



# Perturbations of generators of $C_0$ -semigroups and resolvent decay

Charles J.K. Batty<sup>a</sup>, Sebastian Król<sup>b,\*</sup>,<sup>1</sup>

<sup>a</sup> St John's College, Oxford, OX4 1PY, England, UK

<sup>b</sup> Faculty of Mathematics and Computer Science, Nicolas Copernicus University, ul. Chopina 12/18, 87-100 Torun, Poland

## ARTICLE INFO

### Article history:

Received 9 September 2009

Available online 28 January 2010

Submitted by A. Lunardi

### Keywords:

Perturbation

Resolvent

$C_0$ -semigroup

## ABSTRACT

We obtain new stability results for those properties of  $C_0$ -semigroups which admit characterisation in terms of decay of resolvents of infinitesimal generators on vertical lines, e.g. analyticity, Crandall–Pazy differentiability or immediate norm continuity in the case of Hilbert spaces. As a consequence we get a generalisation of the Kato–Neuberger theorem on approximation of the identity. Finally, we present examples shedding a new light on resolvent characterisation of eventually differentiable  $C_0$ -semigroups for which differentiability is stable under bounded perturbations.

© 2010 Elsevier Inc. All rights reserved.

## 1. Introduction

In [20] R. Phillips initiated the study of permanence properties of  $C_0$ -semigroups under bounded perturbations. He proved, in particular, that properties like immediate norm continuity or immediate compactness persist under bounded perturbations. However, he also showed that this is not the case for eventual norm continuity or eventual differentiability of  $C_0$ -semigroups, see [20, Theorem 5.2].

Subsequently, many results were obtained showing that certain regularity properties of  $C_0$ -semigroups are preserved under natural classes of unbounded perturbations, see e.g. [5,6,22,2], [11, Sect. 13], [9, Sect. 3]. Recently, the permanence of immediate norm continuity under the Miyadera–Voigt, Desch–Schappacher and Batty–Kaiser–Weis classes of perturbations was treated by T. Mátrai in [16].

Note, however, that most of the results on permanence of semigroup properties obtained so far depend on some form of relative boundedness of perturbations with respect to semigroup generators and, in particular, impose domain restrictions.

In this paper we propose a new approach to the study of permanence of properties of  $C_0$ -semigroups which can be characterised by decay of resolvents of generators on vertical lines. The approach does not require any relative boundedness (domain) assumptions which are often too restrictive in applications.

Our stability conditions are formulated in terms of asymptotic behaviour of the difference of resolvents of two generators in the right half-plane or on the asymptotic behaviour of the difference of  $C_0$ -semigroups as  $t \rightarrow 0^+$ . More precisely, let  $\varphi : (0, \infty) \rightarrow (0, \infty)$  be such that  $\varphi(s) \rightarrow \infty$  as  $s \rightarrow \infty$  and let  $A$  be the generator of a  $C_0$ -semigroup on a Banach space  $X$  such that  $\{y_i : |y| > y_0\} \subset \rho(A)$  for some  $y_0 > 0$  and

$$M_{A,\varphi} := \limsup_{|y| \rightarrow \infty} \|\varphi(|y|)R(y_i, A)\| < \infty.$$

\* Corresponding author.

E-mail addresses: charles.batty@sjc.ox.ac.uk (C.J.K. Batty), rachman@mat.uni.torun.pl (S. Król).

<sup>1</sup> The second author was partially supported by the Marie Curie “Transfer of Knowledge” programme, project “TODEQ”, and by a MNiSzW grant No. N201384834.

In Corollary 2.4 of Section 2, which is a consequence of a more general result stated in Theorem 2.2, we show that if  $C$  is the generator of a  $C_0$ -semigroup  $S$  such that the difference of resolvents of  $A$  and  $C$  satisfies

$$\limsup_{|y| \rightarrow \infty} \|\varphi(|y|)[R(\varphi(|y|) + yi, A) - R(\varphi(|y|) + yi, C)]\| < \frac{1}{1 + M_{A,\varphi}},$$

then

$$\limsup_{|y| \rightarrow \infty} \|\varphi(|y|)R(yi, C)\| < \infty.$$

In particular, for  $\varphi(s) = s$  or  $\varphi(s) = s^\beta$ ,  $0 < \beta < 1$ , we obtain stability results for analytic  $C_0$ -semigroup or the Crandall–Pazy class of differentiable  $C_0$ -semigroups [4], respectively.

As a consequence of our approach we obtain an analogue of the well-known Kato–Neuberger theorem on approximation of the identity operator. Namely, if  $C_0$ -semigroups  $T$  and  $S$  are close enough to each other so that  $\limsup_{t \rightarrow 0} \|T(t) - S(t)\| < 1$ , then  $T$  and  $S$  share the same properties which allow description in terms of asymptotics of resolvents of their generators on vertical lines, see Corollary 2.5. For instance,  $T$  is analytic if and only if  $S$  is analytic, and, by a result of P. You [23] (see also [8]), if  $X$  is a Hilbert space, then  $T$  is immediately norm continuous if and only if  $S$  is immediately norm continuous.

In Section 3 we study the class  $\mathcal{IP}(X)$  of generators of eventually differentiable  $C_0$ -semigroups on a Banach space  $X$  for which differentiability of the semigroup is stable under bounded perturbations. This class was characterised by P. Iliey in [12]. It was also shown in [12] that the resolvent of  $A \in \mathcal{IP}(X)$  satisfies

$$\limsup_{|y| \rightarrow \infty} \|(\log |y|)^\delta R(yi, A)\| < \infty \quad \text{for every } \delta \in (0, 1),$$

see [12, Sect. 5]. Recasting the result by R. Chill and Y. Tomilov [3, Lemma 4.13] in the present setting one can infer that for  $A \in \mathcal{IP}(X)$  even stronger decay of resolvents holds, namely

$$\limsup_{|y| \rightarrow \infty} \left\| \frac{\log |y|}{\log \log |y|} R(yi, A) \right\| < \infty. \quad (1.1)$$

Using techniques due to M. Renardy [21] and B. Doytchinov, W. Hrusa and S. Watson [7], we show that the rate of decay of resolvents in (1.1) cannot be improved in general if  $A \in \mathcal{IP}(X)$ . Thus for  $A \in \mathcal{IP}(X)$  the estimate (1.1) is optimal, see Section 3 for more details.

## 2. Main results

In this section we extend the results from [15] on the stability of asymptotic behaviour of resolvents on vertical lines.

We start by introducing some notation and terminology which will be useful for the sequel. Throughout the paper let  $X$  be a complex Banach space and let  $\mathcal{L}(X)$  (resp.  $\mathcal{C}(X)$ ) be the set of all bounded (resp. closed) linear operators on  $X$ . Denote by  $\mathcal{G}(X)$  the set of generators of  $C_0$ -semigroups on  $X$ .

**Definition 2.1.** For a function  $\varphi : (a, \infty) \rightarrow (0, \infty)$ ,  $a > 0$ , and an operator  $A \in \mathcal{C}(X)$  we shall write  $A \sim \varphi$  if  $\{yi : |y| > y_0\} \subset \rho(A)$  for some  $y_0 > 0$  and

$$M_{A,\varphi} := \limsup_{|y| \rightarrow \infty} \|\varphi(|y|)R(yi, A)\| < \infty.$$

Furthermore, let  $\mathcal{G}_\varphi(X) := \{A \in \mathcal{G}(X) : A \sim \varphi\}$  and denote by  $\mathcal{G}_0(X)$  the set of generators of  $C_0$ -semigroups whose resolvents go to zero along the imaginary axis, i.e.,

$$\mathcal{G}_0(X) := \left\{ A \in \mathcal{G}(X) : A \sim \varphi \text{ for some } \varphi : (0, \infty) \rightarrow (0, \infty) \text{ with } \lim_{s \rightarrow \infty} \varphi(s) = \infty \right\}.$$

Note that if  $\varphi$  is given by  $\varphi(s) = s^\beta$ ,  $s > 0$ , then for  $0 < \beta < 1$  the set  $\mathcal{G}_\varphi(X)$  is contained in the Crandall–Pazy class of generators of differentiable  $C_0$ -semigroups and for  $\beta = 1$   $\mathcal{G}_\varphi(X)$  is the class of generators of holomorphic  $C_0$ -semigroups (see e.g. [1, Corollary 3.7.18]). Recall also that if  $\varphi(s) = \log s$ ,  $s > 1$ , and if  $A \sim \varphi$ , then a  $C_0$ -semigroup  $T$  generated by  $A$  is differentiable on  $(2M_{A,\varphi}, \infty)$  (see e.g. [18] and [13, Theorem 3.1.7]). Furthermore, if  $X$  is a Hilbert space then  $\mathcal{G}_0(X)$  is, in fact, the class of generators of immediately norm continuous  $C_0$ -semigroups (see [23] or [8]). We refer e.g. to [9, 1, 19, 4, 13] for more information concerning these classes and their role in the theory of evolution equations.

Moreover, set

$$M_{A,C,\varphi} := \limsup_{|y| \rightarrow \infty, y \in \mathbb{R}} \|\varphi(|y|)[R(\varphi(|y|) + yi, A) - R(\varphi(|y|) + yi, C)]\|$$

for  $A, C \in \mathcal{C}(X)$  with  $\{\lambda \in \mathbb{C} : \operatorname{Re} \lambda > \omega\} \subset \rho(A) \cap \rho(C)$  for some  $\omega \in \mathbb{R}$  and  $\varphi : (0, \infty) \rightarrow (0, \infty)$  such that  $\varphi(s) \rightarrow \infty$  as  $s \rightarrow \infty$ . Now we state our main perturbation result.

**Theorem 2.2.** Let  $A$  and  $C$  be closed linear operators on a Banach space  $X$  such that  $\{\lambda \in \mathbb{C}: \operatorname{Re} \lambda > \omega\} \subset \rho(A) \cap \rho(C)$  for some  $\omega \in \mathbb{R}$ . Let  $\varphi: (0, \infty) \rightarrow (0, \infty)$  be an arbitrary function such that  $\varphi(s) \rightarrow \infty$  as  $s \rightarrow \infty$ . If  $A \sim \varphi$  and

$$M_{A,C,\varphi} < \frac{1}{1 + M_{A,\varphi}}, \quad (2.1)$$

then  $C \sim \varphi$  and, moreover, if  $M_{A,C,\varphi} = 0$ , then  $M_{A,\varphi} = M_{C,\varphi}$ .

The proof of Theorem 2.2 is an immediate consequence of the following elementary lemma.

**Lemma 2.3.** Let  $X$  be a Banach space and let  $B \in \mathcal{L}(X)$  and  $D \in \mathcal{C}(X)$ . Then if  $0 \in \rho(I + B) \cap \rho(I + D)$  and  $\delta := \|(I + B)^{-1} - (I + D)^{-1}\| < \|I + B\|^{-1}$ , then  $D$  is bounded and

$$\|Dx\| \leq \frac{\|B\| + \delta\|I + B\|}{1 - \delta\|I + B\|} \|x\| \quad \text{for all } x \in \mathcal{D}(D).$$

The proof of Lemma 2.3 follows immediately from the following equality:

$$Dx = Bx - (I + B)((I + D)^{-1} - (I + B)^{-1})(I + D)x, \quad x \in \mathcal{D}(D).$$

**Proof of Theorem 2.2.** Without loss of generality we can assume that  $\omega < 0$ . Indeed, the properties that  $A \sim \varphi$  and  $C \sim \varphi$  and the values of  $M_{A,\varphi}$ ,  $M_{A,C,\varphi}$  are unchanged when  $A$ ,  $C$  and  $\varphi$  are replaced by  $A - \omega'$ ,  $C - \omega'$  and  $\varphi - \omega'$ , respectively, for  $\omega' > \omega$ .

By our assumptions there exists  $y_0 > 0$  such that

$$\sup_{|y| > y_0} \|\varphi(|y|)[R(\varphi(|y|) + yi, A) - R(\varphi(|y|) + yi, C)]\| < \frac{1}{1 + \sup_{|y| > y_0} \|\varphi(|y|)R(yi, A)\|}. \quad (2.2)$$

Note that for  $|y| > 0$

$$\begin{aligned} & \|\varphi(|y|)[R(\varphi(|y|) + yi, A) - R(\varphi(|y|) + yi, C)]\| \\ &= \|I - (yi - A)R(\varphi(|y|) + yi, A) - I + (yi - C)R(\varphi(|y|) + yi, C)\| \\ &= \|((\varphi(|y|) + yi - A)R(yi, A))^{-1} - ((\varphi(|y|) + yi - C)R(yi, C))^{-1}\| \\ &= \|(\varphi(|y|)R(yi, A) + I)^{-1} - (\varphi(|y|)R(yi, C) + I)^{-1}\| =: \delta_y. \end{aligned}$$

Hence by (2.2) we have that

$$\sup_{|y| > y_0} \delta_y < \frac{1}{\sup_{|y| > y_0} \|I + \varphi(|y|)R(yi, A)\|}.$$

Now by Lemma 2.3, applied to the operators  $B_y := \varphi(|y|)R(yi, A)$  and  $D_y := \varphi(|y|)R(yi, C)$ ,  $|y| > y_0$ , we get that

$$\limsup_{|y| \rightarrow \infty} \|\varphi(|y|)R(yi, C)\| \leq \frac{M_{A,\varphi} + M_{A,C,\varphi}(1 + M_{A,\varphi})}{1 - M_{A,C,\varphi}(1 + M_{A,\varphi})}.$$

Thus  $C \sim \varphi$ . Moreover,  $M_{C,\varphi} \leq M_{A,\varphi}$ , if  $M_{A,C,\varphi} = 0$ . Since  $M_{A,C,\varphi} = M_{C,A,\varphi}$ , it follows that  $M_{A,\varphi} = M_{C,\varphi}$ .  $\square$

The next corollary shows that Theorem 2.2 essentially characterises  $\mathcal{G}_\varphi(X)$ .

**Corollary 2.4.** Let  $X$  be a Banach space and let  $\varphi: (0, \infty) \rightarrow (0, \infty)$  be an arbitrary function such that  $\varphi(s) \rightarrow \infty$  as  $s \rightarrow \infty$ . If  $A \in \mathcal{G}_\varphi(X)$  and  $C \in \mathcal{G}(X)$ , then the following assertions are equivalent.

- (i)  $C \in \mathcal{G}_\varphi(X)$ .
- (ii) There exists  $\eta > 0$  such that  $M_{A,C,\eta\varphi} < \frac{1}{1 + M_{A,\eta\varphi}}$ .
- (iii) There exists a function  $\eta: (0, \infty) \rightarrow (0, \infty)$  such that  $\eta(\varepsilon) = O(\varepsilon)$  as  $\varepsilon \rightarrow 0^+$  and  $M_{A,C,\eta(\varepsilon)\varphi} < \varepsilon$  for every  $\varepsilon > 0$ .

**Proof.** (i)  $\Rightarrow$  (iii): Assume that  $A, C \in \mathcal{G}_\varphi(X)$ . Then there exists  $y_0 > 0$  such that  $\{yi: |y| > y_0\} \subset \rho(A) \cap \rho(C)$  and

$$\sup_{|y| > y_0} \|\varphi(|y|)R(yi, A)\| < M_{A,\varphi} + 1, \quad \sup_{|y| > y_0} \|\varphi(|y|)R(yi, C)\| < M_{C,\varphi} + 1.$$

Let  $M_1 := \max(M_{A,\varphi} + 1, M_{C,\varphi} + 1)$  and

$$M_2 := \max\left(\sup_{\operatorname{Re} \lambda > \omega} \|\operatorname{Re} \lambda R(\lambda, A)\|, \sup_{\operatorname{Re} \lambda > \omega} \|\operatorname{Re} \lambda R(\lambda, C)\|\right),$$

where  $\omega := \max(\omega(T) + 1, \omega(S) + 1)$  and  $\omega(T), \omega(S)$  denote the growth bounds of  $C_0$ -semigroups  $T, S$  generated by  $A$  and  $C$ , respectively. Set

$$\eta(\varepsilon) := \frac{\varepsilon}{2(M_2 + 1)M_1}, \quad \varepsilon > 0.$$

By the resolvent equation we have that

$$R(\lambda, A) = R(\operatorname{Im} \lambda i, A)(I - \operatorname{Re} \lambda R(\lambda, A)),$$

hence

$$\|R(\lambda, A)\| \leq \|R(\operatorname{Im} \lambda i, A)\| (1 + \|\operatorname{Re} \lambda R(\lambda, A)\|), \quad (2.3)$$

for every  $\lambda \in \mathbb{P} := \{z: \operatorname{Re} z > \omega, |\operatorname{Im} z| > y_0\}$ . Moreover, for every  $\varepsilon > 0$  there exists  $y_\varepsilon > y_0$  such that  $\eta(\varepsilon)\varphi(|y|) + yi \in \mathbb{P}$  for all  $|y| > y_\varepsilon$ . By (2.3) we obtain that

$$\|\eta(\varepsilon)\varphi(|y|)R(\eta(\varepsilon)\varphi(|y|) + yi, A)\| \leq \|\eta(\varepsilon)\varphi(|y|)R(yi, A)\|(M_2 + 1) < \frac{\varepsilon}{2}$$

for every  $|y| > y_\varepsilon$ . The same considerations show that

$$\|\eta(\varepsilon)\varphi(|y|)R(\eta(\varepsilon)\varphi(|y|) + yi, C)\| < \frac{\varepsilon}{2} \quad \text{for every } |y| > y_\varepsilon.$$

Therefore the function  $\eta$  satisfies the required conditions.

(iii)  $\Rightarrow$  (ii): Since  $\eta(\varepsilon)M_{A,\varphi} = M_{A,\eta(\varepsilon)\varphi} \rightarrow 0$  as  $\varepsilon \rightarrow 0$ ,

$$\varepsilon < \frac{1}{1 + M_{A,\eta(\varepsilon)\varphi}}$$

for small  $\varepsilon$ . The implication (ii)  $\Rightarrow$  (i) is an immediate consequence of Theorem 2.2.  $\square$

Recall that, according to Kato–Neuberger's theorem, if a  $C_0$ -semigroup  $S$  satisfies the condition  $\limsup_{t \rightarrow 0^+} \|S(t) - I\| < 2$ , then  $S$  is holomorphic (see [14] or [19, Sect. 2, Corollary 5.7] for the proof). This result was extended in [15, Corollary 3.4] to the following form. If  $A$  is the generator of a holomorphic  $C_0$ -semigroup  $T$  (i.e.  $A \sim \varphi$  with  $\varphi(s) := s, s > 0$ ) and  $C$  is the generator of a  $C_0$ -semigroup  $S$  such that

$$\limsup_{t \rightarrow 0^+} \|S(t) - T(t)\| < \frac{1}{\limsup_{t \rightarrow 0^+} \|(I + T(t))^{-1}\|} =: k(T), \quad (2.4)$$

then  $S$  is holomorphic (i.e.  $C \sim \varphi$  for  $\varphi(s) := s, s > 0$ ). As the next consequence of Theorem 2.2 we obtain an analogue of the Kato–Neuberger result for the class of  $C_0$ -semigroups generated by operators from  $\mathcal{G}_\varphi(X)$  with  $\varphi$  such that  $\varphi(s) \rightarrow \infty$  as  $s \rightarrow \infty$ . In particular, we show that in (2.4) one can replace the constant  $k(T)$  depending on  $T$  by the absolute constant 1.

**Corollary 2.5.** *Let  $T$  and  $S$  be  $C_0$ -semigroups on a Banach space  $X$  with generators  $A$  and  $C$ , respectively. Assume that*

$$\limsup_{t \rightarrow 0^+} \|T(t) - S(t)\| < 1. \quad (2.5)$$

*If  $A \in \mathcal{G}_0(X)$  or  $C \in \mathcal{G}_0(X)$ , then there exist  $M > 0$  and  $s_0 > 0$  such that*

$$\frac{1}{M} \|R(si, A)\| \leq \|R(si, C)\| \leq M \|R(si, A)\| \quad (2.6)$$

*for every  $|s| > s_0$ . In particular, for every function  $\varphi: (0, \infty) \rightarrow (0, \infty)$ ,  $A \in \mathcal{G}_\varphi(X)$  if and only if  $C \in \mathcal{G}_\varphi(X)$ .*

**Proof.** We only present the proof of the existence of a constant  $M > 0$  for which (2.6) holds for large positive values of  $s$ . Arguing similarly one can easily complete the proof.

Assume that  $A \in \mathcal{G}_0(X)$ . Let  $\varepsilon > 0$  be such that

$$\frac{1}{1 + \varepsilon} > \limsup_{t \rightarrow 0^+} \|T(t) - S(t)\|. \quad (2.7)$$

Let  $\varphi_A(s) := \frac{\varepsilon}{\|R(si, A)\|}$ ,  $s > y_0$ , for some  $y_0 > 0$ . Using the reasoning from the proof of [1, Proposition 4.1.3] one can show that

$$\limsup_{\operatorname{Re} \lambda \rightarrow \infty} \|\operatorname{Re} \lambda [R(\lambda, A) - R(\lambda, C)]\| \leq \limsup_{t \rightarrow 0^+} \|T(t) - S(t)\|. \quad (2.8)$$

By our assumption  $\varphi_A(s) \rightarrow \infty$  as  $s \rightarrow \infty$ , hence

$$M_{A,C,\varphi_A}^+ \leq \limsup_{\operatorname{Re} \lambda \rightarrow \infty} \|\operatorname{Re} \lambda [R(\lambda, A) - R(\lambda, C)]\| \quad (2.9)$$

where  $M_{A,C,\varphi_A}^+ := \limsup_{s \rightarrow \infty} \|\varphi_A(s)[R(\varphi_A(s) + si, A) - R(\varphi_A(s) + si, C)]\|$ .

Combining (2.9), (2.8) and (2.7) we get that

$$M_{A,C,\varphi_A}^+ < \frac{1}{1 + \varepsilon} = \frac{1}{1 + M_{A,\varphi_A}^+},$$

where  $M_{A,\varphi_A}^+ := \limsup_{s \rightarrow \infty} \|\varphi_A(s)R(si, A)\| = \varepsilon$ . By the obvious modification of the proof of Theorem 2.2 we obtain that  $M_{C,\varphi_A}^+ := \limsup_{s \rightarrow \infty} \|\varphi_A(s)R(si, C)\| < \infty$ . Therefore there exists  $y_1 > 0$  such that

$$\|R(si, C)\| \leq (M_{C,\varphi_A}^+ + 1)\varepsilon^{-1} \|R(si, A)\|, \quad s > y_1. \quad (2.10)$$

In particular,  $\|R(si, C)\| \rightarrow 0$  as  $s \rightarrow \infty$ . Set  $\varphi_C(s) := \frac{\varepsilon}{\|R(si, C)\|}$ ,  $s > y_1$ . Arguing similarly as above one can show that

$$\|R(si, A)\| \leq (M_{A,\varphi_C}^+ + 1)\varepsilon^{-1} \|R(si, C)\|, \quad s > y_2, \quad (2.11)$$

for some  $y_2 > 0$ . Combining (2.10) and (2.11) we see that there exists a constant  $M > 0$  such that (2.6) holds for  $s > \max(y_1, y_2)$ .  $\square$

**Remark 2.6.** Note that the constant 1 in (2.5) of Corollary 2.5 is optimal for every class  $\mathcal{G}_\varphi(X)$  with  $\varphi$  such that  $\varphi(s) \rightarrow \infty$  as  $s \rightarrow \infty$ . For  $\varphi(s) := s$ ,  $s > 0$ , this was mentioned in the remark before [15, Theorem 3.3]. However, the corresponding example was not given there explicitly. In general, we can also simply illustrate this optimality by multiplication operators.

Indeed, let  $\psi : \mathbb{R}_+ \rightarrow (0, \infty)$  be an arbitrary continuous function such that  $\psi(s) \rightarrow \infty$  as  $s \rightarrow \infty$ . Let  $A$  and  $C$  be the multiplication operators on  $X := L^2(\mathbb{R}_+)$  defined by the functions  $q(s) := -\psi(s) + si$ ,  $s \geq 0$  and  $p(s) := si$ ,  $s \geq 0$ , respectively. Since  $\|R(yi, A)\| = \frac{1}{\operatorname{dist}(yi, q(\mathbb{R}_+))}$ ,  $y \in \mathbb{R}$ ,  $A \in \mathcal{G}_0(X)$  and moreover we can control the rate of decay of resolvent of  $A$  on vertical lines by an appropriate choice of  $\psi$ . In other words, for a given function  $\varphi$  with  $\varphi(s) \rightarrow \infty$  as  $s \rightarrow \infty$ , there exists an appropriate function  $\psi$  such that  $A \in \mathcal{G}_0(X) \setminus \mathcal{G}_\varphi(X)$ .

However, for the  $C_0$ -semigroups  $T$  and  $S$  generated by  $A$  and  $C$ , respectively, we have that

$$\|T(t) - S(t)\| = \sup_{s \geq 0} |e^{tp(s)} - e^{tq(s)}| = \sup_{s \geq 0} |e^{sti}| |1 - e^{-t\psi(s)}| = 1,$$

for every  $t > 0$ .

Note also that for arbitrary generators  $A$  and  $C$  of holomorphic  $C_0$ -semigroups (2.6) holds for some  $M > 0$ .

The next corollary is an immediate consequence of Corollaries 2.4 and 2.5 and corresponding results on regularity properties of  $C_0$ -semigroups, see e.g. [1, Corollary 3.7.18], [19, Sect. 2, Theorem 4.9], [23,10].

**Corollary 2.7.** Let  $X$  be a Banach space and let  $\varphi : (0, \infty) \rightarrow (0, \infty)$  be a function such that  $\varphi(s) \rightarrow \infty$  as  $s \rightarrow \infty$ . Let  $T$  and  $S$  be  $C_0$ -semigroups on  $X$  with generators  $A$  and  $C$ , respectively. Assume that  $A \sim \varphi$  and

$$M_{A,C,\varphi} < \frac{1}{1 + M_{A,\varphi}} \quad \text{or} \quad \limsup_{t \rightarrow 0^+} \|T(t) - S(t)\| < 1.$$

Then the following assertions hold.

- (i) If  $\varphi(s) = s$ ,  $s > 0$ , then  $S$  is holomorphic.
- (ii) If  $\log s = o(\varphi(s))$  as  $s \rightarrow \infty$ , then  $S$  is immediately differentiable. In particular, if  $\varphi(s) = s^\beta$ ,  $s > 0$ , for some  $\beta \in (0, 1)$ , then  $S$  belongs to the Crandall–Pazy class of differentiable  $C_0$ -semigroups.
- (iii) If  $\varphi(s) = \log s$ ,  $s > 1$ , then  $S$  is eventually differentiable.
- (iv) If  $X$  is a Hilbert space, then  $S$  is immediately norm continuous.
- (v) If  $X = L_p(\Omega, \mu)$ , where  $p \in (1, \infty)$  and  $(\Omega, \mu)$  is a  $\sigma$ -finite measure space, and  $S$  is positive, then  $S$  is immediately norm continuous.

### 3. Permanence of differentiability of $C_0$ -semigroups

Let  $\mathcal{IP}(X)$  be the class of generators of eventually differentiable  $C_0$ -semigroups on a Banach space  $X$  for which differentiability is stable under bounded perturbations, i.e.,  $A \in \mathcal{IP}(X)$  if  $A + B$  generates an eventually differentiable  $C_0$ -semigroup for all  $B \in \mathcal{L}(X)$ .

Recently, P. Iley showed that  $A \in \mathcal{IP}(X)$  if and only if the norm of the resolvent of  $A$  is bounded on a particular exponential region, see [12, Theorem 4.1] for a precise formulation. In [12, Sect. 5.1] she also proved that if  $A \in \mathcal{IP}(X)$  then

$$M_{A, (\log)^\delta} := \limsup_{|y| \rightarrow \infty} \|(\log |y|)^\delta R(yi, A)\| < \infty \quad \text{for every } \delta \in (0, 1),$$

i.e.,  $\mathcal{IP}(X) \subset \bigcap_{\delta \in (0, 1)} \mathcal{G}_{\log^\delta}(X)$ . Note that [3, Lemma 4.13] provides better information on the rate of decay of the resolvent of  $A \in \mathcal{IP}(X)$ , namely

$$M_{A, \frac{\log}{\log \log}} := \limsup_{|y| \rightarrow \infty} \left\| \frac{\log |y|}{\log \log |y|} R(yi, A) \right\| < \infty.$$

Therefore, for every Banach space  $X$ ,

$$\mathcal{G}_{\log}(X) \subset \mathcal{IP}(X) \subset \mathcal{G}_{\frac{\log}{\log \log}}(X), \quad (3.1)$$

where the left inclusion is given in [18, Corollary 4.1].

Observe also that  $\mathcal{G}_{\log}(X) \subset \mathcal{IP}(X)$  is optimal in the following sense. For  $X := l_2(\mathbb{N}, \mathbb{C})$  and for an arbitrary function  $\varphi : (0, \infty) \rightarrow (0, \infty)$  such that  $\varphi(s) = o(\log(s))$  as  $s \rightarrow \infty$  we have that

$$\mathcal{G}_\varphi(X) \setminus \mathcal{IP}(X) \neq \emptyset, \quad (3.2)$$

see, e.g., [17, Sect. A-II, Example 1.28.b)]. Hence, a natural question arises. *Is  $\mathcal{IP}(X)$  equal to  $\mathcal{G}_{\log}(X)$ ?*

Remark that a positive answer would imply that all permanence results from the previous section are consequences of permanence of eventual differentiability under bounded perturbations.

In this section we show that the right inclusion in (3.1) is also optimal. More precisely, we prove that for  $X := l_2(\mathbb{N}, \mathbb{C})$  and arbitrary function  $\varphi : (0, \infty) \rightarrow (0, \infty)$ , such that  $\frac{\log(s)}{\log \log(s)} = o(\varphi(s))$  as  $s \rightarrow \infty$ ,

$$\mathcal{IP}(X) \setminus \mathcal{G}_\varphi(X) \neq \emptyset. \quad (3.3)$$

Note that (3.3) shows that  $\mathcal{IP}(X) \neq \mathcal{G}_{\log}(X)$ , in general, and that [3, Lemma 4.13], which is stated for arbitrary vector-valued holomorphic functions, cannot be improved. Moreover, combining (3.2) and (3.3) we see that, in general,  $\mathcal{IP}(X)$  cannot be characterised in terms of decay of resolvents on vertical lines, i.e.,  $\mathcal{IP}(X) \neq \mathcal{G}_\varphi$  for every function  $\varphi : (0, \infty) \rightarrow (0, \infty)$  such that  $\varphi(s) \rightarrow \infty$  as  $s \rightarrow \infty$ .

**Theorem 3.1.** *There exist a Hilbert space  $X$  and an operator  $A \in \mathcal{IP}(X)$  such that*

$$M_{A, \frac{\log}{\log \log}} > 0. \quad (3.4)$$

*In particular,  $A \notin \mathcal{G}_\varphi(X)$  whenever  $\frac{\log s}{\log \log s} = o(\varphi(s))$  as  $s \rightarrow \infty$ .*

In the proof we shall make use of Renardy's construction from [21] and some results due to Doytchinov, Hrusa and Watson from [7].

**Proof.** Let  $X := (\bigoplus_{n \in \mathbb{N}} l_2^{m(n)})_2$  and  $A := \bigoplus_{n \in \mathbb{N}} A_n$ , where

$$A_n := \begin{pmatrix} -n + ip(n) & 0 & \cdots & 0 & 0 \\ n & -n + ip(n) & \cdots & 0 & 0 \\ 0 & n & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -n + ip(n) & 0 \\ 0 & 0 & \cdots & n & -n + ip(n) \end{pmatrix}_{m(n) \times m(n)}$$

and  $m : \mathbb{N} \rightarrow \mathbb{N}$ ,  $p : \mathbb{N} \rightarrow (0, \infty)$ .

First we shall formulate conditions on  $m$  and  $p$  implying that  $A \in \mathcal{IP}(X)$ .

(i) If  $m$  and  $p$  satisfy

$$(1) \quad \lim_{n \rightarrow \infty} \frac{n}{\log p(n)} = \infty, \quad (2) \quad \lim_{n \rightarrow \infty} \frac{2^{m(n)} m(n)^{\frac{3}{2}}}{n p(n)} = 0, \quad (3) \quad \lim_{n \rightarrow \infty} \frac{p(n+1) - p(n)}{4n} > 1, \quad (3.5)$$

then the operator  $A$  generates an immediately differentiable  $C_0$ -semigroup on  $X$  (see the proof of [21, Theorem 1] for details).

(ii) Assuming (3.5), if there exists  $N \in \mathbb{N}$  such that

$$\log(m(n)2^{m(n)}(2n + p(n))) \leq 2 \log p(n) \quad (3.6)$$

for all  $n \in \mathbb{N}$ ,  $n \geq N$ , and  $\delta_N > 0$  such that

$$\sup_{n \in \mathbb{N}} \left( \log(m(n)2^{m(n)}(2n + p(n))) - \frac{nt}{2} \right) \leq \sup_{n \geq N} \left( \log(m(n)2^{m(n)}(2n + p(n))) - \frac{nt}{2} \right) \quad (3.7)$$

for all  $t \in (0, \delta_N)$ , then by [7, Lemma 1] one can show that

$$\log \|AT(t)\| \leq \frac{1}{2} \sup_{n \geq N} (4 \log p(n) - nt) \quad \text{for all } t \in (0, \delta_N), \quad (3.8)$$

where  $T$  denotes a  $C_0$ -semigroup generated by  $A$  (see also the proof of [7, Theorem 2, Claims 1 and 2]). Hence if, in addition,

$$\limsup_{t \rightarrow 0^+} \frac{t \sup_{n \geq N} (4 \log p(n) - nt)}{\log \frac{1}{t}} < \infty, \quad (3.9)$$

then by [7, Theorem 1] we have  $A \in \mathcal{IP}(X)$ .

Now we shall show that if the functions  $m$  and  $p$  satisfy the condition

$$\limsup_{n \rightarrow \infty} \frac{m(n) \log p(n)}{n \log \log p(n)} > 0, \quad (3.10)$$

then (3.4) holds for  $A$ .

Indeed, observe that

$$(ip(n) - A_n)^{-1} = \frac{1}{n} \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 1 & 1 & \cdots & 0 & 0 \\ 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \cdots & 1 & 0 \\ 1 & 1 & \cdots & 1 & 1 \end{pmatrix},$$

so

$$\begin{aligned} \|(ip(n) - A)^{-1}\|_{\mathcal{L}(X)} &= \sup_{k \in \mathbb{N}} \|(ip(n) - A_k)^{-1}\| \geq \|(ip(n) - A_n)^{-1}\| \\ &= \sup_{\|x\|_{l_2^{m(n)}}=1, x \in l_2^{m(n)}} \|(ip(n) - A_n)^{-1}x\|_{l_2^{m(n)}} \\ &\geq m(n)^{-\frac{1}{2}} \|(ip(n) - A_n)^{-1}(1, \dots, 1)\|_{l_2^{m(n)}} \\ &= \frac{1}{n\sqrt{m(n)}} \|(1, 2, \dots, m(n))\|_{l_2^{m(n)}} \\ &= \frac{1}{n\sqrt{m(n)}} \left( \sum_{k=1}^{m(n)} k^2 \right)^{\frac{1}{2}} = \frac{1}{n} \left( \frac{(m(n)+1)(2m(n)+1)}{6} \right)^{\frac{1}{2}} \end{aligned}$$

for all  $n \in \mathbb{N}$ . Therefore

$$\frac{1}{n} \left( \frac{(m(n)+1)(2m(n)+1)}{6} \right)^{\frac{1}{2}} \frac{\log p(n)}{\log \log p(n)} \leq \left\| \frac{\log p(n)}{\log \log p(n)} (ip(n) - A)^{-1} \right\|$$

for large  $n \in \mathbb{N}$  with  $p(n) > 1$ . Hence (3.10) implies (3.4).

By the above considerations it is sufficient to show that there exist functions  $m$  and  $p$  for which (3.5), (3.6), (3.7), (3.9) and (3.10) hold.

As an example of such functions we can consider, e.g.,  $m$  and  $p$  defined by

$$m(n) := [\tilde{m}(n + e^2)], \quad p(n) := \tilde{p}(n + e^2), \quad n \in \mathbb{N}, \quad (3.11)$$

where

$$\tilde{m}(x) := \sqrt{x}(\log \sqrt{x})^{\frac{1}{2}}, \quad \tilde{p}(x) := e^{\tilde{m}(x)} = e^{\sqrt{x}(\log \sqrt{x})^{\frac{1}{2}}}, \quad x > e^2.$$

We can also consider the functions used in the proof of [7, Theorem 3], i.e.,  $\tilde{m}(n) := [\frac{1}{4}nf^{-1}(n)]$  and  $\tilde{p}(n) := e^{\frac{1}{4}nf^{-1}(n)}$ ,  $n \in \mathbb{N}$ , where  $f(t) := \frac{2\log \frac{1}{t}}{t^2}$ ,  $t \in (0, \frac{1}{2}]$ .

The proof of (3.5)–(3.10) for functions  $m$  and  $p$  (or  $\tilde{m}$  and  $\tilde{p}$ ) is elementary, so for the convenience of the reader we include only its more technical parts. First note that in order to show that  $m$  and  $p$  satisfy (3.5)–(3.10) it is sufficient to prove that continuous versions of these conditions hold for  $\tilde{m}$  and  $\tilde{p}$ . The conditions (3.5)(1) and (3.5)(2) are immediate consequences of L'Hospital's rule. For (3.5)(3) note that

$$\frac{\tilde{p}(x+1) - \tilde{p}(x)}{x} = \frac{(e^{\tilde{m}(x+1) - \tilde{m}(x)} - 1) e^{\tilde{m}(x)} (\tilde{m}(x+1) - \tilde{m}(x))}{\tilde{m}(x+1) - \tilde{m}(x)} \frac{1}{x} \quad (3.12)$$

for  $x > e^2$ , where

$$\tilde{m}(x+1) - \tilde{m}(x) \leq \frac{\sqrt{x} \log \frac{x+1}{x}}{(\log \sqrt{x})^{\frac{1}{2}}} + \frac{(\log \sqrt{x+1})^{\frac{1}{2}}}{\sqrt{x+1}} \rightarrow 0 \quad \text{as } x \rightarrow \infty.$$

Hence

$$\lim_{x \rightarrow \infty} \frac{e^{\tilde{m}(x+1) - \tilde{m}(x)} - 1}{\tilde{m}(x+1) - \tilde{m}(x)} = 1.$$

For the second factor in (3.12) we have the following estimate:

$$\frac{e^{\tilde{m}(x)} (\tilde{m}(x+1) - \tilde{m}(x))}{x} \geq \frac{e^{\sqrt{x}}}{x^2} x (\tilde{m}(x+1) - \tilde{m}(x)),$$

where

$$x(\tilde{m}(x+1) - \tilde{m}(x)) \geq \frac{x^2(\log \sqrt{x+1} - \log \sqrt{x})}{2\sqrt{x+1}(\log \sqrt{x+1})^{\frac{1}{2}}} + \frac{x \log \sqrt{x+1}}{2\sqrt{x+1}(\log \sqrt{x+1})^{\frac{1}{2}}} \rightarrow \infty$$

as  $x \rightarrow \infty$ . Hence (3.5)(3) holds for  $m$  and  $p$ . Since

$$\lim_{x \rightarrow \infty} \frac{\tilde{m}(x) 2^{\tilde{m}(x)} (2x + \tilde{p}(x))}{\tilde{p}(x)^2} = \lim_{x \rightarrow \infty} \frac{\tilde{m}(x) 2^{\tilde{m}(x)} (2x + e^{\tilde{m}(x)})}{e^{2\tilde{m}(x)}} = 0,$$

so there exists  $N \in \mathbb{N}$  such that (3.6) holds for all  $n \geq N$ . Now the existence of  $\delta_N > 0$  such that (3.7) holds for all  $t \in (0, \delta_N)$  follows from the fact that the sequence  $(\log(m(n) 2^{m(n)} (2n + p(n))))_{n \in \mathbb{N}}$  is strictly increasing. Hence (3.8) holds for some  $N \in \mathbb{N}$  and  $\delta_N > 0$ .

Consider the function  $f : (e^2, \infty) \rightarrow \mathbb{R}$  given by

$$f(x) := 4\tilde{m}(x) - (x - e^2)t,$$

where  $t \in (0, \delta_N)$ . Then

$$f'(x) = \frac{2}{\sqrt{x}} \left( (\log \sqrt{x})^{\frac{1}{2}} + \frac{1}{2(\log \sqrt{x})^{\frac{1}{2}}} \right) - t$$

and the function

$$\varphi(x) := \frac{2}{\sqrt{x}} \left( (\log \sqrt{x})^{\frac{1}{2}} + \frac{1}{2(\log \sqrt{x})^{\frac{1}{2}}} \right), \quad x > e^2,$$

is positive, strictly decreasing and  $\varphi(x) \rightarrow 0$  as  $x \rightarrow \infty$ . Let  $0 < \delta < \min(\delta_N, \varphi(e^2))$ , then for every  $t \in (0, \delta)$  the point  $\varphi^{-1}(t)$  is the unique maximum point of the function  $f$ . Hence

$$\frac{1}{2} \sup_{n \geq N} (4 \log p(n) - nt) \leq \frac{1}{2} \sup_{x > e^2} f(x) = \frac{1}{2} f(\varphi^{-1}(t)). \quad (3.13)$$

Let  $x_t := \varphi^{-1}(t)$  for  $t \in (0, \delta)$ , then



$$\begin{aligned} \frac{tf(x_t)}{\log \frac{1}{t}} &= \frac{\varphi(x_t)f(x_t)}{\log \frac{1}{\varphi(x_t)}} = \frac{(4\sqrt{x_t}(\log \sqrt{x_t})^{\frac{1}{2}} - (x_t - e^2)\varphi(x_t))\varphi(x_t)}{\log \frac{1}{\varphi(x_t)}} \\ &= \frac{4\log \sqrt{x_t} + \frac{4e^2 \log \sqrt{x_t}}{x_t} + \frac{4e^2}{x_t} + \frac{e^2}{x_t \log \sqrt{x_t}} - \frac{1}{\log \sqrt{x_t}}}{\log \sqrt{x_t} - \log 2 - \log((\log \sqrt{x_t})^{\frac{1}{2}} + \frac{1}{2(\log \sqrt{x_t})^{\frac{1}{2}}})}. \end{aligned}$$

Since  $x_t \rightarrow \infty$  as  $t \rightarrow 0^+$ , one has

$$\lim_{t \rightarrow 0^+} \frac{tf(x_t)}{\log \frac{1}{t}} = 4.$$

By (3.13) we see that (3.9) holds, so  $A \in \mathcal{IP}(X)$ . It is straightforward to verify that (3.10) is true for the functions  $m$  and  $p$ .  $\square$

**Remark 3.2.** Denote by  $\widetilde{\mathcal{IP}}(X)$  the subset of  $\mathcal{IP}(X)$  containing generators of  $C_0$ -semigroups for which immediate differentiability is stable under bounded perturbations (see [12, Theorem 4.2] for a characterisation of this class). One can show that for  $m$  and  $p$  given by (3.11),

$$\lim_{n \rightarrow \infty} \frac{n}{\log p(n)} \left(1 - n^{-\frac{1}{m(n)}}\right) = 2.$$

It follows that the operator  $A$  constructed in the proof of Theorem 3.1 does not belong to  $\widetilde{\mathcal{IP}}(X)$  (see [7, Theorem 4]). We do not know whether there exists  $A \in \widetilde{\mathcal{IP}}(X)$  satisfying (3.4). However, by modification of the proof of [7, Theorem 2] one can show that for every function  $\gamma : (0, \infty) \rightarrow (0, \infty)$  with  $\gamma(s) \rightarrow \infty$  as  $s \rightarrow \infty$  there exists  $A_\gamma \in \widetilde{\mathcal{IP}}(l_2)$ , defined by Renardy's construction, such that  $A_\gamma \notin \mathcal{G}_{\frac{\gamma \log}{\log \log}}(l_2)$ . We provide here only a sketch of the construction of appropriate functions  $m_\gamma$  and  $p_\gamma$ . First note that without loss of generality we can additionally assume that  $\gamma$  is continuous, increasing and has the following properties:  $\gamma(s) = 1$  for all  $s \in (0, 1]$ ,  $\frac{1+\log}{\gamma}$  is increasing and  $\lim_{s \rightarrow \infty} \frac{\log s}{\gamma(s)} = \infty$ . Let  $f : (0, \infty) \rightarrow (0, \infty)$  be given by

$$f(t) := \frac{\log^+(\frac{1}{t}) + 1}{t^2(\gamma(\frac{1}{t}))^{\frac{1}{2}}}.$$

Then  $f$  is decreasing, continuous and onto  $(0, \infty)$ . Following [7] define functions  $m_\gamma$  and  $p_\gamma$  by  $m_\gamma(n) := [\frac{1}{4}nf^{-1}(n)]$  and  $p_\gamma(n) := e^{\frac{1}{4}nf^{-1}(n)}$  for  $n \geq 1$ . Elementary calculations show that these functions satisfy (3.5) and

$$\lim_{n \rightarrow \infty} \frac{m_\gamma(n) \log(p_\gamma(n)) \gamma(p_\gamma(n))}{n \log \log(p_\gamma(n))} = \infty,$$

i.e.,  $A_\gamma \notin \mathcal{G}_{\frac{\gamma \log}{\log \log}}(l_2)$ . Moreover using arguments from the proof of [7, Theorem 2] one can show that

$$\limsup_{t \rightarrow 0^+} \frac{t \log \|A_\gamma T_\gamma(t)\|}{\log \frac{1}{t}} = 0,$$

where  $T_\gamma$  denotes a  $C_0$ -semigroup generated by  $A_\gamma$ . Then [7, Theorem 1] implies that  $A_\gamma \in \widetilde{\mathcal{IP}}(l_2)$ .

## References

- [1] W. Arendt, C.J.K. Batty, M. Hieber, F. Neubrander, Vector-Valued Laplace Transforms and Cauchy Problems, Monogr. Math., vol. 96, Birkhäuser, Basel, 2001.
- [2] W. Arendt, A. Rhandi, Perturbation of positive semigroups, Arch. Math. 56 (1991) 107–119.
- [3] R. Chill, Y. Tomilov, Operators  $L^1(\mathbb{R}_+) \rightarrow X$  and the norm continuity problem for semigroups, J. Funct. Anal. 256 (2009) 352–384.
- [4] M.G. Crandall, A. Pazy, On the differentiability of weak solutions of a differential equation in Banach space, J. Math. Mech. 18 (1969) 1007–1016.
- [5] W. Desch, W. Schappacher, On relatively bounded perturbations of linear  $C_0$ -semigroups, Ann. Sc. Norm. Super. Pisa Cl. Sci. 11 (1984) 327–341.
- [6] W. Desch, W. Schappacher, Some perturbation results for analytic semigroups, Math. Ann. 281 (1988) 157–162.
- [7] B. Doytchinov, W. Hurs, S. Watson, On perturbations of differentiable semigroups, Semigroup Forum 54 (1997) 100–111.
- [8] O. El-Mennaoui, K.J. Engel, On the characterization of eventually norm continuous semigroups in Hilbert spaces, Arch. Math. 63 (1994) 437–440.
- [9] K.J. Engel, R. Nagel, One-Parameter Semigroups for Linear Evolution Equations, Grad. Texts in Math., vol. 194, Springer, 2000.
- [10] V. Goersmeyer, L. Weis, Norm continuity of  $C_0$ -semigroups, Studia Math. 134 (1999) 169–178.
- [11] E. Hille, R.S. Phillips, Functional Analysis and Semigroups, Amer. Math. Soc., Providence, RI, 1957.
- [12] P.S. Iley, Perturbations of differentiable semigroups, J. Evol. Equ. 7 (2007) 765–781.
- [13] P.S. Iley, Differentiability of  $C_0$ -semigroups, PhD thesis, Oriel College, University of Oxford, 2008.
- [14] T. Kato, A characterization of holomorphic semigroups, Proc. Amer. Math. Soc. 25 (1970) 495–498.
- [15] S. Król, Perturbations of holomorphic semigroups, J. Evol. Equ. 9 (2009) 449–468.

- [16] T. Mátrai, On perturbations preserving the immediate norm continuity of semigroups, J. Math. Anal. Appl. 341 (2008) 961–974.
- [17] R. Nagel (Ed.), One-Parameter Semigroups of Positive Operators, Lecture Notes in Math., vol. 1184, Springer, Berlin, 1984.
- [18] A. Pazy, On the differentiability and compactness of semi-groups of linear operators, J. Math. Mech. 17 (1968) 1113–1141.
- [19] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, Springer, Berlin, 1983.
- [20] R.S. Phillips, Perturbation theory for semi-groups of linear operators, Trans. Amer. Math. Soc. 74 (1953) 199–221.
- [21] M. Renardy, On the stability of differentiability of semigroups, Semigroup Forum 51 (1995) 343–346.
- [22] J. Voigt, On resolvent positive operators and positive  $C_0$ -semigroups on  $AL$ -spaces, Semigroup Forum 38 (1989) 263–266.
- [23] P. You, Characteristic conditions for a  $C_0$ -semigroups with continuity in the uniform operator topology for  $t > 0$  in Hilbert space, Proc. Amer. Math. Soc. 116 (1992) 991–997.