The Astrophysical Journal, 886:50 (7pp), 2019 November 20 https://doi.org/10.3847/1538-4357/ab4b49
© 2019. The American Astronomical Society. All rights reserved.

# Modified Friedmann Equations via Conformal Bohm–de Broglie Gravity

G. Gregori[1], B. Reville[2], and B. Larder[1]
[1] Department of Physics, University of Oxford, Parks Road, Oxford OX1 3PU, UK; gianluca.gregori@physics.ox.ac.uk
[2] Max-Planck-Institut für Kernphysik, Postfach 10 39 80, D-69029 Heidelberg, Germany
*Received 2019 May 23; revised 2019 October 4; accepted 2019 October 4; published 2019 November 19*

## Abstract

We use an alternative interpretation of quantum mechanics, based on the Bohmian trajectory approach, and show that quantum effects can be included in the classical equation of motion via a conformal transformation on the background metric. We apply this method to the Robertson–Walker metric to derive a modified version of Friedmann's equations for a universe consisting of scalar, spin-zero, massive particles. These modified equations include additional terms that result from the nonlocal nature of matter and appear as an acceleration in the expansion of the universe. We see that the same effect may also be present in the case of an inhomogeneous expansion.

*Key words:* cosmology: theory – dark energy – dark matter

## 1. Introduction

The equations governing quantum mechanics have been known for nearly 100 years, but the task of reconciling them with the equations of classical motions remains unsolved (see, e.g., Hiley & Muft 1995). The problem is usually expressed in terms of the trajectory (or path) that a particle follows. While in ordinary Newtonian theory a particle moves along a well-defined path (a concept that can be extended to curved space too), this is not true anymore in orthodox (or the Copenhagen interpretation) quantum mechanics. This is because it is impossible to simultaneously measure the position and momentum of the particle. There is, however, an alternative possibility, called Bohmian mechanics or the de Broglie–Bohm interpretation (Bohm 1952; Bohm et al. 1987), whereby the particle follows a definite path, different from the classical one, determined not only by the action of local potential, as in ordinary classical mechanics, but also by the nonlocal feedback of the particle's wave function on its own motion (Bohm 1952).

It has been shown, in the non-relativistic case, that Bohmian quantum mechanics yields the same results as orthodox quantum mechanics (Bohm 1952), and indeed both approaches can be developed from the usual Schrödinger equation, albeit with a different physical interpretation. Experiments have also demonstrated that the de Broglie–Bohm trajectories can have a sound physical interpretation if full non-locality is accounted for (Mahler et al. 2016). Extensions to quantum fluids (Haas 2011; Cross et al. 2014), field theory (Dürr et al. 2004; Carroll 2007), and curved spacetime (Holland 1992; Shojai & Shojai 2006; Dürr et al. 2014) have also been presented.

Here we adopt one of the latter approaches (see Carroll 2005 for a review) and develop a formalism that permits a reinterpretation of the classical path by including the effects of Bohm's nonlocal potential on the particle's motion. This is achieved in the usual manner, through suitable modification of the action for a free particle of mass $m$,

$$\mathcal{S} = -m \int \sqrt{g_{\mu\nu} u^\mu u^\nu}\, ds, \tag{1}$$

where $u^\mu = dx^\mu/ds$ is the dimensionless 4-momentum and $g_{\mu\nu}$ the spacetime metric (we use units, unless specified otherwise, where $c = \hbar = 1$). The metric is taken to have the signature $(+,-,-,-)$.

The effects of an external field can be incorporated into the above action via a modified metric: $g_{\mu\nu} \rightarrow \tilde{g}_{\mu\nu}$. As discussed in Goldstein (1950), there is no single way to modify the metric in Equation (1), and in fact, any function of a world scalar (in this case $u_\mu u^\mu$) that leads to the correct equations of motion is acceptable. The same is also true for actions that account for the particle's interaction with external forces by including an additive term, provided that due care is taken to ensure Lorentz invariance. Zee (2013), for example, exploits a number of different approaches to introduce the effects of electromagnetic and gravitational fields into the action. As an example, it is well known that a simple modification of the Minkowski metric tensor

$$\eta_{\mu\nu} \rightarrow \tilde{\eta}_{\mu\nu} = \mathrm{diag}(1 + 2V/m, -1, -1, -1), \tag{2}$$

for a scalar potential $V$, provides a way to reproduce Newton's law in the non-relativistic limit. We will employ a similar approach below in order to introduce quantum effects into more complex metrics.

## 2. Quantum Effects as a Conformal Transformation of the Metric

Writing Schrödinger's equation, for a particle of mass $m$, in polar form, $\psi = |\psi| \exp(\mathrm{i}S)$, with real function $S(\boldsymbol{x}, t)$, Bohm (1952) establishes the set of coupled equations

$$\partial_t |\psi|^2 + \boldsymbol{\nabla} \cdot (\boldsymbol{v}\, |\psi|^2) = 0 \tag{3}$$

and

$$\partial_t S + \frac{1}{2m}(\nabla S)^2 - \frac{1}{2m}\frac{\nabla^2 |\psi|}{|\psi|} + V_{\rm ext} = 0, \tag{4}$$

where we have set $\boldsymbol{p} \equiv \boldsymbol{\nabla} S = m\boldsymbol{v} = m\, d\boldsymbol{x}/dt$, and $V_{\rm ext}$ is the external potential acting on the particle. The former equation corresponds to conservation of probability density, while the latter is recognized as the Hamilton–Jacobi equation. The analogy with classical mechanics is self-evident after defining the so-called Bohm potential

$$V_{\rm B} = -\frac{1}{2m}\frac{\nabla^2|\psi|}{|\psi|}. \quad (5)$$

Thus, in Bohm's interpretation, the particle moves as guided by its own wave function. In the limit of $V_B \ll V_{\rm ext}$ the trajectory is indistinguishable from its classical one.

As mentioned earlier, an alternative viewpoint is to rewrite (1) in the Minkowski metric as

$$\mathcal{S} = -m\int\sqrt{\tilde{g}_{\mu\nu}u^\mu u^\nu}\,ds, \quad (6)$$

where following Zee (2013) we set

$$\tilde{g}_{\mu\nu} = \left(1 + \frac{2V}{m}\right)\eta_{\mu\nu}. \quad (7)$$

To order $|d\boldsymbol{x}/dt|^2$, ignoring additive constants, the above action gives the classical Lagrangian

$$L \simeq \frac{m}{2}\left(\frac{d\boldsymbol{x}}{dt}\right)^2 - V, \quad (8)$$

with associated Hamiltonian

$$\mathcal{H} = \frac{m}{2}\left(\frac{d\boldsymbol{x}}{dt}\right)^2 + V = \frac{|\boldsymbol{p}|^2}{2m} + V. \quad (9)$$

Exploiting our prior knowledge of the Bohm potential, i.e., letting $V = V_{\rm ext} + V_{\rm B}$, and carrying out the standard substitution $\mathcal{H} = -\partial_t S$ it is immediately observed that (9) reproduces (4) as expected. Thus, if we acknowledge the presence of the Bohm potential in a given system, it is possible to include its effect in macroscopic systems via an appropriate transformation of the metric. Recent work that exploits this approach (Carroll 2005) tacitly assumes the equations already describe an ensemble or cloud of particles. We attempt to make this concept more precise in the next section.

## 3. The Many-body Version of the Bohm Potential

While we have so far considered the motion of a single test particle, it is clear that we cannot neglect the effects of all the other particles in the system. A full theoretical description of this quantum system must take into account effects such as non-locality and correlations on all the scales of interest. A continuum description of such quantum gas is often more convenient, and is usually done in terms of an average density by replacing the product of many-body wave functions with an appropriate density operator into the relevant equations of motion. In previous work (Carroll 2005), the transition from a microscopic to a continuum (macroscopic) description of the Bohm's potential was implicitly assumed.

In order to justify a continuum approach in the Bohm picture more rigorously, let us consider an isolated quantum system of $N$ particles of mass $m$, described by a many-body wave function $\psi = \psi(x_1, x_2, \ldots, x_N)$, where $x_i$ is the spatial coordinate of the $i$th particle. For simplicity, in the following, we take $V_{\rm ext} = 0$. The $N$-body Bohm potential can be written as (Bohm 1952):

$$V^{(N)} = -\frac{1}{2m}\sum_{i=1,N}\frac{\nabla_i^2|\psi|}{|\psi|}, \quad (10)$$

where $\nabla_{\rm i}$ is the gradient with respect to the $i$th particle coordinates. It is first assumed that the many-body wave function can be written as a product of single-particle wave functions: $\psi(x_1, x_2, \ldots, x_N) = \varphi_1(x_1)\varphi_2(x_2)\ldots\varphi_N(x_N)$, where $\varphi_i(x_i)$ is the $i$th particle wave function. The expectation value of the Bohm potential is given by

$$\begin{aligned}\langle V\rangle &= \int dx_1\int dx_2\ldots\int dx_N\, V^{(N)}|\psi|^2\\ &= \sum_{i=1,N}\int dx_i\, V_i^{(N)}(x_i)|\varphi_i(x_i)|^2\\ &= \int dx\sum_{i=1,N} V_i^{(N)}(x)|\varphi_i(x)|^2,\end{aligned} \quad (11)$$

where

$$V_i^{(N)}(x) = -\frac{1}{2m}\frac{\nabla^2|\varphi_i(x)|}{|\varphi_i(x)|} \quad (12)$$

is the single-particle Bohm potential. The total particle density is $n(x) = \sum_i|\varphi_i|^2 = \sum_i n_i$, where $n_i$ is the probability distribution for the $i$th particle. This gives

$$\begin{aligned}\langle V\rangle &= \int dx\sum_{i=1,N} n_i\left(-\frac{1}{2m}\frac{\nabla^2\sqrt{n_i}}{\sqrt{n_i}}\right)\\ &\approx \int dx\, n(x)\left(-\frac{1}{2m}\frac{\nabla^2\sqrt{n}}{\sqrt{n}}\right),\end{aligned} \quad (13)$$

where the last step applies when each particle's wave function significantly extends over their interparticle separation, so the single-particle's amplitudes are assumed to be the same throughout space (that is, $|\varphi_i(x)| \equiv |\varphi_0(x)|$). This is the linearization approximation of quantum hydrodynamics (Manfredi & Haas 2001; Mitcha et al. 2015; Moldabekov et al. 2018), and it strictly applies when the number density of particles varies smoothly in space. However, it can also be shown (Mitcha et al. 2015; Moldabekov et al. 2018) that even in the presence of strong interparticle correlations, Equation (13) continues to apply approximately, to order $\mathcal{O}(1)$.

By comparing Equations (11) and (13) we see that the many-body Bohm potential can thus be written as a function of a single spatial coordinate as

$$V(x) = -\frac{1}{2m}\frac{\nabla^2 n^{1/2}}{n^{1/2}}. \quad (14)$$

The many-body quantum potential discussed here and its application to extended systems of electrons and ions has recently been tested against more established solutions of the Schödinger's equation based on density function theory (DFT; Hohenberg & Kohn 1964; Kohn & Sham 1965). In fact, the quantum potential approach shows excellent agreement with DFT, with substantial reduction in computational speed (Larder et al. 2019).

The complexity of quantum mechanics is not circumvented in Bohm's interpretation; the particles' trajectories are determined by this nonlocal potential, which depends on the particles' wave function itself. However, what Bohm's description offers is a recipe that allows us to obtain quantum corrections to the classical equation.

## 4. The Relativistic Version of the Bohm Potential

Extensions to cosmological models also requires a relativistic and covariant treatment for relevant dark matter candidates. To explore the Bohm potential associated with such dark matter candidates, we apply the previous analysis to the Klein–Gordon equation (Carroll 2005; Nikolić 2005, 2007; Shojai & Shojai 2006)

$$\square\,\psi + m^2\psi = 0\,, \tag{15}$$

where $\square = g^{\mu\nu}\nabla_\mu\nabla_\nu$ is the d'Alembertian, and $\nabla_\mu$ the covariant derivative with respect to the metric $g_{\mu\nu}$. In the Minkowski metric $\square = \partial_t^2 - \nabla^2$.

Following de Broglie (1960), writing the Klein–Gordon equation in polar form, and separating out the real part gives

$$g^{\mu\nu}\nabla_\mu S\ \nabla_\nu S = \left(1 + \frac{2V_{\rm B}}{m}\right)m^2 \equiv \mathcal{M}^2, \tag{16}$$

where $V_B$ is now in covariant form

$$V_{\rm B} = \frac{1}{2m}\frac{\square|\psi|}{|\psi|}. \tag{17}$$

The question, however, arises as Equation (16) cannot be adopted as the relativistic Hamilton–Jacobi equation (Holland 1992; Carroll 2005). The fact that $\mathcal{M}$ is not positive definite is a serious concern, and furthermore (16) leads to tachyonic solutions, which we must exclude on physical grounds—although tachyonic Bohm theories have been discussed in the literature (González-Díaz 2004). The correct approach is that suggested by Shojai & Shojai (2006), giving

$$\mathcal{M}^2 = m^2\exp\left(\frac{2V_{\rm B}}{m}\right). \tag{18}$$

This reduces to the previous case in the weak field limit.

To see how the exponential form for the effective mass $\mathcal{M}$ arises, we follow the approach of Shojai & Shojai (2006). By minimizing the relativistic action, we easily obtain the equation of motion for a particle of variable mass $\mathcal{M} = \mathcal{M}(x)$ in the Minkowski metric

$$\mathcal{M}\frac{du_\mu}{d\tau} = (\eta_{\mu\nu} - u_\mu u_\nu)\partial^\nu\mathcal{M}, \tag{19}$$

where $d\tau^2 = dx_\mu dx^\mu$ is the proper time. By taking the non-relativistic limit (i.e., assuming small velocities and $t = \tau$), the space components of the equation of motion can be written as

$$\mathcal{M}\frac{d^2\boldsymbol{x}}{dt^2} = -\nabla\mathcal{M}, \tag{20}$$

or, alternatively, as

$$m\frac{d^2\boldsymbol{x}}{dt^2} = -\nabla\left(m\ln\frac{\mathcal{M}}{\mathfrak{m}}\right), \tag{21}$$

with $m$ as the bare mass, and $\mathfrak{m}$ as some arbitrary mass scale. Now, if we require that Equation (21) describes the motion of a quantum particle of mass $m$, then consistency with the Hamiltonian (9) implies

$$\mathcal{M} = \mathfrak{m}\exp\left(-\frac{1}{2m^2}\frac{\nabla^2|\psi|}{|\psi|}\right). \tag{22}$$

The relativistic generalization would then lead to

$$\mathcal{M} = \mathfrak{m}\exp\left(\frac{1}{2m^2}\frac{\square^2|\psi|}{|\psi|}\right), \tag{23}$$

which is the same as (18) if we choose the mass scale $\mathfrak{m} \equiv m$.

As pointed out by de Broglie (1960; chap. 10), the inclusion of quantum effects are entirely equivalent to the change of the spacetime metric $g_{\mu\nu} \to \tilde{g}_{\mu\nu} = (\mathcal{M}^2/m^2)g_{\mu\nu}$, that is, a conformal transformation. The approach of Shojai & Shojai (2006) generalizes this result to ensure it satisfies the correct non-relativistic limit. Identifying the four momentum $P_\mu = \nabla_\mu S$, the relativistic energy equation $g^{\mu\nu}P_\mu P_\nu = m^2$ is thus recovered as the weak quantum potential limit of a more general equation $\tilde{g}^{\mu\nu}\tilde{\nabla}_\mu S\ \tilde{\nabla}_\nu S = m^2$, where $\tilde{\nabla}_\mu$ is the covariant derivative with respect to the metric $\tilde{g}_{\mu\nu}$.

Semi-relativistic approaches, where the quantum potential is simply added to the equation of motions without a conformal transformation, have also been studied (Das 2014), and lead to solutions that are different from standard cosmology (Ali & Das 2015).

Following the same arguments as earlier, we can extend this previous result to the many-body system, such that the conformal transformation we should apply has the form

$$\tilde{g}_{\mu\nu} = e^Q\, g_{\mu\nu}, \tag{24}$$

where

$$Q = \frac{1}{m^2}\frac{\square n^{1/2}}{n^{1/2}}. \tag{25}$$

Thus, given $Q$, quantum effects on classical bodies moving in spacetime can be calculated via this prescription. In the next section, this is applied to the Einstein field equations.

## 5. Extension to Cosmology

We proceed by setting $g_{\mu\nu}$ to be the Robertson–Walker metric of an expanding universe. The line element in such a metric is given by

$$\begin{aligned} ds^2 &= dt^2 - R^2(t)\left[\frac{dr^2}{1-\kappa r^2} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right] \\ &\equiv dt^2 - R^2(t)d\Omega^2, \end{aligned} \tag{26}$$

where $R(t)$ is the scale factor, set to unity at present time, and $\kappa$ is the curvature parameter. This allows some simple scaling relations for the quantum potential $Q$ to be obtained. In the homogeneous approximation, the density is a function of time only. For the Robertson–Walker metric the d'Alembertian operator is

$$\square n^{1/2} = \frac{1}{R^3}\frac{\partial}{\partial t}\left(R^3\frac{\partial}{\partial t}\right)n^{1/2}\ . \tag{27}$$

Assuming a matter-like field $n \sim 1/R^3$, the quantum potential has the form

$$Q = -\frac{3}{2}\left(\ddot{R}R/\dot{R}^2 + \frac{1}{2}\right)H^2/m^2\ , \tag{28}$$

where one can identify $q_{\rm d} = -\ddot{R}R/\dot{R}^2$ as the usual deceleration parameter and $H = \dot{R}/R$ is the Hubble parameter.

We now consider the conformal transformation $\tilde{g}_{\mu\nu} = e^Q\, g_{\mu\nu}$ on the Robertson–Walker metric. From the above consideration, we see that the conformal factor has a non-trivial dependence on time. In fact, its scaling with time has been discussed in the literature (Canuto et al. 1977; Maeder 2017a)—while it is required by all scale-invariant gravity theories, it has been only justified via arguments needed to match the present value for the cosmological constant (Canuto & Lee 1977). The modified Einstein field equation now reads as (Canuto et al. 1977; Maeder 2017a)

$$\tilde{\mathcal{R}}_{\mu\nu} - \frac{1}{2}\tilde{g}_{\mu\nu}\tilde{\mathcal{R}} = 8\pi G\tilde{\mathcal{T}}_{\mu\nu} + \Lambda\tilde{g}_{\mu\nu}, \tag{29}$$

where $\tilde{\mathcal{R}}_{\mu\nu}$ is the Ricci tensor with respect to the modified metric $\tilde{g}_{\mu\nu}$, $\tilde{\mathcal{R}}$ is the Ricci scalar, $\tilde{\mathcal{T}}_{\mu\nu}$ is the energy–momentum tensor, and $\Lambda$ is the effective cosmological constant (Weinberg 1989). Since the left side of Equation (29) is manifestly scale-invariant, the same must be true for the right side, in particular the energy–momentum tensor (Canuto et al. 1977; Maeder 2017a), that is, $\tilde{\mathcal{T}}_{\mu\nu} = \mathcal{T}_{\mu\nu}$. The stress tensor, in general, contains two terms (Carroll 2007): $\tilde{\mathcal{T}}_{\mu\nu} = \tilde{\mathcal{T}}^{(M)}_{\mu\nu} + \tilde{\mathcal{T}}^{(Q)}_{\mu\nu}$. The first one is related to the matter contribution, $\mathcal{T}^{(M)}_{\mu\nu} = (p+\rho)u_\mu u_\nu - pg_{\mu\nu}$(for a perfect fluid), where $p$ is the pressure and $\rho$ is the energy density of the matter. A consequence of the scale invariance of the energy–momentum tensor is that both the pressure and energy density of matter are not scale-invariant, as discussed in Maeder (2017a).

The second term of the energy–momentum tensor, $\tilde{\mathcal{T}}^{(Q)}_{\mu\nu}$, instead arises from the energy density of the quantum potential. This is a quantum gravity modification, implying that quantum effects back react on the metric. As these terms appear as corrections of order $\mathcal{O}(m^2/m_P^2)$ (Pinto-Neto & Struyve 2018), where $m_P = 2.4 \times 10^{18}$ GeV is the reduced Planck mass, we can ignore them for particle masses much smaller than $m_P$.

The effective cosmological constant contains two terms, $\Lambda = \Lambda_E + 8\pi G\langle\rho_{\rm vac}\rangle$. The first one is a constant. The second term, instead, is the vacuum energy contribution that arises from field theory. Estimating the actual value for $\langle\rho_{\rm vac}\rangle$ requires further approximations. In Minkowski spacetime, such an estimate can be obtained by summing up the zero-point contributions for each normal mode of a scalar field of mass $m$; we have (Weinberg 1989)

$$\langle\rho_{\rm vac}\rangle = \int_0^{k_c}\frac{4\pi k^2 dk}{(2\pi)^3}\frac{1}{2}\sqrt{k^2+m^2} \approx \frac{k_c^4}{16\pi^2}, \tag{30}$$

with $k_c \gg m$ as some wavelength cutoff, which is usually taken to be set by the reduced Planck length, that is $k_c = 1/\ell_P = (8\pi G)^{-1/2}$. Because cosmological observations tell us that $\Lambda \ll 8\pi G\langle\rho_{\rm vac}\rangle$ (Weinberg 1989), then it must be that $\Lambda_E$ and $8\pi G\langle\rho_{\rm vac}\rangle$ cancel out each other to a very high degree of accuracy, and, in fact, $\Lambda = 0$ may be a good approximation (Hawking 1984; Coleman 1988; Weinberg 1989; Sahni & Krasiński 2008). The latter statement can be justified from a thermodynamic argument (Bucher & Spergel 1999; Volovik 2006). If we consider a vacuum universe, the total pressure is $p = -\Lambda$. Interestingly, the same relation applies for a quantum fluid at zero temperature, i.e., in the absence of any finite temperature excitations. If there are no external forces acting on the fluid, then it must be $p = 0$ and so $\Lambda = 0$ (Volovik 2001, 2006). While $\Lambda = 0$ is still hypothetical, it remains plausible in different approaches of quantum gravity (Hawking 1984; Coleman 1988; Ng & van Dam 1990; Barrow & Shaw 2011).

The solution of Einstein field Equations (29) gives modified Friedmann equations (Canuto et al. 1977; Maeder 2017a):

$$\frac{8\pi G}{3}\rho = \frac{\kappa}{R^2} + \frac{\dot{R}^2}{R^2} + 2\frac{\dot{\lambda}\dot{R}}{\lambda R} + \frac{\dot{\lambda}^2}{\lambda^2} - \frac{\Lambda\lambda^2}{3}\ , \tag{31}$$

$$-8\pi Gp = \frac{\kappa}{R^2} + 2\frac{\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} + 2\frac{\ddot{\lambda}}{\lambda} + 4\frac{\dot{\lambda}\dot{R}}{\lambda R} - \frac{\dot{\lambda}^2}{\lambda^2} - \Lambda\lambda^2, \tag{32}$$

where $\lambda^2 = e^Q$. It is easy to see that the modified Friedmann's equations, (31) and (32), reduce to the usual ones if $\lambda = 1$. This is perhaps more transparent if we combine the equations to read

$$\frac{\dot{R}^2}{R^2} = \frac{8\pi G}{3}\rho + \frac{\tilde{\Lambda}}{3} - 2\frac{\dot{\lambda}\dot{R}}{\lambda R} - \frac{\dot{\lambda}^2}{\lambda^2} - \frac{\kappa}{R^2}, \tag{33}$$

$$\frac{\ddot{R}}{R} = -\frac{4\pi G}{3}(\rho + 3p) + \frac{\tilde{\Lambda}}{3} - \frac{\dot{\lambda}\dot{R}}{\lambda R} + \frac{\dot{\lambda}^2}{\lambda^2} - \frac{\ddot{\lambda}}{\lambda}, \tag{34}$$

with $\tilde{\Lambda} = \Lambda\lambda^2$.

We note that similar equations also appear in quantum gravity theories where the infinite degrees of freedom associated to the solution of the Wheeler–DeWitt equation are collapsed under symmetry constraints, a so-called gravity-matter mini-superspace (Vink 1992; Pinto-Neto & Struyve 2018). The simplest mini-superspace model is described by the metric

$$ds^2 = \mathcal{N}^2(t)dt^2 - R^2(t)d\Omega^2, \tag{35}$$

where $\mathcal{N}(t)$ is the lapse function (which sets the time gauge). Quantum effects are then introduced via the Wheeler–DeWitt equation, leading to a modified equation of motion of the form (Pinto-Neto & Struyve 2018)

$$\frac{\dot{R}^2}{2\mathcal{N}^2R^2} = \frac{\Lambda}{6} - \frac{\kappa}{2R^2} + Q_{\rm WDW}, \tag{36}$$

where for simplicity we have assumed an empty universe, and $Q_{\rm WDW}$ is the Wheeler–DeWitt quantum potential. Equation (36) can be recast in the same form as (33) if we take $\mathcal{N}(t) = 1$, and we set

$$Q_{\rm WDW} \equiv -\frac{\dot{\lambda}\dot{R}}{\lambda R} - \frac{1}{2}\frac{\dot{\lambda}^2}{\lambda^2} + \frac{\Lambda(\lambda^2-1)}{6}. \tag{37}$$

However, Equation (33) is also reproduced if we take $\mathcal{N}(t) = \lambda(t)$, and make a transformation $R(t) \to \lambda(t)R(t)$ in (35), meaning that quantum effects are equivalent to a coordinate change of the metric, as noted earlier. This also means that, at the level of our approximations, the backreaction of quantum effects onto the metric is ignored.

Equations (33) and (34) are very similar to those of standard cosmology, with some noticeable differences. First, there is now a modified cosmological constant $\tilde{\Lambda}$, which also depends on the quantum potential. Second, there is an additional term that produces an acceleration, as with dark energy, but it is not related to the properties of the vacuum. Instead, this new term depends on the non-locality of quantum interactions of a universe filled with matter.

Equation (33) can be rewritten in a more familiar form as

$$\Omega_m + \Omega_\Lambda + \Omega_\kappa + \Omega_Q = 1, \quad (38)$$

where

$$\Omega_m = \frac{8\pi G\rho}{3H^2} = \frac{\rho}{\rho_c}, \quad (39)$$

$$\Omega_\Lambda = \frac{\tilde{\Lambda}}{3H^2}, \quad (40)$$

$$\Omega_\kappa = -\frac{\kappa}{R^2H^2}, \quad (41)$$

$$\Omega_Q = -\frac{\dot{Q}}{H} - \frac{\dot{Q}^2}{4H^2}, \quad (42)$$

with $\rho_c$ being the critical density of matter. Observations suggest $\Omega_m \sim 0.3$ at the present time, accounting for both luminous and dark matter (Riess et al. 1998; Perlmutter et al. 1999; Goobar & Leibundgut 2011; Krauss & Chaboyer 2003).

The implications of Equation (38) in the context of conformal gravity have been discussed in detail in Maeder (2017a, 2017b). Here, we consider for simplicity a universe where, in the weak field limit, $\tilde{\Lambda} \sim \Lambda = 0$, and similarly the space curvature term is set to zero ($\kappa = 0$). The latter approximation is consistent with both observational limits (Goobar & Leibundgut 2011) and existing inflationary models (Guth 1981). Proceeding with this assumption that $\Omega_\Lambda,\ \Omega_\kappa \ll 1$, Equation (38) can be conveniently rewritten in the form

$$\Omega_m = \left(1 + \frac{\dot{Q}}{2H}\right)^2, \quad (43)$$

implying that $\dot{Q} < 0$, as $\Omega_m < 1$. Using our previous definition of $Q$, from Equation (28), the above can be expressed as

$$\frac{d}{dt}\left[\left(\frac{1}{2} - q_\mathrm{d}\right)H^2\right] = \frac{4}{3}m^2H(1 - \Omega_m^{1/2}), \quad (44)$$

a nonlinear third-order differential equation in $R$. Numerical integration of this equation requires three boundary conditions, at least one of which is uncertain. Nevertheless, we can still comment on the general behavior. As the right side of this equation is clearly positive, consistency requires that

$$\frac{\dot{q}_\mathrm{d}}{H} < (q_\mathrm{d} + 1)(2q_\mathrm{d} - 1), \quad (45)$$

indicating a number of different interesting regimes:

(i) $q_\mathrm{d} > 1/2$: in this case, the universe is decelerating, and consistent solutions for both positive and negative $\dot{q}_\mathrm{d}$ can be found. Such large deceleration parameters, however, are disfavored by current observations.

(ii) $q_\mathrm{d} < -1$: i.e., the universe is accelerating rapidly. $\dot{q}_\mathrm{d}$ again can have both positive and negative values but previously, this regime is also disfavored by observations.

(iii) $-1 \leqslant q_\mathrm{d} \leqslant 1/2$: this is physically the most interesting regime, and can be subdivided further into two sub-cases: one accelerating ($q_\mathrm{d} < 0$) and one decelerating ($q_\mathrm{d} > 0$). However, $\dot{q}_\mathrm{d} < 0$ for both implying that a transition from case (iii) to case (i) is not possible. A transition to case (ii) cannot be ruled out.

Current observations favor the latter scenario, with $q_\mathrm{d} < 0$ (Riess et al. 1998; Perlmutter et al. 1999), although this conclusion is not universally accepted (Colin et al. 2018). Proceeding on the assumption that the universe is in fact accelerating, we conclude that the term $\Omega_Q$ behaves as an acceleration that opposes gravity. It is equivalent to what is usually referred to as dark energy (Riess et al. 1998; Perlmutter et al. 1999). In this context, as pointed out earlier, it represents the background energy associated with the nonlocal quantum nature of matter.

Using dimensional analysis on Equation (44), one can infer the present value of the dark energy density $\Omega_{DE}^0 = \Omega_Q^0 \sim H_0^2/m^2 \sim \mathcal{O}(1)$, where $H_0 = 70\ \mathrm{km\ s^{-1}\ Mpc^{-1}} = 2 \times 10^{-33}$ eV (Weinberg 1989). This would imply the existence of a field particle of mass $m \equiv m_g \sim 2 \times 10^{-33}$ eV. We note that this result is in fact analogous to that obtained in quintessence (scalar field) models of dark energy (Copeland et al. 2006), whereby the present value of the cosmological constant is explained by a scalar field of mass $m_\phi \equiv m_g \sim H_0$. The existence of such a field is hypothetical; nevertheless, the value given for its mass is consistent with the more stringent bounds on the graviton mass (Goldhaber & Nieto 2010). There are, however, limits to this interpretation, as it gives the present time a very special place in assigning the mass of such a particle, set only by the current value of the Hubble constant, an example of fine-tuning.

## 6. Beyond the Isotropic and Homogeneous Universe

The model presented so far is still quite ideal. In particular, it is assumed that the universe is homogeneous and isotropic. However, it is clear from observations that the universe, mostly at smaller scales, is neither homogeneous nor isotropic. Locally, it consists of gravitationally bound structures, such as clusters of galaxies, and voids. Because the equations of general relativity are nonlinear, it is plausible that local density inhomogeneities could produce significant changes to the cosmological evolution as described by the standard Friedmann's equations. To investigate the effects of an inhomogeneous distribution of mass, let us consider the case of a spherically symmetric dust universe (as seen from our location at the center) described by the Lemaître–Tolman–Bondi metric. To further simplify the analysis, as we did before for the case of the Robertson–Walker metric, we take $\Lambda = 0$ and $\kappa = 0$. The line element is thus (Enqvist 2008)

$$ds^2 = dt^2 - [\partial_r F(r, t)]^2 dr^2 + [F(r, t)]^2(d\theta^2 + \sin^2\theta\, d\phi^2), \quad (46)$$

where $F(r, t)$ is some (generally unknown) function of the radial coordinate and time. To constrain the model, we now assume that the function $F(r, t)$ can be parameterized in the following form: $F(r, t) = R(t)[r + f(r)]$, where $R(t)$ is the same scale factor as that in the Robertson–Walker metric and $f(r)$ is a function that describes the departure from homogeneity. This reduces to the Robertson–Walker metric (for $\Lambda = 0$ and $\kappa = 0$) if $f(r) = 0$.

As we have done before for the Robertson–Walker metric, in the presence of nonlocal effects induced by the quantum nature of matter, we also need to include the conformal factor $\lambda^2 = e^{Q(r,t)}$, where we have explicitly included both a radial and time dependence in the Bohm potential. With this metric, the solution of Einstein's field equation gives a modified version of Equation (33):

$$\frac{\dot{R}^2}{R^2} = \frac{8\pi G}{3}\rho - \frac{(\partial_t Q)\dot{R}}{R}\left[\frac{(\partial_t Q)R}{4\dot{R}} + 1\right] + \frac{\{2 + 4f'(r) + 2[f'(r)]^2 - [r + f(r)]f''(r)\}(\partial_r Q)}{3R^2[r + f(r)][1 + f'(r)]^3} + \frac{(\partial_r Q)^2 + 4(\partial_r^2 Q)}{12R^2[1 + f'(r)]^2}. \tag{47}$$

If the conformal factor is independent of the radial coordinate, Equation (43) is recovered. However, in an inhomogeneous universe, the conformal factor will, in general, depend on density, so fluctuations also act toward changing the local expansion rate (Buchert 2000, 2001; Alnes et al. 2006), but in a way that is not trivial and depends on the exact structure of those perturbations.

## 7. Dark Energy in an Inhomogeneous Universe

To simplify Equation (47) still further, let us take $Q(r, t) = Q(t)[1 + \ell_q(r)]$ and assume that fluctuations in the metric are uncorrelated with fluctuations of the quantum potential. Such fluctuations are associated with random density perturbations. Locally, however, there are inhomogeneities and anisotropies in the distribution of matter, and a transition to statistical homogeneity is reached only at sufficiently large scales. In standard cosmology this scale is taken to be ∼100 Mpc (Hogg et al. 2005), consistent with baryon oscillations from the Sloan Digital Sky Survey (Eisenstein et al. 2005). On the other hand, there is still the possibility that inhomogeneities of the matter distribution persist at distances exceeding ∼300 Mpc (Colin et al. 2018), which has implications for the determination of the Hubble constant (Heß & Kitaura 2016; Colin et al. 2018). Thus, when averaging over a volume larger than this, we must have $\langle \ell_q(r)\rangle = \langle \partial_r \ell_q(r)\rangle = \langle \partial_r^2 \ell_q(r)\rangle = \langle f(r)\rangle = \langle f'(r)\rangle = 0$. Finally, we take the spatial fluctuations to be small in amplitude, that is, $|f'(r)| < 1$ and $|\ell_q(r)| < 1$. Thus, the only non vanishing terms in Equation (47) result in

$$\Omega_m - \frac{\dot{Q}^2\langle \ell_q^2\rangle}{4H^2} + \frac{Q^2\langle(\partial_r \ell_q)^2\rangle}{12R^2H^2} = \left(1 + \frac{\dot{Q}}{2H}\right)^2. \tag{48}$$

Hence, as discussed earlier, using dimensional analysis, $|Q| \approx H^2/m^2$, $|\dot{Q}| \approx H^3/m^2$, and $|\partial_r \ell_q| \approx \delta/L$, where $L$ is the scale of the fluctuations of the field $m$, and $\delta = \langle \ell_q^2\rangle^{1/2}$. Inhomogeneities dominates the total dark energy density only if $L \ll \delta/mR$. If we take this to be the case, the dark energy density has a much simplified relation:

$$\Omega_{DE} \approx \frac{\delta^2 H^2}{12L^2R^2m^4}. \tag{49}$$

One possibility we explore now is that this hypothetical field $m$ is due to an ultralight axion. The axion is a pseudo-Nambu–Goldstone boson of the broken $U(1)$ Peccei–Quinn symmetry (Peccei & Quinn 1977; Wilczek 1978), which was introduced to explain the absence of CP violation in strong interactions. Extensions of this model to axion-like particles (Jaeckel & Ringwald 2010) can admit the existence of ultralight scalars or pseudo-scalars. The axion remains effectively massless until the universe cools below some critical temperature, and after that it acquires a mass $m_a$ and starts oscillating with wavelength $\propto 1/m_a$. The energy density of these oscillations is of the order of the critical density of the universe, hence the "invisible axion" is a well-motivated candidate for the dark matter (Abbott & Sikivie 1983; Dine & Fischler 1983; Preskill et al. 1983). Because of its small coupling, the axion field decouples with the baryonic matter and behaves as a Bose–Einstein condensate.

The properties of axionic dark matter (with $m_a \sim 8 \times 10^{-23}$ eV giving the best fit to observations) have been investigated in recent structure formation simulations (Schive et al. 2014). Interestingly, those simulations seem to explain the mass distribution around dwarf galaxies, which instead cannot be reproduced correctly by structure formation calculations where dark matter is made by cold non-interacting classical particles (Schive et al. 2014). A characteristic of axionic dark matter simulations is a fuzziness of the mass distribution. This is set by the axion Compton wavelength $\lambda_{Ca} = (2\pi/m_a T_a)^{1/2}$, where $T_a = [2\pi/\zeta(3/2)m_a]n_a^{2/3}$ is the critical temperature of the Bose–Einstein condensate (Pitaevskii & Stringari 2016), $\zeta(3/2) = 2.61$ is the Riemann zeta function and $n_a = \rho_{\rm DM}/m_a$ is the present-day number density of axions, with $\rho_{\rm DM} = 9.6 \times 10^{-12}$ eV$^{-4}$ as the comoving dark matter mass density (Patrignani & Particle Data Group 2016).

We do not know much about the size of the density perturbations. If we assume they are typically of the same order as the fluctuations in temperature seen by the cosmic microwave background, then $\delta \sim 10^{-5}$ (Wright 2004). Setting $L \sim \lambda_{Ca}$, the equation for the dark matter density at the present epoch becomes

$$\Omega_{\rm DE}^0 \approx \frac{\delta^2 H_0^2 \rho_{\rm DM}^{2/3}}{12\,\xi(3/2) m_a^{14/3}}. \tag{50}$$

Requiring $\Omega_{\rm DE}^0 \approx 0.7$, this gives $m_a \sim 10^{-18}$ eV. Such (and even smaller) axion masses are, in fact, what is needed for small-scale structure formation (Schive et al. 2014) and they may be detected by oscillations in pulsar timings (De Martino et al. 2017). Moreover, small ultralight scalars in the range $10^{-21}$ eV $\lesssim m_a \lesssim 10^{-17}$ eV have been invoked to explain the cosmological origin of magnetic fields (Choi et al. 2018) (see also Miniati et al. 2018).

## 8. Summary and Conclusions

In this work we have argued that the interpretation of the modified Friedmann's equation in conformal gravity including inhomogeneities in matter is perhaps clarified if the conformal factor is determined by the quantum potential of Bohmian mechanics. In particular, these results suggest that the acceleration term in the expansion of the universe need not be set by the vacuum energy but rather by the non-locality of matter. This is by no means a solution of the cosmological constant problem, as the solution is reliant on a term that requires the existence of as yet undiscovered particle fields.

The authors thank Prof Subir Sarkar (University of Oxford) for stimulating discussions related to this work. This research was supported in part by the Engineering and Physical Sciences Research Council, UK (EP/M022331/1, EP/N014472/1, EP/N002644/1, EP/P010059/1).

All data that support the findings of this study are available from the authors upon request.

## ORCID iDs

G. Gregori https://orcid.org/0000-0002-4153-0628
B. Reville https://orcid.org/0000-0002-3778-1432

## References

Abbott, L., & Sikivie, P. 1983, PhLB, 120, 133
Ali, A., & Das, S. 2015, PhLB, 741, 276
Alnes, H., Amarzguioui, M., & Grøn, O. 2006, PhRvD, 73, 083519
Barrow, J. D., & Shaw, D. J. 2011, PhRvL, 106, 101302
Bohm, D. 1952, PhRv, 85, 166
Bohm, D., Hiley, B., & Kaloyerou, P. 1987, PhR, 144, 321
Bucher, M., & Spergel, D. M. 1999, PhRvD, 60, 043505
Buchert, T. 2000, GReGr, 32, 105
Buchert, T. 2001, GReGr, 33, 1381
Canuto, V., Hsieh, S. H., & Adams, P. J. 1977, PhRvL, 39, 429
Canuto, V., & Lee, J. 1977, PhLB, 72, 281
Carroll, R. 2005, arXiv:gr-qc/0501045
Carroll, R. 2007, TMP, 152, 904
Choi, K., Kim, H., & Sekiguchi, T. 2018, PhRvL, 121, 031102
Coleman, S. 1988, NuPhB, 310, 643
Colin, J., Mohayaee, R., Rameez, M., & Sarkar, S. 2018, arXiv:1808.04597
Copeland, E. J., Sami, M., & Tsujikawa, S. 2006, IJMPD, 15, 1753
Cross, J. E., Reville, B., & Gregori, G. 2014, ApJ, 795, 59
Das, S. 2014, PhRvD, 89, 084068
de Broglie, L. 1960, Nonlinear Wave Mechanics: A Causal Interpretation (Amsterdam: Elsevier)
De Martino, I., Broadhurst, T., Tye, S.-H. H., et al. 2017, PhRvL, 119, 221103
Dine, M., & Fischler, W. 1983, PhLB, 120, 137
Dürr, D., Goldstein, S., Norsen, T., Struyve, W., & Zanghì, N. 2014, RSPSA, 470, 20130699
Dürr, D., Goldstein, S., Tumulka, R., & Zanghì, N. 2004, PhRvL, 93, 090402
Eisenstein, D. J., Zehavi, I., Hogg, D. W., et al. 2005, ApJ, 633, 560
Enqvist, K. 2008, GReGr, 40, 451
Goldhaber, A. S., & Nieto, M. M. 2010, RvMP, 82, 939
Goldstein, H. 1950, Classical Mechanics (Reading, MA: Addison-Wesley)
González-Díaz, P. F. 2004, PhRvD, 69, 103512
Goobar, A., & Leibundgut, B. 2011, ARNPS, 61, 251
Guth, A. H. 1981, PhRvD, 23, 347
Haas, F. 2011, Quantum Plasmas: An Hydrodynamic Approach (Berlin: Springer)
Hawking, S. 1984, PhLB, 134, 403
Heß, S., & Kitaura, F.-S. 2016, MNRAS, 456, 4247
Hiley, B. J., & Muft, A. A. 1995, The Ontological Interpretation of Quantum Field Theory Applied in a Cosmological Context (Berlin: Springer), 141
Hogg, D. W., Eisenstein, D. J., Blanton, M. R., et al. 2005, ApJ, 624, 54
Hohenberg, P., & Kohn, W. 1964, PhRv, 136, B864
Holland, P. R. 1992, FoPh, 22, 1287
Jaeckel, J., & Ringwald, A. 2010, ARNPS, 60, 405
Kohn, W., & Sham, L. J. 1965, PhRv, 140, A1133
Krauss, L. M., & Chaboyer, B. 2003, Sci, 299, 65
Larder, B., Gericke, D., Richardson, S., et al. 2019, SciA, in press (arXiv:1811.08161)
Maeder, A. 2017a, ApJ, 834, 194
Maeder, A. 2017b, ApJ, 849, 158
Mahler, D. H., Rozema, L., Fisher, K., et al. 2016, SciA, 2, e1501466
Manfredi, G., & Haas, F. 2001, PhRvB, 64, 075316
Miniati, F., Gregori, G., Reville, B., & Sarkar, S. 2018, PhRvL, 121, 021301
Mitcha, D., Graziani, F., & Bonitz, M. 2015, CoPP, 55, 437
Moldabekov, Zh. A., Bonitz, M., & Ramazanov, T. S. 2018, PhPl, 25, 031903
Ng, Y. J., & van Dam, H. 1990, PhRvL, 65, 1972
Nikolić, H. 2005, FoPhL, 18, 549
Nikolić, H. 2007, JPhCS, 67, 012035
Patrignani, C. & Particle Data Group 2016, ChPhC, 40, 100001
Peccei, R., & Quinn, H. 1977, PhRvL, 38, 1440
Perlmutter, S., Aldering, G., Goldhaber, G., et al. 1999, ApJ, 517, 565
Pinto-Neto, N., & Struyve, W. 2018, arXiv:1801.03353
Pitaevskii, L., & Stringari, S. 2016, Bose–Einstein Condensation and Superfluidity (Oxford: Oxford Univ. Press)
Preskill, J., Wise, M. B., & Wilczek, F. 1983, PhLB, 120, 127
Riess, A. G., Filippenko, A. V., Challis, P., et al. 1998, AJ, 116, 1009
Sahni, V., & Krasiński, A. 2008, GReGr, 40, 1557
Schive, H.-Y., Chiueh, T., & Broadhurst, T. 2014, NatPh, 10, 496
Shojai, A., & Shojai, F. 2006, PhyS, 64, 413
Vink, J. C. 1992, NuPhB, 369, 707
Volovik, G. E. 2001, PhR, 351, 195
Volovik, G. E. 2006, arXiv:gr-qc/0612134
Weinberg, S. 1989, RvMP, 61, 1
Wilczek, F. 1978, PhRvL, 40, 279
Wright, E. L. 2004, arXiv:astro-ph/0305591
Zee, A. 2013, Einstein Gravity in a Nutshell (Princeton, NJ: Princeton Univ. Press)