Thesis Title:

Some Problems in Algebraic Topology.

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Systems of Higher Order Cohomology Operations in the p-Torsion-Free Category.

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by

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Contents.

Abstract.

In this thesis, we establish a pair of systems of higher order cohomology operations that act on the Z_p-ordinary cohomology of spaces that are free of p-torsion. These "pyramids" of operations are generated by the p-divisibility of certain sums of "pseudo" primary cohomology operations that operate on the p-local cohomology of p-torsion-free spaces.

The properties of these higher order operations allow us to prove theorems that either generalise or improve (in the sense of decreasing indeterminacy) several results in the literature.

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He has given freely of his time, his knowledge and his perspective. He has been a great source of guidance and encouragement and I am most deeply grateful to him.

This thesis is dedicated to my parents.

Chapter 1: *A General Introduction and Overview.*

It is the purpose of this thesis to introduce a rather different approach to the study of cohomology operations of higher order. The foundation for our method, which involves divisibility of the Chern character by powers of an arbitrary prime, p, is due to J.F. Adams. In his introduction to [2], Adams makes a general reference to this type of procedure.

At the heart of our approach lies the "pseudo" primary cohomology operation of (3.2.3), Θ_J^q. Considering only p-torsion-free spaces, we identify the p-local cohomology with the p-local K-theory by means of the collapsing Atiyah-Hirzebruch spectral sequence, [14]. Using various components of the Chern character to return to cohomology, and certain integrality information to assure ourselves that it is p-local cohomology to which we are returning, we define our "pseudo" operation, Θ_J^q.

The result is a homomorphism of p-local cohomology groups that fails to obey the naturality condition required of strict primary cohomology operations; hence, these operations are called pseudo. This lack of naturality is, however, controlled and can be calculated by means of an explicit formula, (3.2.9). It is this expression which not only allows us to calculate the extent to which Θ_J^q deviates from commutativity with respect to maps of arguments, but also to calculate, in a straightforward manner, the indeterminacy of a set of associated higher order, "genuine" cohomology operations.

This association is done as follows. Given a p-torsion free space with an action of Θ_J^q on its cohomology (p-local), one can examine the "p-divisibility" of the image of the pseudo operation. By dividing this image by the highest power of p that allows us to remain in the p-local environment, and reducing mod p, we produce a higher order cohomology operation associated with Θ_J^q. This is done explicitly in (3.3.2) and (3.3.8), above. The indeterminacy of this operation is then, as already mentioned, calculated with the help of the formula that determines the deviation from naturality of the pseudo operation.

It turns out, that not only a single higher order operation may be associated with Θ_J^q (assuming some divisibility conditions) but an entire "pyramid" of higher order operations. A typical diagram of such a pyramid is shown in (3.3.33). The name, pyramid, moreover, was not casually chosen, but, rather, intended to suggest a close relationship to the pyramids of higher order operations defined and constructed by C.R.F. Maunder in [27] and [28]. Indeed, when certain assumptions are made with respect to the category of the arguments, these two systems can be shown to coincide: (3.3.37).

This entire structure, pseudo operations and pyramids of higher order genuine operations, dualises in the sense of [35]. The resulting duals, (3.4.1) and (3.4.16), moreover, turn out, curiously enough, to exhibit rather nicer behaviour with respect to indeterminacy than their original versions did. This less indeterminate behaviour was not "free", however; as with most advantages in mathematics, it was "purchased". Here, the price was a more restrictive set of defining conditions.

This thesis, then, is concerned with the establishment of these parallel systems of cohomology operations, the derivation of the various properties of these systems, and the exploitation of these properties in the form of explicit calculations.

The presentation of this material takes the following form. In Chapter II, we begin with a brief overview of the study of cohomology operations from the classical viewpoint. A short exposition on generalised cohomology theories and spectra is given, followed by a general definition of a cohomology operation. Higher order cohomology operations, constructed using the classical Postnikov tower approach, and using the more recent techniques of C.R.F. Maunder are considered.

Chapter III is concerned with the introduction of our pseudo primary operations and their associated higher order pyramids. The properties of these systems and those of their duals are discussed extensively, here.

The last chapter, IV, is devoted to the exploitation of the various properties derived in chapter III and to the establishment of some new properties concerning compositions of operations and product behaviour. One of the main results of this chapter is the derivation of a generalised higher order Cartan-like formula. For orders greater than 2, this is a completely new result (although of limited application because of its very restrictive hypotheses) and for order exactly 2, it is a major improvement over other results in the literature, in the sense that the indeterminacy is substantially reduced.

Other applications and calculations are given in chapter IV that either improve, generalise or simplify results in the literature. This last chapter is concluded with a section on the $e_{\mathbb{C}}$-invariant of [5] and its relation to our systems of operations. A consideration of the famous Hopf-invariant one problem is offered as an application of this approach.

Chapter II: Cohomology Operations from the Classical Viewpoint.

§1. *Introduction*:

This chapter shall concern itself with offering a background survey on cohomology operations. Its purpose is to remind the reader of the basic terms and concepts involved and to present a general and somewhat heuristic overview of the study of these operations.

We outline our exposition in the following manner. We shall begin with the defining of a cohomology theory in generalised terms. This will involve the notions of an Ω-spectrum and the resulting representable cohomology theory. An axiomatisation implying the more classical axiom system of Eilenberg and Steenrod [16] will be discussed and examples of such theories will be given. The notion of a primary (stable) cohomology operation in a given theory will be introduced. This shall constitute §2.

In section three, we will turn our attention to higher order cohomology operations and their classical method of construction involving Postnikov towers. Functional cohomology operations will be discussed and their relationship to secondary operations shall be established.

In section four, a somewhat more recent approach to higher order cohomology operations will be considered, that of C.R.F. Maunder. His work, involving chain complexes and resulting systems of pyramids of higher order cohomology operations will be discussed and an example of such a system will be given.

§2. *Generalised Cohomology Theories and Operations*:

Let $\overline{C}$ denote the category of base pointed CW complexes and homotopy classes of maps.

2.2.1. Definitions: (i) A spectrum, E, is a sequence of objects, $E_n \in \overline{C}$, indexed by the integers, together with morphisms, e_n: $SE_n \to E_{n+1}$. Here, SX indicates the suspension of $X \cong S^1 \wedge X$.
(ii) The morphism, e_n, determines an adjoint morphism, e_n', by virtue of the equivalence, $[SX, Y] \leftrightarrow [X, \Omega Y]$. In the case that e_n': $E_n \to \Omega E_{n+1}$ is a homotopy equivalence, we call E an Ω-spectrum.

Let us consider several

2.2.2. Examples: (i) Let X be an object in $\overline{C}$. To X we may associate a spectrum, X, defined by $X_n = S^n \wedge X$ (written as S^nX) for $n \geq 0$. For $n < 0$, we set $X_n = *$, the point space. This is clearly a spectrum, but not an Ω-spectrum.
(ii) Letting $X = S^m$ in the above example, gives us the sphere spectrum: $(S^m)_n = S^{m+n}$ for $(m+n) \geq 0$ and $*$ for $(m+n) < 0$. This, too, is a spectrum, but not an Ω-spectrum.
(iii) Let K(G,n) denote the Eilenberg-MaçLane space of type (G,n). Here, G is is an Abelian group and n is an integer. That is to say, K(G,n) is the space such that $\pi_i(K(G,n)) \cong G$ when $i = n$ and is zero otherwise. By noting that $\Omega K(G,n+1)$ is homotopy equivalent to K(G,n), we see that $K(G,*)$, defined by $K(G,*)_n = K(G,n)$, is an Ω-spectrum.
(iv) Let U denote the infinite unitary group $= \bigcup_{n \geq 1} U(n)$.

We write BU for the classifying space of U. We define a spectrum $\mathcal{BU}$ by setting $BU_{2n} = BU \times \mathbb{Z}$ and $BU_{2n+1} = U$. By virtue of the usual equivalence, $U \simeq \Omega BU$ and the Bott isomorphism, we conclude that $\mathcal{BU}$ is an Ω-spectrum.

Given an Ω-spectrum, one may define the representable generalised cohomology theory associated with it, in the following manner.

<u>*2.2.3. Definitions*</u>: (i) Let $\mathcal{C}$ denote the full subcategory of $\bar{\mathcal{C}}$ comprised of finite, pointed CW complexes and their morphisms. Let $\mathcal{E}$ be a(n) (Ω-)spectrum and assume that $X \in \mathcal{C}$. Define the reduced (representable) generalised, or extraordinary, cohomology theory associated with $\mathcal{E}$ by $\tilde{E}^q(X) = \varinjlim_n [X, \Omega^n E_{n+q}]$. If $\mathcal{E}$ was chosen to be an Ω-spectrum, this general definition restricts to $\tilde{E}^q(X) = [X, E_q]$ and we say $\tilde{E}^*(\ )$ is a representable cohomology theory.
(ii) Let us now define the unreduced or free cohomology theory associated with a spectrum $\mathcal{E}$, as follows. Let X^+ denote $X \cup x_0$ (where x_0 is taken to be the base point). Then we define $E^q(X) = \tilde{E}^q(X^+)$ The injection map, $i: x_0 \to X$ induces a map, $i^*: E^q(X) \to E^q(x_0)$, for all q. We may now recover our reduced theory by defining it to be the kernel of i^*. Because the corresponding exact sequence splits naturally, we get that

$$E^q(X) \cong \tilde{E}^q(X) \oplus E^q(x_0) \cong \tilde{E}^q(X) \oplus \tilde{E}^q(S^0). \ [20]$$

<u>*2.2.4. Proposition*</u>: A representable generalised cohomology theory defined by (2.2.3) satisfies the following three axioms:

(i) (Homotopy) Let f and g be morphisms in C such that $f \simeq g$. Then the induced maps, f^* and g^* are equivalent.

(ii) (Suspension) Let σ denote the suspension homomorphism, $\sigma: E^n(X) \to E^{n+1}(SX)$. Then σ is a natural equivalence.

(iii) (Exactness) Let $Y \xrightarrow{f} X \xrightarrow{i} (X \cup_f CY) = C_f$ be a cofibre sequence in C. Then the resulting (induced) cohomology sequence, $\tilde{E}^n(Y) \xleftarrow{f^*} \tilde{E}^n(X) \xleftarrow{i^*} \tilde{E}^n(C_f)$ is exact for all $n \geq 0$.

Proof: This follows from our definition of Ω-spectrum. We refer the reader to [20] for further details. □

2.2.5. Remarks: (i) These axioms can be shown to imply the more classical axioms for cohomology theories of [16] with the exception of the dimension axiom: $E^n(*) = 0$ for all $n \geq 1$. It is well known that there is only one (up to equivalence) cohomology theory that obeys all of the axioms of (2.2.4) as well as the dimension axiom and that is ordinary singular cohomology, $H^*(X)$. All other theories are, consequently, called extraordinary.

(ii) The interested reader is invited to consult [20] and [7] for a more detailed treatment of this subject.

(iii) The Ω-spectra of (2.2.2) (iii and iv) determine ordinary cohomology (with coefficients in the group G) and complex K-theory, respectively.

(iv) Any generalised cohomology theory, then, is a contravariant functor from the category C to the category AB, of Abelian groups and homomorphisms.

Now we are in a position to discuss the notion of a primary cohomology operation within the context of a given theory, $E^*(\ )$. Firstly we point out that extraordinary cohomology theories, as with the ordinary case, may be endowed with a system of coefficients other than the "natural" one, namely $\Pi_*(E)$. This is done by smashing the spectrum, E, with a suitable Moore space [6]. In light of this fact, we shall write $EG^*(\ )$ to denote that extraordinary cohomology theory associated with Ω-spectrum E with coefficients in G.

2.2.6. Definition: Let E be an Ω-spectrum defining the generalised cohomology theory, $E^*(\ )$. A primary cohomology operation (in $E^*(\ )$) of type (n,G; m,H) (for $n,m \in \mathbb{Z}$ and G and H Abelian groups) is a family of functions $\Theta_X\colon EG^n(X) \to EH^m(X)$, one for each $X \in \mathcal{C}$ satisfying the naturality condition that if $f\colon Y \to X$ is a morphism in $\mathcal{C}$ then $f^*\Theta_X = \Theta_Y f^*$. Moreover, we call such an operation stable if it commutes with the suspension homomorphism, σ. By (2.2.5-iv), we see that a primary cohomology operation is a natural transformation of contravariant functors from $\mathcal{C}$ to $\mathcal{AB}$.

Regarding (2.2.6), one should notice that, by virtue of (2.2.3), one may classify all operations of type (n,G;m,H), with respect to a certain theory, $E^*(\ )$, by noting that there is a one-to-one correspondence between the elements of the set of such operations and the elements of $EH^m(EG_n)$ [32].

We also point out that no additional restrictions were placed upon the algebraic behaviour of a primary cohomology operation by our definition, (2.2.6). In particular, such an operation need not be a homomorphism, in general.

We conclude this section with several

2.2.7. Examples: (i) The most well known of all algebras of operations (note that $E^*(E)$ forms an algebra) is the Steenrod algebra, mod p (p any prime), denoted by A_p. This is isomorphic to $H\mathbb{Z}_p^*(K\mathbb{Z}_p,*)$. The elements are polynomials in the Steenrod reduced powers, P^i (or Sq^i when p = 2), as well as the Bockstein homomorphism , and are subject to the Adem relations [32]. The operations are both stable and homomorphisms.

(ii) The set of homotopy classes of maps, [BU,BU], classify the operations of complex (or unitary) K-theory. These contain the Adams operations, Ψ^k, which are also homomorphisms but of trivial grading.

(iii) The algebra of operations on BP-cohomology is the subject of much recent work and can be found described in [40]. Further background can be obtained from [15].

§3. *Operations of Higher Order.*

This section is devoted to the classical construction of cohomology operations of order higher than one, that is to say, operations that are, in some sense, "liftings" of primary operations. This approach engages the services of an inductive homotopy theoretic tool known as the Postnikov tower (or system). We begin with a description of this object.

For the sake of simplicity, we shall describe the basic workings of a Postnikov tower with only one "k-invariant" at every stage. The towers most generally used in the descriptions of higher order cohomology operations, on the other hand, involve "multiple k-invariants". As the fundamental concepts coincide, we limit ourselves to the simpler case for expositional purposes.

In order to apply the very useful technique of induction to homotopy theory, it is required to find a series of homotopy invariants that "build up" a space, step by step. The obvious first thought would be the skeleta of a given space. These suffer from the unfortunate fact that they lack the key property, that of being a homotopy invariant. Indeed, one need not look further than the sphere for a counter-example.

The homotopy invariant that succeeds where the n-skeleton fails is the n-type of a space. The inductive build-up of a space by considering successively greater n-types is known as a Postnikov tower for the given space. More precisely:

2.3.1. Definition: Let f and g be two morphisms from X to Y in $\overline{C}$. We shall say that f and g are n-homotopic if for every complex of dimension $\leq$ n, K, say, and for every map

$$\varphi: K \to X,$$

the compositions $f\varphi$ and $g\varphi$: $K \to X \to Y$ are homotopic.

2.3.2. Definition: We shall say that two spaces in $\overline{C}$ have the same n-homotopy type if there exist maps f: $X \to Y$ and g: $Y \to X$ such that fg is n-homotopic to the identity map on Y and gf is n-homotopic to the identity map on X.

2.3.3. Definition: Let X and Y be in $\overline{C}$. We say X and Y have the same n-type if their n-skeleta, X^n and Y^n, have the same (n-1)-homotopy type.

2.3.4. Proposition: If X and Y are elements of $\overline{C}$ and they have the same n-type, then they have the same m-type for all m $\leq$ n.

Proof: This is clear. If f: $X^n \to Y^n$ and g: $Y^n \to X^n$ are such that

fg (gf, respectively) is (n-1) homotopic to the identity on Y (X, respectively), then $f \mid X^m$ and $g \mid Y^m$ provide the (m-1)-homotopy. □

2.3.5. Corollary: When $n = \infty$, (2.3.4) tells us that every pair of complexes in $\overline{C}$ with the same homotopy type are of the same m-type for all finite m. Thus, n-type is a homotopy invariant of a complex. In order to apply these concepts to the construction of a Postnikov tower, we will require the following series of results which we state without proof. The interested reader is encouraged to consult chapter 13 of [32] for more details.

2.3.6. Theorem: Suppose X and Y are of the same n-type. Let K be a complex of dimension $\leq$ n-1. Then there is a one-to-one correspondence of sets between [K,X] and [K,Y].

2.3.7. Corollary: If X and Y are of the same n-type, then $\Pi_i(X) \cong \Pi_i(Y)$ for all $i < n$.

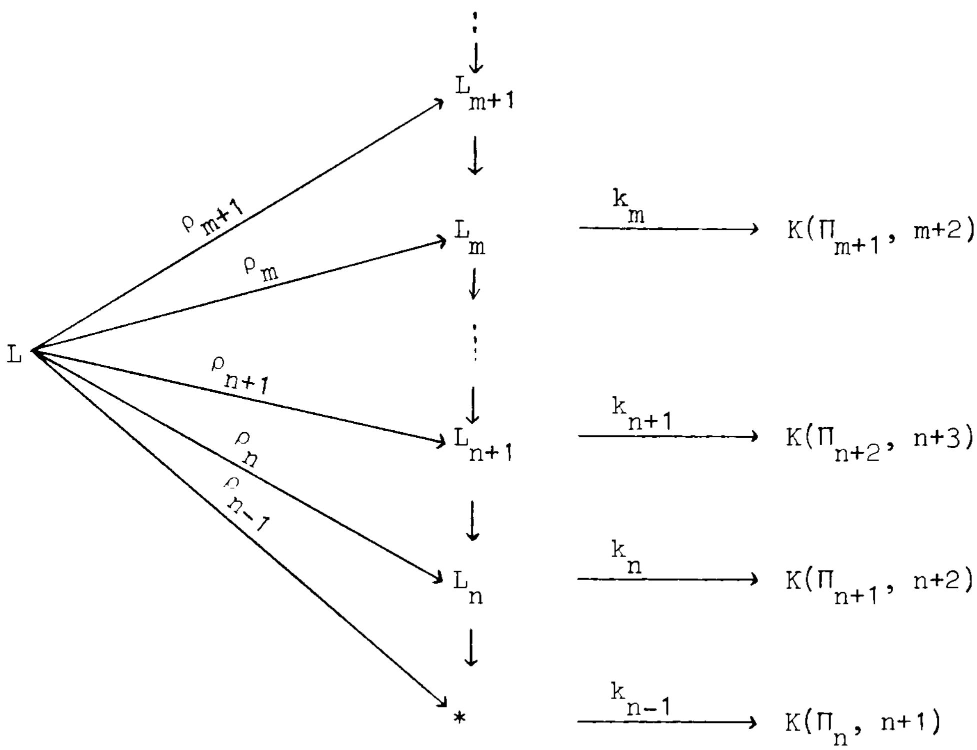
The converse of (2.3.7) is false but we do have the following.

2.3.8. Theorem: (J.H.C. Whitehead): Let f: $X^n \to Y^n$ be in $\overline{C}$ such that the induced homomorphisms on homotopy are isomorphisms for all $i < n$. That is to say, suppose f_*: $\Pi_i(X^n) \overset{\cong}{\to} \Pi_i(Y^n)$. Then X and Y have the same n-type.

2.3.9. Proposition: Let L be an (n-1) connected complex such that $\Pi_n(L) = \Pi_n$. It follows that L and $K(\Pi_n, n)$ have the same (n+1)-type.

The definition of a Postnikov tower is basically the extension of (2.3.9). It is the inductive construction of a space with (n+2)-type of L, the (n+3)-type, and so on. In what follows we let Π_i denote the i^{th} homotopy group of L, the (n-1) connected complex of (2.3.9). Consider the following diagram:

(2.3.10)



<u>2.3.11. *Definition*</u>: In the notation of (2.3.10) and (2.3.9) we say that the diagram (2.3.10) represents a Postnikov tower for L if it satisfies the following conditions:

(i) Each L_m ($m \geq n$) has the same $(m+1)$-type as L and there is a map, $\rho_m: L \to L_m$, inducing isomorphisms $\Pi_i(L) \overset{\cong}{\to} \Pi_i(L_m)$ for all $i \leq m$.

(ii) $\Pi_i(L_m) = 0$ for $i > m$.

(iii) L_{m+1} is the induced fibre space over L_m, induced by the map k_m (the m^{th} k-invariant) from the standard path-loop fibration over $K(\Pi_{m+1}, m+2)$. Thus, by (2.2.5-iii), k_m is an element of $H\Pi^{m+2}_{m+1}(L_m)$. It is defined to be the image of the fundamental class in $H\Pi^{m+1}_{m+1}(F)$ under the transgression, where F is the fibre of ρ_m.

(iv) (2.3.10) is homotopy commutative.

2.3.12. Remarks: (i) m^{th} k-invariant, k_m, of (2.3.11-iii) is chosen to be in the subset of $H\Pi^{m+2}_{m+1}(L_m)$ determined by the kernel of ρ^*_m. The element $\rho^*_m(k_m) = k_m\rho_m$ is in $H\Pi^{m+2}_{m+1}(L_m)$ and is the obstruction to lifting ρ_m to L_{m+1}. Asking that k_m be an element of kernel (ρ^*_m), then, is equivalent to assuring the existence of the extension map, ρ_{m+1}.

(ii) The Postnikov tower given by (2.3.11) and (2.3.10) is somewhat less general in character than the exposition that preceeded it,in the sense that we have restricted ourselves, here, to a particular cohomology theory, ordinary cohomology. This could have been presented in a more general setting where E_n's of the Ω-spectrum $\mathcal{E}$ would replace the $K(\Pi,n)$'s of the singular cohomology spectrum. An example of a Postnikov tower constructed in another theory is given in [38] where the spectrum considered was, $\mathcal{BP}$, the Brown Peterson spectrum [15].

We remain in the case of ordinary cohomology for the purposes of exposition. The reader should bear in mind, however, that the generalised definition is quite parallel to this particular case and we invoke the spirit of (2.3.12-ii) throughout the remainder of this section.

In these terms, then, one may define a secondary cohomology operation as the lifting of a primary operation by one stage in a Postnikov tower. Consider the following diagram <u>We tacitly assume stability throughout</u>.(See [32], chapter 16.):

(2.3.13)

$$\begin{array}{ccccc} K(H,m-1) & \xrightarrow{i} & E & \xrightarrow{\varphi} & K(F,1) \\ & \tilde{u} \nearrow & \downarrow p & & \\ X & \xrightarrow{u} & K(G,n) & \xrightarrow{\Theta} & K(H,m) \end{array}$$

In (2.3.13) we have $X \in \overline{C}$. G, H, and F are Abelian groups. The element, u, of $HG^n(X)$ is represented by a map, u (by abuse of notation and by virtue of (2.2.5-iii)). The map, Θ, is a primary cohomology operation of type (n,G; m,H) and the fibration: $K(H,m-1) \xrightarrow{i} E \xrightarrow{p} K(G,n)$ is induced from the principal path-loop fibration over K(H,m) by Θ. Let us assume that $u \in \ker \Theta$ (the kernel of the map Θ) and let us suppose that we are given a cohomology class $\varphi \in HF^1(E)$. With these hypotheses we have:

<u>*2.3.14. Definition*</u>: The secondary cohomology operation, Φ, associated with the two-stage Postnikov tower in (2.3.13) and the class φ, is defined on any class $u \in \ker \Theta$, and takes values in the coset space, $HF^1(X)/Q$, determined by $\Phi(u) = \tilde{u}^*(\varphi)$, for all maps $\tilde{u}$ such that $p\tilde{u} \simeq u$. The Q, representing the indeterminacy of the secondary operation, Φ, is given by the image of the primary operation, $i^*(\varphi)$, over all maps from X to the fibre of p, K(H,m-1).

Should the class φ, the so-called universal example, lie in the kernel of i^*, then the indeterminacy of the operation would be zero. We shall return to this point below but take the opportunity here to remark that, in general, Q will not be the zero subgroup.

Cohomology operations are divided into two rather distinct types of objects: primary and all those of order higher than one. As we shall see later (see (2.3.19)), the operations of order higher than two are just generalisations of the above procedure which defined the secondary order.

With this in mind, we consider some features contrasting higher order operations with those on the primary level, using our secondary operation as a model for all the higher orders.

To begin with, higher order operations are generally defined on some restricted subset of the "original" cohomology group, and not the entire group, as was the case with primary operations. Secondly, higher order operations are only well defined modulo some amount of indeterminacy, whereas primary ones take a uniquely defined value, that is to say a well-defined element in place of a coset. The requirement that the operations be strictly natural is also (generally) dropped in the higher order cases. This stipulation is usually softened into a naturality condition in which the indeterminacy of a map of spaces applied to an operation is contained in the indeterminacy of the operation applied to the map. This shall be explained in more detail, below. In particular, see A3 of (2.4.2.).

A word or two about the value of Q in (2.3.14) is in order. The indeterminacy of the operation defined by (2.3.14) is due to the choice involved in picking the lifting map, $\tilde{u}$, such that $p\tilde{u} \simeq u$. We were assured of the existence of $\tilde{u}$ by the fact that $u \in \ker\theta$. The choice of particular $\tilde{u}$, however, may vary. Indeed, it may range over $\tilde{u}$ plus any element in the kernel of

$$p_{\#}: [X,E] \to [X,K(G,n)].$$

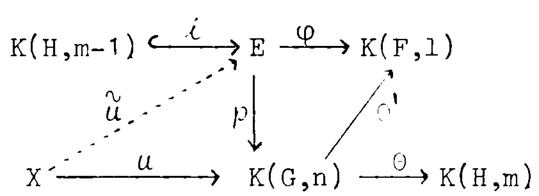
By exactness, these values are precisely equal to the image of

$$i_{\#}: [X,K(H,m-1)] \to [X,E].$$

It follows, then, that $\tilde{u}^*(\varphi)$, the value of the secondary operation, above, may vary by anything in the image of $i^*(\varphi)$. In these terms, we see that Φ of (2.3.14) is defined on the kernel of a primary operation, $\ker(\Theta)$, and takes values in the coset space defined by the cokernel of another primary operation, $\mathrm{cok}\,(i^*(\varphi))$.

This property of having indeterminacy is an essential distinguishing factor between a primary operation and all higher order operations. For example, should the given cohomology element, φ, of (2.3.13) be contained in the kernel of i^*, then our secondary operation, Φ, becomes, essentially, a primary operation defined on the kernel of Θ. We see this as follows. To say that $\varphi \in \ker i^*$ is equivalent to saying that it is in the image of p^*. Setting φ, therefore, equal to $p^*\Theta'$, for some $\Theta' \in HF^1(K(G,n))$ gives the following version of (2.3.13):

(2.3.15)



It follows from this construction that $\varphi\tilde{u} \simeq \Theta' u$. Consequently, our secondary operation, $\Phi(u)$, becomes $\tilde{u}^*(\varphi) \simeq u^*(\Theta') \simeq \Theta' u$ with zero indeterminacy. Thus, Φ becomes a primary operation, Θ', restricted to the kernel of Θ.

It turns out that there is a very strong relationship between secondary operations and relations between primary operations. Consider the following diagram:

(2.3.16)
$$\begin{array}{ccccccc} K(H,m-1) & \xrightarrow{i} & E & \xrightarrow{\varphi} & K(F,l) & & \\ & {\scriptstyle \tilde{u}} \nearrow & \downarrow {\scriptstyle p} & & & & \\ X & \xrightarrow{u} & K(G,n) & \xrightarrow{\Theta} & K(H,m) & \xrightarrow{\Psi} & K(F,l+1) \end{array}$$

We wish to show that the secondary operation, Φ, associated with this two-stage Postnikov tower and cohomology class, φ, corresponds precisely to a relation between primary operations: $\Psi\ \theta = 0$. Firstly, let us assume that Φ is defined. Thus, the hypotheses of (2.3.14) are assumed to be satisfied. Under our stability hypothesis, we may suppose that φi represents the first desuspension of the primary stable operation Ψ, denoted ${}^1\Psi$. Let τ indicate the transgression of the fibration: $K(H,m-1) \xrightarrow{i} E \xrightarrow{p} K(G,n)$. It is well known that primary operations commute with τ [32] and so we have a commutative diagram:

(2.3.17)
$$\begin{array}{ccc} HH^{m-1}(K(H,m-1)) & \xrightarrow{\tau} & HH^{m}(K(G,n)) \\ \downarrow {\scriptstyle {}^1\Psi = \varphi i} & & \downarrow {\scriptstyle \Psi} \\ HF^{l}(K(H,m-1)) & \xrightarrow{\tau} & HF^{l+1}(K(G,n)) \end{array}$$

By (2.3.17), we have that $0 = \tau(i^*(\varphi)) = \tau(\varphi i(\iota_{m-1})) = \tau({}^1\Psi(\iota_{m-1})) = \Psi\tau(\iota_{m-1}) = \Psi\Theta$.

Let us suppose, conversely, that $\Psi\Theta = 0$. We construct our Postnikov system as before but without the map, φ. It is precisely this relation between primary operations that assures us of the existence of the cohomology element, φ (this element is known as the universal example of the secondary operation). Because we know that $0 = \Psi\Theta = \tau({}^1\Psi(\iota_{m-1}))$ we may suppose that ${}^1\Psi(\iota_{m-1})$ is in the image of i^*. Thus, we have ${}^1\Psi = i^*(\varphi)$ for some element, $\varphi \in HF^l(E)$. This allows us to fill in the map, φ, and fully reconstruct the diagram (2.3.16) that gave us the secondary operation, Φ. It must be noted, however, that this procedure does not determine φ uniquely. We are in the following situation:

(2.3.18)

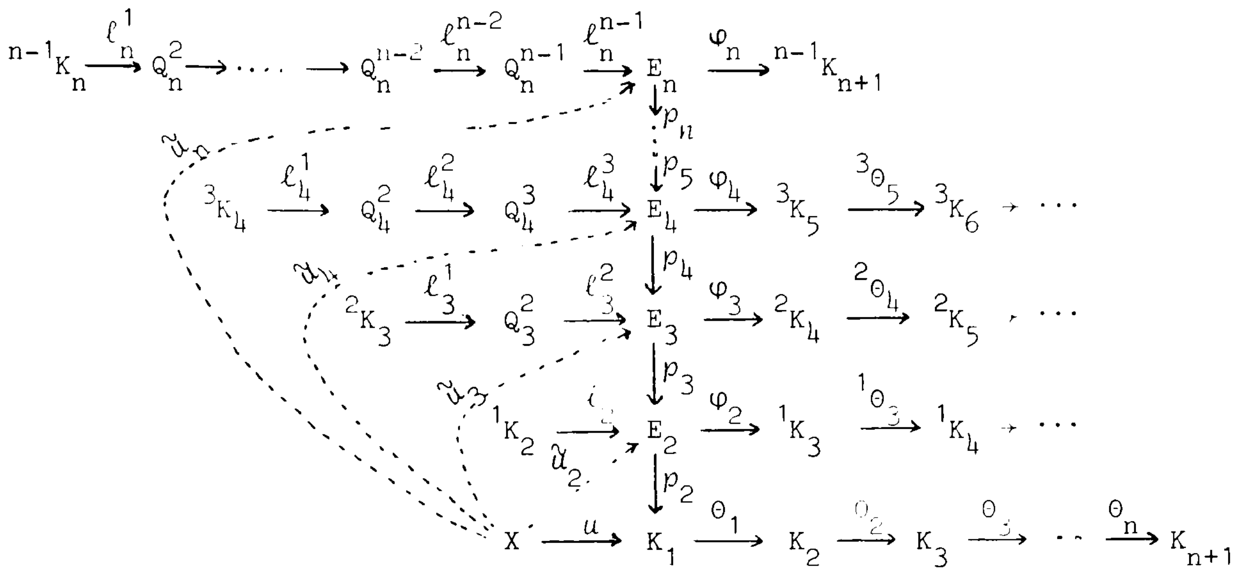
$$\begin{array}{ccccc} HH^{m-1}(K(H,m-1)) & \xrightarrow{{}^1\Psi} & HF^l(K(H,m-1)) & \xrightarrow{\tau} & HF^{l+1}(K(G,n)) \\ & & \nearrow i^* & & \\ HF^l(K(G,n)) & \xrightarrow{p^*} & HF^l(E) & & \end{array}$$

It is clear from (2.3.18) that we have only determined φ modulo some element in the image of p^*, thus, modulo, a primary operation.

There is one thing further that we wish to point out about (2.3.16). We stated that there is a strong relationship between secondary operations and relations between primary ones. The diagram points out that there is also a degree shift involved. The composite of primary operations $\Psi\ \Theta$ had degree $l+1$ whereas the associated secondary operation had degree l.

As we have pointed out above, this procedure generalises to define n^{th} order cohomology operations. Consider the diagram:

(2.3.19)



In (2.3.19) we let K_i denote the Eilenberg-MacLane Space, $K(G_i, m_i)$, where $\{G_i\}$ is a set of Abelian groups and $\{m_i\}$ a set of natural numbers, with $m_i \leq m_{i+1}$ for all i such that, $1 \leq i \leq n+1$. The maps Θ_i are primary operations of type $(m_i, G_i; m_{i+1}, G_{i+1})$. The left superscript on maps and spaces denotes the appropriate desuspension. The map u is a given cohomology element $\in \ker(\Theta_1) \subset HG_1^{m_1}(X)$. We assume, moreover, that the primary operations, Θ_i, are such that $\Theta_{j+1}\Theta_j$ is nulhomotopic for all j, $1 \leq j \leq n$. The existence of the higher universal examples, the φ_i is assumed for the moment. We shall return to this point shortly. The space Q_j^i represents the fibre of the composition $(p_{j-i+1} \circ p_{j-i+2} \circ \ldots \circ p_{j-1} \circ p_j)$. The ℓ_j^i's, then, are inclusions of successive fibres and the composition $(\ell_j^{j-1} \circ \ell_j^{j-2} \circ \ldots \circ \ell_j^1)$ is homotopy equivalent to ι_j, the inclusion of ${}^{j-1}K_j$ in E_j.

In these terms, the secondary operation of (2.3.14) would be $\Phi = \Phi_2 = \tilde{u}_2^*(\varphi_2)$ modulo the indeterminacy equal to the image of $\iota_2^*(\varphi_2)$ over all maps $[X, {}^1K_2]$. Suppose, now, that $\varphi_2 \circ \tilde{u}_2$ is nulhomotopic. Then a lifting, $\tilde{u}_3$, exists. We may, then, provided we are given the universal example, φ_3, define the tertiary operation associated with the Postnikov tower, (2.3.19), by $\Phi_3(u) = \tilde{u}_3^{\,*}(\varphi_3)$ modulo the image of $\varphi_3 \circ \ell_3^2[X,Q_3^2]$ plus the image of $\varphi_3 \circ \ell_3^2 \circ \ell_3^1[X,{}^2K_3]$

In general we have:

2.3.20 Definition: Under the above hypotheses, together with the requirements that Φ_i is defined for all i such that $1 \leq i \leq n-1$, that $\varphi_{n-1} \circ \tilde{u}_{n-1} \simeq 0$, and that we are given a map φ_n, the n^{th} order cohomology operation associated with the Postnikov tower, (2.3.19), is defined by $\Phi_n(u) = \tilde{u}_n^{\,*}(\varphi_n)$ modulo

$$\bigoplus_{i=0}^{n-3} \varphi_n \circ \ell_n^{n-1} \circ \ldots \circ \ell_n^{n-1-i}[X,Q_n^{n-1-i}] \oplus \varphi_n\, \iota_n[X,{}^{n-1}K_n].$$

In this procedure, we required two things at every stage. Firstly, we asked that $\varphi_l \circ \tilde{u}_l$ be nulhomotopic to allow us to lift our map, u, one stage higher in the tower.

We also asked that we be given a map, φ_{l+1}, called the universal example. By the remarks preceeding (2.3.18), however, it can be seen that this is equivalent to asking that ${}^{l-1}\theta_{l+1}\varphi_l \simeq 0$. The property that $\theta_{i+1}\,\theta_i \simeq 0$ for $1 \leq i \leq l+1$ is not, in general,sufficient to assure us of the existence of these universal examples, at least not above the second order and not in ordinary cohomology.

An example of a generalised theory in which the existence of higher (above the second level) universal examples is guaranteed is BP cohomology [15]. Depending upon the "prime of definition" of $BP^*(\ )$, an entire series of universal examples, $\{\varphi_i\}$, are assured of existence given a series of pairwise nulhomotopies, $\theta_{i+1}\,\theta_i \simeq 0$. The property of $BP^*(\ )$ that is responsible for this behaviour is called "q-sparseness", where $q = 2(p-1)$, and p is the "prime of definition". We shall not pursue this point any further here and we refer the interested reader to §1 of [38] for further details.

We conclude this section with a brief discussion of functional cohomology operations. Consider the following diagram:

(2.3.21)
$$\begin{array}{ccccccc} K(H,m-1) & \xrightarrow{i} & E & & & & \\ \uparrow u' & & \uparrow \tilde{u} & \searrow p & & & \\ Y & \xrightarrow{f} & X & \xrightarrow{u} & K(G,n) & \xrightarrow{\Theta} & K(H,m) \end{array}$$

Notice that (2.3.21) is just (2.3.16) without the assumption that we are given an element, φ, or equivalently, without the requirement that we have a relation between primary operations. To this situation we have added a space, Y, and a map of it into X. It is assumed that $f^*(u) = 0$ and we retain the hypothesis that $\Theta(u) \simeq 0$. As before, the facts that $u \in \ker(\Theta)$ and that $p\colon E \to K(G,n)$ is induced from the path-loop fibration over K(H,m) by Θ give us the fact that we may lift u to $\tilde{u}$ such that $p\tilde{u} \simeq u$. It follows, moreover, that since uf is nulhomotopic, then $\tilde{u}f$ is in the kernel of

$$p_{\#}\colon [Y,E] \to [Y,K(G,n)].$$

By exactness, we know that $\tilde{u}f$ lies in the image of

$$i_{\#}\colon [Y,K(H,m-1)] \to [Y,E],$$

Thus, there exists a class, u', say, in [Y,K(H,m-1)] such that $iu' \simeq \tilde{u}f$. In these terms we have:

2.3.22. Definition: Let Θ_f denote the functional cohomology operation determined by (2.3.21). It is an operation defined on the kernel of a primary operation, Θ, intersected with the kernel of the induced map of $f\colon Y \to X$. The functional operation takes values in the coset space

$HH^{m-1}(Y)/Q$. The coset image of Θ_f is the class of u' modulo the indeterminacy, Q, computed as follows:

(2.3.23)
$$\begin{array}{ccccccccc} & & HH^{m-1}(X) & \xrightarrow{i} & [X,E] & \xrightarrow{p_\#} & HG^n(X) & \xrightarrow{\Theta} & HH^m(X) \\ & & \downarrow f^* & & \downarrow f^* & & \downarrow f^* & & \\ HG^{n-1}(Y) & \xrightarrow{{}^1\Theta} & HH^{m-1}(Y) & \xrightarrow{i_\#} & [Y,E] & \xrightarrow{p_\#} & HG^n(Y) & & \end{array}$$

The standard diagram chase, beginning with an element, u, in $HG^n(X)$ such that $\Theta(u) \simeq 0$ and $f^*(u) = 0$, yields an element, $u' \in HH^{m-1}(Y)$, which is well-defined modulo the sum of the images of f^* and ${}^1\Theta$, the first desuspension of our original primary operation, Θ. This is the value of the indeterminacy, Q.

After considering the similarities in the defining diagrams for Θ_f and Φ, it should come as no surprise that these two operations are, indeed, closely related. There are two classical results in this direction which are known as the two Peterson-Stein formulae [34]. We state both of these formulae here, in the notation developed above, and we refer the reader to [32] and [34] for proof and further details.

2.3.24. Theorem: (Peterson-Stein's 1st formula): With the hypotheses and notation of (2.3.16) and (2.3.21) we have that $f^*(\Phi(u)) = {}^1\Psi\Theta_f(u)$ in $HF^1(Y)/Q$ where $Q = f^* \operatorname{Im}({}^1\Psi)$.

In the notation and hypotheses of (2.3.24) we alter the requirements slightly to replace our element u by another element $v \in HG^n(X)$ such that the composition $\Theta v f$ is nulhomotopic. (That is to say, we have relaxed our assumptions on our domain of definition so that we no longer require that $\Theta(\nu) \simeq 0$ and $f^*(\nu) = 0$.) Under these conditions we have:

2.3.25 Theorem (Peterson-Stein's 2nd formula):

$$\Phi(f^*(\nu)) = \Psi_f(\Theta(\nu)) \in HF^1(Y)/Q$$

where $Q = {}^i\Psi_\# \, HH^{m-1}(Y) \oplus f^* \, HF^1(X)$.

§4. The Pyramids of Maunder:

In [27], C.R.F. Maunder developed an axiomatic system of higher order cohomology operations (again, we will be dealing with ordinary cohomology) that associates certain types of chain complexes with "pyramids" of operations of steadily increasing order. This section is concerned with offering a brief account of these results. We begin with the definitions of the basic tools and concepts.

Let

$$(2.4.1) \qquad C_N \xrightarrow{d_N} C_{N-1} \xrightarrow{d_{N-1}} \cdots \longrightarrow C_1 \xrightarrow{d_1} C_0$$

be a chain complex of length N. By this we shall mean a set of N+1, finitely generated, free, left A_p modules, C_i, $(0 \le i \le N)$, together with maps, d_i, of degree zero, satisfying the usual condition that $d_i d_{i+1} = 0$ for all $i \in [1,N-1]$. It is assumed throughout that $(C_i)_q$, the q^{th} dimensional component of C_i is zero for $q < i$.

It turns out that it will not be possible to build up a theory in which every such chain complex determines a pyramid of operations, at least not for $N > 2$. The subset of such complexes as (2.4.1) for which this can be done, that is the set of all chain complexes which associate in a natural way to a pyramid of higher order cohomology operations, is called the set of admissible chain complexes.

We shall return to this point shortly, when we offer a definition of an admissible chain complex (See (2.4.3).).

Assuming for the moment that our chain complex given in (2.4.1) is admissible, we shall associate with it, by considering all the subchain complexes of (2.4.1), one N^{th} order operation, two $(N-1)^{th}$ order operations, three $(N-2)^{th}$ order operations, and so on down to N primary operations. We call this collection of cohomology operations a pyramid of operations. The definitions of these operations will be inductive with an M^{th} order operation being defined on the kernel of an operation of order (M-1). Given an admissible chain complex of length N, in other words, we shall define a set of operations, $\{\Phi^{r,s}\}$, of order r-s, where $N \geq r > s \geq 0$. Such an operation will be defined on a certain map (called the augmentation): $\varepsilon: C_s \to H\mathbb{Z}_p^*(X)$, $X \in \overline{C}$, and its value will be an equivalence class of maps: $\eta: C_r \to H\mathbb{Z}_p^*(X)$.

We define our operations inductively. Consider an augmented chain complex of length one: $C_1 \xrightarrow{d_1} C_0 \xrightarrow{\varepsilon} H\mathbb{Z}_p^*$. Let us define a primary operation associated with it (since all length one chain complexes are admissible we need make no further restrictions) by $\Phi^{1,0}(\varepsilon) = \varepsilon d_1$. This is sufficient to start the induction. We have:

2.4.2. Definition: We shall say that $\{\Phi^{r,s}\}$, $N \geq r > s \geq 0$, is a stable pyramid of operations associated with the admissible chain complex of length N given by (2.4.1) if the following axioms are satisfied:

A0: If $r-s < N$, then the pyramid of operations, $\{\Phi^{u,v}\}$ with $r \geq u > v \geq s$, is associated with the subcomplex of (2.4.1):

$$C_r \xrightarrow{d_r} C_{r-1} \xrightarrow{d_{r-1}} \dots \xrightarrow{d_{s+1}} C_s.$$

A1: $\Phi^{N,0}(\varepsilon)$ is defined for those A_p-maps, ε: $C_0 \to H\mathbb{Z}_p^{\ *}(X)$, such that $\Phi^{r,0}(\varepsilon)$ is the zero coset for $N > r > 0$.

A2: If ε is such a map, and is of degree l, then $\Phi^{N,0}(\varepsilon)$ is defined to be an equivalence class of maps, $[\eta]$: $C_N \to H\mathbb{Z}_p^{\ *}(X)$ of degree l-N+1, where two maps are considered to be equivalent, in this sense, if their difference lies in the image of $\Phi^{N,1}$. (We say that $\mathrm{Im}\Phi^{N,1}$ is the indeterminacy of $\Phi^{N,0}$. It denotes the set of maps in $\Phi^{N,1}(\xi)$ as ξ runs over all suitable maps, that is, all A_p-maps ξ: $C_1 \to H\mathbb{Z}_p^{\ *}(X)$ of degree l-1 such that $\Phi^{N,1}(\xi)$ is defined).

A3: Let g be a morphism in $\overline{C}$, g: $Y \to X$, Let ε be as in A1. Then: $g^*\ (\Phi^{N,0}(\varepsilon)) \subseteq \Phi^{N,0}(g^*\varepsilon)$.

A4: Let σ^*: $H\mathbb{Z}_p^{\ m+1}(SX) \to H\mathbb{Z}_p^{\ m}\ (X)$ be the suspension isomorphism. Then: $\sigma^*(\Phi^{N,0}(\varepsilon)) = \Phi^{N,0}(\sigma^*\varepsilon)$.

(Note that we have chosen ε: $C_0 \to H\mathbb{Z}_p^{\ *}(SX)$ such that $\Phi^{N,0}(\varepsilon)$ is defined).

A5: Let (X,Y) be a pair of spaces. Let η: $C_0 \to H\mathbb{Z}_p^{\ *}(X)$ be a map of degree m such that $\Phi^{N-1,0}(\eta)$ is defined. Let ξ: $C_{N-1} \to H\mathbb{Z}_p^{\ *}(X,Y)$ be such that $j^*\ \xi \in \Phi^{N-1,0}(\eta)$. Let $i^*\eta = \varepsilon^{\bullet}$: $C_0 \to H\mathbb{Z}_p^{\ *}(Y)$. Then, with δ^* equal to the right coboundary:

$$(-1)^{m+1}\ \xi\ d_N \in \delta^*\ (\Phi^{N,0}(\varepsilon)).$$

In the beginning of this section, we mentioned that not every chain complex may be associated with a pyramid of operations in the sense of (2.4.2). We suggested that the necessary criterion for such an association was that of admissibility.

We define this as follows:

2.4.3. Definition: We define a chain complex of the form (2.4.1) to be admissible if it satisfies the following inductively presented property:

(i) Every chain complex of length one is defined to be admissible.

(ii) Let $N > 1$. A chain complex of the form (2.4.1) is defined to be admissible if there is a pyramid of operations, $\{\Phi^{r,s}\}$ for $N-1 \geq r > s \geq 0$ associated with this subcomplex of length $N-1$, such that $\Phi^{N,N-1}\,\Phi^{N-1,0} = 0$, identically (that is, with no indeterminacy). Here $\Phi^{N,N-1}$ denotes the primary operation associated (by (i)) to $C_N \xrightarrow{d_N} C_{N-1}$.

Notice that when $N=2$, this condition will always be satisfied. It is evident from (2.4.4), below, that this requirement of admissibility corresponds precisely to the requirement that higher order universal examples exist.

We record, in the interest of completeness, the following basic results regarding pyramids of operations and we refer the interested reader to [27], §2.4, for further details:

2.4.4. Theorem: (i) If a chain complex of the form (2.4.1) has an associated pyramid of operations, $\{\Phi^{r,s}\}$, for $N \geq r > s \geq 0$, then the complex is admissible.

(ii) If the chain complex given by (2.4.1) is admissible, then there exists at least one associated pyramid of operations.

2.4.5. Theorem: Suppose $\Phi_0^{N,0}$ and $\Phi_1^{N,0}$ are two possible "apexes" of the pyramid associated with an admissible complex of the form (2.4.1). Then there exist an $(N-1)^{th}$ order operation, Ψ associated the subcomplex of (2.4.1):

$$C_N \xrightarrow{d'_N} C_{N-2} \xrightarrow{d_{N-2}} C_{N-3} \xrightarrow{d_{N-3}} \cdots \longrightarrow C_0 \overset{\varepsilon}{\to} H\mathbb{Z}_p^{\;*}(X).$$

such that $\Phi_0^{N,0}(\varepsilon) - \Phi_1^{N,0}(\varepsilon) = \Psi(\varepsilon)$ for all suitable ε, modulo the total indeterminacy involved.

In heuristic terms, then, what Maunder has done is to develop a rigorous system that can be seen to embrace the Postnikov tower approach of §3, above. Corresponding to relations between operations, are the differentials of his chain complexes. Parallel to lifting maps one stage in a Postnikov tower, is applying one further differential (provided, in both cases, that certain "admissibility" criteria were met). In the Postnikov system approach, dualising in the sense of Spanier [35] was done by applying the canonical anti-automorphism of A_p, namely χ [37]. Maunder dualises his chain complexes directly (See §4 of [27] and Theorem 4.3.1, in particular.). We shall return to this point, below.

We end this discussion of Maunder's operation by giving an explicit example. A particular pyramid of operations will be constructed that will be shown to be a generalisation (in the sense of order) of a classical secondary operation (See [10],[11], and [17].). This example was constructed by Maunder in [28].

Consider the chain complex, $C(N,r)$, with $N \leq r$, defined as follows: For $N \geq n \geq 0$, we let C_n be a free-graded left A_2 module.

Each C_n has n+2 generators of the form $\{c_n; c_{n,0}; c_{n,1}; \ldots; c_{n,n}\}$ where the dimension of $c_n = 2r + n - 1$ and the dimension of each $c_{n,i} = 3n - 2i$ for $0 \leq i \leq n$. We set $c_0 = 0$ by convention. The differentials are defined by:

$$(2.4.6)\quad d_n(c_n) = Sq^1\,(c_{n-1}) + \sum_{i=0}^{n-1} \chi\, Sq^{2r-2i}(c_{n-1,\,n-i-1})$$

$$(2.4.7)\quad d_n(c_{n,i}) = Sq^1(c_{n-1,i-1}) + Sq^{0,1}(c_{n-1,i}).$$

Here $Sq^{0,1} = Sq^2\, Sq^1 + Sq^1\, Sq^2$ (See [2].) That $d_n\, d_{n+1} = 0$ for all n such that $1 \leq n \leq N-1$ follows from the four Adem relations:

$$(2.4.8)\quad Sq^1\, \chi Sq^{2r} + \chi Sq^{2r-2}\, Sq^{0,1} + \chi Sq^{2r}\, Sq^1 = 0,$$

$$(2.4.9)\quad Sq^{0,1}\, Sq^{0,1} = 0, \quad (2.4.10)\quad Sq^1\, Sq^{0,1} + Sq^{0,1}\, Sq^1 = 0, \text{ and}$$

$$(2.4.11)\quad Sq^1\, Sq^1 = 0.$$

One pyramid of operations that this chain complex defines is described by the following theorem of Maunder. The reader is referred to Theorem 2 of [28] for details. Let us, first, establish some notation.

We shall denote by $\mathbb{Q}_p$ the integers localised at some prime, p. We will write ch_n for the component of the Chern character in dimension 2n. Let $\iota : \mathbb{Q}_p \to \mathbb{Q}$ induce the coefficient homomorphism, ι_* and let $\rho' : \mathbb{Q}_p \to Z_p$ induce the coefficient homomorphism, ρ'_*.

<u>*2.4.12 Definition*</u>: A cohomology class, $x \in H\mathbb{Q}^n(X)$ is said to be integral mod p if it lies in the image of $\iota_* : H\mathbb{Q}_p^n(X) \to H\mathbb{Q}^n(X)$.

We have the following theorem of Adams [2]:

<u>*2.4.13 Theorem*</u>: Let η be a complex vector bundle over a CW complex, X, such that η is trivial when restricted to the (2q-1)-skeleton of

X. Then $p^r\ ch_{q+r(p-1)}\eta$ is integral mod p.

We may now state the theorem of Maunder which defines a pyramid of operations associated with the above chain complex.

2.4.14 Theorem: There exists a pyramid of operations, $\{\Phi^{t,s}\}$, for $N \geq t > s \geq 0$ associated with C(N,r) with the following property: Suppose that X is a (2q-1)-connected CW complex and that $\xi \in K(X)$ is such that $2^{r-N+1}\ ch_{q+r}(\xi)$ is integral mod 2. Then $\Phi^{N,0}(\rho_*^! \circ 1_*^{-1}\ ch_q(\xi))$ is defined and takes the value $\rho_*^! \circ 1_*^{-1}(2^{r-N+1} ch_{q+r}(\xi))$, modulo the indeterminacy.

Chapter III: Algebraic Pyramids of Operations.

§1. *Introduction.*

In this chapter we shall consider an alternative, rather more algebraic, approach to higher order cohomology operations. In chapter II we saw how Maunder ([27] and [28]) developed an axiomatic system of higher order cohomology operations based upon certain relations in the mod p Steenrod algebra, A_p. The result was a formal structure that embraced the Postnikov tower approach to higher order operations and gave properties and existence criteria for cohomology operations of arbitrarily high order.

What we shall establish in this chapter, is an algebraic pyramid approach to higher order cohomology operations, in the p-torsion-free category, coming from increasing p-divisibility of a sort of "pseudo" primary operation. These primary operations will be called "pseudo" because of their failure to exhibit the classically required naturality property. The pseudo nature of the source notwithstanding, the resulting higher order operations are quite proper and satisfy all the usual properties associated with cohomology operations of higher order. This chapter is concerned with making these notions precise.

We fix the notation which we will be using. Unless we state otherwise, we shall be working in the category whose objects are topological spaces with the homotopy type of a CW complex of finite type whose integral (co)homology is free of p-torsion, for some fixed prime, p. The morphisms of our category will be the homotopy classes of continuous maps of such spaces. This category will be denoted by F_p.

Let $\mathbb{Q}_p$ indicate the subring of the rational numbers whose denominators are all relatively prime to p. We will write $H^*(X)$ and $K(X)$ in place of $H\mathbb{Q}_p^*(X)$ and $K\mathbb{Q}_p^0(X)$, the cohomology and unitary K-theory, respectively, of a space X, with coefficients in $\mathbb{Q}_p$. Let Z_p denote the integers modulo p, Z/pZ. The obvious homomorphisms, $\rho : Z \to Z_p$, $\rho' : \mathbb{Q}_p \to Z_p$, $k : \mathbb{Z} \to \mathbb{Q}$, and $k' : \mathbb{Z} \to \mathbb{Q}_p$, induce the coefficient homomorphisms in cohomology: ρ_*, ρ'_*, k_*, and k'_*, respectively.

In our section on K-theory (§2), we shall also be using the following notation. We will define $K_n(X)$ to be the kernel of the homomorphism: $K(X) \to K(X^{n-1})$, where X^i denotes the i^{th} skeleton of X, as usual. In these terms, we write the standard skeletal filtration as follows:

$$K(X) = K_0(X) \supseteq K_1(X) \supseteq K_2(X) \supseteq \cdots \supseteq K_n(X) \supseteq \cdots \supseteq *. \qquad (3.1.1)$$

Because we will be working frequently with the residue classes mod p-1, we shall fix the notation, m = p-1.

We proceed with this chapter in the following manner. Firstly, we shall define our pseudo primary operations. We will relate them to existing homomorphisms of cohomology in F_p, namely those of Hubbuck [22] and [24], and we shall derive some basic properties. Next, we will define an algebraic pyramid of higher order operations, "generated" by the repeated divisibility of these non-natural homomorphisms. The result will be a pyramid of operations that will relate closely to the pyramid system of [27] and [28]. Following this, the dual theory will be developed and the relation to the dual of [27] shall be established. It will be shown, moreover, that these algebraic pyramids contain, as special cases, certain operations of [18], [26], and [17] when they are applied to spaces in our p-torsion-free category, F_p.

§2. *The Pseudo Primary Cohomology Operation.*

Before we can define our pseudo primary operation, we must make several observations about K-theory in our category, F_p. We know from [6] and [24] that p-localised unitary K-theory, that is to say, the extraordinary cohomology theory defined by the spectrum, $BU\mathbb{Q}_p$, splits up into a direct sum of K-theories, one for each of the mod m residue classes. Thus, we have $BU\mathbb{Q}_p \cong \oplus BU\mathbb{Q}_p^i$, where the sum runs over all residue classes: $0,1,\ldots,m-1$. This gives us $K(X) = \oplus K(X)^{(i)}$ (summed over the same classes), the splitting of the K-theory induced by the splitting of the defining spectrum (see chapter II). Such a decomposition of p-localised K-theory is respected by the action of the Adams operations, ψ^k, and it induces a mod m residue class splitting on the skeletal filtration, (3.1.1).

The following theorem, which is due to Adams and Hubbuck ([6], [8], and [24]) makes this more precise.

3.2.1 Theorem: There is a canonical direct sum splitting, $K(X) = \bigoplus_{i=0}^{m-1} K(X)^{(i)}$ such that: (i) each $K(X)^{(i)}$ is closed under the action of ψ^k for each $k \in \mathbb{Z}$; (ii) the associated graded group is defined by: $K_{2n}(X)^{(i)}/K_{2n+1}(X)^{(i)} \equiv G_{2n}K(X)^{(i)}$ and it equals the usual ("unsplit") associated graded group, $G_{2n}K(X)$, if and only if $n \equiv i \bmod m$. If $n \not\equiv i \bmod m$, $G_{2n}K(X)^{(i)} = 0$.

Now, given that the p-local K-theory breaks up into m summands, we may consider the associated split local cohomology. This is related to the split K-theory as follows:

3.2.2 Proposition: There exists an isomorphism, $J : H^{ev}(X) \to K(X)$ for $X \in F_p$ such that: (i) $J(H^{2n}(X)) \subseteq K_{2n}(X)$; (ii) the composition of J with the quotient map, $I_{2n} : K_{2n}(X) \to K_{2n}(X)/K_{2n+1}(X) \cong H^{2n}(X)$, is the identity map on $H^{2n}(X)$; (iii) we may decompose J into a direct sum, $\bigoplus_{i=0}^{m-1} J^{(i)}$ such that $J^{(i)} : H^{2n}(X)^{(i)} \to K_{2n}(X)^{(i)}$ where $H^{2n}(X)^{(i)}$ is defined by $K_{2n}(X)^{(i)}/K_{2n+1}(X)^{(i)}$.

Proof: Let $x_1 \ldots x_t$ be a basis for $H^{2n}(X)^{(i)}$ for some fixed i, $0 \leq i \leq m-1$. Corresponding to the x_j $(1 \leq j \leq t)$ are elements, $u_j \in K_{2n}^{(i)}$ of exact filtration 2n such that $I_{2n}^{(i)}(u_j) = x_j$. Let us define $J^{(i)}$ by $x_j \to u_j$ for $1 \leq j \leq t$. Now define $J = \bigoplus J^{(i)}$ over all i, $0 \leq i \leq m-1$. This gives us, in view of (3.2.1), the desired results. □

From this point onward, we shall only consider "splitting isomorphisms" of this form, namely, those J's that satisfy (3.2.2).

Now we are in a position to define our pseudo primary cohomology operations. These will be homomorphisms on cohomology groups defined on and evaluated in $H^{ev}(F_p)$, the subring of H^* with even grading and arguments taken from our category, F_p.

3.2.3 Definition: Let J be a splitting satisfying (3.2.2) and let u be any element of $H^{2n}(X)$ for $X \in F_p$ and $n \in Z^+$, the non-negative integers. Then, for each $q \geq 0$, we define a pseudo primary cohomology operation of degree q, by $p^q\ ch_{n+qm}\ J(u)$. We shall denote this operation by $\Theta_J^q : H^{2n}(X) \to H^{2n+2qm}(X)$.

Here ch_s denotes the component of the Chern character in dimension 2s. By convention, we set $\Theta_J^q(u) = 0$ for $q < 0$.

There are several things we wish to point out about this definition. By (2.4.13), the value of $p^q\ ch_{n+qm}J(u)$, which will lie in $HQ^{2n+2qm}(X)$, is integral mod p. We may, consequently, identify this rational cohomology class with a Q_p class under the homomorphism, i_*^{-1}.

Secondly, we know that $u \in K_{2n}(X)/K_{2n+1}(X)$ since X was chosen from F_p. Thus, $J(u)$ is a virtual vector bundle that is trivial over the (2n-1)-skeleton of X and, consequently, we are free to invoke Theorem 2 of [2] without requiring that X be (2n-1)-connected.

A third point to be noticed about (3.2.3) is that $J(u)$ lies in $K(X)$ despite the fact that X is in F_p. Although X itself could well be an infinite complex, the element, u, will always be taken from some finite cohomology grading, $2n$, of X. Thus, we shall only be dealing with finite skeleta at every point. This justifies our use of ordinary K-theory as opposed to K-theory (see [12]) even though we are working in F_p.

It is well known that the Chern character is a homomorphism and so, quite clearly, our pseudo operations must be homomorphisms, as well. Of course, J need not be a ring isomorphism, a priori, and so we are not free to conclude that, in general, $\Theta_J^q(u \cup v) = \sum_{i+j=q} \Theta_J^i(u) \cup \Theta_J^j(v)$. We shall return to this point in chapter IV, §3, below.

One thing, further, to observe is that, by (3.2.2) and (3.2.3), one can conclude that the cohomology homomorphism, Θ_J^q, respects the mod m residue class splitting of $BU\mathbb{Q}_p$ and thus of $H^*(X)$.

3.2.4 Proposition: For $X \in F_p$, the following diagram commutes:

$$\begin{array}{ccc} H^{2n}(X) & \xrightarrow{\Theta_J^q} & H^{2n+2qm}(X) \\ \downarrow \rho'_* & & \downarrow \rho'_* \\ HZ_p^{2n}(X) & \xrightarrow{\chi P^q} & HZ_p^{2n+2qm}(X) \end{array}.$$

Proof: This is a direct application of Theorem 2 of [2]. Applying the theorem to (3.2.3) gives the result.
(Here and throughout this paper we denote by χ, the canonical anti-automorphism of the Steenrod algebra [30]. Moreover, we shall always take P^q to mean Sq^{2q} in the case where $p = 2$.) □

The property that will turn out to be very useful in the evaluation of higher order operations derived from Θ_J^q is established in the following manner. Let X and Y be spaces in F_p and let $f : Y \to X$ be a morphism in this category. We may choose splittings of the proper sort (i.e. those satisfying (3.2.2)), J and L, say, taking $H^{ev}(X) \to K(X)$ and $H^{ev}(Y) \to K(Y)$, respectively. We define a homomorphism f_{JL} by requiring the commutativity of the following diagram:

(3.2.5)
$$\begin{array}{ccc} H^{ev}(X) & \xrightarrow{J} & K(X) \\ \downarrow f_{JL} & & \downarrow f^{!} \\ H^{ev}(Y) & \xrightarrow{L} & K(Y) \end{array}$$

As J and L are both isomorphisms, this homomorphism f_{JL} is well defined. Thus, by (3.2.5) we have $Lf_{JL} = f^{!}J$. Taking the Chern character of both sides of this expression we get:

(3.2.6) $$\text{ch } Lf_{JL} = \text{ch } f^{!}J.$$

Recalling that the Chern character is a natural transformation of cohomology functors, we may write (3.2.6) as:

(3.2.7) $$\text{ch } Lf_{JL} = f^{*}\text{ch } J.$$

Because J and L were chosen such that they satisfy (3.2.2), that is, such that they respect the direct sum splitting of the K-theory, (3.2.5) must commute for each residue class, mod $m = p-1$. That is, for some fixed n and some $j \in [o, m-1]$ such that $n \equiv j(m)$, we get the following commutative diagram:

(3.2.5')
$$\begin{array}{ccc} H^{2n}(X)^{(j)} & \xrightarrow{\quad J^{(j)} \quad} & K_{2n}(X)^{(j)} \\ \Big\downarrow \delta_{JL} & & \Big\downarrow \delta^{!} \\ H^{ev}(Y)^{(j)} & \xrightarrow{\quad L^{(j)} \quad} & K_{2n}(Y)^{(j)} \end{array}$$

Now, by (3.2.1), $K_{2n+1}(Y)^{(j)} = K_{2n+2}(Y)^{(j)} = \ldots = K_{2n+2m}(Y)^{(j)}$. Thus δ_{JL} can be written as a sum of linear maps:

(3.2.8) $$\delta_{JL} = \sum_{i \geq 0} \delta_i$$

where each δ_i raises degree by 2im and where $\delta_0 = \delta^*$.

Now, for any fixed n and q we may consider the component of (3.2.7) in the dimension 2n+2qm:

(3.2.7') $$ch_{n+qm}\, L\, \delta_{JL} = \delta^*\, ch_{n+qm}\, J.$$

Multiplying both sides of this last equation by p^q and substituting in the value for δ_{JL} given us by (3.2.8), we see that we have proved the following important formula which measures the deviation from naturality of our pseudo primary operation:

3.2.9 Theorem: Under the above hypotheses:

$$\delta^* \Theta_J^q = \sum_{i=0}^{q} p^{q-i}\, \Theta_L^i\, \delta_{q-i} \; : H^{2n}(X) \to H^{2n+2qm}(Y).$$

Modulo p, of course, (3.2.9) reduces to a statement of strict naturality: $\rho_*^! \delta^* \Theta_J^q = \rho_*^! \Theta_L^q \delta^*$. In view of (3.2.4), this is hardly surprising since the reduced Steenrod powers are strictly natural and, consequently, "genuine primary cohomology operations".

We end this section with a proof of the correspondence between our pseudo operations and the cohomology homomorphisms of Hubbuck ([22] and [24]) which we shall define below:

3.2.10 Theorem: Let $X \in F_p$ and let $u \in H^{2n}(X)$ for some $n \in \mathbb{Z}^+$. Then, $\Theta_J^q(u) = Q_J^q(u)$ of [22] and [24].

Proof: Define a map, Φ_J^k by requiring strict commutativity of the following diagram:

$$(3.2.11) \qquad \begin{array}{ccc} H^{ev}(X) & \xrightarrow{\Phi_J^k} & H^{ev}(X) \\ \downarrow J & & \downarrow J \\ K(X) & \xrightarrow{\psi^k} & K(X) \end{array}$$

From (3.2.11) we may conclude, after applying the Chern character, that

$$(3.2.12) \qquad \mathrm{ch}\ \psi^k J(u) = \mathrm{ch}\ J\ \Phi_J^k(u), \text{ for all } k \in \mathbb{Z}^+.$$

In [22] and [24] Hubbuck has defined homomorphisms of evenly graded $\mathbb{Q}_p$-cohomology, Q_J^q and S_J^q, in terms of this map, Φ_J^k. In [24] (see 2.8 and 2.9) it is shown that, for a given $x_0 \in H^{2n}(X)$ and a given J and k, one has the existence of a unique finite set of elements, x_i, $1 \le i \le t$, with $x_i \in H^{2n+2im}(X)$ such that $x = \sum_{i=0}^{t} p^{-i}\, x_i$ satisfies $\Phi_J^k(x) = k^n x$. Hubbuck then defines the homomorphism $S_J^q : H^{2n}(X) \to H^{2n+2qm}(X)$ by $x_0 \to x_q$. The homomorphism $Q_J^q : H^{2n}(X) \to H^{2n+2qm}(X)$ is defined by the requirement that $S_J(t) = \sum_{q\ge 0} t^q S_J^q$ and $Q_J(t) = \sum_{q\ge 0} t^q Q_J^q$ be formal inverses. Moreover, it is shown (see 2.10 of [24]) that $\Phi_J^k = k^n \sum_{q\ge 0} \sum_{r\ge 0} (1/_p)^q\, S_J^q(k^m/_p)^r\, Q_J^r$. Now, fixing a q and restricting our attention to dimension $2n+2qm$ and substituting in the value for $\Phi_J^k(u)$, the right-hand side of (3.2.12) becomes:

$$(3.2.13) \qquad \mathrm{ch}_{n+qm} J\{k^n \sum_{a=0}^{q} S_J^{q-a}(1/p)^{q-a}\, Q_J^a(k^m/p)^a(u)\}.$$

Expanding this further gives:

$$(3.2.14) \qquad 1/p^q\ \mathrm{ch}_{n+qm} J\{k^n S_J^q(u) + k^{n+m} S_J^{q-1} Q_J^1(u) + \ldots + k^{n+qm} Q_J^q(u)\}.$$

On the other hand, the left side of (3.2.12) becomes [3]:

(3.2.15) $\qquad k^{n+qm}\, ch_{n+qm} J(u).$

Now, multiplying both (3.2.14) and (3.2.15) by p^q and equating the coefficients of k^{n+qm} of both sides gives:

(3.2.16) $\qquad \Theta_J^q(u) = ch_{n+qm}\, J\, Q_J^q(u).$

But $Q_J^q(u)$ is, by definition, a (2n+2qm)-dimensional $\mathbb{Q}_p$-cohomology class. The theorem follows from (3.2.16) after recalling that $ch_{n+qm} J$ of a $\mathbb{Q}_p$-cohomology class in dimension (2n+2qm) must be just that class itself, namely $Q_J^q(u)$.

This completes the proof. □

§3. *Higher Order Cohomology Operations.*

In this section we shall define an algebraic system of pyramids of higher order cohomology operations in the sense of Maunder [27] based upon "sums" of pseudo primary operations of the form , $\sum_{i=0}^{M} \Theta_J^{q-i}$. As such, we will be building a system of pyramids for spaces in the category, F_p. It will be shown, moreover, that when we restrict a particular pyramid of Maunder's [28] to this category, it will coincide, modulo indeterminacy, with those we construct below. Let us begin by establishing some notation.

3.3.1 Definitions: (i) Let $\{u_i\}$ be a vector in the $\mathbb{Q}_p$-cohomology of some space, $X \in F_p$, where $u_i \in H^{2n+2im}(X)$ for i between 0 and some given non-negative integer, M, and where n is some fixed natural number. For a given $q > M \geq 0$, we shall denote a sum of pseudo primary cohomology operations of degree q and type M by the expression, $\sum_{i=0}^{M} \Theta_J^{q-i}$, defined upon a vector, $\{u_i\}$, and taking values in $H^{2n+2qm}(X)$.

(ii) Let $\{u_i\}$ be a vector in the cohomology of X as above. Suppose that $\sum_{i=0}^{M} \Theta_J^{q-i} u_i = y$ in $H^{2n+2qm}(X)$ and that y is divisible by p^{N-M-1} for some integer $N > M$. Then we shall say that $[J,\{u_i\}]$ is a (q,N,M)-pair. (In particular, any $[J,\{u_i\}]$ is always a $(q,N,N-1)$-pair.)

Before we rigorously present the construction of a pyramid based upon the sum, $\sum_{i=0}^{M} \Theta_J^{q-i}$, we offer a somewhat more heuristic discussion of our method of procedure. The general idea is as follows. Assume we are given a vector, $\{x_i\}$ $(0 \leq i \leq M)$, in the $\mathbb{Z}_p$-cohomology of some space, $X \in F_p$. Suppose, moreover, that there exists a splitting isomorphism, J, such that $[J,\{u_i\}]$ forms a (q,N,M)-pair for some $\mathbb{Q}_p$-lifting, $\{u_i\}$ of the given $\mathbb{Z}_p$-vector ,$\{x_i\}$. That is to say, we assume the existence of a splitting, J, and a $\mathbb{Q}_p$-vector, $\{u_i\}$, which reduces to $\{x_i\}$,modulo p, such that the image of the sum of type M and degree q, $\sum_{i=0}^{M} \Theta_J^{q-i} u_i = y$, is divisible by p^{N-M-1} in $H^{2n+2qm}(X)$. Consequently, dividing y by p^{N-M-1} is a valid operation in the context of $\mathbb{Q}_p$-cohomology. Performing this division and then reducing to $\mathbb{Z}_p$-cohomology gives the coset value of our cohomology operation of order (N-M) acting on the vector $\{x_i\}$. We denote this by:

$$(3.3.2) \qquad \Phi_q^{N,M}\{x_i\} = \rho_*^![(\sum_{i=0}^{M} \Theta_J^{q-i} u_i)/p^{N-M-1}]/Q(\Phi_q^{N,M})$$

where J and $\{u_i\}$ run over all possible (q,N,M)-pairs associated to the vector, $\{x_i\}$, in the sense described above.

We make several remarks concerning this description of an $(N-M)^{th}$ order operation. Firstly, one should notice that the value of $\Phi_q^{N,M}\{x_i\}$ is not without indeterminacy, since choices were made in its definition. Variation in $\mathbb{Q}_p$-representative of the given

$\mathbf{Z}_p$-vector and choice of a splitting isomorphism generate this indeterminacy. Just as all, more classically defined (see chapter II) higher order operations, $\Phi_q^{N,M}$ will be evaluated in a coset space, the range modulo the indeterminacy, $Q(\Phi_q^{N,M})$. It will be shown that the value of $Q(\Phi_q^{N,M})$ is the image of the operation, $\Phi_q^{N,M+1}$. It shall be demonstrated, furthermore, that every element in the indeterminacy can be explicitly realised in the sense of (3.3.2). As a result, we shall be able to conclude that any element of the kernel of $\Phi_q^{N,M}$ can be represented by an explicit $(q,N+1,M)$-pair. In this way we shall inductively construct a pyramid of higher order cohomology operations, $\{\Phi_q^{r,s}\}$, with $q \geq N \geq r > s \geq 0$, such that:

(3.3.3) $$\Phi_q^{r,s} : \text{Ker } \Phi_q^{r-1,s} \to \text{Cok } \Phi_q^{r,s+1}.$$

This will be a pyramid shaped collection of operations with a bottom row consisting of primary operations, a second row of secondary operations, a third row of tertiary operations, and so on, up to a peak of a single N^{th} order operation. Domain and range of any operation in the pyramid will be found in the kernel and cokernel, respectively, of operations in rows lower in the pyramid. Before we proceed to make this precise, we establish some definitions and notations.

3.3.4 Definitions: (i) We shall denote by V a countably infinite direct sum of free cyclic $\mathbb{Q}_p$-modules, $\bigoplus_{n,t\in Z^+} V_{t,n}$. Each $V_{t,n}$ is filtered by $V_{t,n} = V_{t,n}^0 = V_{t,n}^1 = \ldots = V_{t,n}^{2n} \supseteq V_{t,n}^{2n+1} = 0$, with generator, ζ_n^t, in exact filtration, 2n.

(ii) Similarly, we shall write W for the countably infinite direct sum of the associated free cyclic graded $\mathbb{Q}_p$-modules, $\bigoplus_{n,t\in Z^+} W_{t,n}^*$, $W_{t,n}^{2n} = V_{t,n}^{2n}/V_{t,n}^{2n+1} \cong V_{t,n}^{2n}$. We write $\xi_n^t \in W_{t,n}^{2n}$ for the image of ζ_n^t under the action of this isomorphism. We set $W_{t,n}^{2k} = 0$ for all $k \neq n$.

(iii) We define a Chern character ch : $V \to W \otimes \mathbb{Q}$ by $ch_n\ \zeta_n^t = \xi_n^t \otimes 1$ and $ch_{n+s}\ \zeta_n^t = 0$ for $s > 0$. We define ch universally by extending linearly.

3.3.5 Remarks: (i) V might be thought of as a direct sum of copies of $\tilde{K}(S^{2n}) \otimes \mathbb{Q}_p$.

(ii) Similarly, one might think of W as a direct sum of copies of $\tilde{H}\mathbb{Z}^{2n}(S^{2n}) \otimes \mathbb{Q}_p$.

(iii) The purely algebraic formulation of (3.3.4) is, of course, perfectly serviceable for our purposes. Nevertheless, the reader may find the geometric interpretation, given above, easier to work with and more elucidating, in view of the fact that the algebra was motivated by the geometry.

3.3.6 Definitions: (i) We extend our notion of splitting isomorphism in the obvious way $J : H^{ev}(X) \oplus W \to K^0(X) \oplus V$. We restrict our attention, however, for all of what follows, to the case where $J(\xi_n^t) = \zeta_n^t$ for "almost all" n and t. That is to say, we only consider splitting isomorphisms in which $J(\xi_n^t) = \zeta_n^t$, with only finitely many exceptions.

(ii) Let ξ_n^t be one of the finite number of exceptions in the above sense. We shall then say that J crosses ξ_n^t. The name was chosen to reflect the heuristic notion that ξ_n^t was allowed to "cross over the direct sum" under the action of J.

Before we proceed with the construction of our pyramid, we record the following lemma for later use:

3.3.7 Lemma: Let J be a splitting isomorphism for $X \in F_p$. Suppose further that $\{g_i\}$, $i \geq 0$ is a collection of linear maps,

$g_i : H^{2n}(X) \to H^{2n+2im}(X)$ with g_0 = the identity map. Then the $\{g_i\}$ determine (uniquely) another splitting, L, for X such that $J(u) = Lg_i(u)$ in exact filtration 2n+2im, for all non-zero $u \in H^{2n}(X)$, for all $i \geq 0$ and for all $n \geq 0$.

Proof: Given such a collection of maps we obtain an automorphism of filtered groups, $\sum_{i\geq 0} g_i : H^{ev}(X) \to H^{ev}(X)$ (see (3.2.9)). We define our new splitting, L, by $L = J(\sum_{i\geq 0} g_i)^{-1} : H^{ev}(X) \to K(X)$. □

Let us now suppose that we are given positive integers, n, N and q with $q \geq N > 0$. Suppose, moreover, that we are given a space, $X \in F_p$, for which at least one (q,r,s)-pair exists for all r and s in the range $N \geq r > s \geq 0$. With this fixed notation we proceed with the inductive construction of the pyramid of cohomology operations, $\{\Phi_q^{r,s}\}$, which acts on the $\mathbb{Z}_p$-cohomology of X.

3.3.8 Construction: (i) The first step is to construct the "base" of our pyramid. This will consist of primary operations of the form, $\Phi_q^{r,r-1}$, for $1 \leq r \leq N$, defined on vectors in the $\mathbb{Z}_p$-cohomology of X. By virtue of (3.3.2) and (3.2.4) we have:

$$\Phi_q^{r,r-1} = \sum_{i=0}^{r-1} \chi\, P^{q-i} : \bigoplus_{i=0}^{r-1} H\mathbb{Z}_p^{2n+2im}(X) \to H\mathbb{Z}_p^{2n+2qm}(X).$$

This is defined for any $\mathbb{Z}_p$-vector and has no indeterminacy.

3.3.8 - ii. The Second Order: Next, we explicitly calculate the second to bottom row of our pyramid, comprised of secondary operations of the form, $\Phi_q^{r,r-2}$, for $2 \leq r \leq N$. Let us pick an r in this range and consider the resulting operation of degree q, type r-2 and order 2.

This secondary operation will not be universally defined (as is true of all higher operations - see chapter II) but will have as domain some subset of $\bigoplus_{i=0}^{r-2} H\mathbb{Z}_p^{2n+2im}(X)$. This subset will consist of vectors, $\{x_i\}$, that lift to $\mathbb{Q}_p$-vectors, $\{u_i\}$ which form $(q,r,r-2)$-pairs with some splitting isomorphisms. That is, $\{x_i\}$ is in this subset if there exists a splitting J, such that $[J,\{u_i\}]$ is a $(q,r,r-2)$-pair and where $\{\rho_*' u_i\} = \{x_i\}$. This is evidently equivalent to the condition that $\{x_i\}$ lie in the kernel of the primary operation, $\Phi_q^{r-1,r-2}$.

Let us then choose a $\mathbb{Z}_p$-vector, $\{x_i\} \in \text{Ker } \Phi_q^{r-1,r-2}$. By definition we know that there exists at least one corresponding $(q,r,r-2)$-pair, $[J,\{u_i\}]$. Thus, we have $\rho_*'[(\sum_{i=0}^{r-2} \Theta_J^{q-i} u_i)/p^0] = 0 \in H\mathbb{Z}_p^{2n+2qm}(X)$. It follows that the sum within the square brackets is divisible by p within the context of $\mathbb{Q}_p$-cohomology. We define the secondary operation by the result of this division and, modulo the indeterminacy, $Q(\Phi_q^{r,r-2})$, we get:

$$\Phi_q^{r,r-2}\{x_i\} = \rho_*'[(\sum_{i=0}^{r-2} \Theta_J^{q-i} u_i)/p] \in H\mathbb{Z}_p^{2n+2qm}(X).$$

The indeterminacy is computed as follows. We note, firstly, that $Q(\Phi_q^{r,r-2})$ has two sources, the choice of $\mathbb{Q}_p$-lifting and the choice of splitting isomorphism. That is to say, our secondary operation is well defined only up to choice of $(q,r,r-2)$-pair. Let us suppose, then, that we have two $(q,r,r-2)$-pairs, both representing the same $\mathbb{Z}_p$-vector, $\{x_i\}$. We denote these two pairs by $[J,\{u_i\}]$ and $[L,\{v_i\}]$. Let us consider, firstly, the effect of choosing two different $\mathbb{Q}_p$-liftings. By assumption we know that $\rho_*' u_i = \rho_*' v_i = x_i$ for all $i \in [0,r-2]$. Thus, for all i in this range, we have:

$$(3.3.9) \qquad u_i - v_i = p w_i$$

for some $w_i \in H^{2n+2im}(X)$. Applying our homomorphism to this relation, we get:

$$(3.3.10) \qquad \sum_{i=0}^{r-2} \Theta_J^{q-i} u_i = \sum_{i=0}^{r-2} \Theta_J^{q-i} v_i + p \sum_{i=0}^{r-2} \Theta_J^{q-i} w_i.$$

Let us keep this in mind and turn our attention, for the moment, to the other source of indeterminacy, the choice of splitting isomorphism. Let L be another splitting isomorphism. Applying (3.2.9) to the right-hand side of (3.3.10) to change from J to L (here $X = Y$ and $\delta^* = \delta_0 =$ the identity map) and considering the result modulo p^2 gives:

$$(3.3.11) \qquad \sum_{i=0}^{r-2} \Theta_L^{q-i} v_i + p\sum_{i=0}^{r-2} \Theta_L^{q-i-1} \delta_1 v_i + p\sum_{i=0}^{r-2} \Theta_L^{q-i} w_i = \sum_{i=0}^{r-2} \Theta_J^{q-i} u_i .$$

This yields, modulo p^2:

$$(3.3.12) \quad \sum_{i=0}^{r-2} [\Theta_J^{q-i} u_i - \Theta_L^{q-i} v_i] = p[\sum_{i=0}^{r-2} \Theta_L^{q-i} w_i + \sum_{i=0}^{r-2} \Theta_L^{q-i-1} \delta_1 v_i].$$

Recalling that $[J,\{u_i\}]$ and $[L,\{v_i\}]$ were both assumed to be $(q,r,r-2)$-pairs, we see that the right-hand side of (3.3.12), when divided by and reduced modulo p, expresses the difference in $\Phi_q^{r,r-2}\{x_i\}$ when defined using different pairs. That is to say

$$(3.3.13) \quad Q(\Phi_q^{r,r-2}) = \operatorname{Im} \rho_*'[\sum_{i=0}^{r-2} \Theta_L^{q-i}(-) + \sum_{i=0}^{r-2} \Theta_L^{q-i-1}(-)] = \operatorname{Im} \Phi_q^{r,r-1},$$

where L ranges over all possible splittings and, consequently, where $\delta_1 v_i$ ranges over all values in $H^{2n+2(i+1)m}(X)$.

Consequently we may conclude that:

$$(3.3.14) \quad \Phi_q^{r,r-2}: \operatorname{Ker} \Phi_q^{r-1,r-2} \subseteq \oplus HZ_p^{2n+2im}(X) \to \operatorname{Cok} \Phi_q^{r,r-1} = HZ_p^{2n+2qm}(X)/Q(\Phi_q^{r,r-2}).$$

<u>3.3.8 *iii. Intermezzo*</u>: The fact that a primary operation is fully determinate is reflected in the relative simplicity of the definitions of first and second order operations. For orders three and higher, the situation complicates somewhat. In order to deal properly with this increased complexity we must have a rather tighter grasp on the indeterminacy. The following series of lemmas and propositions is dedicated to this aim. By way of preparation we fix some notation: as always $X \in F_p$; $\widetilde{W}^* = W^* \otimes \mathbb{Z}_p$; we shall change our notation and

write ξ_i^t in place of ξ_{n+im}^t: x_i and y_i will always be elements in $HZ_p^{2n+2im}(X) \oplus \widetilde{W}^{2n+2im}$ where $\widetilde{W}^{2n}$ denotes $\bigoplus_{t \in Z^+} \widetilde{W}_t^{2n}$; u_i, v_i, w_i and z_i will lie in $H^{2n+2im}(X) \oplus W^{2n+2im}$; $\widetilde{x}_i$ denotes the coset of x_i in $HZ_p^{2n+2im}(X) \oplus \widetilde{W}^{2n+2im}/\widetilde{W}^{2n+2im}$; η_i and γ_i will always be elements of W^{2n+2im}; J, K, L and T will denote splittings.

The next several results will take place in the context of $H^*(X) \oplus W^*$.

3.3.15 Lemma: Suppose we are given a non-zero class, mod p, $u_0 \in H^{2n}(X)$ together with a splitting, J. Suppose, moreover, that we are given a collection of s vectors each of length r, $\{v_i^j\}$, together with s splittings, K^j. Say $v_i^j \in H^{2n+2im}(X)$ for $1 \leq i \leq r$ and $1 \leq j \leq s$. Then, there exists a splitting, L, such that

$$\Theta_L^q u_0 = \Theta_J^q u_0 + \sum_{j=1}^{s} \sum_{i=1}^{r} p^i \Theta_{K^j}^{q-i} v_i^j.$$

Proof: By (3.3.7) we see that it is sufficient to determine the maps, δ_i, which will, in turn, yield a splitting, L, such that $L = \sum_{i \geq 0} J \delta_i$. We define $\delta_i(u_0)$ to be $J^{-1} \sum_{j=1}^{s} K^j v_i^j$, for $1 \leq i \leq r$ and $\delta_i(u_0) = 0$, for $i \geq r+1$. Linearity then defines δ_i on all multiples of u_0. We now set $\delta_i = 0$ on all elements which are linearly independent of u_0, for all $i \geq 1$. This defines an L with the desired properties. □

3.3.16 Lemma: Suppose we are given a 2n-dimensional class, u_0, together with two splittings, L and J. Then there exists a class, v_1, together with a splitting, K, such that $Ju_0 = Ku_0$ and such that

$$\Theta_L^q u_0 = \Theta_J^q u_0 + p\, \Theta_K^{q-1} v_1.$$

Proof: This follows from (3.2.9) and (3.3.7). Writing $J^{-1}L$ as $\Sigma \delta_i$ and $K^{-1}J$ as Σg_i we set $v_1 = \delta_1 u_0 + \xi_1^t$. Now define $g_i(v_1) = -K^{-1}J\delta_{i+1}u_0$, for all $i \geq 1$. Linearity now defines g_i on multiples of v_1. Set $g_i = 0$, for $i \geq 1$, elsewhere. This defines a K for us and gives the result. □

3.3.17 Proposition: Let $\{\eta_i\}$, $-1 \le i \le r$, be given together with a splitting L. Then, there exists a vector $\{u_i\}$, $0 \le i \le r+1$, and a splitting, K, such that:

$$(3.3.18) \quad \sum_{i=-1}^{r} \Theta_L^{q-i} \eta_i = p \sum_{i=0}^{r+1} \Theta_K^{q-i} u_i .$$

Proof: Choose a J such that it does not cross η_{i-1} for all appropriate i. Then by (3.3.16) we have $\Theta_L^{q-i+1}\eta_{i-1} = p\Theta_K^{q-i}u_i + \Theta_J^{q-i+1}\eta_{i-1}$ for each $i \in [0,r+1]$. This gives the result. □

Now we establish the converse to (3.3.17):

3.3.19 Proposition: Let $\{u_i\}$, $0 \le i \le r+1$, be given together with a splitting,K. Then, there exists a $\{\eta_i\}$, $-1 \le i \le r$ and an L such that (3.3.18) holds.

Proof: Choose a J and a vector $\{\eta_i\}$, $-1 \le i \le r$, such that none of the η_i are crossed by J nor are any $\eta_i \equiv 0 \pmod p$. Now we may apply (3.3.15) to get the result. □

3.3.20 Proposition: Let $\{x_i\}$ and $\{y_i\}$, $0 \le i \le r$, be given such that $x_i - y_i \in \widetilde{W}^{2n+2im}$ for each appropriate i. Let $\{u_i\}$ and $\{v_i\}$ be $\mathbb{Q}_p$-representatives of $\{x_i\}$ and $\{y_i\}$, respectively.
Suppose, moreover, that we are given two splittings, J and K. Then, there exists a vector $\{\gamma_i\}$, $-1 \le i \le r$, and a splitting,L, such that:

$$(3.3.21) \quad \sum_{i=-1}^{r} \Theta_L^{q-i}\gamma_i = \sum_{i=0}^{r} [\Theta_J^{q-i}u_i - \Theta_K^{q-i}v_i].$$

Proof: By (3.3.16) we have $\Theta_J^{q-i}u_i - \Theta_K^{q-i}u_i = p\, \Theta_{L_i}^{q-i-1} z_{i+1}$ for each $i \in [0,r]$. Moreover, by hypothesis, we know that $u_i - v_i = \eta_i + p\omega_i$ for some appropriate η_i, ω_i where $i \in [0,r]$. Consequenlty, we have for each $i \in [0,r]$:

$$[\Theta_J^{q-i}u_i - \Theta_K^{q-i}v_i] = \Theta_K^{q-i}\eta_i + p \quad \Theta_K^{q-i}\omega_i + p \quad \Theta_{L_i}^{q-i-1} z_{i+1}.$$

Let $L = \oplus L_i$, for all $i \in [0,r]$.

The resulting L is a splitting because the pieces can be made to be linearly independent, if they are not already so, by adding appropriate elements from $\widetilde{W}^*$. Let us choose a vector, $\{\xi_i^t\}$, $-1 \leq i \leq r$ such that K does not cross it. Then, suppressing the t's and taking sums we get from our above expression:

$$\sum_{i=0}^{r} [\Theta_J^{q-i} u_i - \Theta_K^{q-i} v_i] = \Theta_K^{q+1}\xi_{-1} + p\Theta_K^{q} w_0 + \sum_{i=0}^{r-1} [\Theta_K^{q-i}(\eta_i+\xi_i) + p\Theta_K^{q-i-1} w_{i+1} + p\Theta_L^{q-i-1} z_{i+1}]$$
$$+\Theta_K^{q-r}(\eta_r+\xi_r) + p\Theta_L^{q-r-1} z_{r+1}.$$

Writing $\gamma_{-1} = \xi_{-1}$ and $\gamma_i = \eta_i+\xi_i$ for $0 \leq i \leq r$ gives the result as a consequence of (3.3.15). □

Conversely, we have:

3.3.22 Proposition: Let $\{x_i\}$, $0 \leq i \leq r$, be given together with a $\mathbb{Q}_p$-representative, $\{u_i\}$, and a splitting, J. Suppose, moreover, that we are given a vector $\{\gamma_i\}$, $-1 \leq i \leq r$, and a splitting L. Then, there exists a $\{y_i\}$, $0 \leq i \leq r$, with a representative, $\{v_i\}$ and a splitting, K, such that $x_i - y_i \in \widetilde{W}^{2n+2im}$ and such that (3.3.21) holds.

Proof: Choosing a vector, $\{\xi_i^t\}$, $0 \leq i \leq r$ such that J does not cross it and applying (3.3.17) we may write:

$$\sum_{i=-1}^{r} \Theta_L^{q-i}\gamma_i - \sum_{i=0}^{r} \Theta_J^{q-i} u_i = p \sum_{i=0}^{r+1} \Theta_T^{q-i} z_i - \sum_{i=0}^{r} \Theta_J^{q-i}(u_i - \xi_i).$$

By (3.3.15), this equals $\sum_{i=0}^{r} \Theta_K^{q-i}(\xi_i - u_i)$ for some splitting, K. Setting $v_i = -(\xi_i - u_i)$ gives the result. □

3.3.23 Remark: What (3.3.20) and (3.3.22) tells us is that given a coset, $\{\widetilde{x}_i\}$, $0 \leq i \leq r$, then there exists a vector, $\{\gamma_i\}$, $-1 \leq i \leq r$, together with a splitting, L, such that

$$(3.3.24') \qquad \sum_{i=0}^{r} [\Theta_J^{q-i} u_i - \Theta_K^{q-i} v_i] = \sum_{i=-1}^{r} \Theta_L^{q-i}\gamma_i,$$

where $\{u_i\}$ and $\{v_i\}$ are any two $\mathbb{Q}_p$-liftings of $\mathbf{Z}_p$-vectors, $\{x_i\}$ and

$\{y_i\}$, representing $\{\tilde{x}_i\}$ and where J and K are any two splittings. Moreover, given any such $\{\gamma_i\}$ and L there exist J and K together with $\{u_i\}$ and $\{v_i\}$, two $\mathbb{Q}_p$-liftings of representatives of $\{\tilde{x}_i\}$, such that (3.3.24') holds.

The two earlier propositions, (3.3.17) and (3.3.19) tell us that given an L and a vector, $\{\gamma_i\}$, $-1 \leq i \leq r$, there is a K and a vector, $\{z_i\}$, $0 \leq i \leq r+1$, such that

$$(3.3.24'') \qquad \sum_{i=-1}^{r} \Theta_L^{q-i}\gamma_i = p \sum_{i=0}^{r+1} \Theta_K^{q-i} z_i ,$$

and conversely. Consequently, beginning with a coset, $\{\tilde{x}_i\}$, $0 \leq i \leq r$, in

$$\bigoplus_{i=0}^{r} [HZ_p^{2n+2im}(X) \oplus \tilde{W}^{2n+2im}] \,/\, \bigoplus_{i=0}^{r} \tilde{W}^{2n+2im} \cong \bigoplus_{i=0}^{r} HZ_p^{2n+2im}(X),$$

we may conclude that

$$(3.3.24) \qquad \sum_{i=0}^{r} [\Theta_J^{q-i}u_i - \Theta_K^{q-i}v_i] = p \sum_{i=0}^{r+1} \Theta_T^{q-i} z_i ,$$

where $\{u_i\}$ and $\{v_i\}$ are $\mathbb{Q}_p$-liftings of $\mathbb{Z}_p$-vectors representing $\{\tilde{x}_i\}$ and where any two of the pairs: $[J,\{u_i\}]$, $[K,\{v_i\}]$, $[T,\{z_i\}]$ determine the third. Now we are in a position to keep track of the indeterminacy in a rigorous fashion and, consequently, may proceed with our construction of the higher order operations in the pyramid, $\{\Phi_q^{r,s}\}$.

<u>3.3.8 *iv. The Higher Orders*</u>: Now we shall proceed with our inductive definition of a pyramid of higher order operations. We shall define a linear homomorphism, $\Phi_q^{r,s}$, on the kernel of $\Phi_q^{r-1,s}$, a subgroup of $\bigoplus_{i=0}^{s} HZ_p^{2n+2im}(X) \cong \bigoplus_{i=0}^{s} [HZ_p^{2n+2im}(X) \oplus \tilde{W}^{2n+2im}]/\bigoplus_{i=0}^{s} \tilde{W}^{2n+2im}$. This last identification will be tacitly assumed in what follows and we shall not explicitly distinguish between a vector, $\{x_i\} \in \bigoplus_{i=0}^{s} HZ_p^{2n+2im}(X)$ and its associated coset, $\{\tilde{x}_i\}$. When con-

sidering ranges, we shall identify $HZ_p^*(X)$ with the obvious subgroup of $HZ_p^*(X) \oplus \widetilde{W}^*$. With these identifications in mind we shall define an operation of degree, q, and order, (r-s):

$$(3.3.25) \quad \Phi_q^{r,s} : \text{Ker}\ \Phi_q^{r-1,s} \subseteq \bigoplus_{i=0}^{s} HZ_p^{2n+2im}(X) \to \text{Cok}\ \Phi_q^{r,s+1} = $$
$$= HZ_p^{2n+2qm}(X)/\text{Im}\ \Phi_q^{r,s+1}.$$

Orders one and two have already been defined and clearly possess, together with (3.3.25), the following properties:

(3.3.26) $\quad \text{Ker}\ \Phi_q^{r,s} \subseteq \text{Ker}\ \Phi_q^{r-1,s}$,

(3.3.27) $\quad \text{Im}\ \Phi_q^{r,s} \supseteq \text{Im}\ \Phi_q^{r,s+1}$ (the precise interpretation of this statement will be given below) and

(3.3.28) Let $\{\widetilde{x}_i\}$ be a coset upon which $\Phi_q^{r,s}$ is defined. Then there exists a $\mathbb{Z}_p$-vector, $\{x_i\}$ representing $\{\widetilde{x}_i\}$, which in turn has a $\mathbb{Q}_p$-representative $\{u_i\}$, together with a splitting, J, such that $[J,\{u_i\}]$ forms a (q,r,s)-pair. Modulo its indeterminacy, the value of this $(r-s)^{th}$ order operation is given by:

$$\Phi_q^{r,s}\{x_i\} = \rho_*'[\sum_{i=0}^{s} \Theta_J^{q-i}\ u_i\ /\ p^{r-s-1}].$$

3.3.29 Remark: For the primary case, the above conditions are either vacuous or trivial. For order two, (3.3.25) and (3.3.26) follow directly from the construction. We turn our attention to the remaining two conditions. Property (3.3.28) follows from proposition (3.3.22). Heuristically speaking, (3.3.22) tells us that our indeterminacy is fully realisable, that given an element of a coset, we may "realise" that element by an explicit pair, $[J,\{u_i\}]$.

Property (3.3.27) is to be interpreted in the following strong sense. Given a cohomology class, z, in the image of $\Phi_q^{r,s+1}$ coming from a particular (q,r,s+1)-pair, $[J,\{u_i\}]$, there exists a (q,r,s)-pair,

$[L,\{v_i\}]$ such that $\Phi_q^{r,s}$ defined using L and $\{v_i\}$ gives precisely the same cohomology class, z. We see this as follows:

Let $z = \rho'_*[\sum_{i=0}^{r-1} \Theta_J^{q-i} u_i]$. That is $z \in H\mathbb{Z}_p^{2n+2qm}(X)$ is in the image of the primary operation, $\Phi_q^{r,r-1}$. Now we may write:

$$(3.3.30)\quad p\sum_{i=0}^{r-1} \Theta_J^{q-i} u_i = \sum_{i=0}^{r-3} \Theta_J^{q-i}(pu_i) + \Theta_J^{q-r+2}(pu_{r-2}+\xi_{r-2}^t) + p\Theta_J^{q-r+1}(u_{r-1}),$$

where ξ_{r-2}^t was chosen so that J does not cross it. Applying (3.3.15) to the last two terms of (3.3.30) and defining a vector $\{v_i\}$, $0 \leq i \leq r-2$, by $v_i = pu_i$ for $0 \leq i \leq r-3$ and $v_{r-2} = pu_{r-2}+\xi_{r-2}^t$ gives:

$$\sum_{i=0}^{r-1} \Theta_J^{q-i} u_i = {}^1/_p[\sum_{i=0}^{r-2} \Theta_L^{q-i} v_i],$$

where L = J in the first r-3 dimensions and is determinedby (3.3.15) in the top dimension. This, of course, implies (3.3.27) for secondary operations.

Lastly, we check that our second order operations are, in fact, linear (the primary case is, again, trivial). Suppose we are given two vectors $\{x_i\}$ and $\{y_i\}$, both in the kernel of $\Phi_q^{r,r-1}$. Let $\{u_i\}$ and $\{v_i\}$ be $\mathbb{Q}_p$- representatives and let J and K be splittings. We may assume that u_i and v_i are linearly independent for each $i \in [0,r-1]$, for if not we may replace u_i by $u_i+\xi_i^t$, where J does not cross ξ_i^t. So, we may define a new splitting, L, that agrees with J on the subspace spanned by u_i and that agrees with K on the subspace spanned by v_i. Now we may simply write:

$$\sum_{i=0}^{r-1} \Theta_J^{q-i} u_i + \sum_{i=0}^{r-1} \Theta_K^{q-i} v_i = \sum_{i=0}^{r-1} \Theta_L^{q-i}(u_i+v_i).$$

Now that we have seen that our primary and secondary operations are, indeed, linear homomorphisms which satisfy conditions (3.3.25 - 28) we may proceed with the inductive step.

Let us assume that all the operations of order (b-1) and lower in our pyramid have been defined and satisfy the above conditions. We shall define a typical entry in the b^{th} row, $\Phi_q^{r,r-b}$, as follows:

Let $\{x_i\}$ be an element of Ker $\Phi_q^{r-1,r-b}$. By property (3.3.28), which is assumed to hold by inductive hypothesis, we know that there exists a vector, $\{u_i\}$, and a J such that $\rho'_*[\sum_{i=0}^{r-b} \Theta_J^q u_i/p^{b-2}]$ is in the zero coset.

By (3.3.27) and (3.3.19) we have, mod p:

$$(3.3.31) \quad (\sum_{i=0}^{r-b} \Theta_J^{q-i} u_i)/p^{b-2} = (\sum_{i=0}^{r-b+1} \Theta_K^{q-i} z_i)/p^{b-3} = (p\sum_{i=0}^{r-b+1} \Theta_K^{q-i} z_i)/p^{b-2}$$
$$= (\sum_{i=-1}^{r-b} \Theta_L^{q-i} \eta_i)/p^{b-2}.$$

Now by (3.3.22) we have, mod p:

$$(3.3.32) \quad (\sum_{i=0}^{r-b} \Theta_J^{q-i} u_i - \sum_{i=-1}^{r-b} \Theta_L^{q-i} \eta_i)/p^{b-2} = (\sum_{i=0}^{r-b} \Theta_T^{q-i} \nu_i)/p^{b-2}.$$

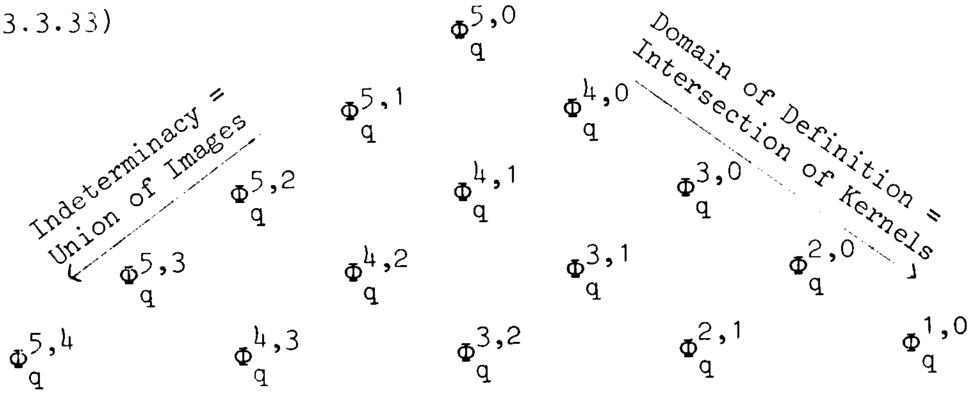
Thus, $\sum_{i=0}^{r-b} \Theta_T^{q-i} \nu_i = 0 \bmod p^{b-1}$. Moreover, $\{\nu_i\}$ is also a representative for $\{\tilde{x}_i\}$ by (3.3.22). We define, consequently, our b^{th} order operation as follows:

$$\Phi_q^{r,r-b}\{x_i\} = \rho'_*[\sum_{i=0}^{r-b} \Theta_T^{q-i} \nu_i/p^{b-1}]/Q(\Phi_q^{r,r-b}).$$

That $Q(\Phi_q^{r,r-b}) = \text{Im}\ \Phi_q^{r,r-b+1}$ follows from (3.3.24) and so condition (3.3.25) is satisfied. Conditions (3.3.26) and (3.3.28) clearly hold and (3.3.27) is seen to be satisfied by a proof completely parallel with that given in (3.3.29). The fact that this is a well-defined linear homomorphism follows as above. This completes the inductive construction of our pyramid of operations, $\{\Phi_q^{r,s}\}$.

Pictorically, we may represent this pyramid, where N = 5, as:

(3.3.33)



3.3.34 Remarks: (i) Diagram (3.3.33) represents a pyramid of higher order operations, generated by the pseudo primary operation, Θ_J^q. Here we have taken $q \geq N = 5$ and allowed r and s to run over all values in the range $5 \geq r > s \geq 0$. By (3.3.8) we see that every operation is defined on the kernel of the operation one step down and to the right of it and has indeterminacy determined by the image of the operation, one step down and to the left.

For example, let us consider $\Phi_q^{5,1}$. It is clear that this operation is defined on $\bigcap_{i=0}^{2} \text{Ker } \Phi_q^{4-i,1} = \text{Ker } \Phi_q^{4,1}$ (nested inclusion) and takes values in the coset space defined by the $HZ_p^{2n+2qm}(X)/\text{Im } \Phi_q^{5,2} = HZ_p^{2n+2qm}(X)/\bigcup_{i=0}^{2} \text{Im } \Phi_q^{5,i+2}$.

Notice that, by virtue of (3.3.27), the indeterminacy of $\Phi_q^{N,M}$ may be expressed in terms of the image of a single operation, namely $\Phi_q^{N,M+1}$.

(ii) The operations in pyramid (3.3.33) are natural and stable under double suspension, as we shall see below:

3.3.35 Lemma: Let $f : Y \to X$ be a morphism of spaces in F_p. Let $\{u_i\}$ be a $\mathbb{Z}_p$-cohomology vector on which $\Phi_q^{r,s}$ is defined. Then: $f^*\Phi_q^{r,s}\{u_i\} = \Phi_q^{r,s}(f^*\{u_i\})$ in $HZ_p^{2n+2qm}(Y)/(\text{indeterminacy})$.

Proof: We shall proceed by induction on the order of the operation, (r-s). When (r-s) = 1, the lemma is obvious. Let us assume that the result is true for all such operations of order up to and including t. Pick an r and an s such that r-s = t+1. By (3.3.8) we have:

(i) $$\delta^*\Phi_q^{r,s} : \text{Ker } \delta^*\Phi_q^{r-1,s} \to \text{Cok } \delta^*\Phi_q^{r,s+1} \quad \text{and}$$

(ii) $$\Phi_q^{r,s}\delta^* : \text{Ker } \Phi_q^{r-1,s}\delta^* \to \text{Cok } \Phi_q^{r,s+1}\delta^*.$$

But $\Phi_q^{r-1,s}$ and $\Phi_q^{r,s+1}$ are both of order t and, consequently, (i) and (ii) have the same domain and range, by inductive hypothesis. Thus, we know that $\delta^*\Phi_q^{r,s}$ and $\Phi_q^{r,s}\delta^*$ are defined under the same circumstances and have a common value of indeterminacy. That these two operations are, in fact, equal follows from applying (3.2.9) and noticing that the difference of the operations, $\delta^*\Phi_q^{r,s} - \Phi_q^{r,s}\delta^*$, is contained in the common value of the indeterminacy of these two operations. This completes the proof. □

3.3.36 Lemma: Let σ^* be the double suspension isomorphism:

$$\sigma^* : H\mathbb{Z}_p^{2n}(X) \to H\mathbb{Z}_p^{2n+2}(S^2 \wedge X).$$

Then, with $\{\tilde{u}_i\}$ as in (3.3.35), we have:

$$\sigma^*\Phi_q^{r,s}\{\tilde{u}_i\} = \Phi_q^{r,s}\{\sigma^*\tilde{u}_i\} \text{ in } H\mathbb{Z}_p^{2n+2qm+2}(S^2 \wedge X),$$

modulo the indeterminacy. (We require that $\Phi_q^{r,s}$ be defined in $\{\sigma^*\tilde{u}_i\}$.)

Proof: That domains and ranges coincide follows by an argument similar to that in (3.3. 35). The last step is obvious. That we used the double suspension in place of the single suspension, reflects the fact that we have defined our operation on odd graded cohomology by identifying $H^{2l+1}(X)$ with $H^{2l+2}(S^1 \wedge X)$. □

Let us consider the chain complex, $C(N,q)$, in the sense of [27] and [28], defined as follows:

Pick a q and an N such that $q \geq N$. Each left A_p-module, C_n $(0 \leq n \leq N)$ is generated by elements $\{c_n;\ c_{n,0};\ c_{n,1};\ldots;c_{n,n}\}$ where the dimension of c_n is $2qm+n-1$ and the dimension of $c_{n,i}$ is $n+2(n-i)m$ for $0 \leq i \leq n$. Define the differentials of $C(N,q)$ by:

(i) $d_n(c_n) = -\beta(c_{n-1}) + \sum_{i=0}^{n-1} \chi P^{q-i}(c_{n-1,n-1-i})$

(ii) $d_n(c_{n,i}) = \beta(c_{n-i,i-1}) + \tau(c_{n-1,i})$.

Here, β denotes the Bockstein homomorphism and τ is $P^1\beta-\beta P^1$ (see [30]). Let $\{\hat{\Phi}^{r,s}\}$, $N \geq r > s \geq 0$, denote the pyramid of operations associated with $C(N,q)$, the mod p version of the pyramid condiered in (2.4.14), above and [28], where it is restricted to the category F_p.

3.3.37 Theorem: Under these hypotheses, $\hat{\Phi}^{r,s} = \Phi_q^{r,s}$, for all r and s in the range $N \geq r > s \geq 0$, modulo indeterminacy.

Proof: That the chain complex, $C(N,q)$ is admissible follows directly from the following four Adem relations and the definitions of the differentials:

(1) $\chi P^q\beta - \beta\chi P^q + \chi P^{q-1}\tau = 0$,

(2) $\tau\tau = 0$,

(3) $\beta\beta = 0$,

(4) $\beta\tau + \tau\beta = 0$.

We point out that $C(N,q)$ becomes the complex of [28] when $p = 2$. (See (2.4.14), above.) Let us remain in this case, momentarily.

We proceed with the proof by induction. In the category, F_p, it is clear that $\hat{\Phi}^{1,0}(\varepsilon) = \chi Sq^{2q}(\varepsilon) = \Phi_q^{1,0}(\varepsilon)$ by (3.2.4). Thus the two primary operations agree. Let us assume, now, that all operations of order less than or equal to s, coincide for some $s \geq 2$. By axioms 1 & 2 of [27] and by (3.3.8), above, we have agreement of domain of definition and image space of $\hat{\Phi}^{s+1,0}$ and $\Phi_q^{s+1,0}$. That these two operations actually agree, follows from (2.4.14) and (3.3.8). That is to say the image of both is precisely the Chern character, divided by 2^s.

If $\hat{\Phi}^{s+1,0} = \Phi_q^{s+1,0}$, we may also conclude that any 2 such (s+1)-order operations coincide by virtue of axiom 0 of [27] and the definition of $\hat{\Phi}^{s+1+a,a}$. This follows simply by re-labelling the modules in C(N,q). This completes the proof for p = 2.

The mod p case, for p odd, is shown in precisely the same way. Here, in place of (2.4.14), we use the mod p version of this theorem. This states that the image of the n^{th} order operation, $\hat{\Phi}^{n,0}$ is just the relevant Chern character divided by p^{n-1} (see §1 of [28]). This is proved in a fashion parallel to that of (2.4.14) (see [28]) but with the mod-p split BU spectrum (see (3.2.1), above).

This completes the proof. □

Proof: This follows directly from (3.2.10), above, the definition, (3.4.1), and the fact that S_J^q and Q_J^q were defined to be formal inverses. See [22] and [24]. □

3.4.5 Corollary: Let f, J, L, X, and Y be as in (3.2.9). Then we have the following formula for the deviation from naturality of our dual pseudo primary operation, $\overline{\Theta}_J^q$:

$$\overline{\Theta}_L^q f^* = \sum_{i=0}^{q} p^{q-i} f_{q-i}\, \overline{\Theta}_J^i : H^{2n}(X) \to H^{2n+2qm}(Y).$$

Let us consider X and Y ∈ F_p which are 2N dual to one another in the sense of [35]. Let J be a splitting isomorphism for $\tilde{H}^{ev}(X)$ and define $\mathcal{J}$, a splitting isomorphism for $\tilde{H}^{ev}(Y)$, by the expression $(Ja,c)_K = (a,\mathcal{J}^{-1}c)$ where (,) is a pairing coming from the duality in cohomology and where $(\ ,\ )_K$ is the corresponding pairing in K-theory (see [37] and [25]). Here we have chosen elements $a \in \tilde{H}^{ev}(X)$ and $c \in \tilde{K}(Y)$. Let us fix, in addition, the element $b \in \tilde{H}^{ev}(Y)$. Let χ be defined by the duality pairing: $(\chi\, \Theta_{\tilde{J}}^q\, b,a) = (b,\ \Theta_J^q\, a)$, with a,b,c in suitable dimensions.

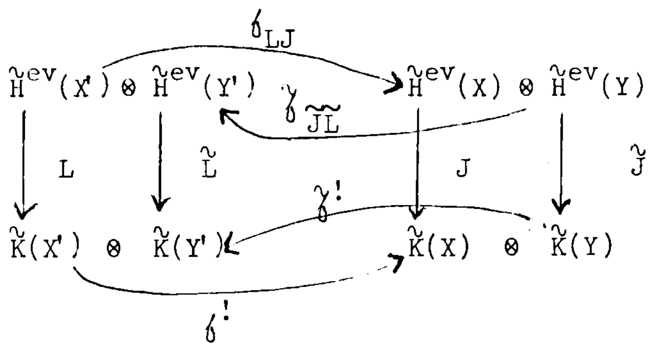
3.4.6 Theorem: With these hypotheses, we have

$$\overline{\Theta}_J^q = \chi\Theta_J^q \quad \text{and} \quad \Theta_J^q = \chi\overline{\Theta}_J^q.$$

Proof: We shall consider the primary cohomology operation, $(\overline{\Theta}_J^q - \chi\Theta_J^q)$ and show that it must be the zero operation for $q \geq 0$. As we shall see, this is actually a "proper" cohomology operation and not pseudo in the sense of (3.2.3) or (3.4.1), as it is natural with respect to morphisms in F_p.

Let L and $\tilde{L}$ (related to one another as J and $\mathcal{J}$ were) be another "pair" of splittings and let X' and Y' be another 2N dual pair of spaces in F_p. Consider the diagram:

(3.4.7)



Here, we have simply taken two copies of (3.2.5) and pieced them together.

Let a', b' and c' be elements of $\widetilde{H}^{ev}(X')$, $\widetilde{H}^{ev}(Y')$ and $\widetilde{K}(Y')$ respectively (again in suitable dimensions). Then, by construction we have:

(3.4.8) $(\tilde{\delta}_i a', b) = (a', \delta_i b)$ where

$$\delta_{LJ} = \Sigma \delta_i \quad \text{and} \quad \gamma_{\widetilde{JL}} = \Sigma \gamma_i .$$

We shall proceed with the proof by induction on the degree of our "operation", $(\bar{\Theta}^q_J - \chi\Theta^q_J)$. When $q = 0$, this is just the identity minus itself, the zero operation. As inductive hypothesis, let us suppose that $(\bar{\Theta}^s_J - \chi\Theta^s_J) = 0$ for all $s \leq q-1$.

Let us consider the effects of a change of splitting isomorphism for this "operation". That is, we look at:

(3.4.9) $$\delta^*(\bar{\Theta}^q_L - \chi\Theta^q_L) = (\bar{\Theta}^q_J\delta^* - p\delta_1\bar{\Theta}^{q-1}_L - p^2\delta_2\bar{\Theta}^{q-2}_L - \dots) - (\delta^*\chi\Theta^q_L)$$

We evaluate the right-most term of (3.4.9) as follows.

By construction and (3.4.8) we know that

$(\delta^*\chi\,\Theta^q_L a', b) = (\chi\,\Theta^q_L a', \gamma^* b) = (a', \Theta^q_{\tilde{L}}\gamma^* b) =$

$(a', (\gamma^*\Theta^q_{\tilde{J}} - p\Theta^{q-1}_{\tilde{L}}\gamma_1 - p^2\Theta^{q-2}_{\tilde{L}}\gamma_2 - \dots)(b)) =$

$((\chi\,\Theta^q_J\delta^* - p\delta_1\chi\,\Theta^{q-1}_L - p^2\delta_2\chi\,\Theta^{q-2}_L - \dots)(a'), b).$

Consequently, the right-most term of (3.4.9) equals

(3.4.10) $$-(\chi\,\Theta^q_J\delta^* - p\delta_1\chi\,\Theta^{q-1}_L - p^2\delta_2\chi\,\Theta^{q-2}_L - \dots).$$

Now, using the inductive hypothesis, we combine (3.4.10) and (3.4.9) to get:

(3.4.11) $$f^*(\bar{\Theta}_L^q - \chi\,\Theta_L^q) = (\bar{\Theta}_J^q - \chi\,\Theta_J^q)f^*.$$

It follows, then, from (3.4.11) and its construction that $(\bar{\Theta}_J^q - \chi\,\Theta_J^q)$ is, in fact, a stable, natural (and thus "proper") primary cohomology operation on the $\mathbb{Q}_p$-cohomology of spaces in $\mathcal{F}_p$. We claim that these characteristics are sufficient to show that such an operation must be the zero operation. We see this as follows.

Firstly, let us consider the action of any stable, natural primary cohomology operation, α, on $H^*(\mathbb{C}P^\infty)$. Let f be a map from $\mathbb{C}P^\infty$ to itself of any degree $k > 1$. Consider the following diagram, where all coefficients are in $\mathbb{Q}_p$:

(3.4.12)
$$\begin{array}{ccc} H^*(\mathbb{C}P^\infty) & \xrightarrow{\ \alpha\ } & H^*(\mathbb{C}P^\infty) \\ \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} \\ H^*(\mathbb{C}P^\infty) & \xrightarrow{\ \alpha\ } & H^*(\mathbb{C}P^\infty) \end{array}$$

It is evident that the commutativity of diagram (3.4.12) under the above hypotheses implies that α must be the zero operation on $H^*(\mathbb{C}P^\infty)$.

From this fact one may deduce that the action of α on the $\mathbb{Q}_p$ cohomology of the n-fold cartesian product of $\mathbb{C}P^\infty$, $(\mathbb{C}P^\infty)^n$, is also zero. Next we deduce that α is zero on $H^*(BU(n))$ by using the naturality of α with respect to the inclusion map of $(\mathbb{C}P^\infty)^n$ in $BU(n)$. Passing to limits, finally, gives us that our operation acts trivially on $H^*(BU)$. To pass from here to all spaces in the category $\mathcal{F}_p$, we consider the following:

(3.4.13)
$$\begin{array}{ccc} H^*(X) & \xleftarrow{\ g^*\ } & H^*(BU) \\ \uparrow{\scriptstyle ch} & & \uparrow{\scriptstyle ch} \\ K(X) & \xleftarrow{\ g^!\ } & K(BU) \end{array}$$

Consider any $X \in F_p$ and let y be an element in $H^{2n}(X)$. Since X is p-torsion free and our coefficients are p-local, we may conclude that there is some element, x, in $K(X)$ such that $p^n\, ch_n(x) = p^n y$ (that is to say there is a splitting isomorphism, J, such that $J(y) = x$.)

Let us assume that $x \in K(X)$ is represented by some map, $g : X \to BU$. It is, then, clear from this construction that if ι denotes the identity element in $K(BU)$, then $g^!(\iota) = x$. Let $p^n\, ch_n(\iota)$ be some element $u \in H^{2n}(BU)$. Then clearly, $g^*(u) = p^n y$ by commutativity of (3.4.13). Thus, for any $X \in F_p$ and any $z \in H^{2n}(X)$ (any $n \in \mathbb{Z}^+$) we have a $v \in H^{2n}(BU)$ such that $g^*(v) = p^n z$ and, consequently:

(3.4.14) $\quad \alpha(p^n z) = p^n \alpha(z) = \alpha(g^*(v)) = g^*\alpha(v) = 0.$

It follows that any natural, stable, primary operation on $H^*(F_p)$ and, in particular $(\overline{\Theta}{}^q_J - \chi\Theta^q_J)$, must be identically zero. The result follows. □

As with Θ^q_J, the dual pseudo primary operation also generates a system of higher order "genuine" cohomology operations over $H\mathbb{Z}_p^*$. This system, however, behaves rather differently from the pyramids of (3.3.8). At the heart of this difference lie the two formulae (3.4.5) and (3.2.9). This seemingly minor contrast gives rise to some fundamental differences in behaviour. Let us begin by establishing some notation:

<u>*3.4.15 Definitions*</u>: (i) Let $\bigoplus_{i=0}^{M} \overline{\Theta}{}^{q-i}_J$ be a direct sum of dual pseudo primary operations defined by (3.4.1). We shall say that such

a sum is of degree q and of type $M \leq q$. In contrast to $\sum_{i=0}^{M} \Theta_J^{q-i}$, our dual "multi-operation" is defined on a single element in $H^{2n}(X)$ for $X \in F_p$ and for some $n \in \mathbb{Z}^+$. Let u be such an element. Then this dual multi-operation of degree q and type M has, as value, a vector of elements in $\bigoplus_{i=0}^{M} H^{2n+2(q-i)m}(X)$.

(ii) Let $q \geq N > M \geq 0$. Let u be any $\mathbb{Q}_p$-cohomology element in $H^{2n}(X)$ such that: $\bigoplus_{i=0}^{M} \overline{\Theta}_J^{q-i}(u) \equiv 0 \bmod p^{N-1-M}$, $\bigoplus_{i=0}^{M} \overline{\Theta}_J^{q-1-i}(u) \equiv 0 \bmod p^{N-1-M}$, $\bigoplus_{i=0}^{M} \overline{\Theta}_J^{q-2-i}(u) \equiv 0 \bmod p^{N-2-M}$, $\bigoplus_{i=0}^{M} \overline{\Theta}_J^{q-3-i}(u) \equiv 0 \bmod p^{N-3-M}$, ..., $\bigoplus_{i=0}^{M} \overline{\Theta}_J^{q-N+M+1-i}(u) \equiv 0 \bmod p$ (recall that $\overline{\Theta}_J^{s}$ for $s < 0$ is defined to be zero, identically). Under such conditions, we say that $[J,u]$ is a (q,N,M)-dual pair.

As above, we shall precede our rigorous presentation with a few heuristic remarks regarding the construction of these higher order operations. We begin with the assumption that we are given a $\mathbb{Z}_p$- cohomology element, $x \in H\mathbb{Z}_p^{2n}(X)$, $X \in F_p$. Suppose, moreover, that we are given a splitting for X, J, that forms a (q,N,M)-dual pair with some $\mathbb{Q}_p$- lifting, u, of x. Then, it follows that $\bigoplus_{i=0}^{M} \overline{\Theta}_J^{q-i}(u) = \{y_i\} \equiv 0 \bmod p^{N-M-1}$ and we are thus free to divide $\{y_i\}$ by p^{N-M-1} in the context of $\mathbb{Q}_p$-cohomology. Doing so and then reducing to $\mathbb{Z}_p$ determines the coset value of our dual $(N-M)^{th}$-order operation acting on x. We shall denote this by:

$$\overline{\Phi}_q^{N,M}(x) = \rho_*'[\bigoplus_{i=0}^{M} \overline{\Theta}_J^{q-i}(u)/p^{N-M-1}]/Q,$$

where $[J,u]$ runs over all suitable dual pairs. As before, the choice of pair generates some indeterminacy in the value of the operation. In contrast to the higher order operations of (3.3.8), however, this indeterminacy turns out to be reasonably straightforward in both its computation and manipulation. In particular, we shall see that $Q(\overline{\Phi}_q^{N,M})$ is equal to the cokernel of $\overline{\Phi}_q^{N-1,M}$. Moreover, it shall become clear that a condition analogous to (3.3.28) will follow directly from the form of the indeterminacy. As such, the involved procedure required to define the "special form" for Q will not be needed in the definition of the dual operations.

Now, for a fixed $q \geq 2$ and a choice of $N \leq q$, we construct a pyramid of cohomology operations, $\{\overline{\Phi}_q^{r,s}\}$ for $q \geq N \geq r > s \geq 0$ in the following inductive manner. We assume, as before, that $X \in F_p$:

3.4.16 Construction: (i) Our pyramid will, again, be constructed upon a base of sums of primary operations: $\overline{\Phi}_q^{r,r-1}$ for $1 \leq r \leq N$. By virtue of (3.4.15) and (3.4.3) we have:

$$(3.4.17) \qquad \overline{\Phi}_q^{r,r-1} = \bigoplus_{i=0}^{r-1} P^{q-i} : H\mathbb{Z}_p^{2n}(X) \to \bigoplus_{i=0}^{r-1} H\mathbb{Z}_p^{2n+2(q-i)m}(X).$$

3.4.16 ii. The Second Order: Before we consider the general case, we explicitly construct the set of secondary operations that comprise the second to bottom row of our pyramid. These will be a set of operations of the form $\overline{\Phi}_q^{r,r-2}$ for $2 \leq r \leq N$.

Let us pick an r in this range and consider the resulting operation. Unlike the primary case, the secondary operation will not be universally defined. Instead, we must consider a $x \in H\mathbb{Z}_p^{2n}(X)$ such that $\overline{\Phi}_q^{r,r-1}(x)$ is zero. By (3.4.15-ii) we see that this subgroup of $H\mathbb{Z}_p^{2n}(X)$ determined by Ker $\overline{\Phi}_q^{r,r-1}$ corresponds, in the obvious way, to the set of (q,r,r-2)-dual pairs, modulo the image of ρ'_*.

Let us suppose, then, that $[J,u]$ is a $(q,r,r-2)$-dual pair. By definition, we may conclude that $\rho'_*[\bigoplus_{i=0}^{r-1} \overline{\Theta}_J^{q-i}(u)]$ is the zero vector in $\bigoplus_{i=0}^{r-1} HZ_p^{2n+2(q-i)m}(X)$. Consequently, we may divide by p to get $\rho'_*[\bigoplus_{i=0}^{r-2} \overline{\Theta}_J^{q-i}u/p] = \overline{\Phi}_q^{r,r-2}(x)$.

This operation is defined on the kernel of a primary operation and takes values in a direct sum of coset spaces $\bigoplus_{i=0}^{r-2} HZ_p^{2n+2(q-i)m}(X)/Q$. The indeterminacy, Q , will be calculated as follows.

As before, we are given two choices to make, once we have an element, x, in the kernel of $\overline{\Phi}_q^{r,r-1}$, the choice of Q_p-lifting of x and the splitting isomorphism to be used.
It will turn out that only the first of these two choices offers any contribution to the indeterminacy. We see this as follows.
Let $[J,u]$ and $[L,v]$ be two possible $(q,r,r-2)$-dual pairs.
Thus, u and v are such that $\rho'_*u = \rho'_*v = x$. As in (3.3.9) we have:

$$(3.4.18) \qquad u-v = pw$$

for some $w \in H^{2n}(X)$. Applying our homomorphism to (3.4.18) we get:

$$(3.4.19) \qquad \oplus\, \overline{\Theta}_J^{q-i}u = \oplus\overline{\Theta}_J^{q-i}v + p\, \oplus\overline{\Theta}_J^{q-i}w \text{ for } 0 \leq i \leq r-1$$

Keeping this in mind, we fix our Q_p-representative, for the moment, and consider the effect of a change in splitting. Let J and L be two possible choices. Using the formula derived in (3.4.5) in the special case where X = Y and where f is the identity morphism, evaluating mod p^2 we get:

(3.4.20) $$\oplus\overline{\Theta}_J^{q-i}u = \oplus\overline{\Theta}_L^{q-i}u + p\oplus\delta_1\overline{\Theta}_L^{q-1-i}u \text{ for } 0 \leq i \leq r-2.$$

Combining (3.4.20) and (3.4.19) and evaluating mod p^2 gives:

(3.4.21) $$\oplus\overline{\Theta}_J^{q-i}u = \oplus\overline{\Theta}_L^{q-i}v + p(\oplus\overline{\Theta}_L^{q-i}w + \oplus\delta_1\overline{\Theta}_L^{q-1-i}v).$$

Now, recalling the hypothesis that $[J,u]$ and $[L,v]$ were two $(q,r,r-2)$-dual pairs, we see, that dividing (3.4.21) by p and reducing mod p, the two possible formulations of $\overline{\Phi}_q^{r,r-2}$ differ by an element in the image of the primary operation, $\overline{\Phi}_q^{r-1,r-2}$. Thus we have:

(3.4.22) $$\overline{\Phi}_q^{r,r-2} : \text{Ker } \overline{\Phi}_q^{r,r-1} \to \text{Cok } \overline{\Phi}_q^{r-1,r-2} = \oplus H\mathbb{Z}_p^*(X)/Q.$$

Notice that the last term in (3.4.21) played no rôle in the evaluation of the indeterminacy. This was a result of the rather more stringent hypotheses (compare (3.4.15-ii) with (3.3.1-ii)) that we required in order to define a higher order dual operation. The effect of these stronger hypotheses is that the variation in splitting isomorphism offers no contribution to the value of Q .

The reason why these more restrictive hypotheses were needed follows from comparing the formulae, (3.4.5) and (3.2.9). In (3.2.9) the application of an "arbitrary" linear homomorphism, δ_i, preceeded the pseudo primary operation. As a result, we were free to regard the image of the composition, $\Theta_L^{q-i}\delta_i$, as simply the image of Θ_L^{q-i}. In (3.4.5) this can no longer be done. Failure to remove the images of the δ_i's from consideration would have resulted in a totally indeterminate higher order cohomology operation.

As remarked above, the general form of the indeterminacy allows us to conclude "total realisability". We see this as follows. Assuming that $[J,u]$ and $[L,v]$ are two $(q,r,r-2)$-dual pairs representing the same x, we may reduce (3.4.21) to:

$$(3.4.23) \qquad \oplus\, \overline{\Theta}_J^{q-i} u - \oplus\, \overline{\Theta}_L^{q-i} v = p\oplus\, \overline{\Theta}_L^{q-i} w.$$

This follows immediately from (3.4.15 - ii) and (3.4.5). What this tells us, of course, is that once we are given a suitable splitting, we may assume that it is unique, provided it came from an appropriate dual pair. Comparing (3.4.23) with (3.3.24) demonstrates that the indeterminacy of the dual operations is considerably simpler than that of the operations of §3. In the context of (3.4.23) it is clear that w and v fully determine one another and that the choice of splitting is removed from consideration altogether.

<u>*3.4.16 - iii. The General Case*</u>: Above, we have constructed primary and secondary operations based upon direct sums of dual pseudo primary operations. We now propose to inductively extend our construction to encompass operations of arbitrarily high order. These will be linear homomorphisms defined upon a subgroup of $H\mathbb{Z}_p^{ev}(X)$ determined by the intersection of kernels of certain operations lower down in the pyramid. The range of a typical operation will be a coset space, a direct sum of cohomology groups factored out by the image of an operation of lower order. That is, we shall define an operation of order s+1, degree q and type (r-s) such that:

$$(3.4.24) \quad \overline{\Phi}_q^{r+1,r-s} : \bigcap_{j=1}^{s} \operatorname{Ker} \overline{\Phi}_q^{r+1,r-s+j} \subseteq H\mathbb{Z}_p^{2n}(X) \to \operatorname{Cok} \overline{\Phi}_q^{r,r-s}.$$

Moreover, the following properties shall hold:

(3.4.25) $\qquad \mathrm{Im}\ \overline{\Phi}_q^{r,s} \supseteq \mathrm{Im}\ \overline{\Phi}_q^{r-1,s}$ in the strong sense of (3.3.27) and

(3.4.26) If χ is a $\mathbb{Z}_p$-cohomology class upon which $\overline{\Phi}_q^{r,s}$ is defined, then there exists a $\mathbb{Q}_p$-lifting of χ, u, and a splitting, J, such that $[J,u]$ forms a (q,r,s)-dual pair. Modulo the indeterminacy, the value of this operation is given by: $\overline{\Phi}_q^{r,s}\chi = \rho_*'[\bigoplus_{i=0}^{s} \overline{\Theta}_J^{q-i} u/p^{r-s-1}]$.

For the primary and secondary cases, these conditions are all obviously satisfied. We wish, now, to consider the general case.

Let us assume, as inductive hypothesis, that we have constructed up to and including the s^{th} row of our pyramid. The top row, then, is made up of operations of the form $\overline{\Phi}_q^{r,r-s}$ for $s \leq r \leq N$. We wish to construct the $(s+1)^{st}$ order operations, $\overline{\Phi}_q^{r+1,r-s}$. To this end, let us consider an arbitrary element in the intersection of kernels, $\bigcap_{j=1}^{s} \overline{\Phi}_q^{r+1,r-s+j}$, χ, say. By (3.4.15 - ii) and (3.4.26) it is clear that we have a $[J,u]$ which forms a (q,r+1,r-s)-dual pair representing χ. We define the coset class value of our $(s+1)^{st}$ order operation by:

$$\overline{\Phi}_q^{r+1,r-s}(\chi) = \rho_*'[\bigoplus_{i=0}^{r-s} \overline{\Theta}_J^{q-i}(u)/p^s]/Q,$$

where the indeterminacy, Q, is generated by the ranging of $[J,u]$ over all possible (q,r+1,r-s)-dual pairs. We compute this as follows. Let $[J,u]$ and $[L,v]$ be two such pairs. Using the same procedure that gave us (3.4.21), above, we get, mod p^{s+1}:

$$(3.4.27) \quad \bigoplus_{i=0}^{r-s} \overline{\Theta}_J^{q-i} u = \bigoplus_{i=0}^{r-s} \overline{\Theta}_L^{q-i} v + \sum_{j=0}^{s-1} p^{j+1} \bigoplus_{i=0}^{r-s} \delta_j \overline{\Theta}_L^{q-j-i} w + \sum_{j=1}^{s} p^j \bigoplus_{i=0}^{r-s} \delta_j \overline{\Theta}_L^{q-j-i} v.$$

The hypotheses reduce (3.4.27) to:

$$(3.4.28) \quad \oplus\overline{\Theta}_J^{q-i} u = \oplus\overline{\Theta}_L^{q-i} v + \sum_{j=0}^{s-1} p^{j+1} \oplus \delta_j \overline{\Theta}_L^{q-j-i} w, \text{ for } 0 \leq i \leq (r-s).$$

This reduces further, by the following inductive process, to:

(3.4.29) $\oplus\overline{\Theta}_J^{q-i}u = \oplus\overline{\Theta}_L^{q-i}v + p\oplus\overline{\Theta}_L^{q-i}w$, for $0 \leq i \leq (r-s)$.

We illustrate this inductive process by way of a specific example, from which it will be evident how to generalise this procedure. This is done in an attempt to keep the generalised notation from obscuring a rather simple technique. Consider the case for $q \geq N=4$ and $M=0$. As hypotheses we have that $\overline{\Theta}_*^q * \equiv 0 \bmod p^3$, $\overline{\Theta}_*^{q-1} * \equiv 0 \bmod p^3$, $\overline{\Theta}_*^{q-2} * \equiv 0 \bmod p^2$, and $\overline{\Theta}_*^{q-3} * \equiv 0 \bmod p$. The symbols * may be replaced by any J's or u's that come from $(q,4,0)$-dual pairs. Let $[J,u]$ and $[L,v]$ be two such pairs. Under these conditions, (3.4.28) becomes:

(3.4.30) $\overline{\Theta}_J^q u = \overline{\Theta}_L^q v + p\overline{\Theta}_L^q w + p^2\delta_1\overline{\Theta}_L^{q-1}w + p^3\delta_2\overline{\Theta}_L^{q-2}w$.

By hypothesis we have:

(3.4.31) $\overline{\Theta}_J^{q-2}u = \overline{\Theta}_L^{q-2}v + p\delta_1\overline{\Theta}_L^{q-3}v + p\overline{\Theta}_L^{q-2}w \equiv 0 \bmod p^2$.

Since $[L,v]$ is a $(q,4,0)$-dual pair we may conclude from (3.4.31) that $\overline{\Theta}_L^{q-2}w \equiv 0 \bmod p$. Similarly:

(3.4.32) $\overline{\Theta}_L^{q-1}v + p\delta_1\overline{\Theta}_L^{q-2}v + p^2\delta_2\overline{\Theta}_L^{q-3}v + p\overline{\Theta}_L^{q-1}w + p^2\delta_1\overline{\Theta}_L^{q-2}w \equiv 0 \bmod p^3$.

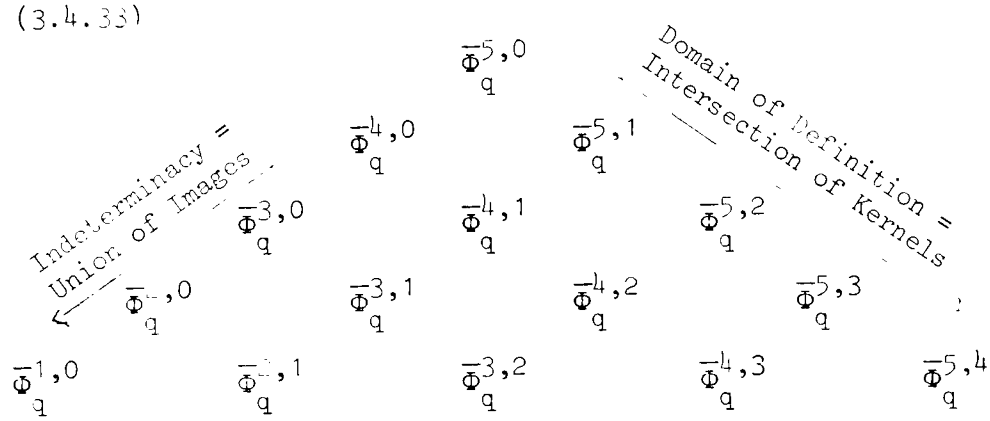
Again, since $[L,v]$ is a $(q,4,0)$-dual pair and since $\overline{\Theta}_L^{q-2}w \equiv 0 \bmod p$, by our previous calculation, we may conclude that $\overline{\Theta}_L^{q-1}w \equiv 0 \bmod p^2$. This removes the last two terms of (3.4.30) from consideration. The generalisation of this procedure reduces (3.4.28) to (3.4.29).

It is now a simple matter to deduce, directly from (3.4.29), that the indeterminacy of $\overline{\Phi}_q^{r+1,r-s}$ will be the union of the images of $\overline{\Phi}_q^{r+1-j,r-s}$ where j ranges over all values in $[1,s]$. From (3.4.25) it follows that (3.4.24) holds for our $(s+1)^{st}$ order operation, that

(3.4.25) and (3.4.26) hold on the $(s+1)^{st}$ order level is obvious by virtue of (3.4.29). It remains to check that our operations are linear. This is, however, trivial since the stringent hypotheses of (3.4.14 - ii) made all splitting isomorphisms essentially the same.

This completes the inductive step and establishes a pyramid of higher order operations which may be depicted as follows (here, we have again chosen $N=5 \leq q$):

(3.4.33)



3.4.34 Remarks: (i) As with (3.3.33), we have pictorially presented a pyramid of higher order cohomology operations for $q \geq N=5$. In comparison with (3.3.33) the dual pyramid is reflected about a vertical line through its centre. After this reflection, the domains and indeterminacies are computed in the same way as in (3.3.33). That the operations in this pyramid are dual to those of (3.3.33) in the sense of [35] follows from (3.4.3) and (3.2.4) in the primary case and from (3.4.6) in the higher order cases. Comparing the constructions of these two pyramids, (3.4.16) and (3.3.8), one sees straight away that the corresponding elements of the two pyramids may not always be defined under the same circumstances. When they both are defined, however, the corresponding elements of the pyramids do, indeed, turn out to be duals of one another.

(ii) It is clear, by construction, that all coset representatives are realisable.

Notice that in contrast to (3.3. 8) the construction to realise the indeterminacy of zero was not needed. For the dual operations, total realisability of the indeterminacy follows for operations of order (N-M) > 2 just as it does in the secondary case. This is a consequence of the mod p^{s+2} congruence, (3.4.29). As was the case with (3.3.34-i), the operations of (3.4.16) enjoy the property of having a "nested" indeterminacy, by virtue of (3.4.25), and so the value of $Q(\bar{\Phi}_q^{N,M})$ will be precisely the image of the operation, $\bar{\Phi}_q^{N-1,M}$.

(iii) The operations in (3.4.33) are natural and stable under double suspension, as we shall see below:

<u>*3.4.35 Lemma*</u>: Let $f : Y \to X$ be a morphism of spaces in F_p. Let $[J,u]$ be a (q,r,s)-dual pair for X. Then there is a dual pair for Y, $[L,f^*u]$, and with $\rho_*' u = x$ we have:

$$f^*\bar{\Phi}_q^{r,s}(x) = \bar{\Phi}_q^{r,s}(f^*x) \text{ in } \bigoplus_{i=0}^{s} H\mathbb{Z}_p^{2n+2(q-i)m}(Y)/(\text{indeterminacy}).$$

<u>*Proof*</u>: This follows directly from the definition of (q,r,s)-dual pair. □

Similarly, we have:

3.4.36 Lemma: Let σ^* be the double suspension isomorphism:

$$\sigma^* : H\mathbb{Z}_p^{2n}(X) \to H\mathbb{Z}_p^{2n+2}(S^2 \wedge X).$$

Then, with $[J,u]$ a (q,r,s)-dual pair and $[L,\sigma^* u]$ another:

$$\sigma^* \overline{\Phi}_q^{r,s}(x) = \overline{\Phi}_q^{r,s}(\sigma^* x) \text{ in } \bigoplus_{i=0}^{s} H\mathbb{Z}_p^{2n+2(q-i)m}(S^2 \wedge X)/(\text{indet.}).$$

Here, as above, $x = \rho_*^! u$.

Let us consider the chain complex $C^*(N,q)$ in the sense of [27] defined as follows (where $q \geq N$):

Each C_n^* $(0 \leq n \leq N)$ is generated by elements $\{c_n; c_{n,0}; \ldots; c_{n,N-n}\}$ where the dimension of c_n is $2qm+N-n-1$ and that of $c_{n,i}$ is $N-n+2m(N-n-i)$ for $0 \leq i \leq N-n$. Define the differentials of $C^*(N,q)$ by:

(i) $d_n^*(c_n) = -\beta(c_{n+1}) + \sum_{i=n}^{N-1} p^{q-i+n}(c_{n+1,N-1-i})$ and

(ii) $d_n^*(c_{n,i}) = \beta(c_{n+1,i-1}) + \tau(c_{n+1,i})$. Here τ is as in (3.3.37).

Notice that $C^*(N,q)$ is the dual complex (in the sense of Theorem 4.3.1 of [27])to $C(N,q)$, the complex of (3.3.37). Let $\{\hat{\Phi}_*^{r,s}\}$ denote the pyramid associated to $C^*(N,q)$ that is dual to $\{\hat{\Phi}^{r,s}\}$ of (3.3.37), for $N \geq r > s \geq 0$. As above, the symbol , ^, signifies our restriction to the p-torsion free category, F_p.

Then:

3.4.37 Theorem: With the above hypotheses, $\hat{\Phi}_*^{r,s} = \overline{\Phi}_q^{r,s}$ for all r and s such that $N \geq r > s \geq 0$, modulo indeterminacy.

Proof: By (3.3.37) we see that it is sufficient to show that $\overline{\Phi}_q^{r,s}$ is dual to $\Phi_q^{r,s}$. We show this inductively.

For order 1, this follows directly from (3.4.6). Assume, then, as inductive hypothesis, that $\overline{\Phi}_q^{r,s}$ is dual to $\Phi_q^{r,s}$ for all r and s such that $N \geq r > s \geq 0$ and $(r-s) \leq 1$.

Formulae (3.4.24) and (3.3.25) tell us that an (l+1) order operation, $\overline{\Phi}_q^{r,s}$, and an (l+1) order operation, $\Phi_q^{r,s}$, are defined on dual domains and take values in dual quotient spaces. Furthermore, the images are just dual operations (by (3.4.6)) divided by powers of p. The result follows. □

3.4.38 Remarks:

We notice, moreover, that, in particular cases, the operations of $\Phi_q^{r,s}$ and $\overline{\Phi}_q^{r,s}$ coincide with several other specific examples of higher order cohomology operations that appear, repeatedly, in the literature. For example , in F_p:

(i) $\overline{\Phi}_q^{2,0}$ is the operation Φ_q of [26] for p an odd prime. That is to say, $\overline{\Phi}_q^{2,0}$ is the stable secondary cohomology operation associated with the following Adem relation:

(3.4.39) $$(P^1\beta)\, P^{q-1} - (q-1)\, \beta P^q - P^q\beta = 0, \text{ for p odd.}$$

(ii) For p = 2, $\overline{\Phi}_{2j}^{2,0}$ is Φ_{4j} of [17] and [18]. Thus, for p and q even in example (i), $\overline{\Phi}_q^{2,0}$ is the operation associated with the mod 2 Adem relation:

(3.4.40) $$Sq^1 Sq^{4j} + (Sq^2 Sq^1) Sq^{4j-2} + Sq^{4j} Sq^1 = 0.$$

(iii) When p = 2 and q is arbitrary $\overline{\Phi}_q^{2,0}$ is the secondary operation studied in [17], [10] and [11] (to name a few) based upon the relation:

(3.4.41) $$Sq^1 Sq^{2q} + (Sq^1 Sq^2 + Sq^2 Sq^1) Sq^{2q-2} + Sq^{2q} Sq^1 = 0.$$

(iv) Lastly, $\Phi_q^{2,0}$ is the operation coming from the dual relation to (3.4.41), namely (2.4.8), when $p = 2$, When p is odd, the defining relation is that dual to (3.4.39). These secondary operations have been considered by Maunder in [28]. (See Theorem (2.4.14) and (3.3.37), above.)

§5. *Pyramid Mixing*

We complete this chapter with a section devoted to the notion of "pyramid mixing". In the previous two sections, pyramids of higher order operations were constructed determined by single pseudo primary operations. In §§3 and 4, above, we considered pyramids, (3.3.33) and (3.4.33), for example, determined by the pseudo operations, Θ_J^q and $\overline{\Theta}_J^q$, respectively. That is to say, these two homomorphisms induced the primary operations in the bottom right and left-hand corners, respectively, of the two above-mentioned pyramids. In this section, we shall consider pyramids of higher order operations determined by rather more general sorts of pseudo primary operations. In order to make these notions precise, we shall require some notation. We remain in the context of F_p throughout this section.

3.5.1 Definitions: (i) We shall, henceforth, refer to the operations (pseudo primary and higher order) of §3, Θ_J^q and $\Phi_q^{r,s}$, (respectively, §4, $\overline{\Theta}_J^q$ and $\overline{\Phi}_q^{r,s}$) as operations of the first (respectively, second) kind. Likewise, we shall call a (q,r,s)-pair (dual pair) a pair of the first (respectively,second) kind.

(ii) Consider the expression:

$$\sum_{i=0}^{S_1} \beta_1 P^{\alpha_1} \Theta_J^{q_1-i} \oplus \sum_{i=0}^{S_2} \beta_2 P^{\alpha_2} \Theta_J^{q_2-i} \oplus \ldots \oplus \sum_{i=0}^{S_t} \beta_t P^{\alpha_t} \Theta_J^{q_t-i},$$

defined upon a vector in $\bigoplus_{i=0}^{l} H^{2n+2im}(X)$, where $l = \max(S_1,\ldots,S_t)$, and taking values in $\bigoplus_{i=0}^{t} H^{2n+2q_i m}(X)$. We assume that the coefficients, β_i, are all relatively prime to p. We shall call this a mixed sum of pseudo primary operations of the first kind, type $(S_1,\ldots,S_t)$, degree $(q_1,\ldots,q_t)$, divisibility $(\alpha_1,\ldots,\alpha_t)$ and with t terms.

(iii) Similarly, we denote a mixed sum of pseudo primary operations of the second kind, type $(S_1,\ldots,S_t)$, degree $(q_1,\ldots,q_t)$, divisibility $(\alpha_1,\ldots,\alpha_t)$ and with t terms by the expression:

$$\bigoplus_{i=0}^{S_1} \beta_1 P^{\alpha_1} \bar{\Theta}_J^{q_1-i} \oplus \bigoplus_{i=0}^{S_2} \beta_2 P^{\alpha_2} \bar{\Theta}_J^{q_2-i} \oplus \ldots \oplus \bigoplus_{i=0}^{S_t} \beta_t P^{\alpha_t} \bar{\Theta}_J^{q_t-i}.$$

This expression, in which all of the β_i are assumed to be prime to p, is defined upon an element in $H^{2n}(X)$ and takes values in $\bigoplus_{j=1}^{t} \bigoplus_{i=0}^{S_j} H^{2n+2(q_j-i)m}(X)$.

3.5.2 Remark: The mixed sums with one term, zero divisibility and $\beta_1 = 1$ correspond to the sums of (3.3.1-i) and (3.4.15-i).

In this section, we shall define pyramids of higher order operations determined by mixed sums of pseudo primary operations. The resulting pyramids will consist of mixed higher order operations. These mixed operations will be of particular value when we consider higher order Adem-like relations and Cartan-like formulae, in the following chapter.

No real generality is lost, of course, by considering expressions of two terms only. Moreover, because we are working with direct sums, the coefficients, β_i , may also be neglected. Furthermore, we may assume without loss of generality that the divisibility of our mixed sum is of the form (0,k). In view of these remarks, let us, henceforth, consider expressions of the form

$$(3.5.3\text{-i}) \qquad \sum_{i=0}^{S} \Theta_J^{q-i} \oplus \sum_{j=0}^{S'} p^k \, \Theta_J^{q'-j} \qquad \text{and}$$

$$(3.5.3\text{-ii}) \qquad \bigoplus_{i=0}^{S} \overline{\Theta}_J^{q-i} \oplus \bigoplus_{j=0}^{S'} p^k \, \overline{\Theta}_J^{q'-j}$$

as models for our mixed sums of the first and second kind, respectively. We do this in an effort to keep the generalised notation of (3.5.1-ii&iii) from obscuring the basic simplicity of our approach.

Let us begin by defining the higher order operations that make up a pyramid based upon a mixed sum of the form (3.5.3-i).

<u>*3.5.4 Definitions*</u>: (i) Let J be a splitting isomorphism and let t be equal to the greater of the two integers, S and S'. Let $\{u_i\}$ be a $\mathfrak{k}_p$-cohomology vector, $u_i \in H^{2n+2im}(X)$, for $0 \leq i \leq t$. Suppose now, that $[J,\{u_i\}]$ forms a (q,r,l)-pair and, simultaneously, a (q',r';l')-pair, both of the first kind, for integers, r,l,r' and l', where (r-l)-(r'-l') = k and where l,l' $\leq$ t. Then, we shall say that $[J,\{u_i\}]$ forms a $(q * q', r * r', l * l')$-mixed pair of the first kind.

(ii) Let $[J,\{u_i\}]$ be such a pair as that given above. We shall define an operation of order (r-l,r'-l'), degree (q,q') and type (l,l') such that its value is a coset, represented by the pair of elements:

$$(3.5.5) \quad \rho'_*[(\sum_{i=0}^{l} \theta_J^{q-i}\, u_i)/p^{r-l-1},\ (\sum_{j=0}^{l'} \theta_J^{q'-j}\, u_j)/p^{r'-l'-1}]$$

in $HZ_p^{2n+2qm}(X) \oplus HZ_p^{2n+2q'm}(X)$. We shall denote this operation by $\Phi_{q*q'}^{r*r',l*l'}\{x_i\}$, where $x_i = \rho'_* u_i$, for all i, $0 \leq i \leq t$.

These "mixed" higher order operations also form pyramids. We shall construct the operations in such a pyramid in an inductive fashion, parallel to that of §3, above.

A typical mixed operation of order (1,1) could be represented by $\Phi_{q*q'}^{r*r',r-1*r'-1}$. This is simply defined to be:

$$\sum_{i=0}^{r-1} \chi P^{q-i} \oplus \sum_{j=0}^{r'-1} \chi P^{q'-j} : \bigoplus_{k=0}^{r'-1} HZ_p^{2n+2km}(X) \to HZ_p^{2n+2qm}(X) \oplus HZ_p^{2n+2q'm}(X),$$

where we have assumed that $r' \geq r$.

We shall, in contrast with §3, allow r and r' to take on values less that 1. For operations with order (r-l,r'-l') with (r-l-1) (respectively, (r'-l'-1)) negative, we define $\Phi_{q*q'}^{r*r',l*l'}$ to be $\Phi_{q'}^{r',l'}$ (respectively, $\Phi_q^{r,l}$) of §3. When both (r-l-1) and (r'-l'-1) are negative, we define our operation to be zero-valued.

Let us assume, as we may do without loss of generality, that $l' \geq l$. We now claim that the remaining operations (those with order (a,b), a and b $\geq$ 1 and one of the two > 1) can be defined, inductively, in a well-defined fashion, consistent with (3.5.5), such that:

$$(3.5.6) \quad \Phi_{q*q'}^{r*r,l*l'} : \operatorname{Ker} \Phi_{q*q'}^{r-1*r'-1,l*l'} \subseteq \bigoplus_{i=0}^{l'} HZ_p^{2n+2im}(X) \to$$
$$\operatorname{Cok} \Phi_{q*q'}^{r*r',l+1*l'+1} = HZ_p^{2n+2qm}(X) \oplus HZ_p^{2n+2q'm}(X)/\operatorname{Im} \Phi_{q*q'}^{r*r',l+1*l'+1}.$$

We shall prove this contention with the help of the following series of lemmas. We use the notation of §3, above.

3.5.7 Lemma: Let $\{\eta_i\}$, $-1 \le i \le l'$ be given, together with a splitting, L. Let $\{u_i\}$, $0 \le i \le l'+1$, and K be the vector and splitting isomorphism that proposition (3.3.17) produces for us such that:

$$(3.5.8) \qquad \sum_{i=-1}^{l'} \theta_L^{q'-i} \eta_i = p \sum_{i=0}^{l'+1} \theta_K^{q'-i} u_i .$$

Suppose, now, that we take the first l+2 components of $\{\eta_i\}$ (recall that we have supposed that $l' \ge l$) together with L. Then, the first l+2 components of $\{u_i\}$, together with K will be such that:

$$(3.5.9) \qquad \sum_{i=-1}^{l} \theta_L^{q-i} \eta_i = p \sum_{i=0}^{l+1} \theta_K^{q-i} u_i .$$

Proof: By (3.3.16), we have $\theta_L^{q-i+1} \eta_{i-1} = p\, \theta_K^{q-i} u_i + \theta_J^{q-i+1} \eta_{i-1}$, for each $i \in [0,l'+1]$. In particular, it is true for each $i \in [0,l+1]$. Choosing a J such that it does not cross η_{i-1}, for all appropriate i, and noticing that the degree, q,played no rôle in the proof of (3.3.17) gives the result. □

3.5.10 Lemma: Propositions (3.3.19), (3.3.20) and (3.3.22) restrict from (l'+1)-vectors to (l+1)-vectors,in the sense of (3.5.7). Moreover they are, in the same sense, independent of the degree, q.

Proof: This follows in the same fashion from the proofs of (3.3.19), (3.3.20) and (3.3.22) as (3.5.7) followed from the proof of (3.3.17). □

3.5.11 Lemma: (i) Suppose we are given $\{u_i\}$ and $\{v_i\}$, two vectors of length l'+1, together with splittings J and K. Then, there is a vector of length l'+2, $\{z_i\}$, and a splitting,L, such that:

$$(3.5.12) \qquad \sum_{i=0}^{l'} [\, \theta_J^{q'-i} u_i - \theta_K^{q'-i} v_i] = p \sum_{i=0}^{l'+1} \theta_L^{q'-i} z_i .$$

(ii) Assume that we are given $\{u_i\}$ and $\{z_i\}$, J and L, all as above. Then there exists a vector, $\{v_i\}$, and a splitting, K, such that (3.5.12) holds.

(iii) Suppose that we are given the first l+1 elements in the vectors $\{u_i\}$ and $\{v_i\}$, given in (i), together with J and K. Then the L and $\{z_i\}$ which (i) gives us, satisfy:

$$(3.5.13) \quad \sum_{i=0}^{l} [\Theta_J^{q-i} u_i - \Theta_K^{q-i} v_i] = p \sum_{i=0}^{l+1} \Theta_L^{q-i} z_i .$$

(iv) Let us assume that only the first l+1 elements of $\{u_i\}$ and the first l+2 elements of $\{z_i\}$ are given, together with J and L. Then the K and $\{v_i\}$ which (ii) produces for us are such that (3.5.13) holds.

Proof: Parts (i) and (ii) have already been proved in propositions (3.3.17), (3.3.19), (3.3.20) and (3.3.22). (See (3.3.24).)

Parts (iii) and (iv) now follow from (i) and (ii) together with lemmas (3.5.7) and (3.5.10). □

We have already constructed $\Phi_{q*q'}^{r*r',l*l'}$ for the cases in which it was not so that both of (r-l) and (r'-l') were greater or equal to 1 and that at least one of the two was strictly greater that 1. Let us assume now, that all our operations of order (r-l,r'-l'), where $\min((r-l),(r'-l')) \le S$, for some $S > 1$, have been constructed in a well-defined manner, consistent with (3.5.5) and (3.5.6). The following theorem will provide the necessary inductive step and show that these operations form well-defined pyramids of arbitrary order.

3.5.14 Theorem: (i) The indeterminacy of $\Phi_{q*q'}^{r*r',l*l'}$ is precisely the image of $\Phi_{q*q'}^{r*r',l+1*l'+1}$.

(ii) Given any $\{x_i\}$, $0 \leq i \leq l'$, in Ker $\Phi_{q*q'}^{r-1*r'-1,l*l'}$, there exists a $(q*q', r*r', l*l')$-mixed pair of the first kind.

Proof: (i) Suppose that $\min((r-l),(r'-l')) = S+1$. Let $[J,\{u_i\}]$ and $[K,\{v_i\}]$ be two possible $(q*q', r*r', l*l')$-mixed pairs of the first kind. Then, by (3.5.11-i&ii), we have:

$$\rho'_*[(\sum_{i=0}^{l} \Theta_J^{q-i}u_i - \Theta_K^{q-i}v_i)/p^{r-l-1}, (\sum_{i=0}^{l'} \Theta_J^{q'-i}u_i - \Theta_K^{q'-i}v_i)/p^{r'-l'-1}] =$$

$$\rho'_*[\sum_{i=0}^{l+1} \Theta_L^{q-i}z_i)/p^{r-l-2}, (\sum_{i=0}^{l'+1} \Theta_L^{q'-i} z_i)/p^{r'-l'-2}] \in \text{Im}\, \Phi_{q*q'}^{r*r',l+1*l'+1}.$$

This concludes the proof of (i).

(ii) For the second part of (3.5.14), let us suppose that $\{x_i\}$, $0 \leq i \leq l'$, is in Ker $\Phi_{q*q'}^{r-1*r'-1,l*l'}$, where we suppose that $\min((r-l),(r'-l')) = S+1$. This says that there exists a J and a $\mathbb{Q}_p$-vector, $\{u_i\}$, $0 \leq i \leq l'$, representing $\{x_i\}$ such that:

$$\rho'_*[(\sum_{i=0}^{l} \Theta_J^{q-i} u_i)/p^{r-l-1}, (\sum_{i=0}^{l'} \Theta_J^{q'-i} u_i)/p^{r'-l'-1}] = (\alpha,\beta) \in \text{Im}\, \Phi_{q*q'}^{r-1*r'-1,l+1*l'+1}.$$

Thus, there exists an L and a vector, $\{z_i\}$, $0 \leq i \leq l'+1$, such that:

$$\rho'_*[(p\sum_{i=0}^{l+1} \Theta_L^{q-i}z_i)/p^{r-l-1}, (p\sum_{i=0}^{l'+1} \Theta_L^{q'-i}z_i)/p^{r'-l'-1}] = (\alpha,\beta).$$

Thus,

$$\rho'_*[(\sum_{i=0}^{l} \Theta_J^{q-i}u_i - p\sum_{i=0}^{l+1} \Theta_L^{q-i}z_i)/p^{r-l-1}, (\sum_{i=0}^{l'} \Theta_J^{q'-i}u_i - p\sum_{i=0}^{l'+1} \Theta_L^{q'-i}z_i)/p^{r'-l'-1}] = (0,0).$$

But, by (3.5.11-ii&iv), this last expression equals:

$$\rho'_*[(\sum_{i=0}^{l} \Theta_K^{q-i}v_i)/p^{r-l-1}, (\sum_{i=0}^{l'} \Theta_K^{q'-i}v_i)/p^{r'-l'-1}] = (0,0).$$

This says that $\{v_i\}$, $0 \leq i \leq l'$, is a $\mathbb{Q}_p$-representative for $\{x_i\}$ which yields a $(q*q, r*r', l*l')$-mixed pair of the first kind when taken together with K. This concludes the proof of (ii) and the theorem. □

3.5.15 Remarks: (i) Heuristically speaking, the point of these mixed operations is their "simultaneity". In order to be defined, they ask that a $\mathbb{Q}_p$-vector and a portion of that same vector form two different pairs, simultaneously, when taken together with one particular splitting isomorphism. This characteristic of these operations will be seen to play a principal rôle in the definition of higher order Cartan-like formulae. We shall return to this point in the next chapter.

(ii) By way of example, we present a diagram of the pyramid based upon the mixed sum of the first kind, type (1,0), degree (4,6) and divisibility (0,3). Here, we choose the maximal order of the pyramid to be (4,1):

$$\Phi_{4*6}^{5*1,1*0}$$

$$\Phi_{4*6}^{5*1,2*1} \qquad \Phi_{4*6}^{4*0,1*0}$$

$$\Phi_{4*6}^{5*1,3*2} \qquad \Phi_{4*6}^{4*0,2*1} \qquad \Phi_{4*6}^{3*-1,1*0}$$

$$\Phi_{4*6}^{5*1,4*3} \qquad \Phi_{4*6}^{4*0,3*2} \qquad \Phi_{4*6}^{3*-1,2*1} \qquad \Phi_{4*6}^{2*-2,1*0}$$

As in (3.3.33), the indeterminacy is determined by the union of the images of operations, below and to the left. The domain is determined by the intersection of kernels of operations, below and to the right. We point out that only the "peak" of our pyramid, $\Phi^{5*1,1*0}_{4*6}$, is a mixed operation in the proper sense. All other operations in this pyramid correspond with operations defined in §3, above. This follows directly from the definition.

Now, we shall turn our attention to higher order mixed operations of the second kind. We shall consider the pyramid based upon a mixed sum of the form given in (3.5.3-ii).

3.5.16 Definitions: (i) Let J be a splitting isomorphism and suppose $u \in H^{2n}(X)$ is such that $[J,u]$ forms a (q,r,l)-pair and, simultaneously a (q',r',l')-pair, both of the second kind, for integers, r, r', l and l', where $(r-l)-(r'-l') = k$. Under these hypotheses, we shall say that $[J,u]$ forms a $(q*q',r*r',l*l')$-mixed pair of the second kind.

(ii) Let $[J,u]$ be such a pair. We shall define an operation of order $(r-l,r'-l')$, degree (q,q') and type (l,l') such that its value is a coset represented by the pair of vectors:

$$\rho'_*[(\bigoplus_{i=0}^{l} \overline{\Theta}^{q-i}_J\, u)/p^{r-l-1},\ (\bigoplus_{j=0}^{l'} \overline{\Theta}^{q'-j}_J\, u)/p^{r'-l'-1}].$$

We shall denote this operation by $\overline{\Phi}^{r*r',l*l'}_{q*q'}(x)$, where $\rho'_*(u) = x$. It will be defined such that:

$$(3.5.17)\quad \overline{\Phi}^{r*r',l*l'}_{q*q'} : \operatorname{Ker} \overline{\Phi}^{r*r',l+1*l'+1}_{q*q'} \subseteq HZ^{2n}_p \to$$

$$\operatorname{Cok} \overline{\Phi}^{r-1*r'-1,l*l'}_{q*q'} = \frac{\bigoplus_{i=0}^{l} HZ^{2n+2(q-i)m}_p(X) \oplus \bigoplus_{j=0}^{l'} HZ^{2n+2(q'-j)m}_p(X)}{\operatorname{Im} \overline{\Phi}^{r-1*r'-1,l*l'}_{q*q'}}.$$

3.5.18 Remarks: (i) As above, we shall allow the value of r or r' to fall below 1. When $(r-l-1)$ (respectively, $(r'-l'-1)$) is negative, we see that setting $\overline{\Phi}_{q*q'}^{r*r',1*1'}$ equal to $\overline{\Phi}_{q'}^{r',1'}$ ($\overline{\Phi}_{q}^{r,1}$, respectively) is consistent with (3.5.16-ii).

(ii) That $\overline{\Phi}_{q*q'}^{r*r',1*1'}$ is well defined on Ker $\overline{\Phi}_{q*q'}^{r*r',1+1*1'+1}$ is seen directly. As the involved procedure of §3 is reflected in the proof of (3.5.14), so is the relatively simple approach of §4 mirrored here. One sees immediately from the fact that the $\mathbb{Q}_p$-classes, ν and ω, fully determine one another in the equation (we use the notation of (3.4.23)):

$$(3.5.19)\quad \overline{\Theta}_J^{q-i}u - \overline{\Theta}_L^{q-i}\nu = p\,\Theta_L^{q-i}\omega,$$

for all appropriate i, that any element giving the zero coset in the range of $\overline{\Phi}_{q*q'}^{r*r',1+1*1'+1}$ yields an explicit (q*q', r*r', 1*1')-mixed pair of the second kind.

(iii) The indeterminacy of $\overline{\Phi}_{q*q'}^{r*r',1*1'}$ is generated by the choice of $\mathbb{Q}_p$-lifting of an element, x, which forms a mixed pair with some splitting, J. As in §4, the choice of J offers no contribution to the indeterminacy, Q. That Q equals the image of the operation, $\overline{\Phi}_{q*q'}^{r-1*r'-1,1*1'}$, follows directly from the parallel calculations of §4, above.

3.5.20 Remark: As in (3.5.15-ii), we present an example of a mixed pyramid. Here our pyramid will be based upon the mixed sum of the second kind, of degree (6,5), type (0,1) and divisibility (0,2). The maximal order in the pyramid will be (4,2):

$$\bar{\Phi}_{6*5}^{4*3,0*1}$$

$$\bar{\Phi}_{6*5}^{3*2,0*1} \qquad \bar{\Phi}_{6*5}^{4*3,1*2}$$

$$\bar{\Phi}_{6*5}^{2*1,0*1} \qquad \bar{\Phi}_{6*5}^{3*2,1*2} \qquad \bar{\Phi}_{6*5}^{4*3,2*3}$$

$$\bar{\Phi}_{6*5}^{1*0,0*1} \qquad \bar{\Phi}_{6*5}^{2*1,1*2} \qquad \bar{\Phi}_{6*5}^{3*2,2*3} \qquad \bar{\Phi}_{6*5}^{4*3,3*4}$$

As in (3.4.33), the indeterminacy is determined by the union of the images of the operations, below and to the left. The domain is determined by the intersections of the kernels of the operations below and to the right of the operation in question. In the above pyramid, the first two rows reduce (in the sense of (3.5.15-i)) to operations of the sort defined in §4. Both here and in (3.5.15-ii), the number of rows which reduce to "unmixed" operations is determined by the divisibility of the sum upon which the pyramid is based.

Chapter IV: *Some Calculations and Applications*.

§1. *Introduction*.

In chapter III we defined a system, or rather a pair of systems of higher order cohomology operations on the p-torsion-free category, F_p. We have shown how, when we restricted to F_p, various (pseudo) primary (see (3.2.10) and (3.4.4)), secondary (see (3.4.38)), and higher order operations (see (3.3.37) and (3.4.37)) in the literature relate to particular elements in our pyramids. Nevertheless, our operations differ from those of the literature in one very important respect: the manner of generation. Whereas higher order operations traditionally come from a set of nested relations in the appropriate algebra of operations, ours are derived from increasing p-divisibility; Postnikov towers are replaced by variously expanded forms of the naturality deviation formulae, (3.2.9) and (3.4.5).

Given such a situation, where two objects with seemingly different sources coincide, a good way to proceed, one might suppose, is to assemble and combine the best features of each manner of generation. This is what we propose to do in this chapter.

For example, a valuable tool for calculation (see [39]), (the determination of which should be an interesting theoretical problem in its own right) would be a "reasonable" product or Cartan-like formula for higher order cohomology operations. Implicit in our use of the word, "reasonable", is a low level of indeterminacy.

Philosophically speaking, the problem with working with higher order operations is not what one can say with them, but, rather, how much indeterminacy is required in order to say it. The overly simplistic nature of this last statement notwithstanding, it is based upon a good deal of truth and it has essentially motivated a substantial amount of what follows.

To return to our example of a product formula, we note that the standard procedure (see [26] and [10],for example) seems to be the calculation of products with the help of functional cohomology operations and the deduction of the result for secondary operations by making use of the second formula of Peterson and Stein [34]. This procedure can, no doubt, be generalised using $(n-1)^{st}$ order functional cohomology operations [33] and their correspondence with n^{th} order cohomology operations. The difficulty is that the Peterson-Stein formula is not very well suited for minimising indeterminacy and, it turns out, one can prove a rather less indeterminate product formula for all orders of operations by using p-divisibility in place of the Postnikov lifting procedure. This is the sort of thing we had in mind when we suggested combining the better features of both systems.

In short, what we propose to do, here, is to develop some properties of the operations defined in the previous chapter and to give some applications thereof. This chapter will be presented as follows. We shall begin with an investigation into the effects of taking compositions of our operations. Next we examine the effect this has on deriving higher order relations from lower order ones. The influence of the coefficients and their divisibility properties will also be discussed. This can, under favourable circumstances, lead to the decomposition of an n^{th} operation in terms of an m^{th} order operation where $m \neq n$. Next, in §3, we shall turn our attention toward the development of a low indeterminate product formula in F_p. Then, in §4, we shall use these properties to make some calculations of the effect of our operations on the cohomology of $\mathbb{C}P^{\infty}$, the cohomology of the Thom space of bundles over $\mathbb{C}P^{\infty}$, and the cohomology of the Thom space of certain specific bundles over an arbitrary space in our category, F_p. We shall conclude this chapter with a brief section on the relation between our operations and the $e_{\mathbb{C}}$-invariant of [5]. This will be used to formulate the "Hopf invariant one" problem in our terms.

§2. *Compositions and Relations*.

We begin with a section concerning the effects of taking sums of compositions of our higher order cohomology operations. Let us adopt the notation that $\overset{(-)}{\Phi}{}^{N,M}_q$ denotes the $(N-M)^{th}$ order operation of either the first or second kind. Thus, in our new notation, we wish to examine relations of the form:

$$(4.2.1) \qquad \sum_{i\in I} \alpha_i \overset{(-)}{\Phi}{}^{N,M}_{q_i} \overset{(-)}{\Phi}{}^{S,T}_{r_i} \equiv 0 \bmod p^R, \text{ for some } R \in \mathbb{Z}^+, \text{ for } \alpha_i \in \mathbb{Q}_p,$$
$$\text{and } i \in I, \text{ some index set.}$$

Before we consider such sums as (4.2.1), however, we must take a look at the individual constituents thereof, simple compositions.

4.2.2 Definition: Let $A = (\bar{\Phi})^{N,M}_q \circ (\bar{\Phi})^{S,T}_r$ be defined on some element or vector of $H^{ev}(X)$, $X \in F_p$. We call A a compound operation and we denote the four possible cases by $A_1 = \bar{\Phi}^{N,M}_q \circ \Phi^{S,T}_r$; $A_2 = \Phi^{N,M}_q \circ \Phi^{S,T}_r$; $A_3 = \bar{\Phi}^{N,M}_q \circ \bar{\Phi}^{S,T}_r$; $A_4 = \Phi^{N,M}_q \circ \bar{\Phi}^{S,T}_r$.

The defining criteria and indeterminacy for these four compound operations are as follows (we adopt the notation that λx_i is some $\mathbb{Q}_p$-lifting of a $\mathbb{Z}_p$-vector, x_i):

(i) A_1 is defined on the set of (r,S,T)-pairs, $[J,\{u_i\}]$, such that $[L,\lambda\Phi^{S,T}_r \rho_*' u_i]$ is a (q,N,M)-dual pair for some splitting, L. Assuming that $u_0 \in H^{2n}(X)$, we have a total indeterminacy value (defined recursively) given by $\bar{\Phi}^{N,M}_q \circ \Phi^{S,T+1}_r \bigoplus_{i=0}^{(T+1)} H^{2n+2im}(X) + \bar{\Phi}^{N-1,M}_q H^{2n+2rm}(X)$.

(ii) A_2 is defined on the set of (r,S,T)-pairs, $[J,\{u_i\}]$, such that $[L,\lambda\Phi^{S,T}_r \rho_*' u_i]$ is a $(q,N,0)$-pair for some L. Notice how A_2 is only defined when M is zero. The total indeterminacy of A_2, assuming that u_0 is in dimension 2n, is $\Phi^{N,0}_q \circ \Phi^{S,T+1}_r \bigoplus_{i=0}^{(T+1)} H^{2n+2im}(X) + \Phi^{N,1}_q \bigoplus_{i=0}^{1} H^{2n+2(r-i)m}(X)$.

(iii) A_3 has, as domain of definition, the set of $(r,S,0)$-dual pairs, $[J,u]$ such that $[L,\lambda\bar{\Phi}^{S,0}_r \rho_*' u]$ is a (q,N,M)-dual pair for some L. Notice how A_3 is only defined when $T = 0$. The total indeterminacy of A_3, with $u \in H^{2n}(X)$, is $\bar{\Phi}^{N,M}_q \circ \bar{\Phi}^{S-1,0}_r H^{2n}(X) + \bar{\Phi}^{N-1,M}_q H^{2n+2rm}(X)$.

(iv) The domain of A_4 is the set of (r,S,T)-dual pairs, $[J,u]$ such that $[L,\lambda\Phi^{N,T}_q \rho_*' u]$ is a (q,N,T)-pair for some splitting, L. Notice that M must equal T for A_4 to be defined. The indeterminacy of A_4, under the assumption that u is in dimension 2n, is given by $\Phi^{N,T}_q \circ \bar{\Phi}^{S-1,T}_r H^{2n}(X) + \Phi^{N,T+1}_q \bigoplus_{i=0}^{(T+1)} H^{2n+2(r-i)m}(X)$.

4.2.3 Proposition: Let us assume that the compound operation A, described above, is defined. Then A is of order (N-M+S-T-1) and A_1 is of type ($|T-M|$) and of degree (r+q). A_2 has type T and degree (r+q). A_3 has type M and degree (r+q). And A_4 has type zero and degree (r+q-T). Here, we use the terms type, order, and degree in a way consistent with (3.3.1) and (3.4.15), above.

Proof: This is obvious and follows directly from the definitions. □

4.2.4 Remarks: There are several points to be made regarding (4.2.3) and (4.2.2).

(i) Firstly, we invite the comparison of (4.2.3) with (3.1.2) of [27].

(ii) The second operation in a compound operation acts as an "indeterminacy filter" in the following sense. The total indeterminacy in a composition of two operations is the indeterminacy of the second operation acting on the range of the first operation plus the second operation acting on the indeterminacy of the first. (See (4.2.2).) Thus, the total indeterminacy of a compound operation ranges from the indeterminacy of the second operation alone, when the second operation is not defined on any element in the indeterminacy of the first, to the full indeterminacy of the second operation plus the value of this operation on the indeterminacy of the first operation. This will occur whenever the second operation is universally defined on the resulting indeterminacy of the first operation.

Let us illustrate this point by considering the case of A_1 of (4.2.2). The total indeterminacy will range from $\bar{\Phi}_q^{N-1,M}\ H^{2n+2rm}(X)$ to this value plus $\bar{\Phi}_q^{N,M} \circ \Phi_r^{S,T+1} \bigoplus_{i=0}^{T+1} H^{2n+2im}(X)$. This last composite portion of the indeterminacy is dependent upon what values of $\Phi_r^{S,T+1} \bigoplus_{i=0}^{T+1} H^{2n+2im}(X)$ admit a (q,N,M)-dual pairing with some splitting, L. This can range between none of the values to all of the values. In this sense, $\overline{\Phi}_q^{N,M}$ may filter out a portion of the indeterminacy of $\Phi_r^{S,T}$.

(iii) This procedure of composing higher order operations corresponds directly to that used in the more standard context of Postnikov towers. Let us consider two secondary operations, Φ and Ψ, defined as follows. (We have restricted ourselves to secondary operations for notational simplicity only. Higher order operations compose in precisely the same way.)

Let α, β, γ and δ be primary operations in ordinary cohomology ($\mathbb{Z}_2$ coefficients, say) of orders a,b,c, and d, respectively. Suppose, moreover, that $\beta \circ \alpha = 0$ and $\delta \circ \gamma = 0$ are two relations that associate with Ψ and Φ, respectively. Let K(n) denote the Eilenberg-MacLane space of type $(\mathbb{Z}_2, n)$. Consider the diagram:

(4.2.5)

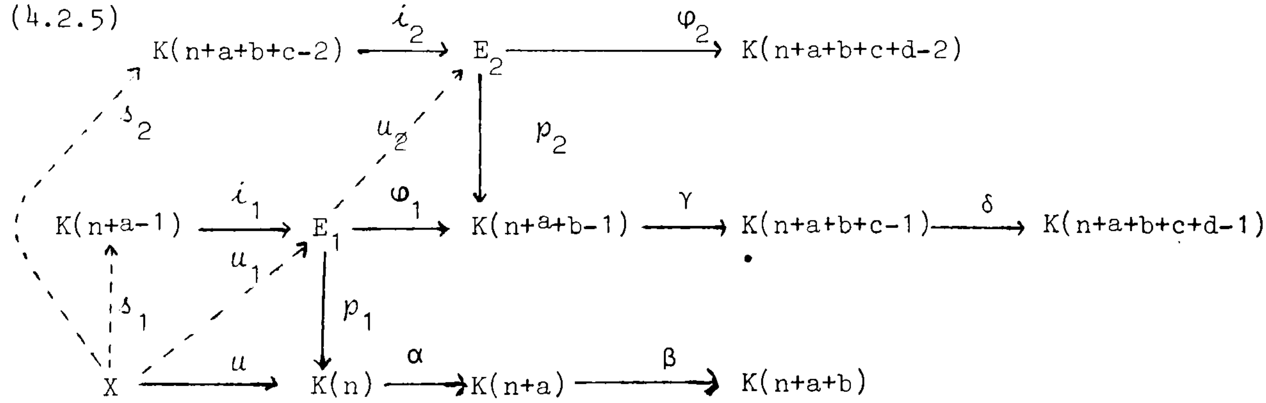


Diagram (4.2.5) represents the composition, $\Phi \circ \Psi$, of two secondary operations. The composition is defined on the subgroup of $H\mathbb{Z}_2^n(X)$ defined by: $\{u \in H\mathbb{Z}_2^n(X)$ such that $u \in \operatorname{Ker} \alpha$ and $\varphi_1 u_1 \in \operatorname{Ker} \gamma\}$. The value of the compound operation is the coset represented by the class

$[\varphi_2 u_2 u_1]$ in the quotient space:

$$H\mathbb{Z}_2^{n+a+b+c+d-2}(X)/(\delta H\mathbb{Z}_2^{n+a+b+c-2}(X) + \Phi\beta H\mathbb{Z}_2^{n+a-1}(X)).$$

(iv) Given the order of a compound operation and, thus, the total p-divisibility, we may uniquely determine the distribution of that p-divisibility (and, thus, the orders of the components of a composition) by means of the following convention. We take as order for the first operation the highest possible order that produces a coset upon which the second operation may be defined (with the "remaining" p-divisibility). All other distributions of order that allow both operations to be defined will produce, necessarily, the zero coset as result.

As an illustration, consider A_4 of (4.2.2). This compound operation is divisible by p to the power (N-M+S-T-2). We consider the highest order of $\overline{\Phi}_r^{*,*}$ such that it is defined, and such that its image coset allows at least one (q,*,*)-pair. In this case we have orders (N-M) and (S-T).

Now, we may turn our attention to relations of the form (4.2.1). Let us begin with considering the pseudo primary form of (4.2.1):

(4.2.6) $$0 \equiv \alpha_1 \, (\overline{\Theta}_J)^{q_1} (\overline{\Theta}_J)^{r_1} + \alpha_2 \, (\overline{\Theta}_J)^{q_2} (\overline{\Theta}_J)^{r_2} + \ldots , \bmod p^R,$$ with $q_i + r_i$ fixed.

For the purpose of (4.2.6) we have chosen a splitting, J. Since a relation in pseudo primary operations (such as (4.2.6)) is a relation for all $u \in H^{ev}(X)$, it clearly holds that it is a relation for any J. Let us, now, consider a very simple sort of relation that is of the form of (4.2.6) but has only two summands and relates a single pseudo primary operation to a single composition:

$$(4.2.7) \qquad \overline{\Theta}_J^q \equiv \alpha\, \overline{\Theta}_J^{q-s}\, \Theta_J^s, \text{ mod } p^R.$$

It shall turn out that only relations of this very restricted form will behave "nicely" with respect to generating higher order relations. In specific cases, more expanded sums may exhibit "nice" behaviour as well, but not generally.

We point out that such relations as (4.2.6) and (4.2.7) do, in fact, exist. The best known examples take the form of the "generalised Adem relations" of Hubbuck [24]. These state, by virtue of (3.2.10) and (3.4.4), that for k, a positive integer relatively prime to p, the following congruence holds:

$$(4.2.8) \qquad (1-k^{qm})\, \overline{\Theta}_J^q + (k^m-k^{qm})\, \overline{\Theta}_J^{q-1}\, \Theta_J^1 + \ldots + (k^{(q-1)m}-k^{qm})\overline{\Theta}_J^1 \Theta_J^{q-1} \equiv 0, \text{ mod } p^q.$$

Now let us return to our special case.

Consider the relation given by (4.2.7). Suppose, now that $[J,u]$ forms a $(q,2,0)$-dual pair. Let us assume, for the time being, α is not p-divisible. We may, then, conclude that $\overline{\Theta}_J^{q-s}\, \Theta_J^s\, u$ is p-divisible. Suppose, moreover, that either $[J,u]$ is a $(s,2,0)$-pair or $[J,\Theta_J^s\, u]$ is a $(q-s,2,0)$-dual pair. This would give rise to one of the two second order relations, respectively:

$$(4.2.9) \qquad \overline{\Phi}_q^{2,0}(\rho_*'u) = (\rho_*'\alpha)\, \overline{\Phi}_{q-s}^{1,0}\, \Phi_s^{2,0}(\rho_*'u) \text{ or } (\rho'\alpha)\, \overline{\Phi}_{q-s}^{2,0}\, \Phi_s^{1,0}(\rho_*'u).$$

This procedure would, in general, yield:

$$(4.2.10) \qquad \overline{\Phi}_q^{A,0}(\rho_*'u) = (\rho_*'\alpha)\, \overline{\Phi}_{q-s}^{B,0}\, \Phi_s^{C,0}(\rho_*'u), \text{ where } B+C = A+1.$$

It is clear that the existence of a $(q,N,0)$-pair of the second kind implies the N^{th} order divisibility of the right-hand side of (4.2.7). That higher order operations (the sum of whose orders is N+1) be defined on the right-hand side of (4.2.7) requires the existence of pairs. As this condition is, in the case of operations of the second kind at least, stronger than mere divisibility, one may not conclude, in general, that (4.2.10) can be obtained from (4.2.7) given a (q,N,0)-pair of the second kind. When the appropriate pairs do exist on the right-hand side of (4.2.7), the orders are then distributed in accordance with our convention, (4.2.4-iv).

As soon as one drops the rigorous restriction that our relation must be of the form of (4.2.7) and considers the generalised relation represented by (4.2.6), rather severe difficulties arise.

Consider the relation:

$$(4.2.11) \qquad \overline{\Theta}^{q}_{J} = \alpha\, \overline{\Theta}^{q-s}_{J}\, \Theta^{s}_{J} + \beta\, \overline{\Theta}^{q-t}_{J}\, \Theta^{t}_{J}, \bmod p^{R}.$$

We maintain the hypothesis that α and β are prime to p. Here, however, it is no longer sufficient to ask that simple pairs of the first or second kind exist on the right-hand side. What we require is a much stronger condition, namely that the two compositions on the right-hand side must be defined, and, in addition, that they be able to be defined simultaneously. That is to say, given the set of (q,A,0)-pairs of the second kind, for a fixed space $X \in F_p$, and a fixed dimension, n, we ask that there exist, at least one, $[J,u]$, say, such that both compositions on the right-hand side of (4.2.11) may be defined with $[J,u]$, each of order A+1. To deal with this situation in a rigorous fashion, we shall have to extend our notation somewhat. We consider the following:

4.2.12 Definition: We shall denote by $\overline{\Phi}^{r,l}_{q} \mathbin{\hat{\oplus}} \overline{\Phi}^{r',l'}_{q'}$, the cohomology operation of order (r-l,r'-l') and degree (q,q') which can be inductively defined such that:

$$\overline{\Phi}^{r,l}_{q} \mathbin{\hat{\oplus}} \overline{\Phi}^{r',l'}_{q'} : \mathrm{Ker}(\overline{\Phi}^{r,l+1}_{q} \mathbin{\hat{\oplus}} \overline{\Phi}^{r',l'+1}_{q'}) \subseteq HZ^{2n-2qm}_{p}(X) \oplus HZ^{2n-2q'm}_{p}(X)$$
$$\to \mathrm{Cok}(\overline{\Phi}^{r-1,l}_{q} \mathbin{\hat{\oplus}} \overline{\Phi}^{r'-1,l'}_{q'}) = \bigoplus_{i=0}^{l'} HZ^{2n-2im}_{p}(X)/\mathrm{Im}(\overline{\Phi}^{r-1,l}_{q} \oplus \overline{\Phi}^{r'-1,l'}_{q'}),$$

where l' is assumed to be greater than or equal to l. The coset value of this operation will be:

$$\rho'_*[\bigoplus_{i=0}^{l} \overline{\Theta}_J^{q-i} u/p^{r-l-1}) + (\bigoplus_{i=0}^{l'} \overline{\Theta}_J^{q'-i} v/p^{r'-l'-1})].$$

Here, we have taken addition to be term-wise. Taking $(r-l-1) = 0 = (r'-l'-1)$ in the above expression gives the precise definition of the operation of order $(1,1)$ and serves to start the inductive description of a pyramid of such opèrations. The inductive step, needed to show that these operations determine a well-defined pyramid, is quite parallel with the arguments of §4 and §5 of the previous chapter, where operations and mixed operations of the second kind were handled. As such, we shall omit a detailed construction.

We are now in a position to deal with the problem of simultaneous definition of compositions, raised above. We do this by combining the mixed operations of chapter III, §5 with the operations of (4.2.12).

Let us suppose that we are given a relation of the form of (4.2.11) on the $\mathbb{C}_p$-cohomology of some space, $X \in F_p$. Suppose, moreover, that there exists a $(q,A,0)$-pair of the second kind, $[J,u]$. Let us assume that $[J,u]$ is also an $(s*t,\ B*B',\ 0*0)$-mixed pair, of the first kind and denote by (x',x'') the image of $\Phi_{s*t}^{B*B',0*0}(x)$, where $\rho'_* u = x$.

4.2.13 Proposition: Under the above hypotheses, if $\overline{\Phi}_{q-s}^{C,0} \hat{\oplus} \overline{\Phi}_{q-t}^{C',0}$ is defined on (x',x''), where $C = A+1-B$ and $C' = A+1-B'$, we have, modulo the total indeterminacy:

$$\overline{\Phi}_q^{A,0}(x) = (\overline{\Phi}_{q-s}^{C,0} \hat{\oplus} \overline{\Phi}_{q-t}^{C',0}) \circ \Phi_{s*t}^{B*B',0*0}(x).$$

Proof: This follows directly from (3.5.4) and (4.2.12). □

4.2.14 Remark: In order to consider the general case represented by the relation (4.2.6), we would have to expand our definition, (4.2.12), to encompass sums longer than two terms of either the first or second kind. As this is more a notational problem than a mathematical one, we consider it sufficient to remark that the obvious generalisation of (4.2.12) yields the obvious generalisation of (4.2.13).

4.2.15 Definition: Suppose we are given a relation of the form, (4.2.6), where all the coefficients, α_i, are relatively prime to p. Suppose, moreover, that there exists a pair, $[J,u]$, that allows each term of the relation to define a composition of operations of total order A (but not A+1). Then, we shall call $[J,u]$ an admissible pair of order A-1 for the given composition (as opposed to product) relation.

4.2.16 Remark: The above definition simply offers a short-hand notation for the hypotheses of (4.2.13). In these terms we would say that (4.2.13) requires the existence of an admissible pair of order A for the relation, (4.2.11). The "non-composed" term of (4.2.11), $\overline{\Theta}^q_J$, may be considered as a "composed" term requiring total order A+1 (in place of A) by composing with the identity (pseudo) operation, Θ^0_J, divided by p^0.

What we propose to show is that the existence of an admissible pair is the necessary and sufficient condition that allows a pseudo primary relation to give use to a higher order relation. That is to say, given the definition of a higher order (composite) operation coming from one of the summands of a relation of the form (4.2.6), we may derive a relation of the form (4.2.1) if and only if we are provided with an admissible pair for the relation in question. Moreover, we claim that when the order with which we are working is two (that is, when we are dividing by p), then the existence of any set of splittings and liftings which allow definition of all the individual terms in a relation implies the existence of an admissible pair of order two for the given relation. We see this as follows.

We lose no real generality by considering a relation of the form (4.2.11). Suppose we are given splitting isomorphisms, J,L,M,N and Q as well as $\mathbb{Q}_p$-classes u,v and w in $H^{2n}(X)$, $X \in F_p$, such that the following conditions hold: (i) $\rho_*' u = \rho_*' v = \rho_*' w$, (ii) $[J,u]$ is a (q,2,0)-pair of the second kind, (iii) either $[M,v]$ is an (s,2,0)-pair of the first kind or $[L,\Theta_M^s v]$ is a (q-s,2,0)-pair of the second kind, and (iv) either $[Q,w]$ is a (t,2,0)-pair of the first kind or $[N,\Theta_Q^t w]$ is a (q-t,2,0)-pair of the second kind.

By virtue of (i), above, we have:

$$(4.2.17)\quad \alpha\, \overline{\Theta}_J^{q-s}\, \Theta_J^s u = \alpha\, \overline{\Theta}_L^{q-s}\, \Theta_M^s v + \alpha\{p\, \overline{\Theta}_L^{q-s}\, \Theta_M^s v' + \sum_{i\geq 1} p^i \delta_i'\, \overline{\Theta}_L^{q-s-i}\, \Theta_J^s (v+pv') + \overline{\Theta}_L^{q-s}(\sum_{j\geq 1} \Theta_M^{s-j} \delta_j (v+pv'))\}$$

and

$$(4.2.18)\quad \beta\, \overline{\Theta}_J^{q-t}\, \Theta_J^t u = \beta\, \overline{\Theta}_N^{q-t}\, \Theta_Q^t w + \beta\{p\, \overline{\Theta}_N^{q-t}\, \Theta_Q^t w' + \sum_{i\geq 1} p^i g_i'\, \overline{\Theta}_N^{q-t-i}\, \Theta_J^t (w+pw') + \overline{\Theta}_N^{q-t}(\sum_{j\geq 1} \Theta_Q^{t-j} g_j (w+pw'))\},$$

where $u = v+pv' = w+pw'$. Modulo p, of course, only the first terms on the right-hand sides of (4.2.17) and (4.2.18) remain. The hypotheses, (ii), (iii) and (iv), above, now imply that $[J,u]$ is an admissible pair of order two for the relation, (4.2.11).

The first of our two claims, that the existence of an admissible pair was the necessary and sufficient condition for the derivation of a higher order relation from a pseudo primary one is an obvious result of the definitions. Evidently, we have proved:

4.2.19 Proposition: (i) Let hypotheses (i-iv) given above be satisfied for a given pseudo primary relation. Then there exists an admissible pair of order two for that relation.

(ii) Let (4.2.6) be a relation in which all of the coefficients, α_i, are prime to p. Then (4.2.6) induces a relation of higher order compositions (all of order A) if and only if there exists an admissible pair, $[J,u]$, of order A-1 for (4.2.6).

We may, now, consider a specific example that illustrates how a primary relation (provided with an admissible pair) may induce a higher order relation. By virtue of (3.4.2), we have:

4.2.20 Proposition: Let $X \in F_p$ and let $[J,u]$ be an admissible pair of order N for (3.4.2). Then, with $\overset{(-)}{\alpha}_t$ denoting the orders determined by (4.2.4-iv) and with $\rho_*' u = x$, we have:

$$(4.2.21) \qquad \bar{\Phi}_q^{N,0}(x) = - \sum_{\substack{i+j=q \\ q>i\geq 0}} \bar{\Phi}_i^{\bar{\alpha}_i,0} \circ \Phi_j^{\alpha_j,0}(x).$$

Proof: This is obvious. □

In order to avoid the inner workings of (4.2.20) which take place in a $\mathbb{Q}_p$-context, we rephrase this last proposition in terms of the higher order $\mathbb{Z}_p$-cohomology operations, defined above. Here, we appeal to the spirit of (4.2.14) and of the early portion of §5, of the previous chapter, where the obvious generalised definitions, in the sense of having more than two "terms", of the operations of (4.2.12) and of mixed operations are left to the reader. The following proposition is equivalent to (4.2.20):

4.2.20' Proposition: Suppose that $\Phi_{1*2*\dots*q}^{\alpha_1*\alpha_2*\dots*\alpha_q,0*\dots*0}$ is defined on some $x \in H\mathbb{Z}_p^{2n}(X)$, $X \in F_p$. Let us suppose, moreover, that $\bar{\Phi}_{q-1}^{\beta_1,0} \hat{\oplus} \bar{\Phi}_{q-2}^{\beta_2,0} \hat{\oplus} \dots \hat{\oplus} \bar{\Phi}_0^{\beta_q,0}$ is defined on the image of the above-mentioned operation, $(x_1,x_2,x_3,\dots,x_q) \in \bigoplus_{i=0}^{q} H\mathbb{Z}_p^{2n+2im}(X)$, where $\alpha_i+\beta_i = N+1$, for all i, $1 \leq i \leq q$. Then, $\bar{\Phi}_q^{N,0}$ is defined on x and, modulo the total indeterminacy, we have:

$$(4.2.21')\quad \overline{\Phi}^{N,0}_{q}(x) = -[(\overline{\Phi}^{\beta_1,0}_{q-1} \mathbin{\hat{\oplus}} \overline{\Phi}^{\beta_2,0}_{q-2} \mathbin{\hat{\oplus}} \ldots \mathbin{\hat{\oplus}} \overline{\Phi}^{\beta_q,0}_{0}) \circ \Phi^{\alpha_1 * \ldots * \alpha_q, 0 * \ldots * 0}_{1*2*\ldots*q}(x)].$$

4.2.22 Remark: It shall become evident that the concept of an admissible pair plays a very important rôle in the theory of higher order operations in the present context.

A heuristic discussion of this concept, then, is in order. In particular, we offer an explanation of what the notion of admissible pair corresponds to in the more standard Postnikov tower context.

Let us consider the particular example of (4.2.21) where $N = p = 2$ and $q = 3$. Let be $u \in H^{2n}(X)$ for $X \in F_2$. Then (3.4.2) becomes:

$$(4.2.23)\quad \overline{\Theta}^{3}_{J} = -[\overline{\Theta}^{1}_{J}\,\Theta^{2}_{J} + \overline{\Theta}^{2}_{J}\,\Theta^{1}_{J} + \Theta^{3}_{J}].$$

The hypothesis that $[J,u]$ be a (3,2,0)-pair of the second kind tells us that $\overline{\Phi}^{2,0}_{3}$ is defined on $\tilde{u} = \rho^{!}_{*}u$. By (3.4.38-iii), this is equivalent to the statement that $\tilde{u}$ is in the kernel of Sq^6 and Sq^4 (recall $X \in F_2$). That $[J,u]$ is an admissible pair of order 2 for (4.2.23) tells us that $\tilde{u} \in \mathrm{Ker}(Sq^6, Sq^4)$ implies that $\tilde{u} \in \mathrm{Ker}\,\chi Sq^6$ and either $\tilde{u} \in \mathrm{Ker}\,\chi Sq^4$ or $\chi Sq^4 \tilde{u} \in \mathrm{Ker}\,Sq^2$ and either $\tilde{u} \in \mathrm{Ker}\,\chi Sq^2$ or $\chi Sq^2 \tilde{u} \in \mathrm{Ker}\,Sq^4$.

Thus, given a relation of "proper" primary cohomology operations one may view the existence of an admissible pair as an assurance that the original relation of the operation is, in fact, a relation of relations of operations together with the requisite universal examples. That the existence of a universal example is guaranteed in the secondary case (this corresponds to the fact that all chain complexes of length two are admissible - see §2.3 and §2.4 of [27]) is reflected by the statement of (4.2.19-i).

Throughout this section, we have been assuming that the coefficients, α_i, of terms in a relation such as (4.2.6) were relatively prime to p. If we drop this condition, we generate another sort of decomposability of higher order operations. Consider:

4.2.24 Example: Let us consider (4.2.8) in the special case where $p = 3$, $q = 3$, $k = 2$ and where $X \in F_3$. Let us assume, moreover, that $H^{2n+i}(X) = 0$ unless $i = 0,4$ or 12. In addition, we ask that $\overline{\Phi}_2^{2,0} \circ \Phi_1^{1,0}$ be defined on some subgroup of $H\mathbb{Z}_3^{2n}(X)$. (Notice that such a space can be constructed for any $n \in \mathbb{Z}$, such that $n \geq 6$. This follows because such an n places us in the stable homotopy range wherein essential attaching maps , for example the Hopf maps, σ and ν, may be found satisfying the necessary criteria.) Under the above assumptions, (4.2.6) becomes:

$$(4.2.25) \qquad (1-2^6)\,\overline{\theta}_J^3 \equiv -(2^2-2^6)\,\overline{\theta}_J^2\,\theta_J^1, \text{ mod } (3^3).$$

This is just:

$$(4.2.26) \qquad (-63)\,\overline{\theta}_J^3 \equiv 60\,\overline{\theta}_J^2\,\theta_J^1, \text{ mod } (27).$$

Dividing both sides of (4.2.26) by 9, as hypotheses assure us we may do, gives (because -7 and 20 are both congruent to -1, mod 3):

$$(4.2.27) \qquad \overline{\Phi}_3^{1,0} = \overline{\Phi}_2^{2,0}\,\Phi_1^{1,0}.$$

In view, of course, of the identifications, (3.3.8-i),(3.4.16-i) and (3.4.38-ii),of the operations defined here with other cohomology operations in the literature, we may rewrite (4.2.27) as

$$(4.2.28) \qquad P^3 = \Phi_4 \circ \chi P^1.$$

The resulting equations, (4.2.27) and (4.2.28), represent a decomposition of a composite of a secondary operation in terms of a primary. Similar decompositions arise whenever the appropriate geometric and number theoretic conditions combine to allow us to associate a coefficient divisibility with the divisibility of an operation.

Once we allow p-divisibility of the coefficients in a relation like (4.2.26), we may no longer talk about admissible pairs, unless, of course, all of the coefficients are precisely divisible by the same power of p. The obvious solution to the problem of dealing with relations with coefficients of varying p-divisibility is to extend our definition of admissible pair to allow compositions of different total orders to be in the same higher order relation but where the total divisibility (that required to allow the definition of a composition of higher order together with the divisibility of the coefficient) remains fixed. In the spirit of (4.2.14), however, we shall not pursue this line of thought, here.

§3. *Products*.

Let us begin with a short examination of the ring structures of the $\mathbb{Q}_p$-modules with which we have been working, $H^{ev}(X)$ and $K(X)$.

Before we can turn our attention to the multiplicative behaviour of our higher order operations, we must consider the effect of our splitting isomorphisms on the product structure of these two modules.

The first point to be made is that it will not be possible, in general, to find a splitting isomorphism, $J : H^{ev}(X) \to K(X)$, that will be a ring isomorphism. However, for any two given elements of positive grading x and y, in $H^{ev}(X)$, it will, indeed, always be possible to find a J such that $J(x)\cdot J(y) = J(xy)$. This is clear since $H^{2n}(X)$ is the associated $2n^{th}$ graded ring of $K(X)$, for $X \in F_p$. In general, however, multiplication in the filtered K-theory of two elements, $J(x)$ and $J(y)$, produces "error terms" in filtration higher than that of $J(x)\cdot J(y)$. However, we do have:

4.3.1 Proposition: (i) Let $x \in H^{2n}(X)$ and $y \in H^{2l}(X)$ for some $X \in F_p$. Let J be any splitting isomorphism satisfying (3.2.2). Then,

$$J(x)\cdot J(y) = J(xy) \bmod K_{2n+2l+1}(X).$$

(ii) $H\mathbb{Q}^{ev}(X) \cong K\mathbb{Q}(X)$ as rings.

Proof: This is obvious. □

In order to be able to deal with these "error terms" in a systematic way, we make the following definition.

4.3.2 Definition: Let $M_i : H^{2n}(X) \times H^{2l}(X) \to H^{2n+2l+2im}(X)$ be the mapping defined by $(u,v) \to J^{-1}(J(u)\cdot J(v))$, restricted to the dimension $2n+2l+2im$, for $i \geq 0$.

It is evident from this definition that

$$(4.3.3) \qquad J^{-1}(J(u)\cdot J(v)) = \sum_{i\geq 0} M_i(u,v)$$

and that $M_o(u,v) = uv$. Before we apply these functions to the problem of developing product formulae for higher order operations, it will be necessary to quote the following:

4.3.4 Lemma: Let J be any splitting isomorphism satisfying (3.2.2) and let u and v be two elements in $H^{ev}(X)$ for $X \in F_p$. Then:

(i) $J\{\sum_{q\geq 0} \overline{\Theta}^q_J(uv).p^{-q}\} = J\{\sum_{q\geq 0} \overline{\Theta}^q_J(u).p^{-q}\}.J\{\sum_{q\geq 0} \overline{\Theta}^q_J(v).p^{-q}\}$

in $K\mathbb{Q}(X)$, and

(ii) $\sum_{q\geq 0} \Theta^q_J[J^{-1}\{J(u).J(v)\}]p^{-q} = \{\sum_{q\geq 0} \Theta^q_J(u)p^{-q}\}.\{\sum_{q\geq 0} \Theta^q_J(v)p^{-q}\}$

in $H\mathbb{Q}^{ev}(X)$.

Proof: This follows directly from (3.2.10), (3.4.4), above, and 2.19 of [24]. □

Applying the definition (4.3.3) to (4.3.4) and fixing a dimension, we get:

4.3.5 Proposition: In the notation of (4.3.4):

(i) $\overline{\Theta}^q_J(u.v) = \sum_{r=0}^{q} \sum_{i+j=q-r} p^r M_r(\overline{\Theta}^i_J(u), \overline{\Theta}^j_J(v))$

(ii) $\sum_{r=0}^{q} p^r\, \Theta^{q-r}_J(M_r(u,v)) = \sum_{i+j=q} \Theta^i_J(u).\Theta^j_J(v).$

4.3.6 Corollary: In the notation of (4.3.4): (i) If J is a ring homomorphism then:

$$\overline{\Theta}^q_J(u.v) = \sum_{i+j=q} \overline{\Theta}^i_J(u)\ \overline{\Theta}^j_J(v).$$

(ii) If J is such that $J(uv) = J(u)\,J(v)$, then:

$$\Theta^q_J(u.v) = \sum_{i+j=q} \Theta^i_J(u).\Theta^j_J(v).$$

What (4.3.5) tells us is that, on the pseudo primary level, a product formula with a general splitting isomorphism contains a series of "error terms", the images of the M_i's, for $i \geq 1$. The main result of this section is that this presence of error terms is a pseudo primary effect, only. That is to say, a pyramid of higher order operations, although built over a base of pseudo operations, exhibits a product behaviour that does not include these error terms.

We shall require the parallel notion to (4.2.15), namely an admissible pair of a given order for a relation of products, in place of compositions:

4.3.7 Definition: Suppose we are given a relation of the form:

$$(\bar{\Theta})^q_J(uv) = \sum_{i=1}^{r} \alpha_i \, (\bar{\Theta})^{q-s_i}_J(u) \cdot (\bar{\Theta})^{s_i}_J(v),$$

where all of the coefficients, α_i, are prime to p. Suppose, moreover, that there exist pairs, $[J,uv]$, $[J,u]$ and $[J,v]$, such that: (i) $[J,uv]$ is a $(q,N,0)$-pair of the first (or second) kind, (ii) $[J,u]$ is a $(q-s_1 * q-s_2 * \dots * q-s_r,\ B_1 * B_2 * \dots * B_r,\ 0 * \dots * 0)$-mixed pair of the first (or second) kind and (iii) $[J,v]$ is a $(s_1 * s_2 * \dots * s_r,\ C_1 * C_2 * \dots * C_r,\ 0 * \dots * 0)$-mixed pair of the first (or second) kind, where $B_i + C_i = N+1$, for $1 \leq i \leq r$. If N is the greatest integer for which this can be done, we shall say that $[J, u * v]$ is an admissible product pair of order N, for the given relation.

Suppose, now, that u and v are elements in $H^{2n}(X)$ and $H^{2l}(X)$, respectively, for some space, $X \in F_p$. As usual, we denote by x and y the respective $\mathbb{Z}_p$-reductions of these elements. What we propose to show is that, under these circumstances, one may generate a pair of "higher order Cartan formulae" provided certain higher order mixed operations are defined on x and on y. More precisely, we have:

4.3.8 Theorem: (i) In the notation given above, let us suppose that there exists some higher order mixed operation,

$$\bar{\Phi}^{B_1*B_2*\ldots*B_{q+1},0*\ldots*0}_{q*q-1*\ldots*0},$$

which we shall denote by $\bar{\Omega}$, which is defined on x. Suppose, moreover, that there exists some

$$\bar{\Phi}^{C_1*C_2*\ldots*C_{q+1},0*\ldots*0}_{0*1*\ldots*q}$$

which we shall denote by $\bar{\Psi}$, which is defined on y, such that $C_i+B_i = N+1$, for $1 \le i \le q+1$. Then, $\bar{\Phi}^{N,0}_q$ is defined on $(x.y)$ and, modulo the total indeterminacy, we have (where $\wedge$ means the sum of the result of termwise multiplication):

(4.3.9.i) $\bar{\Phi}^{N,0}_q(x.y) = \bar{\Omega}(x) \wedge \bar{\Psi}(y)$.

(ii) With the same notation, let us suppose that there exists some operation,

$$\Phi^{B_1*B_2*\ldots*B_{q+1},0*\ldots*0}_{q*q-1*\ldots*0},$$

which we shall denote by Ω, such that it is defined on x. Suppose, moreover, that there exists some

$$\Phi^{C_1*C_2*\ldots*C_{q+1},0*\ldots*0}_{0*1*\ldots*q}$$

which will be denoted by Ψ, such that it is defined on y, where $C_i+B_i = N+1$, for $1 \le i \le q+1$. Then, modulo the total indeterminacy, we have (with $\wedge$ as above):

(4.3.9.ii) $\Phi^{N,0}_q(x.y) = \Omega(x) \wedge \Psi(y)$.

Proof: (i) We shall proceed by induction on the order, N. The primary case is trivial as it is just the ordinary Cartan fomula for the reduced Steenrod powers. Consider the case of order 2. By (4.3.5-i) we have, mod p^2:

$$(4.3.10)\quad \bar{\Theta}_J^q(uv) = \sum_{i+j=q} \bar{\Theta}_J^i(u).\bar{\Theta}_J^j(v) + p \sum_{i+j=q-1} M_1(\bar{\Theta}_J^i(u),\bar{\Theta}_J^j(v)).$$

By hypothesis, we know that every term in (4.3.10) is divisible by p. Which term in a product of terms is actually divided by p is determined by the existence of pairs. In this way we divide (4.3.10) by p to get, modulo p:

$$(4.3.11)\quad \bar{\Phi}_q^{2,0}(xy) = \Sigma\, \bar{\Phi}_i^{B_i,0}(x)\ .\ \bar{\Phi}_j^{C_j,0}(y) + T.$$

Here, for each pair, (i,j), such that $i+j = q$, it is clear that (B_i,C_j) is either (1,2) or (2,1), depending on which term of the product admits a pair of order 2. The value of T is given by:

$$(4.3.12)\qquad T = \rho_*'[\sum_{i+j=q-1} M_1(\bar{\Theta}_J^i(u),\bar{\Theta}_J^j(v))].$$

Let us consider an arbitrary term of T, $M_1(\bar{\Theta}_J^i(u),\bar{\Theta}_J^i(v))$ for $i+j=q-1$. By definition, this is the first "error term" in the product $\bar{\Theta}_J^i(u).\bar{\Theta}_J^j(v)$. But, by hypothesis, $[J,u*v]$ is an admissible product pair of the second kind for (4.3.6-i). It follows that $\bar{\Theta}_J^i(u).\bar{\Theta}_J^j(v)$ is p-divisible. Now, by definition of M_1, we see that this error term must also be p-divisible. This is true for every term within the square brackets in (4.3.12). It follows, then, that T must be zero since it is the mod p reduction of a sum of p-divisible terms. This proves the theorem for second order operations of the second kind.

As inductive hypothesis, let us assume that we have the formula

(4.3.9-i) for $N \leq s$. We wish to show that it holds for (s+1) order operations as well. Consider:

$$(4.3.13) \qquad \bar{\Phi}_q^{N,0}(xy) = \sum_{i+j=q} \bar{\Phi}_i^{B_i,0}(x) \,.\, \bar{\Phi}_j^{C_j,0}(y) + T'.$$

This is (4.3.5-i) evaluated mod p^{s+1}, divided by p^s. As above, T' will be the error terms of operations of order up to and including s. By inductive hypothesis, these terms have "good" Cartan formulae and, hence no error terms. It follows that T' = 0.

This completes the proof of (i).

(ii) The second half of (4.3.8) follows almost immediately from (4.3.5-ii). Evaluating this equation mod p^N and dividing by p^{N-1} gives:

$$(4.3.14) \qquad \Phi_q^{N,0}(xy) + \sum_{r=1}^{q} \Phi_{q-r}^{N-r,0}(M_r(x,y)) = \sum_{i+j=q} \Phi_i^{B_i,0}(x) . \Phi_j^{C_j,0}(y).$$

Since Im $\Phi_{q-r}^{N-r,0}$ is contained in Im $\Phi_q^{N,r}$ and this is contained in the indeterminacy of the right-hand side of (4.3.14), we have the desired result.

This completes the proof of (4.3.8). □

4.3.15 Remarks:

(i) Notice how in both halves of this proof, the actual value of $M_i(\ ,\ )$ was not needed. This, however, should not be surprising as the value of $M_i(\ ,\ )$ is dependent upon J and higher order operations allow J to vary.

(ii) Note that we have avoided the use of operations of type greater than zero in (4.3.8). In the case of operations of the second kind, this was done purely for the sake of notational simplicity. In the case of (4.3.8-ii), the reason is the following one. Suppose that we are given two $\mathbb{Q}_p$-vectors, $\{u_i\}$ and $\{v_j\}$, representing the $\mathbb{Z}_p$-vectors, $\{x_i\}$, $\{y_j\}$, respectively, with $0 \le i \le \alpha$ and $0 \le j \le \beta$. We set M equal to $\alpha+\beta$. Suppose, moreover, that for each pair (i,j) with i and j in their respective index sets, there exist suitable Ω_i and Ψ_j, in the sense of (4.3.8), defined upon x_i and y_j, respectively, with the sum of their order N+1. These hypotheses are sufficient to assure us that $\Phi_q^{N+M,M}$ is defined on the vector, $\sum_{i+j=k} (x_i \cdot y_j)$, $0 \le k \le M$, but are, actually, considerably stronger. They guarantee, namely, that $\Phi_{q-k}^{N,0}$ is defined upon $(x_i \cdot y_j)$, for each suitable pair, (i,j), such that i+j = k, and for each appropriate k. Consequently, we are in a position to apply (4.3.8-ii), repeatedly, once for each pair, (i,j).

(iii) There are several second order "Cartan formulae" in the literature to which (4.3.8) might be compared. Let us consider one in particular, 3.4 of [26]. If we take p to be odd, q = 3 and N = 2 and apply our (4.3.8-i) we get that if $\bar{\Phi}_{3*2*1*0}^{2*2*1*1,0*0*0*0} = \bar{\Omega}$ is defined on $x \in HZ_p^{2n}(X)$ and if $\bar{\Phi}_{0*1*2*3}^{1*1*2*2,0*0*0*0} = \bar{\Psi}$ is defined on $y \in HZ_p^{2l}(X)$, then $\bar{\Phi}_3^{2,0}$ is defined on $x \cdot y$ and that $\bar{\Phi}_3^{2,0}(x \cdot y) = \bar{\Omega}(x) \wedge \bar{\Psi}(y)$ in $HZ_p^{2n+2l+6m}(X)/\mathrm{Im}\ P^3$. If we take these same hypotheses and apply 3.4 of [26] (where we have taken k = 3 and j = 1), we get the same result but with greater indeterminacy. The value of Q in Kobayashi's result is Im $[P^3 + \mathfrak{f}^* HZ_p^{2n+2l+6m}(K(\mathbb{Z},2n) \times K(\mathbb{Z},2l))]$, where $\mathfrak{f}$ is the map $X \to K(\mathbb{Z},2n) \times K(\mathbb{Z},2l)$ defined by $\mathfrak{f}(z) = (g(z),h(z))$, where $g : X \to K(\mathbb{Z},2n)$ (respectively, $h : X \to K(\mathbb{Z},2l)$) is such that $g^*(\gamma^{2n}) = x$ (respectively, $h^*(\gamma^{2l}) = y$) with γ^{2i} the mod p reduction of the fundamental class of $H\mathbb{Z}^{2i}(K(\mathbb{Z},2i))$.

(iv) In [29], Milgram considers a new approach to higher order product formulae, namely the study of smash products of fiberings. The main result of [29] , 4.2.3, seems to agree with (4.3.8-i) (for N = 2 = p) when we restrict to the category F_p. Furthermore, the terms $E^{2n}(k)$ and $E^{2n}(n-k-1)$ in [29] appear to play a rôle parallel to that of our admissible product pairs. (See remark (v), below.)

(v) The result of (4.3.8) can be expressed in other terms. Equivalent with this theorem is the following result:

<u>*4.3.8' Theorem*</u>: (i) With x, y, u and v as above, we suppose that $[J, u * v]$ is an admissible product pair of order N for the relation:

$$(4.3.16\text{-i}) \quad \bar{\Theta}^q_J(u \cdot v) = \sum_{i+j=q} \bar{\Theta}^i_J(u) \cdot \bar{\Theta}^j_J(v).$$

Then, modulo the total indeterminacy, we have:

$$(4.3.17\text{-i}) \quad \bar{\Phi}^{N,0}_q(x \cdot y) = \sum_{i+j=q} \bar{\Phi}^{B_i,0}_i(x) \cdot \bar{\Phi}^{C_j,0}_j(y),$$

where $B_i + C_j = N+1$, for each pair (i,j) such that i+j = q.

(ii) Using the same notation, let us suppose that $[J, u * v]$ is an admissible product pair of order N for the relation:

$$(4.3.16\text{-ii}) \quad \Theta^q_J(u \cdot v) = \sum_{i+j=q} \Theta^i_J(u) \cdot \Theta^j_J(v) - \sum_{r=1}^{q} p^r \Theta^{q-r}_J(M_r(u,v)).$$

Then, modulo the total indeterminacy, we have:

$$(4.3.17\text{-ii}) \quad \Phi^{N,0}_q(x \cdot y) = \sum_{i+j=q} \Phi^{B_i,0}_i(x) \cdot \Phi^{C_j,0}_j(y),$$

where $B_i + C_j = N+1$, for each pair (i,j) such that i+j = q.

(vi) We note that $Q_r(\widetilde{H}^{ev}(F_p)) = \widetilde{H}^{ev}/(\widetilde{H}^{ev} \otimes \ldots \otimes \widetilde{H}^{ev})$ (where we have factored out (r+1)-fold products), the r^{th} indecomposable quotient associated with $\mathbb{Q}_p$-cohomology theory applied to spaces in F_p, is the associated graded ring of $Q_r(\tilde{K}^0(F_p))$. Consequently, our higher order operations, Φ and $\bar{\Phi}$, restrict to indecomposable quotients in $\tilde{H\mathbb{Z}}_p^{ev}(F_p)$.

We conclude this section with a pair of particularly "nice" product formulae for certain higher order operations. Let x, y, u and v be as above. We have:

4.3.18 Theorem: Suppose that $\bar{\Phi}_{2q}^{2q,q}$ is defined on x and on y. Then $\bar{\Phi}_{2q}^{q,0}$ is defined on x, y and on $x \cdot y$ and, modulo the total indeterminacy, one has:

$$\bar{\Phi}_{2q}^{q,0}(x \cdot y) = \bar{\Phi}_{2q}^{q,0}(x) \cdot y + x \cdot \bar{\Phi}_{2q}^{q,0}(y).$$

Proof: The hypotheses imply that the following congruences are satisfied for pairs $[J,u]$ and $[J,v]$:

(i) $\bar{\theta}_J^{2q-i}(u) \equiv \bar{\theta}_J^{2q-i}(v) \equiv 0(p^{q-1})$, for $0 \le i \le q+1$, and

(ii) $\bar{\theta}_J^{2q-j}(u) \equiv \bar{\theta}_J^{2q-j}(v) \equiv 0(p^{2q-j})$, for $q+2 \le j \le 2q-1$.

By (4.3.5-i), we have, modulo p^q:

$$(4.3.19) \quad \bar{\theta}_J^{2q}(u \cdot v) = \sum_{i=0}^{2q} \bar{\theta}_J^{2q-i}(u) \cdot \bar{\theta}_J^{i}(v) + \sum_{r=1}^{q-1} p^r \sum_{i=0}^{2q-r} M_r(\bar{\theta}_J^{2q-r-i}(u), \bar{\theta}_J^{i}(v)).$$

But, by the above congruences, (i & ii), we may rewrite (4.3.19) as:

$$(4.3.20) \quad \bar{\theta}_J^{2q}(u \cdot v) = \bar{\theta}_J^{2q}(u) \cdot v + u \cdot \bar{\theta}_J^{2q}(v), \text{ modulo } p^q.$$

Moreover, the hypotheses imply that each term in (4.3.20) is individually divisible by p^{q-1}. Stronger yet, they guarantee that each term in (4.3.20) admits the definition of $\bar{\Phi}_{2q}^{q,0}$. The result now follows from the definitions. □

4.3.21 Remark: Taking q = 2 in (4.3.18) yields the results of theorem 8.4 of [10]. As in (4.3.15-iii), however, the value of the indeterminacy has been reduced in our case. The differences in our value of Q and Adem's is, once again, the image of $\delta^* H^{2n+2l+2qm}(K(\mathbb{Z},2n) \times K(\mathbb{Z},2l))$.

A similar result to (4.3.18), couched in rather different terms, is the following. We take x, y, u and v as above. Suppose that $[J,u]$ and $[J,v]$ are both $(2q{*}q,\ q{*}q,\ 0{*}0)$-mixed pairs of the second kind, for some splitting isomorphism, J. Then we have:

4.3.22 Theorem: With the hypotheses given above, $[J, u * v]$ is an admissible product pair for the relations:

(i) $$\overline{\Theta}_J^{2q}(u \cdot v) = \sum_{i+j=2q} \overline{\Theta}_J^{i}(u) \cdot \overline{\Theta}_J^{j}(v) \quad \text{and}$$

(ii) $$\overline{\Theta}_J^{q}(u \cdot v) = \sum_{i+j=q} \overline{\Theta}_J^{i}(u) \cdot \overline{\Theta}_J^{j}(v).$$

Moreover, modulo the total indeterminacy, we have (where $\cdot$ is interpreted as being distributive over pairs):

$$\overline{\Phi}_{2q*q}^{q*q,0*0}(x \cdot y) = \overline{\Phi}_{2q*q}^{q*q,0*0}(x) \cdot y + x \cdot \overline{\Phi}_{2q*q}^{q*q,0*0}(y).$$

Proof: That $[J,u]$ and $[J,v]$ are both $(2q{*}q,\ q{*}q,\ 0{*}0)$-mixed pairs implies the following set of congruences:

(i) $$\overline{\Theta}_J^{2q-i}(u) \equiv \overline{\Theta}_J^{2q-i}(v) \equiv 0(p^{q-i}), \text{ for } 1 \leq i \leq q-1,$$

(ii) $$\overline{\Theta}_J^{q-i}(u) \equiv \overline{\Theta}_J^{q-i}(v) \equiv 0(p^{q-i}), \text{ for } 1 \leq i \leq q-1 \quad \text{and}$$

(iii) $$\overline{\Theta}_J^{2q}(u) \equiv \overline{\Theta}_J^{2q}(v) \equiv \overline{\Theta}_J^{q}(u) \equiv \overline{\Theta}_J^{q}(v) \equiv 0(p^{q-1}).$$

The result now follows from these congruences in a similar fashion to the way in which the result of (4.3.18) followed from the set of congruences listed in the proof (4.3.18). □

§4. *Some Calculations.*

In this section we shall be considering several applications of our higher order operations. These shall take the form of calculations on explicit cohomology elements. It will be convenient (and merciful) to restrict ourselves to the simplest case, when the type of our operation is zero. For higher types, the results can be readily generalised by taking sums.

We begin this short, rather eclectic, section by establishing some machinery. Let us, for the moment, restrict ourselves to the case where p = 2 and where the degree of our operation is even. Under these circumstances, (4.2.8) becomes:

$$(4.4.1) \qquad \sum_{i=0}^{2q-1} (k^i - k^{2q})\, \bar{\Theta}_J^{2q-i}\, \Theta_J^i \equiv 0 \pmod{2^{2q}},$$

for all odd k. By (3.4.2), this is equivalent with:

$$(4.4.2) \qquad \sum_{i=0}^{2q} k^i\, \bar{\Theta}_J^{2q-i}\, \Theta_J^i \equiv 0 \ (2^{2q}).$$

Now, since (4.4.2) is valid for all odd k, we may choose k = 1 and k = -1. Writing out (4.4.2) for these two values of k and adding them together yields:

$$(4.4.3) \qquad \sum_{i=0}^{q} \bar{\Theta}_J^{2q-2i}\, \Theta_J^{2i} \equiv 0 \pmod{2^{2q-1}}. .$$

Let us, now, apply this relation in a particular situation. Let X be a space in F_2 and let ξ be a complex k-bundle over X. We denote by T the Thom space of ξ and by u the $\mathbb{Q}_2$-Thom class (the image of the $\mathbb{Z}$-Thom class under coefficient inclusion). It follows from (4.4.3) that, mod 2^{2q-1}, we have:

$$(4.4.4) \qquad \bar{\Theta}_J^{2q} u = \sum_{i=1}^{q} 2^{2i}\, \bar{\Theta}_J^{2q-2i}\, ch_{k+2i}\, J(u).$$

Now, let T_i denote the i^{th} Todd polynomial. Using the well-known result that the Chern character commutes with the Thom isomorphism applied to the inverse of the Todd polynomial, we see that we have proved the following:

4.4.5 Theorem: With the above notation, modulo 2^{2q-1}, we have:

$$\bar{\Theta}_J^{2q} u = [\sum_{i=1}^{q} 2^{2i}\, \bar{\Theta}_J^{2q-2i}\, T_{2i}^{-1}(\xi)] \cup u$$

in $H^{2k+4q}(T)$.

4.4.6 Corollary: (i) With the above notation, let us suppose that the Chern classes of ξ are, for integral classes a,b,c and d: $c_1 = 4a$, $c_2 = 4b$, $c_3 = 2c$ and $c_4 = 4d$. Then the following third order operation is defined on $x = \rho_*^{\prime} u$:

$$\bar{\Phi}_4^{3,0}(x) = \rho_*^{\prime}[Sq^4(b) + d] \cup x,$$

modulo its indeterminacy.

(ii) With the above notation and $c_1 = 2a$, $c_2 = 2b$, $c_3 = c$ and $c_4 = 2d$ we have, modulo the indeterminacy:

$$\bar{\Phi}_4^{2,0}(x) = \rho_*^{\prime}[Sq^4(b) + d + ac] \cup x \quad .$$

Let us remain in the context of $p = 2$ and let us consider a particular space in F_2, namely $\mathbb{C}P^{\infty}$. We let z denote the image of the canonical two-dimensional generator under coefficient inclusion, $\mathbb{Z} \to \mathbb{Q}_2$. We shall denote the mod 2 reduction of z by $\tilde{z}$. We record the following obvious result for later use:

4.4.7 Proposition: If $\bar{\Phi}_q^{N,0}\tilde{z}^k$ is defined to be the non-zero element in $H\mathbb{Z}_2^{2q+2k}(\mathbb{C}P^{\infty})$ then the indeterminacy of $\bar{\Phi}_q^{N,0}$ is zero.

Proof: This follows directly from the defining criteria of $\bar{\Phi}_q^{N,0}$ and (3.4.25). □

Using (4.4.5), we may generalise theorem A of [18] (we use the above notation):

4.4.8 Theorem: Let k be some positive integer and consider the bundle, $k\eta$, where η is the canonical line bundle over $\mathbb{C}P^{\infty}$. Suppose that there is some integer q such that: (i) $2q < k$, (ii) $\binom{k}{t}$, the binomial coefficient, is divisible by 2^N, for all t, $1 \leq t \leq 2q-1$ and (iii) $\binom{k}{2q}$ is divisible by 2^{N-1} but not by 2^N. Then $\bar{\Phi}^{N,0}_{2q}(\tilde{z}^k)$ is defined and, with no indeterminacy, we have:

$$\bar{\Phi}^{N,0}_{2q}(\tilde{z}^k) = \tilde{z}^{k+2q}.$$

Proof: Note that the total Chern class of $k\eta$ is $\Sigma c_i = (1+z)^k$ and that $c_i(k\eta) = \binom{k}{i}z^i$. Furthermore, the Thom space of $k\eta = T(k\eta) \cong \mathbb{C}P^{\infty}/\mathbb{C}P^{k-1}$. Denoting by ι the collapsing map : $\mathbb{C}P^{\infty} \to T(k\eta)$, we see that $\iota^*(u) = z^k$, with u, the Thom class. Now, applying (4.4.5) in this situation, we get:

$$(4.4.9) \quad \bar{\Theta}^{2q}_J u = [D + 2^{N-1}z^{2q}] \cup u,$$

where D is some polynomial in the c_i, $1 \leq i \leq 2q-1$, and, by the hypotheses, is divisible by 2^N. The hypotheses, also assure us that $[J,u]$ is a $(2q,N,0)$-pair of the second kind. Dividing by 2^{N-1}, reducing mod 2 and applying ι^* to (4.4.9) gives that $\bar{\Phi}^{N,0}_{2q}(\tilde{z}^k) = \tilde{z}^{k+2q}$. That the indeterminacy vanishes follows from (4.4.7). This completes the proof. □

4.4.10 Corollary: (i) $\bar{\Phi}^{3,0}_4(\tilde{z}^8)$ is defined and, with zero indeterminacy, is equal to $\tilde{z}^{12}$.

(ii) $\bar{\Phi}^{2,0}_2(\tilde{z}^4)$ is defined and, with zero indeterminacy, is equal to $\tilde{z}^6$.

We remain in the context of $HZ_2^*(\mathbb{C}P^\infty)$. Let $\alpha(q)$ be the usual function which assigns to any natural number the number of ones in its dyadic expansion. We continue to use the above notation. We have:

4.4.11 Theorem: $\Phi_q^{\alpha(q+1),0}\tilde{z}$ is defined and equal to $\tilde{z}^{q+1}$, with no indeterminacy.

Proof: Let J be the splitting isomorphism for $\mathbb{C}P^\infty$ that assigns γ, the Hopf bundle over $\mathbb{C}P^\infty$, to the generator, z. The value of the Chern character in dimension 2(q+1) of γ is then, clearly, $z^{q+1}/(q+1)!$. Using the fact that $\nu_2(r!) = r-\alpha(r)$, (where $\nu_p(s)$ denotes the power of p in the expression resulting from writing s as a product of powers of distinct primes) and the definition of Θ_J^q we get:

$$(4.4.12) \qquad \Theta_J^q z = 2^{\alpha(q+1)-1} z^{q+1}.$$

Consequently, we may conclude that [J,z] is a $(q,\alpha(q+1),0)$-pair of the first kind. Thus, clearly,

$$(4.4.13) \qquad \Phi_q^{\alpha(q+1),0}\tilde{z} = \tilde{z}^{q+1}, \text{ for all } q \geq 1.$$

Now we must show that the indeterminacy is zero. By (3.3.34-i) we see that it is sufficient to show that

$$(4.4.14) \qquad \rho'_*[(\Theta_J^q z + \Theta_J^{q-1} z^2)/2^{\alpha(q+1)-2}]^{\cdot} = 0.$$

Now, J is clearly a ring isomorphism so we may apply (4.3.6-ii) to calculate $\Theta_J^{q-1}z^2$. Using (4.4.12), we see that (4.4.14) is equivalent to the following statement:

$$(4.4.15) \quad k = 2^{\alpha(q+1)-1} + \sum_{i=0}^{q-1} 2^{\alpha(q-i)+\alpha(i+1)-2} \text{ is divisible by } 2^{\alpha(q+1)-1}.$$

We consider the two cases, q being even or odd, separately. Firstly, let q be an even natural number. Then q-1 is odd and k becomes:

(4.4.16) $$2^{\alpha(q+1)-1} + \sum_{i=0}^{[\frac{q-1}{2}]} 2^{\alpha(q-i)+\alpha(i+1)-1},$$

where [x] denotes the integer part of x. Since $\alpha(a)+\alpha(b) \geq \alpha(a+b)$, we have shown (4.4.15) to hold.

Now let q be an odd number. In this case we may write k as:

(4.4.17) $$2^{\alpha(q+1)-1} + \sum_{i=0}^{(\frac{q-1}{2}-1)} 2^{\alpha(q-i)+\alpha(i+1)-1} + 2^{2\alpha(\frac{q-1}{2}+1)-2}.$$

Consider the value $2\alpha(\frac{q-1}{2}+1)-2$. Substituting the value 2n+1 for q, this becomes $2(\alpha(n+1)-1)$. We may now conclude the validity of (4.4.15) in the case q is odd by noting that:

(4.4.18) $$2(\alpha(n+1)-1) \geq \alpha(2(n+1))-1 = \alpha(n+1)-1.$$

This completes the proof. □

4.4.19 Remark: One might be tempted to think that (4.4.11) could be extended, using (4.3.8-ii), to the following:

(4.4.20) $$\Phi_q^{\alpha(q+r)-\alpha(r)+1,0}\tilde{z}^r = \tilde{z}^{q+r}, \text{ mod } 0.$$

Applying the "higher order Cartan formula" can produce additional divisibility, however. An easy counter-example to (4.4.20) is $\Phi_3^{1,0}\,\tilde{z}^3 = 0$.

We conclude this section with a result of a rather different sort from the ones above. Let E_8 denote the exceptional Lie group and let X be a finite H-space such that, as an algebra one has:

(4.4.21) $$H\mathbb{Z}_3^*(X) \cong H\mathbb{Z}_3^*(E_8) \cong E(x_3,x_7,x_{15},x_{19},x_{27},x_{35},x_{39},x_{47}) \otimes T_3(x_8,x_{20})$$

where E denotes an exterior algebra and T_3 denotes a polynomial algebra, truncated at height 3. Given an algebra, A, we shall write

PA and QA for the primitives and indecomposables of A, respectively. We claim that the algebra structure alone, is sufficient to prove that E_8 is "irreducible mod 3", that is to say, E_8 is not 3-equivalent to a product of non-trivial spaces. For the exceptional Lie group, irreducibility was shown in [11½]. In [25½], Kane shows via a rather complicated BP argument that the algebra structure alone, is sufficient to demonstrate irreducibility. We shall deduce this fact as a corollary of the following:

4.4.22 Theorem: With X as above, $\bar{\Phi}_3^{2,0}$ is defined on a generator, a_{46}, of $PH\mathbb{Z}_3^{46}(\Omega X)$ and takes the coset of a_{58}, a generator of $PH\mathbb{Z}_3^{58}(\Omega X)$, as value.

Proof: Let a_{22} be a generator in $PH\mathbb{Z}_3^{22}(\Omega X)$. In [25½] it is shown that one has the following relation:

$$P^0(a_{22}) = a_{58}. \tag{4.4.23}$$

Let us pass to $\mathbb{Q}_3$-coefficients and write b_{22} and b_{58} for $\mathbb{Q}_3$-liftings of a_{22} and a_{58}, respectively. Now, let k be any integer prime to 3 and q any natural number. It follows from 2.14 of [24] and (4.2.8) that:

$$(1-k^{qm})\,\bar{\Theta}_J^q - \sum_{i=1}^{q-1} k^{im}\,T^{q-i}(k)\,\bar{\Theta}_J^i \equiv 0(3^q), \tag{4.4.24}$$

where $T^q(k)$ denotes the pseudo operation given in (4.2.8):

$$T^q(k) = (1-k^{qm})\,\bar{\Theta}_J^q + (k-k^{qm})\,\bar{\Theta}_J^{q-1}\Theta_J^1 + \ldots + (k^{(q-1)m}-k^{qm})\,\bar{\Theta}_J^1\Theta_J^{q-1}. \tag{4.2.8'}$$

Applying (4.4.24) to b_{22} and taking q = 9 gives:

$$(1-k^{18})\,\bar{\Theta}_J^9\,b_{22} - k^{12}T^3(k)\,\bar{\Theta}_J^6\,b_{22} \equiv 0(3^4). \tag{4.4.25}$$

This follows since $PH^{50}(\Omega X) = PH^{54}(\Omega X) = 0$, by coprimitivity. By

(4.4.25) and (4.4.23) we see that there exists a $b_{46} \in PH^{46}(\Omega X)$ such that

$$(4.4.26) \qquad (1-k^6)\, \bar{\theta}_J^3\, b_{46} = (1-k^{18})\, b_{58}, \ \text{mod } (3^4).$$

Now since $\nu_3(1-k^{qm}) = 1+\nu_3(qm)$, (4.4.26) becomes

$$(4.4.27) \quad \bar{\theta}_J^3\, b_{46} = 3b_{58}, \ \text{mod } (9).$$

Since $PH^{54}(\Omega X) = 0$, we see that $[J, b_{58}]$ forms a (3,2,0)-pair of the second kind. This gives the result. □

4.4.28 Corollary: The X given above is not 3-equivalent with a product of non-trivial spaces.

Proof: If X were to be reducible mod 3, then x_{47} would have to be primitive. This would contradict the previous theorem. □

§5. *The $e_{\mathbb{C}}$-Invariant and the Hopf Invariant.*

We conclude this chapter with a brief section on the $e_{\mathbb{C}}$-invariant of Adams [5],[21] and its formulation in terms of our pseudo primary and higher order operations. The Hopf invariant, which is very closely related to the more general $e_{\mathbb{C}}$-invariant, will be discussed and the famous "Hopf invariant one" problem will be considered.

Let X be a CW complex of the following form:

$$(4.5.1) \qquad X = S^{2n_1} \cup e^{2n_2} \cup e^{2n_3} \cup \ldots \cup e^{2n_s}$$

for the $n_i \in \mathbb{Z}^+$ such that $n_1 < n_2 < \ldots < n_s$. Let h_i denote a generator of $H\mathbb{Z}^{2n_i}(X) \cong \mathbb{Z}$, for $1 \leq i \leq s$. These elements are of exact CW filtration $2n_i$ and we may, consequently, consider the images under some splitting isomorphism J. Say $J(h_i) = \bar{h}_i$ for $1 \leq i \leq s$. (We will deliberately confuse h_i and $k'_*(h_i)$). The reduced unitary K-theory of X is a direct sum of s copies of $\mathbb{Z}$.

The total Chern character of any of these elements, $\bar{h}_i$, is of the form:

$$(4.5.2) \quad ch(\bar{h}_i) = e_{i,s}h_s + e_{i,s-1}h_{s-1} + e_{i,s-2}h_{s-2} + \dots + e_{i,i+1}h_{i+1} + h_i$$

In (4.5.2), $e_{i,j}/\mathbb{Z}$ denotes the $e_{\mathbb{C}}$-invariant of the map that attaches the n_j^{th} cell in X to the i^{th} skeleton of X. Clearly $e_{i,j} = 0$ for $i > j$ and $e_{i,j}$ is the coefficient of h_j in the image of the n_j^{th} component of the Chern character of a class of exact filtration $2n_i$. Thus:

$$(4.5.3) \quad e_{i,j}h_j = ch_{n_i+(n_j-n_i)}J(h_i) = {}^1/p^{(n_j-n_i)}\ \theta_J^{(n_j-n_i)}(h_i).$$

This gives an integrality condition on the unreduced $e_{\mathbb{C}}$-invariant, namely that $p^{(n_j-n_i)}\ e_{i,j} \in \mathbb{Q}_p$.

4.5.4 Example: In 7.4 of [5], Adams considers $X = \mathbb{C}P^2$ and shows that $e_{1,2} = \frac{1}{2}$. Our formulation (4.5.3) shows that $2e_{1,2}h_2 = \theta_J^1(h_1) = h_2$ (for $p = 2$).

In the case where $s = 2$, we call $\rho'_*p^{(n_2-n_1)}\ e_{1,2}$ the mod p Hopf invariant for X. We have just seen in (4.5.4) an example of a space with mod 2 Hopf invariant one. Notice that we need not reduce mod $\mathbb{Z}$ because the action of ρ'_* serves to identify all the elements in a mod $\mathbb{Z}$ coset.

Let us, now, formulate the well known "Hopf invariant one" problem in terms of what we have just established above. We consider the case where p = 2 and refer the reader to [23] for the parallel proof when p is odd. For the even case, we also cite [22].

It is a classical result of Steenrod [36] that:

4.5.5 Theorem: Let K be a two cell complex = $S^q \cup_f e^{2q}$. That is, K is a q-sphere with a (2q)-cell attached by a map f, representing a class α in $\pi_{2q-1}(S^q)$. Then, the following are equivalent:

(i) $Sq^q : H\mathbb{Z}_2^q(K) \to H\mathbb{Z}_2^{2q}(K)$ is zero.

(ii) There is no element of Hopf invariant one in $\pi_{2q-1}(S^q)$.

It is also classical that if an element of Hopf invariant one does exist in $\pi_{2q-1}(S^q)$, then q must be of the form 2^s, for some $s \geq 0$ [9]. This follows because the Sq^q, for $q = 2^s$, are the only non-decomposable reduced Steenrod powers.

The famous result of Adams [1] regarding the Hopf invariant one problem will follow as a corollary of:

4.5.6 Theorem: Let $X \in F_p$ be such that $\dot{H}\mathbb{Z}_2^{n+2(q-s)}(X) = 0$ for $1 \leq s \leq 2+\nu_2(q)$ for q an even integer and such that $H\mathbb{Z}_2^{n+2(q-1)}(X) = 0$ for q odd. (Here, $\nu_p(q)$ is the function that assigns to an integer q, and a prime, p, the exponent of p when q is decomposed as a product of primes.) Then $\Phi_q^{2,0}$ is defined on every element of $H\mathbb{Z}_2^n(X)$.

Before we proceed with the proof, we recall the following technical lemma, due to Adams [4]:

4.5.7 Lemma: Let k be an odd integer equal to the generator of $G_8/\pm 1$ for q even and let k be a generator of G_4 if q is odd. (Here G_s denotes the group of units in $\mathbb{Z}_s$.) Then:

(i) $\nu_2(k^q-1) \equiv 2+\nu_2(q)$ for even q, and

(ii) $\nu_2(k^q-1) \equiv 1$ for odd q.

Proof of 4.5.6: Case I: n and q are even.

By the generalised Adem relations (see (4.2. 8), above) we have:

$$(4.5.8) \qquad (1-k^q)\,\overline{\Theta}^q_J = \sum_{i=0}^{q-1} k^i T^{q-i}(k)\,\overline{\Theta}^i_J\ , \bmod 2^q.$$

Let y be a $\mathbb{Q}_2$ representative of $\tilde{y}$ in $H\mathbb{Z}^n_2(X)$. By the vanishing hypotheses of (4.5.6) we may truncate (4.5.8) to get:

$$(4.5.9) \qquad (1-k^q)\,\overline{\Theta}^q_J(y) \equiv \sum_{i=0}^{q-3-\nu_2(q)} k^i T^{q-i}(k)\,\overline{\Theta}^i(y)\ , \bmod 2^q.$$

Choose k to be an odd generator of $G_8/\pm 1$. Now by (4.2. 8), the right hand side of (4.5.9) is zero mod $2^{3+\nu_2(q)}$. But (4.5.7-i) tells us that the left side of (4.5.9) is zero mod $2^{2+\nu_2(q)}$. Dividing (4.5.9) by $2^{2+\nu_2(q)}$, therefore, tells us that $\overline{\Theta}^q_J(y) \equiv 0 \bmod 2$. By (3.4.6) and (3.3.1), we obtain the desired result.

Case II: n is even and q is odd.

This is virtually the same as case I where (4.5.9) becomes:

$$(4.5.10) \qquad (1-k^q)\,\overline{\Theta}^q_J(y) = \sum_{i=0}^{2} k^i T^{q-i}(k)\,\overline{\Theta}^i_J(y),\ \bmod 2^q.$$

The left hand side is zero mod 2 and the right hand side is zero mod 4 by (4.5.7-ii). The result follows as in case I.

Case III: n is odd and q is either even or odd.

Since our pseudo operations are stable, we may consider the suspension of X in place of X. This reduces to either case I or II.

This completes the proof. □

4.5.11 Corollary: Unless n = 1,2,4 or 8, there exists no element of Hopf invariant one in $\pi_{2n-1}(S^n)$.

Proof: This classical result of Adams follows from (4.5.6) because $2+\nu_2(q) < q$ only when $2q > 8$. Setting $n = 2q$ shows that the only possible cases where Hopf invariant one may exist is when n = 2,4,8 or in the trivial case where n = 1. □

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