

Effective Whitney Stratification of Real Algebraic Varieties

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ABSTRACT. We describe new algorithms to compute Whitney stratifications of real algebraic varieties. Using either conormal or polar techniques, these algorithms stratify a complexification of a given real variety. We then show that the resulting stratification can be described by real polynomials. We also extend these methods to stratification problems involving the so-called full semialgebraic sets as well as real algebraic maps.

1. Introduction

Real varieties arise more naturally in applications than their complex counterparts. Unfortunately, the geometry of a real variety is not readily revealed by its defining polynomials: for instance, every real variety is the vanishing locus of a single polynomial, whereas the minimal number of defining polynomials forms an important (and non-trivial) invariant for complex varieties. As such, one often seeks methods to subdivide a given variety into simpler regions which fit together coherently. And indeed, cylindrical algebraic decomposition [2, 5, 6, 13, 47] provides precisely such a subdivision into contractible cells. One of our goals in this work is to outline what we view as a key step in an alternative framework for studying broad classes of applied and computational problems in real algebraic geometry.

Whitney Stratifications. There are many advantages to obtaining a cell decomposition of a variety — it becomes possible, for instance, to directly compute the Euler characteristic, and (with a bit more work) the Betti numbers. On the other hand, cells, being homeomorphic images of open balls, are rather rigid objects. Cylindrical algebraic decomposition tends to be impractical on larger (or more complicated) varieties precisely because one often has to perform many subdivisions in order to obtain cells; similarly, the number of these cells can also be doubly exponential relative to the number of variables [5]. A more general and flexible paradigm is furnished by the stratifications introduced by Whitney [46], which play an essential role in microlocal geometry and stratified Morse theory [17, 33].

Definition 1.1. A pair (M, N) of smooth submanifolds of \mathbb{R}^n satisfies Whitney's **Condition (B)** if the following property holds at every point $q \in N$. Given any pair of sequences $\{p_k\} \subset M$ and $\{q_k\} \subset N$ with $\lim p_k = q = \lim q_k$, if the limiting tangent space and the limiting secant line

$$T := \lim_{k \rightarrow \infty} T_{p_k} M \quad \text{and} \quad \ell := \lim_{k \rightarrow \infty} [p_k, q_k]$$

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both exist, then $\ell \subset T$.

A Whitney stratification of a subset $X \subset \mathbb{R}^n$ is any locally-finite decomposition

$$X = \coprod_{\alpha} M_{\alpha}$$

into smooth, connected nonempty manifolds $M_{\alpha} \subset X$ called *strata*, so that every pair (M_{α}, M_{β}) satisfies Condition (B). It is known that every (real or complex) algebraic variety X admits a Whitney stratification such that for each dimension i the union X_i of all strata of dimension $\leq i$ forms a subvariety (see e.g. [45, page 36–38]). One may therefore identify any such stratification of a k -dimensional variety X with a flag

$$X_{\bullet} = (\emptyset \subset X_0 \subset X_1 \subset \cdots \subset X_{k-1} \subset X_k = X) \quad (1)$$

of subvarieties, with the implicit understanding that its i -strata are connected components of $X_i - X_{i-1}$.

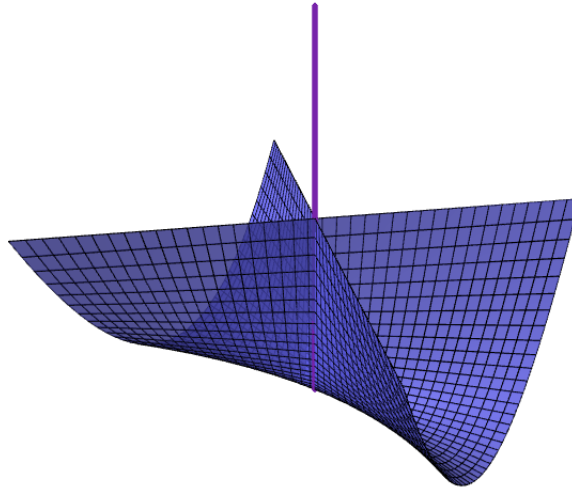


FIGURE 1. A depiction of the Whitney Umbrella, which is commonly used to illustrate equisingularity.

One key consequence of imposing Condition (B) on strata pairs is *equisingularity*: a sufficiently small tubular neighborhood around a given stratum forms a (locally trivial) stratified fiber bundle over that stratum [36, Cor 10.6]. As a consequence, small neighborhoods in X of two points lying in the same stratum M_{α} are stratified homeomorphic¹ to each other. Consider, for instance, the (real) *Whitney Umbrella* from Figure 1. This is the variety $X \subset \mathbb{R}^3$ defined by $x^2 - y^2z = 0$, whose singular locus is the (positive) Z -axis (depicted as a purple line in the figure). If we take a small neighbourhood in X of a point $(0, 0, z)$ lying on the Z -axis, then its topology changes depending on whether z is positive, negative or zero. Thus, the origin must constitute a stratum separate from the rest of the Z -axis in any valid Whitney stratification of X .

¹A homeomorphism $\phi : X \rightarrow Y$ of stratified spaces is said to be stratified whenever it maps each stratum $S \subset X$ diffeomorphically onto a stratum $\phi(S) \subset Y$.

This Paper. The main contribution of our work here is a suite of practical algorithms for constructing Whitney stratifications of real algebraic varieties and related objects. Our prior work [24]² describes an effective algorithm **WhitStrat** for constructing Whitney stratifications of complex varieties; this algorithm relies on primary decompositions of conormal fibers (and hence, on Gröbner basis computation). Here we introduce a new algorithm **WhitneyPolar** for computing complex stratifications; it is based on the theory of polar varieties [14] and exhibits better complexity and runtimes than **WhitStrat**.

In the introductory remarks to [24], we highlighted the lack of Gröbner basis techniques over \mathbb{R} as a primary obstacle to performing similar stratifications for real algebraic varieties and semialgebraic sets. We overcome this obstacle here by examining, for any real variety $X \subset \mathbb{R}^n$, an associated complex variety $X(\mathbb{C})$ — this is obtained by treating (any set of) the real polynomials which define X as elements of the ring $\mathbb{C}[x_1, \dots, x_n]$. Having constructed $X(\mathbb{C})$, we obtain an effective Whitney stratification of X via the following two steps:

- (1) we show in Theorem 4.3 that if $X_\bullet(\mathbb{C})$ is any Whitney stratification of $X(\mathbb{C})$ for which each $X_i(\mathbb{C})$ is generated by real polynomials, then the corresponding real varieties X_i form a valid Whitney stratification of X ; and moreover,
- (2) we show in Corollary 4.4 that if a complex variety $Y \subset \mathbb{C}^n$ is generated by real polynomials, then running either **WhitStrat** or **WhitneyPolar** on Y produces a Whitney stratification Y_\bullet for which each Y_i is also generated by real polynomials.

These two results open the door for stratifying real algebraic varieties via effective complex techniques. Unlike other efforts in this direction (see for instance [1] which stratifies hypersurfaces in \mathbb{R}^3), the algorithm described here works in general for all dimensions. Moreover, the same techniques also readily adapt to the more general setting of *full semialgebraic sets* (see Definition 5.1).

Here we also investigate stratifications of algebraic maps $f : X \rightarrow Y$ between real varieties. This amounts to a pair of Whitney stratifications X_\bullet of X and Y_\bullet of Y such that f sends each stratum of X_\bullet submersively to a stratum of Y_\bullet . If f is a *proper map* (this always holds if X is compact, for instance), then it forms a *stratified fiber bundle* with respect to X_\bullet and Y_\bullet . In other words, the restriction $f^{-1}(N) \rightarrow N$ is an ordinary fiber bundle for each stratum $N \subset Y$. As a consequence, the homeomorphism type of the fiber $f^{-1}(y)$ depends only on the stratum N and is independent of the choice of point $y \in N$. To compute such a stratification of f , we once again proceed by first treating it as a complex algebraic map $f_{\mathbb{C}} : X(\mathbb{C}) \rightarrow Y(\mathbb{C})$ and then using algorithms from [24]. We then show in Theorem 4.5 that the resulting stratifications of $X(\mathbb{C})$ and $Y(\mathbb{C})$ consist of varieties defined by real polynomials, and that the corresponding real varieties furnish the desired stratification of f .

Unfortunately, the requirement that $f : X \rightarrow Y$ be proper fails quite often in cases of interest for general algebraic maps. One exception to this rule is when f is a *dominant morphism* between varieties of the same dimension (see Definition 4.6). For such a morphism

²The Arxiv version [24] cited here incorporates both the original paper [25] and its subsequent correction [28] in one document.

f , we are able to partially recreate the stratified fiber bundle property in Theorem 4.9 by carefully analysing and decomposing the locus of points at which f fails to be proper.

Roadmap. The rest of this paper is organised as follows. In Section 2 we summarise relevant results and algorithms from [24] for stratifying complex varieties. The only new material here lies in Section 2.3, where we are able to remove a genericity hypothesis for map stratifications. Section 3 introduces a new **WhitneyPolar** algorithm for complex Whitney stratifications which is based on polar varieties. In Section 4 we show how both our old and new complex stratification algorithms may be used to stratify real varieties and algebraic morphisms between them. Our task in Section 5 is to similarly stratify basic closed semialgebraic sets whose inequality loci are open subsets of the ambient space — these often arise in practical contexts owing to boundedness or positivity constraints. Finally, in Section 6, we examine the experimental performance of **WhitneyPolar**.

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2. Whitney Stratifications from Conormal Ideals

In this section we begin with a brief review of the stratification algorithms for complex varieties and their morphisms. Section 2.1 and Section 2.2 are a review of the results of [24], restated in a form better suited to the current presentation. On the other hand, Section 2.3 is mostly new and makes a key generalization of the map stratification algorithm of [24, Section 6]; it allows us to drop the genericity requirement from [24, Definition 6.3], so that the Thom-Boardman flag of the given polynomial map $f : X \rightarrow Y$ need not intersect X transversely³. Instead, we consider arbitrary polynomial maps $f : X \rightarrow Y$ between complex affine varieties $X \subset \mathbb{C}^n$ and $Y \subset \mathbb{C}^m$, and derive (in Lemma 2.5) an expression for the locus of smooth points of X where the rank of f 's Jacobian drops below its generic value.

A central role in our generalisation is played by a new type of stratification, which we have called *Whitney-Thom-Boardman* stratifications of X along f (see Definition 2.6). Here the decomposition of X into strata is well behaved with respect to the Thom-Boardman singularities of f .

2.1. Stratification of Complex Varieties. Consider a pure k -dimensional complex variety, that is a variety all of whose irreducible components are of dimension k , $X = \mathbf{V}_{\mathbb{C}}(I_X) \subset \mathbb{C}^n$ defined by a radical ideal $I_X = \langle f_1, \dots, f_r \rangle$ in $\mathbb{C}[x_1, \dots, x_n]$. Let X_{reg} denote the (open, dense) manifold of smooth points in X and let $X_{\text{sing}} = X - X_{\text{reg}}$ be the singular locus of X ;

³See [16, Chapter VI] for a detailed treatment of Thom-Boardman flags.

recall X_{sing} is a closed proper subvariety of X , see e.g. [43, Section 6.1] or [37, page 26–27]. The *conormal variety* of X is the subvariety of $X \times \mathbb{P}^{n-1}$ given by

$$\mathbf{Con}(X) = \overline{\{(p, \xi) \mid p \in X_{\text{reg}} \text{ and } T_p X_{\text{reg}} \subset \xi^\perp\}}. \quad (2)$$

In other words, $\mathbf{Con}(X)$ is obtained by closing the set of pairs (p, ξ) consisting of points $p \in X_{\text{reg}}$ and $\xi \in \mathbb{P}^{n-1}$ such that the inner product $\langle \xi, v \rangle$ is zero for all $v \in T_p X_{\text{reg}}$. We note the later coordinate is taken to be projective as we identify all constant multiples of a particular unit direction vector defining the normal to a plane. The conormal variety comes endowed with a canonical projection map $\kappa_X : \mathbf{Con}(X) \rightarrow X$ induced by the coordinate projection $\mathbb{C}^n \times \mathbb{P}^{n-1} \rightarrow \mathbb{C}^n$. Work in the ring $\mathbb{C}[x_1, \dots, x_n][[\xi_1, \dots, \xi_n]]$ and set

$$\mathcal{K} = \begin{bmatrix} \xi_1 & \cdots & \xi_n \\ \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix} \quad \text{and} \quad \mathcal{J} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}.$$

The conormal variety $\mathbf{Con}(X) = \mathbf{V}_{\mathbb{C}}(I_{\mathbf{Con}(X)})$ is defined by the radical ideal obtained via the saturation $I_{\mathbf{Con}(X)} = (I_X + K) : J^\infty$ — here K is the ideal generated by all $(n - k + 1) \times (n - k + 1)$ minors of the matrix \mathcal{K} , while J is similarly defined by the $(n - k) \times (n - k)$ minors of the matrix \mathcal{J} . Note that the singular locus X_{sing} is precisely $\mathbf{V}_{\mathbb{C}}(J) \cap X$. A definition of ideal saturation and a discussion of how it is computed may be found in books such as [9, §4.4]. The following result [24, Theorem 3.1] forms the basis for efficient Whitney stratification of a given variety X . Recall that given an ideal I its associated primes are the radicals of the ideals appearing in a minimal primary decomposition of I , see books such as [9, 37] for definitions and see [10] for a survey of methods for the computation of associated primes of an ideal via Gröbner basis.

THEOREM 2.1. *Let $X \subset \mathbb{C}^n$ be a pure dimensional variety, let $Y \subset X_{\text{sing}} = (X - X_{\text{reg}})$ be a nonempty irreducible subvariety defined by a radical ideal I_Y , and set $I_{\kappa_X^{-1}(Y)} := I_{\mathbf{Con}(X)} + I_Y$. Let $\{P_1, \dots, P_s\}$ be the associated primes of $I_{\kappa_X^{-1}(Y)}$, let $\sigma \subset \{1, 2, \dots, s\}$ be the set of indices i with $\dim \kappa_X(\mathbf{V}_{\mathbb{C}}(P_i)) < \dim Y$ and let*

$$A := \left[\bigcup_{i \in \sigma} \kappa_X(\mathbf{V}_{\mathbb{C}}(P_i)) \right] \cup Y_{\text{sing}}.$$

Then the pair $(X_{\text{reg}}, Y - A)$ satisfies Condition (B).

We first describe a subroutine, called **Decompose**, which implements this theorem for a given subvariety $Y \subset X_{\text{sing}}$. In particular, it computes the dimension of the elimination ideal arising from each primary component of $I_{\kappa_X^{-1}(Y)}$, and records those which have dimension less than Y . In effect, this subroutine explicitly constructs subvarieties of Y (not necessarily contained in Y_{sing} where Condition (B) fails with respect to X).

	Decompose (Y, X)
	Input: Algebraic varieties $Y \subset X$ in \mathbb{C}^n , with $d := \dim Y$. Output: A list of subvarieties Y_\bullet of Y .
1	Set $Y_\bullet := (Y_d, Y_{d-1}, \dots, Y_0) := (Y, \emptyset, \dots, \emptyset)$
2	Set $J := I_{\text{Con}(X)} + I_Y$
3	For each primary component Q of a primary decomposition of J
4	Set $K := Q \cap \mathbb{C}[x]$
5	If $\dim V_{\mathbb{C}}(K) < \dim Y$
6	Add $V_{\mathbb{C}}(K)$ to $Y_{\geq \dim V_{\mathbb{C}}(K)}$
7	Return Y_\bullet .

It is shown in [24] that the **WhitStrat** algorithm below correctly computes the Whitney stratification of the variety X , and is guaranteed to terminate in finitely many steps.

	WhitStrat (X)
	Input: A pure k -dimensional variety $X \subset \mathbb{C}^n$. Output: A list of subvarieties X_\bullet of X .
1	Set $X_\bullet := (X_k, X_{k-1}, \dots, X_0) := (X, \emptyset, \dots, \emptyset)$
2	Compute X_{sing} and $\mu := \dim(X_{\text{sing}})$
3	For each irreducible component Z of X_{sing}
4	Add Z to $X_{\geq \dim Z}$
5	For each d in $(\mu, \mu - 1, \dots, 1, 0)$
6	Set $X_\bullet := \text{Merge}(X_\bullet, \text{Decompose}(\text{Pure}_d(X_d), X))$
7	Set $X_\bullet := \text{Merge}(X_\bullet, \text{WhitStrat}(\text{Pure}_d(X_d)))$
8	Return X_\bullet .

The **WhitStrat** function makes use of two elementary subroutines besides **Decompose**; these are invoked in lines 6 and 7:

- (1) Pure_d extracts the purely d -dimensional irreducible components of the input variety; this is accomplished via prime decomposition [10, 12].
- (2) Let V_\bullet and W_\bullet be nested sequences of subvarieties of a common variety, with the length p of V_\bullet larger than the length q of W_\bullet . Then **Merge**(V_\bullet, W_\bullet) creates a new sequence U_\bullet of length p via $U_i := V_i \cup W_{\max(i,q)}$.

2.2. Flag-Subordinate Stratifications. By a *flag* \mathbf{F}_\bullet on a variety $X \subset \mathbb{C}^n$ we mean any finite nested set of subvarieties of the form

$$\emptyset = \mathbf{F}_{-1}X \subset \mathbf{F}_0X \subset \mathbf{F}_1X \subset \dots \subset \mathbf{F}_{\ell-1}X \subset \mathbf{F}_\ell X = X.$$

The integer ℓ is called the *length* of \mathbf{F}_\bullet . There are no additional restrictions on the dimensions of the individual \mathbf{F}_jX and, in particular, we do not require successive differences $\mathbf{F}_iX - \mathbf{F}_{i-1}X$ to be smooth manifolds or to satisfy Condition (B).

Definition 2.2. Let $X \subset \mathbb{C}^n$ be an affine variety and \mathbf{F}_\bullet a flag on X of length ℓ . A Whitney stratification X_\bullet of X is **subordinate** to \mathbf{F}_\bullet if for each stratum $S \subset X$ of X_\bullet there exists some $j = j(S)$ in $\{0, \dots, \ell\}$ satisfying $S \subset (\mathbf{F}_jX - \mathbf{F}_{j-1}X)$.

Our main motivation for considering flag subordinate stratification is to develop algorithms to stratify algebraic maps between varieties. It is shown in [24, §5] that a flag subordinate stratification of a complex algebraic variety can always be computed using the **WhitStratFlag** algorithm described below. There are two main subroutines, the first of which is called **InducedFlag**. This takes as input a subvariety $W \subset X$ and a flag \mathbf{F}_\bullet on X , and outputs the restriction of \mathbf{F}_\bullet to W .

InducedFlag (W, \mathbf{F}_\bullet)	
	Input: A subvariety $W \subset X \subset \mathbb{C}^n$ and a flag \mathbf{F}_\bullet on X of length ℓ . Output: A flag \mathbf{F}'_\bullet on W of length ℓ .
1	Set $\mathbf{F}'_\bullet W := (\mathbf{F}'_\ell W, \dots, \mathbf{F}'_0 W) := (\emptyset, \dots, \emptyset)$
2	For each irreducible component V of W
3	Add V to $\mathbf{F}'_i W$ for all \mathbf{F}'_i where $V \subset \mathbf{F}'_i$
4	For each j with $\dim(\mathbf{F}_j X \cap V) < \dim V$
5	Add $V_j := (\mathbf{F}_j X \cap V)$ to $\mathbf{F}'_i W$ for all \mathbf{F}'_i where $V_j \subset \mathbf{F}'_i$
6	Return $\mathbf{F}'_\bullet W$

The second subroutine is called **DecomposeFlag**; this is a variant of the **Decompose** subroutine described in the preceding subsection. The only difference is that rather than merging the detected subvariety $\mathbf{V}_\mathbb{C}(K)$ with Y_\bullet directly, we first use **InducedFlag** to restrict \mathbf{F}_\bullet to $\mathbf{V}_\mathbb{C}(K)$ and then merge the output with Y_\bullet .

DecomposeFlag (Y, X, \mathbf{F}_\bullet)	
	Input: Varieties $Y \subset X \subset \mathbb{C}^n$ with $d := \dim Y$ and a flag \mathbf{F}_\bullet on X . Output: A list of subvarieties $Y_\bullet \subset Y$.
1	Set $Y_\bullet := (Y_d, Y_{d-1}, \dots, Y_0) := (\emptyset, \dots, \emptyset)$
2	Set $J := I_{\text{Con}(X)} + I_Y$
3	For each primary component Q of a primary decomposition of J
4	Set $K := Q \cap \mathbb{C}[x]$
5	If $\dim \mathbf{V}_\mathbb{C}(K) < \dim Y$
6	Merge Y_\bullet with InducedFlag ($\mathbf{V}_\mathbb{C}(K), \mathbf{F}_\bullet X$)
7	Return Y_\bullet

Finally, here is the promised **WhitStratFlag** algorithm which makes use of these two subroutines. This takes as input a variety X and a flag \mathbf{F}_\bullet defined on X . The output X_\bullet is guaranteed to be a Whitney stratification subordinate to \mathbf{F}_\bullet in the sense of Definition 2.2 — see [24, Sec 5] for details.

	WhitStratFlag (X, \mathbf{F}_\bullet)
	Input: A pure k -dimensional variety $X \subset \mathbb{C}^n$ and a flag \mathbf{F}_\bullet on X . Output: A list of subvarieties $X_\bullet \subset X$.
1	Set $X_\bullet := (X_k, X_{k-1}, \dots, X_0) := (X, \emptyset, \dots, \emptyset)$
2	Compute X_{sing} and $\mu := \dim(X_{\text{sing}})$
3	Set $X_d = X_{\text{sing}}$ for all d in $\{\mu, \mu + 1, \dots, k - 1\}$
4	Merge X_\bullet with InducedFlag ($X_{\text{sing}}, \mathbf{F}_\bullet, X$)
5	For each d in $(\mu, \mu - 1, \dots, 1, 0)$
6	Merge X_\bullet with DecomposeFlag ($X_d, X, \mathbf{F}_\bullet$)
7	Merge X_\bullet with WhitStratFlag (X_d, \mathbf{F}_\bullet)
8	Return X_\bullet .

2.3. Stratifying Complex Algebraic Morphisms. Maps of Whitney stratified spaces are typically required to satisfy additional criteria beyond smoothly sending strata to strata — see [4, Def 3.5.1] or [17, Part I, Ch 1.7] for instance.

Definition 2.3. Let \mathcal{X}_\bullet and \mathcal{Y}_\bullet be Whitney stratifications of topological spaces \mathcal{X} and \mathcal{Y} . A continuous function $\phi : \mathcal{X} \rightarrow \mathcal{Y}$ is **stratified** with respect to \mathcal{X}_\bullet and \mathcal{Y}_\bullet if for each stratum $M \subset \mathcal{X}$ there exists a stratum $N \subset \mathcal{Y}$ satisfying two requirements:

- (1) the image $\phi(M)$ is wholly contained in N ; and moreover,
- (2) the restricted map $\phi|_M : M \rightarrow N$ is a smooth submersion.⁴

The pair $(\mathcal{X}_\bullet, \mathcal{Y}_\bullet)$ is called a stratification of ϕ .

Remark 2.4. The second requirement of Definition 2.3 ensures the following crucial property via Thom’s first isotopy lemma [36, Prop 11.1]. If ϕ is a *proper map* – namely, if the inverse image of every compact subset of Y is compact in X – then for every stratum $N \subset Y$, the restriction of ϕ forms a locally trivial fiber bundle from $\phi^{-1}(N)$ to N .

Fix varieties $X \subset \mathbb{C}^n$ and $Y \subset \mathbb{C}^m$ along with a polynomial map $f : \mathbb{C}^n \rightarrow \mathbb{C}^m$ that sends points of X to points of Y . In [24, Section 6], we described an effective algorithm for stratifying any $f : X \rightarrow Y$ whose *Thom-Boardman flag*⁵ intersects X transversely. Here we will describe a new modification which circumvents this transversality requirement. The first step towards this goal is the following result.

Lemma 2.5. *Let $G := \{g_1, g_2, \dots, g_s\}$ be any set of polynomials defining $X \subset \mathbb{C}^n$ which generate a radical ideal. Consider, at each $x \in X$, the matrix of partial derivatives*

$$\mathbf{M}_G(x) := \begin{bmatrix} \partial g_1 / \partial x_1 & \cdots & \partial g_s / \partial x_1 \\ \vdots & \ddots & \vdots \\ \partial g_1 / \partial x_n & \cdots & \partial g_s / \partial x_n \end{bmatrix}$$

⁴Explicitly, its derivative $d(\phi|_S)_x : T_x M \rightarrow T_{\phi(x)} N$ is surjective at each point x in M .

⁵See [16, Chapter VI].

evaluated at x . For each j in $\{0, 1, \dots, \dim X\}$, let $V_j = V_j(X; f)$ be the subvariety of \mathbb{C}^n generated by all $j \times j$ minors of the augmented matrix $\mathbf{A}(x) := [\mathbf{M}_G(x) \quad \mathbf{J}f(x)^\top]$. Writing $f^* : X_{\text{reg}} \rightarrow \mathbb{C}^m$ for the restriction of f to the smooth points of X , we have

$$\text{rank } \mathbf{J}f^*(x) \leq i \quad \text{if and only if} \quad x \in V_{(n-\dim X)+(i+1)},$$

for all i in $\{0, 1, \dots, \min(\dim X, m)\}$.

PROOF. At each point $x \in X_{\text{reg}}$, the left block of $\mathbf{A}(x)$ equals $\mathbf{M}_G(x)$, whose s columns span the orthogonal complement of $T_x X_{\text{reg}}$ in \mathbb{C}^n . Since this complement necessarily has dimension $(n - \dim X)$ regardless of the chosen x , we obtain

$$\text{rank } \mathbf{A}(x) \geq n - \dim X.$$

The right block, consisting of $\mathbf{J}f(x)^\top$, has columns which span the coimage of $\mathbf{J}f(x)$, i.e., the orthogonal complement of $\ker \mathbf{J}f(x)$ in \mathbb{C}^n . Thus, the following criteria are equivalent:

- (1) $\text{rank } \mathbf{J}f^*(x) = 0$,
- (2) the coimage of $\mathbf{J}f(x)$ is a subspace of the orthogonal complement $T_x X_{\text{reg}}^\perp$,
- (3) the columns of $\mathbf{J}f(x)^\top$ are spanned by those of $\mathbf{M}_G(x)$,
- (4) $\mathbf{A}(x)$ assumes its minimum possible rank of $n - \dim X$.

Similarly, $\text{rank } \mathbf{J}f^*(x) = i$ holds whenever $\text{rank } \mathbf{A}(x) = (n - \dim X) + i$; this rank condition on $\mathbf{A}(x)$ corresponds to the vanishing of all minors of size $(n - \dim X) + i + 1$, as claimed above. \square

In light of the preceding result, we will work with the **rank flags** of f along various subvarieties of X ; given such a subvariety $Z \subset X$, the relevant flag is defined by

$$\mathbf{R}_i Z := Z \cap V_{n-\dim Z+i+1}(Z; f). \quad (3)$$

Here $V_\bullet(Z; f)$ is given in the statement of Lemma 2.5, with $0 \leq i \leq \min(\dim Z, m)$. By construction, the Jacobian of the restriction $f^* : Z_{\text{reg}} \rightarrow \mathbb{C}^m$ has rank $\leq i$ at a point $z \in Z_{\text{reg}}$ if and only if z lies in $\mathbf{R}_i Z$.

Definition 2.6. A **Whitney-Thom-Boardman** (WTB) stratification of X along the map $f : \mathbb{C}^n \rightarrow \mathbb{C}^m$ is any pair $(X_\bullet, \mathbf{S}_\bullet X)$, where X_\bullet is a Whitney stratification of X while $\mathbf{S}_\bullet X$ is a flag on X such that two conditions hold:

- (1) X_\bullet is subordinate to $\mathbf{S}_\bullet X$, and
- (2) for each X_\bullet -stratum $M \subset X$, we have $\mathbf{R}_i \overline{M} \subset \mathbf{S}_i X$ whenever $0 \leq i \leq \dim M$.

The first requirement of this definition forces X_\bullet to depend on $\mathbf{S}_\bullet X$ whereas the second requirement makes $\mathbf{S}_\bullet X$ dependant on X_\bullet . We are therefore compelled to construct both X_\bullet and $\mathbf{S}_\bullet X$ simultaneously via the algorithm described below. It will also be necessary in the sequel to produce WTB stratifications $(X_\bullet, \mathbf{S}_\bullet X)$ for which X_\bullet is subordinate to an auxiliary flag $\mathbf{F}_\bullet X$. Setting $\mathbf{F}_i X = X$ for all i , we obtain a WTB stratification as defined above. To simplify the presentation, we will denote this important special case as **WTBStrat** (X, f) .

	WTBStratFlag ($X, f, \mathbf{F}_\bullet X$)
	Input: $f : X \rightarrow \mathbb{C}^m$ algebraic map with $k := \dim X$; $\mathbf{F}_\bullet X$ a flag on X . Output: A pair of flags $(X_\bullet, \mathbf{S}_\bullet X)$ on X .
1	Set $X_\bullet := (X_k, X_{k-1}, \dots, X_0) := (X, \emptyset, \dots, \emptyset)$
2	Compute $\mathbf{R}_\bullet X$ using Lemma 2.5
3	Set $\mathbf{S}_\bullet X := \mathbf{R}_\bullet X$
4	Compute X_{sing} and $\mu := \dim(X_{\text{sing}})$
5	Compute $\mathbf{R}_\bullet X_{\text{sing}}$ using Lemma 2.5
6	Set $X_d := X_{\text{sing}}$ for all d in $\{\mu, \mu + 1, \dots, k - 1\}$
7	Set $\mathbf{S}_\bullet X := \text{Merge}(\mathbf{F}_\bullet X, \mathbf{R}_\bullet X_{\text{sing}})$
8	Set $X_\bullet := \text{Merge}(X_\bullet, \text{InducedFlag}(X_{\text{sing}}, \mathbf{S}_\bullet X))$
9	For each d in $(\mu, \mu - 1, \dots, 1, 0)$
10	Set $X_\bullet := \text{Merge}(X_\bullet, \text{DecomposeFlag}(X_d, X, \mathbf{S}_\bullet X))$
11	Set $X_\bullet := \text{Merge}(X_\bullet, \text{WTBStratFlag}(X_d, f, \mathbf{F}_\bullet X))$
12	Compute $\mathbf{R}_\bullet X_d$ using Lemma 2.5
13	Set $\mathbf{S}_\bullet X := \text{Merge}(\mathbf{S}_\bullet X, \mathbf{R}_\bullet X_d)$
14	Return $(X_\bullet, \mathbf{S}_\bullet X)$

THEOREM 2.7. *Let $X \subset \mathbb{C}^n$ be a pure dimensional complex algebraic variety, $f : \mathbb{C}^n \rightarrow \mathbb{C}^m$ a polynomial map, and $\mathbf{F}_\bullet X$ a flag on X . Then,*

- (1) **WTBStratFlag**($X, f, \mathbf{F}_\bullet X$) terminates,
- (2) its output $(X_\bullet, \mathbf{S}_\bullet X)$ is a WTB stratification of X along f , and
- (3) the Whitney stratification X_\bullet is subordinate to $\mathbf{F}_\bullet X$.

PROOF. The termination of **WTBStratFlag** follows from noting that the recursive call in Line 9 involves the variety X_d of dimension at most d , with d strictly smaller than $k = \dim X$. And the fact that X_\bullet is a Whitney stratification of X follows from the correctness of the original **WhitStrat** algorithm [24, Sec 4.4]. Moreover, since $\mathbf{F}_\bullet X$ is merged into $\mathbf{S}_\bullet X$ via Line 7, verifying that X_\bullet is \mathbf{F}_\bullet -subordinate can be reduced to checking that it is \mathbf{S}_\bullet -subordinate. Therefore, it suffices to confirm that the output pair $(X_\bullet, \mathbf{S}_\bullet X)$ satisfies the two requirements of Definition 2.6. To this end, consider an i -dimensional stratum M of X_\bullet .

To verify requirement (1), note that M may be uniquely written as $Z - X_{i-1}$, where $Z \subset X_i$ is an irreducible component — in particular, this Z will appear as a V in Line 3 of **InducedFlag** when it is called with first input X_i . Let j be the minimal index satisfying $Z \subset \mathbf{S}_j X$. Then $\dim(Z \cap \mathbf{S}_\ell X) < i$ for every $\ell < j$, and so $Z_\ell = Z \cap \mathbf{S}_\ell X$ is merged with X_\bullet in Line 5 of the **InducedFlag**. Thus, Z_ℓ is contained in X_m for some $m < i$. Since $X_m \subset X_i$, it follows that

$$M \cap (\mathbf{S}_p X - \mathbf{S}_{p-1} X) = \emptyset \text{ whenever } p < j.$$

But since $M \subset \mathbf{S}_j X$ and Z is irreducible, we have $Z \subset \mathbf{S}_j X$, whence $Z \subset \mathbf{S}_p X$ for all $p \geq j$. Thus, M also has empty intersections with $\mathbf{S}_p X - \mathbf{S}_{p-1} X$ for $p > j$, as desired. Turning now to property (2) of Definition 2.6, we observe that $\mathbf{S}_\bullet X$ is initialized to $\mathbf{R}_\bullet X$ in line 3 of

WTBStrat. Subsequently, $\mathbf{R}_\bullet X_d$ is merged into $\mathbf{S}_\bullet X$ for each $d \leq \dim X_{\text{sing}}$. Thus, for every i -stratum $M \subset X$, we have enforced that $\mathbf{R}_j \overline{M} \subset \mathbf{R}_j X_i \subset \mathbf{S}_j X$ whenever $j \leq i$. \square

Given a WTB stratification $(X_\bullet, \mathbf{S}_\bullet X)$ of X along f , we construct a *pushforward* flag \mathbf{B}_\bullet on the codomain variety Y as follows. Writing ℓ for the length of \mathbf{S}_\bullet , define

$$\mathbf{B}_i Y := \begin{cases} f(\mathbf{S}_i X) & i \leq \ell \\ Y & i = \ell + 1 \end{cases}. \quad (4)$$

The map stratification algorithm described below first constructs a \mathbf{B}_\bullet -subordinate stratification Y'_\bullet of Y . It then constructs a WTB stratification $(X'_\bullet, \mathbf{S}'_\bullet X)$ of X along f such that X'_\bullet is subordinate to the flag $\mathbf{F}_i X := f^{-1}(Y_i)$.

	WhitStratMap (X, Y, f)
	Input: Pure dimensional varieties X, Y and a any morphism $f : X \rightarrow Y$. Output: Lists of subvarieties $X_\bullet \subset X$ and $Y_\bullet \subset Y$.
1	Set $(X'_\bullet, \mathbf{S}'_\bullet X) := \mathbf{WTBStrat}(X, f)$
2	Set $\mathbf{B}'_\bullet Y := (\mathbf{B}'_{k+2} Y, \dots, \mathbf{B}'_0 Y) := (Y, \emptyset, \dots, \emptyset)$
3	For each j in $(0, 1, \dots, k)$
4	Set $\mathbf{B}'_j Y := f(\mathbf{S}'_j X)$
5	Set $\mathbf{B}'_{k+1} Y := Y$
6	Set $Y'_\bullet := \mathbf{WhitStratFlag}(Y, \mathbf{B}'_\bullet)$
7	For each i in $(0, 1, \dots, \dim Y)$
8	Set $\mathbf{F}_i X := f^{-1}(Y'_i)$
9	Set $(X'_\bullet, \mathbf{S}'_\bullet X) := \mathbf{WTBStratFlag}(X, f, \mathbf{F}_\bullet)$
10	Set $(X_\bullet, Y_\bullet) := \mathbf{Refine}(X'_\bullet, Y'_\bullet, f)$
11	Return (X_\bullet, Y_\bullet)

The final line invokes a **Refine** subroutine, which will be described later. We first highlight a crucial property of the Whitney stratifications (X'_\bullet, Y'_\bullet) which are invoked in Line 10.

Proposition 2.8. *For every X'_\bullet -stratum $M \subset X$ there exist a Y'_\bullet -stratum $N \subset Y$ satisfying $f(M) \subset N$. Moreover, at each point $x \in M$, the Jacobian $\mathbf{J}f|_M(x) : T_x M \rightarrow T_{f(x)} N$ of the restricted map $f|_M$ has maximal rank:*

$$\text{rank}(\mathbf{J}f|_M(x)) = \min(\dim M, \dim N).$$

PROOF. Noting that X'_\bullet is subordinate to the flag \mathbf{F}_\bullet by Line 9 of **WhitStratMap**, we know that for each stratum M of X'_\bullet there is a number $i := i(M)$ satisfying $M \subset (\mathbf{F}_i X - \mathbf{F}_{i-1} X)$. Now by Line 8, for any such stratum we have $f(M) \subset (Y'_i - Y'_{i-1})$. By the definition of a Whitney stratum we know that M is connected, and so its image under the continuous map f must also be connected; thus there is a unique stratum $N \subset (Y'_i - Y'_{i-1})$ of Y'_\bullet satisfying $f(M) \subset N$, as claimed above. It remains to establish that the Jacobian of $f|_M : M \rightarrow N$ has maximal rank at every point of x .

By Line 6 of **WhitStratMap**, the stratification Y'_\bullet is subordinate to the flag $\mathbf{B}'_\bullet Y$; thus, there exists a number $j := j(N)$ satisfying $N \subset (\mathbf{B}'_j Y - \mathbf{B}'_{j-1} Y)$. From Line 9, we note that \mathbf{B}'_\bullet is the

pushforward of \mathbf{S}''_\bullet , so we obtain

$$f^{-1}(N) \subset \mathbf{S}''_j X - \mathbf{S}''_{j-1} X. \quad (5)$$

Recall that $\mathbf{S}''_j X$ contains $\mathbf{R}_j \overline{M}$ by the second requirement of 2.6 (and similarly for $j - 1$). Therefore, the restricted map $f^* : \overline{M}_{\text{reg}} \rightarrow \mathbb{C}^m$ has Jacobian of constant rank ρ_j given by

$$\rho_j := \min(\dim M, m) - j. \quad (6)$$

Since M is an open subset of $\overline{M}_{\text{reg}}$, for each $x \in M$ we immediately have $\text{rank}(\mathbf{J}f^*(x)) = \rho_j$. Let $\iota : N \hookrightarrow \mathbb{C}^m$ be the inclusion map, and recall that $f|_M : M \rightarrow N$ denotes the restriction of f to M , now viewed as a smooth map to N rather than to \mathbb{C}^m . Thus, $f^* = \iota \circ f|_M$, and the following diagram of derivatives commutes by the chain rule:

$$\begin{array}{ccc} T_x M & \xrightarrow{\mathbf{J}f|_{M(x)}} & T_{f(x)} N \\ & \searrow \mathbf{J}f^*(x) & \downarrow \mathbf{J}\iota(f(x)) \\ & & \mathbb{C}^m \end{array}$$

Since ι is an embedding, its Jacobian is injective, so in particular the rank of $\mathbf{J}f|_{M(x)}$ also equals ρ_j . Thus, we immediately have $\rho_j \leq \min(\dim M, \dim N)$ and it only remains to establish the opposite inequality. There are two cases to consider —

- (1) **if $\dim M \leq \dim N$:** Note that $\dim N \leq m$ holds by the fact that N is a submanifold of \mathbb{C}^m . Thus, we have $\dim M \leq \dim N \leq m$, and so (6) gives $\rho_j = \dim M - j$ for some $j \geq 0$, whence $\rho_j \geq \dim M$.
- (2) **if $\dim M > \dim N$:** Since \mathbf{B}'_\bullet is the pushforward of \mathbf{S}''_\bullet , we have $\mathbf{B}'_k Y = f(\mathbf{S}''_k X)$ for all k . And since $\mathbf{S}''_k X$ is part of a WTB stratification of X along f by Line 1, we know from Definition 2.6 that the Jacobian of f has constant rank ρ_k on (the smooth part of) the difference $\Delta_k := \mathbf{S}''_k X - \mathbf{S}''_{k-1} X$. By the implicit function theorem, we therefore have $\rho_k = \dim \mathbf{B}'_k Y$ at every k for which $\Delta_k \neq \emptyset$. Since the nonempty set $f^{-1}(N)$ lies in Δ_j by (5), we therefore have $\dim \mathbf{B}'_j Y = \rho_j$. Finally, since N is an open subset of $f(\Delta_j) = \mathbf{B}'_j Y - \mathbf{B}'_{j-1} Y$, we obtain $\rho_j = \dim N$.

Thus, in both cases, ρ_j equals $\min(\dim M, \dim N)$, and $\mathbf{J}f|_M(x)$ has maximal rank. \square

The stratifications X'_\bullet and Y'_\bullet obtained after the conclusion of Line 9 of **WhitStratMap** do not generally satisfy the requirements of Definition 2.3. In particular, the crucial Jacobian-surjectivity requirement fails whenever $\dim M < \dim N$. As established above, in this case the Jacobian is injective rather than surjective. To fix this defect, we consider the set of problematic strata-pairs $\mathcal{P} = \mathcal{P}(X'_\bullet, Y'_\bullet)$ given by:

$$\mathcal{P} := \{(M, N) \mid f(M) \subset N \text{ with } \dim M < \dim N\}.$$

The **Refine** subroutine (which is invoked in Line 10 of **WhitStratMap**) fixes this defect. We note that a similar subroutine also appears in [24]; the main difference between the two **Refine** subroutines is that the new one calls **WTBStrat** instead of **WhitStrat**.

	Refine ($X'_\bullet, Y'_\bullet, f$)
	Input: Stratifications X'_\bullet, Y'_\bullet of pure dimensional varieties X and Y and a morphism $f : X \rightarrow Y$. Output: Lists of subvarieties $X_\bullet \subset X$ and $Y_\bullet \subset Y$.
1	For each $(S, R) \in \mathcal{P}(X'_\bullet, Y'_\bullet)$ with $\dim R$ maximal
2	Set $Y_\bullet^+ := Y'_\bullet$
3	Set $d := \dim \overline{f(S)}$
4	Add $\overline{f(S)}$ to $Y_{\geq d}^+$
5	Merge Y_\bullet^+ with WhitStrat ($\text{Pure}_d(Y'_d)$)
6	For each $\ell = (d, d-1, \dots, 1, 0)$
7	For each irreducible $W \subset \overline{Y_\ell^+ - Y'_\ell}$ and $S' \in \mathfrak{S}_\ell(X'_\bullet)$
8	If $Z \cap S' \neq \emptyset$ for an irreducible $Z \subset f^{-1}(W)$
9	Set $r := \dim Z$
10	Add Z to $X_{\geq r}'$
11	Merge X'_\bullet with WTBStrat ($\text{Pure}_r(X'_r), f$)
12	Set $Y'_\bullet = Y_\bullet^+$
13	Recompute $\mathcal{P}(X'_\bullet, Y'_\bullet)$
14	Return (X'_\bullet, Y'_\bullet)

From Theorem 2.7 and Proposition 2.8, we obtain Proposition 2.9 below, which is analogous to [24, Proposition 6.7]. The novelty here lies in the fact that the result below applies to arbitrary polynomial maps between affine complex varieties, and does not impose any genericity requirements. The proof, however, is essentially identical to that of [24, Proposition 6.7].

Proposition 2.9. *The **Refine** subroutine terminates, and its output (X_\bullet, Y_\bullet) constitutes a valid stratification (as in Definition 2.3) of the polynomial map $f : X \rightarrow Y$.*

PROOF. In [24, Proposition 6.7] this result is obtained for generic mappings $f : X \rightarrow Y$, as defined in [24, Definition 6.3], using the construction of the **Refine** subroutine along with a result ([24, Proposition 6.6]) analogous to that of Proposition 2.8 in the case of generic maps. The new **Refine** subroutine above is identical to that in [24, Section 6] other than the new version uses **WTBStrat** at Line 11, rather than **WhitStrat**. The argument is then identical to that of [24, Proposition 6.7], except we employ the more general result of Proposition 2.8 in place of [24, Proposition 6.6]. \square

3. Whitney Stratifications from Polar Varieties

Consider a complex affine variety $X \subset \mathbb{C}^n$. In this section we present a new algebraic condition to identify (a superset of) those points in a subvariety $Y \subset X_{\text{sing}}$ where Whitney's Condition (B) fails with respect to X . This criterion replaces Theorem 2.1 and leads to a new algorithm for computing Whitney stratifications. This new algorithm is based on *polar varieties* [14, 40].

The polar approach has two key advantages: first, one can perform all computations in the ring $\mathbb{C}[x_1, \dots, x_n]$ rather than working with the additional n variables ζ_1, \dots, ζ_n . And second, one is no longer required to compute any primary decompositions. Both aspects confer significant practical benefits, as is illustrated in practical tests in Section 6; the later point would also be expected to have significant implications for a worst case complexity analysis of the algorithm (though this is not carried out here). On the other hand, the new polar algorithm has the disadvantage of being probabilistic⁶.

3.1. Polar Varieties and Condition (B). Let $X \subset \mathbb{C}^n$ be a pure d -dimensional complex algebraic variety; we recall that the (Zariski open) subset of smooth points is denoted X_{reg} , and at each smooth point $p \in X_{\text{reg}}$ there is a well-defined d -dimensional tangent space $T_p X_{\text{reg}}$, which we treat as a linear subspace of \mathbb{C}^n . Consider a flag L_\bullet of length d

$$L_\bullet = (L_1 \supset \cdots \supset L_{d-1} \supset L_d),$$

where each $L_i \subset \mathbb{C}^n$ is an $(n - i)$ -dimensional linear subspace.

Definition 3.1. For each i in $\{1, \dots, d\}$, the codimension i **polar variety** of X along the flag L_\bullet is defined as the closure

$$P_i(X; L_\bullet) := \overline{\{p \in X_{\text{reg}} \mid \dim(T_p X_{\text{reg}} \cap L_i) \geq d - i + 1\}}.$$

We define $P_0(X; L_\bullet) := X$.

The fact that each $P_i(X; L_\bullet)$ is an algebraic variety will be established in the next subsection, where we describe explicit generating equations. For now, let us note that if X is irreducible then $P_i(X, L_\bullet)$ is irreducible for all i , and additionally for a fixed flag of linear spaces L_\bullet we obtain a corresponding **polar flag**

$$P_d(X, L_\bullet) \subset \cdots \subset P_0(X, L_\bullet) = X, \tag{7}$$

(see e.g. [14, Remark 3.14]).

Remark 3.2. One often finds the adjective *local* preceding polar varieties in the relevant literature, as for instance in [14]. To align this definition with ours, one chooses a flag of generic linear spaces $F_\bullet = (F_0 \subset \cdots \subset F_d)$ with $\dim F_i = i$ through a given point $y \in X$, and then defines L_i as the orthogonal complement of F_i in \mathbb{C}^n .

Our interest in polar varieties stems from a classical result of Teissier relating Condition (B) to polar multiplicities of points of X . In order to state this result, let us recall that the **Hilbert-Samuel multiplicity** of an irreducible subvariety $V \subset X$ is the (integer) coefficient $m_V X > 0$ of $[X]$ in the Segre class $s(X, V)$ — see [15, Sec 4.3]. These multiplicities may be computed numerically using the algorithm of [19, Theorem 5.3] and even inferred from local point samples as described in [27]. Here is Teissier's polar multiplicity criterion [14, Theorem 4.13] for Condition (B).

⁶In the sense that its correctness depends on a choice of random constants being truly random, as will be explained later.

THEOREM 3.3. *Let $Y \subset X$ be a pure dimensional subvariety of a pure dimensional variety X . The following properties are equivalent:*

- (1) *The manifolds $(X_{\text{reg}}, Y_{\text{reg}})$ satisfy Whitney's Condition (B).*
- (2) *Every point $y \in Y_{\text{reg}}$ admits an open neighbourhood $U \subset Y_{\text{reg}}$ such that for every $z \in U$ and generic flag L_\bullet , the sequence of Hilbert-Samuel multiplicities*

$$m_\bullet(X, z) := \left(m_z X, m_z P_1(X, L_\bullet), \dots, m_z P_{d-1}(X, L_\bullet) \right),$$

is constant.

In particular, if we let $Y' \subset Y_{\text{reg}}$ be the (necessarily open) set of all points which satisfy the second property, then (X_{reg}, Y') satisfies Condition (B).

3.2. Computing (Equations of) Polar Varieties. As in the previous section, let $X \subset \mathbb{C}^n$ be a purely d -dimensional complex algebraic variety. Its polar varieties may be defined similarly to the conormal variety from (2), but – crucially, from an algorithmic perspective – one only needs to work in the coordinate ring of \mathbb{C}^n . In particular, we define

$$\mathcal{K}_i = \begin{bmatrix} c_1^{(0)} & \cdots & c_n^{(0)} \\ \vdots & \ddots & \vdots \\ c_1^{(i)} & \cdots & c_n^{(i)} \\ \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix} \quad \text{and} \quad \mathcal{J} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix},$$

where the $c_j^{(i)}$ are general constants for $i \in \{0, \dots, d-1\}$. Let K_i be the ideal generated by all $(n-d+i+1) \times (n-d+i+1)$ minors of the matrix \mathcal{K}_i , and let J be the ideal defined by the $(n-d) \times (n-d)$ minors of the matrix \mathcal{J} . Then

$$P_{d-i}(X, L_\bullet) = \mathbf{V}_{\mathbb{C}}((K_i + I_X) : J^\infty), \quad (8)$$

where L_i is the orthogonal complement of the linear space spanned by the first i rows of \mathcal{K}_i .

Remark 3.4. The following statements pertain to practical computation of polar varieties.

- (1) The ideal saturation required in (8) may be preformed via a single elimination calculation using (a probabilistic variation of) the Rabinowitsch trick. Namely, if $J = \langle g_1, \dots, g_\nu \rangle$ and we working in the ring $\mathbb{C}[x_1, \dots, x_n, T]$ and consider the ideal

$$\mathcal{I}_i := K_i + I_X + \left\langle 1 - T \sum_{j=1}^{\nu} \lambda_j g_j \right\rangle$$

for general constants $\lambda_i \in \mathbb{C}$. Then, we have

$$(K_i + I_X) : J^\infty = \mathbb{C}[x_1, \dots, x_n] \cap \mathcal{I}_i.$$

For more details, see [21].

- (2) In practice, the general constants are replaced by random ones and hence the computation detailed above becomes inherently probabilistic. In our setting this is no great loss, as our polar varieties already depend on (essentially random) choices of linear polynomials.
- (3) Alternatively, the polar variety $P_i(X, L_\bullet)$ may be computed by a non-probabilistic elimination in the following way:
- Consider any matrix D whose columns span L_i .
 - Let $a = (a_1, \dots, a_i)^T$ be a vector of indeterminates.
 - The projection of $\{(x, a) \mid x \in X, \mathcal{J}Da = 0, a \neq 0\}$ to X is, by definition⁷, the polar variety $P_i(X, L)$.
 - We compute the defining ideal of the projection above using saturation and elimination:

$$(I_X + (\langle \mathcal{J}Da \rangle : \langle a \rangle^\infty)) \cap \mathbb{C}[x].$$

3.3. Polar Varieties and Equi-multiplicity. Our new algorithm to compute a Whitney stratification will make use of the criterion of Teissier from Theorem 3.3. This criterion in turn is phrased in terms of the constancy of the Hilbert-Samuel multiplicity of points relative to a list of varieties; the property of having constant multiplicity relative to a variety for all points in a Zariski open set is often called *equi-multiplicity* [29]. The following lemma gives a simple sufficient condition for equi-multiplicity; in spirit it follows many related results which have been summarized in Theorems 2.2.2 and Theorem 2.2.5 of the Appendix of [29]. The main difference is that here we use a multiplicity formula from [19], which lends itself to a particularly elementary proof.

Lemma 3.5. *Consider varieties $Y \subset X \subset \mathbb{C}^n$, with X pure dimensional and Y nonempty, irreducible, and different from $P_i(X, L_\bullet)$ for all i . Let $\mu = m_Y X$ be the Hilbert-Samuel multiplicity of Y in X , and let r be the codimension $\dim X - \dim Y$. Then for all points $z \in Y_{\text{reg}} - P_r(X, L_\bullet)$, we have $m_z X = \mu$.*

PROOF. First note that multiplicity is unchanged under projective closure, hence we may replace the setup in the statement of the Lemma with $Y \subset X \subset \mathbb{P}^n$. By [19, Sec 2.1.4] we may further suppose that $Y = \mathbf{V}_{\mathbb{C}}(f_1, \dots, f_s)$ where all the f_i have the same degree. Let λ_{ij} be an $r \times s$ matrix of generic complex numbers, and for each $i \in \{1, \dots, r\}$ define the polynomial $Q_i = \sum_j \lambda_{ij} f_j$. Setting $W = \mathbf{V}_{\mathbb{C}}(Q_1, \dots, Q_r)$, Theorem 5.3 of [19] gives

$$\deg(Y)\mu = \deg(X) \deg(W) - \deg(\overline{(X \cap W) - Y}). \quad (9)$$

Fix a point $z \in Y_{\text{reg}}$, and let $A_z \subset \mathbb{P}^n$ be a linear space of codimension $\dim(Y)$ which contains z . We now examine how (9) changes when all varieties in sight are intersected with A_z . For generic choices of A_z , it is known (by [15, Proposition 8.4] for instance) that $\deg(V \cap A_z) = \deg(V)$ holds for $V \in \{W, X, Y\}$. We remark that a sufficient condition for

⁷Note that Da for $a \neq 0$ is an element in L_i ; now the condition $\exists a \in \mathbb{C}^i$ such that $\mathcal{J}Da = 0$ is satisfied exactly by those points $x \in X$ for which the intersection $\kappa_x^{-1}(x) \cap L_i$ is nonempty.

A_z to be generic in the sense required here is for A_z to be transverse to X in the punctured neighborhood of z .

Now, by [26, Proposition 3.3], for generic A_z we also have

$$m_z X = m_{Y \cap A_z}(X \cap A_z).$$

Therefore,

$$m_z X - \mu = \frac{\deg\left(\overline{(X \cap W) - Y}\right) - \deg\left(\overline{(X \cap W \cap A_z) - (Y \cap A_z)}\right)}{\deg Y}. \quad (10)$$

Next, we seek to establish that the numerator on the right side vanishes unless z lies in $P_r(X; L_\bullet)$. To this end, let us recall that the numbers $\{\lambda_{ij}\}$ which define W are generic. Therefore we may safely assume that $\dim W = n - r$, and that $\dim(X \cap W) = \dim Y$, with the latter equality holding due to transversality. Therefore – at least when A_z is generic – both $X \cap W \cap A_z$ and $Y \cap A_z$ are finite sets of cardinality $\deg(X \cap W)$ and $\deg Y$ respectively. Returning to (10), we conclude that $m_z X = \mu$ holds unless A_z fails to intersect X_{reg} transversely near z . But since A_z might well equal $L_{\dim Y}$ in the flag L_\bullet , we conclude that $m_z X \neq \mu$ forces $z \in P_r(X; L_\bullet)$. \square

We now employ the preceding result to construct a new Whitney stratification algorithm. It will be convenient to represent an X_i in the flag X_\bullet as a list of irreducible varieties (or equivalently, as a list of prime ideals) whose union equals X_i . In this way we will update intermediate values of X_i by appending new components to this list. For clarity, we will use $\text{List}(X_i)$ to denote the list of components whose union equals X_i . We will also assume that a random flag L_\bullet (as required for Definition 3.1) has been chosen in advance.

WhitneyPolar (X)	
	Input: An algebraic variety X .
	Output: X_\bullet , the flag in (1) corresponding to the Whitney stratification of X .
1	Set $d := \dim(X)$;
2	Initialize X_\bullet to be a flag of length $d + 1$ with each $\text{List}(X_i) := \emptyset$
3	For each equidimensional component $Z \subset X$, append Z to $\text{List}(X_{\dim(Z)})$
4	For i from 0 to $d - 1$ do
5	For j from 1 to $d - i$ append $P_j(X_{d-i}, L_\bullet)$ to $\text{List}(X_{d-i-j})$
6	Compute $Y := (\bigcup_{j=0}^i X_{d-j})_{\text{sing}}$
7	For each equidimensional component of $Z \subset Y$, append Z to $\text{List}(X_{\dim(Z)})$
8	For j from 0 to $i - 1$
9	$V := X_{d-i} \cap P_{i-j}(X_{d-j}, L_\bullet)$
10	For each equidimensional component $W \subset V$, append W to $\text{List}(X_{\dim(W)})$
11	Return X_\bullet .

Remark 3.6. A recent discussion of algorithms and software to perform equidimensional decomposition (along with a new algorithm for this task) can be found, for example, in [11]. Note that alternatively, depending on the implementation context, one may choose

to take irreducible components above everywhere (i.e. in Lines 3, 7, and 10) rather than equidimensional ones.

Remark 3.7. The **WhitneyPolar** algorithm presents the flag X_\bullet as a union of parts, i.e., a list of components that have the same dimension but are not necessarily irreducible. One may wish to pursue two further goals:

- (1) design an algorithm to compute the actual connected components in the Whitney stratification, or
- (2) produce a refinement of the Whitney stratification which ensures that each component is connected.

These tasks depend in practice on whether one works over the real or complex numbers. The latter can be achieved (over \mathbb{C}) by decomposing every component in the output into irreducible components.

Corollary 3.8. *The algorithm **WhitneyPolar** is correct.*

PROOF. The first three lines of the algorithm merely populate the flag X_\bullet with the equidimensional components of X . To prove the correctness of the algorithm, it suffices to show that during the i -th iteration of the **For** loop on Line 4, each $(d - i)$ -dimensional part of the stratification, $\tilde{X} \in \text{List}(X_{d-i})$ has been correctly processed. Specifically, we show that at step i the algorithm generates a (finite) list of lower-dimensional subvarieties $Y \subset \tilde{X}$ such that the union of the generated varieties Y contains all points y which are either not smooth in X , or for which the following criterion holds: for every $j > d - i$ and equidimensional component $Z \in \text{List}(X_j)$ that contains \tilde{X} , the multiplicity sequence $m_\bullet(Z, y)$ differs from the expected sequence $m_\bullet(Z, x)$ for generic $x \in \tilde{X}$.

Line 5 records all points on all polar subvarieties of \tilde{X} ; in regards to Lemma 3.5, this deals separately with the case where Y in the Lemma is a polar subvariety (by ensuring all of these are already removed). Lines 6-7 handle singular points of \tilde{X} . Finally, Lines 8-10 excise the points that are not singular *and* do not lie on a polar subvariety, but have to be removed to guarantee the conclusion of Lemma 3.5 (note that we identify a variety containing all points where Lemma 3.5 can fail for some component of our pure dimensional variety X_j all at once, rather than one irreducible component at a time). \square

Remark 3.9. The reader is warned that the stratifications produced by **WhitneyPolar** may not be minimal. There are two reasons for this lack of optimality — first, the condition (from Lemma 3.5) which we use to ensure equi-multiplicity is sufficient but not necessary condition. And second, since our polar varieties are defined globally (rather than locally at a point), they inevitably include various random points arising from the choice of L_\bullet which are not in fact necessary. We are aware of two antidotes to this — one may either generate a suboptimal stratification and then coarsen it using the algorithm of [23, Sec 3.3], or alternately, one may replace each polar variety in **WhitneyPolar** with the intersections of several global polar varieties (of the same codimension), each taken along different generic flags L_\bullet .

4. Real Stratifications

Let $\mathbb{R}[x_1, \dots, x_n]$ be the ring of real polynomials in n indeterminates, and fix a radical ideal I of this ring. By definition, the vanishing locus $X := \mathbf{V}(I)$ constitutes a real algebraic subvariety of \mathbb{R}^n . Let $\{f_1, f_2, \dots, f_r\}$ be any finite set of polynomials which generate I . Since each f_i is automatically a polynomial in $\mathbb{C}[x_1, \dots, x_n]$, there is a complex variety $\mathbf{V}_{\mathbb{C}}(f_1, \dots, f_r) \subset \mathbb{C}^n$ generated by the f_i . Although this variety depends on $\{f_i\}$ rather than on X , we will denote it by $X(\mathbb{C})$ in order to emphasise that it is a complexified version of X . Let X_{reg} denote the manifold of smooth points in X . In this section $\dim_{\mathbb{R}}(X)$ will denote the dimension of manifold X_{reg} and $\dim_{\mathbb{C}}(X(\mathbb{C}))$ will denote the dimension of the manifold $(X(\mathbb{C}))_{\text{reg}}$.

4.1. Real Varieties. Our immediate goal here is to show that certain Whitney stratifications of $X(\mathbb{C})$ induce Whitney stratifications of X . We let $\iota : \mathbb{R}^n \hookrightarrow \mathbb{C}^n$ be the embedding of real points in complex Euclidean space.

Lemma 4.1. *Let $X \subset \mathbb{R}^n$ be a real algebraic variety.*

- (1) *The embedding ι identifies X with the real points of $X(\mathbb{C})$.*
- (2) *Assume that $\dim_{\mathbb{R}} X$ equals $\dim_{\mathbb{C}} X(\mathbb{C})$. If $\iota(p)$ is a smooth point of $X(\mathbb{C})$ for some $p \in X$, then p is a smooth point of X .*

PROOF. The first assertion is a tautology. Turning to the second assertion, set $d := \dim_{\mathbb{C}} X(\mathbb{C}) = \dim_{\mathbb{R}} X$. Since the roots of real polynomials occur in complex conjugate pairs, the variety $X(\mathbb{C})$ is invariant under complex conjugation and $\iota(X)$ equals the fixed point set of this conjugation. Noting that $\iota(p)$ is a smooth point by assumption, the tangent space $T_{\iota(p)}X(\mathbb{C})$ exists and has complex dimension d ; this tangent space also inherits invariance under complex conjugation. Thus, $T_{\iota(p)}X(\mathbb{C})$ is the complexification of a real d -dimensional vector space V whose elements consist of all real tangent vectors at $\iota(p)$; this V is evidently isomorphic to $T_p X$, as desired. \square

Every finite descending chain I_{\bullet} of radical ideals in $\mathbb{R}[x_1, \dots, x_n]$

$$I_0 \triangleright I_1 \triangleright \dots \triangleright I_m = I$$

produces an ascending flag $X_{\bullet} := \mathbf{V}(I_{\bullet})$ of subvarieties of X :

$$X_0 \subset X_1 \subset \dots \subset X_m = X.$$

The next result shows that successive differences of X_{\bullet} inherit a smooth manifold structure from the successive differences of $X_{\bullet}(\mathbb{C})$.

Proposition 4.2. *Let $W \subset Z$ be a pair of real algebraic varieties in \mathbb{R}^n . If the difference $Z(\mathbb{C}) - W(\mathbb{C})$ is either empty or a smooth i -dimensional complex manifold, then $M := (Z - W)$ is either empty or a smooth i -dimensional real manifold.*

PROOF. There are two cases to consider — either the image $\iota(Z)$ lies entirely within the singular locus $Z(\mathbb{C})_{\text{sing}}$, or there exists some $p \in M$ with $\iota(p) \in Z(\mathbb{C})_{\text{reg}}$. In the first case, since $Z(\mathbb{C}) - W(\mathbb{C})$ is smooth, we know that $Z(\mathbb{C})_{\text{sing}}$ lies entirely within $W(\mathbb{C})$ and hence

that $\iota(Z) \subset \iota(W)$; but since we have assumed $W \subset Z$, we must have $W = Z$, whence M is empty. On the other hand, let p be a point in $(Z - W)$ for which $\iota(p)$ is a smooth point of $Z(\mathbb{C})$. We may safely assume that the generating ideal of Z is prime in $\mathbb{C}[x_1, \dots, x_n]$ by passing to the irreducible component which contains $\iota(p)$. It now follows from [35, Theorem 12.6.1] or [20, Theorem 2.3] that $(Z - W)$ has dimension i . Finally, Lemma 4.1 ensures that $(Z - W)$ is a smooth real i -manifold. \square

It follows from the above result that if $X_\bullet(\mathbb{C})$ is a Whitney stratification of $X(\mathbb{C})$, then the successive differences of X_\bullet are either empty or smooth manifolds of the expected dimension. We show below these successive differences also satisfy Condition (B).

THEOREM 4.3. *Let I_\bullet be a descending chain of radical ideals in $\mathbb{R}[x_1, \dots, x_n]$. If the flag $X_\bullet(\mathbb{C}) := \mathbf{V}_{\mathbb{C}}(I_\bullet)$ constitutes a Whitney stratification of $X(\mathbb{C})$, then the corresponding flag $X_\bullet := \mathbf{V}(I_\bullet)$ yields a Whitney stratification of X .*

PROOF. Consider a connected component $M \subset (X_i - X_{i-1})$, and let $V \subset X_i$ be the irreducible component which contains M . Similarly, let $W \subset X_i(\mathbb{C})$ be the irreducible component which contains $\iota(M)$ and define $M_{\mathbb{C}} := W - X_{i-1}(\mathbb{C})$. We note that $\iota(M)$ forms an open subset of $M_{\mathbb{C}}$, which must in turn be an i -stratum of $X(\mathbb{C})$. Similarly, consider a nonempty connected $N \subset (X_j - X_{j-1})$ with $i > j$ and analogously define $N_{\mathbb{C}} \subset X_j(\mathbb{C})$. We will show that the pair (M, N) satisfies Condition (B). To this end, consider a point $q \in N$ along with sequences $\{p_k\} \subset M$ and $\{q_k\} \subset N$ which converge to q . Letting ℓ_k denote the secant line $[p_k, q_k]$ and T_k the tangent plane $T_{p_k}M$, we assume further that the limits $\ell = \lim \ell_k$ and $T = \lim T_k$ both exist.

Let $\ell_k(\mathbb{C})$ be the secant line $[\iota(p_k), \iota(q_k)]$ in \mathbb{C}^n and let $T_k(\mathbb{C})$ be the tangent space $T_{\iota(p_k)}M_{\mathbb{C}}$. Since $\iota(p_k)$ and $\iota(q_k)$ are real points for all k , the linear equations defining both $\ell_k(\mathbb{C})$ and $T_k(\mathbb{C})$ as varieties are exactly the same as those defining ℓ_k and T_k , respectively. Thus, the limits $\ell(\mathbb{C})$ and $T(\mathbb{C})$ both exist because the corresponding real limits exist – one may view these as limits of real sequences inside a complex Grassmannian – and they are defined by the same algebraic equations as their counterparts ℓ and T . By definition of secant lines, the image $\iota(w)$ of any $w \in \ell$ is a real point of $\ell(\mathbb{C})$. Since the pair $(M_{\mathbb{C}}, N_{\mathbb{C}})$ satisfies Condition (B) by assumption, we know that $\ell(\mathbb{C}) \subset T(\mathbb{C})$, whence $\iota(w)$ must be a real point of $T(\mathbb{C})$. Since $\iota(M)$ is an open subset of $M_{\mathbb{C}}$, we have $T_{\iota(p_k)}M_{\mathbb{C}} = T_{\iota(p_k)}\iota(M)$ for all k ; and by the proof of Lemma 4.1, the real points of $T_{\iota(p_k)}\iota(M)$ are identified with $T_{q_k}N$. Thus, $\iota(T)$ contains all the real points of $T(\mathbb{C})$, including $\iota(w)$. Since ι is injective, we have $w \in T$ as desired. \square

Corollary 4.4. *Let X be a real algebraic variety. If either the **WhitStrat** algorithm from Section 2.1 or the **WhitneyPolar** algorithm from Section 3 are applied to $X(\mathbb{C})$, then the output produces a Whitney stratification of X .*

PROOF. From an algebraic point of view our input to both the **WhitStrat** and the **WhitneyPolar** algorithm is a list of polynomials in the ring $\mathbb{R}[x_1, \dots, x_n]$ ⁸ which define the real variety $X \subset \mathbb{R}^n$ and the corresponding complex variety $X(\mathbb{C}) \subset \mathbb{C}^n$; in the algorithms we

⁸Note, in a realistic computational setting the input is likely to actually be a list of polynomials in $\mathbb{Q}[x_1, \dots, x_n]$, and all subsequent algebraic operations will occur in this ring.

then perform algebraic operations starting with the polynomials in this list. We need only show that the concrete algebraic operations performed in both the **WhitStrat** algorithm and the **WhitneyPolar** algorithm leave the coefficient field of all intermediate polynomials unchanged. To see this we list the algebraic operations performed, the conclusion then follows from an analysis of the underlying algorithms in standard references such as [9].

First we list the geometric operations which occur in the **WhitStrat** algorithm along with their algebraic counterparts in the usual algebraic-geometric dictionary, see e.g. [9].

- (1) Intersection of varieties, which corresponds to addition of polynomial ideals.
- (2) Computation of the singular locus of a variety, from Section 2.1 (on page 5) this corresponds to computing the radical of the polynomial ideal given by taking the appropriately sized minors of the Jacobian matrix of the defining equations.
- (3) Computing the conormal variety, from Section 2.1 (on page 5) this corresponds the computation of minors of a matrix of polynomials (most of which arise from partial derivatives), and the addition and saturation of polynomial ideals.
- (4) Computing the irreducible components of a variety, which corresponds to prime decomposition of ideals.
- (5) Computing the primary decomposition of ideals.

Addition of ideals is in effect appending polynomials to a list, which leaves the coefficient field unchanged. Polynomial differentiation and the taking of minors of polynomial matrices clearly leave the coefficient field unchanged as well. Ideal saturation as well as the computation of the radical and the prime and primary decomposition can be accomplished via Gröbner basis computation, see [9] for the saturation computation and [12] for the radical, prime and primary decomposition computations. From a standard presentation of Buchberger's algorithm to compute Gröbner basis, e.g. [9, Chapter 2, §8], we see this leaves the coefficient field unchanged. Hence all algebraic operations in items (1)–(5) leave the coefficient field unchanged and the conclusion follows.

We note that the list of algebraic options above includes computing the radical of an ideal, so for the purposes of this argument it does not matter if the input list of polynomials defines a radical ideal or not.

Similarly, the **WhitneyPolar** algorithm performs items (1), (2). The additional operations are:

- (6) Equidimensional decomposition of a variety (or we could use irreducible decomposition in the relevant lines, in which case this is covered by (5)).
- (7) The computation of polar varieties; by equation (8) in Section 3.2 this corresponds the computation of minors of a matrix of polynomials (most of which arise from partial derivatives), and the addition and saturation of polynomial ideals.

It can be seen from [12] that equidimensional decomposition leaves the coefficient field unchanged and, as discussed above, this is also true of the polynomial differentiation, the computation of minors of matrices of polynomials, and saturation of polynomial ideals. The conclusion follows. \square

4.2. Real Algebraic Morphisms. Consider algebraic varieties $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$, and let $f : X \rightarrow Y$ be an algebraic morphism. We recall that this amounts to an m -tuple of real polynomials

$$\left(f_1(x_1, \dots, x_n), f_2(x_1, \dots, x_n), \dots, f_m(x_1, \dots, x_n) \right)$$

whose evaluation at a point of X yields a point of Y . Since each f_i is automatically a complex polynomial, there is an evident morphism $f_{\mathbb{C}} : X(\mathbb{C}) \rightarrow Y(\mathbb{C})$ of complex algebraic varieties. Let I_{\bullet} and J_{\bullet} be descending chains of radical ideals in $\mathbb{R}[x_1, \dots, x_n]$ and $\mathbb{R}[y_1, \dots, y_m]$ respectively so that $X_{\bullet}(\mathbb{C}) := \mathbf{V}_{\mathbb{C}}(I_{\bullet})$ and $Y_{\bullet}(\mathbb{C}) := \mathbf{V}_{\mathbb{C}}(J_{\bullet})$ constitute Whitney stratifications of $X(\mathbb{C})$ and $Y(\mathbb{C})$ respectively. It follows from Theorem 4.3 that $X_{\bullet} := \mathbf{V}(I_{\bullet})$ is a Whitney stratification of X while $Y_{\bullet} := \mathbf{V}(J_{\bullet})$ is a Whitney stratification of Y .

THEOREM 4.5. *If $f_{\mathbb{C}}$ is stratified with respect to $X_{\bullet}(\mathbb{C})$ and $Y_{\bullet}(\mathbb{C})$, then f is stratified with respect to X_{\bullet} and Y_{\bullet} .*

PROOF. Let $M \subset X$ be a nonempty connected component of the i -stratum $X_i - X_{i-1}$, and let $M_{\mathbb{C}}$ be the i -stratum of $X_{\bullet}(\mathbb{C})$ which contains $\iota(M)$. By definition, the image $f_{\mathbb{C}}(M_{\mathbb{C}})$ contains $f(M)$ in its locus of real points. Since $f_{\mathbb{C}}$ is stratified with respect to $X_{\bullet}(\mathbb{C})$ and $Y_{\bullet}(\mathbb{C})$, the first requirement of Definition 2.3 guarantees the existence of a single stratum $N_{\mathbb{C}} \subset Y$ which contains $f_{\mathbb{C}}(M_{\mathbb{C}})$. Thus, $f(M)$ lies in the locus of real points of $N_{\mathbb{C}}$. Letting N denote the stratum of Y_{\bullet} corresponding to $N_{\mathbb{C}}$, we know that the real locus of $N_{\mathbb{C}}$ equals N , whence we obtain $f(M) \subset N$ and it remains to show that the restriction of f to M yields a submersion. Let x be any point of M , and note that $\iota(x)$ lies in $M_{\mathbb{C}}$. Since $f_{\mathbb{C}}|_{M_{\mathbb{C}}}$ is a submersion, its Jacobian at $\iota(x)$ is a surjective linear map from the tangent space to $M_{\mathbb{C}}$ at $\iota(x)$ to the tangent space to $N_{\mathbb{C}}$ at $f_{\mathbb{C}} \circ \iota(x)$. But by construction, $f_{\mathbb{C}} \circ \iota$ equals f . Thus, we have

$$\text{rank}_{\mathbb{C}}(\mathbf{J}f_{\mathbb{C}}|_{M_{\mathbb{C}}}(\iota(x))) = \dim_{\mathbb{C}} N_{\mathbb{C}}.$$

To conclude the argument, we note that the derivative arising on the left side of the above equality may be represented by the Jacobian matrix of f at x , and the rank of this matrix is preserved under field extension to \mathbb{C} . On the other hand, by Proposition 4.2 we know that the complex dimension of $N_{\mathbb{C}}$ equals the real dimension of N . Thus, our equality becomes

$$\text{rank}_{\mathbb{R}}(\mathbf{J}f|_M(x)) = \dim_{\mathbb{R}} N,$$

as desired. □

Assume that a morphism $f : X \rightarrow Y$ has been stratified as described in Theorem 4.5. It is readily checked that the image $f(\overline{M})$ of the closure of a stratum $M \subset X$ is not an algebraic subvariety of Y in general — the best that one can expect is that $f(\overline{M})$ will be semialgebraic. It is important to note, in the context of the above theorem, that we do not obtain semialgebraic descriptions of such images.

4.3. Dominant Morphisms between Equidimensional Varieties. Let $X \subset \mathbb{K}^n$ and $Y \subset \mathbb{K}^m$ be algebraic varieties defined by ideals I_X and I_Y over a field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Let $\mathbb{K}[X]$ denote the coordinate ring $\mathbb{K}[x_1, \dots, x_n]/I_X$ and similarly for Y ; any morphism of varieties $f : X \rightarrow Y$ canonically induces a contravariant ring homomorphism $f^* : \mathbb{K}[Y] \rightarrow \mathbb{K}[X]$.

Definition 4.6. Let $f : X \rightarrow Y$ be a morphism of algebraic varieties over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.

- (1) we say that f is **dominant** if f^* is a monomorphism (or equivalently, if the image $f(X)$ is dense in Y).
- (2) we say that f is **finite** if it is dominant, and moreover, if f^* gives $\mathbb{K}[X]$ the structure of an integral extension of $\mathbb{K}[Y]$.

It is a classical result, see e.g. [39, Chapter II, Corollary 6.7], that if $f : X \rightarrow Y$ is finite in the above sense, then it is also a proper map for any field (see Remark 2.4 for the significance of this result in our context). Over $\mathbb{K} = \mathbb{C}$, a dominant morphism is finite if and only if it is proper.

Throughout this section, $f : X \rightarrow Y$ will denote a morphism between real algebraic varieties of the same dimension d ; we will further require that the map $f_{\mathbb{C}} : X(\mathbb{C}) \rightarrow Y(\mathbb{C})$ is dominant and that $\dim_{\mathbb{C}} X(\mathbb{C}) = \dim_{\mathbb{C}} Y(\mathbb{C}) = d$.

Definition 4.7. The **Jelonek set** of f is the subset $\mathbf{Jel}(f) \subset Y(\mathbb{C})$ consisting of all points y for which there exists a sequence $\{x_k\} \subset X(\mathbb{C})$ satisfying both

$$\lim_{k \rightarrow \infty} |x_k| = \infty \quad \text{and} \quad \lim_{k \rightarrow \infty} f_{\mathbb{C}}(x_k) = y.$$

It is shown in [31] that $\mathbf{Jel}(f)$ is either empty or an algebraic hypersurface of $Y(\mathbb{C})$; a Gröbner basis algorithm for computing the Jelonek set is given in [44]. It follows from this algorithm that if the Jelonek set is non-empty, then it is defined by a polynomial with real coefficients. Finally, it is shown in [31] that $\mathbf{Jel}(f)$ is precisely the locus of points at which f fails to be finite. Thus, if we define

$$V(\mathbb{C}) := \mathbf{Jel}(f) \quad \text{and} \quad W(\mathbb{C}) := \overline{f^{-1}(V)},$$

then the restriction of f forms a proper map $(X(\mathbb{C}) - W(\mathbb{C})) \rightarrow (Y(\mathbb{C}) - V(\mathbb{C}))$ — see [32, Proposition 6.1] for details. Note that the polynomials defining $V(\mathbb{C})$ and $W(\mathbb{C})$ are real; take V to be the real zero set of the polynomials defining $V(\mathbb{C})$, and similarly let W be the real zero set of the polynomials defining $W(\mathbb{C})$. It follows immediately that the restriction of f to the difference $(X - W)$ constitutes a proper map to the difference $(Y - V)$.

Definition 4.8. The **Jelonek flag** of $f : X \rightarrow Y$ is a pair $(W_{\bullet}, V_{\bullet})$ of flags, both of length $d = \dim X = \dim Y$:

$$\begin{aligned} \emptyset &= W_{-1} \subset W_0 \subset \cdots \subset W_d = X \\ \emptyset &= V_{-1} \subset V_0 \subset \cdots \subset V_d = Y, \end{aligned}$$

defined via reverse-induction on $i \in \{d-1, d-2, \dots, 1, 0\}$ as follows. Starting with V_{d-1} as the real points of $\mathbf{Jel}(f)$, we

- (1) let W_i be $\overline{f^{-1}(V_i)}$, and
- (2) let V_{i-1} be the real points in $\mathbf{Jel}(f|_{W_i} : W_i \rightarrow V_i)$.

By construction of the Jelonek flag, at each i we have the following alternative:

- (1) **if $V_i(\mathbb{C})$ is nonempty**, then $\dim V_i(\mathbb{C}) = \dim W_i(\mathbb{C}) = i$ and the restriction of f forms a proper map $(V_i - V_{i-1}) \rightarrow (W_i - W_{i-1})$; otherwise,

- (2) if $V_i(\mathbb{C})$ is empty, then $f|_{W_i} : W_i \rightarrow V_i$ is not dominant; on the other hand, $f|_{W_{i-1}}$ is dominant, but with $\dim W_{i-1}(\mathbb{C}) > \dim V_{i-1}(\mathbb{C})$.

THEOREM 4.9. *Let (W_\bullet, V_\bullet) be the Jelonek flag of $f : X \rightarrow Y$. Assume that X_\bullet is a Whitney stratification of X subordinate to W_\bullet and that Y_\bullet is a Whitney stratification of Y subordinate to V_\bullet such that f is stratified with respect to X_\bullet and Y_\bullet (in the sense of Definition 2.3). Whenever $\dim V_i = \dim W_i$ holds, we have that for every Y_\bullet -stratum $R \subset Y$ with $R \subset V_i - V_{i-1}$, the map $f|_{f^{-1}(R)} : f^{-1}(R) \rightarrow R$ is a locally trivial fiber bundle.*

PROOF. Since the stratification Y_\bullet of Y is subordinate to V_\bullet , for each stratum $R \subset Y$ we have that $R \subset V_i - V_{i-1}$ for some i , and also since X_\bullet is subordinate to W_\bullet we also have $f^{-1}(R) \subset W_i - W_{i-1}$. If $\dim(V_i) = \dim(W_i)$ holds, then the map $f : (W_i - W_{i-1}) \rightarrow (V_i - V_{i-1})$ is proper. An appeal to the semialgebraic version of Thom's first isotopy lemma [8, Theorem 1] achieves the desired result. \square

5. Full Semialgebraic Stratifications

A (basic, closed) *semialgebraic set* is any subset $B \subset \mathbb{R}^n$ which can be expressed as an intersection of the form $B := X \cap C$ where X is a real algebraic subvariety of \mathbb{R}^n , while the set C , called an *inequality locus*, is given as follows:

$$C := \{x \in \mathbb{R}^n \mid g_i(x) \geq 0 \text{ for } 0 \leq i \leq k\}. \quad (11)$$

Here the g_i 's are a finite collection of polynomials in $\mathbb{R}[x_1, \dots, x_n]$. By convention, when the number of inequalities k equals zero, we have $C = \mathbb{R}^n$. Thus, every algebraic variety is automatically a semialgebraic set in the above sense. The sets X and C are not uniquely determined for a given B in general — we may, for instance, safely remove any polynomial generator $f : \mathbb{R}^n \rightarrow \mathbb{R}$ from the defining ideal of X while adding $f \geq 0$ and $-f \geq 0$ to the inequality locus. It is therefore customary to omit X entirely and simply define B as the set of points which satisfy a collection of polynomial inequalities. We find it convenient to write $B = X \cap C$ here because this allows us to isolate a relevant sub-class of semialgebraic sets.

Definition 5.1. A semialgebraic set $B \subset \mathbb{R}^n$ is called **full** if it admits an inequality locus C of the form (11), with the additional requirement that its subset

$$C^\circ := \{x \in \mathbb{R}^n \mid g_i(x) > 0 \text{ for } 0 \leq i \leq k\}$$

is an n -dimensional smooth manifold whose closure equals C .

The following example highlights an important class of full semialgebraic sets which arise in the Morse theoretic study of complex analytic varieties — see [17, II.2].

Example 5.2. Whitney showed in [46] that every complex analytic variety $W \subset \mathbb{C}^n$ admits a Whitney stratification W_\bullet where each intermediate W_i is a complex analytic subvariety. Associated to each i -stratum $M \subset W_i - W_{i-1}$ is a stratified space $\mathcal{L} = \mathcal{L}_M$, which is called the **complex link** of M and defined as follows. Consider a point $p \in M$ and let $A \subset \mathbb{C}^n$ be a generic $(n - i)$ -dimensional affine subspace which intersects W transversely at a point $p \in M$. Let $B_\epsilon(p)$ denote the disk containing all points within a sufficiently small radius

$\epsilon > 0$ around p with respect to the standard euclidean distance in \mathbb{C}^n . Finally, let $\pi : A \rightarrow \mathbb{C}$ be a generic linear form sending p to zero. For some positive $\delta \ll \epsilon$, one defines

$$\mathcal{L}_M := W \cap B_\epsilon(p) \cap \{\pi = \delta\}.$$

The stratified homeomorphism type of \mathcal{L}_M depends only on the stratum M , and not on the auxiliary choices of p, A, ϵ, π and δ . Complex links provide *normal Morse data* for critical points of stratified Morse functions $f : W \rightarrow \mathbb{R}$. Although such a \mathcal{L}_M will not be a complex (or even real) variety in general, it is always a full semialgebraic subset of the affine space $\{\pi = \delta\}$. In particular, its inequality locus is the disk $B_\epsilon(p) \cap \{\pi = \delta\}$.

Given such an inequality locus C of a full semialgebraic set, we call C° its interior and define its boundary as the difference

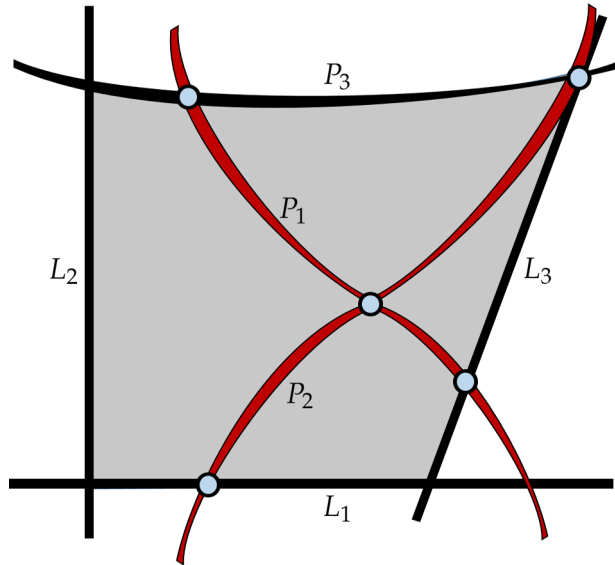
$$\partial C := C - C^\circ.$$

This boundary is a semialgebraic subset of the real algebraic variety

$$Y_C := \mathbf{V} \left(\prod_1^k g_i \right). \quad (12)$$

We adopt the usual convention that the product over the empty set equals 1, which forces $\partial C = Y_C = \emptyset$ when $k = 0$.

Example 5.3. Below is a toy example of a full semialgebraic set $B = X \cap C$ inside \mathbb{R}^2 . Here X is the union of red parabolas P_1 and P_2 while C is the region bounded by the black lines L_1, L_2, L_3 and the black parabola P_3 :



The interior C° has been shaded gray, so B consists of that part of $P_1 \cup P_2$ which lies within the closure of the gray region. The variety Y_C is the union of curves L_1, L_2, L_3 and P_3 . The boundary ∂C consists of the four highlighted points which lie on this union of curves.

Our next result establishes that every full semialgebraic set $B = X \cap C$ inherits a Whitney stratification from Whitney stratifications of the real algebraic varieties X and $X \cap Y_C$. (We

recall for the reader's convenience that flag-subordinate stratifications have been described in Section 2.2).

THEOREM 5.4. *Let $B = X \cap C$ be a full semialgebraic set, with Y_C as in (12), and let X_\bullet be a Whitney stratification of X . If Y_\bullet is a Whitney stratification of $X \cap Y_C$ which is subordinate to the flag $X_\bullet \cap Y_C$, then setting*

$$B_i := (X_i \cup Y_i) \cap C$$

produces a Whitney stratification of B .

PROOF. Since B is full, we know that the interior C° of its inequality locus is a smooth open n -dimensional submanifold of \mathbb{R}^n . This means any open neighborhood of X , which is contained in C° , inside \mathbb{R}^n is identical to the same neighborhood of X inside C° and therefore, the intersections $X_i \cap C^\circ$ form a Whitney stratification X'_\bullet of $X \cap C^\circ$. Alternatively the fact that $X_i \cap C^\circ$ forms a Whitney stratification X'_\bullet of $X \cap C^\circ$ follows since C° is full dimensional, meaning the intersection $X \cap C^\circ$ is transverse and hence the conclusion follows because the Whitney stratification of two spaces which intersect transversely is given by their intersection, [7], see also [17, Chapter 1, §1.2, (5)].

Let Y'_\bullet be the subset of Y_\bullet -strata which intersect ∂C . Since C is the disjoint union of C° and ∂C , it follows that the union of X'_\bullet -strata and Y'_\bullet -strata partitions B . It remains to check that Condition (B) holds for those strata pairs (M, N) of this union for which N intersects the closure of M . There are now three cases to consider, of which the two easy ones are handled as follows:

- (1) if both M and N are strata of X'_\bullet , then Condition (B) holds because both are full-dimensional open subsets of X_\bullet -strata by construction, and X_\bullet is assumed to be a Whitney stratification.
- (2) if M is a Y'_\bullet -stratum, then the fact that N intersects the closure of M forces N to be contained in $X \cap \partial C$, since both X and ∂C are closed subsets of \mathbb{R}^n . Thus, N must also be a Y'_\bullet -stratum in which case Condition (B) holds because Y'_\bullet is Whitney.

Turning now to the third case, assume that M is an X'_\bullet -stratum and N is a Y'_\bullet -stratum. By construction, M must be (a connected component of) the intersection $M_* \cap C^\circ$ for some X_\bullet -stratum M_* . Since C° is n -dimensional, the tangent spaces $T_x M$ and $T_x M_*$ coincide for every x in M . Fix a point $p \in N$, and let N_* be the unique X_\bullet -stratum containing p in its interior. Since Y_\bullet is chosen subordinate to $X_\bullet \cap Y_C$, the Y_\bullet -strata are refinements of X_\bullet -strata, so N must be obtained by removing some (possibly empty) set from $N_* \cap \partial C$. It follows that N is a subset of N_* in a small ball around p . Finally, (M, N) must satisfy Condition (B) at p because (M_*, N_*) satisfy Condition (B) at p . \square

The stratifications obtained in Theorem 5.4 provide a complete description of the flag B_i . Using the techniques of [20, 34, 42], one can perform various fundamental algorithmic tasks involving such strata. These include testing whether the i -stratum $B_i - B_{i-1}$ is empty for each i , and sampling points from the non-empty strata.

Remark 5.5. Using Theorem 5.4 in practice on a full semialgebraic set $B = X \cap C$ requires constructing a stratification of the real variety $X \cap Y_C$; this is somewhat unsatisfactory from

an efficiency perspective, since X and Y_C may intersect far away from B — returning to Example 5.3, note that P_1 and P_2 might intersect L_2 at points far from B . We are not aware of any existing technique which can easily overcome such suboptimality.

6. Performance Analysis and Comparison

In this section we illustrate the performance of the **WhitneyPolar** algorithm of Section 3. The results have been collected in Table 1.

6.1. Other Methods. We have compared **WhitneyPolar** to all known methods for computing practical Whitney stratifications in practice; all of these are based on the conormal stratification procedure reviewed in Section 2.1. This comparison occupies the first three columns of Table 1. For reference, we also give the run time for computing a Cylindrical Algebraic Decomposition (CAD) for each given variety. Conversely, we do not include the method of [30] in our results because it was unable to stratify the Whitney umbrella – which is the simplest nontrivial input – after running for 24 hours. We also implemented the algorithm of [38, 41] for the Whitney umbrella, using the standard `QuantifierElimination` package in Maple. This code ran for slightly longer than 24 hours, using over 50 GB of RAM (on a workstation with an Intel Xeon W-3365 processor and 1000 GB of RAM). It ultimately crashed with a "Stack Limit Reached" error, even if we allowed unlimited stack size.

6.2. Sample Code. An example of the snippet of code which is run in Macaulay2 [18] applied to example X_1 from Table 1 is given below⁹:

```
needsPackage "WhitneyStratifications"
R=QQ[x_1..x_4]
I=ideal(x_1^6+x_2^6+x_1^4*x_3*x_4+x_3^3)
elapsedTime V=whitneyStratify(I)
elapsedTime W=whitneyStratify(I, AssocPrimes=>false)
elapsedTime Vpol=whitneyStratifyPol(I, Algorithm=>"msolve")
```

In the code snippet above the stratification V is computed using the method of [24], the stratification W is computed using the method of [23], and the stratification V_{pol} is computed using the **WhitneyPolar** algorithm introduced above in Section 3.

6.3. Gröbner Basis Computation. For all Whitney stratification algorithms demonstrated in Table 1, we have provided the run time both over \mathbb{F}_{32749} , a finite field of characteristic 32749, and over the rationals. In practice, efficient implementations of Gröbner basis algorithms over \mathbb{Q} often first compute over different large finite fields, and then employ rational reconstruction (this is used in [3] for instance). A similar multi-modular philosophy could be applied to Whitney stratification computation as a whole. On larger examples, the timings in the table seem to indicate there could be a significant benefit to doing so.

⁹Note that running this code successfully requires Macaulay2 version 1.25.06 or higher and version 2.23 or higher of the `WhitneyStratifications` [22] package.

6.4. Hardware. All computations in the first three columns of Table 1 were run using Version 2.23 of the WhitneyStratifications [22] package in Macaulay2 [18] version 1.25.06. The last column used the CylindricalDecomposition command in Mathematica and was run with Wolfram 14.2.1. All computations were carried out on a laptop with an Intel Ultra 7 processor (258V) and 32 GB of RAM. The **WhitneyPolar** algorithm uses the msolve [3] Gröbner basis library (version 0.9.0) via the Msolve Macaulay2 package. Computations which used greater than 32GB we also run again on a desktop with 64GB of RAM available; in *all* cases these computations again exhausted all available RAM before the end of the 11 hour run. On all runs, **WhitneyPolar** used no more that 400MB of memory.

6.5. The Input Varieties. Below we give the equations for the examples presented in Table 1. The Whitney Cusp and Umbrella are the simplest examples of singular varieties whose Whitney stratification can not be obtained by computing iterated singular loci. Examples X_1 through X_4 are taken from [23]; these are more complicated and also have non-trivial Whitney stratifications.

$$X_1 = \mathbf{V} \left(x_1^6 + x_2^6 + x_1^4 x_3 x_4 + x_3^3 \right) \subset \mathbb{C}^4 \quad (13)$$

$$X_2 = \mathbf{V} \left(x_1^2 x_3 - x_2^2, x_2^4 - x_1 x_2^2 - x_3 x_4^2, x_1^2 x_2^2 - x_1^3 - x_4^2 \right) \subset \mathbb{C}^4 \quad (14)$$

$$X_3 = \mathbf{V} \left((x_1^3 - 2x_1^2 x_4 + x_1^3 + x_5^2 x_4 + x_2^2 x_1 - x_2 x_1 x_3 + x_5^3) \right) \subset \mathbb{C}^5 \quad (15)$$

$$\begin{aligned} X_4 = \mathbf{V} \left(x_1^2 x_2^2 x_3 - 2x_1^2 x_2 x_3 x_4 - 2x_1^2 x_2 x_3 x_5 + x_1^2 x_3 x_4^2 + 2x_1^2 x_3 x_4 x_5 + x_1^2 x_3 x_5^2 - 2x_1 x_2^3 x_3 + \right. \\ 4x_1 x_2^2 x_3 x_4 + 4x_1 x_2^2 x_3 x_5 - x_1 x_2^2 x_4 - 2x_1 x_2 x_3 x_4^2 - 6x_1 x_2 x_3 x_4 x_5 - 2x_1 x_2 x_3 x_5^2 + \\ x_1 x_2 x_4^2 + x_1 x_2 x_4 x_5 + 2x_1 x_3 x_4^2 x_5 + 2x_1 x_3 x_4 x_5^2 - x_1 x_4^2 x_5 + x_2^4 x_3 - 2x_2^3 x_3 x_4 - \\ 2x_2^3 x_3 x_5 + x_2^3 x_4 + x_2^2 x_3 x_4^2 + 4x_2^2 x_3 x_4 x_5 + x_2^2 x_3 x_5^2 - 2x_2^2 x_4^2 - x_2^2 x_4 x_5 - 2x_2 x_3 x_4^2 x_5 - \\ \left. 2x_2 x_3 x_4 x_5^2 + x_2 x_4^3 + 2x_2 x_4^2 x_5 + x_3 x_4^2 x_5^2 - x_4^3 x_5 \right) \subset \mathbb{C}^5 \end{aligned} \quad (16)$$

Examples X_5 and X_6 are also constructed to have non-trivial Whitney stratifications.

$$X_5 = \mathbf{V} \left(x_6 x_7^2 + x_7^3 + x_6^2 - x_1 x_7 - x_7^2 + x_2, x_1^2 x_2^2 - x_1^3 + x_1 x_7^2 - x_3 x_7^2 - x_5^2 \right) \subset \mathbb{C}^7 \quad (17)$$

$$\begin{aligned} X_6 = \mathbf{V} \left(3x_2 x_3 x_5 + 2x_1 x_4 x_5 - x_2 x_4 x_5 - x_3 x_4 x_5 - 2x_1 x_5 x_6 - 5x_3 x_5 x_6 + 2x_4 x_5 x_6, \right. \\ x_2 x_4^2 - x_1 x_4 x_5 - 3x_2 x_3 x_6 + x_1 x_5 x_6 + 3x_3 x_5 x_6 - x_4 x_5 x_6, \\ x_2^2 x_4 - x_1 x_2 x_5 + 2x_1 x_2 x_6 - x_2 x_3 x_6 - 2x_2 x_4 x_6 + x_3 x_5 x_6, \\ \left. 3x_2^2 x_3 + 2x_1 x_2 x_4 - x_2 x_3 x_4 - x_1 x_2 x_5 - 6x_2 x_3 x_6 + x_3 x_5 x_6 \right) \subset \mathbb{C}^6 \end{aligned} \quad (18)$$

These examples exhibit varying levels of complexity in terms of degree of generators, number of variables, density of polynomials, etc. Our next example X_7 is the determinantal variety defined by the 3×3 minors of the matrix

$$\begin{bmatrix} x_1 & x_4 & x_7 & x_6 \\ x_2 & x_5 & x_5 & x_7 \\ x_3 & x_3 & x_4 & x_1 \end{bmatrix}.$$

Explicitly, we have

$$\begin{aligned} X_7 = \mathbf{V}(& x_1x_4x_5 - x_3x_5x_6 + x_4x_5x_6 - x_4^2x_7 - x_1x_5x_7 + x_3x_7^2, \\ & x_1^2x_5 + x_2x_4x_6 - x_3x_5x_6 - x_1x_2x_7 - x_1x_4x_7 + x_3x_7^2, \\ & x_2x_4^2 + x_1x_3x_5 - x_3x_4x_5 - x_3x_5x_6 + x_4x_5x_6 - x_2x_3x_7 - x_4^2x_7 - x_1x_5x_7 + x_3x_5x_7 + x_3x_7^2, \\ & x_1x_2x_4 - x_2x_3x_6 + x_2x_4x_6 - x_1x_2x_7 + x_1x_3x_7 - x_1x_4x_7 - x_3x_4x_7 + x_3x_7^2) \subset \mathbb{C}^7 \end{aligned} \quad (19)$$

The last two varieties in Table 1 have Whitney stratifications given by iterated singular loci; these have been included as control examples. The first of these is smooth and has been obtained by intersecting 2 random quadrics in 10 variables. The second one, called the *Barth Sextic* is naturally written in 3 variables, however we require an additional variable to represent the golden ratio $\frac{\sqrt{5}+1}{2}$ algebraically. Thus, our version of the sextic is represented by an ideal of degree 12 in 4 variables.

Input	Assoc. Primes [24]		Block Order [23]		WhitneyPolar		CAD
	\mathbb{F}_{32749}	Q	\mathbb{F}_{32749}	Q	\mathbb{F}_{32749}	Q	
Whitney Umbrella	0.07s	0.07s	0.05s	0.05s	0.06s	0.06s	0.4s
Whitney Cusp	0.09s	0.09s	0.07s	0.07s	0.06s	0.06s	0.4s
X_1 , see (13)	4.8s	2.9s	4.5s	2.7s	0.15s	2.5s	0.4s
X_2 , see (14)	9.5s	58.9s	9.2s	58.3s	0.1s	0.2s	0.4s
X_3 , see (15)	1.1s	1.6s	0.6s	0.9s	0.2s	0.3s	0.9s
X_4 , see (16)	<i>m</i>	<i>m</i>	20.7s	747.2s	1.9s	3.4s	<i>m</i>
X_5 , see (17)	<i>m</i>	<i>m</i>	<i>m</i>	<i>m</i>	1.6s	77.2s	<i>t</i>
X_6 , see (18)	<i>t</i>	<i>t</i>	<i>t</i>	<i>t</i>	1.4s	3.9s	<i>m</i>
X_7 , see (19)	<i>t</i>	<i>t</i>	<i>t</i>	<i>t</i>	52.5s	208.8s	<i>m</i>
Smooth	0.08s	0.4s	0.08s	0.4s	0.08s	0.4s	<i>t</i>
Barth Sextic	0.6s	0.6s	0.6s	0.6s	0.6s	0.6s	12717.7s

TABLE 1. The first three columns show run times of various practical Whitney stratification algorithms on several example varieties; the run time for Cylindrical Algebraic Decomposition (CAD) computation is also included in the last column. Computations which failed to terminate after 11 hours are denoted *t*. Computations which exhausted 64GB of memory and terminated are denoted *m*. For each run we highlight the fastest algorithm in **bold**.

6.6. The Results. Regarding the experiments in Table 1, we note that the practical benefits of the new **WhitneyPolar** algorithm become particularly apparent on larger examples with denser polynomials, e.g. X_4 , X_5 , X_6 and X_7 . This is true both in comparison to the other Whitney stratification methods from [23, 24], as well as when compared to alternative methods to study systems of real polynomials such as Mathematica’s highly refined CAD implementation.

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