reason for this is the possible non–uniqueness of generalised minima which, in turn, is a consequence of the recession parts in the relaxed variational integral. Indeed, even if $f\in\mathrm{C}^2(\mathbb{R}^{N\times n})$ is strictly convex, the recession function $f^\infty\colon\mathbb{R}^{N\times n}\to\mathbb{R}$ is positively 1–homogeneous and thus never strictly convex. In consequence, if a minimiser does not have vanishing singular part with respect to Lebesgue measure, uniqueness in general fails as f and f^∞ act on two mutually singular parts of the gradients. To achieve uniqueness, one must therefore genuinely rule out the singular parts of minima. This has been achieved recently by Beck & Schmidt [34] by sophisticated use of the Ekeland variational principle in the negative Sobolev space $\mathrm{W}^{-1,1}$ for the borderline case $\mu=3$. Referring the reader for the precise outline to [34], the general streamline is this: Starting from an arbitrary minimising sequence $(u_k)\subset u_0+\mathrm{W}^{1,1}_0(\Omega;\mathbb{R}^N)$, the Ekeland variational principle yields another minimising sequence (v_k) that is $\mathrm{W}^{-1,1}$–close to (u_k) and has the same weak*–limit. This new sequence (v_k) is then shown to be a sequence of almost minimisers to suitably stabilised functionals and thus can be proved to belong to $\mathrm{W}^{2,2}_{\mathrm{loc}}$. At this point it is possible to adapt Bildhauer's approach and hence, by arbitrariness of (u_k), uniqueness and the aforementioned higher regularity results follow at once. However, it needs to be stressed that Beck & Schmidt's method of proof also relies on a version of the local boundedness assumption. Further, assuming Uhlenbeck, i.e., radial structure of the integrands, stronger results such as $\mathrm{C}^{1,\alpha}$–regularity of generalised minima can be achieved; see [40, 43, 35].

Remark 5.1 (The Case $\mu > 3$)**.** *By the results gathered in Table 5.1, it is obvious that $\mu = 3$ (an instance of which is the usual area integrand, see Example 5.3 below, displays a borderline case. If we allow for non–autonomous integrands $f\colon \Omega \times \mathbb{R}^{N\times n} \to \mathbb{R}$ satisfying a suitable generalisation of the usual μ–ellipticity condition, then* Bildhauer *[43] proved by a similar method as in the degenerate weighted setting given in [126] that such integrands indeed may produce generalised minimisers in $(\mathrm{BV}\setminus\mathrm{W}^{1,1})(\Omega;\mathbb{R}^N)$. However, for the corresponding counterexample to $\mathrm{W}^{1,1}$–regularity, x–dependence seems a crucial ingredient, and to the best of our knowledge it is not known whether – even in the presence of radial structure – μ–ellipticity can give $\mathrm{W}^{1,1}$–regularity of generalised minima provided $\mu > 3$.*

5.1.2. Obstructions in the BD–case Going back to functionals $\mathfrak{F}$ as given by (5.1), the main difficulty lies in the appearance of the *full difference quotients* when aiming for higher integrability estimates. Testing the Euler–Lagrange equation of a suitably stabilised functional with the canonical choice $\varphi := \Delta^{-}_{s,h}(\rho^2\Delta^{+}_{s,h}v_k)$, where $\Delta^{\pm}_{s,h}v(x) := \frac{1}{h}(v(x\pm he_s)-v(x))$ are the forward or backward difference quotients, respectively, and v_k is a minimiser for the stabilised functional, we need to uniformly control the L^1–norms of (∇v_k) locally. By Ornstein's Non–Inequality, $\Delta^{+}_{s,h}v$ cannot even be bounded locally in L^1 for $v \in \mathrm{BD}$, and thus the suitable device hence is to work with finite differences instead of difference quotients and to establish estimates for carefully chosen Besov–norms of the symmetric gradients.

To employ this method, we in turn utilise the regularity of the dual solution (cp. Section 5.4.1 for this terminology) and its interplay with the primal problem. In order to state our results in Section 5.3, we now discuss the requisite notion of μ–ellipticity.

5.2. The Scale of μ–elliptic Integrands

Throughout the present chapter we shall further assume that $f \in \mathrm{C}^2(\mathbb{R}^{n\times n}_{\mathrm{sym}})$ is a μ–*elliptic* (and thus, in particular, convex) integrand. This notion of ellipticity originates from a series of papers by Bildhauer, Fuchs and Mingione [40, 42, 119, 45] in their study of minimisation problems of the type (5.3), where $\boldsymbol{\varepsilon}$ was replaced by the full gradients and which we recall here for completeness:

Definition 5.2 (μ–ellipticity)**.** *Let $1 < \mu < \infty$. A C^2–integrand $f\colon \mathbb{R}^{n\times n}_{\mathrm{sym}} \to \mathbb{R}$ is called* μ–elliptic *if and only if there exist $0 < \lambda \leqslant \Lambda < \infty$ such that*

$$\lambda\frac{|\mathbf{A}|^2}{(1+|\mathbf{B}|^2)^{\frac{\mu}{2}}} \leqslant \langle f''(\mathbf{B})\mathbf{A},\mathbf{A}\rangle \leqslant \Lambda\frac{|\mathbf{A}|^2}{(1+|\mathbf{B}|^2)^{\frac{1}{2}}} \tag{5.5}$$

holds for all $\mathbf{A},\mathbf{B} \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$.

This notion canonically carries over to functions f defined on $\mathbb{R}^{N\times n}$ with the obvious modifications. As a direct consequence of the definition, μ–elliptic integrands are

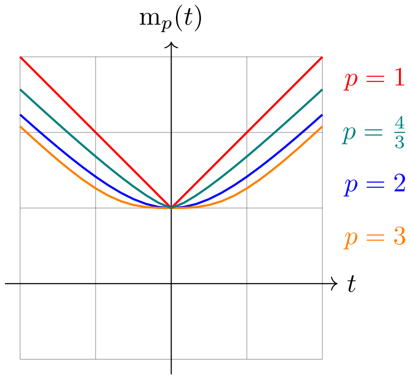

Figure 5.1: The scale $(\mathrm{m}_p)_{1\leqslant p<\infty}$. For $p > 2$, the graph of m_p is very flat at zero, thus leading to degeneracy of its second derivatives, $\mathrm{m}_p''(0) = 0$, whereas the second derivatives become singular for $|t| \searrow 0$ if $1 < p < \infty$. In the borderline case $p = 1$, ellipticity is completely lost also away from zero.

automatically strongly convex. In particular, we have some control on the ellipticity ratio

$$\mathbf{B} \mapsto \frac{\text{largest eigenvalue of } f''(\mathbf{B})}{\text{smallest eigenvalue of } f''(\mathbf{B})}, \qquad \mathbf{B} \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$$

of the variational integrands. This, in turn, is necessary to establish higher Sobolev regularity of minima. However, note that the ellipticity ratio degenerates as $|\mathbf{B}| \to \infty$.

Before discussing several examples of μ–elliptic integrands, let us briefly comment in detail why the case $\mu = 1$ is excluded. To this end, consider the function $f\colon \mathbb{R}_{\geqslant 0} \to \mathbb{R}$ given by $f(t) := t\log(\mathrm{e}+t)$, $t \in \mathbb{R}_{\geqslant 0}$. Then there holds for $t > 0$

$$f'(t) = \log(e+t) + \frac{t}{\mathrm{e}+t}, \quad \text{and} \quad f''(t) = \frac{1}{\mathrm{e}+t} + \frac{\mathrm{e}}{(\mathrm{e}+t)^2}.$$

Hence, taking a mollification[1] of $\widetilde{f} := f(|\cdot|)$, it is easily seen that $\widetilde{f}$ is a 1–elliptic integrand, however, it is not of linear but $L\log L$–growth. This is typical indeed, and so 1–elliptic integrands fall outside the scope of linear growth functionals. On the other hand, the variational problems as considered here are covered by the work of Fuchs & Seregin [116] and Frehse & Seregin [113] with if we consider 1–elliptic integrands; see Section 5.8 below.

Before we pass to various examples demonstrating the richness and also the limitations of the scale of μ–ellipticity, let us remark the typical (p,q)–behaviour of μ–elliptic integrands on the level of second derivatives (as $\mu > 1$ in general). By this, we mean that such the Hessians of the integrands have different growth bounds from below and above. This also indicates that in view of regularity it is more appropriate to make use of approaches that especially arise in the (p,q)–context, see [176] and [40, Chapters 3 and 4] for more information.

Example 5.3 (The Area–type Integrands)**.** *Given $1 \leqslant p < \infty$, the p–th area integrand $\mathrm{m}_p\colon \mathbb{R}^N \to \mathbb{R}$ is given by*

$$\mathrm{m}_p(\xi) := \big(1 + |\xi|^p\big)^{\frac{1}{p}}, \qquad \xi \in \mathbb{R}^N,$$

[1] Since $f(|\cdot|)$ is not differentiable, it needs to be mollified in order to serve as a possible C^2–μ–elliptic integrand.

and the usual area integrand is retrieved by setting $p = 2$, and is $\mu = 3$–elliptic as can be seen from the estimate (see [40, Rem. 4.3])

$$\frac{c_1}{\sqrt{1+|\mathbf{A}|^2}}\Big(|\mathbf{B}|^2 - \frac{\langle \mathbf{A},\mathbf{B}\rangle}{1+|\mathbf{A}|^2}\Big) \leqslant \langle \mathrm{m}_2''(\mathbf{A})\mathbf{B},\mathbf{B}\rangle \leqslant \frac{c_2}{\sqrt{1+|\mathbf{A}|^2}}\Big(|\mathbf{B}|^2 - \frac{\langle \mathbf{A},\mathbf{B}\rangle}{1+|\mathbf{A}|^2}\Big)$$

for all $\mathbf{A},\mathbf{B}\in\mathbb{R}^N$ and two independent constants $c_1, c_2 > 0$. Within the scale of convex integrands $(\mathrm{m}_p)_{1\leqslant p<\infty}$, the area integrand m_2 is the only member to be μ–elliptic for some suitable μ. Indeed, as pointed out by Schmidt *[211, p. 7], if*[2] *$p > 1$, then there exists $C_p > 0$ such that*

$$\begin{aligned} C_p^{-1}|\mathbf{Z}|^{p-2}|\mathbf{Y}|^2 &\leqslant \langle \mathrm{m}_p''(\mathbf{Z})\mathbf{Y},\mathbf{Y}\rangle \leqslant C_p|\mathbf{Z}|^{p-2}|\mathbf{Y}|^2 && \text{if } |\mathbf{Z}|\leqslant 1,\\ C_p^{-1}|\mathbf{Z}|^{-1-p}|\mathbf{Y}|^2 &\leqslant \langle \mathrm{m}_p''(\mathbf{Z})\mathbf{Y},\mathbf{Y}\rangle \leqslant C_p|\mathbf{Z}|^{-1}|\mathbf{Y}|^2 && \text{if } |\mathbf{Z}|> 1. \end{aligned}$$

In particular, if $p > 2$, then m_p'' vanishes at $\mathbf{Z} = 0$ (i.e., degenerates) whereas if $1 < p < 2$, then $\mathrm{m}_p''(\mathbf{Z})$ becomes singular as $|\mathbf{Z}| \searrow 0$. This is illustrated in Figure 5.2.

Example 5.4 (Obtaining μ–elliptic integrands)**.** *As a slight modification of [40, Ex. 3.9], a family of μ–elliptic integrands is obtained in the following way. Fix $1 < \mu < \infty$ and put*

$$f_\mu(A) := \widetilde{f}_\mu(|A|) \quad \textit{with} \quad \widetilde{f}_\mu(t) := \int_0^t\int_0^s \frac{1}{(1+|\sigma|^2)^{\mu/2}} + \frac{1}{(1+|\sigma|^2)^{1/2}}\,\mathrm{d}\sigma\,\mathrm{d}s. \tag{5.6}$$

Then clearly $\frac{\mathrm{d}^2}{\mathrm{d}t^2}\widetilde{f}_\mu(t) = (1+t^2)^{-\frac{\mu}{2}} + (1+|t|^2)^{-\frac{1}{2}}$ and so f_μ satisfies (5.5). Note that the difference of f_μ as given here to Bildhauer*'s example is that in [40, Ex. 3.9] the term $(1+|\sigma|^2)^{-\frac{1}{2}}$ in the integrand appearing in (5.6) is omitted and hence results in an integrand whose Hessian shows the same growth from above and below even though it trivially satisfies (5.5).*

We next record some consequences of the μ–ellipticity condition. In particular, μ–elliptic integrands are of linear growth.

Lemma 5.5 (Bildhauer, [40, Rem. 4.2])**.** *Let $f \in \mathrm{C}^2(\mathbb{R}^{N\times n})$ be a μ–elliptic integrand which has a global minimum at $\xi = 0$ with $f(0) = 0$. Then the following hold:*

(a) *Then there exist constants $c_1, c_2 > 0$ such that for all $\xi\in\mathbb{R}^{N\times n}$ there holds*

$$\langle f'(\xi),\xi\rangle \geqslant c_1\langle\xi\rangle - c_2.$$

(b) *There exist constants $c_3,\dots,c_6 > 0$ such that for all $\xi\in\mathbb{R}^{N\times n}$ there holds*

$$c_3|\xi| - c_4 \leqslant f(\xi) \leqslant c_5|\xi| + c_6.$$

Proof. Ad (a). Let $\xi\in\mathbb{R}^{N\times n}$. We compute, using $f'(0) = 0$,

$$\begin{aligned} \langle f'(\xi),\xi\rangle = \langle f'(\xi)-f'(0),\xi\rangle &= \int_0^1 \Big\langle \frac{\mathrm{d}}{\mathrm{d}t}f'(t\xi),\xi\Big\rangle\,\mathrm{d}t \\ &= \int_0^1 \langle f''(t\xi)\xi,\xi\rangle\,\mathrm{d}t \geqslant \lambda\int_0^1 \frac{|\xi|^2}{(1+|t\xi|^2)^{\frac{\mu}{2}}}\,\mathrm{d}t \\ &\geqslant \lambda|\xi|\left(\int_0^{|\xi|}\frac{\mathrm{d}s}{(1+s^2)^{\frac{\mu}{2}}}\right) \geqslant \widetilde{c}\lambda\mathbb{1}_{\{|\xi|\geqslant 1\}}, \end{aligned} \tag{5.7}$$

[2] The case $p = 1$ is excluded since $\mathrm{m}_1(\xi) - 1 = |\xi|$ which is a positively 1–homogeneous function and hence not even strictly convex.

where $\widetilde{c} := \int_0^1 \frac{\mathrm{d}s}{(1+s^2)^{\frac{\mu}{2}}}$. Therefore, $\langle f'(\xi),\xi\rangle \geqslant \widetilde{c}\lambda(|\xi|-1)$, and since $\langle\xi\rangle \leqslant 1+|\xi|$, we deduce $\langle f'(\xi),\xi\rangle \geqslant \widetilde{c}\lambda\langle\xi\rangle - 2\widetilde{c}\lambda$, and the proof of (a) is complete. Ad (b). Let $\xi\in\mathbb{R}^{N\times n}$. Using $f(0)=0$ and noting that $\langle f'(\xi),\xi\rangle \geqslant 0$ as demonstrated in (5.7), we estimate for $0<a<1$

$$
\begin{aligned}
f(\xi) = f(\xi)-f(0) &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} f(t\xi)\,\mathrm{d}t = \int_0^1 t\langle f'(t\xi),t\xi\rangle\,\mathrm{d}t \\
&\geqslant \int_a^1 t\big(c_1\langle t\xi\rangle - c_2\big)\,\mathrm{d}t \\
&\geqslant c_1 a\int_a^1 \langle t\xi\rangle\,\mathrm{d}t - c_2\int_a^1 t\,\mathrm{d}t = \frac{c_1 a}{|\xi|}\int_{a|\xi|}^{|\xi|}\langle s\rangle\,\mathrm{d}s - c_2\left[\frac{1}{2}-\frac{a^2}{2}\right].
\end{aligned}
$$

Now note that

$$
\frac{c_1 a}{|\xi|}\int_{a|\xi|}^{|\xi|}\langle s\rangle\,\mathrm{d}s \geqslant \frac{c_1 a}{|\xi|}\int_{a|\xi|}^{|\xi|} s\,\mathrm{d}s = \frac{1}{2}c_1 a(1-a^2)|\xi|,
$$

and from here the lower bound claimed in (b) is clear. The upper bound is similar. □

5.3. Main Result

We now proceed by explaining the main results and the structure of this chapter, the latter being depicted in Figure 5.2. Our overall aim is to establish an analogue of Bildhauer's theorem as reported on above. To this end, we employ a vanishing viscosity approach and firstly show that if[3] $u_0\in \mathrm{W}^{1,2}(\Omega;\mathbb{R}^n)$, then the sequence (v_j) of minimisers of the stabilised functionals

$$
\mathfrak{F}_j[v] := \int_\Omega f(\varepsilon(v))\,\mathrm{d}x + \frac{1}{2j}\int_\Omega |\varepsilon(v)|^2\,\mathrm{d}x \qquad \text{over } u_0+\mathrm{W}_0^{1,2}(\Omega;\mathbb{R}^n) \tag{5.8}
$$

converge to a generalised minimiser $u\in \mathrm{GM}(\mathfrak{F};u_0)$ in the sense of Chapter 4. As a generalisation of (5.4), we then have the following result:

Theorem 5.6. *Let $\Omega\subset\mathbb{R}^n$ be an open and bounded Lipschitz domain and let $u_0\in\mathrm{LD}(\Omega)$ be a given Dirichlet datum. Then there exists a number $1<\mu_{\max}<2$, $\mu_{\max}=\mu_{\max}(n)$ with the following property:*

Let $f\colon \mathrm{C}^2(\mathbb{R}^{n\times n}_{\mathrm{sym}})\to\mathbb{R}$ be a μ–elliptic variational integrand with $1<\mu<\mu_{\max}$ satisfying the linear growth hypothesis (5.2). If the sequence (v_j) of viscosity approximations satisfies

$$
\sup_{j\in\mathbb{N}} \|v_j\|_{\mathrm{BMO}(K;\mathbb{R}^n)} < \infty \qquad \textit{for all Lipschitz } K\Subset\Omega, \tag{LBMO}
$$

then there exists a generalised minimiser $u\in\mathrm{GM}(\mathfrak{F};u_0)$ which is of class $\mathrm{W}^{1,p}_{\mathrm{loc}}(\Omega;\mathbb{R}^n)$ for some $p>1$.

In particular, we have $\mu_{\max}\geqslant 1+\frac{3}{2n}$.

The previous theorem is the natural extension of Bildhauer's work as mentioned above, in particular, condition (5.4). The result will be established in Section 5.7 as an interplay of higher regularity of the dual solution σ to be introduced and discussed in the next paragraph and improved fractional differentiability of the viscosity approximations based on the local uniform BMO–assumption (LBMO).

The required tools to accomplish the proof will be given in the sections which precede Section 5.7. In Section 5.4, we examine as minor modification of Seregin's work [212]–[215] the $\mathrm{W}^{1,2}_{\mathrm{loc}}$–regularity of the dual solution and characterise the interplay of the solution

[3] The $\mathrm{W}^{1,2}$–assumption on the Dirichlet data is no essential restriction, cp. Remark 5.8.

$$\begin{array}{ccc}
\text{Characterisation of } \sigma & & \text{Impact of Caccioppoli} \\
\text{Theorem 5.18} & & \text{Section 5.6} \\
\uparrow & & \uparrow \\
\text{Regularity of the dual solution } \sigma & & \text{Interpolation Theorems} \\
\text{Section 5.4} & & \text{Section 5.5} \\
& \searrow \quad \swarrow & \\
& \text{Proof of Theorem 5.6} & \\
& \text{Section 5.7} &
\end{array}$$

Figure 5.2: Structure of the present chapter and dependencies of the single sections.

of the primal and dual problems in the sense of the recent study [36] of Beck & Schmidt. For this, we introduce a symmetric version of an up–to–the–boundary pairing as has been done in the BV–case in [36, Sec. 7]. The latter, in turn, is reminiscent of ideas of Anzellotti [21].

In a second step, relaxing the local boundedness hypotheses (5.4) to the local BMO–hypothesis (LBMO) requires an interpolation result for $\mathrm{BMO} \cap \mathrm{BD}$. This is given in Section 5.5, and to round off the discussion of improvement of higher fractional Sobolev regularity for u (*not* $\varepsilon(u)$), we moreover provide interpolation estimates for spaces $\mathrm{BD} \cap \mathrm{L}^q$ with $q > \frac{n}{n-1}$. These estimates then turn out useful in the intermediate Section 5.6, where we discuss the impact of Caccioppoli–type inequalities on higher integrability. Most notably, we show that Caccioppoli–type inequalites (of the first kind) cannot yield higher integrability results for $\varepsilon(u)$ in the linear growth context, but that they *do* on the scale of fractional Sobolev spaces $\mathrm{W}^{s,p}$, $0 < s < 1$. This is in contrast with the superlinear growth situation, where higher integrability can be inferred from Caccioppoli–type inequalities by use of the Sobolev inequality and Gehring's Lemma. After combining the results of Sections 5.4 and 5.5 which yield the proof of Theorem 5.6, we discuss the aforementioned uniqueness of generalised minima. In particular, we have the following result:

Theorem 5.7. *In the situation of Theorem 5.6, let Ω be an open, bounded and convex Lipschitz subset of $\mathbb{R}^n$ and suppose that generalised minima of the variational integral $\mathfrak{F}$ are unique modulo rigid deformations. If one generalised minimiser u attains the correct boundary values in the sense that* $\mathrm{Tr}(u-u_0) = 0$ $\mathcal{H}^{n-1}$*–a.e. on $\partial\Omega$, then* $\mathrm{GM}(\mathfrak{F}; u_0) = \{u\}$.

The chapter then is concluded by an outlook of how the situation simplifies when we leave the linear growth regime in Section 5.8.

5.4. On the Regularity of the Dual Solution

In this section we give a $\mathrm{W}^{1,2}$–regularisation approach to the regularity and uniqueness of the dual solution which is reminiscent of the foundational work of Seregin [212, 213, 214, 215] and its adaption due to Bildhauer & Fuchs [41]; also see Bildhauer's monograph [40, Chpts. 2.2 and 2.4] and the treatise of Fuchs & Seregin [115]. Throughout, we tacitly assume $u_0 \in \mathrm{W}^{1,2}(\Omega; \mathbb{R}^n)$ for the given Dirichlet datum. This, however, is no essential restriction:

Remark 5.8. *In the same vein as in [40, Rem. 2.5], the assumption of* $\mathrm{W}^{1,2}$*–Dirichlet data is not restrictive but a bit more involved from a technical perspective. If we assume $u_0 \in \mathrm{LD}(\Omega)$, then we pick an approximating sequence $(u_{0,l}) \subset \mathrm{C}^\infty(\overline{\Omega}; \mathbb{R}^n)(\subset \mathrm{W}^{1,2}(\Omega; \mathbb{R}^n))$ which converges strongly in $\mathrm{LD}(\Omega)$ to u_0. In this situation, we would replace the factor $(\frac{1}{2j})$ in the viscosity term (5.8) by some $\delta_{j,l} > 0$ sufficiently small. For a detailled discussion in the gradient case (which, in this respect, does* not *differ from the symmetric* *gradient case as any implications arising from Korn's inequality in* $\mathrm{W}^{1,2}$ *are not used in the limit passage* $j \to \infty$*), see [44].*

5.4.1. The primal and the dual problem In this section we apply the general framework of convex duality as exposed in Chapter 2.4 to the variational principle (1.1). Henceforth let $f \in \mathrm{C}^2(\mathbb{R}^{n\times n}_{\mathrm{sym}})$–integrand satisfy[4] the linear growth assumption (LG).

In view of the minimisation problem of interest, we note that $f\colon \mathbb{R}^{n\times n}_{\mathrm{sym}} \to \mathbb{R}$ is lower semicontinuous and convex. Consequently, by the duality relation (2.5) from Chapter 2 we find that if $\partial f(\xi) \neq \emptyset$ for some $\xi \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$, then $\eta \in \partial f(\xi)$ if and only if $f(\xi)+f^*(\eta) = \langle \eta, \xi\rangle$. Now, because f is assumed to be of class C^2, the preceeding relation with $\eta = f'(\xi)$ implies

$$f(\xi) + f^*(f'(\xi)) = \langle f'(\xi), \xi\rangle \qquad \text{for all } \xi \in \mathbb{R}^{n\times n}_{\mathrm{sym}}. \tag{5.9}$$

In this situation, we define the *Lagrangian* $\ell\colon (u_0 + \mathrm{LD}_0(\Omega)) \times \mathrm{L}^\infty(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}}) \to \mathbb{R}$ by

$$\begin{aligned} \ell(w,\chi) = \ell(u_0+\varphi,\chi) &:= \int_\Omega \langle \chi, \boldsymbol{\varepsilon}(w)\rangle \,\mathrm{d}x - \int_\Omega f^*(\chi)\,\mathrm{d}x \\ &= \ell(u_0,\chi) + \int_\Omega \langle \chi, \boldsymbol{\varepsilon}(\varphi)\rangle\,\mathrm{d}x, \end{aligned} \tag{5.10}$$

where $(w,\chi) = (u_0+\varphi,\chi) \in (u_0 + \mathrm{LD}_0(\Omega)) \times \mathrm{L}^\infty(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})$. The *dual functional* $\mathscr{R}\colon \mathrm{L}^\infty(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}}) \to \overline{\mathbb{R}}$ now is given by

$$\mathscr{R}[\chi] := \inf\big\{\ell(w,\chi)\colon\; w \in u_0 + \mathrm{LD}_0(\Omega)\big\}, \qquad \chi \in \mathrm{L}^\infty(\Omega;\mathbb{R}^{n\times n}). \tag{5.11}$$

The *dual problem* then is given by

$$\text{to maximise } \mathscr{R} \text{ over } \mathrm{L}^\infty(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}}), \tag{5.12}$$

and in this terminology,

$$\text{to minimise } \mathfrak{F}[v] := \int_\Omega f(\boldsymbol{\varepsilon}(v))\,\mathrm{d}x \text{ over } \mathcal{D}_{u_0} \tag{5.13}$$

is referred to as the *primal problem*. Note that if $\chi \in \mathrm{L}^\infty(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})$ is not (row–wise) *solenoidal* or *divergence–free* in the sense of distributions, that is, χ does not verify

$$\int_\Omega \langle \chi, \boldsymbol{\varepsilon}(\varphi)\rangle\,\mathrm{d}x = 0 \qquad \text{for all } \varphi \in \mathrm{C}^1_c(\Omega;\mathbb{R}^n), \tag{5.14}$$

then $\mathscr{R}[\chi] = -\infty$. In fact, choosing any $\varphi \in \mathrm{C}^1_c(\Omega;\mathbb{R}^n)$ for which $\int_\Omega \langle \chi, \boldsymbol{\varepsilon}(\varphi)\rangle\,\mathrm{d}x > 0$ and considering the sequence $(\varphi_k) \subset \mathrm{C}^1_c(\Omega;\mathbb{R}^n)$ with $\varphi_k := -k\varphi$, we find

$$\ell(w_k,\chi) := \ell(u_0+\varphi_k,\chi) = \ell(u_0,\chi) - k\int_\Omega \langle \chi, \boldsymbol{\varepsilon}(\varphi)\rangle\,\mathrm{d}x \to -\infty, \qquad k \to \infty.$$

Hence we obtain the dichotomic statement that

$$\mathscr{R}[\chi] = \begin{cases} -\infty & \text{if } \operatorname{div}(\chi) \neq 0 \text{ in the sense of distributions,} \\ \ell(u_0,\chi) & \text{if } \operatorname{div}(\chi) = 0 \text{ in the sense of distributions.} \end{cases} \tag{5.15}$$

As a consequence, when maximising the dual functional we can safely stick to distributionally solenoidal competitors. In the sequel we shall give a precise description of the relation between the primal and the dual problem. For the time being, it suffices to focus on the following observation.

[4] For the purposes of the present section, it suffices to assume mere continuity on f. However, as we shall only need the C^2–case later on, it suffices to focus on the latter. We refer the reader to [36] for possibly non–C^1–smooth variational problems and their dual variational principles.

Lemma 5.9. *There holds*

$$\inf_{u_0+\mathrm{LD}_0(\Omega)} \mathfrak{F}[w] \geqslant \sup_{\chi\in \mathrm{L}^\infty_{\mathrm{div}}(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})} \mathscr{R}[\chi], \tag{5.16}$$

where $\mathrm{L}^\infty_{\mathrm{div}}(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})$ *denotes the linear space of all* $v\in \mathrm{L}^\infty(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})$ *which are solenoidal in the sense of distributions.*

Proof. First, let us note that by (5.9), $f(\boldsymbol{\varepsilon}(w)(x)) = \langle \chi(x), \boldsymbol{\varepsilon}(w)(x)\rangle - f^*(\chi(x))$ holds for $\mathscr{L}^n$–a.e. $x\in\Omega$ and every $\chi\in \mathrm{L}^\infty(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})$. Therefore, integrating the previous identity with respect to x and passing to the supremum over all $\chi\in \mathrm{L}^\infty_{\mathrm{div}}(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})$ gives

$$\mathfrak{F}[\boldsymbol{\varepsilon}(w)] \geqslant \sup_{\chi\in \mathrm{L}^\infty_{\mathrm{div}}(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})} \int_\Omega \langle \chi, \boldsymbol{\varepsilon}(w)\rangle - f^*(\chi)\,\mathrm{d}x = \sup_{\chi\in \mathrm{L}^\infty_{\mathrm{div}}(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})} \ell(w,\chi).$$

Hence have for all $w = u_0+\varphi \in u_0 + \mathrm{LD}_0(\Omega)$

$$\mathfrak{F}[w] \geqslant \sup_{\chi\in \mathrm{L}^\infty_{\mathrm{div}}(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})} \ell(w,\chi) \geqslant \sup_{\chi\in \mathrm{L}^\infty_{\mathrm{div}}(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})} \inf_{v\in u_0+\mathrm{LD}_0(\Omega)} \ell(v,\chi) = \sup_{\chi\in \mathrm{L}^\infty_{\mathrm{div}}(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})} \mathscr{R}[\chi].$$

Passing to the infimum over all such w yields (5.16). □

5.4.2. Viscosity Approximations In what follows, we consider viscosity approximations as in (5.8). More precisely, for $j\in\mathbb{N}$, put

$$\mathcal{F}_j[v] := \mathfrak{F}[v] + \frac{1}{2j}\int_\Omega |\boldsymbol{\varepsilon}(v)|^2\,\mathrm{d}x, \qquad v\in \mathcal{D}_{u_0} := u_0 + \mathrm{W}^{1,2}_0(\Omega;\mathbb{R}^n). \tag{5.17}$$

Clearly, by the quadratic term, each $\mathcal{F}_j$ is strictly convex and by the direct method in conjunction with the L^2–Korn inequality, it is straightforward to show that each $\mathcal{F}_j$ has a unique minimiser $v_j\in\mathcal{D}_{u_0}$. We shall now derive some a priori estimates on (v_j) to demonstrate that (v_j) is a minimising sequence for $\mathfrak{F}$. Since v_j minimises $\mathcal{F}_j$ and $u_0\in\mathcal{D}_{u_0}$, we find

$$\mathfrak{F}[v_j] + \frac{1}{2j}\int_\Omega |\boldsymbol{\varepsilon}(v_j)|^2\,\mathrm{d}x = \mathcal{F}_j[v_j] \leqslant \mathcal{F}_j[u_0] \leqslant c < \infty.$$

By coerciveness of f, this implies that (v_j) is uniformly bounded in $\mathrm{LD}(\Omega)$ together with

$$\sup_{j\in\mathbb{N}} \int_\Omega |\boldsymbol{\varepsilon}(v_j)|\,\mathrm{d}x < \infty \quad\text{and}\quad \sup_{j\in\mathbb{N}} \frac{1}{2j}\int_\Omega |\boldsymbol{\varepsilon}(v_j)|^2\,\mathrm{d}x < \infty. \tag{5.18}$$

We define $f_j(\xi) := f(\xi) + \frac{1}{2j}|\xi|^2$ for $\xi\in\mathbb{R}^{n\times n}_{\mathrm{sym}}$ and put

$$\sigma_j := f_j'(\boldsymbol{\varepsilon}(v_j)) = f'(\boldsymbol{\varepsilon}(v_j)) + \frac{1}{j}\boldsymbol{\varepsilon}(v_j), \quad \tau_j := f'(\boldsymbol{\varepsilon}(v_j)), \qquad j\in\mathbb{N}. \tag{5.19}$$

As f is C^2 and of linear growth, we deduce that f' is bounded and hence $(f'(\boldsymbol{\varepsilon}(v_j)))$ is bounded in $\mathrm{L}^\infty(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})$. By the Banach–Alaoglu theorem, there exists $\tau\in\mathrm{L}^\infty(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})$ and a non–relabelled subsequence such that

$$\tau_j \overset{*}{\rightharpoonup} \tau \text{ in } \mathrm{L}^\infty(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}}) \text{ as } j\to\infty. \tag{5.20}$$

On the other hand, (5.18) implies

$$\|j^{-1}\boldsymbol{\varepsilon}(v_j)\|_{\mathrm{L}^2(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})} = \Big(\frac{1}{j}\cdot \underbrace{\frac{1}{j}\int_\Omega |\boldsymbol{\varepsilon}(v_j)|^2\,\mathrm{d}x}_{\leqslant c \text{ by } (5.18)}\Big)^{\frac{1}{2}} \leqslant \frac{c}{j^{\frac{1}{2}}} \to 0 \tag{5.21}$$

as $j \to \infty$ and so we obtain upon passing to a suitable non–relabelled subsequence

$$\sigma_j \rightharpoonup \sigma \text{ in } \mathrm{L}^2(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}}) \qquad \text{as } j\to\infty \tag{5.22}$$

for some $\sigma \in \mathrm{L}^2(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})$. Consequently, since weak*–convergence in L^∞ implies weak convergence in L^2 on bounded domains, we find that $\sigma=\tau$. Our next objective is to demonstrate that $\sigma \in \mathrm{L}^\infty(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})$ is a solution of the dual problem as introduced in the previous subsection.

Theorem 5.10. *Let $\sigma \in \mathrm{L}^\infty(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})$ be given by (5.22). Then the following holds:*

(a) *σ is solenoidal in the sense of distributions on Ω.*

(b) *σ is a solution of the dual problem.*

(c) $\sup_{\mathrm{L}^\infty_{\mathrm{div}}(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})} \mathscr{R} = \inf_{u_0+\mathrm{LD}_0(\Omega)} \mathfrak{F}$.

Proof. Ad (a). Since v_j is a minimiser of $\mathcal{F}_j$, it satisfies the Euler–Lagrange equation

$$\int_\Omega \langle f_j'(\boldsymbol{\varepsilon}(v_j)),\boldsymbol{\varepsilon}(\varphi)\rangle\,\mathrm{d}x = 0 \qquad \text{for all } \varphi\in \mathrm{W}^{1,2}_0(\Omega;\mathbb{R}^n). \tag{5.23}$$

Hence, letting $\varphi\in \mathrm{C}^\infty_c(\Omega;\mathbb{R}^n)$ be arbitrary but fixed, we may pass to the limit $j\to\infty$ in (5.23) and use the convergence $f_j'(\boldsymbol{\varepsilon}(v_j)) \rightharpoonup \tau=\sigma$ in $\mathrm{L}^2(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})$ to find $\int_\Omega\langle\sigma,\boldsymbol{\varepsilon}(\varphi)\rangle\,\mathrm{d}x = 0$.

Ad (b). We employ the duality relation $\langle f'(\xi),\xi\rangle = f^*(f'(\xi))+f(\xi)$ for $\xi\in\mathbb{R}^{n\times n}_{\mathrm{sym}}$ with $\xi=\boldsymbol{\varepsilon}(v_j)$ to obtain $\langle\tau_j,\boldsymbol{\varepsilon}(v_j)\rangle = f^*(\tau_j)+f(\boldsymbol{\varepsilon}(v_j))$. Hence,

$$\begin{aligned}
\mathcal{F}_j[v_j] &= \int_\Omega f(\boldsymbol{\varepsilon}(v_j))\,\mathrm{d}x + \frac{1}{2j}\int_\Omega |\boldsymbol{\varepsilon}(v_j)|^2\,\mathrm{d}x\\
&= \int_\Omega \langle\tau_j,\boldsymbol{\varepsilon}(v_j)\rangle - f^*(\tau_j)\,\mathrm{d}x + \frac{1}{2j}\int_\Omega |\boldsymbol{\varepsilon}(v_j)|^2\,\mathrm{d}x\\
&= \int_\Omega \langle\sigma_j,\boldsymbol{\varepsilon}(v_j)\rangle - f^*(\tau_j)\,\mathrm{d}x - \frac{1}{2j}\int_\Omega |\boldsymbol{\varepsilon}(v_j)|^2\,\mathrm{d}x = (*),
\end{aligned}$$

and since $v_j=u_0+w_j$ with some $w_j\in \mathrm{W}^{1,2}_0(\Omega;\mathbb{R}^n)$, we use the distributional (row–wise) solenoidality of σ_j (which holds as v_j minimises $\mathcal{F}_j$) to conclude

$$\int_\Omega \langle\sigma_j,\boldsymbol{\varepsilon}(w_j)\rangle\,\mathrm{d}x = 0,$$

and therefore

$$(*) = \int_\Omega \langle\sigma_j,\boldsymbol{\varepsilon}(u_0)\rangle - f^*(\tau_j)\,\mathrm{d}x - \frac{1}{2j}\int_\Omega |\boldsymbol{\varepsilon}(v_j)|^2\,\mathrm{d}x. \tag{5.24}$$

We conclude by Lemma 5.9 in the first step that

$$\begin{aligned}
\sup_{\mathrm{L}^\infty_{\mathrm{div}}(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})}\mathscr{R} &\leqslant \inf_{u_0+\mathrm{LD}_0(\Omega)}\mathfrak{F} \leqslant \inf_{u_0+\mathrm{W}^{1,2}_0(\Omega;\mathbb{R}^n)}\mathfrak{F}\\
&\leqslant \inf_{u_0+\mathrm{W}^{1,2}_0(\Omega;\mathbb{R}^n)}\mathcal{F}_j \quad (\text{by definition of } \mathcal{F}_j)\\
&= \mathcal{F}_j[v_j]\\
&= \int_\Omega \langle\sigma_j,\boldsymbol{\varepsilon}(u_0)\rangle - f^*(\tau_j)\,\mathrm{d}x - \frac{1}{2j}\int_\Omega|\boldsymbol{\varepsilon}(v_j)|^2\,\mathrm{d}x \qquad (\text{by } (5.24))\\
&= -\frac{1}{2j}\int_\Omega |\boldsymbol{\varepsilon}(v_j)|^2\,\mathrm{d}x + \int_\Omega\langle\tau_j,\boldsymbol{\varepsilon}(u_0)\rangle - f^*(\tau_j)\,\mathrm{d}x\\
&\quad + \frac{1}{j}\int_\Omega\langle\boldsymbol{\varepsilon}(v_j),\boldsymbol{\varepsilon}(u_0)\rangle\,\mathrm{d}x,
\end{aligned} \tag{5.25}$$

where we have used the very definition of τ_j and σ_j in the last step. Since $v \mapsto -\int_\Omega f^*(v)\,\mathrm{d}x$ is upper semicontinuous with respect to weak*–convergence in $\mathrm{L}^\infty(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})$ as consequence of convexity of f, we obtain by $\tau_j \overset{*}{\rightharpoonup} \tau$ as $j\to\infty$ and $\boldsymbol{\varepsilon}(u_0)\in\mathrm{L}^1(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})$ that

$$\limsup_{j\to\infty}\int_\Omega \langle \tau_j,\boldsymbol{\varepsilon}(u_0)\rangle - f^*(\tau_j)\,\mathrm{d}x \leqslant \int_\Omega \langle \tau,\boldsymbol{\varepsilon}(u_0)\rangle - f^*(\tau)\,\mathrm{d}x. \tag{5.26}$$

Moreover, as a consequence of (5.18), we obtain

$$\left|\frac{1}{j}\int_\Omega \langle \boldsymbol{\varepsilon}(v_j),\boldsymbol{\varepsilon}(u_0)\rangle\,\mathrm{d}x\right| \leqslant \|j^{-1}\boldsymbol{\varepsilon}(v_j)\|_{\mathrm{L}^2(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})}\|\boldsymbol{\varepsilon}(u_0)\|_{\mathrm{L}^2(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})}\to 0 \tag{5.27}$$

as $j\to\infty$. Collecting estimates, we find by sending $j\to\infty$ in (5.25) and incorporating (5.26) and (5.27) that

$$\begin{aligned}\sup_{\mathrm{L}^\infty_{\mathrm{div}}(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})}\mathscr{R} &\leqslant \int_\Omega \langle \tau,\boldsymbol{\varepsilon}(u_0)\rangle - f^*(\tau)\,\mathrm{d}x + \limsup_{j\to\infty}\left(-\frac{1}{2j}\int_\Omega |\boldsymbol{\varepsilon}(v_j)|^2\,\mathrm{d}x\right)\\ &\leqslant \sup_{\mathrm{L}^\infty_{\mathrm{div}}(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})}\mathscr{R} + \limsup_{j\to\infty}\left(-\frac{1}{2j}\int_\Omega |\boldsymbol{\varepsilon}(v_j)|^2\,\mathrm{d}x\right) \leqslant \sup_{\mathrm{L}^\infty_{\mathrm{div}}(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})}\mathscr{R},\end{aligned}$$

where in the ultimate estimate we have used the admissibility of $\tau\in\mathrm{L}^\infty_{\mathrm{div}}(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})$ as competitor for the dual problem which, in turn, is ensured by (a). This shows that

$$\limsup_{j\to\infty}\left(-\frac{1}{2j}\int_\Omega |\boldsymbol{\varepsilon}(v_j)|^2\,\mathrm{d}x\right) = 0 \tag{5.28}$$

and hence $\tau\in\mathrm{L}^\infty_{\mathrm{div}}(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})$ is a solution of the dual problem. This also directly implies (c). The proof is complete. □

From the above proof we further extract that (v_j) is a minimising sequence for $\mathfrak{F}$. A standard finite difference approach, which we give here for completeness of presentation, yields (non–uniform) higher Sobolev regularity for the sequence of viscosity approximations (v_j):

Lemma 5.11. *Suppose that the variational integrand $f\in\mathrm{C}^2(\mathbb{R}^{n\times n}_{\mathrm{sym}})$ satisfies*

$$0 < \langle f''(\xi)\eta,\eta\rangle \leqslant \Lambda\frac{|\eta|^2}{1+|\xi|} \qquad \textit{for all } \xi,\eta\in\mathbb{R}^{n\times n}_{\mathrm{sym}},$$

where $\Lambda>0$ is a fixed constant. Then the sequence (v_j) as constructed above belongs to $\mathrm{W}^{2,2}_{\mathrm{loc}}(\Omega;\mathbb{R}^n)$.

Proof. Let $x_0\in\Omega$ be arbitrary and let $r>0$ such that $\mathrm{dist}(x_0,\partial\Omega)>2r$ and hence $\mathrm{B}(x_0,2r)\subset\Omega$. We pick a cut–off function $\rho\in\mathrm{C}^1_c(\Omega;[0,1])$ with $\mathbb{1}_{\mathrm{B}(x_0,r)}\leqslant\rho\leqslant\mathbb{1}_{\mathrm{B}(x_0,2r)}$. For sufficiently small $h\in\mathbb{R}$ and a standard unit vector $e_s\in\mathbb{S}^{n-1}$, the function $\varphi := \Delta^-_{s,h}(\rho^2\Delta_{s,h}v_j)$ is admissible in the weak Euler–Lagrange equation (5.23). Here we recall that for a mapping $w\colon\mathbb{R}^n\to\mathbb{R}^m$ the forward and backward difference quotients $\Delta^\pm_{s,h}w(x)$ at $x\in\mathbb{R}^n$ are then given by

$$\Delta^\pm_{s,h}w(x) := \frac{w(x\pm he_s)-w(x)}{h}, \quad \text{and we set } \Delta_{s,h} := \Delta^+_{s,h}.$$

Inserting this choice for φ as above into (5.23), we hence obtain by integration by parts for difference quotients

$$\int_\Omega \langle \Delta_{s,h}f'(\boldsymbol{\varepsilon}(v_j)),\boldsymbol{\varepsilon}(\rho^2\Delta_{s,h}v_j)\rangle\,\mathrm{d}x + \frac{1}{j}\int_\Omega \langle \Delta_{s,h}\boldsymbol{\varepsilon}(v_j),\boldsymbol{\varepsilon}(\rho^2\Delta_{s,h}v_j)\rangle\,\mathrm{d}x = 0.$$

We find by the product rule for $\boldsymbol{\varepsilon}$ and after expanding terms that

$$
\begin{aligned}
\mathbf{I}+\mathbf{II} &:= \int_\Omega \langle \Delta_{s,h} f'(\boldsymbol{\varepsilon}(v_j)), \rho^2 \Delta_{s,h}\boldsymbol{\varepsilon}(v_j)\rangle \,\mathrm{d}x + \frac{1}{j}\int_\Omega \rho^2 \langle \Delta_{s,h}\boldsymbol{\varepsilon}(v_j), \Delta_{s,h}\boldsymbol{\varepsilon}(v_j)\rangle \,\mathrm{d}x \\
&\leqslant -\int_\Omega \langle \Delta_{s,h} f'(\boldsymbol{\varepsilon}(v_j)), 2\rho\,\mathrm{D}\rho \odot \Delta_{s,h} v_j\rangle \,\mathrm{d}x \\
&\quad - \frac{1}{j}\int_\Omega \langle \Delta_{s,h}\boldsymbol{\varepsilon}(v_j), 2\rho\,\mathrm{D}\rho \odot \Delta_{s,h} v_j\rangle \,\mathrm{d}x =: \mathbf{III}+\mathbf{IV}.
\end{aligned} \tag{5.29}
$$

Now note that since $f \in \mathrm{C}^2(\mathbb{R}^{n\times n}_{\mathrm{sym}})$ is a convex function with everywhere strictly positive definite Hessian, the bilinear forms

$$
\mathcal{A}^{(j)}_{x,h,s}[\xi,\eta] := \int_0^1 \langle f''(\boldsymbol{\varepsilon}(v_j)(x) + th\Delta_{s,h}\boldsymbol{\varepsilon}(v_j)(x))\xi, \eta\rangle \,\mathrm{d}t
$$

for fixed $x \in \Omega$, are positive definite. Invoking the fundamental theorem of calculus, we see that

$$
\begin{aligned}
\Delta_{s,h} f'(\boldsymbol{\varepsilon}(v_j))(x) &= \frac{1}{h}\big(f'(\boldsymbol{\varepsilon}(v_j)(x) + h\Delta_{s,h}\boldsymbol{\varepsilon}(v_j)(x)) - f'(\boldsymbol{\varepsilon}(v_j)(x))\big) \\
&= \frac{1}{h}\int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}\Big(f'(\boldsymbol{\varepsilon}(v_j)(x) + th\Delta_{s,h}\boldsymbol{\varepsilon}(v_j)(x))\Big)\,\mathrm{d}t \\
&= \Big(\int_0^1 f''(\boldsymbol{\varepsilon}(v_j)(x) + th\Delta_{s,h}\boldsymbol{\varepsilon}(v_j)(x))\,\mathrm{d}t\Big)\Delta_{s,h}\boldsymbol{\varepsilon}(v_j)(x) \\
&= \mathcal{A}^{(j)}_{x,h,s}[\Delta_{s,h}\boldsymbol{\varepsilon}(v_j)(x), \cdot].
\end{aligned}
$$

In consequence, to estimate the latter two terms, we use Young's inequality applied to the bilinear forms $\mathcal{A}^{(j)}_{x,h,s}$ to obtain for some suitably small $\varepsilon > 0$ and constants $C(\varepsilon) > 0$

$$
\begin{aligned}
|\mathbf{III}| + |\mathbf{IV}| &\leqslant \varepsilon \int_\Omega \rho^2 \mathcal{A}^{(j)}_{x,h,s}[\Delta_{s,h}\boldsymbol{\varepsilon}(v_j), \Delta_{s,h}\boldsymbol{\varepsilon}(v_j)]\,\mathrm{d}x \\
&\quad + \frac{\varepsilon}{j}\int_\Omega |\rho\Delta_{s,h}\boldsymbol{\varepsilon}(v_j)|^2\,\mathrm{d}x \\
&\quad + C(\varepsilon)\int_\Omega \mathcal{A}^{(j)}_{x,h,s}[2\,\mathrm{D}\rho\odot\Delta_{s,h}v_j, 2\,\mathrm{D}\rho\odot\Delta_{s,h}v_j]\,\mathrm{d}x \\
&\quad + \frac{C(\varepsilon)}{j}\int_\Omega |2\,\mathrm{D}\rho\odot\Delta_{s,h}v_j|^2\,\mathrm{d}x =: \mathbf{III}_1 + \mathbf{III}_2 + \mathbf{III}_3 + \mathbf{III}_4.
\end{aligned}
$$

At this stage we go back to (5.29). We note that

$$
\mathbf{I} = \int_\Omega \rho^2 \mathcal{A}^{(j)}_{x,h,s}[\Delta_{s,h}\boldsymbol{\varepsilon}(v_j), \Delta_{s,h}\boldsymbol{\varepsilon}(v_j)]\,\mathrm{d}x \;\text{ and }\; \mathbf{II} = \frac{1}{j}\int_\Omega \rho^2|\Delta_{s,h}\boldsymbol{\varepsilon}(v_j)|^2\,\mathrm{d}x.
$$

We then choose $\varepsilon > 0$ so small such that $\mathbf{III}_1$ can be absorbed into $\mathbf{I}$ and $\mathbf{III}_2$ into $\mathbf{II}$. We fix this choice of $\varepsilon > 0$ which in turn fixes $C(\varepsilon) > 0$. Since $\mathbf{I} \geqslant 0$, we consequently get

$$
\begin{aligned}
\frac{1}{j}\int_\Omega |\rho\Delta_{s,h}\boldsymbol{\varepsilon}(v_j)|^2\,\mathrm{d}x \leqslant C(\mathbf{I}+\mathbf{II}) &\leqslant C(\varepsilon)\int_\Omega \mathcal{A}^{(j)}_{x,h,s}[2\,\mathrm{D}\rho\odot\Delta_{s,h}v_j, 2\,\mathrm{D}\rho\odot\Delta_{s,h}v_j]\,\mathrm{d}x \\
&\quad + \frac{C(\varepsilon)}{j}\int_\Omega |2\,\mathrm{D}\rho\odot\Delta_{s,h}v_j|^2\,\mathrm{d}x \\
&\leqslant C\int_\Omega |\,\mathrm{D}\rho|^2|\Delta_{s,h}v_j|^2\,\mathrm{d}x,
\end{aligned}
$$

where we used that $\langle f''(z)\xi,\xi\rangle \leqslant C|\xi|^2$ holds by assumption. In conclusion, since $v_j \in \mathrm{W}^{1,2}(\Omega;\mathbb{R}^n)$ by Korn's inequality, the right hand side of the last inequality is bounded uniformly in $s \in \{1,...,n\}$ and $0<h<\frac{1}{2}\operatorname{dist}(x_0,\partial\Omega)$. Therefore, we obtain that

$$\sup_{|h|<\frac{1}{2}\operatorname{dist}(x_0,\partial\Omega)} \frac{1}{j}\int_\Omega \rho^2|\Delta_{s,h}\boldsymbol{\varepsilon}(v_j)|^2\,\mathrm{d}x<\infty,$$

and so $\boldsymbol{\varepsilon}(v_j)\in \mathrm{W}^{1,2}_{\mathrm{loc}}(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})$ by arbitrariness of x_0 and $s\in\{1,...,n\}$. Now, since $\partial_i\boldsymbol{\varepsilon}(v_j)=\boldsymbol{\varepsilon}(\partial_i v_j)$ for all $i\in\{1,...,n\}$, we use Korn's inequality to conclude $\partial_i v_j \in \mathrm{W}^{1,2}_{\mathrm{loc}}(\Omega;\mathbb{R}^n)$ for all $i\in\{1,...,n\}$ and so $v_j\in\mathrm{W}^{2,2}_{\mathrm{loc}}(\Omega;\mathbb{R}^n)$ as claimed. The proof is complete. □

5.4.3. Regularity and Uniqueness for the Dual Solution The purpose of this section is to prove the $\mathrm{W}^{1,2}_{\mathrm{loc}}$–regularity of the dual solution obtained by the viscosity approximation procedure, cp. (5.22). To do so, we shall utilise a method firstly employed by Seregin [213, 215]. In the proof of the local Sobolev regularity of the dual solution, we follow the outline of Fuchs & Seregin [117].

Theorem 5.12 ($\mathrm{W}^{1,2}_{\mathrm{loc}}$–Regularity of the Dual Solution)**.** *Let $f\in\mathrm{C}^2(\mathbb{R}^{n\times n}_{\mathrm{sym}})$ be a μ–elliptic integrand of linear growth and let $\sigma\in\mathrm{L}^\infty_{\mathrm{div}}(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})$ be the dual solution given by (5.22). Then there holds*

$$\sigma\in(\mathrm{L}^\infty_{\mathrm{div}}\cap\mathrm{W}^{1,2}_{\mathrm{loc}})(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}}), \tag{5.30}$$

and in particular, for every relatively compact Lipschitz subset K of Ω, the sequence $(\sigma_j)=(f_j'(\boldsymbol{\varepsilon}(v_j)))$ is uniformly bounded in $\mathrm{W}^{1,2}(K;\mathbb{R}^{n\times n}_{\mathrm{sym}})$.

Proof. Step 1. In a first step, we employ an estimate valid for $\mathscr{L}^n$–a.e. $x\in\Omega$. For $j\in\mathbb{N}$ and $\nu\in\mathbb{R}^{n\times n}_{\mathrm{sym}}$, we define bilinear forms $\mathcal{A}[\nu;\cdot,\cdot]$ and $\mathcal{A}_j[\nu;\cdot,\cdot]$ by

$$\mathcal{A}[\nu;\eta,\xi]:=\langle f''(\nu)\eta,\xi\rangle\,, \tag{5.31}$$

$$\mathcal{A}_j[\nu;\eta,\xi]:=\mathcal{A}[\nu;\eta,\xi]+\frac{1}{j}\langle\xi,\eta\rangle,\qquad \nu,\xi,\eta\in\mathbb{R}^{n\times n}_{\mathrm{sym}}. \tag{5.32}$$

Note that for fixed $\nu\in\mathbb{R}^{n\times n}_{\mathrm{sym}}$, both $\mathcal{A}[\tau;\cdot,\cdot]$ and $\mathcal{A}_j[\nu;\cdot,\cdot]$ for all $j\in\mathbb{N}$ define symmetric positive definite bilinear forms and thus we have a suitable Cauchy–Schwarz inequality at our disposal. Consequently, recalling that $\sigma_j=f'(\boldsymbol{\varepsilon}(v_j))+\frac{1}{j}\boldsymbol{\varepsilon}(v_j)$ for all $j\in\mathbb{N}$, we obtain $\mathscr{L}^n$–a.e.

$$\begin{aligned}
|\,\mathrm{D}\sigma_j|^2 &= \sum_{k=1}^n\langle\partial_k\sigma_j,\partial_k\sigma_j\rangle=\sum_{k=1}^n\left\langle f''(\boldsymbol{\varepsilon}(v_j))\partial_k\boldsymbol{\varepsilon}(v_j)+\frac{1}{j}\partial_k\boldsymbol{\varepsilon}(v_j),\partial_k\sigma_j\right\rangle\\
&=\sum_{k=1}^n\mathcal{A}_j[\boldsymbol{\varepsilon}(v_j);\partial_k\boldsymbol{\varepsilon}(v_j),\partial_k\sigma_j]\\
&\leqslant\sum_{k=1}^n\left(\mathcal{A}_j[\boldsymbol{\varepsilon}(v_j);\partial_k\boldsymbol{\varepsilon}(v_j),\partial_k\boldsymbol{\varepsilon}(v_j)]\right)^{\frac{1}{2}}\left(\mathcal{A}_j[\boldsymbol{\varepsilon}(v_j);\partial_k\sigma_j,\partial_k\sigma_j]\right)^{\frac{1}{2}} && \text{(C–S)}\\
&\leqslant C(\varepsilon)\sum_{k=1}^n\mathcal{A}_j[\boldsymbol{\varepsilon}(v_j);\partial_k\boldsymbol{\varepsilon}(v_j),\partial_k\boldsymbol{\varepsilon}(v_j)]+\varepsilon\sum_{k=1}^n\mathcal{A}_j[\boldsymbol{\varepsilon}(v_j);\partial_k\sigma_j,\partial_k\sigma_j] && \text{(Y)}\\
&=:\mathcal{J}_1^{(j)}+\mathcal{J}_2^{(j)},
\end{aligned} \tag{5.33}$$

with an obvious definition of $\mathcal{J}_1^{(j)}$ and $\mathcal{J}_2^{(j)}$; here, (C–S) or (Y) indicate the use of the Cauchy–Schwarz or Young inequalities, respectively, and $\varepsilon>0$ shall be fixed later on.

We now estimate, using uniform boundedness of f'' and $j \geqslant 1$,

$$\begin{aligned}\mathcal{J}_2^{(j)} &= \varepsilon \sum_{k=1}^{n} \Big(\langle f''(\boldsymbol{\varepsilon}(v_j))\partial_k\sigma_j, \partial_k\sigma_j\rangle + \frac{1}{j}\langle \partial_k\sigma_j, \partial_k\sigma_j\rangle \Big) \\ &\leqslant \varepsilon(1+\Lambda)\sum_{k=1}^{n} |\partial_k\sigma_j|^2 = (1+\Lambda)\varepsilon|\,\mathrm{D}\sigma_j|^2,\end{aligned}$$

where $\Lambda > 0$ is the second parameter appearing in the definition of μ–ellipticity, cp. (5.5). Therefore, inserting this estimate into (5.33) and choosing $\varepsilon > 0$ suitably small, we may absorb the term $(1+\Lambda)\varepsilon|\,\mathrm{D}\sigma_j|^2$ into the overall left hand side. With this choice of ε, we thus arrive at

$$|\,\mathrm{D}\sigma_j|^2 \leqslant C\mathcal{J}_1^{(j)} = \sum_{k=1}^{n} \mathcal{A}_j[\boldsymbol{\varepsilon}(v_j); \partial_k\boldsymbol{\varepsilon}(v_j), \partial_k\boldsymbol{\varepsilon}(v_j)], \tag{5.34}$$

and here $C > 0$ is independent of $x \in \Omega$. To establish the claimed $\mathrm{W}^{1,2}_{\mathrm{loc}}$–regularity of σ, it hence suffices to show that for all relatively compact subsets $U \Subset \Omega$ with $\mathrm{dist}(U; \partial\Omega) > 0$ there holds

$$\sup_{j\in\mathbb{N}} \int_U \mathcal{J}_1^{(j)}\,\mathrm{d}x < \infty. \tag{5.35}$$

The verification of (5.35) in turn is the aim of the next three steps.

Step 2. By Lemma 5.11, we have $(v_j) \in \mathrm{W}^{2,2}_{\mathrm{loc}}(\Omega;\mathbb{R}^n)$. Hence, integrating by parts, we obtain for all $\varphi \in \mathrm{C}^\infty_c(\Omega;\mathbb{R}^n)$ and all $\ell \in \{1,...,n\}$ that

$$\int_\Omega \langle \partial_\ell\sigma_j, \boldsymbol{\varepsilon}(\varphi)\rangle\,\mathrm{d}x = 0. \tag{5.36}$$

We note that as above, $\partial_\ell\sigma_j = f''(\boldsymbol{\varepsilon}(v_j))\partial_\ell\boldsymbol{\varepsilon}(v_j)$ and so in particular $\partial_\ell\sigma_j \in \mathrm{L}^2_{\mathrm{loc}}(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})$. Now let $\varphi \in \mathrm{W}^{1,2}_0(\Omega;\mathbb{R}^n)$ be arbitrary with compact support in Ω and choose $(\varphi_m) \subset \mathrm{C}^\infty_c(\Omega;\mathbb{R}^n)$ such that $\varphi_m \to \varphi$ strongly in $\mathrm{W}^{1,2}(\Omega;\mathbb{R}^n)$ as $m \to \infty$. Then we see by

$$\begin{aligned}\left|\int_\Omega \langle \partial_\ell\sigma_j, \boldsymbol{\varepsilon}(\varphi)\rangle\,\mathrm{d}x\right| &= \left|\int_\Omega \langle \partial_\ell\sigma_j, \boldsymbol{\varepsilon}(\varphi-\varphi_m)\rangle\,\mathrm{d}x\right| \\ &\leqslant \|\partial_\ell\sigma_j\|_{\mathrm{L}^2(\mathrm{spt}(\varphi),\mathbb{R}^{n\times n}_{\mathrm{sym}})}\|\varphi-\varphi_m\|_{\mathrm{L}^2(\Omega;\mathbb{R}^n)} \to 0, \qquad m\to\infty,\end{aligned}$$

that (5.36) also holds for all $\varphi \in \mathrm{W}^{1,2}_0(\Omega;\mathbb{R}^n)$ which vanish outside a compact set contained in Ω.

Step 3. To establish (5.35), we shall use the weak Euler–Lagrange equation (5.36) satisfied by σ_j, $j \in \mathbb{N}$. Let $k \in \{1,...,n\}$ and let $\mathrm{B} \subset \Omega$ be an arbitrary open ball with $\mathrm{dist}(\mathrm{B};\partial\Omega) > 0$. We choose a cut–off function $\rho \in \mathrm{C}^1_c(\Omega;[0,1])$ such that $\rho \equiv 1$ on B. Then, since $v_j \in \mathrm{W}^{2,2}_{\mathrm{loc}}(\Omega;\mathbb{R}^n)$ by Lemma 5.11, we obtain that $\varphi := \rho^2\partial_k(v_j - a_j) =: \rho^2\partial_k w_j$ belongs to $\mathrm{W}^{1,2}_0(\Omega;\mathbb{R}^n)$, has compact support inside Ω and hence qualifies as a competitor map in (5.36). Here, $a_j \in \mathcal{R}(\Omega)$ is an arbitrary rigid deformation to be specified later on, and w_j is defined in the obvious manner. Writing $A = (A^{im})$ for an $(n\times n)$–matrix A and denoting the l–th component of a vector $u \in \mathbb{R}^n$ by $u^{(l)}$, the preceding equation gives

$$\sum_{k,i,m}\int_\Omega (\partial_k\sigma_j^{im})\boldsymbol{\varepsilon}^{im}(\rho^2\partial_k w_j)\,\mathrm{d}x = 0,$$

where the sum is taken over all indices $k,i,m \in \{1,...,n\}$. We now implement the splitting of [117] in detail and hereafter firstly write out the symmetric gradient to obtain

$$\sum_{k,i,m}\int_\Omega (\partial_k\sigma_j^{im})(\partial_i(\rho^2\partial_k w_j^{(m)}) + \partial_m(\rho^2\partial_k w_j^{(i)}))\,\mathrm{d}x = 0. \tag{5.37}$$

Thus we find by expanding terms

$$\sum_{k,i,m}\int_\Omega(\partial_k\sigma_j^{im})((\partial_i\rho^2)\partial_k w_j^{(m)}+\rho^2\partial_{ki}w_j^{(m)}) + (\partial_k\sigma_j^{im})((\partial_m\rho^2)\partial_k w_j^{(i)}+\rho^2\partial_{km}w_j^{(i)})\,\mathrm{d}x=0.$$

Consequently, regrouping terms we find

$$\begin{aligned}2\sum_{k,i,m}\int_\Omega(\partial_k\sigma_j^{im})(\rho^2\boldsymbol{\varepsilon}^{im}(\partial_k v_j))\,\mathrm{d}x &= -\sum_{k,i,m}\int_\Omega(\partial_k\sigma_j^{im})((\partial_i\rho^2)\partial_k w_j^{(m)}+(\partial_m\rho^2)\partial_k w_j^{(i)})\,\mathrm{d}x\\ &= -\sum_{k,i,m}\int_\Omega(\partial_k\sigma_j^{im})((\partial_i\rho^2)\partial_k w_j^{(m)}+(\partial_i\rho^2)\partial_m w_j^{(k)})\,\mathrm{d}x\\ &+\sum_{k,i,m}\int_\Omega(\partial_k\sigma_j^{im})((\partial_i\rho^2)\partial_m w_j^{(k)}+(\partial_m\rho^2)\partial_i w_j^{(k)})\,\mathrm{d}x\\ &-\sum_{k,i,m}\int_\Omega(\partial_k\sigma_j^{im})((\partial_m\rho^2)\partial_k w_j^{(i)}+(\partial_m\rho^2)\partial_i w_j^{(k)})\,\mathrm{d}x\\ &=:\mathbf{I}+\mathbf{II}+\mathbf{III}.\end{aligned}$$

Ad **I** and **III**. Let us note that since the indices i,m run over all numbers $1,...,n$, we have $\mathbf{I}=\mathbf{III}$. Moreover, we note that the artificial terms leading to the appearance of **II** are just introduced to have the symmetric gradient appearing, that is, terms which are conveniently controllable. In consequence, defining ${}_j\Theta_k:=({}_j\Theta_k^{im})$ and ${}_j\widetilde{\Theta}_k:=({}_j\widetilde{\Theta}_k^{im})$ with

$$\begin{aligned}{}_j\Theta_k^{im}&:=(\partial_m\rho^2)\boldsymbol{\varepsilon}^{ik}(w_j),\\ {}_j\widetilde{\Theta}_k^{im}&:=2(\partial_m\rho)\boldsymbol{\varepsilon}^{ik}(w_j),\qquad k,i,m\in\{1,...,n\},\end{aligned}$$

we consequently find

$$\begin{aligned}|\mathbf{I}+\mathbf{III}|\leqslant 2|\mathbf{III}| &= 2\left|\sum_{k,i,m}\int_\Omega(\partial_k\sigma_j^{im})(\partial_m\rho^2)(\partial_k w_j^{(i)}+\partial_i w_j^{(k)})\,\mathrm{d}x\right|\\ &\leqslant 4\left|\sum_{k,i,m}\int_\Omega(\partial_k\sigma_j^{im})(\partial_m\rho^2)\boldsymbol{\varepsilon}^{ik}(w_j)\,\mathrm{d}x\right| = 4\left|\sum_{k=1}^n\int_\Omega\langle\partial_k\sigma_j,{}_j\Theta_k\rangle\,\mathrm{d}x\right|\\ &=:\mathbf{IV}.\end{aligned}$$

We now employ the definition of σ_j in conjunction with the bilinear forms $\mathcal{A}[\boldsymbol{\varepsilon}(v_j);\cdot,\cdot]$ and $\mathcal{A}_j[\boldsymbol{\varepsilon}(v_j);\cdot,\cdot]$. Then we obtain, working from the definition of σ_j,

$$\begin{aligned}\mathbf{IV}&\leqslant 4\left|\sum_{k=1}^n\int_\Omega\mathcal{A}_j[\boldsymbol{\varepsilon}(v_j);\partial_k\boldsymbol{\varepsilon}(v_j),{}_j\Theta_k]\,\mathrm{d}x\right|\\ &\leqslant 4\sum_{k=1}^n\Big(\frac{1}{j}\int_\Omega|\langle\partial_k\boldsymbol{\varepsilon}(v_j),{}_j\Theta_k\rangle|\,\mathrm{d}x+\int_\Omega\mathcal{A}[\boldsymbol{\varepsilon}(v_j);\partial_k\boldsymbol{\varepsilon}(v_j),{}_j\Theta_k]\Big)\,\mathrm{d}x=:\mathbf{IV}',\end{aligned}$$

which, using the elementary bound $|{}_j\Theta_k|\leqslant 2\rho|\nabla\rho|\,|\boldsymbol{\varepsilon}(v_j)|$ and the Cauchy–Schwarz inequality, can be estimated by

$$\begin{aligned}
\mathbf{IV}' &\leqslant 8\sum_{k=1}^{n}\Big(\frac{1}{j}\int_\Omega |\partial_k\boldsymbol{\varepsilon}(v_j)|\rho|\nabla\rho|\,|\boldsymbol{\varepsilon}(v_j)|\,\mathrm{d}x + \int_\Omega \mathcal{A}[\boldsymbol{\varepsilon}(v_j);\partial_k\boldsymbol{\varepsilon}(v_j),{}_j\Theta_k]\Big)\,\mathrm{d}x \\
&\leqslant 8\sum_{k=1}^{n}\Big(\frac{1}{j}\int_\Omega \rho^2|\partial_k\boldsymbol{\varepsilon}(v_j)|^2\,\mathrm{d}x\Big)^{\frac{1}{2}}\Big(\frac{1}{j}\int_\Omega |\nabla\rho|^2|\boldsymbol{\varepsilon}(v_j)|^2\,\mathrm{d}x\Big)^{\frac{1}{2}} \\
&+ 8\sum_{k=1}^{n}\Big(\int_\Omega \rho^2\mathcal{A}[\boldsymbol{\varepsilon}(v_j);\partial_k\boldsymbol{\varepsilon}(v_j),\partial_k\boldsymbol{\varepsilon}(v_j)]\,\mathrm{d}x\Big)^{\frac{1}{2}}\Big(\int_\Omega \mathcal{A}[\boldsymbol{\varepsilon}(v_j);{}_j\widetilde{\Theta}_k,{}_j\widetilde{\Theta}_k]\Big)\,\mathrm{d}x\Big)^{\frac{1}{2}} =: \mathbf{V}.
\end{aligned}$$

To estimate $\mathbf{V}$, we note that

$$\langle\partial_k\sigma_j,\partial_k\boldsymbol{\varepsilon}(v_j)\rangle = \frac{1}{j}|\partial_k\boldsymbol{\varepsilon}(v_j)|^2 + \mathcal{A}[\boldsymbol{\varepsilon}(v_j);\partial_k\boldsymbol{\varepsilon}(v_j),\partial_k\boldsymbol{\varepsilon}(v_j)].$$

On the other hand, utilising the a–priori–estimate (5.18) and the upper bound in the definition of μ–ellipticity together with $|{}_j\widetilde{\Theta}_k| \leqslant C(\rho)|\boldsymbol{\varepsilon}(v_j)|$ (recall that $\boldsymbol{\varepsilon}(w_j) = \boldsymbol{\varepsilon}(v_j)$, we find both

$$\sup_{j\in\mathbb{N}}\frac{1}{j}\int_\Omega |\nabla\rho|^2|\boldsymbol{\varepsilon}(v_j)|^2\,\mathrm{d}x < \infty \quad\text{and}\quad \sup_{j\in\mathbb{N}}\int_\Omega |\boldsymbol{\varepsilon}(v_j)|\,\mathrm{d}x < \infty,$$

and thus moreover

$$\begin{aligned}
\sup_{j\in\mathbb{N}}\int_\Omega \mathcal{A}[\boldsymbol{\varepsilon}(v_j);{}_j\widetilde{\Theta}_k,{}_j\widetilde{\Theta}_k]\,\mathrm{d}x &\leqslant \sup_{j\in\mathbb{N}}\int_\Omega \Lambda\frac{|{}_j\widetilde{\Theta}_k|^2}{1+|\boldsymbol{\varepsilon}(v_j)|}\,\mathrm{d}x \\
&\leqslant \sup_{j\in\mathbb{N}}\int_\Omega C(\Lambda,\rho)|\boldsymbol{\varepsilon}(v_j)|\,\mathrm{d}x \leqslant C(\Lambda,\rho)
\end{aligned}$$

We therefore obtain by Young's inequality for $\delta > 0$ to be fixed later

$$\begin{aligned}
|\mathbf{I}+\mathbf{III}| \leqslant \mathbf{V} &\leqslant 8\sum_{k=1}^{n}\Big(\int_\Omega \rho^2\,\langle\partial_k\sigma_j,\partial_k\boldsymbol{\varepsilon}(v_j)\rangle\,\mathrm{d}x\Big)^{\frac{1}{2}}\times \\
&\times\Big(\Big(\frac{1}{j}\int_\Omega |\nabla\rho|^2|\boldsymbol{\varepsilon}(v_j)|^2\,\mathrm{d}x\Big)^{\frac{1}{2}} + \Big(\int_\Omega \mathcal{A}[\boldsymbol{\varepsilon}(v_j);{}_j\widetilde{\Theta}_k,{}_j\widetilde{\Theta}_k]\Big)\,\mathrm{d}x\Big)^{\frac{1}{2}}\Big) \qquad (5.38) \\
&\leqslant \delta\sum_{k=1}^{n}\int_\Omega \rho^2\,\langle\partial_k\sigma_j,\partial_k\boldsymbol{\varepsilon}(v_j)\rangle\,\mathrm{d}x + C(\delta,\Lambda,\rho,n).
\end{aligned}$$

Ad **II**. By symmetry of σ_j, i.e., $\sigma_j^{im} = \sigma_j^{mi}$ for all $i,m \in \{1,...,n\}$ and all $j \in \mathbb{N}$, and a permutation of indices, it suffices to estimate the term

$$2\left|\sum_{k,i,m}\int_\Omega \partial_k\sigma_j^{im}\partial_i\rho^2\partial_m w_j^{(k)}\,\mathrm{d}x\right| = |\mathbf{II}|.$$

Integrating by parts twice yields

$$\begin{aligned}\sum_{k,i,m}\int_\Omega \partial_k\sigma_j^{im}\partial_i\rho^2\partial_m w_j^{(k)}\,\mathrm{d}x &= -\sum_{k,i,m}\int_\Omega \sigma_j^{im}\partial_k(\partial_i\rho^2\partial_m w_j^{(k)})\,\mathrm{d}x\\ &= -\sum_{k,i,m}\int_\Omega \sigma_j^{im}(\partial_{ik}\rho^2\partial_m w_j^{(k)}+\partial_i\rho^2\partial_{mk}w_j^{(k)})\,\mathrm{d}x\\ &= \sum_{k,i,m}\int_\Omega \partial_m(\sigma_j^{im}\partial_{ik}\rho^2)w_j^{(k)}+\partial_m(\sigma_j^{im}\partial_i\rho^2)\partial_k w_j^{(k)}\,\mathrm{d}x\\ &= \sum_{k,i,m}\int_\Omega \partial_m\sigma_j^{im}\partial_{ik}\rho^2 w_j^{(k)}+\sigma_j^{im}\partial_{ikm}\rho^2 w_j^{(k)}\,\mathrm{d}x\\ &+\sum_{k,i,m}\int_\Omega \partial_m\sigma_j^{im}\partial_i\rho^2\partial_k w_j^{(k)}+\sigma_j^{im}\partial_{im}\rho^2\partial_k w_j^{(k)}\,\mathrm{d}x\\ &=: \mathbf{II}_1+\ldots+\mathbf{II}_4,\end{aligned}$$

where $\mathbf{II}_1,\ldots,\mathbf{II}_4$ are defined in the obvious manner. Note that by the $\mathrm{W}^{2,2}_{\mathrm{loc}}$–regularity of v_j, this is a valid computation. The crucial point in this calculation is that the only derivatives that apply to w_j appear in the form $\partial_k w_j^{(k)}$ (and are decoupled from the (i,m)–components), and summation over $k\in\{1,\ldots,n\}$ corresponds to taking the divergence of w_j. Since the divergence operator appears as the matrix trace of the symmetric gradient, we do have uniform control over these components indeed. To estimate the single terms, we go back to the equation satisfied by σ. More precisely, we record from (5.23) that each σ_j is row–wise solenoidal in the sense of distributions, and hence we deduce by symmetry of σ_j

$$\mathbf{II}_1+\mathbf{II}_3=\sum_{k,i,m}\int_\Omega \partial_m\sigma_j^{im}(\partial_{ik}\rho^2)w_j^{(k)}+\sum_{k,i,m}\int_\Omega \partial_m\sigma_j^{im}\partial_i\rho^2\partial_k w_j^{(k)}\,\mathrm{d}x=0.$$

Leaving $\mathbf{II}_2$ for later, we estimate

$$\mathbf{II}_4=\sum_k\int_\Omega\Big(\sum_{i,m}\sigma_j^{im}\partial_{im}\rho^2\Big)\partial_k w_j^{(k)}\,\mathrm{d}x=\int_\Omega\langle\sigma_j,\nabla^2\rho^2\rangle\,\mathrm{div}(w_j)\,\mathrm{d}x.$$

In conclusion, we find

$$|\mathbf{II}|\leqslant 2\left|\int_\Omega\langle\sigma_j,\nabla^2\rho^2\rangle\,\mathrm{div}(w_j)\,\mathrm{d}x\right|+2\left|\sum_{k,i,m}\int_\Omega\sigma^{im}\partial_{ikm}\rho^2 w_j^{(k)}\,\mathrm{d}x\right|.$$

We recall that by definition there holds $\sigma_j:=f'(\boldsymbol{\varepsilon}(v_j))+j^{-1}\boldsymbol{\varepsilon}(v_j)$, so

$$\begin{aligned}\left|\int_\Omega\langle\sigma_j,\nabla^2\rho^2\rangle\,\mathrm{div}(w_j)\,\mathrm{d}x\right| &\leqslant C(\rho)\int_\Omega |f'(\boldsymbol{\varepsilon}(v_j))|\,|\mathrm{div}(w_j)|+\frac{1}{j}|\boldsymbol{\varepsilon}(v_j)|\,|\mathrm{div}(w_j)|\,\mathrm{d}x\\ &\leqslant C(\rho)\Big(\|\tau_j\|_{\mathrm{L}^\infty(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})}\int_\Omega|\boldsymbol{\varepsilon}(w_j)|\,\mathrm{d}x+\frac{1}{j}\int_\Omega|\boldsymbol{\varepsilon}(v_j)|^2\,\mathrm{d}x\Big)\\ &\leqslant C(\rho),\end{aligned}$$

where we have employed the uniform estimates (5.18)ff.. On the other hand, we have

$$\begin{aligned}\left|\sum_{k,i,m}\int_\Omega \sigma_j^{im}\partial_{ikm}\rho^2 w_j^{(k)}\,\mathrm{d}x\right| &= \left|\sum_{k,i,m}\int_\Omega \tau_j^{im}\partial_{ikm}\rho^2 w_j^{(k)}\,\mathrm{d}x\right| \\ &\quad + \frac{1}{j}\left|\sum_{k,i,m}\int_\Omega \boldsymbol{\varepsilon}^{im}(v_j)\partial_{ikm}\rho^2 w_j^{(k)}\,\mathrm{d}x\right| \\ &\leqslant C(\rho)\|\tau_j\|_{\mathrm{L}^\infty(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})}\|w_j\|_{\mathrm{L}^1(\Omega;\mathbb{R}^n)} + \mathbf{VI}_j, \\ &\leqslant C(\rho) + \mathbf{VI}_j,\end{aligned}$$

where we have used the a–priori–estimates (5.18)ff. again and have put

$$\mathbf{VI}_j := \frac{1}{j}\left|\sum_{ikm}\int_\Omega \boldsymbol{\varepsilon}^{im}(v_j)\partial_{ikm}\rho^2 w_j^{(k)}\,\mathrm{d}x\right|.$$

Consequently, we obtain

$$|\mathbf{II}| \leqslant C(\rho) + \sup_{j\in\mathbb{N}}\mathbf{VI}_j \tag{5.39}$$

and therefore we find with $\delta > 0$ (which we recall to have arranged to be fixed later) and synthesising constants,

$$\begin{aligned}\sum_{k,i,m}\int_\Omega (\partial_k\sigma_j^{im})(\rho^2\boldsymbol{\varepsilon}^{im}(\partial_k v_j))\,\mathrm{d}x &\leqslant |\mathbf{I}+\mathbf{II}+\mathbf{III}| \\ &\overset{(5.38)}{\leqslant} \delta\sum_{k,i,m}\int_\Omega \rho^2\partial_k\sigma_j^{im}\partial_k\boldsymbol{\varepsilon}^{im}(v_j)\,\mathrm{d}x + C(\delta,\Lambda,\rho,n) + |\mathbf{II}| \\ &\overset{(5.39)}{\leqslant} C\delta\sum_{k,i,m}\int_\Omega \rho^2\partial_k\sigma_j^{im}\partial_k\boldsymbol{\varepsilon}^{im}(v_j)\,\mathrm{d}x + C(\delta,\Lambda,\rho,n) \\ &\quad + \sup_{j\in\mathbb{N}} C\mathbf{VI}_j.\end{aligned}$$

At this point, we choose $\delta > 0$ so small such that the first term on the very right hand side of the equation can be absorbed into the left hand side. We fix this choice of δ and hereafter end up with

$$\sum_{k,i,m}\int_\Omega (\partial_k\sigma_j^{im})(\rho^2\boldsymbol{\varepsilon}^{im}(\partial_k v_j))\,\mathrm{d}x \leqslant C(\delta,\Lambda,\rho,n) + \sup_{j\in\mathbb{N}} C\mathbf{VI}_j. \tag{5.40}$$

Step 4. Conclusion. We start by recalling that the overall claim of the theorem follows provided we can show (5.35). Clearly, here it suffices to argue for balls and so pick a ball B and a cut–off function ρ as done in step 3. Since the integrand in the definition of $\mathcal{J}_1^{(j)}$ is non–negative, we obtain

$$\begin{aligned}\int_{\mathrm{B}} \mathcal{J}_1^{(j)}\,\mathrm{d}x \leqslant \int_\Omega \rho^2\mathcal{J}_1^{(j)}\,\mathrm{d}x &= \sum_{k,i,m}\int_\Omega (\partial_k\sigma_j^{im})(\rho^2\boldsymbol{\varepsilon}^{im}(\partial_k v_j))\,\mathrm{d}x \\ &\leqslant C(\delta,\Lambda,\rho,n) + \sup_{j\in\mathbb{N}} C\mathbf{VI}_j,\end{aligned}$$

and we finish the proof by showing that $\sup_{j\in\mathbb{N}} C\mathbf{VI}_j < \infty$. We recall that we still have the freedom of choosing the rigid deformations $a_j \in \mathcal{R}(\Omega)$ as introduced in the beginning of step 3. For each $j \in \mathbb{N}$, we pick $a_j \in \mathcal{R}(\Omega)$ in a way such that

$$\int_\Omega |w_j|^2\,\mathrm{d}x \overset{\mathrm{Def}}{=} \int_\Omega |v_j - a_j|^2\,\mathrm{d}x \leqslant C\int_\Omega |\boldsymbol{\varepsilon}(v_j)|^2\,\mathrm{d}x, \tag{5.41}$$

and here, $C > 0$ does not depend on j. We then have

$$
\begin{aligned}
|\mathbf{IV}_j| &\leqslant C(\rho)\frac{1}{j}\int_\Omega |\boldsymbol{\varepsilon}(v_j)||w_j|\,\mathrm{d}x \\
&\leqslant C(\rho)\Big(\frac{1}{j}\int_\Omega |\boldsymbol{\varepsilon}(v_j)|^2\,\mathrm{d}x\Big)^{\frac{1}{2}}\Big(\frac{1}{j}\int_\Omega |w_j|^2\,\mathrm{d}x\Big)^{\frac{1}{2}} \overset{(5.41)}{\leqslant} C(\rho)\Big(\frac{1}{j}\int_\Omega |\boldsymbol{\varepsilon}(v_j)|^2\,\mathrm{d}x\Big),
\end{aligned}
$$

and the very last term is uniformly bounded in j by (5.18). The proof is complete. □

We conclude this subsection with two remarks, one regarding the full gradient situation and one regarding a deterioration of ellipticty.

Remark 5.13 (Gradient Case)**.** *In the gradient case, the regularity of the dual solution is slightly easier to achieve. For the Dirichlet problem on* BV*, the corresponding result due to* Bildhauer & Fuchs *[41] is an immediate consequence of* Seregin*'s strategy. We shall revisit the* BV*–situation in Chapter 6.6.4 within the framework of Neumann–type problems.*

Remark 5.14 (Degeneracy of f'')**.** *The reader will have noticed that in the proof of Theorem 5.12 we have not used the lower bound for the second derivatives provided by the definition of μ–ellipticity, cp.* (5.5)*. In fact, this implies that the dual solution still belongs to $\mathrm{W}^{1,2}_{\mathrm{loc}}(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})$ provided f is C^2–strictly convex, of linear growth and satisfies $\langle f''(\xi)\eta,\eta\rangle \leqslant \Lambda|\eta|^2/(1+|\xi|)$ for all $\xi,\eta\in\mathbb{R}^{n\times n}_{\mathrm{sym}}$ with a constant $\Lambda > 0$.*

5.4.4. Characterisation of the Dual Solution We recall that $f\in\mathrm{C}^2(\mathbb{R}^{n\times n}_{\mathrm{sym}})$ is assumed to be a convex integrand of linear growth. The purpose of the present section is to give the representation of the dual solution in terms of the solution of the primal problem (5.13). Here it crucially matters whether the symmetric gradient of the latter is absolutely continuous with respect to $\mathscr{L}^n$. We begin with the following lemma, where we from now on assume that the boundary datum satisfies $u_0\in\mathrm{LD}(\Omega)$.

Lemma 5.15 (Duality Relations, LD–case)**.** *Let $u\in u_0+\mathrm{LD}_0(\Omega)$ be a minimiser of the primal problem* (5.13) *and let $\sigma\in\mathrm{L}^\infty_{\mathrm{div}}(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})$ be the unique solution of the dual problem. Then we have*

$$\sigma = f'(\boldsymbol{\varepsilon}(u)) \qquad \mathscr{L}^n\text{-a.e. in }\Omega.$$

Proof. Following the duality relations (2.5), we obtain $\langle f'(\boldsymbol{\varepsilon}(u)),\boldsymbol{\varepsilon}(u)\rangle = f^*(f'(\boldsymbol{\varepsilon}(u))) + f(\boldsymbol{\varepsilon}(u))$ $\mathscr{L}^n$–a.e.. As $u\in\mathrm{LD}(\Omega)$ minimises $\mathfrak{F}$, we have for all $\varphi\in\mathrm{LD}_0(\Omega)$

$$\int_\Omega \langle f'(\boldsymbol{\varepsilon}(u)),\boldsymbol{\varepsilon}(\varphi)\rangle\,\mathrm{d}x = 0.$$

Young's inequality applied to the lower semicontinuous and convex function f gives $\langle\sigma,\boldsymbol{\varepsilon}(u)\rangle \leqslant f^*(\sigma)+f^{**}(\boldsymbol{\varepsilon}(u)) = f^*(\sigma)+f(\boldsymbol{\varepsilon}(u))$ $\mathscr{L}^n$–a.e.. Equivalently, we have $\langle\sigma,\boldsymbol{\varepsilon}(u)\rangle - f^*(\sigma)\leqslant f(\boldsymbol{\varepsilon}(u))$ $\mathscr{L}^n$–a.e., and this implies for *any* $\widetilde{\sigma}\in\mathrm{L}^\infty_{\mathrm{div}}(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})$ that

$$
\begin{aligned}
\mathscr{R}[\widetilde{\sigma}] \leqslant \int_\Omega \langle\widetilde{\sigma},\boldsymbol{\varepsilon}(u_0)\rangle - f^*(\widetilde{\sigma})\,\mathrm{d}x &\leqslant \int_\Omega \langle\widetilde{\sigma},\boldsymbol{\varepsilon}(u)\rangle - f^*(\widetilde{\sigma})\,\mathrm{d}x \\
&\leqslant \int_\Omega f(\boldsymbol{\varepsilon}(u))\,\mathrm{d}x = \int_\Omega \langle f'(\boldsymbol{\varepsilon}(u)),\boldsymbol{\varepsilon}(u)\rangle - f^*(f'(\boldsymbol{\varepsilon}(u)))\,\mathrm{d}x \\
&\leqslant \int_\Omega \langle f'(\boldsymbol{\varepsilon}(u)),\boldsymbol{\varepsilon}(u_0)\rangle - f^*(f'(\boldsymbol{\varepsilon}(u)))\,\mathrm{d}x \\
&= \mathscr{R}[f'(\boldsymbol{\varepsilon}(u))],
\end{aligned}
$$

where the first and last equalities are a consequence of $\boldsymbol{\varepsilon}(u) = \boldsymbol{\varepsilon}(u_0)+\boldsymbol{\varepsilon}(w)$ for some $w\in\mathrm{LD}_0(\Omega)$ and solenoidality of both $\widetilde{\sigma}$ and $f'(\boldsymbol{\varepsilon}(u))$. Passing to the supremum over all $\widetilde{\sigma}\in\mathrm{L}^\infty_{\mathrm{div}}(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})$ shows the claim. □

We note that the pairing $\int_\Omega \langle \widetilde{\sigma}, \boldsymbol{\varepsilon}(u)\rangle \,\mathrm{d}x$ that occurred in the proof of the preceding lemma makes sense because $\boldsymbol{\varepsilon}(u)$ was assumed to belong to $\mathrm{L}^1(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})$. However, if $u \in \mathrm{BD}(\Omega)$, we need to proceed more carefully to give sense to such terms. In analogy with Beck & Schmidt's adaptation [36, Sec. 7] of Anzellotti's pairing (cp. [21]) in the full gradient case, we introduce a pairing $[\![\cdot,\cdot]\!]_{\mathrm{sym}}$ between[5] $\mathrm{L}^\infty_{\mathrm{div}}(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})$–maps and certain elements of $\mathcal{M}_{<\infty}(\overline{\Omega};\mathbb{R}^{n\times n}_{\mathrm{sym}})$. To motivate the definition, firstly assume that $u \in u_0 + \mathrm{LD}_0(\Omega)$ for some given $u_0 \in \mathrm{LD}(\Omega)$. Then, for any $\widetilde{\sigma} \in \mathrm{L}^\infty_{\mathrm{div}}(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})$ we may invoke the integration by parts formula in LD to deduce for all $\varphi \in \mathrm{C}^\infty_c(\mathbb{R}^n)$

$$\begin{aligned}\int_\Omega \langle \varphi\widetilde{\sigma}, \boldsymbol{\varepsilon}(u)\rangle \,\mathrm{d}x &= \int_\Omega \langle \varphi\widetilde{\sigma}, \boldsymbol{\varepsilon}(u-u_0)\rangle \,\mathrm{d}x + \int_\Omega \langle \varphi\widetilde{\sigma}, \boldsymbol{\varepsilon}(u_0)\rangle \,\mathrm{d}x \\ &= -\int_\Omega \langle \mathrm{div}(\varphi\widetilde{\sigma}), u-u_0\rangle + \int_\Omega \langle \varphi\widetilde{\sigma}, \boldsymbol{\varepsilon}(u_0)\rangle \,\mathrm{d}x \\ &= -\int_\Omega \langle \widetilde{\sigma}, \nabla\varphi \odot (u-u_0)\rangle + \int_\Omega \langle \varphi\widetilde{\sigma}, \boldsymbol{\varepsilon}(u_0)\rangle \,\mathrm{d}x.\end{aligned} \tag{5.42}$$

as $u-u_0 \in \mathrm{LD}_0(\Omega)$. The upshot of the previous calculation now is that on the right hand side all terms are well–defined (recall that $u_0 \in \mathrm{LD}(\Omega)$). For fixed $\widetilde{\sigma} \in \mathrm{L}^\infty_{\mathrm{div}}(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})$, we may therefore consider the continuous extension $\overline{\mathcal{G}}$ to $\mathrm{C}_0(\mathbb{R}^n)$ of the functional

$$\mathcal{G}_{u,u_0,\widetilde{\sigma}}\colon\ \mathrm{C}^\infty_c(\mathbb{R}^n) \ni \varphi \mapsto -\int_\Omega \langle \widetilde{\sigma}, \nabla\varphi \odot (u-u_0)\rangle + \int_\Omega \langle \varphi\widetilde{\sigma}, \boldsymbol{\varepsilon}(u_0)\rangle \,\mathrm{d}x \in \mathbb{R} \tag{5.43}$$

with respect to the L^∞– (and so in this situation, the supremum) norm. Using (5.42), we moreover see that we can also define $\mathcal{G}$ for $u \in \mathrm{BD}_{u_0}(\overline{\Omega})$: In fact, given such u, we choose an approximating sequence $(u_j) \subset \mathrm{C}^\infty(\Omega;\mathbb{R}^n) \cap \mathrm{LD}(\Omega)$ as in Lemma 10.9. In particular, we have $u_j \to u$ strictly in $\mathrm{BD}(\Omega)$ as $j \to \infty$ and $\mathrm{Tr}(u_j) = \mathrm{Tr}(u)$ $\mathcal{H}^{n-1}$–a.e. on $\partial\Omega$ for all $j \in \mathbb{N}$. Then we obtain by (5.42) for all $\varphi \in \mathrm{C}^\infty_c(\mathbb{R}^n)$

$$\begin{aligned}&-\int_\Omega \langle \widetilde{\sigma}, \nabla\varphi \odot (u_j-u_0)\rangle + \int_\Omega \langle \varphi\widetilde{\sigma}, \boldsymbol{\varepsilon}(u_0)\rangle \,\mathrm{d}x \\ &\qquad\qquad \to -\int_\Omega \langle \widetilde{\sigma}, \nabla\varphi \odot (u-u_0)\rangle + \int_\Omega \langle \varphi\widetilde{\sigma}, \boldsymbol{\varepsilon}(u_0)\rangle \,\mathrm{d}x =: \mathcal{G}_{u,u_0,\widetilde{\sigma}}[\varphi],\end{aligned}$$

as $j \to \infty$ as strict convergence implies L^1–convergence on the level of functions themselves, and it is easy to see that this definition for $u \in \mathrm{BD}_{u_0}(\overline{\Omega})$ is independent of the particular approximating sequence (u_j). For any $\varphi \in \mathrm{C}^\infty_c(\mathbb{R}^n)$, we moreover obtain the bound

$$\begin{aligned}|\mathcal{G}_{u,u_0,\widetilde{\sigma}}[\varphi]| &\overset{(5.42)}{=} \lim_{j\to\infty} \left|\int_\Omega \langle \varphi\widetilde{\sigma}, \boldsymbol{\varepsilon}(u_j)\rangle \,\mathrm{d}x\right| \\ &\leqslant \lim_{j\to\infty} \big(\sup_{x\in\mathbb{R}^n} |\varphi(x)|\big)\|\widetilde{\sigma}\|_{\mathrm{L}^\infty(\Omega;\mathbb{R}^{n\times n})}\|\boldsymbol{\varepsilon}(u_j)\|_{\mathrm{L}^1(\Omega;\mathbb{R}^{n\times n})} \\ &= \big(\sup_{x\in\mathbb{R}^n} |\varphi(x)|\big)\|\widetilde{\sigma}\|_{\mathrm{L}^\infty(\Omega;\mathbb{R}^{n\times n})}|\,\mathrm{E}u|(\Omega).\end{aligned} \tag{5.44}$$

Here we have used that strict convergence implies strong L^1–convergence in the first step and the convergence of total masses in the final step. As a consequence of (5.44), we obtain the following lemma.

Lemma 5.16. *For fixed $(u,\widetilde{\sigma}) \in \mathrm{BD}_{u_0}(\overline{\Omega}) \times \mathrm{L}^\infty_{\mathrm{div}}(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})$, define $\overline{\mathcal{G}}_{u,u_0,\widetilde{\sigma}}$ as the continuous extension of $\mathcal{G}_{u,u_0,\widetilde{\sigma}}$ as given by (5.43) with respect to the L^∞–norm. Then there exists a unique finite signed Radon measure $\mu \in \mathcal{M}_{<\infty}(\overline{\Omega};\mathbb{R})$ such that $\overline{\mathcal{G}}_{u,u_0,\widetilde{\sigma}}[\varphi] = \int_{\mathbb{R}^n} \varphi \,\mathrm{d}\mu$ holds for all $\varphi \in \mathrm{C}_0(\mathbb{R}^n)$.*

[5] In general, it is *not* possible to define such a pairing between arbitrary L^∞–maps and Radon measures which arise as the (symmetric) gradients of weakly differentiable functions. However, it is possible to weaken the assumption of distributional solenoidality, cp. [21, Def. 1.4] for such generalisations.

Proof. By (5.44), $\overline{\mathcal{G}}_{u,u_0,\widetilde{\sigma}}$ is well–defined and thus can be defined as the continuous extension of $\mathcal{G}_{u,u_0,\widetilde{\sigma}}$ with respect to the sup–norm on $\mathrm{C}_0(\mathbb{R}^n)$. The claim is then an immediate consequence of the Riesz representation theorem. □

Put differently, the following definition is thus sensible.

Definition 5.17 (Symmetric up–to–the–boundary Anzellotti–type pairing)**.** *Let $\Omega \subset \mathbb{R}^n$ be an open and bounded Lipschitz domain and let $u_0 \in \mathrm{LD}(\Omega)$ be a given Dirichlet datum. Given $(u,\sigma) \in \mathrm{BD}_{u_0}(\overline{\Omega}) \times \mathrm{L}^{\infty}_{\mathrm{div}}(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})$, we define $[\![\sigma,\boldsymbol{\varepsilon}(u)]\!]_{\mathrm{sym}}$ as the unique signed Radon measure $\mu \in \mathcal{M}_{<\infty}(\overline{\Omega};\mathbb{R})$ such that*

$$\int_{\overline{\Omega}} \varphi \,\mathrm{d}\mu = \mathcal{G}_{u,u_0,\sigma}[\varphi] = \int_{\Omega} \langle \varphi\sigma, \boldsymbol{\varepsilon}(u_0)\rangle \,\mathrm{d}x - \int_{\Omega} \langle \sigma, \big((u-u_0)\odot\nabla\varphi\big)\rangle \,\mathrm{d}x \tag{5.45}$$

for all $\varphi \in \mathrm{C}^1_c(\mathbb{R}^n)$.

The pairing $[\![\cdot,\cdot]\!]_{\mathrm{sym}}$ as introduced in the the preceding definition turns out important as will be clear from the following theorem. This, in turn, is the main result of the present subsection.

Theorem 5.18. *Suppose that Ω is an open and bounded Lipschitz domain in $\mathbb{R}^n$ and $f \in \mathrm{C}^2(\mathbb{R}^{n\times n}_{\mathrm{sym}})$ is a convex integrand satisfying the linear growth hypothesis (5.2). Then for a pair*

$$(u,\sigma) \in \mathrm{BD}_{u_0}(\overline{\Omega}) \times \mathrm{L}^{\infty}_{\mathrm{div}}(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}}),$$

the following are equivalent:

(a) *u is a generalised minimiser of the primal problem (5.13) and σ is a maximiser of the dual problem (cp. (5.12)).*

(b) *Denoting $\mathrm{E}v = \mathscr{E}v\mathscr{L}^n \llcorner \Omega + \mathrm{E}^s v$ the Radon–Nikodym decomposition of $\mathrm{E}v$ for $v \in \mathrm{BD}(\Omega)$ (cp. (4.13)), $\mathscr{E}u$ satisfies any of the relations*

$$f(\mathscr{E}u) = \langle \sigma, \mathscr{E}u\rangle - f^*(\sigma) \qquad \mathscr{L}^n\text{-a.e. in } \Omega, \tag{5.46}$$
$$\sigma = f'(\mathscr{E}u) \qquad \mathscr{L}^n\text{-a.e. in } \Omega, \tag{5.47}$$
$$\mathscr{E}u = (f^*)'(\sigma) \qquad \mathscr{L}^n\text{-a.e. in } \Omega, \tag{5.48}$$

and $\mathrm{E}^s u$ simultaneously satisfies

$$f^{\infty}\Big(\frac{\mathrm{d}\mathrm{E}^s u}{\mathrm{d}|\,\mathrm{E}^s u|}\Big) = \frac{\mathrm{d}[\![\sigma,\mathrm{E}u]\!]^s_{\mathrm{sym}}}{\mathrm{d}|\,\mathrm{E}^s u|} \qquad |\,\mathrm{E}^s u|\text{-a.e. on } \overline{\Omega}. \tag{5.49}$$

The preceding theorem generalises [36, Thm. 2.1] to the symmetric gradient case. In a first step, we give the

Proof that (a) implies (5.46)–(5.48). Let $u \in \mathrm{BD}_{u_0}(\overline{\Omega})$ be a generalised minimiser. By Lemma 10.9, we find a sequence $(u_k) \subset u_0|_{\Omega} + \mathrm{C}^{\infty}_c(\Omega;\mathbb{R}^n)$ which converges area–strictly to u on $\mathrm{BD}(\overline{\Omega})$ and

$$\mathscr{E}u_k \to \mathscr{E}u \qquad \mathscr{L}^n\text{-a.e. in } \Omega \tag{5.50}$$

as $k \to \infty$. Then (u_k) is a minimising sequence for $\mathfrak{F}$. In fact, by Reshetnyak's continuity theorem in the form of Lemma 4.22 we have

$$\begin{aligned}\mathfrak{F}[u_k] = \int_{\Omega} f(\boldsymbol{\varepsilon}(u_k))\,\mathrm{d}x &\to \int_{\Omega} f(\mathscr{E}u)\,\mathrm{d}\mathscr{L}^n + \int_{\overline{\Omega}} f^{\infty}\Big(\frac{\mathrm{d}\mathrm{E}^s u}{\mathrm{d}|\,\mathrm{E}^s u|}\Big)\,\mathrm{d}|\,\mathrm{E}^s u| \\ &= \overline{F}_{u_0}[u] \leqslant \inf_{\mathcal{D}_{u_0}} \mathfrak{F}.\end{aligned}$$

Hence we find a non–relabelled subsequence such that we have $\mathfrak{F}[u_k] \leqslant \inf_{\mathcal{D}_{u_0}} \mathfrak{F} + \frac{1}{k}$ for all $k \in \mathbb{N}$. Now let $\sigma \in \mathrm{L}^{\infty}_{\mathrm{div}}(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})$ be a fixed Borel representative of a dual solution[6]. We put, for $x \in \Omega$ and $\xi \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$, $f_\sigma(x,\xi) := f(\xi) - \langle \sigma(x), \xi\rangle$ and hereafter obtain by $\mathrm{div}(\sigma) = 0$

$$
\begin{aligned}
\int_\Omega f_\sigma(x,\boldsymbol{\varepsilon}(u_k))\,\mathrm{d}x &= \int_\Omega f(\boldsymbol{\varepsilon}(u_k)) - \langle \sigma, \boldsymbol{\varepsilon}(u_k)\rangle \,\mathrm{d}x = \int_\Omega f(\boldsymbol{\varepsilon}(u_k)) - \langle \sigma, \boldsymbol{\varepsilon}(u_0)\rangle \,\mathrm{d}x \\
&\leqslant (\inf_{\mathcal{D}_{u_0}} \mathfrak{F}) + \frac{1}{k} - \int_\Omega \langle \sigma, \boldsymbol{\varepsilon}(u_0)\rangle \,\mathrm{d}x \\
&\leqslant \frac{1}{k} - \int_\Omega f^*(\sigma)\,\mathrm{d}x.
\end{aligned}
\tag{5.51}
$$

By Young's inequality (cp. Chapter 2.4.1), we have $\langle \sigma(x), \xi\rangle \leqslant f(\xi) + f^*(\sigma(x))$ for $\mathscr{L}^n$–a.e. elements $x \in \Omega$ and $\xi \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$. We therefore obtain for $\mathscr{L}^n$–a.e. $x \in \Omega$ and all $\xi \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$

$$
-f^*(\sigma(x)) \leqslant f(\xi) - \langle \sigma(x), \xi\rangle \stackrel{\mathrm{Def}}{=} f_\sigma(x,\xi). \tag{5.52}
$$

In consequence, this yields for all $\Theta \in \mathrm{L}^1(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})$

$$
-\int_\Omega f^*(\sigma(x))\,\mathrm{d}x \leqslant \int_\Omega f_\sigma(x,\Theta(x))\,\mathrm{d}x. \tag{5.53}
$$

We now intend to apply the Ekeland variational principle in the form as given in Corollary 2.34. To do so, let (Ω_k) be a sequence of open and bounded Lipschitz subsets of Ω which is (i) increasing in the sense of set inclusion and (ii) exhausts Ω in the sense that $\bigcup_{k\in\mathbb{N}} \Omega_k = \Omega$. In this situation, for each $k \in \mathbb{N}$, the space

$$
\mathrm{L}^2_{\Omega_k}(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}}) := \left\{\psi \in \mathrm{L}^2(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})\colon\ \psi \equiv 0 \text{ on } \Omega\setminus\Omega_k\right\}(\subset \mathrm{L}^1(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}}))
$$

is easily seen to be a norm–closed subspace of $\mathrm{L}^2(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})$ and hence is Banach on its own right. We hence infer particularly for every $\Theta \in \boldsymbol{\varepsilon}(u_k) + \mathrm{L}^2_{\Omega_k}(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}}) \subset \mathrm{L}^1(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})$ that

$$
\int_\Omega f_\sigma(x,\boldsymbol{\varepsilon}(u_k))\,\mathrm{d}x \stackrel{(5.51)\&(5.53)}{\leqslant} \frac{1}{k} + \int_\Omega f_\sigma(x,\Theta(x))\,\mathrm{d}x
$$

and hence, by passing to the infimum over all such maps Θ

$$
\int_\Omega f_\sigma(x,\boldsymbol{\varepsilon}(u_k))\,\mathrm{d}x \leqslant \frac{1}{k} + \inf_{\Theta\in\boldsymbol{\varepsilon}(u_k)+\mathrm{L}^2_{\Omega_k}(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})} \int_\Omega f_\sigma(x,\Theta(x))\,\mathrm{d}x.
$$

At this stage we use Corollary 2.34 to obtain for each $k \in \mathbb{N}$ mappings $v_k \in \boldsymbol{\varepsilon}(u_k) + \mathrm{L}^2_{\Omega_k}(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})$ and $\tau_k \in \mathrm{L}^\infty(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})$ such that

$$
\|v_k - \boldsymbol{\varepsilon}(u_k)\|_{\mathrm{L}^2(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})} \leqslant \frac{1}{\sqrt{k}}, \tag{5.54}
$$

$$
\tau_k(x) = f'_\sigma(x, v_k(x)) \qquad \text{for } \mathscr{L}^n-\text{a.e. } x \in \Omega, \tag{5.55}
$$

$$
\int_\Omega \langle \tau_k, \psi\rangle\,\mathrm{d}x < \frac{2}{\sqrt{k}}\|\psi\|_{\mathrm{L}^2(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})} \qquad \text{for all } \psi \in \mathrm{L}^2_{\Omega_k}(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}}). \tag{5.56}
$$

We extract information from these convergences as follows.

[6] At this stage uniqueness is not proven yet.

(i) If we pass to the supremum over all $\psi \in \mathrm{L}^2_{\Omega_k}(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})$ with $\|\psi\|_{\mathrm{L}^2(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})} \leqslant 1$ in (5.56), then we obtain

$$\|\tau_k\|_{\mathrm{L}^2(\Omega_k;\mathbb{R}^{n\times n}_{\mathrm{sym}})} \leqslant \frac{2}{\sqrt{k}}.$$

Now fix any relatively compact subset $K \Subset \Omega$. Then we find $l \in \mathbb{N}$ such that for all $k \geqslant l$ there holds $K \subset \Omega_k$. Hence for all such k we obtain

$$\int_K |\tau_k|^2 \,\mathrm{d}x \leqslant \int_\Omega |\tau_k|^2 \,\mathrm{d}x \leqslant \frac{4}{k} \to 0$$

as $k \to \infty$ and so, by arbitrariness of K, $\tau_k \to 0$ as $k \to \infty$ in $\mathrm{L}^2_{\mathrm{loc}}(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})$.

(ii) By (5.54), we have $v_k - \boldsymbol{\varepsilon}(u_k) \to 0$ as $k \to \infty$ in $\mathrm{L}^2(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})$ and hence, passing to a non–relabelled subsequence, we may assume that $v_k - \boldsymbol{\varepsilon}(u_k) \to 0$ pointwise $\mathscr{L}^n$–a.e. in Ω. On the other hand, we have $\boldsymbol{\varepsilon}(u_k) \to \mathscr{E}u$ pointwise $\mathscr{L}^n$–a.e. by (5.50). Hence we conclude that $v_k \to \mathscr{E}u$ pointwise $\mathscr{L}^n$–a.e. in Ω as $k \to \infty$. In combination with (i) from above, this yields $f'_\sigma(\mathscr{E}u(x)) = 0$ for $\mathscr{L}^n$–a.e. $x \in \Omega$.

(iii) *Conclusion.* We have $f'_\sigma(\mathscr{E}u(x)) = f'(\mathscr{E}(u)(x)) - \sigma(x)$ and hence, by (ii), $\sigma(x) = f'(\mathscr{E}u(x))$ for $\mathscr{L}^n$–a.e. $x \in \Omega$. This is (5.47). To show (5.46) and (5.48), we recall the duality relations (cp. (2.5)): If $h\colon \mathbb{R}^m \to \mathbb{R}\cup\{+\infty\}$ is a proper convex function, we have $\xi^* \in \partial h(\xi)$ if and only if $h(\xi) + h^*(\xi^*) = \langle \xi^*, \xi\rangle$. Applying this to $h = f$ and $\xi := \mathscr{E}u(x)$ and $\xi^* := \sigma(x)$ pointwise in x immediately yields

$$f^*(\sigma(x)) + f(\mathscr{E}u(x)) = \langle \sigma(x), \mathscr{E}u(x)\rangle$$

for $\mathscr{L}^n$–a.e. $x \in \Omega$. This is (5.46). Now note that under the conditions imposed on f, we have $f = f^{**}$ and hence, exploiting the 'if–and–only–if' statement of the aforementioned duality relation we also obtain (5.48).

The proof is complete. □

Anzellotti's pairing

From both a conceptual and historical perspective, it is important to compare the up–to–the–boundary pairing as introduced in Definition 5.17 to that which appears as a symmetric version of Anzellotti's original pairing, cp. [21]. This will also prove useful in the analysis of the absolutely continuous part (with respect to $\mathscr{L}^n$) of $[\![\widetilde{\sigma}, \mathrm{E}u]\!]_{\mathrm{sym}}$. For this we shall require a variant of the usual Gauß–Green formula and refer the reader in advance to Lemma 10.11 from the appendix. We begin with the following lemma, thereby examining the situation for test maps compactly supported in Ω.

Lemma 5.19. *In the situation of Definition 5.17 we have*

$$\int_{\overline{\Omega}} \varphi \,\mathrm{d}[\![\sigma, \mathrm{E}u]\!]_{\mathrm{sym}} = -\int_\Omega \langle \sigma, u \odot \nabla\varphi\rangle \,\mathrm{d}x \qquad \text{for all } \varphi \in \mathrm{C}^\infty_c(\Omega).$$

Proof. We have by use of the Gauß–Green formula as given in Lemma 10.11 (applied to $w = u_0$ and $\psi = \varphi\sigma$)

$$\begin{aligned}\int_{\overline{\Omega}} \varphi \,\mathrm{d}[\![\sigma, \mathrm{E}u]\!]_{\mathrm{sym}} &= \int_\Omega \langle \varphi\sigma, \boldsymbol{\varepsilon}(u_0)\rangle \,\mathrm{d}x - \int_\Omega \langle \sigma, ((u-u_0)\odot\nabla\varphi)\rangle \,\mathrm{d}x \\ &= \int_\Omega \langle \varphi\sigma, \boldsymbol{\varepsilon}(u_0)\rangle \,\mathrm{d}x - \int_\Omega \langle \sigma, u\odot\nabla\varphi\rangle \,\mathrm{d}x + \int_\Omega \langle \sigma, u_0\odot\nabla\varphi\rangle \,\mathrm{d}x \\ &= -\int_\Omega \langle \mathrm{div}(\varphi\sigma), u_0\rangle \,\mathrm{d}x - \int_\Omega \langle \sigma, u\odot\nabla\varphi\rangle \,\mathrm{d}x + \int_\Omega \langle \sigma, u_0\odot\nabla\varphi\rangle \,\mathrm{d}x.\end{aligned}$$

Now note that by solenoidality of σ,

$$\int_\Omega \langle \operatorname{div}(\varphi\sigma), u_0\rangle \,\mathrm{d}x = \sum_{ij}\int_\Omega \partial_j(\varphi\sigma_{ij})(u_0)_i = \sum_{ij}\int_\Omega (\varphi\partial_j\sigma_{ij} + (\partial_j\varphi)\sigma_{ij})(u_0)_i$$
$$= \sum_{ij}\int_\Omega ((\partial_j\varphi)\sigma_{ij})(u_0)_i =: (*),$$

where summation is understood with respect to $i,j\in\{1,...,n\}$. Moreover, we obtain by symmetry of σ

$$\int_\Omega \langle \sigma, u_0\odot\nabla\varphi\rangle\,\mathrm{d}x = \frac{1}{2}\sum_{ij}\int_\Omega \sigma_{ij}\big((u_0)_i\partial_j\varphi + (u_0)_j\partial_i\varphi\big) = \sum_{ij}\int_\Omega ((\partial_j\varphi)\sigma_{ij})(u_0)_i = (*).$$

The proof is complete. □

Note that the preceding lemma shows that if we only admit test functions which are compactly supported *in* Ω, then the symmetric up–to–the–boundary pairing introduced in Definition 5.17 reduces to the symmetrised version of Anzellotti's original pairing as introduced in [21, Def. 1.4]. Also note that it is important that we only work with symmetric, divergence–free competitor maps σ. Now define for $\sigma\in\mathrm{L}^\infty_{\operatorname{div}}(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})$ and $u\in\mathrm{BD}(\Omega)$ the measure $[\![\sigma,\mathrm{E}u]\!]^c_{\mathrm{sym}}\colon \mathrm{C}^\infty_c(\Omega)\to\mathbb{R}$ by

$$\langle [\![\sigma,\mathrm{E}u]\!]^c_{\mathrm{sym}},\varphi\rangle_{\mathscr{D}'\times\mathscr{D}} = \int_\Omega \varphi[\![\sigma,\mathrm{E}u]\!]^c_{\mathrm{sym}} := -\int_\Omega \langle\sigma, u\odot\nabla\varphi\rangle\,\mathrm{d}x, \qquad \varphi\in\mathrm{C}^\infty_c(\Omega). \tag{5.57}$$

By Lemma 5.19, we then have

$$\langle [\![\sigma,\mathrm{E}u]\!]^c_{\mathrm{sym}},\varphi\rangle_{\mathscr{D}'\times\mathscr{D}} = \langle [\![\sigma,\mathrm{E}u]\!]_{\mathrm{sym}},\varphi\rangle_{\mathscr{D}'\times\mathscr{D}} \tag{5.58}$$

whenever $\varphi\in\mathrm{C}^\infty_c(\Omega)$.

Analysis of the absolutely continuous part

The following lemmas now systematically generalise [21, Thm. 1.5– 1.9, Thm. 2.4]. For this, we recall $u_0\in\mathrm{LD}(\Omega)$ and that $\Omega\subset\mathbb{R}^n$ is assumed to have Lipschitz boundary throughout. The goal of this section is as follows: We firstly show that in the situation of the previous section, $[\![\sigma,\mathrm{E}u]\!]^c_{\mathrm{sym}}$ is absolutely continuous with respect to $|\,\mathrm{E}u|$. Hence the density of $[\![\sigma,\mathrm{E}u]\!]^c_{\mathrm{sym}}$ with respect to $|\,\mathrm{E}u|$ exists, and our overall aim is to express its absolutely continuous part (with respect to $\mathscr{L}^n$) in terms of $\frac{\mathrm{d}\mathrm{E}^{ac}u}{\mathrm{d}|\,\mathrm{E}u|}$. Keeping in mind (5.58), this will also apply to the the absolutely continuous part of the Riesz representative of the up–to–the–boundary pairing. Henceforth, let $u\in\mathrm{BD}(\Omega)$ and $\sigma\in\mathrm{L}^\infty_{\operatorname{div}}(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})$.

Lemma 5.20. *Let $A\subset\Omega$ be open. Then we have for all $\varphi\in\mathrm{C}^\infty_c(A)$*

$$\left|\int_A \varphi\,\mathrm{d}[\![\sigma,\mathrm{E}u]\!]^c_{\mathrm{sym}}\right| \leqslant (\sup_A|\varphi|)\|\sigma\|_{\mathrm{L}^\infty(\Omega;\mathbb{R}^{n\times n})}|\,\mathrm{E}u|(A). \tag{5.59}$$

Hence $[\![\sigma,\mathrm{E}u]\!]^c_{\mathrm{sym}}$ extends to a (not relabelled) continuous linear functional

$$[\![\sigma,\mathrm{E}u]\!]^c_{\mathrm{sym}}\colon \mathrm{C}_0(\Omega)\to\mathbb{R}$$

and thus is a Radon measure on $\mathscr{B}(\Omega)$ which equally satisfies (5.59) for all open sets $A\subset\Omega$ and all $\varphi\in\mathrm{C}_0(A)$.

In conclusion, both $[\![\sigma,\mathrm{E}u]\!]^c_{\mathrm{sym}}$ and $|[\![\sigma,\mathrm{E}u]\!]^c_{\mathrm{sym}}|$ are absolutely continuous with respect to $|\,\mathrm{E}u|$ in Ω and for all Borel subsets $B\in\mathscr{B}(\Omega)$ and open sets $A\subset\Omega$ we have

$$\left|\int_B \mathrm{d}[\![\sigma,\mathrm{E}u]\!]^c_{\mathrm{sym}}\right| \leqslant \int_B \mathrm{d}\,|[\![\sigma,\mathrm{E}u]\!]^c_{\mathrm{sym}}| \leqslant \|\sigma\|_{\mathrm{L}^\infty(A;\mathbb{R}^{n\times n}_{\mathrm{sym}})}\int_B \mathrm{d}\,|\,\mathrm{E}u|. \tag{5.60}$$

Proof. Let $\varphi \in \mathrm{C}_c^\infty(A)$ be arbitrary but fixed. We start by approximating $u \in \mathrm{BD}(\Omega)$ by the sequence $(u_j) \subset \mathrm{C}^\infty(\Omega;\mathbb{R}^n) \cap \mathrm{BD}(\Omega)$ with respect to the strict topology such that $\mathrm{Tr}(u_j) = \mathrm{Tr}(u)$ holds $\mathcal{H}^{n-1}$–a.e. on $\partial\Omega$ for all $j \in \mathbb{N}$. Let $U \subset A$ be an open set with $\mathrm{spt}(\varphi) \subset U \Subset A$. We then have by the integration by parts formula of Lemma 10.11 for all $j \in \mathbb{N}$ and by use of $\mathrm{div}(\sigma) \equiv 0$

$$\begin{aligned}\left|\int_\Omega \varphi \,\mathrm{d}[\![\sigma, \mathrm{E}u_j]\!]_{\mathrm{sym}}\right| &= \left|\int_\Omega \langle \sigma, u_j \odot \nabla\varphi\rangle \,\mathrm{d}x\right| \\ &= \left|\int_\Omega \langle \sigma, \boldsymbol{\varepsilon}(\varphi u_j) - \varphi\boldsymbol{\varepsilon}(u_j)\rangle \,\mathrm{d}x\right| = \left|\int_U \langle \sigma, \varphi\boldsymbol{\varepsilon}(u_j)\rangle \,\mathrm{d}x\right| \\ &\leqslant \|\sigma\|_{\mathrm{L}^\infty(A;\mathbb{R}^{n\times n})}(\sup_A |\varphi|)|\,\mathrm{E}u_j|(U).\end{aligned}$$

Now we send $j \to \infty$ and thereby obtain by strict convergence

$$\begin{aligned}\limsup_{j\to\infty}\left|\int_\Omega \varphi \,\mathrm{d}[\![\sigma, \mathrm{E}u_j]\!]_{\mathrm{sym}}\right| &\leqslant \|\sigma\|_{\mathrm{L}^\infty(A;\mathbb{R}^{n\times n})}(\sup_A |\varphi|)|\,\mathrm{E}u|(\overline{U}) \\ &\leqslant \|\sigma\|_{\mathrm{L}^\infty(A;\mathbb{R}^{n\times n})}(\sup_A |\varphi|)|\,\mathrm{E}u|(A) \qquad (\text{as } U \Subset A).\end{aligned}$$

But since $u_j \to u$ strongly in $\mathrm{L}^1(\Omega;\mathbb{R}^n)$ as $j \to \infty$, we have

$$\begin{aligned}\left|\int_\Omega \varphi \,\mathrm{d}[\![\sigma, \mathrm{E}u]\!]_{\mathrm{sym}}^c\right| &= \left|\int_\Omega \langle \sigma, u \odot \nabla\varphi\rangle \,\mathrm{d}x\right| \\ &= \lim_{j\to\infty}\left|\int_\Omega \langle \sigma, u_j \odot \nabla\varphi\rangle \,\mathrm{d}x\right| = \lim_{j\to\infty}\left|\int_\Omega \varphi \,\mathrm{d}[\![\sigma, \mathrm{E}u_j]\!]_{\mathrm{sym}}^c\right|\end{aligned}$$

and hence (5.59) follows. In consequence, $[\![\sigma, \mathrm{E}u]\!]_{\mathrm{sym}}^c$ extends to a continuous linear functional on $\mathrm{C}_0(\Omega)$ and thus can be represented by a Radon measure by the Riesz representation theorem. From here and (5.59), the conclusions of the lemma are immediate. □

In the situation of the preceding lemma, the Radon–Nikodym theorem implies that for each pair $(u, \sigma) \in \mathrm{BD}(\Omega) \times \mathrm{L}_{\mathrm{div}}^\infty(\Omega;\mathbb{R}_{\mathrm{sym}}^{n\times n})$ there exists a $|\,\mathrm{E}u|$–measurable density function $\theta(\cdot, \sigma, \mathrm{E}u)\colon \Omega \to \mathbb{R}$ such that

$$\begin{aligned}&\frac{\mathrm{d}[\![\sigma, \mathrm{E}u]\!]_{\mathrm{sym}}^c}{\mathrm{d}|\,\mathrm{E}u|} = \theta(\cdot, \sigma, \mathrm{E}u) \qquad |\,\mathrm{E}u| - \text{a.e. in } \Omega, \\ &\|\theta(\cdot, \sigma, \mathrm{E}u)\|_{\mathrm{L}_{|\,\mathrm{E}u|}^\infty(\Omega)} \leqslant \|\sigma\|_{\mathrm{L}^\infty(\Omega;\mathbb{R}_{\mathrm{sym}}^{n\times n})}.\end{aligned} \tag{5.61}$$

Here, $\mathrm{L}_{|\,\mathrm{E}u|}^\infty(\Omega)$ denotes the space of mappings $\Omega \to \mathbb{R}$ which are measurable and essentially bounded with respect to $|\,\mathrm{E}u|$. We now discuss a continuity result for the pairing just introduced.

Lemma 5.21. *Let $(u, \widetilde{\sigma}) \subset \mathrm{BD}(\Omega) \times \mathrm{L}_{\mathrm{div}}^\infty(\Omega;\mathbb{R}_{\mathrm{sym}}^{n\times n})$ and suppose that $(u_j) \subset \mathrm{C}^\infty(\Omega;\mathbb{R}^n) \cap \mathrm{BD}(\Omega)$ is a sequence which converges strictly to u in $\mathrm{BD}(\Omega)$ as $j \to \infty$. Then we have*

$$\int_\Omega \mathrm{d}[\![\widetilde{\sigma}, \mathrm{E}u_j]\!]_{\mathrm{sym}}^c \to \int_\Omega \mathrm{d}[\![\widetilde{\sigma}, \mathrm{E}u]\!]_{\mathrm{sym}}^c \qquad \text{as } j \to \infty.$$

Proof. Let $\varepsilon > 0$ be arbitrary. We may assume $\|\widetilde{\sigma}\|_{\mathrm{L}^\infty(\Omega;\mathbb{R}_{\mathrm{sym}}^{n\times n})} > 0$ as otherwise there is nothing to prove. Since $|\,\mathrm{E}u|$ is a finite Radon measure, we find an open set $A \Subset \Omega$ such that $|\,\mathrm{E}u|(\Omega \setminus A) < \varepsilon/(2\|\widetilde{\sigma}\|_{\mathrm{L}^\infty(\Omega;\mathbb{R}_{\mathrm{sym}}^{n\times n})})$ and a function $\rho \in \mathrm{C}_c^\infty(\Omega;[0,1])$ such that $\rho \equiv 1$

on A (and so, by continuity, $\rho \equiv 1$ on $\overline{A}$). We then estimate, writing $1 = \rho + (1-\rho)$,

$$\begin{aligned}\left|\int_\Omega \mathrm{d}[\![\widetilde{\sigma}, \mathrm{E}u_j]\!]^c_{\mathrm{sym}} - \int_\Omega \mathrm{d}[\![\widetilde{\sigma}, \mathrm{E}u]\!]^c_{\mathrm{sym}}\right| &\leqslant \left|\int_\Omega \rho\,\mathrm{d}[\![\widetilde{\sigma}, \mathrm{E}u_j]\!]^c_{\mathrm{sym}} - \int_\Omega \rho\,\mathrm{d}[\![\widetilde{\sigma}, \mathrm{E}u]\!]^c_{\mathrm{sym}}\right| \\ &\quad + \int_\Omega (1-\rho)\,\mathrm{d}\,|[\![\widetilde{\sigma}, \mathrm{E}u_j]\!]^c_{\mathrm{sym}}| \\ &\quad + \int_\Omega (1-\rho)\,\mathrm{d}\,|[\![\widetilde{\sigma}, \mathrm{E}u]\!]^c_{\mathrm{sym}}| =: \mathbf{I}_j + \mathbf{II}_j + \mathbf{III}.\end{aligned}$$

Since $\rho \equiv 1$ on A, we have by Lemma 5.20 and strict convergence

$$\begin{aligned}\limsup_{j\to\infty}(\mathbf{II}_j + \mathbf{III}) &\leqslant \limsup_{j\to\infty} \int_{\Omega\setminus A} \mathrm{d}\,|[\![\widetilde{\sigma}, \mathrm{E}u_j]\!]^c_{\mathrm{sym}}| + \int_{\Omega\setminus A} \mathrm{d}\,|[\![\widetilde{\sigma}, \mathrm{E}u]\!]^c_{\mathrm{sym}}| \\ &\leqslant \limsup_{j\to\infty} \|\widetilde{\sigma}\|_{\mathrm{L}^\infty(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})}|\,\mathrm{E}u_j|(\Omega\setminus A) + |[\![\widetilde{\sigma}, \mathrm{E}u]\!]^c_{\mathrm{sym}}|(\Omega\setminus A) \\ &\leqslant 2\|\widetilde{\sigma}\|_{\mathrm{L}^\infty(\Omega;\mathbb{R}^{n\times n})}|\,\mathrm{E}u|(\overline{\Omega}\setminus A) < \varepsilon.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\mathbf{I}_j &= \left|\int_\Omega \langle \widetilde{\sigma}, u_j \odot \nabla\rho\rangle\,\mathrm{d}x - \int_\Omega \langle \widetilde{\sigma}, u \odot \nabla\rho\rangle\,\mathrm{d}x\right| \\ &\leqslant \|\widetilde{\sigma}\|_{\mathrm{L}^\infty(\Omega;\mathbb{R}^{n\times n})}\|\nabla\rho\|_{\mathrm{L}^\infty(\Omega;\mathbb{R}^{n\times n})}\|u - u_j\|_{\mathrm{L}^1(\Omega;\mathbb{R}^n)} \to 0\end{aligned}$$

as $j \to \infty$ because strict convergence includes L^1–convergence on the level of the functions themselves. By arbitrariness of $\varepsilon > 0$, we may send $\varepsilon \searrow 0$ and the proof is complete. □

We now give an explicit characterisation of $\theta(\cdot, \widetilde{\sigma}, \mathrm{E}u)$.

Lemma 5.22. *Let $\widetilde{\sigma} \in \mathrm{L}^\infty_{\mathrm{div}}(\Omega; \mathbb{R}^{n\times n}_{\mathrm{sym}})$ and $u \in \mathrm{BD}(\Omega)$. Then we have*

$$\theta(x, \widetilde{\sigma}, \mathrm{E}u) = \left\langle \widetilde{\sigma}(x), \frac{\mathrm{dE}u}{\mathrm{d}|\,\mathrm{E}u|}(x)\right\rangle \qquad \textit{for } |\,\mathrm{E}u|^{ac} - \textit{a.e. } x \in \Omega. \tag{5.62}$$

Proof. Note that the claimed characterisation amounts to requiring for all $B \in \mathscr{B}(\Omega)$

$$\begin{aligned}\int_B \theta(x, \widetilde{\sigma}, \mathrm{E}u)\frac{\mathrm{d}|\,\mathrm{E}u|^{ac}}{\mathrm{d}\mathscr{L}^n}(x)\,\mathrm{d}\mathscr{L}^n &\overset{(5.62)}{=} \int_B \left\langle \widetilde{\sigma}(x), \frac{\mathrm{dE}u}{\mathrm{d}|\,\mathrm{E}u|}(x)\right\rangle \mathrm{d}|\,\mathrm{E}u|^{ac} \\ &= \int_B \left\langle \widetilde{\sigma}(x), \frac{\mathrm{dE}^{ac}u}{\mathrm{d}|\,\mathrm{E}u|^{ac}}(x)\right\rangle \frac{\mathrm{d}|\,\mathrm{E}u|^{ac}}{\mathrm{d}\mathscr{L}^n}(x)\,\mathrm{d}\mathscr{L}^n \\ &= \int_B \left\langle \widetilde{\sigma}(x), \frac{\mathrm{dE}^{ac}u}{\mathrm{d}\mathscr{L}^n}(x)\right\rangle \mathrm{d}\mathscr{L}^n\end{aligned} \tag{5.63}$$

Moreover, since $\mathrm{E}^{ac}u$ is not concentrated on $\partial\Omega$, we can restrict ourselves to showing (5.63) for $B \in \mathscr{B}(\Omega)$ with $B \Subset \Omega$.

Let $\mathfrak{E}_{ac}, \mathfrak{E}_s \in \mathscr{B}(\Omega)$ with $\mathfrak{E}_{ac} \cup \mathfrak{E}_s = \Omega$, $\mathfrak{E}_{ac} \cap \mathfrak{E}_s = \emptyset$ and $\mathrm{E}^{ac}u(\mathfrak{E}_s) = \mathrm{E}^s u(\mathfrak{E}_{ac}) = 0$. Let $\varepsilon > 0$ be arbitrary but fixed and pick, by inner regularity of Radon measures, a compact set $K \Subset \mathfrak{E}_s$ such that

$$\int_{\mathfrak{E}_s\setminus K} \mathrm{d}|\,\mathrm{E}^s u| < \varepsilon. \tag{5.64}$$

Let $\mathfrak{B}_0$ be a compact subset of $\mathfrak{E}_{ac}$. Accordingly, we can find an open set $\mathfrak{L}$ with Lipschitz boundary such that $\mathfrak{B}_0 \subset \mathfrak{L} \subset \Omega \setminus K$ and

$$\int_{\overline{\mathfrak{L}\setminus\mathfrak{B}_0}} \mathrm{d}|\,\mathrm{E}u| < \varepsilon. \tag{5.65}$$

We further note that

$$\int_{\mathfrak{L}} \mathrm{d}|\,\mathrm{E}u|^s \leqslant \int_{\mathfrak{L}\setminus\mathfrak{B}_0} \mathrm{d}|\,\mathrm{E}u|^s + \int_{\mathfrak{E}_{ac}} \mathrm{d}|\,\mathrm{E}u|^s = \int_{\mathfrak{L}\setminus\mathfrak{B}_0} \mathrm{d}|\,\mathrm{E}u| < \varepsilon \qquad \text{(by (5.65))}. \tag{5.66}$$

At this point, recalling that $\mathfrak{L}$ has Lipschitz boundary, we choose according to Lemma 10.9 (with U replaced by $\mathfrak{L}$) a sequence $(u_j) \subset \mathrm{C}^\infty(\mathfrak{L};\mathbb{R}^n) \cap \mathrm{BD}(\mathfrak{L})$ together with $\mathrm{Tr}_{\partial\mathfrak{L}}(u_j) = \mathrm{Tr}_{\partial\mathfrak{L}}(u)$.

We then apply Lemma 5.21 to conclude

$$\int_{\mathfrak{L}} \theta(\cdot,\widetilde{\sigma},\mathrm{E}u)\,\mathrm{d}|\,\mathrm{E}u| = [\![\widetilde{\sigma},\mathrm{E}u]\!]^c_{\mathrm{sym}}(\mathfrak{L}) = \lim_{j\to\infty} [\![\widetilde{\sigma},\mathrm{E}u_j]\!]^c_{\mathrm{sym}}(\mathfrak{L}) = \lim_{j\to\infty} \int_{\mathfrak{L}} \langle \widetilde{\sigma}, \boldsymbol{\varepsilon}(u_j)\rangle\,\mathrm{d}\mathscr{L}^n$$

We write

$$\begin{aligned}\int_{\mathfrak{B}_0} \theta(x,\widetilde{\sigma},\mathrm{E}u)\,\mathrm{d}|\,\mathrm{E}u| &- \int_{\mathfrak{B}_0} \langle\widetilde{\sigma}(x),\mathrm{E}^{ac}u\rangle\,\mathrm{d}\mathscr{L}^n \\ &= \int_{\mathfrak{L}} \theta(x,\widetilde{\sigma},\mathrm{E}u)\,\mathrm{d}|\,\mathrm{E}u| - \int_{\mathfrak{L}} \langle\widetilde{\sigma}(x),\mathrm{E}^{ac}u\rangle\,\mathrm{d}\mathscr{L}^n \\ &\quad - \int_{\mathfrak{L}\setminus\mathfrak{B}_0} \theta(x,\widetilde{\sigma},\mathrm{E}u)\,\mathrm{d}|\,\mathrm{E}u| + \int_{\mathfrak{L}\setminus\mathfrak{B}_0} \langle\widetilde{\sigma}(x),\mathrm{E}^{ac}u\rangle\,\mathrm{d}\mathscr{L}^n.\end{aligned} \tag{5.67}$$

Keeping this identity in mind, we estimate

$$\begin{aligned}\Big|\int_{\mathfrak{L}} \theta(x,\widetilde{\sigma},\mathrm{E}u)\;\mathrm{d}|\,\mathrm{E}u| &- \int_{\mathfrak{L}} \langle\widetilde{\sigma}(x),\mathrm{E}^{ac}u\rangle\,\mathrm{d}\mathscr{L}^n\Big| \\ &\overset{\text{Lemma 5.21}}{=} \lim_{j\to\infty}\Big|\int_{\mathfrak{L}} \theta(x,\widetilde{\sigma},\mathrm{E}u_j)\,\mathrm{d}\mathrm{E}u_j - \int_{\mathfrak{L}} \langle\widetilde{\sigma}(x),\mathrm{E}^{ac}u\rangle\,\mathrm{d}\mathscr{L}^n\Big| \\ &\overset{\text{Lemma 10.9}}{\leqslant} \lim_{j\to\infty} \|\widetilde{\sigma}\|_{\mathrm{L}^\infty(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})} \int_{\mathfrak{L}} \Big|\frac{\mathrm{dE}u_j}{\mathrm{d}\mathscr{L}^n} - \frac{\mathrm{dE}u}{\mathrm{d}\mathscr{L}^n}\Big|\,\mathrm{d}\mathscr{L}^n \\ &\leqslant \|\widetilde{\sigma}\|_{\mathrm{L}^\infty(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})} \int_{\mathfrak{L}} \mathrm{d}|\,\mathrm{E}^s u| < \varepsilon\|\widetilde{\sigma}\|_{\mathrm{L}^\infty(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})}.\end{aligned} \tag{5.68}$$

In a second step, we note that

$$\Big|\int_{\mathfrak{L}\setminus\mathfrak{B}_0} \langle\widetilde{\sigma}(x),\mathrm{E}^{ac}u\rangle\,\mathrm{d}\mathscr{L}^n\Big| \leqslant \|\widetilde{\sigma}\|_{\mathrm{L}^\infty(\Omega;\mathbb{R}^{n\times n})} \int_{\overline{\mathfrak{L}\setminus\mathfrak{B}_0}} |\,\mathrm{E}u| \leqslant \varepsilon\|\widetilde{\sigma}\|_{\mathrm{L}^\infty(\Omega;\mathbb{R}^{n\times n})}. \tag{5.69}$$

Thirdly, we have by (5.60)

$$\begin{aligned}\Big|\int_{\mathfrak{L}\setminus\mathfrak{B}_0} \theta(x,\widetilde{\sigma},\mathrm{E}u)\,\mathrm{d}|\,\mathrm{E}u|\Big| &\leqslant \|\widetilde{\sigma}\|_{\mathrm{L}^\infty(\Omega;\mathbb{R}^{n\times n})} \int_{\overline{\mathfrak{L}\setminus\mathfrak{B}_0}} |\,\mathrm{E}u| \\ &\leqslant \varepsilon\|\widetilde{\sigma}\|_{\mathrm{L}^\infty(\Omega;\mathbb{R}^{n\times n})}.\end{aligned} \tag{5.70}$$

Passing to the modulus on the left and right side of (5.67) and applying the triangle inequality to the resulting right hand side, estimates (5.68)–(5.70) together yield

$$\Big|\int_{\mathfrak{B}_0} \theta(x,\widetilde{\sigma},\mathrm{E}u)\,\mathrm{d}|\,\mathrm{E}u| - \int_{\mathfrak{B}_0} \langle\widetilde{\sigma}(x),\mathrm{E}^{ac}u\rangle\,\mathrm{d}\mathscr{L}^n\Big| \leqslant 3\varepsilon\|\widetilde{\sigma}\|_{\mathrm{L}^\infty(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})}.$$

From here we deduce that (5.63) is valid for all compact sets $B = \mathfrak{B}_0 \subset \mathfrak{E}_{ac}$. But since $[\![\widetilde{\sigma},\mathrm{E}u]\!]^c_{\mathrm{sym}}$ is a Radon measure, this transfers to all Borel subsets of Ω and the proof is complete. □

With this in mind, we now turn to the up–to–the–boundary–pairing as introduced in Definition 5.17.

Lemma 5.23. *In the situation of Definition 5.17 and denoting the Radon–Nikodym decomposition of $[\![\widetilde{\sigma}, \mathrm{E}u]\!]_{\mathrm{sym}}$ with respect to $\mathscr{L}^n$ by $[\![\sigma, \mathrm{E}u]\!]_{\mathrm{sym}} = [\![\sigma, \mathrm{E}u]\!]_{\mathrm{sym}}^{ac} + [\![\sigma, \mathrm{E}u]\!]_{\mathrm{sym}}^{s}$, we have*

$$[\![\sigma, \mathrm{E}u]\!]_{\mathrm{sym}}^{ac} = \langle \sigma, \mathscr{E}u\rangle \mathscr{L}^n \llcorner \overline{\Omega} \tag{5.71}$$

and the total variation measure $|[\![\sigma, \mathrm{E}u]\!]_{\mathrm{sym}}|$ satisfies

$$|[\![\sigma, \mathrm{E}u]\!]_{\mathrm{sym}}|(A) \leqslant \|\sigma\|_{\mathrm{L}^\infty(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})} |\,\mathrm{E}u|(A) \qquad \text{for all } A \in \mathscr{B}(\overline{\Omega}). \tag{5.72}$$

Moreover, the density $\frac{\mathrm{d}[\![\sigma,\mathrm{E}u]\!]_{\mathrm{sym}}}{\mathrm{d}|\,\mathrm{E}u|}$ exists $|\,\mathrm{E}u|$–a.e., and we have

$$[\![\sigma, \mathrm{E}u]\!]_{\mathrm{sym}}(\overline{\Omega}) = \int_\Omega \langle \sigma, \boldsymbol{\varepsilon}(u_0)\rangle \,\mathrm{d}x. \tag{5.73}$$

Proof. As argued above, the pairings $[\![\widetilde{\sigma}, \mathrm{E}u]\!]_{\mathrm{sym}}$ and $[\![\sigma, \mathrm{E}u]\!]_{\mathrm{sym}}^{c}$ coincide when being tested against functions compactly supported *within* Ω, cp. Lemma 5.19. This immediately yields (5.71) because of Lemma 5.22, and (5.72) because of Lemma 5.20. It remains to justify (5.73). Let $\varphi \in \mathrm{C}_c^\infty(\mathbb{R}^n; [0,1])$ such that $\varphi \equiv 1$ on $\overline{\Omega}$. Then we obtain by (5.45)

$$\int_{\overline{\Omega}} \mathrm{d}[\![\sigma, \mathrm{E}u]\!]_{\mathrm{sym}} = \int_{\overline{\Omega}} \varphi \,\mathrm{d}[\![\sigma, \mathrm{E}u]\!]_{\mathrm{sym}} = \int_\Omega \langle \varphi\sigma, \boldsymbol{\varepsilon}(u_0)\rangle \,\mathrm{d}x = \int_\Omega \langle \sigma, \boldsymbol{\varepsilon}(u_0)\rangle \,\mathrm{d}x,$$

and this is the claim. The proof is complete. □

Analysis of the singular part

We now turn to the singular part of $[\![\sigma, \mathrm{E}u]\!]_{\mathrm{sym}}$ with respect to $\mathscr{L}^n$ and hereafter systematically generalise [36, Thm. 7.2, Cor. 7.3].

Lemma 5.24. *Let $u \in \mathrm{BD}_{u_0}(\overline{\Omega})$ and $\widetilde{\sigma} \in \mathrm{L}^\infty_{\mathrm{div}}(\Omega; \mathbb{R}^{n\times n}_{\mathrm{sym}})$. Suppose that x_0 is a Lebesgue point of both*

$$\frac{\mathrm{dE}u}{\mathrm{d}|\,\mathrm{E}u|} \quad \text{and} \quad \frac{\mathrm{d}[\![\widetilde{\sigma}, \mathrm{E}u]\!]_{\mathrm{sym}}}{\mathrm{d}|\,\mathrm{E}u|} \qquad \text{with respect to } |\,\mathrm{E}u| \text{ on } \overline{\Omega}.$$

Let $\mathcal{K}$ be a non–empty, closed and convex subset of $\mathbb{R}^{n\times n}_{\mathrm{sym}}$. If $\widetilde{\sigma} \in \mathcal{K}$ $\mathscr{L}^n$–a.e. in a neighbourhood of x_0 in Ω, then there exists $\widetilde{\sigma}_0 \in \mathcal{K}$ such that

$$\frac{\mathrm{d}[\![\widetilde{\sigma}, \mathrm{E}u]\!]_{\mathrm{sym}}}{\mathrm{d}|\,\mathrm{E}u|}(x_0) = \left\langle \widetilde{\sigma}_0, \frac{\mathrm{dE}u}{\mathrm{d}|\,\mathrm{E}u|}(x_0)\right\rangle. \tag{5.74}$$

Proof. Following [36, Thm. 7.2], we divide the proof into three steps.

Reduction. We may suppose that $0 \in \mathcal{K}$. In fact, if not, then by non–emptiness of $\mathcal{K}$, we find $z_0^* \in \mathcal{K}$. Now note that for all $\varphi \in \mathrm{C}_c^\infty(\mathbb{R}^n)$ we have by the definition of the up–to–the–boundary pairing, cp. (5.45),

$$\begin{aligned}\langle [\![\widetilde{\sigma} - z_0^*, \mathrm{E}u]\!]_{\mathrm{sym}}, \varphi\rangle_{\mathscr{D}'\times\mathscr{D}} &= \langle [\![\widetilde{\sigma}, \mathrm{E}u]\!]_{\mathrm{sym}}, \varphi\rangle_{\mathscr{D}'\times\mathscr{D}} \\ &\quad - \int_\Omega \varphi\langle z_0^*, \boldsymbol{\varepsilon}(u_0)\rangle - \langle z_0^*, (u-u_0)\odot\nabla\varphi\rangle \,\mathrm{d}x.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\int_\Omega \langle \varphi z_0^*, \boldsymbol{\varepsilon}(u_0)\rangle \,\mathrm{d}x - \int_\Omega \langle z_0^*, (u-u_0)\odot\nabla\varphi\rangle \,\mathrm{d}x &= \int_\Omega \langle \varphi z_0^*, \boldsymbol{\varepsilon}(u_0)\rangle \,\mathrm{d}x \\ &\quad - \int_\Omega \langle z_0^*, \mathrm{E}(\varphi(u-u_0)) - \varphi\,\mathrm{E}(u-u_0)\rangle \\ &= -\int_\Omega \langle z_0^*, \mathrm{E}(\varphi(u-u_0)) - \varphi\,\mathrm{E}u\rangle = \int_\Omega \langle z_0^*, \varphi\,\mathrm{E}u\rangle,\end{aligned}$$

the ultimate step being valid by integration by parts and $\operatorname{div}(z_0^*)=0$. Hence we have

$$\langle[\![\widetilde{\sigma}-z_0^*,\mathrm{E}u]\!]_{\mathrm{sym}},\varphi\rangle_{\mathscr{D}'\times\mathscr{D}}=\langle[\![\widetilde{\sigma},\mathrm{E}u]\!]_{\mathrm{sym}},\varphi\rangle_{\mathscr{D}'\times\mathscr{D}}-\int_\Omega\langle z_0^*,\varphi\,\mathrm{E}u\rangle$$

which amounts to saying

$$[\![\widetilde{\sigma}-z_0^*,\mathrm{E}u]\!]_{\mathrm{sym}}=[\![\widetilde{\sigma},\mathrm{E}u]\!]_{\mathrm{sym}}-\langle z_0^*,\mathrm{E}u\rangle. \tag{5.75}$$

Therefore, replacing $\mathcal{K}$ by $\mathcal{K}':=\{z^*-z_0^*\colon z^*\in\mathcal{K}\}$ and $\widetilde{\sigma}$ by $\widetilde{\sigma}-z_0^*$, the proof below yields the result corresponding to (5.74) for $\widetilde{\sigma}-z_0^*$ and hence gives, for some $\widetilde{\sigma}_0^*=\widetilde{\sigma}_0-z_0^*\in\mathcal{K}'$ with $\widetilde{\sigma}_0\in\mathcal{K}$,

$$\begin{aligned}\frac{\mathrm{d}[\![\widetilde{\sigma},\mathrm{E}u]\!]_{\mathrm{sym}}}{\mathrm{d}|\,\mathrm{E}u|}(x_0)-\left\langle z_0^*,\frac{\mathrm{dE}u}{\mathrm{d}|\,\mathrm{E}u|}(x_0)\right\rangle &\overset{(5.75)}{=}\frac{\mathrm{d}[\![\widetilde{\sigma}-z_0^*,\mathrm{E}u]\!]_{\mathrm{sym}}}{\mathrm{d}|\,\mathrm{E}u|}(x_0)=\left\langle\widetilde{\sigma}_0^*,\frac{\mathrm{dE}u}{\mathrm{d}|\,\mathrm{E}u|}(x_0)\right\rangle\\ &=\left\langle\widetilde{\sigma}_0,\frac{\mathrm{dE}u}{\mathrm{d}|\,\mathrm{E}u|}(x_0)\right\rangle-\left\langle z_0^*,\frac{\mathrm{dE}u}{\mathrm{d}|\,\mathrm{E}u|}(x_0)\right\rangle,\end{aligned}$$

and the respective cancellations then show (5.74). The justification of our reduction to $0\in\mathcal{K}$ is complete.

Step 1. Approximation. By the refined area–strict approximation result, Lemma 10.9, we find a sequence $(w_j)\subset u_0|_\Omega+\mathrm{C}_c^\infty(\Omega;\mathbb{R}^n)$ such that

$$w_j\to u\quad\text{strongly in}\,\mathrm{L}^1(\Omega;\mathbb{R}^n), \tag{5.76}$$

$$\mathrm{E}w_j\overset{*}{\rightharpoonup}\mathrm{E}u\quad\text{weak* in }\mathcal{M}_{<\infty}(\overline{\Omega};\mathbb{R}^{n\times n}_{\mathrm{sym}}) \tag{5.77}$$

$$\sqrt{1+|\,\mathrm{E}w_j|^2}(\overline{\Omega})\to\sqrt{1+|\,\mathrm{E}u|^2}(\overline{\Omega}) \tag{5.78}$$

$$\mathscr{E}w_j\to\mathscr{E}u\quad\mathscr{L}^n-\text{a.e. in }\Omega \tag{5.79}$$

as $j\to\infty$. We now use $w_j\in u_0+\mathrm{LD}_0(\Omega)$ to obtain for all $\varphi\in\mathrm{C}_c^\infty(\mathbb{R}^n)$

$$\begin{aligned}\int_\Omega\langle\varphi\widetilde{\sigma},\boldsymbol{\varepsilon}(w_j)\rangle\,\mathrm{d}x&\overset{(5.42)}{=}-\int_\Omega\langle\widetilde{\sigma},\nabla\varphi\odot(w_j-u_0)\rangle+\int_\Omega\langle\varphi\widetilde{\sigma},\boldsymbol{\varepsilon}(u_0)\rangle\,\mathrm{d}x\\ &\overset{(5.76)}{\longrightarrow}-\int_\Omega\langle\widetilde{\sigma},\nabla\varphi\odot(u-u_0)\rangle+\int_\Omega\langle\varphi\widetilde{\sigma},\boldsymbol{\varepsilon}(u_0)\rangle\,\mathrm{d}x\\ &=\int_{\overline{\Omega}}\varphi\,\mathrm{d}[\![\widetilde{\sigma},\mathrm{E}u]\!]_{\mathrm{sym}}\qquad\text{as }j\to\infty,\end{aligned}$$

where the last equality holds by definition, cp. (5.45). Smooth approximation of $\mathbb{1}_{\mathrm{B}(x_0,R)}$ yields by use of (5.72)

$$[\![\widetilde{\sigma},\mathrm{E}u]\!]_{\mathrm{sym}}(\overline{\Omega}\cap\mathrm{B}(x_0,R))=\lim_{j\to\infty}\int_{\Omega\cap\mathrm{B}(x_0,R)}\langle\widetilde{\sigma},\boldsymbol{\varepsilon}(w_j)\rangle\,\mathrm{d}x \tag{5.80}$$

provided $|\,\mathrm{E}u|(\overline{\Omega}\cap\partial\mathrm{B}(x_0,R))=0$. We note that with $|\,\mathrm{E}u|$ being a *finite* Radon measure, the latter condition $|\,\mathrm{E}u|(\overline{\Omega}\cap\partial\mathrm{B}(x_0,R))=0$ is satisfied for $\mathscr{L}^1$–a.e. (in fact, up to a countable subset *every*) radius $R>0$. In all of what follows, we assume that $R>0$ is chosen in a way such that this condition is satisfied throughout.

Step 2. Projection estimates. For x_0 fulfilling the assumptions of the lemma, we put $z:=\frac{\mathrm{dE}u}{\mathrm{d}|\,\mathrm{E}u|}(x_0)$ and record $|z|=1$. We denote $\Pi_{z^\perp}\colon\mathbb{R}^{n\times n}_{\mathrm{sym}}\to\mathbb{R}^{n\times n}_{\mathrm{sym}}$ the orthogonal projection onto $\{z\}^\perp$. Now, using the notation $a_+,a_-\in\mathbb{R}_{\geqslant0}$ for the positive or negative parts of a function $a\colon\mathbb{R}^m\to\mathbb{R}$, respectively, we find

$$\begin{aligned}\langle\widetilde{\sigma},\boldsymbol{\varepsilon}(w_j)\rangle&=\langle(\widetilde{\sigma}-\Pi_{z^\perp}\widetilde{\sigma})+\Pi_{z^\perp}\widetilde{\sigma},(\boldsymbol{\varepsilon}(w_j)-\Pi_{z^\perp}\boldsymbol{\varepsilon}(w_j))+\Pi_{z^\perp}\boldsymbol{\varepsilon}(w_j)\rangle\\ &=\langle(\widetilde{\sigma}-\Pi_{z^\perp}\widetilde{\sigma}),(\boldsymbol{\varepsilon}(w_j)-\Pi_{z^\perp}\boldsymbol{\varepsilon}(w_j))\rangle+\langle\Pi_{z^\perp}\widetilde{\sigma},\Pi_{z^\perp}\boldsymbol{\varepsilon}(w_j)\rangle\\ &=\langle\widetilde{\sigma},z\rangle\langle z,\boldsymbol{\varepsilon}(w_j)\rangle_+-\langle\widetilde{\sigma},z\rangle\langle z,\boldsymbol{\varepsilon}(w_j)\rangle_-+\langle\Pi_{z^\perp}\widetilde{\sigma},\Pi_{z^\perp}\boldsymbol{\varepsilon}(w_j)\rangle\\ &=:\mathbf{I}_z^{(j)}+\mathbf{II}_z^{(j)}+\mathbf{III}_z^{(j)},\end{aligned}$$

where $\mathbf{I}_z^{(j)}$–$\mathbf{III}_z^{(j)}$ are defined in the obvious manner.

Ad $\mathbf{I}_z^{(j)}$. We put

$$\mathfrak{M} := \max\{\langle z^*, z\rangle \colon z^* \in \mathcal{K},\ |z^*| \leqslant \|\widetilde{\sigma}\|_{\mathrm{L}^\infty(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})}\}. \tag{5.81}$$

Since $0 \in \mathcal{K}$, $\mathfrak{M} \geqslant 0$, and $\mathfrak{M} \leqslant \|\widetilde{\sigma}\|_{\mathrm{L}^\infty(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})}$. By assumption, it is possible to choose $R > 0$ so small such that $\widetilde{\sigma} \in \mathcal{K}$ holds $\mathscr{L}^n$–a.e. on $\Omega \cap \mathrm{B}(x_0, R)$. By the final part of Lemma 10.9 applied to $g := \langle z, \cdot\rangle_+$ we consequently find by use of the elementary inequality[7]

$$\langle z, w\rangle_+ = \langle z, w\rangle + \langle z, w\rangle_- \leqslant \langle z, w\rangle + |z - w|$$

for vectors $z, w \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$ with $|z| = |w| = 1$ that

$$\begin{aligned}
\limsup_{j\to\infty}\left|\int_{\Omega\cap\mathrm{B}(x_0,R)} \langle\widetilde{\sigma}, z\rangle\langle z, \boldsymbol{\varepsilon}(w_j)\rangle_+ \,\mathrm{d}x\right| &\leqslant \mathfrak{M}\lim_{j\to\infty}\int_{\Omega\cap\mathrm{B}(x_0,R)} \langle z, \boldsymbol{\varepsilon}(w_j)\rangle_+ \,\mathrm{d}x \\
&= \mathfrak{M}\int_{\Omega\cap\mathrm{B}(x_0,R)} \left\langle z, \frac{\mathrm{d}\mathrm{E}u}{\mathrm{d}|\mathrm{E}u|}\right\rangle_+ \mathrm{d}|\mathrm{E}u| \\
&= \mathfrak{M}\int_{\Omega\cap\mathrm{B}(x_0,R)} \left\langle z, \frac{\mathrm{d}\mathrm{E}u}{\mathrm{d}|\mathrm{E}u|}\right\rangle \mathrm{d}|\mathrm{E}u| \\
&\quad + \mathfrak{M}\int_{\Omega\cap\mathrm{B}(x_0,R)} \left|\frac{\mathrm{d}\mathrm{E}u}{\mathrm{d}|\mathrm{E}u|} - z\right| \mathrm{d}|\mathrm{E}u|.
\end{aligned}$$

Ad $\mathbf{II}_z^{(j)}$. Applying the final part of Lemma 10.9 to $g := \langle z, \cdot\rangle_-$ and recalling the elementary estimate $\langle w, z\rangle_- \leqslant |z - w|$ for all $z, w \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$ with $|z| = |w| = 1$ from above, we find

$$\begin{aligned}
\limsup_{j\to\infty}\left|\int_{\Omega\cap\mathrm{B}(x_0,R)} \langle\widetilde{\sigma}, z\rangle\langle z, \boldsymbol{\varepsilon}(w_j)\rangle_- \,\mathrm{d}x\right| &\leqslant \|\widetilde{\sigma}\|_{\mathrm{L}^\infty(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})}\lim_{j\to\infty}\int_{\Omega\cap\mathrm{B}(x_0,R)} \langle z, \boldsymbol{\varepsilon}(w_j)\rangle_- \,\mathrm{d}x \\
&= L\int_{\overline{\Omega}\cap\mathrm{B}(x_0,R)} \left\langle z, \frac{\mathrm{d}\mathrm{E}u}{\mathrm{d}|\mathrm{E}u|}\right\rangle_- \mathrm{d}|\mathrm{E}u| \\
&\leqslant L\int_{\overline{\Omega}\cap\mathrm{B}(x_0,R)} \left|\frac{\mathrm{d}\mathrm{E}u}{\mathrm{d}|\mathrm{E}u|} - z\right| \mathrm{d}|\mathrm{E}u|.
\end{aligned}$$

Ad $\mathbf{III}_z^{(j)}$. Since $\Pi_{z^\perp}$ is an orthogonal projection, we have the operator norm bound $\|\Pi_{z^\perp}\| \leqslant 1$ and therefore $|\Pi_{z^\perp}v| = |\Pi_{z^\perp}(v - z)| \leqslant |v - z|$ for all $v \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$ (recall that $\Pi_{z^\perp}z = 0$). Hence we obtain with $L := \|\widetilde{\sigma}\|_{\mathrm{L}^\infty(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})}$

$$\begin{aligned}
\limsup_{j\to\infty}\left|\int_{\Omega\cap\mathrm{B}(x_0,R)} \langle\Pi_{z^\perp}\widetilde{\sigma}, \Pi_{z^\perp}\boldsymbol{\varepsilon}(w_j)\rangle \,\mathrm{d}x\right| &\leqslant \limsup_{j\to\infty}\left|\int_{\Omega\cap\mathrm{B}(x_0,R)} |\widetilde{\sigma}|\,|\Pi_{z^\perp}\boldsymbol{\varepsilon}(w_j)| \,\mathrm{d}x\right| \\
&\leqslant L\limsup_{j\to\infty}\left|\int_{\Omega\cap\mathrm{B}(x_0,R)} |\Pi_{z^\perp}\boldsymbol{\varepsilon}(w_j)| \,\mathrm{d}x\right| \\
&= L\int_{\overline{\Omega}\cap\mathrm{B}(x_0,R)} \left|\Pi_{z^\perp}\Big(\frac{\mathrm{d}\mathrm{E}u}{\mathrm{d}|\mathrm{E}u|}\Big)\right| \mathrm{d}|\mathrm{E}u| \\
&\leqslant L\int_{\overline{\Omega}\cap\mathrm{B}(x_0,R)} \left|\frac{\mathrm{d}\mathrm{E}u}{\mathrm{d}|\mathrm{E}u|} - z\right| \mathrm{d}|\mathrm{E}u|
\end{aligned}$$

[7] This reduces to $\langle z, w\rangle_- \leqslant |z - w|$ for $|z| = |w| = 1$ and is trivial if $\measuredangle(z, w) \in (-\frac{\pi}{2}, \frac{\pi}{2})$. If $\measuredangle(z, w) \in [\frac{\pi}{2}, \frac{3\pi}{2})$, then $|z - w| \geqslant 1$ but $\langle z, w\rangle_- \leqslant 1$.

Step 3. Limiting behaviour as $R \searrow 0$ and conclusion. We now gather estimates to find

$$\begin{aligned}\llbracket\widetilde{\sigma}, \mathrm{E}u\rrbracket_{\mathrm{sym}}(\overline{\Omega}\cap \mathrm{B}(x_0,R)) \leqslant \mathfrak{M}\langle z, \mathrm{E}u\rangle(\overline{\Omega}\cap \mathrm{B}(x_0,R)) \\ + CL\int_{\overline{\Omega}\cap\mathrm{B}(x_0,R)}\left|\frac{\mathrm{dE}u}{\mathrm{d}|\,\mathrm{E}u|}-z\right|\mathrm{d}|\,\mathrm{E}u|.\end{aligned} \tag{5.82}$$

Dividing the previous inequality by $|\,\mathrm{E}u|(\overline{\Omega}\cap \mathrm{B}(x_0,R))$ (which, in turn, is admissible because x_0 is assumed to be a Lebesgue point for both $\mathrm{E}u$ with respect to $|\,\mathrm{E}u|$, for $\llbracket\widetilde{\sigma}, \mathrm{E}u\rrbracket_{\mathrm{sym}}$ with respect to $|\,\mathrm{E}u|$ and $x_0\in\mathrm{spt}(|\,\mathrm{E}u|)$) and sending $R\searrow 0$ thereby yields

$$\begin{aligned}\frac{\mathrm{d}\llbracket\widetilde{\sigma}, \mathrm{E}u\rrbracket_{\mathrm{sym}}}{\mathrm{d}|\,\mathrm{E}u|}(x_0) &= \lim_{R\searrow 0}\frac{\mathrm{d}\llbracket\widetilde{\sigma}, \mathrm{E}u\rrbracket_{\mathrm{sym}}(\overline{\Omega}\cap\mathrm{B}(x_0,R))}{|\,\mathrm{E}u|(\overline{\Omega}\cap\mathrm{B}(x_0,R))} \\ &\overset{(5.82)}{\leqslant} \lim_{R\searrow 0}\mathfrak{M}\left\langle z, \frac{\mathrm{E}u(\overline{\Omega}\cap\mathrm{B}(x_0,R))}{|\,\mathrm{E}u|(\overline{\Omega}\cap\mathrm{B}(x_0,R))}\right\rangle \\ &\quad + CL\underbrace{\lim_{R\searrow 0}\frac{1}{|\,\mathrm{E}u|(\overline{\Omega}\cap\mathrm{B}(x_0,R))}\int_{\overline{\Omega}\cap\mathrm{B}(x_0,R)}\left|\frac{\mathrm{dE}u}{\mathrm{d}|\,\mathrm{E}u|}-z\right|\mathrm{d}|\,\mathrm{E}u|}_{=0\text{ as } z=\frac{\mathrm{dE}u}{\mathrm{d}|\,\mathrm{E}u|}(x_0)} \\ &\leqslant \mathfrak{M}|z|^2\leqslant\mathfrak{M}\quad(\text{as } |z|=1).\end{aligned}$$

As a consequence of the definition of $\mathfrak{M}$ (cp. (5.81)), there exists $\sigma_{\mathfrak{M}}\in\mathcal{K}$ such that

$$\frac{\mathrm{d}\llbracket\widetilde{\sigma}, \mathrm{E}u\rrbracket_{\mathrm{sym}}}{\mathrm{d}|\,\mathrm{E}u|}(x_0)\leqslant\langle\sigma_{\mathfrak{M}}, z\rangle.$$

Similarly as for $\mathbf{I}_z^{(j)}$, but now estimating from below, yields $\sigma_m\in\mathcal{K}$ with

$$\frac{\mathrm{d}\llbracket\widetilde{\sigma}, \mathrm{E}u\rrbracket_{\mathrm{sym}}}{\mathrm{d}|\,\mathrm{E}u|}(x_0)\geqslant\langle\sigma_{m}, z\rangle,$$

and so a suitable convex combination $\widetilde{\sigma}_0\in K$ of $\sigma_{\mathfrak{M}}$ and σ_m then is seen to satisfy (5.74). The proof is complete. □

Lemma 5.25. *Let $u\in\mathrm{BD}_{u_0}(\overline{\Omega})$ and $\sigma\in\mathrm{L}^{\infty}_{\mathrm{div}}(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})$ be such that $f^*(\sigma)<\infty$ $\mathscr{L}^n$–a.e. in Ω. Then there holds*

$$\frac{\mathrm{d}\llbracket\sigma, \mathrm{E}u\rrbracket_{\mathrm{sym}}}{\mathrm{d}|\,\mathrm{E}u|}\leqslant f^{\infty}\left(\frac{\mathrm{dE}u}{\mathrm{d}|\,\mathrm{E}u|}\right)\qquad |\,\mathrm{E}u|\text{-a.e. on }\overline{\Omega}. \tag{5.83}$$

Proof. Firstly, it is sufficient to establish (5.83) for every $x_0\in\overline{\Omega}$ which is a Lebesgue point (with respect to $|\,\mathrm{E}u|$) for both $\frac{\mathrm{dE}u}{\mathrm{d}|\,\mathrm{E}u|}$ and $\frac{\mathrm{d}\llbracket\sigma,\mathrm{E}u\rrbracket_{\mathrm{sym}}}{\mathrm{d}|\,\mathrm{E}u|}$.

For any $t>0$ and any $x\in\Omega$ and $z\in\mathbb{R}^{n\times n}_{\mathrm{sym}}$, we have by Young's inequality

$$\langle\sigma(x), z/t\rangle\leqslant f(z/t)+f^*(\sigma(x))\quad\text{and so}\quad\langle\sigma(x), z\rangle - tf^*(\sigma(x))\leqslant tf(z/t).$$

Sending $t\searrow 0$ in the previous inequality yields by definition of f^{∞}

$$\langle\sigma(x), z\rangle\leqslant f^{\infty}(z)\qquad\text{for all } z\in\mathbb{R}^{n\times n}_{\mathrm{sym}} \tag{5.84}$$

whenever $f^*(\sigma(x))<\infty$ (which, by assumption, holds $\mathscr{L}^n$–a.e.). In conclusion, $\sigma(x)\in\partial f^{\infty}(0)$ with the subdifferential ∂f^{∞} at zero. Let $\varepsilon>0$ be arbitrary. Then we have $\sigma(x)\in N_{\varepsilon}(\partial f^{\infty}(0))$ for all x in a neighbourhood of $x_0\in\Omega$. We then apply Lemma 5.24 with $\mathcal{K}=\overline{N_{\varepsilon}(\partial f^{\infty}(0))}$ (where for a set $A\subset\mathbb{R}^m$, $N_{\varepsilon}(A):=\{x\in\mathbb{R}^m\colon\ \mathrm{dist}(x,A)<\varepsilon\}$ denotes the ε–neighbourhood of A) and so there exists $\widetilde{\sigma}_0\in\overline{N_{\varepsilon}(\partial f^{\infty}(0)}$ such that

$$\frac{\mathrm{d}\llbracket\sigma, \mathrm{E}u\rrbracket_{\mathrm{sym}}}{\mathrm{d}|\,\mathrm{E}u|}(x_0)=\left\langle\widetilde{\sigma}_0, \frac{\mathrm{dE}u}{\mathrm{d}|\,\mathrm{E}u|}(x_0)\right\rangle.$$

Therefore, we obtain that there exists a subgradient $\sigma^* \in \partial f^\infty(0)$ such that $|\sigma^* - \widetilde{\sigma}_0| < \varepsilon$ and hence, by $|\frac{\mathrm{d}\mathrm{E}u}{\mathrm{d}|\,\mathrm{E}u|}(x_0)| \leqslant 1$,

$$\begin{aligned}\frac{\mathrm{d}[\![\sigma, \mathrm{E}u]\!]_{\mathrm{sym}}}{\mathrm{d}|\,\mathrm{E}u|}(x_0) &= \left\langle \sigma^*, \frac{\mathrm{d}\mathrm{E}u}{\mathrm{d}|\,\mathrm{E}u|}(x_0) \right\rangle + \left\langle \widetilde{\sigma}_0 - \sigma^*, \frac{\mathrm{d}\mathrm{E}u}{\mathrm{d}|\,\mathrm{E}u|}(x_0) \right\rangle \\ &\leqslant \left\langle \sigma^*, \frac{\mathrm{d}\mathrm{E}u}{\mathrm{d}|\,\mathrm{E}u|}(x_0) \right\rangle + \varepsilon \left| \frac{\mathrm{d}\mathrm{E}u}{\mathrm{d}|\,\mathrm{E}u|}(x_0) \right| \\ &\leqslant \varepsilon + f^\infty\left(\frac{\mathrm{d}\mathrm{E}u}{\mathrm{d}|\,\mathrm{E}u|}(x_0) \right) \qquad \text{(by (5.84))}.\end{aligned}$$

Accordingly, we may send $\varepsilon \searrow 0$ to conclude. ☐

After these preparations, we eventually come to the

Proof of Theorem 5.18. We split the proof into the assertions for the absolutely continuous and singular parts. Also, we recall that we have already established that (a) implies (5.46)–(5.48).

The absolutely continuous part. By Young's inequality, we directly find

$$f(\mathscr{E}u) \geqslant \langle \sigma, \mathscr{E}u \rangle - f^*(\sigma) \tag{5.85}$$

$\mathscr{L}^n$–a.e. in Ω.

The singular part. We start by noting that $\frac{\mathrm{d}\mathrm{E}u}{\mathrm{d}|\,\mathrm{E}^s u|} = \frac{\mathrm{d}\mathrm{E}u}{\mathrm{d}|\,\mathrm{E}u|}$ holds $|\,\mathrm{E}^s u|$–a.e. in $\overline{\Omega}$. As a consequence of Lemma 5.23, we further obtain that $|[\![\sigma, \mathrm{E}u]\!]^s_{\mathrm{sym}}|(A) \leqslant \|\sigma\|_{\mathrm{L}^\infty(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})} |\,\mathrm{E}^s u|(A)$ for all $A \in \mathscr{B}(\overline{\Omega})$. Hence we obtain

$$\frac{\mathrm{d}[\![\sigma, \mathrm{E}u]\!]^s_{\mathrm{sym}}}{\mathrm{d}|\,\mathrm{E}^s u|} = \frac{\mathrm{d}[\![\sigma, \mathrm{E}u]\!]_{\mathrm{sym}}}{\mathrm{d}|\,\mathrm{E}u|} \qquad |\,\mathrm{E}^s u|\text{-a.e. on}\,\Omega$$

and, in particular, the left side is well–defined. In this situation, we obtain by Lemma 5.25

$$\frac{\mathrm{d}[\![\sigma, \mathrm{E}u]\!]^s_{\mathrm{sym}}}{\mathrm{d}|\,\mathrm{E}^s u|} \leqslant f^\infty\Big(\frac{\mathrm{d}\mathrm{E}^s u}{\mathrm{d}|\,\mathrm{E}^s u|} \Big) \qquad |\,\mathrm{E}^s u|\text{-a.e. on}\,\Omega \tag{5.86}$$

provided $f^*(\sigma) < \infty$ holds $\mathscr{L}^n$–a.e. in Ω.

By Theorem 5.10 (c), we know $\sup_{\mathrm{L}^\infty_{\mathrm{div}}(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})} \mathscr{R} = \inf_{\mathcal{D}_{u_0}} \mathfrak{F}$, where $\mathcal{D}_{u_0} := u_0 + \mathrm{LD}_0(\Omega)$. On the other hand, we record from Theorem 4.23 (cp. (4.24)) that $\inf_{\mathcal{D}_{u_0}} \mathfrak{F} = \min_{\mathrm{BD}(\Omega)} \widetilde{\mathfrak{F}}_{u_0}$ (with the corresponding notation from Chapter 4.3.2). Hence, letting $u \in \mathrm{BD}(\Omega)$ be a solution of the relaxed primal problem, i.e., $\widetilde{\mathfrak{F}}_{u_0}[u] = \min_{\mathrm{BD}(\Omega)} \widetilde{\mathfrak{F}}_{u_0}$ and $\sigma \in \mathrm{L}^\infty_{\mathrm{div}}(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})$ be a solution of the dual problem, we get

$$\begin{aligned}\widetilde{\mathfrak{F}}_{u_0}[u] = \sup_{\mathrm{L}^\infty_{\mathrm{div}}(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})} \mathscr{R} &= \inf_{\mathcal{D}_{u_0}} \mathfrak{F} \\ &= \int_\Omega \langle \sigma, \boldsymbol{\varepsilon}(u_0) \rangle - f^*(\sigma)\,\mathrm{d}x \\ &\overset{(5.73)}{=} [\![\sigma, \mathrm{E}u]\!]_{\mathrm{sym}}(\overline{\Omega}) - \int_\Omega f^*(\sigma)\,\mathrm{d}x \\ &\overset{(5.71)}{=} \int_\Omega \langle \sigma, \mathscr{E}u \rangle\,\mathrm{d}x + [\![\sigma, \mathrm{E}u]\!]^s_{\mathrm{sym}}(\overline{\Omega}) - \int_\Omega f^*(\sigma)\,\mathrm{d}x \\ &= \int_\Omega \big(\langle \sigma, \mathscr{E}u \rangle - f^*(\sigma) \big)\,\mathrm{d}x + [\![\sigma, \mathrm{E}u]\!]^s_{\mathrm{sym}}(\overline{\Omega}),\end{aligned} \tag{5.87}$$

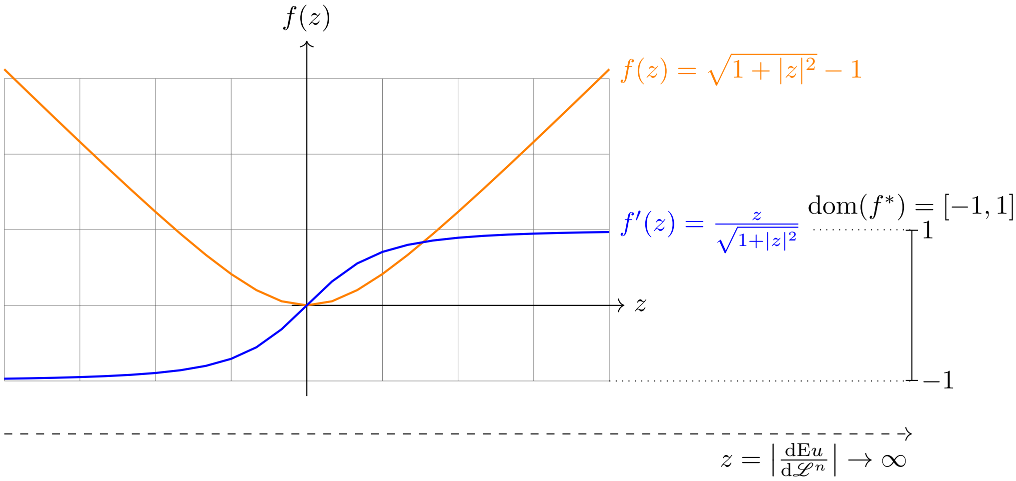

Figure 5.3: The LD– versus BD–regularity in terms of the dual problem. When the densities $|\,\mathrm{E}u(\mathrm{B}(x_0,r))/\mathscr{L}^n(\mathrm{B}(x_0,r))|$ blow up as $r \searrow 0$, then $f'(\mathrm{E}u(\mathrm{B}(x_0,r))/\mathscr{L}^n(\mathrm{B}(x_0,r)))$ converges to an element of the boundary of the effective domain of f^* and vice versa, cp. Remark 5.27.

the last identity being valid by simple regrouping. Keeping (5.85) in mind, we expand the very left hand side of the previous equation to find

$$\begin{aligned}\int_\Omega f(\mathscr{E}u)\,\mathrm{d}x + [\![\sigma, \mathrm{E}u]\!]^s_{\mathrm{sym}}(\overline{\Omega}) &\geqslant \int_\Omega \big(\langle \sigma, \mathscr{E}u\rangle - f^*(\sigma)\big)\,\mathrm{d}x + [\![\sigma, \mathrm{E}u]\!]^s_{\mathrm{sym}}(\overline{\Omega}) \\ &= \int_\Omega f(\mathscr{E}u)\,\mathrm{d}x + \int_{\overline{\Omega}} f^\infty\Big(\frac{\mathrm{dE}u}{\mathrm{d}|\,\mathrm{E}^s u|}\Big)\,\mathrm{d}|\,\mathrm{E}^s u| \\ &\overset{(5.86)}{\geqslant} \int_\Omega f(\mathscr{E}u)\,\mathrm{d}x + [\![\widetilde{\sigma}, \mathrm{E}u]\!]^s_{\mathrm{sym}}(\overline{\Omega}).\end{aligned}$$

This, in turn, implies

$$\int_\Omega \underbrace{\big(\langle \sigma, \mathscr{E}u\rangle - f^*(\sigma) - f(\mathscr{E}u)\big)}_{\leqslant 0 \text{ by } (5.85)\ \mathscr{L}^n\text{-a.e.}}\,\mathrm{d}x \geqslant 0, \tag{5.88}$$

which is possible if and only if $\langle \sigma, \mathscr{E}u\rangle - f^*(\sigma) - f(\mathscr{E}u) = 0$ $\mathscr{L}^n$–a.e. in Ω. With this information at our disposal, we then get

$$[\![\sigma, \mathrm{E}u]\!]^s_{\mathrm{sym}}(\overline{\Omega}) = \int_{\overline{\Omega}} f^\infty\Big(\frac{\mathrm{dE}u}{\mathrm{d}|\,\mathrm{E}^s u|}\Big)\,\mathrm{d}|\,\mathrm{E}^s u| \tag{5.89}$$

so that

$$\int_{\overline{\Omega}} \underbrace{\Big(f^\infty\Big(\frac{\mathrm{dE}u}{\mathrm{d}|\,\mathrm{E}^s u|}\Big) - \frac{\mathrm{d}[\![\widetilde{\sigma}, \mathrm{E}u]\!]^s_{\mathrm{sym}}}{\mathrm{d}|\,\mathrm{E}^s u|}\Big)}_{\geqslant 0 \text{ by } (5.86)\ |\,\mathrm{E}^s u|\text{-a.e.}}\,\mathrm{d}|\,\mathrm{E}^s u| = 0$$

and hence

$$f^\infty\Big(\frac{\mathrm{dE}u}{\mathrm{d}|\,\mathrm{E}^s u|}\Big) - \frac{\mathrm{d}[\![\widetilde{\sigma}, \mathrm{E}u]\!]^s_{\mathrm{sym}}}{\mathrm{d}|\,\mathrm{E}^s u|} = 0 \quad |\,\mathrm{E}^s u|\text{-a.e.} \tag{5.90}$$

Working from here, we conclude as follows.

- We have already proven that (a) implies (5.46)–(5.48). Now, if the pair (u,σ) satisfies (a), then (5.87) is in action. From this we deduce that $f^*(\sigma)<\infty$ must necessarily hold $\mathscr{L}^n$–a.e. in Ω. This in turn implies validity of (5.86) and hence allows the conclusions of (5.88), (5.89) and thus (5.90).

- On the other hand, we see that if $(u,\sigma)\in\mathrm{BD}_{u_0}(\overline{\Omega})\times\mathrm{L}^\infty_{\mathrm{div}}(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})$ satisfy (5.46)–(5.49), then (5.87) holds. We always have $\sup_{\mathrm{L}^\infty_{\mathrm{div}}(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})}\mathscr{R}=\inf_{\mathcal{D}_{u_0}}\mathfrak{F}$ and thus

$$\begin{aligned}\int_\Omega\langle\boldsymbol{\varepsilon}(u_0),\sigma\rangle-f^*(\sigma)\,\mathrm{d}x&\leqslant\sup_{\mathrm{L}^\infty_{\mathrm{div}}(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})}\mathscr{R}\\&\leqslant\inf_{\mathcal{D}_{u_0}}\mathfrak{F}\\&\leqslant\widetilde{\mathfrak{F}}_{u_0}[u]\overset{(5.87)}{=}\int_\Omega\langle\boldsymbol{\varepsilon}(u_0),\sigma\rangle-f^*(\sigma)\,\mathrm{d}x\end{aligned}$$

  Hence we must have

$$\sup_{\mathrm{L}^\infty_{\mathrm{div}}(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})}\mathscr{R}=\int_\Omega\langle\boldsymbol{\varepsilon}(u_0),\sigma\rangle-f^*(\sigma)\,\mathrm{d}x$$

  and thus

$$\inf_{\mathcal{D}_{u_0}}\mathfrak{F}=\int_\Omega f(\mathscr{E}u)\,\mathrm{d}x+\int_{\overline{\Omega}}f^\infty\Big(\frac{\mathrm{dE}u}{\mathrm{d}|\,\mathrm{E}^s u|}\Big)\,\mathrm{d}|\,\mathrm{E}^s u|.$$

  In consequence, u is a solution of the primal problem (5.13) and σ a solution of the dual problem (5.12).

The proof is complete. □

As a corollary of Theorem 5.18, we obtain the following uniqueness assertion.

Corollary 5.26. *In the situation of Theorem 5.18, the existence of a generalised minimiser for the primal problem implies uniqueness of the dual solution.*

Proof. If $u\in\mathrm{BD}_{u_0}(\overline{\Omega})$ is a generalised minimiser and $\sigma\in\mathrm{L}^\infty_{\mathrm{div}}(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})$ is a solution of the dual problem, then σ is uniquely determined by (5.47). The proof is complete. □

We finish this paragraph with two remarks.

Remark 5.27 (The importance of $\partial\mathrm{dom}(f^*)$)**.** *Let us remark that for the vast majority of convex problems (e.g., radially symmetric ones), a duality criterion for the absolute continuity of* Eu *can be derived in the following sense. Suppose for simplicity $f(z)=\sqrt{1+|z|^2}-1$ as depicted in Figure 5.3. If $u\in\mathrm{BD}(\Omega)$ is a generalised minimiser of the variational principle (5.1) and we have $\mathrm{E}^s u\equiv 0$, then $u\in\mathrm{LD}(\Omega)$ and $\sigma=f'(\boldsymbol{\varepsilon}(u))$ with f' as displayed in Figure 5.3. However, if we now assume that $u\in(\mathrm{BD}\setminus\mathrm{LD})(\Omega)$ and that $x_0\in\Omega$ is a point in the support of the singular part $\mathrm{E}^s u$. Then we have by Lebesgue's differentiation theorem (cp. Theorem 2.3) for $|\,\mathrm{E}^s u|$–a.e. such x_0*

$$\limsup_{r\searrow 0}\alpha_r:=\limsup_{r\searrow 0}\frac{|\,\mathrm{E}u|(\mathrm{B}(x_0,r))}{\mathscr{L}^n(\mathrm{B}(x_0,r))}=+\infty,\tag{5.91}$$

and from here we see that if we consider $f'(\alpha_r)$ as a local approximation of the dual solution σ, then $|\alpha_r|\to\infty$ and hence $f'(\alpha_r)\to 1$ as $r\searrow 0$. As a consequence, we obtain that for singular points x_0, we have $\sigma(x_0)\in\{-1,1\}=\partial\mathrm{dom}(f^*)$. This is illustrated in Figure 5.3, and we refer the reader to [36] for related ideas.

Remark 5.28 (Diminishing the smoothness of f)**.** *The reader will have noticed that most of the arguments outlined above do not rely on the* C^2*–regularity of the variational integrands. Indeed, it is possible to reduce the regularity of the integrands* f *as follows (cp. [36, Thm. 2.1] in conjunction with the above modifications): If* $f\colon \mathbb{R}^{n\times n}_{\mathrm{sym}} \to \mathbb{R}$ *is only a convex function of linear growth (the latter meaning now that there exists a constant* $L > 0$ *such that* $|f(\xi)| \leqslant L(1 + |\xi|)$ *for all* $\xi \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$*), then for a pair* $(u, \sigma) \in \mathrm{BD}_{u_0}(\overline{\Omega}) \times \mathrm{L}^{\infty}_{\mathrm{div}}(\Omega; \mathbb{R}^{n\times n}_{\mathrm{sym}})$ *we have the equivalence of (a) and (b)', where (b)' now is constituted by (5.46), (5.49) and* $\sigma \in \partial f(\mathscr{E}u)$ $\mathscr{L}^n$*–a.e. in* Ω *instead of (5.47) and* $\mathscr{E}u \in \partial f^*(\sigma)$ $\mathscr{L}^n$*–a.e. in* Ω*.*

Since we do not deal with the non–differentiable context in this thesis, we shall not make use of this in the sequel.

5.5. An Embedding for BD ∩ BMO

In this section we prove an embedding result for BD ∩ BMO into certain fractional Sobolev spaces which will constitute a substantial part of the proof of the main theorem. The proof combines an argument firstly utilised by Dorronsoro [86] and an embedding result of BD into certain fractional Sobolev spaces. Let us recall that a locally integrable map $u\colon \mathbb{R}^n \to \mathbb{R}^N$ belongs to $\mathrm{BMO}(\mathbb{R}^n; \mathbb{R}^N)$ if and only if its sharp maximal function given by

$$(\mathcal{M}^\sharp u)(x) := \sup_{\substack{Q\,\text{cube}\\ x\in Q}} \fint_Q |u - (u)_Q|\,\mathrm{d}y, \qquad x \in \mathbb{R}^n, \tag{5.92}$$

belongs to $\mathrm{L}^\infty(\mathbb{R}^n)$, cp. Chapter 2.6. The main result of this section is then as follows.

Theorem 5.29. *Let* $n \geqslant 2$*. Let* $1 < p < \infty$ *and* $\varepsilon > 0$ *such that*

$$0 < \varepsilon < \min\left\{ \frac{(n-1)\big(1-\frac{1}{p}\big)}{1+pn-p}, \frac{1}{p} \right\}. \tag{5.93}$$

Then

$$\mathrm{BD}(\mathbb{R}^n) \cap \mathrm{BMO}(\mathbb{R}^n; \mathbb{R}^n) \subset \mathrm{W}^{\frac{1}{p}-\varepsilon, p}(\mathbb{R}^n; \mathbb{R}^n). \tag{5.94}$$

Before embarking on the proof of Theorem 5.29, we wish to make some remarks.

Remark 5.30 ([48, Rem. 3])**.** It is important to note that the preceding theorem is *false* for $\varepsilon = 0$. Indeed, if it was true in this case, the injections $\mathrm{W}^{1,1}(\mathbb{R}^n; \mathbb{R}^n) \hookrightarrow \mathrm{BD}(\mathbb{R}^n)$ and $\mathrm{L}^\infty(\mathbb{R}^n; \mathbb{R}^n) \hookrightarrow \mathrm{BMO}(\mathbb{R}^n; \mathbb{R}^n)$ would yield $(\mathrm{W}^{1,1} \cap \mathrm{L}^\infty)(\mathbb{R}^n; \mathbb{R}^n) \hookrightarrow \mathrm{W}^{\frac{1}{p},p}(\mathbb{R}^n; \mathbb{R}^n)$. However, as pointed out by Bourgain, Brezis and Mironescu [48], this embedding in general fails: Indeed, a localisation argument would then yield $(\mathrm{W}^{1,1} \cap \mathrm{L}^\infty)((-1,1)) \hookrightarrow \mathrm{W}^{\frac{1}{p},p}((-1,1))$ for any $1 < p < \infty$. Consider the sequence (u_k) given by

$$u_k(x) := \begin{cases} 1 & \text{if } -1 < x \leqslant -\frac{1}{2k} \\ 2kx & \text{if } -\frac{1}{2k} \leqslant x \leqslant \frac{1}{2k} \\ 1 & \text{if } \frac{1}{2k} \leqslant x < 1. \end{cases}$$

Then (u_k) is uniformly bounded both in $\mathrm{W}^{1,1}((-1,1))$ and $\mathrm{L}^\infty((-1,1))$. Moreover, it converges weakly* in $\mathrm{BV}((-1,1))$ to sgn which, however, does not belong to $\mathrm{W}^{\frac{1}{2},2}((-1,1))$.

To prove the theorem, we first recall from Chapter 2.6 the mean value characterisation of Besov spaces $\mathrm{B}^s_{p,q}$ from Theorem 2.25. This characterisation is useful for the proof of Theorem 5.29 as it allows to access the additional BMO–regularity which in turn is defined in terms of maximal operators. As we shall see later (cp. Theorem 5.37), for other but related statements it is more useful to work with different characterisations of Besov spaces.

Lemma 5.31. *Let $0 < s < 1$ and $1 \leqslant p < \infty$. A function $u \in \mathrm{L}^p(\mathbb{R}^n)$ belongs to $\mathrm{W}^{s,p}(\mathbb{R}^n)$ if and only if*

$$[u]^*_{s,p} := \left(\int_{\mathbb{R}^n} \int_0^\infty \left| \sup_{\substack{Q \ni x \\ |Q|=t^n}} \fint_Q \frac{|u-(u)_Q|^p}{t^s} \,\mathrm{d}\xi \right|^p \frac{\mathrm{d}t}{t} \,\mathrm{d}x \right)^{\frac{1}{p}} < \infty. \tag{5.95}$$

Moreover, the expression on the left is equivalent to the usual Gagliardo–seminorm $[\cdot]_{s,p}$.

Proof. It suffices to note that $\mathrm{B}^s_{p,p} \simeq \mathrm{W}^{s,p}$ for all $0 < s < 1$ and $1 < p < \infty$. By Theorem 2.25, the claim follows. □

The supremum appearing in the integrand of (5.95) is taken over all cubes having sidelength t and containing x. For the following it is important to bound such quantities in terms of centered maximal operators, and hence we briefly pause and give the required modification. We hereafter recall from [77] that for a locally integrable map $h\colon \mathbb{R}^n \to \mathbb{R}^N$ its *sharp maximal operator of order* $0 < \alpha \leqslant 1$ is given by

$$h^\sharp_\alpha(x) := (\mathcal{M}^\sharp_\alpha h)(x) := \sup_{Q \text{ cube centered at } x} \frac{1}{\ell(Q)^{n+\alpha}} \int_Q |h-(h)_Q| \,\mathrm{d}y, \qquad x \in \mathbb{R}^n, \tag{5.96}$$

where $\ell(Q)$ is the sidelength of the cube Q.

Lemma 5.32. *For each $\alpha > 0$ there exists a number $C = C(n,\alpha) > 0$ such that for all $u \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^n)$, all $x_0 \in \mathbb{R}^n$ and all $t > 0$ we have*

$$\frac{1}{t^\alpha} \sup \left\{ \fint_Q |u-(u)_Q| \,\mathrm{d}y \colon\ Q \ni x_0,\ \mathscr{L}^n(Q) = t^n \right\} \leqslant C(\mathcal{M}^\sharp_\alpha u)(x_0).$$

provided both sides are well–defined and finite.

Proof. Fix $x_0 \in \mathbb{R}^n$ and let $\widetilde{Q}$ be an arbitrary cube with $x \in \widetilde{Q}$ and $\mathscr{L}^n(Q) = t^n$. It is easy to see that there exists $K = K(n) > 0$ such that the cube $Q = Q(x_0, Kt)$ contains $\widetilde{Q}$. We then obtain

$$\begin{aligned}
\frac{1}{t^\alpha} \fint_{\widetilde{Q}} |u-(u)_{\widetilde{Q}}| \,\mathrm{d}y &= \frac{1}{|\widetilde{Q}|^{1+\frac{\alpha}{n}}} \int_{\widetilde{Q}} |u-(u)_{\widetilde{Q}}| \,\mathrm{d}x \leqslant \frac{K^{n+\alpha}}{|Q|^{1+\frac{\alpha}{n}}} \int_Q |u-(u)_{\widetilde{Q}}| \,\mathrm{d}x \\
&\overset{\text{Jensen}}{\leqslant} \frac{K^{n+\alpha}}{|Q|^{1+\frac{\alpha}{n}}} \int_Q \fint_{\widetilde{Q}} |u(x)-u(y)| \,\mathrm{d}y \,\mathrm{d}x \\
&\leqslant \frac{K^{n+\alpha}}{|Q|^{1+\frac{\alpha}{n}}} \frac{K^n}{|Q|} \int_Q \int_{\widetilde{Q}} |u(x)-u(y)| \,\mathrm{d}y \,\mathrm{d}x \\
&\leqslant \frac{K^{2n+\alpha}}{|Q|^{1+\frac{\alpha}{n}}} \fint_Q \int_Q |u(x)-u(y)| \,\mathrm{d}y \,\mathrm{d}x \\
&\leqslant \frac{K^{2n+\alpha}}{|Q|^{1+\frac{\alpha}{n}}} \fint_Q \int_Q |u(x)-(u)_Q+(u)_Q-u(y)| \,\mathrm{d}y \,\mathrm{d}x \\
&\leqslant \frac{2K^{2n+\alpha}}{|Q|^{1+\frac{\alpha}{n}}} \int_Q |u(x)-(u)_Q| \,\mathrm{d}x \leqslant 2K^{2n+\alpha}(\mathcal{M}^\sharp_\alpha u)(x_0).
\end{aligned}$$

We may therefore put $C(n,\alpha) := 2K^{2n+\alpha}$. The proof is complete. □

The second ingredient is an embedding result of BD into fractional Sobolev spaces:

Lemma 5.33. *Let $n \geqslant 2$ and $0 < s < 1$. Then* $\mathrm{BD}(\mathbb{R}^n) \hookrightarrow \mathrm{W}^{s,\frac{n}{n-1+s}}(\mathbb{R}^n;\mathbb{R}^n)$.

Proof. By Theorem 4.11(b), there exists a constant $C > 0$ such that

$$\|\varphi\|_{\dot{\mathrm{W}}^{s,n/(n-1+s)}(\mathbb{R}^n;\mathbb{R}^n)} \leqslant C\|\boldsymbol{\varepsilon}(\varphi)\|_{\mathrm{L}^1(\mathbb{R}^n;\mathbb{R}^{n\times n}_{\mathrm{sym}})}$$

holds for all $\varphi \in \mathrm{C}_c^\infty(\mathbb{R}^n;\mathbb{R}^n)$, where $\dot{\mathrm{W}}^{s,\frac{n}{n-1+s}}(\mathbb{R}^n;\mathbb{R}^n)$ denotes the respective homogeneous fractional Sobolev space. For the general statement, let $u \in \mathrm{BD}(\Omega)$ and choose a sequence $(u_k) \subset \mathrm{C}_c^\infty(\mathbb{R}^n;\mathbb{R}^n)$ such that $u_k \to u$ strictly and pointwisely $\mathscr{L}^n$–a.e. as $k \to \infty$. Then we obtain, using Fatou's lemma,

$$\begin{aligned}\|u\|_{\dot{\mathrm{W}}^{s,\frac{n}{n-1+s}}(\mathbb{R}^n;\mathbb{R}^n)} &\leqslant \liminf_{k\to\infty} \|u_k\|_{\dot{\mathrm{W}}^{s,\frac{n}{n-1+s}}(\mathbb{R}^n;\mathbb{R}^n)} \\ &\leqslant C \liminf_{k\to\infty} \|\boldsymbol{\varepsilon}(u_k)\|_{\mathrm{L}^1(\mathbb{R}^n;\mathbb{R}^{n\times n}_{\mathrm{sym}})} = C|\mathrm{E}u|(\mathbb{R}^n).\end{aligned}$$

Now we use the fact that on $\mathrm{C}_c^\infty(\mathbb{R}^n;\mathbb{R}^n)$, the homogeneous fractional Sobolev norm is equivalent to the Gagliardo seminorm and arrive at the desired estimate $[u]_{\mathrm{W}^{s,p}(\mathbb{R}^n;\mathbb{R}^n)} \leqslant C|\mathrm{E}u|(\mathbb{R}^n)$. Finally, since $1 < \frac{n}{n-1+s} < \frac{n}{n-1}$, we have $\mathrm{L}^1 \cap \mathrm{L}^{\frac{n}{n-1}} \hookrightarrow \mathrm{L}^{\frac{n}{n-1+s}}$ by standard interpolation on L^p–spaces. Then we use $\mathrm{BD}(\mathbb{R}^n) \hookrightarrow (\mathrm{L}^1 \cap \mathrm{L}^{\frac{n}{n-1}})(\mathbb{R}^n)$ which is available by Theorem 4.11(a). In conclusion, we obtain $\|u\|_{\mathrm{L}^{\frac{n}{n-1+s}}(\mathbb{R}^n;\mathbb{R}^n)} \leqslant C\|u\|_{\mathrm{BD}(\mathbb{R}^n)}$ and in conjunction with the first part of the proof, establishes $\|u\|_{\mathrm{W}^{s,\frac{n}{n-1+s}}(\mathbb{R}^n;\mathbb{R}^n)} \leqslant C\|u\|_{\mathrm{BD}(\mathbb{R}^n)}$. The proof is complete. □

We now come to the

Proof of Theorem 5.29. Let $u \in (\mathrm{BD} \cap \mathrm{BMO})(\mathbb{R}^n)$. For fixed $x \in \mathbb{R}^n$, a cube Q of sidelength t and $x \in Q$, we denote $g_{x,t,Q}(\xi) := u(\xi) - (u)_Q$, $\xi \in \mathbb{R}^n$. We remark in advance that, given $1 < q < \infty$,

$$\|u\|_{\mathrm{BMO}_q} := \sup_{Q\,\mathrm{cube}} \left(\fint_Q |u-(u)_Q|^q \,\mathrm{d}x \right)^{\frac{1}{q}}$$

defines an seminorm on $\mathrm{BMO}(\mathbb{R}^n;\mathbb{R}^n)$ which is equivalent to that given by (5.92), see [232]. For $1 \leqslant p < \infty$, we recall the (inhomogeneous) *Calderón space* $\mathscr{C}^{\alpha,p}(\mathbb{R}^n)$ defined by

$$\mathscr{C}^{\alpha,p}(\mathbb{R}^n) := \left\{ v \in \mathrm{L}^p(\mathbb{R}^n) \colon\ \mathcal{M}^\sharp_\alpha v \in \mathrm{L}^p(\mathbb{R}^n) \right\}$$

and equip it with the canonical norm $\|v\|_{\mathscr{C}^{\alpha,p}} := \|v\|_{\mathrm{L}^p} + \|v^\sharp_\alpha\|_{\mathrm{L}^p}$. Then, by Theorem 2.24, we have the continuous embedding $\mathscr{C}^{\alpha,p}(\mathbb{R}^n) \hookrightarrow \mathrm{B}^{\alpha,p}_\infty(\mathbb{R}^n)$ with the corresponding Besov–Nikolskii–spaces $\mathrm{B}^{\alpha,p}_\infty(\mathbb{R}^n)$. By standard embeddings of these spaces, we obtain for each $0 < \theta < \alpha$

$$\mathrm{W}^{\alpha,p}(\mathbb{R}^n) \simeq \mathrm{B}^{\alpha,p}_p(\mathbb{R}^n) \hookrightarrow \mathrm{B}^{\alpha,p}_\infty(\mathbb{R}^n) \hookrightarrow \mathrm{B}^{\alpha-\theta,p}_1(\mathbb{R}^n) \hookrightarrow \mathrm{B}^{\alpha-\theta,p}_p(\mathbb{R}^n) \simeq \mathrm{W}^{\alpha-\theta,p}(\mathbb{R}^n) \tag{5.97}$$

with the fractional Sobolev spaces $\mathrm{W}^{s,p} \simeq \mathrm{B}^{s,p}_p$; see, e.g., [231]. Our aim is to show

$$\mathscr{C}^{\frac{1}{p}-\varepsilon,\frac{n}{n-1+\frac{1}{p}-\varepsilon}}(\mathbb{R}^n) \hookrightarrow \mathrm{W}^{\frac{1}{p}-\varepsilon,p}(\mathbb{R}^n) \tag{5.98}$$

where p and ε are as in the theorem. Since

$$(\mathrm{BD} \cap \mathrm{BMO})(\mathbb{R}^n) \overset{\text{Lemma 5.33}}{\hookrightarrow} (\mathrm{W}^{s,\frac{n}{n-1+s}} \cap \mathrm{BMO})(\mathbb{R}^n) \overset{(2.15)}{\hookrightarrow} (\mathscr{C}^{s,\frac{n}{n-1+s}} \cap \mathrm{BMO})(\mathbb{R}^n), \tag{5.99}$$

this will imply the claim.

Proof of (5.98). Our argument is centered around the Dorronsoro–type characterisation of the Sobolev spaces $\mathrm{W}^{s,p}$, Lemma 5.31. In the situation of the theorem, we put $s := \frac{1}{p} - \varepsilon$. For $\mathscr{L}^n$–a.e. $x \in \mathbb{R}^n$, let $\delta(x) > 0$ be arbitrary. We split the right hand side term of (5.95) as

$$\begin{aligned} |[u]^*_{s,p}|^p &:= \int_{\mathbb{R}^n} \int_0^{\delta(x)} \left| \sup_{\substack{Q \ni x \\ |Q|=t^n}} \frac{1}{t^s} \fint_Q |g_{x,t,Q}(\xi)|\,\mathrm{d}\xi \right|^p \frac{\mathrm{d}t}{t}\,\mathrm{d}x \\ &+ \int_{\mathbb{R}^n} \int_{\delta(x)}^{\infty} \left| \sup_{\substack{Q \ni x \\ |Q|=t^n}} \frac{1}{t^s} \fint_Q |g_{x,t,Q}(\xi)|\,\mathrm{d}\xi \right|^p \frac{\mathrm{d}t}{t}\,\mathrm{d}x =: \int_{\mathbb{R}^n} \mathbf{I}_\delta(x) + \mathbf{II}_\delta(x)\,\mathrm{d}x \end{aligned}$$

with an obvious definition for $\mathbf{I}_\delta(x)$ and $\mathbf{II}_\delta(x)$. Firstly, we have

$$\begin{aligned} \mathbf{I}_\delta(x) &= \int_0^{\delta(x)} \left| \sup_{\substack{Q \ni x \\ |Q|=t^n}} \frac{1}{t^{s+\varepsilon}} \fint_Q |g_{x,t,Q}(\xi)|\,\mathrm{d}\xi \right|^p \frac{\mathrm{d}t}{t^{1-\varepsilon p}} \\ &\leqslant C \int_0^{\delta(x)} (u^\sharp_{s+\varepsilon}(x))^p \frac{\mathrm{d}t}{t^{1-\varepsilon p}} \qquad \text{(by Lemma 5.32)} \\ &\leqslant \frac{C}{\varepsilon p} \delta(x)^{\varepsilon p} (u^\sharp_{s+\varepsilon}(x))^p. \end{aligned}$$

On the other hand, we have by definition of $\mathcal{M}^\sharp$

$$\begin{aligned} \mathbf{II}_\delta(x) &= \int_{\delta(x)}^{\infty} \left| \sup_{\substack{Q \ni x \\ |Q|=t^n}} \fint_Q |g_{x,t,Q}(\xi)|\,\mathrm{d}\xi \right|^p \frac{\mathrm{d}t}{t^{1+sp}} \\ &\leqslant C \int_{\delta(x)}^{\infty} (u^\sharp(x))^p \frac{\mathrm{d}t}{t^{1+sp}} \qquad \text{(by Lemma 5.32)} \\ &\leqslant \frac{C}{sp} \delta(x)^{-sp} (u^\sharp(x))^p. \end{aligned}$$

Collecting estimates, we therefore find

$$[u]^*_{s,p} \leqslant C(s,p,\varepsilon) \int_{\mathbb{R}^n} \delta(x)^{\varepsilon p} (u^\sharp_{s+\varepsilon}(x))^p + \delta(x)^{-sp}(u^\sharp(x))^p\,\mathrm{d}x. \tag{5.100}$$

We choose for $\mathscr{L}^n$–a.e. $x \in \mathbb{R}^n$

$$\delta(x) := \left(\frac{u^\sharp(x)}{u^\sharp_{s+\varepsilon}(x)} \right)^{\frac{1}{s+\varepsilon}}$$

and note that we may assume without loss of generality that $u^\sharp_{s+\varepsilon}(x) > 0$ since u is constant otherwise and thus the claim is trivial. Inserting this choice of δ into (5.100), we obtain

$$[u]^*_{s,p} \leqslant C(s,p,\varepsilon) \int_{\mathbb{R}^n} (u^\sharp(x))^{\frac{\varepsilon p}{s+\varepsilon}} (u^\sharp_{s+\varepsilon}(x))^{p - \frac{\varepsilon p}{s+\varepsilon}}\,\mathrm{d}x \leqslant C \|u^\sharp\|^{\frac{\varepsilon p}{s+\varepsilon}}_{\mathrm{L}^\infty(\mathbb{R}^n)} \int_{\mathbb{R}^n} (u^\sharp_{s+\varepsilon})^{\frac{ps}{s+\varepsilon}}\,\mathrm{d}x = (**).$$

Now note that by Lemma 5.35 below, the fractional maximal functions are log–convex in their smoothness indices. As a consequence, we obtain for $0 < t < s$ and $1 < q < \infty$

$$(\dot{\mathscr{C}}^{s,q} \cap \mathrm{BMO})(\mathbb{R}^n) \hookrightarrow \dot{\mathscr{C}}^{t,\frac{sq}{t}}(\mathbb{R}^n) \qquad \text{for all } 0 < t < s. \tag{5.101}$$

Indeed, write $t = \lambda \cdot 0 + (1-\lambda)s$ with $0 < \lambda < 1$ to deduce for $1 < q < \infty$ and any $v \in (\dot{\mathscr{C}}^{s,q} \cap \mathrm{BMO})(\mathbb{R}^n)$ that

$$(v_t^\sharp(x))^r \leqslant (v_0^\sharp(x))^{\lambda r}(v_s^\sharp(x))^{(1-\lambda)r}$$

and hence (5.101) follows since $(1-\lambda)s = t$ implies $r = sq/t$.

We return to the estimation of $(**)$. Because $0 < \varepsilon p < 1$, we have $0 < 1-\varepsilon p < 1$. By assumption, we have $s + \varepsilon = 1/p \in (0,1)$ and $ps/(s+\varepsilon) = p(1-\varepsilon p)$. Define $\Phi(\theta) := \theta n/(n-1+\theta)$, $\theta \in (0,1)$. Since $n \geqslant 2$, there holds $|\Phi| < 1$ and we have both $\lim_{\theta\searrow 0}\Phi(\theta) = 0$ and $\lim_{\theta\nearrow 1}\Phi(\theta) = 1$. Clearly, Φ is continuous and hence $\Phi\colon (0,1) \to (0,1)$ is bijective. Therefore, choosing $\widetilde{\theta} := (1-\varepsilon p)(n-1)/(n-1+\varepsilon p)$, we see that $\Phi(\widetilde{\theta}) = 1-\varepsilon p$. With this choice of $\widetilde{\theta}$, the embedding (5.101) gives

$$(\dot{\mathscr{C}}^{\widetilde{\theta},n/(n-1+\widetilde{\theta})} \cap \mathrm{BMO})(\mathbb{R}^n) \hookrightarrow \dot{\mathscr{C}}^{\frac{1}{p},p(1-\varepsilon p)}(\mathbb{R}^n)$$

since $\widetilde{\theta} > \frac{1}{p}$: Indeed, since $0 < \varepsilon p < 1 - \frac{1}{p}$ by assumption, we deduce $p(1-\varepsilon p) - 1 > 0$, and therefore with $\gamma = \varepsilon p$

$$\begin{aligned}\widetilde{\theta} = \frac{(1-\gamma)(n-1)}{n-1+\gamma} > \frac{1}{p} &\Leftrightarrow (p-\gamma p)(n-1) > n-1+\gamma \Leftrightarrow pn - \gamma pn - p + \gamma p > n-1+\gamma \\ &\Leftrightarrow pn - p - n + 1 > \gamma + \gamma pn - \gamma p = \gamma(1+pn-p) \\ &\Leftrightarrow \frac{pn-p-n+1}{1+pn-p} > \gamma = \varepsilon p \\ &\Leftrightarrow \frac{n-1-n/p+1/p}{1+pn-p} > \varepsilon\end{aligned}$$

which is true by assumption 5.93. The proof is complete. □

Remark 5.34. The preceding proof, in particular, the choice of x–dependent δ, uses the so–called *Hedberg trick* as explained in [7, Prop. 3.1.2].

In the proof of Theorem 5.29, we used the log–convexity of the fractional maximal operator with respect to its index, a fact whose proof we give now:

Lemma 5.35. *The function $s \mapsto \mathcal{M}_s^\sharp v(x)$ is* log*–convex on $(0,1)$ for any locally integrable function $u\colon \mathbb{R}^n \to \mathbb{R}$. That is, for any $s,t \in (0,1)$ and any $\lambda \in (0,1)$ there holds*

$$\mathcal{M}^\sharp_{\lambda s+(1-\lambda)t}v(x) \leqslant (\mathcal{M}_s^\sharp v(x))^\lambda(\mathcal{M}_t^\sharp v(x))^{1-\lambda}. \tag{5.102}$$

Proof. Let $v \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^n)$. If the right side of (5.102) is infinite, there is nothing to prove, so we may assume without loss of generality that $\mathcal{M}_s^\sharp v(x), \mathcal{M}_t^\sharp v(x) < \infty$. Let $Q \subset \mathbb{R}^n$ be a non–degenerate cube. Then we have

$$\begin{aligned}\frac{1}{\ell(Q)^{n+\lambda s+(1-\lambda)s}}\int_Q |v-(v)_Q|\,\mathrm{d}x &= \frac{1}{\ell(Q)^{\lambda(n+s)}}\left(\int_Q |v-(v)_Q|\,\mathrm{d}x\right)^\lambda \\ &\quad\times \frac{1}{\ell(Q)^{(1-\lambda)(n+t)}}\left(\int_Q |v-(v)_Q|\,\mathrm{d}x\right)^{1-\lambda} \\ &\leqslant (\mathcal{M}_s^\sharp v(x))^\lambda(\mathcal{M}_t^\sharp v(x))^{1-\lambda}.\end{aligned}$$

We then pass to the supremum over all cubes Q to deduce the claim. □

For the sake of better traceability, we explicitely note the following.

Corollary 5.36. *Let $1 < p < \infty$ and $\varepsilon > 0$ such that $p\varepsilon < 1$. Then for any $N \geqslant 1$ we have*

$$(\mathrm{BV} \cap \mathrm{BMO})(\mathbb{R}^n;\mathbb{R}^N) \subset \mathrm{W}^{\frac{1}{p}-\varepsilon,p}(\mathbb{R}^n;\mathbb{R}^N). \tag{5.103}$$

The previous corollary follows along the same lines as the proof of Theorem 5.29, now using the standard Sobolev embedding $\mathrm{BV}(\mathbb{R}^n;\mathbb{R}^N) \hookrightarrow \mathrm{W}^{\theta,\frac{n}{n-1+\theta}}(\mathbb{R}^n;\mathbb{R}^N)$ for $0<\theta<1$. This slightly improves the embedding $(\mathrm{BV}\cap\mathrm{L}^\infty)(\mathbb{R}^n) \hookrightarrow \mathrm{B}^{1/p,p}_{\infty}(\mathbb{R}^n)$ for $1<p<\infty$ as given in Lemma 38.1 in Tartar's monograph [231].

Also note that the embedding $(\mathrm{BD}\cap\mathrm{L}^\infty)_{\mathrm{loc}}(\mathbb{R}^n) \hookrightarrow \mathrm{W}^{1/p-\varepsilon,p}_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^n)$ for all $1<p<\infty$ and sufficiently small $\varepsilon>0$ is much easier than the statement of Theorem 5.29. Indeed, let $u\in(\mathrm{BD}\cap\mathrm{L}^\infty)_{\mathrm{loc}}(\mathbb{R}^n)$ and fix $K\Subset\mathbb{R}^n$. We then estimate

$$
\begin{aligned}
[u]^p_{\mathrm{W}^{\frac{1}{p}-\varepsilon,p}(K;\mathbb{R}^n)} &= \iint_{K\times K}\frac{|u(x)-u(y)|^p}{|x-y|^{n+1-\varepsilon p}}\,\mathrm{d}(x,y)\\
&\leqslant \|u\|^{p-1}_{\mathrm{L}^\infty(K;\mathbb{R}^n)}\iint_{K\times K}\frac{|u(x)-u(y)|}{|x-y|^{n+1-\varepsilon p}}\,\mathrm{d}(x,y)\\
&\leqslant \|u\|^{p-1}_{\mathrm{L}^\infty(K;\mathbb{R}^n)}[u]_{\mathrm{W}^{1-\varepsilon p,1}(K;\mathbb{R}^n)},
\end{aligned}
$$

and note that $[u]_{\mathrm{W}^{1,1-\varepsilon p}(K;\mathbb{R}^n)}<\infty$ as $\mathrm{BD}_{\mathrm{loc}}\hookrightarrow \mathrm{W}^{s,1}_{\mathrm{loc}}$ for all $0<s<1$. Next we discuss a variant of Theorem 5.29 which exploits additional L^p–integrability.

Theorem 5.37 (Improved Sobolev Embedding)**.** *Given $\frac{n}{n-1}<r\leqslant\infty$, put*

$$
X^r_{\mathrm{loc}}(\mathbb{R}^n) := \begin{cases}\mathrm{L}^r_{\mathrm{loc}}(\mathbb{R}^n) & \text{if } 1\leqslant r<\infty,\\ \mathrm{BMO}_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^N) & \text{if } r=\infty.\end{cases}
$$

Then for any $1<p<\frac{n}{n-1}$ there holds $(\mathrm{BD}_{\mathrm{loc}}\cap X^r_{\mathrm{loc}})(\mathbb{R}^n)\subset \mathrm{W}^{s,p}_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^N)$ with $s<s_{\max}(p,r)$, where

$$
s_{\max}(p,r) := \frac{(r-p)}{p(r-1)} \qquad \text{if } \frac{n}{n-1}<r<\infty \quad \text{and} \quad s_{\max}(p,r):=\frac{1}{p} \qquad \text{if } r=\infty.
$$

Proof. The proof for $r=\infty$ has already been given and we henceforth focus on the case $\frac{n}{n-1}<r<\infty$. For this proof, it is convenient to employ the finite differences characterisation of the Nikolskii spaces $\mathrm{B}^s_{p,\infty}$, cp. Chapter 2.5.3. Let $r>\frac{n}{n-1}$. Let $K\subset\mathbb{R}^n$ be relatively compact with Lipschitz boundary ∂K. Let $s\in\{1,...,n\}$ and let $h\in\mathbb{R}$ with $|h|<1$. We estimate for $0<\theta<1$ and $1<q<\infty$ by Hölder's Inequality

$$
\begin{aligned}
\int_K|\tau_{s,h}u(x)|^p\,\mathrm{d}x &= \int_K|\tau_{s,h}u(x)|^{p\theta}|\tau_{s,h}u(x)|^{p(1-\theta)}\,\mathrm{d}x\\
&\leqslant\left(\int_K|\tau_{s,h}u(x)|^{p\theta q}\,\mathrm{d}x\right)^{\frac{1}{q}}\left(\int_K|\tau_{s,h}u(x)|^{p(1-\theta)q'}\,\mathrm{d}x\right)^{\frac{1}{q'}}.
\end{aligned}
\tag{5.104}
$$

At this stage we choose $0<\theta<1$ and $1<q<\infty$ such that $p(1-\theta)q'=r$. This implies

$$
q'=\frac{r}{p(1-\theta)} \quad\Rightarrow\quad q=\frac{\frac{r}{p(1-\theta)}}{\frac{r}{p(1-\theta)}-1}=\frac{r}{r-p(1-\theta)}\Rightarrow\quad pq\theta=\frac{rp\theta}{r-p(1-\theta)}. \tag{5.105}
$$

We now estimate

$$
\left(\int_K|\tau_{s,h}u(x)|^{p(1-\theta)q'}\,\mathrm{d}x\right)^{\frac{1}{q'}}\leqslant C\left(\int_{K+\mathrm{B}(0,2)}|u(x)|^r\,\mathrm{d}x\right)^{\frac{p(1-\theta)}{r}}. \tag{5.106}
$$

At this stage, we put $\theta:=(r-p)/(p(r-1))$. Since $1<p<n/(n-1)$, we certainly have $\theta>0$. On the other hand,

$$
\theta<1\Leftrightarrow r-p<p(r-1)=pr-p\Leftrightarrow p>1,
$$

and the latter is satisfied by assumption. For this choice of θ, we have

$$pq\theta \overset{(5.105)}{=} \frac{rp\frac{r-p}{p(r-1)}}{r-p(1-\frac{r-p}{p(r-1)})} = \frac{rp\frac{r-p}{p(r-1)}}{r-(p-\frac{r-p}{(r-1)})} = \frac{r(r-p)}{(r-p)(r-1)+(r-p)} = 1$$

and

$$q = \frac{r-1}{r-p}. \tag{5.107}$$

We now use the local embedding $\mathrm{BD}_{\mathrm{loc}} \hookrightarrow \mathrm{W}^{1-\varepsilon,1}_{\mathrm{loc}}$ for any $\varepsilon \in (0,1)$ to conclude because of $\mathrm{W}^{1-\varepsilon,1}_{\mathrm{loc}} \hookrightarrow (\mathrm{B}^{1-\varepsilon}_{1,\infty})_{\mathrm{loc}}$ that

$$\left(\int_K |\tau_{s,h}u(x)|^{p\theta q}\,\mathrm{d}x\right)^{\frac{1}{q}} \overset{(5.107)}{=} \left([u]_{\mathrm{B}^{1-\varepsilon}_{1,\infty}(K;\mathbb{R}^n)}|h|^{1-\varepsilon}\right)^{\frac{r-p}{r-1}} \leqslant [u]^{\frac{r-p}{r-1}}_{\mathrm{B}^{1-\varepsilon}_{1,\infty}(K;\mathbb{R}^n)}|h|^{(1-\varepsilon)\frac{r-p}{r-1}}.$$

In conclusion, we obtain by gathering estimates

$$\begin{aligned}\int_K |\tau_{s,h}u(x)|^p\,\mathrm{d}x &\leqslant C\|u\|^{p(1-\theta)}_{\mathrm{L}^r(U;\mathbb{R}^n)}[u]^{\frac{r-p}{r-1}}_{\mathrm{B}^{1-\varepsilon}_{1,\infty}(K;\mathbb{R}^n)}|h|^{(1-\varepsilon)\frac{r-p}{r-1}}\\ &\leqslant C\|u\|^{p(1-\theta)}_{\mathrm{L}^r(U;\mathbb{R}^n)}[u]^{\frac{r-p}{r-1}}_{\mathrm{W}^{1-\varepsilon,1}(K;\mathbb{R}^n)}|h|^{(1-\varepsilon)\frac{r-p}{r-1}}\end{aligned}$$

By arbitrariness of K, we conclude

$$u \in (\mathrm{B}^{(1-\varepsilon)\frac{r-p}{p(r-1)}}_{p,\infty})_{\mathrm{loc}}.$$

Let $\varepsilon' > 0$ be given. Then for $0 < \varepsilon < \varepsilon'$ we have

$$(\mathrm{B}^{(1-\varepsilon)\frac{r-p}{r-1}}_{p,\infty})_{\mathrm{loc}} \hookrightarrow \mathrm{W}^{(1-\varepsilon')\frac{r-p}{r-1},p}_{\mathrm{loc}}. \tag{5.108}$$

Hence, whenever $0 < s < \frac{r-p}{p(r-1)}$, we firstly determine $\varepsilon' > 0$ such that there holds

$$s = (1-\varepsilon')\frac{r-p}{p(r-1)}$$

and hence choosing $0 < \varepsilon < \varepsilon'$, the claim follows from (5.108). The proof is complete. □

Even though the preceding result is stated for $\mathrm{BD}_{\mathrm{loc}}$, it is easily seen to hold for $\mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^N)$, too, and we shall use this fact in the sequel without further explicit notification.

Remark 5.38. *Note the following important distinction for fractional Sobolev spaces: By Remark 2.15, we do not have $\mathrm{W}^{s,q}_{\mathrm{loc}} \subset \mathrm{W}^{s,p}_{\mathrm{loc}}$ provided $q > p$ and $0 < s < 1$ which is in stark contrast with the situation for $s \in \mathbb{N}_0$. However, we* do *have $\mathrm{W}^{s,p}_{\mathrm{loc}} \subset \mathrm{W}^{t,p}_{\mathrm{loc}}$ for $s > t$ and fixed p. This is a consequence of the estimate $|z|^{-n-tp} \leqslant |z|^{-n-sp}$ for $|z| < 1$. Therefore, to speak of* improvement *in the preceding theorem indeed, we need to fix p and show the improvement on the level of s.*

Let us further comment on the exponents appearing in Theorem 5.37: Firstly, if $1 < p < \frac{n}{n-1}$ is fixed, then $\mathrm{BD}_{\mathrm{loc}} \hookrightarrow \mathrm{W}^{\frac{n-np+p}{p},p}_{\mathrm{loc}}$. Now note that if $\frac{n}{n-1} < r < \infty$, then the exponent $s(p)$ provided by Theorem 5.37 is determined by $s_{\max}(p,r) = (r-p)/(p(r-1))$. We then have

$$\begin{aligned}s_{\max}(p,r) > \frac{n-np+p}{p} &\Leftrightarrow \frac{r-p}{r-1} > n-np+p \Leftrightarrow r-p > rn-rnp+rp-n+np-p\\ &\Leftrightarrow r > \frac{n(p-1)}{1-n+p(n-1)} = 1.\end{aligned}$$

We now study the limiting cases and hence define, for fixed $1 < p < \frac{n}{n-1}$, $\kappa(r) := s_{\max}(p,r) = (r-p)/(p(r-1))$ for $\frac{n}{n-1} < r < \infty$. Then we have

$$\frac{\mathrm{d}\kappa(r)}{\mathrm{d}r} = \frac{p(p-1)}{p^2(r-1)^2} > 0.$$

Hence $\kappa\colon (\frac{n}{n-1},\infty) \to \mathbb{R}$ is increasing. We have

$$\lim_{r\searrow\frac{n}{n-1}} \kappa(r) = \frac{\frac{n}{n-1}-p}{p\big(\frac{n}{n-1}-1\big)} = \frac{n-np+p}{p}$$

which is the optimal smoothness for the Sobolev embedding of BD into fractional Sobolev spaces $\mathrm{W}^{\theta,p}$ for fixed p. On the other hand, it is obvious that $\lim_{r\nearrow\infty}\kappa(r) = \frac{1}{p}$ which, in turn, is the exponent given by Theorem 5.29.

5.6. The Caccioppoli Inequality and its Consequences

In the superlinear growth regime, the Caccioppoli inequality is often the instrumental ingredient to yield higher integrability of minima. Caccioppoli type inequalities are in some sense reverse Poincaré–type inequalities and, on a heuristic level, rule out the possiblity for minima to oscillate too much. As we shall briefly recall below, this can be boosted by use of Gehring's lemma to rule out too big concentrations of the gradients. Thus, this line of action is particularly tempting to be applied to linear growth problems, where concentration phenomena are an essential motivation in passing to the spaces BV or BD instead of remaining with $\mathrm{W}^{1,1}$ or LD, respectively. However, as shall be proven below in Lemma 5.43, this approach cannot be made to work in a fruitful way.

To begin with, fix $1<p<2$ and let $g\colon \mathbb{R}^{N\times n}\to\mathbb{R}_{\geqslant 0}$ be a convex C^2–integrand of p–growth, i.e., there exist $0<c_1\leqslant c_2<\infty$ such that $c_1|\xi|^p \leqslant g(\xi)\leqslant c_2(1+|\xi|^p)$ holds for all $\xi\in\mathbb{R}^{N\times n}$. Moreover, let the second derivatives satisfy the ellipticity condition

$$c_3\frac{|\xi|^2}{(1+|\zeta|^2)^{\frac{2-p}{2}}} \leqslant \langle g''(\zeta)\xi,\xi\rangle \leqslant c_4\frac{|\xi|^2}{(1+|\zeta|^2)^{\frac{2-p}{2}}}$$

for all $\xi,\zeta\in\mathbb{R}^{N\times n}$ and some fixed constants $0<c_3\leqslant c_4<\infty$. Then there exists a constant $c>0$ such that for any local minimiser $u\in\mathrm{W}^{1,p}(\Omega;\mathbb{R}^N)$ of the functional $\mathcal{G}[v]:=\int_\Omega g(\mathrm{D}v)\,\mathrm{d}x$, any $x_0\in\Omega$ and $r>0$ with $0<2r<\mathrm{dist}(x_0,\partial\Omega)$ there holds

$$\fint_{\mathrm{B}(x_0,r)} |\,\mathrm{D}u|^p\,\mathrm{d}x \leqslant c\fint_{\mathrm{B}(x_0,2r)}\left|\frac{u-(u)_{x_0,2r}}{r}\right|^p\mathrm{d}x. \tag{Cacc}$$

The preceding inequality (Cacc) is of Caccioppoli–type. Now note that p itself arises as the Sobolev exponent of $q:=np/(n+p)$, and there holds $q\geqslant 1$ if and only if $p\geqslant n/(n-1)$. Moreover, note that $q<p$ regardless of the specific value of $1\leqslant p<n$. For $n/(n-1)\leqslant p<n$, we obtain by use of (Cacc) and the Sobolev embedding theorem

$$\begin{aligned}\left(\fint_{\mathrm{B}(x_0,r)}|\,\mathrm{D}u|^p\,\mathrm{d}x\right)^{\frac{1}{p}} &\overset{(\mathrm{Cacc})}{\leqslant} c\left(\fint_{\mathrm{B}(x_0,2r)}\left|\frac{u-(u)_{x_0,2r}}{r}\right|^p\mathrm{d}x\right)^{\frac{1}{p}}\\ &\leqslant c\left(\fint_{\mathrm{B}(x_0,2r)}|\,\mathrm{D}u|^{\frac{np}{n+p}}\,\mathrm{d}x\right)^{\frac{n+p}{np}}.\end{aligned}$$

The exponent on the right is smaller than that appearing on the left side and in this sense we have obtained a reverse Hölder inequality with increasing supports. On the other hand, if $1 < p < n/(n-1)$, we use Jensen's inequality to obtain

$$\left(\fint_{\mathrm{B}(x_0,r)} |\mathrm{D}u|^p \,\mathrm{d}x\right)^{\frac{1}{p}} \overset{\text{(Cacc) and Jensen}}{\leqslant} \left(\fint_{\mathrm{B}(x_0,2r)} \left|\frac{u-(u)_{x_0,2r}}{r}\right|^{\frac{n}{n-1}} \mathrm{d}x\right)^{\frac{n-1}{n}} \leqslant c \fint_{\mathrm{B}(x_0,2r)} |\mathrm{D}u| \,\mathrm{d}x.$$

Consequently, invoking the following lemma of Gehring–type (here stated in the version of Giaquinta & Modica, cp. [128, Thm. 6.6]), we are in position to deduce that there exists some fixed $\varepsilon > 0$ such that $\mathrm{D}u \in \mathrm{L}^{p+\varepsilon}_{\mathrm{loc}}(\Omega;\mathbb{R}^{N\times n})$.

Lemma 5.39 (Gehring's Lemma)**.** *Given $R > 0$, denote Q_R a cube of sidelength $2R$. Let $f \in \mathrm{L}^1(Q_R;\mathbb{R}_{\geqslant 0})$ and assume that for all cubes $Q \subset \widetilde{Q} \Subset Q_R$ we have*

$$\fint_Q f(x)\,\mathrm{d}x \leqslant B\left(\left(\fint_{\widetilde{Q}} f^m \,\mathrm{d}x\right)^{\frac{1}{m}} + \fint_{\widetilde{Q}} g\,\mathrm{d}x\right), \tag{5.109}$$

where $B > 0$ is a constant, $0 < m < 1$ and $g \in \mathrm{L}^s(Q_R;\mathbb{R}_{\geqslant 0})$ for some $s > 1$. Then there exists $r > 1$ such that $f \in \mathrm{L}^r(Q_{R/2})$ and moreover

$$\fint_{Q_{R/2}} f^r \,\mathrm{d}x \leqslant c\left(\left(\fint_{Q_R} f\,\mathrm{d}x\right)^r + \fint_{Q_R} g^r \,\mathrm{d}x\right). \tag{5.110}$$

We shall now study as to which extent a Caccioppoli–type inequality in the linear growth setting can be made to work to yield higher integrability results. Recall by the above, in the superlinear growth regime it immediately gives higher integrability of the gradients, whereas in the linear growth context we merely have the informal principle that the Caccioppoli inequality leads to an

$$\boxed{\text{improvement of all (fractional) derivatives up to, but not including, } s=1.}$$

Note that although in the sequel, this is only stated for symmetric–convex problems, it is easy to see that it equally applies to the gradient setting, too.

Proposition 5.40. *Let $f\colon \mathbb{R}^{n\times n}_{\mathrm{sym}} \to \mathbb{R}_{\geqslant 0}$ be a symmetric–convex C^1–integrand for which there exist $\lambda_1, \lambda_2 > 0$ such that*

$$\langle f'(\mathbf{Z}), \mathbf{Z}\rangle \geqslant \lambda_1\langle \mathbf{Z}\rangle - \lambda_2 \qquad \textit{for all } \mathbf{Z} \in \mathbb{R}^{n\times n}_{\mathrm{sym}}. \tag{5.111}$$

Then there exists a constant $C > 0$ with the following property: If $u \in \mathrm{LD}(\Omega)$ is a minimiser of the variational problem (5.1), then for all rigid deformations $a \in \mathcal{R}(\Omega)$, any $x_0 \in \Omega$ and all $r > 0$ with $2r < \operatorname{dist}(x_0,\partial\Omega)$ there holds

$$\fint_{\mathrm{B}(x_0,r)} |\boldsymbol{\varepsilon}(u)|\,\mathrm{d}x \leqslant C\fint_{\mathrm{B}(x_0,2r)} \left|\frac{u-a}{r}\right| \mathrm{d}x + C, \tag{5.112}$$

where $C > 0$ is a constant which does not depend on u, a, x_0 and r.

Proof. Pick a smooth cut–off function $\rho \in \mathrm{C}^1_c(\mathrm{B}_{2r};[0,1])$ with $\mathbb{1}_{\mathrm{B}_r} \leqslant \rho \leqslant \mathbb{1}_{\mathrm{B}_{2r}}$, where $\mathrm{B}_t := \mathrm{B}(x_0,t)$, such that $|\mathrm{D}\rho| \leqslant C/r$ for some $C > 0$. We wish to remark that since $\mathrm{E}^s u \equiv 0$ by assumption, we have the Euler–Lagrange equation

$$\int_\Omega \langle f'(\boldsymbol{\varepsilon}(u)), \boldsymbol{\varepsilon}(\varphi)\rangle \,\mathrm{d}x = 0 \qquad \text{for all } \varphi \in \mathrm{LD}_0(\Omega). \tag{5.113}$$

Let $a \in \mathcal{R}(\Omega)$ be an arbitrary rigid deformation. Then $\varphi := \rho(u-a)$ is an admissible test function, and we obtain after a routine estimation

$$\begin{aligned}\int_{\mathrm{B}_{2r}} \lambda_1\rho\langle\varepsilon(u)\rangle - \lambda_2\rho\,\mathrm{d}x &\overset{(5.111)}{\leqslant} \int_{\mathrm{B}_{2r}} \langle f'(\varepsilon(u)), \rho\varepsilon(u)\rangle\,\mathrm{d}x \\ &= -\int_\Omega \langle f'(\varepsilon(u)), \mathrm{D}\rho\odot(u-a)\rangle\,\mathrm{d}x \leqslant C\int_{\mathrm{B}_{2r}}\left|\frac{u-a}{r}\right|\mathrm{d}x,\end{aligned}$$

where we have used that f has bounded first derivatives. After dividing by r^n, we hence obtain by $|\cdot| \leqslant \langle\cdot\rangle$

$$\lambda_1 ⨍_{\mathrm{B}_r} |\varepsilon(u)|\,\mathrm{d}x \leqslant C ⨍_{\mathrm{B}_{2r}}\left|\frac{u-a}{r}\right|\mathrm{d}x + C\lambda_2$$

from where the claim is immediate. The proof is complete. ☐

Corollary 5.41. *In the situation of the previous lemma, there exists a constant $\delta > 0$ with the following property: If $u \in \mathrm{LD}(\Omega)$ is a minimiser of the variational problem (1.1), then there exists $\delta > 0$ such that $u \in \mathrm{L}^{1^*+\delta}_{\mathrm{loc}}(\Omega)$, where $1^* = n/(n-1)$ is the Sobolev conjugate exponent of 1.*

Proof. Let $x_0 \in \Omega$ and $r > 0$ with $0 < 2r < \operatorname{dist}(x_0, \partial\Omega)$ be arbitrary. The usual Poincaré–Sobolev inequality for LD (cp. Chapter 4.2.3) asserts that there exists $a \in \mathcal{R}(\mathrm{B}(x_0, r))$ such that $\|u-a\|_{\mathrm{L}^{\frac{n}{n-1}}(\mathrm{B}(x_0,r);\mathbb{R}^n)} \leqslant Cr\|\varepsilon(u)\|_{\mathrm{L}^1(\mathrm{B}(x_0,r);\mathbb{R}^{n\times n})}$ with $C > 0$ independent of $r > 0$ and u. Picking this $a \in \mathcal{R}(\mathrm{B}(x_0, 2r))$ (which, as an affine map can be extended to Ω), we apply Proposition 5.40 with the rigid deformation a and thereby obtain by use of the Sobolev–Poincaré–Inequality (and the shorthand $\mathrm{B}_s := \mathrm{B}(x_0, s)$)

$$\begin{aligned}\left(⨍_{\mathrm{B}_r}|u|^{\frac{n}{n-1}}\,\mathrm{d}x\right)^{\frac{n-1}{n}} &\leqslant \left(⨍_{\mathrm{B}_r}|u-a|^{\frac{n}{n-1}}\,\mathrm{d}x\right)^{\frac{n-1}{n}} + \left(⨍_{\mathrm{B}_r}|a|^{\frac{n}{n-1}}\,\mathrm{d}x\right)^{\frac{n-1}{n}} \\ &\overset{(*)}{\leqslant} Cr⨍_{\mathrm{B}_r}|\varepsilon(u)|\,\mathrm{d}x + C⨍_{\mathrm{B}_r}|a|\,\mathrm{d}x \\ &\overset{(**)}{\leqslant} Cr⨍_{\mathrm{B}_r}|\varepsilon(u)|\,\mathrm{d}x + C⨍_{\mathrm{B}_r}|u|\,\mathrm{d}x \\ &\overset{(5.112)}{\leqslant} C⨍_{\mathrm{B}_{2r}}|u-a|\,\mathrm{d}x + C⨍_{\mathrm{B}_{2r}}|u|\,\mathrm{d}x + Cr \\ &\leqslant C⨍_{\mathrm{B}_{2r}}|u|\,\mathrm{d}x + C\operatorname{diam}(\Omega).\end{aligned} \tag{5.114}$$

For the estimate $(*)$, we used that $\mathcal{R}(\Omega)$ is a finite dimensional vector space and equivalence of all norms, whereas for $(**)$, we used the inequality

$$⨍_{\mathrm{B}_r}|a|\,\mathrm{d}x \leqslant C⨍_{\mathrm{B}_r}|u|\,\mathrm{d}x$$

which follows from the L^1–stability estimate of Theorem 3.25. Finally, the very last estimate in (5.114) follows equally from the aforementioned stability estimate, finite dimensionality of $\mathcal{R}(\mathrm{B}(x_0, 2r))$ and scaling as above. In this situation, the claim is an immediate consequence of the usual Gehring Lemma in the form due to Giaquinta & Modica, cp. Lemma 5.39. The proof is complete. ☐

Note that a self–improvement of integrability can be derived without referring to the Euler–Lagrange equation but minimality only (see the work of Giaquinta [122]) and can be used to prove that any BD–minimiser if of class $\mathrm{L}^{1^*+\delta}_{\mathrm{loc}}(\Omega;\mathbb{R}^n)$ for some $\delta > 0$ indeed. As such, the proof also reveals that this regularity improvement also applies to non–strictly convex problems which in general might produce very irregular minimisers revealing jumps, say.

Remark 5.42 (Gehring versus Difference Quotients)**.** In general, Gehring's lemma proves particularly powerful in the quasiconvex context rather than for convex problems. This is so because in the convex setup, we can make use of the Euler–Lagrange equation and henceforth difference quotients. The corresponding coercive estimates appearing in higher integrability proofs in turn crucially rely on the positive definiteness of the Hessian of the integrand, a condition which is not satisfied by genuinely quasiconvex integrands. On the other hand, by the convex integration approach of Müller & Šverák [190], the Euler–Lagrange equation itself cannot yield regularity results; also compare with Székelyhidi [230] in the polyconvex setting and with Kristensen & Taheri [164] for the relation of extremals (i.e., solutions of the corresponding Euler–Lagrange equation) and (local) minima.

We proceed by showing that the Caccioppoli–Inequality itself cannot yield higher integrability of the gradients or symmetric gradients, respectively:

Lemma 5.43. *Let $v := \mathrm{sgn}$ be the sign–function on the interval $(-1,1)$. Then v satisfies a Caccioppoli–type inequality in the sense that there exists $c > 0$ such that for all $z \in (-1,1)$ and $r > 0$ such that $z \pm 2r \in (-1,1)$ there holds*

$$|\,\mathrm{D}v|(\mathrm{B}(z,r)) \leqslant \frac{c}{r}\int_{z-2r}^{z+2r} |v - (v)_{\mathrm{B}(z,2r)}|\,\mathrm{d}x. \tag{5.115}$$

Proof. We shall prove the existence of such $c > 0$ by making a case distinction. Firstly note that $|Dv| = 2\delta_0$, so if $0 \notin \mathrm{B}(x,r)$, then $|Dv|(\mathrm{B}(x,r)) = 0$ and there is nothing to prove. Hence we may assume without loss of generality $|x| < r$ and moreover shall tacitly assume that $x \in (-1,1)$ and $r > 0$ are chosen in a way such that $\mathrm{B}(x,2r) \subset (-1,1)$, too. *First case:* $x = 0$. For any $0 < r < 1$ we have $(v)_{\mathrm{B}(0,r)} = 0$ and thus if $c \geqslant 1$, we conclude

$$\frac{c}{r}\int_{\mathrm{B}(x,r)} |v(y) - (v)_{0,2r}|\,\mathrm{d}y = \frac{c}{r}\int_{\mathrm{B}(x,r)} |\,\mathrm{sgn}(y)|\,\mathrm{d}y = \frac{c}{r}2r = 2c \overset{!}{\geqslant} 2 = |Dv|(\mathrm{B}(0,r)).$$

Second case: $x > 0$. We recall that $x < r$. First, we compute the mean value of v over $\mathrm{B}(x,2r)$ as

$$\begin{aligned}(v)_{x,2r} = \frac{1}{4r}\int_{x-2r}^{x+2r} \mathrm{sgn}(y)\,\mathrm{d}y &= \frac{1}{4r}\left(\int_{x-2r}^{0}(-1)\,\mathrm{d}y + \int_{0}^{x+2r}(+1)\,\mathrm{d}y\right) \\ &= \frac{1}{4r}\left((x-2r)+(x+2r)\right) = \frac{x}{2r}.\end{aligned}$$

Now, since $0 < x < r$, $\frac{x}{r} < 1$ and thus $0 < \frac{x}{2r} < \frac{1}{2}$. Consequently,

$$|\,\mathrm{sgn}(y) - (v)_{x,2r}| = |\,\mathrm{sgn}(y) - \frac{x}{2r}| = \begin{cases} 1 - \frac{x}{2r} & \text{if } y > 0, \\ 1 + \frac{x}{2r} & \text{if } y < 0 \end{cases}$$

and therefore, if $c > \frac{2}{3}$,

$$\begin{aligned}
\frac{c}{r}\int_{\mathrm{B}(x,2r)} |v(y)-(v)_{x,2r}|\,\mathrm{d}y &= \frac{c}{r}\left(\int_{x-2r}^{0}(1+\frac{x}{2r})\,\mathrm{d}y + \int_{0}^{x+2r}(1-\frac{x}{2r})\,\mathrm{d}y\right)\\
&= \frac{c}{2r^2}\left((2r+x)(2r-x)+(2r-x)(x+2r)\right)\\
&= \frac{c}{2r^2}(4r^2-x^2+4r^2-x^2) = c\Big(4-\Big(\frac{x}{r}\Big)^2\Big)\\
&\geqslant 3c \quad (\text{as } 0<\frac{x}{r}<1)\\
&\overset{!}{\geqslant} 2 = |Dv|(\mathrm{B}(x,r)).
\end{aligned}$$

Third Case: $x<0$. It is possible to argue by symmetry, but for the reader's convenience we give the precise computation. The condition $|x|<r$ translates into $-x<r$, and we have

$$(v)_{x,2r} = \frac{1}{4r}\int_{x-2r}^{x+2r}\mathrm{sgn}(y)\,\mathrm{d}y = \frac{1}{4r}(-(0-(x-2r))+(x+2r)) = \frac{x}{2r}.$$

Now, we have analogously (recall that $0<-\frac{x}{2r}<\frac{1}{2}$)

$$|\,\mathrm{sgn}(y)-(v)_{x,2r}| = |\,\mathrm{sgn}(y)-\frac{x}{2r}| = \begin{cases} 1+\frac{x}{2r} & \text{if } y<0,\\ 1-\frac{x}{2r}, & \text{if } y\geqslant 0.\end{cases}$$

In conclusion, again recalling $|x|<r$

$$\begin{aligned}
\frac{c}{r}\int_{x-2r}^{x+2r} |\,\mathrm{sgn}(y)-(v)_{x,2r}|\,\mathrm{d}y &= \frac{c}{r}\left(\int_{x-2r}^{0}\big(1+\frac{x}{2r}\big)\,\mathrm{d}y + \int_{0}^{x+2r}\big(1-\frac{x}{2r}\big)\,\mathrm{d}y\right)\\
&= \frac{c}{2r^2}\big((2r-x)(2r+x)+(x+2r)(2r-x)\big)\\
&= \frac{c}{2r^2}(4r^2-x^2+4r^2-x^2) = c\big(4-\big(\frac{x}{r}\big)^2\big)\\
&= 3c \overset{!}{\geqslant} 2,
\end{aligned}$$

which is fulfilled if $c>\frac{3}{2}$.

In conclusion, gathering estimates, we obtain that if $c>\frac{3}{2}$, then estimate (5.115) is fulfilled by $v=\mathrm{sgn}$, and the proof is complete. □

The preceding proof is in line with Buckley & Koskela's proof of the non–availability of Sobolev inequalities in the sublinear regime as given in [55], a fact on which we briefly report now. It is well–known that if $\Omega\subset\mathbb{R}^n$ is an open and bounded Lipschitz domain, then for each $1\leqslant q\leqslant n/(n-1)$ and $p=1$ there holds

$$\inf_{a\in\mathbb{R}^N}\left(\int_{\Omega}|u-a|^q\,\mathrm{d}x\right)^{\frac{1}{q}} \leqslant c\left(\int_{\Omega}|\,\mathrm{D}u|^p\,\mathrm{d}x\right)^{\frac{1}{p}} \tag{5.116}$$

for all $u\in\mathrm{W}^{1,1}(\Omega;\mathbb{R}^N)$, where $c=c(p,q,\Omega,N)>0$ is a constant. Such an inequality does not extend to $0<p<1$ without imposing additional conditions on the admissible maps $u\in\mathrm{W}^{1,1}(\Omega;\mathbb{R}^N)$ in such an inequality. To see this, Buckley & Koskela considered the case $\Omega=(-1,1)$ and put, for $\varepsilon>0$

$$u_\varepsilon(x) := \begin{cases} 0 & \text{if } x\leqslant -\varepsilon,\\ \varphi(x/\varepsilon) & \text{if } |x|<\varepsilon,\\ 1 & \text{if } x>\varepsilon,\end{cases}$$

where $\varphi\colon [-1,1] \to [0,1]$ is a differentiable function with $\varphi(-1) = 0$, $\varphi(1) = 1$ and $\lim_{t\searrow -1} \varphi'(t) = \lim_{t\nearrow 1} \varphi'(t) = 0$. Assuming (5.116) to hold for some $0 < p < 1$, we compute by a change of variables

$$\int_{-1}^{1} |\nabla u_\varepsilon|^p \,\mathrm{d}x = \varepsilon^{1-p} \int_{-1}^{1} |\varphi'(y)|^p \,\mathrm{d}y \to 0, \qquad \varepsilon \searrow 0.$$

To arrive at the desired contradiction, observe that for every $a \in \mathbb{R}$ there either holds $|a| \geqslant \frac{1}{2}$ or $|a-1| \geqslant \frac{1}{2}$ so that

$$\frac{1}{2^q}(1-\varepsilon) \leqslant \inf_{a\in\mathbb{R}} \int_{-1}^{1} |u_\varepsilon - a|^q \,\mathrm{d}x.$$

However, by (5.116), we must have $\inf_{a\in\mathbb{R}} \int_{-1}^{1} |u_\varepsilon - a|^q \,\mathrm{d}x \to 0$ as $\varepsilon \searrow 0$, a contradiction.

On the other hand, note that the above Lemma 5.43 in conjunction with Gehring's Lemma 5.39 yields another proof of the non–availability of the sublinear Sobolev inequality. Indeed, considering suitable mollifications $v_\varepsilon\colon (-1,1) \to \mathbb{R}$ of the sgn–function, it is easy to see that the single v_ε's satisfy a uniform Caccioppoli inequality in the sense that there exists a constant $c > 0$ such that for all $z \in (-1,1)$ and $r > 0$ such that $(z-2r, z+2r) \subset (-1,1)$ there holds $\int_{\mathrm{B}(z,r)} |\mathrm{D}v_\varepsilon| \,\mathrm{d}x \leqslant (c/r) \int_{\mathrm{B}(z,2r)} |v_\varepsilon - (v_\varepsilon)_{z,r}| \,\mathrm{d}x$ uniformly in $\varepsilon > 0$. So, if the sublinear Sobolev–Poincaré inequality were valid, then Gehring's Lemma would automatically yield that $(\mathrm{D}v_\varepsilon)$ is locally uniformly bounded in L^p for some fixed $p > 1$ and thus, by the De Valleé–Poussin criterion, converges weakly to a locally integrable function. Since the weak*–limit of the $\mathrm{D}v_\varepsilon$'s however is $-2\delta_0$, we have arrived at a contradiction.

The failure of (5.116) for general $0 < p < 1$ served as motivation to study the (weak) reverse Hölder classes RH_q^Ω and WRH_q^Ω. These are defined as the sets of functions $w \in \mathrm{L}^q_{\mathrm{loc}}(\Omega; \mathbb{R}_{\geqslant 0})$, $w \not\equiv 0$ such that $\|w\|_{\mathrm{L}^q(Q)} \leqslant C\|w\|_{\mathrm{L}^p(\sigma Q)}$ for all cubes Q such that $\sigma' Q \subset$, where $1 < \sigma \leqslant \sigma'$ in the case of WRH and $1 = \sigma < \sigma'$ in the case of RH.

Theorem 5.44. *Let $u \in \mathrm{LD}(\Omega)$ be a generalised minimiser of the variational principle (5.1), where $f\colon \mathbb{R}^{n\times n}_{\mathrm{sym}} \to \mathbb{R}$ is a convex C^1–variational integrand which satisfies (5.111). Then for any $1 < p < \frac{n}{n-1}$ there exists*

$$s > \frac{n - np + p}{p}$$

such that $u \in \mathrm{W}^{s,p}_{\mathrm{loc}}(\Omega; \mathbb{R}^n)$.

Proof. By Corollary 5.41, $u \in \mathrm{L}^q_{\mathrm{loc}}(\Omega; \mathbb{R}^n)$ for some $q > \frac{n}{n-1}$. Now we use Theorem 5.37 to conclude. □

To conclude, it is not clear to us how the Caccioppoli inequality might yield higher integrability (possibly on the fractional scale) beyond the interpolation argument outlined above. Namely, estimating further using Jensen's inequality,

$$\begin{aligned}
\int_{\mathrm{B}(x_0,r)} |\mathrm{D}u| \,\mathrm{d}x &\leqslant c \int_{\mathrm{B}(x_0,r)} \frac{|u(x) - (u)_{x_0,r}|}{r} \,\mathrm{d}x \\
&\leqslant c \int_{\mathrm{B}(x_0,r)} \frac{1}{r} \fint_{\mathrm{B}(x_0,r)} |u(x) - u(y)| \,\mathrm{d}y \,\mathrm{d}x \qquad \text{(by Jensen)} \\
&\leqslant c \iint_{\mathrm{B}(x_0,r)\times \mathrm{B}(x_0,r)} \frac{|u(x) - u(y)|}{|x-y|^{n+1}} \,\mathrm{d}x \,\mathrm{d}y \; (x, y \in \mathrm{B}(x_0,r) \Rightarrow |x-y| \leqslant 2r)
\end{aligned}$$

the right side cannot be bounded unless u is constant as a consequence of Brezis' Lemma, cp. Remark 2.14.

Let us note that in the BV–context, it is possible to show under a C^1–hypothesis on the variational integrand and some other mild structural condition (in particular, milder than radial structure) a so–called *monotonicity formula* for the gradient, cp. [56, 57]. In a simplified form, this reads as

$$\frac{\mathrm{d}}{\mathrm{d}R}\frac{|\,\mathrm{D}u|(\mathrm{B}(x,R))}{R^{n-1}} \geqslant 0 \tag{5.117}$$

given $u \in \mathrm{BV}$ is a minimiser. This implies by standard means that the jump height of u is bounded and, moreover, $u \in \mathrm{BMO}_{\mathrm{loc}}$. Here, given x_0 is a jump point of u, we can locally write $u(x) = (u_+ - u_-) \otimes \nu$ for $\mathscr{H}^{n-1}$–a.e. x in the jump set of u, where u_+ and u_- are the one–sided traces; by the structure theorem for BV–functions, these notions are well–defined. The jump height then is given by $|u_+(x) - u_-(x)|$.

The $\mathrm{BMO}_{\mathrm{loc}}$–regularity then follows from (5.117) by Poincaré's Inequality as for all $x \in \Omega$ and $0 < R < \mathrm{dist}(x, \partial\Omega)$ there holds

$$\fint_{\mathrm{B}(x,R)} |u - (u)_{\mathrm{B}(x,R)}|\,\mathrm{d}y \leqslant CR\frac{|\,\mathrm{D}u|(\mathrm{B}(x,R))}{R^n},$$

and by (5.117), the latter is locally bounded. However, it is not clear to us how to generalise this to the BD–situation.

5.7. Higher Integrability and Discussion of Uniqueness

In this section we combine the results of Theorem 5.12 and the improved embedding from $\mathrm{BD} \cap \mathrm{BMO}$ from Theorem 5.29 to deduce the Sobolev regularity assertion of Theorem 5.6.

5.7.1. Coercive Estimate and Higher Integrability for one generalised minimiser Before we embark on the proof of Sobolev regularity of u_*, we introduce for $M \in \mathbb{N}$ the auxiliary map $V_\alpha \colon \mathbb{R}^M \to \mathbb{R}^M$ by

$$V_\alpha(\xi) := (1 + |\xi|^2)^{\frac{1-\alpha}{2}}\xi, \quad \xi \in \mathbb{R}^M. \tag{5.118}$$

We then have

Lemma 5.45. *Let $1 < \alpha < 2$ and define V_α by (5.118). Then we have for any measurable function $v \colon \mathbb{R}^n \to \mathbb{R}^M$, $h \in \mathbb{R}$ and $e_s \in \mathbb{R}^n$ the estimate*

$$|\tau_{s,h}V_\alpha(v(x))| \sim (1 + |v(x + he_s)|^2 + |v(x)|^2)^{\frac{1-\alpha}{2}}|\tau_{s,h}v(x)|.$$

Moreover, there exists a constant $c > 0$ such that for all $\xi \in \mathbb{R}^M$ there holds

$$\min\{|\xi|, |\xi|^{2-\alpha}\} \leqslant cV_\alpha(\xi).$$

Lastly, if Ω is an open and bounded set and $u \colon \Omega \to \mathbb{R}^M$ satisfies $V_\alpha(u) \in \mathrm{L}^p(\Omega;\mathbb{R}^M)$, then we have

$$\int_\Omega |u|^{(2-\alpha)p}\,\mathrm{d}x \leqslant \mathscr{L}^n(\Omega) + c(p)\int_\Omega |V_\alpha(u)|^p\,\mathrm{d}x,$$

where $c(p) > 0$ is a constant depending only on p.

Proof. By [4, Lemma 2.2], for every $-\frac{1}{2} < \gamma < 0$ and $\mu \geqslant 0$ there exists a constant $c = c(M) > 0$ such that for all $\xi, \eta \in \mathbb{R}^M$ there holds

$$(2\gamma + 1)|\xi - \eta| \leqslant \frac{|(\mu^2 + |\xi|^2)^\gamma\xi - (\mu^2 + |\eta|^2)^\gamma\eta|}{(\mu^2 + |\xi|^2 + |\eta|^2)^\gamma} \leqslant \frac{c(M)}{2\gamma + 1}|\xi - \eta|.$$

Applying this with $\mu = 1$ and $\gamma = (1-\alpha)/2$ yields the claim as $-\frac{1}{2} < \gamma < 0$ if and only if $1 < \alpha < 2$.

Now let $\xi \in \mathbb{R}^M$ with $|\xi| \geqslant 1$ and let $1 < \alpha < 2$. Then $(1-\alpha)/2 < 0$. Hence, since $t \mapsto t^{(1-\alpha)/2}$ is monotonically decreasing on $\mathbb{R}_{>0}$,

$$
\begin{aligned}
|\xi| > 1 \Rightarrow |\xi|^2 > 1 \Rightarrow 2|\xi|^2 > 1 + |\xi|^2 \Rightarrow 2^{\frac{1-\alpha}{2}} |\xi|^{1-\alpha} \leqslant (1+|\xi|^2)^{\frac{1-\alpha}{2}} \\
\Rightarrow |\xi|^{2-\alpha} \leqslant 2^{\frac{\alpha-1}{2}} |V_\alpha(\xi)| \overset{1<\alpha<2}{\leqslant} \sqrt{2}|V_\alpha(\xi)|.
\end{aligned}
$$

Since $1 < \alpha < 2$ and $|\xi| \geqslant 1$, we have $|\xi|^{2-\alpha} \leqslant |\xi|$ and thus $\min\{|\xi|, |\xi|^{2-\alpha}\} \leqslant \sqrt{2}|V_\alpha(\xi)|$ in this case. Now, if $|\xi| < 1$, then

$$
2^{\frac{1-\alpha}{2}} \leqslant (1+|\xi|^2)^{\frac{1-\alpha}{2}} \Rightarrow 2^{\frac{1-\alpha}{2}}|\xi| \leqslant |V_\alpha(\xi)| \Rightarrow |\xi| \leqslant \sqrt{2}|V_\alpha(\xi)|,
$$

and hence we see because of $|\xi| \leqslant |\xi|^{2-\alpha}$ in this regime that $\min\{|\xi|, |\xi|^{2-\alpha}\} \leqslant \sqrt{2}|V_\alpha(\xi)|$ holds, too. Hence $\min\{|\xi|, |\xi|^{2-\alpha}\} \leqslant \sqrt{2}|V_\alpha(\xi)|$ for all $\xi \in \mathbb{R}^M$. Lastly if the measurable function $u\colon \Omega \to \mathbb{R}^M$ is such that $V_\alpha(u) \in \mathrm{L}^p(\Omega;\mathbb{R}^m)$, then we have

$$
\begin{aligned}
\int_\Omega |u|^{(2-\alpha)p}\,\mathrm{d}x &= \int_{\Omega\cap\{|u|\leqslant 1\}} |u|^{(2-\alpha)p}\,\mathrm{d}x + \int_{\Omega\cap\{|u|>1\}} |u|^{(2-\alpha)p}\,\mathrm{d}x \\
&\leqslant \mathscr{L}^n(\Omega) + c(p)\int_\Omega |V_\alpha(u)|^p\,\mathrm{d}x
\end{aligned}
$$

The proof is complete. □

After these preparations, we come to the

Proof of Theorem 5.6. Let $(v_j) \subset u_0 + \mathrm{W}_0^{1,2}(\Omega;\mathbb{R}^n)$ be the sequence of viscosity approximations as constructed in Section 5.4.2. In contrast to the estimates which eventually lead to the Sobolev regularity assertion $(v_j) \subset \mathrm{W}^{2,2}_{\mathrm{loc}}(\Omega;\mathbb{R}^n)$ of Lemma 5.11, we now need to establish estimates which are uniform in j. In doing so, we shall derive a local Nikolskii–type improvement (with the Nikolskii spaces $\mathcal{N}^{\alpha,p} := \mathrm{B}^\alpha_{p,\infty}$, cp. Chapter 2.5.3) and hence recall that for $s \in \{1,...,n\}$, $h \in \mathbb{R}$ and a measurable map $w\colon \mathbb{R}^n \to \mathbb{R}^N$, the forward and backward finite difference operators are given by $\tau^{\pm}_{s,h}w(x) := w(x \pm he_s) - w(x)$, where e_s is the s–th unit vector. We also put $\tau_{s,h} := \tau^+_{s,h}$ for brievity.

Let $x_0 \in \Omega$ and $r > 0$ such that $\mathrm{B}(x_0, 2r) \Subset \Omega$. For $h \in \mathbb{R}$ with $|h| < \mathrm{dist}(x_0, \partial\Omega) - 2r$ we pick a cut–off function $\rho \in \mathrm{C}^1_c(\mathrm{B}(x_0,r);[0,1])$ with $\mathbb{1}_{\mathrm{B}(x_0,r)} \leqslant \rho \leqslant \mathbb{1}_{\mathrm{B}(x_0,2r)}$ and put $\varphi_j := \tau^-_{s,h}(\rho^2\tau_{s,h}v_j)$. By assumption, we have $\varphi_j \in \mathrm{W}^{1,2}_0(\Omega;\mathbb{R}^n)$ and thus φ_j is admissible in the weak Euler–Lagrange equation (5.23). In consequence, we obtain for each $j \in \mathbb{N}$ (recall that $f_j := f + \frac{1}{2j}|\cdot|^2$)

$$
\int_\Omega \langle f'_j(\boldsymbol{\varepsilon}(v_j)), \boldsymbol{\varepsilon}(\tau^-_{s,h}(\rho^2\tau_{s,h}v_j))\rangle\,\mathrm{d}x = 0.
$$

Since $\boldsymbol{\varepsilon}$ and $\tau^-_{s,h}$ commute, we find by discrete integration by parts that

$$
\mathbf{I} := \int_\Omega \langle \tau_{s,h}(f'_j(\boldsymbol{\varepsilon}(v_j))), \rho^2\tau_{s,h}\boldsymbol{\varepsilon}(v_j)\rangle\,\mathrm{d}x = -\int_\Omega \langle \tau_{s,h}(f'_j(\boldsymbol{\varepsilon}(v_j))), 2\rho\,\mathrm{D}\rho \odot \tau_{s,h}v_j\rangle\,\mathrm{d}x =: \mathbf{II}.
$$

Using the μ–ellipticity condition, we will now suitably estimate $\mathbf{I}$ from below. By the fundamental theorem of calculus, we have

$$
\begin{aligned}
\mathbf{I} &\geqslant \int_\Omega \langle f_j'(\varepsilon(v_j)(x+the_s)) - f_j'(\varepsilon(v_j)(x)), \rho^2\tau_{s,h}\varepsilon(v_j)\rangle \,\mathrm{d}x \\
&= \int_\Omega \rho^2 \int_0^1 \left\langle \frac{\mathrm{d}}{\mathrm{d}t} f_j'(\varepsilon(v_j)+t\tau_{s,h}\varepsilon(v_j)), \tau_{s,h}\varepsilon(v_j)\right\rangle \mathrm{d}t\,\mathrm{d}x \\
&= \int_\Omega \rho^2 \int_0^1 \langle f_j''(\varepsilon(v_j)+t\tau_{s,h}\varepsilon(v_j))\tau_{s,h}\varepsilon(v_j), \tau_{s,h}\varepsilon(v_j)\rangle \,\mathrm{d}t\,\mathrm{d}x \\
&\geqslant \int_\Omega \rho^2 \int_0^1 \langle f''(\varepsilon(v_j)+t\tau_{s,h}\varepsilon(v_j))\tau_{s,h}\varepsilon(v_j), \tau_{s,h}\varepsilon(v_j)\rangle \,\mathrm{d}t\,\mathrm{d}x \\
&+ \frac{1}{j}\int_\Omega \rho^2 |\tau_{s,h}\varepsilon(v_j)|^2 \,\mathrm{d}x =: \mathbf{I}_1 + \mathbf{I}_2.
\end{aligned}
$$

It is clear that $\mathbf{I}_2 \geqslant 0$. Since f is assumed to be μ–elliptic, we find

$$
\begin{aligned}
\mathbf{I} \geqslant \mathbf{I}_1 &\geqslant c\int_\Omega \rho^2 \int_0^1 \frac{|\tau_{s,h}\varepsilon(v_j)|^2}{(1+|\varepsilon(v_j)+t\tau_{s,h}\varepsilon(v_j)|^2)^{\frac{\mu}{2}}} \,\mathrm{d}t\,\mathrm{d}x \\
&\geqslant c\int_\Omega \rho^2 \frac{|\tau_{s,h}\varepsilon(v_j(x))|^2}{(1+|\varepsilon(v_j)|^2+|\varepsilon(v_j(x+he_s))|^2)^{\frac{\mu}{2}}} \,\mathrm{d}x,
\end{aligned}
$$

where we have used that for all $0 \leqslant t \leqslant 1$ and $a, b \in \mathbb{R}^{n\times n}$ there holds (with some fixed $C > 0$)

$$
(1+|a+tb|^2)^{\frac{1}{2}} \leqslant C(1+|a|^2+|b|^2)^{\frac{1}{2}}.
$$

We now use the auxiliary estimate given by the Acerbi–Fusco lemma (cp. Lemma 5.45) to conclude for the auxiliary function $V_\alpha(\xi) := (1+|\xi|^2)^{\frac{1-\alpha}{2}}\xi$ with $1 < \alpha < 2$ that

$$
|V_\alpha(\xi) - V_\alpha(\eta)|^2 \sim (1+|\xi|^2+|\eta|^2)^{\frac{2(1-\alpha)}{2}}|\xi-\eta|^2
$$

and hence

$$
\frac{|\xi-\eta|^2}{(1+|\eta|^2+|\xi|^2)^{\frac{\mu}{2}}} \sim \frac{|V_\alpha(\xi)-V_\alpha(\eta)|^2}{(1+|\xi|^2+|\eta|^2)^{\frac{\mu+2(\alpha-1)}{2}}}.
$$

In particular, we find that for $1 < \alpha < 2$ to be determined later (and accordingly a constant $c = c(\alpha) > 0$),

$$
\frac{|\tau_{s,h}V_\alpha(\varepsilon(v_j))|^2}{(1+|\varepsilon(v_j)|^2+|\varepsilon(v_j(x+he))|^2)^{\frac{\mu+2(1-\alpha)}{2}}} \leqslant c\frac{|\tau_{s,h}\varepsilon(v_j(x))|^2}{(1+|\varepsilon(v_j)|^2+|\varepsilon(v_j(x+he))|^2)^{\frac{\mu}{2}}}. \tag{5.119}
$$

Keeping the previous estimate (5.119) in mind, we now pause to estimate **II** conveniently. Here we shall make substantial use of the uniform BMO–hypothesis (LBMO) and the regularity of the dual solution as stated in Theorem 5.12. To be more precise, recalling the notation $\sigma_j := f_j'(\varepsilon(v_j))$ for $j \in \mathbb{N}$, Theorem 5.12 asserts that for any relatively compact Lipschitz subset K of Ω we have $\sup_{j\in\mathbb{N}}\|\sigma_j\|_{\dot{\mathrm{W}}^{1,2}(K;\mathbb{R}^{n\times n})} < \infty$. On the other hand, the local uniform BMO–hypothesis in conjunction with the interpolation result of Theorem 5.29 applied to $p = 2$ yields that for $\varepsilon > 0$ sufficiently small there holds

$$
\sup_{j\in\mathbb{N}} \|v_j\|_{\dot{\mathrm{W}}^{\frac{1}{2}-\varepsilon,2}(K;\mathbb{R}^n)} < \infty. \tag{5.120}
$$

Deferring the precise value of $\varepsilon > 0$ to the end of the proof, we now estimate **II** for $|h| < \mathrm{dist}(x_0, \partial\Omega) - 2r$ by

$$
\begin{aligned}
|\mathbf{II}| &\leqslant C(\rho) \int_{\mathrm{B}(x_0,2r)} |\tau_{s,h}(f_j'(\boldsymbol{\varepsilon}(v_j)))|\,|\tau_{s,h}v_j|\,\mathrm{d}x \\
&\leqslant C(\rho) h^{1+\frac{1}{2}-\varepsilon} \int_{\mathrm{B}(x_0,2r)} |\Delta_{s,h}(f_j'(\boldsymbol{\varepsilon}(v_j)))| \left|\frac{\tau_{s,h}v_j}{h^{\frac{1}{2}-\varepsilon}}\right| \mathrm{d}x \\
&\leqslant C(\rho) h^{\frac{3}{2}-\varepsilon} \left(\int_{\mathrm{B}(x_0,2r)} |\Delta_{s,h}\sigma_j|^2\,\mathrm{d}x\right)^{\frac{1}{2}} \left(\int_{\mathrm{B}(x_0,2r)} \left|\frac{\tau_{s,h}v_j}{h^{\frac{1}{2}-\varepsilon}}\right|^2 \mathrm{d}x\right)^{\frac{1}{2}} \qquad \text{(by Hölder)} \\
&\leqslant C(\rho) h^{\frac{3}{2}-\varepsilon} \|\sigma_j\|_{\dot{\mathrm{W}}^{1,2}(\mathrm{B}(x_0,2r);\mathbb{R}^{n\times n}_{\mathrm{sym}})} \|v_j\|_{\dot{\mathrm{W}}^{\frac{1}{2}-\varepsilon,2}(\mathrm{B}(x_0,2r);\mathbb{R}^n)} \\
&\leqslant C(\rho) h^{\frac{3}{2}-\varepsilon},
\end{aligned}
$$

and here $C(\rho) > 0$ does not depend on $j \in \mathbb{N}$. Gathering estimates, we find by (5.119) and the shorthand

$$
\omega_{j,h,s}(x) := \frac{1}{(1 + |\boldsymbol{\varepsilon}(v_j(x))|^2 + |\boldsymbol{\varepsilon}(v_j(x+he_s))|^2)^{\frac{\mu+2(1-\alpha)}{2}}}
$$

that

$$
\int_{\mathrm{B}(x_0,2r)} \left|\frac{\tau_{s,h}V_\alpha(\boldsymbol{\varepsilon}(v_j))}{h^{\frac{3}{4}-\frac{\varepsilon}{2}}}\right|^2 \omega_{j,h,s}(x)\,\mathrm{d}x \leqslant C(\rho). \tag{5.121}
$$

By the results from Section 5.4.2 (precisely, the first part of (5.18)), (v_j) is uniformly bounded in $\mathrm{LD}(\Omega)$ and thus it is easily seen that $\sup_{j\in\mathbb{N}} \|\omega_{j,h,s}^{-1}\|_{\mathrm{L}^1(\mathrm{B}(x_0,2r))} < \infty$ provided (recall that $1 < \alpha < 2$ is assumed throughout)

$$
\mu + 2(1-\alpha) \leqslant 1, \quad \text{that is,} \quad \frac{\mu+1}{2} \leqslant \alpha (< 2). \tag{5.122}
$$

Henceforth, assuming condition (5.122) to be in action in all of what follows, we obtain by Young's inequality

$$
\begin{aligned}
\int_{\mathrm{B}(x_0,2r)} \left|\frac{\tau_{s,h}V_\alpha(\boldsymbol{\varepsilon}(v_j))}{h^{\frac{3}{4}-\frac{\varepsilon}{2}}}\right| \mathrm{d}x &\leqslant \int_{\mathrm{B}(x_0,2r)} \left|\frac{\tau_{s,h}V_\alpha(\boldsymbol{\varepsilon}(v_j))}{h^{\frac{3}{4}-\frac{\varepsilon}{2}}}\right|^2 \omega_{j,h,s}(x)\,\mathrm{d}x + \int_{\mathrm{B}(x_0,2r)} \frac{\mathrm{d}x}{\omega_{j,h,s}(x)} \\
&\leqslant C(\rho) \qquad \text{(by (5.121) and (5.122))},
\end{aligned}
$$

where $C(\rho) > 0$ again does not depend on $j \in \mathbb{N}$. From here we conclude that the sequence $(V_\alpha(\boldsymbol{\varepsilon}(v_j))|_{\mathrm{B}(x_0,r)})$ is uniformly bounded in $\dot{\mathrm{B}}^{\frac{3}{4}-\frac{\varepsilon}{2},1}_{\infty}(\mathrm{B}(x_0,r);\mathbb{R}^{n\times n}_{\mathrm{sym}})$. At this point, we recall from Chapter 2.5.1 that for all $\delta > 0$ sufficiently small there holds

$$
\dot{\mathrm{B}}^{\frac{3}{4}-\frac{\varepsilon}{2},1}_{\infty}(\mathrm{B}(x_0,2r)) \hookrightarrow \mathrm{L}^q(\mathrm{B}(x_0,r)) \qquad \text{for all } q \leqslant \frac{n}{n-\left(\frac{3}{4}-\frac{\varepsilon}{2}\right)} - \delta = \frac{4n}{4n-3+2\varepsilon} - \delta.
$$

We may therefore deduce that for $\varepsilon > 0$ and $\delta > 0$ sufficiently small, we have

$$
\sup_{j\in\mathbb{N}} \int_{\mathrm{B}(x_0,r)} |V_\alpha(\boldsymbol{\varepsilon}(v_j))|^{\frac{4n}{4n-3+2\varepsilon}-\delta}\,\mathrm{d}x < \infty.
$$

By Lemma 5.45, the previous estimate implies

$$
\sup_{j\in\mathbb{N}} \int_{\mathrm{B}(x_0,r)} |\boldsymbol{\varepsilon}(v_j)|^{q_{\alpha,n,\varepsilon,\delta}}\,\mathrm{d}x := \sup_{j\in\mathbb{N}} \int_{\mathrm{B}(x_0,r)} |\boldsymbol{\varepsilon}(v_j)|^{(2-\alpha)(\frac{4n}{4n-3+2\varepsilon}-\delta)}\,\mathrm{d}x < \infty, \tag{5.123}
$$

with an obvious definition of the exponent $q_{\alpha,n,\varepsilon,\delta} > 0$. In conclusion, if $q_{\alpha,n,\varepsilon,\delta} > 1$, then Korn's inequality yields uniform boundedness of $(\boldsymbol{\varepsilon}(v_j)|_{\mathrm{B}(x_0,r)})$ in $\mathrm{L}^{q_{\alpha,n,\varepsilon,\delta}}(\mathrm{B}(x_0,r);\mathbb{R}^{n\times n}_{\mathrm{sym}})$ and hence, by arbitrariness of $x_0 \in \Omega$ and $r > 0$, the claim follows.

To establish $q_{\alpha,n,\varepsilon,\delta} > 1$, let us note that the latter is equivalent to

$$\alpha < \frac{4n + 3 - 2\varepsilon - 2\delta(4n - 3 + 2\varepsilon)}{4n - \delta(4n - 3 + 2\varepsilon)}.$$

Sending $\varepsilon, \delta \searrow 0$, we find that $q_{\alpha,n,\varepsilon,\delta} > 1$ can be achieved for sufficiently small $\varepsilon, \delta > 0$ if and only if

$$\alpha < 1 + \frac{3}{4n}. \tag{5.124}$$

On the other hand, recalling (5.122), we must therefore have

$$\frac{\mu + 1}{2} \leqslant \alpha < 1 + \frac{3}{4n}, \tag{5.125}$$

an equation which is solvable for $1 < \alpha < 2$ if and only if $(\mu + 1)/2 < 1 + \frac{3}{4n}$. The latter inequality is solvable for $\mu > 1$ if and only if

$$\mu < 1 + \frac{3}{2n}, \tag{5.126}$$

which is exactly the exponent claimed in the theorem. The proof is complete. □

We proceed by extracting additional information out of the above proof. The exponent $q_{\alpha,n,\varepsilon,\delta} > 1$ as given in (5.123) is optimised for the smallest admissible value of α. This, in turn is given by $(\mu+1)/2$ by (5.125) subject to the condition $\mu < 1 + \frac{3}{2n}$ from (5.126).

Now, sending $\alpha \searrow (\mu + 1)/2$ for the admissible range of μ yields by (5.123) that for all $\gamma > 0$ and $\kappa > 0$ suitably small we have

$$\boldsymbol{\varepsilon}(u) \in \mathrm{L}_{\mathrm{loc}}^{(2-\frac{\mu+1}{2}-\gamma)(\frac{4n}{4n-3}-\kappa)}(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}}). \tag{5.127}$$

We consider now the condition (with $\Gamma := \kappa(4n - 3)$)

$$\begin{aligned}
\mu \overset{!}{\leqslant} (2 - \frac{\mu+1}{2} - \gamma)(\frac{4n}{4n-3} - \kappa) &\Leftrightarrow (8n - 6)\mu \leqslant (3 - \mu - 2\gamma)(4n - \Gamma) \\
&\Leftrightarrow 8n\mu - 6\mu + 4n\mu \leqslant 12n - 8\gamma n - \Gamma(3 - \mu - 2\gamma) \\
&\Leftrightarrow \mu(12n - 6) \leqslant 12n - 8\gamma n - \Gamma(3 - \mu - 2\gamma) \\
&\Leftrightarrow \mu \leqslant \frac{2n}{2n-1} - \frac{8\gamma n + \Gamma(3 - \mu - 2\gamma)}{12n - 6}.
\end{aligned}$$

Sending $\gamma, \kappa \searrow 0$ which in turn implies $\Gamma \searrow 0$, we obtain that $\boldsymbol{\varepsilon}(u) \in \mathrm{L}^{\mu}_{\mathrm{loc}}(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})$ provided $\mu < \frac{2n}{2n-1}$. Let us carefully note that this is more restrictive than the bound provided by (5.126): In fact,

$$\frac{2n}{2n-1} < \frac{2n+3}{2n} \Leftrightarrow 4n^2 < 4n^2 - 2n + 6n - 3 \Leftrightarrow \frac{3}{4} < n \overset{n\in\mathbb{N}}{\Leftrightarrow} n \geqslant 1.$$

We now are ready for the proof of the following corollary.

Corollary 5.46. *In the situation of Theorem 5.6, the sequence of viscosity approximations (v_j) is uniformly bounded in every local Nikolskii space $\mathcal{N}^{t,1}_{\mathrm{loc}}(\Omega;\mathbb{R}^n)$ with $0 < t < \frac{3}{4}$ provided*

$$1 < \mu < \frac{2n}{2n-1}. \tag{5.128}$$

Proof. Similarly to the derivation of (5.121) but not introducing the auxiliary V_α–function, we obtain

$$\int_{\mathrm{B}(x_0,r)} \left|\frac{\tau_{s,h}(\boldsymbol{\varepsilon}(v_j))}{h^{\frac{3}{4}-\frac{\varepsilon}{2}}}\right|^2 \frac{1}{(1+|\boldsymbol{\varepsilon}(v_j)(x)|^2+|\boldsymbol{\varepsilon}(v_j)(x+he_s)|^2)^{\frac{\mu}{2}}}\,\mathrm{d}x \leqslant C(\rho). \tag{5.129}$$

Now, by Young's inequality we have for every $\varepsilon > 0$ sufficiently small

$$\begin{aligned}\int_{\mathrm{B}(x_0,r)} \left|\frac{\tau_{s,h}(\boldsymbol{\varepsilon}(v_j))}{h^{\frac{3}{4}-\frac{\varepsilon}{2}}}\right| \mathrm{d}x &\leqslant \int_{\mathrm{B}(x_0,r)} \left|\frac{\tau_{s,h}(\boldsymbol{\varepsilon}(v_j))}{h^{\frac{3}{4}-\frac{\varepsilon}{2}}}\right|^2 \frac{1}{(1+|\boldsymbol{\varepsilon}(v_j)(x)|^2+|\boldsymbol{\varepsilon}(v_j)(x+he_s)|^2)^{\frac{\mu}{2}}}\,\mathrm{d}x \\ &\quad + \int_{\mathrm{B}(x_0,r)} (1+|\boldsymbol{\varepsilon}(v_j)(x)|^2+|\boldsymbol{\varepsilon}(v_j)(x+he_s)|^2)^{\frac{\mu}{2}}\,\mathrm{d}x \\ &=: \mathbf{I}_j + \mathbf{II}_j.\end{aligned}$$

We now find that $\mathbf{I}_j$ is uniformly bounded in $j \in \mathbb{N}$ by (5.129). On the other hand, $\mathbf{II}_j$ is uniformly bounded since by (5.128), (5.127) is in action. Hence for any relatively compact Lipschitz subset K of Ω we have $\sup_j \|\boldsymbol{\varepsilon}(v_j)\|_{\mathrm{L}^\mu(K;\mathbb{R}^{n\times n}_{\mathrm{sym}})} < \infty$ and thus we deduce the claim by arbitrariness of the suitably small chosen paramter $\varepsilon > 0$. The proof is complete. □

5.7.2. Uniqueness of Generalised Minimisers Building on the results of the previous sections, particularly on the proof of the higher Sobolev regularity of generalised minimisers, we now briefly comment on the uniqueness issues addressed in the introduction of this chapter. In general, the failure of uniqueness of minima of variational integrals (5.1) is mostly due to two reasons (compare [34]): Going back to the relaxed functional $\widetilde{\mathfrak{F}}_{u_0}$ given by (4.23), positive homogeneity of f^∞ implies that f^∞ is not strictly convex even if f is. Thus a possible reason for non–uniqueness is the presence of the singular part of minimisers which genuinely only effects the recession parts of $\widetilde{\mathfrak{F}}_{u_0}$. The second reason for non–uniqueness is a possible non–attainment of the correct boundary values which shall be briefly addressed in the end of this section.

These issues clearly do not arise within the class LD, a simple fact which we briefly record now. Also note that we still cannot expect full uniqueness due to the Santi and Finn examples, see Chapter 4.4.

Lemma 5.47. *Let Ω be an open, bounded and connected Lipschitz subset of $\mathbb{R}^n$. In the situation of Theorem 5.6, suppose that $\mathrm{GM}(\mathfrak{F};u_0) \subset \mathrm{LD}(\Omega;\mathbb{R}^n)$. Then for all $u,v \in \mathrm{GM}(\mathfrak{F};u_0)$ there exists a rigid deformation $R \in \mathcal{R}(\Omega)$ such that $u = v + R$ holds $\mathscr{L}^n$–a.e. in Ω.*

Proof. Let $u,v \in \mathrm{GM}(\mathfrak{F};u_0) \subset \mathrm{LD}(\Omega)$ be two generalised minimisers with respect to the prescribed Dirichlet class $\mathcal{D}_{u_0} := u_0 + \mathrm{LD}_0(\Omega)$. We will show $\boldsymbol{\varepsilon}(u) = \boldsymbol{\varepsilon}(v)$, and this will imply the claim: Indeed, since $\boldsymbol{\varepsilon}(w) = 0$ is equivalent to $w \in \mathcal{R}(\Omega)$ provided Ω is connected, we deduce that there exists $R \in \mathcal{R}(\Omega)$ such that $u = v + R$. To prove the claim, suppose that $\boldsymbol{\varepsilon}(u) \neq \boldsymbol{\varepsilon}(v)$ on a measurable set $U \subset \Omega$ with $\mathscr{L}^n(U) > 0$. Then we obtain, using that f is strictly convex, f^∞ is convex and both $\mathrm{E}^s u$ and $\mathrm{E}^s v$ vanish identically in Ω,

$$\begin{aligned}\widetilde{\mathfrak{F}}_{u_0}\left[\tfrac{1}{2}(u+v)\right] &= \int_\Omega f(\mathscr{E}(\tfrac{1}{2}(u+v)))\,\mathrm{d}x + \int_{\partial\Omega} f^\infty\big((u_0 - \tfrac{1}{2}(u+v)) \odot \nu_{\partial\Omega}\big)\,\mathrm{d}\mathcal{H}^{n-1} \\ &< \frac{1}{2}\int_\Omega f(\mathscr{E}u)\,\mathrm{d}x + \frac{1}{2}\int_\Omega f(\mathscr{E}v)\,\mathrm{d}x \qquad \text{(by strict convexity of } f) \\ &+ \frac{1}{2}\int_{\partial\Omega} f^\infty\big((u_0-u)\odot\nu_{\partial\Omega}\big)\,\mathrm{d}\mathcal{H}^{n-1} + \frac{1}{2}\int_{\partial\Omega} f^\infty\big((u_0-v)\odot\nu_{\partial\Omega}\big)\,\mathrm{d}\mathcal{H}^{n-1} \\ &< \frac{1}{2}(\widetilde{\mathfrak{F}}_{u_0}[u] + \widetilde{\mathfrak{F}}_{u_0}[v]) = \min \widetilde{\mathfrak{F}}_{u_0}[\mathrm{BD}(\Omega)],\end{aligned}$$

which is an obvious contradiction. The proof is complete. □

We now come to the

Proof of Theorem 5.7. Let $R \in \mathcal{R}(\Omega) \setminus \{0\}$ be an arbitrary non–zero rigid deformation and denote $\overline{R}$ its continuous extension to $\overline{\Omega}$. Then we have

$$\widetilde{\mathfrak{F}}[u+R] = \overline{\mathfrak{F}}[u] + \int_{\partial\Omega} f^{\infty}\left(-\overline{R} \odot \nu_{\partial\Omega}\right) \mathrm{d}\mathcal{H}^{n-1} \tag{5.130}$$

because $\mathrm{Tr}(u-u_0) = 0$ $\mathcal{H}^{n-1}$–a.e. on $\partial\Omega$. Since the mapping $T\colon \partial\Omega \to \mathbb{R}^{n\times n}_{\mathrm{sym}}$ given by $T(x) := -\overline{R} \odot \nu_{\partial\Omega}$ for $x \in \partial\Omega$ is continuous and $f^{\infty}\colon \mathbb{R}^{n\times n}_{\mathrm{sym}} \to \mathbb{R}_{\geqslant 0}$ is continuous, too, it suffices to show that there exists $z \in \partial\Omega$ such that $|\overline{R}(z) \odot \nu_{\partial\Omega}(z)| > 0$. Indeed, in this case we conclude by homogeneity of f^{∞} and positivity of f that the boundary integral on the right side of (5.130) is strictly positive so that $u+R$ is not a minimiser of $\widetilde{\mathfrak{F}}_{u_0}$ over $\mathrm{BD}(\Omega)$. The proof is the concluded by Proposition 4.26 which provides the required characterisation of generalised minima in terms of $\widetilde{\mathfrak{F}}_{u_0}$. For simplicity, we shall argue for the unit ball $\Omega = \mathrm{B}$ and only sketch the respective generalisation to arbitrary open, bounded and convex Lipschitz domains Ω below. Write $\overline{R}(z) = Az + b$. If $|\overline{R}(z) \odot \nu_{\partial \mathrm{B}}(z)| = 0$ for all $z \in \partial\mathrm{B}$, then $Az \odot \nu_{\partial\mathrm{B}}(z) = -b \odot \nu_{\partial\mathrm{B}}(z)$ for all $z \in \partial\mathrm{B}$. Since $\nu_{\partial\mathrm{B}}(z) = z$ for any $z \in \partial\mathrm{B}$, this particularly implies $Ae_k \odot e_k = -b \odot e_k$ for all $k = 1, ..., n$. These identities imply

$$Ae_k \odot e_k = \begin{pmatrix} 0 & \dots & a_{1k} & 0 & \dots \\ 0 & \dots & \vdots & 0 & \dots \\ a_{1k} & \dots & a_{kk} & \dots & a_{nk} \\ 0 & \dots & \vdots & 0 & \dots \\ 0 & \dots & a_{nk} & 0 & \dots \end{pmatrix} = -\begin{pmatrix} 0 & \dots & b_1 & 0 & \dots \\ 0 & \dots & \vdots & 0 & \dots \\ b_1 & \dots & b_k & \dots & b_n \\ 0 & \dots & \vdots & 0 & \dots \\ 0 & \dots & b_n & 0 & \dots \end{pmatrix} = -b \odot e_k.$$

and hence $a_{jk} = -b_j$ for all $j, k = 1, ..., n$. In consequence, $a_{jj} = -b_j$ for all $j = 1, ..., n$, but by scew–symmetry of A, $a_{jj} = 0$ and thus $b_j = 0$ for all $j = 1, ..., n$. This further implies $a_{jk} = 0$ for all $j, k = 1, ..., n$ and thus $\overline{R} \equiv 0$. If Ω is not a ball, then one may argue similarly, now using the fact that for any open, bounded and convex Lipschitz subset Ω of $\mathbb{R}^n$ there exist linearly independent $z_1, ..., z_n \in \partial\Omega$ such that $\nu_{\partial\Omega}(z_1), ..., \nu_{\partial\Omega}(z_n)$ are linearly independent, too. The proof is complete. □

Lastly, we briefly turn to the second possible source of non–uniqueness which is in the spirit of Santi's example [206], cp. Chapter 4.4. This has been revisited by Beck & Schmidt [34, Thm. 1.16]. Whereas the former deals with scalar problems exclusively, the latter primarily deals with the vectorial case. As a consequence, even in the case where the symmetric gradient operator $\boldsymbol{\varepsilon}$ is replaced by the full gradient in (5.1), non–uniqueness of generalised minima implies non–attainment of the correct boundary values within the framework of the variational principle (5.1) with $\boldsymbol{\varepsilon}$ replaced by D, and we therefore cannot expect uniqueness in general.

5.8. Variations of the Theme: 1–Ellipticity and p-growth problems

In the final part of this chapter we conclude with several generalisations of the topics discussed above. Among others, its aim is to demonstrate the extent to which the linear growth regime represents a borderline case.

5.8.1. 1–elliptic integrands As explained in Section 5.2, 1–elliptic integrands $f\colon X \to \mathbb{R}$ give rise to integrands of $L\log L$–growth; here, X is either the $\mathbb{R}^{N\times n}$– or the $\mathbb{R}^{n\times n}_{\mathrm{sym}}$–matrices. Specifically, in this situation there exist $0 < c_1 \leqslant c_2 < \infty$ such that f satisfies

$$c_1|\mathbf{Z}|\log(\mathrm{e}+|\mathbf{Z}|) \leqslant f(\mathbf{Z}) \leqslant c_2(1+|\mathbf{Z}|)\log(1+\mathrm{e}+|\mathbf{Z}|) \qquad \text{for all } \mathbf{Z} \in X. \tag{5.131}$$

In slight more generality, the situation with the second derivatives f'' satisfying

$$\lambda\frac{|\xi|^2}{1+|\eta|}\leqslant\langle f''(\eta)\xi,\xi\rangle\leqslant\Lambda\frac{\log(1+|\eta|)}{1+|\eta|}|\xi|^2,\qquad \xi,\eta\in X \tag{5.132}$$

for two fixed constants $0<\lambda\leqslant\Lambda<\infty$ has been studied in depth by SEREGIN and collaborators [116, 117, 113] (also see the extensive monograph [115]) as well as by MINGIONE and collaborators [119, 174].

Let us briefly report on this new setting, with particular emphasis on the symmetric gradient case. We recall from Chapter 3.2 that, since $t\mapsto|t|\log(1+|t|)$ is not of class ∇_2, there is *no* Korn–inequality in $L\log L$ in the sense that for any open and bounded set $\Omega\subset\mathbb{R}^n$ there exists a sequence $(v_k)\subset \mathrm{C}_c^\infty(\Omega;\mathbb{R}^n)$ such that

$$\sup_{k\in\mathbb{N}}\int_\Omega|\boldsymbol{\varepsilon}(v_k)|\log(1+|\boldsymbol{\varepsilon}(v_k)|)\,\mathrm{d}x<\infty\quad\text{but}\quad\limsup_{k\to\infty}\int_\Omega|\,\mathrm{D}v_k|\log(1+|\,\mathrm{D}v_k|)\,\mathrm{d}y=\infty.$$

However, still a uniform L^1–bound can be established for the full gradient sequence $(\mathrm{D}v_k)$. This is basically a consequence of STEIN's theorem (cp. Section 2.6) or, equivalently, the YANO extrapolation theorem[8], cp. [246, Thm. XII.4.14]. Regarding the existence of the variational principle

$$\text{to minimise}\quad\mathfrak{F}[v]:=\int_\Omega f(\boldsymbol{\varepsilon}(v))\,\mathrm{d}x\qquad\text{over }\mathcal{D}_{u_0}:=u_0+\mathrm{W}_0^{\boldsymbol{\varepsilon},L\log L}(\Omega;\mathbb{R}^n) \tag{5.133}$$

subject to (5.131) and (5.132), where u_0 is assumed to belong to the Sobolev–Orlicz space $\mathrm{W}^{\boldsymbol{\varepsilon},L\log L}(\Omega;\mathbb{R}^n)$ and $\mathrm{W}_0^{\boldsymbol{\varepsilon},L\log L}(\Omega;\mathbb{R}^n)$ is the closure of $\mathrm{C}_c^\infty(\Omega;\mathbb{R}^n)$ with respect to the $\mathrm{W}^{\boldsymbol{\varepsilon},L\log L}$–norm, cp. Chapter 2.5.6, we then have the following results. First, by the De Vallee–Poussin criterion (e.g., cp. [117, Lem. 2.2]), we have that if (u_k) is a minimising sequence for $\mathfrak{F}$ (and hence, by (5.131), has symmetric gradients uniformly bounded in $L\log L(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})$), then there exists a subsequence $(u_{k(j)})\subset(u_k)$ and some $U\in L\log L(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})$ such that $\mathrm{D}u_k\rightharpoonup U$ in $\mathrm{L}^1(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})$ as $k\to\infty$. It then can be shown that $U=\mathrm{D}u$ for some $u\in\mathrm{W}^{1,1}(\Omega;\mathbb{R}^n)$ and so there is no need to either pass to BV or even BD. Working on the ellipticity assumption (5.132), it can be shown by straightforward adaptation of the arguments of [117] that a suitable minimising sequence (v_k) satisfies

$$\sup_{k\in\mathbb{N}}\int_K|\,\mathrm{D}\sqrt{1+|\boldsymbol{\varepsilon}(u_k)|}|^2\,\mathrm{d}x<\infty\qquad\text{for all Lipschitz subsets }K\Subset\Omega.$$

This especially implies that if $n=2$, then $\sqrt{1+|\boldsymbol{\varepsilon}(u_k)|}$ is uniformly bounded in $\mathrm{BMO}(K)$ for any Lipschitz subset $K\Subset\Omega$ and hence, by the John–Nirenberg inequality, we have uniform boundedness of $\boldsymbol{\varepsilon}(u_k)$ in $\mathrm{L}^p(K;\mathbb{R}^{n\times n}_{\mathrm{sym}})$ for all $1\leqslant p<\infty$ for all such sets K. Using the relevant Sobolev embeddings $\mathrm{W}^{1,2}\hookrightarrow\mathrm{L}^{2n/(n-2)}$ for $n\geqslant 3$, suitably modified statements can be established. In case we have a p–power growth condition at our disposal, the Korn–inequality indeed applies and leads to the results which we shall briefly sketch in the next subsection.

5.8.2. The case $1<p<\infty$ To round off the theme of this chapter, let us finally explain how the situation simplifies in the case when the integrand $f\in\mathrm{C}^2(\mathbb{R}^{n\times n}_{\mathrm{sym}};\mathbb{R}_{\geqslant 0})$ is of p–growth, $1<p<2$. The case $p\geqslant 2$ is similar but easier from a technical perspective. More precisely, we shall work with the following set of assumptions on f:

[8] The YANO extrapolation theorem states the following: If (Ω,μ) is a finite measure space and for all $1<p<p+\varepsilon$ for some $\varepsilon>0$ a linear operator T satisfies $\|T\|_{\mathrm{L}^p\to\mathrm{L}^p}\leqslant C(p-1)^{-\alpha}$ (where $C>0$ and $\alpha>0$ are constants independent of p), then T extends to a bounded linear operator $T\colon L\log^\alpha L(\Omega)\to\mathrm{L}^1(\Omega)$. The result remains valid for quasilinear T, too, cp. [95].

(a) There exist $0 < c_1 \leqslant c_3 < \infty$ and $c_2, c_4 \in \mathbb{R}$ such that

$$c_1|z|^p - c_2 \leqslant f(z) \leqslant c_3|z|^p + c_4 \qquad \text{for all } z \in \mathbb{R}^{n\times n}_{\mathrm{sym}}.$$

(b) There exists $0 < c_5 < \infty$ such that

$$|f'(z)| \leqslant c_5(1+|z|^2)^{\frac{p-1}{2}} \qquad \text{for all } z \in \mathbb{R}^{n\times n}_{\mathrm{sym}}.$$

(c) There exist $0 < c_6 \leqslant c_7 < \infty$ such that

$$c_6\frac{|z|^2}{(1+|\xi|^2)^{\frac{2-p}{2}}} \leqslant \langle f''(\xi)z, z\rangle \leqslant c_7\frac{|z|^2}{(1+|\xi|^2)^{\frac{2-p}{2}}} \qquad \text{for all } z,\xi \in \mathbb{R}^{n\times n}_{\mathrm{sym}}.$$

We then have the following result which follows in a standard way, e.g., from [192].

Theorem 5.48. *Let $f \in \mathrm{C}^2(\mathbb{R}^{n\times n}_{\mathrm{sym}})$ be a symmetric–convex integrand which verifies the conditions (a)–(c) specified above for some $1 < p < 2$ and let $u_0 \in \mathrm{W}^{1,p}(\Omega;\mathbb{R}^n)$ be a given Dirichlet datum. Then the unique minimiser $u \in \mathcal{D}_{u_0} := u_0 + \mathrm{W}^{1,p}_0(\Omega;\mathbb{R}^n)$ of the variational integral*

$$\mathfrak{F}[v] := \int_\Omega f(\boldsymbol{\varepsilon}(v))\,\mathrm{d}x \qquad \text{over } \mathcal{D}_{u_0}$$

satisfies $u \in \mathrm{W}^{2,p}_{\mathrm{loc}}(\Omega;\mathbb{R}^n)$.

Proof. The proof is a minor modification of the strategy outlined in Chapter 5.7. Let $x_0 \in \Omega$ be arbitrary and choose $0 < r < \mathrm{dist}(x_0, \partial\Omega)$ such that $\mathrm{B}(x_0, 2r) \Subset \Omega$. We pick an arbitrary cut–off function $\rho \in \mathrm{C}^\infty_c(\mathrm{B}(x_0, r); [0,1])$ with $\rho|_{\mathrm{B}(x_0,r/2)} = 1$. Let $s \in \{1, ..., n\}$ and $h \in \mathbb{R}$ with $|h| < \frac{1}{2}r$ be arbitrary but fixed. We put $\varphi := \Delta^-_{s,h}(\rho^2\Delta_{s,h}u) \in \mathrm{W}^{1,p}_0(\Omega;\mathbb{R}^n)$ which consequently qualifies as an admissible test in the Euler–Lagrange equation for the minimiser u. Here, $\Delta^\pm_{s,h}u(x) := \frac{1}{h}(u(x \pm he_s) - u(x))$ are the *full* forward– or backwards difference quotients in direction e_s. Following the same scheme as in Chapter 5.7, it is then readily verified that

$$\begin{aligned}\mathbf{I} &:= \int_{\mathrm{B}(x_0,r)} \rho^2 \frac{|\Delta_{s,h}\boldsymbol{\varepsilon}(u)(x)|^2}{(1+|\boldsymbol{\varepsilon}(u)(x)+\boldsymbol{\varepsilon}(u)(x+he_s)|^2)^{\frac{2-p}{2}}}\,\mathrm{d}x \\ &\leqslant \int_{\mathrm{B}(x_0,r)} |f'(\boldsymbol{\varepsilon}(u))|\,|\Delta^-_{s,h}(2\rho\,\mathrm{D}\rho \odot \Delta_{s,h}u)|\,\mathrm{d}x = \mathbf{II}.\end{aligned}$$

We estimate by use of Young's Inequality for $\varepsilon > 0$

$$\mathbf{II} \leqslant C(\varepsilon)\int_\Omega |f'(\boldsymbol{\varepsilon}(u))|^{\frac{p}{p-1}}\,\mathrm{d}x + c\varepsilon\int_\Omega |\Delta^-_{s,h}(2\rho\,\mathrm{D}\rho \odot \Delta_{s,h}u)|^p\,\mathrm{d}x.$$

We note that $2\rho\,\mathrm{D}\rho \odot \Delta_{s,h}u \in \mathrm{W}^{1,p}(\mathrm{B}(x_0,r);\mathbb{R}^n)$. Hence, using Korn's inequality, we obtain with a constant $C > 0$

$$\begin{aligned}
\int_\Omega |\Delta_{s,h}^-(2\rho\,\mathrm{D}\rho\odot\Delta_{s,h}u)|^p\,\mathrm{d}x &\leqslant C\int_{\mathrm{B}(x_0,2r)}|\partial_s(2\,\mathrm{D}\rho\odot\rho\Delta_{s,h}u)|^p\,\mathrm{d}x\\
&\leqslant C\int_{\mathrm{B}(x_0,2r)}|\partial_s\,\mathrm{D}\rho|^p|\rho\Delta_{s,h}u|^p+|\,\mathrm{D}\rho|^p|\partial_s(\rho\Delta_{s,h}u)|^p\,\mathrm{d}x\\
&\leqslant C\int_{\mathrm{B}(x_0,2r)}|\partial_s\,\mathrm{D}\rho|^p|\rho\Delta_{s,h}u|^p+|\boldsymbol{\varepsilon}(\rho\Delta_{s,h}u)|^p\,\mathrm{d}x\\
&\leqslant C\int_{\mathrm{B}(x_0,2r)}|\partial_s\,\mathrm{D}\rho|^p|\rho\Delta_{s,h}u|^p\,\mathrm{d}x\\
&+\int_{\mathrm{B}(x_0,2r)}|\,\mathrm{D}\rho|^p|\rho\Delta_{s,h}\boldsymbol{\varepsilon}(u)+\mathrm{D}\rho\odot\Delta_{s,h}u|^p\,\mathrm{d}x\\
&= C\int_{\mathrm{B}(x_0,2r)}|\partial_s\,\mathrm{D}\rho|^p|\rho\Delta_{s,h}u|^p\,\mathrm{d}x\\
&+C\int_{\mathrm{B}(x_0,2r)}|\rho\Delta_{s,h}\boldsymbol{\varepsilon}(u)+\mathrm{D}\rho\odot\Delta_{s,h}u|^p\,\mathrm{d}x\\
&\leqslant C\int_{\mathrm{B}(x_0,2r)}|\Delta_{s,h}u|^p\,\mathrm{d}x+C\int_{\mathrm{B}(x_0,2r)}|\rho\Delta_{s,h}\boldsymbol{\varepsilon}(u)|^p\,\mathrm{d}x.
\end{aligned}$$

As $1 < p < 2$, we further have by Young's inequality applied to the dual exponents $q = \frac{2}{p} > 1$ and hereafter $q' = \frac{2}{2-p}$ with some $\delta > 0$ to fixed later,

$$\begin{aligned}
\int_{\mathrm{B}(x_0,r)}|\rho\Delta_{s,h}\boldsymbol{\varepsilon}(u)|^p\,\mathrm{d}x &\leqslant \delta\int_{\mathrm{B}(x_0,r)}\rho^2\frac{|\Delta_{s,h}\boldsymbol{\varepsilon}(u)(x)|^2}{(1+|\boldsymbol{\varepsilon}(u)(x)+\boldsymbol{\varepsilon}(u)(x+he_s)|^2)^{\frac{2-p}{2}}}\,\mathrm{d}x\\
&+C(\delta)\int_{\mathrm{B}(x_0,r)}(1+|\boldsymbol{\varepsilon}(u)(x)+\boldsymbol{\varepsilon}(u)(x+he_s)|^2)^{\frac{p}{2}}\,\mathrm{d}x\\
&\leqslant \delta\mathbf{I}+C(\delta),
\end{aligned}$$

as $\boldsymbol{\varepsilon}(u)\in\mathrm{L}^p(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})$. On the other hand, by (H2) and $u\in\mathrm{W}^{1,p}(\Omega;\mathbb{R}^n)$ we have

$$\begin{aligned}
\int_{\mathrm{B}(x_0,r)}|f'(\boldsymbol{\varepsilon}(u))|^{\frac{p}{p-1}} &\leqslant C\int_{\mathrm{B}(x_0,r)}(1+|\boldsymbol{\varepsilon}(u)|^2)^{(p-1)\frac{p}{2(p-1)}}\,\mathrm{d}x\\
&\leqslant C\int_{\mathrm{B}(x_0,r)}(1+|\boldsymbol{\varepsilon}(u)|^2)^{\frac{p}{2}}\,\mathrm{d}x\leqslant C.
\end{aligned}$$

Combining the two estimates, we have established

$$\mathbf{II}\leqslant c(\delta)\mathbf{I}+C(\delta),$$

where $c(\delta)\searrow 0$ and $C(\delta)\nearrow\infty$ as $\delta\to 0$. We now choose $\delta>0$ so small such that $c(\delta)\mathbf{I}$ can be absorbed into $\mathbf{I}$ and fix this choice of δ. We then obtain

$$\int_{\mathrm{B}(x_0,r)}\rho^2\frac{|\Delta_{s,h}\boldsymbol{\varepsilon}(u)(x)|^2}{(1+|\boldsymbol{\varepsilon}(u)(x)+\boldsymbol{\varepsilon}(u)(x+he_s)|^2)^{\frac{2-p}{2}}}\,\mathrm{d}x=\mathbf{I}\leqslant C,$$

where now $C>0$ does not depend on s,h or u. Similarly as above, we now use Young's inequality applied to the dual exponents $q=\frac{2}{p}$ and $q'=\frac{2}{2-p}$ to obtain

$$\begin{aligned}
\int_{\mathrm{B}(x_0,r/2)}|\Delta_{s,h}\boldsymbol{\varepsilon}(u)|^p\,\mathrm{d}x &= \int_{\mathrm{B}(x_0,r/2)}\frac{|\Delta_{s,h}\boldsymbol{\varepsilon}(u)|^p(1+|\boldsymbol{\varepsilon}(u)(x+h)|^2+|\boldsymbol{\varepsilon}(u)(x)|^2)^{\frac{(2-p)p}{4}}}{(1+|\boldsymbol{\varepsilon}(u)(x+he_s)|^2+|\boldsymbol{\varepsilon}(u)(x)|^2)^{\frac{(2-p)p}{4}}}\,\mathrm{d}x\\
&\leqslant C\int_{\mathrm{B}(x_0,r/2)}\frac{|\Delta_{s,h}\boldsymbol{\varepsilon}(u)|^p}{(1+|\boldsymbol{\varepsilon}(u)(x+he_s)|^2+|\boldsymbol{\varepsilon}(u)(x)|^2)^{\frac{2-p}{2}}}\,\mathrm{d}x\\
&+C\int_{\mathrm{B}(x_0,r/2)}(1+|\boldsymbol{\varepsilon}(u)(x+he_s)|^2+|\boldsymbol{\varepsilon}(u)(x)|^2)^{\frac{p}{2}}\,\mathrm{d}x\leqslant C,
\end{aligned}$$

by the above and $u \in \mathrm{W}^{1,p}(\Omega;\mathbb{R}^n)$. In conclusion, by passing $|h| \searrow 0$ and by arbitrariness of $s \in \{1, ..., n\}$, we obtain that $\boldsymbol{\varepsilon}(u) \in \mathrm{W}^{1,p}_{\mathrm{loc}}(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})$ and hence, by Korn's inequality, $u \in \mathrm{W}^{2,p}_{\mathrm{loc}}(\Omega;\mathbb{R}^n)$. The proof is complete. $\square$

We conclude this chapter with a remark on slightly more general growth conditions.

Remark 5.49. *It is possible to obtain similar results in the context of Orlicz function which satisfy the $\Delta_2 \cap \nabla_2$–condition so that, following the results of Chapter 3.2, a suitable Korn–type inequality is available. In this situation, it is convenient to work in the setting of* DIENING & ETTWEIN *[78], where the corresponding suitable generalisations of the ellipticity of f'' and the growth of f' are stated. However, for sake of clarity, we confine ourselves to the p–growth regime.*

CHAPTER 6

Interlude: $\mathrm{W}^{1,1}$–Regularity for the Neumann Problem on BV

Contrary to the previous chapter, where we considered functionals of linear growth which depend on the symmetric gradients, we now return to full gradient functionals and hence work with $\mathrm{W}^{1,1}$ and BV instead of LD and BD. However, unlike the situation as explained in Section 5.1.1, we do not consider the *Dirichlet* but the *Neumann* problem and ask when $\mathrm{W}^{1,1}$–regularity of generalised minima is to expected. Let us note that this is a qualitative but no *quantitive* regularity problem: Indeed, here we do not ask whether a generalised minimiser belongs to $\mathrm{W}^{1,p}_{\mathrm{loc}}$ for some $p > 1$ depending on f but whether the singular part $\mathrm{D}^s u$ vanishes. Recalling Table 5.1, in the Dirichlet case this is solved up to now only for the scale of μ–elliptic integrands with $\mu \leqslant 3$, and here the $\mathrm{W}^{1,1}$–regularity always comes with a higher integrability result. For the Neumann problem (i.e., variational problems with *natural boundary values*) as sketched in this chapter, we can even leave the restrictive scale of μ–elliptic variational integrands and shall prove that in presence of strictly convex variational integrands of radially symmetric type, $\mathrm{W}^{1,1}$–regularity of generalised minima follows.

Structure of the chapter. After explaining the setup and examining coerciveness conditions in Section 6.1–6.3, we collect tools on identifying a certain biting limit in Section 6.4. We then proceed to the proof of the main result in Section 6.5 and consequently study the dual problem to some extent in Section 6.6. Finally, we comment on some explicit higher integrabilty results in the μ–elliptic situation and explain the difficulties in adapting our approach to the BD–situation in Sections 6.7 and 6.8.

6.1. Setup

Let $\Omega \subset \mathbb{R}^n$ be a bounded, simply connected Lipschitz domain and suppose that $f \in \mathrm{C}^1(\mathbb{R}^+_0)$ is a strictly convex function which satisfies $f(0) = f'(0) = 0$ and which is of linear growth, i.e., there exist two constants $0 < c \leqslant C < \infty$ such that

$$cs - C \leqslant f(s) \leqslant C(s+1) \qquad \text{for all } s \in \mathbb{R}^+. \tag{6.1}$$

Given a map $T_0 \in \mathrm{W}^{1,\infty}(\Omega; \mathbb{R}^{N\times n})$ for some $N \geqslant 1$, in the present chapter we study existence and regularity properties of weak solutions of the system

$$\operatorname{div}\left(\frac{f'(|\nabla u|)\nabla u}{|\nabla u|}\right) = \operatorname{div}(T_0) \qquad \text{in } \Omega \tag{6.2}$$

subject to the *Neumann-type* boundary value condition

$$\frac{f'(|\nabla u|)\nabla u}{|\nabla u|} \cdot \nu_{\partial\Omega} = T_0 \cdot \nu_{\partial\Omega}. \tag{6.3}$$

The last equation needs to be understood as equality in $\mathbb{R}^N$, i.e., in the row–wise sense. In this situation, we have different options to come up with an appropriate concept of weak solutions of (6.2) subject to the boundary condition (6.3). Firstly, supposing for the moment that u belongs to the space $\mathrm{C}^2(\overline{\Omega};\mathbb{R}^N)$, we observe that all expressions are well-defined in the classical sense. Thus, applying the inner product to both sides of (6.2) with a regular test function $\varphi \in \mathrm{C}^1(\overline{\Omega};\mathbb{R}^N)$, integrating over Ω and using the integration by parts formula, we obtain

$$\int_{\partial\Omega}\left\langle \frac{f'(|\nabla u|)\nabla u}{|\nabla u|}\cdot \nu_{\partial\Omega},\varphi\right\rangle \mathrm{d}\mathcal{H}^{n-1} - \int_{\Omega}\left\langle \frac{f'(|\nabla u|)\nabla u}{|\nabla u|},\nabla\varphi\right\rangle \mathrm{d}x = \int_{\partial\Omega}\langle T_0\cdot\nu_{\partial\Omega},\varphi\rangle\,\mathrm{d}\mathcal{H}^{n-1} - \int_{\Omega}\langle T_0,\nabla\varphi\rangle\,\mathrm{d}x,$$

with $\nu_{\partial\Omega}$ denoting the outward pointing unit normal field of the boundary $\partial\Omega$. In view of the Neumann-type constraint (6.3), the boundary terms disappear. Combined with a density argument, this motivates the following definition of a weak solution:

Definition 6.1 (Weak Solution)**.** *Let $T_0 \in \mathrm{W}^{1,\infty}(\Omega;\mathbb{R}^{N\times n})$ and suppose that $f \in \mathrm{C}^1(\mathbb{R}^+_0)$ satisfies $f(0) = f'(0) = 0$ and the linear growth assumption* (6.1)*. We say that a function $u \in \mathrm{W}^{1,1}(\Omega;\mathbb{R}^N)$ is a* weak solution *of the system* (6.2) *subject to the Neumann-type boundary constraint* (6.3) *if*

$$\int_{\Omega}\left\langle \frac{f'(|\nabla u|)\nabla u}{|\nabla u|},\nabla\varphi\right\rangle \mathrm{d}x = \int_{\Omega}\langle T_0,\nabla\varphi\rangle\,\mathrm{d}x$$

holds for all functions $\varphi \in \mathrm{W}^{1,1}(\Omega;\mathbb{R}^N)$.

Alternatively, we may rely on the special structure of the system and interpret it as the Euler–Lagrange system associated to the variational problem

$$\text{to minimise}\quad \mathfrak{F}[w] := \int_{\Omega} f(|\nabla w|) - \langle T_0,\nabla w\rangle\,\mathrm{d}x \quad\text{over}\ \mathrm{W}^{1,1}(\Omega;\mathbb{R}^N). \tag{6.4}$$

Studying variations of a minimiser in a standard way on the one hand and employing the convexity of the integrand f on the other hand, we immediately establish the following connection between (6.2), (6.3) and the variational principle (6.4).

Lemma 6.2. *Let $T_0 \in \mathrm{W}^{1,\infty}(\Omega;\mathbb{R}^{N\times n})$ and suppose that $f \in \mathrm{C}^1(\mathbb{R}^+_0)$ is convex and that it satisfies $f(0) = f'(0) = 0$ and the linear growth assumption* (6.1)*. Then a function $u \in \mathrm{W}^{1,1}(\Omega;\mathbb{R}^N)$ is a weak solution of* (6.2) *subject to the Neumann-type boundary constraint* (6.3) (*in the sense of Definition 6.1*) *if and only if it is a minimiser of the variational problem* (6.4)*.*

Although we shall exclusively study weak solutions in all of what follows, we wish to mention for the sake of completeness that it is possible to deduce validity of (6.2) subject to (6.3) provided that a suitable a priori regularity assumption on the solution is made:

Lemma 6.3. *Let $T_0 \in \mathrm{W}^{1,\infty}(\Omega;\mathbb{R}^{N\times n})$ and suppose that $f \in \mathrm{C}^2(\mathbb{R}^+_0)$ satisfies $f(0) = f'(0) = 0$. If $u \in \mathrm{W}^{2,\infty}(\Omega;\mathbb{R}^N)$ is a minimiser of the variational principle* (6.4)*, then it satisfies* (6.2) *and* (6.3) *in the pointwise sense.*

Proof. In view of $u \in \mathrm{W}^{2,\infty}(\Omega;\mathbb{R}^N)$, we may extend $\mathrm{D}u$ to a Lipschitz function on $\overline{\Omega}$. Given $\delta > 0$ sufficiently small, we choose two functions $\eta^{(i)}_\delta, \eta^{(b)}_\delta \in \mathrm{C}^{2,1}(\mathbb{R}^n;[0,1])$ with $\eta^{(i)}_\delta + \eta^{(b)}_\delta \equiv 1$ in $\overline{\Omega}$ such that $\eta^{(i)}_\delta$ is supported in the set

$$\Omega_\delta := \{x \in \Omega\colon\ \mathrm{dist}(x,\partial\Omega) > \delta\}$$

and such that $\eta_\delta^{(b)}$ satisfies $\eta_\delta^{(b)} \equiv 1$ on $\partial\Omega$ and vanishes outside of $\mathbb{R}^n \setminus \Omega_{2\delta}$. By minimality of u, it satisfies the Euler–Lagrange equation, i.e., for all $\varphi \in \mathrm{W}^{1,1}(\Omega;\mathbb{R}^N)$ we have

$$\int_\Omega \left\langle \frac{f'(|\nabla u|)\nabla u}{|\nabla u|}, \nabla\varphi \right\rangle - \langle T_0, \nabla\varphi\rangle \,\mathrm{d}x = 0, \tag{6.5}$$

which, for $\varphi \in \mathrm{W}_0^{1,1}(\Omega;\mathbb{R}^N)$, implies after the application of the integration by parts formula

$$\int_\Omega \left\langle \operatorname{div}\left(\frac{f'(|\nabla u|)\nabla u}{|\nabla u|}\right), \varphi \right\rangle - \langle \operatorname{div}(T_0), \varphi\rangle \,\mathrm{d}x = 0. \tag{6.6}$$

By arbitrariness of $\varphi \in \mathrm{W}_0^{1,1}(\Omega;\mathbb{R}^N)$, we deduce (6.2) for almost every $x \in \Omega$ by use of the Du Bois–Reymond Lemma. In order to prove the validity of the second identity (6.3), we consider $\varphi \in \mathrm{W}^{1,1}(\Omega;\mathbb{R}^N)$. Then, in view of $\eta_\delta^{(i)}\varphi \in \mathrm{W}_0^{1,1}(\Omega;\mathbb{R}^N)$, (6.5), the integration by parts formula and $\eta_\delta^{(b)} = 0$ on $\partial\Omega_{2\delta}$, we obtain

$$\begin{aligned}
0 &= \int_\Omega \left\langle \frac{f'(|\nabla u|)\nabla u}{|\nabla u|}, \mathrm{D}(\eta_\delta^{(b)}\varphi) \right\rangle - \langle T_0, \mathrm{D}(\eta_\delta^{(b)}\varphi)\rangle \,\mathrm{d}x \\
&= \int_{\partial\Omega} \left\langle \Big(\frac{f'(|\nabla u|)\nabla u}{|\nabla u|} - T_0\Big)\cdot \nu_{\partial\Omega}, \varphi \right\rangle \mathrm{d}\mathcal{H}^{n-1} \\
&\quad - \int_\Omega \left\langle \operatorname{div}\left(\frac{f'(|\nabla u|)\nabla u}{|\nabla u|} - T_0\right), \eta_\delta^{(b)}\varphi \right\rangle \mathrm{d}x.
\end{aligned}$$

Then, by the C^2–regularity of f combined with $\nabla u, T_0 \in \mathrm{W}^{1,\infty}(\Omega;\mathbb{R}^{N\times n})$ and by the convergence $\eta_\delta^{(b)}(x) \to 0$ for all $x \in \Omega$ as $\delta \searrow 0$, Lebesgue's theorem on dominated convergence shows that the second term on right-hand side of the previous equation tends to zero as $\delta \searrow 0$. Hence (6.3) follows again by Du Bois–Reymond's Lemma[1] on $\partial\Omega$. □

Let us further note that the above variational principle (6.4) ignores the addition of constants to competitors. To overcome this issue, we shall additionally require minima $u\colon \Omega \to \mathbb{R}^N$ to satisfy the fixing condition

$$(u)_\Omega := \fint_\Omega u \,\mathrm{d}x := \frac{1}{\mathscr{L}^n(\Omega)} \int_\Omega u \,\mathrm{d}x = 0. \tag{6.7}$$

By the linear growth hypothesis (6.1), we have to expect that members of minimising sequences for $\mathfrak{F}$ might develop concentrations and hence the distributional gradients of minima a priori have to be assumed to be matrix–valued Radon measures. The purpose of the present chapter is to demonstrate that, under some sort of attainability condition imposed on the data T_0 and fairly weak ellipticity assumptions on the variational integrand f, the singular part of the gradients of weak solutions of (6.2) subject to (6.3) – or equivalently of minima of the variational problem (6.4) – do in fact vanish. Thus, in this setting, weak solutions genuinely belong to the space $\mathrm{W}^{1,1}(\Omega;\mathbb{R}^N)$ and the relaxation of the problem to $\mathrm{BV}(\Omega;\mathbb{R}^N)$ is indeed not necessary.

6.1.1. Contextualisation: Potential Theory Due to the specific form of the variational problem (6.4), this task appears in the spirit of some sort of non-linear potential theory for linear growth problems whose connection to perhaps more familiar settings we

[1] This requires testing against all continuous and compactly supported functions $\psi \in \mathrm{C}_c^\infty(\partial\Omega;\mathbb{R}^N)$. Such mappings belong to $\mathrm{L}^1(\partial\Omega;\mathbb{R}^N)$ and thus, by a result due to Gagliardo, can be extended to $\mathrm{W}^{1,1}(\Omega;\mathbb{R}^N)$-maps even though the extension operator is *not* continuous (which, however, does not harm the argument).

shall describe now. The variational problem (6.4) formally leads to the Euler–Lagrange system

$$\operatorname{div}\left(\frac{f'(|\nabla u|)\nabla u}{|\nabla u|}\right) = \operatorname{div}(T_0) \qquad \text{in } \Omega. \tag{6.8}$$

Neglecting for a moment the linear growth assumption (6.1) and setting $f(s) = s^p/p$ for some $p \in (1,\infty)$, (6.8) subject to (6.3) corresponds to the weak formulation of the inhomogeneous p-Laplacean Neumann problem

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) = \operatorname{div}(T_0) \tag{6.9}$$

or, equivalently, the minimization problem (6.4), which is solved by some function $u \in \mathrm{W}^{1,p}(\Omega;\mathbb{R}^N)$, as a consequence of the direct method in the calculus of variations. Here, a typical issue is to transfer regularity properties of the data T_0 to the gradient ∇u of the solution, or more specifically to the function $\mathbb{A}_p(\nabla u) := |\nabla u|^{p-2}\nabla u$, which is adapted to the particular growth properties of the elliptic p-Laplacean system under consideration. For instance, it is known (cp. [83]) that $T_0 \in \mathrm{L}^\infty(\Omega;\mathbb{R}^{N\times n})$ implies $\mathbb{A}_p(\nabla u) \in \mathrm{BMO}(\Omega;\mathbb{R}^{N\times n})$ under some regularity assumptions on the domain Ω. This result is optimal in the sense that in general this cannot be improved to $\mathbb{A}_p(\nabla u) \in \mathrm{L}^\infty(\Omega;\mathbb{R}^{N\times n})$: Even in the simplest linear case $p = 2$ the map $\Phi\colon T_0 \mapsto \nabla u$ is a local singular integral of convolution type which maps $\mathrm{L}^\infty \subset \mathrm{BMO} \to \mathrm{BMO}$.

Now, in our situation of f satisfying the linear growth assumption (6.1) and setting $\mathbb{A}_f(z) := f'(|z|)z/|z|$ for $z \in \mathbb{R}^{N\times n}$, a statement like $\mathbb{A}_f(\nabla u) \in \mathrm{L}^\infty(\Omega;\mathbb{R}^{N\times n})$ would be vacuous: Since f' and thus $\mathbb{A}_f$ is automatically bounded by assumption, we would be able to conclude $\mathbb{A}_f(\mathrm{D}u) \in \mathrm{L}^\infty(\Omega;\mathbb{R}^{N\times n})$ without any further efforts *provided* that $\mathrm{D}u$ would be known to exist as a function. Thus, in this regard, the decisive point is to prove the existence of a solution $u \in \mathrm{W}^{1,1}(\Omega;\mathbb{R}^N)$.

6.1.2. On the Dual Problem Let us further remark that due to the convexity of the variational problem of our interest, we may equally well consider the variational problem dual to the primal *Neumann problem* as was accomplished for the symmetric gradient in Chapter 5.4 within the framework of Dirichlet problems. As we shall discuss in detail below (see Section 6.6), this problem now consists in maximising the associated dual functional given by

$$\mathcal{R}[\chi] := \inf\left\{\int_\Omega \langle \chi, \nabla v\rangle - f^*(|T_0 + \chi|)\,\mathrm{d}x\colon\ v \in \mathrm{W}^{1,1}(\Omega;\mathbb{R}^N)\right\}$$

over a suitable class of competitor maps χ. Similarly as done for the Dirichlet problem on BD in Chapter 5.4.4, it is important to link the maximisers of the dual problem with the minimisers of the original (i.e., primal) problem. In this regard, we recall that the representation of the dual solution σ in terms of the primal solution is strongly influenced by the regularity of the latter. In analogy with Lemma 5.15, we may hence represent the dual solution in terms of the minimiser provided the latter belongs to $\mathrm{W}^{1,1}(\Omega;\mathbb{R}^N)$.

Before we embark on a detailled description of our results, we pause for a moment and discuss the main assumption of a suitable coerciveness condition on the functional $\mathfrak{F}$ which will be imposed throughout the chapter.

6.2. Coerciveness

As a crucial assumption of the chapter, we shall require

$$\|T_0\|_{\mathrm{L}^\infty(\Omega;\mathbb{R}^{N\times n})} < f^\infty(1), \tag{6.10}$$

where $f^\infty(1)$ is defined as the limit $\lim_{s\to\infty} f(s)/s$. As by convexity of f, the function $s \mapsto f(s)/s$ is non-decreasing, the limit exists, and in view of (6.1) is indeed finite and strictly positive, with $f^\infty(1) \geqslant f(s)/s$ for all $s \in \mathbb{R}^+$. The significance of this assumption becomes transparent if one studies the coercivity (or its failure) of the functional $\mathfrak{F}$ in the class $\mathrm{W}^{1,1}(\Omega;\mathbb{R}^N)$ with vanishing mean value on Ω. In fact, if $T_0 \in \mathrm{L}^\infty(\Omega;\mathbb{R}^{N\times n})$ satisfies (6.10), then we can first determine R_0 depending only on f and T_0 such that

$$\frac{f(s)}{s} \geqslant \frac{f^\infty(1) + \|T_0\|_{\mathrm{L}^\infty(\Omega;\mathbb{R}^{N\times n})}}{2} \qquad \text{for } s \geqslant R_0$$

and then compute, for an arbitrary $w \in \mathrm{W}^{1,1}(\Omega;\mathbb{R}^N)$, that

$$\begin{aligned}
\mathfrak{F}[w] &= \int_\Omega f(|\nabla w|) - \langle T_0, \nabla w\rangle \,\mathrm{d}x \\
&\geqslant \int_{\{x\in\Omega:\ |\nabla w(x)|\geqslant R_0\}} \Big[\frac{f^\infty(1) + \|T_0\|_{\mathrm{L}^\infty(\Omega;\mathbb{R}^{N\times n})}}{2} - \|T_0\|_{\mathrm{L}^\infty(\Omega;\mathbb{R}^{N\times n})}\Big]|\nabla w| \,\mathrm{d}x \\
&\quad - \int_{\{x\in\Omega:\ |\nabla w(x)|< R_0\}} \|T_0\|_{\mathrm{L}^\infty(\Omega;\mathbb{R}^{N\times n})}|\nabla w| \,\mathrm{d}x \\
&\geqslant \frac{f^\infty(1) - \|T_0\|_{\mathrm{L}^\infty(\Omega;\mathbb{R}^{N\times n})}}{2}\|\nabla w\|_{\mathrm{L}^1(\Omega;\mathbb{R}^{N\times n})} - f^\infty(1)|\Omega|R_0.
\end{aligned}$$

Condition (6.10) thus is an instrumental ingredient to establish the existence of minima, and to further stress its necessity, we wish to supply the following two examples which demonstrate that in absence of (6.10), minima do not need to exist at all. Here, we consequently stick to the scalar case $N = n = 1$.

Example 6.4 (Non-Existence of minima if $\|T_0\|_{\mathrm{L}^\infty(\Omega;\mathbb{R}^{N\times n})} = f^\infty(1)$)**.** *We consider the shifted area-integrand*

$$f(t) := \sqrt{1+|t|^2} - 1 \qquad \textit{for } t \in \mathbb{R}$$

(*which verifies* (6.1) *with* $c = C = 1$), $T_0 \equiv 1$ *and* $\Omega := (-1,1)$. *In this situation, we have* $f^\infty(1) = 1$ *and the functional* $\mathfrak{F}$ *becomes*

$$\mathfrak{F}[v] = \int_{-1}^1 \sqrt{1+|v'|^2} - v' \,\mathrm{d}x - 2, \qquad \textit{for } v \in \mathrm{W}^{1,1}((-1,1)).$$

Furthermore, since $\sqrt{1+|t|^2} \geqslant t$ *for all* $t \in \mathbb{R}$, *we have* $\inf_{\mathrm{W}^{1,1}((-1,1))} \mathfrak{F} \geqslant -2$. *We then define a sequence* (u_k) *of functions in* $\mathrm{W}^{1,1}((-1,1))$ *with vanishing mean value on* $(-1,1)$ *by setting* $u_k(x) := kx$ *for* $k \in \mathbb{N}$. *Inserting* u_k *into* $\mathfrak{F}$ *yields*

$$\mathfrak{F}[u_k] = 2(\sqrt{1+k^2} - k - 1) \to -2 \quad \textit{as } k \to \infty$$

so that $\inf_{\mathrm{W}^{1,1}((-1,1))} \mathfrak{F} = -2$ *indeed. Assuming that a minimiser* $u \in \mathrm{W}^{1,1}((-1,1))$ *of* $\mathfrak{F}$ *exists, we deduce, by positivity of the integrand, that* $1 + |v'|^2 = |v'|^2$ *holds* $\mathscr{L}^1$*-a.e., a contradiction. Therefore, no minimizer of* $\mathfrak{F}$ *exists in* $\mathrm{W}^{1,1}((-1,1))$.

Example 6.5 (Unboundedness of $\mathfrak{F}$ from below if $\|T_0\|_{\mathrm{L}^\infty(\Omega;\mathbb{R}^{N\times n})} > f^\infty(1)$)**.** *In the setting of the previous example, we consider* $T_0 \equiv c$ *for a constant* $c > 1$. *For the same choice of the sequence* $(u_k)_{k\in\mathbb{N}}$, *we then obtain*

$$\mathfrak{F}[u_k] = 2(\sqrt{1+k^2} - ck - 1) \to -\infty, \qquad \textit{as } k \to \infty$$

which in conclusion shows $\inf_{\mathrm{W}^{1,1}((-1,1))} \mathfrak{F} = -\infty$.

In principle, the reasoning employed in Example 6.4 does not genuinely rule out the non-existence of minima for the so-called *relaxed problem*, i.e., the minimisation of a suitable extension of $\mathfrak{F}$ to $\mathrm{BV}(\Omega;\mathbb{R}^N)$. However, even for the relaxed problem the assumption (6.10) turns out to be necessary for minima to exist, see Example 6.25.

Remark 6.6. *Under the assumption (6.10) we can rewrite $T_0 \in \mathrm{W}^{1,\infty}(\Omega;\mathbb{R}^{N\times n})$ as*

$$T_0 = \frac{f'(|S_0|)S_0}{|S_0|} \quad \text{for } S_0 \text{ given as } S_0 := \frac{T_0}{|T_0|}(f')^{-1}(|T_0|).$$

Since f' is strictly increasing with values in $[0, f^\infty(1))$ (thus, invertible on this set), the map S_0 is well-defined. With this identification, assumption (6.10) guarantees that $\operatorname{div} T_0$ on the left-hand side of the system (6.2) is of the same structure as its left-hand side involving the unknown and thus, in principle, can be attained.

6.3. Main Result and Organisation of the Chapter

We pass now to the description of the main results and the organisation of the present chapter. As we explained above, the main objective here is to establish the $\mathrm{W}^{1,1}$–regularity of minima of (6.4) in the presence of radially symmetric integrands. This objective for the *Neumann* problem needs to be contrasted with the corresponding results for the Dirichlet problem on BV. In this respect, a comprehensive overview of available results has been given in Chapter 5.1.1 (cp. Table 5.1). Recalling the notion of μ–ellipticity (cp. Definition 5.2), let us note that even in the radial symmetric case, the only $\mathrm{W}^{1,1}$–regularity results for the Dirichlet problem on BV with μ–elliptic integrands are available for $1 < \mu \leqslant 3$. In particular, it is still an open problem whether minima might genuinely belong to $\mathrm{BV}\setminus\mathrm{W}^{1,1}$ if $\mu > 3$. The same is easily seen to hold for the Dirichlet problem corresponding to (6.4).

For the Neumann problem at our disposal, we shall see that for the $\mathrm{W}^{1,1}$–regularity it is possible to admit a much weaker condition than that of μ–ellipticity and thereby allow for more general degeneracies than power decay of the second derivatives. In this respect, the instrumental ingredient for deriving higher regularity for BV-minima is the notion of *h–monotonicity*. Let $h\colon \mathbb{R}_0^+ \to \mathbb{R}^+$ be continuous and decreasing. We say that a tensor function $\mathcal{B}\colon \mathbb{R}^{N\times n} \to \mathbb{R}^{(N\times n)\times(N\times n)}$ is *h-monotone* if and only if

$$h(|z|)|\zeta|^2 \leqslant \langle \mathcal{B}(z)\zeta,\zeta\rangle \leqslant L\frac{|\zeta|^2}{1+|z|} \tag{6.11}$$

holds for all $z,\zeta \in \mathbb{R}^{N\times n}$ and some constant $L > 0$. Similarly, we say that a variational integrand $F \in \mathrm{C}^2(\mathbb{R}^{N\times n})$ is *h-monotone* if its Hessian F'' is, and that a function $f \in \mathrm{C}^2(\mathbb{R}^+)$ induces an h-monotone variational integrand if $F(\cdot) := f(|\cdot|)$ is h-monotone. It needs to be stressed that the particular choices $h(t) := (1+|t|^2)^{-\mu/2}$ for $1 < \mu < \infty$ lead to the μ-ellipticity.

Our main result now reads as follows.

Theorem 6.7. *Let Ω be a simply connected, bounded Lipschitz domain in $\mathbb{R}^n$ and let $T_0 \in \mathrm{W}^{2,\infty}(\Omega;\mathbb{R}^{N\times n})$ satisfy (6.10). Consider a convex function $f \in \mathrm{C}^2(\mathbb{R}_0^+)$ which satisfies $f(0) = f'(0) = 0$, the linear growth condition (6.1) and which induces an h-monotone variational integrand. Then the following hold:*

(a) *There exists a weak solution $u \in \mathrm{W}^{1,1}(\Omega;\mathbb{R}^N)$ of the system (6.2) subject to the Neumann-type boundary constraint (6.3) in the sense of Definition 6.1, and this weak solution is unique within the class of all admissible competitor maps $v \in \mathrm{W}^{1,1}(\Omega;\mathbb{R}^N)$ that satisfy $(v)_\Omega = 0$.*

(b) *The dual problem possesses a unique solution $\sigma \in (\mathrm{L}^\infty \cap \mathrm{W}^{1,2}_{\mathrm{loc}})(\Omega;\mathbb{R}^{N\times n})$ which, in addition, satifies*

$$\sigma = f'(|\nabla u|)\frac{\nabla u}{|\nabla u|} \qquad \mathscr{L}^n - \text{a.e. in } \Omega.$$

Although the analysis of the Neumann problem is often omitted in the literature, the methods used for the Dirichlet problem can be easily adapted also to our setting. This will be presented in some detail in Section 6.7 and is presented for the sake of completeness. In conclusion, pursuing this approach within the framework of μ–elliptic problems, we retrieve BILDHAUER's higher integrability results for the Neumann problem for $1 < \mu \leqslant 3$.

In the present chapter and hence in the context of the Neumann problem for h–monotone problems, we do *not* use such techniques but rather show directly that BV–minima belong to $\mathrm{W}^{1,1}(\Omega;\mathbb{R}^N)$. Moreover, let us stress that the h–monotonicity condition as given in (6.11) is not restrictive at all and

$$\boxed{\text{is satisfied for any } \mathrm{C}^2\text{–integrand } f \text{ with } f'' > 0.}$$

We refer the reader to Section 6.6.5 for more detail. As mentioned above, we wish to recall from Chapters 2.4 and 5.4 that convexity of the variational integrals $\mathfrak{F}$ allows to employ methods of convex duality and thus to study the *dual problem* (see Section 2.4.1 for a general outline of convex duality). Studying the dual problem in Section 6.6, our first objective here is to identify the correct setup for the problem dual to the variational principle (6.4) and to link the dual formulation to the primal one in a precise manner. This, in turn, is a similar objective as in Chapter 5.4 for the Dirichlet problem on BD.

In particular, we shall connect the dual problem to the relaxed primal problem, i.e., the extension of $\mathfrak{F}$ to $\mathrm{BV}(\Omega;\mathbb{R}^N)$. This will be done in a slightly more general setup than for functionals with radially symmetric integrands, and for the precise statements the reader is referred to Proposition 6.26. Within this setup, we shall prove existence of BV-(*or generalised*) *minima* of the primal (Neumann) problem, i.e., solutions of the relaxed problem.

Theorem 6.7 will be established in Section 6.5. To do so, we employ a vanishing viscosity approach (similar to that of Chapter 5.4.2) in order to obtain specific minimising sequences for which we are in the position to establish suitable bounds on their first and second derivatives. Using the particular structure of the tensor T_0 and h-monotonicity of the integrands f, we shall then demonstrate that the gradients of the solutions to the aforementioned approximate problems converge $\mathscr{L}^n$-a.e. to an L^1-map. The proof is then concluded by showing that this limit map is a gradient of a $\mathrm{W}^{1,1}(\Omega;\mathbb{R}^N)$-function u. This in turn yields a minimiser for the variational principle (6.4). Here the condition on Ω to be simply connected enters through our method of proof. For showing that the limit is a gradient indeed, we use a compatibility condition argument which asserts that the curl-free maps are precisely the gradients on a simply connected domain. In Section 6.6 we study the dual problem and the relaxation of the corresponding variational integrals to the space $\mathrm{BV}(\Omega;\mathbb{R}^N)$ without imposing an Uhlenbeck structure on the integrands. After discussing explicit higher integrability results for μ–elliptic Neumann problems in Section 6.7, we conclude the chapter with a discussion of the symmetric gradient case in Section 6.8.

Remark 6.8 (Heuristics)**.** *From a heuristic perspective, the main theorem of the present chapter can be read as a PDE–constraint on the absolutely continuous part of the gradient of the minimiser u. For an arbitrary* BV*–function $v\colon \Omega \to \mathbb{R}^N$ and its Radon–Nikodym decomposition $\mathrm{D}v = \nabla v\mathscr{L}^n + \mathrm{D}^s v$ (where ∇v denotes the* approximate gradient *of v), ∇v does* not *need to be* curl*–free. The regularity statement of Theorem 6.7(a) thus can be interpreted as that for a minimiser $u \in \mathrm{BV}(\Omega;\mathbb{R}^N)$ of the variational problem at our disposal we indeed have $\mathrm{curl}(\nabla u) = 0$. In consequence, it arises as the gradient of a $\mathrm{W}^{1,1}$–map, and* if *the singular part would not vanish, it would increase the value of the functional, and u would not minimise the variational integral.*

6.4. On the gradient structure

In this section we collect auxiliary estimates and background results that will be useful in the proof of our main result below when identifying an L^1-function with the gradient of a $\mathrm{W}^{1,1}$-function. We begin with recording the following version of Chacon's biting lemma:

Lemma 6.9 (Chacon's biting lemma, [29])**.** *Let (E_k) be a bounded sequence in $\mathrm{L}^1(\Omega;\mathbb{R}^m)$. Then there exist a subsequence $(E_{k(\ell)})$ and a function $E \in \mathrm{L}^1(\Omega;\mathbb{R}^m)$ such that $(E_{k(\ell)})$ converges weakly to E in the biting sense in $\mathrm{L}^1(\Omega;\mathbb{R}^m)$, that is, there exists an increasing sequence (Ω_j) of measurable sets contained in Ω with $\mathscr{L}^n(\Omega \setminus \Omega_j) \to 0$ such that*

$$E_{k(\ell)} \rightharpoonup E \qquad \text{weakly in } \mathrm{L}^1(\Omega_j;\mathbb{R}^m) \text{ as } \ell \to \infty$$

for every fixed $j \in \mathbb{N}$. For this type of convergence, we write $E_{k(\ell)} \xrightharpoonup{b} E$ as $j \to \infty$.

We shall apply Chacon's biting lemma to the gradients of a minimising sequence of the functional $\mathfrak{F}$ in $\mathrm{W}^{1,1}(\Omega;\mathbb{R}^N)$ and hence to gradients of functions in $\mathrm{W}^{1,1}(\Omega;\mathbb{R}^N)$. In order to deduce a gradient structure of the limit, we will show in the first step that the limit is curl-free in the sense of distributions, a concept which we recall now.

Definition 6.10. *We call a function $e \in \mathrm{C}^1(\Omega;\mathbb{R}^n)$* curl-free *if for all $i,j \in \{1,\dots,n\}$ there holds*

$$\partial_j e_i - \partial_i e_j = 0.$$

Similarly, we call a function $e \in \mathrm{L}^1(\Omega;\mathbb{R}^n)$ curl-free in the sense of distributions *if for any $\varphi \in \mathrm{C}^1_0(\Omega)$ and all $i,j \in \{1,\dots,n\}$ there holds*

$$\int_\Omega (e \otimes \nabla)_{ij}\varphi \,\mathrm{d}x := \int_\Omega \big(e_i\partial_j\varphi - e_j\partial_i\varphi\big)\,\mathrm{d}x = 0.$$

Remark 6.11 (On the curl–free condition)**.**

(i) *In order to verify the curl-free condition, we only need to check the condition for all indices $i<j$. Thus we have $n(n-1)/2$ conditions in total. In particular, for $n=2$,* curl *is defined as a scalar function, while for $n=3$ as a 3-dimensional vectorial function.*

(ii) *If $e \in \mathrm{L}^p(\Omega;\mathbb{R}^n)$ for some $p \in [1,\infty]$, then we can take by approximation test functions $\varphi \in \mathrm{W}^{1,q}_0(\Omega)$ for $q \in [1,\infty]$ such that $1/p+1/q=1$. In this case, we find*

$$\int_\Omega (e\otimes\nabla)\varphi\,\mathrm{d}x \leqslant C(n)\|e\|_{\mathrm{L}^p(\Omega;\mathbb{R}^n)}\|\nabla\varphi\|_{\mathrm{L}^q(\Omega)}.$$

If $e = \nabla w$ for some function $w \in \mathrm{W}^{2,1}(\Omega)$, then e is obviously curl-free via the integration by parts formula. However, the gradient structure is not only sufficient, but indeed necessary for the curl-free condition if Ω is a simply connected domain. The precise statement of this Sobolev-type version of the usual Poincaré Lemma is as follows:

Lemma 6.12. *Let $\Omega \subset \mathbb{R}^n$ be a simply connected bounded Lipschitz domain. If a function $E \in \mathrm{L}^1(\Omega;\mathbb{R}^{N\times n})$ is curl-free in the sense of distributions on Ω, then there exists a function $v \in \mathrm{W}^{1,1}(\Omega;\mathbb{R}^N)$ such that $\nabla v = E$ holds $\mathscr{L}^n$-a.e. in Ω.*

Proof. We first note that the statement is clear if $E \in \mathrm{C}^1(\Omega;\mathbb{R}^{N\times n})$ is curl-free in the classical sense. Indeed, in this case, we associate to E the 1-forms $\omega^\alpha := E^\alpha_1\,\mathrm{d}x_1 + \ldots + E^\alpha_n\,\mathrm{d}x_n$, for $\alpha \in \{1,\dots,N\}$, and we observe that the curl-free condition simply means that each ω^α is closed. By means of the classical Poincaré lemma, see e.g. [220], it is therefore exact, i.e. we find 0-forms v^α with $\omega^\alpha = \mathrm{d}v^\alpha$, for each $\alpha \in \{1, \dots, N\}$, which precisely means $\nabla v = E$ in Ω.

The assertion of the lemma now follows by approximation. To this end, let $K \Subset \Omega$ be a simply connected open set. Given $0 < \varepsilon < \frac{1}{2}\operatorname{dist}(K, \partial\Omega)$, the mollifications $E_\varepsilon \colon K + \mathrm{B}(0,\varepsilon) \to \mathbb{R}^{N\times n}$, defined by convolution $E_\varepsilon := \rho_\varepsilon * E$ with a standard mollifying kernel $\rho_\varepsilon(x) := \varepsilon^{-n}\rho(x/\varepsilon)$ for some non-negative, rotationally symmetric function $\rho \in \mathrm{C}^\infty_c(\mathrm{B}(0,1))$ with $\|\rho\|_{L^1(\mathrm{B}(0,1))} = 1$, are well-defined and smooth. Furthermore, for every test function $\psi \in \mathrm{C}^\infty_c(K; \mathbb{R}^{N\times n})$ we get via Fubini's theorem the relation

$$\int_\Omega (\rho_\varepsilon * E_i^\alpha)\partial_j \psi \,\mathrm{d}x = \int_\Omega E_i^\alpha \partial_j(\rho_\varepsilon * \psi)\,\mathrm{d}x$$

for all $i, j \in \{1, \dots, n\}$ and $\alpha \in \{1, \dots, N\}$. As a consequence, E_ε is curl-free in the sense of distributions on K, and thus, by the fundamental theorem of calculus, also in the classical sense. Therefore, by the Poincaré lemma mentioned above, we find a function $v_\varepsilon \in \mathrm{C}^1(K; \mathbb{R}^N)$ with $\nabla v_\varepsilon = E_\varepsilon$ on K, and we may also suppose $(v_\varepsilon)_K = 0$. Due to strong convergence $E_\varepsilon \to E$ in $\mathrm{L}^1(K; \mathbb{R}^{N\times n})$ as $\varepsilon \searrow 0$ by the usual properties of mollifications and by Poincaré's inequality, we see that (v_ε) is a Cauchy sequence in $\mathrm{W}^{1,1}(K; \mathbb{R}^N)$ and hence converges strongly in $\mathrm{W}^{1,1}(K; \mathbb{R}^N)$ to a limit $v_K \in \mathrm{W}^{1,1}(K; \mathbb{R}^N)$. In order to identify $\nabla v_K = E$ a.e. on K we estimate for an arbitrary $\varphi \in \mathrm{C}^\infty_c(K; \mathbb{R}^{N\times n})$

$$\begin{aligned}\left|\int_\Omega \langle \nabla v_K - E, \varphi\rangle \,\mathrm{d}x\right| &= \lim_{\varepsilon\searrow 0}\left|\int_\Omega \langle \nabla v_\varepsilon - E, \varphi\rangle \,\mathrm{d}x\right| \\ &\leqslant \|\varphi\|_{\mathrm{L}^\infty(K;\mathbb{R}^{N\times n})} \lim_{\varepsilon\searrow 0} \|E_\varepsilon - E\|_{\mathrm{L}^1(K;\mathbb{R}^{N\times n})} = 0.\end{aligned}$$

It only remains to justify that we find a function $v \in \mathrm{W}^{1,1}(\Omega; \mathbb{R}^{N\times n})$ such that $\nabla v = E$ holds $\mathscr{L}^n$-a.e. on all of Ω. To this end, we notice that the sets $\Omega_\delta := \{x \in \Omega \colon \operatorname{dist}(x, \partial\Omega) > \delta\}$ are simply connected Lipschitz domains provided that $\delta \in (0, \delta_0)$ for some sufficiently small $\delta_0 > 0$, with $\Omega_\delta \nearrow \Omega$ as $\delta \searrow 0$. Furthermore, we fix $\delta_1 \leqslant \delta_0$ such that $2|\Omega_{\delta_1}| \geqslant |\Omega|$. With the previous arguments we then find, for every $\delta \leqslant \delta_1$, a function $v_\delta \in \mathrm{W}^{1,1}(\Omega; \mathbb{R}^N)$ (extended via the extension operator in $\Omega \setminus \Omega_\delta$) such that $\nabla v_\delta = E$ holds a.e. in Ω_δ, and we may further suppose that $(v_\delta)_{\Omega_{\delta_1}} = 0$. It is easy to see that (v_δ) is a Cauchy sequence in $\mathrm{W}^{1,1}(\Omega; \mathbb{R}^N)$, with a limit function $v \in \mathrm{W}^{1,1}(\Omega; \mathbb{R}^N)$. Arguing via the pointwise convergence of (∇v_δ) for a subsequence (or, alternatively, via the fundamental theorem of calculus as before) we finally end up with the fact that $\nabla v = E$ holds $\mathscr{L}^n$-a.e. in Ω. The proof is complete. □

6.5. Proof of the Main Theorem

6.5.1. Existence of solutions for approximate problems Aiming for the existence of a weak solution of the system (6.2) subject to the Neumann-type boundary constraint (6.3) (or equivalently of a minimiser for the variational principle (6.4)) in the class $\mathrm{W}^{1,1}$, we start to investigate in this section boundedness and convergence properties of a suitable approximating sequence. This sequence, in turn, is obtained by means of a vanishing viscosity-type approach as in Chapter 5. More precisely, on the level of the elliptic system (6.2) we add a Laplacian to the differential operator, or on the level of the functional we add the Dirichlet energy (both with small prefactor) to the functional $\mathfrak{F}$. As a consequence, we can work in these approximations with solutions of class $\mathrm{W}^{1,2}$. It is easy to see that all arguments which are outlined in this section for the functional $\mathfrak{F}$ with radially symmetric integrands f do in fact also apply to more general functionals (as described in (6.31) later on) without the radial structure. However, it is in the subsequent sections when we need to rely on the Uhlenbeck structure of the integrands f in order to obtain the $\mathrm{W}^{1,1}$-regularity as claimed in Theorem 6.7(a).

Let us now introduce in an intermediate step the functionals

$$\mathfrak{F}_k[w] := \mathfrak{F}[w] + (2k)^{-1}\int_\Omega |\nabla w|^2 \,\mathrm{d}x := \int_\Omega f_k(|\nabla w|)\,\mathrm{d}x - \int_\Omega \langle T_0, \nabla w\rangle\,\mathrm{d}x \tag{6.12}$$

for functions $w \in \mathrm{W}^{1,2}(\Omega;\mathbb{R}^N)$ and all $k \in \mathbb{N}$, where we have set $f_k(s) := f(s) + (2k)^{-1}s^2$ for $s \in \mathbb{R}_0^+$. In a first step we establish the existence of a sequence of functions (u_k) in $\mathrm{W}^{1,2}(\Omega;\mathbb{R}^N)$ such that, for each $k \in \mathbb{N}$, the function u_k has vanishing mean value $(u_k)_\Omega = 0$ on Ω and minimises the functional $\mathfrak{F}_k$ among all functions in $\mathrm{W}^{1,2}(\Omega;\mathbb{R}^N)$.

Lemma 6.13. *Let $T_0 \in \mathrm{L}^\infty(\Omega;\mathbb{R}^{N\times n})$ satisfy (6.10). Consider a convex function $f \in \mathrm{C}^1(\mathbb{R}_0^+)$ satisfying $f(0) = f'(0) = 0$ and the linear growth condition (6.1). Then, for every $k \in \mathbb{N}$, the functional $\mathfrak{F}_k$ defined in (6.12) admits a (unique) minimiser $u_k \in \mathrm{W}^{1,2}(\Omega;\mathbb{R}^N)$ satisfying $(u_k)_\Omega = 0$ and*

$$k^{-1}\|\nabla u_k\|^2_{\mathrm{L}^2(\Omega;\mathbb{R}^{N\times n})} + \|\nabla u_k\|_{\mathrm{L}^1(\Omega;\mathbb{R}^{N\times n})} \leqslant C\big(1 + \mathfrak{F}_k[u_k]\big) \tag{6.13}$$

for a constant C depending only on Ω, f and $\|T_0\|_{\mathrm{L}^\infty(\Omega;\mathbb{R}^{N\times n})}$.

Proof. For each fixed $k \in \mathbb{N}$, the existence of the minimiser u_k is a consequence of the direct method of the calculus of variations. In fact, due to assumption (6.10) on T_0 (implying coerciveness, cp. Section 6.2), the functional $\mathfrak{F}$ and thus also each of the functionals $\mathfrak{F}_k$ is bounded from below via

$$(2k)^{-1}\|\nabla w\|^2_{\mathrm{L}^2(\Omega;\mathbb{R}^{N\times n})} + \gamma\|\nabla w\|_{\mathrm{L}^1(\Omega;\mathbb{R}^{N\times n})} \leqslant \mathfrak{F}_k[w] + C|\Omega|$$

for all functions $w \in \mathrm{W}^{1,2}(\Omega;\mathbb{R}^N)$ and $k \in \mathbb{N}$, with constants γ and C depending only on f and $\|T_0\|_{\mathrm{L}^\infty(\Omega;\mathbb{R}^{N\times n})}$. As a consequence, by Poincaré's inequality in the zero-mean version, we find that every minimising sequence $(w_{k,\ell})_{\ell\in\mathbb{N}}$ of $\mathfrak{F}_k$ in the set $\mathcal{C} := \{w \in \mathrm{W}^{1,2}(\Omega;\mathbb{R}^N) \colon (w)_\Omega = 0\}$, i.e., which satisfies $\mathfrak{F}_k[w_{k,\ell}] \to \inf_{\mathcal{C}} \mathfrak{F}_k$ as $\ell \to \infty$, is bounded in $\mathrm{W}^{1,2}(\Omega;\mathbb{R}^N)$. Since the latter space is reflexive, the classical Banach–Alaoglu Theorem gives a non-relabeled subsequence and a limit map $u_k \in \mathrm{W}^{1,2}(\Omega;\mathbb{R}^N)$ such that $w_{k,\ell} \rightharpoonup u_k$ as $\ell \to \infty$, $(u_k)_\Omega = 0$ and the estimate (6.13) are satisfied. Now, since by convexity of its integrand the functional $\mathfrak{F}_k$ is lower semi-continuous with respect to weak convergence in $\mathrm{W}^{1,2}(\Omega;\mathbb{R}^N)$, cp. [67, Theorem 3.23], we obtain $\mathfrak{F}_k[u_k] \leqslant \liminf_{\ell\to\infty} \mathfrak{F}_k[w_{k,\ell}]$ for each $k \in \mathbb{N}$. Thus, taking also advantage of the strict convexity of the integrand of $\mathfrak{F}_k$, we have shown that u_k is indeed the unique minimiser of $\mathfrak{F}_k$ in $\mathcal{C}$, and the proof of the lemma is complete. ☐

Let us note that every minimiser $u_k \in \mathrm{W}^{1,2}(\Omega;\mathbb{R}^N)$ of the functional $\mathfrak{F}_k$ in $\mathrm{W}^{1,2}(\Omega;\mathbb{R}^N)$ also satisfies the Euler–Lagrange system

$$\int_\Omega \langle \mathbb{A}_k(\nabla u_k), \nabla\varphi\rangle\,\mathrm{d}x = \int_\Omega \langle T_0, \nabla\varphi\rangle\,\mathrm{d}x \tag{6.14}$$

for all functions $\varphi \in \mathrm{W}^{1,2}(\Omega;\mathbb{R}^N)$, where for $k \in \mathbb{N}$ the regularised tensor functions $\mathbb{A}_k \colon \mathbb{R}^{N\times n} \to \mathbb{R}^{N\times n}$ are given by

$$\mathbb{A}_k(z) := \mathbb{A}(z) + k^{-1}z := f'(|z|)\frac{z}{|z|} + k^{-1}z, \qquad \text{for all } z \in \mathbb{R}^{N\times n}. \tag{6.15}$$

Indeed, (6.14) is a simple consequence of the facts that the function $u_k + t(\varphi - (\varphi)_\Omega) \in \mathrm{W}^{1,2}(\Omega;\mathbb{R}^N)$ is an admissible competitor for each $t \in \mathbb{R}$ and that $t \mapsto \mathfrak{F}_k[u_k + t\varphi]$ attains its minimum for $t = 0$ (cp. also Lemma 6.2).

Similarly as in [34, Lems. 3.2 and 3.3], we next show that the functional $\mathfrak{F}_k$ is indeed an approximation of the original functional $\mathfrak{F}$ with respect to minimisation in $\mathrm{W}^{1,1}(\Omega;\mathbb{R}^N)$ (i.e., in the sense that the minimisers u_k of $\mathfrak{F}_k$ form a minimising sequence for $\mathfrak{F}$ in $\mathrm{W}^{1,1}(\Omega;\mathbb{R}^N)$). Moreover, we infer a first uniform bound for the sequence (u_k).

Corollary 6.14. *Let $T_0 \in \mathrm{L}^\infty(\Omega;\mathbb{R}^{N\times n})$ with (6.10). Consider a convex function $f \in \mathrm{C}^1(\mathbb{R}_0^+)$ satisfying $f(0) = f'(0) = 0$ and the linear growth condition (6.1). Then the sequence $(u_k)_{k\in\mathbb{N}}$ of minimisers u_k of the functional $\mathfrak{F}_k$ from Lemma (6.13) is a minimising sequence for $\mathfrak{F}$ in $\mathrm{W}^{1,1}(\Omega;\mathbb{R}^N)$ with*

$$\lim_{k\to\infty} \inf_{\mathrm{W}^{1,2}(\Omega;\mathbb{R}^N)} \mathfrak{F}_k = \inf_{\mathrm{W}^{1,1}(\Omega;\mathbb{R}^N)} \mathfrak{F}.$$

Moreover, we have $k^{-1/2}\nabla u_k \to 0$ in $\mathrm{L}^2(\Omega;\mathbb{R}^{N\times n})$ and there holds

$$\sup_{k\in\mathbb{N}} \left\{ \|u_k\|_{\mathrm{W}^{1,1}(\Omega;\mathbb{R}^N)} + k^{-1}\|u_k\|^2_{\mathrm{W}^{1,2}(\Omega;\mathbb{R}^N)} \right\} < \infty. \tag{6.16}$$

Proof. In order to prove the first claim, for a fixed number $\varepsilon > 0$, we choose first a function $v_\varepsilon \in \mathrm{W}^{1,2}(\Omega;\mathbb{R}^N)$ and then an index $k_0 \in \mathbb{N}$ such that

$$\mathfrak{F}[v_\varepsilon] \leqslant \inf_{\mathrm{W}^{1,1}(\Omega;\mathbb{R}^N)} \mathfrak{F} + \frac{\varepsilon}{2} \quad \text{and} \quad (2k_0)^{-1}\|\nabla v_\varepsilon\|^2_{\mathrm{L}^2(\Omega;\mathbb{R}^{N\times n})} \leqslant \frac{\varepsilon}{2}$$

hold. In this way, we obtain by the minimality of u_k for all indices $k \geqslant k_0$

$$\begin{aligned}
\inf_{\mathrm{W}^{1,1}(\Omega;\mathbb{R}^N)} \mathfrak{F} \leqslant \mathfrak{F}[u_k] &\leqslant \mathfrak{F}[u_k] + (2k)^{-1}\|\nabla u_k\|^2_{\mathrm{L}^2(\Omega;\mathbb{R}^{N\times n})} = \mathfrak{F}_k[u_k] \\
&= \inf_{\mathrm{W}^{1,2}(\Omega;\mathbb{R}^N)} \mathfrak{F}_k \\
&\leqslant \mathfrak{F}_k[v_\varepsilon] = \mathfrak{F}[v_\varepsilon] + (2k)^{-1}\|\nabla v_\varepsilon\|^2_{\mathrm{L}^2(\Omega;\mathbb{R}^{N\times n})} \leqslant \inf_{\mathrm{W}^{1,1}(\Omega;\mathbb{R}^N)} \mathfrak{F} + \varepsilon\,,
\end{aligned}$$

and the first assertion follows by arbitrariness of ε. Moreover, from this chain of inequalities, we also read of the strong convergence $k^{-1/2}\nabla u_k \to 0$ in $\mathrm{L}^2(\Omega;\mathbb{R}^{N\times n})$. Finally, in view of $(u_k)_\Omega = 0$, we may apply Poincaré's inequality in the mean value version in the spaces $\mathrm{W}^{1,2}(\Omega;\mathbb{R}^N)$ and $\mathrm{W}^{1,1}(\Omega;\mathbb{R}^N)$ to u_k, and we thus infer the last claim (6.16) as a direct consequence of the estimate (6.13). □

Let us note that the uniform bound (6.16) allows to conclude that there exist a Radon measure $\mathbf{E} \in \mathcal{M}_{<infty}(\Omega;\mathbb{R}^{N\times n})$ and functions $E \in \mathrm{L}^1(\Omega;\mathbb{R}^{N\times n})$, $B \in \mathrm{L}^\infty(\Omega;\mathbb{R}^{N\times n})$ such that, for a suitable non-relabelled subsequence, we have

$$\begin{aligned}
\nabla u_k \mathscr{L}^n \llcorner \Omega &\overset{*}{\rightharpoonup} \mathbf{E} &&\text{in } \mathcal{M}(\Omega;\mathbb{R}^{N\times n}), &(6.17)\\
\nabla u_k &\overset{\mathrm{b}}{\rightharpoonup} E &&\text{in } \mathrm{L}^1(\Omega;\mathbb{R}^{N\times n}), &(6.18)\\
\mathbb{A}(\nabla u_k) &\overset{*}{\rightharpoonup} B &&\text{in } \mathrm{L}^\infty(\Omega;\mathbb{R}^{N\times n}), &(6.19)
\end{aligned}$$

as $k \to \infty$. Indeed, (6.17) is a direct consequence of the usual Banach–Alaoglu theorem applied to $\mathcal{M}_{<\infty}(\Omega;\mathbb{R}^{N\times n}) \cong (\mathrm{C}_0(\Omega;\mathbb{R}^{N\times n}))^*$, whereas (6.18) follows from Chacon's biting Lemma 6.9. Also note that by (6.16) in conjunction with the compact embedding $\mathrm{W}^{1,1}(\Omega;\mathbb{R}^N) \hookrightarrow\hookrightarrow \mathrm{L}^1(\Omega;\mathbb{R}^N)$, we may conclude that there exists $u \in \mathrm{BV}(\Omega;\mathbb{R}^N)$ such that

$$u_k \to u \qquad \text{in } \mathrm{L}^1(\Omega;\mathbb{R}^n) \qquad \text{as } k \to \infty. \tag{6.20}$$

The estimates and convergence results gathered so far will be an instrumental ingredient in proving the existence of a minimiser of the original functional $\mathfrak{F}$. In order to demonstrate the claimed regularity features of this minimiser, we shall turn in the next Section 6.5.2 to suitable a priori estimates regarding the sequence (u_k).

6.5.2. A Priori Estimates We shall next derive suitable a priori estimates which, in particular, will allow us to conclude the pointwise convergence of the sequence $(\nabla u_k)_{k\in\mathbb{N}}$ to its biting-limit E almost everywhere in Ω. We begin by showing that the sequence constructed in the previous section indeed belongs to $\mathrm{W}^{2,2}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$. This is similar to Lemma 5.11.

Lemma 6.15. *Let $T_0 \in \mathrm{W}^{2,\infty}(\Omega;\mathbb{R}^{N\times n})$ with (6.10). Consider a convex function $f \in \mathrm{C}^2(\mathbb{R}^+_0)$ which satisfies $f(0) = f'(0) = 0$, the linear growth condition (6.1) and which induces an h-monotone variational integrand. Then, for each $k \in \mathbb{N}$, the minimiser u_k from Lemma 6.13 satisfies $u_k \in \mathrm{W}^{2,2}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$, and moreover, for every compact set $K \subset \Omega$ there holds*

$$\sup_{k\in\mathbb{N}} \Big\{ \sum_{s=1}^{n} \int_K \langle \mathrm{D}_z\mathbb{A}(\nabla u_k)\partial_s\nabla u_k, \partial_s\nabla u_k\rangle \,\mathrm{d}x + k^{-1}\int_K |\nabla^2 u_k|^2 \,\mathrm{d}x \Big\} < \infty. \tag{6.21}$$

Proof. Let $\eta \in \mathrm{C}^1_0(\Omega;[0,1])$ be a localisation function with $\eta \equiv 1$ on the given, compactly supported subset K of Ω. For $h \in \mathbb{R}\setminus\{0\}$ with $|h| < \operatorname{dist}(K,\partial\Omega)$ and $s \in \{1,\dots,n\}$ we denote by $\Delta_{s,h}$ the finite difference quotient operator with respect to direction e_s and stepsize h, and we then choose $\varphi := \Delta_{s,-h}(\eta^2\Delta_{s,h}u_k) \in \mathrm{W}^{1,2}(\Omega;\mathbb{R}^N)$ as a test function in the Euler–Lagrange equation (6.14). In this way, we obtain with the integration by parts formula for finite difference quotients and the standard one

$$\int_\Omega \langle \Delta_{s,h}(\mathbb{A}_k(\nabla u_k)), \eta^2\Delta_{s,h}\nabla u_k + 2\eta\nabla\eta\otimes\Delta_{s,h}u_k\rangle \,\mathrm{d}x = -\int_\Omega \langle \Delta_{s,h}\operatorname{div}T_0, \eta^2\Delta_{s,h}u_k\rangle \,\mathrm{d}x, \tag{6.22}$$

which is the starting point for the proof of higher Sobolev regularity. For the right-hand side of (6.22) we obtain from standard properties (regarding norm estimates) for finite difference quotients, in view of $T_0 \in \mathrm{W}^{2,\infty}(\Omega;\mathbb{R}^{N\times n})$ and the uniform bound (6.16), the estimate

$$-\int_\Omega \langle \Delta_{s,h}\operatorname{div}T_0, \eta^2\Delta_{s,h}u_k\rangle \,\mathrm{d}x \leqslant C\|T_0\|_{\mathrm{W}^{2,\infty}(\Omega;\mathbb{R}^{N\times n})}\|\nabla u_k\|_{\mathrm{W}^{1,1}(\Omega;\mathbb{R}^{N\times n})} \leqslant C \tag{6.23}$$

with a constant C depending only on Ω, f, $\|T_0\|_{\mathrm{W}^{2,\infty}(\Omega;\mathbb{R}^{N\times n})}$ and η (but independent of $k\in\mathbb{N}$). In order to find some coercivity estimate for the left-hand side (6.22), let us first rewrite

$$\Delta_{s,h}(\mathbb{A}_k(\nabla u_k(x))) = \int_0^1 \mathrm{D}_z\mathbb{A}_k(\nabla u_k(x) + th\Delta_{s,h}\nabla u_k(x))\,\mathrm{d}t\Delta_{s,h}\nabla u_k(x)$$

for $x \in K$. Thus, for shorter notation, we introduce the bilinear form $\mathcal{B}_{k,h}(x)\colon \mathbb{R}^{N\times n}\times\mathbb{R}^{N\times n}\to\mathbb{R}$, for all $k\in\mathbb{N}$ and all $x \in \Omega$ such that $\mathrm{B}(x,h)\subset\Omega$, by

$$\mathcal{B}_{k,h}(x)[\zeta,\tilde\zeta] := \int_0^1 \langle \mathrm{D}_z\mathbb{A}_k(\nabla u_k(x) + th\Delta_{s,h}\nabla u_k(x))\zeta,\tilde\zeta\rangle\,\mathrm{d}t \qquad \text{for } \zeta,\tilde\zeta\in\mathbb{R}^{N\times n}.$$

Note that, by definition, the radial structure and due to the convexity of f with $f'(0)=0$, these bilinear forms are (for all k, x and h as above) symmetric and positive definite, with lower bound $\mathcal{B}_{k,h}(x)[\zeta,\zeta] \geqslant k^{-1}|\zeta|^2$ for all $\zeta\in\mathbb{R}^{N\times n}$. Consequently, applying Young's inequality for the bilinear forms $\mathcal{B}_{k,h}(x)$ and invoking (6.23), we deduce from (6.22) the estimate

$$
\begin{aligned}
&\int_\Omega \eta^2 \mathcal{B}_{k,h}(x)[\Delta_{s,h}\nabla u_k, \Delta_{s,h}\nabla u_k]\,\mathrm{d}x \\
&= -2\int_\Omega \mathcal{B}_{k,h}(x)[\Delta_{s,h}\nabla u_k, \eta\nabla\eta\otimes\Delta_{s,h}u_k]\,\mathrm{d}x - \int_\Omega \langle \Delta_{s,h}\operatorname{div} T_0, \eta^2\Delta_{s,h}u_k\rangle\,\mathrm{d}x \\
&\leqslant \frac{1}{2}\int_\Omega \eta^2\mathcal{B}_{k,h}(x)[\Delta_{s,h}\nabla u_k, \Delta_{s,h}\nabla u_k]\,\mathrm{d}x \\
&\quad + 2\int_\Omega \mathcal{B}_{k,h}(x)[\nabla\eta\otimes\Delta_{s,h}u_k, \nabla\eta\otimes\Delta_{s,h}u_k]\,\mathrm{d}x + C.
\end{aligned}
$$

We may now absorb the first term of the right-hand side into the left-hand side. By (6.11) in conjunction with (6.15), by standard properties of finite difference quotients and by (6.16) we then obtain

$$
\begin{aligned}
k^{-1}\int_\Omega \eta^2|\Delta_{s,h}\nabla u_k|^2\,\mathrm{d}x &\leqslant \int_\Omega \eta^2\mathcal{B}_{k,h}(x)[\Delta_{s,h}\nabla u_k, \Delta_{s,h}\nabla u_k]\,\mathrm{d}x \\
&\leqslant C\int_\Omega |\nabla\eta|^2|\Delta_{s,h}u_k(x)|^2\,\mathrm{d}x + C \leqslant C
\end{aligned}
$$

for a constant C depending only on Ω, f, $\|T_0\|_{\mathrm{W}^{2,\infty}(\Omega;\mathbb{R}^{N\times n})}$, η and k. By choice of the localization function η we thus obtain, for each $k \in \mathbb{N}$, that $\Delta_{s,h}\nabla u_k$ is bounded uniformly for all $h \in \mathbb{R}\setminus\{0\}$ with $|h| < \operatorname{dist}(K, \partial\Omega)$ in $\mathrm{L}^2(K;\mathbb{R}^{N\times n})$, though not uniformly in k. The $\mathrm{W}^{2,2}_{\mathrm{loc}}$-regularity of u_k then follows from the usual difference-quotient type characterisation of $\mathrm{W}^{1,2}$ and the arbitrariness of the compact set $K \subset \Omega$ and of $s \in \{1, \ldots, n\}$.

Once the $\mathrm{W}^{2,2}_{\mathrm{loc}}$-regularity of each function u_k is at our disposal, we may now proceed to the proof of the uniform estimate. To this end, we firstly differentiate the Euler–Lagrange system and redo essentially the same computations as above, but now with the differential ∂_s instead of the difference quotient operator $\Delta_{s,h}$. More precisely, starting from the identity

$$
\int_\Omega \langle \mathrm{D}_z\mathbb{A}_k(\nabla u_k)\partial_s\nabla u_k, \nabla\varphi\rangle\,\mathrm{d}x = \int_\Omega \langle \partial_s T_0, \nabla\varphi\rangle\,\mathrm{d}x
$$

for all functions $\varphi \in W^{1,2}(\Omega;\mathbb{R}^N)$ with compact support in Ω, we choose $\varphi = \eta^2\partial_s u$ with $\eta \in \mathrm{C}^1_0(\Omega;[0,1])$ a localization function on the compact set $K \subset \Omega$ as above. Doing so, we find via Young's inequality (applied to the positive definite bilinear forms $\mathrm{D}_z\mathbb{A}_k(\nabla u_k(x))$ corresponding to $\mathcal{B}_{k,0}(x)$ above) and the integration by parts formula

$$
\begin{aligned}
&\int_\Omega \eta^2\langle \mathrm{D}_z\mathbb{A}_k(\nabla u_k)\partial_s\nabla u_k, \partial_s\nabla u_k\rangle\,\mathrm{d}x \\
&= -2\int_\Omega \eta\langle \mathrm{D}_z\mathbb{A}_k(\nabla u_k)\partial_s\nabla u_k, \nabla\eta\otimes\partial_s u_k\rangle\,\mathrm{d}x + \int_\Omega \langle \partial_s T_0, \nabla(\eta^2\partial_s u)\rangle\,\mathrm{d}x \\
&\leqslant \frac{1}{2}\int_\Omega \eta^2\langle \mathrm{D}_z\mathbb{A}_k(\nabla u_k)\partial_s\nabla u_k, \partial_s\nabla u_k\rangle\,\mathrm{d}x \\
&\quad + 2\int_\Omega \eta^2\langle \mathrm{D}_z\mathbb{A}_k(\nabla u_k)\nabla\eta\otimes\partial_s u_k, \nabla\eta\otimes\partial_s u_k\rangle\,\mathrm{d}x - \int_\Omega \eta^2\langle \partial_s\operatorname{div} T_0, \partial_s u\rangle\,\mathrm{d}x
\end{aligned}
$$

After absorbing the first integral on the right-hand side into the left-hand side, we directly obtain the lower bound given in the statement via the definition (6.15) of $\mathbb{A}_k$, while the remaining terms on the right-hand side of the previous inequality are estimated via (6.11), combined with (6.15) and $T_0 \in \mathrm{W}^{2,\infty}(\Omega;\mathbb{R}^{N\times n})$. This yields

$$\begin{aligned}&\int_\Omega \eta^2 \langle \mathrm{D}_z\mathbb{A}(\nabla u_k)\partial_s\nabla u_k, \partial_s\nabla u_k\rangle \,\mathrm{d}x + k^{-1}\int_\Omega \eta^2|\partial_s\nabla u_k|^2\,\mathrm{d}x\\ &\leqslant \int_\Omega \eta^2\langle \mathrm{D}_z\mathbb{A}_k(\nabla u_k)\partial_s\nabla u_k,\partial_s\nabla u_k\rangle\,\mathrm{d}x\\ &\leqslant C\big(\|\partial_s u_k\|_{\mathrm{L}^1(\Omega;\mathbb{R}^N)} + k^{-1}\|\partial_s u_k\|^2_{\mathrm{L}^2(\Omega;\mathbb{R}^N)}\big)\end{aligned}$$

with a constant C depending only on L, $\|T_0\|_{\mathrm{W}^{2,\infty}(\Omega;\mathbb{R}^{N\times n})}$ and η, but not on k. At this stage, the assertion (6.21) of the lemma follows from the uniform bound (6.16), combined with the arbitrariness of $s \in \{1,\dots,n\}$. $\square$

Remark 6.16. *In the same vein as in Chapter 5, we see by invoking the assumption (6.11) of h-monotonicity on the integrand that the uniform estimate (6.21) appears as a weighted Sobolev-type estimate. Namely, we have for every compact set $K \subset \Omega$*

$$\sup_{k\in\mathbb{N}}\Big\{\int_K h(|\nabla u_k|)|\nabla^2 u_k|^2\,\mathrm{d}x\Big\} < \infty. \tag{6.24}$$

The uniform bound (6.24) constitutes the key ingredient in order to establish the pointwise convergence of the sequence of gradients (∇u_k).

Corollary 6.17. *Under the assumption of Lemma 6.15 we have*

$$\nabla u_k \to E \qquad \mathscr{L}^n\text{-a.e. in } \Omega \quad \text{as } k\to\infty, \tag{6.25}$$

where $E \in \mathrm{L}^1(\Omega;\mathbb{R}^{N\times n})$ is given by the biting limit (6.18).

Proof. We follow the strategy of proof of [38, Section 4.4]. We start by defining an auxiliary function $\tilde{h} \in \mathrm{C}^1(\mathbb{R}^+,\mathbb{R}^+)$ via

$$\tilde{h}(t) := \int_t^\infty \frac{h(s)}{(1+s)^2}\,\mathrm{d}s, \qquad \text{for } t>0.$$

Since h is positive, $\tilde{h}$ is strictly monotonically decreasing and, moreover, since h is decreasing and with (6.11), we have

$$\tilde{h}(t) \leqslant h(t) \leqslant L(1+t)^{-1} \qquad \text{for } t>0.$$

Next, we introduce the functions

$$\alpha_k := \tilde{h}(|\nabla u_k|)\nabla u_k \qquad \text{and} \qquad \beta_k := \tilde{h}(|\nabla u_k|)$$

for $k\in\mathbb{N}$. Obviously, α_k and β_k are bounded, and with the observation

$$|\nabla\alpha_k|^2 + |\nabla\beta_k|^2 \leqslant h(|\nabla u_k|)|\nabla^2 u_k|^2$$

we immediately deduce from (6.24)

$$\sup_{k\in\mathbb{N}}\Big\{\|\alpha_k\|_{\mathrm{L}^\infty(\Omega;\mathbb{R}^{N\times n})} + \|\beta_k\|_{\mathrm{L}^\infty(\Omega)} + \|\alpha_k\|_{\mathrm{W}^{1,2}(K;\mathbb{R}^{N\times n})} + \|\beta_k\|_{\mathrm{W}^{1,2}(K)}\Big\} < \infty$$

for each relatively compact set $K \subset \Omega$. Picking such K with Lipschitz boundary, we find by the compact embedding $\mathrm{W}^{1,2}(K;\mathbb{R}^{N\times n}) \hookrightarrow\hookrightarrow \mathrm{L}^1(K;\mathbb{R}^{N\times n})$, non-relabeled subsequences such that the following convergence results hold:

$$\begin{aligned}\alpha_k &\rightharpoonup \alpha && \text{weakly in } \mathrm{L}^1(\Omega;\mathbb{R}^{N\times n}),\\ \alpha_k &\to \alpha && \text{strongly in } \mathrm{L}^1(K;\mathbb{R}^{N\times n}),\\ \alpha_k &\to \alpha && \mathscr{L}^n\text{-a.e. in } \Omega\\ \beta_k &\rightharpoonup \beta && \text{weakly in } \mathrm{L}^1(\Omega),\\ \beta_k &\to \beta && \text{strongly in } \mathrm{L}^1(K),\\ \beta_k &\to \beta && \mathscr{L}^n\text{-a.e. in } \Omega.\end{aligned}$$

Since $\tilde{h}$ is strictly decreasing on $\mathbb{R}^+$, the inverse $\tilde{h}^{-1}$ exists on the set $\tilde{h}(\mathbb{R}^+)$, is non-negative, decreasing and continuous. Thus, in view of Fatou's lemma and the boundedness of (∇u_k) in $\mathrm{L}^1(\Omega;\mathbb{R}^{N\times n})$ by (6.16), we get

$$\int_\Omega \tilde{h}^{-1}(\beta)\,\mathrm{d}x \leqslant \liminf_{k\to\infty}\int_\Omega \tilde{h}^{-1}(\beta_k)\,\mathrm{d}x = \liminf_{k\to\infty}\int_\Omega |\nabla u_k|\,\mathrm{d}x < \infty.$$

With $\lim_{t\to\infty}\tilde{h}(t) = 0$ and thus $\lim_{t\to 0}\tilde{h}^{-1}(t) = \infty$, we easily deduce that $\beta > 0$ holds $\mathscr{L}^n$-a.e. in Ω. Therefore, we have the pointwise convergence

$$\nabla u_k = \alpha_k/\beta_k \to \alpha/\beta \qquad \mathscr{L}^n\text{-a.e. in } \Omega \text{ as } k\to\infty.$$

On the other hand, (6.18) yields the existence of an increasing sequence (Ω_j) of sets contained in Ω with $\mathscr{L}^n(\Omega\setminus\Omega_j)\to 0$ as $j\to\infty$ and such that ∇u_k converges weakly to E as $k\to\infty$ on every Ω_j. Therefore, because of uniqueness of the limits, we can identify the pointwise limit $\alpha/\beta = E$ as $\mathrm{L}^1(\Omega;\mathbb{R}^{N\times n})$ functions. In conclusion, we arrive at the convergence $\nabla u_k \to E$ $\mathscr{L}^n$-a.e. in Ω, which was the claim (6.25). Moreover, once again by Fatou's lemma, combined with the uniform bound (6.16), we also have the estimate

$$\|E\|_{\mathrm{L}^1(\Omega;\mathbb{R}^{N\times n})} \leqslant \liminf_{k\to\infty}\|\nabla u_k\|_{\mathrm{L}^1(\Omega;\mathbb{R}^{N\times n})} \leqslant C. \qquad \square$$

6.5.3. Existence and Regularity for the Primal Problem We shall now use the a priori estimates of the preceding sections to conclude that there exists a function $v \in \mathrm{W}^{1,1}(\Omega;\mathbb{R}^N)$ such that E as given by the biting limit (6.18) (and which was just identified as the pointwise limit of (∇u_k)) satisfies

$$\nabla u_k \to E = \nabla v \qquad \mathscr{L}^n\text{-a.e. in } \Omega \quad \text{as } k\to\infty. \tag{6.26}$$

To this end, we shall utilise the Poincaré-type Lemma 6.12 (applied here to the N component functions, each of them with values in $\mathbb{R}^n$). Hence, the main difficulty is to show that E is curl-free in the sense of distributions.

Proof of the representation $E = \nabla v$. In what follows, we prove that each function E^α, for $\alpha\in\{1,\dots,N\}$, is curl-free in the sense of distributions, as introduced in Definition 6.10. This means that we need to show

$$\int_\Omega \big(E_i\partial_j\varphi - E_j\partial_i\varphi\big)\,\mathrm{d}x = 0 \tag{6.27}$$

for any fixed test function $\varphi\in \mathrm{C}^1_0(\Omega)$ and all choices of indices $i,j\in\{1,\dots,n\}$. To this end, we set $K := \operatorname{spt}(\varphi)$. We further consider a sequence of functions $(g_\ell)_{\ell\in\mathbb{N}}$ in $\mathrm{C}^\infty_c(\mathbb{R};[0,1])$ with $g_\ell\equiv 1$ in $[-\ell,\ell]$, $g_\ell\equiv 0$ outside of $[-2\ell,2\ell]$ and $|g_\ell'|\leqslant 2\ell^{-1}$ in $\mathbb{R}$, which allows us to estimate the above expression on sublevels of $|E^\alpha|$. In fact, we may now rewrite the expression (6.27) above as

$$\begin{aligned}\left|\int_\Omega \big(E^\alpha_i\partial_j\varphi - E^\alpha_j\partial_i\varphi\big)\,\mathrm{d}x\right| &= \left|\int_\Omega g_\ell(|E|)\big(E^\alpha_i\partial_j\varphi - E^\alpha_j\partial_i\varphi\big)\,\mathrm{d}x\right| \\ &\quad + \left|\int_\Omega \big(1-g_\ell(|E|)\big)\big(E^\alpha_i\partial_j\varphi - E^\alpha_j\partial_i\varphi\big)\,\mathrm{d}x\right| =: \mathbf{I}_\ell + \mathbf{II}_\ell,\end{aligned} \tag{6.28}$$

and noting that $E^\alpha\in \mathrm{L}^1(\Omega;\mathbb{R}^n)$, we find

$$\lim_{\ell\to\infty}\mathbf{II}_\ell \leqslant 2\sup_K|\nabla\varphi|\lim_{\ell\to\infty}\int_{\{|E|\geqslant\ell\}}|E^\alpha|\,\mathrm{d}x = 0.$$

Thus, it remains to show that we also have $\lim_{\ell\to\infty}\mathbf{I}_\ell = 0$. In order to prove this claim, we start by noting that, as a consequence of Lebesgue's dominated convergence theorem, the pointwise convergence $\nabla u_k \to E$ established in Section 6.5.2 implies the strong convergence $g_\ell(|\nabla u_k|)\nabla u_k^\alpha \to g_\ell(|E|)E^\alpha$ in $\mathrm{L}^1(\Omega;\mathbb{R}^n)$ as $k\to\infty$. Since by Lemma 6.15 we have $u_k \in \mathrm{W}^{2,2}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ for every $k\in\mathbb{N}$, we may hence rewrite $\mathbf{I}_\ell$ by the integration by parts formula as

$$\begin{aligned}
\mathbf{I}_\ell &= \lim_{k\to\infty}\Big|\int_\Omega g_\ell(|\nabla u_k|)\big(\partial_i u_k^\alpha \partial_j\varphi - \partial_j u_k^\alpha\partial_i\varphi\big)\,\mathrm{d}x\Big| \\
&= \lim_{k\to\infty}\Big|\int_\Omega \Big(\partial_j(g_\ell(|\nabla u_k|))\partial_i u_k^\alpha - \partial_i(g_\ell(|\nabla u_k|))\partial_j u_k^\alpha\Big)\varphi\,\mathrm{d}x \\
&\qquad\qquad + \int_\Omega g_\ell(|\nabla u_k|)\big(\partial_j\partial_i u_k^\alpha - \partial_i\partial_j u_k^\alpha\big)\varphi\,\mathrm{d}x\Big| \\
&= \lim_{k\to\infty}\Big|\int_\Omega \Big(\partial_j(g_\ell(|\nabla u_k)|)\partial_i u_k^\alpha - \partial_i(g_\ell(|\nabla u_k)|)\partial_j u_k^\alpha\Big)\varphi\,\mathrm{d}x\Big|.
\end{aligned}$$

We next introduce functions $G_\ell\colon \mathbb{R}_0^+\to\mathbb{R}$ by

$$G_\ell(t) := \int_0^t \frac{g_\ell'(s)s}{f'(s)}\,\mathrm{d}s, \qquad \text{for } t\geqslant 0.$$

Firstly, since f is strictly convex with $f'(0)=0$, we note that f' is monotonically increasing with $f'(t)>0$ for all $t>0$. Consequently, the integrand in the definition of G_ℓ is well-defined and supported in $[\ell,2\ell]\subset\mathbb{R}^+$, and we further have the estimate

$$|G_\ell(t)| \leqslant \frac{4\ell}{\ell f'(\ell)}\int_\ell^{2\ell} 1\,\mathrm{d}s \leqslant \frac{4\ell}{f'(\ell)} \leqslant \frac{4\ell}{f'(1)} \qquad \text{for all } t>0 \text{ and } \ell\in\mathbb{N}. \tag{6.29}$$

Using

$$\partial_j(g_\ell(|\nabla u_k|)) = \partial_j(G_\ell(|\nabla u_k|))\frac{f'(|\nabla u_k|)}{|\nabla u_k|},$$

we may then express $\mathbf{I}_\ell$ in terms of $G_\ell(|\nabla u_k|)$ and apply once again the integration by parts formula (as well as the fact that $u_k\in\mathrm{W}^{2,2}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ holds for each $k\in\mathbb{N}$). In this way, we find

$$\begin{aligned}
\mathbf{I}_\ell &= \lim_{k\to\infty}\Big|\int_\Omega \Big(\big(\partial_j\big(G_\ell(|\nabla u_k|)\big)\big)\partial_i u_k^\alpha - \big(\partial_i\big(G_\ell(|\nabla u_k|)\big)\big)\partial_j u_k^\alpha\Big)\frac{f'(|\nabla u_k|)}{|\nabla u_k|}\varphi\,\mathrm{d}x\Big| \\
&\leqslant \lim_{k\to\infty}\Big|\int_\Omega G_\ell(|\nabla u_k|)\Big(\partial_j\Big(\frac{f'(|\nabla u_k|)\partial_i u_k^\alpha}{|\nabla u_k|}\Big) - \partial_i\Big(\frac{f'(|\nabla u_k|)\partial_j u_k^\alpha}{|\nabla u_k|}\Big)\Big)\varphi\,\mathrm{d}x\Big| \\
&\quad + \lim_{k\to\infty}\Big|\int_\Omega G_\ell(|\nabla u_k|)\big(\partial_i u_k^\alpha\partial_j\varphi - \partial_j u_k^\alpha\partial_i\varphi\big)\frac{f'(|\nabla u_k|)}{|\nabla u_k|}\,\mathrm{d}x\Big|.
\end{aligned}$$

Recalling

$$A(z) = \frac{f'(|z|)z}{|z|} \qquad \text{for all } z\in\mathbb{R}^{N\times n},$$

we next estimate $\mathbf{I}_\ell$ in the more convenient form

$$\begin{aligned}
\mathbf{I}_\ell &\leqslant \lim_{k\to\infty}\Big|\int_\Omega \langle \mathrm{D}_z\mathbb{A}(\nabla u_k)\partial_j\nabla u_k, G_\ell(|\nabla u_k|)e_i\otimes e^\alpha\rangle\varphi\,\mathrm{d}x\Big| \\
&\quad + \lim_{k\to\infty}\Big|\int_\Omega \langle \mathrm{D}_z\mathbb{A}(\nabla u_k)\partial_i\nabla u_k, G_\ell(|\nabla u_k|)e_j\otimes e^\alpha\rangle\varphi\,\mathrm{d}x\Big| \\
&\quad + \lim_{k\to\infty} 2\int_\Omega |G_\ell(|\nabla u_k|)||f'(|\nabla u_k|)||\nabla\varphi|\,\mathrm{d}x,
\end{aligned}$$

where $e_1,\dots,e_n$ denote the standard unit basis vectors in $\mathbb{R}^n$ and $e^1,\dots,e^N$ the ones in $\mathbb{R}^N$. Keeping in mind that $\mathrm{D}_z\mathbb{A}(z)$ is a positive definite, symmetric bilinear form, we infer from the Cauchy–Schwarz inequality

$$\begin{aligned}
&\int_\Omega \langle \mathrm{D}_z\mathbb{A}(\nabla u_k)\partial_j\nabla u_k, G_\ell(|\nabla u_k|)e_i\otimes e^\alpha\rangle\varphi\,\mathrm{d}x\\
&\leqslant \Big(\int_\Omega \langle \mathrm{D}_z\mathbb{A}(\nabla u_k)\partial_j\nabla u_k, \partial_j\nabla u_k\rangle|\varphi|\,\mathrm{d}x\Big)^{\frac12}\\
&\qquad\times\Big(\int_\Omega \langle \mathrm{D}_z\mathbb{A}(\nabla u_k)G_\ell(|\nabla u_k|)e_i\otimes e^\alpha, G_\ell(|\nabla u_k|)e_i\otimes e^\alpha\rangle|\varphi|\,\mathrm{d}x\Big)^{\frac12}
\end{aligned}$$

(and analogously with i replaced by j). Thus, employing the a priori estimate (6.21) from Lemma 6.15 (note $K=\operatorname{spt}(\varphi)\Subset\Omega$), the upper bound in assumption (6.11), boundedness of f' and the growth behaviour (6.29), we arrive at

$$\begin{aligned}
\mathbf{I}_\ell &\leqslant C\lim_{k\to\infty}\Big(\int_\Omega (1+|\nabla u_k|)^{-1}|G_\ell(|\nabla u_k|)|^2|\varphi|\,\mathrm{d}x\Big)^{\frac12}\\
&\quad + C\lim_{k\to\infty} 2\int_\Omega |G_\ell(|\nabla u_k|)||\nabla\varphi|\,\mathrm{d}x\\
&\leqslant C\lim_{k\to\infty}\Phi\Big(\int_{\{\ell\leqslant|\nabla u_k|\leqslant 2\ell\}}|\nabla u_k|\,\mathrm{d}x\Big),
\end{aligned}$$

with $\Phi\colon\mathbb{R}_0^+\to\mathbb{R}_0^+$ given by $\Phi(t):=\max\{t^{\frac12},t\}$ and a constant C depending only on the data and φ, but not on ℓ. Finally, the poinwise convergence $\nabla u_k\to E$ allows us to pass to the limit $k\to\infty$, which yields

$$\mathbf{I}_\ell\leqslant C\Phi\Big(\int_{\{|E|\geqslant\ell\}}|E|\,\mathrm{d}x\Big).$$

In view of the integrability of E, this proves $\lim_{\ell\to\infty}\mathbf{I}_\ell=0$. In conclusion, since $\alpha\in\{1,\dots,N\}$ was arbitrary, we have shown the claim (6.27), i.e., that $E\in\mathrm{L}^1(\Omega;\mathbb{R}^{N\times n})$ is curl-free in the sense of distributions. Thus, as Ω is a simply connected Lipschitz domain, Lemma 6.12 provides a mapping $v\in\mathrm{W}^{1,1}(\Omega;\mathbb{R}^N)$ with $\nabla v=E$, and the proof of the representation is complete. □

For the sake of completeness, we now proceed by demonstrating that $v\in\mathrm{W}^{1,1}(\Omega;\mathbb{R}^N)$ – after translation by $(v)_\Omega$ – is actually a solution to system (6.2) subject to Neumann condition (6.3). To this end, we firstly provide the

Proof of the uniqueness assertion of Theorem 6.7(a). Suppose that there are two solutions $u_1,u_2\in\mathrm{W}^{1,1}(\Omega;\mathbb{R}^N)$ to the system (6.2) subject to (6.3), with $(u_1)_\Omega=(u_2)_\Omega=0$ and $u_1\neq u_2$ as L^1 functions, which, by connectedness of Ω, also implies $\nabla u_1\neq\nabla u_2$ as L^1 functions. In view of Lemma 6.2, u_1 and u_2 both solve the variational problem (6.4), i.e., they both minimise $\mathfrak{F}$ in $\mathrm{W}^{1,1}(\Omega;\mathbb{R}^N)$. Choosing $(u_1+u_2)/2\in\mathrm{W}^{1,1}(\Omega;\mathbb{R}^N)$ as competitor, we deduce from the strict convexity of f (implied by the assumption of h-monotonicity) combined with the minimality of u_1 and u_2

$$\mathfrak{F}\Big[\frac{u_1+u_2}{2}\Big]<\frac12\big(\mathfrak{F}[u_1]+\mathfrak{F}[u_2]\big)=\inf_{\mathrm{W}^{1,1}(\Omega;\mathbb{R}^N)}\mathfrak{F},$$

which is a contradiction. Thus the proof of uniqueness is complete. □

We shall now conclude the proof of Theorem 6.7(a) by the

Proof of the solution property of $v - (v)_\Omega$. By Corollary 6.14, we first note that (u_k) is a minimising sequence for $\mathfrak{F}$. Next, by the pointwise convergence (6.26) we obtain $f(|\nabla u_k|) - \langle T_0, \nabla u_k\rangle \to f(|\nabla v|) - \langle T_0, \nabla v\rangle$ in the limit $k \to \infty$, $\mathscr{L}^n$-a.e. in Ω. By the coerciveness condition (6.10), which in turn implies the boundedness of the map $\mathbb{R}^{N\times n} \ni z \mapsto f(|z|) - \langle T_0, z\rangle$ from below, we thus deduce by the generalised version of Fatou's Lemma[2]

$$\mathfrak{F}[v - (v)_\Omega] = \mathfrak{F}[v] \leqslant \liminf_{k\to\infty} \mathfrak{F}[u_k] = \inf_{\mathrm{W}^{1,1}(\Omega;\mathbb{R}^N)} \mathfrak{F}.$$

In conclusion, we have shown that $v-(v)_\Omega$ is a minimiser with vanishing mean value in Ω. Taking advantage of Lemma 6.2, it is also the desired weak solution to the system (6.2) subject to (6.3). This completes the proof of Theorem 6.7(a). $\square$

6.6. The Dual Problem and Relaxation to BV

The purpose of the present section is to firstly introduce the dual problem associated to the (primal) minimization problem (6.4) with an explicit description and secondly to study its connection to (generalised) minima of the primal problem. In doing so, we shall adopt a more general viewpoint and hereafter let $F\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ be a convex and differentiable function that satisfies, for some constants $0 < c \leqslant C < \infty$, the linear growth condition

$$c|z| - C \leqslant F(z) \leqslant C(1 + |z|) \qquad \text{for all } z \in \mathbb{R}^{N\times n}. \tag{6.30}$$

Then, for a map $T_0 \in \mathrm{C}_b(\Omega;\mathbb{R}^{N\times n})$, we define the functional $\mathcal{F}\colon\ \mathrm{W}^{1,1}(\Omega;\mathbb{R}^N) \to \mathbb{R}$ by

$$\mathcal{F}[w] := \int_\Omega f(\nabla w) - \langle T_0, \nabla w\rangle \,\mathrm{d}x, \qquad \text{for } w \in \mathrm{W}^{1,1}(\Omega;\mathbb{R}^{N\times n}). \tag{6.31}$$

6.6.1. Coerciveness As a modification of the coerciveness condition for radially symmetric integrands (6.10), in this section we shall work with the condition

$$\|T_0\|_{\mathrm{L}^\infty(\Omega;\mathbb{R}^{N\times n})} < \min\{f^\infty(z)\colon z \in \mathbb{R}^{N\times n} \text{ with } |z| = 1\}, \tag{6.32}$$

where we recall the *recession function* $f^\infty\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ given by

$$f^\infty(z) := \lim_{t\searrow 0} t f\left(\frac{z}{t}\right) \qquad \text{for all } z \in \mathbb{R}^{N\times n}.$$

We note that f^∞ is strictly positive, finite-valued and convex, as a consequence of the linear growth condition and the convexity of f. Hence, it attains its strictly positive minimum on $\{z \in \mathbb{R}^{N\times n}\colon |z| = 1\}$ (also compare with (6.37) in the non-autonomous setting below). Now, (6.32) is a generalisation of condition (6.10) in the following sense:

Lemma 6.18. *Let $f\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ be a convex function satisfying* (6.30) *and let $T_0 \in \mathrm{L}^\infty(\Omega;\mathbb{R}^{N\times n})$ verify* (6.32)*. Then the functional $\mathcal{F}$ given by* (6.31) *is coercive in the sense that if (w_k) is a sequence in $\mathrm{W}^{1,1}(\Omega;\mathbb{R}^N)$ such that $\|w_k\|_{\mathrm{W}^{1,1}(\Omega;\mathbb{R}^N)} \to \infty$ as $k \to \infty$ and each w_k has vanishing mean value, then $\mathcal{F}[w_k] \to \infty$ as $k \to \infty$.*

Proof. In what follows we fix $\xi_0 \in \mathbb{R}^{N\times n}$ satisfying $|\xi_0| = 1$ and $f^\infty(\xi_0) = \min\{f^\infty(z)\colon z \in \mathbb{R}^{N\times n} \text{ with } |z| = 1\} > 0$, and we further write

$$\|T_0\|_{\mathrm{L}^\infty(\Omega;\mathbb{R}^{N\times n})} = (1 - 3\delta) f^\infty(\xi_0)$$

[2] If (g_k) is a sequence of $\mathbb{R}\cup\{\infty\}$-valued measurable functions on a measure space (Ω, Σ, μ) and there exists a non-negative integrable function g on Ω such that $g_k \geqslant -g$ for all $k \in \mathbb{N}$, then there holds $\int_\Omega \liminf_{k\to\infty} g_k \,\mathrm{d}\mu \leqslant \liminf_{k\to\infty} \int_\Omega g_k \,\mathrm{d}\mu$. This follows from the standard version of Fatou's Lemma by applying the latter to the sequence $(g_k + g)$.

for some number $\delta \in (0, 3^{-1}]$ depending only on f and T_0. We now consider an arbitrary function $w \in \mathrm{W}^{1,1}(\Omega;\mathbb{R}^N)$ with $(w)_\Omega = 0$. In order to evaluate $\mathcal{F}[w]$, we decompose the domain of integration for some $\ell_0 \geqslant 1$ (to be determined later) as

$$\begin{aligned}
\mathcal{F}[w] &= \int_{\{|\nabla w|\leqslant \ell_0\}} f(\nabla w) - \langle T_0, \nabla w\rangle \,\mathrm{d}x + \int_{\{|\nabla w|>\ell_0\}} f(\nabla w) - \langle T_0, \nabla w\rangle \,\mathrm{d}x \\
&\geqslant \int_{\{|\nabla w|\leqslant \ell_0\}} f(\nabla w) - \langle T_0, \nabla w\rangle \,\mathrm{d}x \\
&\quad + \int_{\{|\nabla w|>\ell_0\}} \Big(\frac{1}{|\nabla w|} f\Big(|\nabla w|\frac{\nabla w}{|\nabla w|}\Big) - (1-2\delta) f^\infty(\xi_0)\Big)|\nabla w| \,\mathrm{d}x \\
&\quad + \delta f^\infty(\xi_0) \int_{\{|\nabla w|>\ell_0\}} |\nabla w| \,\mathrm{d}x =: \mathbf{I} + \mathbf{II} + \mathbf{III}.
\end{aligned}$$

For the first term, we obtain via the growth condition (6.30)

$$|\mathbf{I}| \leqslant \big(C(1+\ell_0) + \ell_0 \|T_0\|_{\mathrm{L}^\infty(\Omega;\mathbb{R}^{N\times n})}\big)\mathscr{L}^n(\Omega).$$

We next show that the second term is non-negative, provided that the level ℓ_0 is suitably chosen. To this end, we choose a finite number of points $(\xi_k)_{k\in\{1,\ldots,M\}}$ in $\mathbb{S}^{N\times n-1}$ such that

$$\inf_{k\in\{1,\ldots,M\}} |\xi - \xi_k| \leqslant \frac{\delta}{L} f^\infty(\xi_0) \quad \text{for all } \xi \in \mathbb{S}^{N\times n-1},$$

where L denotes the Lipschitz constant of f. Thus, M depends only on n, N, δ and f. Then, taking into account the fact that $\ell \mapsto f(\ell\xi)/\ell$ is monotonically increasing and converges to $f^\infty(\xi) \geqslant f^\infty(\xi_0)$ as $\ell \nearrow \infty$, we determine $\ell_0 \geqslant 1$ such that

$$\ell^{-1} f(\ell\xi_k) \geqslant (1-\delta) f^\infty(\xi_0) \qquad \text{for all } k \in \{1,\ldots,M\} \text{ and } \ell \geqslant \ell_0\,.$$

Consequently, by the Lipschitz continuity of f, we find

$$\begin{aligned}
\ell^{-1} f(\ell\xi) &\geqslant \ell^{-1} \sup_{k\in\{1,\ldots,M\}} \big[f(\ell\xi_k) - |f(\ell\xi) - f(\ell\xi_k)|\big] \\
&\geqslant \sup_{k\in\{1,\ldots,M\}} \big[\ell^{-1} f(\ell\xi_k) - L|\xi - \xi_k|\big] \geqslant (1-2\delta) f^\infty(\xi_0)
\end{aligned}$$

for all $\xi \in \mathbb{S}^{N\times n-1}$ and $\ell \geqslant \ell_0$. Applying this inequality pointwisely with $\xi = \nabla w/|\nabla w|$, we thus arrive at $\mathbf{II} \geqslant 0$ as claimed. Finally, we observe

$$\mathbf{III} \geqslant \delta f^\infty(\xi_0)\Big[\int_\Omega |\nabla w| \,\mathrm{d}x - \ell_0 \mathscr{L}^n(\Omega)\Big].$$

In conclusion, we have shown

$$\mathcal{F}[w] \geqslant \delta f^\infty(\xi_0) \int_\Omega |\nabla w| \,\mathrm{d}x - C(f, T_0)\ell_0 \mathscr{L}^n(\Omega)$$

for all functions $w \in \mathrm{W}^{1,1}(\Omega;\mathbb{R}^N)$ with $(w)_\Omega = 0$, and in combination with Poincaré's inequality for $\mathrm{W}^{1,1}$-maps with vanishing mean value, this immediately implies the assertion. □

6.6.2. The Dual Problem In a next step, we provide the following lemma where, in particular, the radial structure imposed on $\mathfrak{F}$ given by (6.4) is dropped:

Lemma 6.19 (On the dual functional)**.** *The following holds:*

(a) *For all $v \in \mathrm{W}^{1,1}(\Omega;\mathbb{R}^N)$ we have the representation*

$$\mathcal{F}[v] \geqslant \sup_{\chi\in\mathrm{L}^\infty(\Omega;\mathbb{R}^{N\times n})} \int_\Omega \langle \chi, \nabla v\rangle - f^*(T_0 + \chi) \,\mathrm{d}x.$$

(b) *Given* $\chi \in \mathrm{L}^{\infty}(\Omega;\mathbb{R}^{N\times n})$*, set*

$$\mathcal{R}[\chi] := \inf\left\{\int_{\Omega}\langle \chi,\nabla v\rangle - f^{*}(T_0+\chi)\,\mathrm{d}x\colon\; v\in \mathrm{W}^{1,1}(\Omega;\mathbb{R}^{N})\right\}. \tag{6.33}$$

Then there holds

$$\sup_{\chi\in\mathrm{L}^{\infty}(\Omega;\mathbb{R}^{N\times n})}\mathcal{R}[\chi] = \sup_{\chi\in\mathrm{L}^{\infty}_{\perp,T_0}(\Omega;\mathbb{R}^{N\times n})} -\int_{\Omega} f^{*}(T_0+\chi)\,\mathrm{d}x, \tag{6.34}$$

where we have set

$$\mathrm{L}^{\infty}_{\perp,T_0}(\Omega;\mathbb{R}^{N\times n}) := \left\{\chi\in\mathrm{L}^{\infty}(\Omega;\mathbb{R}^{N\times n})\colon \begin{array}{c}\int_{\Omega}\langle\chi,\nabla v\rangle\,\mathrm{d}x = 0 \text{ for all } v\in\mathrm{W}^{1,1}(\Omega;\mathbb{R}^{N})\\ \chi+T_0\in\mathrm{dom}(f^{*})\,\mathscr{L}^{n}-a.e.\end{array}\right\}.$$

Proof. Ad (a). Let $w\in\mathrm{L}^{1}(\Omega;\mathbb{R}^{N\times n})$ and set $\widetilde{f}(w) := f(w)-\langle T_0,w\rangle$. Since $f\in \mathrm{C}^{1}(\mathbb{R}^{N\times n})$, we conclude by the duality relation (2.5) and $(\mathrm{L}^{1}(\Omega;\mathbb{R}^{N\times n}))^{*}\cong\mathrm{L}^{\infty}(\Omega;\mathbb{R}^{N\times n})$

$$\int_{\Omega} f(w)-\langle T_0,w\rangle\,\mathrm{d}x = \int_{\Omega}\widetilde{f}(w)\,\mathrm{d}x \geqslant \sup_{\chi\in\mathrm{L}^{\infty}(\Omega;\mathbb{R}^{N\times n})}\int_{\Omega}\langle\chi,w\rangle - \widetilde{f}^{*}(\chi)\,\mathrm{d}x.$$

Inserting $w=\nabla v$ for a $\mathrm{W}^{1,1}$-function $v\colon\Omega\to\mathbb{R}^{N}$, (a) follows because for $\mathscr{L}^{n}$-a.e. $x\in\Omega$

$$\begin{aligned}\widetilde{f}^{*}(\chi(x)) &= \sup_{\zeta\in\mathbb{R}^{N\times n}}\langle\chi(x),\zeta\rangle - f(\zeta)+\langle T_0,\zeta\rangle\\ &= \sup_{\zeta\in\mathbb{R}^{N\times n}}\langle\chi(x)+T_0(x),\zeta\rangle - f(\zeta) = f^{*}(\chi(x)+T_0(x)).\end{aligned}$$

Ad (b). Let us note that if $\chi\in(\mathrm{L}^{\infty}\setminus\mathrm{L}^{\infty}_{\perp,T_0})(\Omega;\mathbb{R}^{N\times n})$, then the infimum appearing in the definition of $\mathcal{R}[\chi]$ is $-\infty$. Indeed, for such χ pick $v\in\mathrm{W}^{1,1}(\Omega;\mathbb{R}^{N})$ with $\int_{\Omega}\langle\chi,\nabla v\rangle\,\mathrm{d}x<0$ and define a sequence $(v_k)\subset\mathrm{W}^{1,1}(\Omega;\mathbb{R}^{N})$ by $v_k := kv$. If χ does not qualify as an element of $\mathrm{L}^{\infty}_{\perp,T_0}(\Omega;\mathbb{R}^{N\times n})$ but satisfies $\chi+T_0\in\mathrm{dom}(f^{*})$ $\mathscr{L}^{n}$-a.e. in Ω, then we deduce by $k\int_{\Omega}\langle\chi,\nabla v\rangle\,\mathrm{d}x - \int_{\Omega}f^{*}(T_0+\chi)\,\mathrm{d}x\to-\infty$ as $k\to\infty$ that $\mathcal{R}[\chi]=-\infty$. On the other side, if $\chi(x)+T_0(x)\notin\mathrm{dom}(f^{*})$ for $\mathscr{L}^{n}$-a.e. $x\in\Omega$, then $f^{*}(\chi+T_0)=\infty$ on a set of positive Lebesgue measure. This immediately yields $\mathcal{R}[\chi]=-\infty$ if $\int_{\Omega}\langle\chi,\nabla v\rangle\,\mathrm{d}x=0$ for all $v\in\mathrm{W}^{1,1}(\Omega;\mathbb{R}^{N})$. If the latter condition is not satisfied, then insert the sequence (v_k) as given above to deduce $\mathcal{R}[\chi]=-\infty$ too. This concludes the proof. $\square$

Remark 6.20. *If we consider the respective Dirichlet problem and thus compare with the theory outlined in Chapter 5.4 in the* gradient case, *i.e.,* (6.4) *restricted to competitor maps* $u\in\mathrm{W}^{1,1}_{0}(\Omega;\mathbb{R}^{N})$*, then the supremum in* (6.34) *is taken over all* $\chi\in\mathrm{L}^{\infty}(\Omega;\mathbb{R}^{N})$ *which are divergence-free in the sense of distributions. In this sense, the fact that we allow for a larger set of competitor maps in the Neumann problem than for the Dirichlet problem is reflected by shrinking the set of competitors in the respective dual problems.*

Lemma 6.21 (Inf-Sup-Relations)**.** *Define* $\mathcal{F}$ *and* $\mathcal{R}$ *by* (6.31) *and* (6.33)*, respectively. Then there holds*

$$\inf_{\mathrm{W}^{1,1}(\Omega;\mathbb{R}^{N})}\mathcal{F} = \sup_{\mathrm{L}^{\infty}_{\perp,T_0}(\Omega;\mathbb{R}^{N\times n})}\mathcal{R}.$$

Proof. Firstly note that by the duality relations (2.5), we have for any $w\in\mathrm{W}^{1,1}(\Omega;\mathbb{R}^{N})$ via Lemma 6.19 (b) and then (a)

$$\begin{aligned}\sup_{\mathrm{L}^{\infty}_{\perp,T_0}(\Omega;\mathbb{R}^{N\times n})}\mathcal{R} &= \sup_{\chi\in\mathrm{L}^{\infty}_{\perp,T_0}(\Omega;\mathbb{R}^{N\times n})}\;\inf_{v\in\mathrm{W}^{1,1}(\Omega;\mathbb{R}^{N})}\int_{\Omega}\langle\chi,\nabla v\rangle - f^{*}(T_0+\chi)\,\mathrm{d}x\\ &\leqslant \sup_{\chi\in\mathrm{L}^{\infty}_{\perp,T_0}(\Omega;\mathbb{R}^{N\times n})}\int_{\Omega}\langle\chi,\nabla w\rangle - f^{*}(T_0+\chi)\,\mathrm{d}x = \mathcal{F}[w].\end{aligned}$$

Now pass to the infimum over all $w \in \mathrm{W}^{1,1}(\Omega;\mathbb{R}^N)$ to conclude validity of '$\geqslant$'. The remaining inequality '$\leqslant$' can either be settled by referring to the general theory of convex duality as outlined in Section 6.6.5, or by suitable approximation procedure, for which the reader is referred to Remark 6.23. □

For the time being, we return to the radially symmetric case and next establish a connection between the primal and the dual problem, showing that the sequence (u_k) of minimisers u_k for the functionals $\mathfrak{F}_k$ can be used to construct a maximiser (which is given by $B - T_0$ in our setting) for the dual problem.

Lemma 6.22. *Consider a convex function $f \in \mathrm{C}^2(\mathbb{R}_0^+)$ satisfying $f(0) = f'(0) = 0$ and (6.1), $T_0 \in \mathrm{L}^\infty(\Omega;\mathbb{R}^{N\times n})$ with (6.10), and the sequence (u_k) of minimisers u_k of the functionals $\mathfrak{F}_k$ from Lemma (6.13). Then any weak-$*$ L^∞-cluster point of the sequence $(\mathbb{A}(\nabla u_k) - T_0)$ is a maximiser of the dual problem.*

Proof. Let τ be a weak*–L^∞-cluster point of $(\mathbb{A}(\nabla u_k) - T_0)$, and without loss of generality we may assume that it is actually its weak-$*$ limit. We start by showing that τ belongs to the space $\mathrm{L}^\infty_{\perp,T_0}(\Omega;\mathbb{R}^{N\times n})$ of admissible functions for the dual formulation. To this end, with $k^{-1/2}\nabla u_k \to 0$ in $\mathrm{L}^2(\Omega;\mathbb{R}^{N\times n})$ from Corollary 6.14, we infer with the Euler–Lagrange system (6.14)

$$\begin{aligned}\int_\Omega \langle \tau, \nabla\varphi\rangle \,\mathrm{d}x &= \lim_{k\to\infty} \int_\Omega \langle \mathbb{A}(\nabla u_k) - T_0, \nabla\varphi\rangle \,\mathrm{d}x \\ &= \lim_{k\to\infty} -k^{-1} \int_\Omega \langle \nabla u_k, \nabla\varphi\rangle \,\mathrm{d}x = 0\end{aligned}$$

for any $\varphi \in \mathrm{W}^{1,2}(\Omega;\mathbb{R}^N)$, and thus, by smooth approximation, we obtain

$$\int_\Omega \langle \tau, \nabla\varphi\rangle \,\mathrm{d}x = 0 \qquad \text{for all } \varphi \in W^{1,1}(\Omega;\mathbb{R}^N).$$

Moreover, since $|\mathbb{A}(\nabla u_k)| \in \mathrm{dom}(f^*)$ via (2.5) and Remark 2.8 for all $k \in \mathbb{N}$, $\nabla u_k \in \mathrm{L}^1(\Omega;\mathbb{R}^{N\times n})$ and $|A(\nabla u_k)| \in \partial f(|\nabla u_k|)$, we obtain using the duality relation (2.5) and once again the Euler–Lagrange system (6.14), now applied with $\varphi = u_k$ that

$$\begin{aligned}\mathfrak{F}_k[u_k] &= \int_\Omega f(|\nabla u_k|) - \langle T_0, \nabla u_k\rangle \,\mathrm{d}x + (2k)^{-1} \int_\Omega |\nabla u_k|^2 \,\mathrm{d}x \\ &= \int_\Omega \langle \mathbb{A}(\nabla u_k) - T_0, \nabla u_k\rangle - f^*(|\mathbb{A}(\nabla u_k)|) \,\mathrm{d}x + (2k)^{-1} \int_\Omega |\nabla u_k|^2 \,\mathrm{d}x \\ &= -\int_\Omega f^*(|\mathbb{A}(\nabla u_k)|) \,\mathrm{d}x - (2k)^{-1} \int_\Omega |\nabla u_k|^2 \,\mathrm{d}x.\end{aligned}$$

Since f^* as the pointwise supremum of affine functions is convex and lower semicontinuous, the map $\chi \mapsto -\int_\Omega f^*(|\chi|)\,\mathrm{d}x$ is upper semicontinuous with respect to weak*-convergence in $\mathrm{L}^\infty(\Omega;\mathbb{R}^{N\times n})$, and we therefore deduce from the previous inequailty

$$\lim_{k\to\infty} \mathfrak{F}_k[u_k] \leqslant -\int_\Omega f^*(|T_0 + \tau|) \,\mathrm{d}x. \tag{6.35}$$

Finally, with the duality formula from Lemma 6.21, the approximation property of the functionals $\mathfrak{F}_k$ from Corollary 6.14 and with (6.35), we arrive at

$$\sup_{\mathrm{L}^\infty_{\perp,T_0}(\Omega;\mathbb{R}^{N\times n})} \mathcal{R} \leqslant \inf_{\mathrm{W}^{1,1}(\Omega;\mathbb{R}^N)} \mathfrak{F} = \lim_{k\to\infty} \mathfrak{F}_k[u_k] \leqslant -\int_\Omega f^*(|T_0 + \tau|)\,\mathrm{d}x = \mathcal{R}[\tau], \tag{6.36}$$

and the proof of the lemma is complete. □

Remark 6.23 (On the Inf-Sup Relations)**.** *The preceding lemma allows for an alternative proof of the inf-sup relations as claimed in Lemma 6.21. Indeed, noting that the derivation of (6.36) only relies on the inequality '$\geqslant$' in Lemma 6.21, estimating $\mathcal{R}[\tau] \leqslant \sup_{\mathrm{L}^\infty_{\perp,T_0}(\Omega;\mathbb{R}^{N\times n})} \mathcal{R}$ yields the other inequality '$\leqslant$' by simple use of (6.36).*

Lemma 6.24. *In the situation of Lemma 6.19, if $v \in \mathrm{W}^{1,1}(\Omega;\mathbb{R}^N)$ is a minimiser of the primal problem, then the unique maximiser of the dual problem is given by $\sigma = f'(\nabla v) - T_0$.*

Proof. Firstly note that since v is a minimiser of $\mathcal{F}$, $\int_\Omega \langle \sigma, \nabla v\rangle \,\mathrm{d}x = 0$ is a consequence of the Euler-Lagrange equation satisfied by v. Inserting σ into $\mathcal{R}$ yields

$$\begin{aligned}\mathcal{R}[\sigma] &= -\int_\Omega f^*(T_0+\sigma)\,\mathrm{d}x = -\int_\Omega f^*(f'(\mathrm{D}v))\,\mathrm{d}x \\ &\overset{(2.5)}{=} \int_\Omega f(\mathrm{D}v) - \langle f'(\mathrm{D}v), \mathrm{D}v\rangle \,\mathrm{d}x = \int_\Omega f(\mathrm{D}v) - \langle T_0, \mathrm{D}v\rangle\,\mathrm{d}x = \inf_{\mathrm{W}^{1,1}(\Omega;\mathbb{R}^{N\times n})} \mathcal{F},\end{aligned}$$

the penultimate step being valid because v satisfies the Euler–Lagrange equation of $\mathcal{F}$ (use $\varphi = v$ as a test function). The claim now follows by use of Lemma 6.21. $\square$

6.6.3. Relaxation of the Primal Problem As mentioned in the introduction, the lack of weak compactness of bounded sets in the non-reflexive space $\mathrm{W}^{1,1}(\Omega;\mathbb{R}^N)$ suggests the passage to a space that enjoys better compactness properties. The natural candidate for such a space is given by $\mathrm{BV}(\Omega;\mathbb{R}^N)$, the space of functions of *bounded variation* as introduced and discussed in some detail in Chapter 4.2.

To apply the functional $\mathcal{F}$ which is a priori defined on $\mathrm{W}^{1,1}(\Omega;\mathbb{R}^N)$ to $v \in \mathrm{BV}(\Omega;\mathbb{R}^N)$, it must be suitably relaxed. We set, for $T_0 \in \mathrm{C}(\Omega;\mathbb{R}^{N\times n})$ and $\widetilde{f} := f - \langle T_0, \cdot\rangle$,

$$\overline{\mathcal{F}}[v] := \int_\Omega \widetilde{f}(\nabla v)\,\mathrm{d}x + \int_\Omega \widetilde{f}^\infty\Big(\frac{\mathrm{d}\mathrm{D}^s v}{\mathrm{d}|\mathrm{D}^s v|}\Big)\,\mathrm{d}|\mathrm{D}^s v| \qquad \text{for } v \in \mathrm{BV}(\Omega;\mathbb{R}^N).$$

Here, $\mathrm{D}v = \mathrm{D}^{ac}v + \mathrm{D}^s v = \nabla v \mathscr{L}^n + \frac{\mathrm{d}\mathrm{D}^s v}{\mathrm{d}|\mathrm{D}^s v|}|\mathrm{D}^s v|$ is the Radon–Nikodym decomposition of $\mathrm{D}v$ into its absolutely continuous and singular parts with respect to Lebesgue measure $\mathscr{L}^n$ and $\widetilde{f}^\infty \colon \mathbb{R}^{N\times n} \to \mathbb{R}$ is the *recession function* which is defined as

$$\widetilde{f}^\infty(x,z) := \lim_{\substack{t\searrow 0\\ \widetilde{x}\to x\\ \widetilde{z}\to z}} t\widetilde{f}\Big(\widetilde{x}, \frac{\widetilde{z}}{t}\Big) \qquad \text{for } x\in\Omega,\; z \in \mathbb{R}^{N\times n}. \tag{6.37}$$

Note that, by oocurance of T_0 in $\widetilde{f}$, the recession function now in fact depends on x, too. By continuity of T_0, it is easy to see that this limit exists for all $x \in \Omega$ and $T_0 \in \mathbb{R}^{N\times n}$ and satisfies

$$\widetilde{f}^\infty(x,z) = f^\infty(z) - \langle T_0(x), z\rangle \qquad \text{for all } x \in \Omega,\; z\in\mathbb{R}^{N\times n},$$

where $f^\infty(z) := \lim_{t\searrow 0} tf(z/t)$ for $z \in \mathbb{R}^{N\times n}$ is the usual recession function of a convex and continuous function $f\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ of linear growth, i.e., f verifies (6.30). Also note that by continuity of T_0, $\overline{\mathcal{F}}$ is well-defined indeed[3].

Having introduced the relaxed functional, our next aim is to establish the existence of minima of $\overline{\mathcal{F}}$ as given by (6.38) subject to the generalised coerciveness condition (6.32) on $\mathrm{BV}(\Omega;\mathbb{R}^N)$. In accordance with the terminology for the Dirichlet problem introduced

[3]It is also possible to admit $T_0 \in \mathrm{L}^\infty(\Omega;\mathbb{R}^{N\times n})$ provided its (row–wise) divergence belongs to certain L^p–spaces. This is similar to Chapter 5.4.4, where we considered vanishing divergences, and can be approached by introducing so-called Anzellotti–type pairings between L^∞-functions and gradients of BV-functions, cp. [20] and Chapter 5.4.4.

in Chapter 4.3.2, such minima are also called *generalised minima* for $\mathcal{F}$. Since the latter condition extends that given for radially symmetric integrands f (see (6.10)) in a natural way, we firstly wish to continue Example 6.4, showing that it remains an essential ingredient to establish existence of minima for the relaxed functional too:

Example 6.25 (Example 6.4, continued)**.** In the situation of Example 6.4, observe that $\overline{\mathcal{F}}$ for $\mathcal{F} = \mathfrak{F}$ is given by

$$\overline{\mathcal{F}}[v] = \int_{-1}^{1} \sqrt{1+|v'|^2} - v'\,\mathrm{d}x + \int_{(-1,1)} \left|\frac{\mathrm{d}\mathrm{D}^s v}{\mathrm{d}|\,\mathrm{D}^s v|}\right| - \frac{\mathrm{d}\mathrm{D}^s v}{\mathrm{d}|\,\mathrm{D}^s v|}\,\mathrm{d}|\,\mathrm{D}^s v| - 2,$$

where now $\mathrm{D}v = v'\mathscr{L}^1 \llcorner (-1,1) + \mathrm{D}^s v$ is the Radon–Nikodým decomposition of $\mathrm{D}v$. Since $\overline{\mathcal{F}} = \mathcal{F}$ on $\mathrm{W}^{1,1}((-1,1))$ and the recession part is non-negative, we deduce by Example 6.4 that $\inf_{\mathrm{BV}((-1,1))} \overline{\mathcal{F}} = -2$. Now assume that there exists a function $u \in \mathrm{BV}((-1,1))$ with $\overline{\mathcal{F}}[u] = -2$. Then we obtain

$$\int_{-1}^{1} \sqrt{1+|u'|^2} - u'\,\mathrm{d}x = -\int_{(-1,1)} \left|\frac{\mathrm{d}\mathrm{D}^s u}{\mathrm{d}|\,\mathrm{D}^s u|}\right| - \frac{\mathrm{d}\mathrm{D}^s u}{\mathrm{d}|\,\mathrm{D}^s u|}\,\mathrm{d}|\,\mathrm{D}^s u|.$$

Now, since the left-hand side is non-negative due to $\sqrt{1+|\cdot|^2} \geqslant |\cdot|$ and since the right-hand side is non-positive, both terms actually need to vanish in order to achieve equality. We thus conclude $\sqrt{1+|u'|^2} - u' \equiv 0$ a.s. in $(-1,1)$ (as before in Example 6.4), which yields a contradiction and shows that such a function u cannot exist.

We now turn to the existence part for the primal problem. Here, to avoid a lenghty technical argument, we assume for the sake of simplicity that T_0 vanishes at the boundary, and refer the reader to [37] for the full statement.

Proposition 6.26. *Suppose that $f \in \mathrm{C}(\mathbb{R}^{N\times n})$ is a convex integrand of linear growth and $T_0 \in \mathrm{C}_0(\Omega;\mathbb{R}^{N\times n})$ is such that (6.32) holds. Then the variational principle*

$$\text{to minimise } \overline{\mathcal{F}} \text{ over } \mathrm{BV}(\Omega;\mathbb{R}^{N\times n}) \tag{6.38}$$

possesses a minimiser $v \in \mathrm{BV}(\Omega;\mathbb{R}^N)$. Moreover, v satisfies

$$\overline{\mathcal{F}}[v] = \inf_{\mathrm{W}^{1,1}(\Omega;\mathbb{R}^N)} \mathcal{F} = \sup_{\mathrm{L}^{\infty}_{\perp,T_0}(\Omega;\mathbb{R}^{N\times n})} \mathcal{R}. \tag{6.39}$$

Proof. Similarly as in Section 6.2, we deduce that $\overline{\mathcal{F}}$ is bounded below on $\mathrm{BV}(\Omega;\mathbb{R}^N)$ and by Poincaré's Inequality on BV, we deduce that minimising sequences

$$(v_k) \subset \mathrm{W}^{1,1}(\Omega;\mathbb{R}^{N\times n}) \cap \{v \in \mathrm{L}^1(\Omega;\mathbb{R}^N)\colon\ (v)_\Omega = 0\}$$

are uniformly bounded in $\mathrm{W}^{1,1}(\Omega;\mathbb{R}^N)$. Thus, by the weak*-compactness theorem in BV, there exists $v \in \mathrm{BV}(\Omega;\mathbb{R}^N)$ such that $v_k \overset{*}{\rightharpoonup} v$ as $k\to\infty$. We split

$$\begin{aligned}\overline{\mathcal{F}}[v] &= \int_\Omega f(\nabla v)\,\mathrm{d}x + \int_\Omega f^\infty\Big(\frac{\mathrm{d}\mathrm{D}^s v}{\mathrm{d}|\,\mathrm{D}^s v|}\Big)\,\mathrm{d}|\,\mathrm{D}^s v| - \int_\Omega \langle T_0, \nabla v\rangle\,\mathrm{d}x + \int_\Omega \langle T_0, \frac{\mathrm{d}\mathrm{D}^s v}{\mathrm{d}|\,\mathrm{D}^s v|}\rangle\,\mathrm{d}|\,\mathrm{D}^s v| \\ &=: \overline{\mathcal{F}}_1[v] + \overline{\mathcal{F}}_2[v].\end{aligned}$$

By the Reshetnyak (lower semi-)continuity theorem (cp. Proposition 4.18), we deduce by its lower semicontinuity part that $\overline{\mathcal{F}}_1[v] \leqslant \liminf_{k\to\infty} \overline{\mathcal{F}}_1[v_k]$. On the other side, we note that by definition of weak*-convergence,

$$\overline{\mathcal{F}}_2[v_k] = \langle \mathrm{D}v_k, T_0\rangle_{\mathcal{M}\times\mathrm{C}} \to \langle \mathrm{D}v, T_0\rangle_{\mathcal{M}\times\mathrm{C}} = \overline{\mathcal{F}}_2[v], \qquad k\to\infty,$$

where $\langle\cdot,\cdot\rangle_{\mathcal{M}\times\mathrm{C}}$ is a shorthand for the dual pairing between finite Radon measures and continuous functions on Ω. Summarising, we obtain $\overline{\mathcal{F}}[v] \leqslant \liminf_{k\to\infty}\overline{\mathcal{F}}[v_k] = \inf_{\mathrm{BV}(\Omega;\mathbb{R}^N)}\overline{\mathcal{F}}$ and therefore $v \in \mathrm{BV}(\Omega;\mathbb{R}^N)$ is proven to be a generalised minimiser for $\mathcal{F}$.

Turning to (6.39), we note that by Lemma 6.21 only the first equality requires a proof. First, since $\mathcal{F}$ and $\overline{\mathcal{F}}$ coincide on $\mathrm{W}^{1,1}(\Omega;\mathbb{R}^N)$, the inequality $\overline{\mathcal{F}}[v] = \inf_{\mathrm{W}^{1,1}(\Omega;\mathbb{R}^N)}$ is immediate. To obtain the reverse estimate '$\geqslant$' as well, let $v \in \mathrm{BV}(\Omega;\mathbb{R}^{N\times n})$ be a generalised minimiser for $\mathcal{F}$. By denseness of $(\mathrm{BV}\cap\mathrm{C}^\infty)(\Omega;\mathbb{R}^{N\times n})$ in $\mathrm{BV}(\Omega;\mathbb{R}^N)$ with respect to area–strict convergence, we find a sequence $(v_k)\subset(\mathrm{BV}\cap\mathrm{C}^\infty)(\Omega;\mathbb{R}^N)$ such that $v_k \to v$ area–strictly as $k\to\infty$. By the continuity part of the Reshetnyak theorem, Proposition 4.18, we then obtain

$$\inf_{\mathrm{W}^{1,1}(\Omega;\mathbb{R}^N)} \leqslant \mathcal{F}[v_k] = \overline{\mathcal{F}}[v_k] \to \overline{\mathcal{F}}[v] = \min_{\mathrm{BV}(\Omega;\mathbb{R}^N)}\overline{\mathcal{F}}, \qquad k\to\infty.$$

The proof is complete. □

6.6.4. Existence and Regularity for the Dual Problem We now finish the proof of Theorem 6.7 by giving the

Proof of Theorem 6.7(b). By part (a) of Theorem 6.7 and Lemma 6.24, we obtain that $\sigma := f'(|\mathrm{D}v|)\,\mathrm{D}v/|\mathrm{D}v| - T_0$ is the unique solution of the dual problem. Obviously, $\sigma\in\mathrm{L}^\infty(\Omega;\mathbb{R}^{N\times n})$, and thus it remains to verify the local $\mathrm{W}^{1,2}$-regularity of σ. To do so, we firstly recall that σ is a weak L^2-cluster point of the sequence $(\sigma_k) := (\mathbb{A}_k(\mathrm{D}u_k)-T_0)$ given by (6.15). We then obtain for $j=1,...,n$ with $g_k := f_k(|\cdot|)$ by use of the Cauchy–Schwarz inequality

$$\begin{aligned}
|\partial_j\sigma_k|^2 &= |\langle g_k''(\mathrm{D}u_k)(\partial_j\mathrm{D}u_k) - \partial_j T_0, \partial_j\sigma_k\rangle| \\
&\leqslant \Big(\langle g_k''(\mathrm{D}u_k)\partial_j\mathrm{D}u_k, \partial_j\mathrm{D}u_k\rangle\Big)^{\frac{1}{2}}\Big(\langle g_k''(\mathrm{D}u_k)\partial_j\sigma_k, \partial_j\sigma_k\rangle\Big)^{\frac{1}{2}} \\
&+ \|T_0\|_{\mathrm{W}^{1,\infty}(\Omega;\mathbb{R}^{N\times n})}|\partial_j\sigma_k| \\
&\leqslant \Big(C\Big(\langle g_k''(\mathrm{D}u_k)\partial_j\mathrm{D}u_k, \partial_j\mathrm{D}u_k\rangle\Big)^{\frac{1}{2}} + \|T_0\|_{\mathrm{W}^{1,\infty}(\Omega;\mathbb{R}^{N\times n})}\Big)|\partial_j\sigma_k|.
\end{aligned}$$

Since $\sup_{\mathbb{R}^{N\times n}}|g_k''|$ is uniformly bounded in k by h–monotonicity of f, the claim follows from the a-priori estimate from Lemma 6.15 by shifting the term $|\partial_j\sigma_k|$ to the left hand side of the inequality and integrating over an arbitrary open Lipschitz subset of Ω. The proof is complete. □

6.6.5. Concluding Remarks In their treatise [96], Ekeland and Temam introduced a rather general framework of convex duality into which the Neumann problem on $\mathrm{W}^{1,1}$ as described in the main part of our paper can be embedded in a natural way. To set up this framework conveniently, let V,Y be two topological vector spaces with continuous duals V^*,Y^* and suppose that a functional $\mathscr{F}\colon V\to\overline{\mathbb{R}} := \mathbb{R}\cup\{\infty\}$ can be written as $\mathscr{F}[u] = J(u,\Lambda u)$, where $\Lambda\colon V\to Y$ is a continuous linear mapping and $J\colon V\times Y\to\overline{\mathbb{R}}$ is a given map which splits into $J(u,\Lambda u) = F(u)+G(\Lambda u)$ with two convex functions $F\colon V\to\overline{\mathbb{R}}$ and $G\colon Y\to\overline{\mathbb{R}}$. By [96, Thm. III.4.1.] the dual problem to the minimisation of $\mathscr{F}$ over V is given by the maximisation of the functional $y^*\mapsto -J^*(\Lambda^*y^*,-y^*)(= -F^*(\Lambda^*y^*)-G^*(y^*))$ on Y^*, and if $\inf_V\mathscr{F}\in\mathbb{R}$ and there exists $u_0\in V$ such that $J(u_0,\Lambda u_0)<\infty$ together with $p\mapsto J(u_0,p)$ being continuous at Λu_0, then there holds

$$\inf_{u\in V}\mathscr{F}[u] = \sup_{y^*\in Y^*} -J^*(\Lambda^*y^*,-y^*).$$

Here, $\Lambda^*\colon Y^*\to V^*$ is the adjoint operator of Λ. To apply this abstract setting to the functional $\mathfrak{F}$ as given by (1.1), we set $V := \mathrm{W}^{1,1}(\Omega;\mathbb{R}^N)$, $Y := \mathrm{L}^1(\Omega;\mathbb{R}^{N\times n})$ and

$\Lambda := \mathrm{D}$ be the weak gradient operator. Moreover, we define $F\colon \mathrm{W}^{1,1}(\Omega;\mathbb{R}^N)$ by $F \equiv 0$ and $G\colon \mathrm{L}^1(\Omega;\mathbb{R}^{N\times n})$ by $G(y) := \int_\Omega \widetilde{f}(y)\,\mathrm{d}x$, where as above $\widetilde{f} := f(\cdot) - \langle T_0, \cdot\rangle$. Since $F^*(y^*) = +\infty$ provided $\langle y^*, v\rangle_{\mathrm{L}^\infty\times\mathrm{L}^1} \neq 0$ for some $v \in \mathrm{W}^{1,1}(\Omega;\mathbb{R}^N)$, we deduce by Lemma 6.19(a)

$$F^*(y^*) = \begin{cases} 0 & \text{if } y^* = 0 \\ +\infty & \text{else,} \end{cases}$$

$$G^*(y^*) = \begin{cases} \int_\Omega f^*(T_0 + y^*)\,\mathrm{d}x & \text{if } T_0 + y^* \in \mathrm{dom}(f^*)\,\mathscr{L}^n - \text{a.e.} \\ +\infty & \text{else,} \end{cases}$$

and thus the general theory of convex duality easily implies that

$$\begin{aligned} \inf_{u\in \mathrm{W}^{1,1}(\Omega;\mathbb{R}^N)} \mathscr{F}[u] &= \sup_{\substack{\Lambda^* y^* = 0 \\ T_0 + y^* \in \mathrm{dom}(f^*)\,\mathscr{L}^n - \text{a.e.}}} -G^*(y^*) \\ &= \sup_{\substack{\Lambda^* y^* = 0 \\ T_0 + y^* \in \mathrm{dom}(f^*)\,\mathscr{L}^n - \text{a.e.}}} -\int_\Omega f^*(T_0 + y^*)\,\mathrm{d}x. \end{aligned}$$

Now it suffices to realise that $\Lambda^* y^* = 0$ is equivalent to $\langle y^*, \Lambda v\rangle = 0$ for all $v \in \mathrm{W}^{1,1}(\Omega;\mathbb{R}^N)$ which, by $\Lambda = \mathrm{D}$ in conjunction with $T_0 + y^* \in \mathrm{dom}(f^*)$ $\mathscr{L}^n$-a.e., in turn precisely amounts to requiring $y^* \in \mathrm{L}^\infty_{\perp,T_0}(\Omega;\mathbb{R}^N)$.

We finally come back to the introduction and show that the h–monotonicity condition (6.11) is indeed satisfied for any strictly convex function $f \in \mathrm{C}^2(\mathbb{R})$ with linear growth (LG). To see this, we compute for arbitrary $z, \xi \in \mathbb{R}^{N\times n}$

$$\begin{aligned} \langle \mathcal{B}(z)\xi, \xi\rangle &= \Big\langle \mathrm{D}_z\Big(\frac{f'(|z|)}{|z|}z\Big)\xi, \xi\Big\rangle \\ &= \Big\langle \Big(f''(|z|)\frac{z\otimes z}{|z|^2} + \frac{f'(|z|)}{|z|}\,\frac{|z|^2\,\mathrm{Id}_{N\times n} - z\otimes z}{|z|^2}\Big)\xi, \xi\Big\rangle \\ &= f''(|z|)\frac{(z\cdot\xi)^2}{|z|^2} + \frac{f'(|z|)}{|z|}\,\frac{|z|^2|\xi|^2 - (z\cdot\xi)^2}{|z|^2}. \end{aligned}$$

By assumption, both terms on the right-hand side are non-negative. We now observe that whenever $2(\xi\cdot z)^2 \geqslant |z|^2|\xi|^2$ holds, we have

$$\langle \mathcal{B}(z)\xi, \xi\rangle \geqslant \frac{1}{2} f''(|z|)|\xi|^2$$

while in the opposite case $2(\xi\cdot z)^2 < |z|^2|\xi|^2$ we have

$$\langle \mathcal{B}(z)\xi, \xi\rangle \geqslant \frac{1}{2}\frac{f'(|z|)}{|z|}|\xi|^2.$$

In conclusion, we obtain

$$\langle \mathcal{B}(z)\xi, \xi\rangle \geqslant \frac{1}{2}\min_{0\leqslant s\leqslant |z|}\min\Big\{f''(s), \frac{f'(s)}{s}\Big\}|\xi|^2 := h(|z|)|\xi|^2$$

with an obvious definition of h, which is the claimed lower bound in (6.11). Moreover, h is also decreasing, by its definition and the strict convexity of f with $f'(0) = 0$.

6.7. Higher Integrability in the μ–elliptic case, $1 < \mu \leqslant 3$

In this section we contrast the $\mathrm{W}^{1,1}$–regularity assertion of Theorem 6.7 with *higher integrability* results for μ–elliptic integrands. This is in the spirit of [40, Chpt. 4.2] and as such merely a minor modification of the corresponding proofs therein, but now incorporating the inhomogeneity T_0.

Theorem 6.27. *Let $T_0 \in \mathrm{W}^{2,\infty}(\Omega;\mathbb{R}^{N\times n})$ and suppose that $f \in \mathrm{C}^2(\mathbb{R}^{N\times n})$ is a (possibly non–radially symmetric) μ–elliptic integrand. If the coerciveness criterion (6.32) is satisfied, then the following hold:*

(a) *If $1 < \mu < 1 + \frac{2}{n}$, then one generalised minimiser of (6.38) belongs to $\mathrm{W}^{1,p}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ for some $p > 1$.*

(b) *If $1 < \mu \leqslant 3$ and the local uniform boundedness hypothesis[4] holds, then one generalised minimiser of (6.38) belongs to $\mathrm{W}^{1,L\log L}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$.*

Proof. We confine ourselves to (a) as the proof of (b) follows along the same lines as the proof of [40, Thm. 4.21]. Hence we may assume $\mu < 2$. The coerciveness criterion (6.32) is only required to ensure the existence of a minimiser. Again, we consider viscosity approximations in the same manner as in (6.12) where now the integrand is *not* assumed radially symmetric. Since f is μ–elliptic, the key estimate from Remark 6.16 now takes the form

$$\sup_{k\in\mathbb{N}}\left\{\int_K \frac{|\nabla^2 u_k|^2}{(1+|\nabla u_k|^2)^{\frac{\mu}{2}}}\,\mathrm{d}x\right\} < \infty \qquad \text{for all } K \Subset \Omega. \tag{6.40}$$

Let $\mathrm{B} := \mathrm{B}(x_0,r) \Subset \Omega$ be an arbitrary but fixed ball. We introduce the function $V_\mu(\xi) := (1+|\xi|^2)^{\frac{2-\mu}{4}}$ for $\xi \in \mathbb{R}^{N\times n}$. We compute for $i \in \{1,...,n\}$

$$\partial_i V_\mu(\nabla u_k) = \frac{2-\mu}{4}(1+|\nabla u_k|^2)^{\frac{-\mu-2}{4}}\langle \partial_i \nabla u_k, \nabla u_k\rangle.$$

In conclusion, we have

$$\int_{\mathrm{B}} |\partial_i V_\mu(\nabla u_k)|^2\,\mathrm{d}x \leqslant \left|\frac{2-\mu}{4}\right|^2 \int_{\mathrm{B}} \frac{|\partial_i \nabla u_k|^2}{(1+|\nabla u_k|^2)^{\frac{\mu}{2}}}\,\mathrm{d}x$$

and thus, by (6.40) and arbitrariness of $i \in \{1,...,n\}$,

$$\sup_{k\in\mathbb{N}} \sum_{i=1}^n \int_{\mathrm{B}} |\partial_i V_\mu(\nabla u_k)|^2\,\mathrm{d}x < \infty. \tag{6.41}$$

In consequence, we obtain that $(V_\mu(\nabla u_k))$ is uniformly bounded in $\dot{\mathrm{W}}^{1,2}(\mathrm{B})$. Now, by arbitariness of $\mathrm{B} = \mathrm{B}(x_0,r)$, if

- $n = 2$, we use the embedding $\mathrm{W}^{1,2}_{\mathrm{loc}} \hookrightarrow \mathrm{BMO}_{\mathrm{loc}}$ to conclude that $(V_\mu(\nabla u_k))$ is uniformly bounded in $\mathrm{BMO}(U)$ for any $U \Subset \Omega$. Since $V_\mu(\xi) \sim |\xi|^{\frac{2-\mu}{2}}$ for large values of $|\xi|$, we deduce by the John–Nirenberg lemma (cp. Lemma 2.20) that $(|\nabla u_k|^{\frac{2-\mu}{2}})$ is uniformly bounded in any $\mathrm{L}^p(U)$, $1 < p < \infty$. Since $\mu < 2$, passing to a suitable subsequence thus establishes the claimed regularity assertion: There exists a generalised minimiser which belongs to *every* $\mathrm{L}^p_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$, $1 \leqslant p < \infty$.

- $n \geqslant 3$, we use the embedding $\mathrm{W}^{1,2}_{\mathrm{loc}} \hookrightarrow \mathrm{L}^{\frac{2n}{n-2}}_{\mathrm{loc}}$. Similarly as in the previous substep, we then find
$$\sup_{k\in\mathbb{N}} \int_U |\nabla u_k|^{\frac{2-\mu}{2}\frac{2n}{n-2}} < \infty.$$
We then note
$$\frac{2-\mu}{2}\cdot\frac{2n}{n-2} > 1 \quad \text{if and only if} \quad \mu < 1 + \frac{2}{n},$$
which is the exactly the exponent claimed in (a). Lastly, passing to a suitable subsequence shows that there exists one generalised minimiser of class $\mathrm{W}^{1,q}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ for all $1 \leqslant q < (2-\mu)\frac{n}{n-2}$.

[4] In the sense that for any $K \Subset \Omega$, the viscosity approximations (v_k) satisfy $\sup_{k\in\mathbb{N}} \|v_k\|_{\mathrm{L}^\infty} < \infty$; cp. [40, Ass. 4.11].

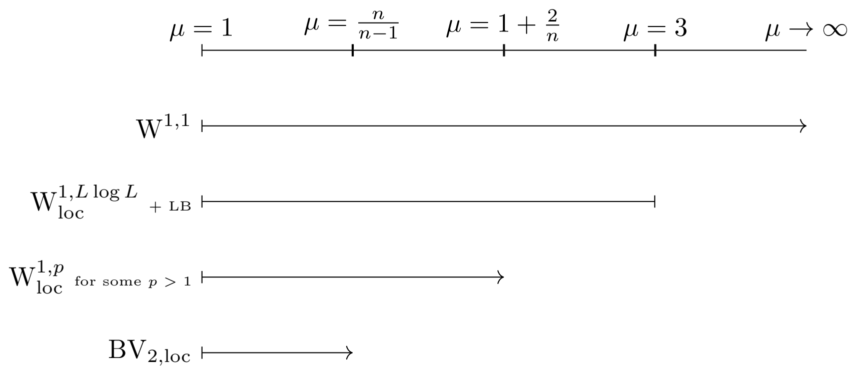

Figure 6.1: The regularity theory for the Neumann problem on BV for radially symmetric integrands and inhomogoneities $T_0 \in \mathrm{W}^{2,\infty}(\Omega;\mathbb{R}^{N\times n})$ within the scale of μ–elliptic integrands. The arrows indicate '*up to*', whereas the vertical dashs indicate '*including*'.

Omitting part (b) as indicated above, the proof is complete. □

Remark 6.28 (Higher Differentiability)**.** *In the situation of (a) of the preceding theorem, it is also possible to gain a higher differentiability result. Namely, working from the higher integrability assertion of (a), we obtain by use of Young's inequality and the fact that* $u_k \in \mathrm{W}^{2,2}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$

$$\sup_{k\in\mathbb{N}} \sum_{i=1}^{n} \int_U |\partial_i \nabla u_k| \,\mathrm{d}x \leqslant \sup_{k\in\mathbb{N}} \sum_{i=1}^{n} \int_U \frac{|\partial_i \nabla u_k|^2}{(1+|\nabla u_k|^2)^{\frac{\mu}{2}}} \,\mathrm{d}x + \sup_{k\in\mathbb{N}} \sum_{i=1}^{n} \int_U (1+|\nabla u_k|^2)^{\frac{\mu}{2}} \,\mathrm{d}x$$

for every $U \Subset \Omega$*. Thus we see that* (u_k) *is uniformly bounded in* $\mathrm{W}^{2,1}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ *if*

- $n = 2$ *and* $1 < \mu < 2$ *is arbitrary.*
- $n \geqslant 3$ *and*

$$\mu < (2-\mu)\frac{n}{n-2}, \quad \textit{i.e.,} \quad \mu < \frac{n}{n-1}.$$

In conclusion, passing to a suitable subsequence establishes the $\mathrm{BV}_{2,\mathrm{loc}}$*–regularity of one generalised minimiser.*

6.8. Comparison with the BD–case

Let us finally compare the situation as described above with the *symmetric gradient* case and explain where the approach used in Section 6.5 seems to be difficult to be apply in this situation. The respective analogue of (6.4) now reads as

$$\text{to minimise} \quad \mathfrak{F}[w] := \int_\Omega f(|\varepsilon(w)|) - \langle T_0, \varepsilon(w)\rangle \,\mathrm{d}x \quad \text{over } \mathrm{LD}(\Omega). \tag{6.42}$$

Similarly as in Section 6.2, a coerciveness criterion in terms of f^∞ can be found. Here, however, we need to take into account that ε not only ignores constants but rigid deformations as well. The approach as explained in Sections 6.4 and 6.5 therefore departs at two points; we keep in mind that Ω still is assumed to be simply connected:

- Recalling the Saint–Venant compatibility conditions from Example 3.6, we need to verify for the corresponding biting limit E (cp. (6.18)) in the symmetric gradient case that it is $\operatorname{curl}\circ\operatorname{curl}$–free in the sense of distributions. In this situation, we are able to argue as in Lemma 6.12 and *could* conclude LD–regularity.

- The foremost difficulty which is not clear to us how to be overcome is to verify that E in fact is $\operatorname{curl}\circ\operatorname{curl}$–free. The underlying reasons for this are two–fold.

  (a) Firstly, the $\operatorname{curl}\circ\operatorname{curl}$–operator is a *second order operator*. For the modification of the proof, this has the following consequence. Going to (6.27), we need to show

$$\int_\Omega \langle E, \operatorname{curl}\circ\operatorname{curl}(\varphi)\rangle\,\mathrm{d}x = 0 \qquad \text{for all } \varphi \in \mathrm{C}_c^\infty(\Omega;\mathbb{R}^{n\times n}). \tag{6.43}$$

  Proceeding as in (6.27)ff., we would similarly introduce the functions g_ℓ in order to use both $E \in \mathrm{L}^1(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})$ and the weak convergences of $(\boldsymbol{\varepsilon}(u_k))$ at our disposal (where (u_k) is the corresponding viscosity approximation sequence). Hence, in analogy with (6.28), we would write for arbitrary $\varphi \in \mathrm{C}_c^\infty(\Omega;\mathbb{R}^{n\times n})$

$$\begin{aligned}\left|\int_\Omega \langle E, \operatorname{curl}\circ\operatorname{curl}(\varphi)\rangle\,\mathrm{d}x\right| &= \left|\int_\Omega \langle g_\ell(|E|)E, \operatorname{curl}\circ\operatorname{curl}(\varphi)\rangle\,\mathrm{d}x\right| \\ &\quad + \left|\int_\Omega \langle (1-g_\ell(|E|))E, \operatorname{curl}\circ\operatorname{curl}(\varphi)\rangle\,\mathrm{d}x\right| =: \widetilde{\mathbf{I}}_\ell + \widetilde{\mathbf{II}}_\ell.\end{aligned}$$

  In a similar way as in (6.28)ff., $\widetilde{\mathbf{II}}_\ell \to 0$ as $\ell\to\infty$. The main problem now stems from the first term $\widetilde{\mathbf{I}}_\ell$. Following the line of proof in the full gradient case, we might either integrate by parts *once* or *twice* to shift one curl or $\operatorname{curl}\circ\operatorname{curl}$ from φ to $g_\ell(|E|)E$. The first option seems to be involved as (compare with (b)) as the operator curl, the symmetric gradient and the Uhlenbeck structure of the integrands are not as easily compatible as for the gradient. The second option, where the full operator $\operatorname{curl}\circ\operatorname{curl}$ is shifted to E reduces to estimates on the *third order* weak derivatives of u_k. In principle, stabilising $\mathfrak{F}$ by a suitable m–Laplacian $(-\Delta)^m := (-\Delta)\circ\dots\circ(-\Delta)$ (m–times), the respective viscosity approximation sequence is sufficiently regular to belong to, say, $\mathrm{W}^{3,2}_{\mathrm{loc}}$. However, in this situation, it is not clear to us how to generalise the corresponding weighted Sobolev–type estimate (cp. (6.24)) in a fruitful manner.

  (b) In general, even though the symmetric gradient has a comparatively simple structure, it still *destroys* the convenient structure when working, e.g., with Uhlenbeck–type problems, where the integrands depend on the modulus of the argument only. This is a general principle and causes difficulties also in the study of the symmetric p–Laplacean operator

$$-\Delta_p v := -\operatorname{div}(|\boldsymbol{\varepsilon}(v)|^{p-2}\boldsymbol{\varepsilon}(v)), \qquad v \in \mathrm{W}^{1,p}(\Omega;\mathbb{R}^n),$$

  or its stabilised version

$$-\widetilde{\Delta}_p v := -\operatorname{div}((1+|\boldsymbol{\varepsilon}(v)|^2)^{\frac{p-2}{2}}\boldsymbol{\varepsilon}(v)), \qquad v \in \mathrm{W}^{1,p}(\Omega;\mathbb{R}^n).$$

CHAPTER 7

Partial Regularity for Symmetric–Convex Problems

Having established $\mathrm{W}^{1,p}_{\mathrm{loc}}$–regularity ($p > 1$) for the Dirichlet problem on BD and $\mathrm{W}^{1,1}$–regularity for the Neumann problem on BV in the previous two chapters, we now turn to the partial regularity of minima. As explained in the introduction, cp. Example 1.4, in the situation of *vectorial* problems we cannot expect full Hölder regularity of minima but only on a set whose complement is relatively closed and has Lebesgue measure zero.

The purpose of the present chapter thus is to establish a partial regularity result for generalised minimisers of the functional (1.1) for *convex* integrands. In doing so, we adapt an approach due to ANZELLOTTI & GIAQUINTA [22] which is particularly designed for *convex* problems and is even applicable to possibly very degenerate convex variational principles. However, as already mentioned in [22], it is not clear how a modification for the proof to work in the strongly quasiconvex case could be accomplished. Therefore, the theory outlined in this chapter will be contrasted with the partial regularity for *strongly symmetric–quasiconvex problems* in Chapter 8: Here we shall consider the substantially weaker convexity notion of quasiconvexity. However, *if* we assume an integrand to be convex *and* to meet the conditions imposed throughout Chapter 8, then we end up with a *convexity* condition that is much stronger than the one in this chapter. So the theory outlined here is independent from the one presented in the final chapter of this thesis.

Structure of the chapter. Starting from a discussion of local BD–minimisers in Section 7.1, we give an overview of partial regularity results and present the main result of this chapter in Section 7.2. We then provide auxiliary estimates for linearised strongly convex systems in Section 7.3, give elementary properties of a V–type function in Section 7.4 and establish a convolution inequality leading to a suitable Poincaré–type inequality in Section 7.5. Finally, we give the proof of the main result of this chapter in Section 7.6 and discuss various outlooks in Sections 7.7 and 7.8.

7.1. Local Minima

The purpose of the present chapter is to establish the partial (Hölder) regularity of *local* minimisers of functionals of the form

$$\mathfrak{F}[u] := \int_\Omega f(\varepsilon(u))\,\mathrm{d}x, \tag{7.1}$$

where $f\colon \mathbb{R}^{n\times n}_{\mathrm{sym}} \to \mathbb{R}$ is a strictly convex C^2–integrand. The notion of local minima which we invoke here is more general than that of minimisers over a given Dirichlet class. In the Sobolev (or LD–) case, a map $u\colon \Omega \to \mathbb{R}^n$ is called a *local minimiser* if and only if for any open Lipschitz subset ω of Ω, u minimises the restricted functional $\mathfrak{F}[u;\omega] := \int_\omega f(\varepsilon(u))\,\mathrm{d}x$ with respect to its own boundary values, i.e.,

$$\mathfrak{F}[u;\omega] \leqslant \mathfrak{F}[u+\varphi;\omega] \qquad \text{for all } \varphi \in \mathrm{LD}_0(\omega).$$

In the BD–situation, we wish to work with the weak*–relaxation of $\mathfrak{F}$ throughout. Hence we briefly pause and introduce the relevant concept first. For a given open Lipschitz subset ω of Ω, $u \in \mathrm{BD}(\Omega)$ and a function $v \in \mathrm{BD}(\omega)$, put

$$\widetilde{\mathfrak{F}}_u[v;\omega] := \int_\omega f(\mathscr{E}v)\,\mathrm{d}x + \int_\omega f^\infty\left(\frac{\mathrm{dE}v}{\mathrm{d}|\,\mathrm{E}^s v|}\right)\mathrm{d}|\,\mathrm{E}^s v| + \int_{\partial\omega} f^\infty(\mathrm{Tr}(u-v)\odot\nu_{\partial\omega})\,\mathrm{d}\mathcal{H}^{n-1},$$

where $\nu_{\partial\omega}$ is the outer unit normal to $\partial\omega$ and

$$\begin{aligned}\mathrm{E}v &= \mathrm{E}^{ac}v + \mathrm{E}^s v\\ &= \mathscr{E}v\mathscr{L}^n \llcorner\omega + \frac{\mathrm{dE}v}{\mathrm{d}|\,\mathrm{E}^s v|}|\,\mathrm{E}^s v|\llcorner\omega + (\mathrm{Tr}(v)\odot\nu_{\partial\omega})\mathcal{H}^{n-1}\llcorner\partial\omega\end{aligned}$$

is the Radon–Nikodym decomposition of the symmetric gradient measure $\mathrm{E}v\llcorner\overline{\omega}$ (cp. Chapter 4.3). We then make the following definition.

Definition 7.1 (Local Minimiser)**.** *An element $u \in \mathrm{BD}(\Omega)$ is called a* local BD–minimiser *for $\mathfrak{F}$ if and only if for every open Lipschitz subset ω of Ω there holds*

$$\widetilde{\mathfrak{F}}_u[u;\omega] \leqslant \widetilde{\mathfrak{F}}_u[v;\omega] \tag{7.2}$$

for all $v \in \mathrm{BD}(\omega)$.

In the terminology of Chapter 4.3.2 and hereafter Definition 4.24 in the framework of Dirichlet problems, we recall that a function $u \in \mathrm{BD}(\Omega)$ is a BD–minimiser (over a given Dirichlet class $\mathcal{D}_{u_0} := u_0 + \mathrm{LD}_0(\Omega)$ for some $u_0 \in \mathrm{LD}(\Omega)$) if and only if $\widetilde{\mathfrak{F}}_{u_0}[u] \leqslant \widetilde{\mathfrak{F}}_{u_0}[v]$ for all $v \in \mathrm{BD}(\Omega)$. Since our main result will be of local nature, we briefly link the two concepts as indicated above.

Remark 7.2 (BD–minima are local BD–minima)**.** *Let $u \in \mathrm{BD}(\Omega)$ be a BD–minimiser of $\mathfrak{F}$. For a given Lipschitz subset ω of Ω and an arbitrary $v \in \mathrm{BD}(\omega)$, we put $w := \mathbb{1}_\omega v + \mathbb{1}_{\Omega\setminus\overline{\omega}}u$. By the classical BD–gluing principle (cp. Theorem 4.8), $w \in \mathrm{BD}(\Omega)$. By definition of BD–minima, we have $\widetilde{\mathfrak{F}}_{u_0}[u] \leqslant \widetilde{\mathfrak{F}}_{u_0}[w]$ and thus, working from the definition of $\widetilde{\mathfrak{F}}_{u_0}$,*

$$\begin{aligned}\widetilde{\mathfrak{F}}_{u_0}[u] &= \int_\omega f(\mathscr{E}u)\,\mathrm{d}x + \int_{\Omega\setminus\overline{\omega}} f(\mathscr{E}u)\,\mathrm{d}x + \int_\omega f^\infty\left(\frac{\mathrm{dE}u}{\mathrm{d}|\,\mathrm{E}^s u|}\right)\mathrm{d}|\,\mathrm{E}^s u|\\ &+ \int_{\Omega\setminus\overline{\omega}} f^\infty\left(\frac{\mathrm{dE}u}{\mathrm{d}|\,\mathrm{E}^s u|}\right)\mathrm{d}|\,\mathrm{E}^s u| + \int_{\partial\Omega} f^\infty(\mathrm{Tr}(u_0-u)\odot\nu_{\partial\Omega})\,\mathrm{d}\mathcal{H}^{n-1}\\ &\leqslant \widetilde{\mathfrak{F}}_{u_0}[w] = \int_\omega f(\mathscr{E}v)\,\mathrm{d}x + \int_{\Omega\setminus\overline{\omega}} f(\mathscr{E}u)\,\mathrm{d}x + \int_\omega f^\infty\left(\frac{\mathrm{dE}v}{\mathrm{d}|\,\mathrm{E}^s v|}\right)\mathrm{d}|\,\mathrm{E}^s v|\\ &+ \int_{\Omega\setminus\overline{\omega}} f^\infty\left(\frac{\mathrm{dE}u}{\mathrm{d}|\,\mathrm{E}^s u|}\right)\mathrm{d}|\,\mathrm{E}^s u| + \int_{\partial\Omega} f^\infty(\mathrm{Tr}(u_0-u)\odot\nu_{\partial\Omega})\,\mathrm{d}\mathcal{H}^{n-1}\\ &+ \int_{\partial\omega} f^\infty(\mathrm{Tr}(u-v)\odot\nu_{\partial\omega})\,\mathrm{d}\mathcal{H}^{n-1},\end{aligned}$$

where $a\odot b := \frac{1}{2}(ab^\mathsf{T} + ba^\mathsf{T})$ for $a,b \in \mathbb{R}^n$ is the symmetrised tensor product. Now, the obvious cancellations yield that u is a local BD–minimiser.

7.2. Main Result

We now pass to the description of our main result which, in turn, is a generalisation of Theorem 1.1 in ANZELLOTTI & GIAQUINTA's article [22]. The main feature of this result is that the convexity assumption is rather weak and does not rely on any uniform bounds on the second derivatives from below. In the terminology of SCHMIDT [211], this is a *local–in–phase–space* $\mathrm{C}^{1,\alpha}$*–regularity result*, and to both highlight its generality and explain its limitations, we firstly contextualise this partial regularity result.

7.2.1. Partial Regularity As we have seen in the introductory chapter (cp. Example 1.4), in the vectorial case we cannot expect full $\mathrm{C}^{1,\alpha}$–regularity of minima. Hence, the correct notion of regularity is that of *partial regularity*, and we now proceed with a formalisation of this concept (cp. [33, Chpt. 4.3], [176, Sec. 4.2] and [128, Chpt. 9]). Let $u \in \mathrm{L}^1_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$.

Given $0<\alpha<1$, we define the *regular set* of u (of order α) by[1]

$$\mathrm{Reg}(u;\alpha) := \big\{x_0 \in \Omega\colon\ u \text{ is of class } \mathrm{C}^{1,\alpha} \text{ in a neighbourhood of } x_0\big\}. \tag{7.3}$$

Accordingly, we define the *singular set* of u (of order α) as the complement of $\mathrm{Reg}(u;\alpha)$ in Ω, i.e., $\mathrm{Sing}(u;\alpha) := \Omega\setminus\mathrm{Reg}(u;\alpha)$. In the following, we say that the mapping u is *partially regular* provided

$$\mathscr{L}^n(\mathrm{Sing}(u;\alpha)) = 0 \quad\text{and}\quad \mathrm{Reg}(u;\alpha) \text{ is open} \tag{7.4}$$

for *all* $0<\alpha<1$. The (full) regular set of u, $\mathrm{Reg}(u)$, consequently is given by

$$\mathrm{Reg}(u) := \big\{x_0\in\Omega\colon\ \forall\, 0<\alpha<1\colon\ u \text{ is of class } \mathrm{C}^{1,\alpha} \text{ in a neighbourhood of } x_0\big\},$$

and partial regularity proofs often[2] reveal as a by–product that $\mathrm{Reg}(u;\alpha)$ actually does not depend on α so that $\mathrm{Reg}(u) = \mathrm{Reg}(u;\alpha)$ for all $0<\alpha<1$. We put $\mathrm{Sing}(u) := \Omega\setminus\mathrm{Reg}(u)$ and further remind the reader of the traditional notations (cp. [128, 33, 176])

$$\boxed{\Omega_u := \mathrm{Reg}(u) \text{ and } \Sigma_u := \mathrm{Sing}(u).}$$

By the Campanato–Meyers characterisation of Hölder continuity (cp. Theorem 2.27), we can translate the definition of the regular and singular sets into mean value integral characterisations. In fact, in a plenty of situations, the singular set can be characterised as

$$\begin{aligned}\Sigma_u = \Big\{x_0\in\Omega\colon\ &\liminf_{r\searrow 0} \fint_{\mathrm{B}(x_0,r)} |\,\mathrm{D}u(x) - (\mathrm{D}u)_{x_0,r}|^p\,\mathrm{d}x > 0\Big\}\\ \cup\Big\{x_0\in\Omega\colon\ &\limsup_{r\searrow 0} |(\mathrm{D}u)_{x_0,r}| = +\infty\Big\}\end{aligned} \tag{7.5}$$

for some given $1<p<\infty$. In the situation at our disposal, we must keep track of the fact that the gradients or symmetric gradients of minima are *measures*. Thus, a characterisation of the singular set as in (7.5) must also include a suitable control the singular parts $\mathrm{D}^s u$ or $\mathrm{E}^s u$, respectively.

Partial regularity results in the convex setting have been witnessed a large number of contributions (see, e.g., [122, 22, 4, 209] to mention a few, and [121, 33, 128] for comprehensive overviews). Regarding problems involving the symmetric gradient, we particularly wish to emphasize Seregin's work [213, 214, 215] and the monograph [115], where also problems with linear growth are considered. However, let us remark that most of the results mentioned above rely on *strong* convexity assumptions in that the bilinear forms induced by the Hessian of the integrands satisfy prescribed lower bounds. For future reference, we stress that this shall *not* be required in the case in the main part of the present chapter.

Since partial regularity is a very weak notion and only requires $\mathscr{L}^n(\Sigma_u) = 0$, Σ_u nevertheless might be very large and thus one aims at quantifying its size. One option to do so is to give bounds on the Hausdorff dimension (as introduced in Chapter 2, cp. (2.2))

[1] Strictly speaking, we would need to require this for a Lebesgue representative of u, and we will tacitly assume this convention in the sequel.

[2] E.g., in the case of autonomous integrands, i.e., those without x–dependence. If x–dependence is involved, then the regularity of the integrand with respect to x crucially matters, cp. [33, Thm. 5.9], for instance.

of the singular set, cp. [123, 175, 178, 159, 160, 162] and MINGIONE's survey articles [176, 177]. Here, the aim is to demonstrate $\dim_{\mathcal{H}}(\Sigma_u) < n$, and as will be established in Chapter 7.8, this can be particularly achieved when dealing with the integrands which are μ–elliptic.

For later reference, let us now give and discuss the partial regularity theorem of ANZELLOTTI & GIAQUINTA, here stated in the version as in SCHMIDT's habilitation thesis [211, Thm. 5.1]:

Theorem 7.3 ([22, Thm. 1.1])**.** *Let $\Omega \subset \mathbb{R}^n$ be open and let $f\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ be a convex integrand such that $\lim_{|z|\to\infty} |f(z)|/|z| < \infty$. Moreover, assume that*

(a) *f is of class C^2 in a neighbourhood of some $z_0 \in \mathbb{R}^{N\times n}$ together with $f''(z_0) > 0$ in the sense that $\langle f''(z_0)\xi, \xi\rangle > 0$ for all $\xi \in \mathbb{R}^{N\times n} \setminus \{0\}$,*

(b) *$u \in \mathrm{BV}_{\mathrm{loc}}$ is a local minimiser of the functional $\overline{F}[v] := \int_\Omega f(\mathrm{D}v)$,*

(c) *$\mathrm{D}u$ has $\mathscr{L}^n$–Lebesgue value z_0 at a point $x_0 \in \Omega$ in the sense of*

$$\lim_{r\searrow 0} \frac{|\,\mathrm{D}u - z_0\mathscr{L}^n|(\mathrm{B}(x_0,r))}{\mathscr{L}^n(\mathrm{B}(x_0,r))} = 0.$$

Then u is of class $\mathrm{C}^{1,\alpha}$ for all $0 < \alpha < 1$ in a neighbourhood of x_0.

The decisive feature of this theorem *besides* convexity is that the remaining conditions on the integrand f are required in an arbitrarily small neighbourhood of the reference point $z_0 \in \mathbb{R}^{N\times n}$ (which, following [211], may be referred to as *almost localised* conditions). In particular, *no* uniform growth condition is imposed on the Hessian f'' so that *any* strictly convex C^2–integrand of linear growth $f\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ satisfies the conditions given in the theorem. We wish to compare this with the substantially stronger convexity condition of μ–ellipticity. In this case, we have an explicit growth bound from below given by $(1+|\xi|^2)^{-\mu/2}|\eta|^2 \lesssim \langle f''(\xi)\eta,\eta\rangle$. By Remark 5.1, there is no positive result on the $\mathrm{W}^{1,1}$–regularity of generalised minima for if $\mu > 3$, but by Theorem 7.3 from above, partial regularity still holds true: Similar to Chapter 6, the theorem allows for arbitrary degeneration of the second derivatives unless C^2–strict convexity is violated. Therefore, whereas higher Sobolev regularity requires a strong convexity condition, partial Hölder regularity survives in the presence of non–uniformly convex integrands.

Let us see how Theorem 7.3 implies partial regularity and hence suppose that f is strictly convex. By Lebesgue's differentiation theorem (cp. Theorems 2.3 and 2.4), for any $\mathbb{R}^m$–valued Radon measure μ on Ω and $\mathscr{L}^n$–a.e. $x_0 \in \Omega$, there exists the finite Lebesgue value

$$z_0 := \lim_{r\searrow 0} \frac{\mu(\mathrm{B}(x_0,r))}{\mathscr{L}^n(\mathrm{B}(x_0,r))}.$$

Applying this to the Radon measure $\mu = \mathrm{D}u$ shows that the conditions on x_0 of Theorem 7.3 are fulfilled at $\mathscr{L}^n$–a.e. element of Ω and so the partial regularity in the sense of (7.4)ff. follows immediately.

However, we emphasize that other than in Chapter 6 (in particular, Theorem 6.7), we do not assume radial symmetry of the gradient. As shown in [35] (also compare [211, Chpt. 4.3]) much stronger results apply in this situation and full regularity can be established. In view of Remark 1.5 (that is, the symmetric gradient destroys the *good* structure provided by radially symmetric integrands), we are mainly interested in a partial than full regularity result when considering local minima of (7.1). This, in turn, is the objective of the next paragraph.

7.2.2. The Main Result, Structure of the Chapter and Discussion of Obstructions We now come to the presentation of the main result of this chapter and hence the adaptation of Theorem 7.3 to the symmetric gradient case.

Theorem 7.4. *Let $f \in \mathrm{C}^2(\mathbb{R}^{n\times n}_{\mathrm{sym}})$ be convex and of linear growth in the sense that $\lim_{|z|\to\infty, z\in\mathbb{R}^{n\times n}_{\mathrm{sym}}} |f(z)|/|z| < \infty$. Suppose that $u \in \mathrm{BD}(\Omega)$ is a local* BD*–minimiser of $\mathfrak{F}$ given by (7.1). If $(x,z) \in \Omega \times \mathbb{R}^{n\times n}_{\mathrm{sym}}$ is such that*

$$\lim_{R\searrow 0}\left[\fint_{\mathrm{B}(x,R)} |\mathscr{E}u - z|\,\mathrm{d}x + \frac{|\,\mathrm{E}^s u|(\mathrm{B}(x,R))}{\mathscr{L}^n(\mathrm{B}(x,R))}\right] = 0 \tag{7.6}$$

and $f''(z)$ is positive definite, then there holds $u \in \mathrm{C}^{1,\alpha}(U;\mathbb{R}^n)$ for a suitable neighbourhood U of x for all $0 < \alpha < 1$.

Note that the condition (7.6) is a reformulation for the symmetric gradient case of part (c) in the gradient case, cp. Theorem 7.3. Similarly as for Theorem 7.3, the preceding theorem immediately implies partial regularity: Indeed, by the Lebesgue differentiation theorem for Radon measures (cp. Theorem 2.3), $\mathscr{L}^n$–a.e. $x \in \Omega$ is a Lebesgue point for $\boldsymbol{\varepsilon}(v)$ and so the assumptions of the previous theorem are satisfied. In conclusion, if we put for $v \in \mathrm{BD}(\Omega)$

$$\widetilde{\Omega}_v := \left\{x \in \Omega\colon\ \lim_{R\searrow 0}\fint_{\mathrm{B}(x,R)} |\mathscr{E}v - (\mathscr{E}v)_{\mathrm{B}(x,R)}|\,\mathrm{d}y + \frac{|\,\mathrm{E}^s v|(\mathrm{B}(x,R))}{\mathscr{L}^n(\mathrm{B}(x,R))} = 0\right\} \tag{7.7}$$

we consequently gain openness of $\widetilde{\Omega}_u$ together with $\mathscr{L}^n(\Omega\setminus\widetilde{\Omega}_u) = 0$. Moreover, we have $\Omega_u = \widetilde{\Omega}_u$. In fact, it actually suffices to establish the requisite decay estimates (which lead to (7.7)) on the level of symmetric gradients. In fact, once this is achieved, we use boundedness of singular integrals of convolution type on the Hölder–Zygmund classes $\mathrm{B}^s_{\infty,\infty}$ (cp. Chapter 2.5.2) to deduce that if $\boldsymbol{\varepsilon}(v)$ is of class $\mathrm{C}^{0,\alpha}$ in a neighbourhood of some $x \in \Omega$, then so is $\mathrm{D}v$ in the same neighbourhood. Using the same method, we also obtain a regularity result for integrands of p–growth, $1 < p < \infty$:

Theorem 7.5. *Let $f \in \mathrm{C}^2(\mathbb{R}^{n\times n}_{\mathrm{sym}})$ be convex and satisfy, for some $1 < p < \infty$, $c_1|\xi|^p \leqslant f(\xi) \leqslant c_2(1+|\xi|^p)$ for all $\xi \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$ with two constants $0 < c_1 \leqslant c_2 < \infty$. Suppose that $u \in \mathrm{W}^{1,p}(\Omega;\mathbb{R}^n)$ is a local minimiser of $\mathfrak{F}$ given by (7.1). If $f''(z)$ is positive definite for any $z \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$, then there exists an open subset Ω_u of Ω such that $\mathscr{L}^n(\Omega\setminus\Omega_u) = 0$ and u is of class $\mathrm{C}^{1,\alpha}$ in a neighbourhood of any of the elements of Ω_u.*

The novelty of the previous result is that, albeit partial regularity results in the superlinear growth regime are well–known, they usually rely on strong convexity assumptions; see [115]. Here, however, we only require positive definiteness of f''.

In proving the previous two theorems, we shall pursue an unified approach for both $p = 1$ and $1 < p < \infty$. To do so, we collect in Section 7.3 preliminary decay estimates for linearised systems and consequently turn to specific types of Poincaré–inequalities in Section 7.5. These shall be instrumental in the proof of Theorems 7.4 and 7.5. This, in turn, is accomplished in Section 7.6. Finally, in Sections 7.7 and 7.8 we investigate the implications for the dual solution (in the sense of convex duality) and the size of the singular set.

As briefly mentioned above, we will directly work on the symmetric gradients and hereafter also establish the partial Hölder regularity for the symmetric gradients. In what follows, we briefly explain why this is equivalent to the partial Hölder regularity of the full gradients.

For this, we firstly record the following lemma and refer the reader to Chapter 2.5.6 for the terminology utilised therein.

Lemma 7.6 ([51, Thm. 1.1]). *Let $\varphi\colon \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ be a Φ–function. Then the following are equivalent:*

(a) *φ is of class $\Delta_2 \cap \nabla_2$.*

(b) *There exists a constant $c > 0$ such that for all balls $\mathrm{B} \subset \mathbb{R}^n$ and all $v \in \mathrm{W}^{1,\varphi}(\mathrm{B};\mathbb{R}^n)$ there holds*

$$\| \mathrm{D}v - (\mathrm{D}v)_{\mathrm{B}}\|_{\mathrm{L}^{\varphi}(\mathrm{B};\mathbb{R}^{n\times n})} \leqslant \|\boldsymbol{\varepsilon}(v) - (\boldsymbol{\varepsilon}(v))_{\mathrm{B}}\|_{\mathrm{L}^{\varphi}(\mathrm{B};\mathbb{R}^{n\times n}_{\mathrm{sym}})}.$$

Lemma 7.7. *Suppose that $u \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^n)$ is such that $\boldsymbol{\varepsilon}(u)$ is of class $\mathrm{C}^{0,\alpha}$ for some $0 < \alpha < 1$ in a neighbourhood of a point $x_0 \in \mathbb{R}^n$. Then $\mathrm{D}u$ is of class $\mathrm{C}^{0,\alpha}$ in a neighbourhood of x_0.*

Proof. The proof can be reduced to the mapping properties of singular integrals on Besov spaces or accomplished by Korn–type inequalities in mean–value form, and we present both approaches.

(i) Let us recall from Chapter 2.5.2 that for $0 < s < 1$, the Hölder spaces admit a Besov–type (which we recall to be referred to as Hölder–Zygmund–) characterisation: $\mathrm{C}^{0,s}(\mathbb{R}^n) \simeq \mathrm{B}^s_{\infty,\infty}(\mathbb{R}^n)$. From Chapter 2.5.4 it follows that translation invariant singular integral operators of convolution type map $\mathrm{B}^s_{\infty,\infty} \to \mathrm{B}^s_{\infty,\infty}$. Let $u \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^n)$ and $0<\alpha<1$ be such that $\boldsymbol{\varepsilon}(u)|_{\mathrm{B}(x_0,r)} \in \mathrm{C}^{0,\alpha}(\mathrm{B}(x_0,r);\mathbb{R}^{n\times n}_{\mathrm{sym}})$ and let $\rho \in \mathrm{C}^1_c(\mathrm{B}(x_0,r);[0,1])$ be such that $\rho \equiv 1$ on $\mathrm{B}(x_0,r/2)$. Then $v := \rho u$ satisfies $\boldsymbol{\varepsilon}(v) \in \mathrm{C}^{0,\alpha}(\mathbb{R}^n;\mathbb{R}^{n\times n}_{\mathrm{sym}})$ and since by Chapter 3.2 the map

$$\mathrm{C}^\infty_c(\mathbb{R}^n;\mathbb{R}^{n\times n}_{\mathrm{sym}}) \ni \boldsymbol{\varepsilon}(\varphi) \mapsto \mathrm{D}\varphi \in \mathrm{C}^\infty_c(\mathbb{R}^n;\mathbb{R}^{n\times n})$$

extends to a translation invariant singular integral operator of convolution type on $\mathrm{C}^{0,\alpha}$, we deduce that v itself must be of class $\mathrm{C}^{1,\alpha}$ on $\mathrm{B}(x_0,r/2)$ (as $v|_{\mathrm{B}(x_0,r/2)} = u|_{\mathrm{B}(x_0,r/2)}$).

(ii) For any $1 < p < \infty$ and any ball B, we have $\mathrm{C}^{0,\alpha}(\overline{\mathrm{B}};\mathbb{R}^n) \simeq \mathcal{L}^{p,n+\alpha p}(\mathrm{B};\mathbb{R}^n)$ with the Campanato spaces $\mathcal{L}^{q,\lambda}$ as introduced in Chapter 2.6.1. This yields in particular $\mathcal{L}^{1,n+\alpha}(\Omega;\mathbb{R}^n) \simeq \mathcal{L}^{p,n+\alpha p}(\Omega;\mathbb{R}^n)$ for $p > 1$. We apply Lemma 7.6 with the particular choice[3] $\varphi(t) := \frac{1}{p}|t|^p$ (which is a Φ–function) together with $\varphi \in \Delta_2 \cap \nabla_2$. Hence,

$$\frac{1}{r^{\alpha p}} \fint_{\mathrm{B}(x_0,r)} |\,\mathrm{D}u - (\mathrm{D}u)_{x_0,r}|^p \,\mathrm{d}x \leqslant c\frac{1}{r^{\alpha p}} \fint_{\mathrm{B}(x_0,r)} |\boldsymbol{\varepsilon}(u) - (\boldsymbol{\varepsilon}(u))_{x_0,r}|^p \,\mathrm{d}x \leqslant C,$$

where $C > 0$ does not depend on $r > 0$. Hence we conclude that $\mathrm{D}u$ itself is of class $\mathrm{C}^{0,\alpha}$ in a neighbourhood of x_0. □

Let us finally remark that our choice of method of proof is due to its *direct* nature, meaning that no argument by contradiction is involved. In the linear growth regime, indirect methods such as the *blow–up* approach are difficult to apply, cp. Chapter 8.3. On the other hand, the method of $\mathbb{A}$–harmonic approximation strictly requires a strong (quasi–)convexity condition which, in the situation as dealt with in Theorems 7.3 or 7.4, is not available.

The approach we use here rests on comparing minima with their mollifications. It therefore exploits, in a crucial manner, the convexity of the integrands. The deeper reason for this is Jensen's inequality (cp. Lemma 2.9) which is available for *convex* functions only[4]. In fact, let $\eta \in \mathrm{C}^\infty_c(\mathbb{R}^n;[0,1])$ be a compactly supported function with $\|\eta\|_{\mathrm{L}^1(\mathbb{R}^n)} =$

[3] Here we need to work with the Luxembourg norm, cp. Chapter 2.5.6. However, if $\varphi(t) := \frac{1}{p}|t|^p$, then its Luxembourg norm equals the usual L^p–norm.

[4] Jensen–type inequalities in the quasiconvex setting are available for Young measures (cp. [188, Thm. 4.7] and the remarks afterwards), but it is not clear to us how to adapt the method as presented here fruitfully to the latter.

1 and put, for $\varepsilon > 0$, $\eta_\varepsilon(x) := \varepsilon^{-n}\eta(x/\varepsilon)$. Then we equally have $\|\eta_\varepsilon\|_{\mathrm{L}^1(\mathbb{R}^n)} = 1$ and thus μ_ε defined as the absolutely continuous measure (with respect to $\mathscr{L}^n$) given by $\mathrm{d}\mu_\varepsilon = \eta_\varepsilon\,\mathrm{d}\mathscr{L}^n$ is a probability measure on $\mathscr{B}(\mathbb{R}^n)$. Hence, by Jensen's inequality, if $u \in \mathrm{L}^1(\mathbb{R}^n;\mathbb{R}^m)$ and $\varphi\colon \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ is convex, then we have for $\mathscr{L}^n$–a.e. $x \in \mathbb{R}^n$

$$\varphi\left((\eta_\varepsilon * u)(x)\right) = \varphi\left(\int_{\mathbb{R}^n} u(x-z)\,\mathrm{d}\mu_\varepsilon(z)\right) \leqslant \int_{\mathbb{R}^n} \varphi(u(x-z))\,\mathrm{d}\mu_\varepsilon(z) = ((\varphi\circ u) * \eta_\varepsilon)(x).$$

The point where the availability of Jensen's inequality enters in the most crucial manner is Lemma 7.20, whose application to the quasiconvex case is not clear (also cp. [22], remark before Lemma 5.1 therein).

7.3. Decay estimates for the linearised systems

Aiming at a comparison with solutions of linear elliptic systems that depend on the symmetric gradients in the partial regularity proof, we collect now decay estimates for such mappings. To this end, let Ω be an open and bounded Lipschitz subset of $\mathbb{R}^n$ and consider for $w \in \mathrm{W}^{1,2}(\Omega;\mathbb{R}^n)$ the variational principle

$$\text{to minimise } \mathfrak{F}[v] := \int_{\mathrm{B}} g(\boldsymbol{\varepsilon}(v))\,\mathrm{d}x \qquad \text{over } v \in w + \mathrm{W}_0^{1,2}(\Omega;\mathbb{R}^n), \tag{7.8}$$

where $g(\xi) := \langle \mathcal{A}\xi,\xi\rangle + \langle b,\xi\rangle + c$ is a polynomial of degree two with arbitrary $b \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$ and $c \in \mathbb{R}$, whereas we assume that $\mathcal{A}\colon \mathbb{R}^{n\times n}_{\mathrm{sym}} \times \mathbb{R}^{n\times n}_{\mathrm{sym}} \to \mathbb{R}$ is bounded and, moreover, elliptic in the sense that there exists $\ell > 0$ such that $\langle \mathcal{A}\xi,\xi\rangle \geqslant \ell|\xi|^2$ holds for all $\xi \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$. As a small variation of [115, Lemma 3.0.5], we provide the following lemma for the reader's convenience:

Lemma 7.8. *There exists a unique solution $u \in w + \mathrm{W}_0^{1,2}(\Omega;\mathbb{R}^n)$ of (7.8). Moreover, this solution satisfies the following:*

(a) *There exists a constant $c = c(n,\ell) > 0$ such that if $\mathrm{B}(z,R) \Subset \Omega$, then for all $0 < r < R/2$ there holds*

$$\int_{\mathrm{B}(z,r)} |\boldsymbol{\varepsilon}(u) - (\boldsymbol{\varepsilon}(u))_{z,r}|^2\,\mathrm{d}x \leqslant c\left(\frac{r}{R}\right)^{n+2} \int_{\mathrm{B}(z,R/2)} |\boldsymbol{\varepsilon}(u) - (\boldsymbol{\varepsilon}(u))_{z,R/2}|^2\,\mathrm{d}x.$$

(b) *There exists a constant $c = c(n,\ell) > 0$ such that if $\mathrm{B}(z,R) \Subset \Omega$ and $\widetilde{u} \in \mathrm{C}^{1,\alpha}(\overline{\Omega};\mathbb{R}^n)$, then*

$$[\boldsymbol{\varepsilon}(u)]_{\mathrm{C}^{0,\alpha}(\mathrm{B}(z,R/2);\mathbb{R}^{n\times n})} \leqslant c[\boldsymbol{\varepsilon}(w)]_{\mathrm{C}^{0,\alpha}(\mathrm{B}(z,R/2);\mathbb{R}^{n\times n})}.$$

Proof. The existence part follows along standard lines, and can be established as in Chapter 5 for quadratic functionals. Let $\mathcal{A}\colon \mathbb{R}^{n\times n}_{\mathrm{sym}} \times \mathbb{R}^{n\times n}_{\mathrm{sym}} \to \mathbb{R}$ be an elliptic bilinear form. In a bit greater generality than actually needed, we consider the linear system

$$-\operatorname{div}(\mathcal{A}(\boldsymbol{\varepsilon}(u))) = -\operatorname{div}(f) \tag{7.9}$$

for $f \in \mathrm{L}^2(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})$. Let $u \in \mathrm{W}^{1,2}(\Omega;\mathbb{R}^n)$ be a weak solution to this equation, i.e., u satisfies

$$\int_\Omega \langle \mathcal{A}\boldsymbol{\varepsilon}(u),\boldsymbol{\varepsilon}(\varphi)\rangle\,\mathrm{d}x = \int_\Omega \langle f,\boldsymbol{\varepsilon}(\varphi)\rangle\,\mathrm{d}x \qquad \text{for all } \varphi \in \mathrm{W}_0^{1,2}(\Omega;\mathbb{R}^n). \tag{7.10}$$

(i) Firstly assume that $f \equiv 0$. Our first goal is to show that $u \in \mathrm{C}^\infty(\Omega;\mathbb{R}^n)$. The proof is a simple variant of the difference–quotient type technique as employed in Chapter 5. Let $x_0 \in \Omega$ and choose $R > 0$ such that $\mathrm{B}(x_0,2R) \Subset \Omega$. We pick a cut–off function

$\rho \in \mathrm{C}_c^1(\mathrm{B}(x_0, 2R); [0,1])$ such that $\rho \equiv 1$ on $\mathrm{B}(x_0, R)$ and $|\nabla\rho| \leqslant \frac{4}{R}$. Let $s \in \{1, ..., n\}$ and define for $|h| < \mathrm{dist}(\mathrm{B}(x_0, 2R); \partial\Omega)$ the test function $\varphi := \Delta_{s,-h}(\rho^2 \Delta_{s,h} u)$. Testing (7.9) with φ and recalling $|\nabla\rho| \leqslant C/R$, we obtain after integration by parts for difference quotients, Young's inequality and using ellipticity of $\mathcal{A}$ that[5]

$$\int_{\mathrm{B}(x_0,R/2)} |\Delta_{s,h}\boldsymbol{\varepsilon}(u)|^2 \,\mathrm{d}x \leqslant \frac{C}{R^2} \int_{\mathrm{B}(x_0,2R)} |\Delta_{s,h} u|^2 \,\mathrm{d}x.$$

By Korn's inequality and the usual difference quotient–type characterisation of $\mathrm{W}^{1,2}$, the right side is uniformly bounded in $|h|$ and so is the left side. Passing $|h| \downarrow 0$ yields validity of the previous inequality with ∂_s instead of $\Delta_{s,h}$. Note that $\partial_s \circ \boldsymbol{\varepsilon}$ is an elliptic second order differential operator and hence, in the terminology of Chapter 3.2, possesses a Green's function. By boundedness of singular integrals of convolution type on L^2 (cp. Chapter 3.2), we obtain $u \in \mathrm{W}^{2,2}_{\mathrm{loc}}(\Omega; \mathbb{R}^n)$ and, noticing that with u also $\partial^\alpha u$ is a weak solution of (7.9) (recall that $f \equiv 0$), we obtain $u \in \mathrm{W}^{k,2}_{\mathrm{loc}}(\Omega; \mathbb{R}^n)$ for any $k \in \mathbb{N}$.

(ii) Let now $0 < r < 2r < R < \mathrm{dist}(x_0, \partial\Omega)$ and $\rho \in \mathrm{C}_c^1(\mathrm{B}(x_0, 2r); [0,1])$ be a cut–off function with $|\nabla\rho| \leqslant C/r$ and $\rho \equiv 1$ on $\mathrm{B}(x_0, r)$. We put $\varphi := \rho^2(u - \mathbf{r})$, where $\mathbf{r} \in \mathcal{R}(\Omega)$ is an arbitrary rigid deformation. Testing (7.9) with $\varphi (\in \mathrm{W}^{1,2}_0(\Omega; \mathbb{R}^n))$ and using Young's inequality yields the Caccioppoli–type inequality

$$\int_{\mathrm{B}(x_0,r)} |\boldsymbol{\varepsilon}(u)|^2 \,\mathrm{d}x \leqslant \frac{C}{r^2} \int_{\mathrm{B}(x_0,2r)} |u - \mathbf{r}|^2 \,\mathrm{d}x, \tag{7.11}$$

where $C = C(n, |\mathcal{A}|, \ell) > 0$ is a constant and still $f \equiv 0$ is assumed. In fact, testing as indicated, we obtain

$$\int_\Omega \langle \mathcal{A}\boldsymbol{\varepsilon}(u), \rho^2\boldsymbol{\varepsilon}(u) + 2\rho\nabla\rho \odot (u - \mathbf{r})\rangle \,\mathrm{d}x = 0.$$

By the ellipticity assumption on $\mathcal{A}$, we have for a constant $C > 0$ and all $\varepsilon > 0$ (with another constant $c = c(\varepsilon) > 0$)

$$\ell \int_\Omega |\rho\boldsymbol{\varepsilon}(u)|^2 \,\mathrm{d}x \leqslant C\varepsilon \int_\Omega |\rho\boldsymbol{\varepsilon}(u)|^2 \,\mathrm{d}x + c(\varepsilon) \int_\Omega |(u - \mathbf{r}) \odot \nabla\rho|^2 \,\mathrm{d}x,$$

and thus, choosing $\varepsilon > 0$ sufficiently small, we may absorb the first term of the right hand side into the left hand side. Consequently, recalling $|\nabla\rho| \leqslant C/r$ yields (7.11).

We now let $\rho \in \mathrm{C}_c^\infty(\mathrm{B}(x_0, r/2); [0,1])$ be a smooth cut–off with $\mathbb{1}_{\mathrm{B}_{r/4}} \leqslant \rho \leqslant \mathbb{1}_{\mathrm{B}_{r/2}}$ (where $\mathrm{B}_r := \mathrm{B}(x_0, r)$) and $|\nabla\rho| \leqslant 8/r$. In a next step, we recall the Korn–type inequality from Proposition 3.9 to conclude after expanding terms that

$$\begin{aligned}
\int_\Omega |(\nabla\rho^2) \otimes (u - (u)_\mathrm{B})|^2 + 2\langle(\nabla\rho^2) \otimes (u - (u)_\mathrm{B}), \rho^2 \,\mathrm{D}u\rangle + |\rho^2 \,\mathrm{D}u|^2 \,\mathrm{d}x \\
&= \int_\Omega |\,\mathrm{D}(\rho^2(u - (u)_\mathrm{B}))|^2 \,\mathrm{d}x \\
&\leqslant C \int_\Omega |\boldsymbol{\varepsilon}(\rho^2(u - (u)_\mathrm{B}))|^2 \,\mathrm{d}x \qquad \text{(by Korn)} \\
&\leqslant C \int_\Omega |(\nabla\rho^2) \odot (u - (u)_\mathrm{B})|^2 \,\mathrm{d}x \\
&\quad + 2C \int_\Omega \langle(\nabla\rho^2) \odot (u - (u)_\mathrm{B}), \rho^2\boldsymbol{\varepsilon}(u)\rangle + |\rho^2\boldsymbol{\varepsilon}(u)|^2 \,\mathrm{d}x.
\end{aligned}$$

[5]These steps are similar but accomplished in a substantially easier way than the finite difference approach of Chapter 5.7.

Since the first term on the very left hand side of the preceding chain of inequalities is positive, we conclude by use of $|a \odot b| \leqslant |a \otimes b|$ for all $a, b \in \mathbb{R}^n$ and Young's inequality for arbitrary $\varepsilon > 0$ (and a corresponding constant $c(\varepsilon) > 0$)

$$\int_\Omega |\rho^2\,\mathrm{D}u|^2\,\mathrm{d}x \leqslant (C + c(\varepsilon))\int_\Omega |(\nabla\rho^2)\otimes(u - (u)_{\mathrm{B}})|^2\,\mathrm{d}x + C\varepsilon\int_\Omega |\rho^2\,\mathrm{D}u|^2\,\mathrm{d}x + (C + c(\varepsilon))\int_\Omega |\rho^2\boldsymbol{\varepsilon}(u)|^2\,\mathrm{d}x.$$

In turn, choosing $\varepsilon > 0$ sufficiently small, we may absorb the second term of the right hand side of the previous inequality into its left hand side. Recalling the Caccioppoli–type inequality (7.11) and applying it with $\mathrm{B}(x_0, r/2)$ instead of $\mathrm{B}(x_0, r)$, we then find

$$\begin{aligned}\int_{\mathrm{B}(x_0,r/4)} |\,\mathrm{D}u|^2\,\mathrm{d}x &\leqslant \int_\Omega |\rho^2\,\mathrm{D}u|^2\,\mathrm{d}x \\ &\leqslant C\int_{\mathrm{B}(x_0,r/2)} |(\nabla\rho^2)\otimes(u - (u)_{\mathrm{B}(x_0,r)})|^2\,\mathrm{d}x \\ &\quad + C\int_{\mathrm{B}(x_0,r/2)} |\boldsymbol{\varepsilon}(u)|^2\,\mathrm{d}x \\ &\leqslant C\int_{\mathrm{B}(x_0,r)} \left|\frac{u - (u)_{\mathrm{B}(x_0,r)}}{r}\right|^2\,\mathrm{d}x,\end{aligned} \tag{7.12}$$

where we additionally used the fact that the rigid deformations contain the constants. It is also clear that the same inequality remains true when $(u)_{\mathrm{B}(x_0,r)}$ is replaced by any constant $a \in \mathbb{R}^n$.

(iii) By Morrey's embedding, $\|u\|_{\mathrm{L}^\infty(\mathrm{B}_{R/2};\mathbb{R}^n)} \leqslant c\|u\|_{\mathrm{W}^{n,2}(\mathrm{B}_{R/2};\mathbb{R}^n)}$. In consequence, iterating steps (i) and (ii) with $\mathbf{r} \equiv 0$, we obtain for $r < R/2$

$$\begin{aligned}\int_{\mathrm{B}_r} |u|^2\,\mathrm{d}x &\leqslant cr^n\|u\|^2_{\mathrm{L}^\infty(\mathrm{B}_{R/2};\mathbb{R}^n)} \\ &\leqslant c(R)r^n\|u\|^2_{\mathrm{W}^{n,2}(\mathrm{B}_{R/2};\mathbb{R}^n)} \leqslant \widetilde{C}(R)(r/R)^n\|u\|^2_{\mathrm{L}^2(\mathrm{B}_R;\mathbb{R}^n)},\end{aligned} \tag{7.13}$$

and a simple rescaling argument yields that $\widetilde{C}(R)$ does not depend on R. Now note that if u is a weak solution, then so is $u - \mathbf{r}$, where $\mathbf{r} \in \mathcal{R}(\Omega)$ is the uniquely determined rigid deformation such that $\int_{\mathrm{B}_r} |u - \mathbf{r}|^2\,\mathrm{d}x \leqslant cr^2\int_{\mathrm{B}_r} |\boldsymbol{\varepsilon}(u)|^2\,\mathrm{d}x$ (as $\mathbf{r}$ is an affine function, it can be extended from $\mathrm{B}(x_0, r)$ to Ω). In consequence, we obtain by use of the Sobolev–Poincaré Inequality in its symmetric gradient form

$$\begin{aligned}\int_{\mathrm{B}_r} |u - \mathbf{r}|^2\,\mathrm{d}x &\leqslant cr^2\int_{\mathrm{B}_r} |\boldsymbol{\varepsilon}(u)|^2\,\mathrm{d}x \\ &\leqslant c\Big(\frac{r}{R}\Big)^2\int_{\mathrm{B}_R} |u - \mathbf{r}|^2\,\mathrm{d}x && \text{(by (7.11))} \\ &\leqslant c\Big(\frac{r}{R}\Big)^{n+2}\int_{\mathrm{B}_{2R}} |u - \mathbf{r}|^2\,\mathrm{d}x && \text{(by (7.13))}.\end{aligned}$$

On the other hand, invoking the usual Poincaré inequality now in conjunction with (7.12), we obtain similarly

$$\int_{\mathrm{B}_r} |u - (u)_r|^2\,\mathrm{d}x \leqslant c\left(\frac{r}{R}\right)^{n+2}\int_{\mathrm{B}_R} |u - (u)_R|^2\,\mathrm{d}x$$

and record that, since with u also $\partial_s u$ for any $s \in \{1, \ldots n\}$ is a weak solution, the previous estimate holds true when consequently replacing u by $\partial_s u$.

Now, put $W(x) := u(x) - (\boldsymbol{\varepsilon}(u))_R x$. By the preceding estimates, we obtain for $0 < r \leqslant \frac{R}{2}$

$$
\begin{aligned}
\int_{\mathrm{B}_r} |\boldsymbol{\varepsilon}(u) - (\boldsymbol{\varepsilon}(u))_r|^2 \,\mathrm{d}x &\overset{a(x):=(\mathrm{D}u)_r x}{\leqslant} \int_{\mathrm{B}_r} |\,\mathrm{D}(u-a)|^2 \,\mathrm{d}x = \int_{\mathrm{B}_r} |\,\mathrm{D}u - (\mathrm{D}u)_r|^2 \,\mathrm{d}x \\
&\leqslant C\left(\frac{r}{R}\right)^{n+2} \int_{\mathrm{B}_{R/2}} |\,\mathrm{D}u - (\mathrm{D}u)_{R/2}|^2 \,\mathrm{d}x \\
&\leqslant C\left(\frac{r}{R}\right)^{n+2} \int_{\mathrm{B}_{R/2}} |\,\mathrm{D}u - \xi|^2 \,\mathrm{d}x = (*).
\end{aligned}
$$

for all $\xi \in \mathbb{R}^{n\times n}$. In the next step, we choose a rigid deformation $\mathbf{r} \in \mathcal{R}(\mathrm{B}(x_0,R))$ such that $\|W - \mathbf{r}\|^2_{\mathrm{L}^2(\mathrm{B}(x_0,R);\mathbb{R}^n)} \leqslant CR^2 \|\boldsymbol{\varepsilon}(W)\|^2_{\mathrm{L}^2(\mathrm{B}(x_0,R);\mathbb{R}^{n\times n}_{\mathrm{sym}})}$, where $C > 0$ does not depend on R. We put $\xi := (\boldsymbol{\varepsilon}(u))_R + \mathrm{D}\mathbf{r}$. We then note that $\mathrm{D}W + (\boldsymbol{\varepsilon}(u))_R = \mathrm{D}u$ and hence

$$
\begin{aligned}
(*) &= C\left(\frac{r}{R}\right)^{n+2} \int_{\mathrm{B}_{R/2}} |\,\mathrm{D}W + (\boldsymbol{\varepsilon}(u))_R - \xi|^2 \,\mathrm{d}x \\
&= C\left(\frac{r}{R}\right)^{n+2} \int_{\mathrm{B}_{R/2}} |\,\mathrm{D}(W - \mathbf{r})|^2 \,\mathrm{d}x = (**).
\end{aligned}
$$

Clearly, by linearity, the map $(W - \mathbf{r})$ is also a weak solution to the system at our disposal (with respect to its own boundary values) and hence satisfies

$$
\begin{aligned}
(**) &= C\left(\frac{r}{R}\right)^{n+2} \frac{1}{R^2} \int_{\mathrm{B}_R} |W - \mathbf{r}|^2 \,\mathrm{d}x \leqslant C\left(\frac{r}{R}\right)^{n+2} \int_{\mathrm{B}_R} |\boldsymbol{\varepsilon}(W)|^2 \,\mathrm{d}x \\
&\leqslant C\left(\frac{r}{R}\right)^{n+2} \int_{\mathrm{B}_R} |\boldsymbol{\varepsilon}(u) - (\boldsymbol{\varepsilon}(u))_R|^2 \,\mathrm{d}x.
\end{aligned}
$$

Adjusting $r, R > 0$ in a suitable way consequently yields the claim.

(iv) We now pass to the case $f \not\equiv 0$. We write $u = v + V$, where $v \in u + \mathrm{W}^{1,2}_0(\mathrm{B}_R;\mathbb{R}^n)$, v solves $\operatorname{div}(\mathcal{A}(\boldsymbol{\varepsilon}(v))) = 0$ in B_R and $V \in \mathrm{W}^{1,2}_0(\mathrm{B}_R;\mathbb{R}^n)$ solves $\operatorname{div}(\mathcal{A}(\boldsymbol{\varepsilon}(V))) = \operatorname{div}(f)$ weakly in B_R. We consequently obtain (as $\int_{\mathrm{B}_R} |\boldsymbol{\varepsilon}(u) - (\boldsymbol{\varepsilon}(u))_{\mathrm{B}_R}|^2 \,\mathrm{d}x \leqslant 2\int_{\mathrm{B}_R} |\boldsymbol{\varepsilon}(u) - a|^2 \,\mathrm{d}x$)

$$
\begin{aligned}
\int_{\mathrm{B}_r} |\boldsymbol{\varepsilon}(u) - (\boldsymbol{\varepsilon}(u))_{\mathrm{B}_r}|^2 \,\mathrm{d}x &\leqslant c\int_{\mathrm{B}_r} |\boldsymbol{\varepsilon}(u) - (\boldsymbol{\varepsilon}(v))_{\mathrm{B}_r}|^2 \,\mathrm{d}x \\
&\leqslant c\int_{\mathrm{B}_r} |\boldsymbol{\varepsilon}(v) - (\boldsymbol{\varepsilon}(v))_{\mathrm{B}_r}|^2 \,\mathrm{d}x + 2\int_{\mathrm{B}_r} |\boldsymbol{\varepsilon}(V)|^2 \,\mathrm{d}x \\
&\leqslant c\Big(\frac{r}{R}\Big)^{n+2} \int_{\mathrm{B}_R} |\boldsymbol{\varepsilon}(v) - (\boldsymbol{\varepsilon}(v))_{\mathrm{B}_R}|^2 \,\mathrm{d}x + c\int_{\mathrm{B}_R} |\boldsymbol{\varepsilon}(V)|^2 \,\mathrm{d}x \\
&\leqslant c\Big(\frac{r}{R}\Big)^{n+2} \int_{\mathrm{B}_R} |\boldsymbol{\varepsilon}(u) - (\boldsymbol{\varepsilon}(u))_{\mathrm{B}_R}|^2 \,\mathrm{d}x + c\int_{\mathrm{B}_R} |\boldsymbol{\varepsilon}(V)|^2 \,\mathrm{d}x.
\end{aligned}
$$

Here we have used that v solves the homogeneous system and hence the previous decay estimates apply. On the other hand, V is an admissible test function for the equation $\operatorname{div}(\mathcal{A}(\boldsymbol{\varepsilon}(V))) = \operatorname{div}(f) = \operatorname{div}(f - (f)_{\mathrm{B}_R})$, and using the ellipticity of $\mathcal{A}$ in conjunction with Young's inequality, we moreover find that $\|\boldsymbol{\varepsilon}(V)\|_{\mathrm{L}^2(\mathrm{B}_R;\mathbb{R}^{n\times n}_{\mathrm{sym}})} \leqslant C(\ell)\|f - (f)_{\mathrm{B}_R}\|_{\mathrm{L}^2(\mathrm{B}_R;\mathbb{R}^{n\times n}_{\mathrm{sym}})}$. Hence,

$$
\int_{\mathrm{B}_r} |\boldsymbol{\varepsilon}(u) - (\boldsymbol{\varepsilon}(u))_{\mathrm{B}_r}|^2 \,\mathrm{d}x \leqslant c\Big(\frac{r}{R}\Big)^{n+2} \int_{\mathrm{B}_R} |\boldsymbol{\varepsilon}(u) - (\boldsymbol{\varepsilon}(u))_{\mathrm{B}_R}|^2 \,\mathrm{d}x + c\int_{\mathrm{B}_R} |f - (f)_R|^2 \,\mathrm{d}x.
$$

In particular, if f is constant, then the last term in the previous inequality vanishes identically and the claim follows.

(v) We now indicate how (b) can be established and firstly note the weak solutions of the systems

$$(\mathrm{S}_1)\begin{cases}-\operatorname{div}(\mathcal{A}(\varepsilon(u)))=0 & \text{in } \Omega,\\ u=w & \text{on } \partial\Omega\end{cases},\qquad (\mathrm{S}_2)\begin{cases}-\operatorname{div}(\mathcal{A}(\varepsilon(\widetilde{u})))=-\operatorname{div}(\mathcal{A}(\varepsilon(w))) & \text{in } \Omega,\\ \widetilde{u}=0 & \text{on } \partial\Omega\end{cases},$$

are linked by $u=\widetilde{u}-w$. Put $f:=\mathcal{A}(\varepsilon(w))$ in (iv) to deduce for a fixed $0<\tau<1$ with $\Phi(s):=\int_{\mathrm{B}_s}|\varepsilon(u)-(\varepsilon(u))_s|^2\,\mathrm{d}x$

$$\Phi(\tau R)\leqslant c\tau^{n+2}\Phi(R)+cR^{n+2\alpha}[\mathcal{A}(\varepsilon(w))]_{\mathcal{L}^{2,n+2\alpha}(\Omega;\mathbb{R}^{N\times n})}.$$

In this situation, the iteration lemma [33, Lemma B.3] completes the proof of the local statement by use of the Campanato characterisation $\mathcal{L}^{p,n+\alpha p}\cong\mathrm{C}^{0,\alpha}$ of Hölder continuity. Gathering the estimates provided so far, the boundary estimates then follow in a standard manner, cp. [128, Chpt. 8.4]. □

7.4. Estimates on an auxiliary map

In the main part of the proof, it is convenient to work with an excess quantity that is defined by use of a function which we discuss in this section. Throughout, we will think of $1\leqslant p<\infty$ as fixed and hereafter put

$$\mathbf{e}(t):=\mathbf{e}_p(t):=(1+|t|^2)^{p/2}-1,\quad t\in\mathbb{R}. \tag{7.14}$$

By slight abuse of notation, we also denote for given $m\in\mathbb{N}$ the function $\mathbf{e}\colon\mathbb{R}^m\to\mathbb{R}_0^+$ defined by

$$\boxed{\mathbf{e}(\xi):=\mathbf{e}_p(|\xi|):=(1+|\xi|^2)^{p/2}-1 \text{ for } \xi\in\mathbb{R}^m,}$$

and no confusions will arise from whether we are dealing with the scalar or vectorial variants throughout. For future reference we collect some preliminary estimates on $\mathbf{e}$.

Lemma 7.9 ([22, Prop. 2.5])**.** *Let $1\leqslant p<\infty$ and define $\mathbf{e}$ by (7.14).*

(i) There exists a positive constant $c=c(p)$ such that for all $\mathbf{A},\mathbf{B}\in\mathbb{R}^m$ and all $t>0$ there holds

$$\mathbf{e}(t\mathbf{A})\leqslant c\,\max\{t^2,t^p\}\mathbf{e}(\mathbf{A})\ \text{ and }\ \mathbf{e}(\mathbf{A}+\mathbf{B})\leqslant c\,(\mathbf{e}(\mathbf{A})+\mathbf{e}(\mathbf{B})).$$

(ii) Given $L>0$, there exist positive constants $C_1=C_1(L,p)$ and $C_2=C_2(L,p)$ such that for all $\mathbf{A}\in\mathbb{R}^m$ with $|\mathbf{A}|\leqslant L$ there holds

$$C_1|\mathbf{A}|^2\leqslant\mathbf{e}(\mathbf{A})\leqslant C_2|\mathbf{A}|^2.$$

(iii) There exists a constant $\ell=\ell(p,m)>0$ such that for all $\mathbf{A}\in\mathbb{R}^m$ there holds

$$\ell\min\{|\mathbf{A}|^p,|\mathbf{A}|^2\}\leqslant\mathbf{e}(\mathbf{A})\leqslant\ell^{-1}\min\{|\mathbf{A}|^p,|\mathbf{A}|^2\}.$$

We finally record the following lemma that helps to encode the Hölder continuity in terms of integrals that depend on $\mathbf{e}$. We state the relevant result for $p=1$ only, but the generalisation to $1<p<\infty$ is straightforward.

Lemma 7.10. *Suppose that $u\in\mathrm{BD}_{\mathrm{loc}}(\mathbb{R}^n)$ satisfies for some $x_0\in\mathbb{R}^n$ and some $0<R\leqslant 1$*

$$\int_{\mathrm{B}(x_0,R)}\mathbf{e}(|\,\mathrm{E}u-(\mathrm{E}u)_{x_0,R}|)\leqslant CR^{n+2\alpha}$$

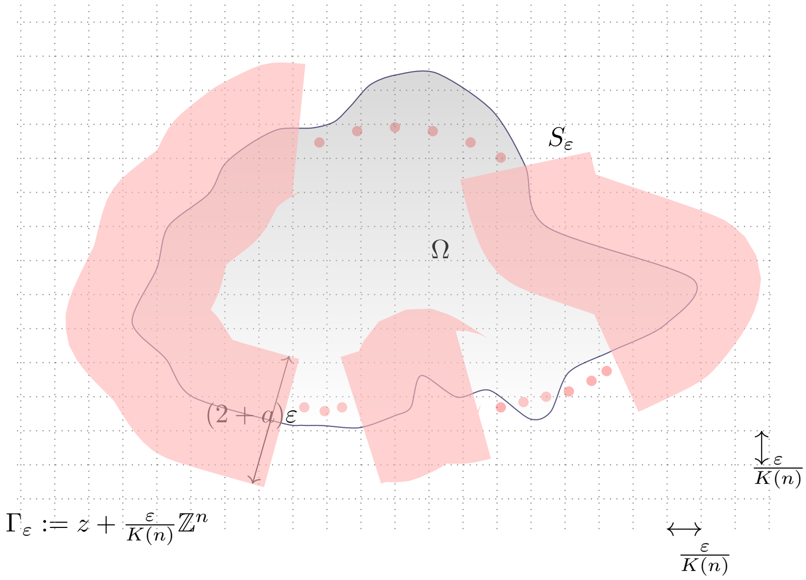

Figure 7.1: Not–to scale construction in the proof of Proposition 7.11.

for some $C>0$. *Then there exists a constant* $C'>0$ *whose value continuously depends on* C *such that*

$$\int_{\mathrm{B}(x_0,R)} |\,\mathrm{E}u - (\mathrm{E}u)_{x_0,R}| \leqslant C'R^{n+\alpha}. \tag{7.15}$$

Here, we have put $(\mathrm{E}u)_{x_0,r} := \mathrm{E}u(\mathrm{B}(x_0,r))/\mathscr{L}^n(\mathrm{B}(x_0,r))$.

Proof. It suffices to argue for $u\in\mathrm{LD}_{\mathrm{loc}}(\mathbb{R}^n)$ first. By the preceding lemma, we have with $b=\ell^{-1}$ that if $|z|>1$, then $|z|\leqslant b\mathbf{e}(z)$, and if $|z|\leqslant 1$, then $|z|^2\leqslant b\mathbf{e}(z)$. Denoting $v:=\boldsymbol{\varepsilon}(u)-(\boldsymbol{\varepsilon}(u))_{x_0,R}$, we split

$$\begin{aligned}
\int_{\mathrm{B}(x_0,R)}|v|\,\mathrm{d}y &= \int_{\mathrm{B}(x_0,R)\cap\{|v|\leqslant 1\}}|v|\,\mathrm{d}y + \int_{\mathrm{B}(x_0,R)\cap\{|v|>1\}}|v|\,\mathrm{d}y\\
&\leqslant \mathscr{L}^n(\mathrm{B}(x_0,R)\cap\{|v|\leqslant 1\})\left(\fint_{\mathrm{B}(x_0,R)\cap\{|v|\leqslant 1\}}|v|^2\,\mathrm{d}y\right)^{\frac{1}{2}}\\
&+ b\int_{\mathrm{B}(x_0,R)\cap\{|v|>1\}}\mathbf{e}(|v|)\,\mathrm{d}y \qquad \text{(by Jensen)}\\
&\leqslant \mathscr{L}^n(\mathrm{B}(x_0,R))^{\frac{1}{2}}b^{\frac{1}{2}}\left(\int_{\mathrm{B}(x_0,R)}\mathbf{e}(|v|)\,\mathrm{d}y\right)^{\frac{1}{2}} + b\int_{\mathrm{B}(x_0,R)}\mathbf{e}(|v|)\,\mathrm{d}y\\
&\leqslant \mathscr{L}^n(\mathrm{B}(x_0,R))^{\frac{1}{2}}b^{\frac{1}{2}}\left(CR^{n+2\alpha}\right)^{\frac{1}{2}} + bCR^{n+\alpha}R^{\alpha}\leqslant C'R^{n+\alpha},
\end{aligned}$$

with $C'=2(b^{\frac{1}{2}}C^{\frac{1}{2}}+bC)$, where we note that the penultimate inequality is valid by assumption (also recall that $R\leqslant 1$). Using a standard area–strict approximation argument, the proof is complete in the BD–case, too. □

7.5. Poincaré–Type Inequalities

The purpose of this section is to provide inequalities of Poincaré–type that help to control the L^1–distance of a BD–map to its mollification in terms of the mollifications radius. In the full gradient setting (or if $p > 1$ by use of Korn's inequality) we can exactly follow the standard approach in establishing such inequalities (cp. [97, Chpt. 4.5.2, Thm. 2]), but this is not possible due Ornstein's Non–Inequality if $p = 1$ (also cp. Remark 7.12 below). On one hand, these inequalities are crucial for the proof of the partial regularity result, Theorem 7.4, but are of separate interest, too. Hence we will discuss various approaches to such inequalities.

As usual, by a *standard mollifier* we understand a positive, radially symmetric and continuous function $\rho\colon \mathrm{B}(0,1) \to \mathbb{R}_{\geqslant 0}$ with $\|\rho\|_{\mathrm{L}^1(\mathrm{B}(0,1))} = 1$. Given $\varepsilon > 0$, we denote the rescaled versions $\rho_\varepsilon(x) := \varepsilon^{-n}\rho(x/\varepsilon)$ for $x \in \mathbb{R}^n$.

Proposition 7.11. *Let $\widetilde{\Omega} \subset \mathbb{R}^n$ be an open Lipschitz subset and suppose that $\Omega \Subset \widetilde{\Omega}$ is a relatively compact subset with $\mathrm{dist}(\Omega, \partial\widetilde{\Omega}) > 0$. Let $\rho \in \mathrm{C}_c^\infty(\mathrm{B}(0,1);[0,1])$ be a standard mollifier. Then for every $a > 0$, there exists a constant $C > 0$ which only depends on ρ and $a > 0$ such that for all $0 < \varepsilon < \mathrm{dist}(\Omega, \partial\widetilde{\Omega})$ and all $u \in \mathrm{BD}(\widetilde{\Omega})$ we have*

$$\int_\Omega |u - \rho_\varepsilon * u| \,\mathrm{d}x \leqslant C\varepsilon \,|\mathrm{E}u|(\overline{N_{a\varepsilon}(\Omega)}), \tag{7.16}$$

where $N_t(\Omega) := \{x \in \tilde{\Omega}\colon\ \mathrm{dist}(x, \partial\Omega) < t\}$ is the t–neighbourhood of Ω.

Proof. We first assume that $u \in \mathrm{LD}(\widetilde{\Omega})$ and shall finally argue by strict denseness to obtain estimate (7.16) for BD–functions. Let $\varepsilon > 0$ be given. In what follows, we consider an arbitrary lattice of the form $\Gamma_\varepsilon := \frac{\varepsilon}{K(n)}\mathbb{Z}^n$ and $\varepsilon > 0$, where $K(n) > 0$ is chosen later on.

We denote $\mathcal{Q}$ the collection of all (closed) cubes Q which can be written as

$$Q = [\frac{\varepsilon}{K(n)}k_1, \frac{\varepsilon}{K(n)}(k_1+1)] \times ... \times [\frac{\varepsilon}{K(n)}k_n, \frac{\varepsilon}{K(n)}(k_n+1)], \qquad (k_1,...,k_n) \in \mathbb{Z}^n$$

and put $\mathcal{Q}' := \{Q \in \mathcal{Q}\colon\ \mathscr{L}^n(Q \cap \Omega) > 0\}$.

Given any cube $Q \subset \mathbb{R}^n$ of sidelength $\ell(Q) > 0$ and $a > 0$, we denote Q_a the cube with the same center as Q but with sidelength $a\ell(Q)$, i.e., if $Q = z + \ell(Q)[-\frac{1}{2},\frac{1}{2}]^n$, then $Q_a := z + a\ell(Q)[-\frac{1}{2},\frac{1}{2}]^n$. Given $Q_1, Q_2 \in \mathcal{Q}$, we say that Q_1, Q_2 are *neighbouring cubes* if and only if $Q_1 \cap Q_2 \neq \emptyset$. Given $Q \in \mathcal{Q}$, we denote $\mathcal{N}(Q)$ the collection of all its neighbouring cubes, and this terminology canonically extends to cubes with edge points in an arbitrary given lattice. Note that for any $Q \in \mathcal{Q}$ there are only finitely many neighbouring cubes in $\mathcal{Q}$, and the maximum number of such cubes only depends on n. For each $Q \in \mathcal{Q}$, we may therefore enumerate

$$\mathcal{N}(Q) = \{\mathfrak{Q}_Q^1, ..., \mathfrak{Q}_Q^k\}$$

where $\mathfrak{Q}_Q^i \in \mathcal{Q}$, $i = 1,...,k$, with k only depending on n. For later purposes it is convenient to arrange the labelling of the single neighbouring cubes as follows, and for this it is convenient to work with the standard lattice. Choose a fixed labelling $\{\mathfrak{Q}^1_{[-\frac{1}{2},\frac{1}{2}]^n}, ..., \mathfrak{Q}^k_{[-\frac{1}{2},\frac{1}{2}]^n}\}$ of the neighbouring cubes with edge points in $\frac{1}{2}\mathbb{Z}^n$ for the unit cube $\mathcal{C} := [-\frac{1}{2},\frac{1}{2}]^n$ (with the obvious meaning). Let $z \in Q$ be the center of the cube Q and define the affine–linear mapping $T_z\colon \mathbb{R}^n \to \mathbb{R}^n$ by $T_z := z + \mathrm{Id}$. Then we arrange $\mathfrak{Q}_Q^i = T_z(\frac{\varepsilon}{K(n)}\mathfrak{Q}^i_{[-\frac{1}{2},\frac{1}{2}]^n})$ for all $i = 1,...,k$.

Given $a > 1$, choose $K(n) > 0$ so large such that for each $Q \in \mathcal{Q}'$ we have $Q + \varepsilon[0,1]^n \subset N_{a\varepsilon}(\Omega)$. Let us note in advance that for any cube $Q \in \mathcal{Q}$ and any $v \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^m;\mathbb{R}^N)$ we have

$$\int_Q |\rho_\varepsilon * v| \,\mathrm{d}x \leqslant \int_{\bigcup\mathcal{N}(Q)} |v| \,\mathrm{d}x. \tag{7.17}$$

Here, $\bigcup\mathcal{N}(Q) := \bigcup_{Q' \in \mathcal{N}(Q)} \mathbf{Q}'$, where $\mathbf{Q}'$ is a cube which is obtained by blowing up Q' by a fixed constant $\kappa(a) > 0$, such that $\bigcup\mathcal{N}(Q)(\subset N_{a\varepsilon}(\Omega))$ contains $\mathrm{spt}(\rho_\varepsilon * u|_Q)$. In particular, note that $\bigcup\mathcal{N}(Q)$ is a cube itself, and we find a constant $\lambda=\lambda(a,n)$ such that $\bigcup\mathcal{N}(Q)=Q_\lambda$ for all $Q\in\mathcal{Q}$.

In view of our main estimate, we firstly decompose

$$\begin{aligned}\int_\Omega |u-\rho_\varepsilon * u|\,\mathrm{d}x &= \sum_{Q\in\mathcal{Q}'}\int_{\Omega\cap Q}|u-\rho_\varepsilon * u|\,\mathrm{d}x\\ &\leqslant \sum_{Q\in\mathcal{Q}'}\int_Q |u-\rho_\varepsilon * u|\,\mathrm{d}x = (*).\end{aligned}$$

To estimate $(*)$, let $Q\in\mathcal{Q}'$ be arbitrary but fixed. We choose a rigid deformation $R_Q\in\mathcal{R}(\mathbb{R}^n)$ such that

$$\begin{aligned}\int_{Q_\lambda}|u-R_Q u|\,\mathrm{d}x &\leqslant C\ell(Q_\lambda)\int_{Q_\lambda}|\boldsymbol{\varepsilon}(u)|\,\mathrm{d}x\\ &= C\lambda\ell(Q)\int_{Q_\lambda}|\boldsymbol{\varepsilon}(u)|\,\mathrm{d}x = C\lambda\varepsilon\int_{Q_\lambda}|\boldsymbol{\varepsilon}(u)|\,\mathrm{d}x.\end{aligned}\tag{7.18}$$

Next we note that since $\mathcal{R}(\mathbb{R}^n)$ is a subspace of the polynomials of degree at most one, and therefore all of its elements are harmonic. Therefore, as a consequence of the mean value property of harmonic maps, convolution with radially symmetric smooth mollifiers is the identity operator on harmonic mappings. As a consequence, for each cube we have $(\rho_\varepsilon * R_Q)|_Q = R_Q$. We now go back to $(*)$ and further estimate

$$\begin{aligned}(*) &= \sum_{Q\in\mathcal{Q}'}\int_Q |u-\rho_\varepsilon * R_Q+\rho_\varepsilon * R_Q-\rho_\varepsilon * u|\,\mathrm{d}x\\ &\leqslant \sum_{Q\in\mathcal{Q}'}\int_Q |u-\rho_\varepsilon * R_Q|\,\mathrm{d}x+\int_Q|\rho_\varepsilon *(u-R_Q)|\,\mathrm{d}x\\ &\leqslant \sum_{Q\in\mathcal{Q}'}\int_Q |u-R_Q|\,\mathrm{d}x+\int_Q|\rho_\varepsilon *(u-R_Q)|\,\mathrm{d}x\\ &\leqslant \sum_{Q\in\mathcal{Q}'}\int_Q |u-R_Q|\,\mathrm{d}x+\int_{Q_\lambda}|u-R_Q|\,\mathrm{d}x \qquad (\text{by } (7.17))\\ &\leqslant 2\sum_{Q\in\mathcal{Q}'}\int_{Q_\lambda}|u-R_Q|\,\mathrm{d}x \qquad (\text{as } Q\subset Q_\lambda \text{ for all } Q\in\mathcal{Q}).\end{aligned}\tag{7.19}$$

At this stage we invoke Poincaré's Inequality, cp. (7.18), to conclude with a constant $C=C(n,a)>0$ and $\kappa(a)>0$

$$\begin{aligned}(*) &\leqslant C\varepsilon\sum_{Q\in\mathcal{Q}'}\int_{Q_\lambda}|\boldsymbol{\varepsilon}(u)|\,\mathrm{d}x = C\varepsilon\sum_{Q\in\mathcal{Q}'}C(n)\int_{\bigcup\mathcal{N}(Q)}|\boldsymbol{\varepsilon}(u)|\,\mathrm{d}x\\ &= C\varepsilon\sum_{j=1}^k\sum_{Q\in\mathcal{Q}'}\int_{\kappa(a)\mathfrak{Q}_Q^j}|\boldsymbol{\varepsilon}(u)|\,\mathrm{d}x\\ &\leqslant C\varepsilon\sum_{j=1}^k\int_{N_{a\varepsilon}(\Omega)}|\boldsymbol{\varepsilon}(u)|\,\mathrm{d}x\\ &\leqslant C\varepsilon\int_{N_{a\varepsilon}(\Omega)}|\boldsymbol{\varepsilon}(u)|\,\mathrm{d}x.\end{aligned}$$

In the penultimate estimate we used that by the specific labelling of the neighbouring cubes, for each $i=1,...,k$ we have $\mathscr{L}^n(\mathfrak{Q}_Q^i\cap\mathfrak{Q}_{Q'}^i)=0$ given $Q,Q'\in\mathcal{Q}'$ and $Q\neq Q'$, and that $\bigcup_{Q\in\mathcal{Q}'}\kappa(a)\mathfrak{Q}_Q^i\subset N_{a\varepsilon}(\Omega)$.

Collecting estimates, the proof is complete for $u \in \mathrm{LD}(\Omega)$. Now let $u \in \mathrm{BD}(\widetilde{\Omega})$ and choose a sequence $(u_j) \subset \mathrm{C}^1(\widetilde{\Omega};\mathbb{R}^n) \cap \mathrm{BD}(\widetilde{\Omega}) \subset \mathrm{LD}(\widetilde{\Omega})$ which converges to u in the strict topology (cp. Chapter 4.2.1). Then we have by $N_\varepsilon(\Omega) \subset \widetilde{\Omega}$

$$\begin{aligned}\int_\Omega |u-\rho_\varepsilon * u|\,\mathrm{d}x &\leqslant \int_\Omega |u-u_j|\,\mathrm{d}x + \int_\Omega |\rho_\varepsilon * (u-u_j)|\,\mathrm{d}x + \int_\Omega |u_j - \rho_\varepsilon * u_j|\,\mathrm{d}x \\ &\leqslant 2\int_{\widetilde{\Omega}} |u-u_j|\,\mathrm{d}x + \int_{N_{a\varepsilon}(\Omega)} |\boldsymbol{\varepsilon}(u_j)|\,\mathrm{d}x =: \mathbf{I}_j + \mathbf{II}_j,\end{aligned}$$

where we have used in the last step that the claimed estimate is already at our disposal for LD–maps. Clearly, we have $\mathbf{I}_j \to 0$ as $j \to \infty$ by strict convergence $u_j \overset{s}{\to} u$. On the other hand, we have

$$\limsup_{j\to\infty} \mathbf{II}_j \leqslant C\varepsilon |\operatorname{E}u|(\overline{N_{a\varepsilon}(\Omega)}).$$

The proof is complete. □

Remark 7.12. *The inequality just proved needs to be put in the context of the usual Poincaré inequality, cp. Chapter 3.5. Without going into the above proof, suppose that the right side of (7.16) vanishes so that $\boldsymbol{\varepsilon}(u) \equiv 0$ $\mathscr{L}^n$–a.e. and hence $u \in \mathcal{R}(N_{a\varepsilon}(\Omega))$. Then the left side is zero too as convolutions with standard mollifiers leave harmonic functions unchanged. On the other hand, it does not seem fruitful to write out the convolution with ρ_ε since in this situation,*

$$|u(x) - \rho_\varepsilon * u(x)| = \left| \int_{\mathbb{R}^n} (u(x-y) - u(x))\rho_\varepsilon(y)\,\mathrm{d}y \right|,$$

and in light of Ornstein's Non–Inequality, the attempt to bound the integral of this expression against the L^1–norm of $\boldsymbol{\varepsilon}(u)$ is not clear to us provided we wish to produce the factor ε on the right side of (7.16) instead of ε^s with $0<s<1$. The latter corresponds to the embedding $\mathrm{BD}_{\mathrm{loc}} \hookrightarrow \mathrm{W}^{s,1}_{\mathrm{loc}}$ for $0<s<1$, but this does not seem good enough for the partial regularity proof below.

We now turn to a version of the inequality (7.16) involving the auxiliary map $\mathbf{e}$ as introduced in the previous section.

Corollary 7.13. *Let $\widetilde{\Omega} \subset \mathbb{R}^n$ be open and suppose that $\Omega \Subset \widetilde{\Omega}$ is a relatively compact subset with $\operatorname{dist}(\Omega, \partial\widetilde{\Omega}) > 0$. Let $\rho \in \mathrm{C}^\infty(\mathrm{B}(0,1);[0,1])$ be a standard mollifier. Then for every $a>1$, there exists a constant $C>0$ which only depends on a and ρ such that for all $0<\varepsilon<\min\{1, \operatorname{dist}(\Omega,\partial\widetilde{\Omega})\}$ and all $u \in \mathrm{BD}(\widetilde{\Omega})$ we have*

$$\int_\Omega \mathbf{e}(|u-\rho_\varepsilon * u|)\,\mathrm{d}x \leqslant C\varepsilon\, \mathbf{e}(|\operatorname{E}u|)(\overline{N_{a\varepsilon}(\Omega)}),$$

where $\mathbf{e}(|\operatorname{E}u|)$ needs to be understood in the sense of convex functions of measures, see Section 4.3, and $\mathbf{e}(t) := \sqrt{1+t^2} - 1$.

The version for p–growth integrands can be obtained in a similar manner. To prove the corollary, we firstly need a Poincaré–type inequality involving the function $\mathbf{e}$. For notational clarity, let us recall that $\mathbf{e}$ can be defined both for scalar and vectorial arguments, and we shall switch between these interpretations whenever convenient.

Lemma 7.14. *Let Ω be an open, connected and bounded Lipschitz subset of $\mathbb{R}^n$. Then there exists a constant $C>0$ only depending on Ω such that for all $u \in \mathrm{BD}(\Omega)$ there holds*

$$\inf_{a\in\mathcal{R}(\Omega)} \int_\Omega \mathbf{e}(|u-a|)\,\mathrm{d}x \leqslant C\int_\Omega \mathbf{e}(|\operatorname{E}u|). \tag{7.20}$$

Proof. The proof evolves around a contradiction argument and follows the lines of [115, Lemma A.3.1]. We firstly prove the lemma for $u \in \mathrm{LD}(\Omega)$. Following [115], we hereafter introduce the space

$$\mathcal{R}^{\perp}(\Omega) := \left\{ w \in \mathrm{LD}(\Omega) \colon \int_{\Omega} \langle w, v \rangle \,\mathrm{d}x = 0 \text{ for all } v \in \mathcal{R}(\Omega) \right\}$$

and consequently prove that there exists a constant $c > 0$ such that

$$\int_{\Omega} \mathbf{e}(|u|)\,\mathrm{d}x \leqslant c \int_{\Omega} \mathbf{e}(|\boldsymbol{\varepsilon}(u)|)\,\mathrm{d}x \qquad \text{for all } u \in \mathcal{R}^{\perp}(\Omega). \tag{7.21}$$

Suppose that (7.21) does *not* hold. Then we find a sequence $(u_k) \subset \mathcal{R}^{\perp}(\Omega)$ such that

$$\int_{\Omega} \mathbf{e}(|u_k|)\,\mathrm{d}x \nearrow \infty, \quad \sup_{k\in\mathbb{N}} \int_{\Omega} \mathbf{e}(|\boldsymbol{\varepsilon}(u_k)|)\,\mathrm{d}x =: b < \infty. \tag{7.22}$$

Now recall that there exist constants $\lambda_1, \lambda_2 > 0$ such that $|t| \geqslant \mathbf{e}(t) \geqslant \lambda_1|t| - \lambda_2$ for all $t \geqslant 0$. We then deduce

$$\int_{\Omega} |u_k|\,\mathrm{d}x \geqslant \int_{\Omega} \mathbf{e}(|u_k|)\,\mathrm{d}x \to \infty$$

and

$$\sup_{k\in\mathbb{N}} \int_{\Omega} |\boldsymbol{\varepsilon}(u_k)|\,\mathrm{d}x \leqslant \sup_{k\in\mathbb{N}} \int_{\Omega} \frac{1}{\lambda_1}(\mathbf{e}(\boldsymbol{\varepsilon}(u_k)) + \lambda_2)\,\mathrm{d}x < \infty.$$

In particular, it is clear that we may assume without loss of generality that $\|u_k\|_{\mathrm{L}^1(\Omega;\mathbb{R}^n)} > 0$ for all $k \in \mathbb{N}$. We put, for $k \in \mathbb{N}$, $v_k := u_k/\|u_k\|_{\mathrm{L}^1(\Omega;\mathbb{R}^n)}$ (which consequently is well–defined). Then we have $\|v_k\|_{\mathrm{L}^1(\Omega;\mathbb{R}^n)} = 1$ for all $k \in \mathbb{N}$ and, on the other hand,

$$\int_{\Omega} |\boldsymbol{\varepsilon}(v_k)|\,\mathrm{d}x = \frac{\|\boldsymbol{\varepsilon}(v_k)\|_{\mathrm{L}^1(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})}}{\|v_k\|_{\mathrm{L}^1(\Omega;\mathbb{R}^n)}} \to 0, \qquad k \to \infty.$$

We consequently record that $(u_k) \subset \mathrm{LD}(\Omega)$ is uniformly bounded. Since $\mathrm{LD}(\Omega) \hookrightarrow\hookrightarrow \mathrm{L}^1(\Omega;\mathbb{R}^n)$ by boundedness of Ω and our regularity assumptions on $\partial\Omega$, we may extract a non–relabelled subsequence and an element $v \in \mathrm{L}^1(\Omega;\mathbb{R}^n)$ such that $v_k \to v$ strongly in $\mathrm{L}^1(\Omega;\mathbb{R}^n)$. It is easily checked that $v \in \mathrm{LD}(\Omega)$, and since $v_k \to v$ strongly in $\mathrm{L}^1(\Omega;\mathbb{R}^n)$,

$$\int_{\Omega} |v|\,\mathrm{d}x = \lim_{k\to\infty} \int_{\Omega} |v_k|\,\mathrm{d}x = 1. \tag{7.23}$$

However, by lower semicontinuity of the total variation with respect to L^1–convergence, we have in this situation $\int_{\Omega} |\boldsymbol{\varepsilon}(v)|\,\mathrm{d}x \leqslant \liminf_{k\to\infty} \int_{\Omega} |\boldsymbol{\varepsilon}(v_k)|\,\mathrm{d}x = 0$ and hence $v \in \mathcal{R}(\Omega)$. Note that it is at this point that we use the connectedness of Ω.

On the other hand, $\mathcal{R}^{\perp}(\Omega)$ is closed with respect to L^1–convergence. In fact, let $(z_j) \in \mathcal{R}^{\perp}(\Omega)$ be such that $z_j \to z$ strongly in $\mathrm{L}^1(\Omega;\mathbb{R}^n)$ for some $z \in \mathrm{L}^1(\Omega;\mathbb{R}^n)$. Let $\varphi \in \mathcal{R}(\Omega)$ be arbitrary. Then we find

$$\left| \int_{\Omega} \langle z, \varphi \rangle \,\mathrm{d}x \right| = \left| \int_{\Omega} \langle z - z_j, \varphi \rangle \,\mathrm{d}x \right| \leqslant \|\varphi\|_{\mathrm{L}^\infty(\Omega;\mathbb{R}^n)} \|z - z_j\|_{\mathrm{L}^1(\Omega;\mathbb{R}^n)} \to 0$$

as $j \to \infty$. Here we used that $\mathcal{R}(\Omega) \subset \mathrm{L}^\infty(\Omega;\mathbb{R}^n)$. So we have established that v from above must also belong to $\mathcal{R}(\Omega)$ and thus $v \in \mathcal{R}(\Omega) \cap \mathcal{R}^{\perp}(\Omega) = \{0\}$. This is obviously at variance with (7.23) and thus (7.21) is proven.

We now pass on to the full statement. Since $\mathcal{R}(\Omega)$ is finite dimensional, it possesses an $\mathrm{L}^2(\Omega;\mathbb{R}^n)$–orthonormal basis $\mathbf{r}_1,...,\mathbf{r}_M$ for some $M\in\mathbb{N}$, and we consider (in a similar way as in Chapter 3.5) the projection $\pi\colon \mathrm{LD}(\Omega)\to\mathcal{R}(\Omega)$ given by

$$\pi(w) := \sum_{i=1}^{M}\langle\mathbf{r}_i,w\rangle_{\mathrm{L}^2(\Omega;\mathbb{R}^n)}\mathbf{r}_i,\qquad w\in\mathrm{LD}(\Omega), \tag{7.24}$$

where $\langle\cdot,\cdot\rangle_{\mathrm{L}^2}$ is the usual L^2–inner product. Note that this is well–defined as $\mathcal{R}(\Omega)\subset\mathrm{L}^\infty(\Omega;\mathbb{R}^n)$. We then claim that for each $w\in\mathrm{LD}(\Omega)$ there holds

$$w-\pi(w)\in\mathcal{R}^\perp(\Omega). \tag{7.25}$$

In fact, let $\widetilde{w}\in\mathcal{R}(\Omega)$. Then we have

$$\int_\Omega\langle\widetilde{w},w-\pi(w)\rangle\,\mathrm{d}x = \sum_{i=1}^{M}\int_\Omega\langle\widetilde{w},\mathbf{r}_i\rangle_{\mathrm{L}^2}\langle\mathbf{r}_i,w-\pi(w)\rangle_{\mathrm{L}^2}.$$

On the other hand, we have for any $i\in\{1,...,M\}$

$$\begin{aligned}\langle\mathbf{r}_i,w-\pi(w)\rangle_{\mathrm{L}^2} &= \langle\mathbf{r}_i,w\rangle_{\mathrm{L}^2}-\langle\mathbf{r}_i,\pi(w)\rangle_{\mathrm{L}^2}\\ &= \langle\mathbf{r}_i,w\rangle_{\mathrm{L}^2}-\sum_{j=1}^{M}\langle\mathbf{r}_i,\mathbf{r}_j\rangle_{\mathrm{L}^2}\langle\mathbf{r}_j,w\rangle_{\mathrm{L}^2}\\ &= \langle\mathbf{r}_i,w\rangle_{\mathrm{L}^2}-\sum_{j=1}^{M}\delta_{ij}\langle\mathbf{r}_j,w\rangle_{\mathrm{L}^2}=0\end{aligned}$$

with the Kronecker symbol δ_{ij}. Hence (7.25) follows. In conclusion, we obtain by applying the first part of the proof to $u-\pi(u)\in\mathcal{R}^\perp(\Omega)$,

$$\inf_{a\in\mathcal{R}(\Omega)}\int_\Omega\mathbf{e}(|u-a|)\,\mathrm{d}x\leqslant\int_\Omega\mathbf{e}(|u-\pi(u)|)\,\mathrm{d}x\overset{(7.21)}{\leqslant}c\int_\Omega\mathbf{e}(\boldsymbol{\varepsilon}(u))\,\mathrm{d}x,$$

and the lemma follows for $u\in\mathrm{LD}(\Omega)$. If $u\in\mathrm{BD}(\Omega)$, then we argue as follows. By smooth approximation (cp. Lemma 4.9), we choose $(u_k)\subset\mathrm{C}^\infty(\Omega;\mathbb{R}^n)\cap\mathrm{BD}(\Omega)$ such that $u_k\to u$ area–strictly as $k\to\infty$. By finite dimensionality of $\mathcal{R}(\Omega)$, for each $k\in\mathbb{N}$, there exists $a_k\in\mathcal{R}(\Omega)$ such that by (7.21)

$$\inf_{a\in\mathcal{R}(\Omega)}\int_\Omega\mathbf{e}(|u_k-a|)\,\mathrm{d}x=\int_\Omega\mathbf{e}(|u_k-a_k|)\,\mathrm{d}x\leqslant C\int_\Omega\mathbf{e}(|\boldsymbol{\varepsilon}(u_k)|)\,\mathrm{d}x$$

Now, since

$$\begin{aligned}\int_\Omega\mathbf{e}(|a_k|)\,\mathrm{d}x &\leqslant C\int_\Omega\mathbf{e}(|u_k-a_k|)\,\mathrm{d}x+\int_\Omega\mathbf{e}(|u_k|)\,\mathrm{d}x\\ &\leqslant c\left(\int_\Omega\mathbf{e}(|u_k|)\,\mathrm{d}x+\int_\Omega\mathbf{e}(|\boldsymbol{\varepsilon}(u_k)|)\,\mathrm{d}x\right)\leqslant C<\infty\end{aligned}$$

for some constant $C>0$ by strict convergence $u_k\to u$ as $k\to\infty$, we conclude similarly as above that $\|a_k\|_{\mathrm{L}^1(\Omega;\mathbb{R}^n)}$ is uniformly bounded in $k\in\mathbb{N}$. By the Heine–Borel theorem, there exists a suitable subsequence $(a_{k(j)})$ of (a_k) and some $a\in\mathcal{R}(\Omega)$ such that $a_{k(j)}\to a$ in *any* norm on $\mathcal{R}(\Omega)$. Now, clearly, $u_{k(j)}-a_{k(j)}\to u-a$ strongly in $\mathrm{L}^1(\Omega)$ and hence, passing to a possibly another subsequence, we may assume without loss of generality $u_{k(j)}-a_{k(j)}\to u-a$ $\mathscr{L}^n$–a.e. as $j\to\infty$. Therefore, by Fatou's lemma,

$$\begin{aligned}\int_\Omega\mathbf{e}(|u-a|)\,\mathrm{d}x &\leqslant\liminf_{j\to\infty}\int_\Omega\mathbf{e}(|u_{k(j)}-a_{k(j)}|)\,\mathrm{d}x\\ &\leqslant C\liminf_{j\to\infty}\int_\Omega\mathbf{e}(|\boldsymbol{\varepsilon}(u_{k(j)})|)\,\mathrm{d}x=C\mathbf{e}(|\,\mathrm{E}u|)(\Omega).\end{aligned}$$

Here, the ultimate equality follows from the continuity part of Reshetnyak's lower semicontinuity theorem (cp. Prop. 4.18) which is available by area–strict convergence $u_k \to u$ as $k \to \infty$. Lastly, we trivially have

$$\inf_{\alpha\in\mathcal{R}(\Omega)} \int_\Omega \mathbf{e}(|u-\alpha|)\,\mathrm{d}x \leqslant \int_\Omega \mathbf{e}(|u-a|)\,\mathrm{d}x \leqslant C\mathbf{e}(|\,\mathrm{E}u|)(\Omega),$$

and the proof is complete. □

Scaling in conjunction with Lemma 7.9(i) immediately yields the following corollary.

Corollary 7.15. *Let $v \in \mathrm{BD}_{\mathrm{loc}}(\mathbb{R}^n)$. Then there exists a constant $C > 0$ such for every ball $\mathrm{B}(x_0, r) \subset \mathbb{R}^n$ there holds*

$$\inf_{\alpha\in\mathcal{R}(\mathrm{B}(x_0,r))} \int_{\mathrm{B}(x_0,r)} \mathbf{e}(|u-\alpha|)\,\mathrm{d}x \leqslant C\max\{r, r^2\} \int_{\overline{\mathrm{B}(x_0,r)}} \mathbf{e}(|\,\mathrm{E}u|).$$

We finally come to the

Proof of Corollary 7.13. We adopt the terminology and notation of the proof of Proposition 7.11 and only indicate the relevant changes. Hence we embark on the proof at (7.19) and obtain equally

$$\begin{aligned}\int_\Omega \mathbf{e}(|u-\rho_\varepsilon * u|)\,\mathrm{d}x &\leqslant \sum_{Q\in\mathcal{Q}'} \int_Q \mathbf{e}(|u-\rho_\varepsilon * R_Q + \rho_\varepsilon * R_Q - \rho_\varepsilon * u|)\,\mathrm{d}x \\ &\leqslant C\sum_{Q\in\mathcal{Q}'} \int_Q \mathbf{e}(|u-\rho_\varepsilon * R_Q|)\,\mathrm{d}x + \int_Q \mathbf{e}(|\rho_\varepsilon * (u-R_Q)|)\,\mathrm{d}x =: (*),\end{aligned}$$

where the second inequality follows from monotonicity and convexity of $\mathbf{e}$ on $\mathbb{R}_{\geqslant 0}$. By Jensen's Inequality, we further obtain

$$\begin{aligned}(*) &\leqslant C\sum_{Q\in\mathcal{Q}'} \int_{Q_\lambda} \mathbf{e}(|u-R_Q|)\,\mathrm{d}x \\ &\leqslant C\max\{\varepsilon, \varepsilon^2\} \sum_{Q\in\mathcal{Q}'} \int_{Q_\lambda} \mathbf{e}(|\,\mathrm{E}u|) \qquad \text{(by Corollary 7.15)},\end{aligned}$$

and at this stage the proof exactly evolves as that of Proposition 7.11. The proof is complete. □

Moreover, let us remark that an inequality in the spirit of Corollary 7.13 involving multiples of u and $\rho_\varepsilon * u$ can be obtained by use of the inequality (for $L > 0$)

$$\inf_{\alpha\in\mathcal{R}(\mathbb{R}^n)} \int_Q \mathbf{e}(L(u-\alpha))\,\mathrm{d}x \leqslant \max\{L\ell(Q), L^2\ell(Q)^2\}\mathbf{e}(|\,\mathrm{E}u|)(Q).$$

In fact, if Q is the unit cube, then by the representation formulas due to Reshetnyak [200], we can write

$$u(x) = \alpha(x) + \int_Q k(x-y)\varepsilon(u)(y)\,\mathrm{d}y, \qquad x \in Q,$$

where $k\colon Q \to \mathscr{L}(\mathbb{R}^{n\times n}_{\mathrm{sym}}; \mathbb{R}^n)$ is a suitable Riesz potential kernel of growth $k(z) \sim |z|^{1-n}$ and $\alpha \in \mathcal{R}(\mathbb{R}^n)$ is a suitable rigid deformation. For $x \in Q$, we define the measure μ_x as the unique measure which is absolutely continuous with respect to $\mathscr{L}^n \llcorner Q$ and has density $|k(x-\cdot)|$ (with respect to $\mathscr{L}^n$). It is then not difficult to see that by the growth of k we have $\mu_x(Q) \approx c\ell(Q)$ for a constant $c > 0$ which only depends on n. In consequence, since $\mu_x/\mu_x(Q)$ is a probability measure on Q for every $x \in Q$, we obtain by Jensen's inequality and Lemma 7.9

$$\begin{aligned}
\int_Q \mathbf{e}(Lu - L\alpha)\,\mathrm{d}x &\leqslant \int_Q \mathbf{e}\left(L\int_Q k(x-y)\boldsymbol{\varepsilon}(u)(y)\,\mathrm{d}y\right)\mathrm{d}x \\
&\leqslant \int_Q \mathbf{e}\left(L\mu_x(Q)\int_Q |k(x-y)|\,|\boldsymbol{\varepsilon}(u)(y)|\frac{\mathrm{d}y}{\mu_x(Q)}\right)\mathrm{d}x \\
&\leqslant \int_Q \max\{((L\mu_x(Q)),(L\mu_x(Q))^2\}\int_Q |k(x-y)|\mathbf{e}(|\boldsymbol{\varepsilon}(u)(y)|)\frac{\mathrm{d}y}{\mu_x(Q)}\,\mathrm{d}x \\
&\leqslant C\max\{((L\ell(Q)),(L\ell(Q))^2\}\int_Q\int_Q |k(x-y)|\frac{\mathrm{d}x}{\ell(Q)}\mathbf{e}(|\boldsymbol{\varepsilon}(u)(y)|)\,\mathrm{d}y \\
&\leqslant C\max\{((L\ell(Q)),(L\ell(Q))^2\}\int_Q \mathbf{e}(|\boldsymbol{\varepsilon}(u)(y)|)\,\mathrm{d}y.
\end{aligned}$$

By translation and the corresponding change of behaviour of k, this estimate is seen to be independent of Q, cp. [200]. With this inequality in mind, a simple adaptation of the proof of Proposition 7.11 yields in the same situation

$$\int_\Omega \mathbf{e}(L(u-\rho_\varepsilon * u))\,\mathrm{d}x \leqslant c(a)\max\{((L\ell(Q)),(L\ell(Q))^2\}\int_{N_{a\varepsilon}(\Omega)} \mathbf{e}(\boldsymbol{\varepsilon}(u))\,\mathrm{d}x, \quad u \in \mathrm{LD}(\Omega),$$

and analogously for $u \in \mathrm{BD}(\Omega)$. Finally, if any of the estimates presented so far is invoked to estimate the differences $u-(u_\delta)_\varepsilon$, then we simply write

$$u-(u_\delta)_\varepsilon = u-u_\delta+u_\delta-(u_\delta)_\varepsilon$$

and apply the gathered estimates to $u-u_\delta$ and $u_\delta-(u_\delta)_\varepsilon$ separately. In the latter case, the resulting integrals involving u_δ are then treated by Jensen's inequality by

$$\int_\Omega g(\mathrm{E}u_\delta) \leqslant \int_{N_\delta(\Omega)} g(\mathrm{E}u).$$

7.6. Partial Regularity

We now provide the proof of Theorem 7.4 and, following the strategy outlined at the beginning of the present chapter, this is only a minor modification of the arguments of [22] in light of the convolution–type Poincaré–inequalities of the previous paragraph. As an important convention, we put

$$X := \begin{cases} \mathrm{BD} & \text{if } p=1, \\ \mathrm{W}^{1,p} & \text{if } 1<p<\infty. \end{cases} \tag{7.26}$$

Moreover, to provide a unifying treatment, let us arrange that if integrals of the form $\int_U f(\boldsymbol{\varepsilon}(u))\,\mathrm{d}x$ with a convex function $f\colon \mathbb{R}^{n\times n}_{\mathrm{sym}} \to \mathbb{R}$ of linear growth and $u \in \mathrm{BD}(U)$ appear, then these integrals are tacitly understood in the sense of functions of measures, see Chapter 4.3. That is, by slight abuse of notation, these are interpreted as $\int_U f(\mathrm{E}u)$. Lastly, to keep notation simple, we shall extensively utilise the symbol '$\lesssim$' as introduced in Chapter 2.1 whenever the precise knowledge of the constants appearing in estimates are not strictly relevant.

7.6.1. The Excess In the subsequent paragraphs we shall work with two different excess quantities. Keeping in mind that $1 \leqslant p < \infty$ is fixed throughout and recalling the definition (7.14) of the auxiliary integrands, we define for a given element $u \in X(\Omega)$, $z \in \Omega$ and $R>0$ such that $\mathrm{B}(z,R) \subset \Omega$,

$$\mathbf{E}(u;z,R) := \int_{\mathrm{B}(z,R)} \mathbf{e}(\boldsymbol{\varepsilon}(u)-(\boldsymbol{\varepsilon}(u))_{z,R})\,\mathrm{d}x \quad \text{and} \quad \widetilde{\mathbf{E}}(u;z,R) := \frac{\mathbf{E}(u;z,R)}{R^n}. \tag{7.27}$$

7.6.2. Decay of Comparison Maps The purpose of this section is to provide decay estimates for Hölder continuous function which, in addition, satisfy a certain smallness condition. Later on in Sections 7.6.3 and 7.6.4 we shall prove that suitable mollifications of minimisers match these conditions and thereby make the results of this section available. Throughout this paragraph, we fix $\mathbf{m} \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$, $\sigma > 0$ and assume that $f \in \mathrm{C}^2(\mathbb{R}^{n\times n}_{\mathrm{sym}})$ satisfies

$$\lambda|\mathbf{Z}|^2 \leqslant \langle f''(\mathbf{m})\mathbf{Z},\mathbf{Z}\rangle \leqslant \Lambda|\mathbf{Z}|^2. \tag{7.28}$$

for some $0 < \lambda \leqslant \Lambda < \infty$ and all $\mathbf{Z} \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$. Moreover, we assume that there exists a bounded and non–decreasing function $\omega\colon \mathbb{R}^+_0 \to \mathbb{R}^+_0$ with $\lim_{t\searrow 0}\omega(t) = 0$ such that for all $\mathbf{m}' \in \mathbb{B}(\mathbf{m},\sigma)$ we have

$$|f''(\mathbf{m}) - f''(\mathbf{m}')| \leqslant \omega(|\mathbf{m}-\mathbf{m}'|). \tag{7.29}$$

Finally, for $0 < r < R$ and $v \in \mathrm{C}^{1,\alpha}(\overline{\mathrm{B}(z,r)};\mathbb{R}^n)$ we put

$$\begin{aligned}
\mathrm{Dev}(v;z,r) &:= \int_{\mathrm{B}(z,r)} f(\boldsymbol{\varepsilon}(v)) - \inf\left\{\int_{\mathrm{B}(z,r)} f(\boldsymbol{\varepsilon}(w))\,\mathrm{d}x\colon \begin{array}{c} w \in \mathrm{C}^{1,\alpha}(\overline{\mathrm{B}(z,r)};\mathbb{R}^n) \\ w = v \text{ on } \partial\,\mathrm{B}(z,r)\end{array}\right\},\\
\mathbf{t}(v;z,r) &:= \sup_{\mathrm{B}(z,r)} |\boldsymbol{\varepsilon}(v)-\mathbf{m}| + 2^\alpha r^\alpha[\boldsymbol{\varepsilon}(v)]_{\mathrm{C}^{0,\alpha}(\overline{\mathrm{B}(z,r)};\mathbb{R}^{n\times n})}
\end{aligned}$$

Notice that Dev is an indicator of how far v is away from minimising the variational integral $\mathfrak{F}$ restricted to $\mathrm{B}(z,r)$. Besides, the function $\mathbf{t}$ will prove useful to find the mentioned smallness condition which is necessary to infer the decay estimate of the Hölder continuous comparison maps. Now we have the following result.

Proposition 7.16. *Let $0 < \alpha < 1$. Then there exists $c_1 > 0$ such that the following holds: If $v \in \mathrm{C}^{1,\alpha}(\mathrm{B}(z,R/2);\mathbb{R}^n)$ satisfies $\mathbf{t}(R/2) < \min\{\sigma/c_1, 1\}$, then there exists a bounded, non–decreasing function $\vartheta\colon \mathbb{R}^+_0 \to \mathbb{R}^+_0$ with $\lim_{t\searrow 0}\vartheta(t) = 0$ such that for all $0 < r < R/2$ we have*

$$\begin{aligned}
\int_{\mathrm{B}(z,r)} |\boldsymbol{\varepsilon}(v) - (\boldsymbol{\varepsilon}(v))_{z,r}|^2\,\mathrm{d}x \lesssim \Big(\frac{r}{R}\Big)^{n+2} &\int_{\mathrm{B}(z,R/2)} |\boldsymbol{\varepsilon}(v) - (\boldsymbol{\varepsilon}(v))_{z,R/2}|^2\,\mathrm{d}x \\
&+ \vartheta(\mathbf{t}(R/2)) \int_{\mathrm{B}(z,R/2)} |\boldsymbol{\varepsilon}(v)-\mathbf{m}|^2\,\mathrm{d}x \\
&+ \mathrm{Dev}(v;z,R/2)
\end{aligned} \tag{7.30}$$

provided $1 \leqslant p \leqslant 2$ and

$$\begin{aligned}
\int_{\mathrm{B}(z,r)} |\boldsymbol{\varepsilon}(v) - (\boldsymbol{\varepsilon}(v))_{z,r}|^p\,\mathrm{d}x \lesssim \Big(\frac{r}{R}\Big)^{n+p} &\int_{\mathrm{B}(z,R/2)} |\boldsymbol{\varepsilon}(v) - (\boldsymbol{\varepsilon}(v))_{z,R/2}|^p\,\mathrm{d}x \\
&+ \vartheta(\mathbf{t}(R/2)) \int_{\mathrm{B}(z,R/2)} |\boldsymbol{\varepsilon}(v)-\mathbf{m}|^2\,\mathrm{d}x \\
&+ \mathrm{Dev}(v;z,R/2)
\end{aligned} \tag{7.31}$$

provided $p \geqslant 2$.

Proof. The strategy of the proof is as follows: First, passing to the second order Taylor polynomial of the integrand f we obtain an integrand g of quadratic growth to whose minimisers we may apply the classical decay estimates for linear elliptic systems. Then we show that the conditions of the present proposition are sufficient for these decay estimates inherit to v in a way such that (7.30) follows. Moreover, we shall only prove (7.30); (7.31) follows in the same way, and we will indicate where slight changes must be incorporated.

We therefore begin by defining the auxiliary integrand $g\colon \mathbb{B}(\mathbf{m},\sigma)\to\mathbb{R}$ through

$$g(\mathbf{Z}) := f(\mathbf{m}) + \langle f'(\mathbf{m}),(\mathbf{Z}-\mathbf{m})\rangle + \frac{1}{2}\langle f''(\mathbf{m})(\mathbf{Z}-\mathbf{m}),(\mathbf{Z}-\mathbf{m})\rangle,\quad \mathbf{Z}\in\mathbb{B}(\mathbf{m},\sigma).$$

Using a Taylor expansion of f up to order two around $\mathbf{m}$, we deduce by (7.29) that

$$|f(\mathbf{Z})-g(\mathbf{Z})| \lesssim \omega(|\mathbf{Z}-\mathbf{m}|)|\mathbf{Z}-\mathbf{m}|^2,\quad \mathbf{Z}\in\mathbb{B}(\mathbf{m},\sigma). \tag{7.32}$$

By Lemma 7.8, the solution h of the auxiliary minimisation problem

$$\text{to minimise } \int_{\mathrm{B}(z,R/2)} g(\boldsymbol{\varepsilon}(w))\,\mathrm{d}x \ \text{ among all } w\in \mathrm{W}^{1,2}_v(\mathrm{B}(z,R/2);\mathbb{R}^n), \tag{7.33}$$

where $\mathrm{W}^{1,2}_v(\mathrm{B}(z,R/2);\mathbb{R}^n) = v + \mathrm{W}^{1,2}_0(\mathrm{B}(z,R/2);\mathbb{R}^n)$, belongs to $\mathrm{C}^{1,\alpha}(\overline{\mathrm{B}(z,R/2)};\mathbb{R}^n)$. In consequence, we have

$$\int_{\mathrm{B}(z,r)} |\boldsymbol{\varepsilon}(h)-(\boldsymbol{\varepsilon}(h))_{z,r}|^2\,\mathrm{d}x \lesssim \Big(\frac{r}{R}\Big)^{n+2}\int_{\mathrm{B}(z,R/2)} |\boldsymbol{\varepsilon}(h)-(\boldsymbol{\varepsilon}(h))_{z,R/2}|^2\,\mathrm{d}x \tag{7.34}$$

for all $0<r<R/2$. Moreover, Lemma 7.8 (b) gives

$$[\boldsymbol{\varepsilon}(h)]_{\mathrm{C}^{0,\alpha}(\overline{\mathrm{B}(z,R/2)};\mathbb{R}^{n\times n})} \lesssim [\boldsymbol{\varepsilon}(v)]_{\mathrm{C}^{0,\alpha}(\overline{\mathrm{B}(z,R/2)};\mathbb{R}^{n\times n})}. \tag{7.35}$$

We will now compare v with h. To this end, we first notice that

$$\begin{aligned}\int_{\mathrm{B}(z,r)} |\boldsymbol{\varepsilon}(v)-(\boldsymbol{\varepsilon}(v))_{z,r}|^2\,\mathrm{d}x \lesssim & \int_{\mathrm{B}(z,r)} |\boldsymbol{\varepsilon}(v)-\boldsymbol{\varepsilon}(h)|^2\,\mathrm{d}x \\ & + \int_{\mathrm{B}(z,r)} |\boldsymbol{\varepsilon}(h)-(\boldsymbol{\varepsilon}(h))_{z,r}|^2\,\mathrm{d}x \\ & + \int_{\mathrm{B}(z,r)} |(\boldsymbol{\varepsilon}(h))_{z,r}-(\boldsymbol{\varepsilon}(v))_{z,r}|^2\,\mathrm{d}x =: \mathbf{I}+\mathbf{II}+\mathbf{III}.\end{aligned}$$

We shall estimate $\mathbf{II}$ by means of (7.34). Keeping this in mind, we turn to the remaining two terms $\mathbf{I}$ and $\mathbf{III}$. By Jensen's inequality, we firstly find

$$\begin{aligned}\mathbf{I}+\mathbf{III} &\lesssim \int_{\mathrm{B}(z,r)} |\boldsymbol{\varepsilon}(v)-\boldsymbol{\varepsilon}(h)|^2\,\mathrm{d}x + \int_{\mathrm{B}(z,r)} \fint_{\mathrm{B}(z,r)} |\boldsymbol{\varepsilon}(v)(y)-\boldsymbol{\varepsilon}(h)(y)|^2\,\mathrm{d}y\,\mathrm{d}x \\ &\lesssim \int_{\mathrm{B}(z,r)} |\boldsymbol{\varepsilon}(v)-\boldsymbol{\varepsilon}(h)|^2\,\mathrm{d}x,\end{aligned}$$

and by the minimality property of h we deduce through (7.34) that

$$\begin{aligned}\mathbf{II} &\lesssim \Big(\frac{r}{R}\Big)^{n+2}\int_{\mathrm{B}(z,R/2)} |\boldsymbol{\varepsilon}(h)-(\boldsymbol{\varepsilon}(h))_{z,R/2}|^2\,\mathrm{d}x \\ &\lesssim \Big(\frac{r}{R}\Big)^{n+2}\int_{\mathrm{B}(z,R/2)} |\boldsymbol{\varepsilon}(v)-(\boldsymbol{\varepsilon}(v))_{z,R/2}|^2\,\mathrm{d}x.\end{aligned}$$

At this stage we note that the case $p\geqslant 2$ is covered by replacing estimate (7.34) by

$$\int_{\mathrm{B}(z,r)} |\boldsymbol{\varepsilon}(h)-(\boldsymbol{\varepsilon}(h))_{z,r}|^p\,\mathrm{d}x \lesssim \Big(\frac{r}{R}\Big)^{n+p}\int_{\mathrm{B}(z,R/2)} |\boldsymbol{\varepsilon}(h)-(\boldsymbol{\varepsilon}(h))_{z,R/2}|^p\,\mathrm{d}x$$

for all $0<r<R/2$; the rest of the proof then goes along the same lines as for $1\leqslant p\leqslant 2$. Turning back to $1\leqslant p\leqslant 2$ and combining the estimates for $\mathbf{I},\mathbf{II}$ and $\mathbf{III}$, we need to

estimate

$$\begin{aligned}\int_{\mathrm{B}(z,R/2)} |\boldsymbol{\varepsilon}(v)-\boldsymbol{\varepsilon}(h)|^2\,\mathrm{d}x &\lesssim \int_{\mathrm{B}(z,R/2)} \langle f''(\mathbf{m})(\boldsymbol{\varepsilon}(v)-\boldsymbol{\varepsilon}(h)),(\boldsymbol{\varepsilon}(v)-\boldsymbol{\varepsilon}(h))\rangle\,\mathrm{d}x \\ &\overset{(*)}{\lesssim} \int_{\mathrm{B}(z,R/2)} g(\boldsymbol{\varepsilon}(v))-g(\boldsymbol{\varepsilon}(h))\,\mathrm{d}x \\ &\lesssim \int_{\mathrm{B}(z,R/2)} g(\boldsymbol{\varepsilon}(v))-f(\boldsymbol{\varepsilon}(v))\,\mathrm{d}x \\ &+\int_{\mathrm{B}(z,R/2)} f(\boldsymbol{\varepsilon}(v))-f(\boldsymbol{\varepsilon}(h))\,\mathrm{d}x \\ &+\int_{\mathrm{B}(z,R/2)} f(\boldsymbol{\varepsilon}(h))-g(\boldsymbol{\varepsilon}(h))\,\mathrm{d}x =: \mathbf{I}_1+\mathbf{I}_2+\mathbf{I}_3.\end{aligned}$$

Let us justify $(*)$. We have

$$\begin{aligned} g(\boldsymbol{\varepsilon}(v)) &= f(\mathbf{m})+\langle f'(\mathbf{m}),\boldsymbol{\varepsilon}(v)-\mathbf{m}\rangle+\frac{1}{2}\langle f''(\mathbf{m})(\boldsymbol{\varepsilon}(v)-\mathbf{m}),(\boldsymbol{\varepsilon}(v)-\mathbf{m})\rangle, \\ g(\boldsymbol{\varepsilon}(h)) &= f(\mathbf{m})+\langle f'(\mathbf{m}),\boldsymbol{\varepsilon}(h)-\mathbf{m}\rangle+\frac{1}{2}\langle f''(\mathbf{m})(\boldsymbol{\varepsilon}(h)-\mathbf{m}),(\boldsymbol{\varepsilon}(h)-\mathbf{m})\rangle\end{aligned}$$

and, by symmetry of $f''(\mathbf{m})$,

$$\begin{aligned}\langle f''(\mathbf{m})(\boldsymbol{\varepsilon}(v)-\mathbf{m}),(\boldsymbol{\varepsilon}(v)-\mathbf{m})\rangle &- \langle f''(\mathbf{m})(\boldsymbol{\varepsilon}(h)-\mathbf{m}),(\boldsymbol{\varepsilon}(h)-\mathbf{m})\rangle \\ &= \langle f''(\mathbf{m})\boldsymbol{\varepsilon}(v),\boldsymbol{\varepsilon}(v)\rangle-\langle f''(\mathbf{m})\boldsymbol{\varepsilon}(h),\boldsymbol{\varepsilon}(h)\rangle \\ &-2\langle f''(\mathbf{m})\mathbf{m},\boldsymbol{\varepsilon}(v-h)\rangle.\end{aligned}$$

Hence we obtain

$$\begin{aligned}\int_{\mathrm{B}(z,R/2)} g(\boldsymbol{\varepsilon}(v))-g(\boldsymbol{\varepsilon}(h))\,\mathrm{d}x &= \int_{\mathrm{B}(z,R/2)} \langle f'(\mathbf{m}),\boldsymbol{\varepsilon}(v-h)\rangle\,\mathrm{d}x \\ &+\frac{1}{2}\int_{\mathrm{B}(z,R/2)} \langle f''(\mathbf{m})\boldsymbol{\varepsilon}(v),\boldsymbol{\varepsilon}(v)\rangle-\langle f''(\mathbf{m})\boldsymbol{\varepsilon}(h),\boldsymbol{\varepsilon}(h)\rangle\,\mathrm{d}x \\ &-\int_{\mathrm{B}(z,R/2)} \langle f''(\mathbf{m})\mathbf{m},\boldsymbol{\varepsilon}(v-h)\rangle\,\mathrm{d}x =: \mathbf{J}_1+\mathbf{J}_2+\mathbf{J}_3.\end{aligned}$$

On the other hand, expanding terms yields

$$\begin{aligned}&\int_{\mathrm{B}(z,R/2)} \langle f''(\mathbf{m})(\boldsymbol{\varepsilon}(v)-\boldsymbol{\varepsilon}(h)),(\boldsymbol{\varepsilon}(v)-\boldsymbol{\varepsilon}(h))\rangle\,\mathrm{d}x \\ &\qquad= \int_{\mathrm{B}(z,R/2)} \langle f''(\mathbf{m})\boldsymbol{\varepsilon}(v),\boldsymbol{\varepsilon}(v)\rangle\,\mathrm{d}x-2\int_{\mathrm{B}(z,R/2)} \langle f''(\mathbf{m})\boldsymbol{\varepsilon}(h),\boldsymbol{\varepsilon}(v)\rangle \\ &\qquad+\int_{\mathrm{B}(z,R/2)} \langle f''(\mathbf{m})\boldsymbol{\varepsilon}(h),\boldsymbol{\varepsilon}(h)\rangle\,\mathrm{d}x \\ &\qquad= \int_{\mathrm{B}(z,R/2)} \langle f''(\mathbf{m})\boldsymbol{\varepsilon}(v),\boldsymbol{\varepsilon}(v)\rangle\,\mathrm{d}x-2\int_{\mathrm{B}(z,R/2)} \langle f''(\mathbf{m})\boldsymbol{\varepsilon}(h),\boldsymbol{\varepsilon}(v-h)\rangle\,\mathrm{d}x \\ &\qquad-\int_{\mathrm{B}(z,R/2)} \langle f''(\mathbf{m})\boldsymbol{\varepsilon}(h),\boldsymbol{\varepsilon}(h)\rangle\,\mathrm{d}x \\ &\qquad= 2\mathbf{J}_2-2\int_{\mathrm{B}(z,R/2)} \langle f''(\mathbf{m})\boldsymbol{\varepsilon}(h),\boldsymbol{\varepsilon}(v-h)\rangle\,\mathrm{d}x =: 2\mathbf{J}_2-\mathbf{J}_4.\end{aligned}$$

It is not difficult to see by an approximation argument that the Euler–Lagrange equation associated with (7.33) reads as

$$\int_{\mathrm{B}(z,R/2)} \langle f''(\mathbf{m})\boldsymbol{\varepsilon}(h),\boldsymbol{\varepsilon}(\varphi)\rangle\,\mathrm{d}x = 0 \qquad \text{for all } \varphi\in \mathrm{W}_0^{1,2}(\mathrm{B}(z,R/2);\mathbb{R}^n). \tag{7.36}$$

Therefore, $v-h\in \mathrm{W}_0^{1,2}(\mathrm{B}(z,R/2);\mathbb{R}^n)$ qualifies as a test map in (7.36) and immediately shows that $\mathbf{J}_4=0$. Moreover, since $v=h$ on $\partial\mathrm{B}(z,R/2)$ and both $f'(\mathbf{m})$ and $f''(\mathbf{m})\mathbf{m}$ are constant, we may integrate by parts to obtain $\mathbf{J}_1=\mathbf{J}_3=0$. In conclusion, we have

$$\frac{1}{2}\int_{\mathrm{B}(z,R/2)}\langle f''(\mathbf{m})\boldsymbol{\varepsilon}(v),\boldsymbol{\varepsilon}(v)\rangle-\langle f''(\mathbf{m})\boldsymbol{\varepsilon}(h),\boldsymbol{\varepsilon}(h)\rangle\,\mathrm{d}x=\int_{\mathrm{B}(z,R/2)}g(\boldsymbol{\varepsilon}(v))-g(\boldsymbol{\varepsilon}(h))\,\mathrm{d}x,$$

which is $(*)$.

The integrals $\mathbf{I}_1,\mathbf{I}_2,\mathbf{I}_3$ will be estimated separately. Among them, we readily obtain through the definition of Dev and h that $\mathbf{I}_2\leqslant \mathrm{Dev}(v;z,R/2)$. Now notice that h solves the auxiliary problem (7.33) and hence we have that $\|\boldsymbol{\varepsilon}(h)-\mathbf{m}\|_{\mathrm{L}^2(\mathrm{B}(z,R/2);\mathbb{R}^{n\times n})}\lesssim\|\boldsymbol{\varepsilon}(v)-\mathbf{m}\|_{\mathrm{L}^2(\mathrm{B}(z,R/2);\mathbb{R}^{n\times n})}$. Therefore we deduce that

$$\begin{aligned}\sup_{\mathrm{B}(z,R/2)}|\boldsymbol{\varepsilon}(h)-\mathbf{m}|&\leqslant\sup_{\mathrm{B}(z,R/2)}|\boldsymbol{\varepsilon}(h)-(\boldsymbol{\varepsilon}(h))_{z,R/2}|+\sup_{\mathrm{B}(z,R/2)}|(\boldsymbol{\varepsilon}(h))_{z,R/2}-\mathbf{m}|\\&\lesssim R^{\alpha}[\boldsymbol{\varepsilon}(h)]_{\mathrm{C}^{0,\alpha}(\mathrm{B}(z,R/2);\mathbb{R}^{Nn})}+\left(\fint_{\mathrm{B}(z,R/2)}|\boldsymbol{\varepsilon}(h)-\mathbf{m}|^2\,\mathrm{d}x\right)^{\frac12}\\&\lesssim R^{\alpha}[\boldsymbol{\varepsilon}(v)]_{\mathrm{C}^{0,\alpha}(\mathrm{B}(z,R/2);\mathbb{R}^{Nn})}+\left(\fint_{\mathrm{B}(z,R/2)}|\boldsymbol{\varepsilon}(v)-\mathbf{m}|^2\,\mathrm{d}x\right)^{\frac12}\\&\lesssim R^{\alpha}[\boldsymbol{\varepsilon}(v)]_{\mathrm{C}^{0,\alpha}(\mathrm{B}(z,R/2);\mathbb{R}^{Nn})}+\sup_{\mathrm{B}(z,R/2)}|\boldsymbol{\varepsilon}(v)-\mathbf{m}|\\&=:c_0\mathbf{t}(v;z,R/2),\end{aligned}$$

where $c_0=c_0(\lambda,\Lambda,n,N,M)>0$ is a constant. At this point we make our definition of $c_1>0$ as it appears in the assumptions of the present proposition by putting $c_1:=c_0$. Then, by assumption, we have that $\boldsymbol{\varepsilon}(h)(x)\in\mathbb{B}(\mathbf{m},\sigma)$ provided $x\in\mathrm{B}(z,R/2)$ and so $|f(\boldsymbol{\varepsilon}(h)(x))-g(\boldsymbol{\varepsilon}(h)(x))|\lesssim\omega(|\boldsymbol{\varepsilon}(h)(x)-\mathbf{m}|)|\boldsymbol{\varepsilon}(h)(x)-\mathbf{m}|^2$ for all $x\in\mathrm{B}(z,R/2)$ by (7.32). In consequence, we obtain for $\mathbf{I}_3$

$$\int_{\mathrm{B}(z,R/2)}f(\boldsymbol{\varepsilon}(h))-g(\boldsymbol{\varepsilon}(h))\,\mathrm{d}x\lesssim\omega(c_0\mathbf{t}(v;z,R/2))\int_{\mathrm{B}(z,R/2)}|\boldsymbol{\varepsilon}(h)-\mathbf{m}|^2\,\mathrm{d}x.$$

The remaining integral $\mathbf{I}_1$ is estimated as follows: By assumption, we have the estimate $\mathbf{t}(v;z,R/2)<\min\{\sigma/c_1,1\}$ so that it holds $\boldsymbol{\varepsilon}(v)(x)\in\mathbb{B}(\mathbf{m},\sigma)$ for all $x\in\mathrm{B}(z,R/2)$. Referring to (7.32) we then conclude

$$\int_{\mathrm{B}(z,R/2)}g(\boldsymbol{\varepsilon}(v))-f(\boldsymbol{\varepsilon}(v))\,\mathrm{d}x\lesssim\omega(\mathbf{t}(v;z,R/2))\int_{\mathrm{B}(z,R/2)}|\boldsymbol{\varepsilon}(v)-\mathbf{m}|^2\,\mathrm{d}x.$$

Collecting estimates, the claim follows for ϑ being a suitable multiple of ω. The proof is complete. □

7.6.3. Mollification In the following, let $\eta\in\mathrm{C}_c^{\infty}(\mathbb{R}^n;[0,1])$ denote the standard mollifier

$$\eta(x):=\tilde{c}\mathbb{1}_{\mathrm{B}(0,1)}(x)\exp\left(-\frac{1}{|x|^2-1}\right),\quad x\in\mathbb{R}^n$$

so that $\mathrm{spt}(\eta)\subset\overline{\mathrm{B}(0,1)}$, with a constant $\tilde{c}>0$ such that $\|\eta\|_{L^1}=1$. Given $u\in\mathrm{L}^1_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ and $\delta,\varepsilon>0$ we define the mollifications u_δ and u_ε by

$$u_\delta(x):=\frac{1}{\delta^n}\int_{\mathrm{B}(x,\delta)}\eta\left(\frac{x-y}{\delta}\right)u(y)\,\mathrm{d}y,\quad u_\varepsilon(x):=\fint_{\mathrm{B}(x,\varepsilon)}u(y)\,\mathrm{d}y,$$

so that u_δ is defined on $\{x \in \Omega\colon\ \operatorname{dist}(x, \partial\Omega) > \delta\}$ and u_ε on $\{x \in \Omega\colon\ \operatorname{dist}(x, \partial\Omega) > \varepsilon\}$. Finally, we put $u_{\delta,\varepsilon} := (u_\varepsilon)_\delta$ which, in consequence, is defined on $\{x \in \Omega\colon\ \operatorname{dist}(x, \partial\Omega) > \delta + \varepsilon\}$. In the next step, we prove that the smallness of the excess guarantees that δ and ε can be adjusted in a way such that Proposition 7.16 applies to $u_{\delta,\varepsilon}$. In what follows, we fix a number $a > 1$ close to 1, e.g., $a = 1 + \frac{1}{1000}$ will do.

Lemma 7.17. *Let $u \in X(\mathrm{B}(x_0, r))$ and put $\mathbf{m} := (\boldsymbol{\varepsilon}(u))_{\mathrm{B}(x_0,r)}$. Fix $0 < \alpha < 1$ and suppose that $\widetilde{\mathbf{E}}(u; x_0, r) < 1$. Then there exist $0 < \gamma < \frac{1}{n+2\alpha}$ and $\beta > 0$ such that if*

$$\delta = \varepsilon = \frac{1}{16a} r \widetilde{\mathbf{E}}(u; x_0, r)^\gamma, \tag{7.37}$$

then $\mathbf{t}(u; x_0, r/2) \lesssim \widetilde{\mathbf{E}}(u; x_0, r)^\beta$.

Proof. First observe that, as a consequence of the elementary estimates for convolutions, we obtain

$$\mathbf{t}\Big(u_{\delta,\varepsilon}; x_0, \frac{r}{2}\Big) \leqslant c\left(1 + \left(\frac{r}{\delta}\right)^\alpha\right) \sup_{x \in \mathrm{B}(x_0, \frac{r}{2}+\delta)} |\boldsymbol{\varepsilon}(u_\varepsilon) - \mathbf{m}|. \tag{7.38}$$

In fact, we have

$$|\boldsymbol{\varepsilon}(u_{\delta,\varepsilon})(x) - \mathbf{m}| = |(\boldsymbol{\varepsilon}(u_\varepsilon) - \mathbf{m})_\delta(x)| \tag{7.39}$$

and hence $\sup_{\mathrm{B}(x_0, r/2)} |\boldsymbol{\varepsilon}(u_{\delta,\varepsilon})(x) - \mathbf{m}| \leqslant \sup_{\mathrm{B}(x_0, \delta + r/2)} |\boldsymbol{\varepsilon}(u_\varepsilon)(x) - \mathbf{m}|$. On the other hand, for any radially symmetric mollifier η there exists a constant $c_\eta > 0$ such that for all $g \in \mathrm{L}^1(\mathbb{R}^n; \mathbb{R}^{n\times n}_{\mathrm{sym}})$ and $\delta > 0$ there holds

$$[\eta_\delta * g]_{\mathrm{C}^{0,\alpha}(\overline{\mathrm{B}(x_0,r)}; \mathbb{R}^{n\times n}_{\mathrm{sym}})} \leqslant \frac{c}{\delta^\alpha} \sup_{\mathrm{B}(x_0, r+\delta)} |g - \xi| \qquad \text{for all } \xi \in \mathbb{R}^{n\times n}_{\mathrm{sym}}. \tag{7.40}$$

To see this, we note

$$\begin{aligned} |\eta_\delta * g(x) - \eta_\delta * g(y)| &= \left| \int_{\mathbb{R}^n} (\eta_\delta(x - z) - \eta_\delta(y - z)) g(z)\,\mathrm{d}z \right| \\ &\leqslant \delta^{-n} \int_{\mathbb{R}^n} \left| \eta\left(\frac{x - z}{\delta}\right) - \eta\left(\frac{y - z}{\delta}\right) \right| |g(z)|\,\mathrm{d}z. \end{aligned}$$

Now note that in the last integral we have

$$z \in (\mathrm{B}(x, \delta) \cup \mathrm{B}(y, \delta))^c \Rightarrow \eta\left(\frac{x - z}{\delta}\right) = \eta\left(\frac{y - z}{\delta}\right) = 0,$$

and hence the integral is actually taken over $\mathrm{B}(x, \delta) \cup \mathrm{B}(y, \delta) \subset \mathrm{B}(x_0, r + \delta)$. This establishes

$$\begin{aligned} |\eta_\delta * g(x) - \eta_\delta * g(y)| &\leqslant \delta^{-n} \int_{\mathrm{B}(x,\delta)\cup\mathrm{B}(y,\delta)} \left| \eta\left(\frac{x - z}{\delta}\right) - \eta\left(\frac{y - z}{\delta}\right) \right| |g(z)|\,\mathrm{d}z \\ &\leqslant (\sup_{\mathrm{B}(x_0, r+\delta)} |g|) \delta^{-n} \int_{\mathrm{B}(x,\delta)\cup\mathrm{B}(y,\delta)} \left| \eta\left(\frac{x - z}{\delta}\right) - \eta\left(\frac{y - z}{\delta}\right) \right| \mathrm{d}z \\ &\leqslant C(\sup_{\mathrm{B}(x_0, r+\delta)} |g|) \frac{|x - y|^\alpha}{\delta^\alpha} \delta^{-n} \int_{\mathrm{B}(0,\delta)} [\eta]_{\mathrm{C}^{0,\alpha}}\,\mathrm{d}z \\ &\leqslant C[\eta]_{\mathrm{C}^{0,\alpha}} (\sup_{\mathrm{B}(x_0, r+\delta)} |g|) \frac{|x - y|^\alpha}{\delta^\alpha} \end{aligned}$$

from where (7.40) is immediate as the left side ignores the addition of constants, and it is obvious that the constant in (7.40) only depends on the α–Hölder seminorm of η. Therefore,

$$r^{\alpha}[\boldsymbol{\varepsilon}(u_{\delta,\varepsilon})]_{\mathrm{C}^{0,\alpha}(\mathrm{B}(x_0,r/2))} \leqslant C\left(\frac{r}{\delta}\right)^{\alpha} \sup_{\mathrm{B}(x_0,r/2+\delta)} |\boldsymbol{\varepsilon}(u_\varepsilon)-\mathbf{m}|. \tag{7.41}$$

In consequence, adding (7.39) and (7.41) yields (7.38), and in order to arrive at the claimed estimate, we must give an estimate for $\sup_{\mathrm{B}(x_0,r/2+\delta)}|\boldsymbol{\varepsilon}(u_\varepsilon)-\mathbf{m}|$.

Now recall that ε and δ are adjusted according to (7.37). Then, by Jensen's inequality and since $\varepsilon<r$,

$$\begin{aligned}\mathbf{e}(\boldsymbol{\varepsilon}(u_\varepsilon)(x)-\mathbf{m}) &\leqslant \fint_{\mathrm{B}(x,\varepsilon)} \mathbf{e}(\boldsymbol{\varepsilon}(u)-\mathbf{m})\,\mathrm{d}y \leqslant \left(\frac{r}{\varepsilon}\right)^{n}\widetilde{\mathbf{E}}(u;x_0,r)\\ &\leqslant 16^n\widetilde{\mathbf{E}}(u;x_0,r)^{1-n\gamma}\leqslant 16^n,\end{aligned}$$

the last estimate being valid due to our assumption $\widetilde{\mathbf{E}}(u;x_0,r)<1$ and

$$1-n\gamma>1-\frac{n}{n+2\alpha}>0 \qquad \text{as } \alpha\in(0,1).$$

Now, for any $K>0$ there exists a constant $c=c(K)$ such that if $|\mathbf{A}|\leqslant K$, then $|\mathbf{A}|^2\leqslant c\,\mathbf{e}(\mathbf{A})$. Applying this with $K=16^n$, we obtain for all $x\in \mathrm{B}(x_0,\frac{\delta}{2}+r)$ that

$$|\boldsymbol{\varepsilon}(u_\varepsilon)(x)-\mathbf{m}|^2 \lesssim \mathbf{e}(\boldsymbol{\varepsilon}(u_\varepsilon)(x)-\mathbf{m}) \lesssim \widetilde{\mathbf{E}}(u;x_0,r)^{1-n\gamma}.$$

Now observe that by (7.38) we have by the specific choice of $\delta=\frac{1}{16}r\widetilde{\mathbf{E}}(u;x_0,r)^{\gamma}$ and $\widetilde{\mathbf{E}}(u;x_0,r)<1$ that

$$\mathbf{t}(u_{\delta,\varepsilon};x_0,r/2)\leqslant c\left(1+\left(\widetilde{\mathbf{E}}(u;x_0,r)\right)^{-\gamma\alpha}\right)\widetilde{\mathbf{E}}(u;x_0,r)^{\frac{1-n\gamma}{2}} \lesssim \widetilde{\mathbf{E}}(u;x_0,r)^{\beta}$$

with $\beta:=\frac{1}{2}(1-n\gamma)-\gamma\alpha>0$ (recall that $0<\gamma<1/(n+2\alpha)$). This proves the claim. □

Corollary 7.18. *In the situation of Lemma 7.17, we have*

$$\begin{aligned}&\int_{\mathrm{B}(x_0,r/2)}|\boldsymbol{\varepsilon}(u_{\delta,\varepsilon})-\mathbf{m}|^2\,\mathrm{d}x\lesssim \mathbf{E}(u;x_0,r), \quad \textit{and}\\ &\int_{\mathrm{B}(x_0,r/2)}|\boldsymbol{\varepsilon}(u_{\delta,\varepsilon})-(\boldsymbol{\varepsilon}(u_{\delta,\varepsilon}))_{\mathrm{B}(x_0,r/2)}|^2\,\mathrm{d}x\lesssim \mathbf{E}(u;x_0,r).\end{aligned}$$

Proof. We recall from Lemma 7.17 that $\mathbf{t}(u_{\delta,\varepsilon};x_0,r/2)\leqslant C\widetilde{\mathbf{E}}(u;x_0,r)^{\beta}\leqslant C$. In particular, there holds $\sup_{x\in\mathrm{B}(x_0,r/2)}|\boldsymbol{\varepsilon}(u_{\delta,\varepsilon})-\mathbf{m}|\leqslant C$. Therefore we have $|\boldsymbol{\varepsilon}(u_{\delta,\varepsilon})(x)-\mathbf{m}|^2\leqslant c\,\mathbf{e}(\boldsymbol{\varepsilon}(u_{\delta,\varepsilon})-\mathbf{m})$ with a constant $c>0$ which only depends on C and $\mathbf{e}$ for every $x\in\mathrm{B}(x_0,r/2)$. We conclude by Jensen's inequality that

$$\begin{aligned}\int_{\mathrm{B}(x_0,r/2)}|\boldsymbol{\varepsilon}(u_{\delta,\varepsilon})-\mathbf{m}|^2\,\mathrm{d}x &\lesssim \int_{\mathrm{B}(x_0,r/2)}\mathbf{e}(\boldsymbol{\varepsilon}(u_{\delta,\varepsilon})-\mathbf{m})\,\mathrm{d}x\\ &\lesssim \int_{\mathrm{B}(x_0,r/2)}(\mathbf{e}(\boldsymbol{\varepsilon}(u)-\mathbf{m}))_{\delta,\varepsilon}\,\mathrm{d}x=\mathbf{E}(u;x_0,r),\end{aligned}$$

the last estimate being valid due to $\frac{r}{2}+\delta+\varepsilon<r$ by assumption. The second inequality directly follows from this and hence the complete statement of the corollary. □

7.6.4. Decay Estimate Having proved suitable decay estimates for smooth comparison maps, we turn now to the proof of how these inherit to the actual minimiser. This is accomplished by introducing a new variational integrand, essentially the first order Taylor approximation of the integrand f, and studying a closely related minimisation problem for which the comparison argument can be carried out conveniently. Combining the results of this section with the decay estimates of the mollified minimisers given in the preceding section, we will be in position to deduce the decay estimate for the minimiser itself.

To do so, we first make some definitions. Let $\mathbf{m} := (\boldsymbol{\varepsilon}(u))_{\mathrm{B}(z,R)}$ and $\sigma > 0$. For $\mathbf{m}' \in \mathbb{B}(\mathbf{m},\sigma)$ we define a new convex integrand $\tilde{f}$ by

$$\tilde{f}(\mathbf{m}'') := f(\mathbf{m}' + \mathbf{m}'') - f(\mathbf{m}') - \langle f'(\mathbf{m}'), \mathbf{m}'' \rangle, \quad \mathbf{m}'' \in \mathbb{R}^{n\times n}_{\mathrm{sym}}. \tag{7.42}$$

For further reference we record that $\tilde{f}$ is convex as well and satisfies both $\tilde{f}(0) = 0$ and $\tilde{f}'(0) = 0$. Moreover, since $f \approx \mathbf{e}$, we also have $\tilde{f} \approx \mathbf{e}$. Given $w\colon \mathrm{B}(z,R) \to \mathbb{R}^n$ and $\mathbf{m}' \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$, we define $\tilde{w}\colon \mathrm{B}(z,R) \to \mathbb{R}^n$ by

$$\tilde{w}(x) := w(x) - \mathbf{m}'(x-z). \tag{7.43}$$

We shall now provide three auxiliary statements which occupy a central position in the proof of the decay estimate. However, we stress that the following preparatory lemmas do *not* take into account minimality of u but strictly rest on the convexity assumption imposed on f. In particular, this is the case for Lemma 7.20 whose generalisation to, e.g., quasiconvex functionals lacks a proof.

Lemma 7.19. *Let $u, v \in X(\mathrm{B}(z,R))$ such that $u = v$ on $\partial\mathrm{B}(z,R)$ in the sense of traces and define $\tilde{f}$ and $\tilde{u}, \tilde{v}$ by (7.42) and (7.43), respectively. Then there holds*

$$\int_{\mathrm{B}(z,R)} f(\boldsymbol{\varepsilon}(u)) - f(\boldsymbol{\varepsilon}(v))\,\mathrm{d}x = \int_{\mathrm{B}(z,R)} \tilde{f}(\boldsymbol{\varepsilon}(\tilde{u})) - \tilde{f}(\boldsymbol{\varepsilon}(\tilde{v}))\,\mathrm{d}x$$

Proof. Writing out the definitions, we obtain

$$\begin{aligned}
\int_{\mathrm{B}(z,R)} \tilde{f}(\boldsymbol{\varepsilon}(\tilde{u})) - \tilde{f}(\boldsymbol{\varepsilon}(\tilde{v}))\,\mathrm{d}x &= \int_{\mathrm{B}(z,R)} f(\mathbf{m}' + \boldsymbol{\varepsilon}(\tilde{u})) - f(\mathbf{m}' + \boldsymbol{\varepsilon}(\tilde{v}))\,\mathrm{d}x \\
&\quad - \int_{\mathrm{B}(z,R)} f'(\mathbf{m}')\boldsymbol{\varepsilon}(\tilde{u}) - f'(\mathbf{m}')\boldsymbol{\varepsilon}(\tilde{v})\,\mathrm{d}x \\
&= \int_{\mathrm{B}(z,R)} f(\boldsymbol{\varepsilon}(u)) - f(\boldsymbol{\varepsilon}(v))\,\mathrm{d}x \\
&\quad - \int_{\mathrm{B}(z,R)} f'(\mathbf{m}')\boldsymbol{\varepsilon}(\tilde{u}) - f'(\mathbf{m}')\boldsymbol{\varepsilon}(\tilde{v})\,\mathrm{d}x.
\end{aligned}$$

Now notice that $\boldsymbol{\varepsilon}(\tilde{u}) - \boldsymbol{\varepsilon}(\tilde{v}) = \boldsymbol{\varepsilon}(u) - \boldsymbol{\varepsilon}(v)$ and so, by the usual Gauss–Green Theorem if $1 < p < \infty$ or its BD–version if $p = 1$ (cp. Theorem 4.8 (b)),

$$\begin{aligned}
\int_{\mathrm{B}(z,R)} f'(\mathbf{m}')(\boldsymbol{\varepsilon}(\tilde{u}) - \boldsymbol{\varepsilon}(\tilde{v}))\,\mathrm{d}x &= \int_{\partial\mathrm{B}(z,R)} \langle f'(\mathbf{m}'), \mathrm{Tr}(u-v) \odot \nu \rangle \,\mathrm{d}\mathcal{H}^{n-1} \\
&\quad - \int_{\mathrm{B}(z,R)} \mathrm{div}((f'(\mathbf{m}')))(u-v)\,\mathrm{d}x = 0
\end{aligned}$$

because $\mathrm{div}((f'(\mathbf{m}'))) = 0$ and, by our assumptions on u and v, $\mathrm{Tr}(u) = \mathrm{Tr}(v)$ $\mathcal{H}^{n-1}$–a.e. on $\partial\mathrm{B}(z,R)$. The claim follows. □

Lemma 7.20. *Let $u \in X(\Omega)$ and let $\varepsilon, \delta, R > 0$ and $z \in \Omega$ be such that $\mathrm{B}(z,R) \Subset \Omega$. Then the following holds:*

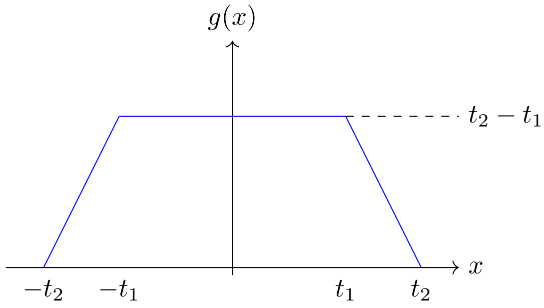

Figure 7.2: The cut–off functions in the first part of the proof of Lemma 7.20.

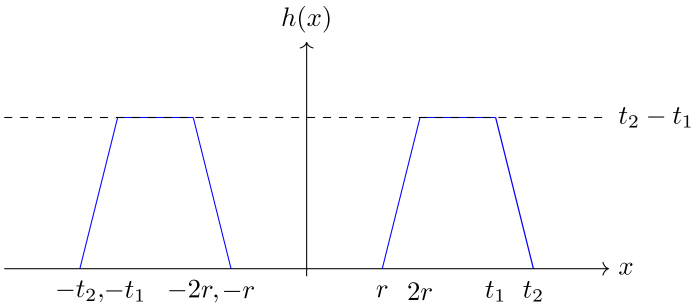

Figure 7.3: The cut–off functions in the second part of the proof of Lemma 7.20.

(i) If $0 < t_1 < t_2 < R - \varepsilon - \delta$*, then there exists* $t \in (t_1, t_2)$ *such that*

$$\int_{\mathrm{B}(z,t)} f(\boldsymbol{\varepsilon}(u_{\varepsilon,\delta})) - f(\boldsymbol{\varepsilon}(u)) \lesssim \frac{\varepsilon + \delta}{t_2 - t_1} \int_{\mathrm{B}(z,R)} f(\boldsymbol{\varepsilon}(u)). \tag{7.44}$$

(ii) If $0 < t_1 < t_2 < R - \varepsilon - \delta$*,* $0 < r < R/4$ *and* $R/2 \leqslant t_1 < R - \varepsilon - \delta$*, then there exist* $r' \in (r, 2r)$ *and* $t' \in (t_1, t_2)$ *with*

$$\int_{\mathrm{B}(z,t') \setminus \mathrm{B}(z,r')} f(\boldsymbol{\varepsilon}(u_{\varepsilon,\delta})) - f(\boldsymbol{\varepsilon}(u)) \lesssim \left(\frac{\varepsilon + \delta}{t_2 - t_1} + \frac{\varepsilon + \delta}{r} \right) \int_{\mathrm{B}(z,R)} f(\boldsymbol{\varepsilon}(u)). \tag{7.45}$$

Proof. Following [22], the proof of (i) or (ii), respectively, is accomplished by considering the auxiliary functions $g, h \colon \mathbb{R}^n \to \mathbb{R}$ given by

$$g(x) := (t_2 - t_1)\mathbb{1}_{\{|x| < t_1\}}(x) + (t_2 - |x|)\mathbb{1}_{\{t_1 \leqslant |x| \leqslant t_2\}}(x), \; x \in \mathbb{R}^n$$

for the first part of the lemma and

$$\begin{aligned} h(x) := \; & \frac{(t_2 - t_1)(|x| - r)}{r} \mathbb{1}_{\{r \leqslant |x| < 2r\}}(x) + (t_2 - t_1)\mathbb{1}_{\{2r \leqslant |x| < t_1\}}(x) \\ & + (t_2 - |x|)\mathbb{1}_{\{t_1 \leqslant |x| < t_2\}}(x), \; x \in \mathbb{R}^n \end{aligned}$$

for the second. In the following we will concentrate on (i) and consequently the use of g; (ii) follows in the same way. The strategy of the proof is to first define

$$\mathbf{F} := \{t \in (t_1, t_2) \colon \text{ inequality (7.44) is not satisfied by } t\}$$

and then to show that $\mathscr{L}^1(\mathbf{F}) \leqslant \frac{1}{2}(t_2 - t_1)$. From this it follows that the one–dimensional Lebesgue measure of the set where (7.44) holds is at least $\frac{1}{2}(t_2 - t_1)$ and so the set must be non–empty; this will imply (i). To prove the estimate for $\mathbf{F}$, we first observe by the definition of g that if $t_1 \leqslant t \leqslant t_2$, then $\mathrm{B}(z,t) = \{x \in \mathrm{B}(z,R) \colon t < g(x)\}$. So we have for any $\psi \in L^1(\mathrm{B}(z,R);\mathbb{R})$ that

$$\begin{aligned}\int_{t_1}^{t_2}\int_{\mathrm{B}(z,t)} \psi(y)\,\mathrm{d}y\,\mathrm{d}t &= \int_{t_1}^{t_2}\int_{\{x\in\mathrm{B}(z,R)\colon\, t<g(x)\}} \psi(y)\,\mathrm{d}y\,\mathrm{d}t \\ &= \int_0^{\infty}\int_{\{x\in\mathrm{B}(z,R)\colon\, t<g(x)\}} \psi(y)\,\mathrm{d}y\,\mathrm{d}t = \int_{\mathrm{B}(z,R)} \psi(y)g(y)\,\mathrm{d}y.\end{aligned}$$

Now recall the notation $\psi_\delta, \psi_\varepsilon$ and $\psi_{\delta,\varepsilon}$ from section 7.6.3. Choosing δ and ε due to the assumptions of the present lemma, we obtain

$$\begin{aligned}\int_{t_1}^{t_2}\int_{\mathrm{B}(z,t)} (\psi-\psi_\delta)\,\mathrm{d}x\,\mathrm{d}t &= \int_{\mathrm{B}(z,R)} (\psi-\psi_\delta)g\,\mathrm{d}x = \int_{\mathrm{B}(z,R)} (g-g_\delta)\psi\,\mathrm{d}x \\ &\leqslant \sup_{\mathrm{B}(z,R)} |g_\delta - g| \int_{\mathrm{B}(z,t_2+\delta)} \psi\ \mathrm{d}x \leqslant \delta \int_{\mathrm{B}(z,t_2+\delta)} \psi\ \mathrm{d}x\end{aligned}$$

and by the same estimation we also get

$$\int_{t_1}^{t_2}\int_{\mathrm{B}(z,t)} (\psi-\psi_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t \leqslant \varepsilon \int_{\mathrm{B}(z,t_2+\varepsilon)} \psi\ \mathrm{d}x.$$

Combined with $f(\boldsymbol{\varepsilon}(u_{\delta,\varepsilon})) \leqslant (f(\boldsymbol{\varepsilon}(u_\varepsilon))_\delta$ which holds due to Jensen's inequality by convexity of f, these estimates imply by $t_2+\delta+\varepsilon<R$ that

$$\begin{aligned}\int_{t_1}^{t_2}\int_{\mathrm{B}(z,t)} f(\boldsymbol{\varepsilon}(u_{\delta,\varepsilon})) - f(\boldsymbol{\varepsilon}(u))\,\mathrm{d}x\,\mathrm{d}t &\leqslant \int_{t_1}^{t_2}\int_{\mathrm{B}(z,t)} (f(\boldsymbol{\varepsilon}(u_\varepsilon)))_\delta - f(\boldsymbol{\varepsilon}(u_\varepsilon))\,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \int_{t_1}^{t_2}\int_{\mathrm{B}(z,t)} f(\boldsymbol{\varepsilon}(u_\varepsilon)) - f(\boldsymbol{\varepsilon}(u))\,\mathrm{d}x\,\mathrm{d}t \\ &\leqslant (\delta+\varepsilon)\int_{\mathrm{B}(z,R)} f(\boldsymbol{\varepsilon}(u))\,\mathrm{d}x.\end{aligned}$$

Now, if $t' \in \mathbf{F}$, then

$$\int_{t_1}^{t_2}\int_{\mathrm{B}(z,t)} f(\boldsymbol{\varepsilon}(u_{\delta,\varepsilon})) - f(\boldsymbol{\varepsilon}(u))\,\mathrm{d}x\,\mathrm{d}t \leqslant \frac{t_2-t_1}{2}\int_{\mathrm{B}(z,t')} f(\boldsymbol{\varepsilon}(u_{\delta,\varepsilon})) - f(\boldsymbol{\varepsilon}(u))\,\mathrm{d}x,$$

so integrating over all $t' \in \mathbf{F}$ an regrouping terms yields

$$\mathscr{L}^1(\mathbf{F}) \leqslant \frac{t_2-t_1}{2}.$$

As we explained, this implies (i). Assertion (ii) is proved in the same way and so the claim follows. $\square$

The main difference to the case $p=1$ is given in the following lemma, where Korn's Inequality plays an essential role. In the following, given $z \in \mathbb{R}^n$ and $0<t<s$, we shall use the shorthand

$$A(z;t,s) := \{x \in \mathbb{R}^n\colon\ t < |x-z| < s\}$$

for the annulus centered at z with inner radius t and outer radius s. Also, we recall that $a>1$ is fixed.

Lemma 7.21. *Let $1 \leqslant p < \infty$ and let $u \in X(\Omega)$. Let $x_0 \in \Omega$ and choose $r, \varepsilon, \delta, s, t, L > 0$ such that $r/2 < t < s < r$, $s + \delta + a\varepsilon < r$ and $\mathrm{B}(x_0, r) \subset \Omega$. Then there exists a constant $c > 0$ such that*

$$\int_{A(x_0;t,s)} \mathbf{e}(L\,(u - u_{\delta,\varepsilon}))\,\mathrm{d}x \leqslant c\,\max\{(L\varepsilon)^p, (L\varepsilon)^2\} \int_{\overline{A(x_0;t-a\varepsilon-\delta,s+a\varepsilon+\delta)}} \mathbf{e}(\boldsymbol{\varepsilon}(u))\,\mathrm{d}x. \tag{7.46}$$

Proof. The estimate is an immediate consequence of the results of Section 7.5. □

The next two propositions form the main part of the partial regularity proof. Thus we stress that from now on we are working with the fact that $u \in X(\Omega)$ is a local (BD–)minimiser of the functional (7.1).

Proposition 7.22. *Let $f\colon \mathbb{R}^{n\times n}_{\mathrm{sym}} \to \mathbb{R}$ be a convex C^2–function of p–growth, $1 \leqslant p < \infty$. Then there exists $a > 0$ such that for any minimiser $u \in X(\Omega)$ the following holds: If we have for a ball $\mathrm{B}(z, R) \Subset \Omega$*

$$(\boldsymbol{\varepsilon}(u))_{z,R} = \mathbf{m} \quad \text{and} \quad \widetilde{\mathbf{E}}(u; z, R) < a,$$

and there exists $\sigma > 0$ such that f is of class C^2 in $\mathbb{B}(\mathbf{m}, \sigma)$ verifying

$$|f''(\mathbf{m}') - f''(\mathbf{m})| \leqslant \omega(|\mathbf{m} - \mathbf{m}'|) \quad \text{for all } \mathbf{m}' \in \mathbb{B}(\mathbf{m}, \sigma) \tag{7.47}$$

with a non–decreasing function $\omega\colon \mathbb{R}^+_0 \to \mathbb{R}^+_0$ which, in addition, satisfies $|\omega(t)| \leqslant M = M(|\mathbf{m}|)$ for all $t \in \mathbb{R}_{\geqslant 0}$ and $\lim_{t\searrow 0} \omega(t) = 0$, such that we have

$$\lambda|\mathbf{Z}|^2 \leqslant \langle f''(\mathbf{m})\mathbf{Z}, \mathbf{Z}\rangle \leqslant \Lambda|\mathbf{Z}|^2 \tag{7.48}$$

for two constants $0 < \lambda \leqslant \Lambda < \infty$ and all $\mathbf{Z} \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$, then there exists a constant $c > 0$ and a bounded and non–decreasing function $h\colon \mathbb{R}^+_0 \to \mathbb{R}^+_0$ with $\lim_{t\searrow 0} h(t) = 0$ such that

$$\mathbf{E}(u; z, r) \leqslant c\mathbf{E}(v; z, 2r) + ch(\widetilde{\mathbf{E}}(u; z, R))(1 + (R/r)^{n+1})\mathbf{E}(u; z, R) \tag{7.49}$$

holds for all $0 < r < R/4$. Here we have set $v := u_{\delta,\varepsilon}$ with δ, ε adjusted according to Lemma 7.17 applied to the radius $R > 0$.

Proof. We split the proof into two steps.

Step 1. Preliminaries. Let $0 < r < R/4$ and put $\mathbf{m}' := (\boldsymbol{\varepsilon}(v))_{\mathrm{B}(z,r)}$. Due to our choice of δ and ε, Lemma 7.17 applied to the radius $R > 0$ yields

$$\sup_{\mathrm{B}(z,\frac{R}{2})} |\boldsymbol{\varepsilon}(v) - \mathbf{m}| + R^{\alpha}[\boldsymbol{\varepsilon}(v)]_{\mathrm{C}^{0,\alpha}(\mathrm{B}(z,\frac{R}{2});\mathbb{R}^{n\times n}_{\mathrm{sym}})} \leqslant \widetilde{c}\widetilde{\mathbf{E}}(u; z, R)^{\beta}$$

for $\beta > 0$ as in Lemma 7.17 (note that both $\widetilde{c} > 0$ and $\beta > 0$ in the previous inequality depend on α and γ as in Lemma 7.17, however, these are thought as fixed throughout). In consequence, we have because of $\mathrm{B}(z, r) \subset \mathrm{B}(z, \frac{R}{2})$

$$|\mathbf{m}' - \mathbf{m}| \leqslant \fint_{\mathrm{B}(z,r)} |\boldsymbol{\varepsilon}(v) - \mathbf{m}|\,\mathrm{d}x \leqslant \sup_{\mathrm{B}(z,\frac{R}{2})} |\boldsymbol{\varepsilon}(v) - \mathbf{m}| \leqslant c\widetilde{\mathbf{E}}(u; z, R)^{\beta}.$$

Thus we may choose $0 < a < \frac{1}{2}$ sufficiently small such that $\widetilde{\mathbf{E}}(u; z, R) < a$ yields $|\mathbf{m} - \mathbf{m}'| < \sigma/2$. Since $\mathbb{B}(\mathbf{m}', \frac{\sigma}{2}) \subset \mathbb{B}(\mathbf{m}, \sigma)$, (7.47) and (7.48) continue to hold in $\mathbb{B}(\mathbf{m}', \frac{\sigma}{2})$. Let us go back to the definition of $\widetilde{f}$ as given in (7.42), where we have the appearance of $\mathbf{m}'$. Here we have the estimate $\widetilde{f} \approx \mathbf{e}$, that is, there exist two constants $c^{(1)}, c^{(2)} > 0$ such that $c^{(1)}\mathbf{e}(\xi) \leqslant \widetilde{f}(\xi) \leqslant c^{(2)}\mathbf{e}(\xi)$, and it is easily seen that these constants do not depend on the specific point $\mathbf{m}'$.

Step 2. Comparison Estimates. We observe that, as a consequence of Jensen's inequality and $\widetilde{f} \approx \mathbf{e}$, for all $0 < r < \frac{R}{2}$ there holds

$$\begin{aligned}\mathbf{E}(u; z, r) &= \int_{\mathrm{B}(z,r)} \mathbf{e}(\boldsymbol{\varepsilon}(u) - (\boldsymbol{\varepsilon}(u))_{\mathrm{B}(z,r)})\,\mathrm{d}x \\ &\lesssim \int_{\mathrm{B}(z,r)} \mathbf{e}(\boldsymbol{\varepsilon}(u) - \mathbf{m}') + \int_{\mathrm{B}(z,r)} \mathbf{e}((\boldsymbol{\varepsilon}(u))_{\mathrm{B}(z,r)} - \mathbf{m}')\,\mathrm{d}x \\ &\lesssim \int_{\mathrm{B}(z,r)} \mathbf{e}(\boldsymbol{\varepsilon}(u) - \mathbf{m}')\,\mathrm{d}x \lesssim \int_{\mathrm{B}(z,r)} \widetilde{f}(\boldsymbol{\varepsilon}(\tilde{u}))\,\mathrm{d}x.\end{aligned} \tag{7.50}$$

Now fix $R/2 < t < s < \frac{7}{8}R$ and define $\rho \in \mathrm{C}_c(\mathrm{B}(z, R); [0, 1])$ by

$$\rho(x) := \frac{2}{s-t}(|x| - t)\mathbb{1}_{\{t \leqslant |x| \leqslant (s+t)/2\}}(x) + \mathbb{1}_{\{|x| > (s+t)/2\}}(x), \quad x \in \mathrm{B}(z, R). \tag{7.51}$$

Then we have $\widetilde{v} + \rho(\tilde{u} - \tilde{v}) \in X(\Omega)$ and, in particular, $\widetilde{v}|_{\partial \mathrm{B}(z,R)} = \widetilde{u}|_{\partial \mathrm{B}(z,R)}$. By Lemma 7.19, $\widetilde{u}$ minimises the functional

$$w \mapsto \int_{\mathrm{B}(z,R)} \widetilde{f}(\boldsymbol{\varepsilon}(w))\,\mathrm{d}x \tag{7.52}$$

with respect to its own boundary values. Using this in the second step, we deduce

$$\begin{aligned}\int_{\mathrm{B}(z,s)} \tilde{f}(\boldsymbol{\varepsilon}(\tilde{u}))\,\mathrm{d}x &\leqslant \left(\int_{\mathrm{B}(z,r)} + \int_{\mathrm{B}(z,t)\setminus\mathrm{B}(z,r)} + \int_{\mathrm{B}(z,s)\setminus\mathrm{B}(z,t)}\right) \tilde{f}(\boldsymbol{\varepsilon}(\tilde{u}))\,\mathrm{d}x \\ &\leqslant \int_{\mathrm{B}(z,r)} \tilde{f}(\boldsymbol{\varepsilon}(\tilde{v}))\,\mathrm{d}x + \int_{\mathrm{B}(z,t)\setminus\mathrm{B}(z,r)} \tilde{f}(\boldsymbol{\varepsilon}(\tilde{v}))\,\mathrm{d}x \\ &\quad + \int_{\mathrm{B}(z,s)\setminus\mathrm{B}(z,t)} \tilde{f}(\boldsymbol{\varepsilon}(\tilde{v} + \rho(\tilde{u} - \tilde{v})))\,\mathrm{d}x.\end{aligned}$$

We recall estimate (7.50) and regroup terms to obtain

$$\begin{aligned}\mathbf{E}(u; z, r) \lesssim &\int_{\mathrm{B}(z,r)} \tilde{f}(\boldsymbol{\varepsilon}(\tilde{v}))\,\mathrm{d}x + \int_{\mathrm{B}(z,t)\setminus\mathrm{B}(z,r)} \tilde{f}(\boldsymbol{\varepsilon}(\tilde{v})) - \tilde{f}(\boldsymbol{\varepsilon}(\tilde{u}))\,\mathrm{d}x \\ &+ \int_{\mathrm{B}(z,s)\setminus\mathrm{B}(z,t)} \tilde{f}(\boldsymbol{\varepsilon}(\tilde{v} + \rho(\tilde{u} - \tilde{v}))) - \tilde{f}(\boldsymbol{\varepsilon}(\tilde{u}))\,\mathrm{d}x =: \mathbf{I} + \mathbf{II} + \mathbf{III}.\end{aligned}$$

We now estimate the single terms. In view of $\mathbf{I}$, we directly work from the definition of $\tilde{f}$ and $\tilde{v}$. Then it follows by $\tilde{f} \approx \mathbf{e}$ and the definition of $\mathbf{m}'$ that

$$\mathbf{I} = \int_{\mathrm{B}(z,r)} \tilde{f}(\boldsymbol{\varepsilon}(v) - \mathbf{m}')\,\mathrm{d}x \lesssim \mathbf{E}(v; z, r). \tag{7.53}$$

The term $\mathbf{II}$ is already in a convenient form and shall be dealt with later. We turn to $\mathbf{III}$; recalling the product rule (i.e., $\boldsymbol{\varepsilon}(\varphi u) = \varphi\boldsymbol{\varepsilon}(u) + \nabla\varphi \odot u$ for $\varphi\colon \mathbb{R}^n \to \mathbb{R}$ and $u\colon \mathbb{R}^n \to \mathbb{R}^n$), monotonicity of $\mathbf{e}$ together with Lemma 7.9(i), we then have

$$\int_{A(z;t,s)} \tilde{f}(\boldsymbol{\varepsilon}(\tilde{v} + \rho(\tilde{u} - \tilde{v})))\,\mathrm{d}x \lesssim \int_{A(z;t,s)} \mathbf{e}(\boldsymbol{\varepsilon}(\tilde{v})) + \mathbf{e}(\boldsymbol{\varepsilon}(\tilde{u})) + \mathbf{e}\left(\frac{\tilde{u} - \tilde{v}}{s - t}\right)\mathrm{d}x.$$

Keeping in mind $s < \frac{7}{8}R$ and therefore $s + \varepsilon + \delta < s + \frac{1}{8}R \leqslant R$, we see that $A(z; t - 2\varepsilon, s + 2\varepsilon) \subset \Omega$ and so it is admissible to estimate

$$\int_{A(z;t,s)} \mathbf{e}(\boldsymbol{\varepsilon}(\tilde{v}))\,\mathrm{d}x \leqslant \int_{A(z;t-2\varepsilon,s+2\varepsilon)} \mathbf{e}(\boldsymbol{\varepsilon}(\tilde{u}))\,\mathrm{d}x \lesssim \int_{A(z;t-2a\varepsilon,s+2a\varepsilon)} \tilde{f}(\boldsymbol{\varepsilon}(\tilde{u}))\,\mathrm{d}x.$$

Here we have used Jensen's inequality and implicitely the fact that $\delta = \varepsilon$. Combining the last inequality with Lemma 7.21, it follows that

$$\begin{aligned}\int_{A(z;t,s)} \tilde{f}(\boldsymbol{\varepsilon}(\tilde{v} + \rho(\tilde{u} - \tilde{v})))\,\mathrm{d}x \lesssim \int_{A(z;t-2a\varepsilon,s+2a\varepsilon)} \tilde{f}(\boldsymbol{\varepsilon}(\tilde{u}))\,\mathrm{d}x \\ + \max\left\{\frac{\varepsilon^2}{(s-t)^2}, \frac{\varepsilon^p}{(s-t)^p}\right\} \int_{A(z;t-2a\varepsilon,s+2a\varepsilon)} \tilde{f}(\boldsymbol{\varepsilon}(\tilde{u}))\,\mathrm{d}x.\end{aligned}$$

Overall, we get the intermediate estimate

$$\begin{aligned}\mathbf{III} &\leqslant \int_{\mathrm{B}(z,s)\setminus \mathrm{B}(z,t)} \tilde{f}(\boldsymbol{\varepsilon}(\tilde{v} + \rho(\tilde{u} - \tilde{v}))) + \tilde{f}(\boldsymbol{\varepsilon}(\tilde{u}))\,\mathrm{d}x \\ &\lesssim \int_{A(z;t-2a\varepsilon,s+2a\varepsilon)} \tilde{f}(\boldsymbol{\varepsilon}(\tilde{u}))\,\mathrm{d}x + \max\left\{\frac{\varepsilon^2}{(s-t)^2}, \frac{\varepsilon^p}{(s-t)^p}\right\} \int_{A(z;t-2a\varepsilon,s+2a\varepsilon)} \tilde{f}(\boldsymbol{\varepsilon}(\tilde{u}))\,\mathrm{d}x\end{aligned}$$

and hence, putting together our preliminary estimates for $\mathbf{I}, \mathbf{II}$ and $\mathbf{III}$, we end up with

$$\begin{aligned}\mathbf{E}(u;z,r) \lesssim \mathbf{E}(v;z,r) &+ \int_{\mathrm{B}(z,t)\setminus \mathrm{B}(z,r)} \tilde{f}(\boldsymbol{\varepsilon}(\tilde{v})) - \tilde{f}(\boldsymbol{\varepsilon}(\tilde{u}))\,\mathrm{d}x \\ &+ \int_{A(z;t-2a\varepsilon,s+2a\varepsilon)} \tilde{f}(\boldsymbol{\varepsilon}(\tilde{u}))\,\mathrm{d}x \\ &+ \max\left\{\frac{\varepsilon^2}{(s-t)^2}, \frac{\varepsilon^p}{(s-t)^p}\right\} \int_{A(z;t-2a\varepsilon,s+2a\varepsilon)} \tilde{f}(\boldsymbol{\varepsilon}(\tilde{u}))\,\mathrm{d}x\end{aligned} \tag{7.54}$$

being valid for any $0 < r < R/2$. The aim of the remaining proof is to utilise this inequality for certain choices of s, t. For $l \in \mathbb{R}_0^+$ denote the integer part of l by $\lfloor l \rfloor$, i.e.,

$$\lfloor l \rfloor := \max\{l' \in \mathbb{N}_0 \colon\ l' \leqslant l\}$$

and define $K := \lfloor 25(2\widetilde{\mathbf{E}}^{\gamma/2}(u;z,R))^{-1} \rfloor$ with $0 < \gamma < 1/(n+2\alpha)$ given by Lemma 7.17. For $k \in \{1, ..., K\}$, put

$$a_k := \frac{5}{8}R + k\frac{R}{400}\widetilde{\mathbf{E}}^{\gamma/2}(u;z,R)$$

so that $a_k \in [\frac{5}{8}R, \frac{7}{8}R]$. Due to Lemma 7.20(ii), the choice $r < R/4$ implies that for each $k = 1, ..., K$ there exists $t_k \in (a_{8k-1}, a_{8k})$ and $r_k \in (r, 2r)$ with

$$\begin{aligned}\int_{A(z;r_k,t_k)} \tilde{f}(\boldsymbol{\varepsilon}(\tilde{v})) - \tilde{f}(\boldsymbol{\varepsilon}(\tilde{u}))\,\mathrm{d}x &\lesssim \varepsilon \max\left\{(a_{8k} - a_{8k-1})^{-1}, r^{-1}\right\} \int_{\mathrm{B}(z,R)} \tilde{f}(\boldsymbol{\varepsilon}(\tilde{u}))\,\mathrm{d}x \\ &\lesssim \max\{\widetilde{\mathbf{E}}^{\gamma/2}(u;z,R), (R/r)\widetilde{\mathbf{E}}^{\gamma}(u;z,R)\}\times \\ &\times \int_{\mathrm{B}(z,R)} \tilde{f}(\boldsymbol{\varepsilon}(\tilde{u}))\,\mathrm{d}x.\end{aligned} \tag{7.55}$$

We can choose $L > 0$ so small such that for $k = 1, ..., K$ the annuli

$$\mathcal{A}_k := \mathrm{B}(z, s_k + 2a\varepsilon) \setminus \mathrm{B}(z, t_k - 2a\varepsilon), \quad s_k := t_k + LR\widetilde{\mathbf{E}}^{\gamma/2}(u;z,R)$$

are pairwise disjoint and subsets of $\mathrm{B}(z,R)$; for instance, the choice $L = \frac{1}{400}$ will do. We can therefore conclude that

$$\int_{\mathcal{A}_1} \tilde{f}(\boldsymbol{\varepsilon}(\tilde{u}))\,\mathrm{d}x + ... + \int_{\mathcal{A}_K} \tilde{f}(\boldsymbol{\varepsilon}(\tilde{u}))\,\mathrm{d}x \leqslant \int_{\mathrm{B}(z,R)} \tilde{f}(\boldsymbol{\varepsilon}(\tilde{u}))\,\mathrm{d}x$$

and so we find $k' \in \{1, ..., K\}$ such that

$$\int_{\mathcal{A}_{k'}} \tilde{f}(\boldsymbol{\varepsilon}(\tilde{u}))\,\mathrm{d}x \leqslant K^{-1} \int_{\mathrm{B}(z,R)} \tilde{f}(\boldsymbol{\varepsilon}(\tilde{u}))\,\mathrm{d}x \lesssim \widetilde{\mathbf{E}}^{\gamma/2}(u;z,R) \int_{\mathrm{B}(z,R)} \tilde{f}(\boldsymbol{\varepsilon}(\tilde{u}))\,\mathrm{d}x. \tag{7.56}$$

Going back to (7.54), the choice $r = r_{k'}$, $t = t_{k'}$ and $s = s_{k'}$ now immediately yields

$$\begin{aligned} \mathbf{E}(u;z,r) \lesssim \mathbf{E}(v;z,2r) & \\ & + \max\{\widetilde{\mathbf{E}}^{\gamma/2}(u;z,R), (R/r)\widetilde{\mathbf{E}}^{\gamma}(u;z,R)\}\times \\ & \times \int_{\mathrm{B}(z,R)} \tilde{f}(\boldsymbol{\varepsilon}(\tilde{u}))\,\mathrm{d}x \qquad (\text{by } (7.55)) \\ & + \widetilde{\mathbf{E}}^{\gamma/2}(u;z,R) \int_{\mathrm{B}(z,R)} \tilde{f}(\boldsymbol{\varepsilon}(\tilde{u}))\,\mathrm{d}x \qquad (\text{by } (7.56)) \\ & \lesssim \mathbf{E}(v;z,2r) + (1+R/r)\widetilde{\mathbf{E}}^{\gamma/2}(u;z,R) \int_{\mathrm{B}(z,R)} \tilde{f}(\boldsymbol{\varepsilon}(\tilde{u}))\,\mathrm{d}x \end{aligned} \tag{7.57}$$

for all $r < R/2$. This is not yet the decay estimate (7.49), so we shall provide a suitable estimate on the last factor on the right side of (7.57). By Lemma 7.9 we conclude

$$\begin{aligned} \int_{\mathrm{B}(z,R)} \tilde{f}(\boldsymbol{\varepsilon}(\tilde{u})) & \lesssim \int_{\mathrm{B}(z,R)} \mathbf{e}(\boldsymbol{\varepsilon}(u) - \mathbf{m}') \\ & \lesssim \Big(\int_{\mathrm{B}(z,R)} \mathbf{e}(\boldsymbol{\varepsilon}(u) - \mathbf{m}) \Big) + R^n \mathbf{e}(\mathbf{m} - \mathbf{m}') \\ & \lesssim \Big(\int_{\mathrm{B}(z,R)} \mathbf{e}(\boldsymbol{\varepsilon}(u) - \mathbf{m}) \Big) + R^n \mathbf{e} \left(\fint_{\mathrm{B}(z,r)} \boldsymbol{\varepsilon}(v) - \mathbf{m}\,\mathrm{d}x \right) \\ & \lesssim \Big(\int_{\mathrm{B}(z,R)} \mathbf{e}(\boldsymbol{\varepsilon}(u) - \mathbf{m}) \Big) + \left(\frac{R}{r} \right)^n \int_{\mathrm{B}(z,r)} \mathbf{e}(|(u-\mathbf{m})_{\delta,\varepsilon}|)\,\mathrm{d}x \\ & \lesssim \mathbf{E}(u;z,R) + (R/r)^n \int_{\mathrm{B}(z,R)} \mathbf{e}(\boldsymbol{\varepsilon}(u) - \mathbf{m}) \\ & \lesssim (1 + (R/r)^n)\mathbf{E}(u;z,R), \end{aligned}$$

the penultimate estimate being valid due to $r + \varepsilon + \delta < R$. Combining this estimate with (7.57), we obtain

$$\begin{aligned} \mathbf{E}(u;z,r) & \lesssim \mathbf{E}(v;z,2r) + h(\widetilde{\mathbf{E}}(u;z,R))\,(1+(R/r)^n)\,(1+R/r)\mathbf{E}(u;z,R) \\ & \lesssim \mathbf{E}(v;z,2r) + h(\widetilde{\mathbf{E}}(u;z,R))\,\big(1 + (R/r)^{n+1}\big)\,\mathbf{E}(u;z,R) \end{aligned} \tag{7.58}$$

we have set $h(t) := t^{\gamma/2}$. Since h obviously is non–decreasing and moreover satisfies $\lim_{t\searrow 0} h(t) = 0$, (7.58) implies the claim. □

Proposition 7.23. *In the situation of Proposition 7.22 we have*

$$\mathrm{Dev}(v;z,R/2) \lesssim h(\widetilde{\mathbf{E}}(u;z,R))\mathbf{E}(u;z,R).$$

Proof. We fix $R/2 < t < s < R$ and a constant $a' > 0$. We then put

$$\begin{aligned} \mathcal{A} &:= \{w \in \mathrm{W}^{1,\infty}(\mathrm{B}(z,t);\mathbb{R}^n)\colon\ w = \tilde{v} \text{ on } \partial\mathrm{B}(z,t)\} \\ \mathcal{B} &:= \{w \in \mathrm{W}^{1,\infty}(\mathrm{B}(z,s)\setminus\mathrm{B}(z,t);\mathbb{R}^n)\colon\ w = \tilde{v} \text{ on } \partial\mathrm{B}(z,t)\cup\partial\mathrm{B}(z,s)\} \end{aligned}$$

and find $w_1 \in \mathcal{A}$ and $w_2 \in \mathcal{B}$ such that

$$\begin{aligned} \int_{\mathrm{B}(z,t)} \tilde{f}(\boldsymbol{\varepsilon}(w_1))\,\mathrm{d}x & \leqslant \inf_{w\in\mathcal{A}} \int_{\mathrm{B}(z,t)} \tilde{f}(\boldsymbol{\varepsilon}(w))\,\mathrm{d}x + a' \\ \int_{\mathrm{B}(z,s)\setminus\mathrm{B}(z,t)} \tilde{f}(\boldsymbol{\varepsilon}(w_2))\,\mathrm{d}x & \leqslant \inf_{w\in\mathcal{B}} \int_{\mathrm{B}(z,s)\setminus\mathrm{B}(z,t)} \tilde{f}(\boldsymbol{\varepsilon}(w))\,\mathrm{d}x + a'. \end{aligned}$$

By Lemma 7.19, we have

$$\operatorname{Dev}\left(\tilde{v}; z, \frac{R}{2}\right) := \int_{\mathrm{B}(z,\frac{R}{2})} \widetilde{f}(\boldsymbol{\varepsilon}(\widetilde{v})) - \inf\left\{\int_{\mathrm{B}(z,R/2)} \widetilde{f}(\boldsymbol{\varepsilon}(w))\,\mathrm{d}x\colon \begin{array}{c} w \in \mathrm{C}^{1,\alpha}(\mathrm{B}(z,\frac{R}{2});\mathbb{R}^n) \\ w = \widetilde{v} \text{ on } \partial\mathrm{B}(z,R/2) \end{array}\right\}$$

Since w_1, w_2 are Lipschitz and coincide on $\partial\mathrm{B}(z,t)$, we deduce that $w_3 := \mathbb{1}_{\overline{\mathrm{B}(z,t)}}w_1 + \mathbb{1}_{\mathrm{B}(z,s)\setminus\mathrm{B}(z,t)}w_2$ belongs to $\mathrm{W}^{1,\infty}(\mathrm{B}(z,s);\mathbb{R}^n)$. We then obtain

$$\begin{aligned}\operatorname{Dev}(\tilde{v}; z, t) &\leqslant \int_{\mathrm{B}(z,t)} \tilde{f}(\boldsymbol{\varepsilon}(\tilde{v}))\,\mathrm{d}x - \int_{\mathrm{B}(z,t)} \tilde{f}(\boldsymbol{\varepsilon}(w_1))\,\mathrm{d}x + a' \\ &= \int_{\mathrm{B}(z,s)} \tilde{f}(\boldsymbol{\varepsilon}(\tilde{v}))\,\mathrm{d}x - \int_{\mathrm{B}(z,s)\setminus\overline{\mathrm{B}(z,t)}} \tilde{f}(\boldsymbol{\varepsilon}(\tilde{v}))\,\mathrm{d}x - \int_{\mathrm{B}(z,t)} \tilde{f}(\boldsymbol{\varepsilon}(w_1))\,\mathrm{d}x + a' \\ &\leqslant \int_{\mathrm{B}(z,s)} \tilde{f}(\boldsymbol{\varepsilon}(\tilde{v}))\,\mathrm{d}x - \int_{\mathrm{B}(z,s)} \tilde{f}(\boldsymbol{\varepsilon}(w_3))\,\mathrm{d}x + 2a'.\end{aligned}$$

Since $\operatorname{Dev}(\tilde{v}, \cdot)$ is not decreasing, we find

$$\operatorname{Dev}(\tilde{v}; z, R/2) \leqslant \int_{\mathrm{B}(z,s)} \tilde{f}(\boldsymbol{\varepsilon}(\tilde{v})) - \tilde{f}(\boldsymbol{\varepsilon}(\tilde{u}))\,\mathrm{d}x + \int_{\mathrm{B}(z,s)} \tilde{f}(\boldsymbol{\varepsilon}(\tilde{u})) - \tilde{f}(\boldsymbol{\varepsilon}(w_3))\,\mathrm{d}x + 2a'.$$

Next, if we let ρ be the function given by (7.51), we similarly obtain by exploiting minimality of $\tilde{u}$ that

$$\begin{aligned}\int_{\mathrm{B}(z,s)} \tilde{f}(\boldsymbol{\varepsilon}(\tilde{u})) - \tilde{f}(\boldsymbol{\varepsilon}(w_3))\,\mathrm{d}x &\leqslant \int_{\mathrm{B}(z,s)} \tilde{f}(\boldsymbol{\varepsilon}(w_3 + \eta(\tilde{u} - \tilde{v}))) - \tilde{f}(\boldsymbol{\varepsilon}(w_3))\,\mathrm{d}x \\ &\leqslant \int_{A(z;t,s)} \tilde{f}(\boldsymbol{\varepsilon}(w_3 + \eta(\tilde{u} - \tilde{v}))) - \tilde{f}(\boldsymbol{\varepsilon}(w_3))\,\mathrm{d}x \\ &\lesssim \int_{A(z;t,s)} \mathbf{e}(\boldsymbol{\varepsilon}(w_3)) + \mathbf{e}(\boldsymbol{\varepsilon}(\tilde{u})) + \mathbf{e}(\boldsymbol{\varepsilon}(\tilde{v})) \\ &\quad + \mathbf{e}\left(\frac{\tilde{u} - \tilde{v}}{s - t}\right)\mathrm{d}x.\end{aligned}$$

Now note that $w_3 = w_2$ on $\mathrm{B}(z,s) \setminus \mathrm{B}(z,t)$ and therefore

$$\int_{A(z;t,s)} \mathbf{e}(\boldsymbol{\varepsilon}(w_3))\,\mathrm{d}x \lesssim \int_{A(z;t,s)} \tilde{f}(\boldsymbol{\varepsilon}(w_3))\,\mathrm{d}x \lesssim \int_{A(z;t,s)} \tilde{f}(\boldsymbol{\varepsilon}(\tilde{v}))\,\mathrm{d}x + a'.$$

Going back to the estimate of **III** in the proof of the preceding Proposition 7.22, we see that the same estimates used there also apply to the present problem, yielding

$$\int_{\mathrm{B}(z,s)} \tilde{f}(\boldsymbol{\varepsilon}(\tilde{u})) - \tilde{f}(\boldsymbol{\varepsilon}(w_3))\,\mathrm{d}x \lesssim \int_{A(z,t-2a\varepsilon,s+2a\varepsilon)} \tilde{f}(\boldsymbol{\varepsilon}(\tilde{u}))\,\mathrm{d}x + a'.$$

Because $a' > 0$ was assumed arbitrary and the constants which implicitely appear in the above estimates do not depend on a, we may pass $a' \to 0$ in these estimates. In combination with our above preliminary estimate on $\operatorname{Dev}(v; z, R/2)$, we then obtain

$$\operatorname{Dev}(v; z, R/2) \lesssim \int_{\mathrm{B}(z,s)} \tilde{f}(\boldsymbol{\varepsilon}(\tilde{v})) - \tilde{f}(\boldsymbol{\varepsilon}(\tilde{u}))\,\mathrm{d}x + \int_{A(z,t-2a\varepsilon,s+2a\varepsilon)} \tilde{f}(\boldsymbol{\varepsilon}(\tilde{u}))\,\mathrm{d}x.$$

In the same way as it is done in [22], we now view this last estimate as the substitute of (7.54) in the proof of Proposition 7.22. Working from this observation and employing the same strategy to find suitable choices of s and t as outlined in the proof of Proposition 7.22, the statement of the present proposition follows. □

In the remaining part of this section we shall combine the preceding two propositions 7.22 and 7.23 with Lemma 7.17 and so obtain the desired decay estimate that will be the main ingredient for proving the partial regularity result.

Proposition 7.24. *Let $f\colon \mathbb{R}^{n\times n}_{\mathrm{sym}} \to \mathbb{R}$ be a convex C^2–function of p–growth, $1 \leqslant p < \infty$. Suppose that $\mathbf{m} \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$ and $\sigma > 0$ are such that*

$$|f''(\mathbf{m}') - f''(\mathbf{m})| \leqslant \omega(|\mathbf{m} - \mathbf{m}'|) \quad \textit{for all } \mathbf{m}' \in \mathbb{B}(\mathbf{m}, \sigma) \tag{7.59}$$

with a non–decreasing function $\omega\colon \mathbb{R}^+_0 \to \mathbb{R}^+_0$ which, in addition, satisfies $|\omega(t)| \leqslant M = M(|\mathbf{m}|)$ for all $t \in \mathbb{R}^+_0$ and $\lim_{t\searrow 0}\omega(t) = 0$, such that we have

$$\lambda|\mathbf{Z}|^2 \leqslant \langle f''(\mathbf{m})\mathbf{Z}, \mathbf{Z}\rangle \leqslant \Lambda|\mathbf{Z}|^2 \tag{7.60}$$

for two constants $0 < \lambda \leqslant \Lambda < \infty$ and all $\mathbf{Z} \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$. There exists $\tau > 0$ depending on the data with the following property: If for $z \in \Omega$ and $R > 0$ with $\mathrm{B}(z, R) \subset \Omega$ a function $u \in X(\Omega)$ minimises

$$\mathfrak{F}_R[u] := \int_{\mathrm{B}(z,R)} f(\boldsymbol{\varepsilon}(u))\,\mathrm{d}x$$

with its own boundary values, satisfies $(\boldsymbol{\varepsilon}(u))_{z,R} = \mathbf{m}$ and the excess satisfies $\widetilde{\mathbf{E}}(u; z, R) < \tau$, then for any $0 < r < R/2$ we have the estimate

$$\mathbf{E}(u; z, r) \lesssim ((r/R)^{n+2} + \overline{\vartheta}(\widetilde{\mathbf{E}}(u; z, R))(1 + (R/r)^{n+1}))\mathbf{E}(u; z, R) \tag{7.61}$$

with a bounded an non–decreasing function $\overline{\vartheta}\colon \mathbb{R}^+_0 \to \mathbb{R}^+_0$ such that $\lim_{t\searrow 0}\overline{\vartheta}(t) = 0$.

Proof. We may assume that $0 < \tau < 1$; the precise choice of τ will be determined throughout the proof. Let $u \in X(\Omega)$ be a local minimiser of $\mathfrak{F}$. We choose the mollification parameters δ and ε as in Lemma 7.17 so that $\mathbf{t}(u_{\delta,\varepsilon}; z, R/2) \leqslant c'\widetilde{\mathbf{E}}^{\beta}(u; z, R)$, where $\beta > 0$ is a constant provided by Lemma 7.17. Hence, going back to Proposition 7.16 and taking τ so small such that $\widetilde{\mathbf{E}}(u; z, R) < \tau$ implies $\mathbf{t}(u_{\delta,\varepsilon}; z, R/2) < \min\{\sigma/c_1, 1\}$, we conclude that $v = u_{\delta,\varepsilon}$ satisfies

$$\begin{aligned}
\mathbf{E}(v; z, r) &\lesssim \Big(\frac{r}{R}\Big)^{n+2}\mathbf{E}(v; z, R/2) + \vartheta(\mathbf{t}(v; z, R/2))\int_{\mathrm{B}(z,R/2)} |\boldsymbol{\varepsilon}(v) - \mathbf{m}|^2\,\mathrm{d}x \\
&\quad + \mathrm{Dev}(v; z, R/2) \\
&\lesssim ((r/R)^{n+2} + \vartheta(\widetilde{\mathbf{E}}(u; z, R)))\mathbf{E}(u; z, R) + \mathrm{Dev}(v; z, R/2),
\end{aligned}$$

where the second estimate is due to Corollary 7.18. Diminishing τ further if necessary, we can assume in addition that $0 < \tau < a$ with the smallness constant $a > 0$ from Proposition 7.22. We then have, since $r < R/4$ implies that $2r < R/2$,

$$\begin{aligned}
\mathbf{E}(u; z, r) &\leqslant c\mathbf{E}(v; z, 2r) + ch(\widetilde{\mathbf{E}}(u; z, R))(1 + (R/r)^{n+1})\mathbf{E}(u; z, R) \\
&\lesssim ((r/R)^{n+2} + \vartheta(\widetilde{\mathbf{E}}(u; z, R)) + ch(\widetilde{\mathbf{E}}(u; z, R))(1 + (R/r)^{n+1}))\mathbf{E}(u; z, R) \\
&\quad + \mathrm{Dev}(v; z, R/2) \\
&\lesssim ((r/R)^{n+2} + \vartheta(\widetilde{\mathbf{E}}(u; z, R)) + ch(\widetilde{\mathbf{E}}(u; z, R))(1 + (R/r)^{n+1}))\mathbf{E}(u; z, R) \\
&\quad + h(\widetilde{\mathbf{E}}(u; z, R))\mathbf{E}(u; z, R),
\end{aligned}$$

the last estimate being valid due to Proposition 7.23. Regrouping terms and defining $\overline{\vartheta}$ in the obvious manner, we immediately infer the claim. $\square$

7.6.5. Iteration and Conclusion. In this section we shall show how an iteration of the estimates provided in Proposition 7.24 yields Theorem 7.4. Regarding conditions (i)–(iii) in the following theorem, we see that they are clearly satisfied provided f is a convex C^2–integrand having positive definite Hessian on the entire space $\mathbb{R}^{n\times n}_{\mathrm{sym}}$. Due to Lebesgue's Differentiation Theorem, condition (iii) will be satisfied at $\mathscr{L}^n$–any point $x_0 \in \Omega$ so that the partial regularity assertion of Theorem 7.4 will immediately follow.

Theorem 7.25. *Let $f\colon \mathbb{R}^{n\times n}_{\mathrm{sym}} \to \mathbb{R}$ be a convex function of p–growth, $1 \leqslant p < \infty$ and let $u \in X(\Omega)$ be a local minimiser of the functional $\mathfrak{F}$ in Ω. Let $x_0 \in \Omega$ be and $\mathbf{m} \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$ be such that*

(i) f is of class $\mathrm{C}^2(U)$ in an open neighbourhood U of $\mathbf{m}$,

(ii) there exist $\lambda, \Lambda > 0$ such that for all $\mathbf{Z} \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$ we have

$$\lambda|\mathbf{Z}|^2 \leqslant \langle f''(\mathbf{m})\mathbf{Z}, \mathbf{Z}\rangle \leqslant \Lambda|\mathbf{Z}|^2,$$

(iii) $\mathbf{m}$ is a Lebesgue point for $\boldsymbol{\varepsilon}(u)$ at x_0, that is,

$$\lim_{r\to 0}\left(\fint_{\mathrm{B}(x_0,r)} |\mathscr{E}u - \mathbf{m}|^p \,\mathrm{d}x + v(p)\frac{|\mathrm{E}^s u|(\mathrm{B}(x_0,r))}{\mathscr{L}^n(\mathrm{B}(x,r))}\right) = 0,$$

where $v(1) = 1$ and $v(p) = 0$ for $1 < p < \infty$.

Then there exists a neighbourhood $\mathcal{O}$ of x_0 in which the minimiser u is of class $\mathrm{C}^{1,\alpha}$ for all $0 < \alpha < 1$.

The statement of the theorem is of course a compact form of Theorems 7.4 and 7.5.

Proof. We follow closely the original proof in [22]. First we justify that (i)–(iii) imply the assumptions of Proposition 7.24: Condition (i) ensures that we find $\sigma > 0$ such that f'' is uniformly continuous on $\mathbb{B}(\mathbf{m}, 3\sigma)$. Therefore we find a modulus of continuity $\omega\colon \mathbb{R}^+_0 \to \mathbb{R}^+_0$, i.e., a non–decreasing, bounded and concave function with $\lim_{t\searrow 0}\omega(t) = 0$, such that $|f''(\mathbf{m}') - f''(\mathbf{m}'')| \leqslant \omega(|\mathbf{m}' - \mathbf{m}''|)$ holds for all $\mathbf{m}', \mathbf{m}'' \in \mathbb{B}(\mathbf{m}, 3\sigma)$. Referring to condition (i) and diminishing σ if necessary, we may further assume without loss of generality that

$$\lambda|\mathbf{Z}|^2 \leqslant \langle f''(\mathbf{m}')\mathbf{Z}, \mathbf{Z}\rangle \leqslant \Lambda|\mathbf{Z}|^2$$

holds for all $\mathbf{m}' \in \mathbb{B}(\mathbf{m}, 3\sigma)$ with two constants $0 < \lambda \leqslant \Lambda \leqslant \infty$. Moreover, we see by the triangle inequality that these estimates also hold in $\mathbb{B}(\mathbf{m}', \sigma)$ for any $\mathbf{m}' \in \mathrm{B}(\mathbf{m}, 2\sigma)$. In conclusion, Proposition 7.24 is available and provides us with $\tau > 0$, a non–decreasing and bounded function $\overline{\vartheta}\colon \mathbb{R}^+_0 \to \mathbb{R}^+_0$ and a constant $c > 0$ such that for all $0 < r < R/2$ we have with $z = x_0$

$$\mathbf{E}(u; z, r) \leqslant c((r/R)^{n+2} + \overline{\vartheta}(\widetilde{\mathbf{E}}(u; z, R))(1 + (R/r)^{n+1}))\mathbf{E}(u; z, R) \tag{7.62}$$

provided

$$(\boldsymbol{\varepsilon}(u))_{z,R} = \mathbf{m}' \text{ and } \widetilde{\mathbf{E}}(u; z, R) < \tau. \tag{Small}$$

We shall use this to prove the following intermediate claim from which the assertion of the theorem is immediate:

Claim. Let $0 < \alpha < 1$ be arbitrary. Then there exists $\tau' > 0$ and $\tau_0 > 0$ such that the following holds: If $\mathrm{B}(x, R) \subset \Omega$ and $\mathbf{m}' \in \mathbb{B}(\mathbf{m}, \sigma)$ are such that $(\boldsymbol{\varepsilon}(u))_{x,R} = \mathbf{m}'$ and $\widetilde{\mathbf{E}}(u; x, R) < \min\{\tau, \tau'\}$, then there holds $\widetilde{\mathbf{E}}(u; x, \tau_0^j R) \leqslant \tau_0^{2\alpha j}\widetilde{\mathbf{E}}(u; x, R)$ for all $j \in \mathbb{N}_0$.

We will verify the claim by induction on j. To do so, we put $\mathbf{m}_j := (\boldsymbol{\varepsilon}(u))_{x,\tau_0^j R}$ and show that for all $j \in \mathbb{N}_0$ we have that

$$|\mathbf{m}_j - \mathbf{m}'| < \sigma \sum_{k=0}^{j} 2^{-k} \quad \text{and} \quad \widetilde{\mathbf{E}}(u; x, \tau_0^j) \leqslant \tau_0^{2\alpha j} \widetilde{\mathbf{E}}(u; x, R). \tag{7.63}$$

Before embarking on the induction, we give a suitable candidate for τ_0 and τ'. From Lemma 7.9 we know that there are two constants $C_1, C_2 > 0$ such that $\mathbf{e}(\mathbf{A}+\mathbf{B}) \leqslant C_1(\mathbf{e}(\mathbf{A}) + \mathbf{e}(\mathbf{B}))$ for all $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$ and if $|\mathbf{A}| \leqslant 3$, then $\mathbf{e}(\mathbf{A}) \leqslant C_2|\mathbf{A}|^2$. Recalling the constant $c > 0$ from (7.62), we put

$$\tau_0 := \min\{2^{-\frac{1}{\alpha}}, (2C_1)^{-\frac{1}{2}}, \frac{1}{2}\sigma(2C_1C_2)^{-\frac{1}{2}}, (2c)^{\frac{1}{2\alpha-2}}\}. \tag{7.64}$$

Having defined τ_0, we pick $\tau' > 0$ such that $\tau' < \tau_0^{n+2}$ and $0 \leqslant t < \tau'$ implies

$$\overline{\vartheta}(t) \leqslant \frac{1}{2}\tau_0^{2n+3}, \tag{7.65}$$

where $\overline{\vartheta}$ is the function appearing in estimate (7.62). The latter property can be achieved because we have that $\lim_{t\searrow 0}\overline{\vartheta}(t) = 0$.

We turn back to the induction. The intial step is certainly fulfilled. Now suppose that (7.63) holds for $j \in \mathbb{N}$. We record that in this case $|\mathbf{m}_j - \mathbf{m}'| < 2\sigma$ holds as well and, since $\tau_0 < 1$, $\widetilde{\mathbf{E}}(u; x, \tau_0^j R) \leqslant \tau$. We conclude that (7.62) holds with R being replaced by $\tau_0^j R$ and all $0 < r < \tau_0^j R/2$. Now notice that with our choice of τ_0 and τ', we also have that

$$\widetilde{\mathbf{E}}(u; x, \tau_0^{j+1} R) \leqslant \tau',\ \tau_0^{-2-2n}\overline{\vartheta}(\widetilde{\mathbf{E}}(u; x, \tau_0^j R)) \leqslant \frac{1}{2},\ \text{and}\ \tau_0^{2-2\alpha} \leqslant 1/(2c). \tag{7.66}$$

Therefore, using (7.62), we deduce that

$$\begin{aligned}
\widetilde{\mathbf{E}}(u; v, \tau_0^{j+1}R) &= \tau_0^{-nj-n} R^{-n} \mathbf{E}(u; x, \tau_0^{j+1}R) \\
&\leqslant c\tau_0^{-n}(\tau_0^{n+2} + \overline{\vartheta}(\widetilde{\mathbf{E}}(u; x, \tau_0^j R))(1 + \tau_0^{-n-1}))\widetilde{\mathbf{E}}(u; x, \tau_0^j R) \\
&\leqslant c(\tau_0^2 + \overline{\vartheta}(\widetilde{\mathbf{E}}(u; x, \tau_0^j R))(\tau_0^{-n} + \tau_0^{-2n-1}))\widetilde{\mathbf{E}}(u; x, \tau_0^j R) \\
&\leqslant c\tau_0^{2-2\alpha}\tau_0^{2\alpha}(1 + \overline{\vartheta}(\widetilde{\mathbf{E}}(u; x, \tau_0^j R))(\tau_0^{-n-2} + \tau_0^{-2n-3}))\widetilde{\mathbf{E}}(u; x, \tau_0^j R) \\
&\leqslant c\tau_0^{2-2\alpha}\tau_0^{2\alpha}(1 + \overline{\vartheta}(\widetilde{\mathbf{E}}(u; x, \tau_0^j R))(2\tau_0^{-2n-3}))\widetilde{\mathbf{E}}(u; x, \tau_0^j R)\ (\text{as}\ \tau_0 < 1) \\
&\leqslant 2c\tau_0^{2-2\alpha}\tau_0^{2\alpha}\widetilde{\mathbf{E}}(u; x, \tau_0^j R) \qquad\qquad (\text{by (7.65)}) \\
&\leqslant \tau_0^{2\alpha}\widetilde{\mathbf{E}}(u; x, \tau_0^j R) \leqslant \tau_0^{2\alpha(j+1)}\widetilde{\mathbf{E}}(u; x, R)\ (\text{by (7.66)}),
\end{aligned}$$

where the last inequality holds by induction hypothesis. We now prove the remaining part of the claim. For this we abbreviate $\mathrm{B}_j := \mathrm{B}(x, \tau_0^j R)$, $j \in \mathbb{N}$. We estimate, using Jensen's inequality and Lemma 7.9(i),

$$\begin{aligned}
\mathbf{e}(\mathbf{m}_{j+1} - \mathbf{m}_j) &\leqslant \fint_{\mathrm{B}_{j+1}} \mathbf{e}(\mathbf{m}_{j+1} - \boldsymbol{\varepsilon}(u)(y) + \boldsymbol{\varepsilon}(u)(y) - \mathbf{m}_j)\,\mathrm{d}y \\
&\leqslant C_1\left(\fint_{\mathrm{B}_{j+1}} \mathbf{e}(\boldsymbol{\varepsilon}(u)(y) - \mathbf{m}_{j+1})\,\mathrm{d}y + \tau_0^{-n}\fint_{\mathrm{B}_j} \mathbf{e}(\mathbf{m}_j - \boldsymbol{\varepsilon}(u)(y))\,\mathrm{d}y\right) \\
&\leqslant C_1(\widetilde{\mathbf{E}}(u; x, \tau_0^{j+1}R) + \tau_0^{-n}\widetilde{\mathbf{E}}(u; x, \tau_0^j R)) \\
&\leqslant C_1(\tau_0^{2\alpha(j+1)} + \tau_0^{-n}\tau_0^{2\alpha j})\widetilde{\mathbf{E}}(u; x, R) \\
&\leqslant C_1\tau_0^{2\alpha j}(\tau_0^{2\alpha} + \tau_0^{-n})\widetilde{\mathbf{E}}(u; x, R) \leqslant 2C_1\tau_0^{2\alpha j}\tau_0^{-n}\widetilde{\mathbf{E}}(u; x, R).
\end{aligned}$$

By our choice of τ_0 and τ' we have $\widetilde{\mathbf{E}}(u;x,R) \leqslant \tau' < \tau_0^{n+2}$ and $\tau_0^2 \leqslant (2C_1)^{-1}$. Therefore,

$$2C_1\tau_0^{2\alpha j}\tau_0^{-n}\widetilde{\mathbf{E}}(u;x,R) \leqslant 2C_1\tau_0^{2\alpha j+2} \leqslant \tau_0^{2\alpha j}$$

and so we obtain $\mathbf{e}(\mathbf{m}_{j+1}-\mathbf{m}_j) \leqslant 1$. But since $|\cdot|^2 = (\mathbf{e}(\cdot)+1)^{2/p}-1$ we also get $|\mathbf{m}_{j+1}-\mathbf{m}_j|^2 \leqslant 2^{2/p}-1 \leqslant 9$ so that $|\mathbf{m}_{j+1}-\mathbf{m}_j| \leqslant 3$. Again recall that if $|\mathbf{A}| \leqslant 3$, then $\mathbf{e}(\mathbf{A}) \leqslant C_2|\mathbf{A}|^2$ and thus

$$|\mathbf{m}_{j+1}-\mathbf{m}_j|^2 \leqslant C_2\mathbf{e}(\mathbf{m}_{j+1}-\mathbf{m}_j) \leqslant 2C_1C_2\tau_0^{2\alpha j}\tau_0^2.$$

We observe that by our definition of τ_0 there holds $2C_1C_2\tau_0^2 \leqslant \sigma^2/4$ and $\tau_0 < 2^{-1/\alpha}$. Taking this into account, we then gain

$$|\mathbf{m}_{j+1}-\mathbf{m}_j| \leqslant \frac{1}{2}\tau_0^{\alpha j}\sigma \leqslant 2^{-j-1}\sigma.$$

In conclusion, using the induction hypothesis we infer that

$$\begin{aligned}|\mathbf{m}_{j+1}-\mathbf{m}'| &\leqslant |\mathbf{m}_j-\mathbf{m}'| + |\mathbf{m}_j-\mathbf{m}'| \\ &\overset{\text{(IH)}}{\leqslant} \left(\sigma\sum_{k=0}^{j}2^{-k}\right) + \sigma 2^{-(j+1)},\end{aligned}$$

and thereby have proved the claim completely. An procedure now yields that there exists $C>0$ which only depends on α and τ_0 such that for all $0<r<R$ there holds

$$\mathbf{E}(u;x,r) \leqslant C\left(\frac{r}{R}\right)^{n+2\alpha}\mathbf{E}(u;x,R) \tag{7.67}$$

provided (Small) is in action. In fact, let $0<r<R$ be arbitrary. Then we find $j\in\mathbb{N}$ such that $\tau_0^{j+1}R < r \leqslant \tau_0^jR$ and thus, by the intermediate claim,

$$\begin{aligned}\widetilde{\mathbf{E}}(u;x,r) &\leqslant (\tau_0^jR/r)^n\widetilde{\mathbf{E}}(u;x,\tau_0^jR) \lesssim (\tau_0^jR/r)^n\tau_0^{2\alpha j}\widetilde{\mathbf{E}}(u;x,R) \\ &\leqslant \tau_0^{-n-2\alpha}\tau_0^{2\alpha(j+1)}\widetilde{\mathbf{E}}(u;x,R) \leqslant C\,(r/R)^{2\alpha}\widetilde{\mathbf{E}}(u;x,R),\end{aligned}$$

where $C=C(\alpha,\tau_0)>0$ is a constant.

The task of the proof hence is to justify validity of (7.67) and hereafter (Small) in a neighbourhood of x_0 with the same choice of $R>0$ which we without loss of generality assume to be contained in $\mathrm{B}(x_0,R)$. For this it is necessary to distinguish the case $p=1$ and $1<p<\infty$. We begin with the latter.

Case $1<p<\infty$. To demonstrate (Small) in a neighbourhood of x_0, we note by (iii) there holds

$$\lim_{R\searrow 0}\fint_{\mathrm{B}(x_0,R)}\boldsymbol{\varepsilon}(u)\,\mathrm{d}y = \mathbf{m} \quad\text{and}\quad \lim_{R\searrow 0}\widetilde{\mathbf{E}}(u;x_0,R)=0.$$

and hence (Small) follows for some $R>0$, and by continuity of the mappings $x\mapsto \fint_{\mathrm{B}(x,R)}\boldsymbol{\varepsilon}(u)\,\mathrm{d}y$ and $x\mapsto\widetilde{\mathbf{E}}(u;x,R)$, we conclude that (Small) hold in a neighbourhood of x_0 with the same $R>0$.

Case $p=1$. The continuity argument as invoked above does not apply here as $\mathrm{E}u$ might be a possibly singular measure with respect to $\mathscr{L}^n$. However, we attempt to reduce to this case. Note that (iii) equally reads as

$$\begin{aligned}&\lim_{R\searrow 0}\fint_{\mathrm{B}(x_0,R)}|\mathscr{E}u-\mathbf{m}'|\,\mathrm{d}\mathscr{L}^n = 0,\\ &\lim_{R\searrow 0}\fint_{\mathrm{B}(x_0,R)}\mathscr{E}u\,\mathrm{d}\mathscr{L}^n = \mathbf{m}',\\ &\lim_{R\searrow 0}\frac{1}{R^n}\int_{\mathrm{B}(x_0,R)}\mathrm{d}|\,\mathrm{E}^su| = 0.\end{aligned}$$

Next observe that for all $\xi \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$ there holds

$$\frac{\mathbf{e}(t\xi)}{t} = \sqrt{t^{-2}+|\xi|^2} - t^{-1} \to |\xi|, \qquad t \nearrow \infty$$

and so $\mathbf{e}^{\infty}(z) = |z|$. We then obtain

$$\lim_{R\searrow 0}\left(\fint_{\mathrm{B}(x_0,R)} \mathbf{e}(\mathscr{E}u - (\mathrm{E}u)_{\mathrm{B}(x_0,R)})\,\mathrm{d}y + \frac{1}{R^n}\int_{\mathrm{B}(x_0,R)} \mathrm{d}|\,\mathrm{E}^s u|\right) = 0. \tag{7.68}$$

In consequence, we find $R > 0$ such that each of the following hold:

$$\left|\fint_{\mathrm{B}(x_0,R)} \mathscr{E}u\,\mathrm{d}\mathscr{L}^n - \mathbf{m}\right| < \frac{\sigma}{2}, \tag{7.69}$$

$$\frac{1}{(2R)^n}\int_{\mathrm{B}(x_0,2R)} \mathrm{d}|\,\mathrm{E}^s u| < \frac{1}{2^{n+1}}\min\{\sigma,\tau,\tau'\}, \tag{7.70}$$

$$\fint_{\mathrm{B}(x_0,R)} \mathbf{e}(\mathscr{E}u - (\mathrm{E}u)_{\mathrm{B}(x_0,R)})\,\mathrm{d}\mathscr{L}^n < \frac{1}{2}\min\{\tau,\tau'\}. \tag{7.71}$$

Similarly as above, we may conclude for the absolutely continuous part that (7.69) and (7.71) continue to hold for all $x \in \mathcal{O}$ (instead of x_0) for a neighbourhood $\mathcal{O} \subset \mathrm{B}(x_0,R)$ of x_0. Let $x \in \mathcal{O}$ be arbitrary. We then find by (7.69) and (7.70) that

$$\begin{aligned}\left|\frac{1}{R^n}\int_{\mathrm{B}(x,R)} \mathrm{dE}u - \mathbf{m}\right| &\leqslant \left|\fint_{\mathrm{B}(x,R)} \mathscr{E}u - \mathbf{m}\,\mathrm{d}\mathscr{L}^n\right| + \frac{1}{R^n}\int_{\mathrm{B}(x,R)} \mathrm{d}|\,\mathrm{E}^s u| \\ &\leqslant \frac{\sigma}{2} + \frac{2^n}{(2R)^n}\int_{\mathrm{B}(x_0,2R)} \mathrm{d}|\,\mathrm{E}^s u| \leqslant \sigma\end{aligned}$$

and similarly $R^{-n}\mathbf{E}(u;x,R) \leqslant \min\{\tau,\tau'\}$, now using (7.70) and (7.71). The proof of (7.67) is complete.

We finally note that by Lemma 7.10 u must be of class $\mathrm{C}^{1,\alpha}$ in a neighbourhood of x_0, and this concludes the proof. □

Remark 7.26. *Interestingly, even the radially symmetric area–type integrands corresponding to* $\mathrm{m}_p(t) := (1+|t|^p)^{\frac{1}{p}}$, $t \in \mathbb{R}$, *do not allow an application of Theorems 7.3 and 7.4 if* $p \neq 2$. *This has been pointed out by* Schmidt *[210] in the* BV*–context and thus the above proof in the symmetric gradient context requires modification; indeed, if* $p > 2$, *then* $\mathrm{m}_p''(0) = 0$, *and if* $1 < p < \infty$, *the second derivative of* m_p *at the origin does not exist. In this situation, the integrands exhibit a similar behaviour as the* p*–Dirichlet integrands* $z \mapsto \frac{1}{p}|z|^p$ $(z \in \mathbb{R}^{N\times n})$, *and so it is possible to accomplish a* p*–harmonic approximation–type strategy.*

7.7. Partial Regularity of the Dual Solution

In this paragraph we briefly discuss the partial regularity of the solution of the dual problem (in the sense of Chapter 5.4) associated with the primal Dirichlet problem

$$\text{to minimise } \mathfrak{F}[v] := \int_{\Omega} f(\varepsilon(v))\,\mathrm{d}x \qquad \text{over } \mathcal{D}_{u_0} := u_0 + \mathrm{LD}_0(\Omega), \tag{7.72}$$

where $u_0 \in \mathrm{LD}(\Omega)$ is a given boundary datum. Here, f is assumed to satisfy the following set of assumptions:

(H1) $f \in \mathrm{C}^2(\mathbb{R}^{n\times n}_{\mathrm{sym}})$ is of linear growth so that there exists $c_1, c_3 > 0$ and $c_2 \in \mathbb{R}$ such that $c_1|\xi| + c_2 \leqslant f(\xi) \leqslant c_3(1+|\xi|)$ for all $\xi \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$.

(H2) f is strictly convex and there exists $\Lambda > 0$ such that for all $\xi, z \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$ there holds

$$0 < \langle f''(\xi)z, z\rangle \leqslant \Lambda \frac{|z|^2}{1+|\xi|}. \tag{7.73}$$

If we already knew that all solutions u of the primal problem belong to $\mathrm{LD}(\Omega)$, then we could represent the dual solution as $\sigma = f'(\boldsymbol{\varepsilon}(u))$, and assuming that $u \in \mathrm{LD}(\Omega)$ is partially regular in the sense of Section 7.2.1, we conclude by $\sup_{\xi\in\mathbb{R}^{n\times n}_{\mathrm{sym}}} |f''(\xi)| < \infty$ that f' is Lipschitz and therefore

$$\begin{aligned} |\sigma(x) - \sigma(y)| = |f'(\boldsymbol{\varepsilon}(u)(x)) - f'(\boldsymbol{\varepsilon}(u)(y))| &= C(\mathrm{Lip}(f'))|\boldsymbol{\varepsilon}(u)(x) - \boldsymbol{\varepsilon}(u)(y)| \\ &\leqslant C(\mathrm{Lip}(f'), x_0, r)|x-y|^{\alpha} \end{aligned} \tag{7.74}$$

provided x, y are elements of a ball $\mathrm{B}(x_0, r)$ in which u is of class $\mathrm{C}^{1,\alpha}$. If, however, $u \in \mathrm{BD}(\Omega)$, then we cannot use the above representation of the dual solution and so we aim at showing the following: If $x_0 \in \mathrm{Reg}(u;\alpha)$, then $\sigma = f'(\boldsymbol{\varepsilon}(u))$ in a neighbourhood of x_0. In this situation, we may apply (7.74) and conclude as above. More precisely, we have the following result which relies on the regularisation procedure for the dual solution as employed in Chapter 5.4.2 and hereafter mimics the approach of Bildhauer & Fuchs (cp. [41] and [40, Chpt. 2.3.2]). For a different approach by means of a relaxed version of a minimax inequality, we refer to the work of Seregin [212, 213, 214, 215].

Proposition 7.27. *Let Ω be an open and bounded Lipschitz subset of $\mathbb{R}^n$ and let the integrand $f\colon \mathbb{R}^{n\times n}_{\mathrm{sym}} \to \mathbb{R}$ satisfy (H1) and (H2) from above. Given $u_0 \in \mathrm{LD}(\Omega)$, let $\overline{u} \in \mathrm{BD}(\Omega)$ be the* BD*–minimiser of the variational principle* (7.72) *which is obtained by the vanishing viscosity approach outlined in* (5.17) *ff.. Then the solution $\sigma \in \mathrm{L}^{\infty}_{\mathrm{div}}(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})$ of the dual variational problem (in the sense of Chapter 5.4.1, cp.* (5.12)*) is locally of class $\mathrm{C}^{0,\alpha}$ on*

$$\mathrm{Reg}(\overline{u}) := \Omega\setminus\Sigma_{\overline{u}} := \left\{ x_0 \in \Omega\colon\; \lim_{r\searrow 0} \frac{|\,\mathrm{E}\overline{u} - z|(\mathrm{B}(x_0,r))}{r^n} = 0 \textit{ for some } z \in \mathbb{R}^{n\times n}_{\mathrm{sym}} \right\}.$$

for every $0 < \alpha < 1$.

In the proof of the preceding proposition, we shall need the following auxiliary fact which is recovered from [40, Rem. 2.26] in a straightforward manner.

Lemma 7.28 ([40, Rem. 2.26])**.** *Let $f\colon \mathbb{R}^{n\times n}_{\mathrm{sym}} \to \mathbb{R}$ satisfy (H1) and (H2). Then for any $z \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$ we have*

$$\lim_{\substack{|\xi|\to\infty \\ \xi\in\mathbb{R}^{n\times n}_{\mathrm{sym}}}} \langle f'(\xi) - f'(z), \xi - z\rangle = \infty.$$

We now come to the

Proof of Proposition 7.27. Following Remark 5.8, it is not restrictive to assume $u_0 \in \mathrm{W}^{1,2}(\Omega;\mathbb{R}^n)$. This enables us to utilise the machinery of Chapter 5.4. Stabilising the functional as given in (7.72) as in (5.17), we obtain a sequence of viscosity approximations $(v_j) \subset u_0 + \mathrm{W}^{1,2}_0(\Omega;\mathbb{R}^n)$ which converge to some BD–minimiser $\overline{u} \in \mathrm{BD}(\Omega)$ in the weak*–sense. Our aim is to show (for a not relabelled subsequence)

$$\boldsymbol{\varepsilon}(v_j)(x) \to \boldsymbol{\varepsilon}(\overline{u})(x) \qquad \text{for } \mathscr{L}^n\text{-a.e. } x \in \mathrm{Reg}(\overline{u}) \tag{7.75}$$

as $j \to \infty$. This will imply the claim of the proposition: Since f' is continuous, we have

$$f'(\boldsymbol{\varepsilon}(v_j)(x)) \to f'(\boldsymbol{\varepsilon}(\overline{u})(x)) \qquad \text{for } \mathscr{L}^n\text{-a.e. } x \in \mathrm{Reg}(\overline{u}). \tag{7.76}$$

Here, by slight abuse of notation, we will equivalently write $\mathscr{E}\overline{u}$ for $\boldsymbol{\varepsilon}(u)$ and vice versa (note that the singular part $\mathrm{E}^s\overline{u}$ vanishes on the regular set and hence $\mathscr{E}u$ in fact arises as the symmetric gradient of an LD–map on this set). Fix any non–empty open subset $\mathcal{O} \subset \mathrm{Reg}(\overline{u})$. Since $|f'(\boldsymbol{\varepsilon}(v_j))|$ is uniformly bounded by $\mathrm{Lip}(f)$, by Lebesgue's theorem on dominated convergence we find that the convergence $f'(\boldsymbol{\varepsilon}(v_j))|_{\mathcal{O}} \to f'(\boldsymbol{\varepsilon}(\overline{u}))|_{\mathcal{O}}$ as $j \to \infty$ is actually strongly in $\mathrm{L}^2(\mathcal{O};\mathbb{R}^{n\times n}_{\mathrm{sym}})$. On the other hand, we have $f'(\boldsymbol{\varepsilon}(v_j)) \rightharpoonup \sigma$ in $\mathrm{L}^2(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})$ as $j \to \infty$ (for a suitable non–relabelled subsequence). In conjunction with (7.76) we consequently have $\sigma(x) = f'(\boldsymbol{\varepsilon}(\overline{u}))$ $\mathscr{L}^n$–a.e. in $\mathrm{Reg}(\overline{u})$, and at this stage we may invoke (7.74) to conclude.

We now turn to the proof of (7.75) and hence recall the notation $\sigma_j := f'(\boldsymbol{\varepsilon}(v_j)) + \frac{1}{j}\boldsymbol{\varepsilon}(v_j)$, $j \in \mathbb{N}$.

Step 1. Let $U \subset \mathrm{Reg}(\overline{u})$ be an arbitrary open subset with $\overline{U} \subset \mathrm{Reg}(\overline{u})$. Then $\boldsymbol{\varepsilon}(\overline{u}) \in \mathrm{C}^{0,\alpha}(\mathrm{Reg}(\overline{u});\mathbb{R}^{n\times n}_{\mathrm{sym}})$ for any $0<\alpha<1$. Let $\varphi \in \mathrm{C}^1_c(U;\mathbb{R}^n)$ be arbitrary. Since $\mathrm{E}^s\overline{u} \equiv 0$ on U, we have by Corollary 10.14 from the appendix

$$\int_U \langle f'(\mathscr{E}\overline{u}), \boldsymbol{\varepsilon}(\varphi)\rangle \,\mathrm{d}x = 0. \tag{7.77}$$

By an approximation argument, we conclude that (7.77) remains valid for all $\varphi \in \mathrm{LD}_0(U)$.

Step 2. Let $\rho \in \mathrm{C}^\infty_c(U;[0,1])$ be arbitrary and consider the sequence (w_j) given by $w_j := \rho^2(v_j - \overline{u})$. We obviously have $(w_j) \subset \mathrm{W}^{1,2}_0(U;\mathbb{R}^n) \subset \mathrm{LD}_0(U)$ and so, by step 1 and $\int_\Omega \langle \sigma_j, \boldsymbol{\varepsilon}(w_j)\rangle \,\mathrm{d}x = 0$ for all $j \in \mathbb{N}$, we deduce

$$\int_\Omega \langle \sigma_j - f'(\mathscr{E}\overline{u}), \boldsymbol{\varepsilon}(\rho^2(v_j - \overline{u}))\rangle \,\mathrm{d}x = 0 \qquad \text{for all } j \in \mathbb{N}. \tag{7.78}$$

We rewrite the preceding equation as

$$\begin{aligned}
\int_U \langle f'(\boldsymbol{\varepsilon}(v_j)) - f'(\mathscr{E}\overline{u}), \boldsymbol{\varepsilon}(v_j) - \mathscr{E}\overline{u}\rangle \rho^2 \,\mathrm{d}x &+ \frac{1}{j}\int_U \langle \boldsymbol{\varepsilon}(v_j), \rho^2(\boldsymbol{\varepsilon}(v_j) - \mathscr{E}\overline{u})\rangle \,\mathrm{d}x \\
&= -\int_U \langle \sigma_j, 2\rho\nabla\rho \odot (v_j - \overline{u})\rangle \,\mathrm{d}x \\
&\quad + \int_U \langle f'(\mathscr{E}\overline{u}), 2\rho\nabla\rho \odot (v_j - \overline{u})\rangle \,\mathrm{d}x \\
&\Leftrightarrow: \mathbf{I}_j + \mathbf{II}_j = -\mathbf{III}_j + \mathbf{IV}_j.
\end{aligned}$$

We treat the single terms separately, and we leave $\mathbf{I}_j$ untouched for the time being.

Ad $\mathbf{II}_j$. We recall from (5.28) that

$$\frac{1}{j}\int_\Omega |\boldsymbol{\varepsilon}(v_j)|^2 \,\mathrm{d}x \to 0 \quad \text{and hence} \quad \frac{1}{j}\boldsymbol{\varepsilon}(v_j) \to 0 \qquad \text{strongly in } \mathrm{L}^2(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}}) \tag{7.79}$$

as $j \to \infty$. Since $\mathscr{E}\overline{u}|_U \in \mathrm{L}^\infty(U;\mathbb{R}^{n\times n}_{\mathrm{sym}})$, we hence obtain

$$|\mathbf{II}_j| \leqslant \frac{1}{j}\int_U |\boldsymbol{\varepsilon}(v_j)|^2\rho^2 \,\mathrm{d}x + C(\rho,U)\|\mathscr{E}\overline{u}\|_{\mathrm{L}^\infty(U;\mathbb{R}^{n\times n}_{\mathrm{sym}})}\|j^{-1}\boldsymbol{\varepsilon}(v_j)\|_{\mathrm{L}^2(U;\mathbb{R}^{n\times n}_{\mathrm{sym}})} \to 0$$

as $j \to \infty$.

Ad $\mathbf{III}_j$. By the definition of σ_j, we have

$$\begin{aligned}
\mathbf{III}_j &= -\int_U \langle f'(\boldsymbol{\varepsilon}(v_j)), 2\rho\nabla\rho \odot (v_j - \overline{u})\rangle \,\mathrm{d}x - \frac{1}{j}\int_U \langle \boldsymbol{\varepsilon}(v_j), 2\rho\nabla\rho \odot (v_j - \overline{u})\rangle \,\mathrm{d}x \\
&=: \mathbf{III}_j^{(1)} + \mathbf{III}_j^{(2)}.
\end{aligned}$$

Let us recall that $v_j \to \overline{u}$ strongly in $\mathrm{L}^1(\Omega;\mathbb{R}^n)$. Since $(f'(\boldsymbol{\varepsilon}(v_j)))$ is uniformly bounded in $\mathrm{L}^\infty(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})$, we obtain

$$|\mathbf{III}_j^{(1)}| \leqslant C(\rho)\|f'(\boldsymbol{\varepsilon}(v_j))\|_{\mathrm{L}^\infty(\Omega;\mathbb{R}^{n\times n})}\|v_j-\overline{u}\|_{\mathrm{L}^1(\Omega;\mathbb{R}^n)} \to 0$$

as $j\to\infty$. Note that this reasoning also covers $\mathbf{IV}_j$ and verifies $|\mathbf{IV}_j|\to 0$ as $j\to\infty$. Finally, we have by $\overline{u}|_U \in \mathrm{L}^\infty(U;\mathbb{R}^n)$ and (7.79)

$$\begin{aligned}|\mathbf{III}_j^{(2)}| &\leqslant C(\rho)\left|\frac{1}{j}\int_U \langle \boldsymbol{\varepsilon}(v_j), 2\rho\nabla\rho\odot(v_j-\overline{u})\rangle\,\mathrm{d}x\right| \\ &\leqslant C(\rho)\left(\frac{1}{j}\int_U|\boldsymbol{\varepsilon}(v_j)|^2\,\mathrm{d}x\right)^{\frac12}\left(\frac{1}{j}\int_U|v_j|^2\,\mathrm{d}x\right)^{\frac12} + \|\overline{u}\|_{\mathrm{L}^2(U;\mathbb{R}^n)}\left(\frac{1}{j^2}\int_U|\boldsymbol{\varepsilon}(v_j)|^2\,\mathrm{d}x\right)^{\frac12} \\ &\to 0, \qquad \text{as } j\to\infty,\end{aligned}$$

provided we have

$$\sup_{j\in\mathbb{N}}\left(\frac{1}{j}\int_U|v_j|^2\,\mathrm{d}x\right)^{\frac12} < \infty. \tag{7.80}$$

To see (7.80), we note that since $v_j \in u_0 + \mathrm{W}_0^{1,2}(\Omega;\mathbb{R}^n)$, we may write $v_j = u_0+\varphi_j$ for some $\varphi_j\in\mathrm{W}_0^{1,2}(\Omega;\mathbb{R}^n)$. Hence we obtain by Korn's inequality

$$\begin{aligned}\frac{1}{j}\int_U|v_j|^2\,\mathrm{d}x &\leqslant C\left(\frac{1}{j}\int_\Omega|u_0|^2\,\mathrm{d}x + \frac{1}{j}\int_\Omega|\varphi_j|^2\,\mathrm{d}x\right) \\ &\leqslant C\left(\frac{1}{j}\int_\Omega|u_0|^2\,\mathrm{d}x + \frac{1}{j}\int_\Omega|\boldsymbol{\varepsilon}(\varphi_j)|^2\,\mathrm{d}x\right) \\ &= C\left(\frac{1}{j}\int_\Omega|u_0|^2\,\mathrm{d}x + \frac{1}{j}\int_\Omega|\boldsymbol{\varepsilon}(v_j-u_0)|^2\,\mathrm{d}x\right) \\ &\leqslant C\left(\frac{1}{j}\int_\Omega|u_0|^2\,\mathrm{d}x + \frac{1}{j}\int_\Omega|\boldsymbol{\varepsilon}(v_j)|^2\,\mathrm{d}x\right) \to 0\end{aligned}$$

as $j\to\infty$ by (7.79). Gathering estimates, we have therefore proved

$$\lim_{j\to\infty}\mathbf{I}_j = \lim_{j\to\infty}\int_U\langle f'(\boldsymbol{\varepsilon}(v_j)) - f'(\mathscr{E}\overline{u}), \boldsymbol{\varepsilon}(v_j)-\mathscr{E}\overline{u}\rangle\rho^2\,\mathrm{d}x = 0. \tag{7.81}$$

Since f is convex and of class C^2, we have $\langle f'(\xi)-f'(\eta),\xi-\eta\rangle \geqslant 0$ for all $\xi,\eta\in\mathbb{R}^{n\times n}_{\mathrm{sym}}$, and hence the integrand in (7.81) is non–negative on $\mathrm{spt}(\rho)$. Hence we deduce that, by possibly passing to another subsequence,

$$\langle f'(\boldsymbol{\varepsilon}(v_j)) - f'(\mathscr{E}\overline{u}), \boldsymbol{\varepsilon}(v_j)-\mathscr{E}\overline{u}\rangle \to 0$$

$\mathscr{L}^n$–a.e. on $\mathrm{spt}(\rho)$ and, by arbitrariness of ρ, $\mathscr{L}^n$–a.e. on U as $j\to\infty$. By Lemma 7.28 we have that for $\mathscr{L}^n$–a.e. $x\in U$ the sequence $(\boldsymbol{\varepsilon}(v_j)(x))$ is bounded and so, by strict convexity of f, we obtain (7.75). The proof is complete. □

7.8. Return to μ–ellipticity and Bounding $\dim_{\mathcal{H}}(\Sigma_u)$

In this section we combine the higher integrability results of Chapter 5 with the partial regularity result, Theorem 7.4, to give upper bounds for the Hausdorff dimension of the singular set Σ_u. Henceforth, let us assume that f is a C^2–regular μ–elliptic integrand with $1<\mu<3$.

7.8.1. The Full Gradient Case In this situation, we may invoke Remark 6.28 to find by the measure density lemma that $\dim_{\mathcal{H}}(\Sigma_u) \leqslant n-1$ if $n \geqslant 2$ and $\mu < \frac{n}{n-1}$. In fact, Remark 6.28 implies that for this range of μ, $(\partial_s \nabla u_k)$ is bounded in $\mathrm{L}^1_{\mathrm{loc}}(\Omega;\mathbb{R}^{N\times n})$ for every $s \in \{1,...,n\}$. Hence (u_k) converges to some $u \in \mathrm{BV}_{2,\mathrm{loc}}(\Omega;\mathbb{R}^N)$ in the weak*–sense. Let $x_0 \in \Sigma_u$ be arbitrary. Then we find by Poincaré's Inequality in BV

$$\fint_{\mathrm{B}(x_0,r)} |\nabla u - (\nabla u)_{\mathrm{B}(x_0,r)}|\,\mathrm{d}x \leqslant Cr^{1-n}|\partial_s \mathrm{D}u|(\overline{\mathrm{B}(x_0,r)}). \tag{7.82}$$

We now define the Radon measure $\mu := |\mathrm{D}^2 u|$. Note that it is not at our disposal whether μ is of finite total variation, however, this does not affect the argument. But, in notational line with Lemma 2.2, $E^{n-1} := \{x \in \Omega\colon \limsup_{r\searrow 0} \mu(\mathrm{B}(x,r))/r^{n-1} > 0\}$. Then, by Lemma 2.2, $\dim_{\mathcal{H}}(E^{n-1}) \leqslant n-1$. We then note that by (7.82) and the suitably modified characterisation of the singular set (cp. (7.7)) in the full gradient case that

$$x_0 \in \Sigma_u \Leftrightarrow \limsup_{r\searrow 0} \int_{\mathrm{B}(x_0,r)} |\nabla u - (\nabla u)_{\mathrm{B}(x_0,r)}|\,\mathrm{d}x > 0 \Rightarrow x_0 \in E^{n-1}.$$

Hence, $\Sigma_u \subset E^{n-1}$ and by monotonicity of $\dim_{\mathcal{H}}$ with respect to set inclusions,

$$\dim_{\mathcal{H}}(\Sigma_u) \leqslant \dim_{\mathcal{H}}(E^{n-1}) \leqslant n-1.$$

Note that by [34, Thm. 1.10], $u \in \mathrm{W}^{1,1}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ holds for *every* generalised minimiser provided $1 < \mu \leqslant 3$, and so the dimension bound $\dim_{\mathcal{H}}(\Sigma_u) \leqslant n-1$ in particular holds for every generalised minimiser in the range of μ specified above (note that this depends on n).

7.8.2. The Symmetric Gradient Case In the symmetric gradient case, we can only rely on some higher fractional differentiability instead of the full second order differentiability. We hence combine Theorems 2.16 and Corollary 5.46 to the deduce the following result:

Theorem 7.29. *In the situation of Theorem 5.6 with a μ–elliptic integrand $f \in \mathrm{C}^2(\mathbb{R}^{n\times n}_{\mathrm{sym}})$ with $1 < \mu < 2n/(2n-1)$, the particular generalised minimiser $u \in \mathrm{GM}(\mathfrak{F};u_0)$ obtained as the weak*–limit of the viscosity approximation sequence (v_j) satisfies the following:*

(a) *There exists an open set $\Omega_0 \subset \Omega$ such that $u \in \mathrm{C}^{1,\alpha}_{\mathrm{loc}}(\Omega_0;\mathbb{R}^n)$ and $\Sigma_u := \Omega\setminus\Omega_0$ satisfies $\mathscr{L}^n(\Sigma_u) = 0$.*

(b) *We have the Hausdorff dimension bound*

$$\dim_{\mathcal{H}}(\Sigma_u) \leqslant n - \frac{3}{4}. \tag{7.83}$$

Proof. Instead of the direct application of the measure density lemma, we now make use of Mingione's modification, cp. Lemma 2.16. Imitating the above strategy for the full gradient and to directly make use of the higher fractional differentiability provided by Corollary 5.46, we would need to introduce a fractional order–type BV–space. This is so because $\mathcal{N}^{\alpha,1}$ itself is non–reflexive. To overcome this problem, we consequently embed $\mathcal{N}^{\alpha,1}$ into a fractional Sobolev space $\mathrm{W}^{\theta,p}$ with $p > 1$ and then proceed by Lemma 2.16 to reduce the Hausdorff dimension of Σ_u.

To do so, note that by a localised variant of Lemma 2.10 we obtain for all $0 < \alpha < 1$, $1 \leqslant p < \infty$ and $\delta > 0$ with $2\delta \ll \alpha$ that $\mathcal{N}^{\alpha,1}_{\mathrm{loc}} = (\mathrm{B}^{\alpha,1}_{\infty})_{\mathrm{loc}} \hookrightarrow \mathcal{N}^{\alpha-\delta,p}_{\mathrm{loc}}$ with the exponent p given by $p = p(n,\delta) = \frac{n}{n-\delta}$. In consequence, we obtain for this choice of p

$$\mathcal{N}^{\alpha,1}_{\mathrm{loc}} \simeq (\mathrm{B}^{\alpha,1}_{\infty})_{\mathrm{loc}} \hookrightarrow (\mathrm{B}^{\alpha-\delta,p(n,\delta)}_{\infty})_{\mathrm{loc}} \hookrightarrow (\mathrm{B}^{\alpha-2\delta,p}_{p(n,\delta)})_{\mathrm{loc}} \simeq \mathrm{W}^{\alpha-2\delta,p(n,\delta)}_{\mathrm{loc}}. \tag{7.84}$$

Now, in the situation of Theorem 5.6, we conclude by Corollary 5.46 and (7.84) that for each $0 < s < \frac{3}{4}$ and each relatively compact Lipschitz subset $\omega \Subset \Omega$, the sequence $(\boldsymbol{\varepsilon}(v_j))$ is uniformly bounded in $\mathrm{W}^{s-2\delta,p(n,\delta)}_{\mathrm{loc}}$. Hence, for each such ω we may extract a non–relabelled subsequence such that for some $\mathbf{v}_\omega \in \mathrm{W}^{s-2\delta,p(n,\delta)}(\omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})$ there holds $\boldsymbol{\varepsilon}(v_j) \rightharpoonup \mathbf{v}_\omega$. Clearly, $\mathbf{v}_\omega$ must coincide with $\boldsymbol{\varepsilon}(u)|_\omega$ and hence $\boldsymbol{\varepsilon}(u) \in \mathrm{W}^{s-2\delta,p(n,\delta)}(\omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})$. Using the characterisation (7.7) of $\Sigma_u := \Omega \setminus \Omega_u$, by Lemma 2.16 we obtain

$$\dim_{\mathcal{H}}(\Sigma_u) \leqslant n - (s-2\delta)p(n,\delta) = n - (s-2\delta)\frac{n}{n-\delta}.$$

Since this is valid for any $s < \frac{3}{4}$ and all sufficiently small $\delta > 0$, we may send $s \nearrow \frac{3}{4}$ and $\delta \searrow 0$ to find $\dim_{\mathcal{H}}(\Sigma_u) \leqslant n - \frac{3}{4}$. The proof is complete. □

Remark 7.30. *It is natural to expect that for linear growth problems with* C^2*–integrands we do have* $\dim_{\mathcal{H}}(\Sigma_u) \leqslant n-1$*. However, for this it seems necessary to obtain suitable estimates for the second derivatives. Unfortunately, it is not clear to us how the method to yield the Sobolev regularity improvement from Chapter 5.7 can yield estimates beyond fractional derivatives.*

CHAPTER 8

Partial Regularity for Symmetric–Quasiconvex Problems

In the final chapter of this thesis we leave the convex regime as dealt with in the preceding chapters and investigate the corresponding adequate notions of *quasiconvexity* and *rank–one convexity*. Since our integrands depend on the symmetric and not the full gradients, these are given by the *symmetric* quasiconvexity and convexity with respect to directions contained in the symmetric rank–one cone $\mathbb{R}^n \odot \mathbb{R}^n$. Here we focus on two results: Our first aim is to establish a *full* $\mathrm{C}^{1,\alpha}$–regularity result for minimisers of strongly symmetric rank–one convex variational integrals of linear growth subject to a smallness condition on the Dirichlet data. In the second and main part of this chapter, we give a partial regularity result for strongly symmetric–quasiconvex integrals of linear growth. To some extent, this amplifies the results of the previous chapter to the quasiconvex regime. Most crucially, however, it extends the results of KRISTENSEN and the author [134] obtained in the BV–setting to the symmetric gradient case. As we shall discuss in detail below, it hereby contributes to the partial regularity program initiated by EVANS in his fundamental study [99].

Structure of the chapter. After describing and contextualising our results in Section 8.1 and explaining the requisite framework for the Dirichlet problem in the (symmetric) quasiconvex situation in Section 8.2, we discuss the main obstructions to be encountered in this chapter in Section 8.3. After gathering auxiliary tools in Section 8.4, in Section 8.5 we then establish the aforementioned regularity result for symmetric rank–one integrands. Finally, in Sections 8.6–8.9 we accomplish the proof of the partial regularity for strongly symmetric quasiconvex functionals of linear growth.

8.1. Description of Results

Let $n \geqslant 2$ and Ω be an open and bounded subset of $\mathbb{R}^n$ with Lipschitz boundary $\partial\Omega$. In a similar vein as in the previous chapters, we study for a given Dirichlet datum $u_0 \in \mathrm{LD}(\Omega)$ minimisers of the variational integral

$$\mathfrak{F}[v] := \int_\Omega f(\boldsymbol{\varepsilon}(v))\,\mathrm{d}x \qquad \text{over } \mathcal{D}_{u_0} := \mathrm{LD}_{u_0}(\Omega) := u_0 + \mathrm{LD}_0(\Omega). \tag{8.1}$$

As distinct from the preceding chapters, we admit the variational integrand $f\colon \mathbb{R}^{n\times n}_{\mathrm{sym}} \to \mathbb{R}$ to be symmetric quasiconvex or symmetric rank–one convex. These notions will be properly introduced and discussed below. Unless otherwise stated, we further require now that $f\colon \mathbb{R}^{n\times n}_{\mathrm{sym}} \to \mathbb{R}$ is of *linear growth* in the sense that there exists a constant $L > 0$ such that

$$|f(z)| \leqslant L(1+|z|) \qquad \text{for all } z \in \mathbb{R}^{n\times n}_{\mathrm{sym}}. \tag{LG'}$$

Let us carefully note that unlike for a plenty of convex integrands f with $f'' > 0$ in the sense of positive definite bilinear forms satisfying a quantified decay of the second derivatives, (LG') in general does not imply a coercive bound $c_1|z| - c_2 \leqslant f(z)$ for all $z \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$ with two constants $c_1, c_2 > 0$ if we pass to weaker semiconvexity notions.

8.1.1. Setup We begin with recalling and strengthening the concept of *symmetric quasiconvexity* which already has been introduced in Definition 1.3. Here and in the following we use the equivalent notations

$$\boxed{V(z) := \sqrt{1+|z|^2} - 1 \text{ for } z \in \mathbb{R}^{n\times n}_{\mathrm{sym}},}$$

and the reader will notice that this is the same as the auxiliary function $\mathbf{e}$ as utilised in the previous chapter.

Definition 8.1 (Strong Symmetric Quasiconvexity)**.** *We say that a continuous integrand $f\colon \mathbb{R}^{n\times n}_{\mathrm{sym}} \to \mathbb{R}$ is* symmetric quasiconvex *if and only if for all $A \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$ and all $\varphi \in \mathrm{W}^{1,\infty}_0(Q;\mathbb{R}^n)$ (where $Q := (0,1)^n$ is the open unit cube) there holds*

$$f(A) \leqslant \int_Q f(A + \varepsilon(\varphi))\,\mathrm{d}x. \tag{8.2}$$

We say that f is strongly symmetric–quasiconvex *provided there exists $\ell > 0$ such that $F\colon \mathbb{R}^{n\times n}_{\mathrm{sym}} \to \mathbb{R}$ given by $F(z) := f(z) - \ell V(z)$ for $z \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$ is symmetric quasiconvex.*

*In this situation, F is also said to be ℓ–*strongly symmetric quasiconvex*.*

Before we continue, let us briefly compare the strong symmetric quasiconvexity with strong convexity conditions. Every convex integrand is quasiconvex, too, but if a convex integrand is *strongly (symmetric–)quasiconvex*, then this has implications on its second derivatives.

Remark 8.2 (Convex integrands in the situation of Definition 8.1)**.** *Suppose that $f\colon \mathbb{R}^{n\times n}_{\mathrm{sym}} \to \mathbb{R}$ is a convex function which is strongly symmetric–quasiconvex in the sense of Definition 8.1. If $F = f - \ell V(\cdot)$ is* convex*, we obtain by passing to the second derivatives that f is essentially μ–elliptic with $\mu = 3$ (cp. Example 5.3). In conclusion, we may interpret the strong symmetric quasiconvexity as the quasiconvex analogue of 3–ellipticity in the convex situation. As we have seen in Chapters 5.2 and 6, this needs to be contrasted by mere* C^2–strict *convexity: If we only work under a strict convexity assumption without specifying the coerciveness rate for the second derivatives, then the implications of Definition 8.1 are* not *at our disposal.*

From the definition as given above, it is not clear whether such integrands exist. However, as shall be briefly discussed in Section 8.2, such integrands are ubiquitous and can be obtained from the construction of functions which are genuinely quasiconvex and of linear growth as accomplished by Müller [189] and Zhang [249].

In light of Rindler's lower semicontinuity results for symmetric–quasiconvex functionals on BD (cp. [202] and Section 8.2 below), symmetric quasiconvexity is the natural condition for variational integrals of the form (8.1) to possess the requisite lower semicontinuity properties for the direct method to yield existence of minima, e.g., in the context of Dirichlet problems. Mere (symmetric) quasiconvexity itself, however, cannot yield higher (Hölder–) regularity results. Instead, it is the *strong* version which makes higher regularity available, and we briefly pause to discuss the relevance of the latter for partial regularity.

8.1.2. Contextualisation: Regularity and quasiconvexity In a plenty of settings, the partial regularity proofs for non–convex multiple integrals are usually centered around a linearisation strategy combined with a Caccioppoli–type inequality (cp. Chapter 5.6 for a related discussion in the convex setting), both of which exploit the strong quasiconvexity in an essential way.

The first partial regularity proof for strongly quasiconvex variational integrals is due to EVANS [99] by adapting the *blow–up method* (cp. Section 8.3 for more detail) in the superquadratic growth regime. As a major achievement, the decisive point in his proof is the derivation of a Caccioppoli–type inequality: Essentially, it is the only point in the proof where both minimality and strong quasiconvexity are utilised simultaneously. Closely following EVANS' original approach, we shall also prove a Caccioppoli–type inequality below in Section 8.8, and hence refer the reader to this section for more information on the strategy. Let us emphasize that it is not clear to us how to arrive at such an inequality when only a strong rank–one convexity condition is imposed; see the following subsection for this terminology.

The strong rank–one convexity, however, is the essential point in employing the aforementioned linearisation strategy. Roughly speaking, we aim for comparing minima on small balls with corresponding solutions to linear systems of the type $\operatorname{div}(\mathbb{A}\mathrm{D}v) = 0$ or $\operatorname{div}(\mathbb{A}\boldsymbol{\varepsilon}(v)) = 0$, where $\mathbb{A}$ is strictly related to the second derivatives of f. So the comparison system – although not convex – is elliptic in the sense of *Legendre–Hadamard*, and its solutions equally satisfy good decay estimates.

We will use variants of these ideas in the $\mathbb{A}$–harmonic approximation technique[1] or –linearisation strategy, respectively, for proving partial regularity. This will be explained in detail in Sections 8.1.4 and 8.3, but beforehand we turn to a *full* regularity result in the next subsection.

8.1.3. A Regularity Theorem for strongly symmetric–rank–one convex problems In this section we present a regularity result which is taylored for integrands which satisfy a strong convexity condition with respect to directions contained in the symmetric rank–one cone $\mathbb{R}^n \odot \mathbb{R}^n$ (and are hereafter called *strongly symmetric–rank–one convex*). This particularly covers the strong symmetric quasiconvexity.

In what follows, we assume that $F\colon \mathbb{R}^{n\times n}_{\mathrm{sym}} \to \mathbb{R}$ is an integrand which satisfies the following set of assumptions:

(H1) $F \in \mathrm{C}^{3,1}_{\mathrm{loc}}(\mathbb{R}^{n\times n}_{\mathrm{sym}})$.

(H2) F is of linear growth, i.e., there exist $0 < c_1 \leqslant c_3 < \infty$ and $c_2 \in \mathbb{R}$ such that $c_1|z| - c_2 \leqslant f(z) \leqslant c_3(1+|z|)$ holds for all $z \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$.

(H3) F is *strongly symmetric–rank–one convex*, i.e., there exists $1 < \mu < \infty$ and $\ell > 0$ such that

$$\ell\frac{|a|^2|b|^2}{(1+|z|^2)^{\frac{\mu}{2}}} \leqslant \langle F''(z)(a\odot b), a\odot b\rangle \qquad \text{holds for all } a,b\in\mathbb{R}^n \text{ and } z\in\mathbb{R}^{n\times n}_{\mathrm{sym}}.$$

We also refer to this hypothesis as *strong symmetric Legendre–Hadamard condition*.

Based on these assumptions, our first result of this chapter reads as follows.

Theorem 8.3. *Let $\Omega \subset \mathbb{R}^n$ be an open and bounded domain with $\mathrm{C}^{1,1}$–boundary $\partial\Omega$ and let $0 < \alpha < 1$ be given. Given a variational integrand $F\colon \mathbb{R}^{n\times n}_{\mathrm{sym}} \to \mathbb{R}$ satisfying* (H1)–(H3) *from above, there exists $\varepsilon_0 > 0$ with the following property: If $g \in \mathrm{C}^{1,\alpha}(\overline{\Omega};\mathbb{R}^n)$ satisfies $\|g\|_{\mathrm{C}^{1,\alpha}(\overline{\Omega};\mathbb{R}^n)} < \varepsilon_0$, then there exists a* LD*–minimiser $u \in \mathcal{D}_g := g + \mathrm{LD}_0(\Omega)$ of the variational integral*

$$\mathfrak{F}[v;\Omega] := \int_\Omega F(\boldsymbol{\varepsilon}(v))\,\mathrm{d}x \qquad \textit{over } \mathcal{D}_g. \tag{8.3}$$

[1]To avoid notational confusion, let us remark that here and in the remainder of the chapter, $\mathbb{A}$ is *not* utilised to denote differential operators as has been done in Chapters 3–5. Instead, we shall use $\mathbb{A}$ for bilinear forms, and this will be clear from the context throughout.

Moreover, we have $u \in \mathrm{C}^{1,\alpha}(\overline{\Omega};\mathbb{R}^n)$*, and there exists a constant* $c = c(\varepsilon_0, F) > 0$ *such that*

$$\|u\|_{\mathrm{C}^{1,\alpha}(\overline{\Omega};\mathbb{R}^n)} \leqslant c\|g\|_{\mathrm{C}^{1,\alpha}(\overline{\Omega};\mathbb{R}^n)}. \tag{8.4}$$

The proof of Theorem 8.3 will be given in Chapter 8.5 and is centered around the Banach space version of the implicit function theorem. For the convenience of the reader, this is recalled in Chapter 10.5 from the appendix.

Since we argue from an abstract perspective and as will become clear in the proof, we are only able to fit the above theorem into the framework of the implicit function theorem if we have *local Lipschitz continuity of the third derivatives* of the integrand. Even though this assumption might seem peculiar, it is not clear to us to which extent Theorem 8.3 survives if this condition is dropped. Also note that a similar condition arises in a different application of the inverse function theorem in the work of Zhang [248].

On the other hand, we have no effective control over the smallness parameter ε_0 but rather have to take what the implicit function theorem gives us. However, accepting a possibly very small such parameter, the theorem in particular yields the existence of a LD–minimiser and so reflects the crucial impact of both the regularity and smallness of the Dirichlet data on the regularity of minima.

8.1.4. Partial Regularity for strongly symmetric quasiconvex problems In this section, we present the second main result of this chapter, Theorem 8.4 below. The result itself is the extension of a long list of partial regularity results to symmetric quasiconvex, linear growth functionals and covers a very special yet open case in this program. With the full gradient case being dealt with only recently in [134], we wish to point out that beforehand, the most general[2] result regarding quasiconvex functionals was due to Diening et al. [82]. In this work, partial regularity for minima of strongly quasiconvex variational integrals of Orlicz growth was established, provided the integrand satisfies the $\Delta_2 \cap \nabla_2$–condition. From our discussion in Chapter 2.5.6, it is however seen that the results from [82] do not apply to quasiconvex integrands from the class $L\log L$. Still, from a compactness viewpoint, the latter is substantially better behaved than L^1–based spaces or spaces of Radon measures. Hence there is indeed a crucial difference between the linear growth situation and the setting of [82], and we aim at closing this gap in what follows.

Main Result

We now impose the following set of hypothesis on the variational integrand $f\colon \mathbb{R}^{n\times n}_{\mathrm{sym}} \to \mathbb{R}$:

(H4) f is of class $\mathrm{C}^{2,1}_{\mathrm{loc}}(\mathbb{R}^{n\times n}_{\mathrm{sym}})$.

(H5) f is of *linear growth*, by which we understand now that for some $L > 0$ there holds

$$|f(z)| \leqslant L(1+|z|) \qquad \text{for all } z \in \mathbb{R}^{n\times n}_{\mathrm{sym}}. \tag{8.5}$$

(H6) f is ℓ–strongly symmetric quasiconvex in the sense of Definition 8.1.

We note that, as a consequence of Lemma 3.28 (also see [53, Sec. 5]), symmetric rank–one or quasiconvex functions of linear growth are Lipschitz on $\mathbb{R}^{n\times n}_{\mathrm{sym}}$. This property thus is implicit in (H6).

Our second main result of this chapter then reads as follows.

[2] Here, the term '*generality*' refers to the Orlicz growth behaviour of integrands. In other settings, so for instance, (p,q)–integrands or systems with u–dependence, we refer the reader to, e.g., Schmidt [207, 208] and Hamburger [142], especially to [141, 143] for proofs that do not utilise Caccioppoli–type inequalities.

Theorem 8.4. *Let Ω be an open and bounded Lipschitz subset of $\mathbb{R}^n$ and let $u_0 \in \mathrm{LD}(\Omega)$. Moreover, let $f\colon \mathbb{R}^{n\times n}_{\mathrm{sym}} \to \mathbb{R}$ be an integrand satisfying* (H4)–(H6) *from above and put, for $v \in \mathrm{BD}(\Omega)$,*

$$\begin{aligned}\widetilde{\mathfrak{F}}_{u_0}[v] := \int_\Omega f(\mathscr{E}v)\,\mathrm{d}x + \int_\Omega f^\infty\Big(\frac{\mathrm{dE}^s v}{\mathrm{d}|\,\mathrm{E}^s v|}\Big)\,\mathrm{d}|\,\mathrm{E}^s v| \\ + \int_{\partial\Omega} f^\infty(\mathrm{Tr}_{\partial\Omega}(u_0 - v)\odot \nu_{\partial\Omega})\,\mathrm{d}\mathcal{H}^{n-1},\end{aligned} \tag{8.6}$$

where f^∞ is the strong *recession function to be defined in Chapter 8.2 below. If $u \in \mathrm{BD}(\Omega)$ is a* BD*–minimiser for $\mathfrak{F}$ in the sense that*

$$\widetilde{\mathfrak{F}}_{u_0}[u] \leqslant \widetilde{\mathfrak{F}}_{u_0}[v] \qquad \text{for all } v \in \mathrm{BD}(\Omega),$$

then there exists an open subset Ω_u of Ω with $\mathscr{L}^n(\Omega\setminus\Omega_u) = 0$ and for every $x_0 \in \Omega_u$, u is of class $\mathrm{C}^{1,\alpha}$ in a neighbourhood of x_0 for any $0<\alpha<1$.

In particular, denoting $\Sigma_u = \Omega\setminus\Omega_u$, we have

$$\begin{aligned}\Sigma_u &= \Sigma_u^1 \cup \Sigma_u^2 \\ &:= \Big\{x_0 \in \Omega\colon\ \liminf_{r\searrow 0} \fint_{\mathrm{B}(x_0,r)} |\mathscr{E}u - (\mathscr{E}u)_{\mathrm{B}(x_0,R)}|\,\mathrm{d}x + \frac{|\,\mathrm{E}^s u|(\mathrm{B}(x_0,r))}{r^n} > 0\Big\} \\ &\cup \Big\{x_0\in\Omega\colon\ \limsup_{r\searrow 0}\Big|\frac{\mathrm{E}u(\mathrm{B}(x_0,r))}{\mathscr{L}^n(\mathrm{B}(x_0,r))}\Big| = \infty\Big\}.\end{aligned} \tag{8.7}$$

This theorem will be established in several stages. In Section 8.6 we provide so–called Fubini–theorems for BD–maps which shall turn out essential in the implementation of our linearisation strategy. In fact, the requisite higher integrability of suitable $\mathbb{A}$–harmonic comparison maps stems from the differentiability properties of the restrictions of minima to $\mathscr{L}^1$–almost all spheres with fixed center. Having gathered auxiliary tools and linear comparison estimates in Section 8.4, we establish a Caccioppoli–type inequality in Section 8.8. With this at our disposal, we then employ an $\mathbb{A}$–harmonic comparison argument to establish the partial regularity in Sections 8.7 and 8.9. The chapter is then closed by remarks on extensions of the work presented so far.

We conclude this section by commenting on the method of proof which we invoke to establish Theorem 8.4.

8.1.5. The method of $\mathbb{A}$–harmonic approximation In proving the partial regularity for strongly symmetric quasiconvex integrands on BD, we make use of a linearisation strategy by explicitely constructing comparison maps. This is a direct version in the spirit of the $\mathbb{A}$–*harmonic approximation* technique, an approach with origins in the works of Allard [12, 13] and Simon [216] in the context of geometric measure theory, and has been adapted to partial regularity proofs by Duzaar et al., cp. [89, 90, 91, 92]. There are also a considerable number of recent applications to the problems with Orlicz growth or manifold constraints, see, among others, [32, 31, 82, 146, 209]. Let us also mention that Beck's recent monograph [33] contains a comprehensive overview of the $\mathbb{A}$–harmonic approximation and other linearisation techniques, cp. [33, Chpts. 4.3.2 and 5.2].

To display the distinction from our method later on, let us briefly sketch the idea of the usual $\mathbb{A}$–harmonic approximation strategy. We again aim at an excess decay estimate for the symmetric gradients of minima. By use of Campanato's lemma, Theorem 2.27, and Lemma 7.7, this yields the Hölder continuity of minima. This decay estimate is then proven to be available for $\mathscr{L}^n$–a.e. point in Ω and thereby establishes the partial regularity. To arrive at the excess decay, we compare the solution on small balls with the solution of an $\mathbb{A}$–harmonic system. The overarching principle now is the following: *If* the excess quantity is small for a given radius, *then* the minimiser is *almost* $\mathbb{A}$*–harmonic* on the given ball for a suitable bilinear form $\mathbb{A}$ (which, in turn, corresponds to the second derivative of the integrand f). This feature is usually referred to as *approximate $\mathbb{A}$–harmonicity* of minima. In this situation, the so–called $\mathbb{A}$–*approximation lemma* asserts that if a mapping is almost $\mathbb{A}$–harmonic, then it is close to an $\mathbb{A}$–harmonic mapping, where closeness mostly is measured in the L^p–distance[3]. If f is in a suitable sense quasiconvex, the bilinear form $\mathbb{A}$ is strongly Legendre–Hadamard and hence $\mathbb{A}$–harmonic mappings satisfy good decay estimates. To establish the decay estimate, we consequently switch from estimates for the (symmetric) gradients to the minimisers themselves by a Caccioppoli–type inequality and are then able to compare the minimisers with the $\mathbb{A}$–harmonic maps provided by the linearisation argument employed before. In this way, the decay estimates for the latter maps are inherited to the given original minimiser and the proof concludes.

8.2. Lower Semicontinuity for Symmetric–Quasiconvex Problems

For completeness of exposition and since it is not dealt with in Chapter 4, we now briefly comment on the Dirichlet problem on BV and BD for (strongly symmetric) quasiconvex problems. Whereas the lower semicontinuity in the convex situation is treated by use of the Reshetnyak–type theorems (cp. Chapter 4.3, Proposition 4.18), the quasiconvex situation is substantially more intricate. For BV, the situation has been treated by Ambrosio & Dal Maso [15] and by Fonseca & Müller [107] (also cp. [108]) first, thereby extending the seminal work of Acerbi & Fusco [2] in the $\mathrm{W}^{1,p}$–case, $p > 1$. Toward a relatively complete theory, Kristensen & Rindler [163] extended the aforementioned works in the BV–case to signed integrands.

For future reference, let us record the following result on integral representations for lower semicontinuous envelopes in the version as given by Rindler [201] for the gradient case:

Theorem 8.5 ([15, 107], Gradient Case)**.** *Let $\Omega \subset \mathbb{R}^n$ be an open and bounded Lipschitz domain and let $f\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ satisfy the following set of assumpitons:*

(i) *f is continuous,*

(ii) *$|f(\xi)| \leqslant M(1+|\xi|)$ for some $M > 0$ and all $\xi \in \mathbb{R}^{N\times n}$,*

(iii) *the strong recession function $f^\infty(\xi) := \lim_{t\to\infty} t^{-1} f(t\xi)$ exists for all $\xi \in \mathbb{R}^{N\times n}$,*

(iv) *f is quasiconvex.*

Then the functional

$$\begin{aligned}\mathcal{F}[u] := &\int_\Omega f(\nabla u)\,\mathrm{d}x + \int_\Omega f^\infty\Big(\frac{\mathrm{d}\mathrm{D}^s u}{\mathrm{d}|\,\mathrm{D}^s u|}\Big)\,\mathrm{d}|\,\mathrm{D}^s u| \\ &+ \int_{\partial\Omega} f^\infty(\mathrm{Tr}(u)|_{\partial\Omega} \otimes \nu_{\partial\Omega})\,\mathrm{d}\mathcal{H}^{n-1}, \qquad \text{for } u \in \mathrm{BV}(\Omega;\mathbb{R}^N),\end{aligned}$$

is lower semicontinuous on $\mathrm{BV}(\Omega;\mathbb{R}^N)$ with respect to weak–convergence. Here, as usual, $\mathrm{D}u = \nabla u \mathscr{L}^n + \mathrm{D}^s u$ is the (Lebesgue–)Radon–Nikodym decomposition of $\mathrm{D}u$, and $\nu_{\partial\Omega}$ denotes the outer unit normal to $\partial\Omega$.*

Some comments are in place. Firstly, the original approach of [15] consisted in identifying $\mathcal{F}$ as a suitable lower semicontinuous envelope. Hence the main difficulty stemmed from establishing this identification rather than proving lower semicontinuity. Secondly, we wish to note since f is assumed quasiconvex, it is particularly rank–one convex and hence the strong recession function exists on the rank–one cone. Since in the definition of

[3] In many situations, this can even be improved to L^p–closeness on the level of gradients. This enhanced version makes use of the Lipschitz truncation technique, cp. [82, 84].

the functional $\mathcal{F}$ the recession function only operates on the density of the singular parts with respect to their own total variation measures, by ALBERTI's Rank–One Theorem we are in fact allowed to work with the strong recession function throughout. However, note that for general quasiconvex integrands, the strong recession function $f^\infty(\xi)$ need not exist for $\xi \in \mathbb{R}^{N\times n} \setminus (\mathbb{R}^N \otimes \mathbb{R}^n)$ and so the passage to the *upper* or *lower* recession functions can be of use. These are defined by passing to the lim sup or lim inf instead of lim, respectively, and we shall treat this in more detail in the symmetric gradient situation below.

Originally, the main part of the proof is centered around a *blow–up* argument (cp. [108]) and ALBERTI's Rank–One Theorem. Just recently it has be proven in [76] that a symmetric rank–one theorem is available for BD, too (cp. Chapter 4), and building this recent generalisation, a more general theory in the spirit of FONSECA & MÜLLER's $\mathcal{A}$–quasiconvexity (cp. Chapter 3) is available (see [24]).

However, as a symmetric rank–one theorem was not available until recently, the main novelty of RINDLER's lower semicontinuity result below was that it overcomes the traditional use of ALBERTI's rank–one theorem. To state it, we make some extensions of the recession function concepts to the symmetrised case. Given a continuous function $f\colon \mathbb{R}^{n\times n}_{\mathrm{sym}} \to \mathbb{R}$, we define the *strong recession function* by

$$f^\infty(\xi) := \lim_{t\to\infty} \frac{f(t\xi)}{t}, \qquad \xi \in \mathbb{R}^{n\times n}_{\mathrm{sym}} \tag{8.8}$$

which is analogous to the gradient situation. If f is merely assumed symmetric quasiconvex, this limit need not exist in general. However, it *does exist* on the symmetric rank–one cone $\mathbb{R}^n \odot \mathbb{R}^n$. In fact, since symmetric quasiconvexity implies symmetric rank–one convexity, f is convex with respect to directions out of the symmetric rank–one cone and hence we may then argue as for the convex case. Consequently , if we wish to come up with a notion of recession functions, then we can pass to *upper* and *lower* recession functions f^∞_{up} and f^∞_{low} given by

$$\begin{aligned} f^\infty_{\mathrm{up}}(\xi) &:= \limsup_{t\to\infty} t^{-1} f(\xi/t), \\ f^\infty_{\mathrm{low}}(\xi) &:= \liminf_{t\to\infty} t^{-1} f(\xi/t), \qquad \text{for } \xi \in \mathbb{R}^{n\times n}_{\mathrm{sym}}. \end{aligned}$$

We now have the following result:

Theorem 8.6 ([202, Thm. 1])**.** *Let $\Omega \subset \mathbb{R}^n$ be an open and bounded Lipschitz domain and let $f \in \mathrm{C}(\mathbb{R}^{n\times n}_{\mathrm{sym}})$ be a symmetric–quasiconvex integrand which satisfies*

(i) $|f(\xi)| \leqslant M(1+|\xi|)$ *for some $M > 0$ and all $\xi \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$,*

(ii) *the strong recession function f^∞ exists on $\mathbb{R}^{n\times n}_{\mathrm{sym}}$.*

Then the functional

$$\widetilde{\mathfrak{F}}[u] := \int_\Omega f(\mathscr{E}u)\,\mathrm{d}x + \int_\Omega f^\infty\left(\frac{\mathrm{dE}u}{\mathrm{d}|\mathrm{E}^s u|}\right)\mathrm{d}|\mathrm{E}^s u| + \int_{\partial\Omega} f^\infty(\mathrm{Tr}(u)|_{\partial\Omega} \odot \nu_{\partial\Omega})\,\mathrm{d}\mathcal{H}^{n-1}$$

for $u \in \mathrm{BD}(\Omega)$ is lower semicontinuous with respect to weak–convergence in the space* $\mathrm{BD}(\Omega)$.

Similarly as above, we may work with the strong recession function as a consequence of the DE PHILIPPIS–RINDLER Theorem. Working from Theorem 8.6 and extending the machinery given in Chapter 4.3.2, we now briefly sketch how the existence theory for linear growth functionals in the symmetric–quasiconvex setting is accomplished. Let $u_0 \in \mathrm{LD}(\Omega)$ be given and put $\mathcal{D}_{u_0} := u_0 + \mathrm{LD}_0(\Omega)$. For a symmetric–quasiconvex variational integrand $f\colon \mathbb{R}^{n\times n}_{\mathrm{sym}} \to \mathbb{R}$ of linear growth, we say that $u \in \mathrm{BD}(\Omega)$ is a *generalised* *minimiser* of $\mathfrak{F}$ given by (8.1) over $\mathcal{D}_{u_0}$ provided u appears as the L^1–limit of a $\mathfrak{F}$–minimising sequence $(u_k) \subset \mathcal{D}_{u_0}$. In analogy with Chapter 4.3.2, we denote the set of all generalised minima by $\mathrm{GM}(\mathfrak{F}; u_0)$. Subject to the hypotheses of Theorem 8.6, the following are then equivalent:

(a) $u \in \mathrm{GM}(\mathfrak{F}; u_0)$.

(b) $u \in \mathrm{BD}(\Omega)$ is a BD–minimiser (subject to the Dirichlet data u_0) in the sense that $\widetilde{\mathfrak{F}}_{u_0}[u] \leqslant \widetilde{\mathfrak{F}}_{u_0}[v]$ holds for all $v \in \mathrm{BD}(\Omega)$. Here, the functional $\widetilde{\mathfrak{F}}_{u_0}$ is given by

$$\begin{aligned}\widetilde{\mathfrak{F}}_{u_0}[v] := & \int_\Omega f(\mathscr{E}v)\,\mathrm{d}x + \int_\Omega f^\infty\left(\frac{\mathrm{dE}v}{\mathrm{d}|\,\mathrm{E}^s v|}\right)\mathrm{d}|\,\mathrm{E}^s v| \\ & + \int_{\partial\Omega} f^\infty(\mathrm{Tr}(u_0 - v)|_{\partial\Omega} \odot \nu_{\partial\Omega})\,\mathrm{d}\mathcal{H}^{n-1}, \qquad v \in \mathrm{BD}(\Omega).\end{aligned}$$

(c) Given any open Lipschitz subset $\widetilde{\Omega}$ of $\mathbb{R}^n$ with $\Omega \Subset \widetilde{\Omega}$ and picking any $\widetilde{u}_0 \in \mathrm{LD}(\widetilde{\Omega} \setminus \overline{\Omega})$ with $\mathrm{Tr}(\widetilde{u}_0)|_{\partial\widetilde{\Omega}} = 0$ and $\mathrm{Tr}(\widetilde{u}_0)|_{\partial\Omega} = \mathrm{Tr}(u_0)|_{\partial\Omega}$ $\mathcal{H}^{n-1}$–a.e. on the respective boundaries, we have

$$\overline{\mathfrak{F}}_{\widetilde{\Omega}}[w_u] \leqslant \overline{\mathfrak{F}}_{\widetilde{\Omega}}[w_v] \qquad \text{for all } v \in \mathrm{BD}(\Omega).$$

Here we have put, for $v \in \mathrm{BD}(\Omega)$,

$$w_v \colon \widetilde{\Omega} \to \mathbb{R}^n, \quad w_v := \begin{cases} v & \text{in } \Omega, \\ \widetilde{u}_0 & \text{in } \widetilde{\Omega} \setminus \overline{\Omega}, \end{cases}$$

and, for $w \in \mathrm{BD}(\widetilde{\Omega})$,

$$\overline{\mathfrak{F}}_{\widetilde{\Omega}}[w] := \int_{\widetilde{\Omega}} f(\mathscr{E}w)\,\mathrm{d}x + \int_{\widetilde{\Omega}} f^\infty\left(\frac{\mathrm{dE}w}{\mathrm{d}|\,\mathrm{E}^s w|}\right)\mathrm{d}|\,\mathrm{E}^s w|.$$

This can be seen similarly to the proof of Proposition 4.26. The only essential difference, however, lies in the direction '(b)⇒(a)' or, equivalently, '(c)⇒(a)'. Given a BD–minimiser $u \in \mathrm{BD}(\Omega)$, we need to find a minimising sequence $(u_k) \subset \mathcal{D}_{u_0}$ which converges to u strongly in $\mathrm{L}^1(\Omega; \mathbb{R}^n)$. Here we appeal to the (symmetric) area–strict smooth approximation result for BD, Lemma 4.9, and find $(u_k) \subset u_0|_\Omega + \mathrm{C}_c^\infty(\Omega; \mathbb{R}^n)(\subset \mathcal{D}_{u_0})$ such that $u_k \to u$ in the area–strict sense on Ω (and hence trivially in $\mathrm{L}^1(\Omega; \mathbb{R}^n)$) as $k \to \infty$. To conclude that (u_k) is a minimising sequence, we require the continuity of $\widetilde{\mathfrak{F}}_{u_0}$ with respect to (symmetric) area–strict convergence.

As it will be required independently in the partial regularity proof below, we give the following lemma on area–strict continuity of symmetric rank–one convex variational integrals of linear growth. In the form as given here, it appears as the canonical generalisation of [163, Thm. 4] to symmetric rank–one convex variational integrals, and we refer the reader to [53, Sec. 5] for the corresponding statement in a more general context.

Lemma 8.7. *Let $f \colon \mathbb{R}^{n \times n}_{\mathrm{sym}} \to \mathbb{R}$ be a continuous integrand of linear growth at infinity, i.e., satisfies $|f(\xi)| \leqslant M(1 + |\xi|)$ for all $\xi \in \mathbb{R}^{n \times n}_{\mathrm{sym}}$ and a constant $M > 0$. Moreover, assume that f is symmetric rank–one convex. Then the functional*

$$\mathscr{F} \colon u \mapsto \int_\Omega f(\mathscr{E}u)\,\mathrm{d}\mathscr{L}^n + \int_\Omega f^\infty_{\mathrm{up}}\Big(\frac{\mathrm{dE}^s u}{\mathrm{d}|\,\mathrm{E}^s u|}\Big)\,\mathrm{d}|\,\mathrm{E}^s u|, \qquad u \in \mathrm{BD}(\Omega),$$

is continuous on $\mathrm{BD}(\Omega)$ with respect to the area–strict topology. That is, if $u, u_1, u_2, ... \in \mathrm{BD}(\Omega)$ are such that $u_k \overset{\langle\cdot\rangle}{\to} u$, then $\mathscr{F}[u_k] \to \mathscr{F}[u]$ as $k \to \infty$.

Proof. We give the quick proof for the reader's convenience and follow closely [163]. Given $m \in \mathbb{N}$, we make use of the terminology explained in the appendix, Chapter 10.3, and recall that an integrand $f\colon \mathbb{R}^{m\times n} \to \mathbb{R}$ belongs to the class $\mathbf{E}(\mathbb{R}^{m\times n})$ provided the function $\xi \mapsto (1-|\xi|)f(x,(1-|\xi|)^{-1}\xi)$ for $\xi \in \mathbb{B}^{m\times n}$ (i.e., the open unit ball in $\mathbb{R}^{m\times n}$), can be extended to a continuous function on the closure of $\mathbb{B}^{m\times n}$). From [10, Lem. 2.3], we record that if $g\colon \mathbb{R}^{m\times n} \to \mathbb{R}$ is a lower semicontinuous integrand for which there exists $l > 0$ such that $g(\xi) \geqslant -l(1+|\xi|)$ for all $\xi \in \mathbb{R}^{m\times n}$ and we define

$$h_g(\xi) := \liminf_{\substack{t\to\infty\\ \xi'\to\xi}} \frac{g(t\xi')}{t},$$

then there exists a sequence (g_k) of class $\mathbf{E}(\mathbb{R}^{m\times n})$ such that

$$\sup_{k\in\mathbb{N}} g_k(\xi) = g(\xi) \quad \text{and} \quad \sup_{k\in\mathbb{N}} g_{k,\mathrm{up}}^{\infty}(\xi) = h_g(\xi) \qquad \text{for all } \xi \in \mathbb{R}^{m\times n}.$$

In what follows, we choose $m \in \mathbb{N}$ such that $\mathbb{R}^{n\times n}_{\mathrm{sym}} \cong \mathbb{R}^{m\times n}$. Let (u_j) be a sequence in $\mathrm{BD}(\Omega)$ with $u_j \to u$ in the (symmetric) area–strict sense on Ω as $j \to \infty$. By the aforementioned result applied to $g = f$, we obtain a sequence of integrands $g_k \in \mathbf{E}(\mathbb{R}^{n\times n}_{\mathrm{sym}})$ to which we may apply the Reshetnyak–type continuity result from Lemma 10.10 from the appendix. This yields

$$\begin{aligned}\liminf_{j\to\infty} \mathscr{F}[u_j] &\geqslant \liminf_{j\to\infty} \int_\Omega g_k(\mathscr{E}u_j)\,\mathrm{d}x + \int_\Omega g_{k,\mathrm{up}}^{\infty}\Big(\frac{\mathrm{dE}^s u_j}{\mathrm{d}|\,\mathrm{E}^s u_j|}\Big)\,\mathrm{d}|\,\mathrm{E}^s u_j| \\ &= \int_\Omega g_k(\mathscr{E}u)\,\mathrm{d}x + \int_\Omega g_{k,\mathrm{up}}^{\infty}\Big(\frac{\mathrm{dE}^s u}{\mathrm{d}|\,\mathrm{E}^s u|}\Big)\,\mathrm{d}|\,\mathrm{E}^s u|.\end{aligned}$$

Since the left hand side does not depend on k, we may pass to the supremum on the right hand side to obtain by monotone convergence

$$\int_\Omega f(\mathscr{E}u)\,\mathrm{d}x + \int_\Omega h_f^{\infty}\Big(\frac{\mathrm{dE}^s u}{\mathrm{d}|\,\mathrm{E}^s u|}\Big)\,\mathrm{d}|\,\mathrm{E}^s u| \leqslant \liminf_{j\to\infty} \mathscr{F}[u_j]. \tag{8.9}$$

An analogous argument with $g = -f$ consequently yields

$$\limsup_{j\to\infty} \mathscr{F}[u_j] \leqslant \int_\Omega f(\mathscr{E}u)\,\mathrm{d}x - \int_\Omega h_{-f}^{\infty}\Big(\frac{\mathrm{dE}^s u}{\mathrm{d}|\,\mathrm{E}^s u|}\Big)\,\mathrm{d}|\,\mathrm{E}^s u|. \tag{8.10}$$

Our aim is to show that

$$f_{\mathrm{up}}^{\infty}(\xi) = h_f(\xi) = -h_{-f}(\xi) \qquad \text{for all } \xi \in \mathbb{R}^n \odot \mathbb{R}^n, \tag{8.11}$$

with the symmetric rank–one cone $\mathbb{R}^n \odot \mathbb{R}^n$, i.e., all those $\xi \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$ which can be written as $\xi = a \odot b$ for some $a, b \in \mathbb{R}^n$. To see that this implies the claim indeed, note that by (4.12) ff., $\frac{\mathrm{dE}^s u}{\mathrm{d}|\,\mathrm{E}^s u|} \in \mathbb{R}^n \odot \mathbb{R}^n$ $|\,\mathrm{E}^s u|$–a.e., and hence we obtain by (8.11) in conjunction with (8.9), (8.10) that

$$\begin{aligned}\limsup_{j\to\infty} \mathscr{F}[u_j] &\overset{(8.10)}{\leqslant} \int_\Omega f(\mathscr{E}u)\,\mathrm{d}x - \int_\Omega h_{-f}^{\infty}\Big(\frac{\mathrm{dE}^s u}{\mathrm{d}|\,\mathrm{E}^s u|}\Big)\,\mathrm{d}|\,\mathrm{E}^s u| \\ &\overset{(8.11)}{=} \int_\Omega f(\mathscr{E}u)\,\mathrm{d}x + \int_\Omega h_f^{\infty}\Big(\frac{\mathrm{dE}^s u}{\mathrm{d}|\,\mathrm{E}^s u|}\Big)\,\mathrm{d}|\,\mathrm{E}^s u| \\ &\overset{(8.9)}{\leqslant} \liminf_{j\to\infty} \mathscr{F}[u_j],\end{aligned}$$

and hence the lim inf and lim sup must be a limit indeed and the claim follows. Toward this aim, we note that

$$-h_{-f}(\xi) = \limsup_{\substack{t\to\infty\\ \xi'\to\xi}} \frac{f(t\xi')}{t}, \qquad \xi \in \mathbb{R}^{n\times n}_{\mathrm{sym}}.$$

Now note that

$$\frac{f(t\xi')}{t} = \frac{f(t\xi') - f(t\xi)}{t} + \frac{f(t\xi)}{t} =: \mathbf{I}_t + \mathbf{II}_t.$$

Since symmetric rank–one convex functions of linear growth are Lipschitz on $\mathbb{R}^{n\times n}_{\mathrm{sym}}$, we immediately find $|\mathbf{I}_t| \leqslant C|\xi' - \xi| \to 0$ as $\xi' \to \xi$. On the other hand, we have $\mathbf{II}_t \to f^{\infty}_{\mathrm{up}}(\xi) = f^{\infty}(\xi)$ as $t \to \infty$ by definition and ξ is of symmetric rank–one structure. The proof is complete. □

Based on the previous lemma, we then obtain for the sequence $(u_k) \subset \mathcal{D}_{u_0}$ that $w_{u_k} \to w_u$ symmetric area–strictly on $\widetilde{\Omega}$, too, and therefore

$$\overline{\mathfrak{F}}_{\widetilde{\Omega}}[w_{u_k}] \to \overline{\mathfrak{F}}_{\widetilde{\Omega}}[w_u] \qquad \text{as } k \to \infty.$$

Now it suffices to note that $u \in \mathrm{BD}(\Omega)$ is a BD–minimiser if and only if w_u is a minimiser for $\overline{\mathfrak{F}}_{\widetilde{\Omega}}$ over all w_v for $v \in \mathrm{BD}(\Omega)$ to conclude. By the direct method and Theorem 8.6, it is now straightforward to establish the existence of a generalised minimiser provided f is suitably coercive, e.g., is bounded below. Finally, we have

$$\inf_{\mathcal{D}_{u_0}} \mathfrak{F} = \min_{\mathrm{BD}(\Omega)} \widetilde{\overline{\mathfrak{F}}}_{u_0}.$$

Here, '$\geqslant$' is obvious as $\widetilde{\overline{\mathfrak{F}}}_{u_0}|_{\mathcal{D}_{u_0}} = \mathfrak{F}|_{\mathcal{D}_{u_0}}$ and '$\leqslant$' follows as in the proof of Theorem 4.23 (ii), now using Lemma 8.7 from above.

We conclude this section by recalling the approach of Zhang [249] to briefly justify that the class of non–convex integrands specified by Definition 8.1 is non–empty. The rough outline in the *full* gradient setting is this: Pick $A, B \in \mathbb{R}^{N\times n}$ which are not rank–one connected, that is, $\mathrm{rk}(A-B) \geqslant 2$. Then the function $\varphi\colon \mathbb{R}^{N\times n} \ni \xi \mapsto \mathrm{dist}(\xi, \{A, B\})$ is not convex. If we accordingly pass to the quasiconvexification $Q\varphi$ of φ, i.e., the largest quasiconvex function below φ, then $Q\varphi$ can be shown to be quasiconvex, not convex, and of linear growth.

In the symmetric–convex setting, we choose $A, B \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$ which are not symmetric–rank–one connected (that is, there exist no two vectors $a, b \in \mathbb{R}^n$ for which there holds $A - B = a \odot b$) and then proceed as above with the obvious modifications. This yields a symmetric quasiconvex function $Q\varphi\colon \mathbb{R}^{n\times n}_{\mathrm{sym}} \to \mathbb{R}$. We then choose a monotone and convex C^2–function $h\colon \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ for which the function $t \mapsto h(t) - \ell V(t) = h(t) - \ell(\sqrt{1+|t|^2} - 1)$ is convex and eventually define $f := h \circ (Q\varphi)$. Then it is not too difficult to demonstrate that f is not convex, and since for all $A \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$ and all $\psi \in \mathrm{W}^{1,\infty}_0(Q; \mathbb{R}^n)$ we have

$$\begin{aligned} f(A) = h((Q\varphi)(A)) &\leqslant h\Big(\int_Q (Q\varphi)(A + \varepsilon(\psi))\,\mathrm{d}x\Big) \\ &\leqslant \int_Q (h \circ (Q\varphi))(A + \varepsilon(\psi))\,\mathrm{d}x = \int_Q f(A + \varepsilon(\psi))\,\mathrm{d}x, \end{aligned}$$

where we used monotonicity of h and symmetric quasiconvexity of $Q\varphi$ in the second and Jensen's inequality for the convex function h in the third step. To achieve the required regularity, it suffices to mollify f via convolution with a standard mollifier.

8.3. Obstructions and Comparison with other Strategies

Before we proceed to the main part of this chapter and give the proofs of Theorems 8.3 and 8.4 from above, we wish to point out the main difficulties in establishing these results and reason our choice of the particular linearisation strategy.

Often, partial regularity proofs employ a *higher integrability* argument. In the convex setting (cp. Chapter 5), it is possible to obtain higher integrability of the gradients of minima by variants of Nirenberg & Shiffman's difference quotient technique. This has been outlined to some extent in Chapters 5 and 6. However, we recall that for this method not only positive definiteness of the integrands' Hessians f'' but also quantifiable notions of convexity are required. From a conceptual view and also recalling Remark 5.42, this is ruled out for quasiconvex integrands as the Hessian is *not* positive definite everywhere.

A different option for establishing the higher integrability is the use of Gehring's Lemma in conjunction with a Caccioppoli–type inequality. We have argued in Chapter 5.6 for the *convex* situation that the Caccioppoli inequality in itself cannot yield higher integrability of the (symmetric) gradients. In the setting of Chapter 5, this was not of primary importance as a finite difference approach still could yield some higher integrability, but as we reasoned above, this is not available in the quasiconvex situation.

From a heuristic viewpoint, the Caccioppoli inequalities rule out *oscillations* of minima. If we are in the p–growth setting with $1 < p < \infty$, then we follow the procedure as sketched in Chapter 5.6 to combine the Caccioppoli inequality with the Sobolev inequality to obtain a *reverse* Hölder inequality. This, in turn, rules out too large singularities. That is to say, for minima of suitably elliptic variational integrals, control over oscillations gives control over L^p–concentrations, $p > 1$. As the example as given in Chapter 5.6 shows, in the linear growth regime these two phenomena seem to be decoupled instead.

Following the study of Faraco & Kristensen [103], the key to the regularity of minima is compactness, the latter usually being an implication of the Caccioppoli inequality. In some sense, this is also the case in our situation. However, in our setting, the compactness does *not* stem from uniform higher integrability assertions on particular minimising sequences as is ruled out by the above discussion. Consequently, we need to choose a method of proof that overcomes the explicit use of higher integrability, and this is offered by the $\mathbb{A}$–harmonic approximation technique as briefly sketched above. To conclude with, we wish to elaborate in more detail where other proof strategies seem to cause adamant difficulties.

Following the terminology of [33, Chpt. 4], the $\mathbb{A}$–harmonic approximation method is a so–called *direct*[4] approach to the regularity, i.e., does not rely in a crucial way on indirect arguments that use proof by contradiction. Put differently, it utilises a minimum of compactness principles and thus is a priori well–suited for our aims in this chapter. To explain the difficulties to be encountered in different approaches, we briefly sketch three other approaches and indicate where their application is not clear to us to yield the partial regularity.

- *The method of blow–up.* This indirect approach was established by De Giorgi [74] and Almgren [12] in their study of regularity of minimal surfaces first. Here we assume towards a contradiction that the excess decay does *not* hold. Based on this assumption, we modify and, in the terminology of [164], *zoom into* the integrand to obtain a sequence of mappings defined on the unit ball (the so–called associated *blow–up sequence*) which converges weakly (or in the weak*–sense, respectively) to a limit mapping. This limit mapping then is proven to solve a strongly rank–one convex system and therefore satisfies the decay estimates available for such systems. In this situation, the crucial step consists in upgrading the weak (or weak*–) convergence to strong convergence of a suitable subsequence. This is usually accomplished by a Caccioppoli–type inequality. In fact, converging strongly, up to finitely many members of the blow–up sequence must satisfy a suitable excess decay dictated by that of the limit mapping. This, in turn, yields a contradiction with the assumption on the blow–up sequence to *violate* such a decay. Therefore, the excess decay for the minimiser must hold and an iteration scheme concludes the proof.

  The main problem which we discover in the quasiconvex linear growth setting is

[4] Apart from a possibly indirect proof of the $\mathbb{A}$–harmonic approximation lemma, cp. [33, Chpt. 4]. However, note that there is also a direct proof of the $\mathbb{A}$–harmonic approximation lemma due to Diening et al., cp. [82], making the $\mathbb{A}$–harmonic approximation method a fully direct strategy.

this: At the relevant stage of the proof, the blow–up sequence can only be proven to converge to a limit mapping in the weak*–sense and whose (symmetric) gradient merely might be a measure. Since the Caccioppoli–inequality in itself cannot yield local higher integrability of the minimiser, it is not clear to us how to deduce that the aforementioned weak*–limit indeed qualifies as a $\mathrm{W}^{1,1}$–map, cp. [176]. As a consequence, we are not in position to deduce that the weak*–limit satisfies a linear strongly rank–one convex equation.

- *The Anzellotti–Giaquinta–type comparison strategy.* This approach, which is based on comparison with mollifications, has been explained in detail in Chapter 7. For completeness, we recall that the decisive point for the proof scheme not to work in the quasiconvex situation is Jensen's inequality, and it is not clear to us how to fruitfully remove this obstruction.

- *The p– and φ–harmonic approximations.* The linearisation strategies pursued in the previous methods work because of the strong Legendre–Hadamard condition of f, i.e., f'' is positive definite with respect to directions contained in the (symmetric) rank–one cone. If, however, the second derivatives degenerate at zero, say, then the comparison with linear systems is not available anymore. In case the second derivatives degenerate in a quantitative precise way and for instance behave like those of the p–Dirichlet integrand (or, in the context of Orlicz functions, the φ–Dirichlet integrand), then still a comparison with p–harmonic mappings can be established. By [237, 238], the latter also enjoy good decay properties. This method is known as p– or φ–harmonic approximation technique, cp. [93, 94] and [80, 81, 82]. Since we, however, will permanently assume strong Legendre–Hadamard conditions on the integrands, there will be no need for this method; however, compare Remark 7.26.

8.4. Auxiliary Estimates

In this preparatory section we gather various auxiliary estimates that will play important roles in the proofs of the main theorems of this chapter. This encompasses estimates on shifted integrands, two iteration lemmas and some linear comparison estimates. The latter are designed for the application to quasiconvex or rank–one convex problems. In this respect, they need to be contrasted with the corresponding estimates for convex problems as given Chapter 7.3, however, here our focus is on different aspects of the decay of solutions.

8.4.1. Shifted Integrands and the V–function In order to implement the linearisation strategy in the main part of the partial regularity proof, we introduce for $f\colon \mathbb{R}^{n\times n}_{\mathrm{sym}} \to \mathbb{R}$ satisfying (H4)–(H6) from Section 8.1.4 and $w \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$ the integrands

$$\begin{aligned} f_w(\xi) &:= f(\xi+w) - f(w) - \langle f'(w),\xi\rangle \\ &\Big(= \int_0^1 (1-t)\langle f''(w+t\xi)\xi,\xi\rangle\,\mathrm{d}t\Big), \qquad \xi\in\mathbb{R}^{n\times n}_{\mathrm{sym}}, \end{aligned} \tag{8.12}$$

and remind the reader of the function $V\colon \mathbb{R}^{n\times n}_{\mathrm{sym}} \to \mathbb{R}$ defined by $V(\xi) := \sqrt{1+|\xi|^2}-1$ for $\xi \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$.

Lemma 8.8 (Auxiliary Assertions, [134, Lems. 3.1, 3.2])**.** *For all $w, z \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$ we have (with an obvious interpretation for $w = 0$ or $z = 0$)*

$$\langle V''(w)z,z\rangle = \frac{1+|w|^2-|w|^2\Big(\frac{w}{|w|}\cdot\frac{z}{|z|}\Big)^2}{(1+|w|^2)^{\frac{3}{2}}} \quad \text{and} \quad V_w(z) \geqslant \frac{1}{16}\frac{V(z)}{(1+|w|^2)^{\frac{3}{2}}}. \tag{8.13}$$

Moreover, for each $m > 0$ there exists a constant $c = c(m) \in [1, \infty)$ with the following properties: If $f\colon \mathbb{R}^{n\times n}_{\mathrm{sym}} \to \mathbb{R}$ satisfies hypotheses (H4)–(H6), *then for all $z \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$ and all $w \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$ with $|w| \leqslant m$ there holds*

(i) $|f_w(z)| \leqslant cLV(z)$,

(ii) $|f_w'(z)| \leqslant cL\min\{|z|, 1\}$,

(iii) $|f_w''(0)z - f_w'(z)| \leqslant cLV(z)$.

and for all $w \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$ we have for all open balls $\mathrm{B} \subset \mathbb{R}^n$

$$\frac{\ell}{c}\int_{\mathrm{B}} V(\boldsymbol{\varepsilon}(\varphi))\,\mathrm{d}x \leqslant \int_{\mathrm{B}} f_w(\boldsymbol{\varepsilon}(\varphi))\,\mathrm{d}x \qquad \text{for all } \varphi \in \mathrm{LD}_0(\mathrm{B}). \tag{8.14}$$

Proof. All assertions apart from (8.14) are taken from [134, Lems. 3.1, 3.2]. To see (8.14), let $\mathrm{B} \subset \mathbb{R}^n$ be an open ball and let $\varphi \in \mathrm{LD}_0(\mathrm{B})$ be arbitrary. From (H6) and hence the ℓ–strong symmetric quasiconvexity of f we deduce

$$\begin{aligned}
\int_{\mathrm{B}} \frac{V(\boldsymbol{\varepsilon}(\varphi))}{(1+|w|^2)^{\frac{3}{2}}}\,\mathrm{d}x &\overset{(8.13)}{\leqslant} 16\int_{\mathrm{B}} V_w(\boldsymbol{\varepsilon}(\varphi))\,\mathrm{d}x \\
&= 16\int_{\mathrm{B}} V(w+\boldsymbol{\varepsilon}(\varphi)) - V(w)\,\mathrm{d}x - 16\underbrace{\int_{\mathrm{B}} \langle V'(w), \boldsymbol{\varepsilon}(\varphi)\rangle\,\mathrm{d}x}_{=0} \\
&\overset{(\mathrm{H6})}{\leqslant} \frac{16}{\ell}\int_{\mathrm{B}} f(w+\boldsymbol{\varepsilon}(\varphi)) - f(w)\,\mathrm{d}x - \frac{16}{\ell}\underbrace{\int_{\mathrm{B}} \langle f'(w), \boldsymbol{\varepsilon}(\varphi)\rangle\,\mathrm{d}x}_{=0} \\
&= \frac{16}{\ell}\int_{\mathrm{B}} f_w(\boldsymbol{\varepsilon}(\varphi))\,\mathrm{d}x.
\end{aligned}$$

Here the underbraced integrals vanish by the Gauss–Green theorem and the fact that $\varphi|_{\partial\mathrm{B}} = 0$. We then multiply the inequality with $(1+|w|^2)^{\frac{3}{2}}$ and use $|w| \leqslant m$ to conclude. The proof is complete. ∎

Also, the following lemma is a reformulation of Lemma 7.9 which is convenient for our purposes.

Lemma 8.9 (Estimates on the V–function)**.** *There exist constants $c_1, c_2, c_3 > 0$ such that there holds*

$$c_1\min\{|z|, |z|^2\} \leqslant V(z) \leqslant \min\{|z|, |z|^2\}, \tag{8.15}$$

$$V(\lambda z) \leqslant c_2\lambda^2 V(z) \qquad \text{for all } \lambda \geqslant 1, \tag{8.16}$$

$$V(z+w) \leqslant c_3(V(z)+V(w)) \qquad \text{for all } z, w \in \mathbb{R}^m. \tag{8.17}$$

8.4.2. Iteration Lemmas We now record two iterations lemmas that are instrumental in the derivation of Caccioppoli–type inequalities.

Lemma 8.10. *Let $0 < \theta < 1$, $A \geqslant 1$ and $R > 0$. Assume that $\Phi, \Psi\colon (0, R] \to \mathbb{R}$ are non–negative functions, where Φ is bounded, Ψ is decreasing with $\Psi(\frac{t}{2}) \leqslant A\Psi(t)$ for all $t \in (0, R]$ and that $\Phi(r) \leqslant \theta\Phi(s) + \Psi(s-r)$ holds for all $r, s \in [\frac{R}{2}, R]$ with $r < s$. Then there exists a constant $C = C(\theta, A) > 0$ such that*

$$\Phi\Big(\frac{R}{2}\Big) \leqslant C\Psi(R). \tag{8.18}$$

The preceding lemma can be found in [134] but appears as a straightforward modification of the argument outlined in Giusti's monograph [128, Lem. 6.1].

Lemma 8.11 ([121, Chpt. 5, Lem. 3.1])**.** *Let $r>0$ and let $g\colon[\frac{r}{2},r]\to[0,\infty)$ be a bounded function such that for all $\frac{r}{2}<t<s<r$ there holds*

$$g(t)\leqslant\theta g(s)+\frac{A}{(s-t)^a}$$

for some non–negative constants $A,a,\theta>0$ such that $0<\theta<1$. Then there exists $c=c(\theta)>0$ such that

$$g\Big(\frac{r}{2}\Big)\leqslant c(\theta)\frac{A}{r^a}.$$

8.4.3. Linear Comparison Estimates: A Lemma by Maz'ya and Shaposhnikova, revisited In what follows, we call a bilinear form $\mathbb{A}\colon\mathbb{R}^{n\times n}_{\mathrm{sym}}\times\mathbb{R}^{n\times n}_{\mathrm{sym}}\to\mathbb{R}$ *strongly Legendre–Hadamard* (or *strongly symmetric rank–one convex*) provided there exists $\lambda>0$ such that for all $\xi,\eta\in\mathbb{R}^n$ there holds

$$\lambda|\xi\odot\eta|^2\leqslant\mathbb{A}[\xi\odot\eta,\xi\odot\eta]. \tag{8.19}$$

The aim of this section is to transfer the results of [171], which are originally stated for strong Legendre–Hadamard operators defined on $\mathbb{R}^{N\times n}$ and hereafter strongly rank–one convex, to the symmetric setting. Here, we recall that a bilinear form $\widetilde{\mathbb{A}}\colon\mathbb{R}^{N\times n}\times\mathbb{R}^{N\times n}\to\mathbb{R}$ is *strongly Legendre–Hadamard* if and only if there exists a constant $\nu>0$ such that

$$\nu|\xi\otimes\eta|^2\leqslant\widetilde{\mathbb{A}}[\xi\otimes\eta,\xi\otimes\eta]\qquad\text{for all }\eta\in\mathbb{R}^N,\ \xi\in\mathbb{R}^n. \tag{8.20}$$

To reduce to the setting considered in [171], let $\mathbb{A}\colon\mathbb{R}^{n\times n}_{\mathrm{sym}}\times\mathbb{R}^{n\times n}_{\mathrm{sym}}\to\mathbb{R}$ be strongly Legendre–Hadamard. We define a bilinear form $\widetilde{\mathbb{A}}\colon\mathbb{R}^{n\times n}\times\mathbb{R}^{n\times n}\to\mathbb{R}$ by setting

$$\widetilde{\mathbb{A}}[\eta,\xi]:=\mathbb{A}[\eta^{\mathrm{sym}},\xi^{\mathrm{sym}}],\qquad\eta,\xi\in\mathbb{R}^{n\times n}, \tag{8.21}$$

where for $A\in\mathbb{R}^{n\times n}$, $A^{\mathrm{sym}}:=\frac{1}{2}(A+A^{\mathsf{T}})$ denotes its symmetric part. We firstly have the following reduction lemma.

Lemma 8.12. *There exists a constant $\Lambda>0$ such that for all $a,b\in\mathbb{R}^n$ there holds*

$$\Lambda|a|\,|b|\leqslant|a\odot b|\leqslant|a\otimes b|\leqslant\Lambda^{-1}|a|\,|b|. \tag{8.22}$$

Proof. We argue by Fourier multipliers and henceforth denote $\boldsymbol{\varepsilon}[\xi]$ the symbol of the symmetric gradient operator evaluated at the Fourier variable $\xi\in\mathbb{R}^n$. The second and third inequality are clear. For the first inequality, it is not restrictive to assume $a\neq0$ so that, by ellipticity of $\boldsymbol{\varepsilon}$, $\boldsymbol{\varepsilon}^*[a]\boldsymbol{\varepsilon}[a]$ is invertible. Hence,

$$|a|b=|a|(\boldsymbol{\varepsilon}^*[a]\boldsymbol{\varepsilon}[a])^{-1}\boldsymbol{\varepsilon}^*[a]a\odot b. \tag{8.23}$$

Since the map $\mathbb{R}^n\setminus\{0\}\ni\xi\mapsto|\xi|(\boldsymbol{\varepsilon}^*[\xi]\boldsymbol{\varepsilon}[\xi])^{-1}\boldsymbol{\varepsilon}^*[\xi]\in\mathscr{L}(\mathbb{R}^n;\mathbb{R}^{n\times(n\times n)})$ is homogeneous of degree zero, it is bounded. Thus, if we put $c_1^{-1}:=\sup_{\xi\neq0}||\xi|(\boldsymbol{\varepsilon}^*[\xi]\mathbb{A}[\xi])^{-1}\boldsymbol{\varepsilon}^*[\xi]|$, then we obtain by (8.23) $|a|\,|b|\leqslant c_1^{-1}|a\odot b|$, and the proof of (8.22) is complete. □

Then we have for all $\xi,\eta\in\mathbb{R}^n$

$$\widetilde{\mathbb{A}}[\xi\otimes\eta,\xi\otimes\eta]=\mathbb{A}[\xi\odot\eta,\xi\odot\eta]\geqslant\lambda|\xi\odot\eta|^2\geqslant\lambda|\xi|^2|\eta|^2\overset{(8.22)}{\geqslant}\lambda\Lambda|a\otimes b|^2.$$

So $\widetilde{\mathbb{A}}$ is strongly Legendre–Hadamard on $\mathbb{R}^{n\times n}$. Moreover, note that for all $\varphi\in\mathrm{C}^1_c(\Omega;\mathbb{R}^n)$ there holds

$$\begin{aligned}-\int_\Omega\langle\operatorname{div}(\widetilde{\mathbb{A}}\,\mathrm{D}u),\varphi\rangle\,\mathrm{d}x&=\int_\Omega\langle\widetilde{\mathbb{A}}\,\mathrm{D}u,\mathrm{D}\varphi\rangle\,\mathrm{d}x\\&=\int_\Omega\langle\mathbb{A}\boldsymbol{\varepsilon}(u)),\boldsymbol{\varepsilon}(\varphi)\rangle\,\mathrm{d}x=-\int_\Omega\langle\operatorname{div}(\widetilde{\mathbb{A}}\boldsymbol{\varepsilon}(u)),\varphi\rangle\,\mathrm{d}x.\end{aligned}$$

We shall frequently refer to this as *reduction principle* for obtaining strong Legendre–Hadamard systems from strong symmetric Legendre–Hadamard systems. Based on this principle, we can now turn to the following result.

Theorem 8.13. *Let $\mathbb{A}$ be a symmetric strongly Legendre–Hadamard bilinear form on $\mathbb{R}^{n\times n}_{\mathrm{sym}}$ and let $Lv := -\operatorname{div}(\mathbb{A}\varepsilon(v))$. Then for each $k \in \mathbb{N}$, $1 < q < \infty$ and any open ball $\mathrm{B} \subset \mathbb{R}^n$, the mapping*

$$\Phi\colon\ \mathrm{W}^{k,q}(\mathrm{B};\mathbb{R}^n) \ni u \mapsto (L(u), \mathrm{Tr}_{\partial\mathrm{B}}\, u) \in \mathrm{W}^{k-2,q}(\mathrm{B};\mathbb{R}^n) \times \mathrm{W}^{k-\frac{1}{q},q}(\partial\mathrm{B};\mathbb{R}^n) \tag{8.24}$$

is a topologically linear isomorphism.

The previous theorem is a consequence of a very general statement on layer potentials due to Maz'ya & Shaposhnikova [171, Lem. 3.2], and we briefly pause to fit Theorem 8.13 into its framework. Let $\Omega \subset \mathbb{R}^n$ be an open and bounded Lipschitz domain. For future reference, we hereafter assume that for every boundary point $x_0 \in \partial\Omega$ there exists a neighbourhood U and a Lipschitz function $g\colon \mathbb{R}^{n-1} \to \mathbb{R}$ such that

$$U \cap \Omega = U \cap \{(x, y)\colon\ x \in \mathbb{R}^{n-1},\ y > g(x)\}. \tag{8.25}$$

Moreover, let $1 \leqslant p < \infty$, $0 < \alpha < \infty$ and $k \in \mathbb{N}$. We then define the *weighted Sobolev space* $\mathrm{W}^{k,\alpha}_p(\Omega;\mathbb{R}^N)$ as the linear space of all $u \in \mathrm{L}^p(\Omega;\mathbb{R}^N)$ which are k–times weakly differentiable and for which the norm

$$\|u\|_{\mathrm{W}^{k,\alpha}_p(\Omega;\mathbb{R}^N)} := \|u\|_{\mathrm{L}^p(\Omega;\mathbb{R}^N)} + \Big(\int_\Omega \mathrm{dist}^{p\alpha}(x,\partial\Omega)|\nabla^k u(x)|^p\,\mathrm{d}x\Big)^{\frac{1}{p}}$$

is finite. As usual, $\mathring{\mathrm{W}}^{k,\alpha}_p(\Omega;\mathbb{R}^N)$ denotes the completion of $\mathrm{C}^\infty_c(\Omega;\mathbb{R}^N)$ with respect to the $\mathrm{W}^{k,\alpha}_p(\Omega;\mathbb{R}^N)$–norm. Also, we denote $\mathrm{W}^{-k,\alpha}_p(\Omega;\mathbb{R}^N) := (\mathring{\mathrm{W}}^{k,\alpha}_p)'(\Omega;\mathbb{R}^N)$. We then record the following proposition.

Proposition 8.14 ([171, Lem. 3.2])**.** *Let $1 < p < \infty$ and let $0 < \alpha + \frac{1}{p} < 1$. Then, for any open Lipschitz domain in $\mathbb{R}^n$ for which the Lipschitz constants of the functions g do not exceed a sufficiently small constant, and any strongly Legendre–Hadamard bilinear form $\mathbb{A}\colon \mathbb{R}^{N\times n}\times\mathbb{R}^{N\times n} \to \mathbb{R}_{\geqslant 0}$, the mapping*

$$\mathrm{W}^{1,\alpha}_p(\mathrm{B};\mathbb{R}^N) \ni u \mapsto (-\operatorname{div}(\mathbb{A}\nabla u), \mathrm{Tr}_{\partial\mathrm{B}}\, u) \in \mathrm{W}^{-1,\alpha}_p(\Omega;\mathbb{R}^N) \times \mathrm{W}^{1-\alpha-\frac{1}{p},p}(\partial\mathrm{B};\mathbb{R}^N)$$

is a topologically linear isomorphism.

Some comments are in order. Actually, [171, Lem. 3.2] is stated for the Laplacian instead of the operator $u \mapsto \operatorname{div}(\mathbb{A}\nabla u)$, but according to the remark on page 106 of [171], the previous result is also available for any strongly Legendre–Hadamard operator. This equally applies to Proposition 8.15 below. Also, even though $\mathbb{A}$ is defined as bilinear form, we may identify it with its representation matrix $\mathbb{A} \in \mathbb{R}^{(N\times n)\times(N\times n)}$ and so make sense of the expression $\operatorname{div}(\mathbb{A}\nabla u)$. Finally, notice that every ball satisfies the hypotheses of the proposition as its boundary is of class C^∞. Hence for every boundary point we can find a neighbourhood in which the corresponding Lipschitz constant is suitably small, and then use a compactness argument to conclude.

Now, based on our reduction principles for symmetric strong Legendre–Hadamard systems, we obtain Theorem 8.13 for $k = 1$ and $\alpha = 0$ (note that if $\alpha = 0$, then the Sobolev spaces $\mathrm{W}^{k,\alpha}_p$ reduce to the usual Sobolev spaces $\mathrm{W}^{k,p}$). The following proposition is originally stated for domains with boundaries of classes $\mathrm{W}^{\ell,p}$ or $\mathrm{MW}^{\ell,p}$. A boundary $\partial\Omega$ is said to be of class $\mathrm{W}^{\ell,p}$ (for $p(\ell-1) > n-1$) provided any function g appearing in (8.25) belongs to $\mathrm{W}^{\ell,p}(\mathbb{R}^{n-1})$. On the other hand, the boundary is said to be of class M^ℓ_p (for $p(\ell-1) \leqslant n-1$) provided any function g appearing in (8.25) belongs to $\mathrm{C}^{0,1}(\mathbb{R}^{n-1})$ and satisfies $\partial_j g \in \mathrm{MW}^{\ell-1,p}(\mathbb{R}^{n-1})$ for all $j=1,...,n-1$. The so–called *Sobolev multiplier class* $\mathrm{MW}^{\lambda,p}(\mathbb{R}^{n-1})$ (for $\lambda>0$ and $1\leqslant p<\infty$) is defined as the collection of all $\gamma\colon\mathbb{R}^{n-1}\to\mathbb{R}$ for which $v:=\gamma u$ is well–defined for every $u\in\mathrm{W}^{\lambda,p}(\mathbb{R}^{n-1})$, belongs to $\mathrm{W}^{\lambda,p}(\mathbb{R}^{n-1})$ and the norm $\|\gamma\|_{\mathrm{MW}^{\lambda,p}(\mathbb{R}^{n-1})}:=\sup\{\|\gamma u\|_{\mathrm{W}^{\lambda,p}(\mathbb{R}^{n-1})}\colon\ \|u\|_{\mathrm{W}^{\lambda,p}(\mathbb{R}^{n-1})}\leqslant 1\}$ is finite; see [172, Chpt. 2.1] for a detailled treatment of such spaces. Moreover, the class $M_p^\ell(\delta)$ is defined for $\delta>0$ provided any function g appearing in (8.25) satisfies $\|\partial_j g\|_{\mathrm{MW}_p^\ell(\mathbb{R}^{n-1})}\leqslant\delta$. For it is only of relevance that C^∞–domains are both $\mathrm{W}^{\ell,p}$– and $\mathrm{MW}^{\ell,p}$–domains for all valid choices of $\ell>0$ and $1\leqslant p<\infty$.

Proposition 8.15 ([171, Thm. 3.1])**.** *Let $1<p<\infty$ and let $\alpha=1-\{\ell\}-\frac{1}{p}$, where $\ell>1$ is non–integer and $\{\ell\}$ denotes non–integer part, i.e., $\ell:=\ell-\lfloor\ell\rfloor$. Suppose that $\partial\Omega$ is of class $\mathrm{W}^{\ell,p}$ for $p(\ell-1)>n-1$ and $\partial\Omega\in M_p^\ell(\delta)$ for some $\delta=\delta(n,p,\ell)$ if $p(\ell-1)\leqslant n-1$. Moreover, let $\mathbb{A}\colon\mathbb{R}^{N\times n}\times\mathbb{R}^{N\times n}\to\mathbb{R}_{\geqslant 0}$ be a strongly Legendre–Hadamard bilinear form.*

Then the mapping

$$\mathrm{W}_p^{\lfloor\ell\rfloor+1,\alpha}(\Omega;\mathbb{R}^N)\ni u\mapsto(-\operatorname{div}(\mathbb{A}\nabla u),\mathrm{Tr}_{\partial\mathrm{B}}\,u)\in\mathrm{W}_p^{\lfloor\ell\rfloor-1,\alpha}(\Omega;\mathbb{R}^N)\times\mathrm{W}^{\ell,p}(\partial\mathrm{B};\mathbb{R}^N)$$

is a topologically linear isomorphism.

With this theorem at our disposal and the reduction principles outlined above, Theorem 8.13 follows at once.

8.4.4. Linear comparison estimates: Decay Since the previous section provides us with solution properties of strongly (symmetric) rank–one convex systems but no quantitative estimates, we now give the estimates that will be required in the proof of Theorem 8.4. The following Lemma 8.16 of Weyl–type should be well–known. However, we could not find a precise reference in the form needed here and so we provide it for the convenience of the reader.

Lemma 8.16 (Weyl)**.** *Let $\Omega\subset\mathbb{R}^n$ be an open Lipschitz domain and let $\mathbb{A}\colon\mathbb{R}^{n\times n}_{\mathrm{sym}}\times\mathbb{R}^{n\times n}_{\mathrm{sym}}\to\mathbb{R}$ be a symmetric bilinear form which is strongly Legendre–Hadamard on the symmetric rank–one cone, i.e., there exists $\lambda>0$ such that for all $a,b\in\mathbb{R}^n$ there holds*

$$\mathbb{A}[a\odot b,a\odot b]\geqslant\lambda|a|^2|b|^2. \tag{8.26}$$

Suppose that $u\in\mathrm{LD}(\Omega)$ satisfies

$$\int_\Omega\mathbb{A}[\boldsymbol{\varepsilon}(u),\boldsymbol{\varepsilon}(\varphi)]\,\mathrm{d}x=0\qquad\text{for all }\varphi\in\mathrm{C}_0^1(\Omega;\mathbb{R}^n). \tag{8.27}$$

Then the following hold:

(i) *$u\in\mathrm{C}^\infty(\Omega;\mathbb{R}^n)$,*

(ii) *There exists a constant $c=c(n,\lambda,M)>0$ (where $M:=\max_{z\in\mathbb{B}}|\mathbb{A}[z,z]|$ with the unit ball $\mathbb{B}$ in $\mathbb{R}^{n\times n}_{\mathrm{sym}}$) such that for all $A\in\mathbb{R}^{n\times n}_{\mathrm{sym}}$*

$$\sup_{\mathrm{B}(x_0,\frac{R}{2})}|\nabla u-A|+R\sup_{\mathrm{B}(x_0,\frac{R}{2})}|\nabla^2 u|\leqslant C\fint_{\mathrm{B}(x_0,R)}|\nabla u-A|\,\mathrm{d}x. \tag{8.28}$$

Proof. The proof is a modification of [63, Prop. 2.1], taking into account the Legendre–Hadamard condition on the symmetric rank–one cone.

We firstly suppose that $u\in\mathrm{W}^{1,2}_{\mathrm{loc}}(\Omega;\mathbb{R}^n)$. We show $u\in\mathrm{W}^{2,2}_{\mathrm{loc}}(\Omega;\mathbb{R}^n)$ which in turn implies (i): In fact, since with u every $\partial^\beta u$ for $\beta\in\mathbb{N}$ solves (8.27), we may succesively apply the improved Sobolev regularity to each $\partial^\beta u$ and thereby conclude. We firstly claim that for all $\varphi \in \mathrm{C}^1_c(\mathbb{R}^n;\mathbb{R}^n)$ there holds

$$\lambda \int_{\mathbb{R}^n} |\boldsymbol{\varepsilon}(v)|^2 \,\mathrm{d}x \leqslant \int_{\mathbb{R}^n} \mathbb{A}[\boldsymbol{\varepsilon}(v),\boldsymbol{\varepsilon}(v)]\,\mathrm{d}x \qquad \text{for all } v \in \mathscr{S}(\mathbb{R}^n;\mathbb{R}^n). \tag{8.29}$$

By smooth approximation, it then easily follows that (8.29) also holds for $\mathrm{W}^{1,2}_0(\mathbb{R}^n;\mathbb{R}^n)$ and, in particular, all $\mathrm{W}^{1,2}_0(\Omega;\mathbb{R}^n)$ (when elements of $\mathrm{W}^{1,2}_0(\Omega;\mathbb{R}^n)$ are viewn as elements of $\mathrm{W}^{1,2}_0(\mathbb{R}^n;\mathbb{R}^n)$ which are trivially extended to the entire $\mathbb{R}^n$). Assuming (8.29) for the time being, we let $\mathrm{B}' \Subset \mathrm{B}'' \Subset \Omega$ be non–empty balls and choose a cut–off function $\rho \in \mathrm{C}^1_c(\Omega;[0,1])$ which satisfies $\mathbb{1}_{\mathrm{B}'} \leqslant \rho \leqslant \mathbb{1}_{\mathrm{B}''}$. By a smooth approximation argument, $\varphi := \rho^2 u \in \mathrm{W}^{1,2}_0(\Omega;\mathbb{R}^n)$ clearly qualifies as a test map in (8.27) and thus we obtain by rearranging and Young's inequality for some constant $C>0$

$$\int_\Omega \mathbb{A}[\rho\boldsymbol{\varepsilon}(u),\rho\boldsymbol{\varepsilon}(u)]\,\mathrm{d}x \leqslant C\int_\Omega \mathbb{A}[\nabla\rho\odot u,\nabla\rho\odot u]\,\mathrm{d}x. \tag{8.30}$$

Now, by Korn's inequality and (8.29),

$$\begin{aligned}
\int_{\mathbb{R}^n} |\rho\,\mathrm{D}u|^2 + 2\langle\rho\,\mathrm{D}u,\nabla\rho\otimes\mathrm{D}u\rangle + |\nabla\rho\otimes u|^2\,\mathrm{d}x &= \int_{\mathbb{R}^n} |\,\mathrm{D}(\rho u)|^2\,\mathrm{d}x \leqslant C\int_{\mathbb{R}^n} |\boldsymbol{\varepsilon}(\rho u)|^2\,\mathrm{d}x \\
&\leqslant \frac{C}{\lambda}\int_{\mathbb{R}^n} \mathbb{A}[\boldsymbol{\varepsilon}(\rho u),\boldsymbol{\varepsilon}(\rho u)]\,\mathrm{d}x \qquad (\text{by (8.29)}) \\
&\leqslant \frac{C}{\lambda}\int_{\mathbb{R}^n} \mathbb{A}[\rho\boldsymbol{\varepsilon}(u),\rho\boldsymbol{\varepsilon}(u)] + 2\mathbb{A}[\rho\boldsymbol{\varepsilon}(u),\nabla\rho\odot u] + \mathbb{A}[\nabla\rho\odot u,\nabla\rho\odot u]\,\mathrm{d}x \\
&\leqslant \frac{C}{\lambda}\int_{\mathbb{R}^n} \mathbb{A}[\rho\boldsymbol{\varepsilon}(u),\rho\boldsymbol{\varepsilon}(u)] + \mathbb{A}[\nabla\rho\odot u,\nabla\rho\odot u]\,\mathrm{d}x \\
&\leqslant \frac{C}{\lambda}\int_{\mathbb{R}^n} \mathbb{A}[\nabla\rho\odot u,\nabla\rho\odot u]\,\mathrm{d}x
\end{aligned}$$

the penultimate inequality following by Young's inequality, whereas the last inequality is a consequence of (8.30). In consequence, since $|\nabla\rho\otimes u|^2 \geqslant 0$, we obtain after a simple rearrangement, Young's inequality, boundedness of $\mathbb{A}$ and the fact that $|a\odot b| \leqslant |a\otimes b|$ for all $a,b\in\mathbb{R}^n$,

$$\int_{\mathbb{R}^n} |\rho\,\mathrm{D}u|^2\,\mathrm{d}x \leqslant \varepsilon\int_{\mathbb{R}^n} |\rho\,\mathrm{D}u|^2\,\mathrm{d}x + C(\varepsilon)\int_{\mathbb{R}^n} |\nabla\rho\otimes u|^2\,\mathrm{d}x. \tag{8.31}$$

Choosing $\varepsilon>0$ sufficiently small, we may absorb the first term of the right hand to the left hand side. Now let $\beta\in\mathbb{N}^n_0$ with $|\beta|=1$ be arbitrary. Obviously, (8.31) also applies to $\partial^\beta u$ instead of u, and since $\partial^\beta u \in \mathrm{L}^2_{\mathrm{loc}}(\Omega;\mathbb{R}^n)$, we have established $u\in\mathrm{C}^\infty(\Omega;\mathbb{R}^n)$ by the reduction scheme outlined above, and thus (i) follows.

Ad (ii). Now let $x_0\in\Omega$ and $R>0$ with $\mathrm{B}(x_0,R)\Subset\Omega$. Let $\frac{R}{2}<t<s<R$ be arbitrary. Applying the Caccioppoli–type inequality (8.31) to the particular choice of ρ with $\rho\equiv 1$ on $\mathrm{B}(x_0,t)$, $\mathrm{spt}(\rho)\subset\mathrm{B}(x_0,s)$ and $|\nabla\rho|\leqslant C/(s-t)$ yields

$$\int_{\mathrm{B}(x_0,t)} |\,\mathrm{D}u|^2\,\mathrm{d}x \leqslant \frac{c}{(s-t)^2}\int_{\mathrm{B}(x_0,s)} |u|^2\,\mathrm{d}x. \tag{8.32}$$

Now let $m\in\mathbb{N}$ be arbitrary, $\beta=(\beta_1,...,\beta_n)\in\mathbb{N}^n_0$ a multi–index with $|\beta|=m$ and define, for $j=1,...,m$, $\mathrm{B}_{(j)} := \mathrm{B}(x_0,t+\frac{j}{m}(s-t))$. Succesive application of (8.32) consequently yields

$$\begin{aligned}
\int_{\mathrm{B}(x_0,t)} |\partial^\beta u|^2\,\mathrm{d}x &\overset{(8.29)}{\leqslant} \frac{c}{\frac{1}{m^2}(s-t)^2}\int_{\mathrm{B}_{(1)}} |\partial^{\beta^{(1)}}u|^2\,\mathrm{d}x \\
&\overset{(8.29)}{\leqslant} \frac{c^2}{\frac{1}{m^4}(s-t)^4}\int_{\mathrm{B}_{(2)}} |\partial^{\beta^{(2)}}u|^2\,\mathrm{d}x \leqslant (...) \leqslant \frac{c^m m^{2m}}{(s-t)^{2m}}\int_{\mathrm{B}(x_0,s)} |u|^2\,\mathrm{d}x,
\end{aligned}$$

where for each β, $\beta^{(j)}$ denotes a multi–index of length $(m-j)$. Now assume $R=1$ for the moment; we shall scale back later to obtain the relevant statement. So let $\frac{1}{2}<t<s<1$. By Morrey's embedding $\mathrm{W}^{n,2}(\mathrm{B}_t)\hookrightarrow\mathrm{L}^\infty(\mathrm{B}_t)$ we obtain (with a constant $C>0$ independent of u,s,t)

$$\begin{aligned}\|u\|^2_{\mathrm{L}^\infty(\mathrm{B}(x_0,t);\mathbb{R}^n)} &\leqslant C\|u\|^2_{\mathrm{W}^{n,2}(\Omega;\mathbb{R}^n)}\leqslant C\sum_{|\gamma|\leqslant n}\int_{\mathrm{B}(x_0,t)}|\partial^\gamma u|^2\,\mathrm{d}x\\ &\leqslant C(n)\Big(\sum_{|\gamma|\leqslant n}\frac{1}{(s-t)^{2|\gamma|}}\Big)\int_{\mathrm{B}(x_0,s)}|u|^2\,\mathrm{d}x\\ &\leqslant (n+1)c(n)\frac{\|u\|_{\mathrm{L}^\infty(\mathrm{B}(x_0,s);\mathbb{R}^n)}}{(s-t)^{2n}}\int_{\mathrm{B}(x_0,s)}|u|\,\mathrm{d}x\\ &\leqslant \frac{1}{2}\|u\|^2_{\mathrm{L}^\infty(\mathrm{B}(x_0,s);\mathbb{R}^n)}+\frac{\widetilde{C}(n)}{(s-t)^{4n}}\Big(\int_{\mathrm{B}(x_0,s)}|u|\,\mathrm{d}x\Big)^2.\end{aligned}$$

Hence, if we put $g(r):=\|u\|^2_{\mathrm{L}^\infty(\mathrm{B}(x_0,r);\mathbb{R}^n)}$ for $0<r<1$, the preceding inequality becomes

$$g(t)\leqslant\frac{1}{2}g(s)+\frac{C}{(s-t)^{4n}}\Big(\int_{\mathrm{B}(x_0,s)}|u|\,\mathrm{d}x\Big)^2. \tag{8.33}$$

It is obvious that the function g meets the requirements of Lemma 8.11 and thus we obtain a constant $c>0$ such that

$$\sup_{\mathrm{B}(x_0,\frac{1}{2})}|u|^2=g\Big(\frac{1}{2}\Big)\leqslant C\Big(\int_{\mathrm{B}(x_0,1)}|u|\,\mathrm{d}x\Big)^2.$$

Taking the root on both sides of the previous equation, applying it to balls or radii $\frac{r}{2}$ or r, respectively, yields by scaling that there exists a constant $c>0$ independent of u and (x_0,r) such that

$$\sup_{\mathrm{B}(x_0,\frac{r}{2})}|\,\mathrm{D}u|\leqslant c\fint_{\mathrm{B}(x_0,r)}|\,\mathrm{D}u|\,\mathrm{d}x, \tag{8.34}$$

where we again used the fact that with u also $\partial^\gamma u$ for any $\gamma\in\mathbb{N}_0^n$ is a solution of (8.27). It is then not difficult to see that a similar argument yields for $\vartheta>\frac{1}{2}$ that there exists a constant $c(\vartheta)>0$ such that

$$\sup_{\mathrm{B}(x_0,\frac{r}{2})}|\,\mathrm{D}u|\leqslant c(\vartheta)\fint_{\mathrm{B}(x_0,\vartheta r)}|\,\mathrm{D}u|\,\mathrm{d}x. \tag{8.35}$$

We now come to the derivation of (8.28). Working from (8.35), we have by (8.32)

$$\begin{aligned}\sup_{\mathrm{B}(x_0,\frac{r}{2})}|\,\mathrm{D}^2u| &\leqslant c\fint_{\mathrm{B}(x_0,\frac{3}{4}r)}|\,\mathrm{D}^2u|\,\mathrm{d}x\leqslant c\Big(\fint_{\mathrm{B}(x_0,\frac{3}{4}r)}|\,\mathrm{D}^2u|^2\,\mathrm{d}x\Big)^{\frac{1}{2}}\\ &\leqslant\frac{c}{r}\Big(\fint_{\mathrm{B}(x_0,\frac{5}{6}r)}|\,\mathrm{D}u|^2\,\mathrm{d}x\Big)^{\frac{1}{2}}\\ &\leqslant\frac{c}{r}\big(\sup_{\mathrm{B}(x_0,\frac{5}{6}r)}|\,\mathrm{D}u|^{\frac{1}{2}}\big)\Big(\fint_{\mathrm{B}(x_0,\frac{5}{6}r)}|\,\mathrm{D}u|\,\mathrm{d}x\Big)^{\frac{1}{2}}\leqslant\frac{c}{r}\fint_{\mathrm{B}(x_0,r)}|\,\mathrm{D}u|\,\mathrm{d}x.\end{aligned}$$

Multiplying the previous by r, setting $\vartheta=1$ in (8.35) and adding the resulting inequalities yield (8.28) for $z=0$. For the general case, it suffices to note that if $u\in\mathrm{W}^{1,2}_{\mathrm{loc}}(\Omega;\mathbb{R}^n)$ satisfies (8.27), then so does $x\mapsto u(x)-zx$ and so the proof is complete under the $\mathrm{W}^{1,2}_{\mathrm{loc}}(\Omega;\mathbb{R}^n)$–assumption on u and (8.29).

Let us now firstly verify (8.29). To do so, we adapt the proof of [128, Lem. 5.1]. Let $v \in \mathscr{S}(\mathbb{R}^n;\mathbb{R}^n)$ be arbitrary. We recall from Lemma 3.14 that $\widehat{\boldsymbol{\varepsilon}(v)}(\xi) = -\mathrm{i}\,\widehat{v}(\xi)\odot\xi$ for all $\xi\in\mathbb{R}^n$. Suppose for the moment that $\eta = \Re(\eta)+\mathrm{i}\,\Im(\eta)\in\mathbb{C}^n$. Then we obtain by symmetry of $\mathbb{A}$ and noting that $\langle\cdot,\cdot\rangle$ is the *real* inner product that

$$
\begin{aligned}
\langle \mathbb{A}(\eta\odot\xi),\overline{\eta\odot\xi}\rangle &= \langle \mathbb{A}(\eta\odot\xi),\overline{\eta}\odot\xi\rangle \\
&= \langle \mathbb{A}((\Re(\eta)+\mathrm{i}\,\Im(\eta))\odot\xi),(\Re(\eta)-\mathrm{i}\,\Im(\eta))\odot\xi\rangle \\
&= \langle \mathbb{A}\Re(\eta)\odot\xi,\Re(\eta)\odot\xi\rangle + \mathrm{i}\langle\mathbb{A}(\Im(\eta)\odot\xi),(\Re(\eta)\odot\xi)\rangle \\
&\quad - \mathrm{i}\langle\mathbb{A}(\Re(\eta)\odot\xi),(\Im(\eta)\odot\xi)\rangle + \langle\mathbb{A}(\Im(\eta)\odot\xi),(\Im(\eta)\odot\xi)\rangle \\
&= (\langle\mathbb{A}(\Re(\eta)\odot\xi),\Re(\eta)\odot\xi\rangle + \langle\mathbb{A}(\Im(\eta)\odot\xi),(\Im(\eta)\odot\xi)\rangle) \\
&\quad + \mathrm{i}(\langle\mathbb{A}(\Im(\eta)\odot\xi),(\Re(\eta)\odot\xi)\rangle - \langle\mathbb{A}(\Re(\eta)\odot\xi),(\Im(\eta)\odot\xi)\rangle).
\end{aligned} \tag{8.36}
$$

Therefore,

$$
\Re\big(\langle \mathbb{A}(\eta\odot\xi),\overline{\eta\odot\xi}\rangle\big) = (\langle\mathbb{A}(\Re(\eta)\odot\xi),\Re(\eta)\odot\xi\rangle + \langle\mathbb{A}(\Im(\eta)\odot\xi),(\Im(\eta)\odot\xi)\rangle)
$$

and hence, using the strong Legendre–Hadamard condition on the symmetric rank–one cone,

$$
\Re\big(\langle \mathbb{A}(\eta\odot\xi),\overline{\eta\odot\xi}\rangle\big) \geqslant \lambda(|\Re(\eta)|^2+|\Im(\eta)|^2)|\xi|^2 = \lambda|\eta|^2|\xi|^2. \tag{8.37}
$$

By Parseval's identity, we finally obtain

$$
\begin{aligned}
\int_{\mathbb{R}^n}\mathbb{A}[\boldsymbol{\varepsilon}(v),\boldsymbol{\varepsilon}(v)]\,\mathrm{d}x &= \Re\Big(\int_{\mathbb{R}^n}\langle\widehat{\mathbb{A}\boldsymbol{\varepsilon}(v)},\overline{\widehat{\boldsymbol{\varepsilon}(v)}}\rangle\,\mathrm{d}x\Big) \\
&= \Re\Big(\int_{\mathbb{R}^n}\langle\mathbb{A}\widehat{v}(\xi)\odot\xi,\overline{\widehat{v}(\xi)}\odot\xi\rangle\,\mathrm{d}\xi\Big) \\
&\geqslant \lambda\int_{\mathbb{R}^n}|\xi|^2|\widehat{v}(\xi)|^2\,\mathrm{d}x \qquad \text{(by (8.37))} \\
&= \lambda\int_{\mathbb{R}^n}|\widehat{\mathrm{D}v}(\xi)|^2\,\mathrm{d}\xi \\
&= \lambda\int_{\mathbb{R}^n}|\mathrm{D}v|^2\,\mathrm{d}x \qquad \text{(by Parseval)} \\
&\geqslant \lambda\int_{\mathbb{R}^n}|\boldsymbol{\varepsilon}(v)|^2\,\mathrm{d}x,
\end{aligned}
$$

the last estimate being valid as the symmetric gradient can always be estimated against the full gradient. The proof of (8.29) is complete.

To finish the full proof, we need to pass from $\mathrm{W}^{1,2}_{\mathrm{loc}}$ to $\mathrm{W}^{1,1}$ or LD, respectively. To do so, let $u\in\mathrm{LD}(\Omega)$ be a solution of (8.27) and let $\varepsilon>0$ be sufficiently small such that $\Omega_\varepsilon := \{x\in\Omega\colon\ \mathrm{dist}(x,\partial\Omega)>\varepsilon\}\neq\emptyset$. We put $u_\varepsilon := \rho_\varepsilon * u\in\mathrm{W}^{1,2}_{\mathrm{loc}}(\Omega_\varepsilon;\mathbb{R}^n)$. It is easily seen that if $\varphi\in\mathrm{C}^1_c(\Omega_\varepsilon;\mathbb{R}^n)$, then

$$
\int_{\Omega_\varepsilon}\langle\mathbb{A}\boldsymbol{\varepsilon}(u_\varepsilon),\boldsymbol{\varepsilon}(\varphi)\rangle\,\mathrm{d}x = 0.
$$

In particular, for any $\mathrm{B}(x_0,2r)\Subset\Omega_\varepsilon$ we have by (8.32) and (8.28) with a constant $c>0$

$$
\begin{aligned}
\int_{\mathrm{B}(x_0,\frac{r}{2})}|\mathrm{D}u_\varepsilon|^2\,\mathrm{d}x &\leqslant \frac{c}{r^2}\int_{\mathrm{B}(x_0,r)}|u_\varepsilon|^2\,\mathrm{d}x \qquad \text{(by (8.32))} \\
&\leqslant \frac{c}{r^2}\big(\sup_{\mathrm{B}(x_0,r)}|u_\varepsilon|\big)\int_{\mathrm{B}(x_0,r)}|u_\varepsilon|\,\mathrm{d}x \\
&\leqslant \frac{c}{r^{n+2}}\Big(\int_{\mathrm{B}(x_0,2r)}|u_\varepsilon|\,\mathrm{d}x\Big)^2.
\end{aligned}
$$

Fix such $r > 0$. By the previous inequality, we then find by uniform boundedness of $(u_\varepsilon|_{\mathrm{B}(x_0,2r)})$ in $\mathrm{L}^1(\mathrm{B}(x_0,2r);\mathbb{R}^n))$ that

$$\sup_{0<\varepsilon<1}\int_{\mathrm{B}(x_0,\frac{r}{2})}|\,\mathrm{D}u_\varepsilon|^2\,\mathrm{d}x \leqslant C(r).$$

On the other hand, we find by a similar but easier argument that $(u_\varepsilon|_{\mathrm{B}(x_0,\frac{r}{2});\mathbb{R}^n)})$ is uniformly bounded in $\mathrm{L}^\infty(\mathrm{B}(x_0,\frac{r}{2});\mathbb{R}^n)$. Hence, by standard arguments we then deduce that the weak–$\mathrm{W}^{1,2}(\mathrm{B}(x_0,\frac{r}{2});\mathbb{R}^n)$–limit of (u_ε) exists and must coincide with u. Therefore, by arbitrariness of x_0 and sufficiently small $r > 0$, $u \in \mathrm{W}^{1,2}_{\mathrm{loc}}(\Omega;\mathbb{R}^n)$, and we may now apply the first part of the proof to conclude. $\square$

8.4.5. Linear comparison estimates: Hölder Scale As an essential ingredient for concluding the proof of Theorem 8.3, we now supply an existence and regularity result for strong Legendre–Hadamard systems. We henceforth let $A\colon \mathbb{R}^{n\times n}_{\mathrm{sym}} \times \mathbb{R}^{n\times n}_{\mathrm{sym}} \to \mathbb{R}$ be a bilinear form (which we assume to be represented by a matrix $\widetilde{A} \in \mathbb{R}^{n^2\times n^2}$) which is strongly elliptic in the sense of Legendre–Hadamard, i.e., there exists $\ell > 0$ such that for all $a \in \mathbb{R}^N$ and $b \in \mathbb{R}^n$ there holds

$$\ell|a|^2|b|^2 \leqslant A(a\odot b, a\odot b). \tag{8.38}$$

To simplify notation, we do *not* distinguish between A and $\widetilde{A}$. Given $V \in \mathcal{L}^{1,n+\alpha}(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})$ with an open and bounded Lipschitz subset Ω of $\mathbb{R}^n$, the purpose of the present section is to examine the existence and regularity of weak solutions of the system $\mathrm{div}(A\boldsymbol{\varepsilon}(\varphi)) = \mathrm{div}(V)$ in Ω. Here, we say that $\varphi \in \mathrm{W}^{1,2}_0(\Omega;\mathbb{R}^n)$ is a *weak solution* of $\mathrm{div}(A\boldsymbol{\varepsilon}(\varphi)) = \mathrm{div}(V)$ in Ω provided for all $\psi \in \mathrm{W}^{1,2}_0(\Omega;\mathbb{R}^n)$ there holds

$$\int_\Omega \langle A\boldsymbol{\varepsilon}(\varphi),\boldsymbol{\varepsilon}(\psi)\rangle\,\mathrm{d}x = \int_\Omega \langle V,\boldsymbol{\varepsilon}(\psi)\rangle\,\mathrm{d}x. \tag{8.39}$$

Lemma 8.17. *Let Ω be an open and bounded Lipschitz domain in $\mathbb{R}^n$ and let $V \in \mathcal{L}^{1,n+\alpha}(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})$. Moreover, let $A\colon \mathbb{R}^{n\times n}_{\mathrm{sym}} \times \mathbb{R}^{n\times n}_{\mathrm{sym}} \to \mathbb{R}$ be a symmetric bilinear form which is strongly symmetric rank–one convex, that is, A satisfies (8.38). Then there exists a weak solution of the system $\mathrm{div}(A(\nabla\varphi,\cdot)) = \mathrm{div}(V)$ in the sense of (8.39).*

Proof. We define a functional $\mathcal{J}\colon \mathrm{W}^{1,2}_0(\Omega;\mathbb{R}^n) \to \mathbb{R}$ by

$$\mathcal{J}[\psi] := \int_\Omega \frac{1}{2}\langle A\boldsymbol{\varepsilon}(\psi),\boldsymbol{\varepsilon}(\psi)\rangle - \langle V,\psi\rangle\,\mathrm{d}x, \qquad \psi \in \mathrm{W}^{1,2}_0(\Omega;\mathbb{R}^n).$$

Any minimiser of $\mathcal{J}$ over $\mathrm{W}^{1,2}_0(\Omega;\mathbb{R}^n)$ is a weak solution of $\mathrm{div}(A\boldsymbol{\varepsilon}(\varphi)) = \mathrm{div}(V)$ in the sense of (8.39). In fact, let $u \in \mathrm{W}^{1,2}_0(\Omega;\mathbb{R}^n)$ be a minimiser of $\mathcal{J}$. Then we have $u \pm \varepsilon\psi \in \mathrm{W}^{1,2}_0(\Omega;\mathbb{R}^n)$ for all $\varepsilon > 0$ and all $\psi \in \mathrm{W}^{1,2}_0(\Omega;\mathbb{R}^n)$ and hence

$$\int_\Omega \frac{1}{2}(\langle A(u\pm\varepsilon\psi),u\pm\varepsilon\psi\rangle - \langle Au,u\rangle)\,\mathrm{d}x - \int_\Omega \langle V,\pm\varepsilon\psi\rangle\,\mathrm{d}x \geqslant 0.$$

Dividing the previous inequalities by $\varepsilon > 0$ and sending $\varepsilon \searrow 0$ then yields (8.39) by arbitrariness of $\psi \in \mathrm{W}^{1,2}_0(\Omega;\mathbb{R}^n)$. Hence our objective is to demonstrate the existence of a minimiser for $\mathcal{J}$ over $\mathrm{W}^{1,2}_0(\Omega;\mathbb{R}^n)$. In doing so, our first aim is to argue that $\mathcal{J}$ is coercive on $\mathrm{W}^{1,2}_0(\Omega;\mathbb{R}^n)$. In what follows, we denote for $\psi \in \mathrm{W}^{1,2}_0(\Omega;\mathbb{R}^n)$ by $\underline{\psi} \in \mathrm{W}^{1,2}_0(\mathbb{R}^n;\mathbb{R}^n)$ its trivial extension to $\mathbb{R}^n$, and further record that the trivial extension $\underline{V}$ of V to $\mathbb{R}^n$ still belongs to $\mathrm{L}^2(\mathbb{R}^n;\mathbb{R}^{n\times n}_{\mathrm{sym}})$. Passing to the Fourier transform, we therefore obtain by a routine estimation (which is completed similarly as in the previous section) and using the strong symmetric rank–one hypothesis on A that

$$
\begin{aligned}
\mathcal{J}[\psi] &= \int_{\mathbb{R}^n} \frac{1}{2}\langle A\boldsymbol{\varepsilon}(\underline{\psi}), \boldsymbol{\varepsilon}(\underline{\psi})\rangle - \langle \underline{V}, \underline{\psi}\rangle \,\mathrm{d}x \\
&= \Re\Big(\int_{\mathbb{R}^n} \frac{1}{2}\langle A\widehat{\boldsymbol{\varepsilon}(\underline{\psi})}, \overline{\widehat{\boldsymbol{\varepsilon}(\underline{\psi})}}\rangle - \langle \widehat{\underline{V}}, \overline{\widehat{\underline{\psi}}}\rangle \,\mathrm{d}\xi \Big) && \text{(by Parseval)} \\
&= \int_{\mathbb{R}^n} \frac{\ell}{2}|\xi \odot \widehat{\underline{\psi}}(\xi)|^2 - C(\varepsilon)|\underline{V}|^2 - \varepsilon|\xi \odot \widehat{\underline{\psi}}(\xi)|^2 \,\mathrm{d}\xi && \text{(by Young).}
\end{aligned}
$$

At this stage we choose $\varepsilon > 0$ so small such that $\vartheta := \frac{\ell}{2} - \varepsilon > 0$ and then transform back to find by means of $\widehat{\psi}(\xi) = -\,\mathrm{i}\,\xi \odot \widehat{\psi}(\xi)$ and Korn's inequality in the zero–boundary–values version

$$
\begin{aligned}
\mathcal{J}[\psi] &\geqslant \vartheta \int_{\Omega} |\boldsymbol{\varepsilon}(\psi)|^2 \,\mathrm{d}x - C(\varepsilon)\int_{\Omega} |V|^2 \,\mathrm{d}x \\
&\geqslant c\vartheta \int_{\Omega} |\nabla\psi|^2 \,\mathrm{d}x - C(\varepsilon)\int_{\Omega} |V|^2 \,\mathrm{d}x \qquad \text{for all } \psi \in \mathrm{W}_0^{1,2}(\Omega;\mathbb{R}^n).
\end{aligned}
$$

By Poincaré's inequality for $\mathrm{W}_0^{1,2}(\Omega;\mathbb{R}^n)$–maps we then easily obtain $\inf_{\mathrm{W}_0^{1,2}(\Omega;\mathbb{R}^n)} \mathcal{J} > -\infty$ and that minimising sequences for $\mathcal{J}$ are bounded in $\mathrm{W}^{1,2}(\Omega;\mathbb{R}^n)$. Now let $(\psi_k) \subset \mathrm{W}_0^{1,2}(\Omega;\mathbb{R}^n)$ be a minimising sequence for $\mathcal{J}$. By the Banach–Alaoglu theorem, we find some $\psi \in \mathrm{W}_0^{1,2}(\Omega;\mathbb{R}^n)$ such that $\psi_k \rightharpoonup \psi$ weakly in $\mathrm{W}_0^{1,2}(\Omega;\mathbb{R}^n)$ (recall that the trace operator on $\mathrm{Tr}\colon \mathrm{W}^{1,2}(\Omega;\mathbb{R}^n) \to \mathrm{W}^{\frac{1}{2},2}(\partial\Omega;\mathbb{R}^n)$ is continuous for weak convergence in $\mathrm{W}^{1,2}(\Omega;\mathbb{R}^n)$).

As Ball notes [28, Sec. 8(a)], rank–one quadratic forms with constant coefficients are quasiconvex and so in particular lower semicontinuous with respect to weak convergence in $\mathrm{W}_0^{1,2}(\Omega;\mathbb{R}^n)$. By the reduction principle to obtain strong Legendre–Hadamard bilinear forms from symmetric ones, this applies to our situation, and since $\mathrm{W}_0^{1,2}(\Omega;\mathbb{R}^n) \ni \psi \mapsto \langle V, \boldsymbol{\varepsilon}(\psi)\rangle_{\mathrm{L}^2}$ is continuous with respect to weak convergence in $\mathrm{W}_0^{1,2}(\Omega;\mathbb{R}^n)$, it follows that $\mathcal{J}$ is sequentially weakly lower semicontinuous on $\mathrm{W}_0^{1,2}(\Omega;\mathbb{R}^n)$. With the lower semicontinuity of $\mathcal{J}$ with respect to weak convergence in $\mathrm{W}_0^{1,2}(\Omega;\mathbb{R}^n)$ and the fact that minima of $\mathcal{J}$ are weak solutions of (8.39) at our disposal, the claim is immediate. The proof is complete. $\square$

The preceding lemma establishes the existence of a weak solutions of particular systems, and now we turn to their Schauder theory. Here we mainly stick to the work of Simon [217] where a very general Schauder theory is given for *hypoelliptic* (vectorial) differential operators. Let $A[D]$ be a k–th order differential operator of the form

$$A[D] = \sum_{\substack{\alpha \in \mathbb{N}_0^n \\ |\alpha| = k}} \mathbb{A}_\alpha \partial^\alpha,$$

where the $\mathbb{A}_\alpha$ are homomorphisms between the two finite–dimensional real vector spaces V, W. We then say that $A[D]$ is *hypoelliptic* provided for any open and connected set Ω in $\mathbb{R}^n$ we have the implication

$$(u \in \mathrm{L}^2_{\mathrm{loc}}(\Omega;V) \text{ and } A[D]u = 0 \text{ in } \Omega) \Rightarrow u \in \mathrm{C}^\infty(\Omega;V).$$

We now pass to the main regularity result of this section which, though not explicitely stated in [217] in this fashion, can be inferred from [217] as we shall briefly sketch below.

Proposition 8.18 (Simon, [217])**.** *Let $0 < \alpha < 1$ and assume that Ω is an open and bounded domain in $\mathbb{R}^n$ with $\mathrm{C}^{1,\alpha}$–boundary. Moreover, assume that $A\colon \mathbb{R}^{N\times n} \times \mathbb{R}^{N\times n} \to$*

$\mathbb{R}$ is a bilinear form which is strongly elliptic in the sense of Legendre–Hadamard and let $f \in \mathrm{C}^{0,\alpha}(\Omega;\mathbb{R}^{N\times n})$. If $u \in \mathrm{W}_0^{1,2}(\Omega;\mathbb{R}^N)$ is a weak solution of the system

$$\begin{cases} -\operatorname{div}(A\nabla u) = -\operatorname{div}(f) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

in the sense that for all $\varphi \in \mathrm{W}_0^{1,2}(\Omega;\mathbb{R}^N)$ there holds

$$\int_\Omega \langle A\nabla u, \nabla\varphi\rangle = \int_\Omega \langle f, \nabla\varphi\rangle \,\mathrm{d}x,$$

then we have $u \in \mathrm{C}^{1,\alpha}(\overline{\Omega};\mathbb{R}^N)$.

The preceding proposition can be retrieved from [217] as follows. First note that although the main part of [217] is stated for equations only, it is mentioned in [217, Sec. 9] that the estimatations remain valid in the vectorial case (provided hypoellipticity holds true). Splitting the approach into interior and boundary estimates, it is then established that for each $\varepsilon > 0$ there exists $C_\varepsilon > 0$ with the following property: If $x_0 \in \Omega$ and $R > 0$ are such that $\mathrm{B}(x_0,R) \Subset \Omega$, then there holds

$$\begin{aligned} R^\alpha[\nabla u]_{\mathrm{C}^{0,\alpha}(\mathrm{B}(x_0,\theta R);\mathbb{R}^{N\times n})} \leqslant C_\varepsilon\big(R^\alpha[f]_{\mathrm{C}^{0,\alpha}(\mathrm{B}(x_0,R);\mathbb{R}^{N\times n})} + R^{-\frac{n}{2}}\|\nabla u\|_{\mathrm{L}^2(\mathrm{B}(x_0,R);\mathbb{R}^{N\times n})}\big) \\ + \varepsilon\big(\sup_{\mathrm{B}(x_0,R)} |\nabla u| + R|u|\big). \end{aligned}$$

Here and in what follows, these estimates work provided $\operatorname{div}(A\nabla u)$ is hypoelliptic, and by the results of the previous Section 8.4.4, this is the case if A is strongly rank–one convex. Following the Lemma after [217, Thm. 2], it is then possible to discard the term $\varepsilon(\sup_{\mathrm{B}(x_0,R)} |\nabla u| + R|u|)$ and consequently obtain the required local estimate. Lastly, as argued in [217, Sec. 7], it is also possible to obtain the requisite estimates for half–balls, and then a standard covering argument leads to the required Schauder estimates.

By the routine reduction outlined above to lift strongly symmetric rank–one convex bilinear forms to strongly rank–one convex bilinear forms (cp. Section 8.4.3), it is now also immediate to deduce that if $u \in \mathrm{W}_0^{1,2}(\Omega;\mathbb{R}^n)$ is a weak solution of the symmetric strong Legendre–Hadamard system (8.39) with $f = V \in \mathrm{C}^{1,\alpha}(\overline{\Omega};\mathbb{R}^{n\times n}_{\mathrm{sym}})$, then $u \in \mathrm{C}^{1,\alpha}(\overline{\Omega};\mathbb{R}^n)$. On the other hand, Lemma 8.17 ensures the existence of a weak solution, and so we obtain that for any $V \in \mathrm{C}^{1,\alpha}(\overline{\Omega};\mathbb{R}^{n\times n}_{\mathrm{sym}})$ there exists a $\mathrm{C}^{1,\alpha}(\overline{\Omega};\mathbb{R}^n)$–solution to (8.39).

8.5. An ε–Regularity Result for symmetric–rank–1–convex Problems

In this section we give the proof of Theorem 8.3. This, in turn, is taylored for integrands which are strongly symmetric–rank one convex. The reader will notice that the corresponding arguments inherit in a straightforward manner to the $\mathbb{A}[D]$–setting introduced in Chapter 3.

As we mentioned after Theorem 8.3, the proof presented here relies on the implicit function theorem on Banach spaces. Its applicability relies on a number of lemmas to be discussed and proven next. Let us firstly remark that if $u \in g + \mathrm{LD}_0(\Omega)$ is a minimiser of the variational integral $\mathfrak{F}$ from (8.3), then it satisfies the weak Euler–Lagrange equation

$$\int_\Omega \langle F'(\boldsymbol{\varepsilon}(u)), \boldsymbol{\varepsilon}(\psi)\rangle \,\mathrm{d}x = 0 \qquad \text{for all } \psi \in \mathrm{LD}_0(\Omega).$$

Hence, assuming $u \in \mathrm{C}^{1,\alpha}(\Omega;\mathbb{R}^n)$ and recalling $g \in \mathrm{C}^{1,\alpha}(\overline{\Omega};\mathbb{R}^n)$, this is equivalent to the mutually equivalent systems (Sys$_1$) and (Sys$_2$) given by

$$\begin{cases} -\operatorname{div}(F'(\boldsymbol{\varepsilon}(u))) = 0 & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega \end{cases} \tag{Sys$_1$}$$

$$\begin{cases} -\operatorname{div}(F'(\boldsymbol{\varepsilon}(\varphi + g))) = 0 & \text{in } \Omega, \\ \varphi = 0 & \text{on } \partial\Omega, \end{cases} \tag{Sys$_2$}$$

and we note that $u \in \mathrm{C}^{1,\alpha}(\overline{\Omega};\mathbb{R}^n)$ is a solution of (Sys$_1$) if and only if $\varphi = u - g \in \mathrm{C}^{1,\alpha}(\overline{\Omega};\mathbb{R}^n)$ with $\varphi|_{\partial\Omega} = 0$ solves (Sys$_2$).

For the following, it is convenient to work with the Campanato–type characterisation of spaces $\mathrm{C}^{m,\lambda}(\overline{\Omega};\mathbb{R}^N)$ given by Theorem 2.28. We recall that the Campanato–type space $\mathcal{L}_1^{1,n+1+\alpha}(\Omega;\mathbb{R}^N)$ for $0 < \alpha < 1$ are defined as all those $v \in \mathrm{L}^1(\Omega;\mathbb{R}^N)$ for which the function $\Omega \ni x \mapsto \mathcal{M}^{\#}_{n+1+\alpha,1,\Omega}v(x) \in \mathbb{R}$ given by

$$\mathcal{M}^{\#}_{n+1+\alpha,1,\Omega}v(x) := \sup_{r>0} \inf_{\pi \in \mathcal{P}_1(\Omega\cap\mathrm{B}(x,r);\mathbb{R}^N)} \min\{1,r\}^{-n-1-\alpha} \int_{\Omega\cap\mathrm{B}(x,r)} |f - \pi| \,\mathrm{d}x$$

is essentially bounded. Here, $\mathcal{P}_1(\mathrm{B}(x,r);\mathbb{R}^n)$ are the affine–linear, $\mathbb{R}^n$–valued maps on $\mathrm{B}(x,r)$. We then recall that as Ω is assumed to have boundary of class $\mathrm{C}^{1,1}$, it is Ahlfors regular and hence we deduce from Theorem 2.28 that for all $0 < \alpha < 1$ there holds

$$\boxed{\mathcal{L}_1^{1,n+1+\alpha}(\Omega;\mathbb{R}^N) \simeq \mathrm{C}^{1,\alpha}(\overline{\Omega};\mathbb{R}^N).}$$

To fit the problem at our disposal into the implicit function theorem framework, we begin with the following definition.

Definition 8.19. *Given an open and bounded domain $\Omega \subset \mathbb{R}^n$ with $\mathrm{C}^{1,1}$–boundary $\partial\Omega$, we define (with the divergence understood in the row–wise distributional sense)*

$$Z := \left\{ \operatorname{div}(V)\colon\ V \in \mathcal{L}^{1,n+\alpha}(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}}) \right\}$$

and endow it with norm

$$\|\operatorname{div}(V)\|_Z := \inf\{\|W\|_{\mathcal{L}^{1,n+\alpha}(\Omega;\mathbb{R}^{n\times n})}\colon\ W \in \mathcal{L}^{1,n+\alpha}(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}}),\ \operatorname{div}(V - W) \equiv 0\}.$$

Before we proceed, we briefly wish to argue that $(Z, \|\cdot\|_Z)$ is a Banach space. For this we formulate a slightly more general result:

Lemma 8.20. *Let $(X, \|\cdot\|_X)$ be a Banach space, $(Y, \mathcal{T}_Y)$ a topological vector space and $T\colon X \to Y$ a continuous linear operator. Then the linear space $\mathcal{Z} := \{Tx\colon\ x \in X\}$ is complete with respect to the norm $\|Tx\|_{\mathcal{Z}} := \inf\{\|\widetilde{x}\|_X \colon \widetilde{x} \in X,\ Tx = T\widetilde{x}\}$.*

Proof. We claim that $\mathcal{Z} \cong X/\ker(T)$. Define $\pi\colon \mathcal{Z} \to X/\ker(T)$ by

$$\pi(z) := x + \ker(T) \qquad \text{if } z = Tx.$$

This is a well–defined mapping. Clearly, π is linear, and if $z = Tx = Tx'$, then $x - x' \in \ker(T)$ and hence $\pi(T(x-x')) = (x-x')+\ker(T) = \ker(T) = 0$ and hence $\pi(Tx) = \pi(Tx')$. Now let $z \in \mathcal{Z}$ be such that $\pi(z) = 0$. Writing $z = Tx$, this means $x + \ker(T) = \ker(T)$ and so $x \in \ker(T)$. In conclusion, $x \in \ker(T)$, so $\|Tx\|_{\mathcal{Z}} = \|0\|_{\mathcal{Z}} = 0$ and injectivity follows. Surjectivity holds by definition: Let $x + \ker(T) \in X/\ker(T)$ and put $z := Tx$; then trivially $\pi(z) = x + \ker(T)$. The respective isometric property of π follows from the fact that $\|\cdot\|_{\mathcal{Z}}$ is actually the quotient norm. Precisely, substituting $x - u = x'$ in the third step (where $z = Tx$)

$$\begin{aligned} \|\pi(z)\|_{X/\ker(T)} = \|x + \ker(T)\|_{X/\ker(T)} &:= \inf\{\|x - u\|_X\colon\ u \in \ker(T)\} \\ &= \inf\{\|x'\|_X \colon T(x' - x) = 0\} \\ &= \|z\|_{\mathcal{Z}}. \end{aligned}$$

So π is an isometric bijective mapping, hence by the inverse mapping theorem,

$$\pi^{-1}\colon X/\ker(T)\to\mathscr{Z}$$

is an isometric bijective mapping as well, and so $\mathscr{Z}\cong X/\ker(T)$. Since T is continuous, $\ker(T)$ is closed in X and so, because X is Banach, $X/\ker(T)$ is Banach too. Since $\mathscr{Z}\cong X/\ker(T)$, $\mathscr{Z}$ must be a Banach space, too. The proof is complete. □

Corollary 8.21. *The space $(Z,\|\cdot\|_Z)$ as introduced in Definition 8.19 is Banach.*

Proof. We aim at applying Lemma 8.20 to $(X,\|\cdot\|_X)=(\mathcal{L}^{1,n+\alpha}(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}});\|\cdot\|_{\mathcal{L}^{1,n+\alpha}(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})})$ and $(Y,\mathcal{T}_Y)=(\mathscr{D}'(\Omega;\mathbb{R}^n),\mathrm{w}^*)$, that is, the $\mathbb{R}^n$–valued distributions on Ω equipped with the weak*–topology. Clearly, $(X,\|\cdot\|_X)$ is Banach, and hence it remains to show that $T:=\operatorname{div}\colon\mathcal{L}^{1,n+\alpha}(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})\to\mathscr{D}'(\Omega;\mathbb{R}^n)$ is continuous.

To this end, let $u,u_1,u_2,...\in\mathcal{L}^{1,n+\alpha}(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})$ be such that $\|u-u_k\|_{\mathrm{L}^{1,\alpha}(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})}\to 0$ as $k\to\infty$. Then, for fixed $\varphi\in\mathscr{D}(\Omega;\mathbb{R}^n)$ we obtain, using $\mathrm{C}^{0,\alpha}(\overline{\Omega};\mathbb{R}^j)\simeq\mathcal{L}^{1,n+\alpha}(\Omega;\mathbb{R}^j)$,

$$\begin{aligned}|\langle\operatorname{div}(u)-\operatorname{div}(u_k),\varphi\rangle_{\mathscr{D}'\times\mathscr{D}}|=|\langle u-u_k,\boldsymbol{\varepsilon}(\varphi)\rangle_{\mathscr{D}'\times\mathscr{D}}|&\leqslant\|u-u_k\|_{\mathrm{L}^\infty(\Omega;\mathbb{R}^{n\times n})}\|\nabla\varphi\|_{\mathrm{L}^\infty(\Omega;\mathbb{R}^{n\times n})}\\&\leqslant\|u-u_k\|_{\mathrm{C}^{0,\alpha}(\Omega;\mathbb{R}^{n\times n})}\|\nabla\varphi\|_{\mathrm{L}^\infty(\Omega;\mathbb{R}^{n\times n})}\\&\leqslant c\|u-u_k\|_{\mathcal{L}^{1,n+\alpha}(\Omega;\mathbb{R}^{n\times n})}\|\nabla\varphi\|_{\mathrm{L}^\infty(\Omega;\mathbb{R}^{n\times n})}\to 0,\quad\text{as }k\to\infty.\end{aligned}$$

Hence div is continuous with respect to the weak*–topology, and the claim now from the definition of Z and Lemma 8.20. □

Let us now put

$$\begin{aligned}X&:=\mathcal{L}^{1,n+1+\alpha}_1(\Omega;\mathbb{R}^n),\\Y&:=\mathcal{L}^{1,n+1+\alpha}_1(\Omega;\mathbb{R}^n)\cap\{\varphi\colon\varphi|_{\partial\Omega}=0\},\end{aligned}$$

both spaces equipped with the $\mathcal{L}^{1,n+1+\alpha}_1(\Omega;\mathbb{R}^N)$–norm. Since $u\in X\simeq\mathrm{C}^{1,\alpha}(\overline{\Omega};\mathbb{R}^n)$, we find $|\boldsymbol{\varepsilon}(u)(x)-\boldsymbol{\varepsilon}(u)(y)|\leqslant C|x-y|^\alpha\leqslant C\operatorname{diam}(\Omega)^\alpha$ and hence $\boldsymbol{\varepsilon}(u)(\Omega)\subset\mathbb{B}_R:=\mathbb{B}(0,R)$ for some sufficiently large $R>0$. In conclusion, because F' is locally Lipschitz, we deduce by

$$|F'(\boldsymbol{\varepsilon}(u)(x))-F'(\boldsymbol{\varepsilon}(u)(y))|\leqslant L(R)|\boldsymbol{\varepsilon}(u)(x)-\boldsymbol{\varepsilon}(u)(y)|\leqslant CL(R)|x-y|^\alpha$$

that $F'(\boldsymbol{\varepsilon}(u))\in\mathrm{C}^{0,\alpha}(\overline{\Omega};\mathbb{R}^{n\times n}_{\mathrm{sym}})\simeq\mathcal{L}^{1,n+\alpha}(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})$. In conclusion, $-\operatorname{div}(F'(\boldsymbol{\varepsilon}(u)))\in Z$ and hence, the operator $\mathcal{F}$ acting on $(g,\varphi)\in X\times Y$ by

$$\mathcal{F}(g,\varphi):=-\operatorname{div}(F'(\boldsymbol{\varepsilon}(\varphi)+\boldsymbol{\varepsilon}(g)))\tag{8.40}$$

maps $\mathcal{F}\colon X\times Y\to Z$. Our next intermediate claim is formulated in the following lemma.

Lemma 8.22. *Define $\mathcal{F}$ by (8.40). Then $\mathcal{F}$ is continuously Fréchet differentiable with total differential*

$$D\mathcal{F}(g_0,\varphi_0)[g,\varphi]=-\operatorname{div}(F''(\boldsymbol{\varepsilon}(\varphi_0)+\boldsymbol{\varepsilon}(g_0))(\boldsymbol{\varepsilon}(\varphi)+\boldsymbol{\varepsilon}(g))),\tag{8.41}$$

where $\varphi,\varphi_0\in X$ and $g,g_0\in Y$.

Proof. Given $(g_0,\varphi_0)\in X\times Y$ arbitrary but fixed, let $(g,\varphi)\in X\times Y$ be arbitrary. We then obtain with $u_0=g_0+\varphi_0$ and $u=g+\varphi$

$$\begin{aligned}\mathbf{J}:=\mathcal{F}(g_0+g,\varphi_0+\varphi)-\mathcal{F}(g_0,\varphi_0)+\operatorname{div}(F''(\boldsymbol{\varepsilon}(\varphi_0)+\boldsymbol{\varepsilon}(g_0))(\boldsymbol{\varepsilon}(g)+\boldsymbol{\varepsilon}(\varphi)))\\&=-\operatorname{div}\big(F'(\boldsymbol{\varepsilon}(u_0)+\boldsymbol{\varepsilon}(u))-F'(\boldsymbol{\varepsilon}(u_0))-F''(\boldsymbol{\varepsilon}(u_0))\boldsymbol{\varepsilon}(u)\big)\\&=-\operatorname{div}\Big(\int_0^1(F''(\boldsymbol{\varepsilon}(u_0)+t\boldsymbol{\varepsilon}(u))-F''(\boldsymbol{\varepsilon}(u_0)))\boldsymbol{\varepsilon}(u)\,\mathrm{d}t\Big)\\&=:-\operatorname{div}(\mathbf{J}_0)\end{aligned}$$

so that, in particular, $\mathbf{J}$ can be written as a divergence. Our aim is to estimate

$$\begin{aligned}\|\mathbf{J}\|_Z &= \inf\left\{\|\widetilde{\mathbf{J}}\|_{\mathcal{L}^{1,n+\alpha}(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})}\colon\ \operatorname{div}(\mathbf{J}-\widetilde{\mathbf{J}})=0\right\}\\ &\leqslant \|\mathbf{J}_0\|_{\mathcal{L}^{1,n+\alpha}(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})} \leqslant C\|\mathbf{J}_0\|_{\mathrm{C}^{0,\alpha}(\overline{\Omega};\mathbb{R}^{n\times n}_{\mathrm{sym}})},\end{aligned} \tag{8.42}$$

and to control the very last term. By assumption, $F\in \mathrm{C}^{3,1}_{\mathrm{loc}}(\mathbb{R}^{n\times n}_{\mathrm{sym}})$, and therefore both F'' and F''' are locally Lipschitz. Since without loss of generality, $\|\boldsymbol{\varepsilon}(u)\|_{\mathrm{C}^{0,\alpha}(\overline{\Omega};\mathbb{R}^{n\times n}_{\mathrm{sym}})}\leqslant 1$, there exists $R>0$ such that for all $x,y\in\overline{\Omega}$ and $t\in[0,1]$ there holds $\boldsymbol{\varepsilon}(u_0)(x)+t\boldsymbol{\varepsilon}(u)(x),\boldsymbol{\varepsilon}(u_0)(x),\boldsymbol{\varepsilon}(u_0)(y)+t\boldsymbol{\varepsilon}(u)(y),\boldsymbol{\varepsilon}(u_0)(y)\in\mathbb{B}_R$, and we shall denote $L_R>0$ the maximum of the Lipschitz constants of the restrictions of F'' and F''' to $\overline{\mathbb{B}_R}$, respectively.

We firstly note that for any $x\in\overline{\Omega}$ there holds

$$\begin{aligned}|\mathbf{J}_0(x)| &\leqslant \int_0^1 |F''(\boldsymbol{\varepsilon}(u_0)(x)+t\boldsymbol{\varepsilon}(u)(x))-F''(\boldsymbol{\varepsilon}(u_0)(x))|\,\mathrm{d}t|\boldsymbol{\varepsilon}(u)(x)|\\ &\leqslant L_R|\boldsymbol{\varepsilon}(u)(x)|^2\leqslant L_R\|\boldsymbol{\varepsilon}(u)\|^2_{\mathrm{C}^{0,\alpha}(\overline{\Omega};\mathbb{R}^{n\times n}_{\mathrm{sym}})}.\end{aligned} \tag{8.43}$$

Now we proceed by suitably estimating the α–th Hölder seminorm of $\mathbf{J}_0$. For this we note in advance that if $F_1\colon U\to\mathbb{R}^{m\times j}$ and $F_2\colon U\to\mathbb{R}^j$ (with U being an arbitrary subset of $\mathbb{R}^n$) are α–Hölder continuous, then we have for all $x,y\in U$

$$\begin{aligned}|F_1(x)F_2(x)-F_1(y)F_2(y)| &= |F_1(x)F_2(x)-F_1(y)F_2(x)+F_1(y)F_2(x)-F_1(y)F_2(y)|\\ &\leqslant |F_1(x)-F_1(y)|\,|F_2(x)|+|F_1(y)|\,|F_2(x)-F_2(y)|\\ &\leqslant 2\|F_1\|_{\mathrm{C}^{0,\alpha}(U;\mathbb{R}^{m\times j})}\|F_2\|_{\mathrm{C}^{0,\alpha}(U;\mathbb{R}^j)}|x-y|^\alpha\end{aligned}$$

and therefore

$$[F_1F_2]_{\mathrm{C}^{0,\alpha}(U;\mathbb{R}^m)}\leqslant 2\|F_1\|_{\mathrm{C}^{0,\alpha}(U;\mathbb{R}^{m\times j})}\|F_2\|_{\mathrm{C}^{0,\alpha}(U;\mathbb{R}^j)}. \tag{8.44}$$

Now, given $x,y\in\overline{\Omega}$, we write

$$F_1(x):=\int_0^1 F''(\boldsymbol{\varepsilon}(u_0)(x)+t\boldsymbol{\varepsilon}(u)(x))-F''(\boldsymbol{\varepsilon}(u_0)(x))\,\mathrm{d}t\quad\text{and}\quad F_2(x)=\boldsymbol{\varepsilon}(u)(x),$$

so that $\mathbf{J}_0=F_1F_2$. Consequently, by (8.44),

$$[\mathbf{J}_0]_{\mathrm{C}^{0,\alpha}(\overline{\Omega};\mathbb{R}^{n^2})}\leqslant 2\|F_1\|_{\mathrm{C}^{0,\alpha}(\overline{\Omega};\mathbb{R}^{n^2\times n^2})}\|\boldsymbol{\varepsilon}(u)\|_{\mathrm{C}^{0,\alpha}(\overline{\Omega};\mathbb{R}^{n\times n})}. \tag{8.45}$$

To bound $\|F_1\|_{\mathrm{C}^{0,\alpha}(\overline{\Omega};\mathbb{R}^{n^2\times n^2})}$, we note that for all $x\in\overline{\Omega}$ we have by Lipschitz continuity of F''

$$\begin{aligned}|F_1(x)| &\leqslant \int_0^1 |F''(\boldsymbol{\varepsilon}(u_0)(x)+t\boldsymbol{\varepsilon}(u)(x))-F''(\boldsymbol{\varepsilon}(u_0)(x))|\,\mathrm{d}t\\ &\leqslant L_R|\boldsymbol{\varepsilon}(u)(x)|\leqslant L_R\|\boldsymbol{\varepsilon}(u)\|_{\mathrm{C}^{0,\alpha}(\overline{\Omega};\mathbb{R}^{n\times n})}.\end{aligned} \tag{8.46}$$

Now note that since F is thrice continuously differentiable, we may write

$$\begin{aligned}F_1(x) &= \int_0^1\int_0^1 \frac{\mathrm{d}}{\mathrm{d}s}F''(\boldsymbol{\varepsilon}(u_0)(x)+st\boldsymbol{\varepsilon}(u)(x))\,\mathrm{d}s\,\mathrm{d}t\\ &= \int_0^1\int_0^1 F'''(\boldsymbol{\varepsilon}(u_0)(x)+st\boldsymbol{\varepsilon}(u)(x))\,\mathrm{d}st\,\mathrm{d}t\boldsymbol{\varepsilon}(u)(x)=:\widetilde{F}_1(x)\widetilde{F}_2(x),\end{aligned}$$

where $\widetilde{F}_2(x):=\boldsymbol{\varepsilon}(u)(x)$ and $\widetilde{F}_1(x)$ accordingly is defined in the obvious manner. On the other hand, (8.44) gives

$$[F_1]_{\mathrm{C}^{0,\alpha}(\overline{\Omega};\mathbb{R}^{n^2\times n^2})}\leqslant 2\|\widetilde{F}_1\|_{\mathrm{C}^{0,\alpha}(\overline{\Omega};\mathbb{R}^{n^2\times n^2})}\|\widetilde{F}_2\|_{\mathrm{C}^{0,\alpha}(\overline{\Omega};\mathbb{R}^{n\times n})}. \tag{8.47}$$

In this situation, we find for any $x \in \overline{\Omega}$ by the Lipschitz continuity of F''' on $\mathbb{B}_R$

$$\begin{aligned} |\widetilde{F}_1(x)| &\leqslant \left|\int_0^1\int_0^1 |F'''(\boldsymbol{\varepsilon}(u_0)(x)+st\boldsymbol{\varepsilon}(u)(x))-F'''(0)|\,\mathrm{d}s t\,\mathrm{d}t\right| + |F'''(0)| \\ &\leqslant L_R(|\boldsymbol{\varepsilon}(u_0)(x)|+|\boldsymbol{\varepsilon}(u)(x)|)+|F'''(0)| \\ &\leqslant L_R(\|\boldsymbol{\varepsilon}(u_0)\|_{\mathrm{C}^{0,\alpha}(\overline{\Omega};\mathbb{R}^{n\times n})}+\|\boldsymbol{\varepsilon}(u)\|_{\mathrm{C}^{0,\alpha}(\overline{\Omega};\mathbb{R}^{n\times n})})+|F'''(0)|. \end{aligned} \tag{8.48}$$

Moreover, the Lipschitz continuity of F''' in turn yields for all $x, y \in \overline{\Omega}$

$$\begin{aligned} |\widetilde{F}_1(x)-\widetilde{F}_1(y)| \leqslant \int_0^1\int_0^1 & |F'''(\boldsymbol{\varepsilon}(u_0)(x)+st\boldsymbol{\varepsilon}(u)(x)) \\ & - F'''(\boldsymbol{\varepsilon}(u_0)(y)+st\boldsymbol{\varepsilon}(u)(y))|\,\mathrm{d}s t\,\mathrm{d}t \\ & \leqslant L_R\big(|\boldsymbol{\varepsilon}(u_0)(x)-\boldsymbol{\varepsilon}(u_0)(y)|+|\boldsymbol{\varepsilon}(u)(x)-\boldsymbol{\varepsilon}(u)(y)|\big) \\ & \leqslant L_R\big(\|\boldsymbol{\varepsilon}(u_0)\|_{\mathrm{C}^{0,\alpha}(\overline{\Omega};\mathbb{R}^{n\times n})}+\|\boldsymbol{\varepsilon}(u)\|_{\mathrm{C}^{0,\alpha}(\overline{\Omega};\mathbb{R}^{n\times n})}\big)|x-y|^{\alpha} \end{aligned}$$

and therefore

$$[\widetilde{F}_1]_{\mathrm{C}^{0,\alpha}(\overline{\Omega};\mathbb{R}^{n\times n})} \leqslant L_R\big(\|\boldsymbol{\varepsilon}(u_0)\|_{\mathrm{C}^{0,\alpha}(\overline{\Omega};\mathbb{R}^{n\times n})}+\|\boldsymbol{\varepsilon}(u)\|_{\mathrm{C}^{0,\alpha}(\overline{\Omega};\mathbb{R}^{n\times n})}\big) \tag{8.49}$$

so that, recalling (8.48),

$$\|\widetilde{F}_1\|_{\mathrm{C}^{0,\alpha}(\overline{\Omega};\mathbb{R}^{n\times n})} \leqslant 2L_R\big(\|\boldsymbol{\varepsilon}(u_0)\|_{\mathrm{C}^{0,\alpha}(\overline{\Omega};\mathbb{R}^{n\times n})}+\|\boldsymbol{\varepsilon}(u)\|_{\mathrm{C}^{0,\alpha}(\overline{\Omega};\mathbb{R}^{n\times n})}\big)+|F'''(0)|. \tag{8.50}$$

In the following, we shall drop the domain $\overline{\Omega}$ and the corresponding target spaces appearing in the definition of the relevant norms in the sequel for notational brievity. Because of $\widetilde{F}_2 = \boldsymbol{\varepsilon}(u)$, (8.46) and (8.47) give

$$\begin{aligned} \|F_1\|_{\mathrm{C}^{0,\alpha}} &= \|F_1\|_{\mathrm{L}^\infty}+[F_1]_{\mathrm{C}^{0,\alpha}} \\ &\overset{(8.46)}{\leqslant} L_R\|\boldsymbol{\varepsilon}(u)\|_{\mathrm{C}^{0,\alpha}}+2\|\widetilde{F}_1\|_{\mathrm{C}^{0,\alpha}}\|\widetilde{F}_2\|_{\mathrm{C}^{0,\alpha}} \\ &\overset{(8.50)}{\leqslant} (L_R+2(2L_R\big(\|\boldsymbol{\varepsilon}(u_0)\|_{\mathrm{C}^{0,\alpha}}+\|\boldsymbol{\varepsilon}(u)\|_{\mathrm{C}^{0,\alpha}}\big)+|F'''(0)|))\times \\ &\quad \times \|\boldsymbol{\varepsilon}(u)\|_{\mathrm{C}^{0,\alpha}}. \end{aligned} \tag{8.51}$$

Inserting this estimate into (8.45), we finally obtain because of $\|\boldsymbol{\varepsilon}(u)\|_{\mathrm{C}^{0,\alpha}} \leqslant 1$

$$\begin{aligned} \|\mathbf{J}_0\|_{\mathrm{C}^{0,\alpha}} &= \|\mathbf{J}_0\|_{\mathrm{L}^\infty}+[\mathbf{J}_0]_{\mathrm{C}^{0,\alpha}} \\ &\overset{(8.43)\&(8.45)}{\leqslant} 2L_R\|\boldsymbol{\varepsilon}(u)\|^2_{\mathrm{C}^{0,\alpha}} \\ &\quad +2\|F_1\|_{\mathrm{C}^{0,\alpha}}\|\boldsymbol{\varepsilon}(u)\|_{\mathrm{C}^{0,\alpha}} \\ &= 2L_R\|\boldsymbol{\varepsilon}(u)\|^2_{\mathrm{C}^{0,\alpha}} \\ &\quad +2(L_R+2(2L_R\big(\|\boldsymbol{\varepsilon}(u_0)\|_{\mathrm{C}^{0,\alpha}}+\|\boldsymbol{\varepsilon}(u)\|_{\mathrm{C}^{0,\alpha}}\big)+|F'''(0)|))\times \\ &\quad \times \|\boldsymbol{\varepsilon}(u)\|^2_{\mathrm{C}^{0,\alpha}} \\ &\leqslant C(L_R,\|\boldsymbol{\varepsilon}(u_0)\|_{\mathrm{C}^{0,\alpha}})\|\boldsymbol{\varepsilon}(u)\|^2_{\mathrm{C}^{0,\alpha}}. \end{aligned}$$

We go back to estimate (8.42) and thereby find

$$\|\mathbf{J}\|_Z \leqslant C(L_R,\|\boldsymbol{\varepsilon}(u_0)\|_{\mathrm{C}^{0,\alpha}})\|\boldsymbol{\varepsilon}(u)\|^2_{\mathrm{C}^{0,\alpha}} = o(\|\boldsymbol{\varepsilon}(u)\|_{\mathrm{C}^{0,\alpha}}),$$

and from here the claimed Fréchet differentiability follows by definition. It remains to demonstrate the continuity of the Fréchet differential. For this, let $(g_0, \varphi_0) \in X \times Y$ be arbitrary but fixed and let $(g_0^h, \varphi_0^h) \in X \times Y$ be arbitrary and put $u_0 = g_0 + \varphi_0$ as well as $h_0 := g_0^h + \varphi_0^h$. We need to show that

$$\begin{aligned}\|\operatorname{div}(F''(\varepsilon(u_0+h_0))\varepsilon(u)) &- \operatorname{div}(F''(\varepsilon(u_0))\varepsilon(u))\|_Z \\ &\leqslant c\|F''(\varepsilon(u_0+h_0))\varepsilon(u) - F''(\varepsilon(u_0))\varepsilon(u)\|_{\mathrm{C}^{0,\alpha}} \\ &\leqslant \theta(\|h_0\|_{X\times Y})\|\varepsilon(u)\|_{\mathrm{C}^{0,\alpha}},\end{aligned} \tag{8.52}$$

where $\theta\colon \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ is a function with $\theta(0) = 0$ and $\lim_{t\searrow 0}\theta(t) = 0$. To this end, let $x \in \overline{\Omega}$ be arbitrary. We estimate, using the Lipschitz continuity of F'',

$$\begin{aligned}|(F''(\varepsilon(u_0+h_0)(x)) - F''(\varepsilon(u_0)(x)))\varepsilon(u)(x)| &\leqslant L_R|h_0(x)|\,|\varepsilon(u)(x)| \\ &\leqslant L_R\|\varepsilon(h_0)\|_{\mathrm{C}^{0,\alpha}}\|\varepsilon(u)\|_{\mathrm{C}^{0,\alpha}}.\end{aligned} \tag{8.53}$$

We continue by estimating the corresponding Hölder seminorm and use a strategy similar to that pursued in the proof of the Fréchet differentiability. We write, using $F \in \mathrm{C}^3(\mathbb{R}^{n\times n}_{\mathrm{sym}})$,

$$\begin{aligned}G(x) &:= (F''(\varepsilon(u_0)(x) + \varepsilon(h_0)(x)) - F''(\varepsilon(u_0)(x))) \\ &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}s}F''(\varepsilon(u_0)(x) + s\varepsilon(h_0)(x))\,\mathrm{d}s\,\varepsilon(h_0)(x) \\ &= \int_0^1 F'''(\varepsilon(u_0)(x) + s\varepsilon(h_0)(x))\,\mathrm{d}s\,\varepsilon(h_0)(x) = G_1(x)G_2(x),\end{aligned}$$

where $G_2(x) := \varepsilon(h_0)(x)$. We now estimate for an arbitrary $x \in \overline{\Omega}$

$$|G_1(x)| \leqslant L_R(\|\varepsilon(u_0)\|_{\mathrm{C}^{0,\alpha}} + \|\varepsilon(h_0)\|_{\mathrm{C}^{0,\alpha}}) + |F'''(0)|.$$

Now, letting $x, y \in \overline{\Omega}$ be arbitrary, we estimate

$$\begin{aligned}|G_1(x) - G_1(y)| &\leqslant \int_0^1 |F'''(\varepsilon(u_0)(x) + s\varepsilon(h_0)(x)) - F'''(\varepsilon(u_0)(y) + s\varepsilon(h_0)(y))|\,\mathrm{d}s \\ &\leqslant L_R(\|\varepsilon(u_0)\|_{\mathrm{C}^{0,\alpha}} + \|\varepsilon(h_0)\|_{\mathrm{C}^{0,\alpha}})|x-y|^\alpha\end{aligned}$$

so that $[G_1]_{\mathrm{C}^{0,\alpha}} \leqslant L_R(\|\varepsilon(u_0)\|_{\mathrm{C}^{0,\alpha}} + \|\varepsilon(h_0)\|_{\mathrm{C}^{0,\alpha}})$. Invoking (8.44) we find

$$\|G\|_{\mathrm{L}^\infty} \leqslant (L_R(\|\varepsilon(u_0)\|_{\mathrm{C}^{0,\alpha}} + \|\varepsilon(h_0)\|_{\mathrm{C}^{0,\alpha}}) + |F'''(0)|)\|\varepsilon(h_0)\|_{\mathrm{C}^{0,\alpha}} =: A_1\|\varepsilon(h_0)\|_{\mathrm{C}^{0,\alpha}}$$

and

$$\begin{aligned}[G]_{\mathrm{C}^{0,\alpha}} &\leqslant 2\|G_1\|_{\mathrm{C}^{0,\alpha}}\|\varepsilon(h_0)\|_{\mathrm{C}^{0,\alpha}} \\ &\leqslant 2\Big(L_R(\|\varepsilon(u_0)\|_{\mathrm{C}^{0,\alpha}} + \|\varepsilon(h_0)\|_{\mathrm{C}^{0,\alpha}}) + |F'''(0)| + L_R(\|\varepsilon(u_0)\|_{\mathrm{C}^{0,\alpha}} + \|\varepsilon(h_0)\|_{\mathrm{C}^{0,\alpha}})\Big) \\ &\quad\times \|\varepsilon(h_0)\|_{\mathrm{C}^{0,\alpha}} =: A_2\|\varepsilon(h_0)\|_{\mathrm{C}^{0,\alpha}}.\end{aligned}$$

From the very definition of A_1 and A_2, we see that these quantities remain bounded in the limit passage $\|\varepsilon(h_0)\|_{\mathrm{C}^{0,\alpha}} \searrow 0$ with a bound $C(u_0, L_R) > 0$ only depending on u_0 and L_R. In conclusion, going back to (8.52), we find

$$\begin{aligned}\|F''(\varepsilon(u_0+h_0))\varepsilon(u) - F''(\varepsilon(u_0))\varepsilon(u)\|_{\mathrm{C}^{0,\alpha}} &\leqslant 3\|G\|_{\mathrm{C}^{0,\alpha}}\|\varepsilon(u)\|_{\mathrm{C}^{0,\alpha}} \\ &\leqslant 6C(u_0, L_R)\|\varepsilon(h_0)\|_{\mathrm{C}^{0,\alpha}}\|\varepsilon(u)\|_{\mathrm{C}^{0,\alpha}},\end{aligned}$$

and from here, the claim is immediate. The proof is complete. □

8.5.1. Conclusion After these preparations, we now come to the

Proof of Theorem 8.3. We reduce to the setting of the implicit function theorem (see Theorem 10.20) and hence claim that $\mathcal{F}\colon X\times Y\to Z$ as given above with $(g_0,\varphi_0):=(0,0)\in X\times Y$ satisfies the assumption of Theorem 10.20. Firstly, by Lemma 8.22, $\mathcal{F}\colon X\times Y\to Z$ is a C^1–map with $\mathcal{F}(g_0,\varphi_0)=0$. Moreover, $D_2\mathcal{F}(g_0,\varphi_0):=D(\mathcal{F}(0,\cdot))(0)\colon Y\to Z$ is a bounded linear transformation: For injectivity, let $\varphi\in Y$ satisfy $\varphi\in\ker(D_2\mathcal{F}(0,0))$. Then, using $\varphi|_{\partial\Omega}=0$, integrating by parts yields

$$0=-\int_\Omega\langle\operatorname{div}(F''(0)\boldsymbol{\varepsilon}(\varphi)),\varphi\rangle\,\mathrm{d}x=\int_\Omega\langle F''(0)\boldsymbol{\varepsilon}(\varphi),\boldsymbol{\varepsilon}(\varphi)\rangle\,\mathrm{d}x.$$

By Plancherel's Theorem and the transformation rule $\widehat{\boldsymbol{\varepsilon}(\varphi)}(\xi)=-\mathrm{i}\,\widehat{\varphi}(\xi)\odot\xi$ for all $\xi\in\mathbb{R}^n$ and all[5] $\varphi\in\mathscr{S}(\mathbb{R}^n;\mathbb{R}^n)$, we obtain

$$\begin{aligned}0&=-\int_\Omega\langle F''(0)\boldsymbol{\varepsilon}(\varphi),\boldsymbol{\varepsilon}(\varphi)\rangle\,\mathrm{d}x=-\int_{\mathbb{R}^n}\langle\widehat{F''(0)\boldsymbol{\varepsilon}(\varphi)},\widehat{\boldsymbol{\varepsilon}(\varphi)}\rangle\,\mathrm{d}\xi\\&=\int_{\mathbb{R}^n}\langle F''(0)\widehat{\varphi}(\xi)\odot\xi,\widehat{\varphi}(\xi)\odot\xi\rangle\,\mathrm{d}\xi\\&\geqslant\ell\int_{\mathbb{R}^n}|\widehat{\varphi}(\xi)\odot\xi|^2\,\mathrm{d}\xi=\ell\int_{\mathbb{R}^n}|\widehat{\boldsymbol{\varepsilon}(\varphi)}|^2\,\mathrm{d}\xi=\int_{\mathbb{R}^n}|\boldsymbol{\varepsilon}(\varphi)|^2\,\mathrm{d}x.\end{aligned}$$

Hence, $\boldsymbol{\varepsilon}(\varphi)=0$ and since $\varphi|_{\partial\Omega}=0$, $\varphi\equiv0$. In conclusion, the claimed injectivity follows. Now note that surjectivity of $D(\mathcal{F}(0,\cdot))(0)\colon Y\to Z$ is an immediate consequence of the Schauder estimates recalled in Section 8.4.5: Indeed, since $F''(0)$ gives rise to a strongly symmetric Legendre–Hadamard bilinear form by (H3), given any $V\in\mathcal{L}^{1,n+\alpha}(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})$, there exists $\varphi\in Y$ such that $\operatorname{div}(F''(0)\boldsymbol{\varepsilon}(\varphi))=\operatorname{div}(-V)$ and hence $-\operatorname{div}(F''(0)\boldsymbol{\varepsilon}(\varphi))=\operatorname{div}(V)$ in Ω. Finally, the required boundedness is immediate from Lemma 8.22, and hence we are in position to apply Theorem 10.20. In conclusion, there exists an open ball $\mathbb{B}(g_0,\varepsilon_0)\subset\mathcal{L}^{1,n,1+\alpha}_1(\Omega;\mathbb{R}^n)$ for which there exists a unique map $G\colon\mathbb{B}(g_0,\varepsilon_0)\to Y$ such that $G(g_0)=\varphi_0$ (i.e., $G(0)=0$), $(g,G(g))\in A:=X\times Y$ and $\mathcal{F}(g,G(g))=0$ for all $g\in\mathbb{B}(g_0,\varepsilon_0)=\mathbb{B}(0,\varepsilon_0)$. Rewriting the last condition, however, exactly asserts that whenever $\|g\|_{\mathcal{L}^{1,n+1+\alpha}_1(\Omega;\mathbb{R}^n)}<\varepsilon_0$, then we have with $\varphi_g:=G(g)$

$$0=\mathcal{F}(g,G(g))=-\operatorname{div}(F'(\boldsymbol{\varepsilon}(g)+\boldsymbol{\varepsilon}(G(g)))=-\operatorname{div}(F'(\boldsymbol{\varepsilon}(g)+\boldsymbol{\varepsilon}(\varphi)))\qquad\text{in }\Omega,$$

and because of $\varphi|_{\partial\Omega}=0$, the statement of Theorem 8.3 follows. The proof is complete. □

8.6. The Fubini Property of BD–maps

In this section we provide and discuss a set of Fubini–type properties for fractional Sobolev spaces and the space BD. In the easiest possible sense, a Fubini–type property asserts that if we have a smoothness space $X(\mathbb{R}^n)$, then for $\mathscr{L}^n$–a.e. $x_n\in\mathbb{R}$, the function $\mathbb{R}^{n-1}\ni x'\mapsto u(x',x_n)$ belongs to $X(\mathbb{R}^{n-1})$, too. For instance, suppose that $X(\mathbb{R}^n)=\mathrm{W}^{1,p}(\mathbb{R}^n)$ for $1\leqslant p<\infty$ and let $s\in\{1,\dots,n-1\}$. Then we have

$$\int_{-\infty}^{\infty}\int_{\mathbb{R}^{n-1}}|\partial_s u(x',x_n)|^p\,\mathrm{d}x'\,\mathrm{d}x_n=\int_{\mathbb{R}^n}|\partial_s u|^p\,\mathrm{d}x\leqslant\int_{\mathbb{R}^n}|\nabla u|^p\,\mathrm{d}x,$$

and so $u(\cdot,x_n)\in\mathrm{W}^{1,p}(\mathbb{R}^{n-1})$ for $\mathscr{L}^1$–a.e. $x_n\in\mathbb{R}$. We note here the difference to interior trace theorems: These assert that $\mathrm{W}^{1,p}$–maps have interior traces along *all hyperplanes* Σ *of codimension one* in

$$\begin{cases}\mathrm{W}^{1-\frac{1}{p},p}_{\mathcal{H}^{n-1}}(\Sigma)&\text{if }1<p<\infty,\\\mathrm{L}^1_{\mathcal{H}^{n-1}}(\Sigma)&\text{if }p=1.\end{cases}$$

[5] It is easy to see by approximation that this argument also works for C^1–maps which vanish at the boundary $\partial\Omega$ and thus can be extended to a continuous function vanishing outside of Ω.

For notational clarity, we have used the subscript $\mathcal{H}^{n-1}$ to indicate that these spaces are defined in terms of the $(n-1)$–dimensional Hausdorff measure. The decisive point here is that the significantly improved smoothness which comes from the Fubini–type theorem only holds for *almost all* but *not all* hyperplanes (once these are suitably parametrised).

In view of the applications of this section, we wish to have a Fubini–type for BD for *spheres* (which can be parametrised by their radius) and not hyperplanes parametrised by their signed distance to the origin. For BV, this is more or less standard, and we refer the reader to [134, Lem. 2.3]. Exemplarily, given $u \in \mathrm{BV}(\mathbb{R}^n;\mathbb{R}^N)$, we can bound the $\mathrm{L}^1_{\mathcal{H}^{n-1}}(\partial\mathrm{B}(0,t))$–norm of the *tangential* derivative $\partial_\tau u$ for $\mathscr{L}^1$–a.e. $t>0$ by the total variation $|\mathrm{D}u|(\mathbb{R}^n)$. Most importantly, this yields higher integrability for the restrictions of a BV–function to $\mathscr{L}^1$–a.e. sphere $\partial\mathrm{B}(0,t)$ by use of the Sobolev embedding.

It is difficult to imitate the very last approach for the space BD because it amounts to speaking of a *tangential symmetric derivative* which does not make sense. On the other hand, it is not clear to us whether the tangential derivative can be controlled by $\boldsymbol{\varepsilon}(u)$ at all. This negative result would be very natural in light of Ornstein's Non–Inequality. As such, we aim at obtaining the higher integrability of restrictions to almost every hyperplane or sphere by firstly embedding BD into fractional Sobolev spaces and then employing a Fubini–type property for the latter spaces.

8.6.1. The Fubini Property on Hyperplanes Fubini–type properties have been studied from a different perspective, e.g., in Triebel's monograph [236, Chpt. I.4]. Comparing with [236, Def. I.4.2] and to avoid confusions, given $n \geqslant 2$, $0<p\leqslant\infty$, $0<q\leqslant\infty$ and $s>\sigma_p = n(\frac{1}{p}-1)_+$, the Besov space $\mathrm{B}^s_{p,q}(\mathbb{R}^n)$ is said to have the *Fubini property* if for all $f \in \mathrm{B}^s_{p,q}(\mathbb{R}^n)$ there holds (with '$\sim$' denoting the equivalence of norms)

$$\|f\|_{\mathrm{B}^s_{p,q}(\mathbb{R}^n)} \sim \sum_{j=1}^n \big\|\|f^{x_j}\|_{\mathrm{B}^s_{p,q}(\mathbb{R})}\big\|_{\mathrm{L}^p(\mathbb{R}^{n-1})}. \tag{8.54}$$

Here, for $j\in\{1,...,n\}$, we denote $x_j \mapsto f^{x_j}(x_j) = f(x)$ for $x=(x_1,...,x_n)\in\mathbb{R}^n$ the j–th partial map. In light of our aim explained in the introduction of this section, we would like to switch the roles of $\mathbb{R}$ and $\mathbb{R}^{n-1}$ in (8.54). Moreover, we are merely interested in inequality '$\gtrsim$' in (8.54).

To demonstrate '$\gtrsim$' in (8.54), it is particularly convenient to work with the finite difference characterisation of Besov spaces $\mathrm{B}^s_{p,p}(\mathbb{R}^n)\simeq \mathrm{W}^{s,p}(\mathbb{R}^n)$. However, this approach is not well–suited for the spherical situation, a fact which shall be discussed to some extend in Remark 8.24 below.

Proposition 8.23. *Let $0<s<1$ and $1\leqslant p<\infty$. Then there exists a constant $C>0$ such that for all $u\in\mathrm{W}^{s,p}(\mathbb{R}^n;\mathbb{R}^N)$ the following hold:*

(a) *For $\mathscr{L}^1$–a.e. $x_n\in\mathbb{R}$, the restriction $u(\cdot,x_n)$ belongs to $\mathrm{W}^{s,p}(\mathbb{R}^{n-1};\mathbb{R}^N)$.*

(b) *We have the estimate*

$$\int_{-\infty}^{\infty} [u(\cdot,x_n)]^p_{\mathrm{W}^{s,p}(\mathbb{R}^{n-1};\mathbb{R}^N)}\,\mathrm{d}x_n \leqslant C[u]^p_{\mathrm{W}^{s,p}(\mathbb{R}^n;\mathbb{R}^N)}.$$

Proof. We recall in advance that the Gagliardo seminorm $[\cdot]_{\mathrm{W}^{s,q}(\mathbb{R}^n;\mathbb{R}^N)}$ is equivalent to the seminorm $[\![\varphi]\!]_{\mathrm{W}^{s,q}(\mathbb{R}^n;\mathbb{R}^N)}$ defined by

$$[\![\varphi]\!]^q_{\mathrm{W}^{s,q}(\mathbb{R}^n;\mathbb{R}^N)} := \sum_{j=1}^n \int_0^\infty \|\Delta_{j,h}\varphi\|^q_{\mathrm{L}^q(\mathbb{R}^n;\mathbb{R}^N)}\frac{\mathrm{d}h}{|h|^{1+sq}}$$

in the sense that there exists a constant $c>0$ such that for all $\varphi\in\mathrm{C}^\infty_c(\mathbb{R}^n;\mathbb{R}^n)$ there holds $c^{-1}[\varphi]_{\mathrm{W}^{s,q}(\mathbb{R}^n;\mathbb{R}^N)} \leqslant [\![\varphi]\!]_{\mathrm{W}^{s,q}(\mathbb{R}^n;\mathbb{R}^N)} \leqslant c[\varphi]_{\mathrm{W}^{s,q}(\mathbb{R}^n;\mathbb{R}^N)}$. This is a consequence of the difference–quotient characterisation of Besov spaces $\dot{\mathrm{B}}^s_{p,q}$ and the equivalence $\dot{\mathrm{B}}^s_{p,p}\simeq\dot{\mathrm{W}}^{s,p}$.

Let us write $x = (x_1, ..., x_n) = (x_1, ..., x_{n-1}, x_n) = (x', x_n)$ for a generic element of $\mathbb{R}^n$. We fix $i \in \{1, ..., n-1\}$. In consequence, if $x_n \in \mathbb{R}$ is fixed and we denote $\mathfrak{H}_{x_n} := \{(x', x_n) \colon x' \in \mathbb{R}^{n-1}\}$, we have $x + te_j \in \mathfrak{H}_{x_n}$ for every $x \in \mathfrak{H}_{x_n}$, $j \in \{1, ..., n-1\}$ and $t \in \mathbb{R}$. Then, using the Fubini theorem, we obtain by equivalence of the seminorms just mentioned

$$\begin{aligned}\int_{\mathbb{R}} \int_0^\infty \|\Delta_{i,h} u(\cdot, x_n)\|^p_{\mathrm{L}^p(\mathbb{R}^n;\mathbb{R}^N)} \frac{\mathrm{d}h}{|h|^{1+sp}} \,\mathrm{d}x_n &= \int_0^\infty \int_{\mathbb{R}} \|\Delta_{i,h} u(\cdot, x_n)\|^p_{\mathrm{L}^p(\mathbb{R}^n;\mathbb{R}^N)} \frac{\mathrm{d}x_n \,\mathrm{d}h}{|h|^{1+sp}} \\ &\leqslant \sum_{j=1}^n \int_0^\infty \int_{\mathbb{R}} \|\Delta_{j,h} u(x', \cdot)\|^p_{\mathrm{L}^p(\mathbb{R}^n;\mathbb{R}^N)} \frac{\mathrm{d}x_n \,\mathrm{d}h}{|h|^{1+sp}} \\ &\leqslant C[\![u]\!]^p_{\mathrm{W}^{s,p}(\mathbb{R}^n;\mathbb{R}^N)}.\end{aligned}$$

Hence, summing over all $j = 1, ..., n-1$, we obtain

$$\begin{aligned}\int_{\mathbb{R}} [u(\cdot, x_n)]^p_{\mathrm{W}^{s,p}(\mathbb{R}^{n-1};\mathbb{R}^N)} \,\mathrm{d}x_n &\leqslant c \int_{\mathbb{R}} [\![u(\cdot, x_n)]\!]^p_{\mathrm{W}^{s,p}(\mathbb{R}^{n-1};\mathbb{R}^N)} \,\mathrm{d}x_n \leqslant C[\![u]\!]^p_{\mathrm{W}^{s,p}(\mathbb{R}^n;\mathbb{R}^N)} \\ &\leqslant C[u]^p_{\mathrm{W}^{s,p}(\mathbb{R}^n;\mathbb{R}^N)}\end{aligned}$$

The proof is complete. ☐

Before we continue, we will abbreviate notation and write

$$\boxed{\sigma := \mathcal{H}^{n-1} \text{ and } \mathrm{d}\sigma_x := \mathrm{d}\mathcal{H}^{n-1}(x)}$$

in the rest of this paragraph. To motivate a more intricate but conceptually more appropriate approach to the fractional Fubini–type properties on spheres, we briefly pause and give the following remark.

Remark 8.24. *In view of the Fubini–type property for spheres, it is convenient to work directly on the Gagliardo seminorms because it allows to integrate over spherical layers. Note that the Gagliardo seminorm of a smooth function $u \colon \mathbb{R}^n \to \mathbb{R}^N$ on a sphere $\partial\mathrm{B}(0,t)$ is given by*

$$[u]^p_{\mathrm{W}^{s,p}(\partial\mathrm{B}(0,t);\mathbb{R}^N)} := \iint_{\partial\mathrm{B}(0,t)\times\partial\mathrm{B}(0,t)} \frac{|u(x) - u(y)|^p}{|x-y|^{n+sp-1}} \,\mathrm{d}\sigma_x \,\mathrm{d}\sigma_y. \tag{8.55}$$

This, in turn, is to be integrated with respect to $t \in [0, R]$ and we aim at estimating the resulting quantitity against $[u]^p_{\mathrm{W}^{s,p}(\mathrm{B}(0,R);\mathbb{R}^N)}$. Passing to coordinates $(\omega, r) \in \mathbb{S}^{n-1} \times [0, R]$, we can write (with $\mathrm{B}_t := \mathrm{B}(0, t)$)

$$[u]^p_{\mathrm{W}^{s,p}(\mathrm{B}(0,R);\mathbb{R}^N)} = \int_0^R \int_{\partial\mathrm{B}_t} \int_0^R \int_{\partial\mathrm{B}_r} \frac{|u(x) - u(y)|^p}{|x-y|^{n+sp}} \,\mathrm{d}\sigma_y \,\mathrm{d}r \,\mathrm{d}\sigma_x \,\mathrm{d}t. \tag{8.56}$$

If we integrate (8.55) over radii $t \in [0, R]$, then we have one radius integral and two integrals over spheres (each of the same *radius) emerging, whereas in (8.56), there are two radius integrals and two integrals over* different *spheres emerging. In this sense, the additional exponent (-1) as it appears in the denominator of the integrand in (8.55) is seen to be necessarily responsible for the emergence of the second radius integral and the integral over a different sphere in (8.56). This happens at the cost of losing one power and will eventually lead to the desired Fubini–type property.*

The emergence of the second radius integral in the spherical case amounts to the emergence of a second integral with respect to the signed distance of affine hyerplanes $\{(x', x_n) \colon x' \in \mathbb{R}^{n-1}\}$ in the plane case. As it gives good motivation for the proof in the spherical case, we now give a

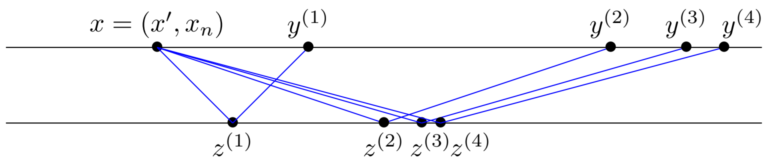

Figure 8.1: The situation of the direct proof of Proposition 8.23 in two dimensions. All the points $x, y^{(1)}, ..., y^{(5)}$ have the same n-th coordinate. To estimate the difference of the mapping evaluated at x and $y^{(i)}$, we create a third point $z^{(i)}$ of fixed n–th coordinate t, say. Then we integrate over a certain interval of such t to create the fourth integral of the Gagliardo seminorm over $\mathbb{R}^n$ on the right hand side of the desired inequality.

Direct proof of Proposition 8.23 by Gagliardo seminorms. We write $(x', x_n), (y', y_n)$ for two generic points in $\mathbb{R}^n$, where $x', y' \in \mathbb{R}^{n-1}$ and $x_n, y_n \in \mathbb{R}$. We have for any $h \in \mathbb{R}^{n-1}$ and any $y_n \in \mathbb{R}$

$$\begin{aligned}|u(x'+h, x_n) - u(x', x_n)|^p \lesssim\ & |u(x'+h, x_n) - u(x'+\tfrac{1}{2}h, y_n)|^p \\ & + |u(x'+\tfrac{1}{2}h, y_n) - u(x', x_n)|^p\end{aligned}$$

and hereafter, integrating the previous inequality with respect to $y_n \in (x_n - \frac{h}{2}, x_n + \frac{h}{2})$,

$$\begin{aligned}|u(x'+h, x_n) - u(x', x_n)|^p \lesssim & ⨍_{x_n-\frac{h}{2}}^{x_n+\frac{h}{2}} |u(x'+h, x_n) - u(x'+\tfrac{1}{2}h, y_n)|^p \\ & + |u(x'+\tfrac{1}{2}h, y_n) - u(x', x_n)|^p \, \mathrm{d}y_n.\end{aligned} \tag{8.57}$$

From a geometric perspective, this is illustrated in Figure 8.1. Now note that with the change of variables $y' = x' + h$ with $h \in \mathbb{R}^{n-1}$ there holds

$$\begin{aligned}
&\int_{-\infty}^{\infty}\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^{n-1}} \frac{|u(x', x_n) - u(y', x_n)|^p}{|x'-y'|^{n+sp-1}} \,\mathrm{d}\sigma_{x'}\,\mathrm{d}\sigma_{y'}\,\mathrm{d}x_n \\
&\leqslant \int_{-\infty}^{\infty}\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^{n-1}} \frac{|u(x'+h, x_n) - u(x', x_n)|^p}{|h|^{n+sp-1}} \,\mathrm{d}\sigma_{x'}\,\mathrm{d}\sigma_{h}\,\mathrm{d}x_n \\
&\overset{(8.57)}{\leqslant} \int_{-\infty}^{\infty}\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^{n-1}} ⨍_{x_n-\frac{h}{2}}^{x_n+\frac{h}{2}} \frac{|u(x'+h, x_n) - u(x'+\frac{1}{2}h, y_n)|^p}{h^{n+sp-1}} \,\mathrm{d}y_n\,\mathrm{d}\sigma_{x'}\,\mathrm{d}\sigma_{h}\,\mathrm{d}x_n \\
&+ \int_{-\infty}^{\infty}\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^{n-1}} ⨍_{x_n-\frac{h}{2}}^{x_n+\frac{h}{2}} \frac{|u(x'+\frac{1}{2}h, y_n) - u(x', x_n)|^p}{h^{n+sp-1}} \,\mathrm{d}y_n\,\mathrm{d}\sigma_{x'}\,\mathrm{d}\sigma_{h}\,\mathrm{d}x_n \\
&\leqslant \int_{-\infty}^{\infty}\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^{n-1}} \int_{x_n-\frac{h}{2}}^{x_n+\frac{h}{2}} \frac{|u(x'+h, x_n) - u(x'+\frac{1}{2}h, y_n)|^p}{h^{n+sp}} \,\mathrm{d}y_n\,\mathrm{d}\sigma_{x'}\,\mathrm{d}\sigma_{h}\,\mathrm{d}x_n \\
&+ \int_{-\infty}^{\infty}\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^{n-1}} \int_{x_n-\frac{h}{2}}^{x_n+\frac{h}{2}} \frac{|u(x'+\frac{1}{2}h, y_n) - u(x', x_n)|^p}{h^{n+sp}} \,\mathrm{d}y_n\,\mathrm{d}\sigma_{x'}\,\mathrm{d}\sigma_{h}\,\mathrm{d}x_n \\
&=: (*) + (**).
\end{aligned}$$

The two integrals are estimated in the same way, and so we focus on the estimation of $(*)$ only. We change variables as follows. Put $z' := x' + h$. Then $x' + \frac{h}{2} = z' - \frac{h}{2}$. Therefore,

$$\begin{aligned}(*) \leqslant C \int_{-\infty}^{\infty}\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^{n-1}} \int_{x_n-\frac{h}{2}}^{x_n+\frac{h}{2}} \frac{|u(z', x_n) - u(z'-\frac{1}{2}h, y_n)|^p}{(|h| + |x_n - y_n|)^{n+sp}} \,\mathrm{d}y_n\,\mathrm{d}\sigma_{z'}\,\mathrm{d}\sigma_{h}\,\mathrm{d}x_n \\ =: (*)'.\end{aligned}$$

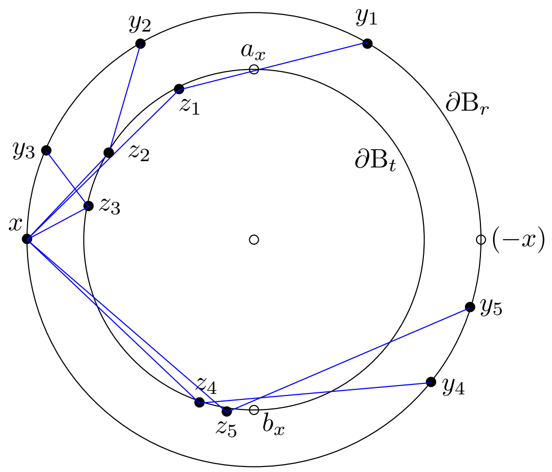

Figure 8.2: The situation in two dimensions for selected points y_i. Excluding the $\mathcal{H}^{n-1}$–nullset $(-x)$, we imitate the construction of Figure 8.1 and project the midpoints of the line segment of x and y_i onto $\partial\,\mathrm{B}_t$. This gives rise to the projections $z_i = \pi_t(x, y_i)$.

Here we have used that

$$y_n \in (x_n - h/2, x_n + h/2) \Rightarrow |x_n - y_n| + h \leqslant 2h \Rightarrow \frac{1}{h^{n+sp}} \leqslant \frac{C}{(|h| + |x_n - y_n|)^{n+sp}}.$$

By Fubini and the change of variables $\xi = z' - \frac{1}{2}h$ and hereafter $h = 2(z' - \xi)$,

$$\begin{aligned}
(*)' &\leqslant C \int_{-\infty}^{\infty} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} \int_{-\infty}^{\infty} \frac{|u(z', x_n) - u(z' - \frac{1}{2}h, y_n)|^p}{(|h| + |x_n - y_n|)^{n+sp}} \,\mathrm{d}y_n \,\mathrm{d}\sigma_{z'} \,\mathrm{d}\sigma_h \,\mathrm{d}x_n \\
&= C \int_{-\infty}^{\infty} \int_{\mathbb{R}^{n-1}} \int_{-\infty}^{\infty} \int_{\mathbb{R}^{n-1}} \frac{|u(z', x_n) - u(z' - \frac{1}{2}h, y_n)|^p}{(|h| + |x_n - y_n|)^{n+sp}} \,\mathrm{d}\sigma_h \,\mathrm{d}y_n \,\mathrm{d}\sigma_{z'} \,\mathrm{d}x_n \\
&= C \int_{-\infty}^{\infty} \int_{\mathbb{R}^{n-1}} \int_{-\infty}^{\infty} \int_{\mathbb{R}^{n-1}} \frac{|u(z', x_n) - u(\xi, y_n)|^p}{(|2(z' - \xi)| + |x_n - y_n|)^{n+sp}} \,\mathrm{d}\sigma_{z'} \,\mathrm{d}y_n \,\mathrm{d}\sigma_{x'} \,\mathrm{d}x_n \\
&\leqslant C \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} \,\mathrm{d}x \,\mathrm{d}y.
\end{aligned}$$

In the very last step, we used a simple change of coordinates argument similar to that from above in conjunction with the fact that $\|\cdot\|$ given by $\|a\| := 2|a'| + |a_n|$ for $a = (a', a_n) \in \mathbb{R}^n$ defines a norm on $\mathbb{R}^n$ which is equivalent to the usual euclidean norm $|\cdot|$ and therefore, in particular, $|\cdot| \leqslant c\|\cdot\|$. By the definition of the Gagliardo seminorms, the proof is complete. □

8.6.2. The Fubini Property on Spheres We now continue the study of Fubini–type properties for fractional Sobolev spaces on spheres. The statement corresponding to Proposition 8.23 now reads as follows:

Proposition 8.25. *Let $0 < s < 1$ and $1 \leqslant p < \infty$. Then there exists a constant $C = C(s, p, n) > 0$ such that for all $x_0 \in \mathbb{R}^n$, $R > 0$ and all $u \in \mathrm{W}^{s,p}(\mathbb{R}^n; \mathbb{R}^N)$ there holds*

$$\int_0^R [u]^p_{\mathrm{W}^{s,p}(\partial \mathrm{B}(x_0,r);\mathbb{R}^N)} \,\mathrm{d}r \leqslant C[u]^p_{\mathrm{W}^{s,p}(\mathrm{B}(x_0,R);\mathbb{R}^N)}, \tag{8.58}$$

that is,

$$\int_0^R \iint_{\partial\mathrm{B}(x_0,r)\times\partial\mathrm{B}(x_0,r)} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp-1}}\,\mathrm{d}\sigma_y\,\mathrm{d}\sigma_x\,\mathrm{d}r \leqslant C\iint_{\mathrm{B}(x_0,R)\times\mathrm{B}(x_0,R)} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,\mathrm{d}x\,\mathrm{d}y. \tag{8.59}$$

Proof. It is no loss of generality to assume $x_0 = 0$. In a first step, we change variables to the unit ball and put $\widetilde{x} = rx$, $\widetilde{y} = ry$. We thereby obtain (recall that $\mathbb{S}^{n-1} := \partial\mathrm{B}(0,1)$)

$$\begin{aligned}(*) := \int_0^R \iint_{\partial\mathrm{B}(x_0,r)\times\partial\mathrm{B}(x_0,r)} & \frac{|u(\widetilde{x})-u(\widetilde{y})|^p}{|\widetilde{x}-\widetilde{y}|^{n+sp-1}}\,\mathrm{d}\sigma_{\widetilde{y}}\,\mathrm{d}\sigma_{\widetilde{x}}\,\mathrm{d}r \\ &= \int_0^R (r^{n-1})^2 \iint_{\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}} \frac{|u(rx)-u(ry)|^p}{|rx-ry|^{n+sp-1}}\,\mathrm{d}\sigma_y\,\mathrm{d}\sigma_x\,\mathrm{d}r \\ &= \iint_{\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}} \int_0^R (r^{n-1})^2 \frac{|u(rx)-u(ry)|^p}{|rx-ry|^{n+sp-1}}\,\mathrm{d}r\,\mathrm{d}\sigma_y\,\mathrm{d}\sigma_x.\end{aligned} \tag{8.60}$$

We aim at modifying the direct proof of Proposition 8.23 using the Gagliardo–seminorms (and not their Besov equivalents). To do so, we divide the proof in two substeps.

Step 1. Let $x \in \mathbb{S}^{n-1}$ be fixed. We start by noting that since a single point is always a nullset with respect to $\mathcal{H}^{n-1}$ provided $n \geqslant 2$, we have

$$\int_{\mathbb{S}^{n-1}} \frac{|u(rx)-u(ry)|^p}{|x-y|^{n-sp+1}}\,\mathrm{d}\sigma_y = \int_{\mathbb{S}^{n-1}\setminus\{-x\}} \frac{|u(rx)-u(ry)|^p}{|rx-ry|^{n-sp+1}}\,\mathrm{d}\sigma_y \tag{8.61}$$

for any $x \in \mathbb{S}^{n-1}$. Given $x \in \mathbb{S}^{n-1}$ and $0 < t < R$, we put

$$\pi_t(x,y) := t\frac{x+y}{|x+y|}, \qquad y \in \mathbb{S}^{n-1}\setminus\{-x\}$$

and we note that the mapping $\Pi_t\colon \mathbb{S}^{n-1}\setminus\{-x\} \to \mathbb{S}^{n-1}$ given by $\Pi_t(y) := \pi_t(x,y)$ is well–defined. The mapping Π_t is the projection of the middle point of the interval $[x,y]$ onto $\partial\mathrm{B}(0,t)$.

Imitating the construction in the hyperplane case discussed above (cp. Figure 8.6.2), we now estimate for arbitrary $x \in \mathbb{S}^{n-1}$ and $y \in \mathbb{S}^{n-1}$

$$|u(rx)-u(ry)|^p \leqslant C(|u(rx)-u(\pi_t(x,y))|^p + |u(ry)-u(\pi_t(x,y))|^p)$$

and hence for all $0 < a(x,y) \leqslant b(x,y) \leqslant R$

$$\begin{aligned}|u(rx)-u(ry)|^p \leqslant C & \fint_{a(x,y)}^{b(x,y)} |u(rx)-u(\pi_t(x,y))|^p\,\mathrm{d}t \\ &+ C\fint_{a(x,y)}^{b(x,y)} |u(ry)-u(\pi_t(x,y))|^p\,\mathrm{d}t.\end{aligned} \tag{8.62}$$

At this point, fix $0 < r \leqslant R$. We then choose $a(x,y) := r(1-\frac{|x-y|}{4})$ and $b(x,y) := r$. This particularly implies

$$|b(x,y)-a(x,y)| = r\frac{|x-y|}{4} \quad\text{and}\quad \frac{r}{2} \leqslant a(x,y) \leqslant b(x,y) = r. \tag{8.63}$$

Moreover, for all $y \in \mathbb{S}^{n-1}\setminus\{-x\}$ we have for $0 < t \leqslant r \leqslant R$

$$\begin{aligned}|rx-\pi_t(x,y)| &= r\left|x-\gamma\frac{x+y}{|x+y|}\right| \qquad (\text{where } \gamma = \frac{t}{r}) \\ &\leqslant r\left|x-\frac{x+y}{|x+y|}\right| + r(1-\gamma)\left|\frac{x+y}{|x+y|}\right| = r\left|x-\frac{x+y}{|x+y|}\right| + r(1-\gamma).\end{aligned}$$

On the other hand, we clearly always have for $y \neq -x$

$$\left| x - \frac{x+y}{|x+y|} \right| \leqslant |x-y|,$$

a formal proof of which is included for completeness in the appendix, cp. Lemma 10.21. In conclusion, we record

$$|rx - \pi_t(x,y)| \leqslant r|x-y| + (r-t). \tag{8.64}$$

Step 2. Combining (8.60), (8.61) and (8.62), we then obtain

$$\begin{aligned}(*) &\leqslant C \iint_{\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}} \int_0^R (r^{n-1})^2 \fint_{r(1-\frac{|x-y|}{4})}^{r} \frac{|u(rx)-u(\pi_t(x,y))|^p}{|rx-ry|^{n+sp-1}} \,\mathrm{d}t\,\mathrm{d}r\,\mathrm{d}\sigma_x\,\mathrm{d}\sigma_y \\ &+ C \iint_{\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}} \int_0^R (r^{n-1})^2 \fint_{r(1-\frac{|x-y|}{4})}^{r} \frac{|u(ry)-u(\pi_t(x,y))|^p}{|rx-ry|^{n+sp-1}} \,\mathrm{d}t\,\mathrm{d}r\,\mathrm{d}\sigma_y\,\mathrm{d}\sigma_x =: \mathbf{I} + \mathbf{II}.\end{aligned}$$

The two integrals are essentially symmetric in x and y (also note that $\pi_t(x,y) = \pi_t(y,x)$), and so it suffices to employ the desired estimate for one of these two integrals. We firstly estimate because of the first part of (8.63)

$$\begin{aligned}\mathbf{I} &= C \iint_{\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}} \int_0^R (r^{n-1})^2 \fint_{r(1-\frac{|x-y|}{4})}^{r} \frac{|u(rx)-u(\pi_t(x,y))|^p}{|rx-ry|^{n+sp-1}} \,\mathrm{d}t\,\mathrm{d}r\,\mathrm{d}\sigma_x\,\mathrm{d}\sigma_y \\ &\leqslant C \iint_{\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}} \int_0^R (r^{n-1})^2 \int_{r(1-\frac{|x-y|}{4})}^{r} \frac{|u(rx)-u(\pi_t(x,y))|^p}{|rx-ry|^{n+sp}} \,\mathrm{d}t\,\mathrm{d}r\,\mathrm{d}\sigma_x\,\mathrm{d}\sigma_y =: \mathbf{J}.\end{aligned}$$

To estimate $\mathbf{J}$, note that if $r(1-\frac{|x-y|}{4}) \leqslant t \leqslant r$, then

$$-t \leqslant r\Big(\frac{|x-y|}{4} - 1\Big) \Rightarrow r - t \leqslant r\frac{|x-y|}{4} \overset{(8.64)}{\Longleftrightarrow} |rx - \pi_t(x,y)| \leqslant \frac{5}{4} r|x-y|. \tag{8.65}$$

Moreover, we note that for such t, we have

$$r(1-\frac{|x-y|}{4}) \leqslant t \leqslant r \Rightarrow (1-\frac{|x-y|}{4}) \leqslant \frac{t}{r} \leqslant 1 \Rightarrow 1 \leqslant \frac{r}{t} \leqslant \frac{1}{1-\frac{|x-y|}{4}} \overset{|x-y|\leqslant 2}{\leqslant} 2. \tag{8.66}$$

We then estimate

$$\begin{aligned}\mathbf{J} &\overset{(8.65)}{\leqslant} C \iint_{\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}} \int_0^R (r^{n-1})^2 \int_{r(1-\frac{|x-y|}{4})}^{r} \frac{|u(rx)-u(\pi_t(x,y))|^p}{|rx-\pi_t(x,y)|^{n+sp}} \,\mathrm{d}t\,\mathrm{d}r\,\mathrm{d}\sigma_x\,\mathrm{d}\sigma_y \\ &= C \iint_{\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}} \int_0^R r^{n-1} \int_{r(1-\frac{|x-y|}{4})}^{r} \frac{|u(rx)-u(\pi_t(x,y))|^p}{|rx-\pi_t(x,y)|^{n+sp}} \Big(\frac{r}{t}\Big)^{n-1} t^{n-1} \,\mathrm{d}t\,\mathrm{d}r\,\mathrm{d}\sigma_x\,\mathrm{d}\sigma_y \\ &\overset{(8.66)}{\leqslant} C \iint_{\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}} \int_0^R r^{n-1} \int_{r(1-\frac{|x-y|}{4})}^{r} \frac{|u(rx)-u(\pi_t(x,y))|^p}{|rx-\pi_t(x,y)|^{n+sp}} t^{n-1} \,\mathrm{d}t\,\mathrm{d}r\,\mathrm{d}\sigma_x\,\mathrm{d}\sigma_y = \mathbf{J}'.\end{aligned}$$

At this point, we change variables and put $z := (x+y)/|x+y|$. In doing so, we must keep track of the Jacobian emerging, but by a geometric argument one can see that this Jacobian has modulus which is bounded only in dependence of n. Exemplarily, in the two–dimensional situation this can be seen by noting that for fixed x, the point $z = z(y)$ runs with half the speed of $y \in \mathbb{S}^{n-1} \setminus \{-x\}$, and for the reader's convenience, this is explained in more detail in Section 10.6. Substituting as indicated, we then obtain

$$\begin{aligned}
\mathbf{J}' &\leqslant C \iint_{\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}} \int_0^R r^{n-1} \int_{r(1-\frac{|x-y|}{4})}^{r} \frac{|u(rx)-u(tz)|^p}{|rx-tz|^{n+sp}} t^{n-1}\,\mathrm{d}t\,\mathrm{d}r\,\mathrm{d}\sigma_x\,\mathrm{d}\sigma_z \\
&\leqslant C \int_0^R \int_0^R \int_{\mathbb{S}^{n-1}} \int_{\mathbb{S}^{n-1}} \frac{|u(rx)-u(tz)|^p}{|rx-tz|^{n+sp}} t^{n-1} r^{n-1}\,\mathrm{d}t\,\mathrm{d}r\,\mathrm{d}\sigma_x\,\mathrm{d}\sigma_z \\
&\overset{(*)}{\leqslant} C \iint_{\mathrm{B}(0,R)\times\mathrm{B}(0,R)} \frac{|u(\widetilde{x})-u(\widetilde{y})|^p}{|\widetilde{x}-\widetilde{y}|^{n+sp}}\,\mathrm{d}\widetilde{x}\,\mathrm{d}\widetilde{y}.
\end{aligned}$$

Here, $(*)$ is a direct consequence of a passage to polar coordinates and the fact that for any $y \in \mathbb{S}^{n-1}$ there holds $\mathbb{S}^{n-1} \setminus \{-y\} \ni x \mapsto (x+y)/|x+y| \in \mathbb{S}^{n-1}$. The proof is complete. □

8.6.3. Consequences for BD By the result from Chapter 4 (cp. Theorem 4.11 (b)), $\mathrm{BD}(\mathbb{R}^n) \hookrightarrow \mathrm{W}^{\theta,n/(n-1+\theta)}(\mathbb{R}^n;\mathbb{R}^n)$ for any $0<\theta<1$. A localised version of this reads as follows.

Lemma 8.26. *Let Ω be an open and bounded Lipschitz domain in $\mathbb{R}^n$. Then for any $0<\theta<1$, there exists a constant $c>0$ such that*

$$\inf_{a\in\mathcal{R}(\Omega)} \|u-a\|_{\mathrm{W}^{\theta,\frac{n}{n-1+\theta}}(\Omega;\mathbb{R}^n)} \leqslant c|\,\mathrm{E}u|(\Omega) \qquad \textit{for all } u \in \mathrm{BD}(\Omega).$$

Proof. Let $v \in \mathrm{LD}(\Omega)$ be arbitrary. It is well–known that there exists a bounded linear extension operator $E\colon \mathrm{LD}(\Omega) \to \mathrm{LD}(\mathbb{R}^n)$ (also see Chapter 3.5). By Van Schaftingen's embedding, $\mathrm{LD}(\mathbb{R}^n) \hookrightarrow \mathrm{W}^{\theta,n/(n-1+\theta)}(\mathbb{R}^n)$ for any $0<\theta<1$. Therefore,

$$\begin{aligned}
\|v\|_{\mathrm{W}^{\theta,\frac{n}{n-1+\theta}}(\Omega;\mathbb{R}^n)} &\leqslant \|Ev\|_{\mathrm{W}^{\theta,\frac{n}{n-1+\theta}}(\mathbb{R}^n;\mathbb{R}^n)} \\
&\leqslant \|Ev\|_{\mathrm{LD}(\mathbb{R}^n)} \\
&\leqslant C\|v\|_{\mathrm{LD}(\Omega)}.
\end{aligned}$$

We pick $a \in \mathcal{R}(\Omega)$ such that $\|u-a\|_{\mathrm{L}^1(\Omega;\mathbb{R}^n)} \leqslant C\|\boldsymbol{\varepsilon}(u)\|_{\mathrm{L}^1(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})}$; such an element exists by $\dim(\mathcal{R}(\Omega))<\infty$. Consequently, applying the above estimate to $v := u-a$, we find

$$\begin{aligned}
\|u-a\|_{\mathrm{W}^{\theta,\frac{n}{n-1+\theta}}(\Omega;\mathbb{R}^n)} &\leqslant C\|u-a\|_{\mathrm{LD}(\Omega)} = C\big(\|u-a\|_{\mathrm{L}^1(\Omega;\mathbb{R}^n)} + \|\boldsymbol{\varepsilon}(u)\|_{\mathrm{L}^1(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})}\big) \\
&\leqslant C\|\boldsymbol{\varepsilon}(u)\|_{\mathrm{L}^1(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})} \qquad \text{(by Poincaré)}.
\end{aligned}$$

Now, for $u \in \mathrm{BD}(\Omega)$, we choose a sequence $(u_k) \subset \mathrm{C}^\infty(\Omega;\mathbb{R}^n) \cap \mathrm{BD}(\Omega)$ which converges strictly and pointwisely $\mathscr{L}^n$–a.e. to u. For each $k \in \mathbb{N}$, we find $a_k \in \mathcal{R}(\Omega)$ such that $\|u_k - a_k\|_{\mathrm{W}^{\theta,n/(n-1+\theta)}(\Omega;\mathbb{R}^n)} \leqslant C\|\boldsymbol{\varepsilon}(u_k)\|_{\mathrm{L}^1(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})}$. By strict convergence $u_k \overset{s}{\to} u$ as $k\to\infty$, there exists a constant $c>0$ such that for all $k\in\mathbb{N}$ there holds

$$\sup_{k\in\mathbb{N}} \int_\Omega |a_k|\,\mathrm{d}x \leqslant c \sup_{k\in\mathbb{N}} \int_\Omega |u_k|\,\mathrm{d}x < \infty.$$

Since $\mathcal{R}(\Omega)$ is a finite dimensional space, we find $a \in \mathcal{R}(\Omega)$ and a subsequence $(a_{k(j)}) \subset (a_k)$ such that $a_{k(j)} \to a$ in *any* norm on $\mathcal{R}(\Omega)$. In consequence, we obtain that $u_{k(j)} - a_{k(j)}$ converges to $u-a$ pointwisely $\mathscr{L}^n$–a.e. as $j\to\infty$ and hence, by Fatou's lemma,

$$\begin{aligned}
\inf_{\alpha\in\mathcal{R}(\Omega)} \|u-\alpha\|_{\mathrm{W}^{\theta,n/(n-1+\theta)}(\Omega;\mathbb{R}^n)} &\leqslant \|u-a\|_{\mathrm{W}^{\theta,n/(n-1+\theta)}(\Omega;\mathbb{R}^n)} \\
&\leqslant \liminf_{j\to\infty} \|u_{k(j)} - a_{k(j)}\|_{\mathrm{W}^{\theta,n/(n-1+\theta)}(\Omega;\mathbb{R}^n)} \\
&\leqslant C \liminf_{j\to\infty} \|\boldsymbol{\varepsilon}(u_{k(j)})\|_{\mathrm{L}^1(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})} \\
&\leqslant C|\,\mathrm{E}u|(\Omega).
\end{aligned}$$

The proof is complete. □

We wish to give a remark on scaling. More precisely, we wish to argue that there exists a constant such that for all $u \in \mathrm{LD}(\mathbb{R}^n)$, $x_0 \in \mathbb{R}^n$ and $R > 0$ there exists $a \in \mathcal{R}(\mathbb{R}^n)$ such that for all $0 < s < 1$ and $p = n/(n-1+s)$ there holds (with $\mathrm{B}_R = \mathrm{B}(x_0, R)$)

$$\left(\fint_{\mathrm{B}_R}\int_{\mathrm{B}_R}\frac{|u_a(x)-u_a(y)|^p}{|x-y|^{n+sp}}\,\mathrm{d}x\,\mathrm{d}y\right)^{\frac{1}{p}} \leqslant CR^{1-s}\fint_{\mathrm{B}_R}|\boldsymbol{\varepsilon}(u)|\,\mathrm{d}x, \tag{8.67}$$

where $C > 0$ neither depends on R nor on x_0. Here, we have put $u_a := u - a$. It is also clear that a corresponding statement is available for BD–maps, too. Determining a rigid deformation $b \in \mathcal{R}(\mathbb{R}^n)$ such that $(*)$ in the following inequality holds on the unit ball due to the preceding lemma, we find

$$\begin{aligned}
&\left(\int_{\mathrm{B}_R}\int_{\mathrm{B}_R}\frac{|u_b(x)-u_b(y)|^p}{|x-y|^{n+sp}}\,\mathrm{d}x\,\mathrm{d}y\right)^{\frac{1}{p}} \\
&\quad = \frac{R^{\frac{2n}{p}}}{R^{\frac{n+sp}{p}}}\left(\int_{\mathrm{B}_1}\int_{\mathrm{B}_1}\frac{|u_b(Rx)-u_b(Ry)|^p}{|x-y|^{n+sp}}\,\mathrm{d}x\,\mathrm{d}y\right)^{\frac{1}{p}} \\
&\quad \overset{(*)}{\leqslant} \frac{R^{\frac{2n}{p}}}{R^{\frac{n+sp}{p}}}R\int_{\mathrm{B}_1}|(\boldsymbol{\varepsilon}(u_b))(Rx)|\,\mathrm{d}x \\
&\quad = R^{\frac{n}{p}}R^{1-s}\fint_{\mathrm{B}_R}|\boldsymbol{\varepsilon}(u)|\,\mathrm{d}x,
\end{aligned}$$

and this in turn determines $a \in \mathcal{R}(\mathbb{R}^n)$. We divide the last inequality by $R^{\frac{n}{p}}$ to consequently deduce (8.67).

We proceed with a consequence of the Fubini–type theorem from above which is substantial in the main part of the proof below:

Theorem 8.27. *Let $n \geqslant 2$ and $0 < \theta < 1$ be arbitrary. Let $x_0 \in \mathbb{R}^n$, $R > 0$ and $u \in \mathrm{BD}_{\mathrm{loc}}(\mathbb{R}^n)$. Then for $\mathscr{L}^1$–almost all radii $0 < r < R$, the restrictions $u|_{\partial\mathrm{B}(x_0,r)}$ are well–defined and belong to the space $\mathrm{W}^{\theta,n/(n-1+\theta)}(\partial\mathrm{B}(x_0,r);\mathbb{R}^n)$.*

Moreover, there exists a constant $C = C(n,\theta) > 0$ (which, in particular, is independent of x_0, R and u), such that for all $0 < s < r \leqslant R$ there exists $t \in (s,r)$ with

$$\begin{aligned}
\left(\fint_{\partial\mathrm{B}(x_0,t)}\int_{\partial\mathrm{B}(x_0,t)}\frac{|u_\alpha(x)-u_\alpha(y)|^{\frac{n}{n-1+\theta}}}{|x-y|^{n-1+\frac{n\theta}{n-1+\theta}}}\,\mathrm{d}\sigma_x\,\mathrm{d}\sigma_y\right)^{\frac{n-1+\theta}{n}} &\leqslant C\frac{r^{1-\theta+\frac{n}{p}}}{t^{\frac{(n-1)(n-1+\theta)}{n}}(s-r)^{\frac{n-1+\theta}{n}}}\times \\
&\quad\times\frac{|\mathrm{E}u|(\mathrm{B}(x_0,r))}{r^n},
\end{aligned} \tag{8.68}$$

where $\alpha \in \mathcal{R}(\mathbb{R}^n)$ is a suitable rigid deformation. Especially, $C > 0$ does not depend on u, s, t, r, R or x_0.

Proof. It is not restrictive to assume $x_0 = 0$. The proof is now split into three parts, ensuring the pointwise evaluation of u $\mathcal{H}^{n-1}$–a.e. on $\mathscr{L}^1$–almost every sphere centered at the origin, a mean–valued–type lemma for Lebesgue functions and the conclusion of the proof.

Step 1. Existence of sufficiently many Lebesgue points. Let $u \in \mathrm{BD}(\mathbb{R}^n)$ and let $0 < R_1 < R_2 < \infty$ be arbitrary. Since $\mathrm{E}u$ is a Radon measure, so is $|\mathrm{E}u|$ and hence the set $I := \{t \in (R_1,R_2)\colon\ |\mathrm{E}u|(\partial\mathrm{B}(0,t)) > 0\}$ is at most countable. Hence $\mathscr{L}^1((R_1,R_2)\setminus I) = \mathscr{L}^1((R_1,R_2)) = R_2 - R_1$. Let $t \in (R_1,R_2)\setminus I$. We recall that, by Lemma 4.16 and since $\partial\mathrm{B}(0,t)$ is a C^1–hypersurface,

$$\mathrm{E}u\llcorner\partial\mathrm{B}(0,t) = (u^+ - u^-)\odot\nu_{\partial\mathrm{B}(0,t)}\mathcal{H}^{n-1}\llcorner\partial\mathrm{B}(0,t) \tag{8.69}$$

with the one–sided Lebesgue limits $u^{\pm}$ and the outer unit normal $\nu_{\partial\mathrm{B}(0,t)}$ to $\partial\mathrm{B}(0,t)$. Therefore we have

$$\int_{\partial\mathrm{B}(0,t)} |(u^+-u^-)\odot\nu_{\partial\mathrm{B}(0,t)}|\,\mathrm{d}\mathcal{H}^{n-1} = |\,\mathrm{E}u|(\partial\mathrm{B}(0,t)) \overset{t\in(R_1,R_2)\setminus I}{=} 0. \tag{8.70}$$

This implies $|(u^+-u^-)\odot\nu_{\partial\mathrm{B}(0,t)}|=0$ $\mathcal{H}^{n-1}$–a.e. on $\partial\mathrm{B}(0,t)$, and since there exists a constant $c=c(n)>0$ such that $|a|\,|b|\leqslant c|a\odot b|$, we conclude that $\widetilde{u}(x_0):=u^+(x_0)=u^-(x_0)$ holds for $\mathcal{H}^{n-1}$–a.e. $x_0\in\partial\mathrm{B}(0,t)$. Pick such a $x_0\in\partial\mathrm{B}(0,t)$. We then have

$$\lim_{r\searrow 0} \fint_{\mathrm{B}(x_0,r)\cap\mathrm{B}(0,t)} |u-\widetilde{u}(x_0)|\,\mathrm{d}\mathscr{L}^n = \lim_{r\searrow 0} \fint_{\mathrm{B}(x_0,r)\cap\overline{\mathrm{B}(0,t)}^c} |u-\widetilde{u}(x_0)|\,\mathrm{d}\mathscr{L}^n = 0. \tag{8.71}$$

We now claim that this implies

$$\lim_{r\searrow 0} \fint_{\mathrm{B}(x_0,r)} |u-\widetilde{u}(x_0)|\,\mathrm{d}\mathscr{L}^n = 0. \tag{8.72}$$

In fact, we have with $\omega_n=\mathscr{L}^n(\mathrm{B}(0,1))$

$$\begin{aligned}
\fint_{\mathrm{B}(x_0,r)} |u-\widetilde{u}(x_0)|\,\mathrm{d}\mathscr{L}^n &= \frac{1}{\omega_n r^n}\int_{\mathrm{B}(x_0,r)\cap\mathrm{B}(0,t)} |u-\widetilde{u}(x_0)|\,\mathrm{d}\mathscr{L}^n \\
&\quad + \frac{1}{\omega_n r^n}\int_{\mathrm{B}(x_0,r)\cap\overline{\mathrm{B}(0,t)}^c} |u-\widetilde{u}(x_0)|\,\mathrm{d}\mathscr{L}^n \\
&= \frac{\mathscr{L}^n(\mathrm{B}(x_0,r)\cap\mathrm{B}(0,t))}{\omega_n r^n} \underbrace{\fint_{\mathrm{B}(x_0,r)\cap\mathrm{B}(0,t)} |u-\widetilde{u}(x_0)|\,\mathrm{d}\mathscr{L}^n}_{\to 0 \text{ by (8.71) as } r\searrow 0} \\
&\quad + \frac{\mathscr{L}^n(\mathrm{B}(x_0,r)\cap\overline{\mathrm{B}(0,t)}^c)}{\omega_n r^n} \underbrace{\fint_{\mathrm{B}(x_0,r)\cap\overline{\mathrm{B}(0,t)}^c} |u-\widetilde{u}(x_0)|\,\mathrm{d}\mathscr{L}^n}_{\to 0 \text{ by (8.71) as } r\searrow 0} \\
&\to 0 \qquad \text{as } r\searrow 0,
\end{aligned}$$

since

$$\lim_{r\searrow 0}\frac{\mathscr{L}^n(\mathrm{B}(x_0,r)\cap\mathrm{B}(0,t))}{\mathscr{L}^n(\mathrm{B}(x_0,r))} = \lim_{r\searrow 0}\frac{\mathscr{L}^n(\mathrm{B}(x_0,r)\cap\overline{\mathrm{B}(0,t)}^c)}{\mathscr{L}^n(\mathrm{B}(x_0,r))} = \frac{1}{2}.$$

In conclusion, $\mathcal{H}^{n-1}$–a.e. $x_0\in\partial\mathrm{B}(0,t)$ is a Lebesgue point for u for $\mathscr{L}^1$–a.e. radius $t\in(R_1,R_2)$. Let us call a sphere $\partial\mathrm{B}(0,t)$ with this property a *Lebesgue sphere* for u.

Step 2. A mean–value–type integral estimate. In a second step, we now make the following

Claim. Let $-\infty<a<b<\infty$ and let $J\subset(a,b)$ be a subset of full Lebesgue measure, i.e., $\mathscr{L}^1((a,b)\setminus J)=0$. Then for every $g\in\mathrm{L}^1((a,b);\mathbb{R}_{\geqslant 0})$ there exists $\xi_0\in J$ which is a Lebesgue point for g and satisfies

$$g(\xi_0)\leqslant\frac{2}{b-a}\int_a^b g\,\mathrm{d}x. \tag{8.73}$$

To see this, we note that $\mathscr{L}^1$–a.e. element of J is a Lebesgue point for g. Assume towards a contradiction that the claim is wrong. Then we find $g\in\mathrm{L}^1((a,b);\mathbb{R}_{\geqslant 0})$ such that for all $\xi_0\in J$ which are Lebesgue points for g there holds

$$g(\xi_0)>\frac{2}{b-a}\int_a^b g\,\mathrm{d}x. \tag{8.74}$$

Since this consequently holds for $\mathscr{L}^1$–a.e. $\xi_0 \in (a,b)$, we infer by integrating with respect to $\xi_0 \in J$

$$2\int_a^b g(y)\,\mathrm{d}y \leqslant \frac{2}{b-a}\int_a^b\int_a^b g\,\mathrm{d}x\,\mathrm{d}y \leqslant \int_a^b g(y)\,\mathrm{d}y.$$

By non–negativity of g, this implies $g \equiv 0$ $\mathscr{L}^1$–a.e. in (a,b). This contradicts (8.74) and the proof of the claim is complete.

Step 3. Conclusion. Let now $0<\theta<1$ be arbitrary and put $p = n/(n-1+\theta)$. We define a function $\lambda\colon (0,R)\to\mathbb{R}_{\geqslant 0}$ for $\mathscr{L}^1$–a.e. $0<r<R$ by

$$\lambda(r) := \iint_{\partial\mathrm{B}(0,r)\times\partial\mathrm{B}(0,r)} \frac{|u(x)-u(y)|^p}{|x-y|^{n+\theta p-1}}\,\mathrm{d}\sigma_x\,\mathrm{d}\sigma_y.$$

Since $\mathrm{BD}(\mathbb{R}^n) \hookrightarrow \mathrm{W}^{\theta,\frac{n}{n-1+\theta}}(\mathbb{R}^n;\mathbb{R}^n)$, we obtain $\lambda \in \mathrm{L}^1((0,R))$ by Proposition 8.25. We put

$$\begin{aligned} J_1 &:= \{t\in(s,r)\colon\ t \text{ is Lebesgue point for } \lambda \text{ with respect to } \mathscr{L}^1\},\\ J_2 &:= \{t\in(s,r)\colon\ \partial\mathrm{B}(0,t) \text{ is a Lebesgue sphere for } u\}. \end{aligned}$$

Then $J := J_1\cap J_2$ has full Lebesgue measure, i.e., $\mathscr{L}^1((s,r)\setminus J) = 0$. Since λ is particularly well–defined on J, we conclude by the above claim that there exists $t\in(s,r)$ such that

$$\begin{aligned} &\iint_{\partial\mathrm{B}(0,t)\times\partial\mathrm{B}(0,t)} \frac{|u(x)-u(y)|^p}{|x-y|^{n+\theta p-1}}\,\mathrm{d}\sigma_x\,\mathrm{d}\sigma_y \\ &\qquad\qquad \leqslant \frac{2}{(s-r)}\int_0^r \iint_{\partial\mathrm{B}(0,\rho)\times\partial\mathrm{B}(0,\rho)} \frac{|u(x)-u(y)|^p}{|x-y|^{n+\theta p-1}}\,\mathrm{d}\sigma_x\,\mathrm{d}\sigma_y\,\mathrm{d}\rho. \end{aligned}$$

Evidently, the same reasoning applies to $u_\alpha = u-\alpha$ for any $\alpha\in\mathcal{R}(\mathbb{R}^n)\subset\mathrm{C}^\infty(\mathbb{R}^n;\mathbb{R}^n)$, too. We now choose a rigid deformation $\alpha\in\mathcal{R}(\mathbb{R}^n)$ such that

$$\left(\fint_{\mathrm{B}_r}\int_{\mathrm{B}_r} \frac{|u_\alpha(x)-u_\alpha(y)|^p}{|x-y|^{n+\theta p}}\,\mathrm{d}x\,\mathrm{d}y\right)^{\frac{1}{p}} \leqslant Cr^{1-\theta}\frac{|\mathrm{E}u|(\mathrm{B}(0,r))}{r^n} \tag{8.75}$$

holds. This is possible by Lemma 8.26 and (8.67). Invoking Proposition 8.25 at $(*)$, we therefore find

$$\begin{aligned} &\left(\iint_{\partial\mathrm{B}(0,t)\times\partial\mathrm{B}(0,t)} \frac{|u_\alpha(x)-u_\alpha(y)|^p}{|x-y|^{n+\theta p-1}}\,\mathrm{d}\sigma_x\,\mathrm{d}\sigma_y\right)^{\frac{1}{p}} \\ &\qquad \leqslant \frac{C}{(s-r)^{\frac{1}{p}}}\left(\int_s^t\iint_{\partial\mathrm{B}(0,\rho)\times\partial\mathrm{B}(0,\rho)} \frac{|u_\alpha(x)-u_\alpha(y)|^p}{|x-y|^{n+\theta p-1}}\,\mathrm{d}\sigma_x\,\mathrm{d}\sigma_y\,\mathrm{d}\rho\right)^{\frac{1}{p}} \\ &\qquad \leqslant \frac{C}{(s-r)^{\frac{1}{p}}}\left(\int_0^r\iint_{\partial\mathrm{B}(0,\rho)\times\partial\mathrm{B}(0,\rho)} \frac{|u_\alpha(x)-u_\alpha(y)|^p}{|x-y|^{n+\theta p-1}}\,\mathrm{d}\sigma_x\,\mathrm{d}\sigma_y\,\mathrm{d}\rho\right)^{\frac{1}{p}} \\ &\qquad \overset{(*)}{\leqslant} C\frac{2^{\frac{1}{p}}}{(s-r)^{\frac{1}{p}}}\left(\iint_{\mathrm{B}(0,r)\times\mathrm{B}(0,r)} \frac{|u_\alpha(x)-u_\alpha(y)|^p}{|x-y|^{n+\theta p}}\,\mathrm{d}x\,\mathrm{d}y\right)^{\frac{1}{p}} \\ &\qquad \leqslant C\frac{r^{\frac{n}{p}}}{(s-r)^{\frac{1}{p}}}\left(\fint_{\mathrm{B}(0,r)}\int_{\mathrm{B}(0,r)} \frac{|u_\alpha(x)-u_\alpha(y)|^p}{|x-y|^{n+\theta p}}\,\mathrm{d}x\,\mathrm{d}y\right)^{\frac{1}{p}} \\ &\qquad \leqslant C\frac{r^{\frac{n}{p}}r^{1-\theta}}{(s-r)^{\frac{1}{p}}}\frac{|\mathrm{E}u|(\mathrm{B}(0,r))}{r^n} \end{aligned}$$

and thus

$$\left(\fint_{\partial\mathrm{B}(0,t)}\int_{\partial\mathrm{B}(0,t)} \frac{|u_\alpha(x)-u_\alpha(y)|^p}{|x-y|^{n+sp-1}}\,\mathrm{d}\sigma_x\,\mathrm{d}\sigma_y\right)^{\frac{1}{p}} \leqslant C\frac{r^{\frac{n}{p}}r^{1-s}}{t^{\frac{n-1}{p}}(s-r)^{\frac{1}{p}}}\frac{|\mathrm{E}u|(\mathrm{B}(0,r))}{r^n}.$$

It is clear that $C>0$ does not depend on u nor R, and so we arrive at (8.68). The proof is complete. ∎

8.7. $\mathbb{A}$–harmonic Approximation and Excess Decay

We now turn to the proof of Theorem 8.4 and so embark on the situation explained therein. Hence, let us now describe, in a more detailled manner than in Chapter 8.1.5, the strategy of the proof. Throughout, we think of the center $x_0 \in \Omega$ of the reference balls as fixed, and write $\mathrm{B}_r := \mathrm{B}(x_0, r)$.

- In a first step, our aim is to find an $\mathbb{A}$–harmonic comparison map, where $\mathbb{A}$ as usual will appear as f'' applied to the mean value of the second derivative of f over a reference ball. Here we firstly select a *good* radius $R > 0$, thereby making the Fubini–type theorem for BD applicable. Hence we are in position to control the $\mathrm{W}^{\frac{1}{n},\frac{n+1}{n}}(\partial\mathrm{B}_R;\mathbb{R}^n)$–seminorm of $u-a$, where a is an affine–linear mapping, in terms of the excess. Most crucially, the $\mathbb{A}$–harmonic comparison maps features higher integrability $q > 1$ on the level of *gradients* in light of Chapter 8.4.3. The overall aim is then to use this gain of higher integrability to establish both the approximate $\mathbb{A}$–harmonicity and the $\mathbb{A}$–harmonic approximation. Other than in [89, 90, 91, 82], we shall essentially accomplish this in *one* single step.
- To avoid manipulations on measures, we use ideas from [85, 34] and employ the Ekeland variational principle to find a LD–map which is not only suitably close to the given BD–minimiser but also satisfies an approximate Euler–Lagrange equation on B_R and attains the same boundary values on $\partial\mathrm{B}_R$ as $(u-a)$.
- Once the two steps from above are accomplished, a more or less standard iteration argument will yield the claimed partial regularity.

Let us remark that, by slight abuse of notation, we will write

$$\int_A \mathrm{d}g(\mu) := \int_A g(\mu)$$

for a convex function g being applied to a (matrix–valued) Radon measure μ on $A \in \mathscr{B}(\mathbb{R}^n)$ whenever convenient.

8.7.1. The Ekeland–type Approximation We fix reference numbers $m_0 > 0$, $\delta > 0$ (to be specified below) and let $x_0 \in \Omega$ be arbitrary but fixed. We firstly recall that for each open Lipschitz subset U of $\mathbb{R}^n$ and[6] $\widetilde{u} \in \mathrm{LD}(U)$, $\mathrm{LD}_{\widetilde{u}}(\Omega) := \widetilde{u} + \mathrm{LD}_0(U)$ is a complete metric space with respect to the symmetric gradient norm $\|v\|_{\mathrm{sym}} := \|\varepsilon(v)\|_{\mathrm{L}^1}$. This is a consequence of the continuity of the trace operator $\mathrm{Tr}\colon \mathrm{LD}(U) \to \mathrm{L}^1_{\mathcal{H}^{n-1}}(\partial U;\mathbb{R}^n)$ with respect to the norm topology.

Starting from the given generalised minimiser $u \in \mathrm{BD}(\Omega)$ and a number $\theta \in (0,1)$, let $R > 0$ be a radius such that $u \in \mathrm{W}^{\theta,n/(n-1+\theta)}(\partial\mathrm{B}(x_0,R);\mathbb{R}^n)$. In the sequel, we shall also write $\mathrm{B}_R := \mathrm{B}(x_0,R)$ for brievity. We employ a two–layer approximation scheme: Firstly we smoothly approximate u in the area–strict topology (while maintaining its trace on $\partial\mathrm{B}_R$) which yields an approximation of u in the space $\mathrm{LD}_{\widetilde{u}}(\mathrm{B}_R)$. In a second step, we then make use of the Ekeland variational principle, and as a main feature of this scheme, the resulting approximations still qualify as almost–minimisers of the original functional.

Proceeding as indicated, we consequently apply the area–strict smooth approximation in BD (cp. Lemma 4.9) to find $\widetilde{w}_\delta \in \mathrm{LD}_{\widetilde{u}}(\mathrm{B}(x_0,R))$ such that

$$\fint_{\mathrm{B}(x_0,R)} \left|\frac{\widetilde{u}-\widetilde{w}_\delta}{R}\right| \mathrm{d}x + \left|\fint_{\mathrm{B}(x_0,R)} \mathrm{d}V(\mathrm{E}\widetilde{u}) - \fint_{\mathrm{B}(x_0,R)} V(\varepsilon(\widetilde{w}))\,\mathrm{d}x\right| \leqslant \delta^2, \tag{8.76}$$

$$\fint_{\mathrm{B}(x_0,R)} \widetilde{f}(\varepsilon(\widetilde{w}))\,\mathrm{d}x \leqslant \fint_{\mathrm{B}(x_0,R)} \mathrm{d}\widetilde{f}(\mathrm{E}\widetilde{u}) + \delta^2, \tag{8.77}$$

[6] If $\widetilde{u} \in \mathrm{BD}(U)$, then we choose an LD–map with the same boundary traces.

where the dash is understood with respect to the Lebesgue measure $\mathscr{L}^n$. By the Ekeland variational principle, Theorem 2.32, we deduce that there exists a mapping $\widetilde{v} \in \mathrm{LD}_{\widetilde{u}}(\mathrm{B}(x_0,R))$ which satisfies

$$\int_{\mathrm{B}(x_0,R)} \widetilde{f}(\boldsymbol{\varepsilon}(\widetilde{v}))\,\mathrm{d}x \leqslant \int_{\mathrm{B}(x_0,R)} \widetilde{f}(\boldsymbol{\varepsilon}(\widetilde{w}))\,\mathrm{d}x, \tag{8.78}$$

$$\fint_{\mathrm{B}(x_0,R)} \left|\frac{\widetilde{v}-\widetilde{w}_\delta}{R}\right|\mathrm{d}x + \fint_{\mathrm{B}(x_0,R)} |\boldsymbol{\varepsilon}(\widetilde{v})-\boldsymbol{\varepsilon}(\widetilde{w})|\,\mathrm{d}x \leqslant \delta, \tag{8.79}$$

$$\int_{\mathrm{B}(x_0,R)} \widetilde{f}(\boldsymbol{\varepsilon}(\widetilde{v}))\,\mathrm{d}x \leqslant \int_{\mathrm{B}(x_0,R)} \widetilde{f}(\boldsymbol{\varepsilon}(\widetilde{\varphi}))\,\mathrm{d}x + \delta\int_{\mathrm{B}(x_0,R)} |\boldsymbol{\varepsilon}\big(\widetilde{v}-\widetilde{\varphi}\big)|\,\mathrm{d}x \tag{8.80}$$

for all $\widetilde{\varphi} \in \mathrm{LD}_{\widetilde{u}}(\Omega)$. Working from here, we deduce the following lemma.

Lemma 8.28. *The mapping $\widetilde{v} \in \mathrm{LD}_{\widetilde{u}}(\mathrm{B}(x_0,R))$ as introduced prior to (8.78) satisfies*

$$\left|\int_{\mathrm{B}(x_0,R)} \langle \widetilde{f}'(\boldsymbol{\varepsilon}(\widetilde{v})),\boldsymbol{\varepsilon}(\varphi)\rangle\,\mathrm{d}x\right| \leqslant \delta\int_{\mathrm{B}(x_0,R)} |\boldsymbol{\varepsilon}(\varphi)|\,\mathrm{d}x, \tag{8.81}$$

$$\left|\int_{\mathrm{B}(x_0,R)} \langle \widetilde{f}''(0)\boldsymbol{\varepsilon}(\widetilde{v}),\boldsymbol{\varepsilon}(\varphi)\rangle\,\mathrm{d}x\right| \leqslant \int_{\mathrm{B}(x_0,R)} (CLV(\boldsymbol{\varepsilon}(\widetilde{v}))+\delta)|\boldsymbol{\varepsilon}(\varphi)|\,\mathrm{d}x \tag{8.82}$$

for all $\varphi \in \mathrm{LD}_0(\mathrm{B}(x_0,R))$.

Proof. Let $\varphi \in \mathrm{LD}_0(\mathrm{B}(x_0,R))$ be arbitrary. Consequently, for every $\varepsilon \in \mathbb{R}\setminus\{0\}$, $\widetilde{\varphi}_\varepsilon^{\pm} := \widetilde{v} \pm \varepsilon\varphi$ qualifies as a competitor in (8.80). Applying (8.80) to this choice of φ, we find

$$\left|\int_{\mathrm{B}(x_0,R)} \frac{\widetilde{f}(\boldsymbol{\varepsilon}(\widetilde{v}))-\widetilde{f}(\boldsymbol{\varepsilon}(\widetilde{v}\pm\varepsilon\varphi))}{\varepsilon}\,\mathrm{d}x\right| \leqslant \delta\int_{\mathrm{B}(x_0,R)} |\boldsymbol{\varepsilon}(\varphi)|\,\mathrm{d}x.$$

In this situation, sending $|\varepsilon| \searrow 0$ yields (8.81). With (8.81) at our disposal, we find

$$\begin{aligned}
\int_{\mathrm{B}(x_0,R)} \langle \widetilde{f}''(0)\boldsymbol{\varepsilon}(\widetilde{v}),\boldsymbol{\varepsilon}(\varphi)\rangle\,\mathrm{d}x &\leqslant \int_{\mathrm{B}(x_0,R)} \langle \widetilde{f}''(0)\boldsymbol{\varepsilon}(\widetilde{v}),\boldsymbol{\varepsilon}(\varphi)\rangle\,\mathrm{d}x \\
&\quad + \underbrace{\int_{\mathrm{B}(x_0,R)} \delta|\boldsymbol{\varepsilon}(\varphi)| - \langle \widetilde{f}'(\boldsymbol{\varepsilon}(\widetilde{v})),\boldsymbol{\varepsilon}(\varphi)\rangle\,\mathrm{d}x}_{\geqslant 0 \text{ by } (8.81)} \\
&\leqslant \int_{\mathrm{B}(x_0,R)} (cLV(\boldsymbol{\varepsilon}(\widetilde{v}))+\delta)|\boldsymbol{\varepsilon}(\varphi)|\,\mathrm{d}x
\end{aligned}$$

and the same obviously is valid for $-\varphi$ instead of φ. This establishes (8.82) and the proof thus is complete. □

8.7.2. $\mathbb{A}$–harmonic approximation Let $b \in \mathcal{R}(\mathbb{R}^n)$ be an arbitrary but fixed rigid deformation, i.e.,
$sg(b) \equiv 0$. Let $0 < \theta < 1$. For $\mathscr{L}^1$–a.e. $0 < R < \mathrm{dist}(x_0,\partial\Omega)$, Theorem 8.27 yields that $u|_{\partial\mathrm{B}(x_0,R)} \in \mathrm{W}^{\theta,n/(n-1+\theta)}(\partial\mathrm{B}(x_0,R);\mathbb{R}^n)$. We apply this to the particular choice $\theta := \frac{1}{n+1}$ which consequently gives

$$p(\theta) = \frac{n}{n-1+\frac{1}{n+1}} = \frac{n(n+1)}{n^2} = \frac{n+1}{n}(>1).$$

In a next step, for the particular choice $\mathbb{A}[\xi,\xi] := \langle \widetilde{f}''(0)\xi,\xi\rangle$ ($\xi \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$), we consider the linear system

$$\begin{cases} -\operatorname{div}(\mathbb{A}\boldsymbol{\varepsilon}(h)) = 0 & \text{in } \mathrm{B}(x_0,R), \\ h = \widetilde{u} & \text{on } \partial\mathrm{B}(x_0,R). \end{cases} \tag{8.83}$$

Put $k=1$ and $q:=1+\frac{1}{n}$. Then $k-\frac{1}{q}=1-\frac{n}{n+1}=\frac{n+1-n}{n+1}=\frac{1}{n+1}=\theta$. In this situation, Theorem 8.13 yields that there exists a unique $h\in \mathrm{W}^{1,1+\frac{1}{n}}(\mathrm{B}(x_0,R);\mathbb{R}^n)$ solving (8.83), and from here it is clear that $\widetilde{h}:=h-b$ is the unique solution of

$$\begin{cases} -\operatorname{div}(\mathbb{A}\boldsymbol{\varepsilon}(\widetilde{h}))=0 & \text{in } \mathrm{B}(x_0,R),\\ \widetilde{h}=\widetilde{u}-b & \text{on } \partial\mathrm{B}(x_0,R). \end{cases} \tag{8.84}$$

We now record some implications of the continuity of the solution operator Φ^{-1} of Theorem 8.13 with focus on its scaling properties. We recall that $\mathrm{d}\sigma_x:=\mathrm{d}\mathcal{H}^{n-1}(x)$ and accordingly for y. Also, we put $\widetilde{u}_b:=\widetilde{u}-b$.

Lemma 8.29. *There exists a constant $C>0$ independent of $R>0$ such that*

$$\left(\fint_{\mathrm{B}_R}|\mathrm{D}\widetilde{h}|^{\frac{n+1}{n}}\,\mathrm{d}x\right)^{\frac{n}{n+1}}\leqslant CR^{-\frac{n}{n+1}}\left(\fint_{\partial\mathrm{B}_R}\int_{\partial\mathrm{B}_R}\frac{|\widetilde{u}_b(x)-\widetilde{u}_b(y)|^{\frac{n+1}{n}}}{|x-y|^{(n-1+\frac{1}{n})}}\,\mathrm{d}\sigma_x\,\mathrm{d}\sigma_y\right)^{\frac{n}{n+1}}. \tag{8.85}$$

Proof. By the choice of the radius $R>0$, $\widetilde{u}|_{\partial\mathrm{B}(x_0,R)}\in \mathrm{W}^{\frac{1}{n},\frac{n+1}{n}}(\partial\mathrm{B}(x_0,R);\mathbb{R}^n)$. We first focus on the case $R=1$ first. The statement will then follow at the end of the proof by scaling, the latter being available by linearity of the problem at our disposal.

Denote $S\colon(\widetilde{u}-b)|_{\partial\mathrm{B}(x_0,1)}\mapsto\widetilde{h}$ the solution operator of the system (8.84). By Theorem 8.13 and the terminology adopted therein, we have $\Phi^{-1}(0,\cdot)=S(\cdot)$, and if we put $T_0=0$ and $T_1=\widetilde{u}-b$, we have $\widetilde{h}=S((T_0,T_1))$ so that

$$\begin{aligned}\|\mathrm{D}\widetilde{h}\|_{\mathrm{L}^{\frac{n+1}{n}}(\mathrm{B}(x_0,1);\mathbb{R}^{n\times n})}&\leqslant\|\widetilde{h}\|_{\mathrm{W}^{1,\frac{n+1}{n}}(\mathrm{B}(x_0,1);\mathbb{R}^n)}\\&=\|S((T_0,T_1))\|_{\mathrm{W}^{1,\frac{n+1}{n}}(\mathrm{B}(x_0,1);\mathbb{R}^n)}\\&\leqslant C\|T_1\|_{\mathrm{W}^{\frac{1}{n+1},\frac{n+1}{n}}(\partial\mathrm{B}(x_0,1);\mathbb{R}^n)}\\&=C\|\widetilde{u}_b\|_{\mathrm{W}^{\frac{1}{n+1},\frac{n+1}{n}}(\partial\mathrm{B}(x_0,1);\mathbb{R}^n)}.\end{aligned} \tag{8.86}$$

Now put

$$\mathbf{u}_b:=(\widetilde{u}_b)_{\partial\mathrm{B}(x_0,1)}:=\fint_{\partial\mathrm{B}(x_0,1)}\widetilde{u}_b\,\mathrm{d}\mathcal{H}^{n-1}(\in\mathbb{R}^n),$$

where the dash is now understood with respect to $\mathcal{H}^{n-1}\llcorner\partial\mathrm{B}(x_0,1)$. Then we have the fractional Poincaré inequality

$$\|\widetilde{u}_b-\mathbf{u}_b\|_{\mathrm{W}^{\frac{1}{n+1},\frac{n+1}{n}}(\partial\mathrm{B}(x_0,1);\mathbb{R}^n)}\leqslant C[\widetilde{u}_b]_{\mathrm{W}^{\frac{1}{n+1},\frac{n+1}{n}}(\partial\mathrm{B}(x_0,1);\mathbb{R}^n)}. \tag{8.87}$$

In fact, we have for $x,y\in\partial\mathrm{B}(x_0,1)$ that $|x-y|\leqslant 2$ and thus, with $\kappa=\frac{1}{n+1}$ and $p=\frac{n+1}{n}$,

$$\begin{aligned}\int_{\partial\mathrm{B}(x_0,1)}|\widetilde{u}_b(x)-\mathbf{u}_b|^p\,\mathrm{d}\sigma_x&=\int_{\partial\mathrm{B}(x_0,1)}\fint_{\partial\mathrm{B}(x_0,1)}|\widetilde{u}_b(x)-\widetilde{u}_b(y)|^p\,\mathrm{d}\sigma_x\,\mathrm{d}\sigma_y\\&\leqslant C\int_{\partial\mathrm{B}(x_0,1)}\int_{\partial\mathrm{B}(x_0,1)}\frac{|\widetilde{u}_b(x)-\widetilde{u}_b(y)|^p}{|x-y|^{n-1}}\,\mathrm{d}\sigma_y\,\mathrm{d}\sigma_x\\&\leqslant C\int_{\partial\mathrm{B}(x_0,1)}\int_{\partial\mathrm{B}(x_0,1)}\frac{|\widetilde{u}_b(x)-\widetilde{u}_b(y)|^p}{|x-y|^{n-1+\kappa p}}\,\mathrm{d}\sigma_y\,\mathrm{d}\sigma_x\end{aligned}$$

and so, since the Sobolev–Slobodeckjii–seminorm ignores addition of constants, we obtain

$$\begin{aligned}\|\widetilde{u}_b-\mathbf{u}_b\|_{\mathrm{W}^{\kappa,p}(\partial\mathrm{B}(x_0,1);\mathbb{R}^n)}&=\|\widetilde{u}_b-\mathbf{u}_b\|_{\mathrm{L}^p(\partial\mathrm{B}(x_0,1);\mathbb{R}^n)}+[\widetilde{u}_b]_{\mathrm{W}^{\kappa,p}(\partial\mathrm{B}(x_0,1);\mathbb{R}^n)}\\&\leqslant C[\widetilde{u}_b]_{\mathrm{W}^{\kappa,p}(\partial\mathrm{B}(x_0,1);\mathbb{R}^n)}.\end{aligned} \tag{8.88}$$

Clearly, $v := \widetilde{h} - \mathbf{u}_b$ is the unique solution of the system

$$\begin{cases} -\operatorname{div}(\mathbb{A}\boldsymbol{\varepsilon}(v)) = 0 & \text{in } \mathrm{B}(x_0,1), \\ v = \widetilde{u}_b - \mathbf{u}_b & \text{on } \partial\mathrm{B}(x_0,1). \end{cases} \tag{8.89}$$

and equally satisfies (8.86) with the obvious modifications, i.e.,

$$\begin{aligned} \|\mathrm{D}\widetilde{h}\|_{\mathrm{L}^{\frac{n+1}{n}}(\mathrm{B}(x_0,1);\mathbb{R}^{n\times n})} &= \|\mathrm{D}v\|_{\mathrm{L}^{\frac{n+1}{n}}(\mathrm{B}(x_0,1);\mathbb{R}^{n\times n})} \\ &\leqslant C\|\widetilde{u}_b - \mathbf{u}_b\|_{\mathrm{W}^{\frac{1}{n+1},\frac{n+1}{n}}(\partial\mathrm{B}(x_0,1);\mathbb{R}^n)} \\ &\leqslant C[\widetilde{u}_b]_{\mathrm{W}^{\frac{1}{n+1},\frac{n+1}{n}}(\partial\mathrm{B}(x_0,1);\mathbb{R}^n)} \qquad \text{(by (8.88))} \end{aligned} \tag{8.90}$$

At this stage, we pass to general radii $R>0$. For this, we use linearity of the problem (8.84). Since $\widetilde{h}$ solves (8.84), the function $w\colon \mathrm{B}(0,1)\to\mathbb{R}^n$ given by $w(x) := \widetilde{h}(Rx)$ is seen to solve

$$\begin{cases} -\operatorname{div}(\mathbb{A}\boldsymbol{\varepsilon}(w)) = 0 & \text{in } \mathrm{B}(x_0,1), \\ w = \psi & \text{on } \partial\mathrm{B}(x_0,1), \end{cases} \tag{8.91}$$

where $\psi\colon \partial\mathrm{B}(x_0,1)\to\mathbb{R}^n$ is given by $\psi(x) = \widetilde{u}_b(Rx)$ for $x\in\partial\mathrm{B}(x_0,1)$. Changing variables, we then see with $p=\frac{n+1}{n}$ and $\kappa = \frac{1}{n+1}$ that

$$\begin{aligned} \left(\int_{\mathrm{B}(x_0,1)} |\mathrm{D}w|^p\,\mathrm{d}x\right)^{\frac{1}{p}} &\overset{(*)}{\leqslant} C\left(\iint_{\partial\mathrm{B}(x_0,1)\times\partial\mathrm{B}(x_0,1)} \frac{|\psi(x)-\psi(y)|^p}{|x-y|^{n-1+\kappa p}}\,\mathrm{d}\sigma_x\,\mathrm{d}\sigma_y\right)^{\frac{1}{p}} \\ &= CR^{\frac{n-1+\kappa p}{p}}R^{-\frac{2(n-1)}{p}}\times \\ &\times\left(\iint_{\partial\mathrm{B}(x_0,1)\times\partial\mathrm{B}(x_0,1)} \frac{|\widetilde{u}_b(Rx)-\widetilde{u}_b(Ry)|^p}{|Rx-Ry|^{n-1+\kappa p}}(R^{n-1})^2\,\mathrm{d}\sigma_x\,\mathrm{d}\sigma_y\right)^{\frac{1}{p}} \\ &\overset{(**)}{=} CR^{\frac{n-1+\kappa p}{p}}R^{-\frac{2(n-1)}{p}}\times \\ &\times\left(\iint_{\partial\mathrm{B}(x_0,R)\times\partial\mathrm{B}(x_0,R)} \frac{|\widetilde{u}_b(x')-\widetilde{u}_b(y')|^p}{|x'-y'|^{n-1+\kappa p}}\,\mathrm{d}\sigma_{x'}\,\mathrm{d}\sigma_{y'}\right)^{\frac{1}{p}}, \end{aligned}$$

where we used the inequality proved in the first part of the proof at $(*)$ and the change of variables $x = Rx'$ and $y = Ry'$ at $(**)$. On the other hand, we have

$$\begin{aligned} R^{1-\frac{n}{p}}\left(\int_{\mathrm{B}(x_0,R)} |\mathrm{D}\widetilde{h}(x')|^p\,\mathrm{d}x'\right)^{\frac{1}{p}} &= R^{1-\frac{n}{p}}\left(\int_{\mathrm{B}(x_0,1)} |(\mathrm{D}\widetilde{h})(Rx)|^pR^n\,\mathrm{d}x\right)^{\frac{1}{p}} \\ &= \left(\int_{\mathrm{B}(x_0,1)} |\mathrm{D}w(x)|^p\,\mathrm{d}x\right)^{\frac{1}{p}}. \end{aligned}$$

Regrouping and rewriting terms, we thereby obtain with a constant $C>0$ which does not depend on R,

$$\begin{aligned} \left(\fint_{\mathrm{B}(x_0,R)} |\mathrm{D}\widetilde{h}(x')|^p\,\mathrm{d}x'\right)^{\frac{1}{p}} &\leqslant CR^{\kappa}R^{-1}\times \\ &\times\left(\fint_{\partial\mathrm{B}(x_0,R)}\int_{\partial\mathrm{B}(x_0,R)} \frac{|\widetilde{u}_b(x')-\widetilde{u}_b(y')|^p}{|x'-y'|^{n-1+\kappa p}}\,\mathrm{d}\sigma_{x'}\,\mathrm{d}\sigma_{y'}\right)^{\frac{1}{p}}. \end{aligned}$$

We then note that $\kappa - 1 = \frac{1}{n+1} - 1 = -\frac{n}{n+1}$, and this concludes the proof. $\square$

8.8. Caccioppoli–type Inequality of the second kind

The crucial point in partial regularity proofs are *Caccioppoli–type Inequalities* and usually it is only here where both quasiconvexity and minimality are utilised. The inequality to be proved below in Proposition 8.30 is not invoked to yield higher integrability which, from a conceptual perspective, seems impossible in light of the results of Chapter 5.6. However, note that in the inequality to be proved next we directly work on $(u-a)$ instead of u, where a is an affine–linear map and so, following the terminology of GIUSTI [128], Proposition 8.30 amounts to a Caccioppoli inequality *of the second kind.*

Proposition 8.30. *Let $M_0 > 0$. Then there exists a constant $c = c(M_0) \in [1, \infty)$ such that if $a\colon \mathbb{R}^n \to \mathbb{R}^n$ is an affine mapping with $|Da| \leqslant M_0$ and $\mathrm{B} = \mathrm{B}(x_0, R) \Subset \Omega$ a ball, then there holds*

$$\int_{\mathrm{B}(x_0,\frac{R}{2})} \mathrm{d}V(\mathrm{E}(u-a)) \leqslant c \int_{\mathrm{B}(x_0,R)} V\Big(\frac{u-a}{R}\Big)\,\mathrm{d}x \tag{8.92}$$

for every generalised minimiser $u \in \mathrm{BD}(\Omega)$.

Proof. The proof evolves around a scheme for establishing Caccioppoli–type inequalities in the quasiconvex setting that is attributable to EVANS (cp. [99, Lem. 3.1]) and hence incorporates the so–called *hole filling trick* of WIDMAN [245]. Recalling the definition of the shifted integrands (cp. (8.12)) , we put $\widetilde{f} := f_{\boldsymbol{\varepsilon}(a)}$ and $\widetilde{u} := u - a$. We then record that $\widetilde{u}$ is a local minimiser the functional

$$\mathcal{F}[v] := \int_\Omega \widetilde{f}(\mathrm{E}v)$$

over $\mathrm{BD}(\Omega)$ in the sense of Chapter 7.1.

Let $\frac{R}{2} < r < s < R$ be arbitrary and choose a cut–off function $\rho \in \mathrm{C}^1_c(\mathrm{B}(x_0,s);[0,1])$ with $\mathbb{1}_{\mathrm{B}(x_0,r)} \leqslant \rho \leqslant \mathbb{1}_{\mathrm{B}(x_0,s)}$ and $|\nabla\rho| \leqslant \frac{2}{s-r}$. Following [99], we define $\varphi := \rho\widetilde{u}$ and $\psi := (1-\rho)\widetilde{u}$ so that $\widetilde{u} = u - a = \varphi + \psi$. We record in advance that

$$\begin{aligned}
\mathrm{E}\varphi &= \rho\frac{\mathrm{dE}u}{\mathrm{d}\mathscr{L}^n}\mathscr{L}^n \llcorner \mathrm{B}(x_0,s) + \mathrm{E}^s u \llcorner \mathrm{spt}(\rho) + \nabla\rho \odot \widetilde{u}\mathscr{L}^n \llcorner \mathrm{spt}(\rho),\\
\mathrm{E}\psi &= (1-\rho)\frac{\mathrm{dE}u}{\mathrm{d}\mathscr{L}^n}\mathscr{L}^n \llcorner \mathrm{B}(x_0,s) + \mathrm{E}^s u \llcorner \mathrm{spt}((1-\rho)) - \nabla\rho \odot \widetilde{u}\mathscr{L}^n \llcorner \mathrm{spt}((1-\rho)).
\end{aligned}$$

We next claim that

$$\frac{\ell}{c}\int_{\mathrm{B}(x_0,s)} \mathrm{d}V(\mathrm{E}\varphi) \leqslant \int_{\mathrm{B}(x_0,s)} \mathrm{d}\widetilde{f}(\mathrm{E}\varphi). \tag{8.93}$$

To see this inequality, note $\varphi|_{\partial\mathrm{B}(x_0,s)} = 0$ and hence we find an approximating sequence $(\varphi_k) \subset \mathrm{C}^\infty_c(\mathrm{B}(x_0,s);\mathbb{R}^n)$ which converges in the (symmetric) area–strict sense on $\mathrm{B}(x_0,s)$ to φ as $k \to \infty$. From Lemma 8.8 we then deduce

$$\frac{\ell}{c}\int_{\mathrm{B}(x_0,s)} V(\boldsymbol{\varepsilon}(\varphi_k))\,\mathrm{d}x \leqslant \int_{\mathrm{B}(x_0,s)} \widetilde{f}(\boldsymbol{\varepsilon}(\varphi_k))\,\mathrm{d}x.$$

Since V is convex, we may invoke the continuity part of RESHETNYAK's theorem, Proposition 4.18 to conclude that the left hand side of the preceding inequality converges to $\frac{\ell}{c}\int_{\mathrm{B}(x_0,s)} \mathrm{d}V(\mathrm{E}\varphi)$. For the right hand side we invoke the continuity result for symmetric rank–one convex functions with respect to area–strict convergence, cp. Lemma 8.7. By area–strict convergence and the fact that symmetric quasiconvexity implies symmetric rank–1–convexity, we hereby obtain (8.93). Note that at this stage it is important to work with area–strict approximation as the corresponding Reshetnyak–type result only yields continuity for this sort of convergence.

Consequently, using minimality of $\widetilde{u}$ with respect to its own boundary values and $\widetilde{u}|_{\partial \mathrm{B}(x_0,s)} = \psi|_{\partial \mathrm{B}(x_0,s)}$ in the second step, we obtain

$$
\begin{aligned}
\frac{\ell}{c}\int_{\mathrm{B}(x_0,r)} \mathrm{d}V(\mathrm{E}\widetilde{u}) &\leqslant \int_{\mathrm{B}(x_0,s)} \widetilde{f}(\mathrm{E}\widetilde{u}) + \int_{\mathrm{B}(x_0,s)} (\widetilde{f}(\mathrm{E}\varphi) - \widetilde{f}(\mathrm{E}\widetilde{u})) \qquad \text{(by (8.93))}\\
&\leqslant \int_{\mathrm{B}(x_0,s)} \widetilde{f}(\mathrm{E}\psi) + \int_{\mathrm{B}(x_0,s)} (\widetilde{f}(\mathrm{E}\varphi) - \widetilde{f}(\mathrm{E}\widetilde{u}))\\
&\leqslant \int_{\mathrm{B}(x_0,s)\setminus \mathrm{B}(x_0,r)} \widetilde{f}(\mathrm{E}\psi) + \int_{\mathrm{B}(x_0,s)\setminus \mathrm{B}(x_0,r)} (\widetilde{f}(\mathrm{E}\varphi) - \widetilde{f}(\mathrm{E}\widetilde{u})),\\
&=: \mathbf{I} + \mathbf{II},
\end{aligned}
$$

where the last inequality holds as φ and $\widetilde{u}$ coincide on $\mathrm{B}(x_0,r)$. We invoke Lemma 8.9 to obtain

$$
\begin{aligned}
\mathbf{I} &\leqslant cL\int_{\mathrm{B}(x_0,s)\setminus \mathrm{B}(x_0,r)} V(\mathrm{E}\psi) = cL\int_{\mathrm{B}(x_0,s)\setminus \mathrm{B}(x_0,r)} V((1-\rho)\,\mathrm{E}\widetilde{u} - \nabla\rho\odot\widetilde{u}\mathscr{L}^n \llcorner \Omega)\\
&\leqslant cL\int_{\mathrm{B}(x_0,s)\setminus \mathrm{B}(x_0,r)} V((1-\rho)\,\mathrm{E}\widetilde{u} - \nabla\rho\odot\widetilde{u}\mathscr{L}^n \llcorner \Omega)\\
&\leqslant 16cL\int_{\mathrm{B}(x_0,s)\setminus \mathrm{B}(x_0,r)} V(\mathrm{E}\widetilde{u}) + 16cL\int_{\mathrm{B}(x_0,s)} V(\nabla\rho\odot\widetilde{u})\,\mathrm{d}x\\
&\leqslant 16cL\int_{\mathrm{B}(x_0,s)\setminus \mathrm{B}(x_0,r)} V(\mathrm{E}\widetilde{u}) + 256cL\int_{\mathrm{B}(x_0,s)} V\Big(\frac{\widetilde{u}}{s-r}\Big)\,\mathrm{d}x.
\end{aligned}
$$

On the other hand, we similarly find

$$
\begin{aligned}
\mathbf{II} &\leqslant \int_{\mathrm{B}(x_0,s)\setminus \mathrm{B}(x_0,r)} \widetilde{f}(\mathrm{E}\varphi) - \widetilde{f}(\mathrm{E}\widetilde{u})\\
&= \int_{\mathrm{B}(x_0,s)\setminus \mathrm{B}(x_0,r)} \widetilde{f}(\rho\,\mathrm{E}\widetilde{u} + \nabla\rho\odot\widetilde{u}\mathscr{L}^n \llcorner \Omega) - \widetilde{f}(\mathrm{E}\widetilde{u})\\
&\leqslant 32cL\int_{\mathrm{B}(x_0,s)\setminus \mathrm{B}(x_0,r)} V(\mathrm{E}\widetilde{u}) + 256cL\int_{\mathrm{B}(x_0,s)} V\Big(\frac{\widetilde{u}}{s-r}\Big)\,\mathrm{d}x.
\end{aligned}
$$

Therefore, gathering estimates, we find

$$
\frac{\ell}{c}\int_{\mathrm{B}(x_0,r)} V(\mathrm{E}\widetilde{u}) \leqslant C(L)\int_{\mathrm{B}(x_0,s)\setminus \mathrm{B}(x_0,r)} V(\mathrm{E}\widetilde{u}) + C(L)\int_{\mathrm{B}(x,s)} V\Big(\frac{\widetilde{u}}{s-r}\Big)\,\mathrm{d}x.
$$

We now apply WIDMAN's *hole–filling trick* and hence add $C(L)\int_{\mathrm{B}(x_0,r)} V(\mathrm{E}\widetilde{u})$ to both sides of the previous inequality and divide the resulting inequality by $(\frac{\ell}{c} + C(L))$. In consequence, letting $\theta := C(L)/(\frac{\ell}{c} + C(L))$ and so $0 < \theta < 1$ yields

$$
\int_{\mathrm{B}(x_0,r)} V(\mathrm{E}\widetilde{u}) \leqslant \theta\int_{\mathrm{B}(x_0,s)} V(\mathrm{E}\widetilde{u}) + \theta\int_{\mathrm{B}(x_0,R)} V\Big(\frac{\widetilde{u}}{s-r}\Big)\,\mathrm{d}x.
$$

From here the conclusion is immediate by Lemma 8.10. The proof is complete. □

8.9. Partial Regularity

We will now provide the proof of the main result of this chapter, Theorem 8.4. This, in turn, is accomplished in two main steps. To fit this approach into the usual framework of $\mathbb{A}$–harmonic linearisation strategies, we recall that the crucial steps of this method consist in

(i) proving that a suitable distance of the minimiser and its $\mathbb{A}$–harmonic approximation can be controlled by the excess. In this sense, if the excess is small, then the minimiser is close to an $\mathbb{A}$–harmonic mapping.

(ii) lifting the closeness from the minimisers themselves to their (symmetric) gradients by use of the Caccioppoli–type inequality.

In our approach, these two points are encapsulated in one step and encoded in the following proposition which is the key to the partial regularity; also see the discussion provided in Remark 8.32 below.

Proposition 8.31. *Let $1 < q < \frac{n+1}{n}$ and $m > 0$. Then there exists a constant $C = C(m,n,q,L,\ell) > 0$ with the following property:*

Suppose that $u \in \mathrm{BD}(\Omega)$ is a generalised local minimiser for $\mathfrak{F}$ under the hypotheses (H4)–(H6) and $\mathrm{B} = \mathrm{B}(x_0,R) \Subset \Omega$ is an open ball with $u|_{\partial\mathrm{B}} \in \mathrm{W}^{\frac{1}{n+1},\frac{n+1}{n}}(\partial\mathrm{B};\mathbb{R}^n)$. Moreover, let $a\colon \mathbb{R}^n \to \mathbb{R}^n$ be an affine–linear mapping with $|\,\mathrm{D}a| \leqslant m$ and denote h the unique solution of the corresponding linear system (8.83) with $\widetilde{u} := u - a$. Then there holds

$$\fint_{\mathrm{B}(x_0,R)} V\Big(\frac{\widetilde{u}-h}{R}\Big)\,\mathrm{d}x \leqslant C\Big(\fint_{\mathrm{B}(x_0,R)} V(\mathrm{E}\widetilde{u})\Big)^q. \tag{8.94}$$

Proof. We fix a ball $\mathrm{B}(x_0,R) \Subset \Omega$ such that the hypotheses of the proposition are in action on $\mathrm{B}(x_0,R)$. Then we employ the Ekeland–type approximation strategy outlined in detail in Section 8.7.1. Hence, given $\delta > 0$ and letting $\widetilde{u} = u - a$, we obtain $\widetilde{v} \in \mathrm{LD}_{\widetilde{u}}(\mathrm{B}(x_0,R)) = \widetilde{u} + \mathrm{LD}_0(\mathrm{B}(x_0,R))$ such that with $\widetilde{f} = f_{\boldsymbol{\varepsilon}(a)}$, Lemma 8.28 is in action. We shall first derive suitable estimates on the approximations $\widetilde{v}$ and eventually send $\delta \searrow 0$ to conclude the claim of the proposition. Accordingly, the proof then evolves in three steps.

Step 1. From Lemma 8.28 (cp. (8.82)) we record that the auxiliary map $\widetilde{v}$ satisfies

$$\left|\int_{\mathrm{B}(x_0,R)} \langle \widetilde{f}''(0)\boldsymbol{\varepsilon}(\widetilde{v}), \boldsymbol{\varepsilon}(\varphi)\rangle\,\mathrm{d}x\right| \leqslant CL\int_{\mathrm{B}(x_0,R)} V(\boldsymbol{\varepsilon}(\widetilde{v}))|\boldsymbol{\varepsilon}(\varphi)|\,\mathrm{d}x + \delta\int_{\mathrm{B}(x_0,R)} |\boldsymbol{\varepsilon}(\varphi)|\,\mathrm{d}x \tag{8.95}$$

for all $\varphi \in \mathrm{W}_0^{1,\infty}(\mathrm{B}(x_0,R);\mathbb{R}^n)$ with a constant $c = c(m,L) > 0$. At this stage, we define $\psi := \widetilde{v} - h$. We scale back to the unit ball and therefore put, for $x \in \mathrm{B}(0,1)$,

$$\Psi(x) := \frac{1}{R}\psi(x_0+Rx), \quad \Phi(x) := \frac{1}{R}\varphi(x_0+Rx), \quad U(x) := \frac{1}{R}\widetilde{v}(x_0+Rx).$$

In this situation, (8.95) becomes with $\mathbb{A} := \widetilde{f}''(0)$

$$\left|\int_{\mathrm{B}(x_0,R)} \langle \mathbb{A}\boldsymbol{\varepsilon}(\Psi), \boldsymbol{\varepsilon}(\Phi)\rangle\,\mathrm{d}x\right| \leqslant CL\int_{\mathrm{B}(x_0,R)} V(\boldsymbol{\varepsilon}(U))|\boldsymbol{\varepsilon}(\Phi)|\,\mathrm{d}x + \delta\int_{\mathrm{B}(x_0,R)} |\boldsymbol{\varepsilon}(\Phi)|\,\mathrm{d}x \tag{8.96}$$

Step 2. We define a truncation operator $T\colon \mathbb{R}^n \to \mathbb{R}^n$ by

$$T(y) := \begin{cases} y & \text{if } |y| \leqslant 1, \\ \frac{y}{|y|} & \text{if } |y| > 1, \end{cases} \qquad y \in \mathbb{R}^n.$$

We note that $T(\Psi) \in \mathrm{L}^\infty(\mathrm{B}(0,1);\mathbb{R}^n)$. Let us now consider the elliptic system

$$\begin{cases} -\operatorname{div}(\mathbb{A}\boldsymbol{\varepsilon}(\mathbf{T})) = T(\Psi) & \text{in } \mathrm{B}(0,1), \\ \mathbf{T} = 0 & \text{on } \partial\mathrm{B}(0,1) \end{cases} \tag{8.97}$$

with its corresponding weak formulation

$$\int_{\mathrm{B}(0,1)} \langle \mathbb{A}\boldsymbol{\varepsilon}(\mathbf{T}), \boldsymbol{\varepsilon}(\varrho)\rangle \,\mathrm{d}x = \int_{\mathrm{B}(0,1)} \langle T(\Psi), \varrho\rangle \,\mathrm{d}x \qquad \text{for all } \varrho \in \mathrm{C}_c^\infty(\Omega;\mathbb{R}^n). \tag{8.98}$$

We then again invoke Theorem 8.13 to conclude that there exists a unique solution $\mathbf{T} \in \mathrm{W}_0^{1,2}(\mathrm{B}(0,1);\mathbb{R}^n)$ of this system, and moreover, for each $n < p < \infty$ we have $\mathbf{T} \in \mathrm{W}^{2,p}(\mathrm{B};\mathbb{R}^n)$ and there exists a constant $C = C(m,n,p,L,\ell) > 0$ such that

$$\int_{\mathrm{B}} |\,\mathrm{D}^2\mathbf{T}|^p \,\mathrm{d}x \leqslant C \int_{\mathrm{B}} |T(\Psi)|^p \,\mathrm{d}x.$$

In this situation, we invoke Morrey's embedding $\mathrm{W}^{1,s}(\mathrm{B}) \hookrightarrow \mathrm{L}^\infty(\mathrm{B})$ provided $s > n$ and obtain for $\mathrm{D}\mathbf{T} \in \mathrm{W}^{1,p}(\mathrm{B};\mathbb{R}^{n\times n})$ the bound

$$\|\,\mathrm{D}\mathbf{T}\|_{\mathrm{L}^\infty(\mathrm{B};\mathbb{R}^{n\times n})} \leqslant C\|\,\mathrm{D}^2\mathbf{T}\|_{\mathrm{L}^p(\mathrm{B};\mathbb{R}^{n\times n}\times\mathbb{R}^n)} \leqslant C\|T(\Psi)\|_{\mathrm{L}^p(\mathrm{B};\mathbb{R}^n)}. \tag{8.99}$$

From this we deduce $\mathbf{T} \in \mathrm{W}_0^{1,\infty}(\mathrm{B}(0,1);\mathbb{R}^n)$. We now claim that

$$\int_{\mathrm{B}(0,1)} \langle \mathbb{A}\boldsymbol{\varepsilon}(\mathbf{T}), \boldsymbol{\varepsilon}(\varrho)\rangle \,\mathrm{d}x = \int_{\mathrm{B}(0,1)} \langle T(\Psi), \varrho\rangle \,\mathrm{d}x \qquad \text{for all } \varrho \in \mathrm{LD}_0(\mathrm{B}(0,1)), \tag{8.100}$$

and in fact, for $\varrho \in \mathrm{LD}_0(\mathrm{B}(0,1))$, both sides of the previous equation are well–defined. Moreover, given $\varrho \in \mathrm{LD}_0(\mathrm{B}(0,1))$; we choose $(\varrho_k) \subset \mathrm{C}_c^\infty(\mathrm{B}(0,1);\mathbb{R}^n)$ such that $\varrho_k \to \varrho$ strongly in $\mathrm{LD}(\mathrm{B}(0,1))$ as $k \to \infty$. We then obtain

$$\begin{aligned}\left|\int_{\mathrm{B}(0,1)} \langle \mathbb{A}\boldsymbol{\varepsilon}(\mathbf{T}), \boldsymbol{\varepsilon}(\varrho)\rangle - \langle T(\Psi), \varrho\rangle \,\mathrm{d}x\right| &= \left|\int_{\mathrm{B}(0,1)} \langle \mathbb{A}\boldsymbol{\varepsilon}(\mathbf{T}), \boldsymbol{\varepsilon}(\varrho-\varrho_k)\rangle - \langle T(\Psi), \varrho-\varrho_k\rangle \,\mathrm{d}x\right| \\ &\leqslant \|\mathbb{A}\|\,\|\boldsymbol{\varepsilon}(\mathbf{T})\|_{\mathrm{L}^\infty}\|\boldsymbol{\varepsilon}(\varrho-\varrho_k)\|_{\mathrm{L}^1} \\ &\quad + \|T(\Psi)\|_{\mathrm{L}^\infty}\|\varrho-\varrho_k\|_{\mathrm{L}^1} \to 0, \qquad k\to\infty,\end{aligned}$$

where $\|\mathbb{A}\|$ is the operator norm of $\mathbb{A}$. In an intermediate step, we further record for $2 \leqslant n < p < \infty$

$$\|T(\Psi)\|_{\mathrm{L}^p}^p = \int_{\mathrm{B}(0,1)} |T(\Psi)|^p \,\mathrm{d}x \leqslant c\int_{\mathrm{B}(0,1)} V(\Psi)\,\mathrm{d}x, \tag{8.101}$$

where we used the fact that for $|a| \leqslant 1$ we have

$$|a|^p \leqslant |a|^2 \leqslant cV(a) \qquad \text{by Lemma 8.9}.$$

Combining (8.101) with (8.99) consequently yields

$$\|\,\mathrm{D}\mathbf{T}\|_{\mathrm{L}^\infty} \leqslant c\Big(\int_{\mathrm{B}(0,1)} |V(\Psi)|\,\mathrm{d}x\Big)^{\frac{1}{p}}. \tag{8.102}$$

Step 3. Recalling (8.15), we find

$$\begin{aligned}\int_{\mathrm{B}(0,1)} V(\Psi)\,\mathrm{d}x &\overset{(8.15)}{\leqslant} \int_{\mathrm{B}(0,1)} \min\{|\Psi|,|\Psi|^2\}\,\mathrm{d}x \\ &= \int_{\mathrm{B}(0,1)} \langle T(\Psi),\Psi\rangle\,\mathrm{d}x \\ &= \int_{\mathrm{B}(0,1)} \langle \mathbb{A}\boldsymbol{\varepsilon}(\mathbf{T}), \boldsymbol{\varepsilon}(\Psi)\rangle\,\mathrm{d}x \\ &= \int_{\mathrm{B}(0,1)} \langle \mathbb{A}\boldsymbol{\varepsilon}(\Psi), \boldsymbol{\varepsilon}(\mathbf{T})\rangle\,\mathrm{d}x \leqslant CL\int_{\mathrm{B}(0,1)} (V(\boldsymbol{\varepsilon}(U))+\delta)|\boldsymbol{\varepsilon}(\mathbf{T})|\,\mathrm{d}x \\ &\leqslant CL\int_{\mathrm{B}(0,1)} (V(\boldsymbol{\varepsilon}(U))+\delta)\,\mathrm{d}x\|\boldsymbol{\varepsilon}(\mathbf{T})\|_{\mathrm{L}^\infty} \\ &\leqslant CL\Big(\int_{\mathrm{B}(0,1)} (V(\boldsymbol{\varepsilon}(U))+\delta)\,\mathrm{d}x\Big)\Big(\int_{\mathrm{B}(0,1)} |V(\Psi)|\,\mathrm{d}x\Big)^{\frac{1}{p}}.\end{aligned} \tag{8.103}$$

We therefore find

$$\Big(\int_{\mathrm{B}(0,1)} V(\Psi)\,\mathrm{d}x\Big)^{1-\frac{1}{p}} \leqslant CL\Big(\int_{\mathrm{B}(0,1)} (V(\boldsymbol{\varepsilon}(U))+\delta)\,\mathrm{d}x\Big) \tag{8.104}$$

and hence, letting $p>n$ be such that $q=\frac{p}{p-1}\in(1,\frac{n+1}{n})$,

$$\int_{\mathrm{B}(0,1)} V(\Psi)\,\mathrm{d}x \leqslant C\Big(\int_{\mathrm{B}(0,1)} V(U)\,\mathrm{d}x\Big)^{q} + C\delta^{q}\mathscr{L}^{n}(\mathrm{B}(0,1))^{q}. \tag{8.105}$$

At this stage we scale back to the original ball to find

$$\fint_{\mathrm{B}(x_0,R)} V\Big(\frac{\widetilde{v}-h}{R}\Big)\,\mathrm{d}x \leqslant C\Big(\fint_{\mathrm{B}(0,R)} V(\boldsymbol{\varepsilon}(\widetilde{v}))\,\mathrm{d}x\Big)^{q} + C\delta^{q}, \tag{8.106}$$

and we note that the constant $C>0$ only depends on m,n,q,L and ℓ. We now intend to send $\delta\searrow 0$. Since the Ekeland–type approximation was accomplished with respect to the symmetric gradient norm after an approximation with respect to the area–strict metric, cp. (8.78)–(8.80), we obtain by the continuity part of Reshetnyak's theorem that

$$\fint_{\mathrm{B}(x_0,R)} V\Big(\frac{\widetilde{u}-h}{R}\Big)\,\mathrm{d}x \leqslant C\Big(\fint_{\mathrm{B}(0,R)} \mathrm{d}V(\mathrm{E}\widetilde{u}))\Big)^{q}$$

The proof is complete. □

Remark 8.32. *It is important to remark that the exponent q as it appears in the previous proposition can be chosen* strictly larger *than one. In the classical works on $\mathbb{A}$–harmonic approximation (cp. [89]–[92] or the exposition of the method in the recent monograph [33]), this corresponds to a suitable linear growth version of approximate $\mathbb{A}$–harmonicity. From a technical perspective, the importance of $q>1$ is given in (8.122), where the a–priori smallness assumption on the excess in turn gives smallness of the critical quantity*

$$\Big(\frac{\mathbf{E}(x_0,R_0)}{R_0^n}\Big)^{q-1}.$$

If we could not use $q>1$ and only had $q=1$ at our disposal, this critical term would equal one and hence would destroy the iteration scheme later on.

The objective of the remainder of the present section is to establish an preliminary excess decay. As usual, this decay shall be iterated in the subsequent paragraph and thereby lead to the partial regularity assertion of Theorem 8.4.

We divide the proof of the excess decay into several steps.

Step 1. *Smallness Assumptions.* Let $M_0>0$ be given and fix a ball $\mathrm{B}_{R_0}=\mathrm{B}(x_0,R_0)\Subset\Omega$ such that

$$|(\mathrm{E}u)_{\mathrm{B}_{R_0}}|<M_0, \qquad \text{where } (\mathrm{E}u)_{\mathrm{B}_{R_0}}=\frac{1}{\mathscr{L}^n(\mathrm{B}_{R_0})}\mathrm{E}u(\mathrm{B}_{R_0}). \tag{8.107}$$

We moreover assume that

$$\fint_{\mathrm{B}(x_0,R_0)} |\,\mathrm{E}u-(\mathrm{E}u)_{\mathrm{B}(x_0,R_0)}|\leqslant 1. \tag{8.108}$$

Hence, if we define the excess $\mathrm{E}(u;x_0,R)$ for $x_0\in\Omega$ and $R>0$ such that $\mathrm{B}(x_0,R)\Subset\Omega$ by

$$\mathrm{E}(u;x_0,R):=\fint_{\mathrm{B}(x_0,R)} \big|\mathscr{E}u(x)-(\mathrm{E}u)_{\mathrm{B}(x_0,R)}\big|\,\mathrm{d}\mathscr{L}^n + \frac{|\,\mathrm{E}^s u|(\mathrm{B}(x_0,R))}{\mathscr{L}^n(\mathrm{B}(x_0,R))},$$

then (8.108) translates into $\mathrm{E}(u;x_0,R_0)\leqslant 1$.

Step 2. *Selection of a good radius.* In a second step, we fix an affine–linear map $a\colon\mathbb{R}^n\to\mathbb{R}^n$ with $\boldsymbol{\varepsilon}(a)=(\mathrm{E}u)_{\mathrm{B}_{R_0}}$. We again follow [134, Sec. 3.4], however, now put

$$\widetilde{u}:=u-a\quad\text{and}\quad\widetilde{f}:=f_{\boldsymbol{\varepsilon}(a)},\tag{8.109}$$

where we explicitely remind the reader of the definition of f_w for $w\in\mathbb{R}^{n\times n}_{\mathrm{sym}}$ as given by (8.12). Starting from $R_0>0$ as given above, we now apply Theorem 8.27. Consequently, we find $R\in(\frac{9}{10}R_0,R_0)$ such that $\widetilde{u}|_{\partial\mathrm{B}(x_0,R)}\in\mathrm{W}^{\frac{1}{n+1},1+\frac{1}{n}}(\partial\mathrm{B}(x_0,R);\mathbb{R}^n)$ together with the corresponding estimate (with $\theta=\frac{1}{n+1}$ and accordingly $p=\frac{n+1}{n}$)

$$\begin{aligned}\inf_{\alpha\in\mathcal{R}(\Omega)}&\left(\fint_{\partial\mathrm{B}(x_0,R)}\int_{\partial\mathrm{B}(x_0,R)}\frac{|\widetilde{u}_\alpha(x)-\widetilde{u}_\alpha(y)|^{1+\frac{1}{n}}}{|x-y|^{(n-1)+\frac{1}{n}}}\,\mathrm{d}\sigma_x\,\mathrm{d}\sigma_y\right)^{\frac{n}{n+1}}\\&\leqslant C\frac{R_0^{1-\theta+\frac{n}{p}}}{R^{\frac{n-1}{p}}(\frac{1}{10}R_0)^{\frac{1}{p}}}\frac{1}{R_0^n}\int_{\mathrm{B}(x_0,R_0)}|\,\mathrm{E}\widetilde{u}|\\&\leqslant C\frac{R_0^{\frac{n}{n+1}}}{R_0^n}\int_{\mathrm{B}(x_0,R_0)}|\,\mathrm{E}u-(\mathrm{E}u)_{\mathrm{B}(x_0,R_0)}|.\end{aligned}\tag{8.110}$$

where $\widetilde{u}_\alpha:=\widetilde{u}-\alpha(=u-a-\alpha)$. Here we have used the uniform comparability of R and R_0 to obtain

$$\frac{R_0^{1-\theta+\frac{n}{p}}}{R^{\frac{n-1}{p}}(\frac{1}{10}R_0)^{\frac{1}{p}}}\leqslant\frac{R_0^{1-\theta+\frac{n}{p}}}{(\frac{9}{10})^{\frac{n-1}{p}}R_0^{\frac{n-1}{p}}(\frac{1}{10}R_0)^{\frac{1}{p}}}\leqslant cR_0^{1-\theta},$$

and here $c>0$ is obviously independent of R_0 and R. We fix the infimiser $b\in\mathcal{R}(\mathbb{R}^n)$ of the left hand side of (8.110) and consequently have $\widetilde{u}_b=\widetilde{u}-b$.

Step 3. *Definition of comparison maps.* We now pick the $\mathbb{A}$–harmonic mapping $\widetilde{h}\colon\mathrm{B}(x_0,R)\to\mathbb{R}^n$ solving

$$\begin{cases}-\operatorname{div}(f''((\mathrm{E}u)_{\mathrm{B}(x_0,R)})\boldsymbol{\varepsilon}(\widetilde{h}))=0&\text{in }\mathrm{B}(x_0,R),\\\widetilde{h}=\widetilde{u}_b&\text{on }\partial\mathrm{B}(x_0,R).\end{cases}\tag{8.111}$$

and put

$$A(x):=\widetilde{h}(x_0)+\langle\mathrm{D}\widetilde{h}(x_0),x-x_0\rangle,\qquad x\in\mathrm{B}(x_0,R),\tag{8.112}$$

together with

$$a_0:=a+A.\tag{8.113}$$

We then obtain

$$
\begin{aligned}
|\,\mathrm{E}a_0| = |\,\mathrm{E}a + \mathrm{E}\widetilde{h}(x_0)| &= |(\mathrm{E}u)_{\mathrm{B}_{R_0}} + (\mathrm{E}\widetilde{h})(x_0)| \\
&\leqslant M_0 + |(\mathrm{E}\widetilde{h})(x_0)| && \text{(by (8.107))} \\
&\leqslant M_0 + \sup_{\mathrm{B}(x_0,R/2)} |\,\mathrm{D}\widetilde{h}| && \text{(as } |\,\mathrm{E}v| \leqslant |\,\mathrm{D}v|\text{)} \\
&\leqslant M_0 + c \fint_{\mathrm{B}(x_0,R)} |\,\mathrm{D}\widetilde{h}|\,\mathrm{d}x && \text{(by (8.28))} \\
&\leqslant M_0 + c\Big(\fint_{\mathrm{B}(x_0,R)} |\,\mathrm{D}\widetilde{h}|^{\frac{n+1}{n}}\,\mathrm{d}x\Big)^{\frac{n}{n+1}} && \text{(by Jensen)} \\
&\leqslant M_0 \\
&\quad + cR^{-\frac{n}{n+1}}\Big(\fint_{\partial\mathrm{B}(x_0,R)}\int_{\partial\mathrm{B}(x_0,R)} \frac{|\widetilde{u}_\alpha(x) - \widetilde{u}_\alpha(y)|^{\frac{n+1}{n}}}{|x-y|^{n-1+\frac{1}{n}}}\,\mathrm{d}\sigma(x)\,\mathrm{d}\sigma(y)\Big)^{\frac{n}{n+1}} && \text{(by Lem. 8.29)} \\
&\leqslant 2M_0 + \frac{c}{R_0^n}\int_{\mathrm{B}(x_0,R_0)} |\,\mathrm{E}u - (\mathrm{E}u)_{\mathrm{B}(x_0,R_0)}| \leqslant 2M_0 + C,
\end{aligned}
$$

where the two last estimations hold because of (8.110) and (8.108). In particular, the constants appearing here do not depend on R or R_0. Summarising, if we put $m := 2M_0 + C$ as in the right side of the previous chain of inequalities, then we obtain

$$
|\,\mathrm{E}a_0| \leqslant m. \tag{8.114}
$$

Step 4. *Comparison estimates.* Let $0 < \sigma < \frac{1}{5}$ be arbitrary and put, for $t > 0$, $\mathrm{B}_t = \mathrm{B}(x_0, t)$. We firstly note by convexity of V and therefore Jensen's inequality,

$$
\begin{aligned}
\int_{\mathrm{B}_{\sigma R_0}} V(\mathrm{E}u - (\mathrm{E}u)_{\mathrm{B}_{\sigma R_0}}) &\leqslant \int_{\mathrm{B}_{\sigma R_0}} V(\mathrm{E}u - \mathrm{E}a_0 + \mathrm{E}a_0 - (\mathrm{E}u)_{\mathrm{B}_{\sigma R_0}}) \\
&\leqslant C\int_{\mathrm{B}_{\sigma R_0}} V(\mathrm{E}u - \mathrm{E}a_0) + C\int_{\mathrm{B}_{\sigma R_0}} (V(\mathrm{E}(u - a_0)))_{\mathrm{B}_{\sigma R_0}}) \\
&\leqslant C\int_{\mathrm{B}_{\sigma R_0}} V(\mathrm{E}(u - a_0)) = C\int_{\mathrm{B}_{\sigma R_0}} V(\mathrm{E}(u - \alpha - a_0))
\end{aligned} \tag{8.115}
$$

for all rigid deformations $\alpha \in \mathcal{R}(\Omega)$. Our next objective is to apply the Caccioppoli–type inequality, Proposition 8.30. Having chosen $m > 0$ as it appears in Proposition 8.30 by (8.114), we find $c = c(m) > 0$ such that (8.92) holds. We then estimate, using (8.115), the Caccioppoli–type inequality in the first and convexity of V in the third step,

$$
\begin{aligned}
\int_{\mathrm{B}_{\sigma R_0}} V(\mathrm{E}u - (\mathrm{E}u)_{\mathrm{B}(x_0,\sigma R_0)}) &\leqslant c(m)\int_{\mathrm{B}_{2\sigma R_0}} V\Big(\frac{\widetilde{u}(x) - \alpha(x) - A(x)}{\sigma R_0}\Big)\,\mathrm{d}x \\
&\leqslant c(m)\int_{\mathrm{B}_{2\sigma R_0}} V\Big(\frac{\widetilde{u}(x) - \alpha(x) - \widetilde{h}(x) - A(x) + \widetilde{h}(x)}{\sigma R_0}\Big)\,\mathrm{d}x \\
&\leqslant C\int_{\mathrm{B}_{2\sigma R_0}} V\Big(\frac{(\widetilde{u} - \alpha) - \widetilde{h}}{\sigma R_0}\Big)\,\mathrm{d}x + C\int_{\mathrm{B}_{2\sigma R_0}} V\Big(\frac{\widetilde{h} - A}{\sigma R_0}\Big)\,\mathrm{d}x \\
&\leqslant \frac{C}{\sigma^2}\int_{\mathrm{B}_R} V\Big(\frac{\widetilde{u}_\alpha - \widetilde{h}}{R}\Big)\,\mathrm{d}x + C\int_{\mathrm{B}_{2\sigma R_0}} V\Big(\frac{\widetilde{h} - A}{\sigma R_0}\Big)\,\mathrm{d}x \\
&=: \mathbf{I} + \mathbf{II}.
\end{aligned}
$$

Here we have used $\mathrm{B}(x_0, 2\sigma R_0) \subset \mathrm{B}(x_0, R)$, uniform comparability of R and R_0 and the fact that $V(\lambda z) \leqslant c\lambda^2 V(z)$ for a constant $c > 0$, all $z \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$ and $|\lambda| \geqslant 1$ (cp. Lemma 8.9). At this stage, we put

$$\boxed{\alpha = b}$$

with $b \in \mathcal{R}(\mathbb{R}^n)$ as picked in Step 2. We begin with the estimation of **I**, and for this purpose let $1 < q < \frac{n+1}{n}$ be arbitrary but fixed. We go back to Proposition 8.31 to obtain

$$\begin{aligned}
\mathbf{I} &= \frac{C}{\sigma^2}\int_{\mathrm{B}_R} V\Big(\frac{\widetilde{u}_b - \widetilde{h}}{R}\Big)\,\mathrm{d}x \\
&= \frac{CR^n}{\sigma^2}\fint_{\mathrm{B}_R} V\Big(\frac{\widetilde{u}_b - \widetilde{h}}{R}\Big)\,\mathrm{d}x \\
&\leqslant C\frac{R_0^n}{\sigma^2}\left(\fint_{\mathrm{B}(x_0,R)} V(\mathrm{E}\widetilde{u})\right)^q,
\end{aligned} \tag{8.116}$$

the last step being valid by uniform comparability of R and R_0.

As to **II**, let $x \in \mathrm{B}(x_0, 2\sigma R_0)$. We employ a pointwise estimate to find by use of Taylor's formula

$$\begin{aligned}
\left|\frac{\widetilde{h}(x) - A(x)}{\sigma R_0}\right| &= \left|\frac{\widetilde{h}(x) - \widetilde{h}(x_0) + \langle \mathrm{D}\widetilde{h}(x_0), x - x_0\rangle}{\sigma R_0}\right| \\
&\leqslant C\big(\sup_{\mathrm{B}(x_0,\frac{R}{2})} |\,\mathrm{D}^2\widetilde{h}|\big)\frac{|x-x_0|^2}{\sigma R_0} \\
&\leqslant C\big(\sup_{\mathrm{B}(x_0,\frac{R}{2})} |\,\mathrm{D}^2\widetilde{h}|\big)\frac{(2\sigma R_0)^2}{\sigma R_0} && (\text{since}\, x \in \mathrm{B}(x_0, 2\sigma R_0)) \\
&\leqslant C\sigma R\big(\sup_{\mathrm{B}(x_0,\frac{R}{2})} |\,\mathrm{D}^2\widetilde{h}|\big) && (\text{since}\, R \leqslant \tfrac{10}{9}R_0) \\
&\leqslant C\sigma \fint_{\mathrm{B}(x_0,R)} |\,\mathrm{D}\widetilde{h}|\,\mathrm{d}x && (\text{by Lemma 8.16}) \\
&\leqslant C\sigma\Big(\fint_{\mathrm{B}(x_0,R)} |\,\mathrm{D}\widetilde{h}|^{\frac{n+1}{n}}\,\mathrm{d}x\Big)^{\frac{n}{n+1}} =: \mathbf{III} && (\text{by Jensen}).
\end{aligned}$$

Similarly as in the estimation following (8.113), we employ Lemma 8.29 to further obtain

$$\begin{aligned}
\mathbf{III} &\leqslant C\sigma\fint_{\mathrm{B}(x_0,R)} |\,\mathrm{E}\widetilde{u}| \overset{\mathrm{Def}}{=} C\sigma \fint_{\mathrm{B}(x_0,R)} |\,\mathrm{E}(u-a)| && (\text{by Lemma 8.29}) \\
&= C\sigma\fint_{\mathrm{B}(x_0,R)} |\,\mathrm{E}u - (\mathrm{E}u)_{\mathrm{B}(x_0,R_0)}| && (\text{since } \mathrm{E}a = (\mathrm{E}u)_{\mathrm{B}(x_0,R_0)}) \\
&= C\sigma\Big(\fint_{\mathrm{B}(x_0,R)} |\,\mathrm{E}u - (\mathrm{E}u)_{\mathrm{B}(x_0,R_0)}|\Big)^{2\cdot\frac{1}{2}} \\
&\leqslant C\sigma\Big(V\Big(\fint_{\mathrm{B}(x_0,R)} |\,\mathrm{E}u - (\mathrm{E}u)_{\mathrm{B}(x_0,R_0)}|\Big)\Big)^{\frac{1}{2}} && (\text{by (8.108) and (8.16)}) \\
&\leqslant C\sigma\Big(\fint_{\mathrm{B}(x_0,R)} V(|\,\mathrm{E}u - (\mathrm{E}u)_{\mathrm{B}(x_0,R_0)}|)\Big)^{\frac{1}{2}} && (\text{by Jensen}).
\end{aligned}$$

Collecting estimates, we obtain

$$\left|\frac{\widetilde{h}(x) - A(x)}{\sigma R_0}\right| \leqslant C\sigma\Big(\fint_{\mathrm{B}(x_0,R)} V(|\,\mathrm{E}u - (\mathrm{E}u)_{\mathrm{B}(x_0,R_0)}|)\Big)^{\frac{1}{2}}. \tag{8.117}$$

We now integrate (8.117) over $\mathrm{B}_{2\sigma R_0}$, yielding

$$
\begin{aligned}
\mathbf{II} &= C\int_{\mathrm{B}_{2\sigma R_0}} V\left(\frac{\widetilde{h}(x)-A(x)}{\sigma R_0}\right)\mathrm{d}x \\
&\overset{(8.117)}{\leqslant} C\int_{\mathrm{B}_{2\sigma R_0}} V\left(C\sigma\Big(\fint_{\mathrm{B}(x_0,R)} V(|\,\mathrm{E}u-(\mathrm{E}u)_{\mathrm{B}(x_0,R_0)}|)\Big)^{\frac{1}{2}}\right)\mathrm{d}x \\
&\leqslant C(\sigma R_0)^n V\Big(\underbrace{\sigma\Big(\fint_{\mathrm{B}(x_0,R)} V(|\,\mathrm{E}u-(\mathrm{E}u)_{\mathrm{B}(x_0,R_0)}|)\Big)^{\frac{1}{2}}}_{=:\Upsilon}\Big)\,\mathrm{d}x \qquad (8.118)\\
&\overset{(8.15)}{\leqslant} C(\sigma R_0)^n\min\{|\Upsilon|,|\Upsilon|^2\} \\
&\overset{(*)}{\leqslant} C\sigma^{n+2}R_0^n\fint_{\mathrm{B}(x_0,R)} V(|\,\mathrm{E}u-(\mathrm{E}u)_{\mathrm{B}(x_0,R_0)}|).
\end{aligned}
$$

To see $(*)$, note that since $V(z)=\sqrt{1+|z|^2}-1\leqslant|z|$ for all $z\in\mathbb{R}^{n\times n}_{\mathrm{sym}}$, we have

$$
\fint_{\mathrm{B}(x_0,R)} V(|\,\mathrm{E}u-(\mathrm{E}u)_{\mathrm{B}(x_0,R_0)}|)\leqslant\fint_{\mathrm{B}(x_0,R)}|\,\mathrm{E}u-(\mathrm{E}u)_{\mathrm{B}(x_0,R_0)}|\overset{(8.108)}{\leqslant}1
$$

and hence, recalling $0<\sigma<\frac{1}{5}$, $|\Upsilon|\leqslant 1$, and therefore $\min\{|\Upsilon|,|\Upsilon|^2\}=|\Upsilon|^2$. We now gather estimates to eventually obtain with

$$
\mathbf{E}(x_0,r):=\int_{\mathrm{B}(x_0,r)} V(\mathrm{E}u-(\mathrm{E}u)_{\mathrm{B}(x_0,r)})\quad\text{and}\quad\widetilde{\mathbf{E}}(x_0,r):=\frac{\mathbf{E}(x_0,r)}{r^n}
$$

that

$$
\begin{aligned}
\mathbf{E}(x_0,\sigma R_0) &\leqslant C\frac{R_0^n}{\sigma^2}\left(\fint_{\mathrm{B}(x_0,R)} V(\mathrm{E}\widetilde{u})\right)^q + C\sigma^{n+2}R_0^n\sigma\fint_{\mathrm{B}(x_0,R)} V(|\,\mathrm{E}\widetilde{u}|) \\
&= \frac{C}{\sigma^2}\left(\frac{\mathbf{E}(x_0,R_0)}{R_0^n}\right)^{q-1}\mathbf{E}(x_0,R_0)+C\sigma^{n+2}\mathbf{E}(x_0,R_0) \qquad (8.119)\\
&= \left(\frac{C}{\sigma^2}\left(\frac{\mathbf{E}(x_0,R_0)}{R_0^n}\right)^{q-1}+C\sigma^{n+2}\right)\mathbf{E}(x_0,R_0).
\end{aligned}
$$

We will now use the previous inequality to deduce a preliminary excess decay.

Proposition 8.33. *Given $0<\alpha<1$, there exist two parameters $\sigma=\sigma(n,L,\ell,\alpha)\in(0,1)$ and $\widetilde{\varepsilon}_0\in(0,1)$ such that if $u\in\mathrm{BD}(\Omega)$ is a local generalised minimiser of the functional $\mathfrak{F}$ given by (8.1) subject to* (H4)–(H5) *satisfies*

$$
\widetilde{\mathbf{E}}(u;x_0,\sigma R_0)\leqslant\sigma^{1+\alpha}\widetilde{\mathbf{E}}(u;x_0,R_0) \qquad (8.120)
$$

for every ball $\mathrm{B}(x_0,R_0)\subset\Omega$ with $0<R_0\leqslant 1$ provided there holds

$$
\widetilde{\mathbf{E}}(u;x_0,R_0)\leqslant\widetilde{\varepsilon}_0^2\quad\textit{and}\quad|(\mathrm{E}u)_{\mathrm{B}(x_0,R_0)}|\leqslant M_0. \qquad (8.121)
$$

Proof. We start by choosing a preliminary $\widetilde{\varepsilon}_0>0$ and $M_0>0$ in a way such that (8.124) imply (8.107) and (8.108). There is no restriction on $M_0>0$. We estimate with $H:=\mathrm{E}u-(\mathrm{E}u)_{\mathrm{B}(x_0,R_0)}$ and the shorthands $A_{R_0}:=\mathrm{B}(x_0,R_0)\cap\{|H|\leqslant 1\}$, $A_{R_0}^c:=$

$\mathrm{B}(x_0, R_0) \cap \{|H| > 1\}$

$$
\begin{aligned}
\int_{\mathrm{B}(x_0,R_0)} |H| &= \int_{A_{R_0}^c} |H| + \int_{A_{R_0}} |H| \\
&= \mathscr{L}^n(A_{R_0}) \fint_{A_{R_0}} |H| + C\int_{A_{R_0}^c} V(|H|) \\
&\leqslant \mathscr{L}^n(A_{R_0}) \left(\fint_{A_{R_0}} |H|^2\right)^{\frac{1}{2}} + C\int_{A_{R_0}^c} V(|H|) \\
&\leqslant C(\mathscr{L}^n(A_{R_0})^{\frac{1}{2}} \left(\int_{\mathrm{B}(x_0,R_0)} V(|H|)\right)^{\frac{1}{2}} + C\int_{\mathrm{B}(x_0,R_0)} V(|H|) \\
&\leqslant CR_0^{\frac{n}{2}} \left(\int_{\mathrm{B}(x_0,R_0)} V(|H|)\right)^{\frac{1}{2}} + C\int_{\mathrm{B}(x_0,R_0)} V(|H|)
\end{aligned}
$$

so that dividing the resulting inequality by R_0^n yields

$$
\begin{aligned}
\fint_{\mathrm{B}(x_0,R_0)} |H| &\leqslant CR_0^{-\frac{n}{2}} \left(\int_{\mathrm{B}(x_0,R_0)} V(|H|)\right)^{\frac{1}{2}} + C\fint_{\mathrm{B}(x_0,R_0)} V(|H|) \\
&\leqslant C\left(\left(\fint_{\mathrm{B}(x_0,R_0)} V(|H|)\right)^{\frac{1}{2}} + \fint_{\mathrm{B}(x_0,R_0)} V(|H|)\right) \\
&\leqslant C(\widetilde{\varepsilon}_0' + \widetilde{\varepsilon}_0'^{\,2}).
\end{aligned}
$$

provided $\fint_{\mathrm{B}(x_0,R_0)} V(|H|) \leqslant \widetilde{\varepsilon}_0'$. We now choose $\widetilde{\varepsilon}_0' > 0$ so small such that the very right hand side of the previous inequality is smaller or equal to 1. At this stage, (8.119) is at our disposal and therefore yields for $0 < \sigma < \frac{1}{5}$

$$
\begin{aligned}
\widetilde{\mathbf{E}}(x_0, \sigma R_0) &\leqslant \left(\frac{C}{\sigma^{n+2}}\left(\frac{\mathbf{E}(x_0, R_0)}{R_0^n}\right)^{q-1} + C\sigma^2\right)\widetilde{\mathbf{E}}(x_0, R_0) \\
&\leqslant \left(\frac{C}{\sigma^{n+2}}(\widetilde{\varepsilon}_0')^{q-1} + C\sigma^2\right)\widetilde{\mathbf{E}}(x_0, R_0).
\end{aligned} \tag{8.122}
$$

We subsequently choose $\sigma > 0$ so small such that with the constant $C > 0$ from the last inequality,

$$\boxed{2C\sigma^2 \leqslant \sigma^{1+\alpha} \text{ and } \widetilde{\varepsilon}_0 := \min\{\sigma^{\frac{n+4}{q-1}}, \widetilde{\varepsilon}_0'\}.}$$

In turn, inserting these choices into (8.122) gives

$$\widetilde{\mathbf{E}}(x_0, \sigma R_0) \leqslant (2C\sigma^2)\widetilde{\mathbf{E}}(x_0, R_0) \leqslant \sigma^{1+\alpha}\widetilde{\mathbf{E}}(x_0, R_0),$$

and this is (8.123). The proof is complete. □

To conclude the proof of Theorem 8.4, we need to iterate the previous lemma. This is accomplished in a standard way and follows, e.g., [33, Lem. 5.8]. More precisely, we have the following result:

Theorem 8.34 (Iteration)**.** *Given $0 < \alpha < 1$, there exists $\varepsilon_0 = \varepsilon_0(n, L, \ell, \alpha, M_0) \in (0, 1)$ such that any generalised local minimiser u of the functional $\mathfrak{F}$ given by (8.1) satisfies for some constant $c = c(n, L, \ell, \alpha, M_0) > 0$*

$$\widetilde{\mathbf{E}}(u; x_0, r) \leqslant c\left(\frac{r}{R}\right)^{2\alpha}\widetilde{\mathbf{E}}(u; x_0, R) \qquad \text{for all } 0 < r \leqslant R \tag{8.123}$$

for every ball $\mathrm{B}(x_0,R)\subset\Omega$ *provided there holds*

$$\widetilde{\mathbf{E}}(u;x_0,R)\leqslant\widetilde{\varepsilon}_0^2 \quad \textit{and} \quad |(\mathrm{E}u)_{\mathrm{B}(x_0,R)}|\leqslant\frac{1}{2}M_0. \tag{8.124}$$

Proof. Under the conditions of the Lemma, let the parameters $\widetilde{\varepsilon}_0>0$ and $\sigma>0$ be given by Proposition 8.33. We firstly choose $\varepsilon\in(0,1)$ such that

$$0<\varepsilon_0\leqslant\min\{\widetilde{\varepsilon}_0,\sqrt{\nu}\frac{M_0}{4}(1-\sigma^\alpha)\sigma^n\}. \tag{8.125}$$

Here $\nu>0$ is an arbitrary but fixed number such that for $|z|\leqslant 1$ there holds $\nu|z|^2\leqslant V(z)$. Such a number exists by Lemma 8.9, and it is clear that we may assume that $0<\nu<1$.

We now consider the statements (S_k), $k\in\mathbb{N}_0$, given by

$$(S_k)\begin{cases}\widetilde{\mathbf{E}}(u;x_0,\sigma^kR)\leqslant\sigma^{(1+\alpha)k}\widetilde{\mathbf{E}}(u;x_0,R),\\ \widetilde{\mathbf{E}}(u;x_0,\sigma^kR)\leqslant\sigma^{2k\alpha}\varepsilon_0^2,\\ |(\mathrm{E}u)_{\mathrm{B}(x_0,\sigma^kR)}|<M_0,\end{cases}$$

and we shall verify these by induction on $k\in\mathbb{N}_0$. The assertion (S_0) is satisfied by assumption. Now assume (S_j) hold for all $j\in\{0,...,k-1\}$. We apply Proposition 8.33 to R_0 replaced by $\sigma^{k-1}R$. Obviously, $\sigma^{k-1}R\leqslant 1$ and, by the second part of (S_{k-1}), $\widetilde{\mathbf{E}}(u;x_0,\sigma^{k-1}R)\leqslant\sigma^{2(k-1)\alpha}\varepsilon_0^2\leqslant\widetilde{\varepsilon}_0^2$. By the third part of (S_{k-1}), $|(\mathrm{E}u)_{\mathrm{B}(x_0,\sigma^{k-1}R)}|<M_0$ is also satisfied, and hence we conclude by (S_{k-1})

$$\begin{aligned}\widetilde{\mathbf{E}}(u;x_0,\sigma^kR)&\leqslant\sigma^{1+\alpha}\widetilde{\mathbf{E}}(u;x_0,\sigma^{k-1}R)\\&\leqslant\sigma^{(1+\alpha)+(1+\alpha)(k-1)}\widetilde{\mathbf{E}}(u;x_0,R)=\sigma^{(1+\alpha)k}\widetilde{\mathbf{E}}(u;x_0,R).\end{aligned}$$

This is the first part of (S_k). Building on this, we further deduce

$$\widetilde{\mathbf{E}}(u;x_0,\sigma^kR)\leqslant\sigma^{(1+\alpha)k}\widetilde{\mathbf{E}}(u;x_0,R)\leqslant\sigma^{(1+\alpha)k}\widetilde{\varepsilon}_0^2\leqslant\sigma^{2\alpha k}\widetilde{\varepsilon}_0^2$$

as $0<\sigma<1$ and $2\alpha k\leqslant(1+\alpha)k$ as the latter is equivalent to $\alpha k\leqslant k$ which is true because of $0<\alpha<1$. We proceed to the third part of (S_k). To see this, we employ a telescope sum argument to find

$$\begin{aligned}|(\mathrm{E}u)_{\mathrm{B}(x_0,\sigma^kR)}|&\leqslant|(\mathrm{E}u)_{\mathrm{B}(x_0,R)}|+\sum_{j=0}^{k-1}|(\mathrm{E}u)_{\mathrm{B}(x_0,\sigma^jR)}-(\mathrm{E}u)_{\mathrm{B}(x_0,\sigma^{j+1}R)}|\\&\leqslant\frac{M_0}{2}+\sum_{j=0}^{k-1}\fint_{\mathrm{B}(x_0,\sigma^jR)}|\,\mathrm{E}u-(\mathrm{E}u)_{\mathrm{B}(x_0,\sigma^{j+1}R)}|\\&\leqslant\frac{M_0}{2}+\sum_{j=0}^{k-1}\fint_{\mathrm{B}(x_0,\sigma^{j+1}R)}|\,\mathrm{E}u-(\mathrm{E}u)_{\mathrm{B}(x_0,\sigma^jR)}|.\end{aligned} \tag{8.126}$$

At this stage we pause and justify how Jensen's inequality in conjunction with the appearance of the V–function can be applied to yield the third part of (S_k). We write

$$
\begin{aligned}
&\fint_{\mathrm{B}(x_0,\sigma^{j+1}R)} |\,\mathrm{E}u - (\mathrm{E}u)_{\mathrm{B}(x_0,\sigma^j R)}| \\
&\qquad = \fint_{\mathrm{B}(x_0,\sigma^{j+1}R)} |\mathscr{E}u - (\mathrm{E}u)_{\mathrm{B}(x_0,\sigma^j R)}|\,\mathrm{d}\mathscr{L}^n + \frac{|\,\mathrm{E}^s u|(\mathrm{B}(x_0,\sigma^{j+1}R))}{(\sigma^{j+1}R)^n} \\
&\qquad = \underbrace{\Big(\fint_{\mathrm{B}(x_0,\sigma^{j+1}R)} |\mathscr{E}u - (\mathrm{E}u)_{\mathrm{B}(x_0,\sigma^j R)}|\,\mathrm{d}\mathscr{L}^n\Big)}_{\leqslant 1}{}^{2\cdot\frac{1}{2}} + \frac{|\,\mathrm{E}^s u|(\mathrm{B}(x_0,\sigma^{j+1}R))}{(\sigma^{j+1}R)^n} \\
&\qquad \leqslant \left(\frac{1}{\nu}\fint_{\mathrm{B}(x_0,\sigma^{j+1}R)} V(|\mathscr{E}u - (\mathrm{E}u)_{\mathrm{B}(x_0,\sigma^j R)}|)\,\mathrm{d}\mathscr{L}^n\right)^{\frac{1}{2}} + \frac{|\,\mathrm{E}^s u|(\mathrm{B}(x_0,\sigma^{j+1}R))}{(\sigma^{j+1}R)^n} \\
&\qquad \leqslant \frac{1}{\sigma^{\frac{n}{2}}}\left(\frac{1}{\nu}\fint_{\mathrm{B}(x_0,\sigma^{j+1}R)} V(|\mathscr{E}u - (\mathrm{E}u)_{\mathrm{B}(x_0,\sigma^j R)}|)\,\mathrm{d}\mathscr{L}^n\right)^{\frac{1}{2}} \\
&\qquad + \frac{1}{\sigma^n}\frac{|\,\mathrm{E}^s u|(\mathrm{B}(x_0,\sigma^{j}R))}{(\sigma^{j}R)^n} \\
&\qquad \leqslant \max\left\{\frac{1}{\sigma^{\frac{n}{2}}},\frac{1}{\sigma^n}\right\}\Big(\frac{1}{\sqrt{\nu}}\widetilde{\mathbf{E}}(u;x_0,\sigma^j R)^{\frac{1}{2}} + \widetilde{\mathbf{E}}(u;x_0,\sigma^j R)\Big) \\
&\qquad \leqslant \max\left\{\frac{1}{\sigma^{\frac{n}{2}}},\frac{1}{\sigma^n}\right\}\Big(\frac{\sigma^{j\alpha}\varepsilon_0}{\sqrt{\nu}} + \sigma^{2j\alpha}\varepsilon_0^2\Big) \leqslant \frac{2}{\sqrt{\nu}\sigma^n}\sigma^{j\alpha}\varepsilon_0
\end{aligned}
$$

as $0<\sigma,\varepsilon_0,\nu<1$.

By our choice of ε_0, we then obtain for every $j\in\{0,...,k-1\}$

$$
\fint_{\mathrm{B}(x_0,\sigma^{j+1}R)} |\,\mathrm{E}u - (\mathrm{E}u)_{\mathrm{B}(x_0,\sigma^j R)}| \leqslant \frac{2}{\sigma^n}\sigma^{j\alpha}\frac{M_0}{4}(1-\sigma^\alpha)\sigma^n = \frac{M_0}{2}(1-\sigma^\alpha)\sigma^{j\alpha}.
$$

We insert this into (8.126) to obtain by using a standard bound on the geometric series

$$
\begin{aligned}
|(\mathrm{E}u)_{\mathrm{B}(x_0,\sigma^k R)}| &\leqslant \frac{M_0}{2} + \sum_{j=0}^{k-1}\frac{M_0}{2}(1-\sigma^\alpha)\sigma^{j\alpha} \\
&< \frac{M_0}{2} + \frac{M_0}{2}(1-\sigma^\alpha)\sum_{j=0}^{\infty}\sigma^{j\alpha} = M_0.
\end{aligned}
$$

In conclusion, the proof of (S_k) is complete. Finally, let $0<r<R$ be arbitrary. Then we find $k\in\mathbb{N}$ with $\sigma^{k+1}R\leqslant r<\sigma^k R$. Let $a>0$ be a constant such that $V(\xi+\eta)\leqslant$

$a(V(\xi) + V(\eta))$ holds for all $\xi, \eta \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$. Then we conclude

$$
\begin{aligned}
\widetilde{\mathbf{E}}(u;x_0,r) &= \frac{1}{r^n}\Big(\int_{\mathrm{B}(x_0,r)} V(\mathscr{E}u - (\mathrm{E}u)_{\mathrm{B}(x_0,r)})\,\mathrm{d}\mathscr{L}^n + |\,\mathrm{E}^s u|(\mathrm{B}(x_0,r))\Big) \\
&\leqslant \fint_{\mathrm{B}(x_0,r)} V(\mathscr{E}u(x) - \fint_{\mathrm{B}(x_0,r)} \mathscr{E}u(y)\,\mathrm{d}y - \frac{\mathrm{E}^s u(\mathrm{B}(x_0,r))}{r^n})\,\mathrm{d}\mathscr{L}^n + \frac{|\,\mathrm{E}^s u|(\mathrm{B}(x_0,r))}{r^n} \\
&\leqslant \fint_{\mathrm{B}(x_0,r)} V(\fint_{\mathrm{B}(x_0,r)} \mathscr{E}u(x) - \mathscr{E}u(y) - \mathrm{E}^s u(\mathrm{B}(x_0,r))\,\mathrm{d}y)\,\mathrm{d}x + \frac{|\,\mathrm{E}^s u|(\mathrm{B}(x_0,r))}{r^n} \\
&\leqslant \fint_{\mathrm{B}(x_0,r)} \fint_{\mathrm{B}(x_0,r)} V(\mathscr{E}u(x) - \mathscr{E}u(y) - \mathrm{E}^s u(\mathrm{B}(x_0,r)))\,\mathrm{d}y\,\mathrm{d}x + \frac{|\,\mathrm{E}^s u|(\mathrm{B}(x_0,r))}{r^n} \\
&\leqslant \fint_{\mathrm{B}(x_0,r)} \fint_{\mathrm{B}(x_0,r)} aV(\mathscr{E}u(x) - \mathscr{E}u(y)) + aV(\mathrm{E}^s u(\mathrm{B}(x_0,r)))\,\mathrm{d}y\,\mathrm{d}x \\
&+ \frac{|\,\mathrm{E}^s u|(\mathrm{B}(x_0,r))}{r^n} \\
&\leqslant a \fint_{\mathrm{B}(x_0,r)} \fint_{\mathrm{B}(x_0,r)} V(\mathscr{E}u(x) - \mathscr{E}u(y))\,\mathrm{d}y\,\mathrm{d}x + (1+a)\frac{|\,\mathrm{E}^s u|(\mathrm{B}(x_0,r))}{r^n},
\end{aligned}
$$

where we have used that $0 < r < 1$ in the last inequality. Now, we have

$$
\fint_{\mathrm{B}(x_0,r)} \fint_{\mathrm{B}(x_0,r)} V(\mathscr{E}u(x) - \mathscr{E}u(y))\,\mathrm{d}y\,\mathrm{d}x \leqslant 2a \fint_{\mathrm{B}(x_0,r)} V(\mathscr{E}u(x) - (\mathrm{E}u)_{\mathrm{B}(x_0,\sigma^k R)})\,\mathrm{d}y\,\mathrm{d}x
$$

which in turn yields with $c(a) = \max\{a^2, 2(a+1)\}$

$$
\begin{aligned}
\widetilde{\mathbf{E}}(u;x_0,r) &\leqslant 2a^2 \fint_{\mathrm{B}(x_0,r)} V(\mathscr{E}u(x) - (\mathrm{E}u)_{\mathrm{B}(x_0,\sigma^k R)})\,\mathrm{d}y\,\mathrm{d}x + (1+a)\frac{|\,\mathrm{E}^s u|(\mathrm{B}(x_0,r))}{r^n} \\
&\leqslant \frac{c(a)}{r^n}\Big(\int_{\mathrm{B}(x_0,r)} V(\mathscr{E}u(x) - (\mathrm{E}u)_{\mathrm{B}(x_0,\sigma^k R)})\,\mathrm{d}x + |\,\mathrm{E}^s u|(\mathrm{B}(x_0,r))\Big) \\
&\leqslant \frac{c(a)}{r^n}\Big(\int_{\mathrm{B}(x_0,\sigma^k R)} V(\mathscr{E}u(x) - (\mathrm{E}u)_{\mathrm{B}(x_0,\sigma^k R)})\,\mathrm{d}x + |\,\mathrm{E}^s u|(\mathrm{B}(x_0,\sigma^k R))\Big) \\
&= \frac{c(a)(\sigma^k R)^n}{r^n}\widetilde{\mathbf{E}}(u;x_0,\sigma^k R) \\
&\leqslant \frac{c(a)(\sigma^k R)^n}{r^n}\sigma^{2k\alpha}\widetilde{\mathbf{E}}(u;x_0,R) \\
&\overset{(*)}{\leqslant} c(a)\sigma^{-n-2\alpha}\left(\frac{r}{R}\right)^{2\alpha}\widetilde{\mathbf{E}}(u;x_0,R).
\end{aligned}
$$

Here, the last estimation can be seen by

$$
\begin{aligned}
\sigma^{k+1}R \leqslant r < \sigma^k R \Rightarrow \sigma^{k+1} \leqslant \frac{r}{R} &\Rightarrow \sigma^{(k+1)(n+2\alpha)} \leqslant \left(\frac{r}{R}\right)^{n+2\alpha} \\
&\Rightarrow \sigma^{(k+1)(n+2\alpha)}\left(\frac{R}{r}\right)^n \leqslant \left(\frac{r}{R}\right)^{2\alpha} \\
&\Rightarrow \sigma^{k(n+2\alpha)}\left(\frac{R}{r}\right)^n \leqslant \sigma^{-n-2\alpha}\left(\frac{r}{R}\right)^{2\alpha},
\end{aligned}
$$

and from here the final statement of the theorem is immediate as $a > 0$ is a constant and σ is fixed. The proof is complete. $\square$

We now come to the

Proof of Theorem 8.4. With Theorem 8.34 at our disposal, we pursue a similar strategy as in Chapter 7 to establish the partial regularity. Other than in the superlinear growth regime, however, we cannot use direct continuity arguments and have to proceed more carefully. Let $x_0 \in \Omega_u := \mathrm{Reg}(u)$ so that

$$\lim_{r\searrow 0} \fint_{\mathrm{B}(x_0,r)} |\mathscr{E}u - (\mathrm{E}u)_{\mathrm{B}(x_0,r)}| \,\mathrm{d}\mathscr{L}^n + \frac{|\,\mathrm{E}^s u|(\mathrm{B}(x_0,r))}{r^n} = 0. \tag{8.127}$$

This trivially implies both

$$\lim_{r\searrow 0} \fint_{\mathrm{B}(x_0,r)} |\mathscr{E}u - (\mathrm{E}u)_{\mathrm{B}(x_0,r)}| \,\mathrm{d}\mathscr{L}^n = 0 \quad \text{and} \quad \lim_{r\searrow 0} \frac{|\,\mathrm{E}^s u|(\mathrm{B}(x_0,r))}{r^n} = 0.$$

Since $V(\cdot) \leqslant |\cdot|$, this yields

$$\lim_{r\searrow 0} \fint_{\mathrm{B}(x_0,r)} V(\mathscr{E}u - (\mathrm{E}u)_{\mathrm{B}(x_0,r)}) = 0.$$

Our aim is to show that the conditions of (8.124) remain valid for all points in a neighbourhood of x_0. We start by noting that for $0 < \delta < 1$ which we assumed sufficiently small but fixed,

$$\sup_{0<r<\delta} |(\mathrm{E}u)_{\mathrm{B}(x_0,r)}| =: M < \infty$$

as a second consequence of $x_0 \in \mathrm{Reg}(u) = \Omega \backslash \Sigma_u$. We then define $M_0 := 2^{8n+1}\max\{M, 1\}$. In consequence, Theorem 8.34 provides us with a radius $R_0 > 0$ and $\varepsilon_0 > 0$ (both of which depend on M_0) such that if $0 < R \leqslant R_0$ is a radius with $\mathrm{B}(x_0, R) \Subset \Omega$ and

$$\widetilde{\mathbf{E}}(u; x_0, R) \leqslant \widetilde{\varepsilon}_0^2 \quad \text{and} \quad |(\mathrm{E}u)_{\mathrm{B}(x_0,R)}| \leqslant \frac{1}{2} M_0,$$

then there holds

$$\widetilde{\mathbf{E}}(u; x_0, r) \leqslant C\Big(\frac{r}{R}\Big)^{2\alpha} \widetilde{\mathbf{E}}(u; x_0, R) \qquad \text{for all } 0 < r < R \tag{8.128}$$

with a constant $C > 0$ independent of $0 < r < R$. We now choose $0 < R < R_0$ in a way such that

$$\widetilde{\mathbf{E}}(u; x_0, R) \leqslant \frac{\varepsilon_0^2}{2\max\{2\cdot 4^{2n}a^2, (1+a)4^n\}} \quad \text{and} \quad |(\mathrm{E}u)_{\mathrm{B}(x_0,R)}| \leqslant M (\leqslant \frac{1}{2} M_0),$$

where again $a > 0$ is an arbitrary but fixed constant such that $V(\xi+\eta) \leqslant a(V(\xi)+V(\eta))$ holds for all $\xi, \eta \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$. This is possible by the definition of $M > 0$ and (8.127). We now show that with $R' := \frac{1}{4}R$ there holds

$$\widetilde{\mathbf{E}}(u; x, R') \leqslant \widetilde{\varepsilon}_0^2 \quad \text{and} \quad |(\mathrm{E}u)_{\mathrm{B}(x,R)}| \leqslant \frac{1}{2} M_0$$

for all $x \in \mathrm{B}(x_0, R')$, and in light of (8.128), this will imply that $\mathrm{E}^s u$ vanishes on $\mathrm{B}(x_0, R')$ and $\mathscr{E}u$ is of class $\mathrm{C}^{0,\alpha}$ on a neighbourhood of x_0. By the usual reduction argument, either using the Campanato–characterisation of Hölder continuity in conjunction with Korn's inequality or the theory of singular integrals on Hölder–Zygmund classes, the claim then follows as in Chapter 7.

We have with $\omega_n = \mathscr{L}^n(\mathrm{B}(0,1))$

$$
\begin{aligned}
&\fint_{\mathrm{B}(x,R')} V(|\mathscr{E}u - (\mathrm{E}u)_{\mathrm{B}(x,R')}|)\,\mathrm{d}\mathscr{L}^n + \frac{|\,\mathrm{E}^s u|(\mathrm{B}(x,R'))}{(R')^n} \\
&\quad\leqslant a \fint_{\mathrm{B}(x,R')} V(|\mathscr{E}u - (\mathscr{E}u)_{\mathrm{B}(x,R')}|)\,\mathrm{d}\mathscr{L}^n + (1+a)\frac{|\,\mathrm{E}^s u|(\mathrm{B}(x,R'))}{(R')^n} \\
&\quad\leqslant \frac{4^{2n}}{\omega_n^2 R^{2n}} \int_{\mathrm{B}(x,R')}\int_{\mathrm{B}(x,R')} V(|\mathscr{E}u(y) - \mathscr{E}u(z)|)\,\mathrm{d}y\,\mathrm{d}z + (1+a)4^n\frac{|\,\mathrm{E}^s u|(\mathrm{B}(x_0,R))}{R^n} \\
&\quad\leqslant \frac{4^{2n}a}{\omega_n^2 R^{2n}} \int_{\mathrm{B}(x_0,R)}\int_{\mathrm{B}(x_0,R)} V(|\mathscr{E}u(y) - \mathscr{E}u(z)|)\,\mathrm{d}y\,\mathrm{d}z \\
&\quad+ (1+a)\cdot 4^n\frac{|\,\mathrm{E}^s u|(\mathrm{B}(x_0,R))}{R^n} \\
&\quad\leqslant \frac{2\cdot 4^{2n}a^2}{\omega_n^2 R^{2n}} \int_{\mathrm{B}(x_0,R)}\int_{\mathrm{B}(x_0,R)} V(|\mathscr{E}u(y) - (\mathrm{E}u)_{\mathrm{B}(x_0,R)}|)\,\mathrm{d}y\,\mathrm{d}z \\
&\quad+ (1+a)4^n\frac{|\,\mathrm{E}^s u|(\mathrm{B}(x_0,R))}{R^n} \\
&\quad\leqslant \max\left\{2\cdot 4^{2n}a^2, (1+a)4^n\right\}\frac{\varepsilon_0^2}{2\max\{2\cdot 4^{2n}a^2,(1+a)4^n\}} \\
&\quad\leqslant \frac{\varepsilon_0^2}{2} < \varepsilon_0^2.
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
|(\mathrm{E}u)_{\mathrm{B}(x,R')}| &\leqslant \left|\fint_{\mathrm{B}(x,R')} \mathscr{E}u\,\mathrm{d}\mathscr{L}^n\right| + \frac{|\,\mathrm{E}^s u|(\mathrm{B}(x,R'))}{(R')^n} \\
&\leqslant \left|\fint_{\mathrm{B}(x,R')} \mathscr{E}u - (\mathrm{E}u)_{\mathrm{B}(x_0,R)}\,\mathrm{d}\mathscr{L}^n\right| + \frac{|\,\mathrm{E}^s u|(\mathrm{B}(x,R'))}{(R')^n} + |(\mathrm{E}u)_{\mathrm{B}(x_0,R)}| \\
&\leqslant 4^n \fint_{\mathrm{B}(x_0,R)} |\mathscr{E}u - (\mathrm{E}u)_{\mathrm{B}(x_0,R)}|\,\mathrm{d}\mathscr{L}^n + 4^n\frac{|\,\mathrm{E}^s u|(\mathrm{B}(x_0,R))}{R^n} + |(\mathrm{E}u)_{\mathrm{B}(x_0,R)}| \\
&\leqslant 4^n\widetilde{\mathbf{E}}(u;x_0,R) + |(\mathrm{E}u)_{\mathrm{B}(x_0,R)}| \\
&\leqslant 4^n\frac{\varepsilon_0^2}{2\max\{2\cdot 4^{2n}a^2,(1+a)4^n\}} + \max\{M,1\} \\
&\leqslant \varepsilon_0^2 + \max\{M,1\} < \max\{M,1\} + 1 < \frac{M_0}{2}
\end{aligned}
$$

because we have with $\widetilde{M} := \max\{M,1\}$

$$
\begin{aligned}
\frac{M_0}{2} = 2^{8n}\widetilde{M} &\geqslant 2^{8n}\widetilde{M} - \widetilde{M} - 1 + \widetilde{M} + 1 = (2^{8n}-1)\widetilde{M} - 1 + (M+1) \\
&\geqslant (2^{8n}-2) + \widetilde{M} + 1 \geqslant \widetilde{M} + 1.
\end{aligned}
$$

Using Lemma 7.10, the proof is complete. □

8.10. Concluding Remarks

We wish to conclude this chapter with various remarks which should help to put the results gathered so far in context with other results in this thesis. By this, we particularly aim at commenting on non–autonomous integrands and dimension reduction in the quasiconvex situation.

- *Non–autonomous integrands.* In principle, it is also possible to slightly modify the proof to yield a partial regularity result for non–autonomous integrands $(x, \xi) \mapsto f(x, \xi)$ under a suitable variant of the quasiconvexity hypothesis considered here. If the integrand, however, is fully non–autonomous in the sense that it depends on x, u and $\mathrm{E}u$, then a reverse Hölder inequality to yield higher integrability of minima seems inevitable. As discussed to some extent in Chapter 5.6, the Caccioppoli–inequality alone cannot yield such a result. In conclusion, it is not clear to us as to which extent the results presented here carry over to variational integrals of the form

$$\mathfrak{F}[u] := \int_{\Omega} f(x, u, \mathrm{E}u).$$

- *Dimension reduction for the singular set.* Not only for quasiconvex linear growth functionals but for quasiconvex functionals in general, the reduction of the Hausdorff dimension of the singular set is still an open problem. To the best of our knowledge, the only result in this direction is due to Kristensen & Mingione [159] by use of a set porosity approach: Here it is shown for superquadratic, strongly quasiconvex variational integrals that the Hausdorff dimension of the singular set of minima is *strictly smaller* than the ambient space dimension *provided* the minima are a–priori assumed to be of class $\mathrm{W}^{1,\infty}$. In particular, it would be desirable to achieve Haudorff dimension bounds without the $\mathrm{W}^{1,\infty}$–hypothesis.

CHAPTER 9

Résumé and Outlook

In this thesis we have given a regularity theory for linear growth problems on the space BD and extended the available regularity theory for such problems on the space on BV. In this concluding paragraph, we summarise these results and discuss possible extensions of the work presented in this thesis which we regard as most relevant.

In particular, we gave conditions on convex variational integrals of linear growth

- on BD to produce generalised minimisers of higher integrability and Sobolev regularity subject to a local boundedness/ BMO assumption imposed on a particular sequence of viscosity approximations. Working on locally bounded generalised minima in the BV–setting, Beck & Schmidt [34] (in turn building on Bildhauer's work [40]) could show that *all* generalised minima of 3–elliptic integrands are of class $\mathrm{W}^{1,L\log L}_{\mathrm{loc}}$. This raises the question as to

  1. which extent the range of μ leading to higher integrability results can be extended in the symmetric gradient situation.

  2. whether the results from [34] in the gradient case can be established without imposing the local boundedness assumption. As briefly sketched in Chapter 6.7, this is in fact possible for $1<\mu<1+\frac{2}{n}$.

  3. how the Ekeland–type approximation strategy of [34] can be utilised to yield higher integrability of the symmetric gradients of *all* generalised minima (subject to a suitable variant of the local BMO–assumption) in the BD–situation.

  4. whether in the full gradient case for the Dirichlet problem, the $\mu=3$–ellipticity in fact displays a borderline case for higher integrability of gradients. We recall from Remark 5.1 that a corresponding counterexample for $\mu>3$ is only available in the non–autonomous setting. In some sense, this would imply a *discontinuity* in the integrability of gradients. Indeed, if there existed a generalised minimiser $u\in(\mathrm{BV}\setminus\mathrm{W}^{1,1})(\Omega;\mathbb{R}^N)$ for $\mu>3$, then the μ–ellipticity scale would show to not be sufficiently fine tuned to yield a continuous deterioration of smoothness (since the gap between $\mathrm{W}^{1,L\log L}$ and $\mathrm{W}^{1,1}$ could be filled, for instance, by other local Sobolev–Orlicz spaces $\mathrm{W}^{1,\varphi}$).

- on BV to produce generalised minima of class $\mathrm{W}^{1,1}$ within the framework of Neumann problems with radially symmetric integrands. As we use Chacon's biting lemma rather than passing to a suitable weak*–limit in a sequence of viscosity approximations and finally prove that the biting limit is curl–free, it is natural to ask

  5. in how far these ideas can be utilised to yield results for the corresponding problems on BD. Here one would need to elaborate on the possible impact of the radial structure of the integrands in the symmetric gradient case and, secondly,

whether the convergences at our disposal suffice to conclude that the biting limit is $\operatorname{curl} \circ \operatorname{curl}$–free.

6. whether the $\mathrm{W}^{1,1}$–regularity for one generalised minimiser can be extended to all generalised minimisers for the respective Neumann problems by use of a Ekeland–type strategy as in [34].

7. whether the assumption on Ω to be simply connected is necessary.

- on BD to possess generalised minima which are $\mathrm{C}^{1,\alpha}$–partially regular. This is accomplished within the framework of non–autonomous integrands, but it can be seen that the results extend without any efforts to x–dependent integrands $f(x,z)$, too. However, in this context we would wish to provide

8. optimal bounds for the Hausdorff dimension of the singular set. This is linked to questions 1 to 3 from above. In principle, we would expect $\dim_{\mathcal{H}}(\Sigma_u) \leqslant n-1$ as an optimal bound, but this does not follow from the Sobolev regularity approach pursued in Chapter 5. On the other hand, it would be interesting to quantify the precise impact of smoothness of $f(x,z)$ in its first argument on the size of the singular set in a similar vein as in Mingione's work [175].

9. an approach that is also available for integrands with u–dependence, i.e., integrands of the form $\Omega \times \mathbb{R}^n \times \mathbb{R}^{n\times n}_{\mathrm{sym}} \ni (x,y,z) \mapsto f(x,y,z)$. In the full gradient case, such integrands have been covered by Schmidt [210].

Finally, for symmetric quasiconvex and rank–one convex functionals,

- we established a *full* Hölder regularity result based on suitable smoothness and smallness assumptions on the Dirichlet data. Here we particularly worked with a $\mathrm{C}^{3,1}_{\mathrm{loc}}$–condition on the variational integrands, and it hereafter would be of interest to inquire

10. in how far this regularity assumption can be weakened to still yield full Hölder continuity of generalised minima.

11. what is the precise deterioration of full Hölder continuity of generalised minima provided the smallness condition on the Dirichlet data is weakened.

- we proved a partial regularity result in the symmetric quasiconvex linear growth setup. In this respect, it would be equally relevant to extend the method to integrands $f(x,u,\mathrm{E}u)$ as in item 9. Here, however, we have to cope with the lack of a higher integrability approach by finite differences. An approach that overcomes the use of Gehring's lemma has been given by Schmidt [209] in the superquadratic growth context, but it is not clear to us how this generalises to the linear growth situation. As it is also open in the superlinear growth regime, it would be of relevance

12. to develop an effective way to estimate the Hausdorff dimension of the singular set of generalised minima in the (symmetric) quasiconvex setting. For generalised minima which are a priori of class $\mathrm{W}^{1,\infty}_{\mathrm{loc}}$, this might be possibly achieved by the set porosity approach due to Kristensen & Mingione [159], and any weakening of the Lipschitz assumption would prove valuable also in the superlinear growth regime.

CHAPTER 10

Appendix

In this chapter we collect various results which were used throughout the main part of the text or put the material presented there in a broader context.

10.1. The Type–(C) and $\mathbb{C}$–Ellipticity Correspondence

Recalling the notions of $\mathbb{R}$– and $\mathbb{C}$–ellipticity from Chapter 3 together with the corresponding notation and terminology, we now pass on to a more detailled discussion and link them to the *type–(C)* condition as introduced in [149]. For the following computation, it is useful to rewrite the operators $\mathbb{A} = \mathbb{A}[D]$ as encountered in Chapter 3 in coordinate notation and henceforth put, for fixed linear mappings $\mathbb{A}_\alpha \colon V = \mathbb{R}^N \to \mathbb{R}^K = W$,

$$\mathbb{A} = \mathbb{A}[D] = \sum_{\alpha=1}^{n} \mathbb{A}_\alpha \partial_\alpha. \tag{10.1}$$

Remark 10.1. *The operator $\mathbb{A}[D] = \mathbb{A}$ is $\mathbb{C}$-elliptic if and only if it is of type (C) in the sense of [149]. More precisely, since $\mathbb{A}_\alpha[\xi]$ is a linear operator from $\mathbb{R}^N$ to $\mathbb{R}^K$ for each $\xi \in \mathbb{R}^n$, we find coefficients $\mathbb{A}_{\alpha,j,k}$ such that*

$$\big(\mathbb{A}[\xi]\eta\big)_k =: \sum_{\alpha=1}^{n}\sum_{j=1}^{N} \mathbb{A}_{\alpha,j,k}\xi_\alpha\eta_j.$$

for every for $\xi \in \mathbb{R}^n$ and $\eta \in \mathbb{R}^N$. Then

$$\mathbb{P}_{j,k}u := \sum_{\alpha=1}^{n} \mathbb{A}_{\alpha,j,k}\partial_\alpha u_j$$

for $k = 1, \ldots, K$ is the family of scalar differential operators as used in [149]. The corresponding symbols are

$$\mathbb{P}_{j,k}(\xi) := \sum_{\alpha=1}^{n} \mathbb{A}_{\alpha,j,k}\xi_\alpha$$

with $j = 1, \ldots, N$ and $k = 1, \ldots, K$. Now according to [149] the family $(\mathbb{P}_k)_k$ is of type (C) if and only if $(\mathbb{P}_{j,k}(\xi))_{j,k}$ has rank K for all $\eta \in \mathbb{C}^n \setminus \{0\}$. Since

$$\sum_{j=1}^{N}\sum_{k=1}^{K} \mathbb{P}_{j,k}(\xi)\eta_j = \sum_{\alpha=1}^{n}\sum_{j=1}^{N}\sum_{k=1}^{K} \mathbb{A}_{\alpha,j,k}\xi_\alpha\eta_j = \mathbb{A}[\xi]\eta$$

this is equivalent to the injectivity of $\mathbb{A}[\xi]$ for all $\eta \in \mathbb{C}^N \setminus \{0\}$, which is exactly the $\mathbb{C}$-ellipticity of $\mathbb{A}$.

10.2. Approximate Differentiability of BD–Functions

In this section we discuss the approximate differentiability of LD– and BD–maps, with particular focus on a Calderón–Zygmund–type approach in Section 10.2.2. This is reminiscent of the so–called *Lipschitz truncation technique*, cp. [5, 112, 79, 52, 84]. Beforehand, we discuss the by now more classical approaches to the approximate differentiability as employed by Hajlasz [140] and Ambrosio, Coscia & Dal Maso [14].

10.2.1. Two approaches to the approximate differentiability In [140], the approximate differentiability of BD–maps is approached as follows. Given a closed subset F of $\mathbb{R}^n$, we define the *Marcinkiewicz integral* as

$$I_*(x) := \int_{\mathbb{R}^n} \frac{\operatorname{dist}(z,F)}{|x-z|^{n+1}}\,\mathrm{d}z.$$

If moreover $\mathscr{L}^n(\mathbb{R}^n \setminus F) < \infty$, then it can be shown (cp. [221]) that $I_*(x) < \infty$ for $\mathscr{L}^n$–a.e. $x \in F$. Secondly, if $E \subset \mathbb{R}^n$ is measurable and $u, v\colon E \to \mathbb{R}$ are measurable functions such that $|u(x)-u(y)| \leqslant |x-y|(v(x)+v(y))$ holds for $\mathscr{L}^n$–a.e. $x,y \in E$, then u is approximately differentiable $\mathscr{L}^n$–a.e. in E.

Let μ a finite Radon measure on $\mathbb{R}^n$ and decompose $\mu = \mu^g + \mu^b$ into a 'good' and 'bad' part in the sense of a Calderón–Zygmund decomposition at given level $t > 0$, i.e., μ^g is an absolutely continuous measure with respect to $\mathscr{L}^n$ whose density is bounded by t. Next it is shown (cp. [140, Thm. 6]) that if $K \in \mathrm{C}^\infty(\mathbb{R}^n \setminus \{0\})$ satisfies $K(x) = |x|^{1-n}K(x/|x|)$ for all $x \neq 0$ and μ is a signed Borel measure on $\mathbb{R}^n$ with finite total variation, then the function $K * \mu$ has an approximate differential $\mathscr{L}^n$–a.e.. This is achieved by realising that $\frac{\mathrm{d}\mu^g}{\mathrm{d}\mathscr{L}^n}$ is essentially bounded and integrable, hence belongs to any L^p, $1 \leqslant p \leqslant \infty$. In turn, $K * \mu^g$ belongs to $\mathrm{W}^{1,p}_{\mathrm{loc}}$ and hence is approximately differentiable $\mathscr{L}^n$. For the bad part, the estimate $|K * \mu^b(x) - K * \mu^b(y)| \leqslant C|x-y|(I_*(x)+I_*(y))$ for $\mathscr{L}^n$–a.e. x,y in the good set (i.e., where the maximal function of μ is smaller or equal than t) is employed. As $t \nearrow \infty$, the good sets exhaust $\mathbb{R}^n$, and the claim follows. As u can be retrieved from the measure $\mathrm{E}u$ by convolution against a Riesz–type potential, the claim follows. This approach has been adapted and generalised to convolution products of Riesz–type kernels and Radon measures by Alberti, Bianchini & Crippa [9].

Remark 10.2 (Comparison with the strategy from below)**.** *To motivate the approach pursued below, let us recall that usually Lipschitz–type truncations (cp. [5]) for* $\mathrm{W}^{1,p}$*–maps* $u\colon \mathbb{R}^n \to \mathbb{R}^n$ *for* $1 < p < \infty$ *rely on the elementary potential or maximal estimate*

$$|u(x)-u(y)| \leqslant C|x-y|\big(\mathcal{M}(\mathrm{D}u)(x) + \mathcal{M}(\mathrm{D}u)(y)\big) \tag{10.2}$$

for $\mathscr{L}^n$*–a.e.* $x, y \in \mathbb{R}^n$*. In consequence, if we put, for some* $\lambda > 0$*,* $\mathcal{O}_\lambda := \{\mathcal{M}(\mathrm{D}u) > \lambda\}$*, then we see by* (10.2) *that* $u|_{\mathbb{R}^n\setminus\mathcal{O}_\lambda}$ *is* $2C\lambda$*–Lipschitz on* $\mathbb{R}^n \setminus \mathcal{O}_\lambda$*. At this stage, we are in position to invoke the Whitney– or Kirszbraun extension theorems (cp. [97, Chpt. 3]) to extend* $u|_{\mathbb{R}^n\setminus\mathcal{O}_\lambda}$ *to a Lipschitz mapping on the entire* $\mathbb{R}^n$ *with the same Lipschitz constant.*

If we aim at working with the full symmetric gradient and $1 < p < \infty$*, then we can still make use of Korn's inequality and work on* $\mathrm{D}u$ *instead of* $\boldsymbol{\varepsilon}(u)$*. However, in general, if we put* $\widetilde{O}_\lambda := \{\mathcal{M}(\boldsymbol{\varepsilon}(u)) > \lambda\}$*, then we cannot use the above approach to conclude that* $u|_{\mathbb{R}^n\setminus\widetilde{O}_\lambda}$ *is Lipschitz and hence also cannot extend it to a Lipschitz function on* $\mathbb{R}^n$*. This also yields the motivation to argue by use of the Marcinkiewicz integrals as above, and below we will show how an appropriate truncation based on the set* $\widetilde{O}_\lambda := \{\mathcal{M}(\boldsymbol{\varepsilon}(u)) > \lambda\}$ *can be made work.*

The strategy of Ambrosio, Coscia & Dal Maso [14, Sec. 7] is different and of separate interest. In fact, it can be employed to approach a stronger notion of differentiability and we now sketch the rough outline. More precisely, as shown in [14, Prop. 7.3], if some $u \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^m)$ satisfies for a point $x \in \mathbb{R}^n$ and $L \in \mathbb{R}^{N\times n}$

$$\lim_{r\searrow 0} \fint_{\mathrm{B}(x,r)} \frac{|u(y)-u(x)-L(y-x)|}{r}\,\mathrm{d}y = 0,$$

then u is approximately differentiable at x with approximate differential L. The aim of [14, Sec. 7] then is to verify that every BD–map satisfies this property at $\mathscr{L}^n$–a.e. $x \in \mathbb{R}^n$ for some $L \in \mathbb{R}^{n\times n}$. Roughly sketched, as a consequence of a Poincaré–type inequality, one is firstly in position to deduce that for $u \in \mathrm{BD}_{\mathrm{loc}}(\mathbb{R}^n)$ $\mathscr{L}^n$–a.e. point $x \in \mathbb{R}^n$ satisfies

$$\lim_{r\searrow 0} \fint_{\mathrm{B}(x,r)} \frac{|\langle u(y)-u(x)-\mathscr{E}u(x)(y-x), y-x\rangle|}{r|x-y|}\,\mathrm{d}y = 0. \tag{10.3}$$

Moreover, $\mathscr{L}^n$–a.e. such point qualifies as a Lebesgue point for $\mathscr{E}u$, and the n–dimensional density of $|\,\mathrm{E}^s u|$ at x vanishes. It is then essential to note that $\langle Ax, x\rangle = \langle A^{\mathrm{sym}}x, x\rangle$ for all $A \in \mathbb{R}^{n\times n}$ and hence (10.3) already contains some information about the full approximate differential. To reconstruct the full approximate differential, it is now KOHN's explicit Poincaré–inequality (cp. [152, Pt.II, Prop. 5.7]) which helps to 'fill up' the full approximate differential by gradients of suitable rigid deformations.

Whereas our proof for LD–maps below is very harmonic–analysis inspired, it makes crucial use of the Poincaré–type inequality for LD–maps and is thus located between the approach of [140] and that of [14].

10.2.2. A Calderón–Zygmund–Type Approach The aim of this section is to give a self–contained proof of the approximate differentiability of LD–maps as claimed below Proposition 4.15. Let $\mu \in \mathcal{M}_{<\infty}(\mathbb{R}^n;\mathbb{R}^{n\times n}_{\mathrm{sym}})$. Given $\lambda > 0$, we define the *bad set* of μ by $\mathcal{O}_\lambda := \{x \in \mathbb{R}^n\colon\ (\mathcal{M}\mu)(x) > \lambda\}$, where $\mathcal{M}\mu$ is the Hardy–Littlewood maximal operator applied to the Radon measure μ, cp. (2.13). Since $\mathcal{M}\mu$ is lower semicontinuous, each $\mathcal{O}_\lambda$ is open. We record from [52, pp. 1914–1915] the following lemma on decomposing sets into Whitney cubes:

Lemma 10.3. *For each $\lambda > 0$, there exists a sequence of cubes (Q_j) (each of which has sidelength r_j) such that*

(w1) $\bigcup_{j\in\mathbb{N}} \overline{Q_j} = \mathcal{O}_\lambda$*,*

(w2) for all $j \in \mathbb{N}$ we have $8\sqrt{n}\ell(Q_j) \leqslant \mathrm{dist}(Q_j, \partial\mathcal{O}_\lambda) \leqslant 32\sqrt{n}\ell(Q_j)$ and if $c_n := 2 + 32\sqrt{n}$, then $(c_nQ_j)\cap(\mathbb{R}^n\setminus\mathcal{O}_\lambda) \neq \emptyset$.

(w3) If the boundaries of Q_j and Q_k touch, then $\frac{1}{2}\ell(Q_k) \leqslant \ell(Q_j) \leqslant 2\ell(Q_k)$.

(w4) For each Q_j there exist at most $(3^n-1)2^n$ cubes Q_k which touch Q_j.

Now put, for $j \in \mathbb{N}$, $Q_j^* := \frac{9}{8}Q_j$ (i.e., Q_j^* has same center and orientation as Q_j but $\frac{9}{8}$ its radius). Then the following hold:

(w5) $\bigcup_{j\in\mathbb{N}} \overline{Q_j^*} = \mathcal{O}_\lambda$*,*

(w6) If Q_j^* and Q_k^* intersect, then the boundaries of Q_j and Q_k touch and $Q_j^* \subset 5Q_k^*$. Moreover, $\ell(Q_j^*) \sim \ell(Q_k^*)$ and $\mathscr{L}^n(Q_k^*\cap Q_j^*) \sim \mathscr{L}^n(Q_k^*) \sim \mathscr{L}^n(Q_j^*)$. Here, as usual, '$\sim$' means that each of the two quantities can be bounded by the other one with constants independent of j and k.

(w7) The family (Q_j^*) is locally 6^n–finite in the sense that for each $k \in \mathbb{N}$ there are only 6^n cubes contained in (Q_j^*) having non–empty intersection with Q_k.

(w8) $\sum_{j\in\mathbb{N}} \mathscr{L}^n(Q_j^*) \leqslant C\mathscr{L}^n(\mathcal{O}_\lambda)$ *with a constant $C = C(n) > 0$.*

Note that the Q_j^*'s overlap and hence for each $j \in \mathbb{N}$ we may pick $\widetilde{\varphi_j} \in \mathrm{C}_c^\infty(Q_j^*)$ such that

$$\mathbb{1}_{\frac{7}{9}Q_j^*} = \mathbb{1}_{\frac{7}{8}Q_j} \leqslant \widetilde{\varphi_j} \leqslant \mathbb{1}_{\frac{9}{8}Q_j} = \mathbb{1}_{Q_j^*}.$$

Note that these functions $\widetilde{\varphi_j}$ can be obtained by a particular smooth cut–off function by translation and dyadic scaling. Now let $L := \sum_{j\in\mathbb{N}} \widetilde{\varphi_j}$ and $\varphi_j := \widetilde{\varphi_j}/L$. Then we have $1 \leqslant L \leqslant 6^n$, $|\nabla L|\mathbb{1}_{Q_j^*} \leqslant C/\ell(Q_j^*)$ for all $j \in \mathbb{N}$.

In conclusion, (φ_j) is a partition of unity subject to (Q_j^*) with the following properties:

(u0) $\sum_{j\in\mathbb{N}} \varphi_j = \mathbb{1}_{\mathcal{O}_\lambda}$ and $\varphi_j \in \mathrm{C}_c^\infty(\mathbb{R}^n)$.

(u1) $\operatorname{spt}(\varphi_j) = Q_j^*$ for all $j \in \mathbb{N}$.

(u2) $\mathbb{1}_{\frac{7}{9}Q_j^*} = \mathbb{1}_{\frac{7}{8}Q_j} \leqslant \varphi_j \leqslant \mathbb{1}_{\frac{9}{8}Q_j} = \mathbb{1}_{Q_j^*}$.

(u3) $|\nabla\varphi_j| \leqslant \frac{c}{\ell(Q_j^*)}\mathbb{1}_{Q_j^*}$ for all $j \in \mathbb{N}$ with a constant $c = c(n) > 0$.

(u4) $|\nabla^2\varphi_j| \leqslant \frac{c}{\ell(Q_j^*)^2}\mathbb{1}_{Q_j^*}$ for all $j \in \mathbb{N}$ with a constant $c = c(n) > 0$.

In what follows, we use the shorthand $r_j := \ell(Q_j^*)$, $j \in \mathbb{N}$. Moreover, as we think of $\lambda > 0$ as arbitrary but fixed, we shall tacitly surpress the occurence of $\lambda > 0$ in the following.

By the LD–Sobolev–Poincaré inequality, for each cube $Q \subset \mathbb{R}^n$ and each $u \in \mathrm{LD}(Q)$ there exists a rigid deformation $\Pi_Q u \in \mathcal{R}(Q)$ such that $\|u-\Pi_Q u\|_{\mathrm{L}^1(Q;\mathbb{R}^n)} \leqslant C\ell(Q)|\,\mathrm{E}u|(Q)$. Here, we use the convention $|\,\mathrm{E}u|(A) := \int_A |\boldsymbol{\varepsilon}(u)|\,\mathrm{d}x$ for $u \in \mathrm{LD}(\mathbb{R}^n)$ and any measurable set A. We apply the setting outlined above to the particular measure μ given by $\mu(A) := \int_A \boldsymbol{\varepsilon}(u)\,\mathrm{d}x$, $A \in \mathscr{B}(\mathbb{R}^n)$.

Let $u \in \mathrm{LD}(\mathbb{R}^n)$. For fixed $\lambda > 0$, let (φ_j) be a partition of unity subject to the covering (Q_j^*) of $\mathcal{O}_\lambda$ obeying (w5)–(w8) from above. We now define

$$u_\lambda := \begin{cases} u & \text{in } \Omega \setminus \mathcal{O}_\lambda, \\ \sum_{j\in\mathbb{N}} \varphi_j \Pi_{Q_j^*} u & \text{in } \mathcal{O}_\lambda. \end{cases} \tag{10.4}$$

Note that, by local finiteness of the Whitney covering, this is a well–posed definition. We note that this can be written conveniently as

$$u_\lambda = u - \sum_j \varphi_j(u - \Pi_{Q_j^*} u). \tag{10.5}$$

To show that this is indeed equivalent to (10.4), observe that if $x \in \mathcal{O}_\lambda$, then

$$\begin{aligned} u(x) - \sum_j \varphi_j(x)(u(x) - \Pi_{Q_j^*}u(x)) &= u(x) - \underbrace{\Big(\sum_j \varphi_j(x)\Big)}_{=1} u(x) + \sum_j \varphi_j(x)\Pi_{Q_j^*}u(x) \\ &= \sum_j \varphi_j(x)\Pi_{Q_j^*}u(x) = u_\lambda(x). \end{aligned}$$

On the other hand, if $x \in \mathbb{R}^n \setminus \mathcal{O}_\lambda$, then $\varphi_j(x) = 0$ for all $j \in \mathbb{N}$ and thus

$$u(x) - \sum_j \underbrace{\varphi_j(x)}_{=0}(u(x) - \Pi_{Q_j^*}u(x)) = u(x).$$

In conclusion, the representation (10.5) holds indeed.

We start by showing that $u_\lambda \in \mathrm{LD}(\mathbb{R}^n)$ provided $u \in \mathrm{LD}(\mathbb{R}^n)$, cp. Lemma 10.5. For this the following lemma (which is a straightforward modification of the arguments outlined in [52, Lem. 2.1]) is instrumental.

Lemma 10.4. *Let $u \in \mathrm{LD}(\mathbb{R}^n)$ and define u_λ for $\lambda > 0$ by (10.4). Then there exists a constant $c = c(n) > 0$ such that the following hold:*

(a) *For all $j \in \mathbb{N}$ and all $y \in Q_j^*$ there holds*

$$\fint_{Q_j^*} \frac{|u - \Pi_{Q_j^*} u|}{\ell(Q_j^*)} \, \mathrm{d}x \leqslant c \frac{|\operatorname{E}u|(Q_j^*)}{\ell(Q_j^*)^n} \leqslant c\mathcal{M}(\operatorname{E}u)(y).$$

(b) *For all $j \in \mathbb{N}$ there holds (with c_n defined by (w2))*

$$\frac{|\operatorname{E}u|(Q_j^*)}{\ell(Q_j^*)^n} \leqslant \frac{|\operatorname{E}u|(c_n Q_j^*)}{\ell(Q_j^*)^n} \leqslant c\lambda.$$

(c) *For all $j, k \in \mathbb{N}$ with $Q_j^* \cap Q_k^* \neq \emptyset$ there holds*

$$\frac{\|\Pi_{Q_j^*} u - \Pi_{Q_k^*} u\|_{\mathrm{L}^\infty(Q_j^*;\mathbb{R}^n)}}{r_j} \leqslant c \left(\fint_{Q_j^*} \frac{|u - \Pi_{Q_j^*} u|}{\ell(Q_j^*)} \, \mathrm{d}x + \fint_{Q_k^*} \frac{|u - \Pi_{Q_k^*} u|}{\ell(Q_k^*)} \, \mathrm{d}x \right).$$

Proof. Ad (a). The first inequality is a reformulation of the Poincaré inequality for LD–maps. The second inequality is a direct consequence of the definition of the Hardy–Littlewood maximal operator. Ad (b). Since $c_n > 1$ is a fixed constant, we have

$$\frac{|\operatorname{E}u|(Q_j^*)}{\ell(Q_j^*)^n} \leqslant c_n^n \frac{|\operatorname{E}u|(c_n Q_j^*)}{\ell(c_n Q_j^*)^n}.$$

On the other hand, by the particular choice of c_n (cp. (w2)), we have that $c_n Q_j$ has non–empty intersection with $\mathbb{R}^n \setminus \mathcal{O}_\lambda$. As a consequence, let $y \in (c_n Q_j) \cap (\mathbb{R}^n \setminus \mathcal{O}_\lambda)$. Then we have for a number $l = l(n) > 0$ that the cube $\widetilde{Q}$ centered at y with sidelength $l c_n Q_j$ properly contains $c_n Q_j$ and therefore

$$\frac{|\operatorname{E}u|(c_n Q_j)}{(c_n \ell(Q_j))^n} \leqslant l^n \frac{|\operatorname{E}u|(\widetilde{Q})}{(l c_n \ell(Q_j))^n} \leqslant l^n \mathcal{M}(\operatorname{E}u)(y).$$

Now recall that $y \in \mathbb{R}^n \setminus \mathcal{O}_\lambda$ implies $\mathcal{M}(\operatorname{E}u)(y) \leqslant \lambda$ and hence (b) follows. Ad (c). We firstly recall that the rigid deformations are a finite dimensional vector space and hence for all $j \in \mathbb{N}$, all norms on $\mathcal{R}(Q_j)$ are equivalent. Now note that $\mathscr{L}^n(Q_j^*) \sim \mathscr{L}^n(Q_k^*) \sim \mathscr{L}^n(Q_j^* \cap Q_k^*)$ for all $j, k \in \mathbb{N}$, where the constants implicit in '$\sim$' do not depend on j, k. This is a consequence of (w6). Hence, by finite dimensionality of $\mathcal{R}(\mathbb{R}^n)$ and scaling, we obtain

$$\int_{Q_j^*} |R| \, \mathrm{d}x \sim \int_{Q_j^* \cap Q_k^*} |R| \, \mathrm{d}x \qquad \text{for all } R \in \mathcal{R}(\mathbb{R}^n). \tag{10.6}$$

We consequently obtain (recall $r_j := \ell(Q_j^*)$)

$$\begin{aligned}
\fint_{Q_j^*} \frac{|\Pi_{Q_j^*} u - \Pi_{Q_k^*} u|}{r_j} \, \mathrm{d}x &\leqslant \frac{C}{|Q_j^*|} \int_{Q_j^* \cap Q_k^*} \frac{|\Pi_{Q_j^*} u - \Pi_{Q_k^*} u|}{r_j} \, \mathrm{d}x \quad (\text{by } (10.6)) \\
&\leqslant \frac{C}{|Q_j^*|} \int_{Q_j^*} \frac{|\Pi_{Q_j^*} u - u|}{r_j} \, \mathrm{d}x + \frac{C}{|Q_j^*|} \int_{Q_j^* \cap Q_k^*} \frac{|\Pi_{Q_k^*} u - u|}{r_j} \, \mathrm{d}x \\
&\leqslant C \fint_{Q_j^* \cap Q_k^*} \frac{|\Pi_{Q_j^*} u - u|}{r_j} \, \mathrm{d}x + C \fint_{Q_k^*} \frac{|\Pi_{Q_k^*} u - u|}{r_k} \, \mathrm{d}x,
\end{aligned}$$

where the last step is valid due to $Q_j^* \cap Q_k^* \subset Q_k^*$ and $\mathscr{L}^n(Q_j^* \cap Q_k^*) \sim \mathscr{L}^n(Q_k^*) \sim \mathscr{L}^n(Q_j^*)$. Finally, by finite dimensionality of $\mathcal{R}(Q)$ and scaling, we obtain that there exists a constant $C > 0$ such that for all non–empty cubes $Q \subset \mathbb{R}^n$ there holds $\|q\|_{\mathrm{L}^\infty(Q;\mathbb{R}^n)} \leqslant C \fint_Q |q| \, \mathrm{d}x$ for all $q \in \mathcal{R}(Q)$. In conjunction with the estimates from above, the proof is complete. $\square$

Lemma 10.5. *Let $u \in \mathrm{LD}(\mathbb{R}^n)$ and define, for $\lambda > 0$, the truncated mapping u_λ by (10.4). Then there holds $u_\lambda \in \mathrm{LD}(\mathbb{R}^n)$, too.*

Proof. We demonstrate $u_\lambda \in \mathrm{L}^1(\mathbb{R}^n;\mathbb{R}^n)$ first. Since $\mathrm{spt}(\varphi_j) \subset Q_j^*$ and $0 \leqslant \varphi_j \leqslant 1$, we have with a finite constant $C = C(n) > 0$

$$\begin{aligned}
\int_{\mathbb{R}^n} |u_\lambda|\,\mathrm{d}x &= \int_{\mathcal{O}_\lambda} |u_\lambda|\,\mathrm{d}x + \int_{\mathbb{R}^n\setminus\mathcal{O}_\lambda} |u_\lambda|\,\mathrm{d}x \\
&\leqslant \int_{\mathcal{O}_\lambda} |u_\lambda|\,\mathrm{d}x + \int_{\mathbb{R}^n\setminus\mathcal{O}_\lambda} |u|\,\mathrm{d}x \\
&\overset{(*)}{\leqslant} \Big(\sum_j \int_{Q_j^*} |\Pi_{Q_j^*}u|\,\mathrm{d}x\Big) + \int_{\mathbb{R}^n} |u|\,\mathrm{d}x \\
&\overset{(**)}{\leqslant} C\Big(\sum_j \int_{Q_j^*} |u|\,\mathrm{d}x\Big) + \int_{\mathbb{R}^n} |u|\,\mathrm{d}x \\
&\leqslant C\int_{\mathbb{R}^n} |u|\,\mathrm{d}x \quad \text{(by local finiteness of the Whitney covering).}
\end{aligned}$$

Here we used at $(*)$ the local finiteness of the sum defining u_λ (i.e., for any $x \in \mathcal{O}_\lambda$ there are only finitely many cubes containing x), and at $(**)$ the stability estimate for projections for operators with finite dimensional nullspace, cp. Theorem 3.25(a). We turn to the level of symmetric gradients. To this end, let us remark that it suffices to establish that $(u_\lambda - u)|_{\mathcal{O}_\lambda} \in \mathrm{LD}_0(\mathcal{O}_\lambda)(:= \overline{\mathrm{C}_c^\infty(\mathcal{O}_\lambda)}^{\|\boldsymbol{\varepsilon}(\cdot)\|_{\mathrm{L}^1(\mathcal{O}_\lambda;\mathbb{R}^{n\times n})}})$, where $\|\boldsymbol{\varepsilon}(\cdot)\|_{\mathrm{L}^1(\mathcal{O}_\lambda;\mathbb{R}^{n\times n})}$ denotes the symmetric gradient norm on $\mathrm{C}_c^\infty(\mathcal{O}_\lambda;\mathbb{R}^n)$. In fact, assuming this, we note that $u_\lambda - u \equiv 0$ outside $\mathcal{O}_\lambda$ and thus

$$u_\lambda = u + (u_\lambda - u) \in \mathrm{LD}(\mathbb{R}^n) + \mathrm{LD}_0(\mathcal{O}_\lambda) \subset \mathrm{LD}(\mathbb{R}^n).$$

Let $x \in \mathcal{O}_\lambda$ be arbitrary but fixed. Then we have

$$\begin{aligned}
\boldsymbol{\varepsilon}(u_\lambda)(x) - \boldsymbol{\varepsilon}(u)(x) &= \boldsymbol{\varepsilon}\Big(\sum_j \varphi_j(x)\Pi_{Q_j^*}u(x)\Big) - \boldsymbol{\varepsilon}\Big(\sum_j \varphi_j(x)u(x)\Big) \\
&= \sum_j \nabla\varphi_j(x) \odot (\Pi_{Q_j^*}u(x) - u(x)) + \sum_j \varphi_j(x)\boldsymbol{\varepsilon}(u)(x) \\
&= \sum_j \mathbf{I}_j + \sum_j \mathbf{II}_j,
\end{aligned}$$

with an obvious definition for $\mathbf{I}_j$ and $\mathbf{II}_j$. Our aim is to show that both sums defining $\mathbf{I}$ and $\mathbf{II}$ converge absolutely in $\mathrm{L}^1(\mathcal{O}_\lambda;\mathbb{R}^{n\times n})$. Namely, since each sum consists of summands which belong to $\mathrm{LD}_0(\mathcal{O}_\lambda)$ and $(\mathrm{LD}_0(\mathcal{O}_\lambda), \|\boldsymbol{\varepsilon}(\cdot)\|_{\mathrm{L}^1(\mathcal{O}_\lambda;\mathbb{R}^{n\times n})})$ is a Banach space, the claim follows. We estimate, using $|\nabla\varphi_j| \leqslant C/r_j$ for all $j \in \mathbb{N}$ and the local finiteness of the Whitney covering,

$$\begin{aligned}
\sum_j \|\mathbf{I}_j\|_{\mathrm{L}^1(\mathcal{O}_\lambda;\mathbb{R}^{n\times n})} &\leqslant C\sum_j \int_{Q_j^*} \left|\frac{u - \Pi_{Q_j^*}u}{r_j}\right|\mathrm{d}x \\
&\leqslant C\sum_j \int_{Q_j^*} |\boldsymbol{\varepsilon}(u)|\,\mathrm{d}x \quad \text{(by Poincaré)} \\
&\leqslant C\int_{\mathcal{O}_\lambda} |\boldsymbol{\varepsilon}(u)|\,\mathrm{d}x \leqslant C\int_{\mathbb{R}^n} |\boldsymbol{\varepsilon}(u)|\,\mathrm{d}x.
\end{aligned}$$

On the other hand, we obtain a slightly easier argument

$$\sum_j \|\mathbf{II}_j\|_{\mathrm{L}^1(\mathcal{O}_\lambda;\mathbb{R}^{n\times n})} \leqslant \sum_j \int_{Q_j^*} |\boldsymbol{\varepsilon}(u)|\,\mathrm{d}x \leqslant C\int_{\mathbb{R}^n} |\boldsymbol{\varepsilon}(u)|\,\mathrm{d}x.$$

This implies the absolute convergence of the series $\sum_j(\mathbf{I}_j+\mathbf{II}_j)$ in $\mathrm{LD}_0(\mathcal{O}_\lambda)$ and the proof is complete. $\square$

Before we come to our main result, Theorem 10.7 below, we need another ingredient provided by the next lemma.

Lemma 10.6. *Given $u\in\mathrm{LD}(\mathbb{R}^n)$ and $\lambda>0$, the map $u_\lambda\colon\mathbb{R}^n\to\mathbb{R}^n$ defined by (10.4) satisfies (with a constant $C>0$ neither depending on u nor λ)*

$$|\boldsymbol{\varepsilon}(u_\lambda)|\leqslant C\lambda \tag{10.7}$$

$\mathscr{L}^n$–a.e. in $\mathbb{R}^n$.

Proof. Let $x\in\mathcal{O}_\lambda^{\mathrm{c}}$. Then $\mathcal{M}(\mathrm{E}u)(x)\leqslant\lambda$. On the other hand, if $x\in\mathcal{O}_\lambda$, then we find a cube Q_k^* with $x\in Q_k^*$. In conclusion,

$$\begin{aligned}\boldsymbol{\varepsilon}(u_\lambda)(x)&=\boldsymbol{\varepsilon}(\sum_j\varphi_j(x)\Pi_{Q_j^*}u(x))-\underbrace{\boldsymbol{\varepsilon}\sum_j\varphi_j(x)\Pi_{Q_k^*}u(x)}_{=0}\\&=\sum_j\varphi_j(x)\underbrace{\boldsymbol{\varepsilon}(\Pi_{Q_j^*}u(x))}_{=0}+\sum_j\nabla\varphi_j(x)\odot(\Pi_{Q_j^*}u(x)-\Pi_{Q_k^*}u(x))\end{aligned}$$

and hence, recalling $|\nabla\varphi_j|\leqslant C/r_j$ and Lemma 10.4 (c),

$$\begin{aligned}|\boldsymbol{\varepsilon}(u_\lambda)(x)|&\leqslant C\sum_{j\colon Q_j^*\cap Q_k^*\neq\emptyset}\frac{\|\Pi_{Q_j^*}u-\Pi_{Q_k^*}u\|_{\mathrm{L}^\infty(Q_j^*;\mathbb{R}^n)}}{r_j}\\&\leqslant C\sum_{j\colon Q_j^*\cap Q_k^*\neq\emptyset}\frac{1}{r_j}\fint_{Q_k^*}|\Pi_{Q_j^*}u-\Pi_{Q_k^*}u|\,\mathrm{d}x\\&\leqslant C\sum_{j\colon Q_j^*\cap Q_k^*\neq\emptyset}\fint_{Q_j^*}\frac{|u-\Pi_{Q_j^*}u|}{r_j}\,\mathrm{d}x\qquad\text{(by Lemma 10.4 (c))}\\&\leqslant C\sum_{j\colon Q_j^*\cap Q_k^*\neq\emptyset}\frac{|\mathrm{E}u|(Q_j^*)}{\ell(Q_j^*)^n}\qquad\text{(by Poincaré).}\end{aligned}$$

At this step, we invoke Lemma 10.4 (b) to find

$$|\boldsymbol{\varepsilon}(u_\lambda)(x)|\leqslant C\sum_{j\colon Q_j^*\cap Q_k^*\neq\emptyset}c\lambda\leqslant C\lambda,$$

the last step being valid to uniform local finiteness of the Whitney covering, cp. (w7). Hence, if $x\in\mathcal{O}_\lambda$, then $|\boldsymbol{\varepsilon}(u_\lambda)(x)|\leqslant C\lambda$, and here $C>0$ does neither depend on u nor λ. On the other hand, if $x\in\mathbb{R}^n\setminus\mathcal{O}_\lambda$, then $|\boldsymbol{\varepsilon}(u_\lambda)(x)|=|\boldsymbol{\varepsilon}(u)(x)|\leqslant\mathcal{M}(\mathrm{E}u)(x)\leqslant\lambda$ by definition of $\mathcal{O}_\lambda$. Gathering estimates and since $u\in\mathrm{LD}(\mathbb{R}^n)$, the proof is complete. $\square$

Our main theorem now reads as follows.

Theorem 10.7. *Let $u\in\mathrm{LD}(\mathbb{R}^n)$ and define u_λ for $\lambda>0$ by (10.4). Then $u_\lambda\in\mathrm{W}^{1,\mathrm{BMO}}_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^n)$ and $\mathscr{L}^n(\{u\neq u_\lambda\})\to0$ as $\lambda\to\infty$.*

Proof. We deduce from the very definition (10.4) that u_λ satisfies $\mathscr{L}^n(\{u\neq u_\lambda\})\leqslant\mathscr{L}^n(\mathcal{O}_\lambda)$. By the Hardy–Littlewood Inequality for measures, cp. Lemma 2.19 , we have with a fixed constant $K>0$

$$\mathscr{L}^n(\{\mathcal{M}(\mathrm{E}u)\geqslant\lambda\})\leqslant\frac{K}{\lambda}|\mathrm{E}u|(\mathbb{R}^n).$$

Hence we easily find $\mathscr{L}^n(\{u \neq u_\lambda\}) \leqslant \frac{K}{\lambda}|\mathrm{E}u|(\mathbb{R}^n) \to 0$ as $\lambda \to \infty$. Thus it remains to establish $u_\lambda \in \mathrm{W}^{1,\mathrm{BMO}}(\mathbb{R}^n;\mathbb{R}^n)$ for all $\lambda > 0$. By Lemma 10.6, we have $|\boldsymbol{\varepsilon}(u_\lambda)| \leqslant C\lambda$ and thus $\boldsymbol{\varepsilon}(u_\lambda) \in \mathrm{L}^\infty(\mathbb{R}^n;\mathbb{R}^{n\times n}_{\mathrm{sym}})$. By the results of Chapter 3.2, we consequently obtain that $\mathrm{D}u_\lambda = \Phi(\mathrm{E}u_\lambda)$, where Φ is a translation invariant singular integral operator of convolution type. This implies $\Phi\colon \mathrm{L}^\infty(\mathbb{R}^n;\mathbb{R}^{n\times n}_{\mathrm{sym}}) \to \mathrm{BMO}(\mathbb{R}^n;\mathbb{R}^{n\times n})$ boundedly and so $\mathrm{D}u_\lambda \in \mathrm{BMO}(\mathbb{R}^n;\mathbb{R}^{n\times n})$. Finally, we claim that $u \in \mathrm{BMO}_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^n)$. By the John–Nirenberg Lemma (cp. Lemma 2.20), $\mathrm{D}u_\lambda \in \mathrm{L}^p_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^{n\times n})$ for any $1 \leqslant p < \infty$. By [170, Chpt. 1.1.2], we have $u_\lambda \in \mathrm{W}^{1,p}_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^n)$ for any $1 \leqslant p < \infty$, and choosing $p > n$ large enough, we can arrange that u_λ is even locally Hölder continuous and the proof is complete. □

We eventually demonstrate that the above implies the approximate differentiability of LD–maps $\mathscr{L}^n$–a.e..

Our aim is to employ Whitney's characterisation of approximate differentiability $\mathscr{L}^n$–a.e. as given in Theorem 2.5. Hence, let $u \in \mathrm{LD}(\mathbb{R}^n)$ and let $\varepsilon > 0$ be arbitrary. To find the Lipschitz mapping as required in Theorem 2.5, we firstly pick $\lambda > 0$ so large such that with the truncations $u_\lambda \in \mathrm{W}^{1,\mathrm{BMO}}_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^n)$ there holds $\mathscr{L}^n(\{u \neq u_\lambda\}) < \varepsilon/3$. Next, we recall from [5] that for every $v \in \mathrm{W}^{1,p}(\mathbb{R}^n;\mathbb{R}^N)$ and every $\delta > 0$ there exists $v_\delta \in \mathrm{W}^{1,\infty}(\mathbb{R}^n;\mathbb{R}^N)$ such that $\mathscr{L}^n(\{v \neq v_\delta\}) < \delta$. Localising if necessary, we may apply this result to $\delta = \varepsilon/3$ and $v = u_\lambda \in \mathrm{W}^{1,\mathrm{BMO}}_{\mathrm{loc}} \subset \mathrm{W}^{1,p}_{\mathrm{loc}}$ with arbitrary $1 < p < \infty$. This yields a map $v_\lambda \in \mathrm{W}^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^n)$ with $\mathscr{L}^n(\{u_\lambda \neq v_\lambda\}) < \varepsilon/3$. We then obtain

$$\begin{aligned}\mathscr{L}^n(\{u_\lambda \neq v_\lambda\}) &= \mathscr{L}^n(\{u_\lambda \neq v_\lambda\} \cap \mathcal{O}_\lambda) + \mathscr{L}^n(\{u_\lambda \neq v_\lambda\} \cap \mathcal{O}_\lambda^c) \\ &= \mathscr{L}^n(\{u_\lambda \neq v_\lambda\} \cap \mathcal{O}_\lambda) + \mathscr{L}^n(\{u \neq v_\lambda\} \cap \mathcal{O}_\lambda^c) \;\; (\text{as } u_\lambda|_{\mathcal{O}_\lambda^c} = u|_{\mathcal{O}_\lambda^c})\end{aligned}$$

and hence

$$\mathscr{L}^n(\{u \neq v_\lambda\} \cap \mathcal{O}_\lambda^c) \leqslant \mathscr{L}^n(\{u_\lambda \neq v_\lambda\}) + \mathscr{L}^n(\{u_\lambda \neq v_\lambda\} \cap \mathcal{O}_\lambda). \tag{10.8}$$

We therefore conclude

$$\begin{aligned}\mathscr{L}^n(\{u \neq v_\lambda\}) &= \underbrace{\mathscr{L}^n(\{u \neq v_\lambda\} \cap \mathcal{O}_\lambda)}_{\leqslant \mathscr{L}^n(\mathcal{O}_\lambda)} + \mathscr{L}^n(\{u \neq v_\lambda\} \cap \mathcal{O}_\lambda^c) \\ &\leqslant \mathscr{L}^n(\mathcal{O}_\lambda) + \mathscr{L}^n(\{u_\lambda \neq v_\lambda\}) + \mathscr{L}^n(\{u_\lambda \neq v_\lambda\} \cap \mathcal{O}_\lambda) \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.\end{aligned}$$

From here and Whitney's theorem, the claim follows.

Remark 10.8 (The BD–case)**.** *Above, we have confined to the* LD*–case. A corresponding result for* BD *is equally possible, but it does not quite follow from the outline for the* LD*–case and one has to truncate more carefully. Namely, one can show that for the truncation* u_λ *which is defined as above that* $\boldsymbol{\varepsilon}(u_\lambda)|_{\mathcal{O}_\lambda}$ *belongs to* $\mathrm{L}^\infty(\mathcal{O}_\lambda;\mathbb{R}^{n\times n}_{\mathrm{sym}})$ *and* $\boldsymbol{\varepsilon}(u_\lambda)|_{\mathcal{O}_\lambda^c}$ *to* $\mathrm{L}^\infty(\mathcal{O}_\lambda^c;\mathbb{R}^{n\times n}_{\mathrm{sym}})$*. However, in the* BD*–case, this does not yet imply that the map* u_λ *has bounded symmetric gradient globally as the symmetric gradient might become singular along* $\partial\mathcal{O}_\lambda$*. Equivalently, this issue can be seen when passing to the dual formulation of the total deformation. We shall address these questions also in the broader context of Hölder quasicontinuity in a future work.*

10.3. Further Approximation and Lower Semicontinuity Results

In this section we collect an approximation and a lower semicontinuity result. The former is in the spirit of Lemma 4.9 but slightly extends it. In this form, the statement is not available in the literature, but it is standard to retrieve it as a merger of the corresponding smooth approximation theorems [36, Thm. 4.1], [21, Lem. 5.1] and the strict approximation result [115, Lem. A.3.1].

Lemma 10.9. *Suppose U is an open and bounded Lipschitz subset of $\mathbb{R}^n$ and let $u_0 \in \mathrm{LD}(U)$. For every $u \in \mathrm{BV}_{u_0}(\overline{U})$ there exists a sequence $(w_k) \subset u_0|_U + \mathrm{C}_c^\infty(U;\mathbb{R}^n) \subset u_0 + \mathrm{LD}_0(U)$ such that*

(a) *$w_k \to u$ strongly in $\mathrm{L}^1(U;\mathbb{R}^n)$,*

(b) *$\boldsymbol{\varepsilon}(w_k)\mathscr{L}^n \llcorner U \overset{*}{\rightharpoonup} \mathrm{E}u \llcorner \overline{U}$ in the weak*–sense for $\mathbb{R}^{n\times n}_{\mathrm{sym}}$–valued finite Radon measures on $\overline{U}$,*

(c) *$|(\mathscr{L}^n, \boldsymbol{\varepsilon}(w_k)\mathscr{L}^n)|(\overline{U}) \to |(\mathscr{L}^n, \mathrm{E}u)|(\overline{U})$,*

(d) *$\boldsymbol{\varepsilon}(w_k) \to \mathscr{E}u$ $\mathscr{L}^n$–a.e. in U,*

(e) *for all $h \in \mathrm{L}^1(U;\mathbb{R}^{n\times n}_{\mathrm{sym}})$ there holds*

$$\int_U |\boldsymbol{\varepsilon}(w_k) - h| \,\mathrm{d}x \to \int_U |\,\mathrm{E}u - h\mathscr{L}^n|$$

as $k \to \infty$. As usual, here $\mathrm{E}v = \mathscr{E}v\mathscr{L}^n + \mathrm{E}^s v$ denotes the Radon–Nikodym decomposition of $\mathrm{E}v$ for a BD–map $v\colon U \to \mathbb{R}^n$. Moreover, for any open set $A \Subset U$ there holds

$$\limsup_{k\to\infty} \int_A |\boldsymbol{\varepsilon}(w_j)| \,\mathrm{d}x \leqslant |\,\mathrm{E}u|(\overline{A}).$$

The following Reshetnyak–type theorem is designed for integrands of the class $\mathbf{E}$ and hereafter does not require convexity. As it is only used in this form at one point, namely the proof of area–strict continuity of rank–one–convex variational integrals in Chapter 8, we include it here. Let $\Omega \subset \mathbb{R}^n$ be open and bounded with Lipschitz boundary. We say that $f\colon \Omega \times \mathbb{R}^{m\times n} \to \mathbb{R}$ belongs to the class $\mathbf{E}(\Omega;\mathbb{R}^{m\times n})$ provided the map $(x,\xi) \mapsto (1-|\xi|)f(x,(1-|\xi|)^{-1}\xi)$ for $x \in \Omega$ and $\xi \in \mathbb{B}^{m\times n}$ with the open unit ball $\mathbb{B}^{m\times n}$ is extendable to a continuous function on $\overline{\Omega \times \mathbb{B}^{m\times n}}$.

Lemma 10.10 ([163, Thm. 5])**.** *Let $f \in \mathbf{E}(\Omega;\mathbb{R}^{m\times n})$ and let $\mu,\mu_1,\mu_2,\ldots$ be $\mathbb{R}^{m\times n}$–valued finite Radon measures such that $\mu_j \to \mu$ in the area–strict sense as $j \to \infty$, i.e., $\mu_j \overset{*}{\rightharpoonup} \mu$ and $\langle \mu_j\rangle(\Omega) \to \langle\mu\rangle(\Omega)$. If $\mu_j = a_j\mathscr{L}^n \llcorner \Omega + \mu_j^s$ and $\mu = a\mathscr{L}^n \llcorner \Omega + \mu^s$ denote the Lebesgue–Radon–Nikodym decompositions, then we have*

$$\int_\Omega f(a_j)\,\mathrm{d}x + \int_\Omega f^\infty\Big(\frac{\mathrm{d}\mu_j^s}{\mathrm{d}|\mu_j^s|}\Big)\,\mathrm{d}|\mu_j^s| \to \int_\Omega f(a)\,\mathrm{d}x + \int_\Omega f^\infty\Big(\frac{\mathrm{d}\mu^s}{\mathrm{d}|\mu^s|}\Big)\,\mathrm{d}|\mu^s|, \text{ as } j \to \infty.$$

Lemma 10.11. *Let $\Omega \subset \mathbb{R}^n$ be an open an bounded Lipschitz set. Then for all $w \in \mathrm{LD}(\Omega)$ and all $\psi \in \mathrm{L}^\infty(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})$ with compact support and $\mathrm{div}(\psi) \in \mathrm{L}^\infty(\Omega;\mathbb{R}^n)$ we have*

$$\int_\Omega \langle \boldsymbol{\varepsilon}(w),\psi\rangle\,\mathrm{d}x = -\int_\Omega \langle w, \mathrm{div}(\psi)\rangle\,\mathrm{d}x. \tag{10.9}$$

Proof. It is clear that (10.9) holds provided we have $w \in \mathrm{LD}(\Omega)$ and, moreover, $\psi \in (\mathrm{C}_c^\infty \cap \mathrm{W}^{1,1})(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})$.

For $1 \leqslant p \leqslant \infty$ and $1 \leqslant q < \infty$, define the space $\mathrm{W}^{\mathrm{div}}_{p,q}(\Omega)$ by

$$\mathrm{W}^{\mathrm{div}}_{p,q}(\Omega) := \big\{\varphi\colon \Omega \to \mathbb{R}^{n\times n}_{\mathrm{sym}}\colon\ \varphi \in \mathrm{L}^p(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}}),\ \mathrm{div}(\varphi) \in \mathrm{L}^q(\Omega;\mathbb{R}^n)\big\}$$

and endow it with norm $\|\varphi\|_{\mathrm{W}^{\mathrm{div}}_{p,q}(\Omega)} := \|\varphi\|_{\mathrm{L}^p(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})} + \|\,\mathrm{div}(\varphi)\|_{\mathrm{L}^q(\Omega;\mathbb{R}^n)}$. Then it is standard to see that (i) $(\mathrm{W}^{\mathrm{div}}_{p,q}(\Omega), \|\varphi\|_{\mathrm{W}^{\mathrm{div}}_{p,q}(\Omega)})$ is a Banach space and (ii) smooth approximation holds in the following sense: If $1 \leqslant p,q < \infty$, then for all $\varphi \in \mathrm{W}^{\mathrm{div}}_{p,q}(\Omega)$ there exists

$(\varphi_k) \subset \mathrm{C}^\infty(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}}) \cap \mathrm{W}^{\mathrm{div}}_{p,q}(\Omega)$ such that $\|\varphi_k - \varphi\|_{\mathrm{W}^{\mathrm{div}}_{p,q}(\Omega)} \to 0$ as $k \to \infty$. If $p = \infty$ and $1 \leqslant q < \infty$, then we find $(\varphi_k) \subset \mathrm{C}^\infty(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}}) \cap \mathrm{W}^{\mathrm{div}}_{p,q}(\Omega)$ with

$$\varphi_j \overset{*}{\rightharpoonup} \varphi \text{ in } \mathrm{L}^\infty(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}}) \text{ and } \operatorname{div}(\varphi_j) \to \operatorname{div}(\varphi) \text{ strongly in } \mathrm{L}^q(\Omega;\mathbb{R}^n) \tag{10.10}$$

as $j \to \infty$. Moreover, if φ vanishes outside a compact subset of Ω, then we can choose the approximating sequence from (ii) to belong to $\mathrm{C}^\infty_c(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}}) \cap \mathrm{W}^{\mathrm{div}}_{p,q}(\Omega)$. By the usual Gauß–Green Theorem for LD, we have (10.9) for $w \in \mathrm{LD}(\Omega)$ and $\varphi \in \mathrm{C}^\infty_c(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})$. Let ψ be as in the lemma and henceforth $\psi \in \mathrm{W}^{\mathrm{div}}_{\infty,n}(\Omega)$. We pick an approximating sequence (ψ_j) satisfying (ii) (i.e., (10.10)) and (iii), where now $p = \infty$, $q = n < \infty$. Accordingly, if $w \in \mathrm{LD}(\Omega) \subset \mathrm{L}^{\frac{n}{n-1}}(\Omega;\mathbb{R}^n)$, then we obtain by (10.10)

$$\begin{aligned}\left|\int_\Omega \langle \boldsymbol{\varepsilon}(w),\psi\rangle + \langle w,\operatorname{div}(\psi)\rangle\,\mathrm{d}x\right| &= \left|\int_\Omega \langle \boldsymbol{\varepsilon}(w),\psi-\psi_j\rangle + \langle w,\operatorname{div}(\psi-\psi_j)\rangle\,\mathrm{d}x\right| \\ &\leqslant \left|\int_\Omega \langle \boldsymbol{\varepsilon}(w),\psi-\psi_j\rangle\right| \\ &\quad + \|u\|_{\mathrm{L}^{\frac{n}{n-1}}(\Omega,\mathbb{R}^n)}\|\operatorname{div}(\psi_j-\psi)\|_{\mathrm{L}^n(\Omega;\mathbb{R}^N)} \to 0, \quad j\to\infty.\end{aligned}$$

The proof is complete. □

10.4. Euler–Lagrange Equations for Measures

In the main part of the thesis we have mentioned at various stages that it is possible to directly set up Euler–Lagrange equations for BV–minimisers and not to work with approximated systems. This has been firstly achieved by Anzellotti [19], and this section is dedicated to exposing the main features of this theory. Also because of applications in the full gradient setting (cp. Chapter 6), the material presented here is stated for BV and taken from [19], where it can be found in much more detail and in slightly greater generality. The only point where our presentation differs from [19], however, is the formulation of the corresponding results for BD, but this is straightforward and only included for completeness[1].

We henceforth stick to the setting of [19] and hereafter consider for an open and bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$ integral functionals

$$\mathfrak{F}[u] := \int_\Omega f(x,\mathrm{D}u)\,\mathrm{d}x + \int_\Omega \langle H(x),u(x)\rangle\,\mathrm{d}x, \tag{10.11}$$

where $f\colon \Omega\times\mathbb{R}^{N\times n} \to \mathbb{R}$ and $H \in \mathrm{L}^n(\Omega;\mathbb{R}^N)$ satisfy the following conditions:

(H1) There exists $M > 0$ such that for all $(x,\xi) \in \Omega\times\mathbb{R}^{N\times n}$ there holds

$$|\xi| \leqslant f(x,\xi) \leqslant M(1+|\xi|).$$

(H2) $f\colon \Omega\times\mathbb{R}^{N\times n} \to \mathbb{R}$ is continuous.

(H3) For each fixed $x \in \Omega$, $\xi \mapsto f(x,\xi)$ is convex.

(H4) $H \in \mathrm{L}^n(\Omega;\mathbb{R}^N)$.

Subject to these conditions, the functional (10.11) is well–defined on $\mathrm{W}^{1,1}(\Omega;\mathbb{R}^N)$ (recall that $\mathrm{W}^{1,1}(\Omega;\mathbb{R}^N) \hookrightarrow \mathrm{L}^{\frac{n}{n-1}}(\Omega;\mathbb{R}^N)$). It is also clear by use of the Sobolev–Strauss embedding $\mathrm{LD}(\Omega) \hookrightarrow \mathrm{L}^{\frac{n}{n-1}}(\Omega;\mathbb{R}^n)$, that the functional is well–defined for $u \in \mathrm{LD}(\Omega)$, where

[1] In fact, as Anzellotti [19, p. 485] writes, 'one could use the general results (...) also to get an Euler equation for functionals of the type $\int_\Omega f(x,\boldsymbol{\varepsilon}(u))$, where u is a vector field of bounded deformation (...). However, we shall not develop such a theory, as it is totally similar to the one (...) for $F(u)$ [in the BV–case]'.

now $N = n \geqslant 2$. In the following, we denote $f_\xi(x,\xi)$ or $f_\xi^\infty(x,\xi)$ the gradients of f or f^∞ with respect to the second variable. Moreover, in all of what follows we tacitly make use of the standard notation for the Radon–Nikodym decompositions of $\mathrm{D}u = \nabla u \mathscr{L}^n + \mathrm{D}^s u$ for $u \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^N)$ and $\mathrm{E}u = \mathscr{E}u\mathscr{L}^n + \mathrm{E}^s u$ for $u \in \mathrm{BD}_{\mathrm{loc}}(\mathbb{R}^n)$. Moreover, we put $f^\infty(x,\xi) := \lim_{t\searrow 0} tf(x,\xi/t)$ for $x \in \Omega$ and $\xi \in \mathbb{R}^{N\times n}$ whenever this limit exists.

Theorem 10.12 ([19, Thm. 3.6])**.** *Assume that $f\colon \Omega\times\mathbb{R}^{N\times n} \to \mathbb{R}$ is such that $\xi \mapsto f(x,\xi)$ is differentiable for every $x \in \Omega$ and suppose f^∞ is differentiable in $(x,\xi) \in \Omega \times (\mathbb{R}^{N\times n} \setminus \{0\})$. Moreover, assume that*

$$|f_\xi(x,\xi)| \leqslant M \quad \text{and} \quad |f_\xi^\infty(x,\xi)| \leqslant M \tag{10.12}$$

for some fixed number $M > 0$ for all $(x,\xi) \in \Omega \times \mathbb{R}^{N\times n}$ or $(x,\xi) \in \Omega \times (\mathbb{R}^{N\times n} \setminus \{0\})$, respectively. Then the functional $\mathfrak{H}\colon \mathrm{BV}(\Omega;\mathbb{R}^N) \to \mathbb{R}$ given by

$$\mathfrak{H}[u] := \int_\Omega f\Big(x, \frac{\mathrm{dD}u}{\mathrm{d}\mathscr{L}^n}\Big)\,\mathrm{d}x + \int_\Omega f^\infty\Big(x, \frac{\mathrm{dD}^s u}{\mathrm{d}|\,\mathrm{D}^s u|}\Big)\,\mathrm{d}|\,\mathrm{D}^s u|, \qquad u \in \mathrm{BV}(\Omega;\mathbb{R}^N)$$

is differentiable at $u \in \mathrm{BV}(\Omega;\mathbb{R}^N)$ in the direction *$\varphi \in \mathrm{BV}(\Omega;\mathbb{R}^N)$ if and only if*

$$|\,\mathrm{D}\varphi|^s \ll |\,\mathrm{D}u|^s, \tag{10.13}$$

and in this case we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\mathfrak{H}[u+t\varphi] &= \int_\Omega \langle f_\xi(x,\nabla u(x)), \nabla\varphi\rangle\,\mathrm{d}x \\ &\quad + \int_\Omega \Big\langle f_\xi^\infty\Big(x, \frac{\mathrm{dD}u}{\mathrm{d}|\,\mathrm{D}u|}(x)\Big), \frac{\mathrm{dD}\varphi}{\mathrm{d}|\,\mathrm{D}\varphi|}(x)\Big\rangle\,\mathrm{d}|\,\mathrm{D}\varphi|^s.\end{aligned}$$

The proofs of this as well as the following theorem will be given at the end of this section. In a next step, we incorporate boundary values to make the theory outlined in the previous lines accesible to the Dirichlet problem on BV or BD, respectively. For this, suppose $u_0 \in \mathrm{L}^1_{\mathcal{H}^{n-1}}(\partial\Omega;\mathbb{R}^N)$ is a given boundary datum.

Theorem 10.13 ([19, Thm. 3.7])**.** *Let Ω be an open and bounded subset of $\mathbb{R}^n$ with Lipschitz boundary. Suppose $f\colon \overline{\Omega}\times\mathbb{R}^{N\times n} \to \mathbb{R}$ is such that $\xi \mapsto f(x,\xi)$ is differentiable for every $x \in \overline{\Omega}$, $\xi \mapsto f^\infty(x,\xi)$ is differentiable in $\xi \in \mathbb{R}^{N\times n} \setminus \{0\}$ for all $x \in \overline{\Omega}$. Moreover, assume that (10.12) is in action. If $u \in \mathrm{BV}(\Omega;\mathbb{R}^N)$ is a minimiser for the integral functional*

$$\begin{aligned}\mathfrak{G}[u] := \int_\Omega f\Big(x, \nabla u\Big)\,\mathrm{d}x &+ \int_\Omega f^\infty\Big(x, \frac{\mathrm{dD}^s u}{\mathrm{d}|\,\mathrm{D}^s u|}\Big)\,\mathrm{d}|\,\mathrm{D}^s u| \\ &+ \int_{\partial\Omega} f^\infty\Big(x, \mathrm{Tr}_{\partial\Omega}(u_0(x)-u(x)) \otimes \nu_{\partial\Omega}\Big)\,\mathrm{d}\mathcal{H}^{n-1},\end{aligned}$$

then we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\mathfrak{G}[u+t\varphi] &= \int_\Omega \langle f_\xi(x,\nabla u(x)), \nabla\varphi\rangle\,\mathrm{d}x \\ &\quad + \int_\Omega \Big\langle f_\xi^\infty\Big(x, \frac{\mathrm{dD}u}{\mathrm{d}|\,\mathrm{D}u|}(x)\Big), \frac{\mathrm{dD}\varphi}{\mathrm{d}|\,\mathrm{D}\varphi|}(x)\Big\rangle\,\mathrm{d}|\,\mathrm{D}^s\varphi| \\ &\quad - \int_{\partial\Omega} \Big\langle f_\xi^\infty\Big(x, \frac{\mathrm{Tr}_{\partial\Omega}(u_0(x)-u(x))}{|\,\mathrm{Tr}_{\partial\Omega}(u_0(x)-u(x))|} \otimes \nu_{\partial\Omega}\Big), \varphi(x) \otimes \nu_{\partial\Omega}(x)\Big\rangle\,\mathrm{d}\mathcal{H}^{n-1} \\ &= 0\end{aligned}$$

for all $\varphi \in \mathrm{BV}(\Omega;\mathbb{R}^N)$ satisfying $|\,\mathrm{D}\varphi|^s \ll |\,\mathrm{D}u|^s$ and $\varphi = 0$ $\mathcal{H}^{n-1}$–a.e. on the set $T_1 := \{x \in \partial\Omega\colon\ \mathrm{Tr}_{\partial\Omega}\, u(x) = \mathrm{Tr}_{\partial\Omega}\, u_0(x)\}$.

For the symmetric gradient case the statement is analogous and reads as follows.

Corollary 10.14. *Let Ω be an open and bounded subset of $\mathbb{R}^n$ with Lipschitz boundary. Suppose $f\colon \overline{\Omega}\times\mathbb{R}^{n\times n}_{\mathrm{sym}}\to\mathbb{R}$ is such that $\xi\mapsto f(x,\xi)$ is differentiable for every $x\in\overline{\Omega}$, $\xi\mapsto f^\infty(x,\xi)$ is differentiable in $\xi\in\mathbb{R}^{n\times n}_{\mathrm{sym}}\setminus\{0\}$ for all $x\in\overline{\Omega}$. Moreover, assume that (10.12) is in action. If $u\in\mathrm{BD}(\Omega)$ is a minimiser for the integral functional*

$$\begin{aligned}\widetilde{\mathfrak{G}}[u] := \int_\Omega f\Big(x,\mathscr{E}u\Big)\,\mathrm{d}x + \int_\Omega f^\infty\Big(x,\frac{\mathrm{d}\mathrm{E}^s u}{\mathrm{d}|\mathrm{E}^s u|}\Big)\,\mathrm{d}|\mathrm{E}^s u| \\ + \int_{\partial\Omega} f^\infty\Big(x,\mathrm{Tr}_{\partial\Omega}(u_0(x)-u(x))\odot\nu_{\partial\Omega}\Big)\,\mathrm{d}\mathcal{H}^{n-1},\end{aligned}$$

then we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\mathfrak{G}[u+t\varphi] &= \int_\Omega\langle f_\xi(x,\mathscr{E}u(x)),\mathscr{E}\varphi\rangle\,\mathrm{d}x \\ &\quad+\int_\Omega\Big\langle f^\infty_\xi\Big(x,\frac{\mathrm{dE}u}{\mathrm{d}|\mathrm{E}u|}(x)\Big),\frac{\mathrm{dE}\varphi}{\mathrm{d}|\mathrm{E}\varphi|}(x)\Big\rangle\,\mathrm{d}|\mathrm{E}^s\varphi| \\ &\quad-\int_{\partial\Omega}\Big\langle f^\infty_\xi\Big(x,\frac{\mathrm{Tr}_{\partial\Omega}(u_0(x)-u(x))}{|\mathrm{Tr}_{\partial\Omega}(u_0(x)-u(x))|}\odot\nu_{\partial\Omega}\Big),\varphi(x)\otimes\nu_{\partial\Omega}(x)\Big\rangle\,\mathrm{d}\mathcal{H}^{n-1} \\ &= 0\end{aligned}$$

for all $\varphi\in\mathrm{BD}(\Omega)$ satisfying $|\mathrm{E}\varphi|^s\ll|\mathrm{E}u|^s$ and $\varphi=0$ $\mathcal{H}^{n-1}$–a.e. on the set $T_1:=\{x\in\partial\Omega\colon\ \mathrm{Tr}_{\partial\Omega}u(x)=\mathrm{Tr}_{\partial\Omega}u_0(x)\}$.

The preceding theorems will follow from general differentiation results on functions of measures to be discussed next. Since this theory is stated for $\mathbb{R}^m$–valued measures (where we evidently are mostly interested in the case $m=Nn$), we now assume that

(H5) $f\colon\Omega\times\mathbb{R}^m\to\mathbb{R}_{\geqslant0}$ is convex,

(H6) for all Borel measurable mappings $\eta\colon\Omega\to\mathbb{R}^m$ the function $\Omega\ni x\mapsto f(x,\eta(x))\in\mathbb{R}$ is Borel measurable,

(H7) for all $x\in\Omega$, there exists the finite limit

$$f^\infty(x,\xi)=\lim_{t\searrow0}tf\Big(x,\frac{\xi}{t}\Big),\qquad \xi\in\mathbb{R}^m. \tag{10.14}$$

Note that in Chapter 4.3, we merely considered autonomous convex functions of measures. In the situation of non–autonomous integrands $f(x,z)$, similar statements are available. Let $\mu\in\mathcal{M}_{<\infty}(\Omega;\mathbb{R}^m)$. As in Chapter 4.3, we define a positive measure $f(\cdot,\mu)$ on the Borel subsets of Ω by

$$f(\cdot,\mu)(B):=\int_B f(x,\mu):=\int_B f\Big(x,\frac{\mathrm{d}\mu}{\mathrm{d}\mathscr{L}^n}\Big)\,\mathrm{d}x+\int_B f^\infty\Big(x,\frac{\mathrm{d}\mu^s}{\mathrm{d}|\mu^s|}\Big)\,\mathrm{d}|\mu^s|,\qquad B\in\mathscr{B}(\Omega).$$

Here, $f^\infty(\cdot,\cdot)$ is the non–autonomous recession function given by (10.14). As usual, $\mu=\mu^{ac}+\mu^s=\frac{\mathrm{d}\mu}{\mathrm{d}\mathscr{L}^n}\mathscr{L}^n+\frac{\mathrm{d}\mu^s}{\mathrm{d}|\mu^s|}|\mu^s|$ is the decomposition of μ into its absolutely continuous and singular part with respect to $\mathscr{L}^n$. Moreover, note that if we have the growth bound

$$|f(x,\xi)|\leqslant c(1+|\xi|), \tag{LG'}$$

then we also have $|f^\infty(x,\xi)|\leqslant c|\xi|$ and the rough bound $\int_\Omega f(x,\mu)\leqslant C(\mathscr{L}^n(\Omega)+|\mu|(\Omega))$. In this situation, the measure $f(x,\mu)$ thus has finite total variation.

We record the following simple lemma.

Lemma 10.15 ([19, Lem. 2.2]). *Let $f\colon \Omega\times\mathbb{R}^m \to \mathbb{R}$ satisfy* (H5)–(H7) *and* (LG').

(a) *If f is positively homogeneous in its second argument, then for all Borel sets $B\subset\Omega$ there holds*

$$\int_B f\Big(x, \frac{\mathrm{d}\mu}{\mathrm{d}|\mu|}(x)\Big)\,\mathrm{d}|\mu| = \int_B f\Big(x, \frac{\mathrm{d}\mu_1}{\mathrm{d}|\mu_1|}(x)\Big)\,\mathrm{d}|\mu_1| + \int_B f\Big(x, \frac{\mathrm{d}\mu_2}{\mathrm{d}|\mu_2|}(x)\Big)\,\mathrm{d}|\mu_2|$$

whenever $\mu=\mu_1+\mu_2$ and the measures μ_1,μ_2 are mutually singular.

(b) *If f is positively homogeneous in its second argument, then for all Borel sets $B\subset\Omega$ there holds*

$$\int_B f(x,\mu) = \int_B f\Big(x, \frac{\mathrm{d}\mu}{\mathrm{d}\alpha}(x)\Big)\,\mathrm{d}\alpha$$

whenever α is a positive measure such that $\mu\ll\alpha$.

Proof. We note that

$$\begin{aligned}\frac{\mathrm{d}\mu}{\mathrm{d}|\mu|} &= \frac{\mathrm{d}\mu_1}{\mathrm{d}|\mu|} = \frac{\mathrm{d}\mu_1}{\mathrm{d}|\mu_1|} \quad |\mu_1| - \text{a.e. in } \Omega,\\ \frac{\mathrm{d}\mu}{\mathrm{d}|\mu|} &= \frac{\mathrm{d}\mu_2}{\mathrm{d}|\mu|} = \frac{\mathrm{d}\mu_2}{\mathrm{d}|\mu_2|} \quad |\mu_2| - \text{a.e. in } \Omega,\end{aligned} \tag{10.15}$$

and since μ_1,μ_2 are mutually singular, there holds $|\mu|=|\mu_1|+|\mu_2|$. Consequently,

$$\int_B f\Big(x, \frac{\mathrm{d}\mu}{\mathrm{d}|\mu|}(x)\Big)\,\mathrm{d}|\mu| = \int_B f\Big(x, \frac{\mathrm{d}\mu}{\mathrm{d}|\mu|}(x)\Big)\,\mathrm{d}|\mu_1| + \int_B f\Big(x, \frac{\mathrm{d}\mu}{\mathrm{d}|\mu|}(x)\Big)\,\mathrm{d}|\mu_2|$$

and we now use (10.15) to conclude (a). For (b), write $\mu=\mu^{ac}+\mu^s$ for the Radon–Nikodym decomposition of μ with respect to $\mathscr{L}^n$ so that still $\mu^{ac},\mu^s\ll\alpha$. Then by (a), the definition of $f(\cdot,\mu)$ and positive homogeneity in the third step we obtain for all Borel subsets $B\subset\Omega$

$$\begin{aligned}\int_B f(x,\mu) &= \int_B f\Big(x, \frac{\mathrm{d}\mu}{\mathrm{d}|\mu|}(x)\Big)\,\mathrm{d}|\mu|\\ &= \int_B f\left(x, \frac{\frac{\mathrm{d}\mu}{\mathrm{d}\alpha}(x)}{\frac{\mathrm{d}|\mu|}{\mathrm{d}\alpha}(x)}\right)\frac{\mathrm{d}|\mu|}{\mathrm{d}\alpha}\,\mathrm{d}\alpha\\ &= \int_B f\Big(x, \frac{\mathrm{d}\mu}{\mathrm{d}\alpha}(x)\Big)\,\mathrm{d}\alpha,\end{aligned}$$

which is (b). The proof is complete. □

Similarly as in Chapter 4, we define the perspective function $f^{\#}\colon \Omega\times\mathbb{R}^m\times[0,\infty)\to\mathbb{R}$ for $(x,\xi)\in\Omega\times\mathbb{R}^m$ by

$$f^{\#}(x,\xi,t) := \begin{cases} f(x,\xi/t)t, & t>0,\\ f^\infty(x,\xi), & t=0.\end{cases}$$

Lemma 10.16 ([19, Thm. 2.3]). *Suppose $f\colon\Omega\times\mathbb{R}^m\to\mathbb{R}$ satisfies* (H6), (H7) *and* (LG'). *Moreover, assume that $|\mu|(\Omega)$ and $\mathscr{L}^n(\Omega)$ are finite. Then we have*

$$\int_B f(x,\mu) = \int_B f^{\#}\left(x, \frac{\mathrm{d}\mu}{\mathrm{d}\alpha}(x), \frac{\mathrm{d}\mathscr{L}^n}{\mathrm{d}\alpha}(x)\right)\mathrm{d}\alpha, \tag{10.16}$$

where α is any positive Borel measure such that $|\mu|+\mathscr{L}^n\ll\alpha$.

Proof. Let $\psi_1\colon \Omega \to \mathbb{R}^m$ and $\psi_2\colon \Omega \to [0,\infty)$ be two Borel functions. By the very construction of $f^{\#}$ we then see that $x \mapsto f^{\#}(x,\psi_1(x),\psi_2(x))$ is Borel measurable, too, and so the right hand side of (10.16) is well–defined. We define new measures $\mu_1 := (\mu, \mathscr{L}^n)$, $\mu_2 := (\mu^{ac}, \mathscr{L}^n)$ and $\mu_3 := (\mu^s, 0)$ which are consequently $\mathbb{R}^m \times \mathbb{R}$–valued. These, in turn, satisfy $\mu_1 = \mu_2 + \mu_3$. Clearly, μ_2 and μ_3 are mutually singular, and by construction $f^{\#}(x,(y,t))$ is positively homogeneous in (y,t). By the preceding lemma, we therefore obtain

$$\begin{aligned}\int_B f^{\#}\left(x, \frac{\mathrm{d}\mu}{\mathrm{d}|\mu_1|}(x), \frac{\mathrm{d}\mathscr{L}^n}{\mathrm{d}|\mu_1|}(x)\right) \mathrm{d}|\mu_1| &= \int_B f^{\#}\left(\left(x, \frac{\mathrm{d}\mu^{ac}}{\mathrm{d}|\mu_1|}(x), \frac{\mathrm{d}\mathscr{L}^n}{\mathrm{d}|\mu_1|}(x)\right)\right) \mathrm{d}|\mu_2| \\ &\quad + \int_B f^{\#}\left(\left(x, \frac{\mathrm{d}\mu^s}{\mathrm{d}|\mu_1|}(x), 0\right)\right) \mathrm{d}|\mu_3| \\ &= \int_B f^{\#}\left(x, \frac{\mathrm{d}\mu^{ac}}{\mathrm{d}\mathscr{L}^n}, 1\right) \mathrm{d}\mathscr{L}^n \\ &\quad + \int_B f^{\infty}\left(x, \frac{\mathrm{d}\mu^s}{\mathrm{d}|\mu^s|}(x)\right) \mathrm{d}|\mu|^s.\end{aligned}$$

Hence (10.16) follows for the choice $\alpha = |\mu_1|$, and we then invoke the second part of the lemma to obtain the statement for all α. The proof is complete. □

Theorem 10.17 ([19, Thm. 2.4])**.** *Assume that $f\colon \Omega \times \mathbb{R}^m \to \mathbb{R}$ is differentiable at all $(x,\xi) \in \Omega \times \mathbb{R}^m$ and that f^∞ is differentiable at all $(x,\xi) \in \Omega \times (\mathbb{R}^m \setminus \{0\})$. Moreover, suppose $|f_\xi(x,\xi)| \leqslant M$ and $|f_\xi^\infty(x,\xi)| \leqslant M$ for a constant $M > 0$ and all $(x,\xi) \in \Omega \times \mathbb{R}^m$ or $(x,\xi) \in \Omega \times (\mathbb{R}^m \setminus \{0\})$, respectively. Then the functional $I[\mu] = \int_\Omega f(x,\mu)$ is differentiable at the point μ in the direction β if and only if*

$$|\beta|^s \ll |\mu|^s,$$

and in this case we have

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} I[\mu + t\beta] = \int_\Omega \langle f_\xi(x,\mu^{ac}(x)), \beta^{ac}(x)\rangle\,\mathrm{d}x + \int_\Omega \left\langle f_\xi^\infty\left(x, \frac{\mathrm{d}\mu^s}{\mathrm{d}|\mu|^s}(x)\right), \frac{\mathrm{d}\beta^s}{\mathrm{d}|\beta|^s}(x)\right\rangle \mathrm{d}|\beta|^s.$$

Proof. Let μ be arbitrary but fixed. Given a Radon measure β, we split

$$\begin{aligned}\beta &= \beta^{ac} + \beta^s \\ &= \beta^{ac} + \beta^{s,ac} + \beta^{s,s},\end{aligned} \tag{10.17}$$

where (i) β^{ac} is the absolutely continuous and β^s the singular part of β with respect to $\mathscr{L}^n$, (ii) $\beta^{s,ac}$ is the absolutely continuous part of β^s with respect to $|\mu|^s$ and $\beta^{s,s}$ is the singular part of β^s with respect to $|\mu|^s$. We have

$$\mu + t\beta = \mu^{ac} + \mu^s + t\beta^{ac} + t\beta^{s,ac} + t\beta^{s,s}$$

and hence, by Lemma 10.15, we obtain by positive homogeneity of f^∞ in its second argument that

$$\begin{aligned}I[\mu + t\beta] &= \int_\Omega f(x, \mu^{ac}(x) + t\beta^{ac}(x))\,\mathrm{d}x \\ &\quad + \int_\Omega f^\infty\Big(x, \frac{\mathrm{d}\mu^s}{\mathrm{d}|\mu|^s}(x) + t\frac{\mathrm{d}\beta^{s,ac}}{\mathrm{d}|\mu|^s}(x)\Big)\,\mathrm{d}|\mu|^s + |t|\int_\Omega f^\infty(x, \beta^{s,s}).\end{aligned} \tag{10.18}$$

We then obviously have

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \int_\Omega f(x, \mu^{ac}(x) + t\beta^{ac}(x))\,\mathrm{d}x = \int_\Omega \langle f_\xi(x,\mu^{ac}(x)), \beta^{ac}(x)\rangle\,\mathrm{d}x.$$

On the other hand, we recall that $\xi \mapsto f^\infty(x,\xi)$ was assumed differentiable in $\xi \neq 0$. Since $\left|\frac{\mathrm{d}\mu^s}{\mathrm{d}|\mu^s|}\right| = 1$ $|\mu^s|$–a.e., we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} &\int_\Omega f^\infty\Big(x, \frac{\mathrm{d}\mu^s}{\mathrm{d}|\mu^s|}(x) + t\frac{\mathrm{d}\beta^{s,ac}}{\mathrm{d}|\mu^s|}(x)\Big)\,\mathrm{d}|\mu^s| \\ &= \int_\Omega \Big\langle f_\xi^\infty\Big(x, \frac{\mathrm{d}\mu^s}{\mathrm{d}|\mu^s|}(x)\Big), \frac{\mathrm{d}\beta^{s,ac}}{\mathrm{d}|\mu^s|}(x)\Big\rangle\,\mathrm{d}|\mu^s| \\ &= \int_\Omega \Big\langle f_\xi^\infty\Big(x, \frac{\mathrm{d}\mu^s}{\mathrm{d}|\mu^s|}(x)\Big), \frac{\mathrm{d}\beta^{s,ac}}{\mathrm{d}|\beta^{s,ac}|}(x)\Big\rangle\,\mathrm{d}|\beta^{s,ac}|.\end{aligned}$$

Lastly,

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0^+}\Big(|t|\int_\Omega f^\infty(x,\beta^{s,s})\Big) = -\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0^-}\Big(|t|\int_\Omega f^\infty(x,\beta^{s,s})\Big) = \int_\Omega f^\infty(x,\beta^{s,s}).$$

Working from here, we see that since the required differentiability of I is ensured if and only if $\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0^+}(\ldots) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0^-}(\ldots)$, we must have

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0^+}\Big(|t|\int_\Omega f^\infty(x,\beta^{s,s})\Big) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0^-}\Big(|t|\int_\Omega f^\infty(x,\beta^{s,s})\Big) \Leftrightarrow \beta^{ss} = 0 \Leftrightarrow \beta^s \ll |\mu^s|,$$

which is the assumption stated in the theorem. The proof thus is complete. □

It is clear that when $f(x,\cdot)$ is positively homogeneous (such as the modulus $f(x,\cdot) = |\cdot|$), then it fails to be differentiable at the origin and hence another ingredient is needed:

Theorem 10.18 ([19, Thm. 2.5])**.** *Assume that for every fixed $x \in \Omega$ the function $\xi \mapsto f(x,\xi)$ is differentiable for all $\xi \neq 0$ and suppose $f(x,0) = 0$ for all $x \in \Omega$ such that $f(x,\cdot)$ is not differentiable at $\xi = 0$. Assume also that $\xi \mapsto f^\infty(x,\xi)$ is differentiable in ξ for all $(x,\xi) \in \Omega \times (\mathbb{R}^m \setminus \{0\})$. Moreover, suppose $|f_\xi(x,\xi)| \leqslant M$ and $|f_\xi^\infty(x,\xi)| \leqslant M$ for a constant $M > 0$ whenever these derivatives exist. Then, if μ, β are such that*

(C1) $\beta^s \ll |\mu|^s$,

(C2) $\beta^{ac} = 0$ $\mathscr{L}^n$–a.e. in $T := \{x \in \Omega\colon\ \mu^{ac}(x) = 0\}$,

then the functional $I[\mu] := \int_\Omega f(x,\mu)$ is differentiable at the point μ in the direction β, and

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} I[\mu + t\beta] = \int_\Omega \langle f_\xi(x,\mu^{ac}(x)), \beta^{ac}(x)\rangle\,\mathrm{d}x + \int_\Omega \Big\langle f_\xi^\infty\Big(x, \frac{\mathrm{d}\mu^s}{\mathrm{d}|\mu^s|}(x)\Big), \frac{\mathrm{d}\beta^s}{\mathrm{d}|\beta^s|}(x)\Big\rangle\,\mathrm{d}\beta^s.$$

holds true. Finally, we have

$$((\mathrm{C1})\text{ and }(\mathrm{C2})) \Leftrightarrow \beta \ll |\mu|. \tag{10.19}$$

Proof. As in the previous proof, we have (10.18). Note that (C1) implies that $\beta^{s,s} = 0$ (cp. (10.17)). Let $x \in \Omega \setminus T$ with T given by (C2). For such x we record

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} f(x, \mu^{ac}(x) + t\beta^{ac}(x)) = \langle f_\xi(x,\mu^{ac}(x)), \beta^{ac}(x)\rangle.$$

Now, for $\mathscr{L}^n$–a.e. $x \in T$, there holds $\beta^{ac} = 0$ by (C2). Hence, for such x we have

$$\frac{\mathrm{d}}{\mathrm{d}t} f(x, \mu^{ac}(x) + t\beta^{ac}(x)) = 0$$

and so, in particular, $\frac{\mathrm{d}}{\mathrm{d}t}|_{t=0} f(x,\mu^{ac}(x)+t\beta^{ac}(x))=0$. From here, the proof concludes as that of Theorem 10.17. Lastly, (10.19) is certainly ensured by $\beta \ll |\mu|$. On the other hand, if $A\in\mathscr{B}(\Omega)$ is such that $|\mu|(A)=0$, then $|\mu|^s(A)=0$ and so $\beta^s(A)=0$ by (C1) as $|\mu|^s(A)=0$. Finally, since $A\subset T$,

$$|\beta|^{ac}(A)\leqslant|\beta|^{ac}(T)=\int_T\frac{\mathrm{d}|\beta|^{ac}}{\mathrm{d}\mathscr{L}^n}\,\mathrm{d}\mathscr{L}^n=0.$$

The proof is complete. $\square$

Based on Theorems 10.17 and 10.18, we can now proceed to the proof of Theorems 10.12 and 10.13.

Proof of Theorem 10.12. Theorem 10.12 follows from Theorem 10.17. In fact, letting $u,\varphi\in\mathrm{BV}(\Omega;\mathbb{R}^N)$ satisfy $|\mathrm{D}\varphi|^s\ll|\mathrm{D}u|^s$, we obtain the statement of the theorem. $\square$

Proof of Theorem 10.13 and Corollary 10.14. This follows as above for Theorem 10.12, however, now incorporating the boundary terms. We exemplarily argue for the symmetric gradient case and write $\mathrm{Tr}=\mathrm{Tr}_{\partial\Omega}$ for simplicity. For $\mathcal{H}^{n-1}$–a.e. $x\in\partial\Omega$ with $x\notin T_1$ we have by positive homogeneity of f^∞

$$f^\infty\Big((\mathrm{Tr}(u_0-u-t\varphi)(x)\odot\nu_{\partial\Omega}(x)\Big)=f^\infty\Big(\frac{\mathrm{Tr}(u_0-u-t\varphi)(x)}{|\mathrm{Tr}(u_0-u)(x)|}\odot\nu_{\partial\Omega}(x)\Big)|\mathrm{Tr}(u_0-u)(x)|$$

for $t\neq0$, and since $x\notin T_1$, we can assume that all terms are well–defined and that $\frac{(\mathrm{Tr}(u_0-u-t\varphi)(x)}{|\mathrm{Tr}(u_0-u)(x)|}>c_x>0$ for all sufficiently small $|t|$. Differentiating with respect to t yields

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}&f^\infty\Big(\frac{\mathrm{Tr}(u_0-u-t\varphi)(x)}{|\mathrm{Tr}(u_0-u)(x)|}\odot\nu_{\partial\Omega}(x)\Big)|\mathrm{Tr}(u_0-u)(x)|\\&=f_\xi^\infty\Big(\frac{\mathrm{Tr}(u_0-u-t\varphi)(x)}{|\mathrm{Tr}(u_0-u)(x)|}\odot\nu_{\partial\Omega}(x)\Big)\frac{(-\varphi(x)\odot\nu_{\partial\Omega}(x))}{|\mathrm{Tr}(u_0-u)(x)|}|\mathrm{Tr}(u_0-u)(x)|.\end{aligned}$$

Using this formula in conjunction with the usual perturbation argument, the claim follows. $\square$

Finally, we come back to the setting of functionals (10.11) and their relaxation to BV or BD, respectively. Noting that

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\int_\Omega\langle H,u+t\varphi\rangle\,\mathrm{d}x=\int_\Omega\langle H,\varphi\rangle\,\mathrm{d}x$$

for all $\varphi\in\mathrm{BV}(\Omega;\mathbb{R}^N)$ or $\varphi\in\mathrm{BD}(\Omega)$ satisfying the requirements of Theorem 10.13 or Corollary 10.14, the Euler–Lagrange equations satisfied by minima $u\in\mathrm{BD}(\Omega)$ of the relaxed functionals are immediate. More precisely, e.g., assuming that the assumptions of Corollary 10.14 on the variational integral in the symmetric gradient case are in action, we have that any minimiser $u\in\mathrm{BD}(\Omega)$ of the functional

$$\begin{aligned}\widetilde{\mathfrak{F}}_{u_0}[v]=&\int_\Omega f(x,\mathscr{E}v)\,\mathrm{d}x+\int_\Omega f^\infty\Big(x,\frac{\mathrm{dE}^s v}{\mathrm{d}|\mathrm{E}^s v|}\Big)\,\mathrm{d}|\mathrm{E}^s v|\\&+\int_{\partial\Omega}f^\infty\big(\mathrm{Tr}_{\partial\Omega}(u_0-v)\odot\nu_{\partial\Omega}\big)\,\mathrm{d}\mathcal{H}^{n-1}+\int_\Omega\langle H,v\rangle\,\mathrm{d}x\end{aligned}$$

satisfies the following Euler–Lagrange equation: For all $\varphi\in\mathrm{BD}(\Omega)$ satisfying $|\mathrm{E}\varphi|^s\ll|\mathrm{E}u|^s$ and $\varphi=0$ $\mathcal{H}^{n-1}$–a.e. on $\{x\in\partial\Omega\colon\ \mathrm{Tr}(u)(x)=\mathrm{Tr}(u_0)(x)\}$, we have

$$\begin{aligned}&\int_\Omega\langle f_\xi(x,\mathscr{E}u(x)),\mathscr{E}\varphi\rangle\,\mathrm{d}x+\int_\Omega\Big\langle f_\xi^\infty\Big(x,\frac{\mathrm{dE}u}{\mathrm{d}|\mathrm{E}u|}(x)\Big),\frac{\mathrm{dE}\varphi}{\mathrm{d}|\mathrm{E}\varphi|}(x)\Big\rangle\,\mathrm{d}|\mathrm{E}^s\varphi|\\&\quad-\int_{\partial\Omega}\Big\langle f_\xi^\infty\Big(x,\frac{\mathrm{Tr}_{\partial\Omega}(u_0(x)-u(x))}{|\mathrm{Tr}_{\partial\Omega}(u_0(x)-u(x))|}\odot\nu_{\partial\Omega}\Big),\varphi(x)\otimes\nu_{\partial\Omega}(x)\Big\rangle\,\mathrm{d}\mathcal{H}^{n-1}=-\int_\Omega\langle H,\varphi\rangle\,\mathrm{d}x.\end{aligned}$$

This is what we want to actually understand by *weak solutions* of the system

$$\begin{cases} -\operatorname{div}(f'(x,\boldsymbol{\varepsilon}(u))) = -H & \text{in } \Omega, \\ u = u_0 & \text{on } \partial\Omega \end{cases}$$

over BD. Lastly, note that coerciveness of the functional at our disposal requires some condition on H. For instance, assuming that $u_0 = 0$, such an additional hypothesis is given by requiring[2] $\|H\|_{\mathrm{L}^n(\Omega;\mathbb{R}^n)} < c_{sob}^{-1}$, where $c_{sob} > 0$ is the optimal constant for the Sobolev–Strauss inequality in $\mathrm{LD}_0(\Omega)$, i.e., $c_{sob} > 0$ is the smallest constant $c > 0$ for which there holds $\|v\|_{\mathrm{L}^{\frac{n}{n-1}}(\Omega;\mathbb{R}^n)} \leqslant c_{\mathrm{sob}}\|\boldsymbol{\varepsilon}(v)\|_{\mathrm{L}^1(\Omega;\mathbb{R}^{n\times n})}$ for all $v \in \mathrm{LD}_0(\Omega)$. In fact, in this case we have for all $\varphi \in \mathrm{LD}_0(\Omega)$ that

$$\begin{aligned} \int_\Omega f(x,\boldsymbol{\varepsilon}(\varphi)) + \langle H,\varphi\rangle \,\mathrm{d}x &\geqslant \int_\Omega |\boldsymbol{\varepsilon}(\varphi)|\,\mathrm{d}x - \|H\|_{\mathrm{L}^n(\Omega;\mathbb{R}^n)}\|\varphi\|_{\mathrm{L}^{\frac{n}{n-1}}(\Omega;\mathbb{R}^n)} \\ &\geqslant \int_\Omega |\boldsymbol{\varepsilon}(\varphi)|\,\mathrm{d}x - c_{sob}\|H\|_{\mathrm{L}^n(\Omega;\mathbb{R}^n)}\|\boldsymbol{\varepsilon}(\varphi)\|_{\mathrm{L}^1(\Omega;\mathbb{R}^n)} \\ &= \|\boldsymbol{\varepsilon}(\varphi)\|_{\mathrm{L}^1(\Omega;\mathbb{R}^{n\times n})}(1 - c_{sob}\|H\|_{\mathrm{L}^n(\Omega;\mathbb{R}^n)}). \end{aligned}$$

It is then seen that if the condition $\|H\|_{\mathrm{L}^n(\Omega;\mathbb{R}^n)} < 1/c_{sob}$ holds, then we have $(1 - c_{sob}\|H\|_{\mathrm{L}^n(\Omega;\mathbb{R}^n)}) > 0$ and we thus have coerciveness on $\mathrm{LD}_0(\Omega)$. In this situation, it is not difficult to show that this coerciveness carries over to BD and hence ensures the existence of a generalised minimiser which satisfies the above Euler–Lagrange equation. Similar conditions can be derived in the situation of general boundary data $u_0 \in \mathrm{L}^1_{\mathcal{H}^{n-1}}(\partial\Omega;\mathbb{R}^n)$.

10.5. Differentiability in Banach Spaces

Throughout, $\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$, but assumed fixed.

Definition 10.19. *Let $(X,\|\cdot\|_X)$ and $(Y,\|\cdot\|_Y)$ be two Banach spaces over $\mathbb{F}$ and let U be an open neighbourhood of the point u in X. Given a mapping $f\colon U \to Y$, we say that the (Fréchet)* derivative *(or* differential*) $Df(u)$ of f at u exists if and only if there is a linear bounded operator $Df(u)\colon X \to Y$ such that*

$$f(u+h) = f(u) + Df(u)h + o(\|h\|_X)$$

holds for all $h \in X$ in an open neighbourhood of $h = 0$ in X. For $k \in \mathbb{N}$, the k–th Fréchet derivatives $D^k f(u)\colon X^k \to Y$ at u are defined inductively. Lastly, we say that $D^k f$ is continuous at $u \in X$ if and only if for each $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that

$$\|D^k f(u+h)(h_1,...,h_k) - D^k f(u)(h_1,...,h_k)\|_Y < \varepsilon\|h_1\|_X ... \|h_k\|_X$$

for all $h \in X$ with $\|h\| < \delta(\varepsilon)$ and $h_1,...,h_k \in X$.

This notion canonically gives rise to spaces $\mathrm{C}^k(X,Y)$ of k–times continuously Fréchet differentiable mappings from X to Y. We now have the following version of the implicit function theorem.

Theorem 10.20 (Implicit Function Theorem, [247, Thm. 4.E])**.** *Let X, Y, W be three Banach spaces, $k \geqslant 1$, $A \subset X \times Y$ an open set (with respect to the usual product topology), $(x_0,y_0) \in A$ and let $f\colon A \to W$ is a C^k–map with $f(x_0,y_0) = 0$. Assume that $D_2 f(x_0,y_0) = D(f(x_0,\cdot))(y_0)\colon Y \to W$ is a bounded invertible linear transformation. Then there exists an open neighbourhood U_0 of x_0 in X such that for all connected open neighbourhoods U of x_0 contained in U_0, there exists a unique continuous map $u\colon U \to Y$ such that*

[2] Sometimes such conditions are referred to as *safety load conditions*, cp. [38].

(a) $u(x_0) = y_0$,

(b) $(x, u(x)) \in A$,

(c) $f(x, u(x)) = 0$ *for all* $x \in U$.

Moreover, u is necessarily of class C^k *and for all* $x \in U$ *we have*

$$Du(x) = -D_2 f(x, u(x))^{-1} D_1 f(x, u(x)).$$

10.6. Miscellaneous

In this final part of the appendix we collect some auxiliary results that were used as minor points in proofs in the main text. First, we give the

Proof of Lemma 2.22. Put $r = q$ in Lemma 2.21. Note that since $t \mapsto t^q$ is a convex function on $\mathbb{R}_0^+$, there exists a constant $c > 0$ such that $(a+b)^q \leqslant c(a^q + b^q)$ for all $a, b \geqslant 0$. Let (Q_k) be a collection of pairwise disjoint cubes with $\mathscr{L}^n(Q_k) = \text{const.}$ such that $\bigcup_k Q_k = \mathbb{R}^n$. We apply inequality (2.14) to each of the Q_k's, raise it to the q–th power and sum up over all $k \in \mathbb{N}$ to obtain

$$\begin{aligned}
\int_{\mathbb{R}^n} |f|^q \,\mathrm{d}x &\leqslant \sum_k \mathscr{L}^n(Q_k) \fint_{Q_k} |f|^q \,\mathrm{d}x \leqslant C \sum_k \mathscr{L}^n(Q_k) \fint_{Q_k} |\mathcal{M}_{Q_k} f|^q \,\mathrm{d}x \\
&\leqslant C \sum_k \mathscr{L}^n(Q_k) \left(\fint_{Q_k} |\mathcal{M}^\sharp_{Q_k} f|^q \,\mathrm{d}x + \left(\fint_{Q_k} |f| \,\mathrm{d}x \right)^q \right) \\
&\overset{\text{Jensen}}{\leqslant} C \sum_k \mathscr{L}^n(Q_k) \left(\fint_{Q_k} |\mathcal{M}^\sharp_{Q_k} f|^q \,\mathrm{d}x + \left(\fint_{Q_k} |f|^p \,\mathrm{d}x \right)^{\frac{q}{p}} \right) \\
&\overset{\mathcal{M}^\sharp_{Q_k} \leqslant \mathcal{M}^\sharp}{=} C \left(\int_{\mathbb{R}^n} |\mathcal{M}^\sharp f|^q \,\mathrm{d}x + \sum_k \mathscr{L}^n(Q_k) \left(\fint_{Q_k} |f|^p \,\mathrm{d}x \right)^{\frac{q}{p}} \right) \\
&\leqslant C \left(\|f\|^{q-p}_{\mathrm{BMO}(\mathbb{R}^n)} \int_{\mathbb{R}^n} |\mathcal{M}^\sharp f|^p \,\mathrm{d}x + \sum_k R^{n(1-\frac{q}{p})} \left(\int_{Q_k} |f|^p \,\mathrm{d}x \right)^{\frac{q}{p}} \right) \\
&\leqslant C \left(\|f\|^{q-p}_{\mathrm{BMO}(\mathbb{R}^n)} \int_{\mathbb{R}^n} |\mathcal{M} f|^p \,\mathrm{d}x + R^{n(1-\frac{q}{p})} \sum_k \left(\int_{Q_k} |f|^p \,\mathrm{d}x \right)^{\frac{q}{p}} \right) \\
&\overset{(***)}{\leqslant} C \left(\|f\|^{q-p}_{\mathrm{BMO}(\mathbb{R}^n)} \int_{\mathbb{R}^n} |f|^p \,\mathrm{d}x + R^{n(1-\frac{q}{p})} \sum_k \left(\int_{Q_k} |f|^p \,\mathrm{d}x \right)^{\frac{q}{p}} \right) = (\dagger),
\end{aligned}$$

where we used at $(***)$ the L^p–boundedness of the maximal operator; recall that $p > 1$. The claim will be deduced as follows. For $k \in \mathbb{N}$, put $a_k := \int_{Q_k} |f|^p \,\mathrm{d}x$. Note that since $f \in \mathrm{L}^p(\mathbb{R}^n)$, we have that $(a_k) \in \ell^1(\mathbb{N})$ and hence $(a_k) \in \ell^{\frac{q}{p}}(\mathbb{N})$ (note that $\frac{q}{p} > 1$). Thus,

$$(\dagger) \leqslant C \left(\|f\|^{q-p}_{\mathrm{BMO}(\mathbb{R}^n)} \int_{\mathbb{R}^n} |f|^p \,\mathrm{d}x + R^{n(1-\frac{q}{p})} \left(\int_{\mathbb{R}^n} |f|^p \,\mathrm{d}x \right)^{\frac{q}{p}} \right)$$

is finite and the proof is complete. $\square$

Lemma 10.21. *Let* $x \in \mathbb{S}^{n-1}$ *and* $y \in \mathbb{S}^{n-1} \setminus \{-x\}$. *Then there holds*

$$\left| x - \frac{x+y}{|x+y|} \right| \leqslant |x - y|. \tag{10.20}$$

Proof. Inequality (10.20) can be reduced to the two–dimensional situation as the origin, x and y span a two dimensional hyperplane which the point $(x+y)/|x+y|$ also belongs to. Moreover, upon rotating $\mathbb{S}^1$, we can assume without loss of generality that $x=(1,0)$ and by symmetry that $y=(y_1,y_2)$ satisfies $y_2>0$. We write $y=(\cos(\varphi),\sin(\varphi))$ for some $\varphi\in(0,\pi)$ (note that $\varphi=\pi$ is ruled out by assumption). We then have $(x+y)/|x+y|=(\cos(\frac{\varphi}{2}),\sin(\frac{\varphi}{2}))$ and hence

$$\begin{aligned}(10.20)\Leftrightarrow &\left|\begin{pmatrix}1-\cos(\frac{\varphi}{2})\\ -\sin(\frac{\varphi}{2})\end{pmatrix}\right|\leqslant\left|\begin{pmatrix}1-\cos(\varphi)\\ -\sin(\varphi)\end{pmatrix}\right|\\ &\Leftrightarrow (1-\cos(\frac{\varphi}{2}))^2+\sin^2(\frac{\varphi}{2})\leqslant(1-\cos(\varphi))^2+\sin^2(\varphi)\\ &\Leftrightarrow 2-2\cos(\frac{\varphi}{2})\leqslant 2-2\cos(\varphi)\Leftrightarrow\cos(\frac{\varphi}{2})\geqslant\cos(\varphi),\end{aligned}$$

and the last inequality is true as $\cos\colon(0,\pi)\to(-1,1)$ is strictly decreasing. The proof is complete. □

We now come to the missing part in the proof of Proposition 8.25. Let us examplarily argue for $n=2$; the case of general n is conceptually the same but technically more demanding. We aim at showing that there exists a constant $c>0$ such that if $f\colon\mathbb{S}^1\to\mathbb{R}_{\geqslant0}$ is an in integrable function on the sphere, then for every $x\in\mathbb{S}^1$ there holds

$$\int_{\mathbb{S}^1}f\Big(\frac{x+y}{|x+y|}\Big)\,\mathrm{d}\sigma_y\leqslant C\int_{\mathbb{S}^1}f(y)\,\mathrm{d}\sigma_y.$$

To this end, we remark that since the unit sphere is parametrised by $\mathbb{S}^1=\{\gamma(\varphi):=(\cos(\varphi),\sin(\varphi))\colon\ \varphi\in[0,2\pi)\}$, we firstly find an angle $\psi\in[0,2\pi)$ such that $x=(\cos(\psi),\sin(\psi))$ and then employ the change of variables formula $y=(\cos(\varphi),\sin(\varphi))$ to find

$$\begin{aligned}\int_{\mathbb{S}^1}f\Big(\frac{x+y}{|x+y|}\Big)\,\mathrm{d}\sigma_y&=\int_0^{2\pi}f\left(\frac{\gamma(\varphi)+x}{|\gamma(\varphi)+x|}\right)|\dot{\gamma}(\varphi)|\,\mathrm{d}\varphi\\ &=\int_0^{2\pi}f\left(\frac{\binom{\cos(\varphi)+\cos(\psi)}{\sin(\varphi)+\sin(\psi)}}{|\binom{\cos(\varphi)+\cos(\psi)}{\sin(\varphi)+\sin(\psi)}|}\right)\underbrace{|\dot{\gamma}(\varphi)|}_{=1}\,\mathrm{d}\varphi.\end{aligned}$$

Though geometrically clear, we now proceed to the formal manipulations on the arguments of the integrands. Using the trigonometric formulas

$$\begin{aligned}&\cos(\alpha-\beta)=\cos(\alpha)\cos(\beta)+\sin(\alpha)\sin(\beta),\qquad\alpha,\beta\in\mathbb{R},\\ &\cos^2(\alpha)=\frac{1}{2}(1+\cos(2\alpha)),\qquad\alpha,\beta\in\mathbb{R},\end{aligned}$$

we compute

$$\begin{aligned}&\left|\begin{pmatrix}\cos(\varphi)+\cos(\psi)\\ \sin(\varphi)+\sin(\psi)\end{pmatrix}\right|\\ &=\sqrt{\cos^2(\varphi)+\cos^2(\psi)+2\cos(\varphi)\cos(\psi)+\sin^2(\varphi)+\sin^2(\psi)+2\sin(\varphi)\sin(\psi)}\\ &=\sqrt{2}\sqrt{1+\cos(\varphi)\cos(\psi)+\sin(\varphi)\sin(\psi)}\\ &=\sqrt{2}\sqrt{1+\cos(\psi-\varphi)}=\sqrt{2}\sqrt{1+\cos\left(2\frac{\varphi-\psi}{2}\right)}=2\left|\cos\left(\frac{\varphi-\psi}{2}\right)\right|.\end{aligned}$$

On the other hand, using the addition theorems

$$\begin{aligned}&\cos(\alpha)+\cos(\beta)=2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right),\\ &\sin(\alpha)+\sin(\beta)=2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right),\qquad\alpha,\beta\in\mathbb{R},\end{aligned}$$

we eventually find

$$\frac{\binom{\cos(\psi)+\cos(\varphi)}{\sin(\psi)+\sin(\varphi)}}{\left|\binom{\cos(\varphi)+\cos(\psi)}{\sin(\varphi)+\sin(\psi)}\right|} = \frac{2}{2\left|\cos\left(\frac{\varphi-\psi}{2}\right)\right|}\begin{pmatrix} \cos\left(\frac{\psi+\varphi}{2}\right)\cos\left(\frac{\psi-\varphi}{2}\right) \\ \sin\left(\frac{\psi+\varphi}{2}\right)\cos\left(\frac{\psi-\varphi}{2}\right) \end{pmatrix}.$$

We slit the sphere into two parts, namely, where $\cos\left(\frac{\varphi-\psi}{2}\right) > 0$ and $\cos\left(\frac{\varphi-\psi}{2}\right) < 0$, and denote the respective set of angles by A_1 and A_2. Note that the set of angles φ where $\cos\left(\frac{\varphi-\psi}{2}\right) = 0$ is a nullset for the one–dimensional Lebesgue measure. On A_1 we then have

$$f\left(\frac{\binom{\cos(\psi)+\cos(\varphi)}{\sin(\psi)+\sin(\varphi)}}{\left|\binom{\cos(\varphi)+\cos(\psi)}{\sin(\varphi)+\sin(\psi)}\right|}\right) = f\left(\begin{pmatrix} \cos\left(\frac{\psi+\varphi}{2}\right) \\ \sin\left(\frac{\psi+\varphi}{2}\right) \end{pmatrix}\right)$$

and thus, changing variables $\vartheta = (\varphi+\psi)/2$, we see by positivity of f that

$$\begin{aligned}
\int_{A_1} f\left(\frac{\binom{\cos(\varphi)+\cos(\psi)}{\sin(\varphi)+\sin(\psi)}}{\left|\binom{\cos(\varphi)+\cos(\psi)}{\sin(\varphi)+\sin(\psi)}\right|}\right)\mathrm{d}\varphi &= \int_{A_1} f\left(\begin{pmatrix} \cos\left(\frac{\psi+\varphi}{2}\right) \\ \sin\left(\frac{\psi+\varphi}{2}\right) \end{pmatrix}\right)\mathrm{d}\varphi \\
&\leqslant 2\int_0^{2\pi} f\left(\begin{pmatrix} \cos(\vartheta) \\ \sin(\vartheta) \end{pmatrix}\right)\mathrm{d}\vartheta \\
&= 2\int_0^{2\pi} f\left(\underbrace{\begin{pmatrix} \cos(\vartheta) \\ \sin(\vartheta) \end{pmatrix}}_{=:\Gamma}\right)\left|\underbrace{\begin{pmatrix} -\sin(\vartheta) \\ \cos(\vartheta) \end{pmatrix}}_{=\dot{\Gamma}}\right|\mathrm{d}\vartheta \\
&\leqslant 2\int_{\mathbb{S}^1} f(z)\,\mathrm{d}\sigma_z
\end{aligned}$$

For A_2, we similarly write

$$\int_{A_2} f\left(\frac{\binom{\cos(\varphi)+\cos(\psi)}{\sin(\varphi)+\sin(\psi)}}{\left|\binom{\cos(\varphi)+\cos(\psi)}{\sin(\varphi)+\sin(\psi)}\right|}\right)\mathrm{d}\varphi \leqslant 2\int_0^{2\pi} f\left(-\begin{pmatrix} \cos(\vartheta) \\ \sin(\vartheta) \end{pmatrix}\right)\mathrm{d}\vartheta,$$

and since with ϑ running from 0 to 2π, $-(\cos(\vartheta), \sin(\vartheta))$ runs over $\mathbb{S}^1$, we equally have

$$\int_{A_2} f\left(\frac{\binom{\cos(\varphi)+\cos(\psi)}{\sin(\varphi)+\sin(\psi)}}{\left|\binom{\cos(\varphi)+\cos(\psi)}{\sin(\varphi)+\sin(\psi)}\right|}\right)\mathrm{d}\varphi \leqslant 2\int_{\mathbb{S}^1} f(z)\,\mathrm{d}\sigma_z.$$

as above. The application in the proof of Proposition 8.25 then follows by setting

$$f(z) := \int_{\mathbb{S}^1} \frac{|u(rx)-u(tz)|^p}{|rx-tz|^{n+sp}}\,\mathrm{d}\sigma_x$$

for fixed r and t.

List of Symbols

Symbol	Meaning
General Notation	
$\langle \cdot, \cdot \rangle$	Euclidean inner product
$\langle \cdot, \cdot \rangle_{X' \times X}$	Duality pairing
$\mathscr{F}$	Fourier transform
f^*	Fenchel conjugate of a function f
∂f	subdifferential of a function f
f^∞	recession function
$\mathcal{M}$	Hardy–Littlewood maximal operator
$\mathcal{M}^\#$	sharp maximal operator
$\mathcal{M}^\flat$	flat maximal operator
$\mathcal{M}_s$	fractional maximal operator
$\mathcal{M}_s^\#$	fractional sharp maximal operator
Measure Theory	
$\mathscr{B}(\Omega)$	Borel σ–algebra on Ω
$\mathcal{M}(\Omega; \mathbb{R}^m)$	$\mathbb{R}^m$–valued Radon measures on Ω
$\mathcal{M}_{<\infty}(\Omega; \mathbb{R}^m)$	finite $\mathbb{R}^m$–valued Radon measures on Ω
$\frac{\mathrm{d}\mu}{\mathrm{d}\nu}$	density of μ with respect to ν
Function Spaces	
L^p	Lebesgue space
L^φ	Orlicz space
C	continuous functions
C_0	continuous functions vanishing at the boundary
C_b	bounded continuous functions
$\mathfrak{P}, \mathcal{P}$	polynomials of arbitrary degree
$\mathfrak{P}_k, \mathcal{P}_k$	polynomials of degree at most k
Smoothness Spaces	
$\mathrm{C}^{m,\alpha}$	Hölder space
$\mathscr{S}$	Schwartz class
C_c^∞	Space of test functions
$\mathcal{L}_k^{p,\lambda}$	k–th order Campanato space
$\mathrm{L}_C^{p,\lambda}$, $\mathcal{L}^{p,\lambda}$	Campanato space
$\mathrm{L}_M^{p,\lambda}$	Morrey space
$\mathrm{B}_{p,q}^\alpha, \mathrm{B}_q^{\alpha,p}$	Besov space
$\mathrm{W}^{k,p}$	Sobolev space
$\mathrm{W}^{\alpha,p}$	Sobolev–Slobodeckjii space ($\alpha \in \mathbb{R}_{>0} \setminus \mathbb{N}$)
$\mathfrak{L}^{\alpha,p}$	Bessel potential space
$\mathcal{N}^{\alpha,p}$	Nikolskii space
$\mathscr{C}^{\alpha,p}$	(sharp) Calderón space
$\mathcal{C}^{\alpha,p}$	(flat) Calderón space
BMO	functions of bounded mean oscillation
BV	functions of bounded variation
BV_k	functions of bounded k–th order variation
BD	functions of bounded deformation

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And in the end
the love you take
is equal to
the love you make.
The Beatles