

Aspects of Massive Spin-2 Effective Field Theories



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Abstract

General relativity describes gravity in terms of an interacting massless spin-2 field—the graviton. This 100-year-old theory has been spectacularly successful in explaining observations. However, theoretical exploration and the cosmological constant problem motivate the study of alternative theories of gravity. Recently, there has been great progress in understanding theories that give the graviton a mass. This thesis considers several aspects of these massive spin-2 effective field theories and related theories.

These theories are first studied from the perspective of scattering amplitudes. The most general $2 \rightarrow 2$ scattering amplitude is constructed for theories containing a single massive graviton or vector. These amplitudes are then used to find the highest strong coupling scales in such theories, assuming a particular scaling of fields and momenta. Generalisations to include additional fields and self-interactions for massive higher-spin fields are also discussed.

Constraints that arise from the existence of an ultraviolet completion are then studied. It is shown using dispersion relation arguments that the pseudo-linear massive spin-2 theory cannot admit an analytic, Lorentz-invariant, and unitary ultraviolet completion, but that such completions are not ruled out for massive vector theories.

The behaviour of massive spin-2 theories under dimensional reduction is also explored. Stability conditions and the lower-dimensional spectrum are derived for the Kaluza-Klein dimensional reduction of a partially massless graviton and a massive graviton on an Einstein product manifold. Additionally, the nonlinear dimensional reduction of the zero modes in dRGT massive gravity is shown to produce a mass-varying massive gravity theory.

Lastly, attempts to construct a version of unimodular gravity containing a massive graviton are discussed. A candidate theory is proposed and is shown to have pathologies. Dimensional reduction is then used to generate massive spin-2 theories with noncanonical kinetic terms and auxiliary fields. These theories are shown to be equivalent to the Fierz-Pauli theory, which provides further evidence for the uniqueness of the kinetic term used in dRGT massive gravity.

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List of Abbreviations

dRGT	de Rham, Gabadadze, and Tolley
DGP	Dvali-Gabadadze-Porrati
EFT	Effective field theory
FLRW	Friedmann-Lemaître-Robertson-Walker
GR	General relativity
IR	Infrared
KK	Kaluza-Klein
QCD	Quantum Chromodynamics
UV	Ultraviolet

The effort to understand the universe is one of the very few things which lifts human life a little above the level of farce and gives it some of the grace of tragedy.

— Steven Weinberg, *The First Three Minutes*

1

Introduction

1.1 Gravity and fields

In fundamental particle physics, nature is described using the language of quantum fields. When formulating the theory of general relativity (GR), Einstein relied on geometrical notions, such as spacetime curvature and geodesics, but it is also useful to think of gravity in field theoretic terms. From this perspective, gravity is described by merely another dynamical field, albeit one that determines the background on which all other fields propagate and interact.

As a classical field theory, GR describes an interacting massless spin-2 field. When quantised, this will give rise to massless spin-2 particles, called gravitons. In the Standard Model of particle physics, there are also massive particles with spin 0, spin 1/2, and spin 1, and the massless spin 1 photon and gluons. The gross features of the world can be largely understood in terms of interactions between these fields of various spins, governed by the strict requirements of Lorentz invariance, locality, and quantum mechanical unitarity.

1.2 Current problems

However, this is not yet a complete picture. There are many outstanding issues in both gravitational physics and particle physics that hint that something is missing

from our description of nature. In the realm of gravity, there is the cosmological constant problem [1–5] and the fact that GR is not renormalisable. The cosmological constant problem arises from the fact that explaining the observed accelerated expansion of the universe with a cosmological constant Λ requires a large but incomplete cancellation between various expected particle physics contributions to the vacuum energy and a bare cosmological constant. More precisely, the observed value of Λ appears to be sensitive to ultraviolet (UV) deformations of physics, which violates the expected decoupling of physics at different energy scales that underlies the effectiveness of effective field theory (EFT). The fact that GR is not renormalisable means that it violates perturbative unitarity at Planckian energies. GR makes perfectly good sense as a low-energy EFT [6, 7], but the theory is not complete. For example, there is no weakly-coupled, predictive description of gravitational scattering at high energies. This problem of quantum gravity is an obstacle to interpreting GR as a fundamental theory valid up to arbitrarily high energies and the only known solution is string theory [8, 9].

In the realm of particle physics, one outstanding problem is the weak-scale hierarchy problem, which is due to the sensitivity of the Higgs boson mass to UV deformations of physics. There is also the problem of dark matter, which might be solved by altering gravity, particle physics, or both. Dark matter is inferred to exist through its gravitational interactions, but proposals to explain dark matter usually invoke beyond the Standard Model particle physics [10].

1.3 Motivations

The successes and challenges of understanding nature in terms of interacting fields suggest that it should be a profitable theoretical exercise to explore the space of allowed field theories. This is possible since Wigner showed that particles in Minkowski spacetime are classified by their spin and their mass [11]. Physical particles have mass that is zero or positive and spin that is an integer or half-integer, and continuous spin particles might also make sense [12]. This restricted set of particle types allows a systematic exploration of the space of interacting field theories.

Gravity is the focus of this thesis, so field theories describing spin-2 particles are the main theories of interest. There is a sense in which GR is the unique low-energy theory of an interacting massless spin-2 field [13–15], so the primary focus of this thesis is massive spin-2 theories. In curved spacetimes there are other possible particle types, such as partially massless spin-2 fields [16–26], which will also be discussed. If the massive spin-2 field is identified with the mediator of the gravitational force, then these theories give a mass to the gravitational field and are called massive gravity. When quantised, such theories are expected to give massive gravitons.¹ A phenomenologically viable graviton mass must be very small, so this entails modifying gravity in the infrared (IR). The cosmological constant is also an IR quantity, so there is some hope that a graviton mass could address the cosmological constant problem. One aspect of this is to use a graviton mass to screen a large bare cosmological constant through a degravitation mechanism [27–30], but concrete realisations of this have other difficulties [31]. An easier goal is to use the graviton mass to produce cosmic acceleration in the absence of a cosmological constant, which does seem possible [31, 32]. For a recent review of cosmology in massive gravity, see [33].

Even apart from fine-tuning problems, it is interesting on purely theoretical grounds to study spin-2 theories. Some interesting field-theoretic phenomena become possible only at high spins, with spin 2 often being a borderline case. For example, in nature there is a massless (or very light) spin-2 particle, the graviton, but we do not observe massless particles with spin greater than two. As discussed in Chapter 3, this is because higher-spin gauge symmetry is incompatible with the universality of flat spacetime gravitational minimal couplings. As another example, consider massive particles; the Standard Model of particle physics contains elementary massive spin-1 particles, the W^\pm and Z^0 bosons, but there are no elementary massive particles with spin 2 or greater. There are hadronic resonances with mass and high spin, described by nonperturbative quantum chromodynamics (QCD), but they do not admit a

¹In this thesis I will often refer to any spin-2 field as a graviton, even when considering classical fields. This is justified since ultimately we have to quantise gravity, given that gravity interacts with quantum matter.

local point-like description for a parametrically large energy range since their masses are above the QCD strong coupling scale. In tree-level string theory, the higher vibrational modes of the string give rise to massive particles with all spins [8]. These higher-spin particles enter in Regge trajectories containing infinitely many particles. In fact, having one massive higher-spin particle requires having infinitely many of them in theories that are well-behaved at high energies [34], which is not the case for lower-spin particles. More primitively, just writing down free Lagrangians for massive particles requires additional auxiliary fields when the spin is greater than two [35, 36], making it more difficult to construct consistent interactions.

Lastly, it is interesting phenomenologically to bound the mass of the graviton. It would be surprising if finite observations could show that the graviton mass is precisely zero, but upper bounds can be placed. As an example, the recent detection of gravitational waves by the LIGO Scientific Collaboration and Virgo Collaboration bounds the graviton mass to be $m \leq 1.2 \times 10^{-22}$ eV [37, 38]. For a recent review of graviton mass bounds, see [39]. In the quest to explain the world, it is important to discover the fundamental properties of particles, such as the graviton mass, and a proper theoretical understanding is a prerequisite for making such discoveries.

1.4 Spin-2 effective field theories

Massive gravity has an interesting history, which will now be briefly reviewed as a way to introduce some important ideas. The linear theory of a massive spin-2 field was discovered by Fierz and Pauli in 1939 [40]. In 1970, it was pointed out by van Dam and Veltman, and Zakharov that the massless limit of the massive spin-2 propagator is not equal to the massless spin-2 propagator [41, 42]. This was thought to demonstrate that there is a discontinuous difference between massive and massless gravity, which would be enough to experimentally rule out a graviton mass. However, the continuity between observables in massive and massless gravity can be restored by considering interactions, as demonstrated by Vainshtein in 1972 [43]. The Vainshtein screening mechanism shows that nonlinearities can become important near a source and they can restore continuity with the massless theory

by screening the effects of the extra degrees of freedom. Later in 1972, Boulware and Deser argued that nonlinearities would also inevitably introduce ghosts into the theory [44]. Ghosts are fields that have a wrong sign kinetic term in the Lagrangian and they correspond to instabilities in a theory if the resulting particles appear in the spectrum. It was believed for a long time that any Lorentz-invariant massive gravity theory would contain ghosts [44, 45], but in 2010 de Rham, Gabadadze, and Tolley (dRGT) succeeded in finding four-dimensional Lorentz-invariant theories that are ghost-free up to quartic order and in a decoupling limit [46, 47]. It was later shown that the ghost is absent to all orders [48, 49]. These theories, called dRGT massive gravity or ghost-free massive gravity, will feature heavily in this thesis.

The dRGT massive gravity theories are not renormalisable. This means that they should be treated as EFTs with some finite range of validity, but within this range they can make good sense as predictive quantum theories. The energy scale at which perturbative unitarity breaks down in an EFT is called the strong coupling scale. For dRGT massive gravity in four dimensions, the strong coupling scale is $\Lambda_3 \equiv (m^2 M_p)^{1/3}$, where m is the graviton mass and M_p is the Planck mass. At the strong coupling scale, perturbative scattering amplitudes become of order one and violate perturbative unitarity bounds. This does not necessarily mean that the theory violates unitarity at this scale, only that perturbation theory is breaking down. If the theory is to remain weakly coupled and unitary, then new degrees of freedom must enter before the strong coupling scale to soften amplitudes at high energies. An alternative is that strong coupling dynamics become important, but in this case it is difficult to calculate anything. The same is true for GR, which is not renormalisable and becomes strongly coupled at the Planck scale [6, 7]. For more on EFTs, see [50, 51] and references therein.

There are interesting effective theories other than dRGT massive gravity that contain massive spin-2 fields, such as the pseudo-linear massive spin-2 theory [52, 53] and partially massless gravity [16–26], both of which will be discussed in this thesis. Some examples of theories that will not be discussed much in this thesis are bigravity [54], multi-gravity [55], Lorentz-violating massive gravity [56],

massive gravity in three dimensions [57, 58], and the Dvali-Gabadadze-Porrati (DGP) braneworld model [59].

1.5 Scattering amplitudes and extra dimensions

When trying to build or understand a theory, there are several ways to proceed. Often it is best to construct an action, by integrating local products of fields over spacetime, and then to use this to evaluate a path integral or to find the equations of motion for the theory. A complementary approach is to directly study the S -matrix. The S -matrix encodes the quantum mechanical probabilities for the scattering of asymptotically free particle states in flat spacetime. Often scattering amplitudes can be written down directly without recourse to a Lagrangian description, making it easier to study certain aspects of a theory. Since S -matrix elements deal directly with the physical degrees of freedom, they avoid many of the redundancies inherent to a Lagrangian description. The first half of this thesis, Chapters 3 and 4, will make use of scattering amplitudes to better understand massive spin-2 EFTs.

Another fruitful way to construct and study theories is by increasing the number of spacetime dimensions. Kaluza and Klein found that four-dimensional GR and electromagnetism could be unified by dimensionally reducing GR in five dimensions [60, 61]. This reduction also results in an infinite tower of massive gravitons of increasing mass, which indicates a deep connection between extra dimensions and massive particles. The second half of this thesis, Chapters 5 and 6, will explore some of the connections between extra dimensions and massive spin-2 EFTs.

1.6 Overview of this thesis

Notation and conventions are presented at the end of this chapter and the remainder of the thesis is divided into six chapters. Chapter 2 contains a review of spin-2 theories and the Galileons. This serves to introduce the main theories of interest for this thesis and sets some further notation. Chapter 3 studies massive spin-2 theories from the perspective of the S -matrix. The most general $2 \rightarrow 2$ scattering

amplitude for a massive spin-2 particle is constructed and the highest possible strong coupling scale is determined. Extensions to higher-spin fields and couplings to additional fields are also discussed. Chapter 4 then considers constraints on massive spin-2 and spin-1 theories from analytic dispersion relations, assuming the existence of an analytic UV completion. Chapter 5 studies theories obtained by a Kaluza-Klein (KK) dimensional reduction of massive and partially massless spin-2 theories. The general spectrum is determined for an Einstein product manifold and the nonlinear zero-mode reduction of dRGT massive gravity on a circle is presented. Chapter 6 then explores whether or not there is a massive modification of unimodular gravity by studying a candidate theory and dimensionally reducing various linear massless theories. Chapter 7 contains the conclusions.

1.7 Notation and conventions

The mostly plus metric signature $\eta_{\mu\nu} = \text{diag}(-, +, +, +, \dots)$ is always used. Natural units where $\hbar = c = 1$ are employed throughout. $M_p = 1/\sqrt{8\pi G}$ denotes the reduced Planck mass. Spacetime indices are usually labelled with Greek letters, μ, ν , etc. The trace of a rank-two tensor $h_{\mu\nu}$ is denoted by h and the Einstein summation convention is used. Indices are symmetrised or antisymmetrised with weight one, e.g., $\partial_{[\mu}A_{\nu]} = (\partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu})/2$. The Riemann tensor is defined by $R^{\mu}{}_{\nu\rho\lambda} = \partial_{\rho}\Gamma^{\mu}_{\nu\lambda} + \dots$ and the Ricci tensor by $R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu}$. The scales Λ_n are defined by $\Lambda_n \equiv (m^{n-1}M_p)^{1/n}$.

2

Spin-2 Theories and the Galileons

This chapter reviews the basics of the field theories that will be used often throughout this thesis. Various linear spin-2 theories are reviewed in Section 2.1, followed by a review of interacting spin-2 theories in Section 2.2, and lastly the scalar Galileons are discussed in Section 2.3.

2.1 Linear spin-2 theories

Linear theories describe the free propagation of particles and are an essential foundation before adding interactions. In this thesis, massless and massive spin-2 degrees of freedom will always be embedded in a rank-two tensor field $h_{\mu\nu}$, although embeddings in other fields are possible [62]. Various linear spin-2 theories will also be discussed in Chapter 6.

2.1.1 Massless spin 2

The linear theory of a massless spin-2 field in flat spacetime is given by the Lagrangian

$$\mathcal{L} = -\frac{1}{2}\partial_\lambda h_{\mu\nu}\partial^\lambda h^{\mu\nu} + \partial_\mu h_{\nu\lambda}\partial^\nu h^{\mu\lambda} - \partial_\mu h^{\mu\nu}\partial_\nu h + \frac{1}{2}\partial_\lambda h\partial^\lambda h. \quad (2.1)$$

The action constructed from the Lagrangian (2.1) is invariant under linearised diffeomorphisms

$$\delta h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu, \quad (2.2)$$

for a vector gauge parameter ξ_μ . The coefficients of (2.1), up to an overall scaling of $h_{\mu\nu}$, are completely fixed by the gauge symmetry (2.2). In d dimensions this theory describes $d(d-3)/2$ degrees of freedom, which is the correct number for the massless spin-2 representation. Another Lagrangian that describes a massless spin-2 field with a rank-two tensor is discussed in Chapter 6.

The most general background metric $\bar{g}_{\mu\nu}$ on which a massless graviton can propagate is an Einstein space, a space that solves the vacuum Einstein equations with a cosmological constant Λ (see below) [63, 64]. The massless spin-2 Lagrangian on such a space is

$$\mathcal{L} = -\frac{1}{2} \nabla_\alpha h_{\mu\nu} \nabla^\alpha h^{\mu\nu} + \nabla_\alpha h_{\mu\nu} \nabla^\nu h^{\mu\alpha} - \nabla_\mu h \nabla_\nu h^{\mu\nu} + \frac{1}{2} \nabla_\mu h \nabla^\mu h + \frac{R}{d} (h_{\mu\nu} h^{\mu\nu} - \frac{1}{2} h^2), \quad (2.3)$$

where indices are raised with the background metric $\bar{g}_{\mu\nu}$, covariant derivatives are those of $\bar{g}_{\mu\nu}$, and the overall factor of $\sqrt{-\bar{g}}$ is omitted. The action constructed from this Lagrangian is invariant under the gauge symmetry

$$\delta h_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu. \quad (2.4)$$

2.1.2 Massive spin 2

The Lagrangian for a massive spin-2 field in flat spacetime was first constructed by Fierz and Pauli [40]. The Fierz-Pauli Lagrangian is given by

$$\mathcal{L}_{\text{FP}} = -\frac{1}{2} \partial_\lambda h_{\mu\nu} \partial^\lambda h^{\mu\nu} + \partial_\mu h_{\nu\lambda} \partial^\nu h^{\mu\lambda} - \partial_\mu h^{\mu\nu} \partial_\nu h + \frac{1}{2} \partial_\lambda h \partial^\lambda h - \frac{1}{2} m^2 (h_{\mu\nu} h^{\mu\nu} - h^2). \quad (2.5)$$

The relative tuning between $h_{\mu\nu} h^{\mu\nu}$ and h^2 is necessary to avoid an additional ghostly degree of freedom. This describes the $(d+1)(d-2)/2$ degrees of freedom of the massive spin-2 representation in d dimensions, which is equal to the number of degrees of freedom of a massless spin-2 field in $d+1$ dimensions.

A massive spin-2 field in curved spacetime can be obtained by adding the Fierz-Pauli mass term to (2.3),

$$\begin{aligned} \mathcal{L} = & -\frac{1}{2}\nabla_\alpha h_{\mu\nu}\nabla^\alpha h^{\mu\nu} + \nabla_\alpha h_{\mu\nu}\nabla^\nu h^{\mu\alpha} - \nabla_\mu h\nabla_\nu h^{\mu\nu} + \frac{1}{2}\nabla_\mu h\nabla^\mu h \\ & + \frac{R}{d}(h_{\mu\nu}h^{\mu\nu} - \frac{1}{2}h^2) - \frac{1}{2}m^2(h_{\mu\nu}h^{\mu\nu} - h^2). \end{aligned} \quad (2.6)$$

For generic values of m , this theory describes the same number of degrees of freedom as (2.5). There is a special nonzero value of the mass,

$$m^2 = \frac{R(d-2)}{d(d-1)} = \frac{2\Lambda}{d-1}, \quad (2.7)$$

for which there are fewer degrees of freedom, corresponding to the partially massless theory described below. The massive graviton is unstable for masses less than (2.7), which is called the Higuchi bound [18].

2.1.3 Partially massless spin 2

When the graviton mass takes the special value (2.7), the Lagrangian (2.6) acquires a scalar gauge invariance that removes the scalar helicity mode of the massive graviton,

$$\delta h_{\mu\nu} = \nabla_\mu \nabla_\nu \alpha + \frac{m^2}{d-2} \bar{g}_{\mu\nu} \alpha, \quad (2.8)$$

where $\bar{g}_{\mu\nu}$ is the background metric and α is a scalar gauge parameter. This describes a partially massless spin-2 field, which in four dimensions has four degrees of freedom. This theory only exists on curved backgrounds and is stable for $R > 0$. Partially massless fields also exist for higher spins. For spin- J there are $J - 1$ different partially massless fields, corresponding to fields with different amounts of gauge symmetry [16–26].

The spin-2 partially massless graviton is of interest due to a possible connection with the cosmological constant and cosmology [65, 66], since the partially massless gauge symmetry (2.8) ties the value of the cosmological constant to the graviton mass, which can be small in a technically natural way. There have been many studies of the properties of the theory and possible nonlinear extensions, although currently no unitary nonlinear theory is known and there are many constraints on such a theory [65–88].

2.2 Nonlinear spin-2 theories

This section reviews interacting spin-2 theories. Only the interactions appropriate to four dimensions will be written down for the massive theories, although there are natural generalisations to higher dimensions, such as the five-dimensional version of dRGT massive gravity discussed in Chapter 5.

2.2.1 GR

The Lagrangian for GR with a cosmological constant Λ is

$$\mathcal{L} = \frac{M_p^2}{2} \sqrt{-g} (R - 2\Lambda), \quad (2.9)$$

where R is the Ricci scalar and g is the determinant of the metric. This is called the Einstein-Hilbert Lagrangian and describes the two interacting degrees of freedom of a massless spin-2 field in four dimensions. The equations of motion following from this Lagrangian are

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 0, \quad (2.10)$$

which are called the vacuum Einstein equations. In d dimensions, the vacuum Einstein equations are equivalent to

$$R_{\mu\nu} = \frac{R}{d} g_{\mu\nu}, \quad \Lambda = \frac{(d-2)}{2d} R, \quad (2.11)$$

and metrics $\bar{g}_{\mu\nu}$ that solve these equations define Einstein spaces. Linearisation of the Einstein-Hilbert Lagrangian around a Minkowski background, $g_{\mu\nu} = \eta_{\mu\nu} + 2h_{\mu\nu}/M_p$, gives the linear theory (2.1) and linearising around an Einstein space, $g_{\mu\nu} = \bar{g}_{\mu\nu} + 2h_{\mu\nu}/M_p$, gives the linear theory (2.3).

2.2.2 dRGT massive gravity

A nonlinear theory of massive gravity can be obtained by adding a zero-derivative potential term to the Einstein-Hilbert Lagrangian,

$$\mathcal{L} = \frac{M_p^2}{2} \sqrt{-g} \left(R - \frac{1}{4} m^2 V(g, h) \right), \quad (2.12)$$

where $h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$ is the massive spin-2 field, $g_{\mu\nu}$ is a dynamical metric, and $\eta_{\mu\nu}$ is a fiducial Minkowski metric. The fiducial metric is necessary to write down local interactions for a massive graviton. The potential can be expanded order by order in $h_{\mu\nu}$ as

$$V(g, h) = V_2(g, h) + V_3(g, h) + V_4(g, h) + \dots, \quad (2.13)$$

where

$$V_2(g, h) = \langle h^2 \rangle - \langle h \rangle^2, \quad (2.14a)$$

$$V_3(g, h) = c_1 \langle h^3 \rangle + c_2 \langle h^2 \rangle \langle h \rangle + c_3 \langle h \rangle^3, \quad (2.14b)$$

$$V_4(g, h) = d_1 \langle h^4 \rangle + d_2 \langle h^3 \rangle \langle h \rangle + d_3 \langle h^2 \rangle^2 + d_4 \langle h^2 \rangle \langle h \rangle^2 + d_5 \langle h \rangle^4. \quad (2.14c)$$

Angled brackets denote tracing with the dynamical metric, e.g.

$$\langle h^3 \rangle = h_\mu^\alpha h_\alpha^\nu h_\nu^\mu = g^{\beta\alpha} g^{\gamma\nu} g^{\mu\sigma} h_{\mu\beta} h_{\alpha\gamma} h_{\nu\sigma}. \quad (2.15)$$

There is no linear tadpole term in (2.13) so that Minkowski spacetime is a background solution and the quadratic term has been chosen so that the linearised theory is given by (2.5). For generic choices of coefficients, this theory describes six degrees of freedom, corresponding to five massive spin-2 degrees of freedom and a Boulware-Deser scalar ghost [44]. The generic theory becomes strongly coupled at the energy scale $\Lambda_5 \equiv (m^4 M_p)^{1/5}$ [89, 90].

It turns out that it is possible to remove the Boulware-Deser ghost by choosing special coefficients in (2.14) [46]. Up to quartic order, the ghost-free coefficients are given by

$$c_1 = 2c_3 + \frac{1}{2}, \quad c_2 = -3c_3 - \frac{1}{2}, \quad (2.16)$$

$$d_1 = -6d_5 + \frac{1}{16}(24c_3 + 5), \quad d_2 = 8d_5 - \frac{1}{4}(6c_3 + 1),$$

$$d_3 = 3d_5 - \frac{1}{16}(12c_3 + 1), \quad d_4 = -6d_5 + \frac{3}{4}c_3. \quad (2.17)$$

It is possible to resum the ghost-free expansion and compactly write the full nonlinear theory as

$$\mathcal{L}_{\text{dRGT}} = \frac{M_p^2}{2} \sqrt{-g} \left(R - 2\Lambda - \frac{m^2}{4} \sum_{n=0}^4 \beta_n e_n \left(\sqrt{g^{-1}\eta} \right) \right), \quad (2.18)$$

where $\eta_{\mu\nu}$ is the flat fiducial metric and e_n are the symmetric polynomials, given by

$$e_0(M) = 1, \quad (2.19a)$$

$$e_1(M) = [M], \quad (2.19b)$$

$$e_2(M) = \frac{1}{2!} \left([M]^2 - [M^2] \right), \quad (2.19c)$$

$$e_3(M) = \frac{1}{3!} \left([M]^3 - 3[M][M^2] + 2[M^3] \right), \quad (2.19d)$$

$$e_4(M) = \frac{1}{4!} \left([M]^4 - 6[M]^2[M^2] + 8[M][M^3] + 3[M^2]^2 - 6[M^4] \right), \quad (2.19e)$$

where $[M]$ denotes the trace of $M^\mu{}_\nu$ [47]. This theory is called dRGT massive gravity after its discoverers, de Rham, Gabadadze, and Tolley.

To have a flat spacetime solution when $\Lambda = 0$, the tadpole term coming from the β 's in (2.18) must cancel, which requires that

$$\beta_0 + 3\beta_1 + 3\beta_2 + \beta_3 = 0. \quad (2.20)$$

Additionally, the graviton mass is given by m if the β 's satisfy

$$\beta_1 + 2\beta_2 + \beta_3 = 8. \quad (2.21)$$

The term proportional to β_4 does not contribute to the equations of motion, so overall the theory has two free parameters in addition to the graviton mass and the cosmological constant. The matrix square root in (2.18) is well defined in perturbation theory, but in general the existence of a real matrix square root is not guaranteed [91].

Another useful form for the dRGT action is

$$\mathcal{L}_{\text{dRGT}} = \frac{M_p^2}{2} \sqrt{-g} \left(R - 2\Lambda + 2m^2 \sum_{n=0}^4 \alpha_n e_n(\mathcal{K}) \right), \quad (2.22)$$

where $\mathcal{K} \equiv \mathbb{I}_4 - \sqrt{g^{-1}\eta}$. The β_n and α_n coefficients are related by

$$\alpha_0 = -\frac{1}{8} (\beta_0 + 4\beta_1 + 6\beta_2 + 4\beta_3 + \beta_4), \quad (2.23)$$

$$\alpha_1 = \frac{1}{8} (\beta_1 + 3\beta_2 + 3\beta_3 + \beta_4), \quad (2.24)$$

$$\alpha_2 = -\frac{1}{8} (\beta_2 + 2\beta_3 + \beta_4), \quad (2.25)$$

$$\alpha_3 = \frac{1}{8} (\beta_3 + \beta_4), \quad (2.26)$$

$$\alpha_4 = -\frac{1}{8} \beta_4. \quad (2.27)$$

One redundant combination of the terms $e_i(\mathcal{K})$ can be used to set $\alpha_0 = 0$ and then cancelling the tadpole contribution when $\Lambda = 0$ requires setting $\alpha_1 = 0$. The graviton mass is given by m when $\alpha_2 = 1$, so in this formulation the two free parameters are α_3 and α_4 .

The dRGT ghost-free theory is an EFT that becomes strongly coupled at the energy scale $\Lambda_3 \equiv (m^2 M_p)^{1/3}$. This energy scale is higher than the strong coupling scale Λ_5 that results from a generic potential. Although this Lagrangian is ghost free as a classical field theory, it is properly thought of as a quantum EFT and hence there are an infinite number of additional interactions generated by quantum corrections at the scale Λ_3 .

The theory (2.18) was proven to be classically ghost free in references [48, 49]. The fiducial metric can be generalised to an arbitrary metric $f_{\mu\nu}$ and the theory remains ghost free [92]. Other proofs of ghost freedom can be found in the references [93–96]. The fiducial metric can also be made dynamical by giving it its own Einstein-Hilbert term. This results in a theory of ghost-free bigravity, sometimes called Hassan-Rosen bigravity, which describes a single interacting massive spin-2 field and a massless spin-2 field [54, 97]. Additional dynamical metrics can also be added and made to interact, giving multi-metric theories [55]. These theories have a simple form when written using vielbein fields and describe multiple massive spin-2 fields interacting with each other and with a single massless spin-2 field, although there are additional subtleties when there are cycles of interactions [55, 98]. For reviews of massive gravity, see [99, 100].

2.2.3 Pseudo-linear massive spin-2

The derivative interactions in dRGT massive gravity are given by the nonlinear Einstein-Hilbert kinetic term. It is also possible to construct an interacting theory of a massive spin-2 field that is instead based on the linear Fierz-Pauli theory (2.5), without the nonlinear derivative interactions of the Einstein-Hilbert term. This gives the so-called pseudo-linear theory of massive spin 2 [52, 53].

In four dimensions there are three possible pseudo-linear terms that can be added to (2.5) such that the total number of degrees of freedom is unchanged,

$$\mathcal{L} = \mathcal{L}_{\text{FP}} + \frac{1}{M_p} \lambda_1 \mathcal{L}_{2,3} + \frac{m^2}{M_p} \lambda_2 \mathcal{L}_{0,3} + \frac{m^2}{M_p^2} \lambda_3 \mathcal{L}_{0,4}, \quad (2.28)$$

where $\lambda_1, \lambda_2, \lambda_3$ are dimensionless coupling constants and the interaction terms are defined below. One of the coupling constants is redundant and can be absorbed into M_p when there is no coupling to matter. The first of these terms, $\mathcal{L}_{2,3}$, is a two-derivative cubic interaction,

$$\mathcal{L}_{2,3} = -\frac{1}{2} \epsilon^{\mu_1 \mu_2 \mu_3 \mu_4} \epsilon^{\nu_1 \nu_2 \nu_3 \nu_4} \partial_{\mu_1} \partial_{\nu_1} h_{\mu_2 \nu_2} h_{\mu_3 \nu_3} h_{\mu_4 \nu_4}, \quad (2.29)$$

where $\epsilon^{\mu_1 \mu_2 \mu_3 \mu_4}$ is the completely antisymmetric tensor. The other two terms, $\mathcal{L}_{0,3}$ and $\mathcal{L}_{0,4}$, are the symmetric polynomials in $h_{\mu\nu}$,

$$\mathcal{L}_{0,3} = e_3(h) = \frac{1}{6} \left([h]^3 - 3[h][h^2] + 2[h^3] \right), \quad (2.30)$$

$$\mathcal{L}_{0,4} = e_4(h) = \frac{1}{24} \left([h]^4 - 6[h]^2[h^2] + 3[h^2]^2 + 8[h][h^3] - 6[h^4] \right), \quad (2.31)$$

where square brackets indicate traces with $\eta_{\mu\nu}$. Like dRGT, the pseudo-linear massive spin-2 theory becomes strongly coupled at the scale Λ_3 .

2.3 Galileons

The Galileon theories are scalar theories that have second-order equations of motion and are invariant under the shift symmetry

$$\phi \rightarrow \phi + c + b_\mu x^\mu, \quad (2.32)$$

where c and b_μ are constant [101]. In four dimensions, there are three possible interaction terms and a kinetic term that satisfy these requirements,

$$\begin{aligned} \mathcal{L} = & -\frac{1}{2} (\partial\phi)^2 + \frac{\alpha_1}{\Lambda^3} (\partial\phi)^2 (\square\phi) + \frac{\alpha_2}{\Lambda^6} (\partial\phi)^2 \left((\square\phi)^2 - (\partial_\mu \partial_\nu \phi)^2 \right) \\ & + \frac{\alpha_3}{\Lambda^6} (\partial\phi)^2 \left((\square\phi)^3 - 3\square\phi (\partial_\mu \partial_\nu \phi)^2 + 2(\partial_\mu \partial_\nu \phi)^3 \right), \end{aligned} \quad (2.33)$$

where $\square \equiv \partial^\beta \partial_\beta$ and α_i are constants. The equations of motion are second-order, even though there is more than one derivative per field in the Lagrangian. The

Galileons describe the self-interactions of the helicity-0 mode of the massive graviton in dRGT or the pseudo-linear massive spin-2 theory in the decoupling limit $m \rightarrow 0$ and $M_p \rightarrow \infty$ with $\Lambda_3 \equiv (m^2 M_p)^{1/3}$ fixed [46, 53]. The decoupling limits also generically contain mixing terms between the helicity-0 and helicity-2 modes.

2.4 Summary of theories

To summarise, there are two interacting ghost-free theories of massive gravity. Adding a mass term that softly breaks diffeomorphism symmetry gives dRGT massive gravity and adding a mass term that softly breaks linearised diffeomorphism symmetry gives the pseudo-linear massive spin-2 theory. These are both EFTs that become strongly coupled at the scale Λ_3 . In the decoupling limits of these theories, the self-interactions of the helicity-0 mode of the massive graviton are described by the Galileon interactions.

3

Scattering Massive Spinning Particles

3.1 Introduction

In this chapter, scattering amplitudes for self-interacting, massive, bosonic, spinning particles are studied to try to find the highest strong coupling scales for various theories.

3.1.1 Scattering amplitudes

Scattering amplitudes are a powerful tool for understanding physical theories. The dream of S -matrix theorists in the 1960s was to use S -matrix consistency conditions to reach a full understanding of the strong interactions [102]. Although this aspect of the program was a failure, since the strong interactions were eventually understood using QCD, the effort did lead to the discovery of string theory [103]. The last few decades have seen a resurgence in the development of scattering amplitude techniques, which for some calculations has enabled progress far beyond what is feasible using traditional field theory methods [104]. One example of the power of amplitude-based arguments is a simple proof of the fact that massless higher-spin particles cannot interact with gravity in flat spacetime [105]. A proof of this fact is given in the appendix to this chapter, using the amplitude formalism presented in Section 3.2. For a review of recent developments in scattering amplitudes, see [106].

Most of the recent progress has focused on the scattering of massless states and fewer techniques have been developed for dealing with massive particles. Even so, it can be advantageous to work with on-shell scattering amplitudes for massive particles, as this avoids many of the redundancies inherent to the Lagrangian formalism. For example, in the Lagrangian there is always the freedom to perform field redefinitions that preserve free states, or to integrate by parts, without affecting the S -matrix. Working directly with on-shell amplitudes avoids these ambiguities and makes classifying interactions much easier. In fact, at the cubic level it is possible to classify all on-shell amplitudes for massive or massless particles in any number of dimensions [107–109]. This result will be used in this chapter to constrain the general S -matrix for massive spin-2 EFTs.

3.1.2 Strong coupling scales

The primary goal of this chapter is to use on-shell amplitudes to find a model-independent bound on the strong coupling scale of massive spin-2 theories, focusing on the strong coupling scale defined by the scattering of a finite number of quanta in flat spacetime.¹ As discussed in Chapter 2, simply adding generic zero-derivative interaction terms to the Einstein-Hilbert action results in an EFT with a cutoff of $\Lambda_5 \equiv (M_P m^4)^{1/5}$ [89]. Partial-wave unitarity bounds imply that perturbative unitarity is violated around this scale, so new degrees of freedom must enter at energies below Λ_5 if unitarity is to be maintained.² In dRGT massive gravity, the ghost-free potential results in the strong coupling scale being raised to Λ_3 . The raising of the strong coupling scale works through a tree-level cancellation of the leading high-energy behaviour between exchange diagrams and contact diagrams. Moreover, using S -matrix arguments Schwartz has shown that Λ_3 is the highest possible strong coupling scale among theories whose kinetic structure is fixed to

¹It has been argued that the strong coupling scale can be raised in nontrivial backgrounds due to environmental dependence [110], although see the discussion in [111]. There are also certain Lorentz-violating backgrounds that raise the strong coupling scale to $\Lambda_2 \equiv (M_P m)^{1/2}$ [112].

²The ghost necessitates that new degrees of freedom enter below the cutoff. If a theory is ghost free, as in dRGT massive gravity, then it is possible for the theory to self-unitarise and remain valid above the strong coupling scale [113].

be Einstein-Hilbert [114]. Similarly, the pseudo-linear interactions (2.28) give an interacting theory with the strong coupling scale Λ_3 , which is higher than the scale that would be expected from naive power counting.

In this chapter it is demonstrated that regardless of the choice of massive spin-2 self-interactions with up to six derivatives, there is no way to raise the strong coupling scale beyond Λ_3 . Additionally, it is shown that the only Λ_3 $2 \rightarrow 2$ scattering amplitudes are the dRGT or pseudo-linear massive spin-2 amplitudes. The calculation allows for all possible Lorentz-invariant interactions that could help raise the scale of strong coupling, including parity-odd terms. Parity-odd terms are included since it is possible that there are parity-violating interactions that can be added to dRGT massive gravity that maintain its high cutoff and do not introduce a ghost, as mentioned in [115]. There is in fact a two-derivative parity-odd cubic interaction, which could be part of a parity-odd theory of massive gravity in four dimensions. In this chapter it is shown that any unitary theory containing this parity-odd amplitude must become strongly coupled by at least the scale $\Lambda_{7/2} \equiv (m^{5/2}M_p)^{2/7}$. The generalisation of this procedure to finding the highest strong coupling scale when there are interactions with additional fields or for higher-spin fields is also discussed.

3.1.3 Finding the highest strong coupling scale

The procedure for finding the highest strong coupling scale is now outlined, using the spin-2 case as an example. To start with, the most general $2 \rightarrow 2$ tree-level scattering amplitude is calculated. This is accomplished by working directly with on-shell amplitudes using a method discussed in [108, 109], which is described in detail below. The general amplitude for spin 2 is defined by $\mathcal{O}(100)$ possible terms and there are 5^4 different external polarisation configurations. Thus, despite being a four point tree-level calculation, finding this general amplitude is computationally very intensive. With the most general massive spin-2 scattering amplitude in hand, a search is made for amplitudes with softer than $\mathcal{O}(s^3)$ high-energy behaviour, where s is the Mandelstam invariant proportional to the square of the centre-of-mass

energy. It turns out that there are no nonzero amplitudes that satisfy this criteria. Moreover, only two amplitudes have $\mathcal{O}(s^3)$ high-energy behaviour: the amplitudes corresponding to the dRGT and pseudo-linear massive spin-2 theories. These $2 \rightarrow 2$ amplitudes become strongly coupled at the scale Λ_3 assuming a particular scaling of fields and momenta, as explained further in Section 3.4. In this sense, this shows that Λ_3 is the highest strong coupling scale for these amplitudes.

3.1.4 Outline of this chapter

A formalism for constructing general on-shell amplitudes is reviewed in Section 3.2. This formalism is then used to find cubic and quartic interactions for spin-0, spin-1, and spin-2 fields in Section 3.3. The on-shell amplitudes are then used to calculate $2 \rightarrow 2$ scattering for spin-0, spin-1, and spin-2 fields and the highest possible strong coupling scales are determined. An extension of this procedure to include the exchange of additional states is discussed in Section 3.5 and the extension to higher-spin fields is discussed in Section 3.6. Some concluding discussion is presented in Section 3.7. The appendix to this chapter reviews some no-go results for interacting massless higher-spin fields in flat spacetime. This chapter is based on unpublished work that was completed in collaboration with Kurt Hinterbichler.

3.2 Constructing on-shell amplitudes

This section reviews a procedure for constructing all on-shell cubic and quartic amplitudes for a collection of interacting bosonic fields. The tree-level amplitude will be written in terms of Lorentz-invariant contractions of the polarisation tensors and momenta of the external particles. An external particle i with spin l_i , mass m_i , and momentum p^i is associated with a symmetric, transverse, and traceless polarisation tensor $\epsilon_{\mu_1 \dots \mu_{l_i}}^i$,

$$p_i^{\mu_1} \epsilon_{\mu_1 \dots \mu_{l_i}}^i = 0, \quad (3.1)$$

$$\eta^{\mu_1 \mu_2} \epsilon_{\mu_1 \dots \mu_{l_i}}^i = 0. \quad (3.2)$$

Since the polarisation tensors are symmetric, each can be replaced with l_i instances of a vector z_μ^i when writing the amplitude, $\epsilon_{\mu_1 \dots \mu_{l_i}}^i \rightarrow z_{\mu_1}^i \dots z_{\mu_{l_i}}^i$. This is just a trick for keeping track of index contractions and does not mean that the physical polarisation tensors can be written as products of vectors. An amplitude coming from a local contact interaction can then be written as a polynomial in contractions of z 's and p 's. These contractions are denoted by $p_{ij} = p_\mu^i p^{j,\mu}$, $z_{ij} = z_\mu^i z^{j,\mu}$, and $zp_{ij} = z_\mu^i p^{j,\mu}$, where repeated instances of i and j are not summed over. The conditions (3.1) and (3.2) become $zp_{ii} = 0$ and $z_{ii} = 0$. There are also the further conditions $z_{ij} = z_{ji}$, $p_{ij} = p_{ji}$, and $p_{ii} = -m_i^2$, since the external momenta are on-shell.

Amplitudes built from products of z_{ij} , zp_{ij} , and p_{ij} will be parity-even. There can also be parity-odd terms that come from contractions involving a single antisymmetric epsilon tensor, $\varepsilon^{\mu_1 \mu_2 \mu_3 \mu_4}$, which will be denoted by, e.g., $\varepsilon(p_1 p_2 z_1 z_2) \equiv \varepsilon^{\mu_1 \mu_2 \mu_3 \mu_4} p_{\mu_1}^1 p_{\mu_2}^2 z_{\mu_3}^1 z_{\mu_4}^2$.

The amplitudes must satisfy additional constraints if there are massless external particles. If particle i is massless then the amplitude must be invariant under the transformation

$$z_i \rightarrow z_i + \chi p_i, \quad (3.3)$$

to first order in χ . This is the on-shell statement of nonlinear gauge invariance, as explained nicely in reference [116], and ensures that spurious polarisations decouple from physical amplitudes. If some of the particles are identical, then the amplitude must also be invariant under additional permutation symmetries. For example, for the scattering of four identical particles the amplitude must have an S_4 symmetry under interchanging particle labels. Lastly, in low dimensions there can be dimensionally-dependent identities that introduce redundancies between various amplitudes. These need to be accounted for when finding an irreducible set of amplitudes.

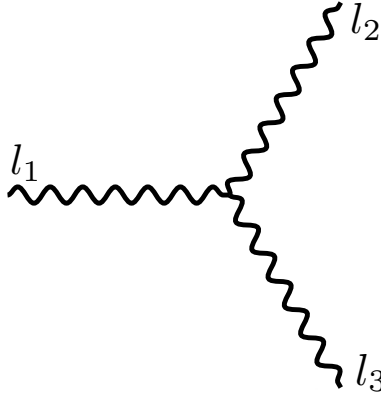


Figure 3.1: A cubic contact interaction for a triplet of bosonic particles with spins (l_1, l_2, l_3) .

3.2.1 Cubic terms

Consider a three-point interaction of particles with spins (l_1, l_2, l_3) , as in Figure 3.1. For each $l_i > 0$, introduce a null vector z^i . Momentum conservation with all momenta incoming gives³

$$p_\mu^1 + p_\mu^2 + p_\mu^3 = 0. \quad (3.4)$$

Equation (3.4) and $p_{ii} = -m_i^2$ imply that $2p_{ij} = m_i^2 + m_j^2 - m_k^2$ for distinct i, j, k , so all contractions p_{ij} can be written in terms of masses. Contracting (3.4) with z^i gives the three relations

$$zp_{12} + zp_{13} = 0, \quad (3.5a)$$

$$zp_{21} + zp_{23} = 0, \quad (3.5b)$$

$$zp_{31} + zp_{32} = 0. \quad (3.5c)$$

This means that there are only three independent contractions zp_{ij} , rather than six. In total, there are therefore six independent Lorentz scalars that can be used to construct the on-shell parity-even cubic amplitude, which can be taken to be z_{12} , z_{13} , z_{23} , zp_{12} , zp_{23} , and zp_{31} . Each z_i must appear l_i times in the

³In general these momenta must be complex for the on-shell cubic amplitude to be non-vanishing. To calculate exchange diagrams these cubic amplitudes must be analytically continued to real off-shell momenta.

cubic amplitude, since each polarisation vector appears once. The amplitude is thus a sum of terms of the form

$$z_{12}^{n_{12}} z_{13}^{n_{13}} z_{23}^{n_{23}} z_{p_{12}}^{m_{12}} z_{p_{23}}^{m_{23}} z_{p_{31}}^{m_{31}}, \quad (3.6)$$

where n_{ij} and m_{ij} are non-negative integers satisfying

$$n_{12} + n_{13} + m_{12} = l_1, \quad (3.7a)$$

$$n_{12} + n_{23} + m_{23} = l_2, \quad (3.7b)$$

$$n_{13} + n_{23} + m_{31} = l_3. \quad (3.7c)$$

Overall factors of i are omitted here, even though these are sometimes needed for the amplitudes (3.6) to derive from Feynman rules applied to a real Lagrangian interaction. These imaginary factors are reintroduced when necessary.

There are a finite number of solutions to the equations (3.7) for a given triplet of spins. If some of the particles are identical, then combinations of the amplitudes that are invariant under permutation symmetries must be found. If some of the particles are massless then there are additional conservation conditions arising from gauge invariance (3.3). In low dimensions there are redundancies amongst the amplitudes that result from dimensionally-dependent constraints. For example, in four dimensions the five vectors $x^I = (z^1, z^2, z^3, p^1, p^2)$ cannot be independent and, for equal mass particles, the vanishing of the Gram determinant gives

$$\begin{aligned} 2 \det_{1 \leq I, J \leq 5} (x^I \cdot x^J) &= 3m^4 z_{12} z_{13} z_{23} + 4z p_{12} z p_{23} z p_{31} (z_{23} z p_{12} + z_{13} z p_{23} + z_{12} z p_{31}) \\ &\quad - 2m^2 (z_{12}^2 z p_{31}^2 + z_{13}^2 z p_{23}^2 + z_{23}^2 z p_{12}^2 + z_{13} z_{23} z p_{12} z p_{23} + z_{12} z_{23} z p_{12} z p_{31} + z_{12} z_{13} z p_{23} z p_{31}) \\ &= 0. \end{aligned} \quad (3.8)$$

There can also be parity-odd cubic terms containing the contractions $\varepsilon(p_1 p_2 z_1 z_2)$, $\varepsilon(p_1 p_2 z_1 z_3)$, $\varepsilon(p_1 p_2 z_2 z_3)$, $\varepsilon(p_1 z_1 z_2 z_3)$ or $\varepsilon(p_2 z_1 z_2 z_3)$. These must be multiplied by parity-even contractions so that the result contains enough z 's. For example, the term multiplying $\varepsilon(p_1 z_1 z_2 z_3)$ must be of the form (3.6) with n_{ij} and m_{ij} solving (3.7) for $l_i \rightarrow l_i - 1$.

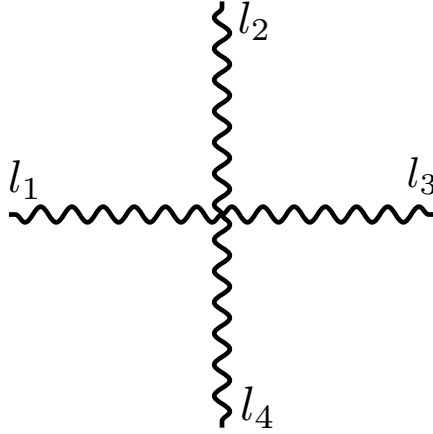


Figure 3.2: The quartic contact interaction for a quartet of bosonic particles with spins (l_1, l_2, l_3, l_4) .

3.2.2 Quartic terms

Now consider a four-point interaction of particles with spins (l_1, l_2, l_3, l_4) , as in Figure 3.2. Momentum conservation with all momenta incoming gives

$$p_\mu^1 + p_\mu^2 + p_\mu^3 + p_\mu^4 = 0. \quad (3.9)$$

Using the conditions (3.9) and $p_{ii} = -m_i^2$, it can be shown that there are only two independent contractions p_{ij} for $i \neq j$, which are taken to be p_{12} and p_{13} . These correspond to the two independent Mandelstam variables s and t for $2 \rightarrow 2$ scattering. Contracting (3.9) with z^i gives the four relations

$$zp_{12} + zp_{13} + zp_{14} = 0, \quad (3.10a)$$

$$zp_{21} + zp_{23} + zp_{24} = 0, \quad (3.10b)$$

$$zp_{31} + zp_{32} + zp_{34} = 0, \quad (3.10c)$$

$$zp_{41} + zp_{42} + zp_{43} = 0. \quad (3.10d)$$

This means that there are only eight independent contractions zp_{ij} , rather than 12. In total, there are 16 independent Lorentz scalars. The amplitude is thus

a sum of terms of the form⁴

$$p_{12}^{k_{12}} p_{13}^{k_{13}} z_{12}^{n_{12}} z_{13}^{n_{13}} z_{14}^{n_{14}} z_{23}^{n_{23}} z_{24}^{n_{24}} z_{34}^{n_{34}} z p_{13}^{m_{13}} z p_{14}^{m_{14}} z p_{21}^{m_{21}} z p_{24}^{m_{24}} z p_{31}^{m_{31}} z p_{32}^{m_{32}} z p_{42}^{m_{42}} z p_{43}^{m_{43}}, \quad (3.11)$$

where k_{ij} , n_{ij} and m_{ij} are non-negative integers satisfying

$$n_{12} + n_{13} + n_{14} + m_{13} + m_{14} = l_1, \quad (3.12a)$$

$$n_{12} + n_{23} + n_{24} + m_{21} + m_{24} = l_2, \quad (3.12b)$$

$$n_{13} + n_{23} + n_{34} + m_{31} + m_{32} = l_3, \quad (3.12c)$$

$$n_{14} + n_{24} + n_{34} + m_{42} + m_{43} = l_4. \quad (3.12d)$$

The k_{ij} are unconstrained by (3.12), so there are on-shell quartic amplitudes with arbitrarily many derivatives. This is unlike the cubic case, where there was a finite number of amplitudes. This will not present a problem here since the aim is to find theories with the highest possible cutoff and this means that only quartic amplitudes with a bounded number of derivatives need to be considered.

The parity-odd quartic amplitudes contain contractions of the form $\varepsilon(p_1 p_2 p_3 z_i)$, $\varepsilon(p_i p_j z_k z_l)$, $\varepsilon(p_i z_j z_k z_l)$, or $\varepsilon(z_i z_j z_k z_l)$. There are 35 such contractions. These must be multiplied by parity-even contractions so that the result contains enough z 's. The term multiplying a contraction of the antisymmetric tensor is of the form (3.11), where n_{ij} and m_{ij} satisfy (3.12) with $l_i \rightarrow l_i - 1$ if the antisymmetric tensor contracts with z^i .

As in the cubic case, permutation symmetries and the presence of massless particles impose extra constraints on the quartic amplitudes and there can be redundancies from dimensionally-dependent identities.

⁴This construction of quartic amplitudes for identical particles differs slightly from that in [108, 109]. Factors of p_{12} and p_{13} are included in the amplitude building blocks and linear combinations are found that contain a bounded number of momenta and that are invariant under all permutations of the external particles. This means that Bose symmetry is inbuilt but the number of momenta must be bounded. In [109], tensor structures without p_{ij} 's are considered and invariance is imposed for the ‘‘kinematic’’ permutations that leave s and t invariant. Each structure is then multiplied by a coefficient that is a function of s and t .

3.3 Amplitudes for self-interacting massive bosons

Now the formalism from Section 3.2 will be used to construct some on-shell amplitudes. The cases considered are single self-interacting massive fields with spin 0, spin 1, and spin 2, all in four dimensions. Spin 2 is the primary case of interest, but the lower spin cases are considered first as simpler examples. For each spin, all cubic amplitudes and quartic amplitudes containing up to a certain number of derivatives are constructed.

3.3.1 Spin 0

Cubic amplitudes

Scalar fields have no indices so there are no z_i 's and no Lorentz-invariant contractions that can be used to build a cubic amplitude. Thus the only solution to the cubic equations (3.7) is $n_{ij} = m_{ij} = 0$, which gives a constant cubic amplitude. For a single self-interacting scalar, this corresponds to the ϕ^3 interaction. Any other cubic interaction can be written in terms of ϕ^3 and higher-order interactions using field redefinitions and integration by parts. For example, the interaction $-2\phi(\partial\phi)^2$ can be written as $\phi^2\Box\phi$ using integration by parts. A useful fact is that using the lowest-order equations of motion in the Lagrangian is equivalent to a field redefinition up to higher-order terms. Since the lowest-order equations of motion for a massive scalar are $\Box\phi = m^2\phi$, a field redefinition can be used to write $\phi^2\Box\phi$ as $m^2\phi^3$ plus interactions that are of higher order in ϕ . Since field redefinitions do not affect the S -matrix [117], this shows that at cubic order the interaction $\phi(\partial\phi)^2$ is equivalent to the ϕ^3 interaction when calculating scattering amplitudes for a massive scalar. If the scalar is massless then $\phi(\partial\phi)^2$ is equivalent to higher-order interactions only.

There are no parity-odd cubic or quartic terms for a scalar since there are only two or three independent momenta and any contraction $\varepsilon(p_i p_j p_k p_l)$ must therefore vanish.

Quartic amplitudes

At the quartic level, there are the Lorentz-invariant contractions p_{12} and p_{13} . A general scalar quartic amplitude is thus expressible as a sum over products of p_{12}

and p_{13} . For a single scalar there are additional crossing symmetry constraints. For example, a single scalar has the constant amplitude corresponding to the ϕ^4 interaction, no two-derivative amplitudes, one four-derivative amplitude given by

$$p_{12}^2 + p_{12}p_{13} + p_{13}^2 - m^2(p_{12} + p_{13}), \quad (3.13)$$

and so on. It is easier to write the scalar quartic amplitudes using the Mandelstam variables s , t and u (defined in Section 3.4). Then the most general crossing-symmetric amplitude can be written as

$$\sum_{i,j \geq 0} \alpha_{ij} (st + su + tu)^i (stu)^j, \quad (3.14)$$

where α_{ij} are dimensionful constants.

3.3.2 Spin 1

Cubic amplitudes

The cubic amplitudes for spin-1 are found by solving (3.7) with $l_1 = l_2 = l_3 = 1$. There are four solutions, giving the amplitudes

$$z_{12}z p_{31}, \quad z_{13}z p_{23}, \quad z_{23}z p_{12}, \quad z p_{12}z p_{23}z p_{31}. \quad (3.15)$$

No combination of these is invariant under all permutations of the particles, so there are no on-shell parity-even cubic amplitudes for a single vector. This implies, for example, that there can be no cubic photon interactions, in accordance with Furry's theorem [118]. There are cubic amplitudes for multiple vectors, such as the Yang-Mills cubic vertex

$$f^{abc} \left(z_{12}^{ab} z p_{31}^c + z_{13}^{ac} z p_{23}^b + z_{23}^{bc} z p_{12}^a \right), \quad (3.16)$$

where f^{abc} is an antisymmetric structure constant. There are also no parity-odd cubic terms for a single vector. The five possible parity-odd structures are

$$\varepsilon(p_1 p_2 z_1 z_2) z p_{31}, \quad \varepsilon(p_1 p_2 z_1 z_3) z p_{23}, \quad \varepsilon(p_1 p_2 z_2 z_3) z p_{12}, \quad \varepsilon(p_1 z_1 z_2 z_3), \quad \varepsilon(p_2 z_1 z_2 z_3), \quad (3.17)$$

but no combination of these is invariant under permuting all particles. In five dimensions there is a single parity-odd cubic amplitude $\varepsilon(p_1 p_2 z_1 z_2 z_3)$, which is generated by the five-dimensional Chern-Simons theory [119].

Quartic amplitudes

To find the parity-even quartic amplitudes, equation (3.12) must be solved with $l_1 = l_2 = l_3 = l_4 = 1$. This calculation is performed on a computer using *Mathematica*. Considering terms with up to four derivatives gives 106 structures. From these 106 parity-even structures there are 10 independent combinations that are symmetric under all permutations of the external particles. The parity-odd terms are found by following the prescription described at the end of Section 3.2. The result is that there are 10 symmetric parity-odd amplitudes but only four of these are independent because of dimensionally-dependent identities. The redundant amplitudes are identified by explicitly evaluating the candidate amplitudes in the centre-of-mass frame and finding combinations that vanish, but an irreducible set could also be found directly using the “scattering frame” method described in reference [109]. The quartic amplitudes are lengthy so their explicit form is not included here.

3.3.3 Spin 2

Cubic amplitudes

The parity-even cubic amplitudes for a spin-2 field are found by solving (3.7) with $l_1 = l_2 = l_3 = 2$. There are 11 solutions in total. Five combinations of these solutions are symmetric under permuting all labels, corresponding to the cubic amplitudes for identical particles. They are given by

$$\mathcal{A}_1 = z_{12}z_{13}z_{23}, \quad (3.18a)$$

$$\mathcal{A}_2 = (z_{23}z p_{12} + z_{13}z p_{23} + z_{12}z p_{31})^2, \quad (3.18b)$$

$$\mathcal{A}_3 = z_{23}^2 z p_{12}^2 + z_{13}^2 z p_{23}^2 + z_{12}^2 z p_{31}^2, \quad (3.18c)$$

$$\mathcal{A}_4 = z p_{12} z p_{23} z p_{31} (z_{12} z p_{31} + z_{23} z p_{12} + z_{13} z p_{23}), \quad (3.18d)$$

$$\mathcal{A}_5 = z p_{12}^2 z p_{23}^2 z p_{31}^2. \quad (3.18e)$$

There are familiar Lagrangian interactions that generate these cubic amplitudes through the usual procedure of generating S -matrix elements from Lagrangian interactions (potentially along with other amplitudes with fewer derivatives). For

example, \mathcal{A}_1 is generated by the cubic potential $h_\mu{}^\nu h_\nu{}^\lambda h_\lambda{}^\mu$, \mathcal{A}_2 is generated by the cubic part of the Einstein-Hilbert Lagrangian, $\mathcal{A}_2 + \mathcal{A}_3$ is generated by the cubic two-derivative pseudo-linear interaction

$$\varepsilon^{\mu\nu\lambda\rho}\varepsilon^{\alpha\beta\gamma\delta}\partial_\mu\partial_\alpha h_{\nu\beta}h_{\lambda\gamma}h_{\rho\delta}, \quad (3.19)$$

\mathcal{A}_4 is generated by the cubic part of the Gauss-Bonnet interaction

$$R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\lambda\rho}R^{\mu\nu\lambda\rho}, \quad (3.20)$$

and \mathcal{A}_5 is generated by the cubic part of the Riemann-cubed interaction

$$R_{\mu\nu}{}^{\alpha\beta}R_{\alpha\beta}{}^{\lambda\rho}R_{\lambda\rho}{}^{\mu\nu}. \quad (3.21)$$

These cubic amplitudes are not all independent in four dimensions since the five vectors z_1, z_2, z_3, p_1 and p_2 cannot be independent. The vanishing of the Gram determinant (3.8) gives

$$4\mathcal{A}_4 = m^2(\mathcal{A}_2 + \mathcal{A}_3) - 3m^4\mathcal{A}_1, \quad (3.22)$$

so the amplitude \mathcal{A}_4 can be written in terms of the others. This corresponds to the fact that the Gauss-Bonnet interaction is a total derivative in four dimensions.⁵

Now consider parity-odd amplitudes. There are naively five different parity-odd cubic amplitudes for spin 2 in four dimensions, but there are only two independent combinations because of dimensionally-dependent identities. This number agrees with the formula for counting independent cubic structures given by Kravchuk and Simmons-Duffin [109]. The irreducible set in four-dimensions is given by

$$\mathcal{B}_1 = z_{13}z_{23}\varepsilon(p_1p_2z_1z_2) - z_{12}z_{23}\varepsilon(p_1p_2z_1z_3) + z_{12}z_{13}\varepsilon(p_1p_2z_2z_3), \quad (3.23a)$$

$$\mathcal{B}_2 = zp_{12}zp_{23}zp_{31}(zp_{31}\varepsilon(p_1p_2z_1z_2) - zp_{23}\varepsilon(p_1p_2z_1z_3) + zp_{12}\varepsilon(p_1p_2z_2z_3)). \quad (3.23b)$$

The term \mathcal{B}_2 is generated by the cubic part of the parity-odd Riemann-cubed interaction

$$\tilde{R}_{\mu\nu}{}^{\alpha\beta}R_{\alpha\beta}{}^{\lambda\rho}R_{\lambda\rho}{}^{\mu\nu}, \quad (3.24)$$

⁵This is consistent with (3.22) because the cubic Gauss-Bonnet term generates not only \mathcal{A}_4 but also terms with fewer derivatives since $\square = m^2$ on shell.

where $\tilde{R}_{\mu\nu}{}^{\alpha\beta} \equiv \varepsilon_{\mu\nu\lambda\rho} R^{\lambda\rho\alpha\beta}$. If the external particles are massless, then (3.23b) is conserved and using a dimensionally-dependent identity it can be written more simply as

$$z p_{12} z p_{23} z p_{31}^2 \varepsilon(p_1 p_2 z_1 z_2), \quad (3.25)$$

which agrees with the form of this amplitude as presented in reference [120]. The term \mathcal{B}_1 is generated by the Lagrangian interaction

$$\varepsilon^{\mu\nu\lambda\rho} \partial_\mu h_{\nu\alpha} \partial_\lambda h_{\rho\beta} h^{\alpha\beta}, \quad (3.26)$$

but I do not know of any theory that utilises this term. It has two derivatives, is not gauge invariant, and exists only in four dimensions, so it is a good candidate cubic interaction for a four-dimensional parity-odd theory of massive gravity. There are no parity-odd cubic amplitudes for identical spin-2 particles in five dimensions, even though the structure $\varepsilon(p_1 p_2 z_1 z_2 z_3)$ is allowed. This is because there is no combination of amplitudes containing $\varepsilon(p_1 p_2 z_1 z_2 z_3)$ that is symmetric under permuting the particles. In greater than five dimensions, there is no way to contract the indices of the antisymmetric tensor for cubic amplitudes without getting zero, so there are no parity-odd terms at all. In three dimensions there is a well-known parity-odd theory of massive gravity, namely topological massive gravity [58].

Finding this irreducible set of parity-odd amplitudes requires using dimensionally-dependent identities. These can be obtained by considering determinants of the form

$$\begin{vmatrix} A_1 & A_2 & A_3 & A_4 & \sum_{i=1}^4 \alpha_i A_i \\ z_1 & z_2 & p_1 & p_2 & z_3 \end{vmatrix} = 0, \quad (3.27)$$

where $z_3 = \alpha_1 z_1 + \alpha_2 z_2 + \alpha_3 p_1 + \alpha_4 p_2$ and A_i are any scalars [108]. For example, choosing $A_i = z_{ij}$ for $j = 1, 2, 3$ gives the three four-dimensional identities

$$z_{13} \varepsilon(p_1 p_2 z_1 z_2) - z_{12} \varepsilon(p_1 p_2 z_1 z_3) - z p_{12} \varepsilon(p_1 z_1 z_2 z_3) = 0, \quad (3.28a)$$

$$z_{12} \varepsilon(p_1 p_2 z_2 z_3) + z_{23} \varepsilon(p_1 p_2 z_1 z_2) - z p_{23} \varepsilon(p_2 z_1 z_2 z_3) = 0, \quad (3.28b)$$

$$z_{13} \varepsilon(p_1 p_2 z_2 z_3) - z_{23} \varepsilon(p_1 p_2 z_1 z_3) + z p_{31} (\varepsilon(p_1 z_1 z_2 z_3) + \varepsilon(p_2 z_1 z_2 z_3)) = 0. \quad (3.28c)$$

These identities can be used to show, for example, that the candidate amplitude

$$(z_{12} z p_{31} - z_{23} z p_{12}) \varepsilon(p_1 z_1 z_2 z_3) + (z_{12} z p_{31} - z_{13} z p_{23}) \varepsilon(p_2 z_1 z_2 z_3) \quad (3.29)$$

is proportional to \mathcal{B}_1 .

Quartic amplitudes

Finding the parity-even quartic amplitudes requires solving (3.12) with $l_1 = l_2 = l_3 = l_4 = 2$. Considering terms with up to six derivatives gives 1518 different structures. Of these 1518 parity-even structures, there are 95 combinations that are symmetric under all permutations of the particles. As with spin 1, all redundancies are eliminated by explicitly evaluating the 95 candidate amplitudes in the centre-of-mass frame and finding linear combinations that vanish. This leaves 69 independent parity-even quartic amplitudes. Similarly, finding all quartic parity-odd amplitudes with up to six derivatives gives 49 independent amplitudes. These quartic amplitudes are exceedingly lengthy so their explicit form is not included here.

3.4 Four-particle scattering

In this section, the on-shell contact amplitudes found in Section 3.3 are used to calculate $2 \rightarrow 2$ scattering amplitudes. Mass scales are discussed in 3.4.1 and the scattering set up is presented in 3.4.2. Then the results of the $2 \rightarrow 2$ amplitude calculations are discussed for spin 0, spin 1, and spin 2.

3.4.1 Scales

To find theories with a nontrivial scattering amplitude and the highest possible strong coupling scale for $2 \rightarrow 2$ scattering, a rule must be specified for including scales in the amplitudes. The choice of scaling used here is that momenta are scaled with the mass m , fields are scaled with M_p , and there is an overall factor of $m^2 M_p^2$. This corresponds to the scaling of amplitudes that are generated by the Lagrangian (2.12). This is a particular choice of scaling and other schemes are possible, which could be interesting to explore in future. With the fixed scaling chosen here, terms of the same form come with the same scale and amplitudes that grow with larger powers of s at high energies have lower strong coupling scales, so raising the cutoff means finding amplitudes that grow with smaller powers of s . This discussion can be framed entirely in terms of finding amplitudes with the softest high-energy

behaviour assuming the presence of certain cubic vertices, rather than talking about strong coupling scales, and this is independent of the scales used.

3.4.2 Set up

The external momenta satisfy $p^2 = -m^2$. Momenta p^3 and p^4 are taken to be outgoing, so momentum conservation gives

$$p_\mu^1 + p_\mu^2 = p_\mu^3 + p_\mu^4. \quad (3.30)$$

In the centre-of-mass frame with scattering angle θ , see Figure 3.3, the momenta can be written as

$$p_\mu^i = (E, 0, p \sin \theta^i, p \cos \theta^i), \quad (3.31)$$

where $i = 1, 2, 3, 4$ labels the external particle, $\theta^1 = 0$, $\theta^2 = \pi$, $\theta^3 \equiv \theta$, $\theta^4 = \theta - \pi$, and $E^2 = m^2 + p^2$. The Mandelstam variables are $s = -(p^1 + p^2)^2$, $t = -(p^1 - p^3)^2$, and $u = -(p^1 - p^4)^2$, which satisfy $s + t + u = 4m^2$. These are related to the scattering angle θ by

$$\cos \theta = 1 - \frac{2t}{4m^2 - s} = \frac{2u}{4m^2 - s} - 1. \quad (3.32)$$

The scalar propagator is

$$\frac{-i}{p^2 + m^2 - i\epsilon}. \quad (3.33)$$

Propagators for higher-spin particles differ from (3.33) by having tensorial terms in the numerator.

A massive vector has three degrees of freedom. For each momentum p_μ , there is a basis of polarisation vectors $\epsilon_\mu^{(a)}$, $a = 1, 2, 3$, that are orthogonal to the momentum, $\epsilon_\mu^{(a)} p^\mu = 0$, and orthonormal, $\epsilon_\mu^{(a)} \epsilon^{*(b)\mu} = \delta^{ab}$. These can be split into two transverse polarisations ($a = 1, 2$) and a longitudinal polarisation ($a = 3$). The longitudinal polarisation is defined such that at high energies it is proportional to the momentum,

$$\epsilon_\mu^{(3)} = \frac{p_\mu}{m} \left(1 + \mathcal{O}\left(\frac{1}{s}\right) \right), \quad (3.34)$$

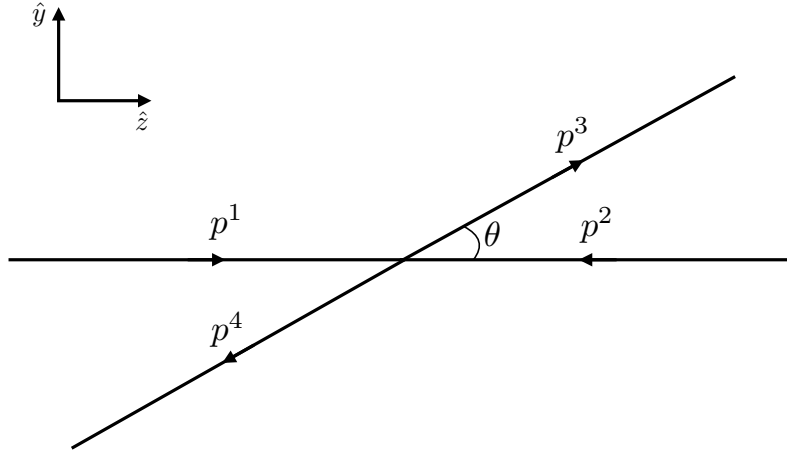


Figure 3.3: Scattering set up in the centre-of-mass frame with scattering angle θ . Particles 1 and 2 are incoming and particles 3 and 4 are outgoing.

which is the Goldstone equivalence limit. The four external polarisation vectors are linear sums of these basis polarisations,

$$\epsilon_\mu^i = \sum_{a=1}^3 \alpha_a^i \epsilon_\mu^{(a)}(p^i), \quad (3.35)$$

where $i = 1, 2, 3, 4$ labels the external particles and α_a^i are normalised coefficients

$$\sum_{a=1}^3 |\alpha_a^i|^2 = 1. \quad (3.36)$$

The polarisation vectors satisfy the completeness relation

$$\Pi_{\mu\nu} = \sum_{a=1}^3 \epsilon_\mu^{(a)} \epsilon_\nu^{(a)*}, \quad (3.37)$$

where the projection tensor $\Pi_{\mu\nu}$ appears in the numerator of the massive vector propagator and is given by

$$\Pi_{\mu\nu} = \eta_{\mu\nu} + \frac{p_\mu p_\nu}{m^2}. \quad (3.38)$$

In the centre-of-mass frame (3.31), a choice of vector polarisations is

$$\epsilon_\mu^{(1)}(p^i) = (0, 1, 0, 0) \quad (3.39a)$$

$$\epsilon_\mu^{(2)}(p^i) = (0, 0, \cos \theta^i, -\sin \theta^i) \quad (3.39b)$$

$$\epsilon_\mu^{(3)}(p^i) = \frac{1}{m} (p, 0, E \sin \theta^i, E \cos \theta^i), \quad (3.39c)$$

where $i = 1, 2, 3, 4$ labels the external particle.

A massive graviton has five degrees of freedom. For each momentum p_μ , define a basis of polarisation tensors $\epsilon_{\mu\nu}^{(a)}$, $a = 1, 2, 3, 4, 5$, that are traceless, $\epsilon_\mu^{(a)\mu} = 0$, orthogonal to the momentum, $\epsilon_{\mu\nu}^{(a)} p^\mu = 0$, and orthonormal $\epsilon_{\mu\nu}^{(a)} \epsilon^{*(b)\mu\nu} = \delta^{ab}$. These can be split into two tensor polarisations ($a = 1, 2$), two vector polarisations ($a = 3, 4$), and a scalar polarisation ($a = 5$). Written in terms of products of the vector polarisations, they are given by

$$\epsilon_{\mu\nu}^{(1)} = \sqrt{\frac{1}{2}} \left(\epsilon_\mu^{(1)} \epsilon_\nu^{(1)} - \epsilon_\mu^{(2)} \epsilon_\nu^{(2)} \right), \quad \epsilon_{\mu\nu}^{(2)} = \sqrt{\frac{1}{2}} \left(\epsilon_\mu^{(1)} \epsilon_\nu^{(2)} + \epsilon_\mu^{(2)} \epsilon_\nu^{(1)} \right), \quad (3.40a)$$

$$\epsilon_{\mu\nu}^{(3)} = \sqrt{\frac{1}{2}} \left(\epsilon_\mu^{(1)} \epsilon_\nu^{(3)} + \epsilon_\mu^{(3)} \epsilon_\nu^{(1)} \right), \quad \epsilon_{\mu\nu}^{(4)} = \sqrt{\frac{1}{2}} \left(\epsilon_\mu^{(2)} \epsilon_\nu^{(3)} + \epsilon_\mu^{(3)} \epsilon_\nu^{(2)} \right), \quad (3.40b)$$

$$\epsilon_{\mu\nu}^{(5)} = \sqrt{\frac{3}{2}} \left(\epsilon_\mu^{(3)} \epsilon_\nu^{(3)} - \frac{1}{3} \Pi_{\mu\nu} \right). \quad (3.40c)$$

The four external polarisation tensors are linear sums of these basis tensors,

$$\epsilon_{\mu\nu}^i = \sum_{a=1}^5 \alpha_a^i \epsilon_{\mu\nu}^{(a)}(p^i), \quad (3.41)$$

where $i = 1, 2, 3, 4$ labels the external particles and α_a^i are normalised coefficients

$$\sum_{a=1}^5 |\alpha_a^i|^2 = 1. \quad (3.42)$$

The polarisation tensors satisfy the completeness relation

$$\Pi_{\mu\nu, \alpha\beta} = \sum_{a=1}^5 \epsilon_{\mu\nu}^{(a)} \epsilon_{\alpha\beta}^{(a)*}, \quad (3.43)$$

where the transverse-traceless tensor $\Pi_{\mu\nu, \alpha\beta}$ appears in the numerator of the massive graviton propagator and is given by

$$\Pi_{\mu\nu, \alpha\beta} = \frac{1}{2} \Pi_{\mu\alpha} \Pi_{\nu\beta} + \frac{1}{2} \Pi_{\mu\beta} \Pi_{\nu\alpha} - \frac{1}{3} \Pi_{\mu\nu} \Pi_{\alpha\beta}. \quad (3.44)$$

3.4.3 Spin 0

Now the simple example of $2 \rightarrow 2$ scattering for a scalar can be presented. The constant cubic vertex λ , which corresponds to the interaction $\lambda\phi^3/6$, contributes to $2 \rightarrow 2$ scattering through s -, t -, and u -channel exchange diagrams. Using the scaling

described in 3.4.1, there is an overall factor of m^2/M_p in the cubic amplitude and λ is dimensionless. The total contribution to $2 \rightarrow 2$ scattering from this cubic vertex is

$$\mathcal{A} = -\lambda^2 \frac{m^4}{M_p^2} \left(\frac{1}{s-m^2} + \frac{1}{t-m^2} + \frac{1}{u-m^2} \right), \quad (3.45)$$

where $u = 4m^2 - s - t$. A quartic amplitude $m^2 f(s, t, u)/M_p^2$ can be added to this, where f is invariant under interchanging s , t , and u ,

$$f(s, t, u) = \alpha_0 + \frac{\alpha_1}{m^4} (s^2 + t^2 + u^2) + \frac{\alpha_2}{m^6} stu + \dots, \quad (3.46)$$

where α_i are constants. Choosing a constant f gives a theory with a bounded fixed-angle amplitude as $s \rightarrow \infty$, so for small couplings perturbative unitarity can be maintained at all energies and there is no strong coupling scale.

3.4.4 Spin 1

Now the more complicated example of a spin-1 field is considered. The simplest interaction for a massive spin-1 field is the quartic potential,

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} m^2 A^2 + \frac{\lambda_1 m^2}{M_p^2} (A^2)^2, \quad (3.47)$$

where $A^2 \equiv A_\mu A^\mu$. The high-energy amplitude for scattering longitudinally polarized states with this interaction is

$$\mathcal{M}(3333) = \frac{2\lambda_1}{\Lambda_2^2} (s^2 + t^2 + u^2) + \mathcal{O}(s), \quad (3.48)$$

where $\Lambda_2 = (mM_p)^{1/2}$ and $\mathcal{M}(abcd)$ denotes the amplitude for the scattering of states with definite basis polarisations (3.39),

$$\epsilon_\mu^{(a)}(p^1) + \epsilon_\mu^{(b)}(p^2) \rightarrow \epsilon_\mu^{(c)}(p^3) + \epsilon_\mu^{(d)}(p^4). \quad (3.49)$$

This amplitude becomes order one and violates perturbative unitarity at energies around the strong coupling scale Λ_2 . The amplitude (3.48) agrees with the power counting estimate: at high energies the longitudinal modes scale according to (3.34), so

$$\mathcal{M}(3333) \sim \frac{\lambda_1 m^2}{M_p^2} \left(\frac{p}{m} \right)^4 \sim \frac{\lambda_1 s^2}{\Lambda_2^2}. \quad (3.50)$$

There are no on-shell cubic amplitudes for a single spin-1 field, so only quartic interactions need to be considered to find the highest strong coupling scale for $2 \rightarrow 2$ scattering. A quartic interaction with $2k$ derivatives will grow fastest at high energies due to the scattering of longitudinal modes, so generically such terms will produce amplitudes that at high energies are $\mathcal{O}(s^{k+2})$,

$$\frac{1}{m^{2k-2}M_p^2}\partial^{2k}A^4 \xrightarrow{\text{generic}} \mathcal{A} \sim \frac{s^{k+2}}{\Lambda_{k+2}^{2(k+2)}}. \quad (3.51)$$

This assumes the scaling of fields and derivatives that was described earlier. Gauge invariance ameliorates the high-energy growth of scattering amplitudes since it ensures that longitudinal modes do not contribute. Gauge invariant interactions should thus give the most modest amplitude growth at high energies for a fixed number of derivatives, due to the scattering of transverse modes. A gauge invariant interaction⁶ with $2k$ derivatives will produce amplitudes that at high energies are $\mathcal{O}(s^k)$,

$$\frac{1}{m^{2k-2}M_p^2}\partial^{2k}A^4 \xrightarrow{\text{gauge inv.}} \mathcal{A} \sim \frac{s^k}{\Lambda_k^{2k}}. \quad (3.52)$$

This means that only interactions with $k \leq 2$ (four derivatives or fewer) need to be considered if the strong coupling scale is to be higher than or equal to Λ_2 . Moreover, any quartic four-derivative terms that contribute at Λ_2 should be gauge invariant. A gauge-invariant quartic interaction with fewer than four derivatives would have a strong coupling scale higher than Λ_2 , but there are no such interactions.

Consider the general quartic amplitude containing up to four derivatives, which was found in Section 3.3. There are fourteen independent structures, so the $2 \rightarrow 2$ scattering amplitude is

$$\mathcal{A}_{\text{spin-1}}^{(4)} = \sum_{j=1}^{14} \beta_j A_j \left(\alpha_a^i, s, t, m \right), \quad (3.53)$$

where α_a^i are the vector polarisation basis coefficients defined in (3.35), the functions A_j correspond to the different quartic amplitudes, and β_j are parameters that must

⁶These interactions blow up when naively taking the massless limit. The correct decoupling limit should preserve the total number of degrees of freedom and requires simultaneously sending $m \rightarrow 0$ and $M_p \rightarrow \infty$ while keeping the lowest interaction scale fixed.

be real if the amplitude derives from a real Lagrangian. The external polarisation vectors appear linearly in the amplitude, so $\mathcal{A}_{\text{spin-1}}$ can be written as a sum of amplitudes for scattering of states with definite polarisations. Each of the four external particles has three possible polarisations, so there are 3^4 different ways to scatter vectors with definite polarisations, although many of these are related by crossing symmetry. These 3^4 scattering processes can be calculated using the explicit form of the amplitudes in (3.53) and the kinematics discussed in Subsection 3.4.2. The results can then be expanded around $s, t = \infty$ to find the high-energy part of the amplitude. The coefficients β_j can then be chosen to make the leading part of the amplitude vanish, which raises the strong coupling scale. This process can be repeated order by order until the only way to raise the strong coupling scale is to make the entire amplitude vanish.

Completing this procedure for the spin-1 theory shows that there are no non-vanishing amplitudes that become strongly coupled above Λ_2 and that there are seven interaction terms that become strongly coupled at Λ_2 . The most general Λ_2 theory is generated by the following interaction Lagrangian:

$$\begin{aligned}
M_p^2 \mathcal{L}_{\text{int}} &= \lambda_1 m^2 (A^2)^2 + \lambda_2 A^2 F_{\mu\nu} F^{\mu\nu} + \lambda_3 A_\mu A_\nu (\partial^\mu A_\lambda \partial^\nu A^\lambda - \partial_\lambda A^\mu \partial^\lambda A^\nu) \\
&+ \frac{\lambda_4}{m^2} (F_{\mu\nu} F^{\mu\nu})^2 + \frac{\lambda_5}{m^2} (\tilde{F}_{\mu\nu} F^{\mu\nu})^2 + \lambda_6 A^2 \tilde{F}_{\mu\nu} F^{\mu\nu} \\
&+ \frac{\lambda_7}{m^2} \tilde{F}_{\mu\nu} F^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta}.
\end{aligned} \tag{3.54}$$

In addition to the quartic potential, there are two parity-even two-derivative terms, two Euler-Heisenberg-like terms, and two parity-odd terms. The three four-derivative terms represent all quartic gauge-invariant terms.⁷ Including quartic terms with up to eight derivatives does not produce any new Λ_2 interactions, in accordance with the discussion above.

⁷These terms appear to blow up as $m \rightarrow 0$, but the correct decoupling limit requires taking $m \rightarrow 0$ and $M_p \rightarrow \infty$ with Λ_2 fixed.

3.4.5 Spin 2

Now consider massive spin-2 scattering amplitudes. The ghost-free dRGT and pseudo-linear theories both become strongly coupled at the scale Λ_3 , as reviewed in Chapter 2. Finding the highest strong coupling scale thus requires looking for massive spin-2 amplitudes that become strongly coupled at the scale Λ_3 or higher. Using the same reasoning as for spin 1, this means that only quartic amplitudes with up to six derivatives and cubic amplitudes with up to four derivatives need to be considered.

In Section 3.3 it was shown that there are three parity-even and one parity-odd cubic amplitudes with up to four derivatives. The general cubic amplitude that needs to be considered is thus of the form

$$\mathcal{A}_{\text{spin-2}}^{(3)} = \sum_{j=1}^4 \gamma_j B_j(\alpha_a^i, s, t, m), \quad (3.55)$$

where α_a^i are the tensor polarisation basis coefficients defined in (3.41), the functions B_j correspond to the various cubic amplitudes, and γ_j are real parameters. There are also 69 parity-even and 49 parity-odd quartic structures with up to six derivatives, so the general quartic amplitude that needs to be considered is

$$\mathcal{A}_{\text{spin-2}}^{(4)} = \sum_{j=1}^{118} \chi_j C_j(\alpha_a^i, s, t, m), \quad (3.56)$$

where the functions C_j correspond to the various quartic amplitudes and χ_j are real parameters.

Each of the external gravitons has five possible basis polarisations so there are 5^4 different polarisation configurations. The $2 \rightarrow 2$ amplitude is calculated for general external polarisations using the amplitudes $\mathcal{A}_{\text{spin-2}}^{(3)}$ and $\mathcal{A}_{\text{spin-2}}^{(4)}$. The resulting amplitude is a complicated function of the 122 parameters γ_j and χ_j , the 20 polarisation coefficients α_a^i , and the variables m , s , and t . This amplitude is then expanded around $s, t = \infty$ and the coefficients γ_j and χ_j are constrained order by order in the Mandelstam variables to cancel the high-energy part of the amplitude and raise the strong coupling scale.

The result of this calculation is that there are no non-vanishing amplitudes that become strongly coupled above the scale Λ_3 . This is one of the main results of this chapter. Another result is that there are exactly two families of amplitudes that become strongly coupled at the scale Λ_3 , each with two parameters plus the freedom to rescale M_p . These amplitudes match precisely the amplitudes of the dRGT and pseudo-linear theories. This shows that there are no additional Λ_3 interactions that can be added to the dRGT or pseudo-linear theories that modify the $2 \rightarrow 2$ scattering amplitude, assuming the scaling of fields described earlier. It also shows that there is no Λ_3 parity-odd theory. There are amplitudes that use the parity-odd cubic amplitude \mathcal{B}_1 (3.23a) and become strongly coupled at Λ_3 , but these require imaginary cubic couplings. Imaginary cubic couplings imply unitarity violation, so these theories are unphysical. There are amplitudes involving \mathcal{B}_1 with real couplings that become strongly coupled at $\Lambda_{7/2}$, so there could exist unitary parity-violating theories at this scale.

The non-vanishing dRGT amplitudes are remarkably simple considering that dRGT contains all the nonlinearities of the Einstein-Hilbert term. The leading high-energy behaviour of the tree-level four-particle S -matrix is defined by the following amplitudes:

$$\mathcal{M}(3344) = \frac{3}{32} (1 - 4c_3)^2 s^3, \quad (3.57a)$$

$$\mathcal{M}(3333) = \mathcal{M}(4444) = \frac{9}{32} (1 - 4c_3)^2 st(s + t), \quad (3.57b)$$

$$\mathcal{M}(1555) = -\frac{1}{\sqrt{3}} (c_3 + 8d_5) st(s + t), \quad (3.57c)$$

$$\mathcal{M}(3355) = \frac{1}{96} s \left(9(1 - 4c_3)^2 s^2 - (7 - 24c_3 + 432c_3^2 + 768d_5) t(s + t) \right), \quad (3.57d)$$

$$\mathcal{M}(4455) = -\frac{1}{96} s \left(9(1 - 4c_3)^2 s^2 + (-5 + 72c_3 - 144c_3^2) t(s + t) \right), \quad (3.57e)$$

$$\mathcal{M}(5555) = \frac{1}{6} \left(1 - 4c_3 + 36c_3^2 + 64d_5 \right) st(s + t). \quad (3.57f)$$

All other non-vanishing terms can be obtained from these ones by crossing symmetry. For example, the amplitude $\mathcal{M}(3535)$ can be obtained from $\mathcal{M}(3355)$ by exchanging the second and third particles, $p^2 \leftrightarrow -p^3$ or $s \leftrightarrow t$, where the minus sign arises

because particle 3 is outgoing. There is a further overall minus sign that comes from conjugating the odd-parity vector polarisation, thus

$$\mathcal{M}(3535) = -\frac{1}{96}t \left(9(1 - 4c_3)^2 t^2 - (7 - 24c_3 + 432c_3^2 + 768d_5) s(s + t) \right). \quad (3.58)$$

The S -matrix for the pseudo-linear massive spin-2 theory can be written down similarly, but there are many more terms.

3.5 Exchanging additional states

This section discusses massive graviton amplitudes containing the exchange of additional states. It was found above that no theory containing a single massive graviton can have a strong coupling scale greater than Λ_3 , but including couplings to other fields might alter this conclusion. In particular, cubic interactions of the form graviton-graviton-matter will contribute to $2 \rightarrow 2$ graviton scattering through exchange diagrams and could cancel the high-energy parts of the amplitude. Such cancellations underly the Higgs mechanism, where exchanging a massive scalar gives UV-finite $2 \rightarrow 2$ scattering amplitudes for massive vectors. In this section, the methods described above are used to construct massive graviton $2 \rightarrow 2$ amplitudes including the exchange of scalars and vectors to determine if this can raise the strong coupling scale. There are no graviton-graviton-fermion cubic couplings due to Lorentz invariance, so scalars and vectors are the only particles with spins less than two that can be exchanged. All parity-even cubic amplitudes and parity-even quartic amplitudes with up to six derivatives will be considered.

3.5.1 Scalar interactions

The formalism of Section 3.2 can be used to find all on-shell cubic amplitudes of the form graviton-graviton-scalar. Such interactions contribute to the massive graviton four-point amplitude through scalar exchange diagrams. Taking the scalar to be particle 3, the general cubic amplitude is

$$z_{12}^{n_{12}} z_{12}^{m_{12}} z_{23}^{m_{23}}, \quad (3.59)$$

where n_{12} , m_{12} and m_{23} are non-negative integers satisfying

$$n_{12} + m_{12} = 2, \quad (3.60)$$

$$n_{12} + m_{23} = 2. \quad (3.61)$$

Solving these equations gives the three amplitudes

$$z_{12}^2, \quad z_{12}z_{p_{12}}z_{p_{23}}, \quad z_{p_{12}}^2z_{p_{23}}^2, \quad (3.62)$$

which are already symmetric under interchanging 1 and 2 since $z_{p_{12}} = -z_{p_{13}}$ and $z_{p_{23}} = -z_{p_{21}}$. These amplitudes are generated by the cubic interactions

$$\frac{1}{2}h_{\mu\nu}h^{\mu\nu}\phi, \quad -\frac{1}{2}h_{\mu\nu}\partial^\nu h^{\mu\lambda}\partial_\lambda\phi, \quad \frac{1}{2}h_{\mu\nu}\partial^\mu\partial^\nu h^{\lambda\rho}\partial_\lambda\partial_\rho\phi. \quad (3.63)$$

Any other cubic interaction can be written in terms of these and higher-point interactions using field redefinitions and integration by parts. The scalar mass can be zero or positive since there is no gauge symmetry associated with a massless scalar.

3.5.2 Vector interactions

In a similar way, all cubic amplitudes of the form graviton-graviton-vector can be found. Taking the vector to be particle 3, the general cubic amplitude is given by (3.6),

$$z_{12}^{n_{12}}z_{13}^{n_{13}}z_{23}^{n_{23}}z_{p_{12}}^{m_{12}}z_{p_{23}}^{m_{23}}z_{p_{31}}^{m_{31}}, \quad (3.64)$$

where n_{ij} and m_{ij} are non-negative integers satisfying

$$n_{12} + n_{13} + m_{12} = 2, \quad (3.65a)$$

$$n_{12} + n_{23} + m_{23} = 2, \quad (3.65b)$$

$$n_{13} + n_{23} + m_{31} = 1. \quad (3.65c)$$

There are seven solutions to these equations, but only two combinations are symmetric under interchanging the gravitons. The symmetric amplitudes are

$$z_{12}(z_{13}z_{p_{23}} - z_{23}z_{p_{12}}), \quad (z_{13}z_{p_{12}}z_{p_{23}}^2 - z_{23}z_{p_{23}}z_{p_{12}}^2). \quad (3.66)$$

These are generated by the interactions

$$h_{\mu\nu}\partial^\mu h^{\nu\lambda}A_\lambda, h_{\mu\nu}\partial^\mu h_{\lambda\rho}\partial^\lambda\partial^\rho A^\nu. \quad (3.67)$$

If the vector is massless then by gauge invariance it must couple to a conserved current. This corresponds to an amplitude that is invariant under $z_3 \rightarrow z_3 + p_3$. No combination of the amplitudes (3.66) has this symmetry, so the vector must be massive. The electromagnetic minimal coupling interaction, given by $zp_{12}z_{23}^2$, is not a counterexample to this conclusion, since this couples a massless photon to a *complex* spin-2 field.

3.5.3 Scattering amplitudes with exchanged states

Now the $2 \rightarrow 2$ massive graviton scattering amplitude can be calculated with the s -, t -, and u -channel exchange of a massive vector and a massive scalar. It is straightforward to find theories coupling massive gravitons to extra fields with a Λ_3 cutoff, so it is more interesting to find amplitudes with a cutoff above Λ_3 . There are some simple Λ_2 amplitudes uncovered by this analysis, all of which involve imaginary couplings and only zero-derivative graviton self-interactions. One example requires coupling the massive graviton to a ghostly vector with mass squared $5m^2/2$ and a healthy scalar with mass squared $5m^2$. These theories are not unitary, but it could still be interesting to try find an off-shell realisation of them.

It is already known that exchanging a ghostly state can raise the cutoff. For example, in quadratic curvature gravity there is one massive graviton and one massless graviton, and precisely one of these is a ghost. This theory is renormalizable [121], since at high energies the interactions of the ghostly and healthy gravitons cancel each other out, as discussed using the Stückelberg formalism in [122]. It is possible to calculate the $2 \rightarrow 2$ massive graviton amplitude including the exchange of a ghostly massless graviton using the methods described in this chapter. This confirms that the strong coupling scale can be raised, in agreement with the expectation from quadratic gravity.⁸

⁸There is a subtlety when using on-shell amplitudes to calculate the exchange of a massless field, since this involves taking the massless particle off-shell. Using on-shell cubic amplitudes

It is also possible to raise the cutoff without violating unitarity by adding additional massive spin-2 fields. For example, dimensionally reducing five-dimensional GR on a circle with radius R gives a four-dimensional theory with massless fields $\phi^{(0)}$, $A_\mu^{(0)}$, and $h_{\mu\nu}^{(0)}$ and a tower of complex massive gravitons $h_{\mu\nu}^{(n)}$ with masses $m_{(n)} = n/R$, where $n = 1, 2, \dots$ labels the mass level. The cutoff is given by the five-dimensional Planck mass, $M_{(5)}$. This is related to the four-dimensional Planck mass $M_{(4)}$ and the mass of the lightest massive graviton $m_{(1)}$ by

$$M_{(5)} \sim \left(M_{(4)}^2 m_{(1)} \right)^{1/3} = \Lambda_{3/2}. \quad (3.68)$$

This theory thus has a strong coupling scale $\Lambda_{3/2} \gg \Lambda_3$, so exchanging the KK states must ameliorate the high-energy behaviour of the $2 \rightarrow 2$ massive graviton scattering amplitudes. For $2 \rightarrow 2$ scattering of $h_{\mu\nu}^{(n)}$, the relevant cubic interactions contain two factors of the field $h_{\mu\nu}^{(n)}$ and a factor of $\phi^{(0)}$, $A_\mu^{(0)}$, $h_{\mu\nu}^{(0)}$, or $h_{\mu\nu}^{(2n)}$, due to conservation of KK number. Thus the cutoff raising is achieved by coupling one massive graviton to another massive graviton with twice the mass and a massless scalar, vector, and graviton. This has been explicitly checked by Schwartz [114]. The photon coupling is possible because the massive gravitons are complex fields. Each massive graviton must couple to a heavier graviton to raise the strong coupling scale of its scattering amplitude, which shows that the infinite tower of massive gravitons is needed to raise the cutoff in the full theory. Some problems with truncating this theory to a finite number of fields are mentioned in reference [123]. KK dimensional reductions are discussed more in Chapters 5 and 6.

It would be interesting in future to include higher-derivative interactions, parity-odd terms, and multiple low-spin fields to constrain in greater generality the possible unitary extensions of massive gravity with a higher strong coupling scale. The amplitude method presented here seems well-suited to this task as it straightforwardly allows for the systematic incorporation of additional interactions.

implicitly requires a modified Faddeev-Popov gauge-fixing function to remove certain cubic terms and this introduces gauge-dependent quartic terms. These gauge-dependent quartic terms can be absorbed into the generic quartic terms used to calculate the amplitude, so in practice this is not an issue.

3.6 Higher spins

It is also interesting to constrain effective theories containing massive bosonic fields with spins greater than two. Writing the free Lagrangian for a massive spin- J field requires additional auxiliary fields. Using only symmetric traceless fields requires auxiliary fields with ranks $0, \dots, J - 2$, in addition to the rank- J tensor describing the massive field [35]. Using symmetric traceful fields, the minimal choice of auxiliary fields requires a symmetric auxiliary tensor with rank $J - 3$, in addition to a traceful rank- J tensor $\phi_{\mu_1 \dots \mu_J}$ [36]. These auxiliary fields vanish on-shell but are necessary to impose constraints so that the on-shell equations of motion are

$$(\square - m^2)\phi_{\mu_1 \dots \mu_J} = 0, \quad (3.69)$$

$$\partial^{\mu_1} \phi_{\mu_1 \dots \mu_J} = 0, \quad (3.70)$$

$$\phi^{\mu_1}_{\mu_1 \mu_3 \dots \mu_J} = 0, \quad (3.71)$$

which describe the degrees of freedom of a massive spin- J particle.

A massless spin- J field can be described using a symmetric rank- J tensor that is doubly traceless using the Fronsdal Lagrangian [124]. These Lagrangians have a higher-spin gauge symmetry generated by a traceless tensor with rank $J - 1$. The massless Lagrangians can be obtained by setting $m = 0$ in the massive Lagrangians [124]. Conversely, the massive theories can be obtained by dimensionally reducing the massless theories [125]. The higher-dimensional gauge symmetry then becomes a Stückelberg symmetry [126] acting on the lower-dimensional massive fields. The dimensional reduction of spin 2 is discussed more in Chapters 5 and 6. For spins greater than two, there is not enough gauge symmetry to remove all the Stückelberg fields, which is why auxiliary fields are required for the massive higher-spin theories. There are also strong constraints on possible interactions of massless higher-spin fields in flat spacetime, as reviewed in the appendix to this chapter. For recent reviews of higher-spin theory, see [127, 128].

This section will present some simple observations about massive higher-spin amplitudes. In particular, a conjecture is made for the highest strong coupling

scale in an EFT containing a single massive spin- J particle. Only zero-derivative parity-even quartic interactions will be considered since these are simpler to work with than derivative and cubic interactions. In the spin-1 and spin-2 cases, zero-derivative quartic interactions are already enough to determine the highest strong coupling scale and the conjecture assumes that this remains true for higher spins.

3.6.1 Spin- J strong coupling scale

Consider zero-derivative four-point contact amplitudes for a massive spin- J field. Since the auxiliary fields vanish on-shell, only the symmetric, on-shell traceless, rank- J tensor $\phi_{\mu_1 \dots \mu_J}$ needs to be considered. The relevant Lagrangian interactions have the schematic form

$$\mathcal{L} \sim \frac{m^2}{M_p^2} (\phi_{\mu_1 \dots \mu_J})^4, \quad (3.72)$$

where the indices can be contracted in various ways. For $J = 1$, the only possible interaction of this form is the quartic vector potential

$$(A_\mu A^\mu)^2. \quad (3.73)$$

For $J = 2$, there are two on-shell quartic potential terms

$$(h_{\mu\nu} h^{\mu\nu})^2, \quad h_\mu{}^\nu h_\nu{}^\lambda h_\lambda{}^\rho h_\rho{}^\mu. \quad (3.74)$$

For $J = 3$, there are three terms

$$(\phi_{\mu\nu\lambda} \phi^{\mu\nu\lambda})^2, \quad \phi_{\mu\nu\lambda} \phi^{\mu\nu\rho} \phi^{\alpha\beta\lambda} \phi_{\alpha\beta\rho}, \quad \phi_{\mu\nu\lambda} \phi^{\mu\alpha}{}_\rho \phi^\nu{}_{\alpha\beta} \phi^{\lambda\rho\beta}. \quad (3.75)$$

These terms are fixed by how the indices of any given tensor are contracted with the remaining tensors. For example, in the second term in (3.75), each tensor contracts two of its indices with one tensor and contracts one index with another tensor, and this specification fixes how all the remaining indices contract. Each term thus corresponds to a partition of J of length three. The number of such partitions is given by the nearest integer to $(J + 3)^2/12$, so this gives the number of zero-derivative on-shell four-point amplitudes for spin J .

The amplitudes generated by (3.72) will generically scale at high energies as

$$\mathcal{A}_J \propto \left(\frac{s}{\Lambda_{2J}^2} \right)^{2J}, \quad (3.76)$$

due to the scattering of longitudinal modes. However, by judiciously choosing the relative coefficients of the quartic terms, the high-energy behaviour of the amplitude can be softened so that the strong coupling scale is raised. For $J = 1$, there is only one interaction term,

$$A_{\mu_1} A_{\mu_2} A_{\mu_3} A_{\mu_4} \eta^{\mu_1 \mu_2} \eta^{\mu_3 \mu_4}, \quad (3.77)$$

and hence the highest strong coupling scale is Λ_2 . As discussed in Section 3.4, this is also the highest possible strong coupling scale when arbitrary interactions are considered. The corresponding amplitude is

$$\mathcal{A}_1 = z_{12} z_{34} + z_{13} z_{24} + z_{14} z_{23}. \quad (3.78)$$

For scattering of longitudinally polarised vectors (the helicity-0 modes), the amplitude gives

$$\mathcal{A}_1^{\text{helicity}-0} \propto s^2 + t^2 + u^2. \quad (3.79)$$

For $J = 2$, the highest strong coupling scale is achieved with the amplitude

$$\mathcal{A}_2 = z_{12}^2 z_{34}^2 + z_{13}^2 z_{24}^2 + z_{14}^2 z_{23}^2 - 2(z_{12} z_{13} z_{24} z_{34} + z_{12} z_{14} z_{23} z_{24} + z_{13} z_{14} z_{23} z_{24}). \quad (3.80)$$

This becomes strongly coupled at Λ_3 , which is also the highest strong coupling scale when considering arbitrary spin-2 interactions. This amplitude is generated by the quartic potential

$$\varepsilon^{\mu_1 \mu_2 \mu_3 \mu_4} \varepsilon^{\nu_1 \nu_2 \nu_3 \nu_4} h_{\mu_1 \nu_1} h_{\mu_2 \nu_2} h_{\mu_3 \nu_3} h_{\mu_4 \nu_4}, \quad (3.81)$$

which is the quartic interaction from the pseudo-linear theory (2.31). The leading part of the helicity-0 scattering amplitude is given by the Galileon four-point amplitude

$$\mathcal{A}_2^{\text{helicity}-0} \propto stu. \quad (3.82)$$

For $J = 3$, the highest strong coupling scale is achieved with the amplitude

$$\mathcal{A}_3 = \mathcal{A}_1 \mathcal{A}_2, \quad (3.83)$$

which interestingly is just the product of the lower spin amplitudes. This amplitude becomes strongly coupled at the scale Λ_5 . This amplitude is generated by the potential

$$\varepsilon^{\mu_1 \mu_2 \mu_3 \mu_4} \varepsilon^{\nu_1 \nu_2 \nu_3 \nu_4} \phi_{\mu_1 \nu_1 \lambda_1} \phi_{\mu_2 \nu_2 \lambda_2} \phi_{\mu_3 \nu_3 \lambda_3} \phi_{\mu_4 \nu_4 \lambda_4} \eta^{\lambda_1 \lambda_2} \eta^{\lambda_3 \lambda_4}. \quad (3.84)$$

The helicity-0 amplitude is also given by the product of the lower-spin amplitudes

$$\mathcal{A}_3^{\text{helicity-0}} \propto stu(s^2 + t^2 + u^2). \quad (3.85)$$

This calculation requires knowing a basis of orthonormal polarisation tensors for a massive spin-3 field. A polarisation basis for higher-spin fields can be found by forming symmetrised products of vector polarisations, subtracting traces, and then normalising the result. For spin-3 this gives

$$\begin{aligned} \epsilon_{\mu\nu\lambda}^{(1)} &= \frac{3}{2} \left(\epsilon_{(\mu}^{(1)} \epsilon_{\nu}^{(1)} \epsilon_{\lambda)}^{(2)} - \frac{1}{3} \epsilon_{\mu}^{(2)} \epsilon_{\nu}^{(2)} \epsilon_{\lambda}^{(2)} \right), & \epsilon_{\mu\nu\lambda}^{(2)} &= \frac{3}{2} \left(\epsilon_{(\mu}^{(2)} \epsilon_{\nu}^{(2)} \epsilon_{\lambda)}^{(1)} - \frac{1}{3} \epsilon_{\mu}^{(1)} \epsilon_{\nu}^{(1)} \epsilon_{\lambda}^{(1)} \right), \\ \epsilon_{\mu\nu\lambda}^{(3)} &= \sqrt{6} \left(\epsilon_{(\mu}^{(1)} \epsilon_{\nu}^{(2)} \epsilon_{\lambda)}^{(3)} \right), & \epsilon_{\mu\nu\lambda}^{(4)} &= \sqrt{\frac{3}{2}} \left(\epsilon_{(\mu}^{(1)} \epsilon_{\nu}^{(1)} \epsilon_{\lambda)}^{(3)} - \epsilon_{(\mu}^{(2)} \epsilon_{\nu}^{(2)} \epsilon_{\lambda)}^{(3)} \right), \\ \epsilon_{\mu\nu\lambda}^{(5)} &= \frac{\sqrt{15}}{2} \left(\epsilon_{(\mu}^{(1)} \epsilon_{\nu}^{(3)} \epsilon_{\lambda)}^{(3)} - \frac{1}{5} \Pi_{(\mu\nu} \epsilon_{\lambda)}^{(1)} \right), & \epsilon_{\mu\nu\lambda}^{(6)} &= \frac{\sqrt{15}}{2} \left(\epsilon_{(\mu}^{(2)} \epsilon_{\nu}^{(3)} \epsilon_{\lambda)}^{(3)} - \frac{1}{5} \Pi_{(\mu\nu} \epsilon_{\lambda)}^{(2)} \right), \\ \epsilon_{\mu\nu\lambda}^{(7)} &= 3\sqrt{10} \left(\epsilon_{\mu}^{(3)} \epsilon_{\nu}^{(3)} \epsilon_{\lambda}^{(3)} - \frac{1}{10} \Pi_{(\mu\nu} \epsilon_{\lambda)}^{(3)} \right), \end{aligned} \quad (3.86)$$

where $\Pi_{\mu\nu}$ is the projector (3.38). The polarisation tensor $\epsilon_{\mu\nu\lambda}^{(7)}$ describes the helicity-0 mode.

For $J = 4$, it is difficult to calculate the full amplitude since there are 9^4 polarisation configurations. The helicity-0 polarisation tensor is given by

$$\epsilon_{\mu\nu\lambda\rho}^{(9)} = \sqrt{\frac{35}{8}} \left(\epsilon_{\mu}^{(3)} \epsilon_{\nu}^{(3)} \epsilon_{\lambda}^{(3)} \epsilon_{\rho}^{(3)} - \frac{6}{7} \Pi_{(\mu\nu} \epsilon_{\lambda}^{(3)} \epsilon_{\rho)}^{(3)} + \frac{3}{35} \Pi_{(\mu\nu} \Pi_{\lambda\rho)} \right). \quad (3.87)$$

Using just this helicity-0 mode, the amplitude with the highest strong coupling scale is found to be

$$\mathcal{A}_4 = \mathcal{A}_2^2, \quad (3.88)$$

which becomes strongly coupled at Λ_6 . This amplitude is generated by the potential

$$\varepsilon^{\mu_1\mu_2\mu_3\mu_4}\varepsilon^{\nu_1\nu_2\nu_3\nu_4}\varepsilon^{\lambda_1\lambda_2\lambda_3\lambda_4}\varepsilon^{\sigma_1\sigma_2\sigma_3\sigma_4}\phi_{\mu_1\nu_1\lambda_1\sigma_1}\phi_{\mu_2\nu_2\lambda_2\sigma_2}\phi_{\mu_3\nu_3\lambda_3\sigma_3}\phi_{\mu_4\nu_4\lambda_4\sigma_4}. \quad (3.89)$$

The helicity-0 scattering amplitude is just the square of the Galileon amplitude

$$\mathcal{A}_4^{\text{helicity-0}} \propto (stu)^2. \quad (3.90)$$

The epsilon structure in (3.89) should guarantee that amplitudes involving states with any polarisation also become strongly coupled at Λ_6 or above, as is suggested by a naive Stückelberg analysis, but these have not been calculated explicitly.

There is now an obvious pattern for general spins $J > 0$. If J is even, the zero-derivative quartic amplitude with the highest strong coupling scale should be a product of spin-2 amplitudes,

$$\mathcal{A}_J = \mathcal{A}_2^{J/2}, \quad (3.91)$$

and become strongly coupled at $\Lambda_{3J/2}$. If J is odd, the zero-derivative quartic amplitude with the highest strong coupling scale should be a product of the spin-2 and spin-1 amplitudes,

$$\mathcal{A}_J = \mathcal{A}_1\mathcal{A}_2^{(J-1)/2}, \quad (3.92)$$

and become strongly coupled at $\Lambda_{(3J+1)/2}$. The conjecture is that these strong coupling scales are still the highest possible for a spin- J theory when arbitrary interactions are allowed. The difference between even and odd spins arises from the fact that odd powers of $\varepsilon(z_1z_2z_3z_4)$ are not symmetric under permuting the particle labels, whereas even powers are symmetric.⁹

This conjecture is analogous to the result of Poratti and Rahman for the highest strong coupling scales for theories of massive higher-spin fields coupled to electromagnetism [129, 130]. They found the model-independent cutoff $\Lambda \sim m/e^{2J-1}$ for massive spin- J particles with mass m and charge e interacting with electromagnetism.

⁹This suggests that for odd J there might exist parity-odd theories with multiple massive spin- J particles $h_{\mu_1\dots\mu_J}^{(A)}$ with a cutoff $\Lambda_{3J/2}$ if there is an antisymmetric structure constant f^{ABCD} .

The ability to write massive higher-spin amplitudes as powers of lower-spin amplitudes is reminiscent of the various $\text{GR} \sim \text{YM}^2$ relations [131, 132]. It would be interesting to see if this property extends to more complicated higher-spin interactions. It would also be interesting to try find off-shell ghost-free realisations of these theories, or to try prove that this is impossible. This would require consistently including auxiliary field interactions.

3.7 Discussion

This chapter has looked at scattering amplitudes in EFTs containing massive bosons. The main results are the following:

1. For a theory with a single massive graviton, there is no way to raise the strong coupling scale above Λ_3 .
2. The only massive graviton $2 \rightarrow 2$ scattering amplitudes with a Λ_3 strong coupling scale are identical to the amplitudes in the dRGT and pseudo-linear theories.
3. There is no unitary theory containing the two-derivative parity-odd cubic vertex (3.26) that has a Λ_3 strong coupling scale, but a theory with a $\Lambda_{7/2}$ strong coupling scale might exist.
4. There are simple Λ_2 amplitudes that should arise from non-unitary theories containing massive particles with spins 0, 1, and 2.
5. The highest strong coupling scale for a single spin- J field is conjectured to be $\Lambda_{[(3J+1)/2]}$.

The scales stated in this chapter all assume a particular scaling of fields and momenta, but the results can be stated in a way that is independent of this scaling. For example, the first result listed above is equivalent to the following: the $2 \rightarrow 2$ scattering amplitude in massive gravity must be $\mathcal{O}(s^3)$ at high energies and no softer behaviour is possible.

There is an apparent conflict between the second result and the results of reference [133], which proposed Λ_3 derivative interactions that could be added to dRGT massive gravity. However, this work ignored the vector modes of the massive graviton and these modes contribute to amplitudes that become strongly coupled at $\Lambda_{7/2}$. The interactions proposed in reference [133] thus cause the theory to become strongly coupled at an energy scale lower than Λ_3 when all modes of the massive graviton are considered.

A weakness of the procedure used in this chapter is that it is only sensitive to the tree-level four-point amplitude. One could imagine some nontrivial theory or interactions that vanish at the four-point level, or look identical to the dRGT or pseudo-linear theories, and this analysis would not discern them. Alternatively, the theory could have higher-point interactions with a lower scale, but that would just mean that the strong coupling bound is not saturated.

The calculations described in this chapter could be generalised to address other questions. For example, it could be checked whether the conjectured higher-spin cutoff is correct for spin 3 when including derivative interactions. Finding theories with a Λ_5 cutoff might then indicate the existence or not of ghost-free interacting spin-3 theories. Another interesting calculation would be to generalise the results of Section 3.5 to constrain possible Higgs-like completions of massive gravity. This is particularly interesting since there are regions of dRGT parameter space that might admit analytic Lorentz-invariant UV completions, as discussed in Chapter 4.

3.8 Appendix A: No-go results for massless higher spins

In this appendix, a proof is given of the fact that gravity and particles coupled to gravity cannot couple to massless bosonic particles with spins greater than two. This result is well-known, see [134, 135] for reviews. This proof, which uses the formalism from Section 3.2, is quite straightforward and cleanly shows why the gravitational minimal coupling interaction is the most important interaction in the soft limit.

Weinberg's soft limit argument is first reviewed [136]. Then it is shown that gravitational minimal coupling interactions are the most important interactions in the soft limit, without assuming that the two particles coupling to the graviton are identical. Next it is shown that massless higher-spin particles have no minimal gravitational couplings. Finally, a summary is given for how these arguments constrain couplings to massless higher-spin particles.

3.8.1 Weinberg's soft limit argument

Consider an amplitude \mathcal{M} describing the scattering of N incoming particles with momenta $\{p_i\}$. These can be particles with arbitrary masses and integer spins. Now consider the amplitude \mathcal{M}' for another process that differs from \mathcal{M} only by the emission of an additional (massless) graviton with momentum q_μ . The goal is to relate \mathcal{M}' to \mathcal{M} in the soft limit $q \rightarrow 0$.

To find \mathcal{M}' , sum over all ways of adding the graviton to the process described by \mathcal{M} . Suppose the graviton lands on the external line i with momentum p_i , mass m_i , spin l_i , and polarisation tensor $\epsilon_{\mu_1 \dots \mu_{l_i}}^i$ and write the original amplitude as

$$\mathcal{M} = \mathcal{M}_i^{\mu_1 \dots \mu_{l_i}} \epsilon_{\mu_1 \dots \mu_{l_i}}^i. \quad (3.93)$$

Adding the graviton requires a cubic vertex containing the graviton, spin- l_i particle, and some internal spin- l'_i particle,

$$\Gamma_{\mu_1 \dots \mu_{l_i} \nu_1 \dots \nu_{l'_i} \alpha \beta}. \quad (3.94)$$

The relevant cubic interactions are the ones that dominate in the soft limit. In the next subsection it is shown that the most important interaction in the soft limit is the minimal coupling interaction $z_{23}^{l_i} z p_{12}^2$, which couples the graviton to two spin- l_i particles. This simplifies the problem enormously since only this interaction needs to be considered,

$$\Gamma_{\mu_1 \dots \mu_{l_i} \nu_1 \dots \nu_{l'_i} \alpha \beta}^i = \kappa_i \eta_{\mu_1 \nu_1} \dots \eta_{\mu_{l'_i} \nu_{l'_i}} p_\alpha^i p_\beta^i, \quad (3.95)$$

where κ_i is some coupling constant. The modified amplitude is then given by

$$\mathcal{M}'_i = \mathcal{M}_i^{\mu_1 \dots \mu_{l_i}} \frac{\prod_{\mu_1 \dots \mu_{l_i} \nu_1 \dots \nu_{l_i}} \Gamma_i^{\nu_1 \dots \nu_{l_i} \lambda_1 \dots \lambda_{l_i} \alpha \beta} \epsilon_{\lambda_1 \dots \lambda_{l_i}}^i \epsilon_{\alpha \beta}, \quad (3.96)$$

where $\epsilon_{\alpha \beta}$ is the graviton polarisation. The numerator of the spin- l_i propagator satisfies the on-shell completeness relation

$$\prod_{\mu_1 \dots \mu_{l_i} \nu_1 \dots \nu_{l_i}} = \sum_a \epsilon_{\mu_1 \dots \mu_{l_i}}^{(a)} \epsilon_{\nu_1 \dots \nu_{l_i}}^{*(a)}, \quad (3.97)$$

where a ranges over a basis of polarisations for the spin- l_i representation. Now substitute in the minimal coupling interaction (3.95) and take the soft limit $q \rightarrow 0$. Using the completeness relation and the orthonormality of the polarisation states

$$\epsilon_{\nu_1 \dots \nu_{l_i}}^{*(a)} \epsilon^{(b) \nu_1 \dots \nu_{l_i}} = \delta^{ab}, \quad (3.98)$$

this gives the result

$$\lim_{q \rightarrow 0} \mathcal{M}'_i = \lim_{q \rightarrow 0} \mathcal{M} \frac{\kappa_i p_i^\alpha p_i^\beta}{2p_i \cdot q} \epsilon_{\alpha \beta}. \quad (3.99)$$

This result has a pole as $q \rightarrow 0$. There are also contributions to \mathcal{M}' from diagrams with the graviton landing on an internal line, but these cannot give poles in the soft limit since their momenta are off-shell and these diagrams are therefore subdominant. In the soft limit \mathcal{M}' is thus obtained by summing over all diagrams with the graviton landing on an external line,

$$\lim_{q \rightarrow 0} \mathcal{M}' = \lim_{q \rightarrow 0} \mathcal{M} \sum_i \frac{\kappa_i p_i^\alpha p_i^\beta}{2p_i \cdot q} \epsilon_{\alpha \beta}. \quad (3.100)$$

Writing $\epsilon_{\alpha \beta} = z_\alpha z_\beta$, gauge invariance for the graviton (3.3) requires invariance under $z_\alpha \rightarrow z_\alpha + \chi q_\alpha$, to first order in χ . This implies the condition

$$\frac{1}{2} z \cdot \sum_i \kappa_i p_i = 0. \quad (3.101)$$

Momentum conservation has already been assumed, $\sum_i p_i = 0$, so (3.101) implies that $\kappa_i = \kappa$. This says that the gravitational minimal coupling constant is the same for all particles, including the graviton itself.

3.8.2 The leading coupling is minimal coupling

The above proof relies crucially on the fact that the minimal coupling vertex (3.95) is the dominant interaction in the soft limit. This is now proven to be true.

Consider a cubic coupling of a graviton to particles with spins l and l' , which are labelled as particles 1, 2, and 3, respectively, and take $l \leq l'$ without loss of generality. When the graviton becomes soft $p_1 \rightarrow 0$, this implies $z p_{31} \rightarrow 0$ and $z p_{23} \rightarrow 0$. Using the formalism of Section 3.2, the on-shell cubic amplitudes in the soft limit must be of the form

$$z_{12}^{n_{12}} z_{13}^{n_{13}} z_{23}^{n_{23}} z p_{12}^{m_{12}}, \quad (3.102)$$

where n_{ij} and m_{12} are non-negative integers satisfying

$$n_{12} + n_{13} + m_{12} = 2, \quad (3.103a)$$

$$n_{12} + n_{23} = l, \quad (3.103b)$$

$$n_{13} + n_{23} = l'. \quad (3.103c)$$

The first equation implies that $n_{13} \leq 2$ and the second equation implies that $n_{23} \leq l$. Combining these conditions with the third equation gives $l' \leq l + 2$. But $l \leq l'$, so there are only the following three cases to consider:

1. $l' = l$
2. $l' = l + 1$
3. $l' = l + 2$

For each of these cases, it will be determined whether there is an amplitude that survives the soft limit and if it can be made invariant under spin-2 gauge transformations, which corresponds to invariance under

$$z_1 \rightarrow z_1 + \chi p_1, \quad (3.104)$$

to first order in χ . To make the soft-limit amplitudes gauge invariant, terms might have to be added that vanish in the soft limit. In general, these extra terms could

have additional factors of momenta compared to the soft-limit amplitudes since the term zp_{12} generates $p_{12} = (m_2^2 - m_3^2)/2$ under (3.104), which reduces the number of momenta in a term. However, if $p_{12} = 0$ then only terms with the same number of momenta can help make the soft amplitudes gauge invariant. It will be assumed for now that $p_{12} = 0$, so only extra terms with the same number of derivatives need to be considered, and the case $p_{12} \neq 0$ will be returned to at the end. The three different cases are now considered.

1. $l' = l$: In this case there are two amplitudes that survive the soft limit,

$$z_{12}z_{13}z_{23}^{l-1}, \quad z_{23}^l z p_{12}^2. \quad (3.105)$$

The first term is the only zero-derivative amplitude and it is not gauge invariant, so it can be ignored for now. The second term is already gauge invariant since it has been assumed that $p_{12} = 0$. This is the minimal coupling amplitude.

2. $l' = l + 1$: In this case there is one amplitude that survives the soft limit,

$$z_{13}z_{23}^l z p_{12}, \quad (3.106)$$

which has one derivative. There are also two one-derivative amplitudes that vanish in the soft limit,

$$z_{13}^2 z_{23}^{l-1} z p_{23}, \quad z_{12}z_{13}z_{23}^{l-1} z p_{31}, \quad (3.107)$$

which could be added to (3.106) to give a gauge invariant amplitude. However, no combination of these amplitudes is invariant under (3.104), so (3.106) cannot be made gauge invariant if $p_{12} = 0$.

3. $l' = l + 2$: This case also has one amplitude that survives the soft limit,

$$z_{13}^2 z_{23}^l. \quad (3.108)$$

This is the only zero-derivative term and it is not gauge invariant.

This shows that if $p_{12} = 0$, then the minimal coupling amplitude is the only gauge invariant amplitude that survives the soft limit. One possible loop-hole is if an amplitude is gauge invariant in four-dimensions by use of the dimensionally-dependent identity

$$m^2 (z_{13}z p_{23} + z_{12}z p_{31})^2 = 2z p_{12}z p_{23}z p_{31} (z_{23}z p_{12} + z_{13}z p_{23} + z_{12}z p_{31}), \quad (3.109)$$

but there are too many derivatives in (3.109) for this to work.

When $p_{12} \neq 0$, there are other gauge invariant amplitudes that survive the soft limit. These require adding soft-limit-vanishing higher-derivative terms to the soft-limit amplitudes found above. The resulting amplitudes are inhomogeneous in derivatives and are included in Metsaev's classification of all cubic interactions [107], but I do not know of any theory that generates these amplitudes. When $l' = l$, adding higher-derivative terms to $z_{12}z_{13}z_{23}^{l-1}$ gives the gauge invariant amplitude

$$z_{23}^{l-1} (p_{12}z_{12} + z p_{12}z p_{23}) (p_{12}z_{13} - z p_{12}z p_{31}). \quad (3.110)$$

The simplest example of this is a massless spin-2 particle coupled to two massive vectors with different masses. This amplitude does not have the required gauge invariance for one of the spin- l particles to be massless, except when $l = 1$. When $l' = l+2$, adding higher-derivative terms to $z_{13}^2 z_{23}^l$ gives the gauge invariant amplitude

$$z_{23}^l (p_{12}z_{13} - z p_{12}z p_{31})^2. \quad (3.111)$$

The simplest example of this is a massless spin-2 particle coupled to a scalar and a massive spin-2 particle. This amplitude does not have the required gauge invariance for the other particles to be massless. Although these interactions survive the soft limit, they are subleading to the minimal coupling interactions in the soft limit. This is because they do not produce poles in the soft limit since the denominator of the propagator contains the finite term $m_3^2 - m_2^2 \neq 0$. This means that these terms can be ignored for Weinberg's soft limit argument and only the gravitational minimal coupling is important.

3.8.3 Massless higher spins cannot couple minimally

Now it is shown that massless higher-spin particles cannot couple minimally to gravity. Under a spin- l gauge transformation, $z_2 \rightarrow z_2 + \chi p_2$ and $z_3 \rightarrow z_3 + \chi p_3$, the minimal coupling amplitude transforms as

$$z_{23}^l z p_{12}^2 \rightarrow z_{23}^l z p_{12}^2 + l \chi z p_{12}^2 z_{23}^{l-1} (z p_{23} - z p_{31}). \quad (3.112)$$

Making this gauge invariant requires adding two-derivative terms that vanish in the soft limit. Terms with more than two derivatives do not need to be considered since these cannot help make the minimal term gauge invariant when all particles are massless. There are six two-derivative terms, given by

$$\begin{aligned} & z_{23}^l z p_{12}^2, \quad z_{13} z_{23}^{l-1} z p_{12} z p_{23}, \quad z_{12} z_{23}^{l-1} z p_{12} z p_{31}, \\ & z_{13}^2 z_{23}^{l-2} z p_{23}^2, \quad z_{12}^2 z_{23}^{l-2} z p_{31}^2, \quad z_{12} z_{13} z_{23}^{l-2} z p_{23} z p_{31}. \end{aligned} \quad (3.113)$$

There are three combinations of these six terms that are invariant under the spin-2 gauge transformation. A general linear combination of the gauge invariant terms is proportional to

$$z_{23}^l z p_{12}^2 + \alpha_1 z_{23}^{l-1} z p_{12} (z_{13} z p_{23} + z_{12} z p_{31}) + \alpha_2 z_{23}^{l-2} (z_{13} z p_{23} + z_{12} z p_{31})^2. \quad (3.114)$$

Under a spin- l gauge transformation, the variation of the combination (3.114) is given by $\chi z_{23}^{l-3} (z p_{23} - z p_{31})$ multiplied by the term

$$z_{23}^2 z p_{12}^2 (l - \alpha_1) + (z_{13} z p_{23} + z_{12} z p_{31}) [z p_{12} z_{23} (\alpha_1 (l - 1) - 2\alpha_2) + \alpha_2 (l - 2) (z_{13} z p_{23} + z_{12} z p_{31})]. \quad (3.115)$$

This term is required to vanish if the spin- l particles are massless. There are the following three cases:

1. $l = 1$: there is spin-1 gauge invariance if $\alpha_1 = 1$ and $\alpha_2 = 0$. This describes massless spin-1 particles minimally coupled to gravity.
2. $l = 2$: there is spin-2 gauge invariance if $\alpha_1 = 2\alpha_2 = 2$. This gives the GR cubic coupling (3.18b), which shows that the graviton in GR is minimally coupled to itself.

3. $l > 3$: there is no choice of α_i that makes the gauge variation vanish.

This proves that massless bosonic particles with spin $l > 2$ cannot couple minimally to gravity.

3.8.4 Putting it all together

Now the previous results can be combined to rule out the coupling of massless higher-spin particles to anything that couples to gravity. It was found above that the graviton minimally couples to itself in GR. By Weinberg's soft limit argument, this implies that every particle that couples to gravity, or couples to a particle that couples to gravity, must also couple minimally to gravity with the same constant. But massless higher-spin particles do not have minimal gravitational couplings, so in flat spacetime they cannot couple to anything that couples to gravity.

4

Positivity Constraints

4.1 Introduction

Not every EFT is created equal. If an EFT contains a ghost at low-energies—a field with the wrong sign kinetic term in the Lagrangian—then the vacuum is completely unstable and the theory is unphysical. Having a ghost at the cutoff of the EFT is permissible, since the unstable degree of freedom cannot be excited within the regime of validity of the theory. The effects of dangerous higher-derivative operators are treated perturbatively in the EFT and any instabilities are assumed to be resolved by the unknown UV completion [137]. This means that the effective theory breaks down at the cutoff, since full unitarity is violated there.

Tachyons are another type of instability. These fields have the wrong sign for the mass squared in the Lagrangian. A tachyon signals that you have expanded around an unstable vacuum and results in instabilities on time-scales of order of the inverse mass. Such instabilities are less severe than ghosts, since the theory can evolve nonperturbatively to a new stable vacuum. This type of instability occurs in bosonic string theory and is remedied by putting supersymmetry on the world-sheet [9].

Ensuring the absence of ghosts and tachyons is a basic consistency condition for field theories. But there are other consistency conditions that can be imposed on an EFT, if certain assumptions are made about a hypothetical UV completion. In

particular, if an EFT admits a Lorentz-invariant UV completion with an analytic S -matrix, then this can impose nontrivial constraints on operator coefficients in the low-energy theory [138]. These constraints are achieved through analytic dispersion relations that relate IR amplitudes to positive integrals, so the resulting constraints are called positivity constraints.

Positivity constraints can be used to determine if massive gravity theories can admit analytic UV completions. The helicity-0 decoupling limit interactions of the dRGT and pseudo-linear massive spin-2 theories are described by the Galileons. It was argued by Adams et al. [138] that the Galileons are unable to satisfy positivity constraints and hence cannot admit an analytic UV completion, so it might have been assumed that massive gravity inherits the same problems. However, it was shown by Cheung and Remmen [139] that dRGT massive gravity can satisfy certain four-particle positivity constraints in a compact region of parameter space, which leaves open the possibility of finding a UV completion. These conditions are necessary but not sufficient for the existence of an analytic UV completion—it is possible that other constraints rule out the remaining dRGT parameter space. Also, strictly speaking, dRGT may already be complete, since the theory is ghost free, but this would require an understanding of strongly coupled quantum effects, along the lines of asymptotic safety [140].

There are several assumptions made about the hypothetical UV completion when deriving these constraints: it must be Lorentz invariant and unitary; it must be analytic, which means that scattering amplitudes are the real boundary values of an analytic function of the complexified kinematic invariants up to poles and branch cuts as dictated by unitarity; and it must satisfy crossing symmetry and polynomial boundedness. Analyticity encodes causality and polynomial boundedness is linked to locality.

In this chapter, the four-particle forward limit positivity constraints are applied to the pseudo-linear theory. These constraints are also applied to self-interacting vector theories with the property that their longitudinal modes have Galileon interactions in the decoupling limit. Both the pseudo-linear massive spin-2 theory

and the vector theories contain the Galileons in a certain limit, but they are simpler than dRGT massive gravity so it might be easier to find UV completions of them if they satisfy the positivity constraints. The result is that the pseudo-linear theory is unable to satisfy the positivity constraints, unlike dRGT, but that certain vector theories can satisfy them. The failure to satisfy these constraints does not mean that the EFT is inconsistent, only that any UV completion is not a local Lorentz-invariant field theory of the usual type. This chapter is based on work that was completed in collaboration with Kurt Hinterbichler and Rachel Rosen [141].

4.2 Analytic dispersion relations

The positivity constraints are derived following the procedure described in [138, 139, 142]. The four-particle amplitude $\mathcal{M}(s, t)$ is considered in the crossing-symmetric forward limit using the kinematic set up described in Subsection 3.4.2, except that the transverse vector polarisations (3.39) are changed as $\epsilon_\mu^{(a)} \rightarrow i\epsilon_\mu^{(a)}$, for $a = 1, 2$, to simplify the crossing relations. The forward limit $\theta \rightarrow 0$ corresponds to $t \rightarrow 0$, as can be seen from (3.32), which is a well-defined limit due to the mass gap. With particles 3 and 4 outgoing, the forward limit implies that $k^1 = k^3$ and $k^2 = k^4$. Crossing symmetry in the forward limit requires that

$$\epsilon_{\mu\nu}^{3*} = \epsilon_{\mu\nu}^1 \quad \text{and} \quad \epsilon_{\mu\nu}^{4*} = \epsilon_{\mu\nu}^2, \quad (4.1)$$

and that the coefficients α_a^i from (3.41) are real [143]. The crossing symmetric forward amplitude in the full UV-complete theory is given by $\mathcal{M}(s, 0)$.

The analytic structure of the crossing-symmetric forward amplitude in the UV theory, $\mathcal{M}(s, 0)$, in the complex s -plane is represented in Figure 4.1. There are simple poles at $s = m^2$ and $s = 3m^2$ and there are branch-cuts along the real axis for $s \geq 4m^2$ and $s \leq 0$, which come from reaching the two-particle production threshold inside loops. There are additional poles and branch cuts on top of these ones coming from multi-particle thresholds and new heavy states, but the end result is one large branch cut.

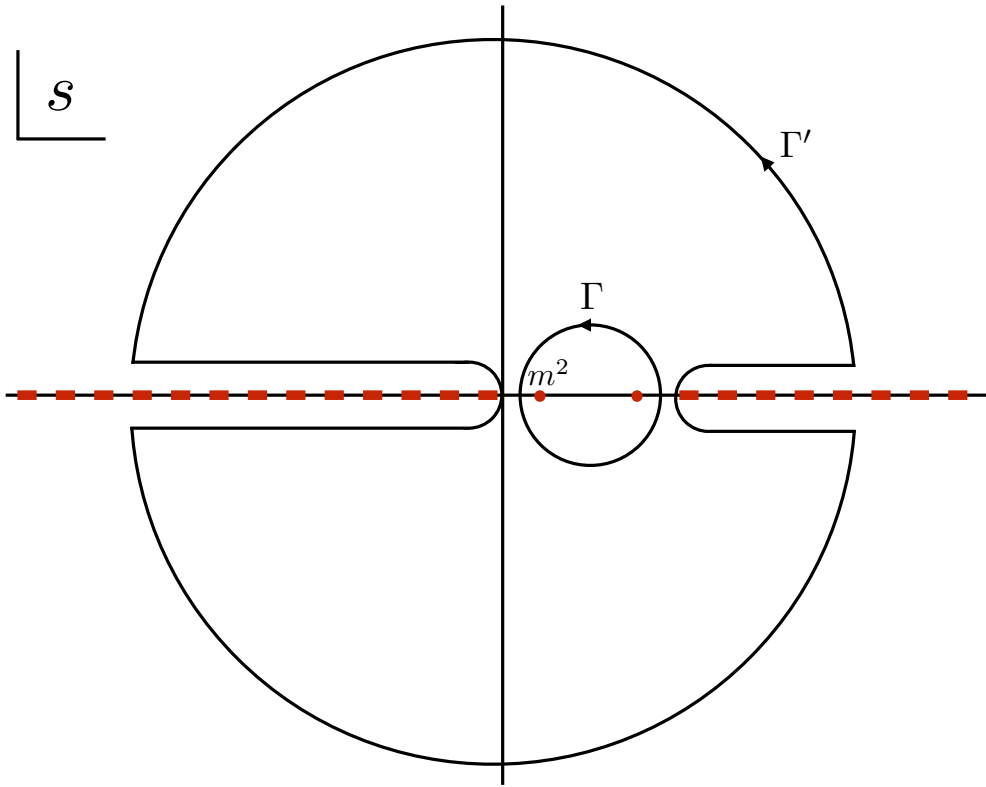


Figure 4.1: Analytic structure of the crossing-symmetric forward amplitude in the complex s -plane. There are simple poles at m^2 and $3m^2$ and branch cuts along the real axis for $s \geq 4m^2$ and $s \leq 0$.

Next consider the contour integral

$$f = \frac{1}{2\pi i} \oint_{\Gamma} ds \frac{M(s, 0)}{(s - \mu^2)^3}, \quad (4.2)$$

where Γ is the contour indicated in Figure 4.1 and μ is an arbitrary mass scale taken to be in the interval $0 < \mu^2 < 4m^2$ and inside the contour. By Cauchy's theorem, the contour can be deformed to another contour Γ' that goes above and below the branch-cuts and includes a large circle at infinity. This theory is unitary and has a mass gap, hence the Froissart bound is satisfied [144, 145]. This bound implies that $\mathcal{M}(s, 0)$ grows slower than s^2 in any complex direction, so the circular contour integral at infinity can be discarded. The integrals across the branch cuts give the discontinuity of the amplitude and by the Schwarz reflection principle this can be related to the imaginary part of the amplitude. The imaginary part is then related to the cross section by the optical theorem. Details are presented

in [139], but the final result is

$$f = \frac{1}{\pi} \int_{4m^2}^{\infty} ds \sigma(s) \left[\frac{s}{(s - \mu^2)^3} + \frac{s}{(s + \mu^2 - 4m^2)^3} \right] \sqrt{1 - \frac{4m^2}{s}}, \quad (4.3)$$

which is a positive quantity in an interacting theory. The expression (4.2) for f can be evaluated in the low-energy EFT, since it only depends on the residues of the poles at low-energies. Demanding positivity of the right-hand-side of (4.2) then gives the required constraints.

4.3 Constraining effective field theories

4.3.1 Pseudo-linear constraints

The positivity constraints are now applied to the pseudo-linear massive spin-2 theory. The theory is defined by the three parameters λ_1 , λ_2 , and λ_3 , as reviewed in Chapter 2. One of these parameters can be set to one by redefining the Planck mass, so there are really only two independent parameters in the absence of matter. The $2 \rightarrow 2$ amplitude is $\mathcal{O}(s^2)$ in the forward limit and it is the s^2 parts of the amplitude that are constrained by positivity. Using notation from Chapter 3, $f(abcd)$ labels f for the scattering configuration $\mathcal{M}(abcd)$. Then two examples of f in the pseudo-linear theory are

$$f(1515) = -\frac{4\lambda_1^2 + 2\lambda_1\lambda_2}{3\Lambda_2^4}, \quad (4.4)$$

$$f(3333) = -\frac{15\lambda_1^2 + 13\lambda_1\lambda_2 + 5\lambda_2^2}{12\Lambda_2^4}. \quad (4.5)$$

There is no choice of λ_1 and λ_2 such that these quantities are both positive, which implies that the pseudo-linear theory cannot satisfy the positivity constraints.

4.3.2 dRGT constraints

It was shown in reference [139] that a certain region of parameter space in dRGT massive gravity can satisfy the positivity constraints. I have reproduced their results, but rather than repeat the details of their calculation, the final result is presented in Figure 4.2. This shows the region of dRGT parameter space that satisfies the constraints.

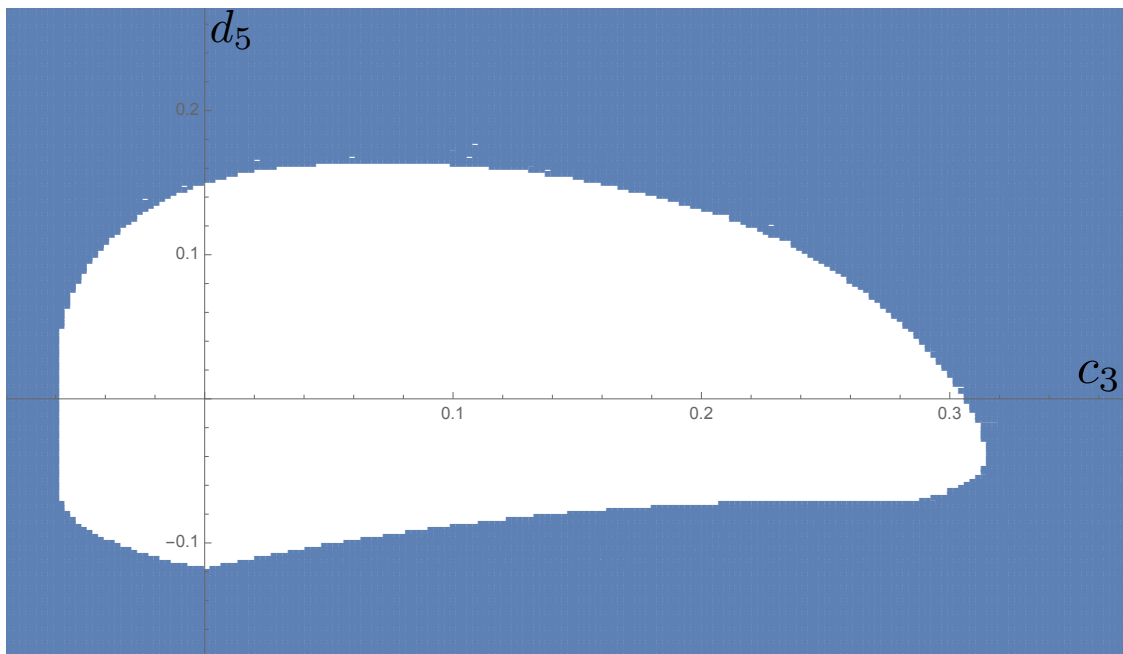


Figure 4.2: The white region shows the dRGT massive gravity parameter space that satisfies the $2 \rightarrow 2$ scattering forward-limit positivity constraints. This plot was produced using the numerical procedure described in [139].

4.3.3 Massive vector constraints

The result of Cheung and Remmen [139] shows that dRGT massive gravity might admit an analytic UV completion. Such a completion would reduce in some limit to the Galileons, so it is interesting to ask whether simpler extensions of the Galileons could also admit UV completions. Theories of ghost-free massive vectors have attracted interest recently [146–154], and there are massive vector theories in which the longitudinal mode of the vector is described by a Galileon in the decoupling limit. The most general two-derivative theory of a massive vector whose helicity-0 mode has Galileon interactions in the decoupling limit is given by

$$\begin{aligned}
\mathcal{L} = & -\frac{1}{4}F_{\mu\nu}^2 + \frac{m^2}{2}A^2 + \lambda_3 \frac{m}{M_p} A^\mu A^\nu \partial_\mu A_\nu + \lambda_4 \frac{1}{M_p^2} A^2 (\partial_\mu A^\mu \partial_\nu A^\nu - \partial_\mu A^\nu \partial_\nu A^\mu) \\
& + \lambda_5 \frac{1}{M_p^2} A_\mu A_\nu \partial_\lambda A^\mu F^{\lambda\nu} + \lambda_6 \frac{1}{2M_p^2} A^2 F_{\mu\nu} F^{\mu\nu} \\
& + \lambda_7 \frac{1}{M_p^2} A_\mu A_\nu (\partial^\mu A_\lambda \partial^\nu A^\lambda - \partial_\lambda A^\mu \partial^\lambda A^\nu), \tag{4.6}
\end{aligned}$$

with $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$. Terms higher than quartic order in the fields are not important since they do not contribute to $2 \rightarrow 2$ scattering. To see the Galileon limit, make the replacement

$$A_\mu \rightarrow A_\mu + \frac{1}{m} \partial_\mu \phi, \quad (4.7)$$

and then take the decoupling limit $m \rightarrow 0$ and $M_p \rightarrow \infty$ with $\Lambda_3 \equiv (m^2 M_p)^{1/3}$ fixed. This gives

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^2 + (\partial\phi)^2 + \frac{\lambda_3}{\Lambda_3^3} \partial^\mu \phi \partial^\nu \phi \partial_\mu \partial_\nu \phi + \frac{\lambda_4}{\Lambda_3^6} (\partial\phi)^2 \left((\square\phi)^2 - \partial_\mu \partial_\nu \phi \partial^\nu \partial^\mu \phi \right), \quad (4.8)$$

which is the action for a cubic and quartic Galileon [101], as reviewed in Chapter 2.

Calculating f for this theory for external vectors with transverse polarisations gives

$$f(1111) = f(2222) = f(1212) = 0, \quad (4.9)$$

which shows that this theory alone cannot satisfy the positivity constraints. However, the constraints are sensitive to higher-derivative operators and such terms can be added without introducing extra degrees of freedom, unlike in the case of the pseudo-linear theory. For example, the Euler-Heisenberg terms,

$$\frac{1}{\Lambda_2^4} \left(c_1 F_\nu^\mu F_\rho^\nu F_\sigma^\rho F_\mu^\sigma + c_2 (F_{\mu\nu} F^{\mu\nu})^2 \right), \quad (4.10)$$

can be added without affecting the decoupling limit. The parity-odd Λ_2 terms from (3.54) could also be added, but the Euler-Heisenberg terms are sufficient to satisfy the positivity constraints. Now calculating f gives

$$f(1111) = f(2222) = \frac{8(c_1 + 2c_2)}{\Lambda_2^4}, \quad f(1212) = \frac{4c_1}{\Lambda_2^4}, \quad (4.11)$$

$$f(1313) = f(2323) = \frac{\lambda_7}{\Lambda_2^4}, \quad f(3333) = -\frac{\lambda_3^2 + 2\lambda_5 - 4\lambda_7}{\Lambda_2^4}, \quad (4.12)$$

which are not inconsistent with positivity. It is possible to find parameter choices for which the amplitude is completely positive, including external states with arbitrary polarisations. For example, choosing

$$c_1 = \frac{1}{4}, \quad c_2 = -\frac{1}{16}, \quad \lambda_3 = 1, \quad \lambda_4 = \frac{1}{2}, \quad \lambda_5 = 1, \quad \lambda_6 = 0, \quad \lambda_7 = 1, \quad (4.13)$$

gives

$$f = \frac{((\alpha_1^1)^2 + (\alpha_2^1)^2 + (\alpha_3^1)^2)((\alpha_1^2)^2 + (\alpha_2^2)^2 + (\alpha_3^2)^2)}{\Lambda_2^4} = \frac{1}{\Lambda_2^4}, \quad (4.14)$$

where α_a^i is the coefficient of basis polarisation a for particle i , as defined in (3.35), and the normalisation condition (3.36) has been used. This f is manifestly positive for all choices of external polarisations.

4.4 Discussion

In this Chapter, positivity constraints have been derived for the pseudo-linear massive spin-2 theory and massive vector theories that have Galileon decoupling limits. It was found that the pseudo-linear theory is unable to satisfy the constraints, unlike dRGT massive gravity. This implies that the Einstein-Hilbert kinetic term is needed for a ghost-free massive gravity theory to satisfy the positivity constraints. In contrast, the massive vector theories are easily able to satisfy the positivity constraints once four-derivative terms are included. No attempt was made to systematically constrain the parameter space of vector theories that satisfy positivity since there is lots of freedom in writing down such theories. Nevertheless, these massive vector theories provide good examples of alternative IR completions of the Galileons that can admit analytic UV completions.

Failure to obey positivity constraints does not mean that a theory is inconsistent. Rather, it means that any UV completion must violate locality, Lorentz invariance, unitarity, or some other assumption usually made in quantum field theory. In contrast, dRGT massive gravity may possess an ordinary, local and Lorentz invariant UV completion for the parameter space discovered by [139] and shown in Figure 4.2.

Only positivity constraints coming from the forward limit of the four-point tree amplitude have been considered here. It is possible that other constraints further reduce the allowed parameter space. One option is to consider constraints on amplitudes away from the forward limit, so that $t \neq 0$. This has been pursued recently in [155, 156], where it was shown that a massive version of the Galileon can satisfy these additional constraints. There are also arguments that additional

constraints arise by studying causality directly in the low-energy EFT. There is less consensus about the trustworthiness of such results [100, 157, 158], but these are claimed to constrain the dRGT parameter space to the line $c_3 = 1/4$ [159]. It might also be possible to constrain higher-point amplitudes. This is tricky since there are $n(n-3)/2$ independent kinematic invariants for an n -point amplitude, so the analytic structure of the amplitude for $n > 4$ would presumably be very complicated.

5

Kaluza-Klein Dimensional Reduction

5.1 Introduction

5.1.1 Dimensional reduction

Extra spacetime dimensions have proven to be a useful tool in theoretical physics since the pioneering work of Nordström, Kaluza, and Klein [60, 61, 160]. Extra dimensions are required for the consistency of string theory [8, 9], they have been used in attempts to address the weak-scale hierarchy problem [161], they play a central role in the AdS/CFT correspondence [162], and they can also lead to interesting IR modifications of gravity. For example, the higher-dimensional brane construction of DGP contains a resonance of massive gravitons and precipitated much progress in the study of massive gravity [59].

Given a higher-dimensional theory, one is usually interested in the implications for four-dimensional physics. For extra dimensions described by some compact internal manifold, the higher-dimensional fields can be decomposed into sums of eigenfunctions of Laplacian operators on the internal manifold and the coefficients of these eigenfunctions correspond to lower-dimensional fields. The hallmark of any KK reduction is an infinite but discrete tower of modes of increasing masses in the lower-dimensional theory, where these masses are determined by the eigenvalues of the Laplacians.

If the higher-dimensional theory contains massless gauge fields, then dimensional reduction produces an infinite tower of lower-dimensional gauge symmetries. Most of these gauge symmetries are algebraic Stückelberg symmetries [126] for massive modes and can be eliminated by a suitable gauge fixing. There can also be massless zero-mode fields that inherit a true gauge symmetry from the higher-dimensional theory. For example, five-dimensional GR when reduced on a circular extra dimension gives a four-dimensional interacting theory with a tower of massive gravitons, a massless graviton, a massless vector, and a scalar. The five-dimensional diffeomorphism invariance becomes the four-dimensional gauge symmetries of the massless graviton and photon, plus Stückelberg symmetries for the infinite tower of massive gravitons. This can be generalised to the reduction of massless gravity on arbitrary compact internal manifolds and the result is a rich spectrum of lower-dimensional fields depending on the topology and geometry of the internal manifold [163]. See [164, 165] for reviews of KK theory.

5.1.2 Outline of this chapter

This chapter studies the dimensional reduction of massive and partially massless spin-2 fields. The lower-dimensional spectrum that arises from a higher-dimensional spin-2 field is computed for a large class of compact internal manifolds. It is found that varying the mass of the higher-dimensional graviton results in qualitatively different lower-dimensional effective theories. One particularly interesting case is when the higher-dimensional field is partially massless, since then there is a higher-dimensional gauge symmetry. It is shown that this gauge symmetry produces an infinite tower of Stückelberg symmetries, just as with diffeomorphism symmetry, but that there is no zero-mode gauge symmetry. This means that the lower-dimensional theory has no partially massless field. The effect of the gauge symmetry on the zero-modes is instead to eliminate the unstable scalar radion mode. The radion is often a source of instability or an undesired fifth force and many mechanisms have been invented to stabilise or screen such light moduli [43, 166–168], so a mechanism to remove the radion completely via a gauge symmetry could be of interest.

Section 5.2 describes the background spacetimes that will be used for the KK reduction and then discusses the linear massive spin-2 theories that will be dimensionally reduced. A generalised Hodge decomposition for decomposing fields and gauge parameters into eigenmodes of various Laplacian operators is then described. This decomposition does not require gauge fixing and works for any internal geometry. Integrating over the extra dimensions and using the orthogonality of the eigenmodes gives actions for the various lower-dimensional fields, which are presented in Section 5.3. These actions are then diagonalised so that the spectrum and stability for different masses and curvatures can be determined. All of the subtleties associated with zero modes, Killing vectors, and conformal Killing vectors of the internal space are accounted for in this procedure. The procedure is taken from Hinterbichler et al. [163], where it was used to study the reduction of massless spin-2 theories.

The effect of interactions under dimensional reduction can also be studied. In Section 5.4, a nonlinear dimensional reduction is constructed for dRGT massive gravity on a circular extra dimension, keeping only the zero-mode interactions. This is the massive version of the nonlinear KK reduction of GR. The result is a four-dimensional theory of massive gravity interacting with a vector and scalar. Projecting out the scalar gives a particular family of mass-varying massive gravity theories [169, 170] with varying α 's [171]. Because these theories arise as consistent truncations of a ghost-free higher-dimensional theory, they are expected to be ghost free. This chapter is based on work that was completed in collaboration with Kurt Hinterbichler [172]

Conventions and notation: the higher-dimensional spacetime is taken to have $D = (d + N)$ dimensions with indices A, B, \dots and coordinates X^A . It is a product of a d -dimensional spacetime \mathcal{M} and an N -dimensional space \mathcal{N} . \mathcal{M} has spacetime indices μ, ν, \dots and coordinates x^μ and \mathcal{N} has spatial indices m, n, \dots and coordinates y^m .

5.2 Set up

5.2.1 Background spacetime

Let us begin by choosing a suitable direct product background metric on which the higher-dimensional massive graviton can propagate. Massless gravitons are only known to consistently propagate on background spacetimes that are Einstein spaces [63, 64]. Partially massless spin-2 fields are known to propagate on Einstein spaces [68, 173] and more recently have been found to consistently propagate on slightly more general backgrounds [174]. There are no restrictions for general massive gravitons, which can propagate on an arbitrary background [175–177]. This chapter will only consider Einstein spaces since the partially massless limit of massive gravity is especially interesting and the more general backgrounds considered by Bernard et al. [174] are still not well understood, but it would also be interesting to see what happens on non-Einstein backgrounds.

Consider then a $D = (d + N)$ -dimensional background metric G_{AB} that satisfies the condition for an Einstein space,

$$R_{AB} = \frac{R_{(D)}}{D} G_{AB}, \quad (5.1)$$

with constant scalar curvature $R_{(D)}$, which is equivalent to demanding that the metric satisfies Einstein's equations

$$R_{AB} - \frac{1}{2} R_{(D)} G_{AB} + \Lambda_{(D)} G_{AB} = 0 \quad (5.2)$$

with cosmological constant

$$\Lambda_{(D)} = \frac{(D - 2)}{2D} R_{(D)}. \quad (5.3)$$

To perform a KK reduction, the background must be in the form of a product space $\mathcal{M} \times \mathcal{N}$, so the background metric is taken to be of the form

$$G_{AB}(x, y) = \begin{pmatrix} g_{\mu\nu}(x) & 0 \\ 0 & \gamma_{mn}(y) \end{pmatrix}, \quad (5.4)$$

where $g_{\mu\nu}$ is the metric on the d -dimensional spacetime \mathcal{M} with coordinates x^μ and γ_{mn} is the metric on the N -dimensional space \mathcal{N} with coordinates y^m . Writing (5.1) in terms of the lower-dimensional metrics and their curvatures gives

$$R_{\mu\nu} = \frac{R_{(d)}}{d} g_{\mu\nu}, \quad R_{mn} = \frac{R_{(N)}}{N} \gamma_{mn}, \quad (5.5)$$

where

$$R_{(d)} = \frac{2d}{d+N-2} \Lambda_{(D)}, \quad R_{(N)} = \frac{2N}{d+N-2} \Lambda_{(D)}, \quad (5.6)$$

which shows that both \mathcal{M} and \mathcal{N} must also be Einstein spaces. Note also that the curvatures satisfy

$$\frac{R_{(D)}}{D} = \frac{R_{(d)}}{d} = \frac{R_{(N)}}{N}. \quad (5.7)$$

The partially massless masses for each Einstein manifold with more than two dimensions are defined by

$$m_{(D)\text{PM}}^2 \equiv \frac{D-2}{D-1} \frac{R_{(D)}}{D}, \quad m_{(N)\text{PM}}^2 \equiv \frac{N-2}{N-1} \frac{R_{(N)}}{N}, \quad m_{(d)\text{PM}}^2 \equiv \frac{d-2}{d-1} \frac{R_{(d)}}{d}. \quad (5.8)$$

For $R < 0$ these squared masses are all negative. When $R > 0$, equation (5.7) implies that the partially massless mass increases with dimension, $m_{(D)\text{PM}}^2 > m_{(d)\text{PM}}^2$.

5.2.2 Linear action

Consider now massive spin-2 perturbations H_{AB} about the Einstein background G_{AB} . These are described by the Fierz-Pauli Lagrangian in D dimensions,

$$\begin{aligned} \frac{\mathcal{L}_{\text{FP},D}}{\sqrt{-G}} = & -\frac{1}{2} \nabla_C H_{AB} \nabla^C H^{AB} + \nabla_C H_{AB} \nabla^B H^{AC} - \nabla_A H_{(D)} \nabla_B H^{AB} + \frac{1}{2} \nabla_A H_{(D)} \nabla^A H_{(D)} \\ & + \frac{R_{(D)}}{D} \left(H_{AB} H^{AB} - \frac{1}{2} H_{(D)}^2 \right) - \frac{1}{2} m^2 \left(H_{AB} H^{AB} - H_{(D)}^2 \right), \end{aligned} \quad (5.9)$$

where ∇_A is the background covariant derivative, $R_{(D)}$ is the constant curvature given by (5.3), and m is the graviton mass. The Lagrangian (5.9) describes different numbers of propagating degrees of freedom depending on the value of the mass and scalar curvature. Only the dimensional reduction for $m^2 > 0$ will be considered since $m^2 < 0$ necessarily implies a tachyon instability and the case $m = 0$ has been discussed in detail elsewhere [163]. When $R > 0$, the Higuchi bound states that

a spin-2 field is only stable if the mass is greater than or equal to the partially massless mass, $m \geq m_{(D)\text{PM}}$ [18]. The different cases considered are the following:

1. $0 < m < m_{(D)\text{PM}}$, $R > 0$. This describes an unstable massive graviton on a positively curved Einstein background with $(D+1)(D-2)/2$ degrees of freedom, one of which is a ghostlike degree of freedom.
2. $m = m_{(D)\text{PM}}$, $R > 0$. This describes a stable partially massless graviton on a positively curved Einstein background which has the scalar gauge symmetry

$$\delta H_{AB} = \nabla_A \nabla_B \alpha + G_{AB} \frac{R_{(D)}}{D(D-1)} \alpha \quad (5.10)$$

and propagates $(D+1)(D-2)/2 - 1$ degrees of freedom.

3. $m > m_{(D)\text{PM}}$. This describes a stable massive graviton on an Einstein background with $(D+1)(D-2)/2$ degrees of freedom. This includes backgrounds with nonpositive curvature, $R \leq 0$.

There are other values of m^2 that are special from the perspective of lower-dimensional physics and these are discussed below.

5.2.3 Symmetric tensor decomposition

In order to find the d -dimensional KK spectrum, the higher-dimensional field must be decomposed into orthonormal eigenfunctions of various Laplacian operators defined on the internal manifold \mathcal{N} . This is the generalisation to arbitrary manifolds of the familiar Fourier decomposition used in the original KK construction and allows for the dependence of fields on the internal space \mathcal{N} to be integrated out. The field H_{AB} breaks up into pieces $H_{\mu\nu}$, $H_{\mu n}$, and H_{mn} , which transform respectively as a scalar, vector and symmetric tensor on the internal manifold. A decomposition is therefore needed for general scalar, vector and symmetric tensor fields into eigenmodes of appropriate Laplacian operators on an arbitrary compact manifold. The appropriate decomposition is given by the Hodge decomposition for scalar

and vector fields and its generalisation to symmetric tensor fields [178, 179], which for the different components of H_{AB} gives

$$\begin{aligned}
H_{\mu\nu} &= \sum_a h_{\mu\nu}^a \psi_a + \frac{1}{\sqrt{\mathcal{V}_n}} h_{\mu\nu}^0, \\
H_{\mu n} &= \sum_i A_\mu^i Y_{n,i} + \sum_a A_\mu^a \nabla_n \psi_a, \\
H_{mn} &= \sum_{\mathcal{I}} \phi^{\mathcal{I}} h_{mn,\mathcal{I}}^{TT} + \sum_{i \neq \text{Killing}} \phi^i (\nabla_m Y_{n,i} + \nabla_n Y_{m,i}), \\
&\quad + \sum_{a \neq \text{conformal}} \phi^a \left(\nabla_m \nabla_n \psi_a - \frac{1}{N} \nabla^2 \psi_a \gamma_{mn} \right) + \sum_a \frac{1}{N} \bar{\phi}^a \psi_a \gamma_{mn} + \frac{1}{N} \frac{1}{\sqrt{\mathcal{V}_N}} \phi^0 \gamma_{mn}.
\end{aligned} \tag{5.11}$$

The fields ψ_a are a basis of positive orthonormal eigenmodes of the scalar Laplacian on \mathcal{N} with positive eigenvalue λ_a (i.e., excluding the zero mode),

$$\int d^N y \sqrt{\gamma} \psi^{a*}(y) \psi_b(y) = \delta_b^a, \tag{5.12}$$

$$\left(\square_{(y)} + \lambda_a \right) \psi^a(y) = 0, \quad \lambda_a > 0, \tag{5.13}$$

and \mathcal{V}_N is the volume of \mathcal{N}

$$\mathcal{V}_N \equiv \int d^N y \sqrt{\gamma}. \tag{5.14}$$

The fields $Y_{n,i}$ are a basis of orthonormal transverse eigenvectors of the vector Laplacian on \mathcal{N} with eigenvalues λ_i ,

$$\int d^N y \sqrt{\gamma} Y_{i,m}^*(y) Y_j^m(y) = \delta_{ij}, \tag{5.15}$$

$$\nabla_{(y)}^n Y_{i,n}(y) = 0, \tag{5.16}$$

$$\Delta Y_{i,n} \equiv -\square_{(y)} Y_{i,n}(y) + R_n^m Y_{i,m}(y) = \lambda_i Y_{i,n}(y), \quad \lambda_i \geq 0. \tag{5.17}$$

The fields $h_{mn,\mathcal{I}}^{TT}$ are a basis of symmetric transverse traceless orthonormal eigenmodes of the Lichnerowicz operator [180] on \mathcal{N} with eigenvalues $\lambda_{\mathcal{I}}$,

$$\nabla^m h_{mn,\mathcal{I}}^{TT}(y) = \gamma^{mn} h_{mn,\mathcal{I}}^{TT}(y) = 0, \tag{5.18}$$

$$\int d^N y \sqrt{\gamma} h_{mn}^{TT,\mathcal{I}*}(y) h_{\mathcal{J}}^{mn,TT}(y) = \delta_{\mathcal{J}}^{\mathcal{I}}, \tag{5.19}$$

$$\Delta_{L,(y)} h_{mn,\mathcal{I}}^{TT}(y) \equiv -\square_{(y)} h_{mn,\mathcal{I}}^{TT}(y) + \frac{2R^{(N)}}{N} h_{mn,\mathcal{I}}^{TT}(y) - 2R_{mpnq} h_{\mathcal{I}}^{pq,TT}(y) = \lambda_{\mathcal{I}} h_{mn,\mathcal{I}}^{TT}(y). \tag{5.20}$$

The coefficients of the eigenmodes— $h_{\mu\nu}^a$, $h_{\mu\nu}^0$, A_μ^i , A_μ^a , $\phi^{\mathcal{I}}$, ϕ^i , ϕ^a , $\bar{\phi}^a$, and ϕ^0 —are d -dimensional fields. The bases of eigenmodes can always be chosen to be real functions; however it is often convenient to allow them to be complex. Ensuring that the physical fields are real then requires a reality condition, e.g, restricting the $\phi^a(x)$ to satisfy

$$\phi^{a*}(x) = \phi^{\bar{a}}(x), \quad (5.21)$$

where \bar{a} denotes some involution on the set of indices a . The sum over i in the decomposition of H_{mn} in (5.11) excludes Killing vectors, which satisfy

$$\nabla_m Y_{n,i}(y) + \nabla_n Y_{m,i}(y) = 0. \quad (5.22)$$

Nontrivial Killing vectors can exist on closed Einstein manifolds only when $R_{(N)} \geq 0$ and they are precisely the transverse eigenmodes of the vector Laplacian with eigenvalue $\lambda_i = 2R_{(N)}/N$, which is the smallest possible eigenvalue for transverse eigenmodes of the vector Laplacian on a closed Einstein manifold with non-negative curvature. Similarly, the sum over a excludes conformal scalars, which satisfy

$$\nabla_m \nabla_n \psi_a(y) - \frac{1}{N} \square_{(y)} \psi_a(y) \gamma_{mn} = 0. \quad (5.23)$$

Conformal scalars exist only on manifolds that are isometric to the sphere [181] and are precisely the eigenmodes of the scalar Laplacian with eigenvalue

$$\lambda_{a=\text{conformal}} = \frac{R_{(N)}}{N-1}. \quad (5.24)$$

By the Lichnerowicz bound [182], (5.24) is the smallest possible eigenvalue for eigenmodes of the scalar Laplacian on a closed Einstein manifold with positive curvature. See Hinterbichler et al. [163] for more details about the decomposition of H_{AB} and why it takes the form it does.

5.2.4 Reduction of the partially massless gauge transformations

When $m = m_{(D)\text{PM}}$, the D -dimensional Lagrangian acquires a partially massless gauge symmetry, (5.10). To find how this gauge symmetry acts on the lower-dimensional fields, the partially massless gauge parameter must be decomposed into eigenmodes of the internal Laplacian,

$$\alpha = \sum_a \alpha^a \psi_a + \frac{1}{\sqrt{\mathcal{V}_N}} \alpha_0. \quad (5.25)$$

Expanding the gauge transformation (5.10) using (5.11) and (5.25) and equating components then gives the following gauge transformations for the lower-dimensional fields:

$$\begin{aligned} \delta h_{\mu\nu}^a &= \nabla_\mu \nabla_\nu \alpha^a + g_{\mu\nu} \frac{R_{(D)}}{D(D-1)} \alpha^a, & \delta A_\mu^a &= \nabla_\mu \alpha^a, \\ \delta h_{\mu\nu}^0 &= \nabla_\mu \nabla_\nu \alpha^0 + g_{\mu\nu} \frac{R_{(D)}}{D(D-1)} \alpha^0, & \delta A_\mu^i &= 0, \\ \delta \bar{\phi}^a &= \left(\frac{NR_{(D)}}{D(D-1)} - \lambda_a \right) \alpha^a, & \delta \phi^a &= \alpha^a, \quad a \neq \text{conformal} \\ \delta \phi^0 &= \frac{NR_{(D)}}{D(D-1)} \alpha^0, & \delta \phi^i &= 0, \quad i \neq \text{Killing} \\ \delta \phi^{\mathcal{I}} &= 0. \end{aligned} \quad (5.26)$$

The higher-dimensional partially massless symmetry acts as a Stückelberg symmetry on some of the lower-dimensional fields. In particular, the gauge parameters α^0 and α^a can be used to gauge fix

$$\phi^0 = \phi^{a \neq \text{conformal}} = \bar{\phi}^{a = \text{conformal}} = 0. \quad (5.27)$$

This completely fixes the gauge symmetry, which indicates already that there will be no partially massless fields in the lower-dimensional theory. Note that some of the gauge symmetry would remain if

$$\lambda_{a = \text{conformal}} = \frac{NR_{(D)}}{D(D-1)}, \quad (5.28)$$

but this is forbidden by (5.7) and (5.24) for $D > N$.

5.3 Dimensionally reduced action

Substituting the decomposition (5.11) into (5.9) and integrating over the extra dimensions using orthogonality gives the d -dimensional action

$$S = \int d^d x \sqrt{-g} (\mathcal{L}_{\mathcal{I}} + \mathcal{L}_i + \mathcal{L}_0 + \mathcal{L}_a), \quad (5.29)$$

where the \mathcal{L} 's are quadratic Lagrangians containing the d -dimensional fields appearing in the decomposition of H_{AB} . The different terms in (5.29) are

$$\begin{aligned} \mathcal{L}_{\mathcal{I}} &= \sum_{\mathcal{I}} -\frac{1}{2} |\partial\phi^{\mathcal{I}}|^2 - \frac{1}{2} \left(\lambda_{\mathcal{I}} + m^2 - 2\frac{R(d)}{d} \right) |\phi^{\mathcal{I}}|^2, \\ \mathcal{L}_i &= \sum_i -\frac{1}{2} |F_{\mu\nu}^i|^2 - \left(\lambda_i - \frac{2R(d)}{d} \right) |A_\mu^i - \partial_\mu\phi^i|^2 - m^2 |A_\mu^i|^2 - \frac{1}{2} m^2 \left(\lambda_i - \frac{2R(d)}{d} \right) |\phi^i|^2, \\ \mathcal{L}_0 &= \varepsilon (h^0)_{m^2} + h^{\mu\nu,0} (\nabla_\mu \nabla_\nu \phi^0 - \square \phi^0 g_{\mu\nu}) + h^0 \phi^0 \left(m^2 - \frac{R(d)}{d} \right) \\ &\quad + \frac{N-1}{2N} \left[(\partial\phi^0)^2 + (\phi^0)^2 (m^2 - m_{(N)\text{PM}}^2) \right], \\ \mathcal{L}_a &= \sum_a \epsilon (h^a)_{m^2 + \lambda_a} - \frac{1}{2} \lambda_a |F_{\mu\nu}^a|^2 + \lambda_a \left(\frac{2R(d)}{d} - m^2 \right) |A_\mu^a|^2 + \frac{N-1}{2N} (|\partial_\mu \bar{\phi}^a|^2 + m^2 |\bar{\phi}^a|^2) \\ &\quad + \frac{1}{2} \left[2\lambda_a h^{\mu\nu,a*} (\nabla_\mu A_\nu^a - g_{\mu\nu} \nabla^\alpha A_\alpha^a) + h^{\mu\nu,a*} (\nabla_\mu \nabla_\nu \bar{\phi}^a - g_{\mu\nu} \square \bar{\phi}^a) \right. \\ &\quad \left. + h^{*a} \bar{\phi}^a \left(\frac{N-1}{N} \lambda_a + m^2 - \frac{R(d)}{d} \right) - 2\lambda_a \frac{N-1}{N} \nabla^\mu A_\mu^{a*} \bar{\phi}^a + \text{c.c} \right] \\ &\quad + \frac{N-1}{N} \lambda_a \left(\lambda_a - \frac{R(N)}{N-1} \right) \left[\frac{N-2}{2N\lambda_a} |\lambda_a \phi^a + \bar{\phi}^a|^2 - \frac{1}{2} |\partial\phi^a|^2 - \frac{1}{2} m^2 |\phi^a|^2 \right. \\ &\quad \left. + ((h^a/2 - \nabla_\mu A_\mu^a) \phi^{a*} + \text{c.c}) \right], \end{aligned} \quad (5.30)$$

where $\epsilon(h)_{m^2}$ is the d -dimensional Fierz-Pauli action (2.5).

5.3.1 Diagonalization

Determining the spectrum and stability of S requires diagonalising \mathcal{L}_i , \mathcal{L}_0 and \mathcal{L}_a with field transformations that depend on m^2 and the curvatures of the spaces involved. The final diagonalised Lagrangians \mathcal{L}_i , \mathcal{L}_0 and \mathcal{L}_a are presented below, with the details of the required field transformations presented in the appendix to this chapter. Although the fields in the diagonalised Lagrangians are combinations

of the fields in the original Lagrangian, they are given the same names to avoid clutter. The following definition is useful:

$$m_k^2 \equiv m^2 + \lambda_k - \frac{2R_{(d)}}{d}, \quad (5.31)$$

where $k = \mathcal{I}, i, 0$, or a and $\lambda_0 = 0$.

Diagonalised \mathcal{L}_i

The diagonalised Lagrangian \mathcal{L}_i is given by

$$\begin{aligned} \mathcal{L}_i = & \sum_{i \neq \text{Killing}} -\frac{1}{2} |F_{\mu\nu}^i|^2 - m_i^2 |A_\mu^i|^2 - \frac{m^2(m_i^2 - m^2)}{m_i^2} \left(|\partial\phi^i|^2 + \frac{1}{2} m_i^2 |\phi^i|^2 \right) \\ & + \sum_{i=\text{Killing}} -\frac{1}{2} |F_{\mu\nu}^i|^2 - m^2 |A_\mu^i|^2. \end{aligned} \quad (5.32)$$

Diagonalised \mathcal{L}_0

When $m \neq m_{(d)\text{PM}}$ and $d > 1$, the diagonalised \mathcal{L}_0 is

$$\mathcal{L}_0 = \epsilon(h^0)_{m^2} - \frac{D-1}{2N(d-1)} \frac{m^2 - m_{(D)\text{PM}}^2}{m^2 - m_{(d)\text{PM}}^2} \left((\partial\phi^0)^2 + m_0^2 (\phi^0)^2 \right). \quad (5.33)$$

When $m = m_{(d)\text{PM}}$,

$$\mathcal{L}_0 = \epsilon(h^0)_{m_{(d)\text{PM}}^2} - \frac{1}{4} m_{(d)\text{PM}}^2 h_0^2. \quad (5.34)$$

When $d = 1$, many of the terms vanish and \mathcal{L}_0 reduces to a nondynamical Lagrangian.

Diagonalised \mathcal{L}_a

When $m_a \neq 0$, $N > 1$, and $d > 1$, the Lagrangian \mathcal{L}_a is

$$\begin{aligned} \mathcal{L}_a = & \sum_a \epsilon(h^a)_{m^2 + \lambda_a} - \frac{m^2 \lambda_a}{m^2 + \lambda_a} \left(\frac{1}{2} |F_{\mu\nu}^a|^2 + m_a^2 |A_\mu^a|^2 \right) \\ & - \sigma_a \left(|\partial\phi^a|^2 + m_a^2 |\phi^a|^2 \right) - \bar{\sigma}_a \left(|\partial\bar{\phi}^a|^2 + m_a^2 |\bar{\phi}^a|^2 \right), \end{aligned} \quad (5.35)$$

where σ_a and $\bar{\sigma}_a$ are functions of λ_a , R , d and N that are defined in the appendix – in particular, $\sigma_{a=\text{conformal}} = 0$. When $m_a = 0$, \mathcal{L}_a can be written as

$$\begin{aligned} \mathcal{L}_a = & \sum_{a, m_a^2=0} \epsilon(h^a)_{2R_{(d)}/d} - \frac{m^2 \lambda_a}{m^2 + \lambda_a} \left(|\partial_\mu A_\nu^a|^2 + \frac{R_{(d)}}{d} |A_\mu^a|^2 \right) \\ & - \frac{(N-1)(D-1) \left(m^2 - m_{(N)\text{PM}}^2 \right) \left(m^2 - m_{(D)\text{PM}}^2 \right)}{2d} |\partial\phi^a|^2. \end{aligned} \quad (5.36)$$

Achieving $m_a = 0$ requires tuning m and λ_a but can occur for $m \leq m_{(N)\text{PM}}$. When $N = 1$, all scalars are conformal and for $d > 1$

$$\begin{aligned} \mathcal{L}_a = & \sum_{a=\text{conformal}} \epsilon(h^a)_{m_a^2} - \frac{m^2 \lambda_a}{m_a^2} \left(\frac{1}{2} |F_{\mu\nu}^a|^2 + m_a^2 |A_\mu^a|^2 \right) \\ & - \frac{dm^4}{2(d-1)m_a^4} \left(|\partial_\mu \bar{\phi}^a|^2 + m_a^2 |\bar{\phi}^a|^2 \right). \end{aligned} \quad (5.37)$$

Lastly, when $d = 1$

$$\mathcal{L}_a = - \frac{N-1}{2N} \sum_{a \neq \text{conformal}} \left(|\partial_\mu \bar{\phi}^a|^2 + m_a^2 |\bar{\phi}^a|^2 \right). \quad (5.38)$$

5.3.2 Spectrum and stability

Now that the action is diagonalised, its spectrum and stability can be determined. The spectrums of $\mathcal{L}_{\mathcal{I}}$ and \mathcal{L}_i are straightforwardly determined, so these are discussed first. Next the spectrums of \mathcal{L}_0 and \mathcal{L}_a are discussed for the three cases where m is less than, equal to, or larger than the D -dimensional partially massless mass $m_{(D)\text{PM}}$. It is assumed that $d > 1$ and $N > 1$ until the end where the special cases $N = 1$ and $d = 1$ are discussed. A summary of these stability results is given at the end of this section.

Spectrum of $\mathcal{L}_{\mathcal{I}}$

The Lagrangian $\mathcal{L}_{\mathcal{I}}$ describes a tower of massive scalars with squared masses given by $m_{\mathcal{I}}^2$, where $m_{\mathcal{I}}^2$ is defined by (5.31). The stability of a scalar mass term depends on the background curvature. For a flat or de Sitter space stability requires $m_{\mathcal{I}}^2 \geq 0$ and for an anti-de Sitter space stability requires

$$m_{\mathcal{I}}^2 \geq \frac{d-1}{4} \frac{R_{(d)}}{d}, \quad (5.39)$$

which is the Breitenlohner-Freedman bound [183, 184]. These bounds hold for maximally symmetric spaces but will also be taken as the conditions for stability for the more general case of an Einstein space. There exist positively curved compact manifolds with large negative eigenvalues of the Lichnerowicz operator, such as the Böhm metrics on products of spheres [185]. This means that the

scalars $\phi^{\mathcal{I}}$ can be tachyonic, but for any given internal space \mathcal{N} there is some smallest graviton mass that stabilises them.

Spectrum of \mathcal{L}_i

The spectrum of \mathcal{L}_i is a tower of massive vectors with squared masses given by m_i^2 and a tower of massive scalars with squared masses given by $m_i^2/2$, one scalar and vector for each non-Killing transverse vector of the internal space, and a massive vector with mass m for each Killing vector of the internal space. For a non-Killing vector the eigenvalue satisfies $\lambda_i > 2R_{(N)}/N$ and this implies that $m_i^2 > m^2 > 0$. Stability of vectors requires only that their mass is positive and hence all fields are stable in this sector.

Spectrum of \mathcal{L}_0 and \mathcal{L}_a for $0 < m < m_{(D)\text{PM}}$

Consider the spectrum of \mathcal{L}_0 and \mathcal{L}_a for the mass range $0 < m < m_{(D)\text{PM}}$, which is relevant when $D > 2$ and $R > 0$. This corresponds to a higher-dimensional theory with an unstable graviton, since its mass falls below the Higuchi bound, so instabilities are expected in the lower-dimensional theory. The spectrum consists of a zero-mode massive scalar and a zero-mode massive graviton, plus a massive scalar, vector and tensor for each conformal scalar. There is also a tower of massive gravitons, a tower of massive vectors, and two towers of massive scalars, one field in each of the towers for each nonconformal scalar in the internal space. The stability of this spectrum for each spin is discussed below.

Spin 2: The zero-mode graviton has mass m and thus by the Higuchi bound is unstable for $m < m_{(d)\text{PM}}$ and stable for $m > m_{(d)\text{PM}}$. When $m = m_{(d)\text{PM}}$, the zero-mode sector Lagrangian takes the unusual form presented in equation (5.34). A 3 + 1 analysis reveals that for $d = 4$ this Lagrangian describes four partially massless modes in addition to two scalar modes, one of which is a ghost.¹ In fact,

¹Another way to see the ghost once it is known that there are six degrees of freedom is to perform the Stückelberg transformation

$$h_{\mu\nu}^0 \rightarrow h_{\mu\nu}^0 + \nabla_\mu \nabla_\nu \varphi + g_{\mu\nu} \frac{m_{(d)\text{PM}}^2}{(d-2)} \varphi, \quad (5.40)$$

which introduces a new scalar field φ . This transformation has the form of a partially massless

it is not possible to transform (5.34) to a set of decoupled kinetic terms of familiar fields. The quadratic terms describe a new kind of irreducible representation that is a combination of a spin 2 and a spin 0 on curved space; this is a field theoretic manifestation of a so-called “extended module” which mixes spin 0 and spin 2, seen in the Hilbert space of the non-unitary \square^2 conformal field theory in three dimensions [186]. An equivalent form of (5.34) occurs in the holographic dual of the \square^2 conformal field theory [187]. The heavier spin-2 fields have squared masses $m^2 + \lambda_a > 0$ and are always stable.

Spin 1: For $m_a \neq 0$ the massive vectors have mass m_a , defined in (5.31), so their stability depends on m and λ_a . For $m < m_{(N)\text{PM}}$, the massive vectors corresponding to conformal scalars are tachyonic since

$$m_{a=\text{conformal}}^2 = m^2 - m_{(N)\text{PM}}^2 < 0, \quad (5.42)$$

and the massive vectors corresponding to nonconformal scalars can be tachyonic or stable depending on the sign of m_a^2 . For $m > m_{(N)\text{PM}}$ the massive vectors are all stable. When $m_a = 0$ the spatial components of the vector field describe $d - 1$ scalars with squared masses equal to $R_{(d)}/d$, which violate the Breitenlohner-Freedman stability bound when $R < 0$ and $d < 5$, and the temporal component of the vector field is a scalar ghost.

Spin 0: The zero-mode scalar—the radion mode—is tachyonic for $m < m_{(d)\text{PM}}$ and ghostly for $m_{(d)\text{PM}} < m < m_{(D)\text{PM}}$. When $m = m_{(d)\text{PM}}$ the zero-mode scalar gets folded into the graviton field giving (5.34). The scalars coming from the conformal modes are tachyonic for $m < m_{(N)\text{PM}}$ and ghostly for $m_{(N)\text{PM}} < m < m_{(D)\text{PM}}$. The scalars in the two towers, corresponding to nonconformal scalars in the internal transformation so only affects the $|h_0|^2$ term. The resultant Lagrangian is then invariant under the partially massless Stückelberg symmetry

$$\delta h_{\mu\nu} = \nabla_\mu \nabla_\nu \alpha + g_{\mu\nu} \frac{m_{(d)\text{PM}}^2}{(d-2)} \alpha, \quad \delta \varphi = -\alpha. \quad (5.41)$$

Now rescale $\varphi \rightarrow \varphi/m$ and take the decoupling limit $m \rightarrow 0$ while preserving the partially massless relation $m = m_{(d)\text{PM}}$. In this limit, the partially massless kinetic term breaks up into massless tensor and vector modes and the surviving scalar term is higher derivative $\sim \varphi \square^2 \varphi$, which shows that there is a scalar ghost.

space, are both tachyonic if $m_a^2 < 0$ and have one healthy scalar and one ghost if $m_a^2 > 0$. When $m_a = 0$ there is one massless scalar if $a \neq$ conformal and no scalars if $a =$ conformal, since then $m = m_{(N)\text{PM}}$. The scalar spectrum coming from the nonconformal modes of the internal space is summarised in Figure 5.1.

Spectrum of \mathcal{L}_0 and \mathcal{L}_a for $m = m_{(D)\text{PM}}$

Now consider the case when the higher-dimensional spin-2 field is partially massless, $m = m_{(D)\text{PM}}$. The spectrum consists of a tower of massive gravitons, a tower of massive vectors, and a tower of massive scalars. There is also a massive graviton and massive vector for each conformal mode, and a zero-mode massive graviton. All of these fields are stable. The effect of the higher-dimensional partially massless gauge symmetry is to remove a tower of scalar fields, including the would-be unstable radion, which is consistent with the gauge fixing (5.27). There are no partially massless fields in the lower-dimensional spectrum since the lightest massive graviton has mass $m_{(D)\text{PM}}$ and this is larger than $m_{(d)\text{PM}}$.

Spectrum of \mathcal{L}_0 and \mathcal{L}_a for $m_{(D)\text{PM}} < m$

Lastly, consider the case of a stable higher-dimensional massive graviton, $m > m_{(D)\text{PM}}$. This includes all cases when $R \leq 0$. The field content is the same as for $m < m_{(D)\text{PM}}$. The zero-mode scalar is tachyonic for $m_{(D)\text{PM}}^2 < m^2 < 2R_{(d)}/d$ and nontachyonic for $m^2 > 2R_{(d)}/d$. The scalar is massless when $m^2 = 2R_{(d)}/d$. All other fields are stable.

Spectrum when $d = 1$ or $N = 1$

When $N = 1$, i.e. when \mathcal{N} is the circle S^1 with radius r , there are no symmetric transverse traceless tensors on \mathcal{N} so $\mathcal{L}_{\mathcal{I}}$ is zero. There are no non-Killing transverse vectors and the only Killing vector is the constant vector on the circle, so \mathcal{L}_i contains a single vector with mass m , which is a massive version of the original KK photon. There is a zero-mode massive graviton and a zero-mode massive scalar, both with mass m . The interactions of these zero-mode fields are found in Section 5.4 by

dimensionally reducing dRGT massive gravity. There is also a tower of massive gravitons, vectors and scalars all with squared masses given by

$$m_a^2 = m^2 + \frac{a}{r^2}, \quad (5.43)$$

where $a = 1, 2, 3, \dots$. When $d = 1$ there are massive scalars for each Lichnerowicz mode, each non-Killing mode and each nonconformal mode.

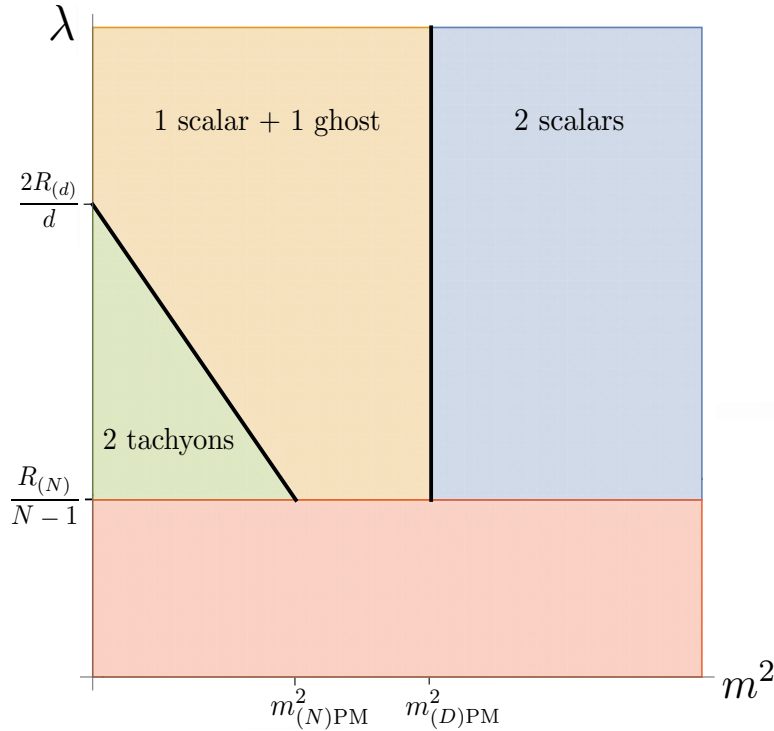


Figure 5.1: Plot of the field content of the nonconformal scalar sector in the (m^2, λ) plane for $R > 0$, $d > 1$, $N > 2$. The lower rectangle corresponds to eigenvalues at or below the Lichnerowicz bound. The rightmost region has two healthy scalars, the adjacent region has one healthy scalar and one ghost, and the triangular region has two tachyons. The angled line is $m_a = 0$ and the vertical line is $m = m_{(D)PM}$, which has one healthy scalar.

Summary of stability results

Avoiding the radion instability requires either $m^2 \geq 2R_{(D)}/D$ or $m^2 = m_{(D)PM}^2$. The partially massless value thus corresponds to an isolated line of stability in the (m^2, R) plane since the partially massless gauge symmetry renders the unstable

radion pure gauge. There may be other instabilities depending on the size of m^2 and the spectrum of eigenvalues of the Lichnerowicz operator on the internal space.

5.4 Interactions

So far only quadratic actions have been considered. Including interactions requires starting with the nonlinear D -dimensional version of dRGT massive gravity. dRGT massive gravity itself can be derived from “dimensional deconstruction” of Einstein gravity [89, 188, 189], which can be thought of as a kind of discrete KK reduction. Since there is no known self-interacting theory of a single partially massless spin-2 field [65, 72, 74, 77], this case cannot be addressed nonlinearly.

It is possible, if tedious, to include interactions in the lower-dimensional theory by substituting the decomposition of H_{AB} order by order into the higher-dimensional interactions and integrating over the resulting products of eigenmodes. Doing this to all orders gives a lower-dimensional rewriting of the higher-dimensional theory. For example, in the original KK setup, $D = 5$ general relativity with a single compact extra dimension gives a $d = 4$ theory with a massless graviton, vector and scalar, all interacting with an infinite tower of massive gravitons. Similarly, based on the results of the previous section, reducing $D = 5$ dRGT massive gravity should give a $d = 4$ theory of an infinite number of massive gravitons, vectors, and scalars, all interacting with one another.²

² What is the strong coupling scale of this theory? The strong coupling scale in $(n + 4)$ -dimensional massive gravity is given by $\Lambda_{\frac{n+6}{n+2}} \equiv \left(m^{\frac{4}{n+2}} M_{(n+4)}\right)^{\frac{n+2}{n+6}}$, where $M_{(n+4)}$ is the $(n + 4)$ -dimensional Planck mass. The $d = 4$ Planck mass is given by $M_{(4)}^2 \sim M_{(n+4)}^{n+2} R^n$, where R is the characteristic length scale of the internal space. This gives

$$\Lambda_{\frac{n+6}{n+2}} \sim \left(\frac{\Lambda_3^6}{R^n}\right)^{\frac{1}{n+6}}, \quad (5.44)$$

where $\Lambda_3 \equiv (m^2 M_{(4)})^{\frac{1}{3}}$ is the usual strong coupling scale of massive gravity in $d = 4$. Consistency of the EFT requires that the inverse size of the internal space is smaller than the cutoff, $R^{-1} < \Lambda_{\frac{n+6}{n+2}}$. Together with (5.44) this implies that $\Lambda_{\frac{n+6}{n+2}} < \Lambda_3$, so dimensionally reducing in this way cannot raise the strong coupling scale above Λ_3 . As mentioned earlier, dimensionally reducing the massless theory gives a theory with the cutoff $\Lambda_{3/2}$.

One tractable way to study nonlinear interactions is to consider a single compact extra dimension and to truncate to the zero modes. This truncation is consistent because translations in the extra dimension act as a global $U(1)$ symmetry on the four-dimensional fields, and the zero modes (which are constants on the circle) are the only singlets under this $U(1)$.

This procedure will be briefly reviewed for the case of GR before applying it to massive gravity. The five-dimensional spacetime is taken to be $M^4 \times S^1$, the product of four-dimensional Minkowski space and a circle of length L . The five-dimensional Einstein-Hilbert action is

$$S = \frac{M_{(5)}^3}{2} \int d^4x \int_0^L dy \sqrt{-G} R(G), \quad (5.45)$$

where G_{AB} is the higher-dimensional metric and $M_{(5)}$ is the five-dimensional Planck mass. The zero-mode truncation is accomplished by substituting the following ansatz for the metric

$$G_{AB}(x) = \begin{pmatrix} g_{\mu\nu}(x) + \phi^2(x) A_\mu(x) A_\nu(x) & \phi^2(x) A_\mu(x) \\ \phi^2(x) A_\nu(x) & \phi^2(x) \end{pmatrix}. \quad (5.46)$$

Substituting this into the action gives

$$S = \frac{M_{(5)}^3}{2} L \int d^4x \sqrt{-g} \phi \left(R - \frac{1}{4} \phi^2 F_{\mu\nu}^2 \right). \quad (5.47)$$

Going to the Einstein frame and canonically normalising the scalar by defining $\tilde{g}_{\mu\nu} \equiv \phi g_{\mu\nu}$ and $\phi \equiv e^{-\psi}$ gives

$$S = \frac{M_{(4)}^2}{2} \int d^4x \sqrt{-\tilde{g}} \left(\tilde{R} - \frac{1}{4} e^{-3\psi} F_{\mu\nu}^2 - \frac{3}{2} (\partial\psi)^2 \right), \quad (5.48)$$

where the four-dimensional Planck mass is given by $M_{(4)}^2 = M_{(5)}^3 L$. This describes an interacting massless graviton, vector and scalar.

Now consider five-dimensional dRGT massive gravity on the spacetime $M^4 \times S^1$. The action is given by a slight generalisation of (2.18),

$$S = \frac{M_{(5)}^3}{2} \int d^4x \int_0^L dy \sqrt{-G} \left(R(G) - \frac{m^2}{4} \sum_{n=0}^5 \beta_n e_n \left(\sqrt{G^{-1}\eta} \right) \right), \quad (5.49)$$

where η_{MN} is the five-dimensional fiducial metric, which is taken to be flat. e_n are the symmetric polynomials (2.19), where e_5 is given by

$$e_5(M) = \frac{1}{5!} \left([M]^5 - 10[M]^3[M^2] + 15[M][M^2]^2 + 20[M]^2[M^3] - 20[M^2][M^3] - 30[M][M^4] + 24[M^5] \right). \quad (5.50)$$

The term in (5.49) proportional to β_5 does not contribute to the equations of motion. Furthermore, imposing that flat spacetime (or $M^4 \times S^1$) is a solution and that m is the graviton mass gives the conditions

$$\beta_0 + 4\beta_1 + 6\beta_2 + 4\beta_3 + \beta_4 = 0, \quad (5.51)$$

$$\beta_1 + 3\beta_2 + 3\beta_3 + \beta_4 = 8, \quad (5.52)$$

which leaves a three-parameter family of theories.

The general dimensional reduction will be considered for the simple case of (5.49) given by the ‘‘minimal model’’ of [190], which corresponds to $\beta_1 = 8$, $\beta_0 = -32$, and the other β_n vanishing. It is also convenient to use the vielbein formulation, following [55]. The action for the five-dimensional minimal model in vielbein form is

$$S = \frac{M_{(5)}^3}{2} \int d^4x \int_0^L dy |E| \left(R(E) + 8m^2 - 2m^2 E^M{}_A \delta_M^A \right), \quad (5.53)$$

where E_M^A and δ_M^A are the vielbeins for G_{MN} and η_{MN} , respectively,

$$G_{MN} = E_M^A E_N^B \eta_{AB}, \quad \eta_{MN} = \delta_M^A \delta_N^B \eta_{AB}, \quad (5.54)$$

$|E|$ is the determinant of E_M^A , and $E^M{}_A$ is the inverse vielbein

$$E^M{}_A E_N^A = \delta_N^M. \quad (5.55)$$

The vielbein and inverse vielbein corresponding to (5.46) are

$$\hat{E}_M^A = \begin{pmatrix} e_\mu^a(x) & \phi(x) A_\mu(x) \\ 0 & \phi(x) \end{pmatrix}, \quad \hat{E}^M{}_A = \begin{pmatrix} e^\mu{}_a(x) & 0 \\ -e^\nu{}_a(x) A_\nu(x) & \phi^{-1}(x) \end{pmatrix}, \quad (5.56)$$

where e_μ^a and $e^\mu{}_a$ are the vielbein and inverse vielbein for $g_{\mu\nu}$. Putting a general vielbein in the diagonal form (5.56) requires a local rotation of four-dimensional

spacetime into the fifth dimension, which is not possible when there is a mass term since local Lorentz invariance is broken. A general vielbein may thus be written as

$$E_M^A = \Lambda^{*A}{}_B \hat{E}_M^B, \quad (5.57)$$

where \hat{E}_M^B is upper triangular and $\Lambda^{*A}{}_B$ defines a local rotation. The required rotation can be parametrised in terms of a vector B^a by writing the generator as

$$\omega^A{}_B = \begin{pmatrix} 0 & B^a \\ -B_b & 0 \end{pmatrix}, \quad (5.58)$$

where $\Lambda^{*A}{}_B = (e^\omega)^A{}_B$ and $B_b \equiv \eta_{ba} B^a$. Defining $B^2 \equiv B_a B^a$ and $|B| \equiv \sqrt{B^2}$, the rotation matrix can be written as

$$\Lambda^{*A}{}_B = \delta^A{}_B + \frac{(\omega^2)^A{}_B}{B^2} (1 - \cos |B|) + \omega^A{}_B \frac{\sin |B|}{|B|}. \quad (5.59)$$

The dimensionally reduced action is now obtained by substituting (5.57) into (5.53) and using (5.59). The Einstein-Hilbert kinetic term is locally Lorentz invariant so B_a only appears through the potential. Defining $B^\mu = e^\mu{}_a B^a$, so that $B^2 = B_\mu B^\mu$, the dimensionally reduced theory can be written as

$$S = \frac{M_{(4)}^2}{2} \int d^4x |e| \phi \left[R(e) - \frac{1}{4} \phi^2 F_{\mu\nu}^2 + 8m^2 - 2m^2 \left(\delta_\mu{}^a e^\mu{}_a + e^\nu{}_a \delta_\mu{}^a \frac{B^\mu B_\nu}{B^2} (\cos |B| - 1) + \phi^{-1} \cos |B| + A_\mu B^\mu \frac{\sin |B|}{|B|} \right) \right], \quad (5.60)$$

where $\delta_\mu{}^a$ is the vielbein for the four-dimensional flat fiducial metric.

The field B_μ appears in (5.60) without derivatives and is thus an auxiliary field that should be integrated out using its equations of motion. It is difficult to integrate out B exactly, but it can be done to any desired order in powers of the fields. For example, to quadratic order

$$S = \int d^4x \left[\epsilon(h)_{m^2} - \frac{1}{4} F_{\mu\nu}^2 - 2m^2 (A_\mu B^\mu - B^2) - (\partial\psi)^2 - m^2 \psi^2 \right], \quad (5.61)$$

where $\phi \equiv e^{-\psi}$ and the fields $h_{\mu\nu}$ and ϕ have been demixed. Eliminating B using its equations of motion gives

$$S = \int d^4x \left[\epsilon(h)_{m^2} - \frac{1}{4} F_{\mu\nu}^2 - \frac{1}{2} m^2 A^2 - (\partial\psi)^2 - m^2 \psi^2 \right], \quad (5.62)$$

which describes a massive graviton, vector and scalar with mass m . This agrees with the zero-mode spectrum obtained in Subsection 5.3.2 for $N = 1$ and $d = 4$.

The theory (5.60) is a massive scalar-vector-tensor theory that derives from the five-dimensional minimal model. A massive scalar-tensor theory can be obtained by setting $A_\mu = 0$, which is classically consistent since A_μ does not appear linearly after integrating out B_μ . When $A_\mu = 0$, the auxiliary field's equations of motion are solved by $B_\mu = 0$, which corresponds to the simple metric ansatz

$$G_{AB}(x) = \begin{pmatrix} g_{\mu\nu}(x) & 0 \\ 0 & \phi^2(x) \end{pmatrix}. \quad (5.63)$$

Using the ansatz (5.63), the dimensional reduction for the full three-parameter higher-dimensional dRGT theory (5.49) can be readily found. The result is

$$S = \frac{M_{(4)}^2}{2} \int d^4x \sqrt{-g} \phi \left(R(g) - \frac{m^2}{4} \sum_{n=0}^4 \left(\beta_n + \frac{\beta_{n+1}}{\phi} \right) S_n \left(\sqrt{g^{-1}\eta} \right) \right), \quad (5.64)$$

where the S_5 term has been dropped since it becomes a total derivative in four dimensions. Again, the parameter β_5 does not contribute to the equations of motion. In Einstein frame the action is

$$S = \frac{M_{(4)}^2}{2} \int d^4x \sqrt{-\tilde{g}} \left(\tilde{R} - \frac{3}{2} (\partial\psi)^2 - \frac{m^2}{4} \sum_{n=0}^4 e^{\frac{(4-n)}{2}\psi} (\beta_n e^{-\psi} + \beta_{n+1}) S_n \left(\sqrt{\tilde{g}^{-1}\eta} \right) \right), \quad (5.65)$$

where $\tilde{g}_{\mu\nu} \equiv \phi g_{\mu\nu}$ and $\phi \equiv e^{-\psi}$. The action (5.65) defines a theory with three parameters in addition to the graviton mass and cosmological constant.³ At quadratic order around flat spacetime, (5.65) describes a free graviton and scalar, both with mass m . Because (5.65) arises as a consistent truncation of dRGT ghost-free massive gravity, it is also expected to be ghost free. It is an example of a type of mass-varying massive gravity [169, 170] in which the graviton mass and dRGT parameters are promoted to functions of a scalar field. Cosmological perturbations of this general class of theories were studied in [171].

³In deriving (5.65), a flat fiducial metric was used and the conditions (5.51) and (5.52) were imposed, but the final result makes sense for fiducial metrics and backgrounds that are not flat.

5.5 Discussion

This chapter has studied the KK spectrum that results from dimensionally reducing a massive spin-2 field on an Einstein product space. The dimensionally reduced spectrum and stability conditions were determined for a general massive graviton.

One special value of the mass corresponds to a higher-dimensional partially-massless graviton. It was found that a higher-dimensional partially massless symmetry does not result in a lower-dimensional partially massless graviton, which shows that dimensional reduction destroys partial masslessness. Rather, the partially massless symmetry acts to eliminate a tower of scalars, including the massless radion. The radion, like other moduli, is a light scalar, so consistency with observations usually requires realistic models to have a screening mechanism [43, 167, 168]. The removal of such light modes by gauge symmetry might be useful as a mechanism to avoid such light scalars without screening.

Another interesting theory resulted when the graviton mass was tuned to match the lower-dimensional partially massless value. This case produced a Lagrangian that is not diagonalisable and describes a novel irreducible de Sitter representation [186, 187].

In addition, the nonlinear zero-mode interactions were found for the dimensional reduction of dRGT massive gravity on a circle. This gave a three-parameter family of interacting massive scalar-vector-tensor theories. Projecting out the vector produced a type of mass-varying massive gravity. These theories should be ghost free since they descend from a ghost-free theory. It would be interesting to check this directly and to further explore their properties.

5.6 Appendix A: Diagonalisation transformations

This appendix describes the transformations needed to diagonalise the Lagrangians in (5.29) and discusses some special cases. When there are multiple transformations they are performed in the order listed. It is useful to define

$$m_k^2 \equiv m^2 + \lambda_k - \frac{2R_{(d)}}{d}, \quad (5.66)$$

where $k = \mathcal{I}, i, 0$, or a and $\lambda_0 = 0$.

Transformations for \mathcal{L}_i

To diagonalise \mathcal{L}_i for $i \neq \text{Killing}$, transform

$$A_\mu^i \rightarrow A_\mu^i + \frac{1}{m_i^2} (m_i^2 - m^2) \nabla_\mu \phi^i. \quad (5.67)$$

Since $\lambda_{i \neq \text{Killing}} > 2R_{(d)}/d$ for $i \neq \text{Killing}$, it follows that $m_i > m$ and hence this transformation is well defined.

Transformations for \mathcal{L}_0

When m is not equal to $m_{(d)\text{PM}}$ and $d > 1$, \mathcal{L}_0 is diagonalised by

$$h_{\mu\nu}^0 \rightarrow h_{\mu\nu}^0 - \frac{1}{(d-1)(m^2 - m_{(d)\text{PM}}^2)} \left(\left(m^2 - \frac{R_{(d)}}{d} \right) g_{\mu\nu} \phi^0 - \nabla_\mu \nabla_\nu \phi^0 \right). \quad (5.68)$$

This transformation is not defined when $m = m_{(d)\text{PM}}$. This is because the $h_{\mu\nu}^0$ kinetic term in the original action acquires a partially massless symmetry and hence the $h^0 \phi^0$ term cannot be unmixed by a transformation of the form (5.68), since this demixing transformation takes precisely the form of a partially massless gauge transformation. When $m = m_{(d)\text{PM}}$ the derivative scalar-tensor term can be eliminated by transforming

$$h_{\mu\nu}^0 \rightarrow h_{\mu\nu}^0 + \frac{1}{m^2} \nabla_\mu \nabla_\nu \phi^0; \quad (5.69)$$

then the scalar kinetic term can be removed using a partially massless transformation, and finally the $h^0 \phi^0$ term can be removed by redefining ϕ^0 . This leaves

$$\mathcal{L}_0 = \epsilon (h^0)_{m_{(d)\text{PM}}^2} - \frac{1}{4} m_{(d)\text{PM}}^2 |h_0|^2 + \frac{m_{(d)\text{PM}}^2}{(d-2)^2} |\phi^0|^2. \quad (5.70)$$

The scalar is now decoupled and its equations of motion just set $\phi^0 = 0$, so it can be dropped, giving (5.34).

Transformations for \mathcal{L}_a

The case $N > 1$ is considered first. When $d > 1$ and $m_a^2 \neq 0$, \mathcal{L}_a can be diagonalised with the following transformations

$$\begin{aligned} h_{\mu\nu}^a \rightarrow & h_{\mu\nu}^a - \frac{N-1}{N(d-1)(\lambda_a + m^2 - m_{(d)\text{PM}}^2)} \left[\lambda_a \left(\lambda_a - \frac{R_{(N)}}{N-1} \right) \left(g_{\mu\nu} \phi^a + \frac{d-2}{\lambda_a + m^2} \nabla_\mu \nabla_\nu \phi^a \right) \right. \\ & + \left. \left(\lambda_a + \frac{Nm^2 - R_{(N)}}{N-1} \right) g_{\mu\nu} \bar{\phi}^a - \frac{m^2 N + (D-2)\lambda_a}{(N-1)(\lambda_a + m^2)} \nabla_\mu \nabla_\nu \bar{\phi}^a \right] \\ & + \frac{\lambda^a}{\lambda_a + m^2} (\nabla_\mu A_\nu + \nabla_\nu A_\mu), \end{aligned} \quad (5.71)$$

$$A_\mu^a \rightarrow A_\mu^a + \frac{1}{Nm_a^2} \left((N-1) \left(\lambda_a - \frac{R_{(N)}}{N-1} \right) \partial_\mu \phi^a - \partial_\mu \bar{\phi}^a \right). \quad (5.72)$$

Additionally, when $a \neq$ conformal transform

$$\phi^a \rightarrow \phi^a - \frac{(D-2)m_a^2 + m^2 d}{\lambda_a \left(\lambda_a (D-2) - 2N \frac{R_{(d)}}{d} \right) + (2\lambda_a + m^2 N) (d-1) (m^2 - m_{(d)\text{PM}}^2)} \bar{\phi}^a. \quad (5.73)$$

These transformations give the diagonalised Lagrangian (5.35). The coefficients σ_a and $\bar{\sigma}_a$ for $a \neq$ conformal are given by⁴

$$\begin{aligned} \sigma_a = & \frac{(N-1)\lambda_a \left(\lambda_a - \frac{R_{(N)}}{N-1} \right)}{2N^2(d-1)m_a^2 \left(\lambda_a + m^2 - m_{(d)\text{PM}}^2 \right)} \times \\ & \left(\lambda_a \left(\lambda_a (D-2) - 2N \frac{R_{(d)}}{d} \right) + (2\lambda_a + m^2 N) (d-1) (m^2 - m_{(d)\text{PM}}^2) \right), \end{aligned} \quad (5.74)$$

and

$$\sigma_a \bar{\sigma}_a = \frac{\lambda_a m^2 (N-1)(D-1) \left(\lambda - \frac{R_{(N)}}{N-1} \right) (m^2 - m_{(D)\text{PM}}^2)}{4N^2(d-1)m_a^2 (m^2 + \lambda_a - m_{(d)\text{PM}}^2)}. \quad (5.75)$$

When $a =$ conformal, $\sigma_a = 0$ and

$$\bar{\sigma}_a = \frac{(m^2 - m_{(D)\text{PM}}^2)}{m_a^2 \left(m^2 + \frac{R_{(D)}}{D} \frac{(D-2)}{(d-1)(N-1)} \right)} \frac{m^2 (D-1)}{2N(d-1)}. \quad (5.76)$$

⁴The transformation (5.73) is undefined for certain λ_a when $m < m_{(N)\text{PM}}$. At these points the determinant of the kinetic matrix (5.75) is negative and the Lagrangian can be put in the form (5.35) with $\sigma_a = 1$, $\bar{\sigma}_a = -1$.

When $m_a = 0$, the Lagrangian is diagonalised by transforming $h_{\mu\nu}$ as above and then transforming

$$\bar{\phi}^a \rightarrow \bar{\phi}^a - (N-1)(m^2 - m_{(N)\text{PM}}^2)\phi^a, \quad (5.77)$$

$$A_\mu^a \rightarrow A_\mu^a - \frac{(N-1)(m^2 - m_{(N)\text{PM}}^2)(m^2 + (D-2)\frac{R_{(d)}}{d})}{m^2 d (m^2 - \frac{2R_{(d)}}{d})} \partial_\mu \phi^a - \frac{1}{4} \left(\frac{(N-2)(D-2)}{d m^2 N} - \frac{D}{d m^2 - 2R_{(d)}} + \frac{d-2}{N R_{(d)}} \right) \partial_\mu \bar{\phi}^a, \quad (5.78)$$

$$\bar{\phi}^a \rightarrow \bar{\phi}^a - N \nabla^\mu A_\mu^a. \quad (5.79)$$

Note that when $a = \text{conformal}$, $m_a = 0$ corresponds to $m = m_{(N)\text{PM}}$. These transformations leave (5.36) and a nondynamical term proportional to $|\bar{\phi}^a|^2$.

When $d = 1$, $\mathcal{L}_{a=\text{conformal}}$ is nondynamical. $\mathcal{L}_{a \neq \text{conformal}}$ can be simplified by transforming $h^a \rightarrow h^a + 2\nabla_\mu A_\mu^a$ and $\bar{\phi}^a \rightarrow \bar{\phi}^a - \lambda^a \phi^a$. Both $h_{\mu\nu}^a$ and A_μ^a then appear algebraically and can be integrated out using their equations of motion, which leaves (5.38).

Lastly, consider $N = 1$, i.e. \mathcal{N} is the circle with radius r . In this case $R_{(N)} = 0$, all scalars are conformal scalars, and the scalar Laplacian eigenvalues are given by $\lambda_a = a/r^2$, where a is a positive integer. Assuming $d > 1$, then demixing requires transforming $h_{\mu\nu}^a$ as above, followed by the transformation

$$A_\mu^a \rightarrow A_\mu^a - \frac{1}{\lambda_a + m^2} \partial_\mu \bar{\phi}^a, \quad (5.80)$$

which gives the diagonalised Lagrangian (5.37).

6

Giving Mass to the Unimodular Graviton

6.1 Introduction

6.1.1 Overview

This chapter describes some attempts to give a mass to the graviton in unimodular gravity. Unimodular gravity is a theory that is equivalent to GR except that the cosmological constant receives contributions from an integration constant. This means that in unimodular gravity there is an extra global degree of freedom compared to GR and that the cosmological constant is partly determined by initial conditions. There is a large body of literature consisting of attempts to address the cosmological constant problem using this feature of unimodular gravity [191–196]. However, unimodular gravity does not resolve the problem of the radiative instability of the cosmological constant, as emphasised in the references [5, 197]. The goal of this chapter is instead to see if anything new can be learnt by trying to give mass to the unimodular graviton. This chapter is based on work that was completed in collaboration with Kurt Hinterbichler and Pedro Ferreira [198] and other unpublished work.

6.1.2 Motivations

GR and unimodular gravity both describe a massless spin-2 field, but they do so with different kinetic terms, as reviewed below. Massive gravity is obtained by adding a mass and interaction terms to massless gravity, so different realisations of a massless graviton could allow for different massive theories. In other words, it can be asked whether the noncanonical kinetic term of unimodular gravity can allow for a novel theory of massive gravity. There are known cases where different kinetic terms describe the same free degrees of freedom but lead to different interacting theories. For example, in five dimensions a free massless graviton can be described using a mixed-symmetry field or a symmetric tensor, but interactions only exist for the symmetric tensor [199].

A massive version of unimodular gravity, if it existed, might also treat the cosmological constant differently, compared to a massive theory based on the Einstein-Hilbert kinetic term. A graviton mass can allow for self-accelerating solutions [31, 169, 200–204], so a small graviton mass can be used to explain the observed accelerated expansion of the universe in the absence of a cosmological constant. However, in dRGT massive gravity any large bare cosmological constant is usually set to zero by hand, so not much can be said about this aspect of the cosmological constant problem unless some mechanism for eliminating a large bare cosmological constant can be made to work [27–30]. Thus it would be interesting to find a theory of massive gravity that coupled in a different way to vacuum energy.

6.1.3 Outline

Section 6.2 begins by reviewing unimodular gravity. A candidate action for a massive version of unimodular gravity is then constructed by adding a Lagrange multiplier to dRGT massive gravity and the solutions of this theory are studied. This theory has some appealing features at first glance, but on deeper inspection it seems to be unphysical. In Section 6.3, a more systematic search is made for a massive theory with a novel kinetic term by studying linear theories. In particular, KK dimensional reduction is used to obtain massive linear theories from various

massless theories in higher dimensions. The result is that it is possible to add a mass term to the unimodular graviton by introducing extra auxiliary fields, but the resulting theory is equivalent to the usual massive theory after gauge fixing and relabeling fields. This suggests that there is no new massive gravity theory based on unimodular gravity. While these results are negative, they provide further evidence for the uniqueness of the dRGT structure based on the Einstein-Hilbert kinetic term (see [52, 53, 133, 205–207] for other works exploring different kinetic terms for massive gravity). This chapter concludes with a discussion in Section 6.4. The appendix to this chapter contains details of a calculation for finding Friedmann-Lemaître-Robertson-Walker (FLRW) solutions and cosmological perturbations in the constrained version of dRGT massive gravity.

Conventions: d is the number of spacetime dimensions, which will be kept arbitrary, but because both standard gravity and massive gravity have no local degrees of freedom in $d = 2$, only the case $d \geq 3$ will be considered.

6.2 Nonlinear theories

This section reviews unimodular gravity, showing how the equations of motion reduce to the usual Einstein equations with the cosmological constant containing a contribution from an integration constant. A constrained theory of dRGT massive gravity is then introduced and studied as a candidate massive version of unimodular gravity.

6.2.1 Unimodular gravity

One simple way to write the action for unimodular gravity is to modify the Einstein-Hilbert action with a Lagrange multiplier λ ,

$$S = \frac{M_p^2}{2} \int d^d x \left(\sqrt{-g}(R(g) - 2\Lambda) + \lambda(x)(\sqrt{-g} - \epsilon_0) \right), \quad (6.1)$$

where ϵ_0 is a non-dynamical volume element [208]. The volume element ϵ_0 is a background structure that breaks the diffeomorphism invariance to the subgroup

of volume preserving diffeomorphisms. The equations of motion are

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} - \frac{1}{2}\lambda(x)g_{\mu\nu} = 0, \quad (6.2)$$

together with $\sqrt{-g} = \epsilon_0$. The equation of motion for λ constrains the metric determinant to equal the non-dynamical volume element. Taking the covariant divergence of (6.2) gives $\partial_\nu\lambda = 0$, which implies that λ is an arbitrary integration constant. Thus the equations of motion following from (6.1) are

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda_{\text{tot}}g_{\mu\nu} = 0, \quad (6.3)$$

where $\Lambda_{\text{tot}} = \Lambda - \lambda/2$, plus the additional constraint that $\sqrt{-g} = \epsilon_0$. In GR, the condition $\sqrt{-g} = \epsilon_0$ can be satisfied with a particular choice of gauge. The only difference between GR and unimodular gravity is that the effective cosmological constant Λ_{tot} receives contributions from an integration constant in unimodular gravity and is thus partly determined by initial conditions. This means that locally unimodular gravity is equivalent to GR, even though the gauge transformations are different; the two theories differ only by a global degree of freedom, in the sense that specifying initial data in unimodular gravity requires one additional number as compared to GR [191–193].

6.2.2 Massive unimodular gravity

Based on the action (6.1), a sensible guess for a massive modification of unimodular gravity is

$$\mathcal{L} = \mathcal{L}_{\text{dRGT}} + \frac{M_p^2}{2}\lambda(x) \left(\sqrt{-g} - \sqrt{-f} \right), \quad (6.4)$$

where $\mathcal{L}_{\text{dRGT}}$ is the dRGT massive gravity Lagrangian (2.18) with $\eta_{\mu\nu} \rightarrow f_{\mu\nu}$. This theory is just dRGT massive gravity with the additional constraint that the determinant of the dynamical metric equals the determinant of the nondynamical metric, $\sqrt{-g} = \sqrt{-f}$. Since diffeomorphism invariance is already broken in massive gravity by the non-dynamical metric $f_{\mu\nu}$, no additional background structure is needed to define (6.4). The equations of motion that follow from (6.4) are the traceless part of the dRGT equations of motion.

The solutions of (6.4) are now studied to determine whether or not it is a viable theory. By substituting the flat FLRW ansatz,

$$ds^2 = -N^2(t)dt^2 + a^2(t)d\mathbf{x}^2, \quad (6.5)$$

into (6.4), it is shown in the appendix to this chapter that this theory has flat FLRW solutions, which could make it suitable for cosmology. In dRGT massive gravity there are no flat FLRW solutions, which is also reviewed in the appendix to this chapter. The FLRW solutions in (6.4) have a contribution to the cosmological constant from an integration constant, which is similar to what happens in unimodular gravity, so this theory seems like a good candidate for a massive version of unimodular gravity. Unfortunately, the theory has six degrees of freedom and one of these is a scalar ghost around the flat FLRW solutions. This is shown in the appendix to this chapter by calculating cosmological perturbations about the FLRW solutions. This means that these interesting solutions, and probably the full theory, are unphysical. The next section will discuss whether this pathology can be avoided by trying to construct a consistent linear theory.

6.3 Linear theories

This section presents a more systematic search for a massive version of unimodular gravity. Several massless spin-2 theories are first reviewed. A method for obtaining massive theories from massless ones in higher dimensions through a KK reduction is then reviewed. This method is then applied to noncanonical massless theories and the resulting massive theories are studied.

6.3.1 Flat-space massless theories

First a review is given of the various Lagrangians that describe a massless spin-2 field. The most general Lorentz invariant quadratic Lagrangian for a symmetric tensor field $h_{\mu\nu}$ on flat spacetime containing two derivatives is

$$\mathcal{L} = a_1 \partial_\lambda h_{\mu\nu} \partial^\lambda h^{\mu\nu} + a_2 \partial_\lambda h_{\mu\nu} \partial^\mu h^{\lambda\nu} + a_3 \partial_\mu h^{\mu\nu} \partial_\nu h + a_4 \partial_\mu h \partial^\mu h. \quad (6.6)$$

Among the four parameters a_1, \dots, a_4 , there are two redundancies: an overall scaling of $h_{\mu\nu}$ and the field redefinition $h_{\mu\nu} \rightarrow h_{\mu\nu} + c h \eta_{\mu\nu}$ for constant $c \neq -1/d$ (ensuring invertibility). This leaves a two-parameter family of possibilities. It turns out that there are precisely two choices of coefficients (up to the aforementioned redundancies) that lead to theories propagating precisely the degrees of freedom of a massless spin-2 particle and nothing more [209].¹

Linearised GR

The first possibility for describing a massless spin-2 field is the well-known linearisation of GR, as reviewed in Chapter 2,

$$\mathcal{L}_{\text{GR}} = -\frac{1}{2} \partial_\lambda h_{\mu\nu} \partial^\lambda h^{\mu\nu} + \partial_\mu h_{\nu\lambda} \partial^\nu h^{\mu\lambda} - \partial_\mu h^{\mu\nu} \partial_\nu h + \frac{1}{2} \partial_\lambda h \partial^\lambda h. \quad (6.7)$$

The Lagrangian (6.7) takes the same form in any dimension and is invariant under linearised diffeomorphisms

$$\delta h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu, \quad (6.8)$$

where ξ_μ is a vector gauge parameter.

Weyl-invariant theory

The other possibility is a Weyl-invariant theory,

$$\mathcal{L}_{\text{Weyl}, d} = -\frac{1}{2} \partial_\lambda h_{\mu\nu} \partial^\lambda h^{\mu\nu} + \partial_\lambda h_{\mu\nu} \partial^\mu h^{\lambda\nu} - \frac{2}{d} \partial_\mu h^{\mu\nu} \partial_\nu h + \frac{d+2}{2d^2} \partial_\mu h \partial^\mu h, \quad (6.9)$$

which seems to have been first mentioned by Weyl [210]. The coefficients in (6.9) depend on the dimension of spacetime and the Lagrangian is invariant under the gauge symmetries

$$\delta h_{\mu\nu} = \partial_\mu \xi_\nu^T + \partial_\nu \xi_\mu^T + \chi \eta_{\mu\nu}, \quad \partial^\mu \xi_\mu^T = 0, \quad (6.10)$$

where χ is a scalar gauge parameter and ξ_μ^T is a vector gauge parameter which is transverse, $\partial^\mu \xi_\mu^T = 0$; χ is a linearised Weyl transformation and ξ_μ^T is a linearised

¹Adding a term bh^2 , with arbitrary constant b , to these massless spin-2 theories does not change the number of local propagating degrees of freedom. This term reduces the size of the gauge group in both cases, so that the kinetic terms are no longer completely fixed by the gauge symmetry. Only the cases with maximal gauge symmetry will be considered here.

volume preserving diffeomorphism. The coefficients in (6.9) are completely fixed, up to an overall scaling of $h_{\mu\nu}$, by demanding the gauge symmetry (6.10). This theory can be obtained from linear GR (6.7) by replacing $h_{\mu\nu} \rightarrow h_{\mu\nu} - \frac{1}{d}\eta_{\mu\nu}h$, but this is not an invertible field redefinition so the theories are not equivalent. The difference between the two theories is an extra global degree of freedom. The nonlinear version of this Weyl invariant theory is equivalent to unimodular gravity and is discussed in [209]. There are also generalisations of (6.9) for massless higher-spin fields [211].

The Lagrangian (6.9) describes, in addition to the $d(d-3)/2$ helicity-2 degrees of freedom at each point, an extra global degree of freedom, in the sense that there is one extra global initial condition that needs to be specified to uniquely determine the dynamics. This shows up as an integration constant in the equations of motion and behaves as a tadpole term (a nonzero vacuum expectation value for $h_{\mu\nu}$). This is the linear analogue of the cosmological constant appearing as an integration constant in unimodular gravity.

To see this, first use the χ gauge symmetry to set $h = 0$; this leaves a residual symmetry given by (6.10) with $\chi = 0$. Taking a divergence of the equations of motion then gives $\partial_\rho(\partial^\mu\partial^\nu h_{\mu\nu}) = 0$, which can be integrated once to give

$$\partial^\mu\partial^\nu h_{\mu\nu} = c, \quad (6.11)$$

for some gauge-invariant constant c . Integrating this again gives

$$\partial^\mu h_{\mu\nu} = \frac{c}{d}x_\nu + a_\nu^T(x), \quad (6.12)$$

where x^μ are the spacetime coordinates and $a_\mu^T(x)$ is an arbitrary transverse vector field, $\partial^\mu a_\mu^T(x) = 0$. This arbitrary transverse vector field can be set to zero using the residual gauge symmetry. The equations of motion then give the unsourced Klein-Gordon equation $\square h_{\mu\nu} = 0$, but the tensor is not transverse so it does not directly describe the spin-2 degrees of freedom. The field $\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{c}{2d}x^2\eta_{\mu\nu}$ is transverse but it obeys a sourced Klein-Gordon equation,

$$\square\bar{h}_{\mu\nu} = -c\eta_{\mu\nu} \quad \text{and} \quad \partial^\mu\bar{h}_{\mu\nu} = 0, \quad (6.13)$$

which shows that the integration constant c contributes a tadpole term.

Linear unimodular gravity

Another theory can be obtained from (6.9) by using the Weyl symmetry to fix $h = 0$. This gives an equivalent theory written in terms of a traceless tensor $\tilde{h}_{\mu\nu}$,

$$\mathcal{L}_{\text{uni}} = -\frac{1}{2}\partial_\lambda\tilde{h}_{\mu\nu}\partial^\lambda\tilde{h}^{\mu\nu} + \partial_\lambda\tilde{h}_{\mu\nu}\partial^\mu\tilde{h}^{\lambda\nu}, \quad (6.14)$$

which has the gauge symmetry

$$\delta\tilde{h}_{\mu\nu} = \partial_\mu\xi_\nu^T + \partial_\nu\xi_\mu^T, \quad \partial^\mu\xi_\mu^T = 0. \quad (6.15)$$

This traceless theory is the linearisation of unimodular gravity written without a Lagrange multiplier. There is also a linear theory with a Lagrange multiplier, as in (6.1). The theory (6.14) can be formally obtained from linearised GR by setting $h = 0$, but this is *not* an algebraic gauge fixing—in general it is not legitimate to perform such a gauge fixing in the action—so it should be thought of only as a shorthand way of transforming between different theories.

6.3.2 Flat-space massive theories

The relativistic equations that describe the dynamics of the $(d+1)(d-2)/2$ polarisations of a massive spin-2 particle of mass m are

$$(\square - m^2)h_{\mu\nu} = 0, \quad \partial^\mu h_{\mu\nu} = 0, \quad h = 0. \quad (6.16)$$

How can the above massless Lagrangians be modified to yield the massive spin-2 equations (6.16)? For the case of linearised GR, the answer to this question is to add the term $-\frac{1}{2}m^2(h_{\mu\nu}h^{\mu\nu} - h^2)$ to the massless action [40]. Adding a generic quadratic mass term to the massless Weyl-invariant theory (6.9) or linear unimodular gravity (6.14) always results in a theory with an additional ghost degree of freedom [209],² as can be checked using a Hamiltonian constraint analysis. Hence the Weyl-invariant theory seems to not have a massive deformation.

Below it will be shown that there is a massive spin-2 theory with a Weyl-invariant kinetic term if auxiliary fields are introduced. This theory can be obtained by

²Except the term proportional to h^2 , which gives a Lagrangian that propagates the correct number of massless degrees of freedom, as footnoted in Sec. 6.3.1.

performing a truncated KK reduction on the massless Weyl-invariant theory and it is equivalent to the massive Fierz-Pauli theory after gauge fixing and redefining fields. In what follows, the method for obtaining the massive Fierz-Pauli theory using KK reduction is first reviewed and then this technique is used to find the massive Weyl-invariant theory.

Massive Fierz-Pauli

The Fierz-Pauli massive spin-2 theory is

$$\mathcal{L}_{\text{FP}} = -\frac{1}{2}\partial_\lambda h_{\mu\nu}\partial^\lambda h^{\mu\nu} + \partial_\lambda h_{\mu\nu}\partial^\mu h^{\lambda\nu} - \partial_\mu h^{\mu\nu}\partial_\nu h + \frac{1}{2}\partial_\mu h\partial^\mu h - \frac{1}{2}m^2(h_{\mu\nu}h^{\mu\nu} - h^2). \quad (6.17)$$

As discussed in Chapter 5, dimensionally reducing GR gives a lower-dimensional theory containing an infinite tower of massive spin-2 fields. The Fierz-Pauli theory can be obtained by truncating the linear part of this KK reduced theory to a single massive sector. This is a consistent truncation at the linear level that can be derived automatically by replacing H_{AB} in the higher-dimensional action with

$$H_{\mu\nu}(x, y) = \sqrt{\frac{2m}{\pi}} \cos(my) h_{\mu\nu}(x), \quad (6.18)$$

$$H_{\mu y}(x, y) = \sqrt{\frac{2m}{\pi}} \sin(my) A_\mu(x), \quad (6.19)$$

$$H_{yy}(x, y) = \sqrt{\frac{2m}{\pi}} \cos(my) \phi(x), \quad (6.20)$$

where capital Latin indices are $(d+1)$ -dimensional, Greek indices are d -dimensional, and y is the coordinate of the extra compact dimension that is integrated over using the following orthogonality relations:

$$\int_0^{\frac{\pi}{m}} \cos^2(my) dy = \int_0^{\frac{\pi}{m}} \sin^2(my) dy = \frac{\pi}{2m}, \quad \int_0^{\frac{\pi}{m}} \cos(my) \sin(my) dy = 0. \quad (6.21)$$

Since the higher-dimensional graviton satisfies the higher-dimensional gauge transformation,

$$\delta H_{AB} = \partial_A \Xi_B + \partial_B \Xi_A, \quad (6.22)$$

the lower-dimensional gauge transformations are obtained by expressing Ξ_A in terms of lower-dimensional Fourier modes,

$$\Xi_\mu(x, y) = \sqrt{\frac{2m}{\pi}} \cos(my) \xi_\mu(x), \quad (6.23)$$

$$\Xi_y(x, y) = \sqrt{\frac{2m}{\pi}} \sin(my) \xi(x), \quad (6.24)$$

and equating components.

Reducing the $(d+1)$ -dimensional linearised GR action in this way leads to

$$\mathcal{L}_{\text{GR}, d+1} \rightarrow \mathcal{L} = \mathcal{L}_{\text{FP}, d} - \frac{1}{2} F_{\mu\nu}^2 - 2mh^{\mu\nu}(\partial_\mu A_\nu - \eta_{\mu\nu} \partial^\alpha A_\alpha) + h^{\mu\nu}(\partial_\mu \partial_\nu \phi - \eta_{\mu\nu} \square \phi). \quad (6.25)$$

Redefining $\phi \rightarrow -d\phi$, to simplify later expressions, gives

$$\mathcal{L} = \mathcal{L}_{\text{FP}, d} - \frac{1}{2} F_{\mu\nu}^2 - 2mh^{\mu\nu}(\partial_\mu A_\nu - \eta_{\mu\nu} \partial^\alpha A_\alpha) - dh^{\mu\nu}(\partial_\mu \partial_\nu \phi - \eta_{\mu\nu} \square \phi), \quad (6.26)$$

which has the gauge symmetries

$$\begin{aligned} \delta h_{\mu\nu} &= \partial_\mu \xi_\nu + \partial_\nu \xi_\mu, \\ \delta A_\mu &= \partial_\mu \xi - m \xi_\mu, \\ \delta \phi &= -\frac{2}{d} m \xi. \end{aligned} \quad (6.27)$$

This Lagrangian is just the Fierz-Pauli theory (6.17) after the Stückelberg replacement [126]

$$h_{\mu\nu} \rightarrow h_{\mu\nu} + \frac{1}{m} \partial_\mu A_\nu + \frac{1}{m} \partial_\nu A_\mu + \frac{d}{m^2} \partial_\mu \partial_\nu \phi. \quad (6.28)$$

In this case, A_μ and ϕ are both Stückelberg fields that can be gauge fixed to zero (unitary gauge), giving back (6.17).

Massive Weyl-invariant theory

Performing the KK reduction on the higher-dimensional Weyl-invariant theory (6.9), and making the redefinition

$$\phi \rightarrow \frac{1}{d} h - (d+1)\phi, \quad (6.29)$$

gives the massive Weyl-invariant theory with Stückelberg fields

$$\begin{aligned} \mathcal{L} = & \mathcal{L}_{\text{Weyl}, d} - \frac{1}{2}m^2 \left(h_{\mu\nu}^2 - \frac{1}{d}h^2 \right) \\ & - \frac{1}{2}F_{\mu\nu}^2 - 2mh^{\mu\nu} \left(\partial_\mu A_\nu - \frac{1}{d}\eta_{\mu\nu}\partial \cdot A \right) - 2(d-1)mA^\mu \partial_\mu \phi \\ & - 2h^{\mu\nu} \left(\partial_\mu \partial_\nu \phi - \frac{1}{d}\eta_{\mu\nu}\square\phi \right) - \frac{(d-1)}{2} \left[(d+2)(\partial\phi)^2 - dm^2\phi^2 \right], \end{aligned} \quad (6.30)$$

which has the gauge symmetries

$$\begin{aligned} \delta h_{\mu\nu} &= \partial_\mu \xi_\nu + \partial_\nu \xi_\mu + \chi \eta_{\mu\nu}, \\ \delta A_\mu &= \partial_\mu \xi - m \xi_\mu, \\ \delta \phi &= -\frac{2}{d}m\xi. \end{aligned} \quad (6.31)$$

Here χ descends from the higher-dimensional Weyl gauge parameter, while ξ_μ and ξ descend from the higher-dimensional diffeomorphism parameter Ξ^A . The transverse condition in $(d+1)$ -dimensions, $\partial^A \Xi_A = 0$, constrains the lower-dimensional gauge parameters to satisfy

$$\partial_\mu \xi^\mu + m\xi = 0. \quad (6.32)$$

Both A_μ and ϕ are Stückelberg fields, but—unlike in the massive Fierz-Pauli case—they cannot be eliminated simultaneously, due to the constraint (6.32). There is thus no analogue of the pure unitary gauge found for the Fierz-Pauli theory. The best that can be done is to eliminate either ϕ or A_μ .

Eliminating ϕ

Eliminating ϕ from (6.30) using the $U(1)$ symmetry gives a tensor-vector description of the massive spin-2 field,

$$\mathcal{L} = \mathcal{L}_{\text{Weyl}, d} - \frac{1}{2}m^2 \left(h_{\mu\nu}^2 - \frac{1}{d}h^2 \right) - \frac{1}{2}F_{\mu\nu}^2 - 2mh^{\mu\nu} \left(\partial_\mu A_\nu - \frac{1}{d}\eta_{\mu\nu}\partial \cdot A \right), \quad (6.33)$$

which has the residual gauge symmetries

$$\delta h_{\mu\nu} = \partial_\mu \xi_\nu^T + \partial_\nu \xi_\mu^T + \chi \eta_{\mu\nu}, \quad (6.34)$$

$$\delta A_\mu = -m\xi_\mu^T, \quad (6.35)$$

where ξ_μ^T is transverse, $\partial^\mu \xi_\mu^T = 0$. The transversality of ξ_μ results from the constraint (6.32) once ξ has been eliminated.

The massive spin-2 field is given by the following combination of the fields $h_{\mu\nu}$ and A_μ ,

$$\bar{h}_{\mu\nu} \equiv h_{\mu\nu} + \frac{1}{m} \partial_\mu A_\nu + \frac{1}{m} \partial_\nu A_\mu, \quad (6.36)$$

since, after gauge fixing $\bar{h} = 0$, the equations of motion of (6.33) give

$$(\square - m^2)\bar{h}_{\mu\nu} = 0, \quad \partial^\mu \bar{h}_{\mu\nu} = 0. \quad (6.37)$$

The Lagrangian (6.33) can be obtained from the massive Fierz-Pauli theory (6.17) by the replacement

$$h_{\mu\nu} \rightarrow h_{\mu\nu} - \frac{1}{d} \eta_{\mu\nu} h + \frac{1}{m} \partial_\mu A_\nu + \frac{1}{m} \partial_\nu A_\mu. \quad (6.38)$$

Eliminating A_μ

Alternatively, the vector can be eliminated from (6.30) by using ξ^μ to gauge fix $A_\mu + \frac{d}{2m} \partial_\mu \phi = 0$. This gives a scalar-tensor description of a massive spin-2 theory with the Weyl-invariant kinetic term,

$$\begin{aligned} \mathcal{L} = & \mathcal{L}_{\text{Weyl}, d} - \frac{1}{2} m^2 \left(h_{\mu\nu}^2 - \frac{1}{d} h^2 \right) + (d-2) h^{\mu\nu} \left(\partial_\mu \partial_\nu \phi - \frac{1}{d} \eta_{\mu\nu} \square \phi \right) \\ & + \frac{(d-1)}{2} \left[(d-2) (\partial\phi)^2 + dm^2 \phi^2 \right], \end{aligned} \quad (6.39)$$

which has the residual gauge symmetry

$$\delta h_{\mu\nu} = \chi \eta_{\mu\nu}. \quad (6.40)$$

Note that ξ is not left as a residual gauge parameter because it was constrained in terms of ξ_μ by (6.32).

It is instructive to see how the equations of a massive spin-2 field emerge from this Lagrangian. The ϕ equation of motion is

$$(d-2)\square\phi - dm^2\phi = \frac{(d-2)}{(d-1)} \left(\partial^\mu \partial^\nu h_{\mu\nu} - \frac{1}{d} \square h \right), \quad (6.41)$$

and $\partial^\mu \partial^\nu$ acting on the $h_{\mu\nu}$ equations of motion gives

$$(d-2)\square\left(\partial^\mu\partial^\nu h_{\mu\nu} - \frac{1}{d}\square h\right) + m^2 d\left(\partial^\mu\partial^\nu h_{\mu\nu} - \frac{1}{d}\square h\right) = (d-1)(d-2)\square^2\phi. \quad (6.42)$$

Solving (6.41) for $\partial^\mu\partial^\nu h_{\mu\nu} - \frac{1}{d}\square h$ and substituting it into (6.42) gives $m^4\phi = 0$, which implies that the scalar vanishes. Using $\phi = 0$ in (6.41), and in ∂^μ acting on the $h_{\mu\nu}$ equations of motion, then gives the constraint

$$\partial^\mu h_{\mu\nu} - \frac{1}{d}\partial_\nu h = 0. \quad (6.43)$$

Using $\phi = 0$ and (6.43) in the $h_{\mu\nu}$ equations gives

$$(\square - m^2)\left(h_{\mu\nu} - \frac{1}{d}\eta_{\mu\nu}h\right) = 0. \quad (6.44)$$

Using the symmetry (6.40) to gauge-fix $h = 0$ then gives the standard massive spin-2 equations (6.16). Thus ϕ is not dynamical, but rather its equation of motion combines with the tensor equations of motion to enforce the transversality constraint, $\partial^\mu(h_{\mu\nu} - \frac{1}{d}\eta_{\mu\nu}h) = 0$. However, even though ϕ is ultimately nondynamical, it cannot be directly integrated out of the action because it appears with derivatives.

The theory (6.39) is a Weyl-invariant version of the Fierz-Pauli theory with Stückelberg fields. The Fierz-Pauli theory can be recovered by first gauge fixing $h_{\mu\nu}$ to be traceless using the Weyl symmetry (6.40), $h_{\mu\nu} \rightarrow \tilde{h}_{\mu\nu}$, where $\tilde{h}_{\mu\nu}$ is traceless,

$$\begin{aligned} \mathcal{L} = & -\frac{1}{2}\partial_\lambda\tilde{h}_{\mu\nu}\partial^\lambda\tilde{h}^{\mu\nu} + \partial_\lambda\tilde{h}_{\mu\nu}\partial^\mu\tilde{h}^{\lambda\nu} - \frac{1}{2}m^2\tilde{h}_{\mu\nu}^2 + (d-2)\tilde{h}^{\mu\nu}\partial_\mu\partial_\nu\phi \\ & + \frac{(d-1)}{2}\left[(d-2)(\partial\phi)^2 + dm^2\phi^2\right], \end{aligned} \quad (6.45)$$

and then repackaging $d\phi$ to be the trace of $h_{\mu\nu}$,

$$h_{\mu\nu} \equiv \tilde{h}_{\mu\nu} + \eta_{\mu\nu}\phi, \quad (6.46)$$

which recovers (6.17). Going the other way, the Lagrangian (6.39) can be obtained from the Fierz-Pauli theory (6.17) by the replacement

$$h_{\mu\nu} \rightarrow h_{\mu\nu} - \frac{1}{d}\eta_{\mu\nu}h + \phi\eta_{\mu\nu}, \quad (6.47)$$

which is a Weyl Stückelberg transformation followed by a field redefinition of the scalar. This shows that the massive Weyl and Fierz-Pauli theories are equivalent, unlike the massless theories.

Massive traceless theory

Another description of a massive spin-2 field, which is also equivalent to the Fierz-Pauli theory, can be obtained by dimensionally reducing linearised unimodular gravity (6.14). This gives

$$\begin{aligned} \mathcal{L} = & -\frac{1}{2}\partial_\lambda h_{\mu\nu}\partial^\lambda h^{\mu\nu} + \partial_\lambda h_{\mu\nu}\partial^\mu h^{\lambda\nu} - \frac{1}{2}\partial_\mu h\partial^\mu h - \frac{1}{2}m^2(h_{\mu\nu}h^{\mu\nu} - h^2) \\ & - \frac{1}{2}F_{\mu\nu}^2 - 2mh^{\mu\nu}(\partial_\mu A_\nu - \eta_{\mu\nu}\partial^\alpha A_\alpha), \end{aligned} \quad (6.48)$$

which has the gauge transformations

$$\delta h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu \quad (6.49)$$

$$\delta A_\mu = \partial_\mu \xi - m\xi_\mu, \quad (6.50)$$

and the constraint

$$\partial^\mu \xi_\mu + m\xi = 0. \quad (6.51)$$

The lower-dimensional field $h_{\mu\nu}$ is not traceless, but there is no scalar field present because of the requirement that the higher-dimensional trace vanish. Using the Weyl symmetry to fix $h = 0$ in (6.33) gives the same result as fixing $h = 0$ in (6.48) using the $U(1)$ symmetry.

6.4 Discussion

There are multiple ways to describe a massless graviton. The canonical description uses the Einstein-Hilbert action of GR, but there is also a noncanonical description given by unimodular gravity. Both GR and unimodular gravity describe the same local degrees of freedom, but in unimodular gravity the cosmological constant is partly determined by an integration constant. Ghost-free massive gravity is built on the Einstein-Hilbert kinetic term and describes an interacting massive spin-2 field. This chapter considered the possibility of constructing a massive spin-2 theory that instead uses the noncanonical kinetic term of unimodular gravity.

A naive guess for such a nonlinear theory was proposed and found to have ghost instabilities, despite having simple cosmological solutions. Next some linear theories related to unimodular gravity were reviewed and it was shown how to obtain massive versions of them by KK dimensional reduction. Massive theories based on the noncanonical kinetic term were successfully constructed by using the auxiliary fields that arise from extra dimensions. These theories were seen to be equivalent to the usual Fierz-Pauli theory after gauge fixing and field redefinitions. This suggests that there is no inequivalent way to describe a massive graviton by pursuing a connection with unimodular gravity. This result further supports the uniqueness of the Einstein-Hilbert kinetic term in describing massive gravity [205].

The different theories studied in this chapter can be realised as different limits of a parent theory related to the conformally coupled scalar [212], as discussed in reference [198]. There are also extensions of these theories to curved spacetimes and to include couplings to matter [198]. A nonlinear extension of the massive spin-2 theory using auxiliary fields and the noncanonical kinetic term can also be constructed. This gives a theory equivalent to dRGT massive gravity that has a different kinetic term and contains an auxiliary scalar field.

6.5 Appendix A: FLRW solutions and perturbations

This appendix derives the flat FLRW solutions for the theory (6.4) and then analyses cosmological perturbations about these solutions.

6.5.1 FLRW solutions

Consider the constant determinant dRGT theory defined by the action (6.4), where $\mathcal{L}_{\text{dRGT}}$ is given by (2.22) with $\alpha_0 = \alpha_1 = 0$ and $\alpha_2 = 1$. Substituting a flat FLRW solution,

$$ds^2 = -N^2(t)dt^2 + a^2(t)d\mathbf{x}^2, \quad (6.52)$$

into this action gives

$$\frac{S}{V} = M_p^2 \int dt \left(-\frac{3a\dot{a}^2}{N} - \Lambda N a^3 + 3m^2 (NF(a) - G(a)) + \lambda(Na^3 - 1) \right), \quad (6.53)$$

where

$$F(a) = a(a-1)(2a-1) + \frac{\alpha_3}{3}(a-1)^2(4a-1) + \frac{\alpha_4}{3}(a-1)^3, \quad (6.54)$$

$$G(a) = a^2(a-1) + \alpha_3 a(a-1)^2 + \frac{\alpha_4}{3}(a-1)^3. \quad (6.55)$$

The equations of motion are

$$\frac{\delta \mathcal{L}}{\delta \lambda} = Na^3 - 1 = 0 \implies \dot{N} = -\frac{3N\dot{a}}{a}, \quad (6.56)$$

$$\frac{\delta \mathcal{L}}{\delta N} = \frac{3a\dot{a}^2}{N^2} - \Lambda a^3 + 3m^2 F(a) + \lambda a^3 = 0, \quad (6.57)$$

and

$$\frac{\delta \mathcal{L}}{\delta a} = 21\frac{\dot{a}^2}{N} + 6\frac{a\ddot{a}}{N} - 3\Lambda Na^2 + 3m^2(NF'(a) - G'(a)) + 3\lambda Na^2 = 0, \quad (6.58)$$

where $F'(a) = dF/da$, $G'(a) = dG/da$ and (6.56) has been used to eliminate \dot{N} from (6.58). Combining these gives

$$\ddot{a} = -\frac{m^2}{2a^4} \left(\frac{F'(a)}{a^3} - G'(a) - \frac{3F(a)}{a^4} \right) - \frac{2\dot{a}^2}{a} \quad (6.59)$$

and

$$N \frac{d}{dt} \frac{\delta \mathcal{L}}{\delta N} - \dot{a} \frac{\delta \mathcal{L}}{\delta a} = 3m^2 G' \dot{a} + \dot{\lambda} a^3 N = 0. \quad (6.60)$$

Equation (6.60) implies that $3m^2 G(a) + \lambda$ is equal to a constant, c . In dRGT massive gravity, this same procedure implies that $G(a)$ is constant and hence that a is constant, so that there are no flat FLRW solutions in dRGT [32]. For the modified dRGT theory, there is a Friedmann equation given by

$$H^2 \equiv \left(\frac{\dot{a}}{Na} \right)^2 = \frac{1}{3}(\Lambda - c) - m^2 \left(\frac{F(a)}{a^3} - G(a) \right), \quad (6.61)$$

which shows that there is a contribution to the effective cosmological constant from the arbitrary integration constant. The contributions to the Friedmann equation coming from the mass terms will dominate the energy density at an unacceptably early time unless they are suppressed by assuming that $m^2 \ll \Lambda_{\text{tot}}^2$, where Λ_{tot} is the total effective cosmological constant (including possible matter contributions). Such a small mass is expected to be technically natural since a symmetry is recovered when $m = 0$.

6.5.2 FLRW perturbations

The stability of the flat FLRW solutions is now checked by studying perturbations. The notation for perturbations follows reference [213].

The metric perturbations are split into scalar, transverse vector, and transverse-traceless tensor parts

$$\delta g_{00} = -2N^2\phi, \quad (6.62a)$$

$$\delta g_{0i} = Na(B_i^T + \partial_i B), \quad (6.62b)$$

$$\delta g_{ij} = a^2 \left(h_{ij}^{TT} + \frac{1}{2}(\partial_i E_j^T + \partial_j E_i^T) + 2\delta_{ij}\psi + (\partial_i \partial_j - \frac{1}{3}\delta_{ij}\partial_l \partial^l)E \right), \quad (6.62c)$$

and the Lagrange multiplier field λ is perturbed around the background value λ_0 ,

$$\lambda = \lambda_0 + \delta\lambda. \quad (6.63)$$

Next, the perturbed fields are substituted into the action, expanded in Fourier plane waves, and evaluated to second-order in the perturbations to determine the mass and kinetic stability of the perturbations. The perturbations split into tensor, vector and scalar sectors at this order, so these are considered in turn.

The action for the tensor sector is

$$S = \frac{M_p^2}{8} \int d^3k dt \left(\frac{1}{N^2} |\dot{h}_{ij}^{TT}|^2 - \left(\frac{k^2}{a^2} + M_{GW}^2 \right) |h_{ij}^{TT}|^2 \right), \quad (6.64)$$

where the mass of the tensor modes is given by

$$M_{GW}^2 = \frac{m^2}{a^3} \left[-a(a^4 - 3a + 1) - \alpha_3(2a^5 - a^4 - 8a^3 + 15a^2 - 10a + 2) - \alpha_4(a - 1)^2(a^3 + a^2 - a + 2) \right]. \quad (6.65)$$

The kinetic term always has the correct sign, so that the tensor is never a ghost. In general, the mass term can take either sign, although the tensor mass is never tachyonic around $a \sim 1$ since $M_{GW}^2|_{a=1} = m^2$ for all choices of α_3, α_4 . The tensor mass M_{GW}^2 will always tend to $\pm\infty$ as $a \rightarrow 0$.

The action for the vector sector contains the two transverse vectors B_i^T and E_i^T . There are no time derivatives on B_i^T , hence it is nondynamical and can be eliminated with its equations of motion

$$B_i^T = \frac{k^2 a^4 \dot{E}_i^T}{m^2 f_1(a) + 2k^2}, \quad (6.66)$$

where

$$f_1(a) \equiv \frac{4}{(1+a^4)^2} \left[(3a-2)(1+a^4) + \alpha_3(a-1)(a^4+1)(4a^5-5a^4+a^3+4a-2) + \alpha_4(1-2a+a^2-a^3+4a^4-5a^5+2a^6-a^7+3a^8-3a^9+2a^{10}) \right]. \quad (6.67)$$

The resulting action is

$$S = \frac{M_p^2}{8} \int d^3k dt \left(\frac{\mathcal{T}_V^2}{N^2} |\dot{E}_i^T|^2 - \frac{k^2 M_{\text{GW}}^2}{2} |E_i^T|^2 \right), \quad (6.68)$$

where

$$\mathcal{T}_V^2 \equiv \frac{k^2}{2} \left(\frac{m^2 f_1(a)}{m^2 f_1(a) + 2k^2} \right). \quad (6.69)$$

The sign of the kinetic term is determined by \mathcal{T}_V^2 . If $f_1(a)$ is always positive then the vector is never ghostlike, which is possible for certain choices of the parameters. If $f_1(a_*)$ is negative for some a_* , then keeping the kinetic term positive requires

$$k^2 < \frac{m^2 |f_1(a_*)|}{2}. \quad (6.70)$$

In this case there is thus a critical momentum scale above which the vector is a ghost and to avoid introducing a ghost below the UV cutoff of the EFT Λ_{UV} requires that

$$0 < \Lambda_{\text{UV}}^2 \lesssim \frac{m^2 |f_1(a)|}{2a^2}, \quad (6.71)$$

for all a . This is also possible for certain choices of the parameters. Checking for other instabilities in the vector requires canonically normalising the kinetic term and analysing the resulting dispersion relation. Since there is always a ghost in the scalar sector, this calculation is not included here.

In the scalar action the Lagrange multiplier perturbation appears as

$$\delta\lambda(\phi + 3\psi), \quad (6.72)$$

so its equations of motion enforce the condition $\phi = -3\psi$. Integrating out $\delta\lambda$ by substituting $\phi = -3\psi$ gives an action for three scalar perturbations E , B , ψ . The field B appears in this action without time derivatives so it can also be integrated out using its equations of motion

$$B = \frac{8a^3}{m^2 f_1(a)} \left(\frac{1}{6} a k^2 \dot{E} + a\dot{\psi} + 3\dot{a}\psi \right). \quad (6.73)$$

This leaves two scalar perturbations, for a total of six degrees of freedom. Defining $Y = (\psi, E)^T$, the resultant action can be written in the form

$$S = M_p^2 \int d^3k dt \frac{1}{2} \left(\frac{\dot{Y}^\dagger}{N} \mathcal{K} \frac{\dot{Y}}{N} + \frac{\dot{Y}^\dagger}{N} \mathcal{M} Y + Y^\dagger \mathcal{M}^\dagger \frac{\dot{Y}}{N} - Y^\dagger \Omega^2 Y \right), \quad (6.74)$$

where \mathcal{K} and Ω^2 are symmetric real 2×2 matrices and \mathcal{M} is a real antisymmetric 2×2 matrix. The kinetic matrix \mathcal{K} is given explicitly by

$$\mathcal{K} = \begin{pmatrix} -6 & 0 \\ 0 & \frac{k^4}{6} \end{pmatrix} - \frac{8}{3m^2 f_1(a)} \begin{pmatrix} 6 & k^2 \\ k^2 & \frac{k^4}{6} \end{pmatrix}, \quad (6.75)$$

which has the determinant

$$\det \mathcal{K} = -k^4. \quad (6.76)$$

The negativity of $\det \mathcal{K}$ for all choices of the parameters α_3, α_4 means that this theory always propagates a scalar ghost about the flat FLRW solutions, so these solutions are unstable.

7

Conclusions

The structure of general relativity is largely determined by gauge invariance, which follows from Lorentz invariance and the requirement of describing only the degrees of freedom of a massless spin-2 particle. There is no gauge symmetry required to describe massive spin-2 particles, so it might have been expected that there are correspondingly more interacting theories of massive gravity. However, we have seen in this thesis that it is quite difficult to make a massive graviton interact. The requirement that a classical theory describes only a massive spin-2 particle is so constraining that for a long time it was thought that no interacting theory existed [44, 45]. The discovery of ghost-free Lorentz-invariant theories of massive gravity by de Rham, Gabadadze, and Tolley [46, 47] thus spurred an exciting period of progress in the study of massive spin-2 theories that is still ongoing.

These massive spin-2 theories lie at the intersection of field theory, gravity, and cosmology, and there are several motivations for studying them. Even though general relativity with dark matter and a small cosmological constant can account reasonably well for all cosmological observations, the fine-tuning needed for a small cosmological constant is a hint that something is missing from our understanding of fundamental physics [1]. Theories that modify GR in the IR might therefore be needed to describe our universe. Even if GR is the correct theory of gravity at low energies, we can still hope to better understand GR by trying to modify it. It is

also worthwhile on purely theoretical grounds to try to construct theories using the various particle types available to us. Such pursuits help us to better understand field theories generally and can unexpectedly lead to insights about nature.

This thesis has explored several aspects of massive spin-2 EFTs and their relatives. One overarching theme has been the idea that massive spin-2 theories are highly constrained and not easily deformed without introducing new degrees of freedom. This uniqueness is partly why such theories are interesting when trying to connect to observations.

One aspect of this was seen in Chapter 3 by studying on-shell scattering amplitudes. A method for constructing all on-shell amplitudes was reviewed and used to calculate general $2 \rightarrow 2$ scattering for massive spin-1 and spin-2 particles. It was shown that the $2 \rightarrow 2$ massive graviton amplitude is equal to the ghost-free dRGT or pseudo-linear massive spin-2 amplitudes when the strong coupling scale is raised as high as possible. This indicates that the tree-level ghost-free structure is determined once the Λ_3 strong coupling scale is specified and that the cutoff cannot be raised beyond Λ_3 without additional fields. This calculation also allowed for the possibility of parity-odd interaction terms, including a two-derivative parity-odd cubic amplitude that might hint at the existence of a parity-odd theory of massive gravity in four dimensions. The amplitude calculation showed that any such parity-odd theory that is unitary would have a strong coupling scale lower than Λ_3 .

The scattering amplitude formalism can also fruitfully constrain other theories, such as possible extensions of massive gravity with extra fields and a higher strong coupling scale. By including the exchange of additional states in the $2 \rightarrow 2$ amplitude it can be determined whether or not the cutoff can be raised in a similar way to the Higgs mechanism for massive vector scattering. Some simple examples along these lines were worked out and presented in Chapter 3. Another generalisation that was considered was massive higher-spin theories. A conjecture was made for the highest strong coupling scale for a self-interacting massive spin- J field, based on calculations of zero-derivative higher-spin quartic amplitudes.

Any physical theory must eventually be compared to observations. Fortunately, rather than working out the detailed consequences of every model, it is sometimes possible to rule out large classes of theories using general theoretical considerations. We saw in Chapter 4 how otherwise sensible EFTs could be constrained if certain assumptions were made about a hypothetical UV completion, assumptions that are ultimately informed by observations of low-energy physics. In particular, if dRGT massive gravity is the low energy effective theory of an analytic, Lorentz invariant, and unitary UV completion, then only a certain range of parameter space is permitted [139]. It was shown in Chapter 4 that the pseudo-linear massive spin-2 theory cannot satisfy such constraints, which implies that the Einstein-Hilbert interactions are needed for a ghost-free massive spin-2 theory to satisfy the constraints. It was also demonstrated that massive vector theories that reduce to the Galileons in a decoupling limit can satisfy the constraints, so these theories could be UV-completed by Lorentz invariant field theories. It is possible to further constrain these and other EFTs using similar theoretical considerations [155, 156]. Such tools are a powerful way to constrain the space of field theories, even before comparing directly to experiments.

Interacting theories of massless and massive gravitons are both described by robust theoretical structures and it is perhaps not surprising that there are connections between them. One connection is that massless particles are related to massive particles in one fewer dimension, as suggested by the coincidence of the nontrivially acting little groups. A concrete realisation of this comes from KK dimensional reduction. This connection was explored quite generally in Chapter 5 by studying the KK spectrum of a partially massless graviton and a general massive graviton. The linear dimensional reduction was worked out for Einstein product manifolds using a generalised Hodge decomposition. It was found that higher-dimensional partially massless gauge symmetry has the effect of eliminating a tower of scalar fields and does not lead to a zero-mode partially massless graviton in lower dimensions. The lower-dimensional spectrum was found for a general massive graviton and the stability conditions were discussed. The nonlinear zero-mode

reduction of dRGT massive gravity on a circle was also deduced in Chapter 5. This produced a particular family of mass-varying massive gravity theories that could be of interest due to their higher-dimensional origin.

Lastly, Chapter 6 attempted to find a novel theory of a massive spin-2 field based on a noncanonical kinetic term related to unimodular gravity. A candidate theory was proposed and some of its cosmological solutions were found. These solutions were found to be unstable due to ghostly cosmological perturbations. Next some candidate linear theories were constructed by dimensionally reducing noncanonical massless spin-2 theories and the resulting theories were found to be equivalent to the usual Fierz-Pauli theory. The results of this investigation suggest that any massive gravity theory based on unimodular gravity is either equivalent to ordinary massive gravity or has a ghost degree of freedom. This provides another piece of evidence to support the claim that the Einstein-Hilbert kinetic term is unique in describing massive spin-2 EFTs.

There is still plenty of theoretical terrain left to explore. In massive gravity, one of the major open problems is to find a UV completion of the theory, or alternatively to understand how the theory remains healthy in the strongly coupled regime. Another open problem is to find a unitary interacting theory of a partially massless graviton. Such a theory is constrained by various no-go results, but if it existed it would have many appealing properties and could be relevant for cosmology. Finally, beyond spin 2 lies the higher-spin frontier. Theories with infinitely many interacting massless higher-spin fields are known to exist in AdS [214, 215] and string theory contains infinitely many massive higher-spin states [8], but EFTs containing a finite number of massive higher-spin fields are relatively unexplored. Finding ghost-free massive higher-spin effective theories, or proving that such theories do not exist, would lead to important insights about what kinds of effective theories can and cannot exist.

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