# Materiality in Numerical Cognition

## *Material Engagement Theory and the Counting Technologies of the Ancient Near East*










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Submitted: Trinity Term, 2016

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# MATERIALITY IN NUMERICAL COGNITION

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Using the Material Engagement Theory of Cognitive Archaeologist Lambros Malafouris as its framework, the thesis offers a unique synthesis of data from neuroscience, ethnography, linguistics, and archaeology to outline how number concepts are realized, manipulated, and elaborated. The process is described as an interactivity of psychological processes like numerosity, behaviors that manipulate objects into concept-generating stimuli, and material objects with semiotic qualities distinct from those of language and agency distinct from that of brains and bodies.

The counting technologies of the Ancient Near East (ANE) are then analyzed through archaeological and textual evidence spanning the late Upper Paleolithic to the Bronze Age, from the first realization of number concepts in a pristine original condition to their elaboration into one of the ancient world's greatest mathematical traditions, a foundation for mathematical thinking today. Insights from the way numbers are realized through psychological–behavioral–material interactivity are used to challenge three dominant conceptualizations of ANE numbers: first, the idea that the ANE numerical

lexicon would have counted only to very low numbers; second, that Neolithic tokens were the first counting technology; and third, that numbers were 'concrete' before they became 'abstract'. Considering archaeological evidence from the Epipaleolithic Levant and drawing on linguistic and ethnographic evidence to characterize the regional prehistory, the thesis suggests that the numerical lexicon would have included relatively high numbers prior to the Neolithic; that finger-counting (linguistically attested) and tallies (archaeologically attested) would have preceded tokens; and that numbers are 'abstract' concepts whose content changes in conjunction with the incorporation and use of different material forms. The evidence provided to support these alternatives implies that numbers may have originated in the late Upper Paleolithic and arithmetic early in the Neolithic, pushing the onset of these capabilities further back than is commonly held.

In addition to tallies and tokens, the thesis explores fingers and numerical notations as material artifacts, enabling an analysis of how materiality might structure numerical concepts, influence a number system's capabilities, limitations, and elaboration potential, and affect brains and behavior over cultural spans of time. Insights generated by the case study are then applied to the role of materiality in cognition more generally, including how concepts become distributed across multiple material forms; the reasons why materiality might be transparent (or invisible) in cognition; and the differences between *thinking through* and *thinking about* materiality.

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## ABBREVIATIONS

| | |
|---|---|
| ANE | Ancient Near East; Ancient Near Eastern |
| ANS | Approximate Number System |
| Ashm | Designator for an artifact held in the collection of the Ashmolean Museum, Oxford, England |
| Bisexagesimal System | Archaic counting system (B: cereal, bread, fish, and milk products; B*: rations) |
| bur | Archaic unit of area measurement |
| *ca*. | *circa* (Latin for 'around') |
| CDLI | Cuneiform Digital Library Initiative |
| CE | Common Era |
| Chalcolithic | 4500–3300 BC |
| DV | Designator for an artifact held in the collection of the State Hermitage Museum, St. Petersburg, Russian Federation |
| Early Bronze Age | 3300–2000 BC |
| ED | Early Dynastic period (I/II: 2900–2700 BC; IIIa/IIIb: 2700–2340 BC) |
| Epipaleolithic | 12,000 years before present to 8300 BC |
| ETCSL | Electronic Text Corpus of Sumerian Literature |
| $GAN_2$ System G | Archaic system for counting field measurements |
| *H*. | *Homo* (e.g., *Homo sapiens*) |
| IPS | Intraparietal sulcus |
| JA | Designator for an artifact held in the collection of the National Museum of Syria, Raqqa, Syria |
| JN | Jemdet Nasr period, 3200–3000 BC |
| JRL | Designator for an artifact held in the collection of the University of Manchester, England |
| KYA | Thousand years ago |
| *l*. | Liter |
| Late Upper Paleolithic | 30 to 12 thousand years ago |
| MA | Middle Assyrian period, 1400–1000 BC |

| | |
|---|---|
| MS, MW | Designators for artifacts in private, anonymous collections in Europe |
| NA | Neo-Assyrian period, 911–612 BC |
| NB | Neo-Babylonian period, 626–539 BC |
| Neolithic | 8300–4500 BC |
| NIM | Designator for an artifact held in the collection of the National Museum, Tehran, Iran |
| NMSR | Designator for an artifact held in the collection of the National Museum of Syria, Raqqa, Syria |
| NS | Not Specified |
| $N_{xx}$ | Numerical sign in an archaic number system (e.g., $N_1$, $N_{14}$) |
| OA | Old Akkadian period, 2340–2200 BC |
| OB | Old Babylonian period, 1900–1600 BC |
| Old Assyr. | Old Assyrian period, 1950–1850 BC |
| PW | Pre-writing period, 8500–3500 BC |
| rod | Archaic unit of area measurement |
| sar | Archaic unit of area measurement |
| Sb | Designator for an artifact held in the collection of the Louvre Museum, Paris, France |
| SB92 | Schmandt-Besserat's original token dataset, published in 1992 |
| ŠE system | Archaic system for counting grain (Š: barley; Š': malt; Š": wheat; Š*: barley) |
| Sexagesimal System | Archaic system for counting objects (S: most things; S': dead or spoiled things) |
| SNARC | Spatial–Numerical Association of Response Codes |
| *spp*. | *Species pluralis* (Latin for 'multiple species') |
| three + *x*, five + *x* | Compound number formations (e.g., 'three + two' is 'five') |
| $U_4$ System U | Archaic system for counting units of calendrical time (days, months, years) |
| UM | Designator for an artifact held in the collection of the University of Pennsylvania Museum of Archaeology and Anthropology, Philadelphia, Pennsylvania, USA |
| Uruk *X* | Uruk V period: 3500–3200 BC; IV: 3500–3350 BC; III: 3200–3000 BC |

| | |
|---|---|
| VAT | Designator for an artifact held in the collection of the Vorderasiatisches Museum, Berlin, Germany |
| W | Excavation number for an artifact found at the ancient city of Uruk (modern Warka), Iraq |
| WEIRD | Western, educated, industrialized, rich, and democratic |

# CHAPTER 1

# INTRODUCTION

## 1.1 Purpose

Humans rearrange the physical world, creating novel material combinations that offer new opportunities for brains to notice patterns. Over generations of this interactivity, material forms become increasingly fine-tuned to elicit specific psychological and behavioral responses. Perhaps nowhere else is this relation of long-term change between brains, bodies, and materiality so evident as in numbers, concepts shared by sets of objects arranged to compare their quantity.[1] Understanding such long-term change involves four essential elements. First, it requires a theoretical framework capable of reconciling interdisciplinary data and comparing cognitive states without prejudice. Second, it needs an archaeological sequence of sufficient extent and duration, like that of the Ancient Near East (ANE). Third, it demands the assumptions of the various disciplines providing data be reexamined. Finally, mathematical insights into what numbers are as concepts must ground the whole.

As the selected theoretical framework, Material Engagement Theory,[2] is explored in detail in Chapter 2, I will not comment further on it at this point, except to say I consider it both ideally suited to my purpose and critical to my success. None of the various disciplines investigating numbers have fully reconciled their observations with

[1] Russell 1910; 1920.
[2] Malafouris 2013.

those engaged in the same pursuit, nor have any of them seriously investigated the role of materiality in numerical cognition. Material Engagement Theory provided an optimal vehicle for providing, if not the realization of both ambitions, then at least a start, by allowing insights from neuroscience, ethnography, and linguistics to be synthesized with the material record data of archaeology and the textual evidence of Assyriology. The extent to which the desired synthesis was achieved I attribute to the theoretical framework; the extent to which it fell short, I take upon myself.

My selected case study was the sequence of material technologies used for counting in the ANE. Reconstructing the ANE sequence required evidence from both the archaeological and textual records. The artifactual evidence was drawn from the Levant in the Late Upper Paleolithic/Epipaleolithic (30 thousand years ago to 8300 BC) and Mesopotamia in the Neolithic (8300–4500 BC), Chalcolithic (4500–3300 BC), and Early Bronze Age (3300–2000 BC), space and time spanning the realization of initial number concepts to their elaboration into a complex mathematics. The textual evidence (numerical language and notations) was particularly important because it attested to the likelihood of multiple indigenous number traditions in the region, the similarities of numerical lexicons and organization in ANE and contemporary languages and numbers that warranted inferences about similarities in psychological processing, and the use of fingers as a material counting technology.

There were a number of assumptions to reexamine. The notions that humans have numbers because they have language[3] and that language therefore must be considered first and separately from all other contributing factors[4] can be criticized because not all humans have numbers, and when they do, linguistic labels occur secondarily to manuovisual

[3] Chomsky 2004a; 2004b; Everett 2013a; 2013b; Hurford 1987; 2007.
[4] Everett 2016.

means of accessing numerical intuitions[5]; further, language in holistic context has new explanatory value for numbers. In accepting the assumption of linguistic priority, neuroscience too has overemphasized language (thus shortchanging behavioral and material contributions), as it has the perceptual system for quantity (thereby obscuring the role of categorization and abstraction in number concept formation). Ethnography has dismissed the behaviors of anumeric peoples as mere compensation strategies, rather than recognizing pairing and one-to-one correspondence as the mechanism for generating number concepts. Assyriology has not incorporated the past several decades of research into numerical cognition by linguistics, neuroscience, and ethnography, enabling several assumptions about ANE numbers,[6] now outdated, to remain entrenched (discussed further in Section 1.4). For its part, archaeology has considered the counting technologies of the Levant and Mesopotamia in isolation, disconnecting them as parts of a larger sequence.

I based my understanding of what numbers are as concepts on what mathematicians think they are, for surely no one understands them better than they do. My primary insight into the nature of number came from the work of Bertrand Russell, one of the most established thinkers in the philosophy of mathematics. While there are some inherent shortfalls (e.g., Russell's theory of logical types was crafted to resolve a paradox in set formation,[7] whose further implications I do not explore; it is also representational, a perspective rejected by Material Engagement Theory[8]), his insight into the nature of number is influential in mathematics, accessible for the non-mathematician, and explanatory of behaviors and concept formation in numbers. Further, Russell's concept of number provided a basis for comparing numbers across cultures in a way that avoided characterizing some as 'concrete' and others as 'more abstract', terminology with implicit

[5] Overmann et al. 2013.
[6] Damerow 1996a; 1996b; 2010; 2012; Schmandt-Besserat 1992a; 1992b.
[7] Irvine et al. 2014.
[8] Malafouris 2013.

bias for Westernized constructs.[9] I also drew ideas from mathematical pedagogy, as mathematicians and mathematical educators have unique insights into numbers as abstract concepts (differing somewhat from the way abstraction is conceptualized in psychology) and what might be involved in learning and transforming basic concepts into more advanced ones.

## 1.2 Assumptions and Stipulations

I assumed that fingers and numerical notations could be analyzed as if they were physical devices like tallies and tokens in order to consider qualities like linearity, persistence, and manipulability in the longer sequence of ANE counting technologies. This approach was somewhat non-traditional, as neither fingers nor numerical notations are usually treated as materials; instead, numerical notations are usually thought of in terms of signs and symbols, fingers as biological supports to psychological capabilities like working memory. This assumption entails that the conceptual boundaries between the body and the material (for fingers), and the material and the psychological (for numerical notations), need to tolerate a degree of ambiguity in what they delimit. To bridge this ambiguity, I used Gibson's[10] notion of *affordance*, which is a relation between what an organism (or agent) is capable of and what its environment enables it to do (Chapter 2). Analysis of the affordances provided by the different counting technologies then allowed me to say something about how they were similar (e.g., fingers and tallies, in providing things like linearity) and how they differed (fingers only, in being part of the body, etc.).

I also considered language as being analogous to a material for counting,[11] as language not only provides an access to numerical intuitions that is critical to their realization, manipulation, and elaboration, it also develops and elaborates in conjunction

[9] Bloch 2012; Piaget 1952; Saxe 2012.
[10] Gibson 1977; 1979.
[11] Clark 2006a; 2006b.

with materiality (e.g., in absorbing productive cycles), provides insight into specific psychological processing (as grammatical number displays the characteristic signature of numerosity), and attests to past forms of materiality used in counting (especially the fingers). Indeed, numerical elaboration may be highly dependent upon having several ways of accessing numerical intuitions, with language and multiple forms of materiality with differing characteristics constituting such means of access. That this is the case is suggested by the stability of systems (e.g., body-counting in Papua New Guinea[12]), where both language and the material structure provide an ordinal counting sequence in relatively unmanipulable form.

I did not analyze the specific social conditions under which numerical cognition develops or the social purposes to which numbers are applied, since I assumed that ANE numerical realization, manipulation, elaboration, and use were consistent with and motivated by increased needs for managing expansions of scope and scale in endeavors like administration and trade. Certainly, these topics have been extensively covered by previous scholars.[13] Further, I am not sure that a longer timeline for developing ANE numbers substantially changes conventional accounts of Bronze Age developments in administration and trade. Similarly, observing that early Neolithic peoples would have quantified, for example, stored resources[14] strikes me as unnecessary because they were only doing what other peoples, present and past, are well known to do. That they have instead been characterized, rather exceptionally, as anumeric is the result of some now-outdated assumptions about the archaeological evidence and how numbers work, which I have done my best to update. If my update is successful, I am of the opinion that anyone still interested in defending the exceptional hypothesis has the burden of making the case.

[12] Lean 1992; Saxe 2012; also see Overmann 2015.
[13] E.g., Braidwood et al. 1983; Childe 1936; Hesse 2010; Lloyd 1978; Pollock 1999; Postgate 2013; Robson 2008; Schmandt-Besserat 1982; Steiner et al. 2013.
[14] Goring-Morris et al. 2011.

Lastly, I did not provide an exhaustive catalogue of artifacts used for counting in the Levant and Mesopotamia, since I assumed that representational artifacts would be sufficient to illustrate and analyze the role of such technologies in numerical cognition. Similarly, I did not greatly detail the geographic scope of the artifacts, as some (e.g., finger-counting) cannot be specified on the available evidence; further, in my opinion, the more compelling story, and one that has been hitherto untold, lies in such things as how the interplay of technologies develops the conceptual domain of numbers.

## 1.3 Numbers, Materiality, and the ANE

As concepts, numbers are discrete quantities (e.g., 'one, two, three, …') and the relations between them (2 + 2 = 4), which are manipulated using operations like addition and subtraction. Concepts of numbers, relations, and operations are realized from the interactivity of brains, bodies, and materiality, the components of the cognitive system for numbers.[15] While the functionality of brain and body in numerical cognition (e.g., psychological processes like quantity perception and language; bodily contributions like finger-counting and gesture) has been well studied, materiality is a particularly underappreciated component of the cognitive system for numbers,[16] though it undoubtedly plays a critical role: As the cognitive system involves materiality, the innate sense of quantity is made tangible and given structure, stability, and manipulability.[17] The sense of quantity becomes explicit, discrete, and persistent, affording opportunities to notice similarities and patterns, making it more likely concepts will be codified and endure in the form of artifacts, algorithms, and language.[18]

[15] Malafouris 2013.
[16] Malafouris 2013.
[17] While "the term 'innate' is undoubtedly vexed by its pretheoretic imprecision, [here it] means (1) determined by factors present from birth, (2) genetically inherited, (3) fundamental to representational structure, and (4) shared with other species" (Coolidge et al. 2012, 204–205; also see Malafouris 2010b).
[18] Overmann 2016d. Also see Chapter 2.

The ANE broadly consists of the Levant (the Mediterranean region containing the modern countries of Israel, Jordan, Lebanon, and Palestine, as well as portions of Syria and Turkey) and Mesopotamia (the region between the Tigris and Euphrates rivers encompassing the modern country of Iraq and the neighboring portions of Iran, Syria, and Turkey). The Levant was settled in the Late Upper Paleolithic and Epipaleolithic by small, semi-nomadic groups subsisting through hunting and gathering. Mesopotamia, settled later as climatic change enabled territorial expansion, includes the Neolithic transition marked by increasing sedentism (between 15,000 to 10,000 BC), animal husbandry and small-scale cereal cultivation (by 7000 BC), and the domestication of cattle (after 6500 BC).[19] Once climate change facilitated the rapid, dense settlement of lower Mesopotamia,[20] the Neolithic transition intensified to large populations living in dense urban centers with industrialized construction, massive workforces, and extensive agriculture. The scope and scale of these endeavors would have motivated the use and elaboration of numbers:

> … increasingly large cities, with populations in the tens of thousands, were far too huge and impersonal to be run along the lines of family, trust, and memory. … [By 3000 BC] temple administrators of Uruk adapted token accounting to their increasingly complex needs by developing the means to record not only quantities but the objects of account as well. Thus numeracy became literate for the first time in world history.[21]

The ANE is a particularly appropriate case for studying the role of materiality in numerical cognition, for of all the places where mathematics and writing have been independently invented—Mesopotamia, Egypt, China, and Mesoamerica[22]—Mesopotamia is associated with the longest sequence of materiality and texts for insight into the technologies used for counting. It is arguably the world's only instance in which there is sufficient material and linguistic evidence to reconstruct the development of a numerical system from a pristine original condition to its elaboration into a complex mathematics. It

[19] Robson 2008.
[20] Nissen 1988.
[21] Robson 2008, 28.
[22] Daniels et al. 1996; Houston 2008; Jackson 1981; Senner 1989; Woods 2010.

offers an unparalleled opportunity to examine not just language holistically (i.e., in the context of materiality for numbers) but materiality as well (in the context of language for numbers). Thus, the ANE has the potential to provide unique and important insights into the ways in which different material counting technologies inform the realization, manipulation, and elaboration of number concepts.

Four major counting technologies were used in the ANE:

(1) Fingers would likely have been the first material technology for counting in the ANE, as this would be consistent with the way counting emerges across cultures,[23] as well as evidence of 'five-plus' compound numbers in the Sumerian language[24] and productive cycles of 10 in the Sumerian, Akkadian, and Elamite number systems.[25] Unambiguous linguistic evidence for finger-counting does not appear until the mid-third millennium[26] but presumably represents numerical language developed earlier—though only a relative chronology can be inferred from the available evidence.

(2) Worked bones possibly used as tallies are archaeologically attested throughout the Levant from about 30 to 15 thousand years ago,[27] the place and time from which Mesopotamia would be settled. They are also mentioned in second-millennium texts.[28] Ethnographic parallels show that tallies are the kind of technology that follows finger-counting, likely because of attributes and capabilities they share with fingers. Tallies also represent a critical transition to using materiality as a collaborative medium.

[23] Andres et al. 2008; Ardila 2010; Bender et al. 2012; Menninger 1992; Overmann 2014.
[24] Edzard 1980.
[25] E.g., Blažek 1999; Englund 2004b; Nissen et al. 1993; Wilcke 2005.
[26] E.g., Edzard 1980.
[27] E.g., Coinman 1996.
[28] ETCSL 2005.

(3) Tokens were small, geometrically shaped objects made of clay found throughout Mesopotamia from the ninth to the first millennia.[29] Their most common shapes and sizes correspond to those of the first numerical notations, implying that tokens had similar numerical meanings.[30] Because tokens reflected the relations between things like containers of grain,[31] they were related to each other and not just the commodity they counted (making them two-dimensional[32]), and they were more manipulable than tallies were. These attributes would have helped numbers become more than a counting sequence by supporting the realization of additional numerical relations.

(4) Numerical notations appeared in the archaeological record in the mid-fourth millennium, first as impressions possibly made with tokens, then as proto-cuneiform numerical notations, and finally as numbers written in the cuneiform script. Because they are fixed, notations would have motivated the further explication of relations between numbers and the development of new algorithms[33] to manipulate them. Notations, relations, and algorithms enabled the development of a complex mathematics by the Old Babylonian period,[34] ideas later adopted by the Greeks[35] and ultimately bequeathed to the Western mathematics tradition.

Precursor technologies like tallies and fingers attest to the likelihood that ANE peoples realized, manipulated, and elaborated their numbers in ways typical to extant peoples. However, while there is significant commonality to the way numbers are realized

[29] Schmandt-Besserat 1992a.
[30] Nissen et al. 1993.
[31] Similar to the idea that four quarts equal a gallon.
[32] Chrisomalis 2010.
[33] Algorithms are step-by-step procedures for performing arithmetical or mathematical tasks; they decompose large tasks into a series of small decisions, like which of two numbers is bigger and by how much. Computations performed with abaci, calculators, and computers are algorithms (see Reynolds 1993; Stone 1972), as are written computations performed with pencil and paper (which place a greater demand on the memorization, recall, and application of arithmetic facts) and automated computations performed by computers.
[34] Høyrup 2002a; 2002b.
[35] Høyrup 1994.

cross-cultural, all number systems have unique aspects as well, and ANE numbers are no different in this regard: Numbers vary by things like how high they count; how they are organized, represented, and manipulated; and how their linguistic and material representations might differ from each other.[36] Nor do societies follow an identical path to realizing numbers, as the variability in extant number systems attests. Such variation reflects different choices made among the combinatorial options (e.g., as 'three' may be formed as 'two and one' or 'one and two'), as well as the different forms of materiality used to realize and represent numbers. Thus, ANE number concepts are particular to the ways in which the societies of that place and time realized and understood them.

## 1.4 Three Assumptions about ANE Numbers

I examined three assumptions about ANE numbers with the goal of presenting new interpretations based on the way numbers are realized through psychological, behavioral, and material interactions. First, I did not assume that tokens and early numerical notations were used with few lexical numbers but rather observed that by the time a material technology for counting reaches high numbers, a well-developed numerical lexicon is available. Second, I did not assume tokens were the first material counting technology but instead considered evidence of finger-counting and tallies that supported the idea that ANE peoples invented their numbers in typical ways. Finally, I did not assume that fourth-millennium numerical concepts were 'concrete' instead of 'abstract' or 'second-order'[37] but applied insights from the philosophy of mathematics (i.e., Russell's theory of logical types[38]) to characterize numbers as concepts and examine how such concepts are realized, manipulated, and elaborated through the use of material technologies. Once these assumptions have been challenged, the possibility that ANE numbers may have been

[36] Chrisomalis 2010; Divale 1999.
[37] Damerow 1996a.
[38] Russell 1910; 1920.

realized as early as the Late Upper Paleolithic, simple arithmetic early in the Neolithic, becomes more likely.

### 1.4.1 First Assumption: The ANE Numerical Lexicon Was Restricted

ANE linguists have assumed that the peoples of southern Mesopotamia who invented writing had a restricted numerical lexicon, possibly as late as the middle of the third millennium.[39] A restricted numerical lexicon is one that "does not effectively go above 20."[40] The assumption that the ANE peoples of the late fourth millennium had a restricted numerical lexicon can be traced to the circumstance that the earliest writing (including the numerical impressions that first appear in the late fourth millennium) cannot be definitively ascribed to a particular language,[41] and versions of words for numbers with sufficient phonetic information to determine pronunciation do not appear until the mid-third millennium,[42] many centuries after writing was first invented. Thus, early numerical signs are not associable with a specific language, causing doubt as to whether a numerical lexicon had developed to the capacity of the material technologies in use at the time (i.e., tokens and numerical impressions).

This assumption is difficult to reconcile with the way numerical lexicons develop in extant number systems, where the initial words for numbers in any language are 'one, two, three, many'[43] (i.e., numerical language starts with very low quantities), and they increase in conjunction with the capacity of the material technologies used for counting[44]

[39] E.g., Dahl 2015.
[40] Comrie 2013, 3. Some restricted number systems have as few as two numbers, and the language of the Pirahã people of Amazonian Brazil (also classified as a restricted number system) is said to have no numbers whatsoever (Dryer et al. 2013).
[41] Early writing lacks sufficient semantic and phonetic clues to determine the language that its inventors would have spoken (Hyman 2006). While many scholars have suggested that the language may have been Sumerian, the "most ancient civilization" of the ANE (Suzuki 2009, 8) and the first language that would be associated with writing several centuries after its invention (Veldhuis 2014), there is no unanimity on this, a debate that has been called the Sumerian question (Englund 2006).
[42] Edzard 1980.
[43] Menninger 1992.
[44] E.g., Divale 1999; Overmann 2013b.

(i.e., while the development of language for numbers lags behind the materiality for numbers, it does not do so for long, or create dramatic disparities between materially and linguistically represented numbers). In comparison, Mesopotamian tokens thought to have represented quantities in the several hundreds have been dated to between 8500 and 3500 BC,[45] suggesting there would have been a comparable numerical lexicon in place by that time. Further, restricted number systems are associated with small, relatively isolated societies,[46] while the widespread urbanism, trade, and attendant resource management needs of the late Neolithic suggest that the associated numerical lexicon would have been consistent with the capacity of ANE counting technologies.

Against the view that there was at most a restricted numerical lexicon as recently as the late fourth millennium, I have presented linguistic and archaeological evidence to support the ideas that ANE numbers would likely have existed before the late fourth millennium, possibly for a significant period of time, and would have taken the form of an ordinal counting sequence that would have been coextensive with the capacity of tokens and early numerical impressions.

### 1.4.2 Second Assumption: Tokens Were the First ANE Counting Technology

ANE archaeologists have assumed that Neolithic clay tokens were the first ANE material counting technology.[47] This assumption likely originated because the Mesopotamian archaeological record was considered in relative isolation[48] from the archaeological record of the Levant (eliminating evidence of possible tallies used earlier than tokens and numerical impressions), linguistic evidence in archaic texts (eliminating

[45] University of Oxford 2015.
[46] Divale 1999; Epps et al. 2012; Overmann 2013b.
[47] E.g., Damerow 2010; Schmandt-Besserat 1992a.
[48] In the penultimate chapter of *Before Writing* Volume I (1992a), Schmandt-Besserat described archaeological evidence of possible Levantine tallies, as well as linguistic evidence of possible finger-counting and 'one-two-three' counting, but she did not explore the implications of these as material precursors to tallies and numerical notations.

evidence of finger-counting), and the organization of early numerical impressions (eliminating consideration of the number system complexity that is more consistent with a long period of use than the realization of initial numbers). For example, though Schmandt-Besserat recognized Paleolithic tallies as precursing Neolithic tallies, this was in the context of record-keeping: Tokens were likely "the second step in the development of record keeping, following Paleolithic tallies."[49] The importance and necessity of such one-dimensional precursors were not developed; certainly, it was not likely that they could be, given that the necessary insights from neuroscience, ethnography, and linguistics had themselves not yet been realized.

If the assumption were true and tokens were the first material technology used for ANE counting, it would make Mesopotamian numerical invention a surprisingly exceptional case, given that neuroscientific, ethnographic, and linguistic evidence suggest that fingers are typically the first material technology used. Ethnographic and linguistic data attest to the use of fingers in the form of observed counting behaviors, embodied vocabularies (e.g., 'digit' means both 'finger' and 'number' in many languages), numerical bases (bases 10, 5, and 20 are the most prevalent among the world's number systems[50]), and compound number formation (e.g., a word for 'six' that means 'five plus one', 'seven' as 'five and two', etc.[51]). Neuroscience explains the cross-cultural ubiquity of finger-counting, as well as its persistence even in highly elaborated number systems, as a function of the interaction between the parts of the brain that appreciate quantity and control the fingers[52] (respectively, the intraparietal sulcus [IPS] and angular gyrus).

Instead of assuming that tokens were the first ANE material counting technology, I have presented evidence from neuroscience, ethnography, and linguistics to provide

[49] Schmandt-Besserat 1992b, xii.
[50] Comrie 2013.
[51] Edzard 1980.
[52] E.g., Roux et al. 2003.

insights into the way number concepts are realized generally, and then applied these insights to the ANE material and textual records. Further, I considered artifacts from the Upper Paleolithic Levant and linguistic evidence from third-millennium texts as evidence of tally precursors and finger-counting, respectively, to Neolithic clay tokens and Bronze Age numerical notations.

### 1.4.3 Third Assumption: ANE Numbers Were 'Concrete' before They Were 'Abstract'

ANE scholars have assumed that early token-based accounting represented a 'concrete' concept of number, while the invention of writing—specifically, the separation of three-dimensional clay tokens into two-dimensional numerical impressions and graphic labels—enabled the development of an 'abstract' or 'second-order'[53] concept of number.[54] For example, "the litany is often repeated that the Mesopotamians were incapable of abstract thought, that their languages lacked terms to express concepts."[55] The assumption of a 'concrete–abstract' distinction resulted partly from the significant change in the material representation of number created by writing, which separated numerals from their commodity labels,[56] and partly from the contrast of the early token-based accounting of the late fourth millennium with the complex mathematics of the Old Babylonian period.

The separation of numerals from commodity labels in particular has motivated accounts of early numbers in which ANE peoples purportedly lacked an ability to distinguish counter from counted and used one-to-one correspondence for token-based counting.[57] The term *one-to-one correspondence* generally means that the members of one set of objects are iteratively matched to the members of a second set, with no members

[53] Damerow 1996a.
[54] E.g., Damerow 2010; Malafouris 2010b; Robson 2007; Schmandt-Besserat 1992a.
[55] Glassner 2000, 55.
[56] Amiet 1972; Schmandt-Besserat 1992a.
[57] Schmandt-Besserat 1992a.

unmatched or matched more than once; typically, one set (perhaps the fingers or a tally) is then used to represent the cardinal number of the other set. For example, fingers might be put into one-to-one correspondence with kernels of corn.[58] The idea of an inability to distinguish counter from counted has discounted a plausible alternative: Conjoint representation may have simply been a limitation of the material form, rather than indicating conceptual conflation. Indeed, when a hunter–gatherer counts days or kills with marks on his arm or the ground,[59] he likely does not confuse the marks with either but simply represents the quantity of what he counts with the marks and maintains the knowledge of what he counts separately, either in memory or context. The use of one-to-one correspondence would have constituted a particularly cumbersome and inefficient method of manipulation quite at odds with the capacity and organization of tokens and early numerical notations, as well as the administrative requirements in place by the late fourth millennium.

The 'concrete–abstract' distinction has been described as originating in Piagetian developmental theory,[60] which was based on observations of childhood cognitive development but when applied to cultures divides them into 'primitive' and 'advanced' states and assumes the transition between the two is unilineal progress.[61] For example, writing about the cognitive significance of Neolithic clay tokens, Historian of Science Peter Damerow stated that the ANE Neolithic bore

> … the closest resemblance with what we know from extant indigenous cultures at a stone age level. If, therefore, Piaget's assumption is correct

[58] Intuitively, there is something different between comparing two sets of objects physically present for manipulation and comparing one set that is present with one that is not (i.e., one of the items is distally referenced, or it is conceptual). That is, the ethnographic literature describes people counting days with marks on the body or ground, with 'days' being conceptualizations of diurnal transitions rather than manipulable physical objects. In the case of Mesopotamian numbers, tokens are thought to have represented objects like sheep and grain, but these objects were likely not physically present when the tokens were manipulated. This seems intermediate between the previous two examples, the first involving two present objects (i.e., fingers and corn), the second involving an object that is present and an object that is conceptual (marks and days).
[59] Morgan 1852; Pelleschi 1896.
[60] Chrisomalis 2005.
[61] Damerow 1996b.

> that cognition in such cultures does not exceed the preoperative level, then the Neolithic Revolution in spite of the material progress did not fundamentally change the level of cognition achieved at the end of the Paleolithic Period.[62]

Addressing the assumption that ANE numbers were 'concrete' before becoming 'abstract' is not merely a matter of updating the terminology used to describe it because early and later ANE number concepts undoubtedly differed from one another in significant ways. That is, if the 'concrete–abstract' terminology is inapt or outdated, it nonetheless labels real phenomena. Thus, I have characterized early and later ANE numbers in terms of numerical abstraction, conceptual development, and change according to Russell's concept of what numbers are[63] and how they are realized, manipulated, and elaborated through the perspective of Material Engagement Theory.

## 1.5 Challenges and Limitations

My reconstruction of the materiality used in realizing, manipulating, and elaborating numerical concepts in the ANE was complicated by several factors, including limitations inherent in the archaeological and linguistic evidence, as well as challenges in the use of interdisciplinary data, inferential argumentation, and ethnographic research.

### 1.5.1 Limitations of the Archaeological and Linguistic Evidence

Across the globe, the realization of initial numbers by extant peoples typically involves perishable materials (e.g., fingers, notched wood, knotted string, marks on the body or the ground, etc.).[64] Thus, the material record, considered in isolation of other evidence, is likely to underestimate the time depth for numerical cognition. This

[62] Damerow 1996b, 18.
[63] As will be discussed in Chapter 3, in his theory of logical types, Russell (1910; 1920) defined numbers as concepts of cardinality shared between sets of objects; as such, numbers are abstract from their inception, but as argued here, their conceptual content changes in conjunction with changes in their material realizer(s) and they are materially realized no matter how elaborated they become.
[64] Overmann 2016d.

circumstance motivates an examination of evidence from other domains (e.g., linguistic evidence from ANE texts). However, language produces only a relative chronology, not the absolute chronology made possible with dating techniques. Thus, while Sumerian texts from the third millennium attest to compound number word formations that imply finger-counting and it is reasonable to presume the texts reflected preexisting linguistic practices made explicit by the increasing expressive of the writing system,[65] there are no techniques to derive a more precise chronology, given that data before the fourth-millennium invention of writing is non-existent and Sumerian was a linguistic isolate (i.e., there are no available data or comparisons). The identification of fingers as the first material technology in the ANE therefore rests on the observation that this appears to be the case in extant number systems (as ethnographically and linguistically attested) and is neurologically explainable.

### 1.5.2 Limits of Interdisciplinarity and Inferential Argumentation

Drawing on evidence from other domains, however, introduces problems associated with interdisciplinarity (e.g., different and sometimes conflicting theoretical assumptions between disciplines; the researcher's lack of sufficient familiarity with all evidentiary sources; etc.) and inferential argumentation (i.e., which yields explanations that are 'merely consistent' with the available data rather than conclusions based on testable hypotheses that exclude plausible alternative explanations). To address these, the research was fairly narrowly focused on numerical phenomena, and the data used to interpret ANE numerical cognition was grounded in the experimental or observational findings of neuroscientific, ethnographic, and linguistic research. In addition, the implications of conjoining a representational framework like Russell's theory of logical

[65] Overmann 2016a.

types[66] with a non-representational framework like Material Engagement Theory[67] is discussed briefly in Chapter 3. It was also hoped that an explanation of ANE numerical cognition that was consistent with data from multiple disciplines would be at least as satisfactory, if not more so, as one reached solely on the basis of archaeological data.

Implicit in the inquiry into ancient cognition are the differences between extant and ancient brains. Certainly, even extant brains are not all the same: Cross-cultural psychological research has demonstrated that cognitive domains like perception, reasoning, and memory can exhibit significant variation across and because of cultural contexts, often in unexpected ways; the phenomenon may not be universally appreciated because of a tendency to rely on data on subjects from Western, educated, industrialized, rich, and democratic (WEIRD) cultures, which has yielded an artificially narrowed view of psychological variation.[68] Although WEIRD individuals are only 12% of the world's population, they participate in 96% of the studies published in 2008 in the top psychology journals; further, psychological measures and tests tend to be based on WEIRD cultures and are thus optimized for WEIRD subjects, and as a result they may not accurately assess psychological functions in non-WEIRD subjects.[69] Examples include the initial tests used to assess intelligence; developed by WEIRD researchers, so-called standardized IQ tests assumed a familiarity with WEIRD culture that disadvantaged subjects who were not WEIRD, yielding results that did not take cultural differences (or, for that matter, differences of race, gender, or socio-economic status) into account.[70]

Anthropology has been slow to acknowledge cross-cultural psychological variation, for reasons that include the history of racist assumptions about WEIRD superiority and the growing awareness of the need to protect indigenous rights from

[66] Russell 1910; 1920.
[67] Malafouris 2013.
[68] Henrich et al. 2010.
[69] Henrich et al. 2010.
[70] Henrich et al. 2010.

hegemonic power differentials.[71] These legacies do not demand that all brains be viewed as identical, but rather suggest that cross-cultural psychological variation be considered as revealing shared cognitive potential and the role of materiality in shaping that potential in different ways. As the similarity of form in extant human brains does not preclude culturally mediated variation in psychological functioning, it cannot be assumed that extant and ancient cognition are highly similar, which limits inferences drawn about ancient brains based on data from extant ones. In addition, cognitive science is developing theoretical insights that allow assumptions about the nature of cognition to be questioned—for example, whether cognition is more computational or connectionist, the extent and nature of its representations, and the extent to which the body and materiality contribute. In particular, the idea that cognition is embodied and extended requires questioning the role of the body and materiality in cognition; they may not, after all, be mere indices of, but rather integral to, changes in the brain, as much as changes in the brain are related to behavioral and material developments.

Inferences about ancient brains based on contemporary neuroscientific insights are necessarily functionalist (i.e., unfalsifiable hypotheses because ancient brains are inaccessible to controlled experimentation). Functionalist arguments appear in both strong and weak forms (the latter is pejoratively known as the *just-so story*[72]). Constructing a viable functionalist argument involves several elements. First, the available data must be used appropriately, and certainly, insights from contemporary neuroscience provide the best available insights into human brain functioning and form, including that of ancient brains. Second, the weakness of the functionalist argument must be recognized (as in the caveat provided here). Third, other scientific principles must be applied, such as the

[71] Bloch 2012.

[72] The term comes from the *Just-So Stories* of Rudyard Kipling. An example is the story of "How the Leopard Got His Spots," in which the patterning is instantaneous created when an Ethiopian presses his fingers against the leopard's coat. Thus, a bad functionalist argument, like Kipling's just-so stories, provides a particularly specious and unscientific explanation.

preference for an explanation that is parsimonious (i.e., requires the fewest assumptions) over one that is non-parsimonious. In the present case, the role of materiality in contemporary numerical cognition is subject to falsifiability. If it is not so in ancient numerical cognition, it is nonetheless more parsimonious to assume that ancient numerical cognition (including its psychological component) bears affinities with extant numerical cognition, especially given the similarities of the material and behavioral components between the two, than it is to assume that there is no such affinity.

Along these same lines is the problem of generalization. Given the variability with which knowledge can be distributed within a society, especially novel or recent concepts whose representation has not yet conventionalized, it is risky to extend relatively limited observations to characterize entire cultures and languages, let alone the human species as a whole. This caveat applies equally to generalized explanations of a process like the realization of initial numbers and to specific constructs like *highest number counted*, a frequent measure of number system complexity.[73] Generalizations can be more wrong than right when reapplied to particular societies, and even closely associated individuals within the same cultural and linguistic group can vary widely in their knowledge of numbers. However, it is also true that generalizations can convey an accurate overall pattern that is useful (if appropriately caveated) for comparison with particular cases; for example, in societies with restricted number systems, individuals able to count beyond 20 are improbable.

### 1.5.3 Challenges in Using Ethnographic Research

Using ethnographic data is challenging. One of the complicating factors is that the act of observing changes what is observed: Psychological reactions to social and cultural differences mean that outsider accounts can never truly capture what might have happened

[73] Beller et al. 2008; Divale 1999; Overmann 2013b.

without observation. Another complication is that early contact ethnography, especially accounts recorded before the twentieth century, was almost invariably influenced by Western paradigms of cultural progress and notions of superiority and primitiveness, biasing both what was observed and the terms in which it was recorded.[74] Such bias persisted well into the latter half of the 1900s, as evidenced by the troubling implications of applying Piagetian developmental insights to entire cultures, as previously discussed. Recent observations have been additionally challenged by a growing awareness of indigenous rights to human dignity, national sovereignty, and self-determination, which have necessarily been privileged over the collection and analysis of data, and have generated particular sensitivities (e.g., socio-biological explanations of cultural or behavioral phenomena).[75] Finally, there is a sense that comparing ancient or industrialized peoples to extant cultures may characterize the latter as 'primitive', a tendency that obviously degrades rather than promotes dignity and respects sensitivities. Perhaps the most egregious example of this was the American mathematician Levi Leonard Conant, who stated (among many similar pronouncements in his seminal work, *The Number Concept: Its Origin and Development*) that "those races which are lowest in the scale of civilization, have the feeblest number sense also; or in other words, the least possible power of grasping the abstract idea of number."[76]

Nonetheless, the ethnographic literature, including early contact accounts, can provide valuable insight (albeit insight that must always be taken in context and used with respect) about social conventions and behaviors before much post-contact cultural diffusion had developed. This is especially important in numbers, given the ease with which numerical loan words are acquired. These accounts show that peoples without extensive words for numbers or concepts of quantity nonetheless quantify; they do so the

[74] E.g., Tylor 1871.
[75] Bloch 2012; Chagnon 2013.
[76] Conant 1896, 30.

ways that are consistent with the use of cognitive processes like quantity perception and categorization and behaviors like pairing, matters that demonstrate robust cross-cultural universality.[77] Most importantly, alinguistic quantification behaviors represent the process by which number concepts originate—the material engagements that provide opportunities for numerical concepts to develop and elaborate.

## 1.6 Organization and Outline of the Thesis

I have divided the thesis into eleven chapters organized as follows: The theoretical framework is discussed in Chapter 2, with a focus on how numerical cognition relates to the three central tenants of Material Engagement Theory: extended cognition, material agency, and enactive signification. What numbers are as concepts, as understood through Russell's[78] theory of logical types, is the subject of Chapter 3. Chapters 4 and 5 examine how initial number concepts are realized through the interactivity of brains, bodies, and materiality, using insights from neuroscience, ethnography, and linguistics. The context for interpreting the various material technologies used for counting in the ANE is discussed in Chapter 6. Specifically, language for numbers is analyzed to suggest that the region likely contained multiple indigenous number systems, as well as an extensive period of using numbers prior to the late fourth millennium. The three assumptions about ANE numbers are also addressed in this chapter.

Evidence of the material technologies used for counting is reviewed in Chapters 7 through 9: Evidence of finger-counting is drawn from third-millennium texts, evidence of possible tallies from the archaeological record of the Levant. Tokens are examined through the archaeological record of Mesopotamia, numerical notations through textual evidence and ANE mathematical analyses (especially the work of Mathematical Historians

[77] Henrich et al. 2010.
[78] Russell 1910; 1920.

Jöran Friberg and Jens Høyrup). Chapter 10 explores the role of materiality in numerical cognition, attempting to move the discussion beyond the "broad socio-material contexts motivating [the use of] numbers" as a form of external memory or a cognitive technology, to consider "the interactions where fingers bridge numerosity with materiality and manipulate it into new concept-generating stimuli."[79] Finally, the last chapter looks at whether the insights generated bear on issues of cognition and cognitive archaeology more generally.

Five appendices provide detailed data that would otherwise have significantly disrupted the narrative flow of the chapters. Appendix A presents a review of the literature showing that the psychological ability to appreciate quantity (numerosity) is widely distributed across a range of species. Appendix B examines and compares the frequency distribution of words for ordinal numbers (e.g., 'first, second, third, …') in English, Chinese, Arabic, and Sumerian. Appendix C analyzes the correspondences between numerical impressions and token shapes that form the basis of inferring the numerical meaning of tokens in the pre-writing period (*ca*. 8500–3500 BC). Appendix D provides the sources, references, and shapes of the tokens that were newly added to Schmandt-Besserat's[80] data. Finally, Appendix E offers an analysis of the numerical impressions on clay tablets assigned to the Uruk V period (*ca*. 3500–3350 BC).

[79] Overmann 2016c.
[80] Schmandt-Besserat 1992a.

# CHAPTER 2

# MATERIAL ENGAGEMENT THEORY AND NUMERICAL COGNITION[81]

## 2.1 Material Engagement Theory

The theoretical framework used to examine numerical cognition is Material Engagement Theory,[82] which is ideal for considering the respective contributions of brains, bodies, and materiality, as all of these are involved in numerical cognition. Particularly important is the theoretical framework's ability to integrate materiality as an integral part of cognition.[83] This is similar to the idea that perception involves not only processing in auditory and association cortex but sound waves striking the eardrum and something in the world producing noise, but it additionally assumes greater interactivity and bidirectional change between brains and materiality than is characteristic of the causal linkage of audition. Viewing materiality in this way necessarily redraws the boundaries of cognition: It is no longer something performed by the brain alone but a system that includes the body and the environment (including material culture), characterized by a dynamical interaction among the components that causes change at both componential and systemic levels over short- and long-term time spans.[84] This is not the view of mainstream neuroscience, which has just begun to recognize the body's contribution to cognition and

[81] A version of this chapter was published as Overmann 2016b.
[82] Malafouris 2013.
[83] Malafouris 2013.
[84] Malafouris 2013.

has seemingly not yet realized that materiality has any role, let alone a constitutive one. This focus is reasonable, given the investigative methodologies currently available and the need to limit scope and potential confounds. Nor is it certain that materiality is constitutive of cognition, calling its inclusion in cognition into question; nonetheless, seeing cognition through Material Engagement Theory's paradigmatic lens—considering it in terms of nontraditional boundaries—has the potential to yield new insights into how humans think.[85]

The new boundaries position material culture as an integral part of cognition, not merely its byproduct, assistant, external repository, or source of information over which its computations might range. Once material culture is viewed as part of a larger cognitive system, insights about the functionality of material culture within the system follow, as do insights about how human brains have evolved to exploit material structures. For example, the material environment, broadly speaking, consists of social institutions, cultural practices, and material artifacts. These allow problems to be decomposed into series of smaller tasks, which are more easily solved because they are smaller.[86] This effect is seen in numbers, when for example a complex task like multiplying 8912 by 7252 can be reduced to "a series of simpler problems beginning with 2 × 2."[87] Decomposing problems also makes it easier for multiple individuals to collaborate in solutions, opening up new possibilities for outcomes, in addition to the variation gained through "difference, localism, and choice"[88] and the fact that the involvement of multiple individuals increases the potential of realizing something novel (the 'many heads are smarter than one' effect). Solutions can also take the form of artifacts, making them available to other individuals

[85] Malafouris 2013.
[86] Beer 2003.
[87] Hodder 2012, 35.
[88] Robson 2008, xxii.

and future generations; this opens up further opportunities to refine or apply artifacts to new uses, affording additional possibilities for change.[89]

When material culture enables future individuals to perform tasks and solve new problems, cognitive effort becomes distributed over space and time, one of the major differences in how the human species uses materiality, relative to other species.[90] A calculator, for example, represents inventions of numbers, notations, algorithms, mathematical pedagogy, metallurgy, plastics, electronics, software, data storage, marketing, transportation, and so on, an amount of cognitive effort that no one individual or society could easily formulate and synthesize. Someone using a calculator to perform a mathematical task today is unlikely to be any of the people who invented the things that went into the device, which instantiates all sorts of knowledge, making it available to other individuals as a resource for performing tasks and discovering new applications. That is, someone with a slide rule need not reinvent calculus; he need merely apply the device's ability to manipulate numbers to the problem at hand.

However, as argued here, there are other processes at work, which will be called *persistence of material structure* and *conceptual distribution* (see Chapter 10). The first process is the structuring of concepts by the material forms involved either in their initial realization or through which they have subsequently passed, structuring that continues despite the incorporation of and transition to forms of materiality that lack the quality in question. For example, number concepts acquire linearity and stable order as a function of their realization through the use of material devices like fingers[91] and tallies, and linearity and stable order persist in the way tokens and numerical notations are ordered, even though those represent material forms that are not themselves inherently linear. As the structure of older material forms persists in the use of newer material forms, conceptual

[89] Damerow 2010.
[90] Hutchins 1995.
[91] Gelman et al. 1978.

content changes to incorporate characteristics of both (e.g., in addition to linearity and stable order, number concepts add the manipulability of tokens). The persistence of material structure distributes conceptual content across the range of material forms, past and present, which makes the concepts seem abstract of any particular material form and tends to obscure or make transparent the functionality of the material component of numerical cognition.

Cumulatively, processes of invention, structuring, learning, use, refinement, extension, and conceptual distribution afford opportunities for both artifacts and brains to change, especially when long spans of time are considered. Artifactual designs are refined and extended to become more efficient, capable, and tailored to support new or increasingly specialized cultural purposes, while brain functions become repurposed for cultural inventions (e.g., object recognition in literacy[92]). Over even longer spans of time, new brain functions or regions may come into being (e.g., regions of the human IPS specialized for detailed vision[93]). Modern calculation devices instantiate, in a very meaningful sense, much of this ancestral cognitive activity: Electronic calculators are not invented *ex nihilo* but rather from the number concepts, numerical algorithms and notations; supporting technologies; and repurposed and new brain functions realized through the past and present interactivity, interdependency, and multidirectional change of brains, bodies, and materiality.

## 2.2 Three Central Tenets

Taking the perspective that cognition is a complex system that includes not only brain but body and materiality provides an apt framework for examining numerical cognition, because all three contribute to it, though each in different ways. This

[92] Perfetti et al. 2013.
[93] Orban et al. 2006.

functionality will be viewed through Material Engagement Theory's three central tenets: First, *cognition is extended*. That is, in addition to the brain, the body and materiality not only contribute to but are constitutive of cognition. This positions the body and materiality as meaningful parts of numerical cognition (something seldom granted to materiality in particular, which tends instead to be viewed as merely "the stimulus that triggers or mediates some cognitive process"[94]), while preserving the brain's unambiguous and important role. It also provides a basis for comparing the numerical cognition of ancient and existing peoples, particularly through the material and behavioral components.

Second, in shaping behavior and psychological responses, *materiality has agency*: It influences the other components of cognition through its properties (or *affordances*, a term defined below), thereby shaping numerical system outcomes. This suggests that the materials used to realize and represent numbers have a particular influence on the way they are conceptualized. Third and finally, materiality has a semiotic function (or a meaning potential), making it an *enactive sign*. Simply, things acquire meaning in virtue of what they are and what people do with them. This is important in numerical cognition, where material meaning precedes language for numbers and change in material form can alter how numbers are conceptualized.

These central tenets will be applied to the sequence of material artifacts used as counting technologies in the ANE: fingers, tallies, tokens, and numerical notations. Considering these artifacts as the material component of numerical cognition allows an interpretation of their potential functions, properties, and signification, yielding new insights into numerical cognition, both ancient and extant.

[94] Malafouris 2010a, 265.

### 2.2.1 The Mind Is Extended

The idea that the brain is not all there is to cognition is a philosophical perspective known as the extended mind hypothesis. It is important to clarify that seeing cognition as extended beyond the body does not entail that the brain is uninvolved or unimportant; rather, the brain is a vital part of a complex, dynamic system that also includes the body and materiality.[95] As an illustration, writing a note to assist memory performs the same function as remembering without material assistance. Since they are functionally indistinct, the standard by which the second is deemed cognitive while the first is excluded can be questioned.[96] Once paper, pen, or the iPhone Notes application and acts of writing and reading have been admitted into the human memory system, the definition of cognition has necessarily been expanded from something performed by the brain alone to something encompassing material devices and the activities of interacting with them.[97] This is not to say that brain, body, and materiality play the same role within the cognitive system; rather, they are seen as contributing in different but complementary and mutually interacting ways.

The interaction between parietal functions for quantity perception (numerosity, which is shared by many species) and the fingers (finger gnosia, the ability to know the fingers; haptic perception, the sense of active touch; and neural reactions to tools and graspable objects) may inform another unique human capability: Engaging materiality with the fingers may have allowed humans to express the innate, alinguistic quantity perception in material form, making numerical properties tangible, visible, and manipulable, thereby bridging the mental and physical senses of quantity and facilitating

[95] Conversely, conflating mind or cognition with the brain is known as 'neurocentrism', a "trap … in which the brain is seen as the centre of the human world" (Renfrew et al. 2008, 381).
[96] Clark 2008; Clark et al. 1998.
[97] Estany et al. 2014; Malafouris 2013.

the formation of explicit concepts of number.[98] Few species use paws, hooves, or pads to explore and gain information about the world.[99] The unique capabilities of the hand that enable humans to use it for these purposes are consistent with the amount of sensorimotor cortex dedicated to its sensation and control.[100]

Without the ability to engage materiality with the hands and fingers, human quantity appreciation might otherwise have remained locked within the limits of the perceptual experience of quantity. More severely, a lack of material engagement might also have precluded the formation of explicit concepts of even the small quantities salient within the subitizing range, since the tactile manipulation of objects and reference sets appears integral to the process of realizing higher-order relations of shared cardinality for both subitizable and larger numbers.[101] The engagement of materiality by brains and bodies is thus vital, the very mechanism by which numerosity becomes human numerical cognition.

### 2.2.2 Materiality Has Agency

The second tenet of Material Engagement Theory is the idea that materiality has agency,[102] the capacity of an agent to act in the world. Agency is often conceptualized as something humans have, related to their abilities to form and pursue courses of action (sometimes called free will and often connoting intentionality). However, agency can also be understood as a relational property, something that causes change within a system; thus, material artifacts, in changing human behavior, display agency; thus, an agent need not be

[98] Coolidge et al. 2012; Malafouris 2010b.
[99] Gallagher 2013.
[100] Penfield et al. 1954.
[101] Overmann 2015.
[102] Malafouris 2013.

alive in order to act. For example, a speed bump insists that drivers slow down or risk car damage and the attendant need for repairs.[103]

However, material agency is no mere passive repository of human intentionality, like that of the fellows who design and install speed bumps to moderate or penalize driving behavior. A stone has agency in producing (or not) a sharp, usable edge, making it suitable (or not) for producing a handaxe, affecting outcomes of activities like hunting and informing decisions about whether a group returns to its source area; a totem has agency in enforcing taboos, altering beliefs and behavior in those who acknowledge its power[104]; the sextant has agency in enabling sailors to estimate latitude and longitude, improving their chances of reaching their desired destination.[105] Though neither equivalent to nor interchangeable with human agency, material agency is nonetheless complementary and relationally intertwined with human agency to the extent that it can be difficult to separate the two in any meaningful sense.[106]

While materiality has agency in being able to influence numerical cognition, the specific changes it is able to effect is a function of its *affordances*.[107] As theorized by psychologist James Gibson, affordances are what the environment offers to an agent, and they are found in the interaction between an agent's abilities and its environment, rather than being properties of either. Affordances are inherently related to an agent's abilities, both physical and mental; if a species lacks the ability to exploit a particular affordance, that affordance is not one for that species.[108] A monkey, for example, has little use for a calculator beyond mashing its buttons or using it as a projectile. While such employments undoubtedly fall within the human behavioral repertoire, people can also apply calculators

[103] Malafouris 2013.
[104] Gell 1998.
[105] Hutchins 1995.
[106] Knappett 2005.
[107] Gibson 1977; 1979.
[108] Gibson 1977; 1979.

to mathematical tasks (and certainly, many of them might not be able to perform such calculations without the affordances the calculator provides).

The example of the calculator illustrates the idea that affordances are not just features of the natural environment: They can be encoded in material artifacts, making them available for other individuals and generations and distributing cognitive effort over space and time.[109] When solutions to problems and enablers of task performance take the form of artifacts, their capabilities influence behaviors and outcomes in subsequent performances and problem-solving. An example is the sextant, a navigation instrument that instantiates the "kinds of [wayfinding] knowledge that would be exceedingly difficult to represent mentally."[110] The sextant (as a calculator does for mathematical knowledge) represents navigational knowledge that has been accumulated, applied, and improved by generations of individuals, as well as algorithms implicit in and automated through the artifact's design.

Artifactual structures reflect the requirements of the tasks for which they were created, just as their designs complement and compensate for human psychological and anatomical abilities and limitations. Their structures, designs, and capabilities shape and constrain task performance: The sextant entails that navigation involves measuring celestial altitudes; the calculator solves square roots when buttons that activate preformed algorithms are pressed in certain combinations. Such devices can also offer different and perhaps new possibilities for action, expanding the range of potential outcomes.

What do material agency and affordances look like in numerical cognition? Answering this requires first considering (briefly) what *cognition* is; a general definition is that it is the process of knowing, as well as what the process of knowing produces (i.e.,

[109] Hutchins 1995; Smith 2007.
[110] Hutchins 1995, 96.

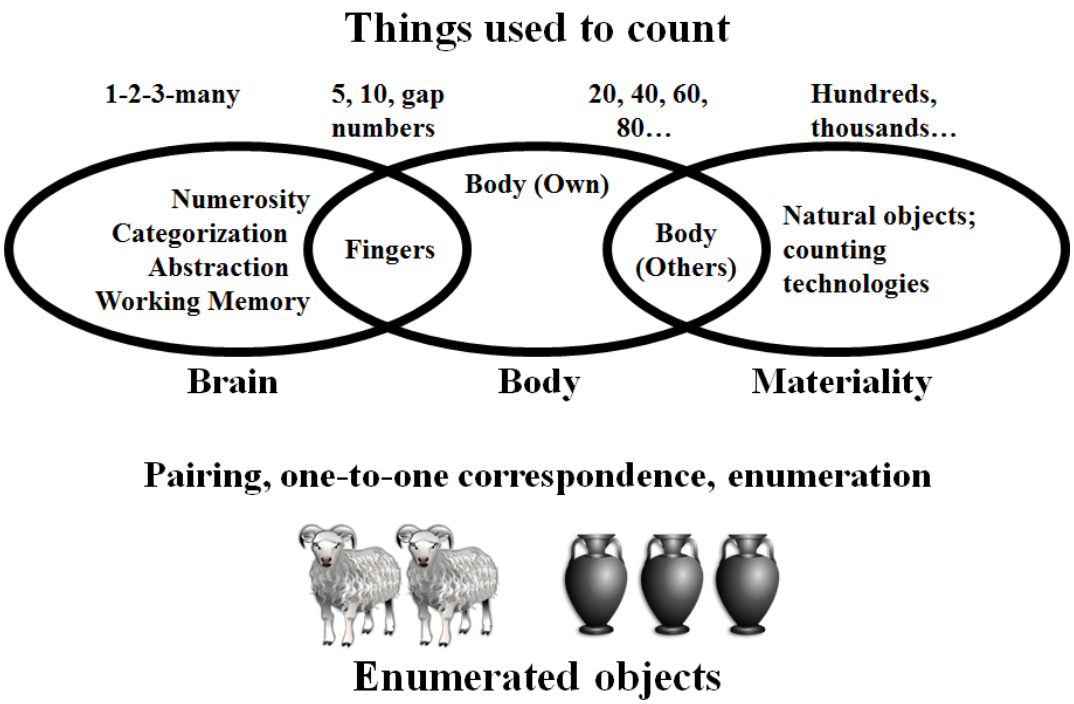



*Figure 2.1*. Conceptual model of counting in numerical cognition, based on the neuroscience of number, ethnographic observations about the use of the body and material devices for counting, linguistic data about characteristics that number words share across languages, and ideas about how the components contribute and interact when considered through Material Engagement Theory. The model is not reflective of any particular society or culture and should not be understood as implying either progress or linearity, but rather the typical changes that occur as materiality is incorporated into numerical cognition. Images from OpenClipArt.com.

Components include brain (left), body (middle), and materiality (right). The brain supports functions such as numerosity and categorization, while the body provides fingers that interact with the brain and materiality through processes like finger gnosia, haptic perception, and neural reactions to tools and graspable objects. Materiality includes both naturally occurring objects used as reference sets and artifacts made for counting—respectively, things like fingers and notched sticks. Brain, body, and materiality overlap to characterize their interactivity and because brains do not occur in a vat, nor bodies in a vacuum.

The model differentiates things used to count (top) from enumerated objects (bottom), with relations between the two achieved through strategies such as pairing (a behavior within the capabilities of non-human primates[111]; and one-to-one correspondence (behavior unique to humans). Highest number counted is related to the things used to count; it increases as materiality is incorporated into the system through the changes that result from the interaction of material affordances and agent abilities.

Number system characteristics suggest their origins. Initial quantity expressions are generally ‘one’, ‘two’, ‘three’, ‘big many’ and ‘small many’,[112] which are consistent with the neurological constraints on quantity perception and attention. Involvement of the body leaves traces in the form of anatomic bases (e.g., decimal) and embodied vocabulary like the word ‘digit’.[113] Material devices used as counting technologies are not necessarily limited to ‘three’ or anatomically patterned; their characteristic signature is the extension of highest number counted.

[111] Conway et al. 2001; Fragaszy et al. 2002.
[112] Menninger 1992.
[113] Comrie 2013.

behavior, concepts, etc.). It is important to realize that cognition is not synonymous with either conscious awareness or the inner linguistic monologue, each of which is only a mere fraction of cognition (i.e., most cognitive activity is neither conscious[114] nor put into words). Numerical cognition, then, produces numerical concepts. These include *numbers*, concepts of discrete quantity like 'one', 'two', and 'three'; *numerical relations*, concepts of how numbers are related to one another (e.g., as 'five' is 'two more' than 'three'); and concepts about how numerical relations might be manipulated by arithmetical operations like addition and subtraction (these definitions are explored in greater detail in Chapter 3).

Numerical cognition is thinking with numbers, a characterization that is not intended to limit its scope to numeracy, or reasoning with numbers, but rather to include the process by which number concepts are realized through the interaction of brains, bodies, and materiality, a model of which is proposed in Figure 2.1. Simply, numerical concepts and words like 'one', 'two', and 'three' result from the interaction of the perceptual system for quantity, the fingers, and material objects; in this regard, fingers act as a critical bridge between the psychological, behavioral, and material components of cognition.

As materiality—both the body and artifacts—is incorporated into the cognitive system for numbers, the innate sense of quantity that is common to many species is given opportunities for explication and elaboration in a way that is unique to humans. Tangibility yields structure and stability and (depending on the materiality that comprises the artifact) a certain amount of manipulability for the quantity instantiated. Once explicit, quantity also becomes discrete, and its material form gives it persistence and allows its representation. These qualities increase the likelihood that someone interacting with the artifact will notice similarities and patterns in the quantity it instantiates, as for example

[114] Kihlstrom 1987; 1989.

each new notch on a tally being one more than the previous notch. Patterns and similarities, in turn, can be codified and thus endure in the form of artifacts, algorithms, and language.

Different counting technologies offer different affordances, each of which shapes or constrains the resultant quantification behavior and numerical system outcomes in different ways. The use of the fingers for counting, for example, appears to be cross-culturally universal, independent of how elaborate their number system might be. This behavior reflects the neurological interaction between the perceptual system for quantity and control of the fingers.[115] Because the fingers interact with materiality and are literally ready to hand, they bridge psychological capabilities like numerosity with material objects that have quantity. They are also typically the first things used as a reference in counting: They instantiate the concept of quantity, making it accessible and communicable.

In addition to being found in emerging number systems, finger-counting persists in highly elaborated ones as well, where it acts as a convenient materiality for enumeration and an aid to memory, likely because of the neurological interaction between the regions of the parietal lobe that perceive quantity and control of the fingers (respectively, the IPS and angular gyrus). In comparison, beads can also be used as a counting technology, have a comparable cross-cultural prevalence, and similarly span the full spectrum of numerical elaboration. For example, the rosary requires neither concepts of nor words for numbers, while beads have been used to count into the tens of thousands by the North American people known as the Pomo.[116] However, many societies make and wear beads, but few count with them, an uneven ratio that perhaps reflects the need to repurpose beads from ornamentation to quantification and the lack of a specific neurological function integrating the material form with numerosity. Fingers and beads also have different capacities for

[115] Reviewed in Overmann 2014.
[116] Barrett 1952.

quantity: Fingers are generally (but not invariably) limited to low numbers, whereas beads (as attested by Pomo counting) have a much greater potential.

A similar differential is found in material capacities for instantiating relations between numbers. Accumulation is something that all early counting technologies appear to have in common: It is easy to extend another finger or add another notch to a tally, knot a string, add a stone to a pile, or mark the ground—at least up to the capacity of the material form: the number of fingers, the surface area for adding notches, the length of string to be knotted, and so on. Fewer of these technologies have a ready potential for subtraction, however. Notches are difficult to move or fill in, though knots can be untied, stones removed, and marks on the ground erased.

Fingers, despite their universality in counting, are seemingly not a factor in developing subtraction (or it might be developed sooner), perhaps because the embodied experience of quantity does not disappear when an extended finger is flexed in the same way that it does when, for example, a stone is removed from the pile. Given the potential of material counting technologies for accumulation (all) and subtraction (some), it cannot be coincidental that addition is generally the first arithmetical operation to emerge, with subtraction developing later. Rather, the order in which arithmetical operations emerge is determined by material agency through the affordances of the material form(s) used for counting.

Regularity, manipulability, and integrity are other characteristics in which material affordances differ. Notches on sticks and knots in strings instantiate the kind of stable-order linearity that fingers afford, a similarity that is perhaps a factor in why such technologies tend to follow finger-counting. By comparison, stones and other noncontiguous objects lack implicit order, but their individual elements can be manipulated in ways that notches, knots, and fingers cannot. Greater manipulability is a

likely factor in extending the recognition of numerical relations beyond plus-one. Four pebbles, for example, can be decomposed and rearranged as 'three' and 'one', 'one' and 'three', or 'two' and 'two', groupings with the potential to illuminate not only the quantity relations involved but their relative equality (i.e., 'two' and 'two' balance in a way that 'three' and 'one' do not). A notched tally, however, maintains its integrity as an instantiated quantity in a way that loose, manipulable counters do not, unless they are somehow contained.

As these examples serve to illustrate, potential outcomes for numerical elaboration may be enabled or constrained by the affordances of particular material forms,[117] just as affordance differences may be potential factors in the incorporation of new material forms. Material agency and the variability admitted through affordance differences may be as significant in influencing numerical system outcomes as the various decisions, applications, and values of the social dimension. Even more critical than particular material affordances, however, may be the contrast between the affordances of multiple material forms, a contrast that potentializes numerical elaboration—to the extent made possible by the affordance contrasts (as discussed below with an example from Papua New Guinea in the section comparing material and linguistic signs).

The psychological functioning of the brain changes as the cognitive system for numbers incorporates materiality. While parietal functions such as numerosity and finger control are significant to the realization of initial discrete quantities, arithmetical and mathematical tasks additionally engage the frontal lobe and executive functions like intentionality and working memory,[118] increasing goal-directed behavior. Demands on various forms of memory may decrease: spatial memory (that is, memory for object

[117] Hodder 2012; Knappett 2005.
[118] Zamarian et al. 2009.

location used as a heuristic in societies with few numbers),[119] working memory, and visuospatial working memory (a component of working memory that supports mental imagery). Reduced demands on memory, along with the visualization and problem-solving facilitated by material engagement, allow freed attentional capacity to be repurposed to things like pattern recognition, categorization, and abstraction, enabling the formation of new concepts. At the same time, interaction with material linearity increases the linearity of the mental number line, and this assists the realization of higher quantities in regularized and productive ways.

The numerical system also changes: Highest number counted goes up as materiality is incorporated; this seems only commonsensical, given that material devices for counting seem like just the very thing to help people count. An increase in highest number counted, in turn, affords an increased likelihood that the relations between numbers will be discovered (this is simply a function of exemplar availability[120]) and codified in the form of devices, algorithms, and language. As numbers become more elaborated, they become more useful, a utility that makes them familiar and habitual. This positions them as a cognitive technology, a body of knowledge that structures understanding and behavior. [121] The more familiar and habitual a cognitive technology becomes, the more likely it is that it will be extended to new domains[122] and become entangled in cultural practices and behaviors.[123] The more ingrained in culture numbers become, the more likely they are to be elaborated; their usefulness may intensify the need to develop and extend them.

[119] Butterworth & Reeve 2008; Butterworth et al. 2011; Butterworth, Reeve, et al. 2008.
[120] Beller et al. 2011.
[121] De Cruz 2008; 2012.
[122] Overmann 2013a; 2013b.
[123] Hodder 2012.

### 2.2.3 Material Signs Are Enactive

The third tenet of Material Engagement Theory is enactive signification, the idea that things acquire meaning in virtue of what they are and what people do with them. Integral to enactive signification is the material sign, which functions semiotically in a manner quite different from that of the linguistic sign: Where a word is ephemeral—speak it, and it is just as quickly gone—the material sign persists; where a string of words occur in a linear sequence, a function of sound production in language, the material sign has no obvious start or stop.[124] Linguistic signs are communicative and symbolic, denoting quantity through arbitrary conventions, while material signs are enactive and instantiate quantity by simply being what they are,[125] objects of a certain quantity in interaction with the perceptual system for quantity and perhaps the fingers.

Language and the linguistic signs that comprise it have been said to form a "longer, slower, and more complex" bridge between communication and cognition, implying that "things, concepts of things, and signs of concepts of things" may constitute a shorter, faster, simpler bridge, one that is more directly connected to thinking,[126] including numerical thinking. Essentially, both are critical to accessing numerical cognition and elaborating numerical intuitions into number systems; however, they differ in more than the speed of the bridge they provide (i.e., in things like their semiotic function, as discussed below), and any account of numerical cognition that does not consider both is necessarily incomplete.

The semiotic differentiae illuminate some differences in the roles of material and linguistic signs in the development of number concepts. By instantiating discrete quantities, the material sign (a term encompassing material counting technologies, including the body) anchors and stabilizes them, facilitating the realization of explicit

[124] Malafouris 2013.
[125] Malafouris 2013.
[126] Brandt 2010, 253.

number concepts and their interrelations.[127] Materiality (including forms of numerical notation as well, as will be discussed) enable the manipulation of numerical relations and algorithms in ways that language does not.[128] Simply, language, counting devices, and numerical notations represent numbers differently (as attested by the different universal properties of language for numbers and numerical notations[129] and the fact that neither considers the properties of non-notational material forms). Algebraic problems in the form of spoken narratives are difficult to solve,[130] a challenge not resolved by writing the language phonographically (i.e., the difference between solving algebraic problems as narratives or as equations).

Material structures and numerical notations, by comparison, enable a more direct manipulation of the logical and spatial relationships that are the essence of numerical cognition and mathematics. If language is a "slow and hesitant" route to numerical cognition,[131] however, linguistic signs nonetheless enable numerical concepts to be communicated, learned, and applied in social contexts in ways not possible through *manuo-visual* means alone (i.e., the tactile and visual perceptual modalities and their interaction with abilities for judging quantity relations).[132] Language has at best "a secondary function with regards to the development of mathematical concepts," although "the verbal communication of mathematical information [undoubtedly] becomes more and more important as the learning process progresses."[133] Further, lexicalization may help number concepts become conceptualized as entities in their own right, allowing them to acquire new properties (as stabilized and represented by material realizers) and become available to be acted upon by new operations and processes.

[127] Coolidge et al. 2012; Hutchins 2005; Malafouris 2013.
[128] Sfard et al. 1994.
[129] Chrisomalis, 2010; Greenberg, 1978.
[130] Sfard et al. 1994.
[131] Sfard et al. 1994, 198.
[132] Ferrari 2003, 1228.
[133] Damerow 2010, 150–151.

Numerosity (the perceptual system for quantity shared with other species) and language (which is not shared) are both essential to numeracy (earlier defined as the ability to reason with numbers). Language per se, however, is insufficient for developing number concepts, or there would be no languages that lack them, as the Pirahã language spoken in Amazonian Brazil famously does.[134] As cognitive functions, numeracy (which depends on numerosity) and language are independent, in that they are doubly dissociable.[135] That is, a person can lose the ability to calculate but retain the ability to converse, and vice versa.[136] Independence is also suggested by the multiplicity of species with numerosity but lacking language and the fact that human infants demonstrate the ability to appreciate quantity before they develop language.[137] Further, material signs precede linguistic signs in ethnographic accounts of initial quantification,[138] and material and linguistic signs differ semiotically. These differences suggest that materiality and language be analyzed both separately and in interaction with each other for their contributions to numerical cognition.

#### 2.2.3.1 Material and Linguistic Signs

An initial analysis of the semiotic differences of materiality and language is shown in Table 2.1. The basic systems (material objects and non-numerical speech) appear in the outermost columns and are related as signified and signifier.[139] From the left toward the middle, the order reflects increasing elaboration in material signs for numbers: material

[134] Dryer et al. 2013; Everett 2005.
[135] Amalric et al. 2016; Ardila et al. 2002; Brannon 2005; Carreiras et al. 2015; Hannagen et al. 2015; Monti et al. 2012; Park et al. 2012; Varley et al. 2005.
[136] Ardila et al. 2002; Varley et al. 2005.
[137] Cantlon et al. 2006; Van Marle et al. 2011.
[138] E.g., Chagnon 2013; Da Silva 1962; Karsten 1935; Murphy et al. 1955; Staden 1928; Turrado Moreno 1945.
[139] That is, the concept *jar* and the word 'jar' are related as signifier and signified (de Saussure 1959), or, in Peircean terms, *interpretant* and *representatem* (Gottdiener 1995; Peirce 1991).

Table 2.1

*Comparison of Material Objects, Numerical Notations, and Speech*

| | Material | | Numerical Notations | | | Speech | | |
|---|---|---|---|---|---|---|---|---|
| Quality | Object | Artifact | Cumulative | Ciphered | Phonographic | Lexical Nr. | Adjective | Noun |
| Example |  |  | III | 3 | three | ‘three’ | ‘-ness’ | ‘jar’ |
| Type | Metrological | Numerical | Numerical | Numerical | Numerical | Numerical | Numerical | Non-Numeric |
| Sense Mode | Tactile/visual | Tactile/visual | Visual | Visual | Visual | Auditory | Auditory | Auditory |
| Persistence | Durable | Durable | Durable | Durable | Durable | Ephemeral | Ephemeral | Ephemeral |
| Sequentiality | No start/stop | [Linear] | [Linear] | Linear | Linear | Ordinal | [Linear] | [Linear] |
| Logic | Enactive | Enactive | Enactive | Communicative | Communicative | Communicative | Communicative | Communicative |
| Representation | Instantiate | Instantiate | Instantiate | Symbolize | Symbolize | Symbolize | Symbolize | Symbolize |
| Composition | Metrological | Quantitative | Quantitative | Quantitative | Qty. or gram. | Qty. or gram. | Grammatical | Grammatical |
| Decomposition | Metrological | Quantitative | Quantitative | Meaningless | Phonemic | Phonemic | Phonemic | Phonemic |
| Conventionality | Low | Low | Low | Medium | Medium | High | High | High |
| Language | Low | Low | Low | Medium | Medium | High | High | High |

*Notes*. Comparison of type, sense mode, semiotic function (persistence, sequentiality, logic, and representation), composition–decomposition, and amount of cultural–linguistic knowledge needed to understand meaning (conventionality and language) in material objects and signs, numerical notations (cumulative, ciphered, and phonographic), and speech (lexical numbers, emerging adjectival quantity terms, and non-numerical language). Developed from concepts of numerical notations and lexical numbers; conventionality and language; and semiotic functions of material and linguistic signs.[137]

Material objects and non-numerical spoken language appear in the outer columns with no qualities in common, though they are related through reference (i.e., as the word ‘jar’ refers to the object *jar*). The other systems generally share qualities with one or more basic systems. The linearity of artifacts and cumulative notations is not so much inherent in the material form as imposed through a combination of factors (e.g., motor efficiencies, etc.) that cumulatively influence linear structure, as previously discussed (the quality has been placed in brackets to highlight this); in comparison, the linearity of ciphered notations is integral to their recognizability as written notations. Similarly, language is linear as a function of sound production, but the order of words is changeable; the expression of number words in order of increasing magnitude (ordinality) is influenced by the same or similar factors (e.g., reduced demand on memory, etc.).

[137] Chrisomalis 2010; Houston 1994; Malafouris 2013.

Objects, counting technologies, and cumulative notations share most characteristics; the only difference is that objects can be related metrologically (through characteristics of capacity, weight, size, measure, etc.), and objects, counting technologies, and written notations involve different motor movements. Ciphered notations share several characteristics with material objects and counting technologies. These similarities perhaps explain why material instantiations of quantity and numerical notations are recognized translinguistically (i.e., they interact with the manuo-visual means of accessing numerical cognition), making them highly likely to diffuse across linguistic and cultural boundaries and generally the first thing to be recognized in unknown scripts.

Overall, language and materiality interact in ways that contribute to the development of numerical cognition. Language provides an access critical to explicating and elaborating numerical concepts in material form, absorbs material characteristics like productive grouping, and helps influences counting sequences toward linear order. Materiality provides the stability, anchoring,[138] and representation that are critical to realizing and expressing numerical concepts in language, and it structures numerical thinking (e.g., linearity, stable order, productive grouping, etc.).

objects (jars), counting technologies (e.g., tallies or tokens), and cumulative and ciphered numerical notations (respectively, the Roman numeral III and the Hindu–Arabic numeral 3).[139] From the right toward the middle, the order reflects increasing elaboration in linguistic signs for numbers: non-numerical language (‘jar’), emergent adjectival number terms (‘threeness’), full-fledged lexical terms for numbers (‘three’), and phonographically written numbers (three).[140]

The semiotic qualities considered in the analysis included type (metrological, related through properties of capacity and weight; numerical, denoting quantity; and non-numerical), sense mode (tactile, visual, or auditory), persistence (durable or ephemeral), sequentiality (no start/stop or linear), logic (enactive or communicative), and representation (instantiate or symbolize). Composition (metrological, quantitative, or grammatical) examined syntactic structure in larger combinations; decomposition (metrological, quantitative, phonemic, or meaningless) explored the meaning of individual elements when separated from a larger group. Finally, conventionality and language (low,

[138] Hutchins 2005.
[139] Chrisomalis (2010) notes the tendency for cumulative numerical systems to precede ciphered ones.
[140] Because phonographic numbers like ‘three’ require a system of writing be in place, they develop after spoken lexical numbers (Chrisomalis 2010), and so they appear to the left of the spoken forms.

medium, or high) examined the amount of cultural or linguistic knowledge required to understand meaning.[141]

The basic systems—material objects and non-numerical speech (or material and non-numerical linguistic signs)—had no qualities in common, despite their signifier–signified relation, and they accessed numerical cognition primarily through manuo-visual and linguistic means, respectively (as was noted earlier, these represent cognitive systems that are doubly dissociable but mutually interacting). Systems adjacent to the basic ones were similar to the basic systems in their semiotic qualities; as adjacent systems became elaborated (with ciphered and phonographic notations containing the greatest mix of semiotic qualities from the two basic systems), they adopted more of the qualities belonging to the opposing basic system. Such hybridization is presumably incremental and non-linear, and depends upon the extensive interaction and interdependence of materiality and language (that is, their interaction and interdependence comprise the mechanism that enables their elaboration). It also seems reasonable to conjecture that not all qualities would be equally changeable (thus, not all transitions would represent change of the same magnitude), and that few transitions likely represent deliberate, consciously thought-through change (though these would not be excluded), but rather combinations of extended usage, discovery, happenstance, and mistakes, and the contributions of many individuals.

As language provides a means of accessing numerical intuitions, it may seem reasonable to propose that material structures can become elaborated in ways that suggest linguistic influence, as for example the acquisition of linearity by material forms used for counting. The converse—the notion that language for numbers might acquire material characteristics like productive grouping—is less intuitive. However, a possible mechanism

[141] Houston 1994.

may be found in the idea that linguistic structure is not so much a matter of innate rules, as proposed by theoretical frameworks like Chomsky's Universal Grammar, but a property that emerges from the interaction of cognitive processes (e.g., numerosity,[142] categorization, abstraction, memory, representation, etc.)[143] that are composed of psychological, behavioral, and material components that influence their structure. Thus, initial counting sequences may become structured linearly because they are realized through material forms that, while not themselves linear, are influenced toward linearity by things like language (whose contribution in this regard is that words used for counting must necessarily occur one after another), economies of the motor movements involved in producing artifacts (which are realized through qualities like regularity), and the preference for accessible data (whose usability is achieved through qualities like representational order, concision, and standardization, which reduce demand on cognitive processes like memory). Similarly, manuo-visual exposure to the productive grouping necessitated by the subitization constraint in material forms like fingers and artifacts may underlie the subsequent emergence of recursive rules for producing lexical terms for numbers (e.g., rules producing names composed of elements and operations, as 'twenty-one' means 'two times ten plus one').

#### 2.2.3.2 Material and Linguistic Affordances

Additional differences are found in the affordances that material and linguistic signs offer to numerical cognition. Linguistic signs for quantity are arranged in ordinal sequences (e.g., 'one, two, three' …), and the enumeration enabled by ordinal sequencing can facilitate the bundling of material signs for quantity (e.g., in which six or ten units of a smaller quantity equal and are exchanged for one unit of a larger quantity). Bundling and

[142] Overmann 2015.
[143] Bybee 2010.

debundling arguably require the ability to count if they are to be performed with reasonable convenience; an alternative would be the use of one-to-one correspondence for every bundling or debundling operation, which seems infeasible, given its laboriousness. Material signs for quantity can also be rearranged and decomposed (to the degree is permitted by the material form, as previously noted), enabling access to the individual elements and their various permutations in a way that linguistic signs do not permit. There is no way to get inside the spoken word 'four' to understand 'two plus two', let alone 'ten minus six', 'twenty divided by five', 'square root of sixteen', or indeed any of the infinity of possible relations.

While the modern reader may understand relations like 'ten minus six' as being implicit in the definitions of numbers, in actuality, such relations and algorithms are unknown until they have been explicated and codified, including ideas about their general extensibility ('–1,365,788 divided by –341,447', for example, also equals 'four'). In such explication and extension, materiality (even in the symbolic form of ciphered negative numbers) is critical to visualizing and stabilizing numerical concepts, and the example illustrates the observation that "the symbol cannot exist without the substance, and the material reality of the substance precedes the symbolic role."[144]

Where the affordances of materiality and language coincide, there may be an increased potential for long-term stability of the numerical system. For example, the Oksapmin people of Papua New Guinea count up to 27 using their bodies.[145] The material structure used for counting is an ordinal counting sequence, similar to the ordinal counting sequence provided by language. The affordances are mutually reinforcing, so the system is geared toward stability, and indeed, body-counting systems have a long history in Papua

[144] Renfrew 2004, 23.
[145] Saxe 2012.

New Guinea.[146] Beyond their mutual reinforcement, there are two other forces that influence long-term stability in the Oksapmin cognitive system for numbers: Language is inherently not 'numbery,'[147] and the material form is not only relatively unmanipulable, until the introduction of Westernized mathematics education and monetary exchange, it was the only material form thus incorporated.[148]

Where the affordances of materiality and language contrast, there may be an opportunity to notice and analyze the differences to gain insight into the nature of each, enabling their exploitation, extension, and elaboration.[149] This is simply the principle that contrast can illuminate differences better than similarities might. However, a differential opportunity for elaboration does not entail that the potential will be realized: Any potential for elaboration is still subject to cultural needs and priorities. No matter how dissimilar the affordances of material and language might be for analytic comparison, a society that does not need or value numbers will not elaborate them to the same extent as a society that finds them vital to its survival.

Such affordance similarities and differences are neither advantageous nor disadvantageous, merely circumstances that inform numerical system outcomes. Further, the potential for contrastive difference to afford opportunities for analysis and change is not the only consideration. For example, Chinese lexical and notational numbers share common structuring (i.e., affordance similarities), a circumstance that may at least

[146] Lean 1992; Saxe 2012.
[147] As noted in Overmann (2016c), "… there's nothing particularly 'numbery' about language. Words lack the cardinality and ordinality of objects and the magnitude that yields numerical order. Their grouping makes phrases and sentences, not productive cycles; their divisibility generates disconnected sounds, not new quantities related to the original. This is why, by itself, language generates no number systems. Numbers are numbery precisely because of their materiality: They are concepts of quantity shared by sets of material objects, explicated and elaborated by manipulating things like fingers and artifacts. Materiality is why number universals are so similar across number systems and so different from those of language. It is why numerical (but not non-numerical) signs can be represented without phonetic values in fully glottographic writing and deciphered in otherwise untranslatable scripts (Overmann 2016[a])."
[148] As Saxe's (2012) work shows, "upon incorporating materiality with differing manipulability, new numerical elaboration quickly result[ed]" in the Oksapmin cognitive system for numbers (Overmann 2016c).
[149] Beller et al. 2011.

partially explain the mathematical task performance advantages of Chinese speakers relative to English speakers, whose lexical and notational numbers are structured differently.[150] Thus, it is not simply affordance similarities or differences per se, but how they interact with psychological and physical abilities to influence potential outcomes in the cognitive system for numbers.

[150] Cantlon et al. 2007; Chrisomalis 2010; Miller et al. 1995.

# CHAPTER 3

# WHAT NUMBERS ARE AS CONCEPTS

## 3.1 Concepts of Numbers and Numbers as Concepts

A fair number of disciplines study numbers: mathematics, obviously; psychology, especially developmental psychology; linguistics; anthropology, ethnography, and ethnomathematics; writing systems, paleography, and Assyriology; and so on. While each has contributed substantially to the understanding of numbers, the picture that has emerged has done so in a somewhat disjointed fashion that has cast relatively little light on the question of numerical origins. Psychologists, for example, in focusing on the brain, have studied it in relative isolation, thereby eliminating most of the external contributions to numerical cognition, which may also have "longer temporalities than neural reactions and ontogenetic developments"[151] that are accordingly difficult to incorporate, both theoretically and methodologically. Anthropologists and ethnographers who have noticed the correlation between the complexity of number systems and material culture more generally[152] have failed to offer mechanisms that are sufficiently explanatory or specific. For its part, Assyriology has not incorporated the insights into numerical cognition that have been generated by other disciplines over the past few decades.

In similar discipline-focused fashion, linguists have attributed the rule-based patterning that governs numbers to language; in its extreme form, this view attributes

[151] Overmann 2016c.
[152] E.g., Divale 1999; Epps et al. 2012.

numerical origins to language. For example, "Without language, no numeracy,"[153] a claim hailed as "prescient,"[154] though Hurford would ultimately weaken it to "Without using language, we still can't go [higher than the subitizing range of about four],"[155] which nonetheless attributes most (rather than all) numbers to language. In either form, the claim is untenable, given its circular reasoning and failure to consider other causal factors. Chomsky envisioned a "computational capacity" (an innate ruleset for language) that underlies both numeracy and language and is responsible for the generativity of both; he includes numeracy because he believes the "ability to deal with the properties of natural numbers is acquired just like languages [are] …. If you are presented with the relevant materials you immediately know it all" and "the fact that you can add one indefinitely, does not have to be taught."[156] However, it is far from established that numeracy is acquired as effortlessly as language, even in numerate cultures; rather, numeracy involves neural reorganizations gained through training (see Chapter 10) and is similar in this regard to literacy.[157] Neither is it certain that the generativity of numeracy and language are the same: Not only do they differ significantly in qualities such as the degree to which reordering yields well-formed expressions,[158] the successor function and productive grouping in numeracy have behavioral–material components with no linguistic counterpart. Finally, making numbers innate has been used to question the human status of peoples without numbers (an implication of Chomsky's view[159]).

In addition, few of those studying the psychological, behavioral, or material aspects of numbers appear to have consulted mathematicians for their ideas on what numbers are as concepts. This omission is curious, given that defining the object of study

[153] Hurford 1987, 305.
[154] Everett 2013b, 81.
[155] Hurford 2007, 91–92.
[156] Chomsky 2004, 47–49.
[157] Overmann 2016a.
[158] Overmann 2016a.
[159] Verran 2000.

seems not only a logical first step but foundationally necessary. However, translating an everyday understanding of number into a formal definition can be difficult. Beyond "It's a quantity" (more synonym than definition) or "It's how many of something there are" (more use than identity), folk definitions might be expressed in terms of numerical relations. Asked to define the number 'six', someone might respond, "It's the number between five and seven," or "It's what you get when you add four plus two." Interestingly, the first statement locates 'six' in an ordinal sequence (i.e., numbers ordered by increasing magnitude), while the second specifies one of the additive combinations that yield 'six' (another being 'three' plus 'three'). These reflect elements of a more formal definition, like the one that specifies number as "a unit belonging to an abstract mathematical system and subject to specified laws of succession, addition, and multiplication."[160] Magnitude ordering, which characterizes ordinality, also characterizes the successor function, which recursively forms new numbers by adding 'one' to any other number (the successor function ($S(n) = n + 1$) differs from ordinality in that the latter does not stipulate the amount of increase between numbers, while the former specifies it as 1), as 6 is 5 + 1 and 7 is 6 + 1. New numbers can also be formed by adding numbers other than 'one', as was the case with 4 + 2 = 6.

Such relations and operations, once known, give each number a unique identity and enable them to be manipulated, even when those performing the manipulations cannot precisely define the rules. For example, 'one' is the only number that yields itself when multiplied or divided by itself, even if these operations are performed an infinite number of times, and it is this characteristic that excludes it from being considered a prime number (i.e., any natural number greater than 'one' that can be divided only by itself and 'one'). 'Two' is the only even prime (all other even numbers are divisible by 'two' and hence are

[160] *Merriam-Webster* 2014a.

not prime), while 'three', the first odd prime, is the only number that is the sum of all the previous positive integers (or 'one plus two'). 'Four' is the first composite number (any positive integer divisible by a number other than itself or 'one'), 'five' the first prime that is the sum of all previous primes, and so on.[161]

From this brief discussion, a definition for number begins to emerge, at least in the way an individual enculturated into the Western mathematical tradition might understand them: A number is a discrete quantity related in specifiable ways to other numbers, with operations like addition enabling the manipulation of those relations. Implicit to the definition is the notion that all numbers have equal status, regardless of the domain to which they apply. These, however, are fairly recent notions of number: For example, if the idea that numbers are real is longstanding,[162] the notion that fractions are numbers too only emerged around the 16th century CE (this conception of fractions is attributed to, among others, Simon Stevin, a Flemish mathematician),[163] making it unlikely that the Mesopotamians thought of fractions as numbers. Similarly, the idea that lengths are numbers was presumably unknown as late as the Old Babylonian period, as area calculations involved separate units and distinct operations (discussed further in Chapter 10).[164] Implicit to both Mesopotamian and more recent notions of number is a lengthy history—and prehistory—in which countless individuals realized and refined numerical concepts, relations, and operations, making them available to future generations. The goal, ultimately, is a historically situated notion of number rooted in known commonalities of origin (i.e., shared psychology and physiology; socio-material similarities) that does not expect the Mesopotamians to have realized anachronistic ideas of numbers.

[161] See Gowers 2002; Higgins 2011; Ore 1948.
[162] Marr 2015.
[163] Gouvêa 2008.
[164] Høyrup 2002b. The idea that lengths were numbers was actively rejected by the first-millennium Greeks who inherited and extended Old Babylonian mathematics, on the basis that a result like $\sqrt{2}$ neither terminates nor repeats, making it literally irrational—hence, it could not be a number (Gouvêa 2008).

Insight into numerical origins comes from two other definitions. The first defines number as "an element … of any of many mathematical systems obtained by extension of or analogy with the natural number system."[165] This highlights the natural numbers[166] as the basis for mathematical elaboration. Concepts of discrete quantity like the natural numbers are not innate, though they draw on capabilities humans share with other species; rather, they are concepts realized, manipulated, and elaborated into simple counting sequences and complex mathematical systems through processes unique to humans. The second refers to mathematics as *the science of patterns*[167]:

> However vague, [this] definition [is] appealing because the origin of all sciences is found in pattern recognition and generation, and mathematics is the most powerful method we have for characterizing those patterns we find in nature as well as generating new ones. Strongly interacting with our mathematical descriptions of nature are those patterns engendered within the exercise of mathematics itself.[168]

Further,

> What the mathematician does is examine abstract 'patterns'—numerical patterns, patterns of shape, patterns of motion, patterns of behavior, and so on. Those patterns can be either real or imagined, visual or mental, static or dynamic, qualitative of quantitative, purely utilitarian or of little more than recreational interest.[169]

These definitions are useful because they evoke cognitive processes like categorization and abstraction that are important to recognizing the patterns inherent in realizing the very first number concepts, as well as elaborating them into highly complex mathematical systems.

## 3.2 How Number Concepts Are Realized and Elaborated

Defining numbers in terms of concepts, relations, and operations does not answer the question of how number concepts (specifically, their most basic form, the natural

[165] *Merriam-Webster* 2014b.
[166] The *natural numbers* are those in the familiar counting sequence: 'one, two, three, …'; mathematicians disagree regarding whether or not 'zero' is rightfully included (Ellenberg 2014).
[167] Devlin 2003.
[168] Marr 2015, 75.
[169] Devlin 2003, 3.

numbers) are realized, manipulated, and elaborated. For this, an important distinction must be introduced: the difference between *number* and *cardinality*. For any group of objects, cardinality is the number that specifies the amount of objects in the group.[170] In a numerate culture, cardinality is discovered through *enumeration*, or counting the members of the group sequentially from 'one' until there are no more to be counted; the final number is the number of objects in the group, or its cardinality. However, though cardinality can be understood as a number and discovered through enumeration, it is nonetheless distinct:

> A particular number [concept] is not identical with any collection of terms having that [cardinal] number: the number 3 is not identical with the trio consisting of Brown, Jones, and Robinson. The number 3 is something which all trios have in common, and which distinguishes them from other collections. A number is something that characterises certain collections, namely, those that have that number.[171]

Problematically, understanding cardinality as a number and using enumeration to discover it presuppose that number concepts exist, begging the question of how they could ever be achieved in the first place,[172] since a method of invention cannot presuppose the existence of that which it invents.[173] For example, language, often asserted to explain how number concepts originate and are organized,[174] is unsatisfactory in that original conditions presumably lack both numerical lexicons and the concepts of cardinality they represent. As language must therefore be excluded as an explanation, it may be that number concepts develop "through the material affordances of … bodies or the agency of material culture and innovation."[175] This proposal for numerical origins is more plausible, considering its consistency with the neuroscientific, ethnographic, and linguistic data (explained in detail in Chapter 4).

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[170] *Merriam-Webster* 2014a.
[171] Russell 1920, 11–12.
[172] Rips et al. 2008.
[173] Malafouris 2013.
[174] E.g., Everett 2013b; Hurford 1987; 2007.
[175] Malafouris 2013, 110.

An *original context* is one in which humans have numerosity, which is widespread phylogenetically and thus fundamental; material culture, which has characterized the hominid lineage for the past few million years; and language, which has characterized *Homo sapiens* for perhaps several hundred thousand years; but no numbers. An original context that is pristine is one in which there is no stimulus diffusion or borrowing of number concepts through contact between groups. Potential insight into the question of how numbers might emerge in pristine original contexts comes from the theory of logical types. Philosopher of mathematics Bertrand Russell, in differentiating the familiar type of numerical elaboration (toward complexity, the orientation of ordinary mathematics) from the less familiar (toward originating principles, the orientation of mathematical philosophy), defined number as *cardinality shared between sets of objects*.[176]

Russell's theory of logical types (Table 3.1) distinguishes individuals (or objects or entities, Type 1) from sets of such objects, which have the property of cardinality[177] (Type 2); sets of sets of objects that share cardinality (Type 3); and sets of Type 3 like the natural numbers (Type 4). Type 3 individuals are concepts of shared cardinality, or numbers. That is, three apples and three fingers both have the property 'threeness', and that shared property is the number 'three', which is common to all trios.[178] Once there are several Type 3 individuals—not just 'three' but the numbers 'one' through 'five' (a small subset of the natural numbers)—the relations between them may start becoming more apparent, as a consequence of magnitude ordering, though as few as two numbers imply relations

[176] Russell 1910; 1920; also see Soames 2003.
[177] Of course, a single individual has cardinality too, as does an absence of individuals. In set theory, these would be, respectively, sets containing one and no members (Lipschutz 1998). Singularity is intuitive with one object, plurality with two or more; often, concepts of 'one' and 'two' appear to be defined against one another (e.g., Closs 1993; Eells 1913a; 1913b). In comparison, 'zero' is a difficult concept to achieve, famously undiscovered by the Romans and not widely adopted in Western mathematics before the 17th century (Kaplan 2000).
[178] Russell 1920.

Table 3.1

*Russell's Theory of Logical Types*[179]

| Type | Entity | Description | Example |
|---|---|---|---|
| 1 | Objects | Individuals, objects, or entities | a pair of objects |
| 2 | Sets of Type 1 | Cardinality of a set of Type 1 | 'twoness' (cardinality of a pair of objects) |
| 3 | Sets of Type 2 | Cardinality shared by sets of Type 2 (a number) | 'two' ('twoness' shared by two pairs of objects) |
| 4 | Sets of Type 3 | A collection of Type 3 entities (a sequence of numbers) | the natural numbers (of which 'two' is a member) |

*Notes*. Russell differentiated objects (Type 1); sets of objects with the property of cardinality (Type 2); sets of sets of objects that share the cardinal property (Type 3, or number), and sets of Type 3 (Type 4). While Russell's analytic framework has a certain intuitive plausibility, it most probably does not accurately depict either neural activity or psychological functioning. This, however, is true of many such constructs, which are nonetheless helpful in generating testable hypotheses and interpreting and organizing empirical results.

like 'greater than', 'less than', and 'not equal to'. To Russell's framework might be added a fifth type, operations on Type 4 relations between numbers (not shown in Table 3.1). The idea that operations like 'addition' are generalized from a myriad of material exemplars is suggested by the circumstance that the southern Mesopotamians had several, rather than unified notions.[180] As late as the Old Babylonian period, there were two additive, two subtractive, and four multiplicative operations.[181]

Russell's definition of number as *cardinality shared by sets of objects* implies that the concepts are realized through a process of comparing sets.[182] Such comparisons are well documented in the ethnographic literature as behaviors like pairing and one-to-one correspondence that do not presuppose numerical concepts but have the potential to generate them. Pairing, which falls within the behavioral repertoire of other primate

[179] Adapted from Soames 2003.
[180] Friberg 2007.
[181] Høyrup 2002b also see Chapter 10.
[182] A linguistic definition of lexical number is: "1. *spoken* 2. *normed expressions* that are used to denote the 3. *exact number* of objects for an 4. *open class of objects* in an 5. *open class of situations* with 6. *the whole speech community* in question" (Hammarström 2010, 11). This definition emphasizes, as Russell's does not, aspects of extension and use (i.e., what numbers apply to and who uses them).

species,[183] physically groups two objects, and putting two pairs together (e.g., a pair of objects to be enumerated and a pair of fingers) enables conceptualization of what the pairs share (i.e., their cardinality, the number 'two'). By comparison, one-to-one correspondence iteratively matches elements across two sets until all of the elements have been matched, with none remaining or matched more than once (Figure 3.1). Neither pairing nor one-to-one correspondence requires concepts of or words for discrete quantities because "it is simpler logically to find out whether two collections have the same number of terms than it is to define what that number is."[184] Both pairing and one-to-one correspondence, however, create opportunities to realize number concepts because a concept of discrete quantity is implicit in a comparison. In addition, both sets instantiate the number being realized (through their cardinality) and thus exemplify it[185]; one (usually the fingers) is often used as a reference set to express the number and for future comparisons.

Historically, the various fields investigating numerical cognition have tended to dismiss behaviors like one-to-one correspondence as mere compensation strategies—things people do because they lack numbers. This characterization, however, misses the point: *Behaviors like one-to-one correspondence are the very process by which number concepts are realized*. Such behaviors are material engagements, the moments of interactivity between brains, bodies, and materiality: psychological processes like numerosity, categorization, and abstraction; behaviors in which the fingers (whose neural control interacts with the perception of quantity[186]) grasp and manipulate objects; and the

[183] Conway et al. 2001.
[184] Russell 1920, 15.
[185] A set of objects instantiates "some particular number. A trio of men, for example, is an instance of the number 3" (Russell 1920, 11).
[186] As will be discussed in detail in Chapter 3, the neurological basis for the use of the fingers for counting arises from the interaction between the IPS, implicated in quantity appreciation, and the angular gyrus,

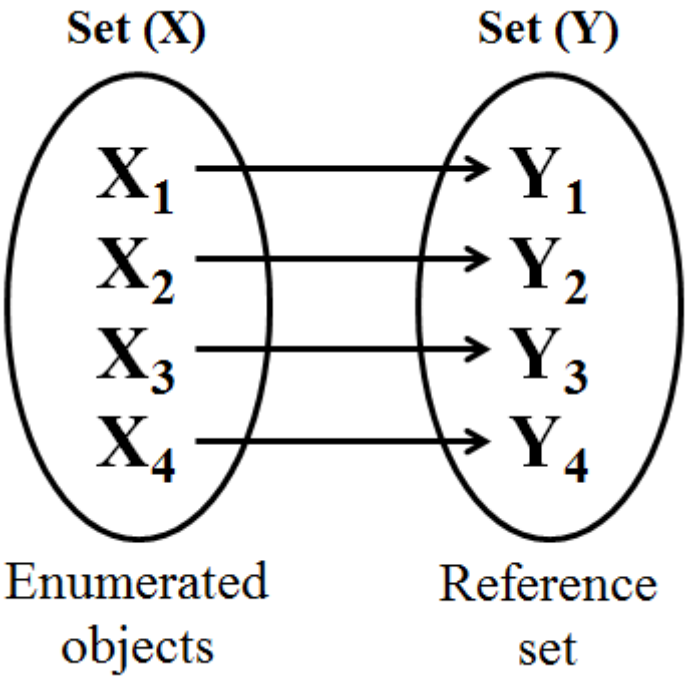



*Figure 3.1*. One-to-one correspondence. Elements of the set of enumerated objects (Set X, left) are iteratively matched to elements of the reference set (Set Y, right): $X_1$ is matched to $Y_1$, $X_2$ to $Y_2$, and so on. The process continues until all of the elements of Set X have been matched (with none left over or matched to elements of Set Y more than once), at which point the reference set instantiates and thus represents the quantity of enumerated objects, without a concept of or a word for the quantity in question being needed.

objects themselves and the properties they provide. The behavior of comparing reveals the cardinality shared by sets of objects, facilitating the interaction of psychological processes and materiality that forms a concept of number, which both sets then instantiate. Such behaviors are also instances of *enactive signification*, the process whereby material objects acquire meaning through use[187]: Just as being graspable, heavy, and durable makes an object appropriate for driving nails and being used for this purpose makes it a 'hammer', objects have properties that make them sets with particular cardinalities, and when the sets are compared, they share their cardinal property, provide opportunities for realizing number concepts, and become references.

implicated in functions like finger gnosia and the retrieval of arithmetic facts from memory (also reviewed in Coolidge et al. 2012; Overmann 2014).

[187] Malafouris 2013.

Human numerical cognition produces number systems whose properties are so similar across cultures and languages that numbers are recognizable, even when nothing else is, in otherwise undeciphered scripts in unknown languages (e.g., Proto-Elamite, Minoan Linear A[188]). This similarity arises from the common psychological capabilities, physical behaviors, and materiality used in realizing number concepts, particularly the interaction of numerosity, the fingers, and material objects. Simply, human psychological, physiological, and material attributes structure the way numbers are expressed and organized, in both material and linguistic forms. Conversely, typical features in the material and linguistic forms of archaic numbers suggest that the ancient peoples who invented and used them also shared the psychological capabilities and physiological behaviors that underlie these structural similarities in other number systems.

Understanding what numbers are as concepts is the only first step; the steps that must follow involve

> … discarding the deeply entrenched assumption of linguistic primacy …, looking more deeply than broad socio-material contexts motivating numbers, and considering longer temporalities than neural reactions and ontogenetic developments. It means incorporating the interactions where fingers bridge numerosity with materiality and manipulate it into new concept-generating stimuli, and considering psychological-behavioral-material interactivity at temporal resolutions capable of discerning change at both systemic and componential levels. This means adopting a framework (e.g., Material Engagement Theory; Malafouris 2013) that views cognition as comprised of interacting brains, bodies, and materiality and encompasses their contributions and characteristics.[189]

## 3.3 Representation and Representationalism

Russell's framework is *representational*: It assumes concepts like numbers are internal mental phenomena, generated by and contained within the brain and distinct from the physical world. Representationalism in its strong form implies that brains reproduce

[188] Englund 2004b; Packard 1974.
[189] Overmann 2016c.

the external world internally. However, while the brain is undeniably important to cognition, the essentiality of materiality to quantity perception, categorization, and abstraction suggests that it is both difficult to separate the internal and external components of cognition and an open question as to exactly how much of the external world human brains actually need to reproduce.[190] For one thing, reproducing the external world internally would require massive storage and enormous computational power, and even then, the result would tend to be slow and inefficient—hardly an advantageous adaptation.[191] For another, there is no need to reproduce the external world internally, since it is always there and readily available for use whenever it is needed. This anti-representationalist insight, originally gained from Artificial Intelligence research, has been dubbed the *007 principle*: the idea that we only know what we need to know, when we need to know it.[192] Application of the principle has decreased the programming required to produce complex behaviors in robots; when extended to organic brains, the principle significantly reduces the amount of information processing and representing that brains are thought to do.[193]

While representationalism in its strong form is likely incorrect,[194] it is probably also not the case that there are no representations at all, the strong form of anti-representationalism. After all, the brain must engage the material world in some fashion through its neural activation patterns,[195] or it would be difficult to explain how humans would ever know anything about it. Representations are also a handy way of explaining phenomena like stimulus intensity and the mental number line. If representations are neither 'all' nor 'none', they logically inhabit the middle ground of 'some', without

[190] E.g., Clark 1997; 2008; Hutto 2013.
[191] Clark 1997.
[192] Clark 1997.
[193] Clark 1997.
[194] Material Engagement Theory specifically rejects the idea of 'internal representations' (Malafouris 2013).
[195] Malafouris 2015.

necessarily being fixed in any part of the middle spectrum. In fact, the amount of representation may vary depending on the activity of any particular moment: Activities that are "representation-hungry," like "reasoning about absent, non-existent, or counterfactual states of affairs,"[196] are probably more representational in nature, while skilled physical activities intently focused on sensation and movement (e.g., making pottery[197]) are perhaps less so.

In whatever amount humans might actually have them, representations may be stabilized by the association of conceptual and material structures.[198] In this view, representations not only depend on material structures—the stuff of the natural environment and material culture—but are partly constituted by them.[199] That is, concept formation may involve the selective projection of content and structure from various domains (e.g., memory, movement, perception) to a mental space where they are integrated or blended, enabling the emergence of content and structure that differ from the inputs giving rise to them.[200] Ideally, some aspects of a blend would remain stable while its other parts were manipulated.[201] Materiality may provide the requisite stability, thus not only supporting but directly related to the complexity of conceptual content and its manipulation. Material structures thus become a meaningful part of the cognitive system: neither passive creations (i.e., material culture as cognitive byproduct) nor mere data repositories (e.g., externalized forms of memory) but the very mechanism whereby concepts are realized, manipulated, and elaborated. Mathematicians have long thought that materiality helps generate mental representations: "Mental representations [of mathematical concepts] are created in the mind [through] concrete representation

[196] Clark et al. 1994, 419.
[197] Malafouris 2013.
[198] Hutchins 2005.
[199] Malafouris 2013.
[200] Fauconnier et al. 1998.
[201] Hutchins 2005.

systems," or systems comprised of materially realized, "external artefacts."[202] Surely, for cognitive abilities like numerosity, categorization, abstraction, and sequence, materiality is essential to transcending quantity perception, discerning higher-order relations, and realizing, expressing, ordering, and manipulating concepts of discrete quantity (discussed in Chapter 4). Materiality not only acts as stimulus and scaffold for the development and operation of these cognitive abilities but partly constitutes the representation of numerical concepts.

Here it is assumed that representations of numerical concepts are partly psychological, partly behavioral, and partly material. The psychological domain includes capabilities like numerosity and the pattern recognition implicit in categorization and abstraction; the behavioral domain encompasses the active manipulations that rearrange materiality into new concept-generating stimuli and bridge the internal, psychological and external, material domains; and the material domain includes objects, whether natural or altered/created by human agency, with quantity that interacts with psychological processes like numerosity, enables behavioral manipulations, and anchors and stabilizes representational content.[203]

Once the need for internal mental representations of the external world is understood as minimal, the domain of interest shifts from the internal domain to active engagements with the world. The physical environment, including the material structures humans create, holds any needed information ready until it is required. The human ability to manipulate those structures yields information and helps create new concepts in the process. The criticality of material engagement is revealed by the process of realizing number concepts, through the interaction of psychological processes (e.g., numerosity, categorization, abstraction), behaviors (pairing, one-to-one correspondence), and material

[202] Dreyfus 1991, 31; also see Kaput 1987.
[203] Hutchins 2005.

structures (enumerated and reference objects). Concept formation, a change to the system that can be described and quantified neurally and behaviorally, requires the interactivity of perceptual experience, material instantiation, and object manipulation. Thus, it is difficult to draw a meaningful demarcation between the neural processes, fingers, material objects, and behavioral interactivity; the boundaries of the system comprising numerical cognition must be drawn in a way that includes them all.

The centrality of materiality to representation and representation to conceptual content (and hence, of materiality to conceptual content) does not imply that significant differences in materiality necessitate significant differences in conceptual content. Nor does it entail that conceptual content (*what* ancient peoples thought) can be determined from materiality alone; were this the case, cognitive archaeology would be a relatively straightforward and simple process of description. However, the centrality of materiality to conceptual content does license inferences about *how* ancient peoples thought from their material culture.[204] Concepts are not mutually unintelligible across differences of material culture; the fact that humans understand each other cross-culturally to a relatively high degree suggests conceptual content overlaps to a large extent. Numbers certainly demonstrate this: Though they vary in their representation and organization (a range of variation that is admittedly fairly narrow), individuals from different material cultures and languages acquire number concepts from other number systems with relative ease (especially in comparison to other conceptual domains, which may be mediated exclusively or primarily by language, rather than manuo-visual abilities in interaction with materiality, as numbers are). However, change in materiality has an effect on conceptual content. Even subtle material differences may admit enough change into conceptual

[204] Malafouris 2013.

content that new behaviors and concepts are enabled, and these in turn create opportunities for new material engagements and psychological responses.

Neither is it certain that multiple levels are necessarily involved (Brandt 2004), as the hierarchy of Types and the idea that concepts become 'more abstract' suggest; rather, it may be that once a concept has formed or changes, the understanding it engenders has priority over other, previous understandings. That is, a number understood through its relations with other numbers (Type 4) differs conceptually from one realized by comparing sets sharing cardinality (Type 3), even though both might be referred to with the linguistic label 'four'. This does not mean the former is 'more' abstract than the latter but that the two, as abstractions, have different content. Their abstract nature is equivalent. It is equally uncertain that the transitions from one Type to another are discrete phenomena; numeral classifiers, which connect number words to enumerated objects based on properties like shape, suggests there may be intermediate Types interposed between those in Russell's typology.

There are two mutually interacting ways to access numerical intuitions, one manuo-visual and the other linguistic, both of which interact with numerosity. Of the two of these means of accessing numerical cognition, manuo-visual mediation between material signs and numerosity is more direct and intuitive (and thus quicker), with language and linguistic signs forming a "longer, slower, and more complex"[205] means of access. Undoubtedly, "[spoken] language is not always the best medium for communicating thought,"[206] especially for structural, spatial, and logical relations and combinations. Both language and manuo-visual means of accessing numerical intuitions interact with constructs like the mental number line (which is influenced toward linearity by both material and linguistic qualities) and with each other to enable their mutual

[205] Brandt 2010, 253.
[206] Boone 1994, 10.

elaboration (as was discussed in Chapter 2). However, what this may imply in terms of internal representations (if any) is unclear. Certainly, the role of materiality in functions like anchoring and stabilizing conceptual content[207] is more implicit to manuo-visual access of material signs than in the access of language to linguistic signs. While this may explain why manuo-visual and linguistic means of access differ in terms of intuitiveness and speed, it remains unclear whether they inform the same or different mental representations of numerical constructs, or what neurological interrelations support their interaction (beyond the suggestion, discussed in Chapter 4, that the interaction is a function of the collocation of parietal regions involved in quantity perception and linguistic metaphor[208]).

Both material and linguistic access to number concepts may in fact be misleading as to the degree that number concepts are similar across number systems, even for the natural numbers that form simple ordinal counting sequences. For example,

> … '*ūnus*, *duo*, *trēs*…' and 'I, II, III…' in Latin; '*ichi*, *ni*, *san*…' and '一, 二, 三 …' in Japanese; and '*tip^na*, *tipnarip*, *bumrip*…' and 'right-hand thumb, right index finger, right middle finger…' in Oksapmin all mean '*one*, *two*, *three*…' and '1, 2, 3…', as they do in English.[209]

However, prior to Western influence, Oksapmin numbers presumably lacked the detailed network of relations between numbers[210] (e.g., 'six' is 'five plus one', 'two times three', 'eight minus three', and 'twenty-four divided by four') whose formation depends upon the incorporation of material forms with the requisite manipulability and representational concision.[211] When explicated and elaborated, such relations give each number a unique identity.[212] Other potential differences in concepts of numbers include whether they comprise a productive or restricted system (i.e., a *productive system* can create new quantities

[207] Hutchins 2005.
[208] Coolidge et al. 2012.
[209] Overmann 2016c.
[210] Saxe 2012.
[211] Overmann 2016c.
[212] Higgins 2011; Ore 1948. Also see Chapter 10.

indefinitely; a *restricted system* reaches a limit),* their material representation(s),[213] how they are manipulated, their degree of manipulability, the kinds of objects for which they can be used in enumeration (animate or inanimate; the use of numeral classifiers),[214] how they are extended to enumerated objects (e.g., pairing, one-to-one correspondence, etc.; this has bearing on how numbers are conceptualized as equivalences, collections, or entities[215]), and who is permitted to learn and apply them (e.g., for the Yupno people of Papua New Guinea, number knowledge is reserved for men[216]). While the quantity is undoubtedly the same in all cases (i.e., as '*trēs*', '*san*', '*bumrip*', 'lll', '三', 'right middle finger', and '3' all share the discrete quantity 'three' in their respective number systems), other conceptual properties can and do vary across number systems, variability that reflects significant differences in psychological–behavioral–material dimensions of use. For example,

> … the Roman numeral system famously lacked a zero and the algorithmic manipulability it can facilitate, Japanese numbers are attached to the objects they enumerate with numeral classifiers in a language spoken by a highly numerate society, and Oksapmin numbers reach their limit at 27 and are relatively unmanipulable, qualities that have changed under Western influence (Saxe 2012; discussed further [in Overmann 2016c]). By comparison, Western numbers have been elaborated, over millennia, to include concepts like zero, algorithmic manipulability, negativity, fractions, areas and lengths, axioms, proofs, set theory, and so on, and to discard any extensional conditions or restrictions (Bender and Beller 2011[a]; Gouvêa 2008; Ifrah 1981; 2000; Selin 2000).[217]

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* Comrie 2013.

[213] This assumes that as materiality is integral to concepts (Hutchins 2005; Malafouris 2013), its form affects the conceptual content of numbers (discussed further in Chapter 10).

[214] As noted in Overmann (2015), in some number systems, lexical numbers are modulated by animacy: Inanimate objects are counted, but animate beings might not be. This phenomenon perhaps reflects the origins of lexical numbers in the manipulation of objects and the tendency to retain pre-numerical heuristics for beings who are known as individuals and thus do not require enumeration (e.g., divine beings, other humans, and herd animals). For example, "herders such as the Nuer [a people of the South Sudan and southwestern Ethiopia] know each of their animals and treat them as individuals but do not count them as a herd" (Schmandt-Besserat 1982, 874, citing Evans-Pritchard 1969). Other number systems (e.g., Japanese) require numeral classifiers that reference object properties to attach lexical numbers to enumerated objects (Aikhenvald 2000).

[215] Discussed further in Chapter 10.

[216] Wassmann et al. 1994.

[217] Overmann 2016c.

## 3.4 Numbers: Real, Innate, or Extended?

Plato, the first-millennium Greek philosopher, held that universals or abstract objects existed independently of the human mind and were discovered rather than invented; extended to numbers, it is the idea that numbers are real and need only be discovered known as *mathematical realism*:

> We might say that in the study of arithmetic there are just the arithmetical truths, there are the properties of the numbers. So the fact that the square root of two is irrational is just like the fact that the sun has helium in it. And if we take a classical view of mathematics, then the truths of mathematics are just truths on a par with the truths of physics even. They are truths that are independent of our ways of thinking. So the theory of mathematics in a sense is independent of any particular techniques that we use for carrying out arithmetical computations or making mathematical constructions.[218]

Undoubtedly, this view has been informed by the high degree of similarity that number concepts share across different numerical systems. Other contributing factors include the strong emphasis on universals in the Western scientific tradition[219] and the cultural hegemony of Western mathematics.[220] While the cross-system similarity of number concepts might perhaps be dismissed as a mere consequence of physics (as is reasonable, given that the conceptual content of numbers is informed, if not determined, by the physical reality of the cardinality of objects in sets), it is more difficult to explain away the similarity of the patterns potentially discoverable within sequences of numbers. For example, given a lengthy sequence of numbers in base 10 and the idea of divisibility, the same primes (numbers divisible only by themselves and 1) will invariably be found; the primes do not vary between number systems. Further, the myriad of mathematical patterns[221] that have been identified,[222] as well as the potential of identifiable patterns thought to remain, attest to a seemingly infinite amount of relations inherent in the natural

[218] Chomsky 2004, 43.
[219] Fasheh 2012.
[220] Greer et al. 2012.
[221] Devlin 2003.
[222] Gowers 2008.

numbers as the basis of mathematical elaboration. Rather than invention, there is an undeniable air of discoverability to such numerical characteristics, as well as in the numbers themselves. Thus, in its strong form, mathematical realism holds that number concepts exist as abstract entities, in the same way Plato thought ideal forms existed as objects immaterial in this world but substantial in their own realm, independent of human rationality but accessible by it.[223] However, mathematical realism in the strong sense remains a matter of some debate.[224]

In polar opposition to the idea that numbers are real entities existing externally to human rationality is the idea that they are innate and internal to the human mind (i.e., more specifically, the brain). From the intuitionist view, "mathematics becomes the study of mental constructions of a certain type."[225] Taking this view, the Dutch mathematician Luitzen Egbertus Jan Brouwer proposed that the realization of number concepts was a matter of "the consciousness of self in time."[226] Brouwer, who was arguing against mathematical realism, took his ideas about the origin of numbers from "the old metaphysical view that individual consciousness is the one and only source of knowledge."[227] In the appendix to the volume of his Cambridge lectures, Brouwer "roughly sketched" the "inner experience"[228] of realizing number concepts as follows:

> twoity;
> twoity stored and preserved aseptically by memory;
> twoity giving rise to the conception of invariable unity;

[223] Maddy 1990.
[224] Russell noted, "It is, of course, exceedingly difficult to decide what is meant by an 'idea'; but let us assume that we know. We are then to suppose that, starting (say) with Socrates, there is the idea of Socrates, and so on *ad inf*. Now it is plain that this is not the case in the sense that all these ideas have actual empirical existence in people's minds. Beyond the third or fourth stage they become mythical. If the argument is to be upheld, the 'ideas' intended must be Platonic ideas laid up in heaven, for certainly they are not on earth. But then it at once becomes doubtful whether there are such ideas. If we are to know that there are, it must be on the basis of some logical theory, proving that it is necessary to a thing that there should be an idea of it. We certainly cannot obtain this result empirically, or apply it … to 'meine Gedankenwelt'—the world of my thoughts" (Russell 1920, 139).
[225] Chomsky 2004, 43.
[226] Brouwer 1981.
[227] Ferreirós 2008, 148–149.
[228] Brouwer 1981, 90.

> twoity and unity giving rise to the conception of unity plus unity;
> threeity as twoity plus unity, and the sequence of natural numbers;
> mathematical systems conceived in such a way that a unity is a mathematical system and that two mathematical systems, stored and aseptically preserved by memory, apart from each other, can be added;
> etc.[229]

Brouwer's process of realizing number concepts starts with the subitizable numbers, which the alternative process of numerical realization outlined in Chapters 4 and 5 does as well. Brouwer understood 'one' through its difference from 'two' and conceived 'three' as compounded from 'two' and 'one' (these are identified in Chapter 4 as only one of several possible combinations for realizing concepts of the subitizable numbers). Moreover, he was likely correct that the idea of 'one' may help the apprehension of 'two' as 'one plus one'. Further, if one assumes Brouwer's 'aseptic preservation' has a meaning akin to 'abstraction', another point of commonality between his realization process and that in Chapters 4 and 5 might be construed. However, Brouwer apparently did not consider ethnographically attested behaviors like pairing or the use of sets of material objects whose cardinality is comparable, making his realization process fully internal to the brain. In addition, he did not differentiate between subitizable and non-subitizable numbers, thereby ignoring the insights that neuroscience and linguistics have brought to the inquiry into numerical origins.

Further, Brouwer also held that mathematical ideas in general were explicitly constructed,[230] making it curious that he exempted the numbers themselves from realization through constructionist processes. Notably, Brouwer's introspectionist theory was written decades before the availability of either neuroscientific insight into brain activity associated with numerical cognition or comparative linguistic and ethnographic data supporting the position that[231]:

[229] Brouwer 1981, 90.
[230] Chabert 2008.
[231] Overmann 2016c.

> … number concepts are explicitly constructed through the interactivity of human psychological, physiological, and behavioral abilities with materiality, a process affording both overall commonality and cultural variability. Given Brower's overall constructivist orientation, one might think he would have been sympathetic to the idea that this applies to numbers too.[232]

In contrast to the ideas that numbers are real, external entities whose existence and properties are discoverable, or innate, internal representations accessed introspectively, is the position adopted here: Numbers are concepts created through the interactivity of brains, bodies, and materiality; as such, numerical cognition is extended and constituted by the interactivity among these components. This extension allows their content to vary between number systems, as well as change over time within any particular system,[233] with materiality accounting for a large portion of the between-and-within-systems variability. Specifically, number concepts are:

> (1) constructed through interactivity among the psychological, behavioral, and material components of numerical cognition; (2) neither wholly external (real) or internal (innate/representational) but extended to include brain, body, and materiality; and (3) their content changes as the components interact and change through their interactivity.[234]

The next chapters examine the neurological and behavioral bases for numbers, which are shared across cultures and languages. Subsequent chapters examine the material basis for numbers in the ANE, focusing on change in material counting technologies, numerical concepts, and the respective contributions of brains, bodies, and materiality to the cognitive system for numbers.

[232] Overmann 2016c.
[233] De Freitas et al. 2014; Malafouris 2013; Núñez 2009; Read 2016.
[234] Overmann 2016c.

# CHAPTER 4

# HOW NUMBERS ORIGINATE

# THROUGH MATERIAL ENGAGEMENT

## 4.1 Realizing Numbers through the Interactivity of Brains, Bodies, and Materiality

Certain characteristics of number systems are both prevalent and similarly structured across cultures and languages: finger-counting, lexicalization, and organization. Finger-counting spans the spectrum of number availability, from restricted to highly elaborated number systems.[235] Across languages, words for numbers share common tendencies, known as universals,[236] and across number systems, numbers are organized by increasing magnitude and productive grouping into cycles and bases.[237] These common characteristics are attributed to the shared psychological ability to appreciate quantity (i.e., numerosity), shared physiological features (e.g., five-fingered hands), and may, as argued here, be because numerical language absorbs material characteristics like productive grouping. That is, numerical organization is structured by the materiality used to realize and represent numbers, and this in turn informs the universal characteristics of both (spoken) number words and (written) numerical notations.

Linguists who argue that such patterning represents rules inherent to language (e.g., Chomsky, Hurford, Everett; discussed in Chapter III) have failed to explain why this

[235] Overmann 2013a; 2013b; 2014.
[236] Aikhenvald 2000; Comrie n.d.; 1989; Greenberg 1978.
[237] Chrisomalis 2010.

must necessarily be the case, given that the patterning is implicit to material engagements (i.e., behaviors that rearrange materiality into groups sized according to things like the amount of fingers on both hands). In other words, they have not successfully defended the idea that the mechanism is necessarily internal (linguistic), in view of the availability and influence of external (material) structures.

The commonalities shared by number systems suggest that different peoples invent similar numbers as a function of what they share: human brains, human bodies, and the material world. It also makes it likely that ANE peoples would have been subject to similar patterning and would have realized their similar numbers (attested by characteristics like finger-counting, lexicalization, and organization) in similar fashion (inferred from archaeological and textual evidence). Certainly, the patterning of grammatical number (discussed in Chapter 6) in Sumerian, Akkadian, and Elamite, attested in third-millennium texts, suggests that ANE peoples had typical quantity perception, as would only be expected, given that numerosity is found in primates and mammals generally (i.e., has a phylogenetic distribution that suggests numerosity is an ancestral capability). ANE peoples also presumably possessed typical human physiology, including five-fingered hands, supported by linguistic evidence of finger-counting (Chapter 6). These similarities of psychological abilities and physical characteristics imply that ANE peoples would have realized their numbers the same way extant peoples do.

Despite substantial commonalities, however, no two numerical systems are identical, suggesting that the mechanism—material engagement—also admits significant variability, as individuals produce different combinations and societies accumulate unique practices, histories, and trajectories of change. This implies that ANE numerical cognition would have had unique aspects as well, differences related to the specific material technologies used to realize, express, and manipulate numerical concepts; the specific

social motivations for elaborating numerical concepts; and the particular decisions made and conventions adopted by the different communities of practice.

## 4.2 Brains: Psychological Processing in Numerical Cognition

The psychological processes critical to numerical cognition include *numerosity*, the ability to appreciate quantity; *categorization*, the ability to classify objects as similar; *abstraction*, the ability to form concepts; and *sequence*, or serial order processing.

### 4.2.1 Numerosity

Only humans (as far as is known) transcend the quantity perception shared with other species—*numerosity*—to discern higher-order relations and form concepts of discrete quantities through material engagement. Numerosity underlies both the human ability to form explicit concepts of discrete quantity (numbers) and reason with them (numeracy). The involvement of numerosity in numbers is demonstrated by activity in the IPS, the part of the brain implicated in appreciating quantity.[238] In numeracy, numerosity's involvement in manipulating symbolic number representations is demonstrated through *distance* and *size effects* (respectively, the accuracy of numerical judgments increases when the interval between numbers is larger and decreases when both numbers are big[239]). Damage to the IPS or regions with which it interacts is also associated with impaired ability to reason with numbers.[240]

The ability to discern *more* from *less* has been demonstrated in mammals, birds, amphibians, fish, and possibly insects (Figure 4.1): For example, salamanders, given a

[238] Fias et al. 2007; Kazui et al. 2000.
[239] Fayol et al. 2005.
[240] Kahn et al. 1991; Roux et al. 2003.

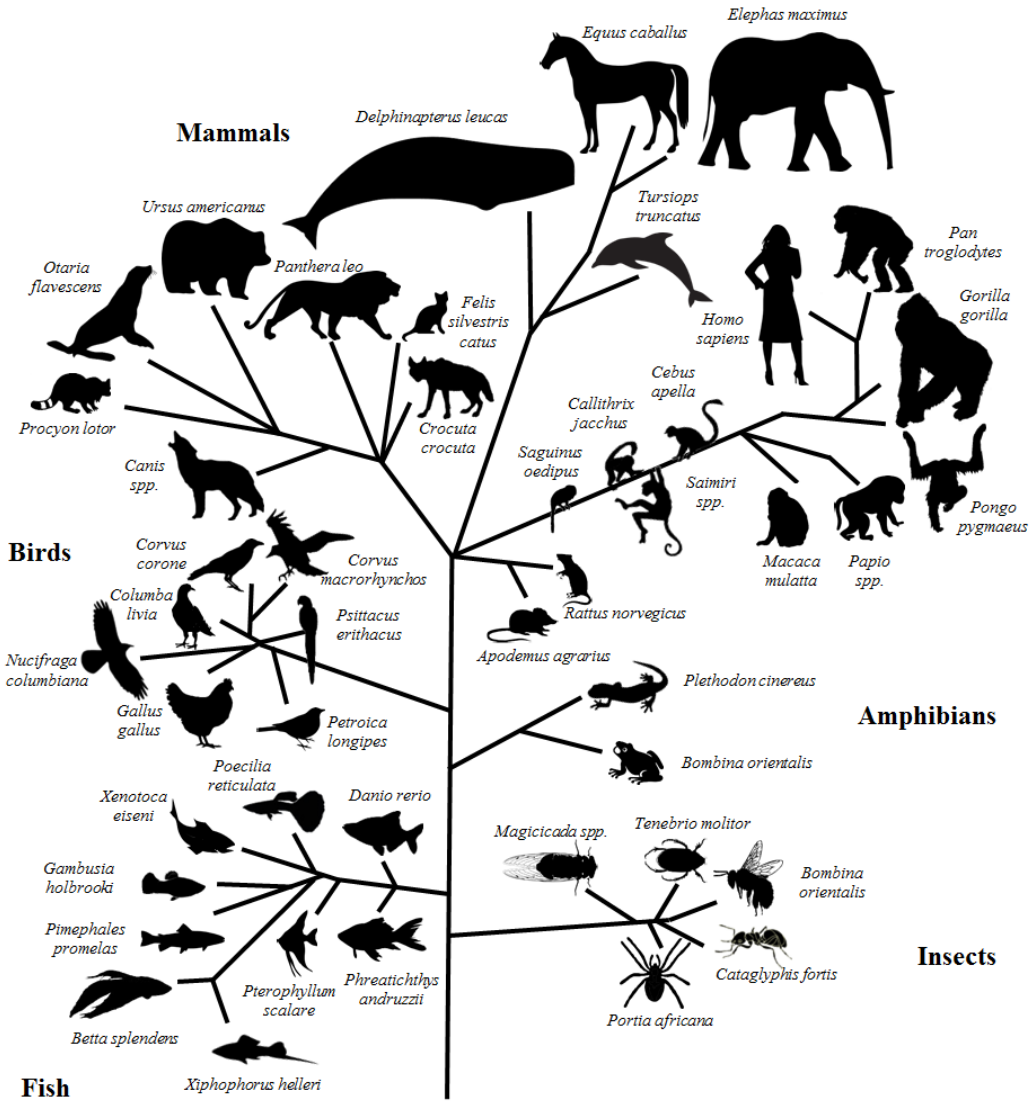


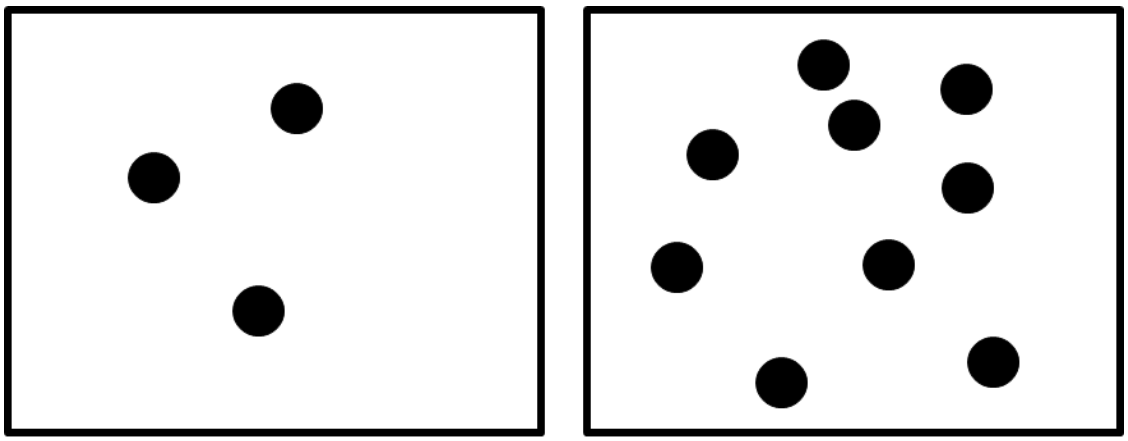

*Figure 4.1*. Animal species demonstrating numerosity (also see Appendix A). Cladistic relations are notional. Key: *spp*. = *Species pluralis* (Latin, 'multiple species'). Images are from the public domain and were sourced from clker.com, OpenClipart.org, ClassroomClipart.com, pixabay.com, and Wikipedia Common.

choice between more flies to eat and fewer, reliably choose more[241]; guppies, given a choice of two groups of fish to join, generally choose the larger.[242] The widespread phylogenetic distribution suggests that the ability to discern quantity in things like food resources, threat potentials, and safety-in-numbers may help a species survive.[243]

Numerosity is functionally divided into *subitization*, the ability to appreciate

[241] Krusche et al. 2010.
[242] Piffer et al. 2012.
[243] Ansari 2008.

quantities up to about 'three' without counting (infrequently, the upper limit is 'four'),[244] and *magnitude appreciation*, the ability to appreciate quantities above 'three' as quantity differentials (i.e., bigger and smaller). Collectively, these are known as the *approximate number system* (ANS)[245] (Figure 4.2). The division reflects attentional constraints on object recognition and individuation, as well as a constant that governs the perception of stimulus intensity. Subitization, which may have emerged from the ability to attend to multiple objects simultaneously,[246] engages attentional resources and is thus constrained by limits on the number of items that can be concurrently distinguished and attended.[247] The Weber–Fechner constant, also known as the law of just-noticeable differences,[248] postulates a logarithmic relation between a physical stimulus and its internal representation that spans multiple sensory modalities (e.g., quantity, temperature, pressure,



**Subitization ('one', 'two', and 'three')** **Magnitude appreciation ('many')**

*Figure 4.2*. The functional divisions of the ANS. Subitization (left) is the ability to appreciate quantities up to 3 without counting, magnitude appreciation (right) the ability to discriminate *more* from *less* in quantities above 3 if the difference falls above the noticeability threshold. Without counting, the quantity on the left is recognizable as 'three', while the quantity on the right is generally characterized as 'many'.

[244] Kaufman et al. 1949.
[245] Piazza et al. 2013.
[246] Piazza 2011.
[247] Alvarez et al. 2007; Burr et al. 2010; Railo et al. 2008.
[248] Fechner 1860; Weber 1834. However, Masin (2009) has argued that the so-called constant lacks a law's invariancy.

weight, sound volume, visual brightness[249]). Simply, stimuli must be sufficiently distinct (i.e., above a certain threshold), or they cannot be told apart.

While the constant can be expressed as a logarithmic function,[250] it can more easily be envisioned as the distance between numbers.[251] In the natural numbers, the *interval* between adjacent numbers is stable (i.e., in all cases, the interval between 1, 2, 3, and 4 is 1), but the *distance* between adjacent numbers, the Weber ratio, decreases rapidly (e.g., the ratios for 1:2, 2:3, and 3:4 are 1, ½, and ⅓, respectively).[252] At quantities of 'three' and below, the subitizable range, the numerical distance is (relatively) large, and thus for subitizable quantities, the constant arguably governs nothing but its upper limit. At numbers higher than 'three' (non-subitizable quantities), numerical distance is too small to be appreciated except as differences between groups, and then the numerical distance must remain above a certain threshold, or it cannot be perceived.

Attentional-resource limits and the Weber–Fechner constant entail that quantities up to 'three' are recognized without counting, quantities above 'three' as bigger or smaller in groups. This perceptual experience of quantity, subserved by common psychological functionality and thus shared across cultures and languages,[253] corresponds exactly to the first number terms to emerge across languages. In languages widely separated by time and distance, lexical terms for numbers emerge as 'one-two-three-many', with some languages containing relative terms for 'many' that mean 'small many' and 'big many'.[254] For example, the Andamanese (islanders in the Bay of Bengal, south of the Indian subcontinent, thought to have migrated there some 60,000 years ago) and the Yaghan (or

[249] Grunwald et al. 2008.
[250] Weber's law postulates "a logarithmic relation between the physical stimulus and its internal representation. $Dp = K*\Delta S/S$: where $Dp$ is the smallest noticeable difference; [K is a constant;] $\Delta S$ is the physical difference and $S$ is the stimulus intensity" (Piazza 2011, 542).
[251] Nieder et al. 2003; Piazza 2011; Pica et al. 2004.
[252] The Weber ratio is "equal to the difference between the two numbers divided by the smaller number" (Halberda et al. 2008, 1457). For example, the ratio 1:2 is the difference between them (2 – 1 = 1) divided by the smaller number, which yields 1; the ratio 2:3 is (3 – 2)/2 = ½, the ratio 3:4 is (4 – 3)/3 = ⅓, etc.
[253] Henrich et al. 2010.
[254] Comrie n.d.; Menninger 1992.

Yamana, residents of Tierra del Fuego, southern South America, who are believed to have inhabited the region for about 10,000 years) are separated by more than 25,000 kilometers over land, yet at the time of contact, both counted only to 'two' or 'three' and indicated higher quantities as 'many':

> [T]he only numerals in the [Andamans] language are those for denoting "one" and "two," and that they have absolutely no word to express specifically any higher figures, but indulge in some such vague term as "several," "many," "numerous," [or] "innumerable"[255]

And,

> La numération parlée s'arrète pour les Fuégiens au nombre trois; au delà, ils désignent toute collection d'hommes ou de choses par les mots *quelques-uns* et *beaucoup*.[256]

The ubiquity of 'one-two-three-many' as the initial numbers[257] suggests that quantity expression is structured by numerosity. The impression is reinforced by the observation that artificial models programmed with only linguistic parameters (e.g., grammar; lexical/syntactic rules) generally fail to converge on solutions that resemble real-world number systems.[258] Simply, initial quantity is expressed according to the way it is perceived. However, the anumeric experience of quantity is mere cardinality, and cardinality is insufficient for realizing number concepts.[259] Peoples are limited to anumeric cardinality until they engage materiality in a way that allows them to form number concepts, a process of categorization and abstraction.

### 4.2.2 Categorization

The cognitive process that groups or differentiates objects according to similarities or dissimilarities of their properties, relations, or functions is known as categorization. The

[255] Man 1932, 32.
[256] Martial 1888, 208; emphasis added.
[257] Comrie 2013; Menninger 1992.
[258] Hurford 1999.
[259] Rips et al. 2008.

ability to categorize is something other species share to a certain degree but in which humans also differ significantly. Along with many other species, humans can make categorical judgments of identity; this is the ability to tell sameness or difference between single elements, as in being able to tell that A is the same as A and that A differs from B.[260] Unlike most other species, humans can also make categorical judgments of relations, sameness and difference in multiple elements: AA and BB share the quality of containing identical elements, while AB, AC, and CD do not.[261] Other great apes can form categories of relations, but for them it is neither intuitive nor effortless (like it is for human children), requires specific training (while it is spontaneous in humans), and even with training never appears to become as powerful.[262] It is doubtful, for example, that chimpanzees could distinguish cross-dimensional relations like recognizing that *aaaa* and BBB are both composed of identical elements, *aaaa* and **cde** share small letter size, and BBB and **cde** (which share neither elements nor size) are both trios.[263] The latter comparison is a categorization of cardinality.

Human categorization exceeds that of even closely related species partly because of our larger brains (brain size, adjusted for body size, correlates with intelligence[264]) and partly because of our ability to exploit materiality. In categorization, material structures provide anchoring and stability for concepts, as well as a continuity that the brain cannot achieve alone.[265] Thus, the incorporation of material culture into the human cognitive system may have enabled our categorization to become more complex than that of other primates. However, the idea that categorization extent and organization increase through material engagement does not imply that they are related to technological complexity,

[260] Christie et al. 2007.
[261] Christie et al. 2007.
[262] Christie et al. 2007; Langer 1986; Thompson et al. 1997; 2000.
[263] Gentner et al. 2010.
[264] Falk 2011.
[265] Fauconnier et al. 1998; Hutchins 2005.

since categorization complexity appears cross-culturally similar across significant technological differences. What does vary culturally are whether categories are based on properties, relations, or functions; the kinds of objects and entities categorized; organizing principles (e.g., coordination; hierarchical); and the amount and boundaries of categories formed.[266] This suggests that species differences in categorization lie in material engagement per se, not in specific forms of materiality. The independence of technological complexity and categorization in turn suggests that expansion of the ability would have begun early in the hominid lineage, perhaps in conjunction with the first tools, around 3 million years ago.

Russell's theory of logical types (Table 3.1) suggests how categorization might inform number concepts. When individual objects (Type 1) are compared, they are categorized as similar or dissimilar in categorical judgments of identity (something other primates can do); this enables the grouping of similar objects into sets (Type 2). Comparing cardinality across sets of objects (Type 3), however, requires the ability to form categorical judgments of relations (something other primates appear unable to do). For example, a Volkswagen and a Fiat are both automobiles (categorization of identity), and the 'twoness' shared by a pair of cars and two fingers is the number 'two'. Admittedly, categorizing two Volkswagen of the same model and color would be easier for a chimpanzee than categorizing two Volkswagen of different models and colors or a Volkswagen and a Fiat, which differ significantly in build; the principle that *the more properties objects share, the easier categorization becomes* applies to humans as well. However, apart from cardinality, sets of automobiles and fingers are dissimilar in most other respects, making a categorical judgment of their cardinal relation likely something only humans could achieve.

[266] Hunn et al. 1984; Sillitoe 2002; Unsworth et al. 2005.

### 4.2.3 Abstraction

A category is an abstraction, and since human categorization abilities exceed those of other species, human capacities for abstraction do as well. As a cognitive ability, abstraction is a process of concept formation that encompasses mechanisms like generalization, decontextualization, synthesis, and reification. *Generalization* is the identification of properties common to sets of objects, the induction from particulars, and the application of inductive insights to new domains; *decontextualization* is the extraction of content from its original circumstances to remove their influence on its meaning; *synthesis* is the combination of parts to form wholes, often in such a way that sums are greater than their constitutive parts; and *reification* is the reinterpretation of processes or relations as permanent entities in their own right, which makes them available to act as inputs to other processes or relations.[267] Abstraction is both a process "in which concepts are seen in a broader context" and the products realized by the process,[268] and it is critical to mathematics, which considers itself to deal with abstract objects, or "objects that are not concrete and tangible."[269] Neuropsychology also views numbers as abstract: "[N]umber processing [is] an abstract and nonspatial cognitive activity."[270] It is, however, an open question (examined in subsequent chapters) whether numbers are 'concrete' before they are 'abstract' or become 'more abstract' as they are elaborated,[271] especially since they are abstract from their onset (i.e., as categorizations of shared cardinality) and materially realized at their most elaborated (e.g., as manipulable symbols).

As categorizations, numbers are abstract because they are realized through abstractive processes. Generalization is involved when number concepts are realized from shared cardinalities, and when they are extended to new objects. Decontextualization and

[267] Dreyfus 1991; Ferrari 2003; Sfard et al. 1994.
[268] Tall 2002, 11.
[269] Gowers 2002, 16.
[270] Fias et al. 2005, 43.
[271] Robson 2007; Schmandt-Besserat 1992a.

synthesis occur when the activity of sequencing quantities for purposes like counting may dissociate them from their former relationships with the objects and sets that gave rise to them; this affords opportunities for conceptualizing them with respect to one another, which in turn affords opportunities for noticing and explicating the relations between them. Finally, reification is implicit when motor activities give rise to notions like counting or quantity relations give rise to concepts of discrete quantities. Reification is challenging, since it requires separating products from their instantiating processes.[272]

Abstraction tends to be thought of as a purely mental process, yet the concepts produced remain closely tied to their material forms:

> On the one hand, a concept is often abstracted from several of its [material] representations, on the other hand representations are always representations of some more abstract concept. When a single representation of a concept is used, attention may be focussed on this instead of the abstract object. However, when several representations are being considered in parallel, the relation to the underlying abstract concept becomes important.[273]

As new material forms are incorporated into the cognitive system for numbers, concepts may seem 'less' material (and thus 'more abstract') than they really are, the idea of conceptual distribution. For example, people enculturated into Western mathematics may think of numbers as 'immaterial' ideas, known through the signs that represent them (e.g., '–258.97'), the algorithms that manipulate them (long division), and the uses to which they are applied (budgets, tax returns, etc.). The immaterial conception, however, ignores the nature of their representation, as signs written, printed or displayed on paper and screens.[274] However, even signs displayed on computers have a materiality that interacts with human psychological and physiological capabilities. The conception also forgets the material history of number as categorizations of cardinality shared between sets of objects, realized through behaviors like pairing and one-to-one correspondence and expressed

[272] Gray et al. 1994.
[273] Dreyfus 1991, 38.
[274] Harel et al. 2002; Kaput 2002.

through material devices like the fingers. Such ignoring and forgetting, of course, are themselves forms of abstraction, as is the "delicate shift of attention" that enables a number or numerical operation to be conceptualized as "an object or property."[275] Such transformation makes numerical concepts available as objects for new processes and applications, thus opening up possibilities for new processes and applications.

### 4.2.4 Sequence

Beyond forming the number concepts themselves, construing the relations between them is essential to their elaboration as counting sequences and mathematical systems: "[T]he essential feature of constructing mathematical knowledge is the creation of relationships, and creating relationships is the hallmark of mathematical problem solving."[276] As was stated earlier, relations emerge with as few as two numbers (e.g., 'more than', 'less than', 'not equal'), so a key difference between number systems is their degree of relational explication and elaboration, matters facilitated by the increased availability of numerical exemplars.[277] Magnitude ordering has "enormous importance in mathematics. Not only the integers, but also rational fractions and all real numbers have an order of magnitude, and this is essential to most of their mathematical properties."[278]

Sequence, the ability to order, informs a wide variety of cognitive functions, including language, perception, executive functioning, organization and timing, learning, and movement.[279] Other scholars view *sequence* as an executive function,[280] positioning it more as a deliberative monitoring process than an implicit function. In numbers, sequence

[275] Mason 1989, 2.
[276] Thompson 1985, 190.
[277] Beller et al. 2011.
[278] Russell 1920, 29. Each type Russell mentions is a larger, more inclusive set of numbers. The integers are the positive natural numbers, their negative counterparts, and zero. Rational numbers are ratios of integers: fractions like ⅛ and numbers with decimals that terminate (9.765) or repeat (6.33333). Real numbers are the rational numbers plus the irrationals, numbers that cannot be written as ratios since they neither terminate nor repeat (e.g., $\pi$, $\sqrt{2}$).
[279] Bapi et al. 2005.
[280] Reviewed in Salthouse 2005.

informs the ordering of counting sequences, which always proceed from low to high, as well as the mental number line,[281] which enables estimations of numerical magnitude. Unsurprisingly, the better someone is at ordering numbers, the better she also tends to be at performing arithmetical tasks.[282] Evidence for dedicated neural circuitry in processing ordinal sequences like the numbers used for counting comes from several sources: the double dissociability of ordinal and cardinal numerical judgements,[283] which demonstrates their independence as cognitive processes; *synesthesia*,[284] a "condition in which a perceptual experience, such as color, is triggered by an unrelated sensory input … [typically in the form of] learned ordinal categories" like numbers; *semantic dementia*, a condition that preserves the ability to construct ordinal sequences when non-ordinal processing is impaired; and the *Spatial–Numerical Association of Response Codes (SNARC) effect*, in which the spatial representation of ordinal sequences influences the allocation of attentional resources.[285]

Sequencing by increasing magnitude is cross-modal, seen not just in quantity perception (which itself encompasses the sensory modes touch, vision, and audition) but also in auditory phenomenon like the musical scale. Multi-modal sequencing is demonstrated in an account of quantification by the Bemba of modern Zambia, who concurrently rank-ordered fingers quantity and voice intonation when counting:

[281] There is considerable debate over the properties of the mental number line, including whether it is innate, culturally acquired, or both (Brannon 2006; Fischer 2008; Lindemann et al. 2011; Núñez 2011; Previtali et al. 2011; Priftis et al. 2006; Sato et al. 2008).
[282] Lyons et al. 2011.
[283] Butterworth et al. 1997.
[284] Synesthesia is thought to be caused by incomplete neural pruning, such that perceptual information from one domain can leak into others (e.g., vision evokes smells, etc.); common forms include number–space synesthesia (numbers appear to occupy positions in space) and grapheme synesthesia (numbers or letters evoke colors) (Cohen Kadosh et al. 2012; Gertner et al. 2013; also see discussion in Wynn, Overmann, Coolidge, et al. 2016).
[285] Pariyadath et al. 2012, 1.

> The Bemba system of numerals is very similar to that employed by the other Bantu peoples—that is to say numbers up to ten are expressed by words and the use of the fingers arranged in a series of different positions, increase in quantity being emphasized by a rising tone of the voice.[286]

The ability to order numbers by increasing magnitude enables the explication and elaboration of numerical relations, a process to which material engagement is critical. For example, 'six' as a numerical word does not mean 'five + one' or 'seven – one', 'eighteen ÷ three', or 'square root of thirty-six', let alone any of the relations that can be characterized through number theory.[287] While some relations are implicit in number words (e.g., 'thirty-one' means 'three tens plus one'), not all those possible are implied (e.g., as 'thirty-one' does not also mean 'twenty-nine plus two' or 'thirty-two minus one'). Unlike number words, however, material expressions of quantity can be composed and manipulated, to the degree permitted by their material form, allowing the individual elements to be accessed and rearranged in different combinations, thereby facilitating the discovery of relations like the plus-one of the successor function, multiplicative grouping, divisibility, and so on.

## 4.3 Bodies: Physiological and Behavioral Contributions to Numerical Cognition

Bodies contribute to numerical cognition through unique aspects of the *H. sapiens* brain, as well as through the movement that enables material engagement, especially that of the fingers.

### 4.3.1 The Parietal Lobes

Though numerosity is widely distributed among species, the distribution also entails that the neural substrate cannot be the same: Fish brains differ from bird brains,

[286] Richards 1939, 203.
[287] Ore 1948.

which differ from mammal brains, and so on, in more than just their size. However, the neural substrate for numerosity appears fairly consistent across the order of primates.[288] From monkeys to great apes (including humans), numerosity is a function of the IPS, a cortical furrow located toward the top of the parietal lobe. In the human brain (Figure 4.3), the parietal lobe has been implicated in an astonishing variety of functions that include (in addition to numerosity and numeracy) control of and information from and about the fingers ('finger gnosia'), multimodal association, spatiality, detailed vision and tool use, motor planning, perceptual–motor coordination, the sense of self and self-representation, various forms of attention, working memory, aspects of long-term memory and language, and intent interpretation.[289]

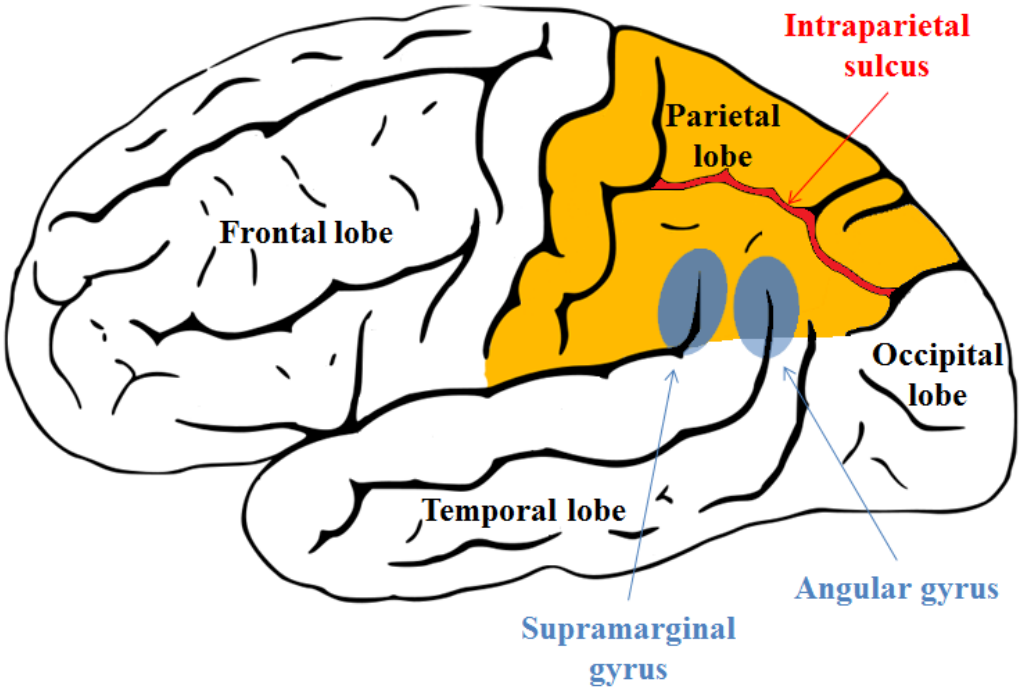



*Figure 4.3*. Diagram of the human brain, showing the major lobes (frontal, temporal, occipital, and parietal). The parietal lobe is highlighted in orange, the IPS in red, and the supramarginal and angular gyri with blue. A lobe is a major part of the cerebrum, while a sulcus is a furrow or groove and a gyrus a ridge on the cerebral surface. Image of parietal lobe from the public domain.

[288] Biro et al. 2001; Choi et al. 2006; Le Gros Clark et al. 1936; Lewis et al. 2000; Tomonaga 2008; Varga et al. 2010.
[289] Ansari et al. 2005; Bruner 2010; Bruner et al. 2014; Cantlon et al. 2006; Diester et al. 2008; Hamilton et al. 2006; Hubbard et al. 2009; Koenigs et al. 2009; Orban et al. 2006.

Possibly *the* single characteristic that best distinguishes the human brain from the brains of other primates (including those of even the most closely related species, the Neandertals) is parietal expansion.[290] While hominid brains became bigger overall in relation to body size, the *H. sapiens* parietal lobes became even bigger still. Further, parietal expansion in *H. sapiens* was vertically oriented, causing the overall shape of the brain to become globularized (i.e., more spherical than is typical for primate brain shape, including those of the Neandertals).

Expansion of the parietal lobe had several important consequences for *H. sapiens*. First, the general relation between the size of a brain region and the adaptive use of its functions relates parietal expansion to important aspects of the human ecological niche,[291] like tool use.[292] Second, globularization may have:

> … decreased inter-regional connectivity, shortening the long connections between brain regions, and consolidating intra-regional connections, increasing within-region connectivity (Rilling & Insel, 1999). Shorter, denser pathways are faster (Gibson & Petersen, 1991; Gibson, 1991) and more efficient in larger brains, which generally "reduce the proportion of connections, especially the most costly long connections" as a consequence of encephalization (Kaas, 2000, p. 17). This trade-off perhaps explains the limited inter-hemispheric connectivity in extant *H. sapiens*; as a percentage of total brain volume, the human corpus callosum is half or a third of that of other primates (Preuss *et al*., 2002). Globularization was perhaps also a factor in avoiding another potential consequence of encephalization, the reduction in the number of neurons (Kaas, 2000). The number of human neurons has remained isometrically proportional to the number found in other primate brains (Azevedo *et al*., 2009), despite encephalization.[293]

Finally, parietal expansion may have co-located regions involved in number (i.e., the IPS) with those supporting control of the fingers, tool use, and language functions like metaphor and inner speech, facilitating their interaction.[294] Thus, numerical cognition draws on evolutionary changes in the brain that are correlated with the earliest tool use in

[290] Bruner 2004; 2010; Bruner et al. 2003; 2010; 2011; 2014.
[291] Rehkämper et al. 1995.
[292] Orban et al. 2006; 2014.
[293] Wynn et al. 2016.
[294] Coolidge et al. 2012; Geva et al. 2011; Seghier 2012.

the hominid lineage and which suggest adaptive morphological change in the human lineage.[295]

The IPS interacts with the supramarginal gyrus, which has been implicated in tool use[296] and, along with Broca's area (a brain region key to the production of speech), inner speech (the internal experience of conscious thought as coherent language).[297] The IPS also interacts with angular gyrus, which has been implicated in higher-level cross-domain thinking (e.g., metaphorizing, often used to characterize numerical concepts[298]), the manipulation of numbers in verbal form, the retrieval of arithmetic facts from memory, and mathematical competence.[299] Both gyri are active when mathematical tasks are performed,[300] supporting functions like the use of the fingers in counting and the verbal expression of numerical concepts in metaphoric language.[301] The IPS represents numbers topographically, which is thought to facilitate higher-level functions such as abstraction,[302] and it may also support the ability to understand symbolic numbers: Neurons habituate to non-symbolic quantity, and this habituation has been proposed to transfer to adjacent neurons to enable them to react to symbolic numbers (effects of training and practice).[303]

In humans, the IPS contains new regions for responding to dimensional aspects of shape, especially for objects held in the hands (called *detailed vision*).[304] Since these brain regions have no homologue in macaques, a lineage of Old World monkeys separated from ours by roughly 25 million years (during which time both lineages would have

[295] Coolidge et al. 2009.
[296] Orban et al. 2014.
[297] Ardila 2011; Geva et al. 2011; Lurito et al. 2000; McGuire et al. 1996; Owen et al. 2004.
[298] Lakoff et al. 2000.
[299] Dehaene et al. 2003; Grabner et al. 2007; Grabner, Ansari, et al. 2009; Grabner, Ischebeck, et al. 2009; Ramachandran 2004.
[300] Dehaene et al. 1999; Zamarian et al. 2009.
[301] Lakoff et al. 2000.
[302] Harvey et al. 2013.
[303] Dehaene 2007.
[304] Orban et al. 2006; 2014.

accumulated evolutionary change),[305] the regions appear to have evolved since the lineages separated.[306] The human IPS has four regions sensitive to two-dimensional shape and three representing central vision; by comparison, macaques have two regions sensitive to dimension in shape and only one for central vision.[307] The IPS also supports the fine motor control of the fingers; together, detailed vision and precision finger control would have greatly improved the ability of hominids[308] to make and use tools, especially tools with multiple or moving components.[309]

Tools highlight the use of the fingers, which appear to be integral to realizing number concepts. Much of the human brain is dedicated to the fingers and hands: knowing where they are, what they are feeling and doing, and controlling their movements. This is illustrated by the Penfield homunculus, whose grotesquely huge hands signify the proportion of cortex dedicated to their motor activity (Figure 4.4).[310] Specific neural functions involving the fingers with application to the formation of number concepts include finger gnosia, haptic perception, and the neural response to grasped and graspable objects.

### 4.3.2 The Fingers

Although numerosity does not 'think' in base ten (or any base), the numerical bases most prevalent in counting systems around the world are decimal, quinary, and vigesimal (respectively, base 10, 5, and 20), numbers whose intuitive anatomical

[305] Bright 2009. Macaques are common subjects for neurological research, since they are not endangered, have a brain size that facilitates neuroanatomical study, and are closely enough related to humans that parallels can be drawn from experiments with techniques too invasive for humans.
[306] Orban et al. 2006.
[307] Orban et al. 2006.
[308] The term *hominid* encompasses extant and extinct bipedal primates, including all species of the genera *Homo* (e.g., *H. sapiens*, *H. neanderthalensis*, *H. erectus*, *H. habilis*, etc.) and the *australopithecines*.
[309] Orban et al. 2014.
[310] Penfield et al. 1954; Penfield et al. 1950.

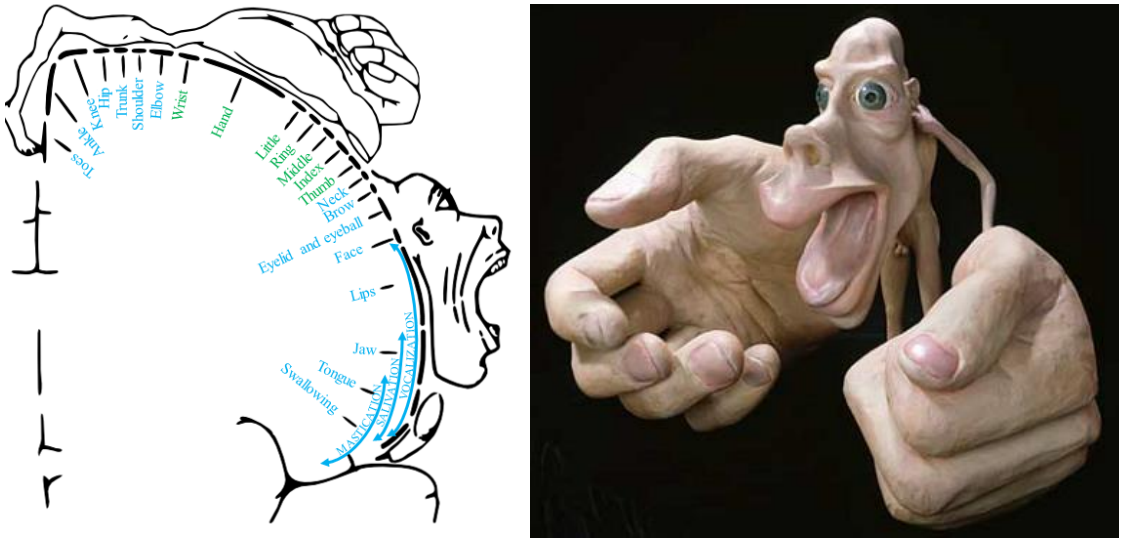



*Figure 4.4*. The cortical homunculus. The original Penfield drawing (left) characterized the amount of cortex dedicated to various motor (shown) and sensory functions (adapted from Penfield et al. 1950, Fig. 115, p. 215 and an image in the public domain). Redrawn as a human image (right), the non-proportional amount of cortex dedicated to the motor control of the hands is apparent (© The Trustees of the Natural History Museum, London).

correspondences[311] suggest a somatic basis for counting,[312] a general tendency to compare the body and materiality to each other to judge the size of each.[313] Embodied vocabularies, also widely found, additionally support the idea of a somatic basis for counting. For example, across languages, terms for 'six' through 'nine' often suggest finger-counting: 'five plus one', 'five plus two', and so on; some do this more explicitly, by referencing the fingers and toes (e.g., the term for 'thirteen' used by the Bororo people of modern Brazil, *ičare būture ivūre boeyadadawūto pugēǰe*, means 'now the one on my foot that is in the middle again', encompassing the fingers of both hands and three toes[314]). The word for 'hand' may also mean 'five', the word 'man', 'ten' or 'twenty'. The verb 'to count' means 'to finger' in the Siberian languages Chukchi and Koryak,[315] and the noun 'digit' means

[311] Menninger 1992; also see Comrie 2013. There may be an anatomical basis for bases 4, 6, 8, and 12 as well: Base 4 may omit the thumb or uses the spaces between the fingers, base 6 can include the wrist, and bases 8 and 12 might either involve the knuckles or the second hand (Bender et al. 2011a; Comrie 2013; Greenberg 1978).
[312] Comrie n.d.
[313] Mattens 2013.
[314] Closs 1993.
[315] Antropova et al. 1956.

both 'finger' and 'number' in many languages (e.g., in addition to English, the languages of the Aimoré people of Brazil and the Hudson Bay Inuit).[316] Such terms, and the ubiquity of anatomically informed base numbers (nearly all known number systems[317]), demonstrate the involvement the anatomy (particularly the fingers and hands) in enumeration and expression, not just in manipulating material forms in cardinal comparisons but as the form of materiality most typically used as the first reference set.[318] Using the fingers to count appears to be cognitively prepotent, a function of the neurological interaction of numerosity, finger gnosia, and haptic perception.

#### 4.3.2.1 Finger Gnosia

How the fingers are involved in numbers starts with finger gnosia, the somatosensory knowledge of the fingers. Finger gnosia is a function of the angular gyrus, a part of the parietal lobe that interacts with the IPS, where numerosity resides.[319] The angular gyrus also forms metaphors for numbers, manipulates numbers in verbal form, and retrieves arithmetic facts from memory.[320] Finger gnosia is also a good predictor of mathematical ability: A person who can tell which finger has been tapped when she cannot see her hands is more likely to be good at mathematical tasks.[321] Damage to the angular gyrus is associated with *finger agnosia*, the impaired ability to recognize the fingers and understand what they are doing; loss of finger-counting, the impaired ability to count with the fingers; and *acalculia*, the impaired ability to perform simple mathematical tasks.[322]

[316] Conant 1896; Richardson 1916.
[317] Bender et al. 2011a; Comrie 2013; Greenberg 1978.
[318] As will be discussed in later chapters, the subsequent incorporations of material artifacts may represent the need to quantify beyond levels feasible with fingers and toes, for reasons like limited capacity (usually ten to twenty per person) and perishability (quantities disappear once the hands assume new positions). Artifacts, by comparison, have greater capacity and representational durability.
[319] Roux et al. 2003.
[320] Dehaene et al. 2003; Grabner et al. 2007; Grabner, Ansari, et al. 2009; Grabner, Ischebeck, et al. 2009; Ramachandran 2004.
[321] Gracia-Bafalluy et al. 2008.
[322] Roux et al. 2003.

These functions are consistent with the interaction between the angular gyrus and the IPS, which underlies the appreciation of finger quantity and likely the tendency to use the fingers as a material technology for enumerating and expressing quantity, as well as manipulating objects to form categorical judgments of shared cardinality. As the fingers are recruited during the earliest realizations of initial numbers, the interaction between the angular gyrus and IPS may not be an effect of numbers but rather a cause.

#### 4.3.2.2 Haptic Perception

In forming initial number concepts, touch may be the primary mode of assessing quantity. For example, Von den Steinen emphasized this for the Bakaïrí, indigenes of Brazil, by noting that "… the Bakaïrí is not in the habit of counting with his eyes. He is still almost entirely limited to the result of touch perception."[323] As a sensory modality, active touch (or haptic perception) is limited, alterable, and intentional: It involves a relatively small perceptual field—only what the fingers can touch—that demands the fingers explore to gain sufficient information.[324] Vision, in contrast, provides a wide perceptual field easily explored through eye movement and attentional shifts, which acquires more information about larger spaces in less time than is possible through touch. However, touch generates an information stream that can be actively altered.[325] By comparison, audition involves an information stream that is highly linear; a perceiver can take action to control or block the source but cannot otherwise alter the sequence in which sound is received. As a result of these characteristics, touch has a great deal of intentionality and goal-directedness, in addition to its support of action in the form of tool manipulation.[326]

[323] Von den Steinen 1894, 505.
[324] Hatwell 2003.
[325] Helbig et al. 2008.
[326] Hatwell 2003.

Touch is complex in integrating sensory cues from what the fingers touch with internal sensory and positional cues about what the fingers are feeling and doing,[327] and it is unique in having sensory dimensions that are simultaneously internal and external (respectively, what people feel when something touches them, and what something people touch feels like to them).[328] While sensing what objects are like by touching them undoubtedly supports the ability to explore and manipulate materiality, feeling what touches the body may be similarly important: The dual nature of touch (its *bipolarity*) may help broker interactions between human psychological abilities and the objects of material environments and cultures.[329] The hands are thus the main interface between human psychological abilities and the world:

> [T]he hand is not simply an instrument for manipulating an externally given objective world by carrying out the orders issued to it by the brain; it is instead one of the main perturbatory channels through which the world touches us, and it has a great deal to do with how this world is perceived and classified.[330]

The efficiency of touch in discerning quantity is comparable to that of sight and hearing.[331] In numerosity judgments of temporally presented stimuli, touch has demonstrated greater accuracy than vision,[332] suggesting possible differences in memory, perhaps in characteristics like persistence (memory trace duration and decay profiles) or noise (simultaneous information competing for limited attentional resources). Short-term persistence appears similar—about 2 seconds—across both modalities[333]; however, the reduced perceptual field for touch may entail less noise, perhaps accounting for the greater

[327] Hatwell 2003.
[328] Mattens 2013.
[329] Mattens 2013.
[330] Malafouris 2013, 60.
[331] Philippi et al. 2008.
[332] Lechelt 1975; Philippi et al. 2008; White et al. 1959.
[333] Shih et al. 2009.

accuracy thereby. Alternatively, accuracy may be a function of neurological interaction (i.e., tactile perception with finger control, finger gnosia, and numerosity).[334]

#### 4.3.2.3 Neural Responses to Grasped and Graspable Objects

When tools are grasped, they function neurologically as if they were extended fingers: Neurons controlling finger movement react to tools as if they were part of the hand; tools literally become part of the individual using them, incorporated into the body schema (the representation of body parts, positions, and movements in space).[335] For example, a stick extends tactile perception to its tip, a phenomenon used by visually impaired people who navigate by cane.[336] In a sighted person, the extension of tactile perception is additionally accompanied by a remapping of visual space, so that things within the extended reach of the tool are perceived as being nearer.[337] In addition, if an object is sized so that it can be grasped, sensitivity to its numerical magnitude increases.[338] Sensitivity to the numerical magnitude of graspable objects is a visual phenomenon, as it can occur before objects are touched.[339] These qualities suggest that humans may be predisposed to making quantity judgments about objects that are touched and manipulated (whether enumerated or used as quantity references), with the fingers supporting both functions as actor and instrument.[340]

#### 4.3.2.4 Finger Movements

Support for the idea that physical activity is an essential component of numerical cognition comes from the phenomenon known as mental abacus, a technique for

[334] Marinthe et al. 2001.
[335] Vaesen 2012.
[336] Malafouris 2008.
[337] Berti et al. 2000; Farnè et al. 2000; Maravita et al. 2001; 2002; 2004.
[338] Ranzini et al. 2011.
[339] Ranzini et al. 2011.
[340] Gallagher 2013.

performing arithmetical operations by manipulating an imaginary device. Practitioners of the technique move their fingers, hands, and arms to perform calculations just as if they slid beads along the rods of an abacus. The technique involves language, movement, and memory (especially visual working memory).[341] Interestingly, movements do not need to be seen visually or carried out physically: Performance appears to be unimpaired when practitioners perform blindfolded or keep their hands still; however, requiring unrelated hand movements dramatically interferes with performance.[342] This suggests that the critical element is motor movement planning, not its execution (both planning and execution of motor movements are functions of the motor cortex, a region of the frontal lobe). More broadly, gesture helps make implicit knowledge more accessible, reduces demands on memory, and improves learning: For example, children who gesture are more likely to solve previously difficult mathematical tasks and incorporate novel problem-solving strategies; gesture is thought to make spatial and numerical intuitions visible and thus accessible.[343]

The observations that an abacus need not be physically present, nor finger movements seen or carried out, have an important implication for mental content: Material engagement is the basis for that content, even in the absence of both materiality and the movements needed to engage it. That is, in the absence of materiality and movement, psychological processes continue function much as if these components were present. This may explain phenomena like daydreaming and imagination, as well as the understanding and participation of mobility-impaired individuals in numeracy.

[341] Frank et al. 2012.
[342] Brooks et al. 2014.
[343] Broaders et al. 2007; Cook et al. 2008; 2010; Goldin-Meadow et al. 2001; 2009; Ping et al. 2010.

#### 4.3.2.5 Finger-counting

The use of the fingers for counting is something that at least some individuals in all cultures do, regardless of how elaborate the number system or highly skilled their proficiency level might be.[344] Fingers are more than a method of manipulating objects and supplementing memory: Their sensation and control are integrated with numerosity through neurological functions such as finger gnosia, their sensory bipolarity makes them ideally suited for brokering interactions between psychological processes and materiality, and they are conveniently (and literally) ready to hand whenever a reference set is needed. Further, nearly everyone has the same hand (i.e., five fingers, opposable thumb, etc.). As a result, across cultures, humans not only count on their fingers and hands, they do so in similar fashion.

A recent study[345] analyzed finger-counting by 33 societies from around the world, with the following results: People generally extend or flex fingers to count with them, though they also point, grasp, or shake them. Finger-counting typically involves a so-called infeasible finger position—flexing the little finger without flexing the ring finger, a position the human hand–forearm musculature is not configured to do and which therefore is not easily performed[346]—demonstrating the prioritization of sequentiality over ease of production. Counting typically starts from the outside (i.e., either the thumb or little finger) and moves across the hand. About half the time, counting is initiated with the left hand, suggesting that the starting hand may be a matter of chance or culture, not a function of handedness or the mental number line. When the second hand is involved, it is positioned with either anatomic or spatial symmetry (respectively, thumbs pointing

[344] Bender et al. 2012; Overmann 2013a; 2013b; 2014.
[345] Overmann 2014.
[346] Lin et al. 2000.

oppositely or in the same direction). The use of the fingers also imposes anatomic cycles (productive grouping) on number systems.[347]

The hands are also used to express, through shape and gesture, concepts (generally spatial or numerical in nature) that are in turn perceived visually by their recipient. The hands "communicate the features of external objects … [by] address[ing] themselves to someone's eyesight."[348] The significance of the communicative use of fingers in expressing quantity, or *finger-montring*,[349] is perhaps underappreciated because of the focus on the other quantificational purposes to which the hands are applied (e.g., as the means of sensing and manipulating enumerated objects, in pairing and one-to-one correspondence, as material instantiations of quantity, in finger-counting, etc.). As philosopher of phenomenology (the study of the structures of experiential consciousness) Filip Mattens notes, "What enables me to have a non-linguistic conversation about an object … is the fact that my hands literally copy its *spatial* features."[350] Not only do the hands play an important role in realizing initial quantity concepts, they also enable the communication of those concepts to others. Importantly, the fingers' role as an expressive medium for communicating quantity information is both non-linguistic (i.e., no language for numbers is required) and translinguistic (it communicates across differences of language).[351]

A final point about finger-counting is that its cross-cultural prevalence and similarities are partly the result of shared brain functionality and partly what the fingers have to offer in terms of their neuroanatomic characteristics (e.g., the interaction of numerosity with the neurological processes governing the fingers; their bipolarity and expressive potential), physical characteristics (availability, capacity), use applications

[347] Bender et al. 2011a; Comrie 2013; Greenberg 1978.
[348] Mattens 2013, 171.
[349] Di Luca et al. 2008.
[350] Mattens 2013, 170; emphasis in original.
[351] Chrisomalis 2010; Mattens 2013.

(object manipulation, memory offload, expression, communication), and logical characteristics (starting/stopping points, sequentiality). In what the fingers have to offer, the role of materiality (which includes the human body as a counting technology) and material engagement in numerical cognition starts to become clear.

# CHAPTER 5

# MATERIAL ENGAGEMENTS:

# BEHAVIORS IN NUMERICAL COGNITION

## 5.1 Before Numbers

Without numbers, numerosity entails an anumeric experience of cardinality in subitizable quantities and magnitude judgments of non-subitizable quantities that alone is insufficient for explaining the human ability to realize number concepts. Some mechanism is needed to wrest discrete concepts of both subitizable and non-subitizable numbers from the perceptual 'feel for' quantity, which Russell's theory of logical types[352] implies may be the comparison of cardinality between sets of objects, an instance of material engagement. Such engagement of materiality allows the innate sense of quantity to be made tangible, a form in which it is more easily explicated, manipulated, and communicated,[353] qualities that enable number concepts to be realized, expressed, and shared[354] in ways that are subject both to the capabilities and limitations of psychological processes (e.g., the attentional limits governing subitization) and the properties or affordances[355] of the material form (its durability, capacity, etc.). The active nature of material engagement is important: It is the ability to manipulate material forms that allows people to shape them and, in turn, allows materiality to engage human psychological

[352] Russell 1910; 1920.
[353] Malafouris 2010b.
[354] Coolidge et al. 2012.
[355] Gibson 1977; 1979.

processes and influence human behavior. Certainly, the active engagement of materiality—not its passive viewing—was implicit in the Bakaïrí vignette (see Chapter 4): They did not simply look at the kernels and fingers and let their brains do all the work; rather, they actively touched, manipulated, and compared the sets of objects to realize numerical concepts.

Despite significant variability to the materiality used for counting across number systems, there is a general pattern: The fingers are typically the first counting technology, likely because of their cognitive prepotence. However, the capacity of fingers for quantifying is relatively limited (usually to five per hand[356]), and their persistence is low (as the hand tires or is eventually needed for other activity). Material forms adopted to expand capacity and improve persistence include stones, knotted strings, and notched sticks or bones (only the latter tends to leave archaeological traces). These forms differ in their degree of manipulability (i.e., easier with stones, more difficult with notches), a factor in the likelihood of performing and explicating operations beyond accumulation. Groups of objects and series of marks are also subject to subitization limits on individuation, a problem of expressive complexity resolved by grouping, either into smaller chunks or encoded forms. Grouping arises from a common need for representational succinctness: Fewer representations are easier to track and manipulate because of the attentional limits on object individuation that characterizes subitization. Even when groups are based on non-subitizable numbers, they are generally composed of smaller, subitizable patterns. Encoded forms include the bundling relations inherent in the different shapes and sizes of ANE numerical tokens. Overall, numerical cognition is a system in which psychological, physical, and material capabilities and characteristics interact and structure numerical outcomes; the incorporation of new material forms

[356] Overmann 2014.

enables new applications and elaborations (which represent changes in brains and behaviors); and social circumstances and requirements motivate further elaboration.

There is a general pattern to the emergence of number terms as well: The first are subitizable (though the specific terms for them differ across cultures and languages because of the variable combinations of materiality used in realizing them). Concepts of and words for 'five' and 'ten' tend to follow, through the use of the fingers of one or two hands as referents. Concepts and words for quantities that fill the gaps between subitizable quantities and those scaffolded by the fingers (e.g., 'four' and 'six' through 'nine') might emerge next, scaffolded either by the intervening fingers[357] or some other form of materiality, formed by adding, multiplying, or subtracting from previously established quantities (another source of variability).[358] Across cultures and languages, this process produces a variety of results. For example, 'six' might be 'two + two + two' (Bakaïrí); 'three + three' (Waimiri, an indigenous language of modern Brazil); 'five + one' (Berik, an Indonesian language); 'four + two' (Mianmin, from Papua New Guinea); or an unanalyzable term like the one it has in English (perhaps originally a compound term that has become unrecognizable through longevity of use, a common lexicalization process).[359] Thus, grouping by five (and thus the use of the hands as referents) is not an exclusive route to building initial quantities; rather, subitizable quantities of objects and the hands are productively combined in ways unique to each social group.

Given that the initial words in any language are subitizable but the bases 10, 5, and 20 are the most prevalent across the world's number systems,[360] productive grouping by fingers and toes may be a secondary phenomenon, similar to the replacement of an initial

[357] The idea that the fingers and hands are used to scaffold concepts of 'five', 'ten', and the 'gap' numbers between subitizable quantities and those scaffolded by the fingers and hands is long-standing (e.g., Conant 1896; Cushing 1892).
[358] Menninger 1992; Pica et al. 2004.
[359] Rosenfelder 2011; Von den Steinen 1894.
[360] Comrie 2013; Greenberg 1978.

compound by an unanalyzable term,[361] the 'downward creep' of regularized numbers into those previously individuated by unique names,[362] and the tendency to replace longer indigenous words with shorter loan words.[363] Grouping that reflects the digits of the hands and feet may be a form of conventionalization, similar to the ways in which language and notations for numbers initially vary but converge over time on community standards through social interactions.[364]

## 5.2 The Behavioral Component of Numerical Cognition

Since artifacts "should not be considered by themselves but always in relation to use within a system of activity,"[365] the behavioral component of numerical cognition is critical to understanding the role of materiality. The behaviors considered here include pairing, one-to-one correspondence, and non-tactile enumeration (i.e., counting with only the assistance of words).

### 5.2.1 Pairing

The behavior known as pairing may be what enables humans to categorize the initial cardinal relations between sets of objects. Pairing is a simple combinatorial strategy in which more complex structures are created by combining simpler ones; it involves a sequential (non-hierarchical) motor action, like stacking one object atop another. Pairing is

[361] The Bakaïrí had a word for 'three', *ahewáo*, which was used "just as frequently" as the compound *ahége tokále* (Von den Steinen 1894, 491). The emergence of a single 'atomic' term and the replacement of compound terms by single terms in other languages suggest that *ahewáo* might have similar potential to eventually replace *ahége tokále*; etymological transparency "tends to be obscured over time and/or through frequent use via the processes of lexicalization that affect linguistic expressions in general" (Epps et al. 2012, 55–56).

[362] *Regularized numbers* are terms like 'thirty-one' ('three tens plus one') formed by lexical rules (Hurford 1975). Across languages, names for lower numbers are irregular (e.g., 'eleven' has a unique name and 'thirteen' reverses the compounding order) and imply a somewhat haphazard invention; higher number names are regular through a 'trick' of lexical naming that emerges subsequently. Where one yields to the other is unique for each language, often overlaps, and generally lowers over time as regularized names displace irregular ones (Greenberg 1978).

[363] Comrie 2013; Greenberg 1978; Sherzer 1983; Richards 1939.

[364] Saxe (2012) provides an example of this process related to the standardization of number words through social interactivity; Powell (1971) an example related to numerical notations.

[365] Dos Santos and Matos 2008, 183.

used to assess cross-species differences in representational complexity during motor actions with implications for grammar and capacities for language;[366] it puts objects (Russell's[367] Type 1) together in a way that enables their apprehension as a group (Type 2) whose property of quantity (or cardinality) is 'twoness'. Pairing also compares groups (two entities of Type 2) that share the property of 'twoness' with each other, affording an opportunity for a categorical judgment of their cardinal relation (Type 3, a number). Conversely, a pair and a single object (also two entities of Type 2) differ in their cardinality, affording an opportunity for a categorical judgment of numerical dissimilarity (also a Type 3 entity). The term 'pairing' is also used to describe the practice of "counting by pairs instead of by units."[368] Pairing in this sense uses a similar logic in that two pairs inform a concept of the number 'two'; it differs in that neither of the two pairs being compared is a body part (i.e., fingers) and in producing units that would be considered 'two' rather than 'one' in other counting systems.

Categorization and pairing are conjoined in a late nineteenth-century description of the Bakaïrí (or Bacairi), an indigenous people of modern Brazil whose highest number with an associated lexical term was 'six'.[369] Asked to count to 'three', the Bakaïrí were observed moving two of the kernels into a pair and then comparing the pair of kernels to a pair of fingers:

> If I put down 3 kernels … [the Bakaïrí] would first touch the pair of kernels, … then on the left grasp fingers V and IV and say "aháge"; then they would grasp the single kernel, shove finger III to the left toward IV and V, say "tokále," and finally announce: "aháge tokále."[370]

The behavior involved in counting to 'two' described by Von den Steinen was neither finger-counting (which would involve sequentially extending or flexing the fingers to

[366] Conway et al. 2001; Fragaszy et al. 2002; Greenfield et al. 1972; Johnson-Pynn et al. 1999.
[367] Russell 1910; 1920.
[368] Tylor 1871, 232; also see Biersack 1982 for an example from Papua New Guinea.
[369] Von den Steinen 1894.
[370] Von den Steinen 1894, 494.

enumerate objects) nor one-to-one correspondence (which would iteratively match elements across sets), but something altogether different—the comparison of the 'twoness' of two sets in a way that suggests facilitating the apprehension of their numerical similarity. The vignette implies that despite their perceptual salience, subitizable quantities are no more intuitive as numerical concepts than higher quantities are but must be similarly made explicit (though they probably undergo the process first because of their perceptual salience). Pairing also yields the atomic quantities seen cross-linguistically in the composition of lexical terms, especially in emerging number systems: Terms for small quantities (i.e., 'one' and 'two') are used to build terms for other quantities, including other subitizable quantities (e.g., 'two' might alternatively be 'one + one', categorization of a pair, or two sets, each composed of single items; 'three' might be 'two + one' or 'one + two', demonstrating variability in the order of elements in compound terms).[371] Such compound terms reinforce the impression that subitizable quantities require explication in a manner similar to that seen in higher quantities: If they were directly given by their perceptual salience, the need to build them indirectly through compounding is somewhat puzzling.

### 5.2.2 One-to-One Correspondence

A related behavior used in realizing quantity concepts, one-to-one correspondence, appears unique to humans. As was the case with pairing, one-to-one correspondence compares the cardinality of sets to ascertain similarity (or dissimilarity), yielding both a cardinal comparison and a reference set used to denote the quantity. The difference between the two is that pairing involves pairs, while one-to-one correspondence can involve any number of items (subject only to reasonable limitations in handling and keeping track of the objects involved). Additionally, pairing establishes groups for

[371] Comrie 2013; Eells 1913a; 1913b; Lean 1992.

possible subsequent comparisons, while one-to-one correspondence need not. Rather, in the latter, items are iteratively matched across groups until the set of enumerated objects is exhausted (note that there need not be an associated lexical term, in which case the quantity is 'that many', which denotes a quantity that is the same as that of the reference, rather than 'how many', which characterizes a discrete quantity).

One-to-one correspondence was observed for the Wathaurung people of southern Australia,[372] whose highest number counted, as described about 40 years later, was 'ten' (the relative dates suggest that the highest lexical number at the time of Morgan's observation might have been even lower)[373]:

> [A] messenger, came to us; he had his arms striped with red clay, to denote the number of days it would take us to reach the tribe he came from; …. The time stated for this march would be fourteen days ….[374]

Each day of his journey, the Wathaurung messenger added a stripe to denote the passage of time. Each stripe thus signified a single day, in one-to-one correspondence between stripes and days. If asked how long the return journey would take, he might point to the stripes to indicate 'that many' days, lacking both a concept of and a word for 'fourteen', though able to distinguish 'that many' (i.e., the quantity) from 'days' (the enumerated object) and 'stripes' (the quantity referent) linguistically. Meaning was contextual: The messenger said that the stripes meant days, rather than, for example, a quantity of successful hunts or a decoration unrelated to quantification. Even if the language had contained a concept of and a word for 'fourteen', the material representation lacked the ability to distinguish enumeration ('that many' or 'fourteen') from enumerated object ('days'). Representational dissociation may be critical to conceptualizing numbers as

[372] Morgan 1852.
[373] R. H. Matthews, referenced in Blake et al. (1998).
[374] Morgan 1852, 48–49.

entities in their own right[375]; the pursuit of greater specificity in the material representation of enumeration and enumerated object may be motivated by increasing ambiguity in contextualized meaning. Ambiguity in the meaning of quantification marks increases with socio-material complexity (i.e., as enumeration involves more people or additional object types). Change in their material expression in turn becomes a factor in how numbers are conceptualized.

### 5.2.3 The Effect of Behaviors with Materiality

Ordinal sequencing is implicit in behaviors with materials that impose linearity through their structure and use, as is the case with fingers and the technologies that typically follow them in the invention, manipulation, and elaboration of number concepts (e.g., tallies, torn leaves, stringed beads, and knotted strings):

> A string of beads is potentially more powerful than notched sticks or knotted strings because it materially instantiates three phenomena (CS1): a one-dimensional series, individuation of items, and an invariant sequence. Each bead occupies an unchanging sequential position along the dimension of the string, though there are two different sequences determined by the direction of movement. If the various beads have different sizes, shapes, or colors, the structure becomes a feral ordinal composition, with each bead occupying a unique, invariant place (until the string is broken). These qualities are inherent to the structure of a string of beads, and do not require that the maker possess an ordinal concept. But stringing and manipulating beads enables anyone to *notice* and even *think about* certain relationships, with beads acting as placeholders, much as modern children use numerical labels. One of the relationships is one-to-one correspondence; beads on a string can be matched with items in another array. From this a simple quantifying device—*without numbers*—could be developed.[376]

[375] As noted in Chapter 1, this change occurred in the ANE but has been characterized as a distinction between 'concrete' and 'abstract' numbers (e.g., Robson 2008; Schmandt-Besserat 1992a). In contrast, as argued here, both conceptualizations are 'abstract' (since numbers are abstractions), but their content as abstractions differs with their material representation.

[376] Wynn et al. 2016. The modern counterpart to the simple, anumeric quantifying device is the rosary, which matches beads to prayers without any need for concepts of or words for discrete quantities. The authors also describe *bootstrapping* as "a metaphor for a particular kind of cognitive development in which a more powerful cognitive system … is self-generated using a less powerful antecedent system" (Wynn et al. 2016).

Since devices like fingers and tallies can also be used for counting in non-linear or randomized fashion, the fact that they are instead used linearly suggests that linearization is beneficial. For example, such devices act as aids to psychological processes like working memory, and linearization minimizes the demand on working memory while concurrently maximizing the accessibility of the quantity information realized or represented by fingers and similar devices. Such benefits would not obtain if the information were randomized.

Making notches (or engaging in analogous activities with similar counting technologies) also represent "opportunities for learning, practice, and the development of skill,"[377] as well as interaction with:

> … the kind of tactile and visual regularity important in numerical patterning. That is, each additional notch bears a specific quantity relationship to the notches preceding and succeeding it. The accumulation of notches also instantiates the 'plus-one' relationship, which forms new quantities by adding one to any preceding quantity, and thereby perhaps provides a manuovisual basis for the idea of the successor function. … These tactile and visual regularities represent opportunities for the application of existing psychological processes (e.g., categorization, working memory) to be applied to functions such as pattern detection and the manipulation of information; existing psychological processes can also be repurposed to new cultural algorithms (Coolidge and Overmann, 2012) and, over time, new psychological processes can emerge (Wynn et al., in press).[378]

### 5.2.4 Non-tactile Enumeration: Using Language as the Reference Set

Counting "with [the] eyes"[379] (i.e., without touching either the enumerated objects or a reference set) is characterized as difficult in an observation about the Mapuche, a people of modern south-central Chile and southwestern Argentina:

> A teacher, during her first days in a Boroa area school, said she wondered why a child was continually bending over. "I found she was doing her addition problems on her toes. Every time there was a number greater than

[377] Overmann 2016d, 7.
[378] Overmann 2016d, 7.
[379] Von den Steinen 1894, 505. Hodder (2012, 122) noted the occurrence of a similar shift from haptic perception to vision in art.

> 10, of course, she had to stoop." An Alepúe [coastal city in Chile] father said with pride, "My boy counts in his head now; he no longer needs an outside thing, like his fingers and toes, to do so."[380]

Since the formation of quantity concepts may involve touch as a primary or parallel sensory modality, transitioning to a single sensory modality in concept application might be complicated by the change from manuo-visual to visual-only sensory information. Other complications presumably include the motivation for such enumeration; availability of a lexical counting sequence; the additional demands on working memory posed by counting without material involvement; formulating a performative technique; discarding previous, more familiar methods; and the difficulty of extending numerical concepts to new objects. Transfer of information between sensory modes, or cross-modal transfer, can be sequential (intermodal) or parallel (bimodal).[381] Which of the two might be the case in forming initial quantity concepts is currently unclear, though the combination of information from multiple senses may "reduce perceptual ambiguity."[382] While touch may be primary, quantity judgments are formed visually and aurally well, and vision is thought to have priority over both tactile sensation and proprioception, especially for the hands.[383] However, whether touch is primary, secondary, or concurrent, counting without touch might predictably be influenced by its omission, complicating the shift to visual counting.

## 5.3 Language

Language is a unique cognitive function that is critically involved in numerical cognition. It has an obvious and important role in making number concepts accessible, manipulable, and communicable,[384] as the related domain of social interaction does more

[380] Hilger 1957, 84.
[381] Hatwell 2003.
[382] Helbig et al. 2008, 236.
[383] Gallagher 2013.
[384] E.g., Xenidou-Dervou et al. 2015.

generally.[385] Language complements material means of accessing, manipulating, and communicating numerical concepts, and both language and materiality may be necessary to attaining high levels of numerical elaboration, as was discussed in Chapter 2. However, material engagement not only precedes linguistic access to numbers (as was implicit in several of the ethnographic vignettes), language by itself appears insufficient for realizing numbers, since there are languages that lack them (e.g., Pirahã, an indigenous language of Amazonian Brazil, perhaps the best-known example of an anumeric language).[386] Anumeric languages may be "one of the few areas in linguistics where present-day languages provide direct insight into the evolution of language."[387] In addition, language attests to the commonality of numerical invention through the 'one-two-three-many' structure of emerging numbers,[388] lexical forms like embodied vocabularies for number, and linguistic features that are structured by numerosity (e.g., grammatical number, a feature of language that distinguishes singulars from plurals; like lexical numbers, it too has a 'one-two-three-many' structure across languages).[389] Such evidence, however, is not unambiguous. Numerical invention is characterized by "multiple origins, diffusion, serial foundation, and forgetting,"[390] complicating any assessment of numerical invention and its associated lexicon, especially since words for numbers are among the most likely and quickest to diffuse from one language to another. As attested by the prevalence of numerical loan words and similar phenomena, numbers are extremely apt to diffuse between groups.[391] This tendency toward diffusion occurs because numbers are useful, especially in contact situations characterized by power differentials; differ between groups, providing contrast that can facilitate elaboration; and are *translinguistic*, or

[385] E.g., Robson 2008; Saxe 2012.
[386] Comrie 2013; Everett 2005.
[387] Comrie n.d., 1.
[388] Comrie 1989; Greenberg 1978; Menninger 1992.
[389] Corbett 2000; Perkins 1992.
[390] Overmann 2015, 13.
[391] E.g., Comrie 1999; 2005.

comprehensible in virtue of being numbers in a way that applies to few other linguistic or conceptual domains. However, linguistic features can suggest only a relative (not an absolute) chronology for the development of numbers, and phonetic and morphological variation and change tend to obscure original linguistic forms over time.

In general, ethnographic observations of quantification by peoples whose languages lack numbers suggests that when numbers are invented, they emerge first in material form—as fingers, gestures, and objects: body parts and things that can be touched or indicated with the body.[392] Lexicalization (the labeling of number concepts with lexical terms) follows, providing an access that facilitates the further explication and elaboration of numerical intuitions. The lag between conceptual realization and lexicalization is unsurprising: There is no evidence to suggest that language mediates between quantity perception and materiality; rather, the recognition of patterns in quantity percepts occur first in the form of psychological—behavioral–material interactions (like the one described earlier for the Bakaïrí), which generate numerical concepts that thereby become available for subsequent linguistic labeling. However, though numerical lexicalization lags material engagement, in general it does not do so for long. That is, though behaviors capable of scaffolding the realization of number concepts precede the first numerical concepts and vocabulary, the first concepts of and words for discrete quantities are subitizable.[393] While it is more than just theoretically possible to create material expressions of numbers for which even a highly numerate language lacks associated words—for example, by adding another zero onto the end of the highest number with a

[392] E.g., Chagnon 2013; Da Silva 1962; Karsten 1935; Murphy et al. 1955; Staden 1928; Turrado Moreno 1945.
[393] Menninger 1992.

lexical term[394]—there is more general correspondence between material and spoken numbers in terms of highest number counted than there is not.[395]

Since lexicalization follows materiality for numbers, it might be thought that spoken numbers might correspond more closely to material expressions of numbers (e.g., written numerals) than is actually the case. While both are organized similarly, they are not identical in terms of their universal characteristics.[396] Some of the variance in universals may arise from the fact that material numbers interact with perceptual modalities (e.g., quantity, tactile, and visuospatial perception) primarily and language secondarily, others from the fundamental differences in the semiotic function of material and linguistic signs. Words are ephemeral, linear, symbolic, and communicative, while materiality is persistent, lacks obvious start–stop points, enactive, and instantiative[397] (differences explored in Chapter 2). For example, where a material expression of number can imply value conventionally, in language such values must be explicitly stated (e.g., the difference between 432, where the numerals are understood to be positionally weighted, from left to right, as hundreds, tens, and units, and 'four hundred and thirty-two', where weights are expressed through the word 'hundred' and the form 'thirty'). The important point is that when the contrast between material and linguistic expressions of number is noticeable, it may highlight qualities of each, increasing the likelihood of numerical explication and elaboration; alternatively, where the contrast is minimal, the result might be a numerical system of long-term stability (e.g., the body-counting systems of Papua New Guinea, where both materiality and language provide an ordinal counting sequence

[394] While a language might lack a succinct term like 'billion' for expressing a number like $14^{2,456,210}$, the number can be expressed linguistically through other conventions, like 'fourteen to the power of two million, four hundred and fifty-six thousand, two hundred and ten'.
[395] Chrisomalis 2010.
[396] See Chrisomalis (2010) for a detailed description of the differences between the universals of written and spoken numbers.
[397] Malafouris 2013.

and have persisted for perhaps thousands of years[398]). A generalization of the principle would be that the more ways there are into numerical cognition, the more likely its elaboration becomes, subject to societal needs that motivate such elaboration.

## 5.4 Variability in Number Concepts between Number Systems and within a Single Number System over Time

Number concepts are abstractions of shared cardinality between sets of objects that have been compared through the active manipulation of materiality (e.g., behaviors like pairing and one-to-one correspondence).[399] Of the compared sets of objects, one consists of the objects being counted and may be designated as the set of enumerated objects, while the other, used to represent the concept of shared cardinality, may be designated as the reference set (i.e., used to compare and represent the cardinal number realized during counting). Within any system of number concepts, the two sets imply a four-part framework that consists of (1) sets of enumerated objects, (2) the reference set(s), (3) the relations mapped between the two, and (4) social norms and expectations governing set membership, relational mapping, and the application and use of the number concepts realized by the system. This framework enables an analysis of the similarities and differences of number concepts between number systems and within a particular number system over time. Between two number systems at any point in time, number concepts can differ in the following ways:

(1) Sets of enumerated objects: The types of objects eligible to comprise a set of enumerated objects, and the characteristics that enable them to be members, can be unconstrained or limited. In the former, numbers are assumed to apply to everything and anything: "… to tastes and to sounds, to apples and to angels, to the ideas of the mind and to the bones of the body. The nature of the things is perfectly

[398] Lean 1992; Saxe 2012.
[399] Overmann 2015.

indifferent ….”[400] However, ‘indifferent application’ is a characteristic of numbers in the Western mathematics tradition, not all number concepts in all number systems. For example, there are number systems in which numbers cannot be used to count animate beings; thus, divinities, people, and herd animals cannot belong to the set of things that can be enumerated.[401]

There is also the question of whether enumerated objects can belong to a single set of things that can be counted with a single set of numbers, or whether they belong to separate sets that must be counted with different sets of numbers. While a number system operating in this fashion might seem at first glance to be unusual, it is pertinent to the case study of the ANE: In early token-based accounting, sets of enumerated objects were represented through individualized systems of counters: one system for the category of people, another for the category of grain, a third for dairy products, and so on.[402] In fact, such division of enumerated objects into separate sets for enumeration may not be uncommon at all, as it is also suggested by the use of numeral classifiers, words that designate properties of the enumerated objects, like their shape.[403] Numeral classifiers appear to be an extensional strategy (i.e., a device that helps apply number concepts to objects that are different from those objects used to realize the concepts initially).[404]

(2) The reference set(s): As will be discussed at length in subsequent chapters, the materiality used to realize, represent, and manipulate number concepts informs the

[400] Whitehead 1911, 9.
[401] As noted in Overmann (2016c), the “enumeration of objects but not animate beings likely relates to the origins of number concepts in the manipulation of inanimate objects and the retention of anumeric strategies (e.g., spatial heuristics; see Butterworth and Reeve 2008; Butterworth, Reeve, and Reynolds 2011) for animate beings like divinities, humans, and herd animals, who might not be enumerated because they are not objects and are known as individuals.”
[402] Nissen et al. 1993.
[403] Aikhenvald 2000.
[404] Overmann 2015.

way they are conceptualized. For example, within a particular number system, number concepts can change over time "from being equivalences ('that many') to collections ('two tokens and two tokens are four tokens') to entities related numerically to similar entities ('two plus two is four'; Gowers 2008)."[405] The material form of the reference set also imposes characteristics like linearity that inform the ways in which number concepts are conceived, "manipulated, their degree of manipulability, and the types of operations they can be used with,"[406] and whether they comprise a productive or restricted system.[407] In subsequent chapters, it will be proposed that the material forms used for counting in the ANE would have helped number concepts transition from being equivalences to collections to entities, and that the capacity of the material form was related (as were social considerations) to the highest number counted.

Another question related to reference set(s) is whether there is one or several for counting enumerated objects. Again, such a number system might seem atypical, but it too is pertinent to ANE counting: For many of the various categories of enumerated objects, the counters that comprised the different reference sets were based on the metrological containers used (e.g., the jars and vessels used for grain and dairy products). This not only created system-unique relations between different units of value, it has creating an impression that the peoples of the ANE may have lacked a unified concept of number (however, doubts that this was likely to have been the case will be raised in Chapter 6).

[405] Overmann 2016c.
[406] Overmann 2016c. "Number concepts represented with the fingers, notches on a tally, or body-counting are less likely to be associated with division and subtraction, operations that may require a more manipulable material form for their realization (Overmann [2016d])." In addition, the highest number to which it is possible to count in a number system is related to the material counting technologies employed (Overmann 2013b; 2016d).
[407] The term 'productive' in this context means that there is no upper limit to the highest number counted, as it can be infinitely extended by adding 'one' to any number; this use of the term contrasts with that of Houston (2004), for whom an 'open' system is relatively accessible because its intelligibility does not require knowledge of a particular language or culture.

(3) Relational mapping between sets of enumerated objects and the reference set(s): There is a strong cross-cultural tendency for relational mapping to occur between one member of a set of enumerated objects and one member of the reference set (e.g., as is the case in counting prayers as the set of enumerated objects with the beads of a rosary as the reference set). However, such mapping is not invariably one-to-one. For example, there are number systems in which the objects being enumerated occur in pairs, rather than singles.[408] Within the cultural system where this counting occurs, it is considered one-to-one (that is, the enumerated pair is considered to be a single unit); however, from the outsider perspective, the relational mapping appears to be one-to-two.

For ANE numbers, it has generally been assumed that one-to-one correspondence was used, which in and of itself is unproblematic (certainly, one-to-one correspondence is assumed implicit in the use of fingers and tallies in this analysis). However, the use of one-to-one correspondence has also been extended, in a rather peculiar sense, to early token-based accounting: That is, not only did one-to-one correspondence in the ANE include the relations between enumerated and reference sets containing identical members (e.g., counting a flock of sheep with marks on a tally, in which one mark means one sheep) but relations between sets containing members with different numerical values. For example, in the ŠE System Š for counting barley,[409] scholars have assumed that an enumerated set of containers of grain (i.e., bowls of size $N_{34}$, jars of size $N_1$, and containers of size $N_{14}$) was also counted through the use of one-to-one correspondence. However, this entails ignoring the relations between the different units of value (e.g., five $N_{34}$ equal one $N_1$, and six $N_1$ equal one $N_{14}$). Carried to its logical conclusion, such a

[408] Bender et al. 2011a.
[409] Nissen et al. 1993.

system would lack a unified concept of number; certainly, its concept of number would radically differ from number concepts as understood in most (if not all) known number systems. Again, the likelihood of whether this was the case is discussed in Chapter 6.

(4) Social norms and expectations: Many of the issues involved in characterizing the members of the enumerated and reference sets, as well as performing the relational mapping between them, are social matters informing the content and governing the application of number concepts:

> Social groups make different decisions for things like the material form(s) used for numbers, how quantities are productively grouped (10-cycles are common but hardly foreordained), what types of objects numbers are used with, who is authorized to know and apply them, and so on, and they have particular circumstances, needs, and priorities motivating the elaboration (or not) of numbers as a cognitive technology. Individuals socially transacting numbers within the group each contribute unique perceptions, comprehensions, decisions, preferences, and mistakes, which inform how social needs are adjudicated and number concepts invented, proliferated, and conventionalized.[410]

As noted in the quotation above, one social matter is the issue of which individuals are permitted to learn and use number concepts. In some societies, the knowledge and use of the number system is considered privledged knowledge (in which case, it is typically reserved to adult males).[411] In the ANE, the demarkation between permitted and proscribed individuals appears to have been not as much a matter of gender as profession (i.e., numbers were more likely to have been known by those performing accounting functions, such as merchants and scribes, who were typically but not exclusively male[412]). However, there is a sense in which the increasing elaboration of the system of numeracy created a defacto exclusion: As the system becomes more complex, it also tends to become "increasingly opaque to

[410] Overmann 2016c.
[411] Wassmann et al. 1994.
[412] Postgate 2013; Robson 2008.

non-initiates."[413] Over time, this will increase the need for greater formalization in the training process, while intensifying the sense that initiates belong to a community of specialized practitioners.

If number concepts can vary across number systems in so many ways, why is it often assumed that they are all the same?

> Variability is obscured because comparative lexical numbers and numerical notations highlight the conceptual component shared across systems. In all number systems, the natural numbers are concepts of discrete quantity arranged in order of increasing magnitude.[414]

Variability between systems is also difficult to detect because there is a tendency to separate the instantiating material technologies into different categories, making them more difficult to compare to one another. For example, finger-counting and numerical notations are often not compared to forms like tallies and abaci, as fingers are not generally considered to be material, and the symbolic nature of representation can dominate discussions of numerical notations.[415] However, a primary reason for cross-cultural variation in number concepts is:

> … the materiality used to realize, represent, and manipulate them. Materiality is integral to concepts in general, where it acts to anchor and stabilize them (Hutchins 2005; Malafouris 2013). Change in material form thus affects conceptual content …. Materiality is important to number concepts in particular, where it performs functions like aiding visualization, decreasing demand on memory, and extending the manageable complexity of calculations (Dreyfus 1991; Kaput 1987). In this regard, however, numerical notations are only one material form among many, and not necessarily even the preferred form for numerical manipulation, as the centuries-long argument over the relative merits of abaci and algorithms attests (Reynolds 1993; Stone 1972).[416]

As was stated in Chapter 1, it is assumed in the present analysis that both fingers and numerical notations are material forms for realizing, representing, and manipulating numbers whose characteristics can be analyzed, compared, and contrasted with those of

[413] Overmann 2016a, 289.
[414] Overmann 2016c.
[415] E.g., Sutliff 2016.
[416] Overmann 2016c.

devices like tallies and tokens. Before this analysis can be presented, however, it is necessary to review the evidence for finger-counting, tallies, tokens, and numerical notations in the Levant and ANE, as well as the issues involved in assessing how numbers would have been conceptualized.

# CHAPTER 6

# CONTEXT FOR AND ASSUMPTIONS ABOUTANE NUMBERS

## 6.1 The ANE as Material Case Study

Following the previous chapters' discussion of how brains, bodies, and behaviors with materiality contribute to the realization and elaboration of numbers, this chapter and the next several focus on the ANE as a case study for materiality in numerical cognition, selected for its potential for reconstructing a particularly long sequence of material counting technologies. Several issues perplex the recovery of the sequence, however, the foremost being that the archaeological record generally does not provide much insight into things like finger-counting. While this shortfall can be addressed by using evidence of ancient languages from archaic texts, such linguistic evidence may tend to underestimate the availability, complexity, and abstractness of early numbers. In conjunction with the limitations of the archaeological record, this has reinforced the impression that tokens were the first counting technology used in the ANE, an interpretation at odds with the way number concepts are typically realized and elaborated (i.e., through counting technologies much less complex than the ANE tokens, like fingers). It has also contributed to an impression that early ANE numbers were 'concrete', a characterization counterposed to

the idea that number, a Type 3 entity,[417] is an abstraction. Thus, to assess early ANE numbers, it is necessary to address both shortfalls and evidence.

Regarding language for numbers, how numbers operate in the archaic languages and notations of Sumerian (primarily), Akkadian, and Elamite are compared to the qualities of extant languages, as known through their universals (i.e., things all or most languages do in common).[418] These extinct languages are known primarily because they are associated with one of the world's earliest original inventions of writing, which they developed into literacy[419] and elaborate mathematics.[420] That is, whereas Sumerian is known through Akkadian transcriptions, Akkadian is additionally understood through its relatedness to other, extant Semitic languages:

> One may say that we see Sumerian through an Akkadian glass darkly, because the values ('Lautwerte') of nearly all signs used in the Sumerian syllabaries of different places and periods have been identified by way of Akkadian syllabic spellings or—additionally—from the so-called tu-ta-ti syllabaries.[421]

Beyond transcriptions, Akkadian itself provides insight into Sumerian:

> [The] influence of Sumerian on Akkadian (and vice versa) is evident in all areas, from lexical borrowing on a massive scale, to syntactic, morphological, and phonological convergence. This has prompted scholars to refer to Sumerian and Akkadian in the third millennium as a 'Sprachbund' [area of linguistic convergence].[422]

Since the earliest writing reflected language so poorly that the one(s) associated with its invention have not been definitively identified[423] and two of the three languages in question (i.e., Sumerian and Elamite) are not known to be related to any other languages,

[417] Russell 1910; 1920.
[418] Comrie (1989) and Greenberg (1978) for spoken language for numbers, Chrisomalis (2010) for written notations for numbers.
[419] Overmann 2016a.
[420] E.g., Friberg 2007; Høyrup 2002b.
[421] Edzard 2003, 7.
[422] Deutscher 2007, 20–21.
[423] Veldhuis 2014.

the evidence that attests numerical language is much more recent than the linguistic phenomena itself, though the amount of time by which this is the case is also unknown.

After reviewing the linguistic evidence, an alternative view of ANE numbers is offered in which there is a greater time depth for numbers, multiple indigenous number systems, and increased availability, complexity, and abstractness of number concepts and attendant numerical vocabulary. This analysis provides important context for the discussion of finger-counting (very early and not geographically specified) and tallies (the Late Upper Paleolithic Levant), as well as issues like the tenuous cultural linkage between the users of these technologies and the later southern Mesopotamian peoples using tokens and numerical notations.

## 6.2 Early ANE Numbers

Insight into early ANE numbers comes from evidence of language and notational change, which suggest the region may have had several indigenous number systems; the intensification of sedentism, material resourcing strategies, and population density characterizing the Neolithic, Chalcolithic, and Bronze Age, characteristics associated with mature number systems and unrestricted lexical numbers in other societies; and the archaic number systems themselves, whose complexity makes it unlikely they would have been the first counting technology or used with a restricted numerical lexicon. Finger-counting is suggested by linguistic evidence of possible embodied vocabulary, and artifacts perhaps used as tallies have been found in the Upper Paleolithic Levant; both are typical early counting technologies in extant number systems and likely precursors to Neolithic numerical tokens and associated artifacts. Finger-counting in particular is also consistent with the observation that the archaeological record tends to underestimate the onset of counting because the materials typically used to realize emerging number concepts are

often either perishable (e.g., fingers, organic materials) or unrecognizable as such (e.g., collections of unmodified pebbles).[424]

### 6.2.1 Evidence of Multiple Indigenous Number Systems

Cultural groups in Mesopotamia included (in addition to the Sumerians, Akkadians, and Elamites) the Amorites, Eblaites, Gutians, Hittites, Hurrians, Kassites, and Urartians[425]; thus, the potential for there to have been multiple indigenous number systems in contact with one another was significant. The idea that there were several indigenous number systems in the ANE is important because the interaction and contrast between systems would have increased the elaboration of numerical concepts. At least one of the systems (i.e., the one associated with the early numerical impressions) would have had an unrestricted numerical lexicon. The likelihood of multiple number systems is supported by evidence of the languages associated with the ANE numerical tradition, as well as the changes the Elamites made to the numerical notations they borrowed from the Sumerians. In the debate over whether Sumerian was the language associated with archaic notations, some scholars (most notably the American Assyriologist Robert Englund) have taken the position that the language was something other than Sumerian, as the readings of certain signs suggest "loans from an unknown language."[426]

The "predominantly Semitic population in Mesopotamia in the 2nd and 1st millennia BC normally counted with decimal numbers, using their own Semitic decimal number words,"[427] suggesting an independently invented number system. Known for having "borrowed the Sumerian form of writing, the Eblaites "knew all the cuneiform

[424] Overmann 2016d.
[425] Myers 2010; Woods 2010.
[426] Englund 1998c, 80.
[427] Friberg 2007, 5.

signs expressing numbers, but they still prefer a system with a decimal base. This appears most clearly in the numbers above 100 which are always expressed by this system and not by the sexagesimal prevalent in Mesopotamia."[428] Other cultural groups may have contributed ternal, decimal, and quinary–vigesimal attributes to the Sumerian sexagesimal number system,[429] adding attributes that perhaps indicate early forms of materiality used within independent numerical traditions. The evidence of multiplicity in early number systems raises a number of issues, including the complications caused by early writing's lack of fidelity to language, whether numerical lexicons were likely to have been restricted (i.e., count no higher than about 20),[430] and what linguistic features imply about numerical lexicons being unrestricted.

Although numerical impressions appeared in the mid-to-late fourth millennium and tokens perhaps go as far back as the ninth millennium, neither provides insight into the numerical lexicon, which is not attested before the third millennium.[431] This is because early numerical impressions and proto-cuneiform (including its number signs) were non-glottographic, able to "be read with similar facility by speakers of different languages, or ... its reading has the character of paraphrase (i.e. two different 'readings' are likely to employ significant differences in word choice of syntactic construction."[432] By comparison, *glottographic* writing uses "visible marks to represent elements of a specific spoken language."[433] Because non-glottographic writing lacks fidelity to language, the language associated with early writing (and thus, with early numbers as well) cannot be identified with certainty.[434] While some scholars have supported the view that the

[428] Pettinato 1981, 183.
[429] Blažek 1999; Wilcke 2005.
[430] Comrie 2013.
[431] Dahl 2015.
[432] Hyman 2006, 234.
[433] Sampson 1999, 895.
[434] Englund 1998c.

language was likely Sumerian,[435] there is significant likelihood that the realization and elaboration of ANE numbers involved several cultural groups,[436] especially since number concepts are known to transfer readily between languages and cultures.

Because early writing is non-glottographic, it provides little insight into lexical numbers (words for numbers like 'one, two, three, …'), ordinal numbers (words for numerical order like 'first, second, third, …'), and grammatical number (singular–plural distinctions) associated with early numeration technologies, creating doubt about the extent to which the people (or peoples) inventing and using the technologies may have had lexical numbers in particular.[437] This lacuna is unfortunate, as language generally provides insight into the early materiality used to realize and elaborate number concepts, especially in the form of embodied vocabulary and features like grammatical number. The later linguistic evidence (i.e., from texts) has been interpreted as meaning that before the third millennium, archaic peoples might have had at most a restricted numerical lexicon.[438] However, the absence of unambiguous lexical and ordinal numbers and grammatical number prior to their appearance in third-millennium texts does not entail that archaic languages necessarily lacked these features, though that is one potential explanation. A plausible alternative is that features like phonetic pronunciation simply cannot be determined from non-glottographic writing:

> Numerals are written with word signs, which do not provide any information about how they are pronounced. Although we have an endless number of numerals in our corpus, we learn preciously little about their linguistic form and structure.[439]

However, this has more to do with a non-glottographic writing system's inability to represent language, a capacity whose development requires centuries of change in both the

[435] Veldhuis 2014.
[436] Dombrowski et al. 1991.
[437] E.g., Dahl 2015.
[438] Dahl 2015.
[439] Jagersma 2010, 241.

materiality of writing and psychological processing,[440] than it does a possible lack of linguistic features.

Similarly, the appearance of lexical and ordinal numbers and grammatical number in third-millennium texts is more likely a function of increased expressiveness in the script (i.e., its capacity to represent "the lexicon, grammar and syntax of a language with sufficient semantic and phonetic clues to link written expressions to a specific language"[441]) than linguistic change in the form of the acquisition of lexical and ordinal numbers and grammatical number. The former is simply more probable, given that the script was becoming increasingly glottographic and the latter typically requires more time to develop than the few centuries separating the appearance of proto-literate numerical impressions and unambiguous numerical language:

> [T]he approximate timeline for developing [grammatical number] suggested by its current geographic distribution [in extant languages] makes it doubtful the Sumerians could have [developed or] borrowed [lexical numbers], complex notations, numerical accounting—and developed [grammatical number]—in the few centuries separating the appearance of proto-cuneiform from the [ordinal numbers] of later texts.[442]

An additional consideration is the fact that numerical notations are non-glottographic from their onset and remain that way, even in highly literate traditions with fully glottographic and highly phonetic scripts (e.g., English).[443] This is because numerical notations (i.e., numerals, the material form typically associated with lexical numbers in the Western mathematical tradition) are fully semantically meaningful, and are more easily represented and manipulated, in non-glottographic form (i.e., the difference between manipulating a semantically meaningful sign for number like '7' in an equation and a glottographic representation like 'seven' in a narrative paragraph). Non-glottographic

[440] Overmann 2016a.
[441] Overmann 2016a, 285.
[442] Overmann 2015, 14.
[443] Overmann 2016a.

representation may suffice for lexical numbers, explaining why phonetic pronunciations of Sumerian lexical numbers are rarely found[444] and adding to the underestimation of their availability. This raises the question of why material representations have not been considered as evidence for lexical numbers, especially since Sumerian ordinal numbers were formed as number signs plus a suffix; when ordinal numbers are formed similarly in English—for example, '4$^{th}$' and 'fourth'—they certainly constitute both a number sign (e.g., '4') and a lexical number ('four') to which a suffix ('-th') has been added.

For these reasons, non-glottographic representation of lexical numbers likely persists well after a script starts glottographicizing, though the ability to additionally represent lexical numbers phonetically (e.g., 'seven' rather than '7') may eventually develop. Interestingly, one of the few rare occurrences of phonetic information for the Sumerian numbers comes from twenty-fourth-century Ebla, where a Semitic language (Eblaite) was spoken,[445] suggesting a non-native speaker may have needed access to the phonetic values of the Sumerian words associated with the number signs. By comparison, ordinal numbers and grammatical number are words and features that, while numerical, are not numbers. Grammatical number is the deictic encoding of referential quantity: It points to the quantity of items referred to in speech.[446] Ordinal numbers express order numerically, though order can also be expressed non-numerically (e.g., 'before' and 'next' rather than 'first' and 'second'). They are more likely to be represented glottographically: For example, ordinal numbers were formed by adding to a number-sign a suffix representing a sound in the Sumerian language: {kamma} (mid-third millennium) or {kam} (late-third millennium),[447] much like ordinal numbers are formed in English by adding the sound *-th* (e.g., the difference between 'four' and 'fourth'). Thus, grammatical

[444] Jagersma 2010.
[445] Edzard 1980.
[446] Perkins 1992.
[447] Jagersma 2010.

number and ordinal numbers appear symptomatic of a script becoming increasingly glottographic.

Regarding the question of the state of the numerical lexicon associated with the early numeration technologies, a further circumstance must be considered: If the people creating and using the archaic number systems had indeed possessed a restricted numerical lexicon[448] while materially representing numbers in the tens of thousands (i.e., unrestricted numbers in material form), they would have been quite unusual in this regard, as no extant number system is known to have such dramatic disparity between its spoken and material numbers. Though material forms for representing and manipulating quantity emerge before linguistic ones, as discussed previously, language for numbers does not lag far behind: Across extant languages and cultures, material and spoken numbers tend to correspond more closely than not. This suggests that as archaic proto-cuneiform counted into the tens of thousands through the material form (certainly, tokens that may have represented the number '600' have been dated to between 8500 and 3500 BC[449]), its users would likely have had a numerical lexicon that corresponded fairly closely to the values they represented materially, even if the words themselves remain unknown through the inadequacies of early writing and numbers persisting in non-glottographic form.

Grammatical number, which appears in third-millennium texts as a function of increasing glottographicalization,[450] suggests that ANE peoples had numerosity: Such singular–plural distinctions are structured similarly across extant languages by numerosity as 'one-two-three-(and rarely four)' or smaller subsets thereof (i.e., subitization), and 'many' or 'big many' and 'small many' (magnitude appreciation). For example, English

[448] Dahl 2015.
[449] University of Oxford 2015.
[450] Jagersma 2010.

expresses grammatical number as 'one-many': *dog* (one) and *dogs* (many)[451]; other languages add a dual, trial, and occasionally a quadral form.[452] Further, the presence of grammatical number in multiple ANE languages supports the idea there were multiple indigenous number systems in the region. Sumerian, Akkadian, and Elamite, three of the oldest languages associated with ANE numbers, had grammatical number, marked through suffixing and reduplication. *Suffixing* indicates plurality by adding to the end of a word (e.g., as *dogs* adds *–s* to *dog*), while *reduplication* repeats a word to indicate plurality (i.e., yielding a plural construction used in Sumerian but not in English like *mountain-mountain*). The Sumerian plural marker {ene} has also been considered a phrase-final *clitic*,[453] a morpheme that acts like a word syntactically but which cannot exist on its own. Suffixing, reduplication, and encliticization are three of eight possible methods for coding nominal plurality in extant languages.[454] Both Sumerian[455] and Elamite[456] distinguished singulars and plurals (a 'one–many' distinction similar to that of English) in animate nouns but not inanimate objects, though the rules governing the expression of plurality and its modulation by animacy in Sumerian have "yet to be established"[457] (by comparison, English is insensitive to animacy and thus applies grammatical number to both). Akkadian had an additional dual form used with a small number of nouns (a 'one-two-many' structure) and was insensitive to animacy.[458]

These differences in grammatical number expression and animacy modulation suggest that the three languages were associated with unique indigenous number

[451] Plural suffixing in English often consists of *-s*, *-es*, and *-ies*. English also uses internal vowel changes (e.g., man/men), retains some formations from Old English (ox/oxen), and adopts some foreign conventions (criterion/criteria).
[452] Corbett 2000; Perkins 1992.
[453] Jagersma 2010.
[454] Dryer 2013.
[455] Edzard 2003; Jagersma 2010; Michalowski 2008; Rubio 2005.
[456] Grillot-Susini 1987; Khačikjan 1998; Stolper 2004.
[457] Wilcke 2010, 26.
[458] Edzard 2003; Goetze 1946; Huehnergard et al. 2008; Sayce 1875.

traditions. This assessment appears even more likely because none of the languages were related: Sumerian and Elamite were linguistic isolates, unrelated to any known languages (including each other), while Akkadian was a Semitic language. Thus, grammatical number (and hence, lexical numbers) likely developed separately in each, even if they used common (or highly similar) material numerical representations. Further, while Akkadian incorporated elements of the Sumerian numerical tradition[459] (e.g., they adopted Sumerian numeral signs and cycles without changing them[460]), their spoken numerals (particularly characteristics like cycles and compounding) suggest they originally had a separate decimal tradition (in contrast to the quinary/vigesimal tradition suggested for Sumerian numbers).[461] For their part, the Elamites adapted the numeral signs they borrowed from the Sumerian system to decimal cycles.[462] These differences also imply distinct numerical traditions (similar cyclical variation has been documented across the number systems of Papua New Guinea[463]).

Grammatical number characterizes most extant languages (91.5% of a sample of 905 languages), appears to develop subsequent to lexical numbers (as there are no extant languages with grammatical number but no lexical numbers whatsoever), and is associated with both restricted and unrestricted lexical numbers.[464] Thus, languages with grammatical number minimally have restricted lexical numbers, and therefore the presence of grammatical number in Sumerian, Akkadian, and Elamite suggest that these languages would have had at least restricted lexical numbers. A further inference can be made from the presence of grammatical number: Across extant languages, grammatical number is

[459] Dombrowski et al. 1991.
[460] Huehnergard et al. 2008; Michalowski 2008.
[461] Dombrowski et al. 1991.
[462] Englund 1998a; 2004b.
[463] Lean 1992.
[464] Overmann 2015.

consistent with numerosity,[465] structuring that likely represents subitization's attentional constraints on object individuation and magnitude appreciation's 'fuzzy' comparison of quantity differences above the subitizing range. In extinct languages, such structuring suggests the presence of numerosity functioning in the same way it does in extant languages and peoples. This bolsters the notion that ANE numerical realization would have occurred in much the same way it does in extant number systems (i.e., influenced by shared psychological and physiological capabilities like numerosity and fingers).

Ordinal numbers, as grammatical number does, appear to develop subsequent to lexical numbers; however, whereas grammatical number occurs with both restricted and unrestricted lexical numbers in extant languages, ordinal numbers in extant languages appear to occur only with unrestricted numbers.[466] Ordinal numbers are also subject to frequency effects in spoken and written language, with the result that subitizable ordinal numbers occur more often than non-subitizable ordinal numbers. For example, in English, the frequency of ordinal numbers use decreases significantly with 'fourth', despite the availability of a well-developed lexicon for non-subitizable ordinal numbers (e.g., seventieth, hundredth, thousandth, billionth, trillionth). The correspondence between usage frequency and numerosity may not be merely coincidental: The ordinal numbers 'first', 'second', and 'third' may be more salient as a function of subitization, while the ordinal numbers 'fourth' and higher, though lexically expressible, may be less conceptually distinguishable in the 'undifferentiated many' of magnitude appreciation.

Corpora for extant languages are realized by data mining extensive bodies of text and produce word frequency lists consisting of hundreds of thousands to several million words for fully literate societies with highly glottographic scripts. By comparison, the

[465] Overmann 2015.
[466] Overmann 2015.

Sumerian corpora is limited to the texts that have been found, translated, and published, yielding a much smaller list of words that also represents a writing system in the process of becoming glottographic for a significant amount of the time considered. There are also considerable differences in the age (modern vs. archaic), coverage timespans (days to years vs. hundreds or thousands of years), and document composition (highly narrative vs. mainly administrative). Nonetheless, Sumerian ordinal numbers frequency is roughly similar to that of English, Chinese, and Arabic (see Appendix B), particularly in the literary corpora (i.e., the type most comparable to modern corpora). If indeed Sumerian ordinal numbers followed the general usage tendency, as this assessment suggests, Sumerian numbers from the third millennium cannot be classified as restricted based solely on ordinal numbers data. It is also worth noting that ordinal numbers in Sumerian were formed by adding an affix to signs for lexical numbers,[467] the method used in Mandarin (第一，第二，第三，…)[468] and an alternate method in English (1st, 2nd, 3rd, …). The similarity of formation suggests that that the numerical component of Sumerian ordinal numbers similarly represented not just number signs but lexical numbers.

Regarding Sumerian lexical numbers, "the numbers 1, 10, 60, and 3600 ha[d] their own word signs, which suggests that they [were] bases in the Sumerian numeral system,"[469] and this implies they were part of an unrestricted numerical lexicon counting to numbers consistent with the forms represented materially. The linguistic evidence therefore suggests that ANE peoples (particularly the Sumerians) were likely to have had unrestricted lexical numbers well before ordinal numbers and grammatical number appeared in third-millennium texts. This evidence, however, establishes only the prior development of unrestricted lexical numbers. It does not establish timelines for the

[467] Jagersma 2010.
[468] Xiao et al. 2004.
[469] Jagersma 2010, 241.

development of lexical numbers, ordinal numbers, or grammatical number in any of the ANE languages, nor does it identify Sumerian as the language associated with proto-literate numerical impressions or proto-cuneiform.

### 6.2.2 Material and Lexical Numbers

Two additional arguments support the idea that Sumerian, Akkadian, and Elamite may have had unrestricted lexical numbers. The first arises from the circumstance that numbers, in both material and linguistic form, are a response to, and thus an indication of, social needs for control of resources, especially across increases of scope and scale:

> … the elaboration of overall complexity in numeral systems—i.e., the development from highly restricted to generative systems that can name precise quantities into the tens, hundreds, or thousands—is motivated (at least in part) by social and cultural practices that encourage counting and/or keeping track of precise quantities. Many authors have suggested that these practices correspond to aspects of social complexity, such as the scale and scope of regional exchange (Winter 1999) and/or division of labor (Greenberg 1978: 291). However, the most frequently proposed correlate to numeral complexity is subsistence pattern—that is, whether a group relies primarily on hunting, gathering, and/or fishing, as opposed to farming and/or pastoralism, for primary food sources ….[470]

In general, restricted number systems are associated with small-scale, traditional societies living in relative isolation[471]; number systems counting into the hundreds or perhaps thousands are associated with larger societies living in villages in conditions of periodic contact and trade with other peoples; and number systems counting into the thousands, tens of thousands, and beyond are associated with large-scale, urbanized societies with well-established trade contact and significant internal complexity to manage.[472] Possible reasons why this correlation exists are explored in Chapter 11.

[470] Epps et al. 2012, 42. Notably, Epps and colleagues found only a weak correlation between specific forms of subsistence practice and number system complexity.
[471] Epps et al. 2012.
[472] Blažek 1999; Chrisomalis 2009; Divale 1999; Epps et al. 2012; Greenberg 1978; Heine 1997; Winter 1999.

The development and elaboration of lexical numbers correspond to increases in possessions and resources needing management, the emergence of value systems motivating the enumeration and accounting of goods, and the invention and use of material technologies for quantification and record-keeping.[473] Development and elaboration are also motivated and intensified by contact with other groups, increasing group size, or both,[474] conditions obtaining in both early Neolithic settlements (around 8500 BC) and village communities (by 7000 BC).[475] The intensive agriculture, monumental architecture, and complex bureaucracy that ultimately characterized the ANE by the late fourth millennium are associated with number systems counting into the tens of thousands and beyond (e.g., Egypt, China, and Mesoamerica[476]).

Given the urbanization and resourcing practices of the late fourth-millennium ANE, the peoples of that place and time would be predicted to be capable of counting to the tens of thousands or higher, levels consistent with their material expression of numbers as numerical impressions and proto-cuneiform numerical signs. Counting systems (both spoken and material) capable of counting into the hundreds or perhaps thousands would be predicted between the ninth and fifth millennia, in conjunction with increasing sedentism and food cultivation of the Neolithic,[477] corresponding well to the numbers thought to be associated with token finds from that period.[478] Finally, the emergence of initial number concepts is predicted to have occurred even earlier, in conjunction with earlier hunter–gatherer modes of existence in the late Upper Paleolithic,[479] which is consistent with finds

[473] Overmann 2013b.
[474] Overmann 2015.
[475] Bar-Yosef 1998.
[476] Chrisomalis 2005; 2010; Damerow 2010; Houston 1994; Ronan et al. 1981.
[477] E.g., Childe 1936; Lloyd 1978; Nashli et al. 2013; Reade 1991; Schmandt-Besserat 1982.
[478] University of Oxford 2015.
[479] Coinman 1996; Gladfelter 1998; Kuhn et al. 2001.

of possible tallies from that period in the Levant and later linguistic evidence suggesting the earlier development of numerical words.

### 6.2.3 Complex Archaic Numbers

Considered by themselves, the number systems of the late fourth millennium suggest that number concepts had existed for a significant period of time. Though there were about a dozen archaic number systems, each with a unique organization of quantity relations between bigger and smaller units, the overall level of complexity was relatively high and in extant number systems would be associated with mature number concepts and lexicons (not emerging or restricted ones). For example, Sexagesimal System S was used for counting most discrete objects: "humans and animals, dairy and textile products, fish, wooden and stone implements, and containers."[480] It was organized in regular cycles of ten and six[481] (see Chapter 8), making it a base-structured system whose ten-cycle suggests finger-counting.[482] Exchanging ten or six smaller units for one higher unit, the process of bundling and debundling, would have required a relative facility with using and manipulating non-subitizable quantities; were this absent, any quantity of tokens above the subitizing range would have effectively been indistinguishable ('many'), necessitating cumbersome comparison with the objects enumerated.[483] Proto-cuneiform Sexagesimal System S also counted up to 36,000,[484] and among the earlier artifacts attributed to the pre-

[480] Nissen et al. 1993, 28. System S was used to enumerate living items, System S' if the enumerated objects were dead or spoiled.
[481] Nissen et al. 1993.
[482] Chrisomalis 2005.
[483] Friberg noted, "One-to-one correspondence was needed only for counting small multiples of a number unit, since there were counting rules requiring that, depending on the circumstances, 2, 3, 4, 5, 6, or 10 units of a certain kind had to be replaced by the next higher unit. *It is a matter of opinion whether or not this counting of small multiples amounted to abstract counting*" (Friberg 1994, 483, emphasis added). Here it is recognized that such bundling–debundling entails the availability of an ordinal counting sequence and relative faculty in recognizing and manipulating non-subitizable quantities.
[484] Nissen et al. 1993.

writing period, enumeration up to 3600 or 60 was not uncommon.[485] It is difficult to envision that a number system spanning such a variety of object types, constructed in such regular cycles, and counting up to such high numbers would have been recently invented or associated with restricted lexical numbers.

Further, lexical rules for naming numbers start to emerge with as few as a dozen numbers, beginning the process of regularizing the conventions for naming quantities based on productive grouping (i.e., regularized constructions yield numbers like '22,' which is compounded from 'two tens and two', and as contrasted with giving individual quantities unique names). Thus, any number system that counts to one hundred or more implies the use of lexical rules, regularized naming conventions, and productive base numbers, including the number system(s) of the ANE. Certainly, the compounding of number words (e.g., *eš-u* or *eweš-ḫaw* as 'three tens' in proto-Sumerian,[486] discernable after the script developed sufficient fidelity to represent language, shows the use of lexical rules, which would have developed previously to the third millennium (it is only possible to infer a relative chronology). In such productive grouping, it is likely that 'three tens' represents addition rather than multiplication, as the former is the earliest operation to emerge across number systems[487] and the latter emerges subsequently because it involves new (manipulable) material forms, as well as new concepts and an increased amount of processing power.[488]

[485] At least nine and possibly 11 of 45 bullae or envelopes with tokens attributed to the pre-writing period contain the sign $N_{45}$ thought to signify 3600 and are likely Sexagesimal System S based on the included numerical signs and their order. The included artifacts were from Susa (Sb 01927, Sb 01932, Sb 01936, Sb 01967); Umma (MS 4631, MS 4634, MS 4637, and possibly MS 4638 and MS 4646); and Uruk (W 20987,11, W 20987,20). (University of Oxford 2015). An additional five such artifacts suggest counting up to 60; they are attributed to Susa (Sb 06350 and NIM—); Tepe Yahya (provenience uncertain; museum number not listed); and Uruk (W 20987,17 and W 20987,18). (University of Oxford 2015).
[486] Blažek 1999; Diakonoff 1983.
[487] Overmann 2016d suggested this reflected material agency.
[488] Kaput et al. 2008.

The method of representing quantity also suggests a long existence for numbers. In Sexagesimal System S, the complex sign for 600 ($N_{48}$) was the sign for 10 ($N_{14}$) superimposed over the sign for the 60 ($N_{34}$), a combination necessitating the mental multiplication (or, less likely, addition) of the values of the two units.[489] As was true of the amounts reached, the use of multiplication (or even addition) to form compounds for their representation is sorely at odds with the idea of a restricted number system.[490] Similarly at odds was the recognition of non-subitizable quantities implicit in fractional representation. ŠE System Š, used for barley, represented fractions by repeating sign elements, including non-subitizable values like 'six' (for $^1/_6$) and 'ten' (for $^1/_{10}$).[491] As was true of bundling–debundling, the meanings of these representations depended on the ability to count the repeated units: They were meaningful in virtue of the ability to recognize non-subitizable quantities, without which any fraction over the subitizing range would effectively be $^1/_{\text{many}}$, indistinguishable from any other non-subitizable fraction and necessitating cumbersome comparisons.

Time was also quantified in one of the archaic number systems of the late fourth millennium. $U_4$ System U was "used to note time and calendar units (i.e., twelve 30-day months to a year)"[492] in an administrative calendar used by the Uruk III period (3100–3000 BC).[493] Notably, peoples with restricted numbers do not quantify time—generally not at all, let alone in non-subitizable amounts like 12 and 30. Instead, they approximate time by natural phenomena, like the height of the sun and stars (approximate time of day

[489] Repetition is an alternative strategy commonly used in numerical notations, with the repeated units added together to obtain a complex sum (Chrisomalis 2010). However, a single $N_{48}$ appears to have meant 600 (Nissen et al. 1993), suggesting multiplication was used instead of repetition and addition.
[490] Chrisomalis noted that a numerical sign like $N_{48}$ was "multiplicative [and] meta-representational"; it "do[es] not count objects, but count[s] counts" (Chrisomalis 2005, 4).
[491] Nissen et al. 1993. In a modern number system, these units would form the denominator, though the Mesopotamians had different notions of fractions and division, notions that would not easily fit with the numerator–denominator distinction.
[492] Nissen et al. 1993, 28.
[493] Englund 1998c. There are also 87 instances of the sign $U_4$ in 70 texts from the Uruk IV period (*ca*. 3350–3200 BC) in the Cuneiform Digital Library Initiative (CLDI) database.

and night); solar and stellar rising/setting positions and floral and faunal trends (approximate time of year); lunar appearance and shape (approximate time of month), and so on.[494] Only after the development of material counting technologies and the realization numbers well above the subitizing range (both material and spoken)[495] does the practice of quantifying time develop through practices like counting lunar and solar appearances, which provide the common basis for the ultimate development of 30-day months and 12-month years across cultures and languages.

Mesopotamian calendars used for scheduling celebrations and ceremonies were also correcting for the mismatch between the solar and lunar cycles by periodically inserting an extra month, "probably … well before the Uruk III period."[496] Such interstitial corrections do not develop initially, but only after an extensive period of time (perhaps several thousands of years) in which solar and lunar cycles are counted (which depends on the availability of numbers), the interstitial mismatch is detected, and a method of adjustment is ultimately worked out. While it is not logically impossible for the Mesopotamians to have developed such timekeeping in a much shorter period of time than is typical in extant peoples, the explanation is not parsimonious. $U_4$ System U thus suggests that the ANE peoples of the late fourth millennium were the recipients of a much longer tradition of counting that would have included unrestricted numbers in both material and spoken forms. Comparing Old Babylonian calendars to the equinox and constellations suggests that "the Sumerians celebrated the beginning of the year at the vernal equinox as early as 5700 B.C., and possibly even earlier," which "presupposes a

[494] Overmann 2013a; 2013b.
[495] Both are attended by social change like increased sedentism and task specialization (Divale 1999; Overmann 2013b).
[496] Englund 1998c, 121.

system of numbers and some form of calculation, so that we may safely say that some kind of arithmetic existed in Babylonia in the 4th or 5th millennium B.C."[497]

The idea that ANE peoples had unrestricted material and spoken numbers by the late fourth millennium corresponds well with the evidence of the material numbers themselves (complexity indicating a mature number system), the level of social needs for numbers (high, indicating the likelihood of complex material and spoken numbers), and the evidence of language as documented in later texts (linguistic features likely developed much earlier than their representation in third-millennium texts). The converse assumption, that number concepts were minimal or non-existent, is crucial to the ideas that one-to-one correspondence was used and that numbers were concrete concepts. Once there are reasons to doubt the 'no numerical language' assumption, the ideas it supports are correspondingly weakened.

## 6.3 One-to-One Correspondence Re-examined

Assuming ANE peoples had a restricted lexicon for numbers well into the third millennium has informed not only the belief that tokens were the first counting technology[498] (unlikely) but also the belief that tokens were used in a one-to-one correspondence fashion with the enumerated objects:

> … the token system was based on the simple principle of one-to-one correspondence. This made it cumbersome, of course, to deal with large quantities of data, since humans can only identify small sets by pattern recognition. There are a few instances of tokens, however, which stood for a collection of items. Among them, the lenticular disk stood for a flock or, presumably, ten sheep; also, the large tetrahedron may have represented a

[497] Smith 1958, 37.
[498] Englund (1998c, 78) claimed "it [wa]s more likely that the Sumerian number word series originated in the inscribed sexagesimal system rather than the other way around," suggesting that the word series was a rather late development (near concurrent with the inscribed sexagesimal system). However, there is no case in which a notational system is known to have scaffolded initial numbers; rather, a notational system typically develops after finger-counting, an initial numerical vocabulary, and the incorporation of a material counting technology such as a tally.

> week's work or the work of a gang, compared to the small tetrahedron, expressing one man-day's work.[499]

The belief that accounting was based on one-to-one correspondence has been additionally supported with the observation that the different token shapes and sizes likely represented containers for commodities like grains and oils, wherein a number of smaller metrological units equaled a larger one.[500] However, the metrological-relations assumption is probably incorrect for Sexagesimal System S, where the first higher unit, $N_{14}$, appears based on ten fingers, rather than a natural or metrologically based quantity of objects like people or sheep. For container-based units in the systems used to account for commodities like grains and oil, anumeric one-to-one correspondence entails that quantity relations would have been based on and manipulated by manually filling or emptying containers into others of greater or lesser sizes, tokens would have merely emulated such containers in appearance, and accounting would presumably have been accomplished by matching tokens to the containers they resembled and represented. Were this so (which is unlikely), the resultant system would have been highly inefficient: As such, it would not have fit the resource management needs of the society and would likely have been either refined so that it did or abandoned and replaced by one with the requisite capacity and efficiency.

One-to-one correspondence tends to be found in emerging number systems with minimal language for or concepts of abstract numbers, where it is behavior that instantiates shared cardinality between sets of objects and thus helps to realize abstract number concepts. It is undoubtedly a cumbersome method of quantification, especially when used to enumerate higher quantities. For example, one-to-one correspondence was described as having been used to take periodic population censuses in the Kingdom of

[499] Schmandt-Besserat 1992a, 162.
[500] Nissen et al. 1993.

Dahomey, West Africa, during the eighteenth and early nineteenth centuries CE.[501] The methodology consisted of placing pebbles in sacks that were labeled to differentiate demographics like gender and age.[502] While placing one pebble per person in an appropriately labeled sack is easily performed even by innumerate people, turning the mass of pebbles collected into a discrete population number would ultimately require some enumeration technique (like the bundling–debundling used in ANE token-based accounting). Lacking some method of higher-number enumeration, a correspondence-based census would only enable estimates of whether the population was increasing or decreasing, based on changes in sack sizes from census to census.

In fact, the rulers of Dahomey do not seem to have used any enumeration techniques beyond sack-size comparisons to estimate the number of their subjects (which may have exceeded 150,000 to 180,000 individuals, as one administrative province alone was calculated to contain 83,645 individuals in 1931).[503] A technique capable of resolving the mass of counters into an intelligible quantity is needed well before the items counted with one-to-one correspondence reach a dozen (let alone hundreds or thousands), as it is difficult to identify or differentiate any quantities over the subitizing range, including the amounts enumerated in ANE token-based accounting. Notably, as early as the mid-to-late fourth millennium, ANE accounting is believed to have been capable of more precise numerical estimations than mere increases or decreases in the sizes of token collections.[504]

Certainly, it is difficult to envision how ANE peoples could have been described as having any grasp of number whatsoever if their manipulations of tokens and numerical impressions required one-to-one correspondence with the objects being enumerated; how they could have done so efficiently to the quantities suggested by even the earliest bullae,

[501] Oppenheim 1959.
[502] Herskovits (1932) was Oppenheim's source.
[503] Herskovits 1932.
[504] Nissen et al. 1993.

envelopes, and tablets; or how such correspondences could even conceivably have been made in the absence of the objects themselves. Scholars who have suggested the use of one-to-one correspondence for token-based accounting[505] do not additionally appear to believe, for example, that sheep were necessarily driven through the accounting houses for their enumeration.[506] A potential alternative, in which commodities and counters were matched elsewhere (i.e., in the sheep-fields) and accounts collected and compiled at a scribal location, implies that some method of bundling–debundling was in use, or it is difficult to explain how the accounts would appear in bundled quantities. Shepherds' pebbles or tallies, which presumably are not divided neatly into groupings of ten and six, would have required such grouping at some point. This alternative scenario merely shifts the bundling–debundling problem elsewhere and does not solve it.

Since ANE token-based accounting appears to have included bundling–debundling operations, it suggests that at least a portion of the population had a fair grasp of number concepts, at least to the extent of the ordinal counting sequence admitted by Jöran Friberg.[507]

## 6.4 Were ANE Numbers Once 'Concrete'?

The idea that the correspondences between tokens and metrological containers entailed enumerating by one-to-one correspondence, a restricted numerical lexicon, and an absence of abstract concepts of numbers has become well entrenched in the literature:

[505] E.g., Damerow 2010; Robson 2008; Schmandt-Besserat 1992a.
[506] On the other hand, Robson (2001, 102) states that numerical tokens are identifiable as such "only when they are found archaeologically with other administrative artefacts," including "standard-sized mass-produced ration bowls." This perhaps implies an expectation that at least some commodities would have been handled in conjunction with associated tokens. Of course, once commodities and tokens are no longer viewed as needing to be placed into one-to-one correspondence, the expectation that the ration bowls be found in the same contexts may well be discarded. Robson's point that accounting contexts can be identified through artifacts like seals and bullae, in addition to tokens, remains valid.
[507] Friberg 1994.

> [T]okens reflected an archaic mode of 'concrete' counting prior to the invention of abstract numbers. This is supported by the fact that there are no tokens to express abstractly numbers such as '1' or '10'. Instead, a particular counter was needed to account for each type of goods: Jars of oil were counted with ovoids, small measures of grain with cones, and large measures of grain with spheres.[508]

Similarly,

> Prior to the invention of the sexagesimal place value system [in the late third millennium], there was no concept of abstract numeration: numbers were thought of not as independent entities but as attributes of concrete objects—the length of a line, for instance, or the quantity of sheep in a flock.[509]

An absence of abstract or unified concepts of numbers is also assumed to have been the case because material representations of quantity and commodity were conjoined within any one numerical system: Proto-literate number signs and tokens encoded both quantity and commodity through their shapes, sizes, and contexts of use.[510] While quantity was understood through values associated with shapes and sizes in certain contexts, as well as by the repetition of those shapes and sizes, token shape and size could also represent commodity type. However, several methods of representing commodity type appear to have been used, only some of which jointly represented quantity. Container-shaped tokens (e.g., jars of oil) conjoined commodity and quantity,[511] as did the systems of plain token (e.g., ŠE system Š for barley),[512] though in different ways: the first by resemblance, the second by a unique shape–value system.

These systems have been interpreted as supporting the idea that there was little conceptual distinction between commodity and quantity, interpreted as a 'concrete' notion of number (the Piagetian roots of this notion were previously noted in Chapter 1). In

[508] Schmandt-Besserat 1992a, 6.
[509] Robson 2007, 75.
[510] Friberg 1994; Schmandt-Besserat 1992a.
[511] Damerow et al. 1995.
[512] Nissen et al. 1993.

comparison, commodity understood contextually,[513] through seals,[514] or by including complex tokens[515] was not conjoined, and these argue against ANE concepts of number being particularly concrete. Despite some representation being separate, the conjoined representation of quantity and commodity within any one system and the variable representation of number across multiple numerical systems have been interpreted as indicating the absence of 'abstract' numbers and the presence of 'concrete' numbers that were additionally fractured (or not unified across multiple systems of representation, as discussed below). However, possible precursor technologies like fingers and tallies necessarily represent commodity type through context or memory, implying that the representation of commodity type through conventions of token shapes and sizes was a later enhancement and not particularly indicative of concrete or fractured concepts of number.

Proto-literate number signs (and probably the plain tokens as well) were organized into about a dozen different metrologically based archaic numerical systems, each with a distinctive hierarchy of numerical values.[516] Across the different number systems, the same numerical sign might have a different value based on its context of use,[517] a characteristic called polyvalency. This the ability of any particular sign to hold different meanings based on its context of use: A numerical sign might represent six smaller units in one context but ten in another[518] (notably, in his review of Schmandt-Besserat's *Before Writing*, Friberg doubted that tokens were used in one-to-one correspondence on the

[513] This system was likely based on the premise that one or more of the trading partners only dealt in a specific item (Englund 1998c; Jasim et al. 1986).
[514] Jasim et al. 1986.
[515] Schmandt-Besserat 1992a.
[516] Nissen et al. 1993.
[517] Damerow 2010.
[518] Robson 2008.

grounds of their polyvalency[519]). As a basis for the lack of abstract numbers, however, polyvalency and context-dependence quickly lead to absurdity:

> If … polyvalence and context-dependence imply an absence of abstract number concepts, then paradoxically, the quasi-literate Uruk accountants would be less numerate than the average Sumerian who did not use texts, only number words.[520]

An alternative explanation is that the distribution of meaning over shapes, sizes, and contexts allowed for the representation of commodity type in a system limited in its representational potential, since writing had not yet been invented. In comparison, precursor counting technologies like fingers or tallies do not differentiate or record the types of objects being enumerated.[521] Instead, identification of enumerated types must be understood contextually or retained in human memory (or marked in some ancillary fashion). Such identification is a limitation of the material systems used for counting, not necessarily a lack of conceptual distinction on the part of the peoples doing the enumeration; on the contrary, they likely understood that when they marked a tally on one occasion, the marks stood for sheep, on another, goats, even if the marks themselves were indistinguishable apart from their context.

Another factor that has reinforced the impression that archaic numbers were 'concrete' arises from variability (both temporally and geographically) in the shapes, sizes, and possible meanings of artifacts classified as tokens.[522] The variability is significant even in the late fourth millennium, the period in which the correspondences between numbers and tokens are the most secure.[523] As earlier time periods are considered, more variability is introduced and the numerical meanings become less certain, culminating in the likelihood that tokens not only lacked any unified meaning across their millennia and

[519] Friberg 1994.
[520] Chrisomalis 2005, 4.
[521] Schmandt-Besserat 1992a.
[522] Englund 1998b; Friberg 1994; Zimansky 1993.
[523] Damerow et al. 1995; Friberg 1994.

geographical distribution of use but that such can never be established.[524] The variability and attendant uncertainty imply that early ANE number concepts were neither abstract nor unified, even by the late fourth millennium.[525]

The notion that differences in meaning distributed across multiple number systems, time, and geographic regions necessarily entails an absence of abstract number concepts leads to another absurd conclusion: It would additionally imply that any peoples with multiple representation systems necessarily lacked abstract number concepts. However, people enculturated into Western mathematics do not believe this to be true of themselves, despite the use of number systems with different bases (base 10 for counting; base 60 for telling time; bases 2, 8, and 16 for programming computers), metrological units and relations (liquids, weight, distance, temperature), numerals (Hindu–Arabic, Roman), and manipulation methods (fingers, tallies, pencil and paper, money, calculators, spreadsheets, etc.). Such distinctions may not represent an absence of abstract concepts as much as they represent change in conceptual content related to the materiality used to represent and manipulate number concepts, as well as conceptual distribution over all the material forms used (discussed further in Chapter 10). In all likelihood, ANE peoples were similarly unconfused:

> With regard to late fourth-millennium Mesopotamian tokens and numerals, the accountants and scribes who used them were able to manage complex administrative tasks, and it is implausible that they did not recognize that '8 sheep' and '8 bushels of grain' had something in common. There is simply no evidence from existing human languages for Bertrand Russell's assertion that, 'It must have required many ages to discover that a brace of pheasants and a couple of days were both instances of the number 2' (Russell 1919, 3).[526]

The notion that numbers become 'more abstract' when they are elaborated and are 'concrete' before then is problematic, not only for its disregard of what numbers are but

[524] Zimansky 1993.
[525] MacGinnis et al. 2014; Schmandt-Besserat 1992a.
[526] Chrisomalis 2009, 502.

also for the unilineal evolution and dominant-culture superiority implicit in applying Jean Piaget's concepts of cognitive development in children to entire cultures. For the main proponent of a Piagetian approach to ANE numbers, Peter Damerow,[527]

> … the archaic numerals represent[ed] the third in a five-stage historical stage-based sequence in the evolution of the number concept, beginning with an unsystematic preoperative stage, and followed by the proto-arithmetic used in the Near East starting around 10,000 BC.[528]

Piaget asserted that children of any and all cultures pass through four identical stages of cognitive development: a *sensorimotor stage* (0–2 years), characterized by experiencing the world through sensation and movement; a *preoperative stage* (2–7 years), characterized by the lack of concrete logic or the ability to manipulate information mentally; a *concrete operational stage* (7–11 years), characterized by logical thinking limited to matters that involved physical manipulation; and a *formal operational stage* (11–16+ years), characterized by abstract thought, metacognition, and complex problem solving.[529] Neo-Piagetian theories add a fifth, higher-order stage. For Piaget, types of societies possessed distinct mentalities, which he called *primitive* for traditional societies and *civilized* for modern ones; he asserted that "primitive mentality [was] a precursor of civilized mentality much as child thought is a developmental precursor of adult thought."[530] Extending Piaget's work to entire cultures has had the effect of characterizing Westernized societies as 'abstract thinkers' and traditional societies as 'concrete thinkers', a hierarchy implicitly biased in favor of Westernized models of thought.[531]

Among the many criticisms of Piaget's work,[532] the developmental acquisition of number concepts is not nearly as rigid as he had suggested,[533] and there is no compelling

[527] Damerow 2010.
[528] Chrisomalis 2005, 2–3.
[529] Piaget 1937; Piaget 1952; Inhelder et al. 1958.
[530] Cole 2003, 86.
[531] E.g., Bloch 2012; Saxe 2012.
[532] E.g., Lourenço et al. 1996.

reason to believe that ontogeny recapitulates phylogeny[534] in the case of number concepts. Certainly, realizing and elaborating numbers in the absence of preexisting concepts differs significantly from the acquisition of number concepts once they do exist, though it is undoubtedly the case that the latter begins shortly after the former. Further, there are no signs of a progression through similar stages in the other great numerate–literate traditions (Egypt, China, and Mesoamerica). If Piaget's theory of cognitive developmental truly applied to how societies and cultures realized number concepts, it would follow that,

> that everywhere [especially 'ancient Egypt, in the ancient Shang Dynasty of China, among the Maya of Mesoamerica and the Inka of Peru'], polyvalent, multiple numerical notation systems would exist just prior to the development of single, monovalent systems. … [Yet, only in Mesopotamia is numerical notation thought to have gone] through a stage of context-dependent or polyvalent numerical notation.[535]

This suggests that polyvalency may simply indicate that the Mesopotamians crafted unique solutions to a fairly common set of problems, including how commodity type and quantity might be represented in material form before the availability of writing and numerical notations.[536]

However problematic notions of 'abstractness' and 'concreteness' might be, they nonetheless highlight a real phenomenon: change in how numbers are conceptualized and applied. Such change is inextricably bound to the materiality used to realize, elaborate, and manipulate numerical concepts; it also encompasses the 'concrete–abstract' distinction often used to characterize the difference between applied (everyday) and theoretical (pure) mathematics (the former is concerned with everyday questions of 'how many' and 'how much', the latter on numerical interrelations). Mesopotamian scribes had made the

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[533] Chrisomalis 2005, 3.
[534] The idea that an organism's lifetime development passes through ancestral stages of species development originated in the late nineteenth century with Ernst Haeckel; recapitulation has a limited applicability to the formation of related species but is generally more incorrect than not (Gould 1967).
[535] Chrisomalis 2005, 3.
[536] Chrisomalis 2005, 3.

transition to a 'pure' mathematics by the Old Babylonian period,[537] a change that was enabled by—and very much dependent upon—the materiality used to realize, manipulate, and elaborate their number concepts.

[537] Friberg 2007; Høyrup 2002a; 2002b; Robson 2007; 2008.

# CHAPTER 7

# FINGERS AND TALLIES

# IN THE LATE UPPER PALEOLITHIC

## 7.1 Precursor Counting Technologies

Understanding how materiality ultimately enabled the elaboration of numerical concepts to achieve a 'pure' mathematics by the Old Babylonian period[538] requires examining the long sequence of ANE counting technologies and the ways in which they complemented, supplemented, and extended each other in realizing and elaborating numerical concepts. Each technology would have shared characteristics with those preceding it, solved one or more of its predecessors' limitations, injected its own characteristics and limitations into the system that is numerical cognition, and shaped the content of abstract numerical concepts. The process of realizing numbers would likely have started with finger-counting for ANE peoples, as it does for extant peoples, because of the prepotent neurological association of numerosity with the fingers, which often involves the fingers as a reference set and leaves characteristic traces like 'five + $x$' number constructions and anatomically informed productive grouping. The process of numerical elaboration would likely have been characterized by the incorporation of a material device like a tally because of affordance similarities (e.g., linearity; see Chapter 10) and differences (capacity for enumeration) relative to fingers.

[538] Robson 2007; 2008.

## 7.2 Finger-counting, Numerosity, and Early Sumerian Numbers

Assessments of early Sumerian numbers have been influenced by the lack of evidence for lexical forms of numbers prior to the third millennium and the assumptions that early ANE numerical concepts were 'concrete' and initially realized through tokens. The alternative view presented here is informed by Material Engagement Theory, which recognizes the embodied nature of numerical cognition inherent in the ubiquitous 'one-two-three-many' structuring of initial numbers (i.e., numerosity) and the universal involvement of the fingers in realizing non-subitizable numbers. Linguistic evidence suggests that the Sumerians and other early ANE peoples had numerosity,[539] and with typical hands and fingers, they were likely to have realized their initial numbers in similar fashion. Certainly, the presence of grammatical number in ABE languages suggests that ANE peoples had the same perceptual experience of quantity, which in turn supports the idea that they would have realized their numbers in much the same way extant peoples do, initially as a 'one-two-three-many' system that transcended the subitization constraint by involving the fingers and which was extended further by incorporating material devices like notched bones (tallies) and stones.[540]

Though initial numbers emerge across languages and cultures as 'one-two-three-many',[541]

> … [subitizable numbers] do not necessarily emerge in consistent order (though magnitude-ordered when expressed in counting sequences). 'Two' might emerge first (cardinality shared by pairs), 'one' next (dissimilarity of pair and single), then 'three' as 'two + one' or the cardinality of trios; other permutations are suggested by linguistic data.[542]

Across number systems, the initial non-subitizable numbers are typically realized in one of two ways: They are either constructed from smaller, subitizable quantities ('three + *x*'

[539] Overmann 2015.
[540] Nissen 1988.
[541] Menninger 1992.
[542] Overmann 2015, 13.

number constructions) or based on the fingers ('five + *x*' number constructions and terms for 'five' and 'ten' that mean 'hand' or 'man': "Where terms for 5 to 10 exist, these are usually based on 'hand', and terms for 10 to 20 on 'foot'; cross-linguistically, these are typical sources for these values"[543]). However, the predominant numerical bases (e.g., 5, 10, 20, about 85% of the 196 languages in the WALS[544]) suggest that when 'three + *x*' number constructions are used, the imposition of decimal, quinary, or vigesimal cycles generally follows at some point, through the need for a productive cycle capable of realizing higher quantities more efficiently; the availability of the hand as a common, neurologically prepotent device for realizing and displaying quantity; and the "within- and between-group interactions that occasion conventionalization."[545]

Sumerian contained both 'three + *x*' and 'five + *x*' number constructions. The rare 'three + *x*' constructions combined peš and bi (respectively, '3' and '1') "in the fashion of Roman numerals"[546]: 'one', 'two', 'three', 'three + one', 'three + two', 'three + three', and so on.[547] While "peš is not the usual Sumerian for '3', it is well attested in lists .... However, bi for '1' seems to occur in this system alone."[548] The 'three + *x*' forms correspond to the forms of later numerical notations, in which wedges were repeated or accumulated in groups of up to three (i.e., the sign for 'five' was composed of three wedges written above two wedges; 'six' was three wedges above three; etc.; by comparison, a quinary 'five + *x*' representation might represent 'five' as five wedges, 'six' as five wedges above one, etc.). Ternary grouping evokes the perceptual limits of numerosity, which constrains the ability to recognize or differentiate quantities over the subitizing range, a problem typically solved by grouping accumulative numerical

[543] Epps et al. 2012, 67.
[544] Comrie 2013; Greenberg 1978; Menninger 1992.
[545] Overmann 2015, 13.
[546] Lambert 1969, 243.
[547] Black 2007.
[548] Lambert 1969, 243.

representations into subitizable amounts. Such grouping entails the use of the fingers in manipulating enumerated objects and as a reference set in the comparisons that instantiate shared cardinality between groups.

Ternary numbers have been interpreted as the remnants of an even more archaic number system or an *emesal*, a dialect variously ascribed to women, eunuchs, literary conventions, or cultural minorities.[549] Certainly, in emerging number systems, ternary counting is sometimes used to build non-subitizable numbers out of subitizable ones,[550] a strategy that allows number system to transcend the subitization constraint on recognizing quantities higher than about three.[551] However, the sources in which the ternary numbers appear are all relatively late: a second-millennium epic (the myth *Enki and Ninmaḫ*), a first-millennium lexical series (*Antagal*), and a first-millennium grammatical list.[552] While "greater antiquity in textual problems never absolutely guarantees more original readings,"[553] nor does a younger age inevitably preclude them, the origins and relative temporality of different numerical forms are, at the very least, difficult to determine on the basis of such sources.

The ternary system has also been interpreted as possibly being a 'women's language' (emesal) that retained numerical forms older than those used by men, as is the case in some extant languages with similar forms.[554] The evidence currently suggests that emesal "occurs only as a literary dialect; that is, as the vehicle of specific literary and cultic genres and as a literary device employed, among other things, to identify female speakers generically, never individually, in a given social role."[555] This does not explain,

[549] Black 2007; Diakonoff 1983; Michalowski 2006; Thureau-Dangin 1928; Whittaker 2002.
[550] Systems employing ternary grouping are listed among the many of Papua New Guinea by Lean (1992).
[551] Lean (1992) and Saxe (2012) provide many examples of such compounding in Papua New Guinea, number systems whose limits suggest they emerged relatively recently.
[552] Lambert 1969.
[553] Lambert 1969, 247.
[554] Diakonoff 1983.
[555] Whittaker 2002, 641; also see Michalowski 2006.

however, why a literary device would include a separate number system, especially one that resembles extant number systems.[556] The explanation favored here is that ternary counting represented either retained archaic number forms (as has been suggested in the literature) or, more plausibly, the counting system of a cultural minority. These alternatives seem more likely than a 'women's language' or 'literary device' because one of the sources of the ternary numbers "contained rituals and at least one incantation" and was a fragment of a more complete text found in Ashurbanipal's library.[557] The topic, archiving, and implied scribal training are more appropriate to a (male) priestly domain than a (female) household one. Alternatively, the ternary system might indicate the incorporation of minorities from cultures with different numerical traditions into Sumerian businesses or households[558] (e.g., slaves), an interpretation consistent with the idea that the region contained multiple indigenous number systems. Beyond such speculations, the ternary counting system certainly reinforces the impression that ANE peoples had the usual perceptual experience of quantity and realized their initial numbers in ways typical among extant number systems.

Sumerian also contains 'five + *x*' number constructions. Lexically, the numbers 'six' through 'nine' are "based—at least in part—upon the fingers of the hand."[559] Damerow assumed that the emergence of a numerical lexicon "merely reflect[ed] the development of more efficient, extralinguistic techniques" like representing numbers materially, especially in the case of forming compounds through arithmetical operations, as is the case in finger-counting and associated 'five + *x*' constructions.[560] These are 'five + *x* constructions: 'six' is *yaš* (*ya*+*aš*), or 'five + one'; 'seven' is *imin* (*ya*+*min*), or

[556] E.g., from Papua New Guinea; see Lean 1992.
[557] Lambert 1969, 243.
[558] Diakonoff 1983.
[559] Powell 1971, 37.
[560] Damerow 2010, 212.

'five + two'; 'eight' is *uššu* (*ya*+*eš*), or 'five + three'; and 'nine' is *ilimmu* (*ya*+*limmu*), or 'five + four'[561]:

> As to the usual Sumerian quintal-vigesimal system, it has long been understood that it is based on counting fingers, 'five' being equivalent to 'hand' or 'fist', or simply to 'all the fingers on the hand'. 'Ten', by the same token, must have meant 'both hands', and 'twenty'—'all four limbs', or 'the whole man'.[562]

Similarly,

> Sumerian numeration seems to have begun—as most primitive systems of counting do—by numbering the fingers, one through five. Then one started over again and added up to ten—though eight is anomalous and of uncertain origin. This process was apparently repeated on the toes until one arrived at twenty. Then, beginning with the hands once more, one counted to twenty again by the same method and arrived at forty, which was called "two twenties." Then, starting back on the hand again, one counted to fifty, calling this stage "two twenties, ten," and, continuing on, one counted again on the feet until one reached sixty.[563]

And,

> That Sargon of Akkad, the ruler of the first state encompassing almost all of Mesopotamia, tried to introduce a uniform—decimally structured—system of weights and measures implies that [the Akkadians] were aware of the decimal base. This supports the idea of pre-existing number knowledge based on fingers.[564]

The lack of complete agreement regarding the etymology of 'eight' as 'five + three'[565] might indicate that an original 'five + three' construction had become a single word.[566] If *ussu* had emerged from an earlier 'five + three' construction, a fairly extensive time depth for counting up to ten would be indicated (i.e., enough time for 'eight' to become a single word whose form had undergone significant diachronic change). Such change would additionally suggest that 'eight' might have been used more

[561] Black 2007; Jagersma 2010. Also see Edzard 1980; Friberg 1986; Powell 1971.
[562] Diakonoff 1983, 90.
[563] Powell 1972, 9.
[564] Damerow 2010, 237. As noted in Chapter 6, Akkadian is known to have had decimal numbers, as Semitic languages generally do, and as noted in Chapter 4, finger-counting informs ten-cycles and the decimal base on number systems (Comrie 2013; Menninger 1992).
[565] Jagersma 2010; Powell 1971.
[566] In Chapter 6, the development of a single term for 'three' was described as occurring for the Bakaïrí words *ahewáo* and *aháge tokále* (Von den Steinen 1894) and suggested as having occurred for lexical terms like 'six' in English.

frequently than 'six', 'seven', or 'nine'. If this were the case, 'eight' might be predicted to occur as a cycle in one or more of the archaic number systems; since it is not, the discontinuity might imply the interaction of multiple numerical traditions.

Number systems in languages and cultures widely separated by time and distance are patterned anatomically, and the early Sumerian numerical sign systems contained a variety of cycles (e.g., 2, 3, 4, 5, 6, 10, and 12)[567] for which it is possible to construe an anatomical basis. Three-cycles are also common in emerging number systems, reflecting the subitization constraint and the construction of non-subitizable quantities out of subitizable ones. The most common numerical bases in extant number systems suggest that embodiment with pentadactylic limbs has a significant canalizing effect on numerical expression because the fingers (and toes) are the typical initial counting technology.[568] Sexagesimal Systems S and S', used to count discrete objects, incorporated 10-cycles[569] that suggest the fingers were used to count metrologically unrelated objects. Finger-counting can also give rise to bases of 4 (finger-counting that omits the thumb), 6 (the fingers plus the wrist), and 8 and 12 (the involvement of the second hand in 4- and 6-cycle counting).[570] Alternatively, small cycles (particularly 2, 3, and 4) might be based on metrological relations between containers, and a 12-cycle might derive from counting the lunar cycles in a year.

As an alternative to finger-counting, Schmandt-Besserat thought that animals were understood as flocks, a container concept.[571] However, whereas containers are related metrologically, objects like sheep are not. There is no inherent 'ten-ness' to a flock of 'sheep' in the same way that four quarts make a gallon; moreover, without some form of counting (i.e., fingers or tallies), there would have been no way to ensure a flock had 10

[567] Nissen et al. 1993.
[568] E.g., Cushing 1892; Menninger 1992.
[569] Nissen et al. 1993.
[570] Bender et al. 2011a; 2012.
[571] Schmandt-Besserat 1992a.

animals, rather than eight, nine, or 11. A similar logic excludes speculation that the initial forms of the Sumerian words for 'one', 'two', and 'three' were derived from the words for 'man', 'woman', and 'house' (i.e., "Les Sumériens auraient compté: «homme», «femme», «maison», c'est-à-dire «homme», «couple», «famille», c'est-à-dire «1, 2, plusieurs»"[572]). Not only does this impose modern conventions on ANE household structure, there is no inherent 'oneness' for 'man', 'twoness' for 'woman' or 'marriage', or 'threeness' for 'house' or 'household' in the same way there is an inherent 'fiveness' to the fingers of one hand.

It is unsurprising that the fingers are generally the first material counting technology across cultures and languages. As was noted in Chapter 3, the hands are the main interface between human psychological abilities and the world,[573] their sensorimotor control interacts with numerosity, and their sensory bipolarity makes them both actor and instrument.[574] As actor, their use in measuring or counting objects of experience represents a fundamental "act of comparison" in which qualities of size and quantity are judged relative to the body.[575] Further, they bridge the numerical intuitions of numerosity with the tactile manipulations of enumerated objects; and they are always available to supplement memory and act as a reference set. As instrument, they are more integrated into the cognitive system than any material object could be, but they share with materiality having affordances[576] that structure numerical cognition.

Assessing the affordances supported by the hand in counting starts with five principles of number systems that are considered to be invented, rather than innate[577]:

[572] Thureau-Dangin 1928, 120.
[573] Malafouris 2013; Tallis 2003; Wilson 1998.
[574] Gallagher 2013.
[575] Mattens 2013, 177.
[576] Gibson 1977; 1979; Norman 1999. As was discussed in Chapter 2, affordances are relations between material properties and an organism's psychological and physical capabilities that afford opportunities for action. Organisms must have the capability to exploit an object property for the capability–property relation to constitute an affordance (Greeno 1994; Scarantino 2003).
[577] Gelman et al. 1978.

> (1) the one-to-one principle, which assigns distinct representations or words to each item counted; (2) the stable-order principle, the idea that order is fixed; (3) the cardinal principle, the idea that the number for the last object in a set represents the set's quantity or cardinality; (4) the abstraction principle, the idea that numbers can be applied to any type of object to be counted; and (5) the order-irrelevance principle, the idea that numeric value is not based on order (Gelman and Gallistel 1986).[578]

The fingers connect an external material scaffold (i.e., the fingers themselves) with internal representations of quantity (numerosity) to support the individuation and categorization of specific quantities shared by sets of fingers and enumerated objects, individuated with unique linguistic labels (the one-to-one principle). Such individuation is initially limited to subitizable quantities but is later scaffolded by materiality (especially the total number of fingers per hand) to include non-subitizable ones. Stable order or ordinality emerges next[579] through the physical structure of the hand, which provides logical starting and ending points and a linear succession that typically starts from one of the outermost fingers, proceeds sequentially through the intermediate fingers, and ends with the other outermost finger (this is not an invariant sequence; finger-counting is also culturally inflected[580]). Cardinality then emerges through the use of cumulative fingers to represent the number of objects in a set. As the ordinality implicit in finger-counting may inhibit the emergence of the final principles,[581] the ideas that object type and enumeration order are irrelevant may require the incorporation of material artifacts into the cognitive system for numbers,[582] something that is likely to be motivated (or not) by a social need for numbers.

As a material device for counting, the fingers are cognitively prepotent because their neural control interacts with numerosity[583]; they are integral to the manipulation of

[578] Overmann 2014, 71.
[579] Andres et al. 2008.
[580] Bender et al. 2012; Domahs et al. 2010; Fischer 2008.
[581] Beller et al. 2011.
[582] Overmann 2014.
[583] Coolidge et al. 2012.

enumerated objects and readily available to act as quantity realizer and reference. The hand's linearity complements the ability to sequence quantity by magnitude, producing concepts of stable order. In terms of semiotic value, the hand instantiates quantities up to five, and its flexibility enables the fingers to act as material signs for quantity in order to reference and communicate numerical values. However, the hand is generally able to reflect only limited quantities, typically no more than 'five' on one hand, though a few finger-counting systems have ultimately counted higher by incorporating the knuckles, finger segments, or multiple finger positions.[584] The hand also has limited persistence in representing or recording quantity (i.e., sooner or later, it will be needed for another purpose), and its durability is relatively limited (the quantity is lost as soon as the hand does something else or the attention is diverted or interrupted).

The 'three + *x*' and 'five + *x*' number constructions imply that the Sumerians realized their initial numbers by using their fingers in a way consistent with the perceptual limits of subitization. While there is no technique that can provide an absolute chronology for this, tokens are found as early as the ninth millennium and tallies even earlier, potential counting technologies that suggest the initial realization of numbers would have occurred even earlier (but not so early that most 'three + *x*' and 'five + *x*' constructions had become unrecognizable through diachronic change). In extant peoples, such constructions are generally associated with emerging number systems and traditional societies living in conditions of relative isolation and unelaborated material culture. Such conditions obtained in the ANE until the end of the Upper Paleolithic.

With the initial realization of numbers, ANE numerical cognition would have changed from an anumeric state to one characterized by counting with the fingers, reaching quantities consistent with numerosity and the material scaffolding of the fingers.

[584] Overmann 2014.

There would likely have been an attendant change to ANE peoples' mental number line similar to that seen in development and ethnographic research. Children enculturated into Western number systems display ontogenetic differences in their ability to place interim quantities along the mental number line: At the age of five and before much training, such estimation is logarithmic (i.e., smaller numbers are widely spaced toward the left endpoint, higher numbers clustered to the right, a distribution evoking the Weber–Fechner law of stimulus intensity governing numerosity); by the time children reach the age of nine, after several years of training in established cultural numbers and arithmetic, their numerical estimations are more regularly spaced along a linear continuum.[585] Research among indigenous peoples with restricted numerical lexicons (e.g., Papua New Guinea; South America) has shown their numerical estimation to be more logarithmic.[586]

While finger-counting helps realize initial (restricted) numbers, the characteristic mental number line associated with finger-counting and restricted lexicons exhibits magnitude ordering but not regularized spacing:

> For instance, if the counting is performed with the hands open and palms up, the spatial distance between the tip of the thumb (for, say, 'one') and the tip of the index finger (for 'two')—corresponding to one unit—is significantly greater than the distance between the index and the middle finger (for 'three'), which also corresponds to one unit. Such mappings then do not satisfy the properties of a Euclidean distance function.[587]

This suggests that although finger-counting helps instantiate concepts of stable order, it has less effect on linearizing the mental number line; such linearization may require a more extensive numerical lexicon, which in turn is a development that is facilitated by the incorporation of material artifacts (e.g., tallies) into the system of numerical cognition.

[585] Siegler et al. 2004.
[586] Dehaene et al. 2008; Núñez et al. 2012.
[587] Núñez et al. 2012, 2. *Euclidean distance* is the shortest distance between two points (i.e., a straight line).

## 7.3 Tallies in the Late Upper Paleolithic Levant

In the ethnographic literature, tallies are the kind of counting technology that typically follows the use of the fingers, likely because they are similar to the fingers in capabilities like accumulation and characteristics like linearity. Tallies represent a shift from counting by comparing things to the body to counting by comparing things to each other, and they have the potential to extend counting sequences to quantities well beyond the capabilities of the hands, toes, and other body parts (and indeed, to inspire the concept of unconstrained counting in a way that the more limited capacity of fingers might not). The historic approach to tallies found in archaeological contexts has been to examine the cutmarks microscopically to determine whether they were made by the same or different tools: Use of the same tool implies cutmarks made at-a-time, suggesting a purpose like decoration; use of different tools, in comparison, implies cutmarks made over time, a temporal distribution more consistent with the purpose of enumeration.

Other characteristics historically considered include whether cutmarks are haphazard or regularized (the second is more consistent with both decoration and enumeration) and how they are disposed on the artifactual surface (a regularized disposition indicates intentionality more than an irregular distribution might). It is important to note that notches on a tally need not be made in a regular or linear fashion; they could just as easily be made randomly across its surface. The fact that they are not randomly disposed but typically linearized is likely a function of influences that include: (1) previous structuring (i.e., from the use of fingers as a counting device); (2) greater accessibility of quantity information (i.e., notches that are regular and linear are more easily accessed for the quantity information they provide than notches that are haphazardly disposed); and (3) motor movement efficiency (i.e., regularized notches involve an economy of movement that making haphazard notches lacks).

When Material Engagement Theory is applied to the tally, the focus shifts from trying to determine artifactual intent to recognizing the cognitive opportunities the artifact provides. For example, notches represent motor actions and the development of skill, as well as the kind of tactile and visual regularities important in numerical patterning. Notches are accumulative but not manipulable, as it is easy to add more notches but relatively difficult to move or erase them; these characteristics make it more likely that addition will emerge before subtraction in the cognitive system for numbers.[588] Each notch is related to the notches that precede and follow it, and a set of notches can be used as a reference, potentially giving rise to notions of plus-one relations (implicit in the tally's manufacture) and the lexicalization of both instantiated quantities and the activities of manufacture and use (e.g., 'to tally', 'to count'). Thus, even if decorative in intent, tallies represent a range of opportunities for numerical thinking, and certainly, they are associated in ethnographic analyses with increases in the highest numbers counted.[589]

Inscribed notches need not entail an associated lexical term for the quantity represented by each new notch—quantity can be 'that many', a comparison to a referent that instantiates quantity, rather than 'how many', the identification of an explicit discrete quantity through an associated lexical label. While tallies require neither words for nor explicit concepts of discrete quantity, they also do not preclude them, as they are associated with extending initial 'one-two-three-many' and finger-based counting systems to much higher quantities. Notches instantiate the stable-order regularity that fingers afford, similarity that is possibly a factor in why tallies and similar technologies tend to follow finger-counting. A tally maintains its integrity as an instantiated quantity, well beyond the persistence capacity of fingers. Further, as notches are difficult to manipulate,

[588] The fingers do not appear to support the idea of subtraction; that is, finger-counting is ubiquitous and typically the first material used to support counting, but subtraction emerges relatively late in numerical systems. Possibly this is because, as part of the body, it is difficult and undesirable to remove the fingers in a way that suggests subtraction.
[589] Overmann 2013a; 2013b.

follow-on material forms will likely be chosen for greater manipulability, as was discussed in Chapter 2. Tallies therefore represent a likely intermediate technology between fingers and Neolithic clay tokens; their archaeological presence in the Late Upper Paleolithic Levant implies an Upper Paleolithic chronology for finger-counting. Certainly, tallies and their attendant increase in numbers is consistent with the changes in material resourcing known to have characterized the Late Upper Paleolithic and early Neolithic, in which dispersed, semi-nomadic peoples subsisted through hunting and gathering and were entering new relations with resources and landscapes (cereal cultivation, animal husbandry, population increases, greater sedentism, etc.).[590]

Archaeological evidence of early tallies in the Southern Mesopotamian plain where writing would later develop is rare. These regions were mostly uninhabited until around 9000 BC, with settlements between 9000 and 6000 BC being small and isolated, making the discovery of sites and artifacts less likely.[591] Further, the climate of the region does not readily preserve artifacts made of organic materials.[592] The result of these deposition and preservation traits means that artifacts made of organic materials are infrequently found at sites located inland, far from the Mediterranean coast[593]:

> [B]one tools and a well-developed bone-working technology were most likely characteristic of the Levantine Upper Paleolithic in general and typical of both the Levantine Aurignacian and the Ahmarian[594]. However, the evidence of such technology has been preserved differentially across the Levant in cave and rockshelter deposits and under unique conditions of preservation and burial at open-air sites in the more xeric environments of the southern and eastern Levant.[595]

Artifacts made of organic materials, however rare, are not unknown in the Levant, the Mediterranean coastal regions where Neolithic farming culture is believed to have

[590] E.g., Braidwood et al. 1983; Childe 1936; Hesse 2010; Lloyd 1978; Pollock 1999; Steiner et al. 2013.
[591] Nissen 1988.
[592] Coinman 1996.
[593] Coinman 1996.
[594] The Aurignacian is the period 45–35 KYA (Mellars 2006), the Ahmarian 39–24 KYA (Gladfelter 1998).
[595] Coinman 1996, 114–115.

originated around 10,000 BC[596]—including some that may have been used as tallies. Notched or incised bones dated to 30,000 thousand years ago (30 KYA) have been discovered at Ha-yonim (modern Israel); similar artifacts have been found at Ksar'Aqil and Jita (both sites are in modern Lebanon; the artifacts date to, respectively, the Upper Paleolithic and 17–15 KYA) and Ain el-Buhira (Jordan, 25–19 KYA).[597] The geographic distribution of the Upper Paleolithic/Epipaleolithic sites is shown in Figure 7.1. Incised bones, generally cattle scapulae, have also been found in multiple sites in Israel, Palestine, Lebanon, Turkey, Iran, and Iraq from the Levantine Aurignacian and throughout the Neolithic, with a variety of purposes attributed to them, including quantificational record-keeping, counting, divination, music, fiber- or leather-working, or tools of an unknown type.*

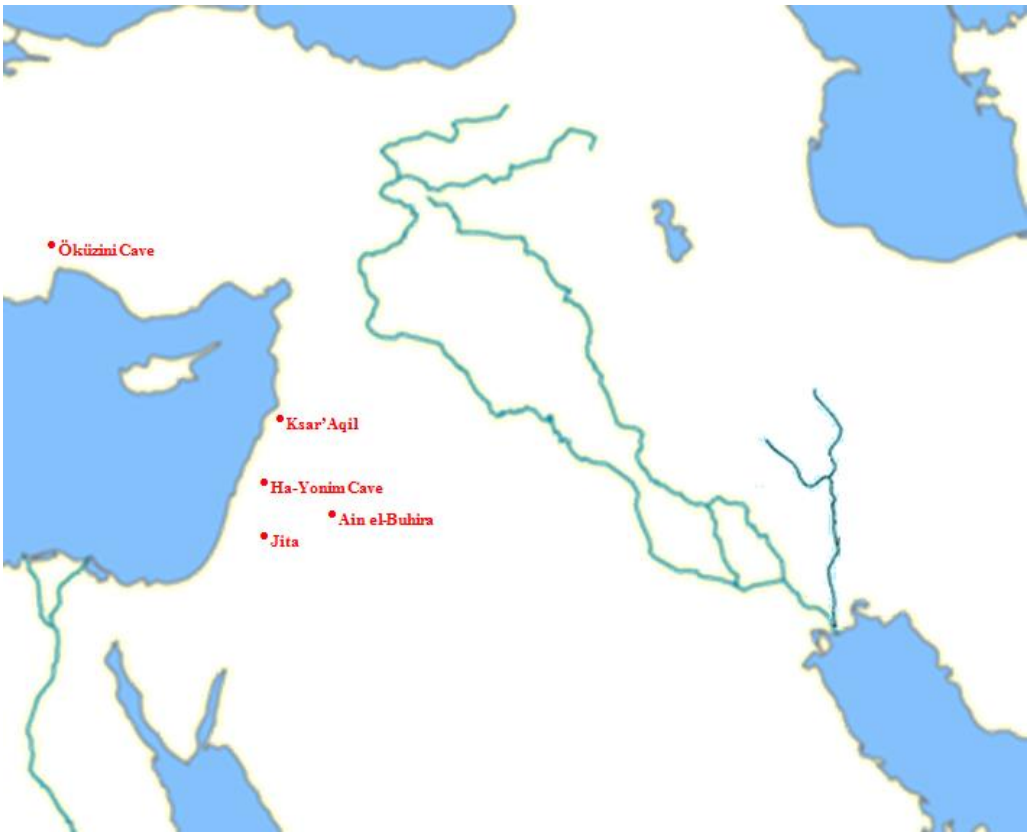



*Figure 7.1*. Map of the ANE showing the geographic distribution of sites from the Late Upper Paleolithic and Epipaleolithic where worked bones possibly used as tallies have been found. Adapted from an image in the public domain.

[596] Hours et al. 1999; Nissen 1988; Stordeur et al. 2002.
[597] Coinman 1996; Copeland et al. 1977; Davis 1974; Tixier 1974.
* Reese 2002.

Although the Ha-yonim and Ksar'Aqil artifacts may have been used as tallies (the first is presented in Figure 7.2, the second in Figure 7.3),[598] the Jita artifact (Figure 7.4) is decorated with diagonal lines and appears more decorative than quantificational in purpose.[599] Similarly, the notches on the lateral edges of the Ain el-Buhira artifact[600] (Figure 7.4) might also have been decorative.




*Figure 7.2*. Four worked bones from Ha-yonim cave (modern Israel). While their function is unknown, "it might be supposed that they served as some method of keeping quantitative records."[601] While the artifacts meet several of the criteria proposed by d'Errico and colleagues[602] in terms of the size and surface disposition of the incised marks, if they did represent counting, they presumably counted to lower numbers than would have been the case for the artifact presented in Figure 7.3. The images above were adapted from Figures 3–6 in Davis (1974, 182).

For those artifacts that are perhaps quantificational in their intent, as may be the case with the Ha-yonim and Ksar'Aqil bones (Figures 7-2 and 7-3) and several of the other notched bones documented in the literature,[603] none appear to have been subjected to the type of microscopic analysis recommended by d'Errico and colleagues.[604] The recommended analysis is intended to establish the temporal distribution of the accumulation of the marks. If the marks were determined to have been made by different tools (and thus, presumably, at different times), their potential use for quantification would be placed on a more certain basis. Alternatively, if the marks were determined to have

[598] Schmandt-Besserat 1992a.
[599] Copeland et al. 1977.
[600] Coinman 1996.
[601] Davis 1974, 181.
[602] d'Errico 1991; 1998; d'Errico et al. 2003.
[603] Davis 1974; Reese 2002.
[604] d'Errico 1991; 1998; d'Errico et al. 2003.

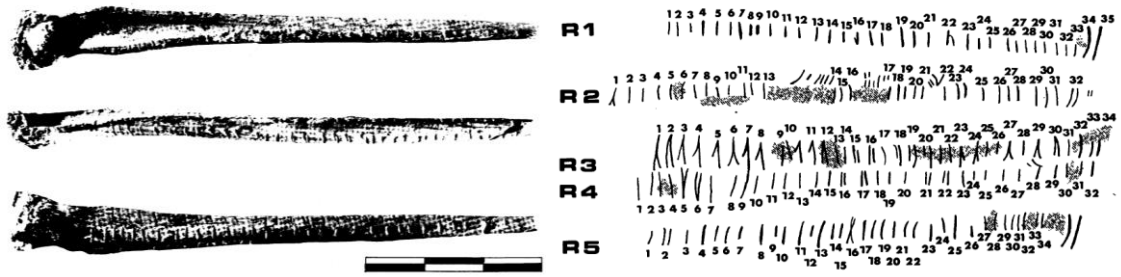



*Figure 7.3*. Worked bone possibly used as a tally (left) from the Late Upper Paleolithic site of Ksar'Aqil in modern Lebanon. Found by French archaeologist Jacques Tixier in the early 1970s, the artifact contains five rows of incised markings. Tixier drew and numbered the incisions in each row (right). While most of the incisions consist of single lines, a minority are V-shaped in appearance and either point to the right (e.g., R2.1) or left (R3.1) of the artifact. Of the worked bones from the Upper Paleolithic Levant for which graphic reproduction is available in the literature, this artifact bears the greatest resemblance to a modern tally. Tixier cautioned that he could not identify whether the artifact served a purpose other than decoration with any certainty ("Dans quel but ces séries de marques ont-elles été incisées? Une tentative pour prouver qu'il s'agit d'autre chose qu'une simple décoration sans arrière-pensée n'est pas de mon ressort"[605]). The rows of incisions are disposed along the length of the artifact to a fairly similar degree, and the incisions are roughly parallel, evenly spaced, and similarly sized with respect to one another, meeting several of the criteria suggested by d'Errico and colleagues.[606] While such characteristics are consistent with both decoration and counting, the number of incisions is about the same for all five rows; further, in several cases, a number of smaller incisions have been interspersed between those of the more typical larger size (e.g., such as those placed between R2.13 and R2.14), "a disposition across the surface of the artefact that suggests their totality was unplanned, indicating temporal separation in their accumulation that would be consistent with quantification (d'Errico 1998)." These characteristics may be more suggestive of counting than decoration. Both figures were adapted from Figures 3 and 4 in Tixier (1974, 190).



*Figure 7.4*. Worked bones from Jita, modern Lebanon[607] (left) and Ain el-Buhira, modern Jordan[608] (right). Compared to the artifact from Ksar'Aqil (modern Lebanon) presented in Figure 7.3, both appear more decorative than quantificational in their intent using the criteria of d'Errico and colleagues. The figure of the Jita artifact was adapted from Figure I.3 in Copeland et al. (1977, 296), the figure of the Ain el-Buhira artifact from Figure 4 in Coinman (1996, 117).

been made by the same tool (and thus, presumably, at the same time), they would be considered to have been more decorative in their purpose.

[605] Tixier 1974, 192.
[606] d'Errico 1991; 1998; d'Errico et al. 2003.
[607] Copeland et al. 1977.
[608] Coinman 1996.

If the artifacts typified by those presented in Figures 7-2 through 7-4 were indeed tallies, their use beyond the Levant in the Late Upper Paleolithic may have persisted until well after the Neolithic. Such artifacts, though rare, are found in later periods.[609] Bookkeeping in the third and fourth millennia may have included "... a number of devices such as tally sticks and tokens."[610] Tallies are referenced in the second millennium: A portion of a text from that period, "The debate between grain and sheep," translates as,

> Every night your count is made and your tally-stick put into the ground, so your herdsman can tell people how many ewes there are and how many young lambs, and how many goats and how many young kids.[611]

Left unexplained in the poem is how the herdsman would have differentiated the different kinds of animals using a tally, a problem that may have been solved by using tokens of different shapes,[612] and which the numerical sign systems solved through systems of modifiers and external markings.

With the incorporation of tallies, ANE numerical cognition would have reached numbers higher than those typically enabled by finger-counting. The mental number line would likely have acquired increased linearity and regularized spacing through the tactile–visual regularities of the tally's notches. Concepts of higher numbers and plus-one accumulation, as well as the lexical naming of higher numbers characterized by compounding, would have been realized through the accumulation of individual notches. Tallies are more durable and transportable than fingers, enabling quantity information to be recorded and stored, enabling its greater use in purposes like food storage and resource management. In terms of semiotic value, tallies instantiate higher quantities, can symbolize quantity as a reference set, and can communicate quantity across time and distance in a way that fingers generally cannot.

[609] Reese 2002.
[610] Englund 2001, 23.
[611] ETCSL 2005, lines 130–133.
[612] Schmandt-Besserat 1992a.

Tallies, however, do not indicate the commodity they record, and notches cannot easily be moved or subtracted; further, cardinal values for non-subitizable numbers are difficult to distinguish because of the perceptual constraints on visually identifying quantities over 'three' (a problem typically solved by grouping, a technique that characterizes the tokens used for accounting). These characteristics would have injected new limitations into the ANE cognitive system for numbers (i.e., new problems to be solved), even as tallies would have provided a third method of accessing numerical concepts. That is, ANE peoples would likely have retained finger-counting (as is the case even in highly numerate cultures) and would have been developing a numerical lexicon. The increase in access methods would have increased the likelihood of subsequent numerical elaboration.

When writing has not (yet) been invented or incorporated by a particular culture, the problem of designating commodity has been solved in a variety of ways. For example, indigenous Peruvian "herdsmen count their beasts by means of beans in a small bag which they always carry with them; the colours of the seeds correspond to the various kinds of animals, cows, calves, and so forth."[613] This solution evokes the use of color in the Incan quipu, the counting/recording device of multiple strands of strings whose knots (method of tying and placement along the string) signified quantity and whose color and position in the strand series signified commodity.[614] These solutions imply that if the Levantine tallies were indeed used for counting, the commodities they represented may have been recorded by similar means, though this is not indicated by the artifacts themselves, and indeed likely could not be, as the materials (e.g., bags, strings, pigments, etc.) used tend to be highly perishable, and other strategies (context, memory) leave few archaeological signs.

[613] Baudin 1961, 39.
[614] Urton 2010.

Such precursors are indicated, however, by the encoding of commodity as shape and size in the Neolithic tokens.

Neolithic Mesopotamia was settled by many peoples, including those from the Late Upper Paleolithic Levant. Some cultural descent is attested by things like architectural similarities and the relatedness of some of the regions' languages (e.g., Akkadian, an East Semitic language, and West Semitic languages like Aramaic spoken in the Levant). At the beginning of the Neolithic, there is evidence of long-distance trade within the region (e.g., highly valued materials like obsidian[615]), implying contact between cultures that may have had different number systems. The ease with which numerical concepts and language are known to diffuse in contact situations makes it probable that material technologies used for counting and associated number words may have spread through the region fairly rapidly. While there appears to be no direct evidence linking the earliest Neolithic settlements using token-based accounting[616] with the earlier Late Upper Paleolithic tallies in the Levant, the view of such cultures as independent, non-overlapping, and self-contained is being displaced by more nuanced perspectives that recognize the Late Upper Paleolithic Levant and Neolithic ANE were characterized by vibrant, expanding cultural networks and increasing interactions both on local and regional scales and across time.[617]

The eighth-millennium settlements where possible tokens have been found—Tell Aswad, Tel Mureybet, and Cheikh Hassan, Syria; Ganj Dareh Tepe and Tepe Asiab, Iran—were characterized by greater sedentism, larger settlement size, new food resourcing methods (e.g., animal husbandry, cereal cultivation), and increased food storage

[615] Obsidian has been found 300 km from the nearest deposits; however, long-distance trade was probably "not a major trigger for the emergence of recording, … [since] the distributions of tokens and of obsidian do not coincide" (Schmandt-Besserat 1982, 874).
[616] Plain tokens of the shapes used in fourth-millennium accounting (i.e., spheres, cones, disks, cylinders, and tetrahedrons), some with size distinctions or modifications, have been found at five early Neolithic sites in Syria and Iran (Schmandt-Besserat 1982).
[617] Hesse 2010; Goring-Morris et al. 2011; Watkins 2008.

requirements, social needs that suggest motivations for accounting[618] and hence, quantification with greater capacity, manipulability, and permanence than material structures like fingers and tallies could provide. The concomitant changes in numerical cognition suggested by tokens and numerical notations are discussed in the next chapters.

In and of themselves, tallies seem merely an intermediate step between fingers and tokens, and yet they are critically important. Before concepts of and names for numbers, the use of such devices facilitates complex behaviors. For example, the clay stripes on the Wathaurung messenger's arm (Chapter 5)—a tally—help him return to his tribe after a two weeks' march,[619] a journey implying significant territorial expansion and increased potential to exploit its resources. While tallies provide greater capacity, manipulability, and permanence than can be achieved with the fingers or body alone, they also represent the incorporation of materiality that is not part of the body into the cognitive system for numbers, as well as the transition to the use of materiality as a collaborative medium. By comparison, number systems that continue to use the body as the primary counting technology are influenced by the social conventions and norms governing the permissibility of interpersonal touch and imitative behaviors (e.g., matters affected by the gender, age, social status, etc. of participants). Simply, bodies are not public and shareable in the same way materiality is. The integration of materiality into cognition and the transition to its use as a collaborative medium bypasses social restraints on bodies and imitative behaviors, providing for greater interpersonal interaction via the sense of touch (significant, given its centrality in realizing number concepts; see Chapters 4 and 5) through a channel of exchange that is more communal and impersonal, and hence, less restricted.

[618] Schmandt-Besserat 1982.
[619] Morgan 1852.

# CHAPTER 8

# NEOLITHIC CLAY TOKENS

## 8.1 Unambiguous Numbers in the ANE

A discussion of why the Neolithic tokens (small, geometrically shaped objects typically made of clay) are thought to have been used as numerical counters does not begin with the tokens themselves but rather with token-shaped impressions on fourth-millennium clay artifacts, whose correspondences to later numerical notations make them the first unambiguous numbers of the ANE. By comparison, for precursor technologies such as tallies, numerical meaning remains ambiguous to some degree, as there is always an element of doubt regarding whether an artifact represented counting or some other purpose like decoration; in finger-counting, there is no way to establish a definitive timeline, as the artifacts leave only linguistic evidence behind (these issues were discussed in Chapter 7).

The clay artifacts showing the critical numerical correspondences[620] are known as bullae (round balls of clay; Figure 8.1) and envelopes (round, hollow clay wrappers). Their surfaces were impressed with shapes corresponding, more or less, to small pieces of molded clay ('tokens') that were often but not invariably contained inside them. The faces of tablets (flattened solid clay slabs) were also impressed with similar marks (Figure 8.1). The proto-literate impressions on bullae, envelopes, and tablets appear to have been made

[620] The first scholar to have speculated on the tokens' numerical meaning was Archaeologist Pierre Amiet (1972). The scholar most widely associated with the idea, Denise Schmandt-Besserat (1992a), developed the idea and researched and documented the associated artifacts.

with the tokens themselves, though other implements were also used, including the fingers; impressing would later be effected by styli made of organic materials (e.g., reed, wood).[621] Considered as a group (see Appendix C), the token–impression correspondences suggest both a high degree of overall commonality and a fair degree of variability by city of use (and possibly over time as well). The data suggest at least five systems for representing quantity: Sexagesimal System S (used for most discrete objects), Bisexagesimal System B (commodities associated with a rationing system), ŠE System Š (grain, particularly barley), ŠE System Š' (grain, possibly germinated barley for beer), and ŠE System Š" (another system for grain, possibly emmer).[622] Other shapes (e.g., square, lens, triangle, bow ties) were possibly used to indicate commodity or ownership. The token–impression correspondences on artifact P285636 from Uruk and P285660 from



*Figure 8.1*. Clay bulla and numerical impressions. (Left) Clay bulla (W 20987, 08; CDLI no. P285630) dated to the pre-writing period; the artifact likely dates toward the more recent end of the range) from the city of Uruk (modern Warka), Iraq (image courtesy of the German Archaeological Institute, Berlin). Its surface is marked with five $N_{64\sim t}$ and two $N_{39\sim t}$, impressions that correspond well to the shapes, sizes, and quantity of tokens it contained, five tetrahedra and two disks.[623] (Right) Clay tablet with numerical impressions (A 21310; CDLI no. P235768) dated to the Uruk V period from the city of Tutub (modern Khafaje), Iraq (image courtesy of the Oriental Institute of the University of Chicago); its obverse face is impressed with one $N_{14}$ and five $N_1$.[624]

[621] Englund 2006, 23.
[622] Nissen et al. 1993. Note that $N_{64}$/TETRA has not been assigned to a system.
[623] Damerow et al. 1995; Schmandt-Besserat 1992a; University of Oxford 2015.
[624] Delougaz et al. 1967; University of Oxford 2015. Also see Frankfort 1936, 25, fig. 19.

Habuba Kabira do not conform to the general patterns shown, anomalies that suggest regional variation or perhaps yet other systems. The data from Umma and Susa also show an interesting variation in representing $N_{45}$ with both cones and SPHERES, possibly because similar impressions might be produced with either a sphere or the base of a cone.

Because the proto-literate impressions correspond closely in their shapes, sizes, and numerical quantities to the number signs of proto-cuneiform[625] (Figure 8.2) and its

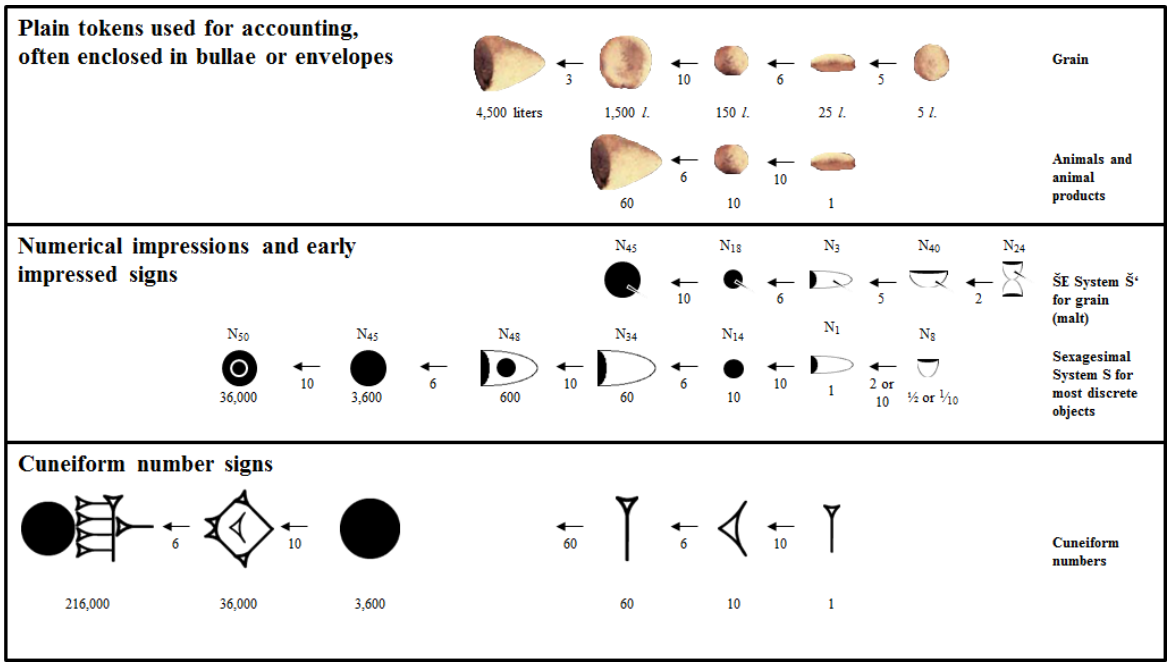



*Figure 8.2.* Plain tokens used for accounting before the invention of writing (8500–3500 BC; top),[626] numerical impressions from the archaic and Uruk IV periods (3500–3200 BC; middle),[627] and cuneiform number signs (the signs shown did not necessarily form a coherent set, as the ones for 3600 and 216,000 are attested as early as 3000 BC, the signs for 10, 3600, 36,000 as late as 2000 BC; bottom).[628] Correspondences of shapes, sizes, and numerical relations between cuneiform numbers and numerical impressions, and numerical impressions with tokens, suggest that tokens had similar numerical meanings. While the overall trend is toward higher numbers, tokens thought to have represented '600' have been dated to the pre-writing period,[629] a quantity that implies the number system was relatively mature even at that time.

[625] Proto-literate number signs received "their distinctive round or oblong shapes because they were impressed signs, formed by pressing a round stylus into wet clay, either at a right angle or obliquely," rather than being shaped specifically to imitate tokens, though the correspondences of shapes between the number signs and plain tokens is striking (Friberg 1994, 484).
[626] Adapted from Englund 2006, 29.
[627] Adapted from Englund 2006; Nissen et al. 1993.
[628] Adapted from the University of Pennsylvania 2015.
[629] University of Oxford 2015.

descendent cuneiform script, they have been interpreted as representing similar numerical information.[630] Comparable correspondences between the proto-literate numerical impressions and tokens contained within or found nearby bullae, envelopes, and tablets suggest that tokens had a similar numerical intent, at least in the form of plain tokens, whose shapes (e.g., cones, spheres, disks, cylinders, and tetrahedra), sizes (usually small and large, with an occasional intermediate size), and quantities (a token of one shape and size might be equal to six or ten of another shape or size) coincide with those of the early number signs:

> There is no reason to doubt that the (mostly) plain tokens enclosed in spherical envelopes belonged to a small number of preliterate systems of "number tokens", very much similar to the now [well-known] protoliterate systems of impressed number notations.[631]

Similarly,

> The only really plausible assumption in [identifying the numerical values and meanings of plain and complex tokens] is that 'plain' tokens were forerunners of impressed protoliterate number signs, while 'complex' tokens were forerunners of incised protocuneiform signs of writing. Even if this basic assumption is correct, one has to admit that almost certainly the values of the plain tokens were temporally and regionally different. Therefore, the most one can ever hope to establish is that the values of the plain tokens in the period immediately preceding the invention of writing were in some way related to the values of the protoliterate number signs, and that the plain tokens just like the protoliterate number signs were organized into separate 'systems', each with its own hierarchical structure.[632]

## 8.2 Issues in Interpreting Tokens as Numerical Counters

What tokens might potentially reveal about ANE numbers prior to the fourth millennium (i.e., the period in which their use to make numerical impressions makes their meaning unambiguous) is challenged by various caveats that limit their interpretation as numeration technologies, including:

[630] Englund 1995; Nissen et al. 1993; Schmandt-Besserat 1992a.
[631] Friberg 1994, 492.
[632] Friberg 1994, 483–484.

- Tokens were generally under-collected or under-documented in historic excavations,[633] creating an insufficiency of data for analysis, as well as the potential for specific finds to skew the results (e.g., sites like Jarmo in northern Iraq, where more than 2000 tokens have been found; these comprised nearly half [48.7%] of Schmandt-Besserat's original dataset).
- Tokens are often found in contexts suggesting that they may have had purposes other than counting (e.g., burial sites imply their use as funerary offerings[634]); or they are found in contexts that are not unambiguously administrative (circumstances that have led to speculations about potential alternative uses, like gaming pieces, children's toys, and sling missiles); or their context has not been documented or their provenience is unknown or suspicious,[635] but they have been classified and treated as numerical tokens nonetheless, simply on the basis that they are small clay objects.[636] Further, there is a general sense that unless tokens are found in contexts that unambiguously point to counting (rather than, for example, in the rubble fill of a wall that may have once been located near a temple), their use in representing numbers will remain a matter of debate. To date, no convincingly unambiguous contexts have been published.
- There is no single typological system for tokens at either the type or subtype level, making it more difficult to compare tokens published by different authors. Further, no standard criteria have been established for classifying tokens in any typological system, or for distinguishing tokens used for counting from small clay objects used for other purposes.

[633] Schmandt-Besserat 1992a.
[634] Englund 1998c.
[635] A considerable portion of artifacts thought to have been used for counting have been traded illegally, including tokens.
[636] Schmandt-Besserat 1992a; Englund (1998c) has been particularly critical of the contextual insufficiencies of token cataloging, by Schmandt-Besserat in particular and archaeology more generally.

- The division of tokens by shape and size into multiple metrological systems (as discerned through their correspondences with late-fourth millennium numerical impressions), as well as the regional and temporal variation in token shapes and sizes (as discerned through analysis of the artifactual record),[637] entail there could not have been a single system of numerical meaning even for the plain tokens.[638]
- Significant disparities between the meanings proposed for complex tokens[639] and the archaeological evidence of commodity prevalence (e.g., complex tokens purported to represent 'sheep' are rare, yet sheep were a common commodity[640]) have created considerable reasonable doubt as to the accuracy of the proposed meanings.
- The best source of numerical correspondences for token meaning, impressed bullae and envelopes, are relatively scarce; when found unsealed or broken, they are often not clearly associable with any tokens found nearby.[641] Further, most sealed artifacts have remained unopened,[642] and imaging, though less destructive, has made only limited progress in identifying contents for comparison to surface marks.[643] Finally, bullae and envelopes with tokens are also not invariably marked externally[644] or have markings that do not correspond as expected to the contents, and might also be empty (even if sealed).[645]

[637] Friberg 1994, 483–484.
[638] Englund 1998c; Friberg 1994.
[639] Complex tokens are incised, punctated, or otherwise modified. Schmandt-Besserat (1992a) interpreted them as prefiguring proto-cuneiform signs for commodities; her interpretation has not been universally accepted (e.g., see the criticism of Friberg 1994; Zimansky 1993).
[640] Zimansky 1993.
[641] Damerow et al. 1995.
[642] Eighty of these artifacts remain intact, nearly two-thirds of those found to date (Englund 2006; Friberg 1994).
[643] Damerow et al. 1995; Englund 2006.
[644] The surfaces of bullae and envelopes could also be plain; impressed with seals; marked only with numerical impressions; or marked with both seals and numerical impressions.
[645] Damerow et al. 1995.

These issues, along with the overlap in the temporal distribution of the various artifactual types, suggest that invention, adoption, and use of new material forms were complexly inter-related but non-linear and comprised multiple numerical systems whose meanings varied by context, region, and time. An approximate chronology for artifacts associated with early numeration is shown in Figure 8.3.

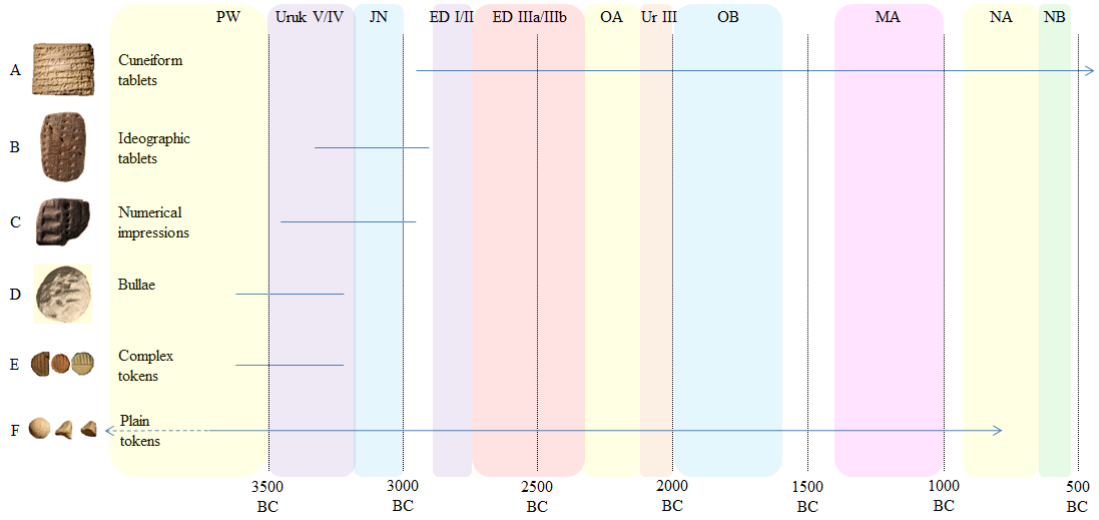



Figure 8.3. Chronology of artifacts used in ANE accounting.[646] Possible plain tokens are found as early as the tenth millennium[647] (this is notionally depicted by the dashed line and left-facing arrow at left side of the line for plain tokens) and as late as the first millennium.[648] The last known inscription in the cuneiform writing dates to approximately 75 CE,[649] while many elements and ideas of the Babylonian number system persist to the present day through their incorporation by ancient Greek mathematicians. Key: PW = Pre-writing period, 8500–3500 BC; Uruk V/IV period, 3500–3200 BC; JN = Uruk III/Jemdet Nasr, 3200–3000 BC; ED = Early Dynastic, ED I/II 2900–2700 BC, ED IIIa/IIIb 2700–2340 BC; OA = Old Akkadian, 2340–2200 BC; OB = Old Babylonian, 1900–1600 BC; MA = Middle Assyrian, 1400–1000 BC; NA = Neo-Assyrian, 911–612 BC; NB = Neo-Babylonian, 626–539 BC. Images: A: cuneiform tablet dated to 1900–1600 BC (JRL 0915; CDLI no. P430825), © the University of Manchester, image courtesy of the CDLI; B: proto-cuneiform tablet dated to 3200–3000 BC (VAT 16773; CDLI no. P000014), Source: © Staatliche Museen zu Berline Vorderasiatisches Museum, Foto: Olaf M. Teßmer; C: bulla dated to 8500–3500 BC (Sb 06350; CDLI no. P281696), © Musée du Louvre, France, image courtesy of the CDLI; D: tablet with numerical impressions dated to 3500–3350 BC (Sb 02313; CDLI no. P281731), © Musée du Louvre, France, image courtesy of the CDLI; E: complex and F: plain tokens dated to 8500–3500 BC (The Schøyen Collection MS 3131; CDLI no. P252140; The Schøyen Collection MS 5067; CDLI no. P254152), The Schøyen Collection, Oslo and London.

[646] Adapted from Englund 2006, 16; also see Englund 1998c; MacGinnis et al. 2014; Schmandt-Besserat 1992a.
[647] Moore 2000.
[648] MacGinnis et al. 2014.
[649] Geller 1997.

## 8.3 Newly Catalogued Token Finds and Their Analysis

It was hypothesized that some of the ambiguity related to the numerical meaning of tokens prior to the fourth millennium might be resolved by examining data on token finds published since Schmandt-Besserat's landmark 1992 publication, *Before Writing* with its comprehensive catalogue of over 8000 tokens and compelling analysis of their importance to early numeration. Accordingly, over 2300 new entries[650] were added to Schmandt-Besserat's[651] token dataset.[652] Though the new entries were not an exhaustive inventory of all tokens found since 1992, they did comprise a representational sample sufficient to demonstrate that new token finds (i.e., published after 1992) have generally continued to be treated much as Schmandt-Besserat described the situation in 1992: Many known finds remain unpublished. Scholars with unpublished token data are often not inclined to share them prior to publication.[653] Authors whose published data lack the details necessary to perform classification either do not have or prefer not to provide more detailed descriptions of the artifacts. Contexts are often under-described or over-interpreted. There is still no standard set of criteria for identifying tokens nor an established typology for classifying them. Finds might be given the generic designation 'token' (rather than 'cone' or 'sphere', etc.) in publication. Data that might assist classification (e.g., images or drawings, measurements such as dimensions and weight, dating estimations, etc.) are not provided as a matter of routine. The interpretations proposed for complex tokens remain contested. Intact bullae and envelopes are still rare; sealed artifacts remain intact, and there

[650] It is estimated that this represents less than half of the token data potentially available, as particularly large token finds have been reported but remain unpublished (e.g., 1100 tokens from Tell Tayinat, Turkey were noted by MacGinnis et al. 2014, another 1600 from Tell Sabi Abyad, Syria were noted by Akkermans et al. 1995; 2006).
[651] Schmandt-Besserat 1992a.
[652] Several of the sources for these were pointed out by Dr. Schmandt-Besserat, whose advice and assistance throughout the project were significant and greatly appreciated. Further, Schmandt-Besserat and the University of Texas at Austin press granted copyright permission for copying the original dataset and converting it to an electronic format suitable for analysis.
[653] The notable exception to this was MacGinnis and colleagues of the Ziyaret Tepe Archaeological Project, whose unreserved willingness to share their data included access to the project's database; they also took the time to answer any questions and provide additional details whenever requested.

has been little progress in discerning their contents through imaging techniques. Finally, there is no central repository for making token data available to scholars comparable to platforms such as the CDLI for texts. While tokens were unlikely to have been a unified, consistent method of representing numbers, given the variability of their forms and use across their temporal and geographic distribution,[654] that variability has certainly been compounded by the way new token finds have generally been treated, reducing any ability to interpret their numerical use prior to the late fourth millennium.

Nonetheless, it is useful to describe the newly catalogued tokens in terms of type, geographic distribution, and temporal span; contrast these characteristics with those of Schmandt-Besserat's original dataset; and analyze the total data for what it might reveal about the use of tokens prior to the fourth millennium. In terms of type (or major shapes),

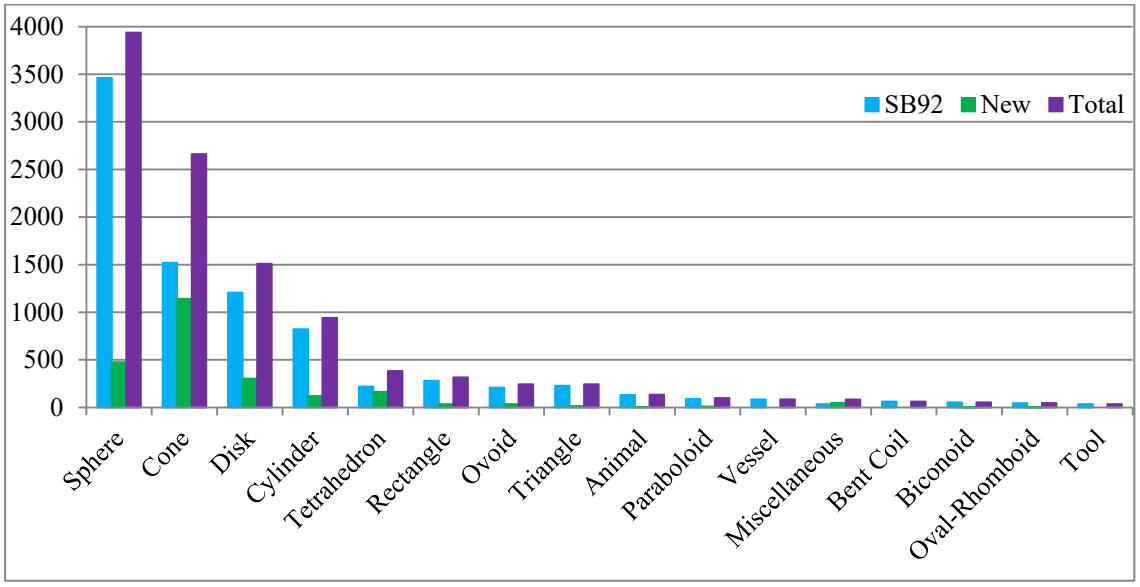



*Figure 8.4*. Comparison of the distribution of tokens by the 16 types established by Schmandt-Besserat in her original dataset (SB92; $n$ = 8477), the newly catalogued tokens (New; $n$ = 2352), and the combined data (Total; $n$ = 10,829). The types are depicted in order of token frequency from high to low (rather than in order of Schmandt-Besserat's numbering schema). Both the original and new datasets showed that the types of plain tokens thought to have been used for counting (e.g., spheres, cones, disks, cylinders, and tetrahedra) were the most common.

[654] Englund 1998b; 1998c.

the new tokens conformed fairly well to the findings Schmandt-Besserat published in 1992: The plain types used for numbers—cones, spheres, disks, cylinders, and tetrahedra—remained the most common (Figure 8.4), consistent with the widely accepted view that the plain tokens were used for numeration.[655]

In contrast with the classifications of tokens in the original dataset, which Schmandt-Besserat performed by examining thousands of artifacts, classification of the new tokens was based on an examination of the literature. This required combining, converting, or applying Schmandt-Besserat's typology as follows:

(1) Some of the new token classifications were derived from the published sources, in which various authors classified their tokens themselves, based on their own, typically unspecified criteria. Several of these individualized typologies shared a high degree of commonality with Schmandt-Besserat's typology, while others diverged from it, occasionally to a large degree. In all such cases, the various authors' classifications were converted to Schmandt-Besserat's typology for the present analysis.

(2) Tokens that were not initially classified by the authors who published them were retroactively classified by applying Schmandt-Besserat's typology to the available pictures, drawings, and descriptive data. Since basing classifications on pictures, drawings, and descriptions is less ideal than basing it on the artifacts themselves, in many cases, the retroactive classifications were coordinated with and verified for accuracy by Schmandt-Besserat.

(3) Across all the new tokens, it was generally not possible to classify by subtype, given the lack of original authorial determinations or the images and details needed to make retroactive classification determinations.

[655] Friberg 1994.

Schmandt-Besserat's original 8477 tokens[656] were distributed between 109 sites in five modern countries: Iran (40 sites), Iraq (41), Palestine (8), Syria (15), and Turkey (5). They had a temporal distribution spanning from the ninth to the first millennia BC (Figure 8.5). In comparison, the newly catalogued tokens came from 26 sites[657] in Iran (8 sites), Iraq (7), Palestine (1), Syria (4), Turkey (5), and Jordan (1), and they were temporally distributed between the tenth millennium BC and the first millennium of the Common Era.[658] In Schmandt-Besserat's original dataset, the earliest places where tokens were found were located in Syria (Cheikh Hassan, Mureybet, and Tell Aswad), Iran (Ganj-Dareh and Tepe Asiab), and Turkey (Beldibi),[659] sites whose "geographic distribution ... along the Fertile Crescent ... suggests that the area where [token-based] recording was practiced first coincides with the region in Southwest Asia where plant and animal domestication first appeared."[660] The geographic distribution of the newly catalogued tokens dated to the tenth to the eighth millennia was consistent with that of the original dataset (Figure 8.6). The temporal distribution of the new tokens differed from that of the original dataset, however, as those found at Ziyaret Tepe in Turkey (22.6% of the new tokens) have been dated to the first millennium.[661] These finds appear to contradict one of Schmandt-Besserat's key conclusions: that the use of tokens decreased from the fourth millennium onward because token-based accounting was displaced by the invention of

[656] This figure is higher than the one for Schmandt-Besserat's published dataset (8162) because it adds tokens that she catalogued but could not published because of copyright restrictions. The two amounts also differ by several minor corrections either pointed out in various reviews of her book or detected by the use of software not available in the early 1990s (e.g., Excel).
[657] Most of these were sites where tokens had previously been found.
[658] These artifacts were from, respectively, the sites of Abu Hureyra, Syria (Moore 2000) and Ziyaret Tepe, Turkey (from the unpublished database of the Ziyaret Tepe Archaeological Expedition).
[659] Schmandt-Besserat 1992a.
[660] Schmandt-Besserat 1982, 872.
[661] MacGinnis et al. 2014.

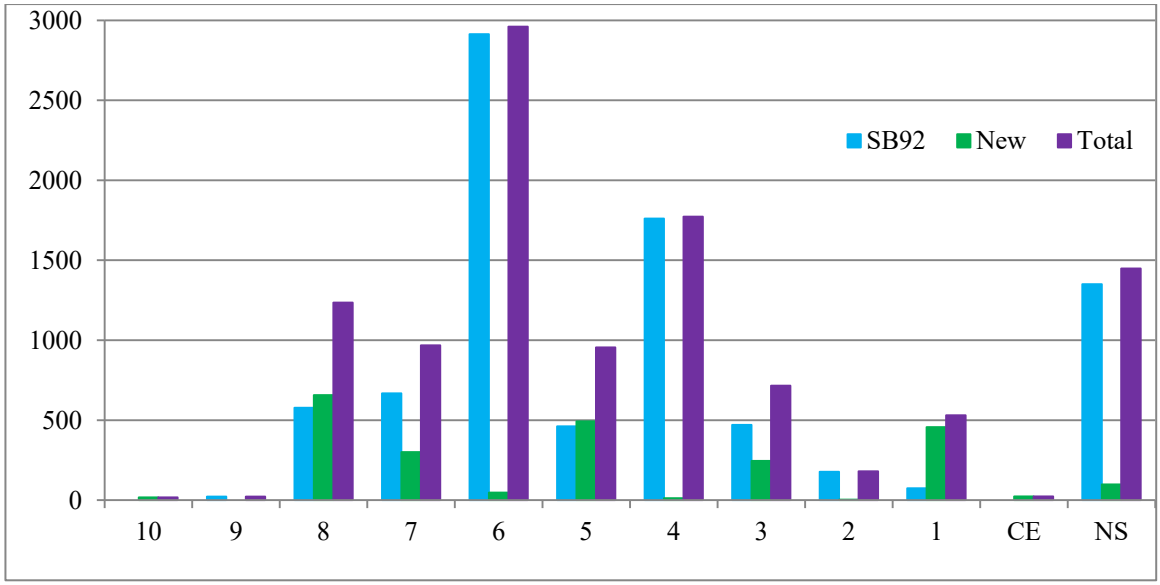



*Figure 8.5*. Comparison of the temporal distribution of tokens in Schmandt-Besserat's original dataset (SB92; *n* = 8477), the newly catalogued tokens (New; *n* = 2352), and the combined data (Total; *n* = 10,829). The time period considered includes the tenth millennium BC to the first millennium of the Common Era (CE). Tokens with unestablished or unpublished dating were categorized as Not Specified (NS).

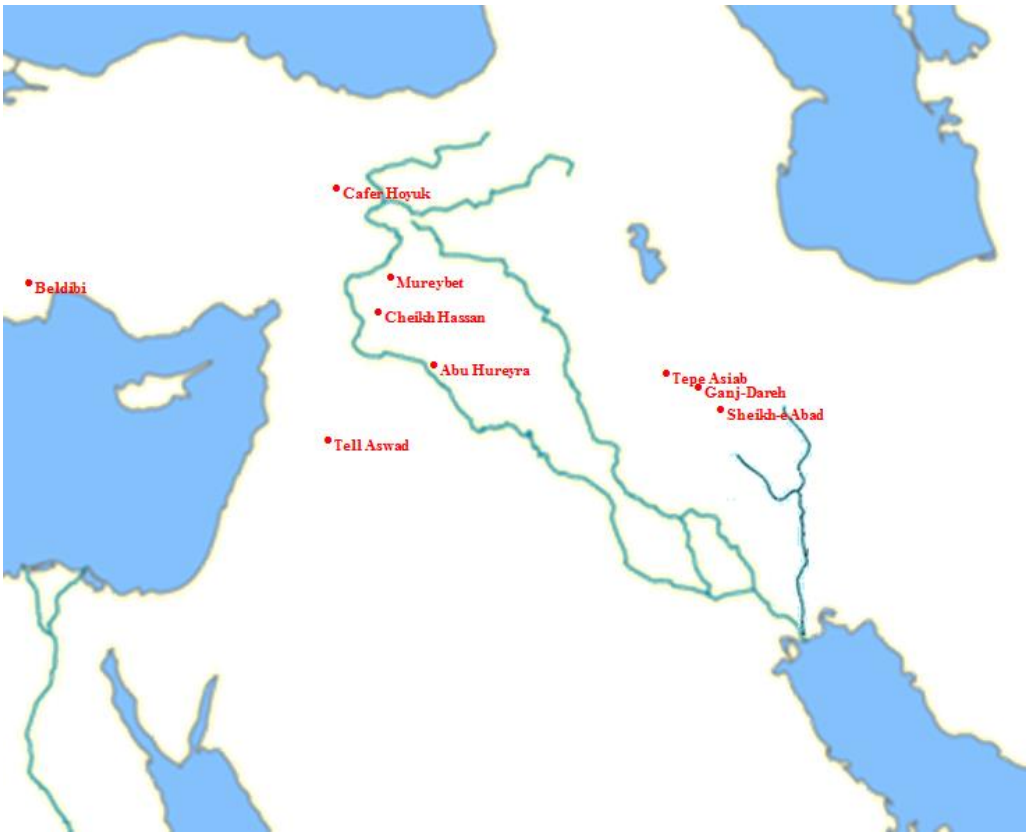



*Figure 8.6*. Map of the ANE showing the geographic distribution of sites from the tenth to the eighth millennia BC where tokens have been found. Adapted from an image in the public domain.

writing[662] (discussed further in Chapter 9). Of note, the first-millennium tokens were found at a site that might be considered somewhat peripheral to the region in which writing was invented and from which it spread, suggesting that it may have been later in adopting the new technology as a function of its distance. The geographic distribution of Schmandt-Besserat's original and the newly catalogued token datasets is compared in Table 8.1.

Regarding the temporal distribution of the five types of plain tokens thought to have been used as numerical counters, there are some suggestions of temporal contiguity in the data (Figure 8.7): When these five plain types (from all sites; with established dating) were graphed, there was a roughly similar pattern of increase and decrease in their

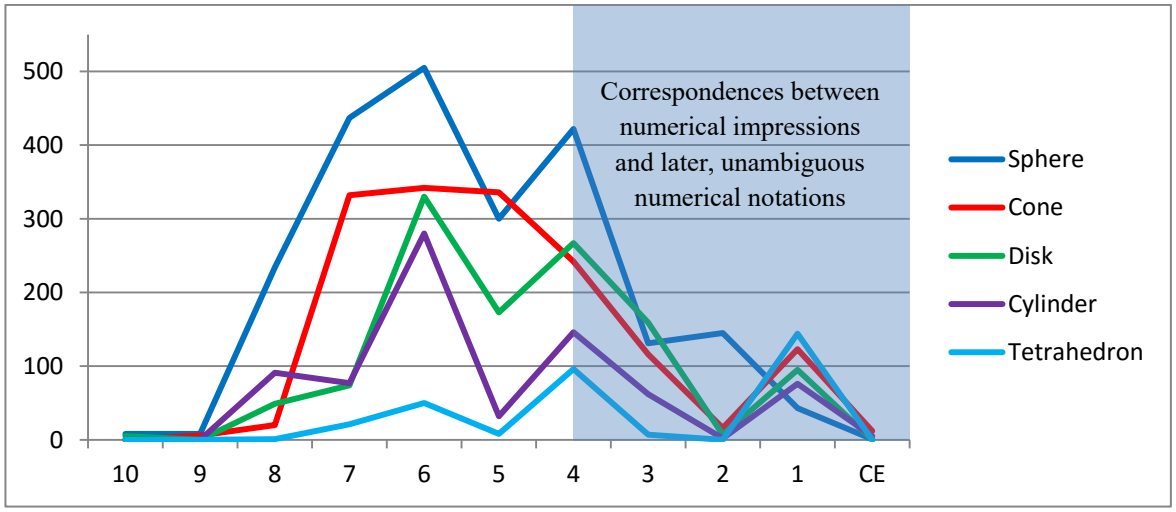



*Figure 8.7*. Temporal distribution of five token types used as numerical counters ($n$ = 6018) with the time frame of their having unambiguous numerical meaning highlighted in blue. The five types show a roughly similar pattern of peaks at the sixth, fourth, and first millennium and decreases at the third and second millennia and the Common Era. The data also show some suggestion that the relative amounts were consistent (i.e., spheres were the most common, tetrahedra the least). However, cones counter these trends somewhat. The sample included 2233 spheres, 1533 cones, 1159 disks, 766 cylinders, and 327 tetrahedra; the sample also omitted 828 eighth-millennium cones from Ganj-Dareh, Iran from the and 1399 sixth-millennium spheres from Jarmo, Iraq, as the large amounts of tokens from these sites were statistical outliers that skewed the results.

[662] MacGinnis et al. 2014.

Table 8.1

*Tokens by Country and Site*

| Iran | | | | Iraq | | | |
|---|---|---|---|---|---|---|---|
| Site | SB92 | New | Total | Site | SB92 | New | Total |
| Anau | 56 | | 56 | Abu Salabikh | | 74 | 74 |
| Bampur | 2 | | 2 | Arpachiyah | 93 | | 93 |
| Belt Cave | 6 | | 6 | Assur | | 16 | 16 |
| Chagha Sefid | 34 | 47 | 81 | Billa | 18 | | 18 |
| Chogha Bonut | 3 | 38 | 41 | Choga Mami | 18 | | 18 |
| Chogha Mish | 65 | 70 | 135 | Eridu | 12 | | 12 |
| Dalma Tepe | 3 | | 3 | Fara | 15 | | 15 |
| Deh Luran | 24 | | 24 | Gird Ali Agha | 3 | | 3 |
| Ganj-Dareh | 575 | 644 | 1219 | Gird Banahilk | 1 | | 1 |
| Geoy Tepe | 1 | | 1 | Hassuna | 3 | | 3 |
| Hajji Firuz | 13 | | 13 | Ischali | 3 | | 3 |
| Istakhr | 1 | | 1 | Jarmo | 2022 | | 2022 |
| Jaffarabad | 70 | | 70 | Jemdet Nasr | 27 | 61 | 88 |
| Jeitun | 81 | | 81 | Khafaje | 10 | | 10 |
| KS34 | 23 | | 23 | Kish | 60 | | 60 |
| KS54 | 7 | | 7 | Larsa | 3 | | 3 |
| KS76 | 2 | | 2 | Maghzaliyah | 8 | | 8 |
| Malyan | 17 | | 17 | Matarrah | 1 | | 1 |
| Moussian | 19 | | 19 | M'lefaat | 2 | | 2 |
| Seh Gabi | 61 | | 61 | Nineveh | 12 | | 12 |
| Sharafabad | 8 | | 8 | Nippur | 26 | | 26 |
| Sheikh-e Abad | | 11 | 11 | Nuzi | 26 | | 26 |
| Sialk | 8 | | 8 | Qal'at'Ana | | 1 | 1 |
| Sorkh-i-Dom | 1 | | 1 | Ras al 'Amiya | 30 | | 30 |
| Susa | 705 | | 705 | Sippar | 1 | | 1 |
| Tal-i-Iblis | 39 | | 39 | Tell Abada | 50 | 180 | 230 |
| Tall-i-Bakun | 56 | 53 | 109 | Tell Agrab | 1 | | 1 |
| Tepe Abdul Hosein | 44 | 42 | 86 | Tell Asmar | 7 | | 7 |
| Tepe Asiab | 193 | | 193 | Tell es-Sawwan | 77 | | 77 |
| Tepe Bouhallan | 4 | | 4 | Tell Hyelri | 5 | | 5 |
| Tepe Farukhabad | 5 | | 5 | Tell Madhhur | | 2 | 2 |
| Tepe Gaz Tavila | 93 | | 93 | Tell Oueili | 8 | 4 | 12 |
| Tepe Giyan | 15 | | 15 | Tell Raschid | 4 | | 4 |
| Tepe Guran | 34 | | 34 | Tell Songor | 1 | | 1 |
| Tepe Hissar | 92 | | 92 | Tell Yalkhi | 5 | | 5 |
| Tepe Muradabad | 1 | | 1 | Tello | 92 | | 92 |
| Tepe Sarab | 340 | | 340 | Telul eth Thalathat | 11 | | 11 |
| Tepe Siahbid | 3 | | 3 | Tepe Gawra | 485 | | 485 |
| Tepe Yahya | 142 | | 142 | Ubaid | 1 | | 1 |
| Tulai | 7 | | 7 | Umm Dabaghiyah | 11 | | 11 |
| Zagheh | 29 | 238 | 267 | Umm Hafriyat | 26 | | 26 |
| Total (Iran, 41 sites) | 2882 | 1143 | 4025 | Uqair | 5 | | 5 |
| | | | | Ur | 107 | | 107 |
| | | | | Uruk | 809 | | 809 |
| | | | | Yarim Tepe | 59 | | 59 |
| | | | | Total (Iraq, 45 sites) | 4158 | 338 | 4496 |

*Notes*. Comparison of the geographic distribution of tokens in Schmandt-Besserat's original dataset (SB92; $n$ = 8477), the newly catalogued tokens (New; $n$ = 2352), and the combined data (Total; $n$ = 10,829). The geographic distribution is similar in both datasets. A breakdown of token shape is provided in Appendix D.

(Continued)

| Palestine | | | |
|---|---|---|---|
| Site | SB92 | New | Total |
| 'Ain Ghazal | 12 | 137 | 149 |
| Beidha | 20 | | 20 |
| Beisamoun | 4 | | 4 |
| Jericho | 3 | | 3 |
| Ktar Tell Kazarei | 1 | | 1 |
| Megiddo | 9 | | 9 |
| Munhatta | 1 | | 1 |
| Tell Aphek | 1 | | 1 |
| Total (Palestine, 8 sites) | 51 | 137 | 188 |

| Turkey | | | |
|---|---|---|---|
| Site | SB92 | New | Total |
| Beldibi | 11 | | 11 |
| Cafer Höyük | | 2 | 2 |
| Can Hasan | 41 | | 41 |
| Çatalhöyük | | 11 | 11 |
| Cayönü Tepesi | 108 | | 108 |
| Gritille | 33 | | 33 |
| Hacınebi | | 12 | 12 |
| Suberde | 66 | | 66 |
| Tell Kurdu | | 4 | 4 |
| Ziyaret Tepe | | 490 | 490 |
| Total (Turkey, 10 sites) | 259 | 519 | 778 |

| Syria | | | |
|---|---|---|---|
| Site | SB92 | New | Total |
| Abu Hureyra | 5 | 17 | 22 |
| Amuq | 17 | 13 | 30 |
| Chagar Bazar | 9 | | 9 |
| Cheikh Hassan | 2 | | 2 |
| Ghoraife | 11 | | 11 |
| Habuba Kabira | 141 | | 141 |
| Hadidi | 147 | | 147 |
| Jebel Aruda | 11 | | 11 |
| Mureybet | 2 | | 2 |
| Ras Shamra | 16 | | 16 |
| Tell Aswad | 320 | | 320 |
| Tell 'Atij | | 21 | 21 |
| Tell Brak | 3 | 86 | 89 |
| Tell Halaf | 5 | | 5 |
| Tell Kannas | 58 | | 58 |
| Tell Ramad | 380 | | 380 |
| Total (Syria, 16 sites) | 1127 | 137 | 1264 |

| Jordan | | | |
|---|---|---|---|
| Site | SB92 | New | Total |
| Es-Sifya | | 78 | 78 |
| Total (Jordan, 1 site) | 0 | 78 | 78 |

temporal distribution, as well as some consistency in the relative amounts of tokens found (i.e., spheres were generally the most common of the five types, tetrahedra the least). However, cones differed somewhat in these characteristics from the other types in the sample, as can be seen in the graph.

The general trend is one of increase in the frequency of token finds from the ninth to the sixth millennia,[663] consistent with the movement of people into southern Mesopotamia and the growth of settlement and population that occurred during the early Neolithic, and decrease from the fourth millennium onward, the period of time in which tokens have unambiguous numerical meaning through their correspondences with later numerical notations (the time period highlighted with blue shading). The significant decrease in token frequency shows, as Schmandt-Besserat concluded,[664] that developments

[663] Robson (2008) estimated the start of tokens at around 6000 BC.
[664] Schmandt-Besserat 1992a.

and transitions in the material technologies used for counting in the late fourth millennium were associated with important and significant changes in the ways numbers were conceptualized (as will be discussed further in the next chapter). The small increase around the first millennium is associated with more peripheral locations and perhaps reflects something of the uneven pace at which technology could have spread in an era when distance was a considerable barrier to the exchange of ideas. There is also an apparent decrease in the frequency of the various clay objects classified as tokens around the fifth millennium, which is difficult to explain regardless of the social purpose the objects may have served. Four of the five plain token types used for numeration show the same fifth-millennium decrease, possibly implying either change from a common cause (like a population decrease or the loss of trade in specific commodities) or a number of causes perhaps affecting a coherent system of use (i.e., specific to counting). The different pattern shown by cones is discussed below but suggests they were relatively unaffected, either by whatever caused the other types to decrease or their non-inclusion in the same system of use.

The graph also suggests that spheres were the most likely shape to be found and classified as tokens, followed by cones, disks, cylinders, and tetrahedra. As was noted, cones conform less to the general distributional profile, relative to the other shapes, which is interesting because cones were fully integrated into the multiple numerical systems that appeared as numerical impressions in the late fourth millennium. The different distribution may relate to the designation of cone-shaped objects as tokens when they may have served other social purposes (e.g., game pieces, etc.). Alternatively, they could have been used as counters for specific commodities (e.g., grain), with the different distribution indicating a standalone use (i.e., cones might have been originally used to account for only grain) that was later combined with other token shapes (cones used as one shape among several in

accounting systems for multiple commodities). Unfortunately, there appears to be no way to determine which of these several possibilities (if any) might have been the case.

The general trends cannot establish whether tokens had similar numerical meaning across a time span of more than 10,000 years, especially given the difficulties associated with identifying any clay object as a token and classifying it as a particular type, known and assumed variability in numerical meanings by places of use and over time, and the ability of similar tokens to hold multiple meanings within a single system of use at any particular time. Certainly, "all attempts to identify the measuring and counting units represented by the pre-literate tokens and impressions have failed so far,"[665] though the basic 10-cycle of Sexagesimal System S is likely related to finger-counting. Some evidence for "the function of tokens ... comes through particular types of archaeological context, as for instance at Tell Sabi Sbyad, a late Neolithic village in north-central Syria … destroyed by fire around 6000 BCE"; the place appears to have been used for storing and processing grain and manufacturing cloth.[666]

Generally, token-based accounting prior to the fourth millennium is thought to have been conducted in much the same way as that reflected by the earliest numerical impressions, with context-dependent numerical value and the use of bundling–debundling,[667] at least in at the time where they were used to create the earliest numerical impressions and for some amount of the pre-writing period that immediately preceded it. However long that span of time may have been, it was undoubtedly shorter than 10,000 years; further, "impressions on envelopes of tokens and the impressions on those numerical tablets that can be dated beyond doubt into the pre-literate period lack the standardization of later numerical notations," demonstrating that tokens were indeed likely to have been a transitional technology between one-to-one correspondence and numerical

[665] Damerow 1996b, 24.
[666] Robson 2008, 34.
[667] Friberg 1994, 483–484.

notations,[668] something that would predictably be attended by things like mistakes in bundling and a lack of standardization that would respectively dwindle and converge over time. These may imply an earlier system of notation that did not involve bundling.[669]

Ultimately, there is no way to tell when or to what degree tokens assumed characteristics like context-dependent numerical value and the use of bundling–debundling. However, these issues might not need to be resolved to recognize tokens as a transition between tallies and numerical notations, probably sharing more characteristics with one or the other depending on when in the span of time between the two is being considered.

## 8.4 The Complexity of Tokens at and before the Late Fourth Millennium

For the remainder of this discussion, the term 'archaic period' will be used inclusively to designate the archaic and Uruk IV periods (3500–3200 BC) and whatever unknown portion of the pre-writing period can be reasonably construed as including token-based accounting similar to that reflected in the earliest numerical impressions. In the archaic period and for at least some portion of the pre-writing period that immediately preceded it, plain tokens are assumed to have been used like the unambiguous numerical impressions they correspond to would be in the late fourth millennium: within over a dozen commodity-specific systems in which any particular token took its numerical value from its shape and size (e.g., sphere or cone; large or small), occurrence within an order based on increasing magnitude (the small spheres that are $N_{14}$ occur between the signs that are $N_1$ and $N_{34}$, where they are larger than the former and smaller than the latter), and context of use (which specified both commodity type and numerical values and relations).

[668] Damerow 1996b, 24. Damerow believed that tokens represented one-to-one correspondence and did not include the tallies as a precursor technology.
[669] Glassner 2000.

For example, in Sexagesimal System S (Figure 8.8), the system of numerical signs used for counting most objects, numerical values were represented by three basic shapes in two sizes, a particular order of specific shapes/sizes, and the combination of some of the signs to create other, complex signs (e.g., as $N_{48}$ combines $N_{34}$ and $N_{14}$; the combined sign was likely an accumulation rather than a multiplication, as the "archaic economic texts do not … contain *a single example of a genuine multiplication*" [670]). Other forms of contextual indication were systems of markings applied to plain tokens, the use of tokens shaped like containers, the inclusion of complex tokens that specified commodity (and perhaps other information as well, like trading partners or origin locales), external markings on the bullae and envelopes containing the tokens, or exchange of tokens between partners whose identity specified commodity in a way that required no overt marking of either the tokens or their containers.

Shape/size combinations also conveyed numerical relations between signs. One sign (and presumably the earlier tokens) of a higher numerical value represented two, three, four, five, six, ten, or twelve signs (or tokens) of a lower numerical value. For example, the unit sign ($N_1$) was related to the next higher value sign ($N_{14}$) by a factor of

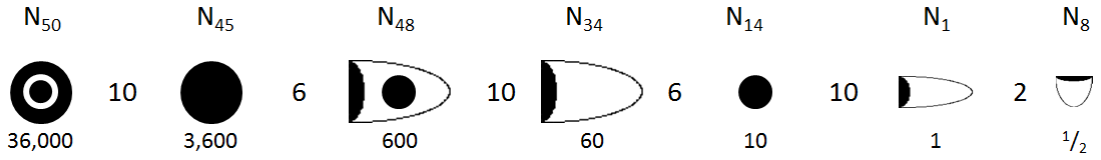



*Figure 8.8*. Sexagesimal System S, used for counting most objects in the archaic period.[671] The top row represents the names for the signs used by Assyriologists (e.g., $N_{14}$), the bottom row the signs' numerical values (10). The numbers in between the signs represent the numerical relations between them (e.g., one $N_{14}$ was equivalent to 10 $N_1$). Because of the correspondences between tokens and early numerical impressions, tokens are thought to have held similar numerical values, at least in the archaic period and some portion of the pre-writing era that preceded it.[672]

[670] Damerow 2010, 236, emphasis in original.
[671] Nissen et al. 1993.
[672] Friberg 1994.

ten, so that ten $N_1$ were equivalent to one $N_{14}$ (and vice versa: one $N_{14}$ was equivalent to ten $N_1$); for its part, $N_{14}$ was related to the next higher value sign ($N_{34}$) by a factor of six, so that six $N_{14}$ made one $N_{34}$.[673] Notably, the numerical values in question are non-subitizable, in both the values of the numerical relations between signs (i.e., factors of six and ten) and in the total values they enabled (ten $N_1$ equals one $N_{14}$, and both mean 'ten'; six $N_{14}$ equals one $N_{34}$, and both mean 'sixty').

In an ordinal counting sequence used with a counting technology like the tally, the possible relations between the counting numbers are *greater than*, *less than*, *not equal to*, *falls between*, and *plus-one*. For example, in the sequence 'one' through 'ten', 'nine' is more than 'eight', less than 'ten', not the same as 'seven', comes between 'eight' and 'ten', and is plus-one of 'eight'. Such relations are potentially realizable because they are implicit in magnitude ordering; this does not necessarily make them explicit. These relations are implicit in an ordinal counting sequence because of its magnitude ordering.[674] It is possible for them to be explicit as an element of the number concept, though they need not necessarily be so and might not be, especially in a more recently emerging number system, where each number word represents a concept of shared cardinality (e.g., 'five') and its place in the ordinal sequence (in this case, the number that occurs between 'four' and 'six'). Whether or not the relations between numbers are explicit, they represent only a small subset of the between-number relations possible within any set of counting numbers (as 'five' can be related to 'three' and 'fifteen' in several ways). The archaic number systems, and presumably the tokens as well, included some of these additional, explicit relations (e.g., 'six of these equal one of those'), presumably in addition to an explicit set of the basic relations inherent in magnitude ordering.

[673] Nissen et al. 1993.
[674] In numbers, "order denotes increasing magnitude, forms sequences, and gives '[n]ot only the integers, but also rational fractions and all real numbers … most of their mathematical properties' (Russell 1920, 29)" (Overmann 2016a, 295).

These additional relations were likely to have been realized from several potential sources: Some are believed to have originally been based on metrological containers.[675] Others were possibly realized by manipulating the tokens themselves, as the tokens were a material form that afforded a degree of manipulability well beyond that possessed by the tallies and fingers that preceded them or the numerical impressions that followed. Another possible source of the numerical relations was the contact and borrowing that would have occurred as different cultural groups with unique indigenous number traditions moved into and settled in the region; the contrast of individuated ways of constructing numbers (e.g., five + *x*, three + *x*) would likely have served to highlight differences in the numerical relations between systems, creating additional possibilities for explicating and incorporating them. Whatever their source, the relations began to adhere to and characterize the tokens themselves, as well as the concept of number they instantiated.

These relations can be analyzed using two dimensions established by anthropologist Stephen Chrisomalis[676] for comparing numerical notations across number systems (his dimensions, their definitions, and examples of the resultant categories are depicted in Figure 8.9). In the first dimension, structure between exponents can be additive or positional, while in the second dimension, structure within exponents can be cumulative, ciphered, or multiplicative. Together, these two dimensions yield five possible classifications for numerical notation systems (the sixth possibility is logically excluded). All known numerical notation systems can be classified according to these two dimensions, attesting to the high degree of similarity between number systems no matter how independently invented (in contrast, a classification system for comparing languages

[675] Nissen et al. 1993. However, as was noted in Chapter 6, metrological containers are unlikely to have been a basis of the between-number relations in Sexagesimal System S, as there is no inherent 'ten-ness' to a group of people in the same way that 'four quarts make a gallon' would (in an anachronistic example).
[676] Chrisomalis 2004; 2010.

| Intraexponential Structure \ Interexponential Structure | **Additive**<br>The sum of the values of each power is taken to obtain the total value of the numeral-phrase. | **Positional**<br>The value of each power must be multiplied by a value dependent on its position before taking the sum of the numeral-phrase. |
|---|---|---|
| **Cumulative**<br>Many signs per power of the base, which are added to obtain the total value of that power. | **Roman (classical)**<br>1434 = MCCCCXXXIIII<br>(1000) + (100 + 100 + 100 + 100) + (10 + 10 + 10) + (1 + 1 + 1 + 1) | **Babylonian cuneiform**<br>1434 = 𒐜 𒐐𒐐𒐐 𒌋𒌋𒌋𒌋 𒐝<br>[(10 + 10 + 1 + 1 + 1) × 60] + [(10 + 10 + 10 + 10 + 10 + 1 + 1 + 1 + 1) × 1] |
| **Ciphered**<br>Only one sign per power of the base, which alone represents the total value of that power. | **Greek alphabetic**<br>1434 = ͵αυλδ<br>1000 + 400 + 30 + 4 | **Khmer**<br>1434 = ๑๔๓๔<br>(1 × 1000) + (4 × 100) + (3 × 10) + (4 × 1) |
| **Multiplicative**<br>Two components per power, unit-sign(s) and a power-sign, multiplied together, give that power's total value. | **Chinese (traditional)**<br>1434 = 一千四百三十四<br>(1 × 1000) + (4 × 1000) + (3 × 10) + (4) | **LOGICALLY EXCLUDED**<br>(Power cannot be both expressed as a sign and assumed as a positional weight.) |

*Figure 8.9*. Chrisomalis' two dimensions for classifying systems of numerical notations. As shown, the later Babylonian cuneiform numbers were a cumulative–positional notation system, while proto-cuneiform (and the earlier numerical impressions and the corresponding tokens as well) was a cumulative–additive system. The difference between proto-cuneiform and the Roman numeral system shown in the graph is that the latter was a single system, while the former consisted of multiple systems. While any one of the proto-cuneiform notation systems is cumulative–additive, when considered together, the number system cannot be said to have had "a base," as there was no single base number (or pair of numbers, as a system with a base and sub-base would have) to unify the multiple systems, though this would occur as the number system continued to develop (see discussion in Chapter 9). Sexagesimal System S, with its alternating pattern of 'six' and 'ten', was the most regularized and likely the basis for the sexagesimal base of the place-value system that started to emerge around 2350 BC.[677]

yields many more than five categories[678]). This profound similarity of organizing numerical notations and the number concepts they represent—across differences of language, culture, precursor counting technologies, need for and explication of number concepts, and notational form—reveals and perhaps explains the translinguistic quality of numbers (previously noted in Chapter 4); it is "why numerical (but not non-numerical) signs can be represented without phonetic values in fully glottographic writing and deciphered in otherwise untranslatable scripts."[679]

Chrisomalis classifies proto-cuneiform numbers as a cumulative–additive system

[677] Robson 2008.
[678] E.g., Comrie 1989; also see discussion in Overmann 2016a.
[679] Overmann 2016a. Though phonetic values can be represented in glottographic writing, numbers are fully intelligible without it, the difference between 'four' (the phonetic representation of the word for the number) and '4' (the non-glottographic representation of the number).

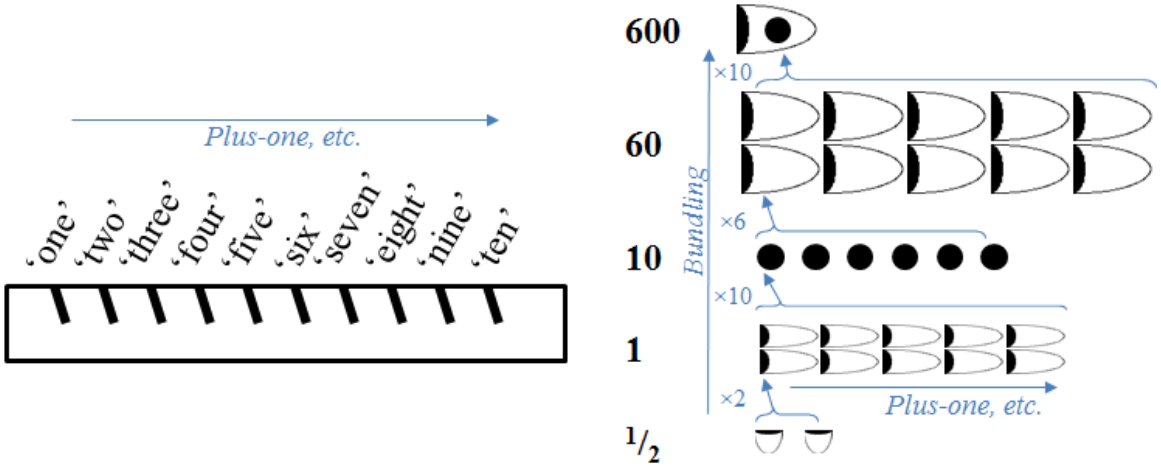



*Figure 8.10*. Exponential dimensions of counting devices. (Left) In a tally associated with lexical terms for an ordinal counting sequence, the potential relations between numbers are one-dimensional (intra-exponential only) and include *greater than*, *less than*, *not equal to*, *falls between*, and *plus-one*. Total value is achieved by accumulating value along the single dimension. (Right) In the archaic Sexagesimal System S for counting most objects,[680] the relations between numbers are explicit and include those of an ordinal counting sequence (intra-exponential, the horizontal axis), as well as the relations implicit in bundling (inter-exponential, the vertical axis). Total value is achieved by accumulating value along both dimensions. The two-dimensional structure of the tokens implies a one-dimensional precursor (such as the tally).

(by comparison, the later Babylonian cuneiform numerical notations were a cumulative–positional system).[681] That is, proto-cuneiform—as well as the earlier numerical impressions they resembled and the tokens they corresponded with, at least in the period immediately preceding the use of early numerical impressions—were two-dimensional notations for number concepts. This stands in stark contrast with the tally, which is a one-dimensional system (Figure 8.10). As the material form associated with the transition from one- to two-dimensional representation of number concepts, tokens were also the scaffold that enabled the number system to be elaborated to include the second representational dimension. In other words, the dimensional structure of the tokens implies an earlier technology with a simpler dimensional structure (e.g., tallies).

[680] Adapted from Nissen et al. 1993.
[681] Chrisomalis 2010.

For using the tokens, as was true of the earliest numerical impressions, "counting rules required that multiples of a given unit … be replaced by the next 'higher' units,"[682] operations known as bundling and debundling. After a group of tokens had been accumulated, these operations allowed them to be reduced and simplified to their lowest, most accessible terms by exchanging multiple tokens of smaller values with single tokens of higher value, providing a functional ability to add and subtract. Bundling–debundling operations imply two preconditions: first, the availability of and faculty with an ordinal counting sequence, as this would be essential to performing the exchange of tokens; second, some method of organizing groups of tokens by type, as this would facilitate the exchange of tokens with greater or lesser value in order to reduce the amount of tokens so that those remaining represented the numerical value in the simplest possible way.

If there were no ordinal counting sequence or faculty using it for enumeration, bundling–debundling would have been difficult—if not impossible—to perform at the quantities suggested to have been used during the pre-writing period, which reached several hundreds. Alternatives to enumeration with an ordinal counting sequence include estimating the size of the respective piles, which would have lacked the computational accuracy that generally characterizes the earliest numerical impressions, and performing the exchanges through one-to-one correspondence, which would have been quite cumbersome and laborious. Both alternatives would have been under significant pressure to achieve greater efficiency and effectiveness, especially as the requirements for using the number system increased along with population, trade, construction, labor pools, the first urban centers, and so on,[683] making it doubtful that an inefficient, ineffective system would have persisted in that form for very long.

[682] Friberg 1994, 483–484.
[683] Englund 2004a.

Similarly, the lack of a method of separating tokens by type and—even more importantly—arranging them in terms of magnitude increase would have made bundling–debundling difficult to perform and increased the number of potential errors. In fact, analysis of mathematical texts from the Old Babylonian period suggests that the earlier tokens might have been used with a counting board or abacus because "additive contributions were no longer identifiable once they had entered the computation—as [is the case with calculations performed with] … the medieval dust abacus or on a counting board, but not in the paper algorithms that were developed by the late medieval *maestri d'abbaco* and which are still with us [today]."[684] Counting boards, while not attested archaeologically,[685] are suggested by the visual appearance of later cuneiform signs; such devices would have organized tokens by quantity, facilitating bundling–debundling operations and the eventual development of place value. Minimally, tokens would have been grouped by shape and size and ordered by magnitude, perhaps less formally than an abacus but certainly more formally than a commingled, random heap. One of the Babylonian signs (SID or ŠID) "may have been derived from a pictograph of … an instrument" like an abacus.[686] After the archaic period,

> [Lexical lists] suggest that the Sumerians used tallying boards made of wood, which being perishable would not have been unearthed in excavations. There is also some evidence that the sign SANGA, designating the chancellor of an economic unit, derives from a pictogram depicting such a tallying board. This, in fact, is supported by the cuneiform sign ŠID which, also having developed from this sign, was employed as an ideogram with the meaning 'account.'[687]

[684] Høyrup 2000, 2. The lack of identifiability of the original contributions once accumulated has been associated with certain types of scribal errors, as well as limitations in the number of places to which a calculation could be carried out (Friberg 1981).
[685] Nissen et al. 1993.
[686] Smith 1958, 40.
[687] Nissen et al. 1993, 134. Also see Crump 1992; Ronan et al. 1981.

However, the idea that an abacus or counting board was used has not been universally accepted; Oppenheim,[688] for example, argued against it.

While counting technologies like tallies enable quantities to be accumulated (but little else[689] because the material form lacks the necessary manipulability[690]) and recorded (albeit with no inherent way to mark commodity type), tokens could be used to accumulate, reduce, group, ungroup, and record quantity in a way that not only kept track of commodity type but also made numerical information accessible[691] (i.e., through bundling/debundling). Accumulation and reduction comprised a functional ability to add and subtract, grouping and ungrouping a functional multiplication and division. However, "a sequence of steps [such as that involved in accumulating and reducing tokens] is not the same as a conceptual understanding [of addition and subtraction]."[692] The arithmetic operations, like numbers, are abstractions realized by generalizing from particulars. In the Western mathematical tradition, arithmetic is an operation that can be performed on any numbers to realize a total that includes them all. In Babylonian mathematics, by comparison, well over a millennium after the late fourth millennium, there were still two types of addition, two types of subtraction, and four types of multiplication[693] (discussed further in Chapter 9), showing that while some generalization of the arithmetical operations had occurred, by no means had it converged on the arithmetical operations in the way they are conceptualized in the modern tradition. In what is probably too severe a characterization, Damerow described token-based accounting as follows:

> … the arithmetical techniques of archaic bookkeeping were in fact techniques without an integrating number construct. Both the meaning of these signs as well as the way they were used do not correspond to what

[688] Oppenheim 1959.
[689] Kaput et al. 2008, 50.
[690] Overmann 2016d.
[691] Glassner 2000; Nissen et al. 1993; Schmandt-Besserat 1992b. The numbers of tokens found in rubble fills also suggests they were discarded once they had served these purposes (Schmandt-Besserat 1992b).
[692] Hutchins 1995, 294.
[693] Høyrup 2002b.

> could be expected if they would have represented numbers or generalized numerically structured concepts such as the abstract concepts of space, time, weight, volume, area and so on.[694]

Nonetheless, the manipulability afforded by tokens and their ability to represent numerical value along two dimensions were significant because they represented both enhancements to the ANE number system and enablers of the further elaboration that would develop. Further, while Damerow was correct in asserting that ANE concepts of numbers and any correspondences to "generalized numerically structured concepts" would have differed substantially from those of the Western mathematical tradition, it is not necessarily the case that they had no idea that things like "space, time, weight, volume, [and] area"[695] could be represented numerically. As was mentioned in Chapter 6, when the representation of time appears in written form, the use of recognizable units (i.e., 30-day months, 12-month years) implies a long prehistory of counting lunar and solar cycles. Volume calculations are implicit in the metrologically based units for commodities like grain.[696] Finally, both land measures (space) and area calculations are represented by an artifact from the city of Uruk (modern Warka, Iraq) that has been dated to around 3250 BC (the Uruk IV period)[697]: "[Now] known as W 19408,76, [it] contains two exercises on calculating the areas of fields [by means of the agrimensors' method[698]]. It is thus the

[694] Damerow 1996b, 29. Damerow used the term *proto-arithmetic* "to designate symbol systems, such as tallies or units of counting [like the tokens], for controlling quantities in direct relationship with the objects and symbols that represent these objects. Such systems of symbols do not represent any kind of context-independent numbers reflecting accumulated experiences achieved in the process of controlling quantities by means of correspondences. Their invention and use may have resulted in the development of arithmetical techniques, but they can be used for controlling quantities of objects, even without any cognitive numerical construct" (Damerow 2012, 158). However, Damerow did not differentiate tallies and tokens in terms of characteristics like exponential dimensionality (i.e., as Chrisomalis does; see Chrisomalis 2004; 2010).
[695] Damerow 1996b, 29.
[696] Nissen et al. 1993.
[697] Nissen et al. 1993. Friberg (2005b) gives the dating as around 3000 BC.
[698] Robson 2007, 73. "W19408,76 uses just three different number signs, all within the length measurement system. Small circles represent 10 rods each, c. 60 meters, while the large D-shaped impressions represent units of 60 rods, c. 360 meters long. The combination of the two—a small circle within a large D—represents 60 × 10 = 600 rods, or c 3.6 km. There are just two other signs on the tablet: a horizontal line representing length and a vertical line representing width. Each side of the tablet thus records two lengths

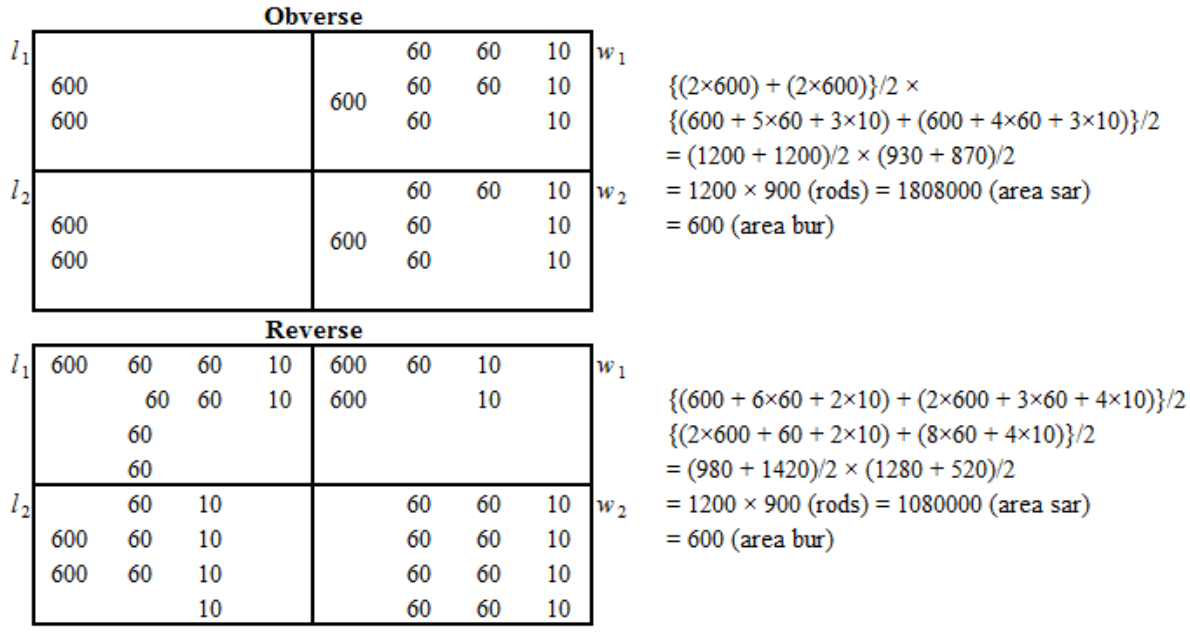



*Figure 8.11*. Two agrimensors' method for calculating the areas of quadrilaterals as represented on the obverse (front) and reverse faces of artifact W 19408,76 (CDLI no. P003118),[699] a mathematical exercise from the Uruk IV period held at the German Archaeological Institute, Berlin, as transcription into Western numerical notations. While none of the lengths of the four sides of each quadrilateral are equal (the field on the reverse is more irregular in this respect than the one described on the obverse), the ANE people performing the calculations in the later fourth millennium apparently had no way to include considerations of acute or obtuse angles[700] created by uneven sides. Thus, there is a degree of error in both calculations that far exceeds those of the closest modern parallels (e.g., inaccuracies of measuring; the number of places to which the calculation is carried out). Since the angles in question span a considerable range of possibilities, there is no way to determine the quadrilateral dimensions intended. However, the ability to calculate area in this manner was sufficiently accurate to fulfill its purposes (e.g., estimating the amount of seed needed to plant the field or its expected yield).

world's oldest piece of recorded mathematics (Nissen et al. 1993, fig. 50; Damerow 2001, 258)."[701] The complexity of the calculation contained on the artifact (Figure 8.11) suggests that such calculations (or at least simpler versions of them) were likely performed prior to the invention of the notational recording the artifact represents.

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and two widths; no areas are marked. But if, in each case, the average width is multiplied by the average length—the so-called agrimensors' method for finding the area of a field, $A = (l_1 + l_2)(w_1 + w_2)/4$—the area found is a conspicuously round number in the contemporaneous area system (1 rod × 1 rod = 1 sar, 1800 sar = 1 bur)."

[699] Adapted from Robson 2007, 74.

[700] Respectively, these are less than the 90° right angle and greater than the right angle but less than the 180° that forms a straight line.

[701] Robson 2007, 73.

If their manipulability enabled tokens to be used for purposes beyond the simple accumulation and reduction of quantities of commodities, it had a significant drawback as well, in that tokens were loose. This meant that any particular calculation or record was easily disrupted (if tokens were dropped, for example) and was not secure (i.e., easily altered). This problem meant that they needed to be contained, especially as their use proliferated, and is the commonly accepted explanation for the development of bullae and envelopes, whose instantiation of quantity would not only make tokens redundant as a method of recordkeeping but also allow their conjoined representation of commodity and quantity to be separated, the critical technological change leading to the development of writing and literacy on the one hand, and numerical notations and mathematics on the other.[702] This is described in Chapter 9, along with the implied changes in ANE numerical cognition.

[702] E.g., Englund 1998c; Malafouris 2010b; Schmandt-Besserat 1992a.

# CHAPTER 9

# NUMERICAL NOTATIONS AND MATHEMATICAL ELABORATION

## 9.1 Numerical Notations and Numerical Cognition

Discussions of numerical notations tend to focus on their "symbolic nature … rather than the material nature of numerical representation more generally."[703] Alternatively, as in the present analysis, focusing on the material nature of numerical notations affords an opportunity to consider the ways in which they are similar and in which they differ from their predecessors, the clay tokens used in the Neolithic ANE. The ways in which tokens and numerical notations are similar in representing numerical information are noteworthy, not least because their correspondences of shape and size provide insight into the meaning of the former (discussed in Chapter 8). Also important are four differences between the two technologies: whether the representation of quantity and commodity is conjoined (tokens) or separate (numerical notations); whether the representation of number is manipulable (tokens) or fixed (numerical notations); whether the representation is concise (numerical notations) or not (tokens); and the method of production (i.e., tokens were shaped by hand, whereas numerical notations would ultimately be handwritten) and its influence on psychological processing.

[703] Overmann 2016c.

The differences in particular had crucial implications for numerical cognition in the ANE, as they would have enabled numbers to be conceptualized as entities, the relations between numbers to be explicated and elaborated, and new algorithms and operations for manipulating numbers and the relations between them to be developed, creating one of the world's great ancient mathematical traditions. In this regard, the ANE mathematical tradition rivals that of Egypt, China, and Mesoamerica; the fact that these four civilizations are also believed to have independently invented writing[704] cannot be unrelated to their mathematical elaboration. Before these characteristics and their consequences are presented, however, an overview of and a brief chronology for numerical notations is necessary, to show how they transformed from tokens, to numerical impressions that looked like the tokens, to proto-cuneiform notations that looked like the numerical impressions, and finally to cuneiform signs for numbers.

## 9.2 Early Numerical Impressions and the Beginning of Writing (Uruk V Period)

As mentioned in Chapter 8, the first numerical impressions resembled the plain tokens used for accounting that preceded them, in terms of their shapes (especially cones, spheres, disks, cylinders), sizes (typically, small and large), and bundling relations (discussed further below). The physical correspondences between the two suggest that tokens were often used to make the impressions, though other implements (e.g., fingers) were used as well.[705] The earliest tablets with numerical impressions have been dated to the Uruk V period. In the CDLI database, there are 193 tablets from the Uruk V period with at least one numerical impression (see Appendix E; the geographic distribution of the sites is shown in Figure 9.1). Of 193 tablets with numerical impressions in the CDLI database as of February 10, 2016, most are relatively simple. For example, 107 tablets

[704] Powell 2009; Senner 1989.
[705] Englund 2006; Schmandt-Besserat 1992a.

(55.4%) contain impressions with a single numerical value, 62 tablets (32.1%) contain numerical impressions of two different values, 18 tablets (9.3%) have three, three tablets (2.6%) have four, and one tablet (0.5%) has five. The majority of tablets (*n* = 137, 71.0%) consist of a single cell or column of numerical values. However, 37 (19.2%) contain two cells, 15 (7.8%) contain three or four cells, and four (2.1%) contain between five and seven cells.

Just over 40% of these tablets contain impressions of more than one numerical value (e.g., $N_{14}$ and $N_1$ is a frequent combination) and show that impressions were placed in order of increasing numerical value (Figure 9.2; the few exceptions to magnitude ordering are discussed below in conjunction with exceeded bundling in area calculations).




*Figure 9.1*. Map of the ANE showing the geographic distribution of sites from the Uruk V period where numerical impressions have been found. Adapted from an image in the public domain.

As was construed for the tokens (see Chapter 8), the first numerical impressions comprised a cumulative–additive numerical notation system, representing numerical value in two dimensions (intra- and inter-exponentially).[706] Commodity and quantity were conjoined, as they had been in token-based accounting, with commodity and other information like ownership specified using a variety of methods (e.g., specific shape–size combinations and the use of seals,[707] as well as the inclusion of signs for commodities like sheep [e.g., UDU on tablet CDLI no. P000852 from the city of Uruk]).

Variability in bundling relations has been interpreted as evidence of the recency of numbers in the ANE (i.e., as consistent with a notion of number that was 'concrete' rather than abstract, the use of one-to-one correspondence, and the lack of a numerical lexicon),[708] so it is important to examine the bundling variability in the numerical tablets from the Uruk V period for what it might reveal in this regard. While most of the quantities of these numerical impressions were consistent with the amounts predicted by the bundling rules associated with the later proto-cuneiform counting systems,[709] there were eight instances that did not (see Table 9.1 and Figure 9.3), representing seven artifacts (3.6%) of the 193 tablets with numerical impressions, as well as five of the 12 cities (41.7%) where the artifacts were found (i.e., the CDLI database for five of the seven artifacts with exceeded bundling was uncertain as to the name of the ancient city listed as the provenience; in these cases, the modern city name given in the CDLI database appears in Table 9.1). Three of the seven artifacts contained numerical impressions of more than one value (e.g., P235759 from Jebel Aruda contained three $N_{34}$, 22 $N_{14}$, and five $N_1$); in all three cases, there was one numerical value whose quantity exceeded the bundling limits

[706] Chrisomalis 2010.
[707] Høyrup 1994.
[708] E.g., Englund 2006.
[709] Nissen et al. 1993.

(e.g., for artifact P235759, the 22 repetitions of the impression for $N_{14}$), while the others conformed to them (three $N_{34}$ and five $N_1$ for the same tablet).

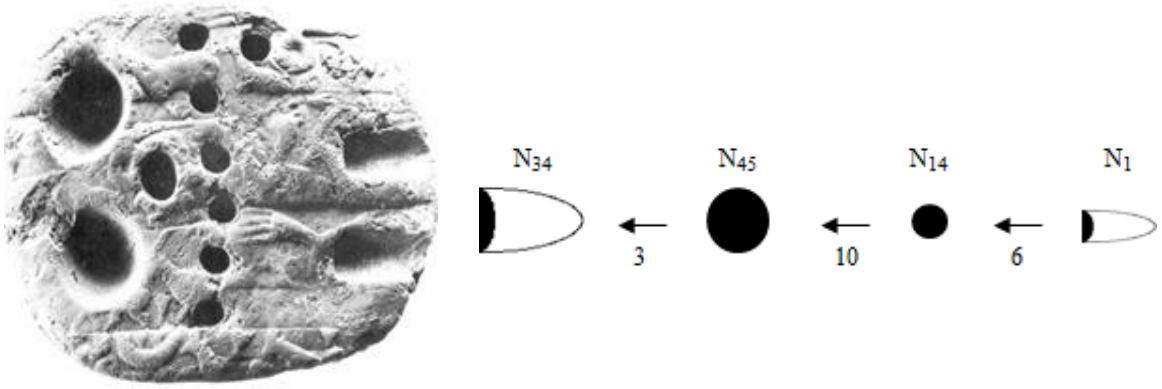



*Figure 9.2*. Magnitude ordering. (Left) Administrative tablet (JA 74-024; CDLI no. P235757) with numerical impressions dated to the Uruk V period, likely from Jebel Aruda (modern Syria), most recently held in the collection of the National Museum of Syria, Raqqa, Syria. The impressions (two $N_{34}$, one $N_{45}$, seven $N_{14}$, and two $N_1$) are ordered by increasing magnitude and probably represented the ŠE system Š for grain (right; redrawn from Nissen et al. 1993).

Table 9.1

*Seven Numerical Tablets from the Uruk V Period that Exceed Bundling Rules*

| Source | CDLI no. | Sign | Number | Unbundled Maximum |
|---|---|---|---|---|
| Tell Jokha | P200010 | $N_{14}$ | 12 | Nine $N_{14}$ in $GAN_2$ System G (area measurements) and ŠE System Š (grain) |
| Susa | P008553 | $N_{14}$ | 14 (two instances) | |
| Tepe Sialk | P009520 | $N_{14}$ | 23 | |
| Jebel Aruda | P235755 | $N_{14}$ | 10 | |
| | P235759 | $N_{14}$ | 22 | |
| | P235760 | $N_{14}$ | 10 | |
| Mari | P235767 | $N_{14}$ | 11 (see note) | |

*Notes*. Seven numerical tablets from the Uruk V period representing eight instances in which expected bundling limits were exceeded. The unbundled maximum is the highest amount of numerical impressions possible without necessitating bundling to the next higher value, if bundling rules are followed. For example, 10 $N_{14}$ should be bundled to one $N_{50}$ in the $GAN_2$ System G for measuring area or one $N_{45}$ in the ŠE System Š for grain.[710] Three of the instances (P235755 and P235760 from Jebel Aruda and P235767 from Mari) were at or just over the bundling limit, while two (P009520 from Tepe Sialk and P235759 from Jebel Aruda) exceeded the limit by more than double. The 11 $N_{14}$ signs on artifact P235767 from Mari might alternatively be read as two columns of four and seven $N_{14}$.

[710] Nissen et al. 1993.

Interestingly, all eight instances of exceeded bundling for the Uruk V numerical tablets involved the impression $N_{14}$. While the reason for this is unknown, a possible explanation might be that the lack of bundling may have been limited to particular trading networks or applications of use. For some of these instances, however, there were several potential applications (e.g., the numerical impressions on artifact P235759 were consistent with Systems S, Š, and B). Where the attributed application was more specific, the exceeded bundling appears to have occurred in the context of an area calculation (e.g., the two instances of exceeded bundling on artifact P008553 from Susa), possibly representing unbundled measurements of the sides of rectangular fields. The notations on artifact P008553 have been interpreted as calculating area: ((40n + 30n)/2) × 140 = 49 iku (as given in the CDLI listing), amounts corresponding to the four $N_{14}$ (40n in the formula), three $N_{14}$ (30n), and 14 $N_{14}$ (140, the multiplier) impressed on the artifact[711]; the formula is anachronistic, as area calculations in the Uruk V period would not have included multiplicative operations. For all instances of exceeded bundling, whatever their actual system of use, the total would be numerically meaningful, given the availability of a sufficient numerical lexicon and ability to count to the indicated amounts (as was suggested to have been the case in Chapters 6 and 7). Rather than implying the use of one-to-one correspondence, as might be consistent with an emerging number system with a restricted lexicon, these considerations suggest that the number system was already relatively mature.

When the Uruk V numerical impressions are compared with their token predecessors, both represented conjoined numerical and commodity information through conventionalized shapes, sizes, and ordering. However, whereas tokens were manipulable,

[711] See Scheil 1923, 25 and Pl. XLVII, the CDLI listing for P008553, and discussion of similar texts from the Uruk IV period in Friberg 1997/1998.

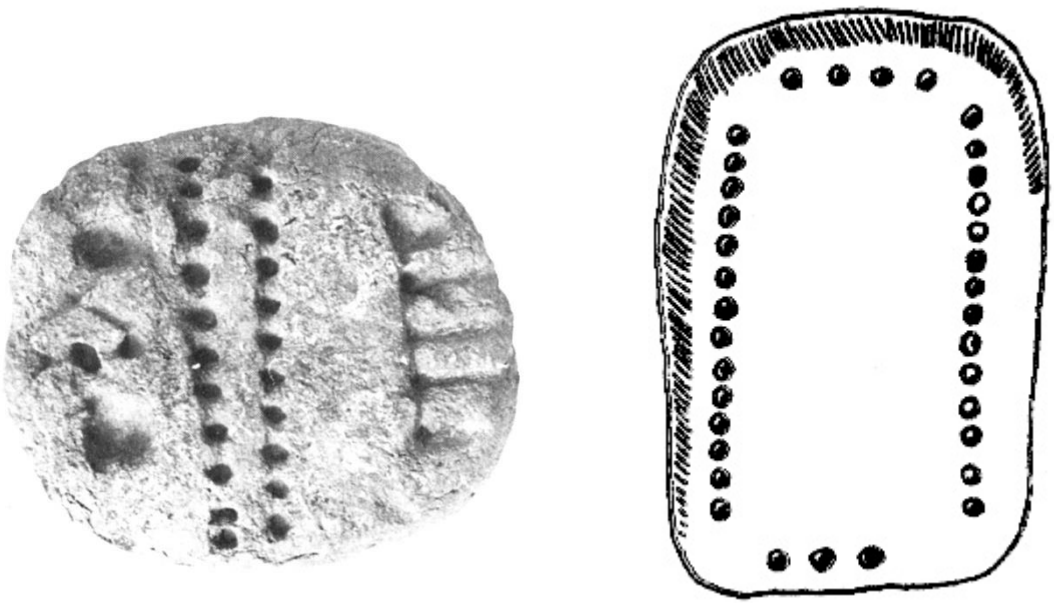


*Figure 9.3*. Artifacts in which the repetition of the sign $N_{14}$ exceeded the expected bundling limits. (Left) Numerical tablet (JA 75-104; CDLI no. P235759) from Jebel Aruda (modern Syria), at one time held in the collection of the National Museum of Syria, Raqqa, Syria. The artifact contains 22 instances of the sign $N_{14}$, well beyond the number expected if bundling rules had been followed. (Right) Numerical tablet (CDLI no. P008553) from Susa (modern Iran), location currently unknown. The artifact contains 14 instances of the sign $N_{14}$ on two sides of the obverse face and has been interpreted as an area calculation.[712]

numerical impressions were fixed: Once they had been impressed onto the surfaces of tablets, they were not alterable to the same degree tokens had been, and thus they could not support accumulation, reduction, grouping, separation, or simplification operations (respectively, these would be addition, subtraction, multiplication, reciprocals, and bundling–debundling). Accordingly, arithmetical operations would still have been performed with tokens,[713] with impressions being used only to record the results until the necessary numerical relations and algorithms had been developed to adjust to the

[712] Adapted from Scheil 1923 Pl. XLVII; see Friberg 1997/1998 for a discussion of similar texts from the Uruk IV period.

[713] As mentioned in Chapter 8, the continued use of tokens for calculations is judged from "from the way the results were transcribed" (Glassner 2000, 188). Additive and subtractive operations "seem to have been performed on a counting board and not in clay" (Høyrup 2002b, 40); "the absence of computations of complex products from the tablets for rough work … have served to 'posit' [that] results [were] obtained on a separate device" and then copied (Høyrup 2002b, 73; also see his p. 195, footnote 222). The use of tokens for calculating is also attested by certain types of scribal errors (e.g., numbers that were "inserted twice into the calculational scheme or device instead of once—which cannot easily happen unless addends disappear from view once they [were] inserted" [Høyrup 2002b, 73], as would be the case if calculations were still performed with tokens), and the circumstance that the relations and algorithms required to perform arithmetical or mathematical calculations on fixed notations (rather than a manipulable technology) needed to be developed.

difference in manipulability inherent in the technological change. Even after relations and algorithms appropriate for fixed notations had been developed, mathematicians would formally debate whether it was better to manipulate devices like abaci or calculate with algorithms designed for written notations as late as the European Renaissance,[714] and questions of which method of calculation is faster or more accurate have persisted to the present, albeit with the device category now including, in addition to abaci, calculators and computers. Both abaci and algorithms continue to be used today; they can be considered different forms of the material engagement that comprises numerical cognition, differentiated by the material forms and the specific behaviors needed to manipulate them. Importantly, the contrasts between the two technologies in terms of their affordances (e.g., manipulability, integrity, etc.) may have served to illuminate aspects of numerical identity and relations between numbers, thereby facilitating their explication and elaboration.

For example, the concurrent use of tokens and impressions would have allowed the result of a calculation to be represented two ways: as instantiated by the tokens used to perform the calculation and as impressed into the surface of a clay tablet to record the result. The latter not only had a greater integrity as a set of marks (relative to a collection of loose tokens), they provided a numerical representation that could be compared to that of the tokens. That is, while tokens could easily be separated into various subgroup combinations (e.g., eight tokens might be separated into two groups of four, or three groups of three, three, and two, etc.), their cardinality as a total set could only be remembered until the point they were reunited (at which point, the subgroups would be lost). By comparison, while numerical notations could not easily be separated, they would have provided a whole-set reference for any subgrouping of tokens. The comparison of manipulable tokens and fixed notations representing the same total number would have

[714] Reynolds 1993; Stone 1972.

enabled the visualization and apprehension of whole–part relations between the two, allowing numerical relations to be explicated and elaborated beyond the level they had reached with token-based accounting. Like an abacus does, tokens would have enabled "the person doing the sums to visualise, summarise and note the processes of the calculation in a way that the Western [algorithmic and written] style perhaps does not; [tokens would have permitted] … a visual, and in this case tactile, representation of the computation."[715]

If impressions were only a little more concise in representing numerical information than tokens had been, they nonetheless would have had integrity as fixed marks on single artifacts that loose tokens, even when contained in bullae or envelopes, had lacked. In addition, the production of impressions on the surfaces of tablets involved movements that were distinct from all those that had preceded them: shaping the cones, spheres, disks, cylinders, and tetrahedra used as numerical tokens, making notches in the bones used as tallies, and rearranging sets of objects to compare their cardinality with that of the fingers. If the productive movements were not yet handwriting, they were nonetheless more similar to it than token-shaping had been.

## 9.3 The Separation of Numerical Notations and Commodity Information with the Invention of Writing (Uruk IV Period)

Within mere centuries of the onset of making impressions that resembled tokens in representing numerical and commodity information, ANE peoples began using pictographs and ideographs to identify commodities (Respectively, these are "figurative and conventional drawings intended to have a communicative function"[716]; in Peircian terms, pictographs are iconic and physically resemble their referents; ideographs are symbolic, so their connection to their referents is arbitrary and depends on social

[715] Pellatt 2007, 19.
[716] Glassner 2000, 84.

agreement). Single picto-/ideographic signs were incised with the sharp end of a stylus and applied as labels to numerical signs made on clay tablets used for accounting purposes.[717] The numerical signs initially continued to be the same metrologically informed combinations of shapes and sizes familiar in token-based accounting, made by impressing objects or the blunt end of a stylus into the surfaces of clay tablets and repeated to indicate quantity (Figure 9.4).[718] This separation of numerical and commodity information—and the writing that enabled it—had several critical outcomes. First, it allowed numerical information to be represented without the complication of commodity information, simplifying the visualization of whole–part relations and allowing attributes like the numerical base to be made explicit and organized in ways that would begin to crystallize as a system of place value by the Old Akkadian period (*ca*. 2350–2250 BC).[719]



*Figure 9.4*. Obverse face of an administrative tablet (W 6066,a; CDLI no. P000742) from the city of Uruk, dated to the Uruk IV period and at one time held at the National Museum of Iraq, Baghdad (image courtesy of the CDLI). The tablet shows the separation of numerical and commodity information as different types of signs organized into cells demarked by horizontal and vertical lines. For example, the cell at the top of the left column contains two $N_{14}$ signs (small circular impressions) and the sign for sheep (a crossed circle); the cell directly below it contains two $N_{14}$ signs and a sign for a vessel or bowl.

[717] Powell 1971. Nissen (1986) noted that Uruk IV artifacts are distinguished from those of the Uruk V period by paleographic methods (as well as context and dating).
[718] Powell 1971.
[719] Robson 2007; 2008. The system of place value appears to have been based on Sexagesimal System S, with its base of 6 and subbase of 10. It functioned "to ease movement between one metrological system and another" (Robson 2008, 16); thus, the structuring of the earlier metrological units appears to have remained influential. The "earliest known firmly dated example of the use of sexagesimal numbers in place value notation" is a mathematical tablet (YBC 1793) dated to the Ur III period (2100–2000 BC) held in the Yale Babylonian Collection (Friberg 2005a; Keiser 1919; Powell 1976).




*Figure 9.5*. Table of reciprocals of 60. (Left) Obverse face of a mathematical tablet in the Sumerian language (DV 14645; CDLI no. P211991), provenience uncertain, dated to the Old Babylonian period; image courtesy of The State Hermitage Museum, St. Petersburg. (Right) The tablet contains a table of reciprocals of 60, numerical relations that comprise a functional multiplication and division (for example, the reciprocal of 2 is 30 because 2 × 30 = 60, 60 ÷ 2 = 30, and 60 ÷ 30 = 2; the reciprocal of 3 is 20; and so on). The numbers have been transcribed into base 60; 7,30 is 7.5 in decimal.[720] The notation for 18 is expressed '20 $la_2$ 2', or '20 minus 2',[721] an instance of 'backward counting',[722] in which a productive magnitude (e.g., '20') is adjusted by decrementing a subitizable amount ('2'). Backward counting was famously used by the Romans (as in IV, 'one from five' for 'four', and IX, 'one from ten' for 'nine').

Second, the use of writing for commodity labels along with impressing for numerical signs started to give numerical notations a concision that would eventually allow for the representation of increasingly large volumes of numerical information: for example, information about length and area measures (2999–2000 BC), reciprocals (2050–

[720] I.e., 7;30 is understood as 7 and $^{30}/_{60}$, which is equal to 7 and ½. It would be expressed exponentially as $(7 \times 60^{0}) + (30 \times 60^{-1}) = (7 \times 1) + (30 \times {}^{1}/_{60}) = 7.5$.
[721] Friberg 2009.
[722] Menninger 1992.

1500 BC; see Figure 9.5), and powers of the integers (2000–1600 BC).[723] The increased accessibility of numerical relations made possible by writing and tables, along with the mandate to recreate and learn the tables as part of scribal training as it became increasingly formalized,[724] would have helped facilitate the conception of numbers as entities. Understanding how writing would have helped numbers to become conceptualized as entities requires a discussion of literacy, including the effects of handwriting and the training of object-recognition processing, and how these informed the elaboration of the number system.

## 9.4 Literacy and Numerical Elaboration

As previously mentioned, the fact that the great mathematical traditions of the ancient world—Mesopotamia, Egypt, China, and Mesoamerica—appeared and flourished in conjunction with the invention of writing cannot be merely coincidental: The co-occurrence of mathematical elaboration and literacy suggests that there is a causal relation between the two. Here it is suggested that it is not just the freeing of quantity representation from commodity that helps realize the achievement of a 'pure' mathematics, at least as far as this had been achieved by the Old Babylonian period (i.e., while unparalleled heights of complexity in calculation methods were realized, the nature of mathematical inquiry remained closely aligned with practical applications).

The 'pure' mathematics practiced in Mesopotamia contrasts with that of the later Greeks who inherited much of their methods.[725] By the Old Babylonian period, Mesopotamian mathematics had become 'pure' "*in substance*" in performing calculations based on quantities and measurements that could not exist in actuality, while remaining "*applied in form*"* because the calculations, however artificial, pertained to matters like

[723] Robson 2007; 2008.
[724] Robson 2003.
[725] Høyrup 1994.
* Høyrup 1994, 8; italics in original.

military construction, agricultural yield, and the positions and movements of celestial bodies. In comparison, Greek mathematicians not only applied "geometry to astronomy … [but also dealt] with spheres in general …. Greek mathematics had become pure in form, i.e., fully abstract, even when it was applied in substance."[726] A more severe characterization states,

> Up to 500 B.C. or thereabout, mathematics was indeed the study of number. This was the period of Egyptian and Babylonian mathematics. In those civilizations, mathematics consisted almost solely of arithmetic. It was largely utilitarian and very much of a 'cook-book' variety. ('Do such and such to a number and you will get the answer.') … [It] was only with the Greeks that mathematics came into being as an area of study, and ceased being a collection of techniques for measuring, counting, and accounting.[727]

As will be argued here, literacy helped achieve a 'pure' mathematics through the effects of handwriting and the training of object-recognition processing in the brain,[728] and by providing an ability to describe and record algorithmic operations and processes. However,

> Having some kind of written numbering system, and using that system to count, as the Sumerians did, is one thing; recognizing a concept of number and investigating the properties of numbers—developing a 'science' of numbers—is quite another. This latter development came much later, when people first began to carry out intellectual investigations of the kind that we would now classify as 'science'.[729]

Arguably, the Sumerians both recognized a concept of number (though it changed considerably over time) and had begun to investigate its properties (as attested by their development of an elaborated mathematics). Whether this investigation rose to the level of 'science' is a matter of opinion. The Greek mathematical tradition is considered to have been scientific, as it pursued knowledge "*systematically* and *for its own sake* (or at least

[726] Høyrup 1994, 8.
[727] Devlin 2003.
[728] Overmann 2016a. The analysis of the development of literacy focused on the material record of script change in the ANE, which has an extent and duration spanning the initial 'this-means-that' associations of the late fourth millennium to the kind of literacy involved in reading the text on this page, which appears to have been realized by the Old Babylonian period, a timeline of about 1500 years.
[729] Devlin 2003, 12.

without any intentions of application) *beyond the level of everyday knowledge*"; by comparison, the Mesopotamian mathematical tradition to have been subscientific in consisting of "*specialists' knowledge* that (at least as a corpus) is acquired and transmitted *in view of its applicability*."[730]

Acquiring literacy in the modern age involves writing by hand enough time per day and enough days over time that specific neural reorganization occurs (it also assumes the availability of a script that has developed the requisite fidelity to language[731]). In reaction to the behavioral–material engagement implicit in producing characters through handwriting and viewing them visually, there are several so-called handwriting effects. First, hand–eye coordination is improved in a way that is specific to handwriting (as contrasted with other skilled hand movements, like flintknapping or playing a musical instrument). The part of the brain that recognizes characters (the fusiform gyrus, about which more will be said below) learns to interact with the part of the brain that controls handwriting movement (Exner's Area),[732] providing "a core recognition of the gesture in the written word."[733] There are also improvements in the abilities to recognize characters despite ambiguity of form and recall written material.[734] The neural reorganizations attendant upon handwriting are likely responsible for changes in scripts that occur over time, including the greater abstractness of character form (changes suggesting improvements to "character individuation and differentiation" and "relaxation of the need to preserve the original iconic form"[735]) and the development of cursive (a script

[730] Høyrup 1994, 25; emphasis in original.
[731] Overmann 2016a.
[732] Pegado et al. 2014.
[733] Stanislas Dehaene, quoted in Konnikova 2014.
[734] James et al. 2012; Longcamp et al. 2005; Mueller et al. 2014; Sülzenbrück et al. 2011.
[735] Overmann 2016a, 298. Such change is apparent in sign chronologies like the one depicted in Figure 9.6.

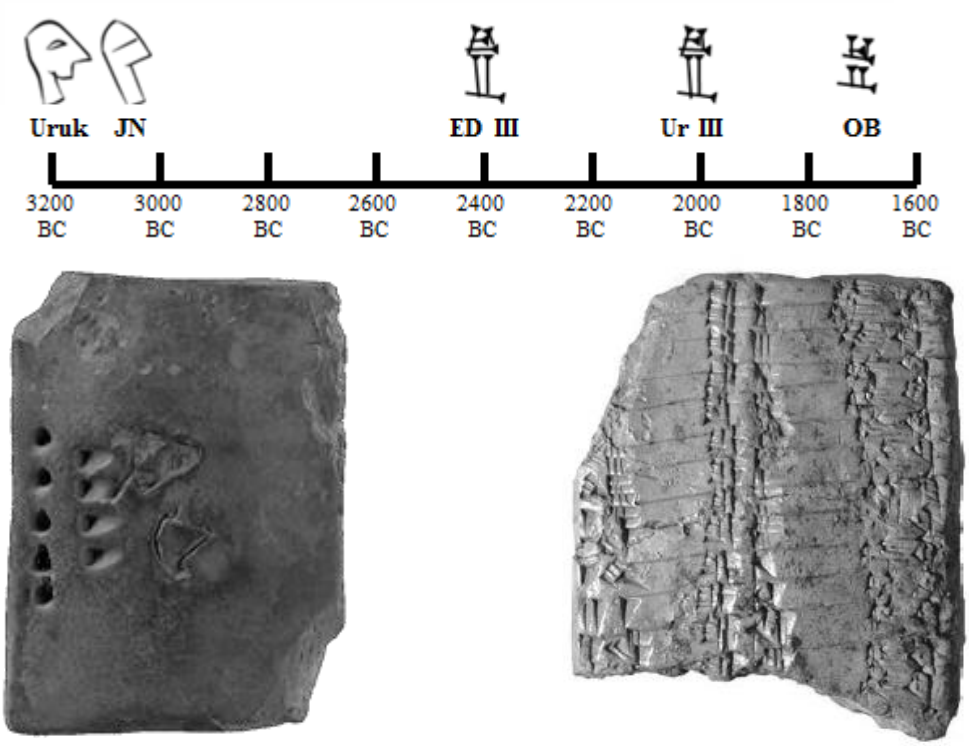



*Figure 9.6*. Comparison of change in numerical and non-numerical signs. (Top) Chronology of the sign SAG showing greater abstractness over time.[736] The sign transitions from an icon recognizable without literacy to an abstract symbol whose differentiation from other symbols would require the training and neural reorganization involved in literacy. Key: JN = Uruk III/Jemdet Nasr, 3200–3000 BC; ED = Early Dynastic III 2700–2340 BC; OB = Old Babylonian, 1900–1600 BC. (Bottom, left) Reverse face of an accounting tablet (W 7227,a/VAT 15002; CDLI no. P000958), from the city of Uruk dated to the Uruk IV period (source: © Staatliche Museen zu Berline Vorderasiatisches Museum, Foto: Olaf M. Teßmer). The proto-cuneiform numerical component consists of five $N_{14}$ and four $N_1$ impressions. (Bottom, right) Obverse face of a mathematical tablet (UM 55-21-356; CDLI no. P257395), from the city of Nippur dated to the Old Babylonian period (image courtesy of the Penn Museum). The tablet is an inverse list of squares; the sixth line is the first one with an unbroken initial number and reads "36 is the square of 6."[737] When these numerical notations are compared to those of the Uruk IV period (left), they are handwritten (rather than impressed) as integral signs (rather than independent repetitions), qualities that suggest that proto-cuneiform numbers were conceptualized as collections, cuneiform numbers as entities. However, since both consist of repeated elements whose meaning derives from element quantity, numerical representations have a consistency across time that contrasts significantly with that of the non-numerical language shown above.

characterized by "abbreviated signs, crowded writing, and unclear sign boundaries,"[738] "characteristics indicative of greater tolerance of ambiguity in recognizing signs"[739]).

[736] Redrawn from Nissen et al. 1993.
[737] Robson 2002, 360.
[738] Veldhuis 2011, 72.
[739] Overmann 2016a, 299.

Change in the fusiform gyrus is also key in literacy. The fusiform gyrus is a part of the temporal lobe that recognizes faces and objects through a combination of their local and global features.[740] In literacy, it becomes trained to recognize characters in the same way, and once trained, the region is known as the Visual Word Form Area. Its activation in literacy is constant across writing systems, languages, cultures, and individuals,[741] though there are differences in related brain activity associated with characteristics of writing systems like their visual complexity (e.g., as in the letters of the Roman alphabet relative to Chinese kanji characters; the former are generally comprised of three or fewer strokes, while the latter has many more than three[742] and is associated with differences in neural activity related to the increased demands placed on phonological processing and working memory by the visual complexity of the characters[743]). The literate brain is also characterized by interaction between the Visual Word Form Area, the regions involved in producing and comprehending speech (Broca's and Wernicke's Areas), and Exner's Area for handwriting.[744]

Simply, as the fusiform gyrus becomes trained, it learns to recognize written characters through increasingly subtle differences. This in turn allows written characters, over time (i.e., centuries), to become more abstract compared to their original picto-/ideographic forms and more similar relative to one another.[745] Such change in written signs for non-numerical language is significant, as easily seen when signs for a particular

[740] Dehaene et al. 2007; 2011; James et al. 2012; McCandliss et al. 2003; but also see Price et al. 2003. Words are recognized, as objects are, through the cumulative features of their elements (Adelman et al. 2010). Dehaene and Cohen (2004) use the term 'neuronal recycling' to describe the co-option of an existing brain function like object recognition for a cultural invention like reading.
[741] Bolger et al. 2005; Carreiras et al. 2007; Frost 2012.
[742] Changizi et al. 2005; Coltheart 2014.
[743] Cantlon et al. 2007; Tan et al. 2005.
[744] Pegado et al. 2014.
[745] For an example of greater abstractness and relative similarity, see Figure 9.6 (top). Greater similarity relative to one another is the difference between original picto-/ideograpic images for eating and grain (e.g., a head and a vessel), where the differences between the signs are significant, and literacy is not required for their differentiation, and the descendent cuneiform signs for the same words, where the differences between signs are subtle and require specific training for their differentiation (Overmann 2016a).

word are compared across time (see Figure 9.6, top). While there is obvious change in numerical signs across a similar span of time, it is less dramatic than that of non-numerical language (Figure 9.6, bottom). Some of the difference relates to the fact that non-numerical language as represented through images is ambiguous to a degree that numerical information is not; as a result, non-numerical language is under a pressure to reduce ambiguity through greater specificity (i.e., by providing semantic and phonetic clues to the words intended) that has no counterpart in numbers (because they are fully meaningful without such semantic and phonetic specificity).[746] Thus, while the representation of number transforms from collections of objects produced by repeated impressions (as in the earliest numerical impressions; the idea is extensible in principle to the motions involved in producing tokens or tallies, etc.) to integral signs produced by handwriting and conceptualized as numerical entities, the representation itself remains composed of repeated elements whose quantity is numerically meaningful.

In interacting with the perceptual system for quantity (and an increasingly linearized mental number line), written representations of numbers are sufficiently meaningful without further elaboration toward greater specificity. This allows numerical notations to represent numbers through the quantity of notational elements (e.g., fingers, tallies, tokens, proto-cuneiform notations, and cuneiform numbers) and to remain non-glottographic (i.e., fully meaningful without representing phonetic clues to morphemes and phonemes[747]) when represented in symbolic form. In processing symbolic numbers presented visually, it has been suggested that the neurons in the IPS that habituate to quantity stimuli presented in non-symbolic form also learn to respond to quantity

[746] Overmann 2016a. This difference is why numbers are translinguistic.
[747] Hyman 2006; Gelb 1980; Powell 2009; Sampson 1999.

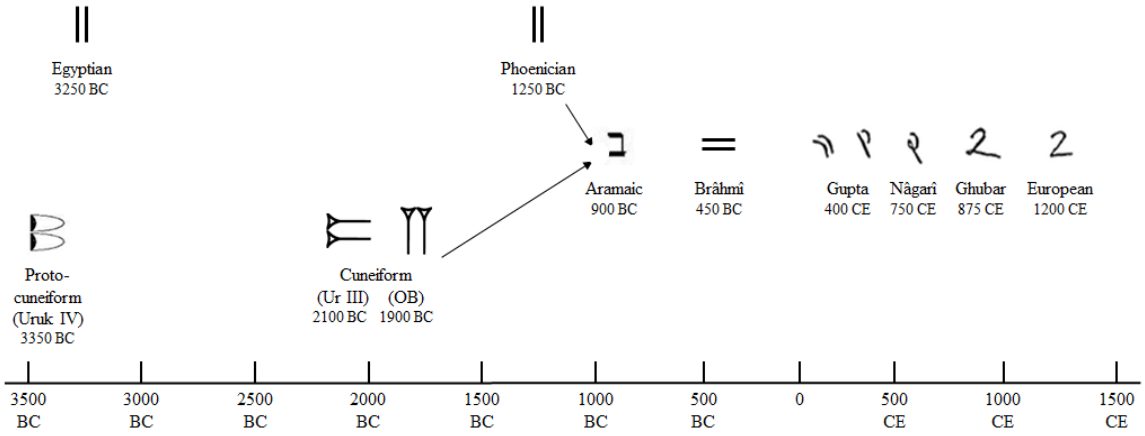



*Figure 9.7.* Script chronology for the modern Western (Hindu-Arabic) numeral '2', a cursive number sign associated with the use of algorithmic calculation methods in Europe.[748] The antecedent numerical notations can be traced back to two original (invented) writing systems, Mesopotamian proto-cuneiform and Egyptian hieroglyphics, that both informed Aramaic numerals.[749] Overall, the transition is marked by a high degree of form conservation in the first several thousand years (3350 to 450 BC) and an increasing abstractness thereafter (400 to 1200 CE). While the reason(s) for this would require future research, the increase in abstractness likely represents the use of different materials in writing (e.g., the surface characteristics of clay necessitate the use of more linear strokes, a constraint that is not inherent in the use of materials like ink on surfaces like papyrus, parchment, and paper[750]), as well as the increased tolerance for ambiguity (a handwriting effect) that facilitates the development of cursive scripts (i.e., scripts characterized by high ambiguity).

presented in symbolic form,[751] providing a neural basis (in conjunction with the training of object-recognition processing) for recognizing written numerals as holistic patterns or objects rather than combinations of elements, a phenomenon that informs the development of ciphered notations[752] (i.e., notations that represent different multiples of a power with different number signs, as the digits 0 through 9 represent the range of multiples of $10^n$, as for example $214 = [2 \times 10^2] + [1 \times 10^1] + [4 \times 10^0]$). There are other differences in brain activity when processing numerical information relative to non-numerical language, as for example the activity of the angular gyrus in recalling arithmetic facts and manipulating

[748] Ifrah 2000.
[749] Information compiled and images adapted from Chrisomalis 2010; Ifrah 2000; CDLI; Pennsylvania Sumerian Dictionary.
[750] Jackson 1981.
[751] Dehaene 2007.
[752] Chrisomalis 2010.

numbers in verbal form,[753] and activity in the IPS both during judgments of semantic distance between number words (symbolic) and numerical distance (non-symbolic).[754] The numerical meaningfulness of element quantity in the visual representation of number (a quality of the material sign, which represents quantity by instantiating it; see Chapter 2) thus enables number signs to remain more constant over time, relative to signs for non-numerical words, a tendency that is displayed in the relative continuity of number sign form over time in the ANE proto-/cuneiform script (Figure 9.7).

Though anchored in psychological processes that allow the form of number signs to remain relatively unchanged across time, number signs also operate as visual objects in much the same way that signs for non-numerical words do in a system of literacy.[755] That is, their written forms are recognized through a combination of local and global elements that allows for a range of abstractness, both in the number of elements and form: For example, the numeral '2' can be written as two lines, vertically as in the Roman numeral II or horizontally as in the Chinese numeral 二, forms that instantiate the represented quantity. The numeral '2' can also be represented by curves and circles, as in the Telugu (India) numeral ౨, and Thai ๒, or with complex combinations, as in the Tamil (India and Sri Lanka) ௨, Korean 이, and Ge'ez (Ethiopia) ፪.

The relative imperviousness to change in the form of written numerals suggests that the elaboration of counting sequences and the token-based accounting system into

[753] Dehaene et al. 2003; Grabner et al. 2007; Grabner, Ansari, et al. 2009; Grabner, Ischebeck, et al. 2009; Ramachandran 2004.
[754] Lussier et al. 2016.
[755] As noted in (Overmann 2016a), there are several additional differences in the visual processing of signs for numbers and non-numerical language: Numerals need not be written, but literacy does not exist without writing. Complex numbers become more accessible if written with some form of separation (e.g., separators like commas or periods, as used in Western numbers; differences of digit representation, as in proto-/cuneiform, where digits are represented by $N_1$ or 𒐕, tens by $N_{14}$ or 𒌋), while the intelligibility of words is degraded by the same means. In addition, object-recognition processing in words adds some error protection (i.e., the phenomenon that makes it difficult to spot typographical errors while proofreading) that in processing numbers yields mistakes (because interpreting one digit as another yields the wrong number) (see Rayner et al. 2006).

mathematics depended on more than conceptualizing numbers as entities; recognizing numbers as written signs (handwriting effects and object-recognition processing); and an increased elaboration and availability of numerical relations (previously discussed). The additional necessary condition is suggested by the co-occurrence of the height of mathematical elaboration and the achievement of true literacy in the Old Babylonian period.[756] Old Babylonian mathematical developments included arithmetic progressions; multiplication and division by reciprocals; powers of the integers; squares, inverse squares, diagonals of squares, and cubes; properties of lengths and areas; quadratic equations for squares, circles, circumferences, areas, diameters, right triangles, and rectangles; and a type of algebra that enabled the manipulation of areas and lines to find unknown sides of squares.[757] The number system had become cumulative–positional[758]; mathematicians had developed the sexagesimal place value system as a way of integrating the different metrological systems.[759]

The coincidence of these developments with the achievement of literacy is not necessarily expected, given that the roots of numeracy extend perhaps as far back as the Upper Paleolithic, and writing was not invented until the late fourth millennium. Numeracy persisted and elaborated across significant changes in the materiality used for realizing, representing, and manipulating numbers. Of these multiple technologies, if numerical notations were associated with the greatest numerical elaboration, they nonetheless only extended number concepts that were not much different than they were when initially realized with the fingers. By comparison, the development of scripts capable of expressing language with fidelity and the psychological change that is literacy were unprecedented. In other words, language became manuovisual for the first time,

[756] Overmann 2016a.
[757] Robson 2007.
[758] Chrisomalis 2010.
[759] Høyrup 1994; Lewy 1949; Seidenberg 1965.

something numbers had always been. Yet, both literacy and numeracy reached an elaborational epiphany at the same time, implying a causal relation.

Certainly, the Old Babylonian period was typified by more mathematical elaboration than had been achieved in previous time periods or would be in succeeding time periods by the Babylonians themselves, as a comparison of the relative attainments clearly reveals.[760] However, rather than inventing new concepts or techniques, Old Babylonian mathematicians were merely doing what practitioners in other fields like divination were doing: "borrowing from oral practices and putting [previously developed ideas and techniques] into order,"[761] applying the new ability and tool—literacy and literate writing—to extending the range of elaboration in numbers as they already knew them. Mathematics was not a discipline "of autonomous thought, [as it had been for the Greeks, as this] had not been known to the Babylonians."[762] Here it is suggested that the causal relation enabling mathematical elaboration was partly technological (e.g., the ability to manipulate and represent wholes and parts made possible by the concurrent use of tokens and notations, as previously discussed) and partly linguistic (i.e., the ability to represent arithmetical and mathematical operations through a combination of numerical notations and signs for non-numerical language).

The latter is not insignificant, as implicit to the modern Western notion of number is the idea that it is something that can undergo operations like addition and division without changing its essential nature: Numbers are things that are related to other like entities not only by numerical relations (e.g., as 2 and 3 are related to 5) but by the operations that can be performed upon those relations (as both $2 + 3$ and $3 + 2 = 5$; $5 - 2 = 3$ and $5 - 3 = 2$, and so on; that is, the relations between 2, 3, and 5 can be manipulated by addition and subtraction). Implicit to the concept of such operations is the

[760] Robson 2007; 2008.
[761] Høyrup 2011, 11.
[762] Høyrup 2011, 17.

idea that numbers can be manipulated by them without changing them in the process: A 'two' to which a 'three' has been added produces a 'five' without any of them becoming different than the numbers they initially were. This, however, is a modern concept of number, one that did not apply to numbers as they were understood in the ANE, even during the mathematical heights achieved in the Old Babylonian period.

In Old Babylonian mathematics, some generalization of the arithmetical operations had occurred, but by no means had it converged on the arithmetical operations in the way they are currently conceptualized in the Western tradition. There were many differences: The Babylonians probably did not regard areas and lengths as numbers; even the later Greek mathematicians, who inherited many concepts from the Babylonians, did not consider areas and lengths to be numbers because such calculations could yield so-called irrational numbers (e.g., √2, which is irrational because it neither terminates nor repeats[763]). ANE integers were not separated from fractions; rather, there was a 'floating' decimal point; there was also no zero, though the practice of leaving a blank space to indicate places without value would be incorporated into the sexagesimal place value system.[764] Before "the inventions of a zero symbol and a separation point, additive ... Sumerian cuneiform [numbers] were easier to read and probably led to fewer mistakes than positional Babylonian cuneiform."[765]

Even simple arithmetic differed. For example, there were two types of addition: In the first, one addend was "absorbed into the other, which so to speak conserve[d] its identity while increasing in magnitude" (Høyrup gives the example of adding interest to a bank account, addition that changes the account's monetary value but not its

[763] Gowers 2008.
[764] Neugebauer 1941.
[765] Rudman 2010, 50.

ownership).[766] In the second, "add[ing] the measuring numbers of two or more addends … seem[ed] to refer to the sum as compounded from still identifiable constituents," a procedure associated with calculating lengths and areas.[767] There were also two subtractions and four different kinds of multiplication, including one that would be considered division in the Western tradition (i.e., multiplication by a number's reciprocal; see Figure 9.5).[768] Significantly, operations were not written as mathematical symbols (e.g., '+', '×', etc.), but as logographic and glottographic symbols for words. For the identity-preserving form of addition, the Sumerogram daḫ and Akkadian term *wasābum* were used and are translated as 'to append'; for the second form of addition, the Sumerian ğar.ğar, word and sign UL.GAR, and Akkadian *kamārum* were used and are translated as 'to accumulate'.[769] This has three implications for the ANE concept of number (discussed using the example of addition but extensible to concepts of subtraction, etc.).

First, while word and signs like daḫ and UL.GAR represented semantic concepts of addition without phonetic value, they were relatively complex signs (i.e., compared to more concise symbols like '+'). This complexity would have imposed demands in reproducing and processing the symbol (i.e., writing it by hand and processing it in visual working memory). Glottographic representations of words developed from picto-/ideograms, icons and symbols that were ambiguous regarding semantic and phonetic value. The trend for written non-numerical language, toward the development of true literacy in the Old Babylonian period,[770] was toward greater expressiveness, something that required increasing specification of both semantic and phonetic value. Thus, even if the trend in writing numerical notations was toward greater concision and compactness

[766] Høyrup 2002b, 19.
[767] Høyrup 2002b, 19. In the arithmetical operation of addition, addends are numbers added together to produce a sum.
[768] Høyrup 2002b.
[769] Høyrup 2002b.
[770] Overmann 2016a.

with a basic conservation of the representational form (i.e., the instantiated quantity), the prevailing trend in the writing system was toward expressiveness gained by specifying additional clues to linguistic value, something that added complexity to symbols for individual words. As non-numerical signs gained greater expressiveness for language, they also functioned linguistically, thus accessing the cognitive system for numbers through the "slow and hesitant"[771] route of language, rather than as concise manuovisual stimuli.

Second, the various words and signs in Sumerian and Akkadian (daḫ, *wasābum*, ğar.ğar, UL.GAR, and *kamārum*) represented multiple additions—*appending* and *accumulating*—rather than the unitary concept of *addition* conveyed by the symbol '+' as used in the modern Western mathematical tradition. However, the conceptual difference between appending/accumulating and adding extends beyond the relational manipulation, though the Babylonian mathematicians were clearly thinking about how the relations between numbers might be manipulated. Also implied is a difference in the way numbers were conceptualized in the ANE, relative to the way they have come to be conceptualized in the Western tradition: A concept of number that sometimes keeps its identity under the condition of accumulation and which sometimes does not differs fundamentally from one that makes no such distinction. Accounts of ANE number concepts that seemingly assume a modern notion of number followed once the representation of quantity was separated from that of commodity fail to recognize that what was realized in actuality was a number concept that differed in several respects from the modern one.

Third, the differences in the concepts of numbers and their relational manipulations reveal the material roots of numerical concepts in the ANE system (e.g., as the operation of appending evokes comingling tokens of the same numerical value, division by reciprocals suggests rearranging tokens into different groups, and the method of

[771] Sfard et al. 1994, 198.

calculating the area of a field implies its measurement with rods and ropes[772]). The overt materiality presents a different argument against the 'concrete–abstract' distinction, which is the observation that even at its most elaborated point (the Old Babylonian period), ANE numerical and mathematical concepts were still very much bound to the material forms that represented them and enabled their manipulation. However, rather than implying that ANE numerical concepts were still more 'concrete' and less 'abstract' than their modern Western counterparts, it suggests that the role of materiality may be differently visible across systems, an idea that is explored in the final chapters.

[772] Høyrup (2002b) describes the method of calculating the area of a field in the Old Babylonian period as involving a 'confrontation' (a measurement with width and no depth) in which the idea of measuring fields by means of rods and ropes is palpable. Notional pieces of the field were then moved around until an area had been calculated.

# CHAPTER 10

# THE ROLE OF MATERIALITY IN NUMERICAL COGNITION IN THE ANE

## 10.1 Materiality in Numerical Cognition

As was discussed in Chapter 2, Material Engagement Theory views materiality as a constitutive component of cognition with semiotic qualities distinct from those of language and agency distinct from that of brains and bodies.[773] The present chapter examines the role of materiality in ANE numerical cognition to answer such questions as whether and to what extent materiality is integral to numerical cognition in general, and what this might imply about numerical cognition in the ANE; how materiality becomes meaningful as number concepts through the perceptual experience of quantity and the active manipulation of material forms, the role of language, and how this might have occurred in the ANE; and how materiality may shape numerical cognition through the agency and affordances of material devices like the fingers, tallies, tokens, and numerical notations used in the ANE. The discussion of material agency includes ideas about conceptual and psychological change, the persistence of material structure across changes in technology, and the distribution of concepts across multiple material forms.

[773] Malafouris 2013. As discussed in Chapter 2, the three central tenets of Material Engagement Theory are *cognition is extended* beyond the brain and body to include materiality; semiotic meaningfulness (the *enactive sign*) derives from what materiality is and what humans do with it; and *materiality has agency* in shaping human behavior and brain function and form over developmental and evolutionary spans of time.

## 10.2 Materiality as Constitutive of Numerical Cognition

Just as it is difficult to envision reading the words on this page without the materiality of writing,[774] it is difficult to envision reasoning with numbers without some form of materiality to realize, represent, and manipulate them. Number concepts result through the interaction of psychological, behavioral, and material capabilities: the ability to perceive quantity, the cardinality of sets of material objects, the comparison of set cardinalities, the categorization and abstraction of concepts of shared cardinalities, and the creation and use of material forms that express and manipulate the concepts. Absent any one of these psychological, behavioral, or material components, numerical cognition would likely not exist. Granting that it is possible to perform at least simple calculations 'in the head', such mental operations nonetheless rely upon the prior existence of explicit concepts of numbers, a process of conceptual realization that necessarily involves materiality (as has been described in previous chapters).

The idea that materiality is constitutive of numerical cognition must be viewed not only in the moment of calculation itself but also over the developmental and evolutionary spans of time appropriate for discerning changes in brain function and form, as well as material forms that accumulate cognitive effort and stimulate change in the other components. Like literacy, numeracy originates in innate psychological responses to material stimuli that are meaningful in virtue of what they are and which become differently meaningful when they are manipulated into new forms: Scripts, which provide sufficient semantic and phonetic clues to a lexicon, grammar, and syntax that written expressions are linked to a specific language,[775] develop through the active manipulation of written signs that stimulate psychological processes like object recognition. In response, psychological processing changes, thereby enabling new behaviors that in turn afford new

[774] Haas 1996; Overmann 2016a.
[775] Overmann 2016a.

material manipulations. Similarly, in numeracy, sets of material objects with cardinality interact with the innate ability to perceive quantity that humans share with many nonhuman species, with concepts of shared cardinality[776] originating in active manipulations of sets of objects that bring them into comparative arrangements and then use the sets to represent the concepts. Psychological processes like the mental number line change in response to interactions with the material forms used to realize and represent numbers, affording opportunities for new behaviors and material manipulations to emerge.

Materiality is often underappreciated as a component of numerical cognition[777] for reasons that include:

> … the Cartesian assumption that mind and matter are distinct substances, the transparency of the functioning of materiality and the body within cognition that seemingly locates cognition in the head, and neuroscience's methodological focus on the brain (Renfrew and Malafouris, 2008; Tallis, 2011).[778]

Alternatively, when their role within the cognitive system for numbers is understood as constitutive, meaningful, and agentive, material artifacts can provide critical insights into present and past ways of thinking about numbers. The material forms documented in the archaeological and textual records of the ANE, for example, imply that these ancient peoples realized, represented, manipulated, and elaborated their concepts of numbers in ways similar to those seen in extant peoples: through the use of their fingers and the incorporation of additional material devices with qualities like higher capacities and greater durability, with concomitant increases in the highest number counted, and in conjunction with developments and expansions of numerical lexicons and the social purposes to which numbers were applied. Numerical elaboration would have been facilitated by the availability of language and enabled by the adoption of multiple material forms (see Chapter 2). However, the sequence of material artifacts used in the ANE

[776] Russell's (1910; 1920) concept of what a number is was discussed in Chapter 3.
[777] Malafouris 2013.
[778] Overmann 2016d, 9.

conceptualization of number is of much greater duration than is typical, as it spans the initial realization of numbers in a pristine original context to their elaboration into a highly complex mathematical system. Analysis of the sequence suggests that ANE peoples would have realized initial concepts of numbers as equivalences and transitioned to concepts of numbers as collections and entities, consistent with change in the material forms used for representation and manipulation. In addition to change in the conceptualization of number, psychological functioning would have changed as well through the interactions with different material forms (described further below).

## 10.3 Material Semiosis in Numerical Cognition

Several linguists have argued that humans have numbers because they have language[779]: "Without language, no numeracy"[780]; "Without using language, we still can't go [higher than the subitizing range of about four]"[781]; "the number faculty is actually a by-product of ... the language faculty"[782], and

> … numeric language serves as "ground-floor conceptual scaffolding for much of human numerosity"[[783]] ... since it apparently allows children and adults to recognize quantities in an exact manner. … If exact quantity recognition for quantities greater than three relies on numeric language [as suggested by Hurford in 2007], this implies that many ancient material technologies that required such quantity recognition must have also relied indirectly on such language ...[784]

Further,

> Speculations about the origin of the mathematical capacity as an abstraction from linguistic operations are not unfamiliar. There are apparent problems,

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780 Hurford 1987, 305.

781 Hurford 2007, 91–92. Hurford's 2007 revision of his 1987 position "admits the possibility of apprehending some quantities—those at least in the subitizing range—without establishing vocabulary for them [first]" (Overmann et al. 2013, 83).

782 Fukui et al. 2004, 15.

783 Everett 2013b, 81. Overmann et al. (2013) faulted this claim by Everett on the grounds that he conflated *numerosity*, the innate ability to perceive quantity that humans share with many species, with *numeracy*, the ability to reason with numbers that depends on both material engagement and language, and he did not understand that numerosity is a psychological capability shared by all extant humans.

784 Everett 2013a, 169.

> including dissociation with lesions and diversity of localization, but the significance of such phenomena is unclear for many reasons (including the issue of possession vs. use of the capacity). There may be something to these speculations, perhaps along the lines just indicated.[785]

And,

> [It does] not seem completely implausible to suppose that at the point where the language faculty with its central property of discrete infinity emerged, the number faculty was also formed in the brain, by extracting from the language faculty the very mechanism needed for discrete infinity, and by abstracting away from various other language particular properties of the language faculty (i.e., those properties that are necessary for connecting sound and meaning). The number faculty would then be latent, but ready to be used as soon as appropriate environmental conditions are met. If this account is on the right track, it presents a case where a mental organ (the number faculty) emerged as a by-product of the emergence of another mental organ (the language faculty).[786]

Language is undoubtedly vital as a way of accessing numerical cognition[787] because it helps explicate and elaborate material semiosis (see Chapter 2) and facilitates the acquisition of number concepts by children and other novices through mechanisms such as bootstrapping and semantic mapping.[788]

> Humans 'think through concrete things, with words being a special case' that complements and explicates material engagement (Roepstorff 2008:2051), even as language assimilates material characteristics like productive grouping (patterning that emerges from 'the dynamical interactions of brain, body, and world' rather than being innate to language [Gibbs and Clark 2012:242–243]).[789]

However, the assumption of linguistic primacy in the origin of numbers ignores the semiotic meaningfulness of the quantity that material objects have in the perceptual experience of quantity.[790] Material objects "*have* and *mean* quantity in virtue of what they

[785] Chomsky 2004b, 8–9. Chomsky (2004a) also implied that peoples with language but not numbers are deficient. This was noted by Verran (2000, 292), who stated that "In Chomsky's view Australian Aborigines, having no elaborated number system, must be considered biologically less than truly human."
[786] Fukui et al. 2004, 15, quoting Chomsky in a 2004 interview.
[787] Clark 2006a; 2006b.
[788] Barner 2012; Carey 2009. As noted in Overmann (2015), acquiring number concepts through language entails the prior existence of both number concepts and lexical labels.
[789] Overmann 2016c.
[790] In Chapter 4, the perceptual experience of quantity (numerosity) was described as having two components: *subitization*, the ability to recognize cardinalities up to three or four without counting, and

are and what humans do with them; their quantity is comprehensible and communicable without language but not without materiality to instantiate and represent it,"[791] as has been well documented in the ethnographic literature (see Chapter 5). Quantity, whether in non-symbolic or symbolic form, is recognized by the perceptual system for quantity, to the extent permitted by the elaboration of the material technologies for numbers, the concepts they represent and manipulate, and an individual's enculturation into and individual aptitude for numeracy. The relation between object quantity and the perceptual experience of quantity exists before there are any words for quantity or concepts to which lexical terms for quantity might be applied. This prior existence of meaningful quantity is thought to be the case because the ability to perceive quantity is demonstrated by many nonhuman species (see Chapter 4). Alinguistic quantity perception is also demonstrated by human neonates, who at only hours to a few days old have neither acquired nor been exposed to much language.[792]

Given the cardinality of objects and a shared, alinguistic ability to perceive cardinality, initial concepts of both subitizable and non-subitizable numbers emerge through material engagement (i.e., bringing sets of objects into arrangements for comparing their cardinality; see Chapter 5) and not through language (though lexical labeling is a crucial but secondary phenomenon). For example,

> … how do discrete quantities like "seven," "eight," or "nine" emerge out of the undifferentiated *many*, the universal term for quantities higher than the subitization range? In [Coolidge et al. 2012], we suggested that this happens through the interaction of numerosity with material culture, which acts to scaffold quantities beyond the subitizing range, enabling their "crystallization" as distinct visual and tactile entities (Pica et al. 2004:503). Indeed, were it true that meaningful higher-number words were a necessary

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*magnitude appreciation*, the ability to recognize cardinalities higher than about four as 'larger' and 'smaller', as long as the difference in quantity fell above a threshold of noticeability.

[791] Overmann 2016c.

[792] Izard et al. 2009. Other experiments have demonstrated the ability to appreciate quantity in human infants up to nine months old (Barth et al. 2006; Brannon et al. 2003; Cantlon et al. 2006; Starkey et al. 1990; Van Marle et al. 2011; Xu et al. 2005). Numeracy and language are also considered to be independent based on their double dissociability as cognitive processes (see Chapter 2).

> prerequisite for developing higher-number concepts, it would be surprising if any society ever developed any.[793]

The interaction of language with material semiosis highlights an important aspect of the elaboration of number concepts, which is its dependence on having multiple ways of accessing them. Differential access can be provided by the two major systems for accessing numerical cognition—manuovisual and linguistic—as was discussed in Chapter 2 with respect to Oksapmin body-counting. Beyond these and perhaps more critical is the access provided by having multiple material forms, whose contrast serves to illuminate exploitable differences and which can inject new qualities into the conceptualization of number, structure that may persist across the incorporation of new material forms (as discussed below).

## 10.4 Material Agency in Numerical Cognition

Concepts of cardinality shared by sets of objects are similar across cultures, languages, and time as a function of "human psychological, physiological, and behavioral interactions with the same material world, [that are] motivated and conventionalized by similar social requirements and transactions."[794] However, these concepts are not identical but rather vary across number systems, primarily because of differences in the material component:

> Materiality is integral to concepts in general, where it acts to anchor and stabilize them (Hutchins 2005; Malafouris 2013). Change in material form thus affects conceptual content (as was suggested [in Chapter 5] for the equivalence–collection–entity transition). Materiality is important to number concepts in particular, where it performs functions like aiding visualization, decreasing demand on memory, and extending the manageable complexity of calculations (Dreyfus 1991; Kaput 1987).*

However, materiality does not act merely as an external support to internal computations, the neurocentric model of numerical cognition. Rather, in the extended-

[793] Overmann et al. 2013, 83.
[794] Overmann 2016c.
* Overmann 2016c.

mind model, it is a constitutive part of numerical cognition and as such influences change in the other components. As explained in Chapter 2, the term 'agency' describes the ability of materiality to influence change in brains and behaviors; materiality has agency through its affordances, or relations between material properties and the psychological and physical abilities that can exploit them.[795] As the following discussion will show, each material form used for counting provides capabilities and structure, as well as limitations that may eventually motivate the incorporation of new material forms. From the available resources in any environment, new material forms will be selected because of affordances they share and capabilities that differ from those of previous forms. New material forms will solve a limitation of a previous form while injecting new limitations of their own. Further, the structure of older material forms may persist in the way newer forms are used. All material forms anchor and stabilize number concepts,[796] making quantity perception tangible and explicit,[797] which gives number concepts structure and helps them persist in the form of artifacts and algorithms. In addition, the incorporation of new material forms enables the elaboration of number concepts because new material forms introduce new affordances, contrasts between forms, and outcome potentials.

### 10.4.1 Continuity and Discontinuity of Material Affordances in ANE Counting Technologies

Given four material counting technologies and a relative sequence in which they occurred in the ANE as reconstructed through textual and archaeological evidence, it not only becomes possible to examine what each technology provides in terms of material affordances but also to analyze and infer how each technology might have influenced the further elaboration of the system of numerical cognition. For example, older material

[795] Gibson 1977; 1979; Malafouris 2013.
[796] Hutchins 2005.
[797] Coolidge et al. 2012; Malafouris 2010b.

counting technologies influenced the structure and capacity of the cognitive system for numbers; structure in turn informed how a new material form was selected, incorporated, and used in the system; and a capacity of an incorporated material form would become a shortfall that motivated the further elaboration of the system. In the sections that follow, the four technologies used in the ANE are discussed in their order of occurrence in terms of the affordances, structure, capacities, shortfalls, and opportunities for elaboration they may have offered. A graphical representation of the continuity and discontinuity of their affordances is shown in Figure 10.1.

### 10.4.2 The Fingers

As discussed in Chapter 4, fingers are where numbers originate because the part of the brain that controls their movement and sensation (the angular gyrus) interacts with the part of the brain that perceives quantity (the IPS).[798] Fingers also manipulate enumerated objects, thereby bridging the psychological, behavioral, and material components of cognition.[799] As number concepts emerge, fingers are often the first material form used to represent them, for several reasons: first, because of the neural interaction between their sensorimotor control and the perception of quantity; second, because gesture helps express ideas (especially when the ideas are difficult to express with language, generally because they either cannot be put into words or there are no words to put them into)[800]; and third, because fingers are so readily available to be used for the representation of numbers.

The fingers of one and both hands are typically used to scaffold and represent concepts of ‘five’ and ‘ten’, often the first non-subitizable numbers to emerge (as quantity

[798] E.g., Geva et al. 2011; Orban et al. 2006; Roux et al. 2003; Seghier 2012.
[799] Gallagher 2013; Mattens 2013.
[800] Broaders et al. 2007; Cook et al. 2008; Cook et al. 2010; Goldin-Meadow et al. 2001; 2009; Rasmussen et al. 2004.

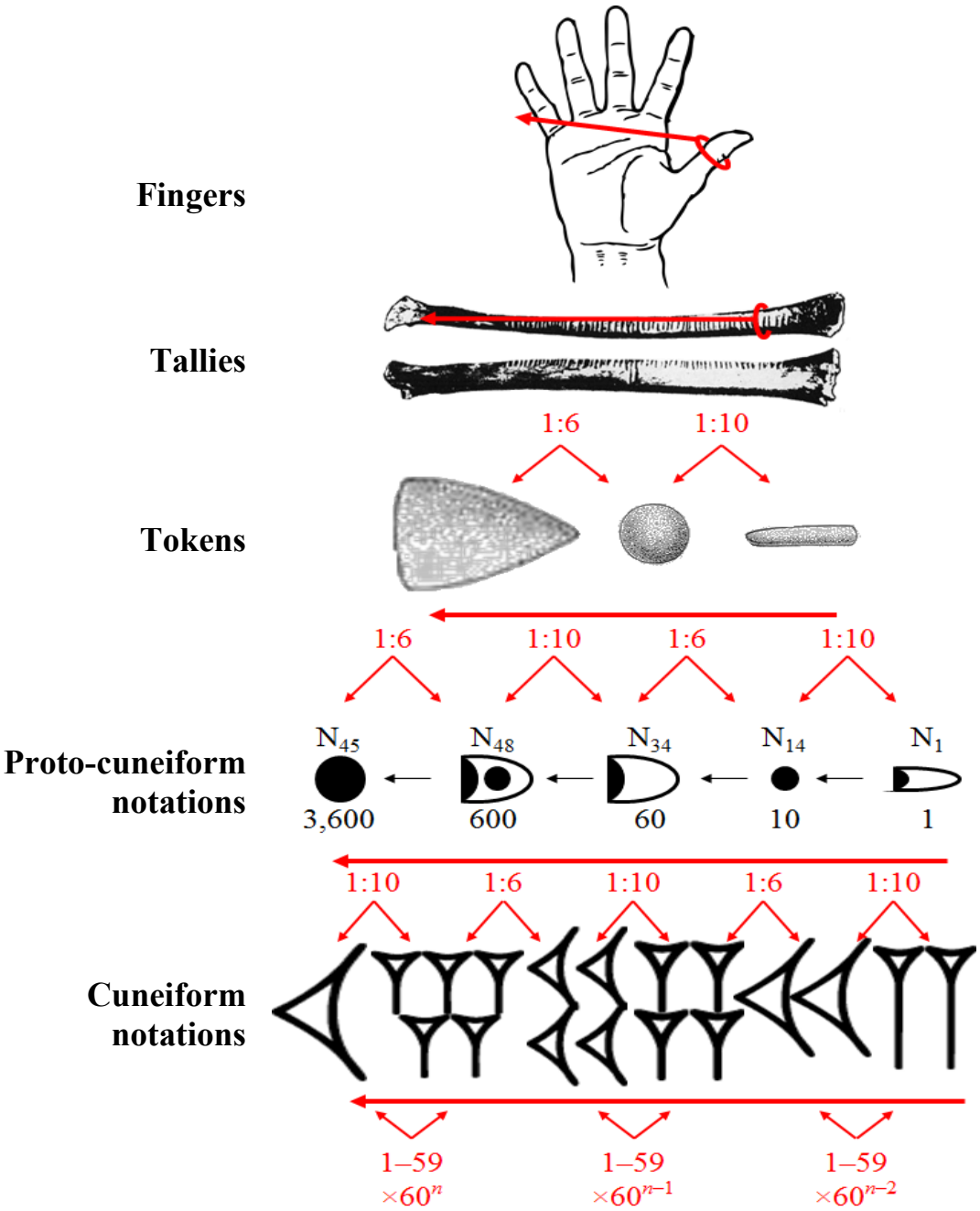



*Figure 10.1*. Continuity and discontinuity of affordances in material counting technologies.[801] Linearity and magnitude ordering are represented by long horizontal arrows, bundling by short diagonal arrows. Linear structure in the visual arrangement of numerical information and the ordering by magnitude increase is an affordance with continuity, as all the material forms have it, though it is implicit in some of them (e.g., fingers and tallies) and imposed on others (tokens and notations). Manipulability is an affordance with discontinuity, as there is relatively little in some forms (e.g., fingers, tallies, and notations) and a high degree in others (tokens); however, manipulability persists in the form of algorithms that allow numbers to be manipulated through methods like mental recalling arithmetic facts and recording interim steps in a calculation. The system for numerical cognition also gains increases in the highest number counted and representational concision as it becomes elaborated.

[801] Image of a hand from the public domain (pixabay.com); image of tally adapted from Beckmann 1971, 10; token images adapted from Englund 2006, 29; proto-cuneiform notations adapted from Nissen et al. 1993; cuneiform notations adapted from Pennsylvania Sumerian Dictionary.

equivalences between the fingers per hand and a set of objects). Given concepts of and words for 'one' through 'three', 'five', and 'ten', the intervening fingers are often used to scaffold concepts of and compound words for the gap numbers (i.e., 'four' and 'six' through 'nine'); this was described in Chapter 4 as a cross-cultural process and in Chapters 6 and 7 as being the case for Sumerian numbers. This use of the fingers introduces productive grouping that enables the formation of new quantities. In enumeration, the fingers are also used linearly in a way that informs how numbers become ordered from small to large,[802] a characteristic that helps make numerical information more accessible (i.e., as compared to random ordering). Linearity and stable order thus emerge as structure in numerical cognition out of the interactivity of its psychological, behavioral, and material components.

If fingers help realize explicit number concepts from the inchoate perception of quantity, they also do not specify what they count (a limitation), with the result that the information must be maintained separately, either in context or in memory. Quantity information maintained in these forms is particularly prone to processes like interruption and forgetting.[803] Fingers also have a relatively limited capacity for enumeration—typically five per hand[804]—and as a material form for representing quantity, they are perishable in the sense that sooner or later, the hands are needed for some other activity. These limitations may motivate the incorporation of new material forms that are more durable and capable of instantiating higher quantities. As the fingers have also introduced

[802] Gelman et al. 1978. Overmann (2014) noted the cross-cultural tendency for finger-counting systems to start with an outside finger (either the thumb or little finger) and work sequentially across the hand.
[803] Wynn et al. 2016.
[804] It is certainly possible to count to more than five on one hand. For example, the capability to count to 9999 using both hands was possible in "the earliest finger-counting system documented with sufficient granularity to enable recreation of specific finger-patterns [was] attributed to the English monk Bede (*ca*. 672–735 CE) in the early eighth century (however, some scholars have noted that Bede's finger-counting system might have been more intellectual exercise than actual custom; see Bragg 1997; Karamanolis 2010; Menninger 1992; Olsen 1982)[, and in practical use, if any, the system was probably not often used for counting beyond 100]" (Overmann 2014, 67). It should be noted that numbers like 9999 and 100 are associated with relatively mature number systems, while the present discussion of the role of fingers is more focused on emerging systems.

linear structure to the cognitive system for numbers, a new form incorporated to address limitations like their lack of capacity and durability are likely to be structured linearly. Indeed, this appears to be the case, as the ethnographic literature describes a variety of linearly structured material forms used in addition to the fingers in counting systems that are relatively new: tallies, body stripes, marks on the ground, torn leaves, knotted strings, and beads.

The fingers help realize concepts of number that are equivalences of quantity between sets of objects, as is implicit in pairing and one-to-one correspondence (see Chapter 5). Numerical systems based on finger-counting tend to count to fairly low numbers (i.e., numbers consistent with the capacity of the material form).[805]

### 10.4.3 Tallies

Tallies are a kind of technology that tends to follow finger-counting, as documented in the ethnographic literature,[806] likely because they have a great deal in common with them: Tallies share the linearity and stable order of fingers, although tallies can be more regular in their spatial increments than fingers are. Like the fingers, tallies do not specify what they count, necessitating that the information be preserved in context or memory, though the quantity component of the information is much less perishable compared to the fingers. Tallies also solve the problems of capacity and persistence that

[805] Overmann 2013a; 2013b; 2014.

[806] Other technologies in this category are knotted strings, beads, torn leaves, body stripes (e.g., such as those made with ochre), and marks on the ground. Like tallies, these share with fingers characteristics like linearity and stable order, are similarly formed through one-to-one correspondence, do not denote what they count, and scaffold the development of concepts of and linguistic terms for numbers (i.e., do not require preexisting numerical concepts or language). Unlike tallies, these technologies may be made of less durable materials and thus may lack tallies' persistence. Collections of loose objects used for counting (e.g., stones, grain) lack the same degree of linearity and stable order but are similarly formed, do not denote or require preexisting numerical concepts, etc.

fingers have, and the motor activity of making notches may enable numerical concepts like accumulation and relations like plus-one to be realized.[807]

Because of the subitization constraint on the perceptual experience of quantity, however, it becomes increasingly difficult to distinguish how many notches there are on a tally whenever there are more than three or four of them. This creates the need to differentiate the notches in some fashion, usually by grouping or separating them into subitizable or recognizable amounts (an example of the latter is the familiar tally grouping in which a single stroke is written, either horizontally or diagonally, across four vertical strokes: 𝍸). Tallies are also not manipulable: It is difficult to move or subtract a notch once it has been made (though notches can of course be grouped or separated as they are made, bundling that may suggest that the number system has become elaborated beyond one-to-one correspondence). This lack of manipulability is a factor in the way numbers and operations are conceptualized in a cognitive system for numbers based on the tally: Numbers are unlikely to have many explicit relations between them (i.e., beyond those noted in Chapter 8: *greater than*, *less than*, etc.), and the most likely operation to be present in the system is accumulation.

Tallies represent the incorporation of materiality into the cognitive system, as well as the transition to the use of materiality as a collaborative medium. They would also have been associated with a change in the way numbers were conceptualized, from *equivalences* between sets of objects to a *collection* of notches used to enumerate sets of objects. Numerical systems based on tallies count to numbers higher than those associated with finger-based systems, as is consistent with the greater capacity of the material form,[808] allowing for an increase in the associated numerical lexicon.

[807] Overmann 2016d.
[808] Overmann 2013a; 2013b.

### 10.4.4 Tokens

As suggested by the earliest numerical impressions and as would be consistent with their efficient and effective use (see Chapter 8), tokens were likely to have been ordered from low to high in numerical value,[809] in the same way that fingers and tallies would have been. However, the change in material form means the source of that linearity and stable order is no longer apparent in the material form itself (i.e., because its manipulability allows for alternative arrangements with the potential for providing similar accessibility to the numerical information being represented). Tokens would have solved the limitations on perceiving non-subitizable quantity through the bundling relations[810] that tokens assumed from metrological units and other sources; as discussed in Chapters 6 and 8, bundling was encoded with conventions of shape and size that additionally represented commodity type, although context (e.g., the exchange of accounting information between known partners; external markings like seals; etc.) also continued to be used[811]; this representation of commodity type resolved a limitation inherent in the use of both fingers and tallies. Tokens were also manipulable to a degree not possible with fingers or tallies. This manipulability enabled the performance of bundling and debundling operations,[812] as well as afforded opportunities to discover operations like subtraction and reciprocation (the token-based method of division), form relations besides plus-one, and develop algorithms for calculating (i.e., the agrimensors' method of approximating the area of a field used in the late fourth millennium[813]). Operations, relations, and algorithms are important because they form the basis of mathematics (see Chapter 9).[814]

[809] Nissen et al. 1993.
[810] Nissen et al. 1993.
[811] E.g., Damerow et al. 1995; Nissen et al. 1993.
[812] Nissen et al. 1993.
[813] Robson 2007.
[814] Gowers 2008.

Unlike tallies, tokens were loose, so they needed to be contained (a limitation). As mentioned in Chapter 8, in the mid-fourth millennium, ANE people began sealing tokens inside bullae and envelopes. The rationale for containment is unknown: While bullae and envelopes would have ensured the integrity of a group of tokens representing the quantity of a particular enumeration, this may not have been the motivation for using them. Other motivations that have been proposed in the literature have included increasing the security of the represented amount (as sealed containers would have been relatively tamper-proof), as well as affording an ability to include a seal to identify commodity, trading partners, or context of use. Whatever the reason, containment meant that tokens could no longer be accessed without breaking their container. Within a relatively short amount of time (mere centuries, as judged by the archaeological record), ANE people began impressing the outsides of bullae with tokens before placing them inside; since the impressions on the outside made the tokens on the inside redundant, bullae were soon flattened out to become tablets with impressions, leading to the ability to represent quantity and commodity separately.[815] Finally, tokens did not represent numerical information concisely, another limitation. Using tokens to make a multiplication table, for example, would have required significant space and have been difficult to read. The information would have been perishable, because loose tokens are easily disarranged, and the information would not have been transportable or easily recreated.

### 10.4.5 Numerical Impressions and Proto-/Cuneiform Numerical Notations

As explained in Chapter 9, the first notations looked like tokens in terms of their shapes and sizes, as well as their bundling relations, because they were impressions made with tokens or objects (like fingers) that produced similar impressions in clay. Notations had the integrity of form that tokens lacked. They also shared linearity and stable order

[815] Malafouris 2010b; 2013; Schmandt-Besserat 1992a.

with their predecessor technologies, but again, the new material form did not impose these attributes (that is, proto-cuneiform impressions would have had the same numerical meaning regardless of their organization, but they were standardly grouped by numerical value and generally arranged so that the quantity of each group was easily discerned, with the groups placed in order of increasing magnitude). Comparison of impressions with tokens would have provided a way to apprehend whole–part relations and thus relations between numbers beyond those that had been elaborated with previous technologies. Like tokens, proto-cuneiform size and shape encoded bundling relations, as well as commodity type. Unlike tokens, notations were fixed, making them suitable only for recording and communicating numerical information. Until the necessary relations between numbers and algorithms needed to manipulate numerical notations were developed, calculations would still have been performed with tokens.

Cuneiform notations had the linearity and stable order of fingers and tallies, as well as the bundling of tokens and proto-cuneiform, which would ultimately develop into a system of place value. Like tallies and tokens, cuneiform was accumulative in how it represented the values ‘one’ through ‘nine’ and ‘ten’ through ‘fifty’. Unlike tokens and proto-cuneiform notations, cuneiform notations separated quantity and commodity into numerals and pictographic labels (with the labels ultimately developing into a script for the Sumerian language).[816] This separation allowed commodity information to be maintained in pictures or script, thereby avoiding the need for using context or memory, while also avoiding the conflation of its representation with that of number. Like any notations, cuneiform was fixed, but unlike proto-cuneiform, it represented numerical information concisely. That concision enabled numerical information to be recorded in tables (e.g., multiplication tables, tables of reciprocals), a material form capable of

[816] Malafouris 2013; Schmandt-Besserat 1992a; also see Overmann 2016a.

containing significant volumes of information that was not perishable and which was transportable and easily recreated. Scribes practiced writing tables as part of their training, and in the process, they learned the numerical relations. This gave scribes more options for calculating than just using tokens: They could also use information from tables or memory, a factor in developing new, complex algorithms for manipulating the relations between numbers.

Explicit relations between numbers, as well as the training of object-recognition processing and handwriting effects implicit in written numerical notations,[817] would have helped scribes think of numbers not as collections, as they were likely to have done with tallies and tokens, but as numerical objects or entities in their own right, entities related to other similar entities in numerical terms (e.g., the way 'sixty' is related to 'two' and 'thirty' or 'three' and 'twenty', relations discovered with tokens and recorded in handwritten cuneiform tablets). A linguistic shift denotes whether numbers are conceived as collections or entities: "two and two are four" and "two plus two is four."[818] Only the wording of the first suggests combining groups of material objects; the second lacks the same connotation. In terms of material representation, numbers as collections or entities differ significantly in their manipulability (collections are manipulable, while entities are fixed); in both cases, the instantiated quantity is distributed across a collection of objects (numbers represented as a group of material objects or a group of materially represented numbers).

The effects of handwriting and trained object-recognition processing would have intensified the conception of numbers as objects, and the development of literacy would have aided thinking about the nature of the manipulations of the relations between them.

[817] Overmann 2016a.
[818] Gowers 2008.

## 10.5 Conceptual Change and Structural Persistence

As concepts of discrete quantity shared by sets of objects,[819] numbers are abstract from their inception. As noted previously, this differs from the idea that numbers are 'concrete' before they become 'abstract', the prevailing characterization of ANE numbers[820] drawn from the developmental theory of Piaget.[821] Always being abstract does not mean that number concepts do not change, as they obviously do, especially in conjunction with change in the material form used to realize, manipulate, and represent them. However, there is also a large overlap in conceptual content across all their material forms: They are all concepts of discrete quantity shared by sets of objects and represented materially.

As concepts of discrete quantity, numbers are initially equivalences realized through behaviors like one-to-one correspondence that match fingers and objects. As forms like tallies are incorporated, numbers become collections that share quantity with enumerated objects, and the source of productive grouping once provided by fingers is no longer apparent. As manipulable forms like tokens are incorporated, numbers begin to share quantity with each other, and the source of their linearity and stable order is no longer apparent. Finally, once numbers become entities related numerically to similar entities, the source of the manipulability needed to develop relations and algorithms is no longer apparent. Simply, attributes like linearity, stable order, productive grouping, and manipulable relations remain implicit in the number concepts themselves, as well as the behaviors that make and use material technologies for counting, but they may no longer implicit in the material forms themselves. These phenomena will be called, respectively, conceptual change and structural persistence.

[819] Russell 1910, 1920.
[820] E.g., Damerow 2010; Robson 2007; Schmandt-Besserat 1992a.
[821] Chrisomalis 2005.

One reason that conceptual change occurs is because the materiality used as the primary method of representing and manipulating number concepts changes. Material change constitutes a form of abstraction (i.e., decontextualization, the extraction of content from its original circumstances to remove their influence on its meaning). Perhaps the most obvious example of such change is the separation of quantity and commodity representation enabled by the invention of writing.[822] The concision of numerical notations enabled the development of numerical tables for representing large volumes of information about the numerical relations between numbers, allowing numbers to be conceptualized as entities related numerically to like entities (a conceptualization that would also have been facilitated and enhanced by the effects of literacy; see Chapter 9). Similarly, the incorporation of tokens into a tally-based number system would have helped foster a concept of number that was more manipulable than an ordinal counting sequence and which contained more relations between numbers than would have been realizable using a tally for representation and manipulation.

Material structure persists for several reasons. One is that the material component, considered across space and time, is really a patchwork out of which new forms can emerge and in which older and newer forms often coexist. That is, the adoption of new material forms is hardly a straightforward process, even if a smooth narrative can be reconstructed after the fact. Adding to this technological variability were the multiple indigenous number systems involved in the origination, manipulation, and elaboration of ANE numbers (see Chapter 6), each with unique forms of the materiality used for numbers. Second, there are neurological reasons that structure persists, like the neural interaction that underlies and perpetuates finger-counting (Chapter 4) and the increased linearization of the mental number line. Third, learning and practice, which represent

[822] Malafouris 2013; Schmandt-Besserat 1992a.

structured interactions with materiality, are also social behaviors that transmit such knowledge between individuals and generations. The ordinal structure of counting sequences in language may also act to reinforce the linearity of manipulable (i.e., non-linear) forms of materiality used for counting. Additionally, when numbers are entities with linear order, productive grouping, and manipulable relations, material structure is implicit in the concepts themselves, and this implicit structure may facilitate the retention and use of older material forms. However, materiality also has agency. What material form is capable of as a substance enables, limits, and structures what people can do with it. The structure of older material forms also influences the structuring potential of newer forms because materiality conditions people to things working in certain ways, and this narrows the range of possible behavioral outcomes (as discussed further in Chapter 11).

### 10.5.1 Change in Brain Function

Interaction with materiality also changes the brain, allowing for the emergence of new behaviors, concepts, and material interactions. For example:

- Grey matter density increases, and synaptic responses are strengthened, changes in form and function that represent learning effects realized from practicing motor movements and memorizing procedural and declarative information (respectively, these are how to perform actions and facts like relations between numbers).[823]
- Parietal functions like numerosity demonstrate training effects, such as the increased acuity of the ANS,[824] formation of, or increased linearity in, the mental number line,[825] and the recognition of numbers in symbolic form.[826] Increased ANS acuity and the linearity of the mental number line in turn facilitate the

[823] Bear et al. 2007.
[824] Piazza et al. 2013.
[825] Fischer 2008; Göbel et al. 2001; Núñez 2011.
[826] Dehaene 2007.

conceptualization of higher quantities in regularized and productive ways, as the recognition of symbolic numbers enables a representational concision that is a factor in the elaboration of numerical concepts (Figure 10.2).

- Parietal regions beyond the IPS (e.g., the angular and supramarginal gyri) become trained in culture-specific patterns of finger-counting[827] and in mediating the recall and manipulation of arithmetical terms and facts.[828]
- Materiality reduces the demands on memory[829]; reduced demands on memory, along with the visualization and problem-solving facilitated by material engagement,[830] allow freed attentional capacity to be repurposed to things like pattern recognition, categorization, and abstraction, enabling the formation of new concepts.
- Arithmetic and mathematical tasks[831] recruit executive functions (Figure 10.2) like attention, working memory, shifting, updating, and inhibition that are important to performing mathematical tasks,[832] which is why there is a correlation between executive functioning and mathematical performance.[833] *Shifting* is the ability to switch flexibly between tasks or mental sets; *updating* is the ability to monitor and manipulate the contents of working memory; and *inhibition* is the ability to override prepotent or dominant responses in pursuit of a goal.[834] A recent study

[827] Dehaene et al. 1999; Zamarian et al. 2009; Domahs et al. 2010. Finger-counting practices not only differ across cultures, they can differ within a culture, perhaps reflecting distinctions of ordinality, cardinality, and one-to-one correspondence (Wasner et al. 2014). However, there are overarching similarities, as documented in Overmann 2014.
[828] Dehaene et al. 2003; Grabner et al. 2007; Grabner, Ansari, et al. 2009; Grabner, Ischebeck, et al. 2009; Ramachandran 2004.
[829] d'Errico 1998.
[830] Malafouris et al. 2014.
[831] Basic arithmetic and complex mathematics activate the same brain regions (independently of the regions involved in language, except during the recall of arithmetic facts); see Amalric et al. 2016.
[832] Bull et al. 2001; 2014; Van der Ven et al. 2012.
[833] Bull et al. 2001; 2014; Van der Ven et al. 2012.
[834] Miyake et al. 2000.

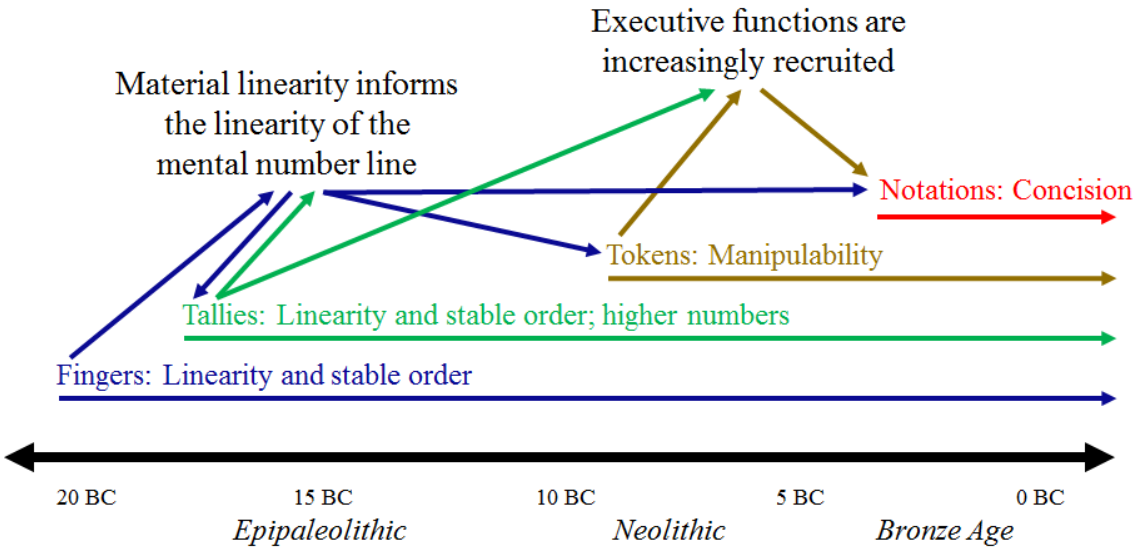



*Figure 10.2*. Interaction with materiality changes the brain. Interaction with linear forms with stable order (e.g., fingers and tallies) increases the linearity of the mental number line; the increased linearity of the mental number line informs the linear structuring of subsequent material forms (tokens and notations). Similarly, as material forms become more capable of representing and manipulating numbers with higher numbers counted and more explicit relations between them, the executive functions are increasingly recruited; the increasing recruitment of the executive functions allows for an increase in goal-directed behaviors.

suggests that of the three, updating has the greatest effect on mathematical task performance.[835]

- The increased recruitment of the executive functions, in turn, increases goal-directed behavior, allowing people to do things like rearrange tokens in new combinations, perform calculations that are more complex, and seek new relations and patterns in numerical data.
- Memorizing and recalling arithmetic facts may also change how calculations are performed (e.g., the difference between calculating with an abacus, whose configuration maintains the result of interim steps in a calculation, or an algorithm in which the interim steps are written[836]), a behavioral and material change related to the availability of arithmetic facts like explicit relations between numbers.

[835] Van der Ven et al. 2012.
[836] Reynolds 1993; Stone 1972.

Changes in psychological processing enable different behaviors with materiality, opening up additional cascading possibilities for change; psychological changes also help perpetuate and intensify systemic change, as systemic change influences change in psychological processing, behaviors, and material forms.

### 10.5.2 Change in the Number System

The number system also changes: Highest number counted goes up as materiality is incorporated[837]; this seems only commonsensical, given that the incorporation of material devices with increased capacity for counting seems like just the thing to facilitate counting to higher numbers.[838] An increase in the highest number counted, in turn, affords an increased likelihood that the relations between numbers will be discovered, explicated,[839] and codified in the form of material artifacts, behaviors, algorithms, and language. In addition, as numbers become more elaborated, they also become more integrated into social purposes, a utility that makes them familiar and habitual. Utility and use position numbers as a cognitive technology, a body of knowledge that structures understanding and behavior.[840] The more familiar and habitual a cognitive technology becomes, the more likely it is that it will be extended to new domains and become entangled in cultural practices and behaviors.[841] The more ingrained in culture numbers become, the more likely they are to be elaborated; their usefulness may intensify the need to develop and extend them.

[837] Overmann 2013a; 2013b.
[838] Overmann 2016d.
[839] Beller et al. 2011; Overmann 2013a.
[840] De Cruz 2008; 2012.
[841] Hodder 2012.

## 10.6 Material Distribution

Thinking of numbers as entities is an interesting conceptual development because they are actually distributed across multiple material forms: the present forms used to represent and apply them, the past forms that gave them some of their characteristics, the objects to which they are applied. For example, a number like '75' in the Hindu–Arabic notations used to represent and apply numbers in the Western mathematical tradition is distributed across the signs for 'seven' and 'five', similar to the way it was once distributed across cuneiform notations (i.e., numerical entities) and tokens (numerical collections). Distribution across past material forms is also implicit in the persistence of older forms, not only in the structure and capabilities they provide but perhaps in their actual use as well: For example, people in Western societies still count with their fingers, make tally marks, and use coins,[842] along with writing numbers on paper and typing them into calculators and computers, applying a conceptualization of number that encompasses all these various ways of representing them, along with the things to which they are applied (e.g., time, distance, speed, cost, temperature, etc.). The persistence of tallies and tokens thousands of years after their first appearance in the archaeological record suggests that Mesopotamian numbers may have been similarly distributed across the different forms of materiality used to represent and manipulate them and to which they were applied.

When numbers are materially distributed, they function as a cognitive technology, structuring how the world is experienced and engaged. In being anchored and stabilized,[843] structured, and distributed by multiple forms of materiality with different affordances that are more implicit in the number concept than in any particular material form used to

[842] Coins are a material form with significant commonality to tokens, particularly in being manipulable and numerically interrelated.
[843] Hutchins 2005.

represent and manipulate them, numbers also become somewhat independent of any one particular form. This independence not only helps numbers function as an abstract conceptual domain but also makes the material component of cognition transparent in its functioning within the cognitive system. Distribution may also explain the correlation between elaboration in numbers and material culture more generally: In addition to reflecting increased social needs for numbers, both may represent the incorporation and distribution of conceptual content across multiple forms of materiality.

Numbers become distributed across multiple material forms as new material forms are incorporated. The incorporation of new material forms is also the mechanism of numerical elaboration, because new forms have different outcome potentials, and the contrast between forms may illuminate useful differences. Numerical concepts elaborate when and because they encompass characteristics of the multiple forms they are distributed across.

In the final chapter, these insights into the role of materiality in numerical cognition are broadened to consider the role of materiality in cognition more generally.

# CHAPTER 11

# IMPLICATIONS AND CONCLUSIONS

## 11.1 Summary

In the final chapter of the thesis, a summary of change in the numerical cognition of the ANE will be presented, and several characteristics of the cognitive system for numbers will be highlighted, along with their implications. Given the suggested criticality of materiality to cognition (numerical and otherwise), reasons why its functionality appears to be generally transparent (i.e., invisible and underappreciated) in this regard will be suggested. Finally, some conclusions about the contributions of the thesis and possible future directions for research in cognitive archaeology will be offered.

## 11.2 Change in ANE Numerical Cognition

Change in the numerical cognition of the ANE is summarized chronologically (as reconstructed archaeologically and inferred from textual evidence) across its multiple dimensions: psychological functioning; behaviors and social context; the material technologies used to realize, represent, and manipulate number concepts; and the number concepts themselves.

### 11.2.1 Finger-counting

Counting with the fingers to realize initial number concepts would likely have been used by small, isolated cultural groups with hunter–gatherer resourcing strategies, as is the

case today, possibly as early as the Late Upper Paleolithic.[844] Because of their cognitive prepotence and ready availability,[845] the fingers would have been used as a reference set, leaving the cycles of 10 and linguistic traces that characterize numbers when they appeared in unambiguous form as numerical impressions in the late fourth millennium and lexical and ordinal numbers in the mid-third millennium.[846] The use of the fingers would have influenced number concepts toward linear structure and stable order[847] with characteristic anatomical grouping (particularly 5- and 10-cycles and compound number word constructions).[848] Interaction with material linearity would have begun to influence the mental number line toward linearity. As representations of quantity, however, fingers would have been limited in the quantities they could attain, perishable as records of quantity, and would not have recorded the type of commodity being enumerated. Numbers would have been concepts of (and words for) equivalences between the fingers and enumerated objects. Numerical relations would have consisted of (some) ordinal sequencing and (perhaps) accumulation. Once a restricted number system emerged with associated lexical terms, grammatical number would have become possible as a linguistic feature.[849]

### 11.2.2 Tallies

Possible tallies are archaeologically attested in the Epipaleolithic Levant and associated with slightly larger, less isolated cultural groups with decreasing nomadism and new resourcing strategies.[850] Counting would have reached quantities higher than those associated with finger-counting through the capacity of the new material form, which

[844] Gladfelter 1998; Kuhn et al. 2001.
[845] Coolidge et al. 2012.
[846] Blažek 1999; Diakonoff 1983; Edzard 1980; Powell 1971.
[847] Gelman et al. 1978.
[848] Comrie 2013; Greenberg 1978; Menninger 1992.
[849] Overmann 2015.
[850] Coinman 1996; Copeland et al. 1977; Davis 1974; Tixier 1974.

would also have influenced numbers toward linear structure and stable order. Notches (and thus quantities) were not grouped, as discerned from the artifacts that have been found, with the result that the subitization constraint on visual distinguishability would have been a limitation on the accessibility of higher quantities represented through tally marks. The mental number line would have been linearized further, and brain regions and functions beyond those of the parietal lobe would have been increasingly recruited. As a record of quantity, the material form would have been more persistent than finger-counting had been, but it would not have recorded commodity type,[851] nor would it have been manipulable. Number concepts would have been quantity comparisons facilitated by the use of the tally as a reference set, and they would have been increasingly adopted for social purposes. Numerical relations would have consisted of *greater than*, *less than*, and *not equal to*, and the operation of accumulation (plus-one) would have become more explicit, influencing the intervals in ordinal sequencing toward regularity.

### 11.2.3 Tokens

Tokens were used in the Neolithic and Chalcolithic by larger, more sedentary groups who were in increasing contact with other cultural groups[852] that likely possessed distinct, independently invented indigenous number systems. Numbers would have been increasingly implicit to and relied upon for social purposes like resource management and trade. The (loose) material form of tokens afforded manipulability that would have enabled the realization of additional numerical relations; the lack of integrity would have necessitated containment, influencing the development of subsequent technological elaborations like bullae and envelopes.[853] The linearized mental number line would have influenced the representation of numbers through the tokens toward linear structure and

[851] Schmandt-Besserat 1992a; 1992b.
[852] Childe 1936; Lloyd 1978; Nashli et al. 2013; Reade 1991.
[853] Malafouris 2010b; Schmandt-Besserat 1992a.

stable order. Productive grouping (bundling) based on the quantities of fingers and metrological units would have resolved the subitization constraint on visual discrimination, helped achieve quantities in the several hundreds and thousands, and resolved the problem of visual discriminability. The executive functions of the frontal lobe would have been increasingly engaged, allowing for more goal-directed behavior in using and elaborating numbers. The material form was persistent as a record of quantity and allowed for the representation of commodity type, which was conjoined with that of quantity.[854] Numbers would have been collections instantiated by tokens as reference sets. The highly manipulable form meant that there would have been increased flexibility in appreciating the numerical relations between the bundled amounts instantiated by tokens; this in turn would have allowed for the emergence of new operations (e.g., removal of tokens as a form of subtraction).

### 11.2.4 Numerical Impressions

The numerical impressions of the Early Bronze Age were used by communities of increasingly specialized practitioners operating within large urbanized groups in frequent contact with each other through formal and informal networks supporting large-scale trade, agriculture, and construction.[855] Bullae and envelopes became tablets, and tokens were supplemented with impressions. The impressions were structured linearly with stable order, as influenced by the use of older, linearized technologies and the linearized mental number line. Representations of quantity were persistent, more concise, and possessed a greater integrity of form than their predecessors. However, they were still conjoined with the representation of commodity and bundled by amounts based on fingers and metrological units, forming over a dozen systems for representing quantities of

[854] Schmandt-Besserat 1982; 1992a.
[855] Englund 2004a.

commodities. Numerical impressions were not manipulable; they would have been suitable only for recording. Comparisons of impressions with tokens would have allowed for the realization of whole–part relations between larger and smaller numbers. Number concepts would have been collections instantiated by numerical impressions and tokens as reference sets.

### 11.2.5 Proto-cuneiform Number Signs

The proto-cuneiform signs of the Early Bronze Age were used by specialized practitioners administering the social purposes of massive urbanized groups. The representation of commodity and quantity separated with the invention of writing,[856] affording possibilities for numbers to become more clearly organized. The representation of commodity through picto-/ideographs and quantity through impressions made by styli[857] yielded handwriting effects, greater concision in signs for quantity, and trained object-recognition processing. Representations of quantity were still persistent and had form integrity, and they were becoming more concise than their predecessors through handwriting effects and trained object-recognition processing. Numerical signs were suitable for recording quantity information, but they would have influenced the eventual development of algorithms for manipulating the relations between numbers. Numbers were starting to become entities represented by numerical signs and characterized by numerical relations with like entities.

### 11.2.6 Cuneiform Number Signs

The cuneiform number signs of the Old Babylonian period were used by highly specialized practitioners administering everything from vast bureaucracies associated with

[856] Schmandt-Besserat 1992a.
[857] Cammarosano 2014; Nissen 1986.

state-level projects to informal trade networks and household requirements. Having discarded the representational complication of commodity, numerical organization was free to become simplified, to the extent that a system of place value developed (likely from the 6- and 10-cycles of Sexagesimal System S, as mentioned in Chapter 9). However, the use of the sexagesimal place value system was limited to acting as an interface between different metrological systems,[858] which suggests that their structure remained influential in how numbers were conceptualized. The use of picto-/ideographs had yielded literacy (i.e., both neural reorganization and a script capable of representing language with fidelity),[859] while numerical representations had become concise number signs through similar neural reorganization (i.e., handwriting effects and trained object-recognition processing). Numbers had become entities represented as numerical signs and were characterized by many numerical relations with like entities; the numerical relations could be manipulated through a variety of means, including algorithms that depended on the mental recall of numerical relations. Ideas about manipulations such as accumulation were starting to generalize,[860] though they were not yet unified concepts.

## 11.3 Characteristics of the Cognitive System for Numbers and Their Implications

From the summary of change, several characteristics of the ANE cognitive system for numbers stand out in particular: (1) Number concepts are abstract from their inception but are subject to change, especially in conjunction with new material forms for representing and manipulating them. (2) Number concepts are materially bound throughout, even at their most elaborated. (3) Numerical elaboration is facilitated by the incorporation of multiple material forms. (4) Elaborating the realization of initial number

[858] Robson 2008.
[859] Overmann 2016a.
[860] Høyrup 2002b.

concepts into a mathematical system is a multigenerational process (5) yielding a cultural phenotype (numeracy) that is widely accessible, given conditions of social opportunity and support. (6) Interaction with the materiality of numbers and writing would have intensified the development of specialized communities, identities, and training. Finally, (7) the interaction of psychological capabilities, behavioral capacities, and material affordances forms a cognitive landscape that defines the initial configuration for the cognitive system for numbers, provides structure throughout, injects potential limitations, and narrows the range of possible solutions. These are explored in greater detail below.

### 11.3.1 Numbers Are Abstract Concepts that Undergo Change

Initial number concepts are realized from the active manipulation of sets of material objects that enable the categorization and abstraction of cardinal equivalences (see Chapters 4 and 5). The realization process suggests that number concepts are abstract from their inception, providing a rationale for discarding the 'concrete–abstract' distinction. However, as additional material forms are incorporated into the cognitive system, number concepts change from equivalences to collections and ultimately to entities (as was discussed occurring in the ANE in conjunction with the transition from fingers to tallies, tokens, and numerical notations in Chapters 7 through 9), though the quantity component of the concept remains consistent throughout. This transition suggests that although numerical concepts share significant commonality across number systems, they are not identical. It also implies that conceiving of numbers as numerical entities may require the invention and sufficient repetition of handwritten signs (i.e., so that handwriting effects and trained object-recognition processing may occur).

### 11.3.2 Number Concepts Are Always Materially Bound

Number concepts remain bound to the materiality used to realize, represent, and manipulate them, even when elaborated into a mathematical system. This is exemplified by, for example, the language used to represent the arithmetical operations of the Old Babylonian period,[861] which suggest that the different types of addition reflected distinct material forms of accumulative operations. Such material binding is not unexpected, given materiality's role in anchoring and stabilizing concepts,[862] and provides another rationale for discarding the 'concrete–abstract' distinction. However, as additional material forms are incorporated into the cognitive system, number concepts acquire new materially informed characteristics (e.g., manipulability), structural properties that may persist even when the material form giving rise to them is superseded or de-emphasized by another. Distribution over multiple material forms effectively frees numbers from any particular form, enabling them to operate as a cognitive technology (see Chapter 10).

### 11.3.3 The Importance of Multiple Material Forms to Numerical Elaboration

The incorporation of multiple material forms may facilitate numerical elaboration because new materiality provides additional ways to represent and manipulate number concepts, the potential for novel structure, and material contrasts that may serve to highlight differences and foster insight. This may have bearing on why there is a correlation between numerical elaboration and material culture more generally[863]: Material culture provides opportunities for numerical elaboration, first because it is an implicit part of the interactivity that is numerical cognition:

> … manuovisual stimulation from the manipulation of material forms occasion[s] collective, cumulative change in psychological processing, which over the long term influence[s] brain functionality and form, and

[861] Høyrup 2002b.
[862] Hutchins 2005; Malafouris 2013.
[863] Divale 1999; Epps et al. 2012; Overmann 2013a; 2013b.

> change in psychological processing facilitate[s] the manipulation of materiality into new, stimulating forms with subsequent impact on neurological functioning.[864]

Manipulating existing artifacts into new configurations can provide hints[865] that engage "basic cognitive capacities of pattern recognition and transformation,"[866] providing opportunities for the recognition of relations and the formation of categorizations and abstractions.

Beyond the artifacts used to realize, represent, and manipulate number concepts, material culture more broadly may increase the potential availability of new material forms for representing and manipulating numbers. That is, an unrelated material form may stimulate a novel solution to a limitation (e.g., lack of manipulability) inherent in the artifacts used for numbers. Environments described as "the most innovative" are those "designed to make it easy to encounter and proliferate hints by offering aids like material sheds or idea carts …"[867] These are

> … a collection of artifacts like mechanisms, fabrics, patterns, iconic designs and toys or gadgets that a designer, in the early part of research, is encouraged to visit. Viewing the thousands of items filling the shed, someone in search of inspiration might respond to an odd metallic texture, an unusual gear configuration, a suction device or a mechanism exploiting a physical principle in an uncommon way. The aspect that triggers interest need not be an element that would ever work in the final design or a part of something central to the design. Its role is to provoke an idea that had not been considered before, an idea that helps the designer realign how he or she thinks or that arouses consideration of candidates outside the norm. This operates quite differently than another familiar strategy: using pictures, written descriptions and physical examples of *available solutions*, where the reverse effect often happens. Past ideas, or idea analogs, become fixated. The reviewed solutions bias thinkers to look for variants on

[864] Overmann 2016a, 298.
[865] Kirsh (2014, 13) describes how hints are realized through manipulation using the case of monkeys becoming able to use both a box and a stick to reach a bunch of bananas: "... in the course of walking around, [monkeys will] eventually see the box and bananas line up visually. At that moment, or shortly thereafter, they seem to have a simian aha! and immediately drag the box under the bananas, recover the stick and use it to hit the bunch." Similarly, rearranging the materials used to represent numbers has the potential to yield hints and thus new insights for their users.
[866] Clark 1997, 194.
[867] Kirsh 2014, 12.

> existing solutions or to use existing search spaces, rather than invent new ones.[868]

Essentially, what is depicted as a specialized technique for "break[ing] habitual thinking" in order to make "new connections or [develop] new ideas"[869] is being ascribed here to material culture in a more general, if less intentional, sense.

Material culture also motivates numerical elaboration through the need to apply numbers as a cognitive technology for managing resources. In providing resource management, numbers also enable the resources to reach a level of complexity that might not be attainable in the absence of numbers. Interestingly, the four places where writing is thought to have been independently invented—Mesopotamia, Egypt, China, and Mesoamerica[870]—were all state-level bureaucracies that had elaborated number concepts into mathematical systems, implying both that such mathematical elaboration reflects the social needs for numbers to manage resources and that state-level bureaucracies might not be able to form and reach such scale without numbers as an enabler (certainly, there are no known state-level bureaucracies without complex number systems).

### 11.3.4 Multigenerational Elaboration

The elaboration involved in realizing a mathematical system from initial number concepts is a multigenerational process that results in new, culturally acquired phenotypes. This phenomenon is interesting, since each participant has to acquire the necessary skills, and "the long-term sustainment of behaviours required to develop the necessary psychological and material changes are socially motivated."[871] The elaboration of number concepts, therefore, is inextricably bound to the conditions that foster multigenerational change in the psychological, behavioral, and material components of the cognitive system;

[868] Kirsh 2014, 22.
[869] Kirsh 2014, 15.
[870] Houston 2008; Jackson 1981; Senner 1989.
[871] Overmann 2016a, 298.

if the social commitment to sustaining the specialized material engagements needed to foster the psychological changes that constitute numeracy were to lapse, the cognitive system for numbers would not likely persist (nor would the system that is literacy). The time span and amount of participants implies that numbers, even when elaborated from their initial forms into a mathematical system, are not the invention of genius (unlike the concurrent but independent invention of infinitesimal calculus in the mid-1600s by Isaac Newton and Gottfried Leibniz[872]; such inventions represent enhancements to existing structures of numerical and mathematical concepts developed through generations of communal effort). Passage through the psychological and behavioral capacities of so many individuals yields a cultural phenotype accessible to most (at least to the extent of functional numeracy; the acquisition of higher mathematical systems may require degrees of aptitude and motivation that are less common), with acquisition of the cultural phenotype requiring both social opportunity and support. The idea that formation and acquisition of a cultural phenotype are matters of common psychological and behavioral capacities and social opportunity and support runs counter to the suggestion that these processes may have depended on "individuals who easily mapped symbols onto the ANS to construct more precise and flexible representations of quantity," thereby providing "advantages in some cultural contexts (e.g., trade) that resulted in survival and reproductive advantages that in turn led to the evolutionary refinement" of the cognitive system for numbers.[873]

[872] Smith 1958; Gowers 2002.
[873] Geary et al. 2015, 348.

### 11.3.5 Realization of a Widely Accessible Cultural Phenotype

As a conceptual system, numeracy might be considered 'discovered' or 'invented',[874] terms that evoke notions like 'creativity', 'inspiration', and perhaps even 'genius'. However, none of these terms fully conveys the multigenerational collaborative effort and psychological, behavioral, and material reorganization involved in producing the culturally acquirable phenotype that is numeracy.[875]. The neurofunctional reorganizations of numeracy are the unanticipated outcomes of behaviors with materiality. Manipulating materiality reorganizes neural activity (this is how tools make minds[876]), opening up possibilities for the cognitive system to change further, as incremental change is gained in psychological processes, behavioral repertoires, and material stimuli. As a component of the cognitive system, materiality does not merely accumulate knowledge,[877] distribute cognitive effort,[878] or structure the system (see Chapter 10), but acts as a medium for conceptual expression and exchange, enabling the collaborative development of the system to be distributed over the time needed to realize significant change.

Were unusual aptitudes required,[879] number systems might not ever be realized, since individuals with unusual aptitude are few and far between, and more known for elaborating established conceptual systems (as in the example of Newton and Leibniz with respect to infinitesimal calculus) than establishing them in the first place. Further, with geniuses in charge, number systems might take forms that few could comprehend, except for those who invented them or who similarly possessed the unusual aptitude needed for participation. Yet no unusual abilities are required to participate in (basic) numeracy, making it widely accessible as a cultural phenotype. The critical element in developing an

[874] Arguably, as the cognitive system for numbers elaborates, an initial stage characterized by realization and explication slowly yields to one in which established principles guide exploration and discovery.
[875] Overmann 2015; 2016a; 2016c.
[876] Malafouris 2013.
[877] Smith 2007.
[878] Hutchins 1995.
[879] Geary et al. 2015.

accessible and easily acquired cultural phenotype might be the highly collaborative, multigenerational nature of its development, which ensures that it is realized from mainly ordinary (rather than extraordinary) psychological and behavioral capabilities.

### 11.3.6 Materiality, Number Concepts, and Social Effects

As the cognitive system for numbers became elaborated and literacy developed, the members of the society who directly interacted with the materiality of numbers and writing became a highly specialized community with a profound sense of professional identity.[880] Regarding literacy, these social effects can be ascribed to the increasing undecipherability of the script to anyone not trained to read it[881]; for numeracy, the question of why the social intensification effects may have occurred is less easily answered. Part of the answer might reasonably be found in the idea that the performance of complex mathematical calculations would have required training not only in using the material technologies (e.g., numerical notations, tables, tokens, etc.) but in the relations between numbers and the algorithms for manipulating them as well. The need to acquire knowledge and proficiency may have influenced the intensification of the training needed to become part of the mathematical community and have concurrently acted to exclude anyone without the necessary training and skills.

Another contributing factor may have been the availability of numerical language (i.e., lexical numbers and grammatical number), which would have acted to enable children to acquire basic number concepts and the associated lexical terms at young ages;

[880] Postgate 2013; Robson 2008.

[881] Overmann 2016a. As written characters became abstract (i.e., less identifiable regarding their original picto-/ideographic forms) and contained more clues to phonetic and semantic aspects of language, specific training was required in order to be able to identify local and global features of script characters and differentiate them from one another: "Mastering [literacy] was no longer a simple matter of decoding sign meanings and sound conventions, but entailed reorganizing the brain through specific training and practice, without which the script was just so many undifferentiable marks on clay. The effort of training and practice, and the script's relative meaninglessness without it, would effectively have limited literacy to those with the necessary opportunities and skills and would have been a factor in the formalization of such training."

the acquisition of basic concepts and terms in turn would have facilitated children's acquisition of more complex numerical concepts, under conditions of selection for scribal training (i.e., mainly, though not exclusively, boys[882]). Grammatical number is known to facilitate the acquisition of number concepts and the associated lexical terms through mechanisms like bootstrapping and semantic mapping.[883] However, grammatical number cannot perform this function when it does not exist as a feature of language, as a minority of languages do not.[884] However, some of these languages (e.g., Cantonese) are associated with highly numerate societies. How children acquire number concepts and associated lexical terms in languages without grammatical number is an understudied phenomenon.

### 11.3.7 The Cognitive Landscape

The components of the cognitive system—brains, bodies, and materiality—define a conceptual space or cognitive landscape that determines numerical realization, representation, manipulation, and elaboration. That is, psychological capabilities like numerosity, behavioral capacities such as pairing, and material affordances like the structure of the hand define the initial configuration for numerical cognition, which is why initial number concepts are similar across cultures. Their interactivity is structured by their characteristics, influencing the relations established between the components, systemic performance, and the potential for elaboration. Theoretically, the further elaboration of the system could be achieved by adjusting any of the components. In actuality, psychological capabilities do not change on a whim, especially in the absence of a preexisting cultural system with the potential to implement such change. Behavior may perhaps be a bit more flexible in terms of rapid changeability, but again, the absence of preexisting concepts and goals, as does the determinism implicit in material structure, may act to inhibit change.

[882] Postgate 2013; Robson 2008.
[883] Barner 2012; Carey 2009.
[884] Overmann 2015.

However, over cultural (multigenerational) spans of time, psychological and behavioral change may intensify the overall change trajectory, as was suggested in Chapter 10 for ANE numbers with the linearization of the mental number line and the increasing recruitment of executive functions.

The idea of a cognitive landscape defined by interacting psychological, behavioral, and material components is similar to the idea of skilled performance being a matter of mastering variable relations between psychological processes, motor movements, and material structures to produce invariant results, drawing upon the work of experimental psychologists Tetsushi Nonaka and Blandine Bril with a quadriplegic master calligrapher and Indian stone carvers given glass instead of carnelian to produce beads.[885] While performing the long division algorithm with pencil and paper appears to involve different contributions from the psychological processes, behaviors, and materials than does wielding an inked brush against paper to produce an ornate Kanji character, the idea that skilled performance involves mastering the contributions of brains, bodies, and materiality is common to both, given that the respective contributions differ across the two performances.

Of the three components, materiality may have the greatest change potential, as limited by preexisting structure. However, materiality also motivates change through the limitations it injects, and it structures potential solutions (i.e., as the forms that follow fingers for counting to higher numbers tend to work like the fingers do, but with greater capacity for enumeration). Such structuring may explain at least some of the contiguity of material forms noted by typologists. The typological observation of continuity and gradual evolution in material forms has been described in language borrowed from Linnæus:

[885] Nonaka 2011; 2013; Nonaka et al. 2012; 2014.

"Natura non facit saltus"[886] ('Nature does not make jumps'). For example, "The law that Nature makes no jumps, can be taught by the history of mechanical contrivances …. The knowledge of the facts of evolution, and of the processes of gradual development, is the one great knowledge that we have to inculcate, whether in natural history or in the arts and institutions of mankind …."[887] Material culture was capable of being arranged so that the "successive links of [its] development" could be made apparent.[888] Further, "there is not a single work of man's hand which has not its history of slow and continuous development, capable of being traced back, like branches of a tree, to its junction with others, and so on until the roots of all are found to lie in the simplest contrivances of primeval man."[889]

The structuring of potential solutions is also a limitation, since it narrows the range of possibilities to those that bear affinity to previous solutions. This may be why the presentation of unrelated material forms has become a technique for sparking novel solutions.[890]

## 11.4 Why Materiality Is Transparent within Human Cognition

Although materiality appears critical to numerical cognition (and by extension, to cognition more broadly), it is not generally appreciated as an integral part of human cognition, numerical or otherwise. Rather, people tend to think of themselves as agents and materiality as the instruments that help them achieve their agentive goals, a characterization that tends to render materiality passive. After all, a hammer does not drive nails into wood on its own; a person adds the necessary movement and intent. Lacking any ability to move on its own or motive to build a house, the hammer seems inert. Even while it is being used and is perhaps the most unambiguously integral to cognition, the hammer

[886] Linnæus 1755, 27.
[887] Pitt-Rivers 1891, 116.
[888] Pitt-Rivers 1891, 116.
[889] Lane Fox 1868, 405.
[890] Kirsh 2014.

neither understands its purpose nor has any capacity for such understanding. The ability to comprehend is one of the key differences between human agency and material agency. However, human comprehension is not absolute. For example, an individual can be part of a distributed collaborative effort without understanding the goals or accomplishments of the whole system, a point made in describing modern navigation (i.e., a system involving multiple collaborating individuals, some of whom are novices lacking both proficiency in task performance and conceptual apprehension of navigational techniques; they are correspondingly assigned low-level tasks performed under close supervision[891]). Similarly, an Inuit can become highly skilled at building and maneuvering kayaks[892] without necessarily understanding fluid mechanics or being able to describe hydrodynamics mathematically, matters an engineer can understand without necessarily being able to build or sail any kind of boat.

When a tool malfunctions or is used unskillfully, it becomes an impediment to purpose, and in this regard may seem even less a part of cognition. If a device like a calculator is recognized as at least providing cognitive support,[893] an artifact like a chair is even less so understood; the former involves a computational goal, as well as a sustained interaction with materiality, while sitting is more cognitively instantaneous (the decision to sit) and then the physical behavior itself. Materiality becomes simply the setting for and props through which people perform their actions and activities; objects become submerged in the background of experience until such time as they are noticed, picked up, and used.

If materiality is indeed a constitutive element of human cognition, as argued by Material Engagement Theory, and critical to numerical cognition, as has been argued in

[891] Hutchins 1995.
[892] Walls et al. 2015.
[893] E.g., d'Errico 1998.

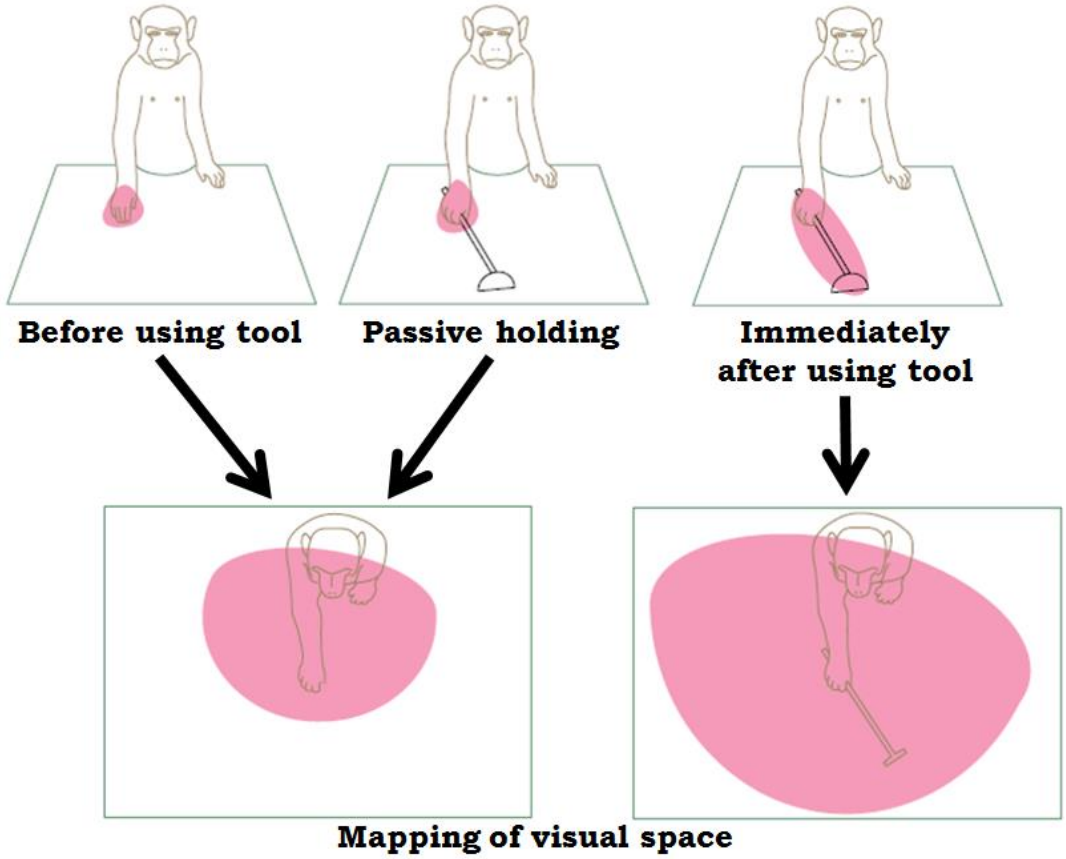



*Figure 11.1*. Mapping of visual space (bottom left) in macaque monkeys before using a rake (top left) and while passively holding a rake (top center). Visual space is extended (bottom right) when the measurement is taken immediately after the rake is used (top right).[894] Note that the requirement to measure brain activity after and not during movement is a limitation of currently available measurement technologies.[895] While humans and monkeys appear to have similar neural substrates for the psychological processing involved, "humans might have such neural machinery present in the brain from birth or early in life, whereas lower primates seem to need a period of training to induce behavioural learning, and electrophysiological and even neurogenetic changes in the animal's parietal cortex."[896]

[894] Adapted from Maravita et al. (2004), Figure 1.
[895] Benjamini et al. 1995.
[896] Maravita et al. 2004, 85. Electrophysiological measurements detect neuronal responses in the form of ion flow. Neurogenetic research with nonhuman primates seeks to identify genes associated with cognition and learning that may be shared with humans. The training required to get macaques to use tools may "mimic" an evolutionary pressure similar to that thought to have expressed tool use in the hominid lineage (and thus in extant humans), a neurogenetic difference between macaques and humans with the potential to "shed light on developmental and evolutionary aspects of higher cognitive functions in primates," including humans (Maravita et al. 2004, 80). The authors focused on the IPS of the parietal lobe because it has been implicated in a wide variety of cognitive functions, including tool use, motor planning, perceptual–motor coordination, various forms of attention and attentional management, numerosity and calculation, visuospatial working memory, spatial representation, spatial coordinate transformation, mental rotation, aspects of long-term memory and language, and intent interpretation and management (Ansari et al. 2005; Bisley et al. 2003; Cantlon et al. 2006; Diester et al. 2008; Freedman et al. 2006; Hamilton et al. 2006; Hubbard et al. 2009; Orban et al. 2014; Orban et al. 2006).

this thesis, its role in this respect is certainly invisible most of the time. Such transparency may have a neurological basis: Neurons adapt to tool use, incorporating the tool into the body schema (or image). For example, the neurons that control finger movements react to tools as if they were part of the hand, allowing them to function as extended fingers.[897] A stick extends tactile perception, the phenomenon used by visually impaired people when they navigate by cane.[898] In a sighted person, use of a stick or similar implement (e.g., a rake) also remaps visual space, so that things within the extended reach of the tool seem nearer.[899] This is shown in Figure 11.1, which compares near-space visual mapping before and after using a tool; simply holding the tool does not have this effect; rather, the tool must be put into action.[900]

Beyond the incorporation of tools into the body schema is the observation that much cognitive activity is unconscious and thus transparent: for example, acts of perception, the formation and recall of memories, learning, information processing, and behavior.[901] No conscious planning goes into, for example, the production of movement or speech; people simply move or talk without deliberately thinking through how arm and hand positions will need to be changed from instant to instant to pick up an object, or the ways in which phonetic, morphemic, lexical, and syntactic elements will need to be arranged and integrated from one to another to form a coherent sentence.[902] Some cognitive processing can become unconscious through automaticity[903] (an example is driving a car by a proficient driver), implying learning and practice. While it is operating as part of the cognitive system, materiality is not only integrated into the body schema; it

[897] Vaesen 2012.
[898] Malafouris 2008.
[899] Maravita et al. 2004.
[900] Maravita et al. 2001; 2004; also see Maravita, Clarke, et al. 2002; Maravita, Spence, et al. 2002.
[901] Kihlstrom 1987; 1989.
[902] There are moments of some awareness, of course, such as producing or avoiding movements based on performative intents or an awareness of social context, or choosing one's words carefully to produce diplomatic speech; nonetheless, such awareness of goals and context is high-level monitoring and does not encompass the totality of planning and execution implicit in movements and language.
[903] Uleman 2005.

also becomes subject to the focus of conscious attention elsewhere, on the human agent's "[g]oals, motives, and self-regulation."[904] Both effects make materiality transparent as a constitutive part of the (typically unconscious) cognitive machinery.

Transparency means that materiality's role in forming, anchoring, stabilizing,[905] and manipulating concepts goes unnoticed, as does the way that materiality distributes cognitive effort over space and time,[906] influences change in human behavior and psychological processing,[907] accumulates knowledge and makes it available to social others,[908] and structures the cognitive system (see Chapter 10). Further, materiality is thought of as supporting mental processing or physical actions, rather than as an integral, constitutive part of cognition; in particular, numerical cognition is neither recognized nor characterized as a system that is comprised of and distributed between psychological processing, interactive behaviors, and material tools. The result is that tools are conceived in passive, instrumental terms, not as integral to cognition or part of a relation of interactive change with behaviors and brains. When the functionality of materiality within cognition is transparent and concepts become distributed across multiple material forms, concepts are seemingly independent of any of them and may thus appear to be 'abstract' within the experience of cognition.

The concepts of other peoples (including those of ancient peoples) may seem 'concrete' by comparison (although in fact they are, as argued here, equally abstractive and materially bound), since the materiality that helps realize, represent, and manipulate them may be all that is accessible to an observer, especially when seen through the archaeological record. This will be characterized as the difference between *thinking through* and *thinking about* materiality. The former entails the integration of the

[904] Uleman 2005, 6.
[905] Hutchins 2005.
[906] Hutchins 1995.
[907] Malafouris 2013.
[908] Smith 2007.

psychological, behavioral, and material components whose interactivity comprises cognition, while the latter occurs when the psychological component is not integrated, either in whole or in part, with the behavioral and material components. The distinction is similar to the difference between the *emic* and *etic* perspective in cultural anthropology: An integrated member of a society is able to draw upon implicit sociocultural knowledge that is unknown to an outside observer, who tends to be disadvantaged through such exclusion.[909] A similar distinction is being offered here between a cognitive system in which the psychological component is fully integrated and one in which it is not.

When materiality is an integral part of a cognitive system that is extended and distributed, thinking is accomplished *through* materiality, both during the (brief) moments when materiality functions as an explicit component of cognition and (more generally) as concepts are distributed across materiality. Thinking *about* materiality occurs when the human agent focuses on the concepts, not the enabling material forms. While this may submerge materiality into the background of experience, the phenomenon of material distribution remains accessible to members of the culture and cognitive system. In comparison, when the materiality of an unfamiliar cognitive system is examined, tools are neither as submerged into the background of experience nor acting to distribute conceptual content across multiple material forms. It may be difficult to think through another culture's materiality, especially when artifacts may bear little resemblance to familiar tools, are perhaps so dissimilar that their purposes are unclear, or the functionality ascribed to them typologically has more to do with the archaeologist's thinking and culture than the ways ancient peoples conceived and used their tools.

What remain available to the archaeologist are the artifacts and the behavioral traces they imply, removed from most of their psychological and conceptual dimensions.

[909] Ferraro et al. 2014; Morris et al. 1999; Pike 1967.

Without the latter, cognition involving only materials and behaviors may appear mechanical (and hence, 'concrete'). However, from inside the cognitive system with full access to the psychological and conceptual dimensions, the materiality was likely as transparent to the ancient peoples (or contemporary others[910]) that used it as it is within any other system viewed from within.

## 11.5 Thesis Contributions and Possible Future Research

In revealing the extent to which numerical cognition involves materiality, the study of numbers has the potential to illuminate the pervasiveness of materiality as an integral component of human cognition more generally. As numbers are multifaceted phenomena, their study requires a theoretical framework like Material Engagement Theory that enables a holistic consideration of the respective contributions of psychological processes like numerosity, abstraction, and language; behaviors that manipulate materials into concept-generating forms; and materials whose quantity is semantically meaningful and whose characteristics influence change in the other components and systemic structure. Injecting considerations of materiality into the study of numbers not only requires a way to reconcile the different time scales considered by psychology, ethnography, linguistics, and archaeology, it also necessitates that archaeology be willing to consider data from other disciplines to an extent that is perhaps unconventional and uncomfortable.

To the degree that it has been successful, the most notable contributions of the present thesis will be found in its holistic consideration of the materiality of numbers in light of what is known from studying the psychological, behavioral, and linguistic behavior of peoples who are extant or relatively recent (i.e., within the last few hundred years); in its implications about the role of materiality from considering the lengthy

[910] See the criticisms of Piaget's (1937) 'abstract–concrete' distinction applied to contemporary others and ancient societies in Chapters 1 and 6.

technological sequence from the ANE; and in its attempt to reconsider some of the entrenched assumptions about how numbers originated in the region. The first injects materiality into the study of numbers, where its agency and semiotic value can be seen as influencing the realization, representation, and elaboration of number concepts and systems, including a considerable portion of their variability across cultures and time. It also reduces the linguistic primacy assumed by many researchers (see Chapter 10); this should not be construed as a claim that language is unimportant in numbers as much as an assertion that both materiality and language work in concert to realize number concepts and systems.

The second, reconstructing the long sequence of ANE counting technologies, enabled an analysis of how material agency and semiotic meaning might influence the psychological and behavioral components of numerical cognition, as well as how material affordances might structure the selection and use of the material forms incorporated into the cognitive system for numbers and works to distribute conceptual content across multiple material forms. The reconstruction also required considering the hand and numerical notations for their materiality, rather than their organic or symbolic attributes. It also argued that the cross-cultural commonality that typifies extant number systems as a function of shared psychological abilities and physiological attributes would have included how the peoples of the ANE realized their number concepts. The latter opened up a discussion of the third contribution of the thesis, reconsidering assumptions about ANE numbers: The Piagetian ‘concrete–abstract’ distinction was discarded in favor of a more nuanced explanation of numerical realization and elaboration. Tokens were argued not to have been the first counting technology, on the basis of their multidimensional

organization[911] and other attributes, and language for numbers was examined to argue that numbers had deeper roots in the region than has previously been thought.

Finally, it is hoped that the thesis might suggest new directions for research into numerical cognition, by positioning materiality as an integral component of the cognitive system for numbers; for cognitive archaeology, as it continues to interpret past numerical cognition through artifacts whose role in quantification might be ambiguous, through the holistic application of data on the components of numerical cognition from extant and recent peoples; and for archaeology more generally, by casting the peoples of the Upper Paleolithic, Epipaleolithic, and Neolithic in a new light, that of emerging and elaborating numeracy, possibly engendering new ways of understanding their societies and interactions with one another.

[911] Chrisomalis 2010.

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# APPENDIX A

# NUMEROSITY IN ANIMAL SPECIES

Table A.1

*Numerosity in Mammals, Birds, Amphibians, Fish, and Insects*

| Common Name | Species Name | References |
|---|---|---|
| *Mammals* | | |
| Baboon | *Papio spp* | Barnard et al. 2013; Smith et al. 2003 |
| Bear | *Ursus americanus* | Vonk et al. 2012 |
| Capuchin | *Cebus apella* | Addessi et al. 2007; 2008; Beran 2008; Beran et al. 2012; Evans et al. 2009; Judge et al. 2005 |
| Cat | *Felis silvestris catus* | Pisa et al. 2009 |
| Chimpanzee | *Pan troglodytes* | Beran 2001, 2012; Boysen et al. 2001; Matsuzawa 2009; Pérusse et al. 1990; Tomonaga 2008; Murofushi 1997 |
| Coyote | *Canis latrans* | Baker et al. 2011 |
| Dog | *Canis lupus familiaris* | Baker et al. 2012; Ward et al. 2007 |
| Dolphin | *Tursiops truncatus* | Jaakkola et al. 2005; Kilian et al. 2003; Yaman et al. 2003 |
| Elephant | *Elephas maximus* | Irie-Sugimoto et al. 2009; Perdue et al. 2012 |
| Gorilla | *Gorilla gorilla* | Anderson et al. 2005; Hanus et al. 2007; Vonk et al. 2013 |
| Horse | *Equus caballus* | Uller et al. 2009 |
| Human | *Homo sapiens* | Haun et al. 2011; Piazza 2011 |
| Hyena | *Crocuta crocuta* | Benson-Amram et al. 2011 |
| Lion | *Panthera leo* | McComb et al. 1994 |
| Macaque | *Macaca mulatta* | Beran 2007; 2008; Beran et al. 2006; Brannon et al. 1998; 2000; 2006; Cantlon et al. 2006; Harris et al. 2007; Hauser et al. 2000; Jordan et al. 2005; 2006; 2008; Merritt et al. 2009 |
| Marmoset | *Callithrix jacchus* | Stevens et al. 2007 |
| Mouse | *Apodemus agrarius* | Panteleeva et al. 2013 |
| Orangutan | *Pongo pygmaeus* | Anderson et al. 2007; Call 2000; Hanus et al. 2007; Shumaker et al. 2001; Vonk 2014 |
| Raccoon | *Procyon lotor* | Davis 1984 |
| Rat | *Rattus norvegicus* | Capaldi et al. 1988; Davis et al. 1986; 1989; Huntley-Fenner 2001; Platt et al. 1971 |
| Sea lion | *Otaria flavescens* | Abramson et al. 2011 |
| Squirrel monkey | *Saimiri spp* | Smith et al. 2003; Terrell et al. 1990; Thomas et al. 1980 |
| Tamarin | *Saguinus oedipus* | Hauser et al. 2002; 2003; Stevens et al. 2007; Uller et al. 2001 |
| Whale | *Delphinapterus leucas* | Abramson et al. 2013 |
| Wolf | *Canis lupus* | Utrata et al. 2012 |

(Continued)

# Appendix A: Numerosity in Animal Species

| *Common Name* | *Species Name* | *References* |
|---|---|---|
| *Birds* | | |
| Chicken | *Gallus gallus* | Rugani et al. 2009; 2013 |
| Crow | *Corvus spp.* | Bogale et al. 2011; Ditz et al. 2016 |
| Nutcracker | *Nucifraga columbiana* | Tornick et al. 2015 |
| Robin | *Petroica longipes* | Garland et al. 2012; Hunt et al. 2008 |
| Parrot | *Psittacus erithacus* | Al Aïn et al. 2009; Pepperberg 2006; 2012; Pepperberg et al. 1991; 2005 |
| Pigeon | *Columba livia* | Brannon et al. 2001; Honig et al. 1989; Olthof et al. 2000; Roberts 2005 |
| *Amphibians* | | |
| Frogs | *Bombina orientalis* | Stancher et al. 2015 |
| Salamander | *Plethodon cinereus* | Krusche et al. 2010; Uller et al. 2003 |
| *Fish* | | |
| Angelfish | *Pterophyllum scalare* | Agrillo, Miletto Petrazzini, et al. 2012; Gómez-Laplaza et al. 2011a; 2011b; 2012; 2013 |
| Blind cavefish | *Phreatichthys andruzzii* | Bisazza et al. 2014 |
| Guppy | *Poecilia reticulata* | Agrillo, Miletto Petrazzini, et al. 2012; Agrillo, Piffer, et al. 2012; Piffer et al. 2012 |
| Minnow | *Pimephales promelas* | Hager et al. 1991 |
| Mosquitofish | *Gambusia holbrooki* | Agrillo et al. 2007; 2008; 2009; 2010; Dadda et al. 2009 |
| Redtail splitfin | *Xenotoca eiseni* | Agrillo, Miletto Petrazzini, et al. 2012; Stancher et al. 2013 |
| Siamese fighting fish | *Betta splendens* | Agrillo, Miletto Petrazzini, et al. 2012 |
| Swordtail | *Xiphophorus helleri* | Buckingham et al. 2007 |
| Zebrafish | *Danio rerio* | Agrillo, Miletto Petrazzini, et al. 2012; Arthur et al. 2001 |
| *Insects* | | |
| Ant | *Cataglyphis fortis* | Wittlinger et al. 2007 |
| Bee | *Bombus terrestris* | Bar-Shai et al. 2011; Dacke et al. 2008; Gross 2011; Gross et al. 2009; Leppik 1953; Pahl et al. 2013; Van Hateren et al. 1990 |
| Beetle | *Tenebrio molitor* | Carazo et al. 2009; 2012 |
| Cicada | *Magicicada spp* | Karban et al. 2000 |
| Spider | *Portia africana* | Jackson et al. 2012; Nelson et al. 2012 |

*Notes*. Numerosity in various animal species, organized by sub-groups: vertebrates (mammals, birds, amphibians, fish) and arthropods (insects), drawn from a survey of the literature on comparative cognition. The tabular information is also depicted graphically in Figure 4.1.

# APPENDIX B

# COMPARISONS OF WORD FREQUENCY FOR ORDINAL NUMBERS IN ENGLISH, CHINESE, ARABIC, AND SUMERIAN

Table B.1

*Ordinal Number Word Frequency in English, Chinese, and Arabic*

| Ordinal Number | Frequency | | |
|---|---|---|---|
| | English | Chinese | Arabic |
| first | 463,566 | 593.92 | 143,626 |
| second | 103,621 | 350.36 | 28,511 |
| third | 67,037 | 145.73 | 34,276 |
| fourth | 23,227 | 63.88 | 16,914 |
| fifth | 13,150 | 36.93 | 11,472 |
| sixth | 8170 | 23.96 | 7968 |
| seventh | 6517 | 30.94 | 6265 |
| eighth | 5814 | 24.95 | 5996 |

*Notes*. The languages shown are American English (language family: Indo-European), Mandarin (Sino-Tibetan), and Arabic (Afro-Asiatic).[912] Though the scales differ, the word frequency for the ordinal numbers 'first' through 'eighth' show similar order (i.e., decreasing frequency from 'first' through 'eighth') and distribution (significant decrease of use frequency around 'third' and 'fourth') in all three corpora, which were formed by searching hundreds of thousands of documents and encompass thousands to millions of words. The tabular data are graphically displayed in Figure B.1.

[912] Davies et al. 2010; Kilgarriff et al. 2014; Xiao et al. 2004.

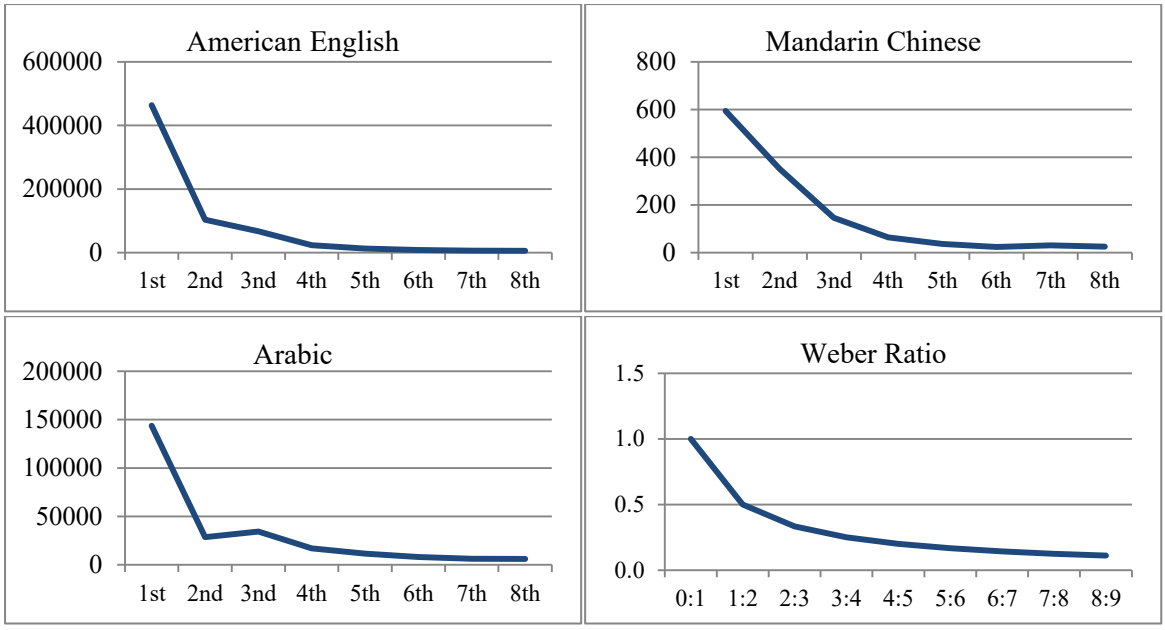



*Figure B.1*. Graphical display of ordinal number word frequency in English, Chinese, and Arabic. After an initial steep decrease around 'third', word frequency for higher ordinals tends to remain low. The frequency distribution in all three corpora resembles that of the Weber ratio (bottom right), suggesting the frequency ordinal numbers are used in language may be related to the ability to appreciate quantity (numerosity). The distribution might suggest restricted number systems, though all three are associated with unrestricted numbers and highly numerate cultures. Note that Arabic has an unusual higher frequency of word use for 'third' than 'second', as well as a small second increase between 'third' and 'fifth'.

Table B.2

*Ordinal Number Word Frequency in Sumerian (2500–1600 BC)*

| Ordinal | Frequency | | | | | | |
|---|---|---|---|---|---|---|---|
| Number | ED IIIb | OA | Lagash II | Ur III | Early OB | OldAssyr. | OB |
| 1st | | 6 | 4 | 2131 | 18 | 205 | 50 |
| 2nd | 2 | 8 | 5 | 2236 | 20 | 23 | 86 |
| 3rd | | 3 | | 2850 | 6 | 20 | 68 |
| 4th | | 3 | 1 | 2904 | 12 | 5 | 63 |
| 5th | | 3 | | 1015 | 7 | 9 | 67 |
| 6th | | 2 | | 1861 | 14 | 5 | 53 |
| 7th | | | | 1755 | 10 | | 70 |
| 8th | | | | 1764 | 11 | 4 | 23 |
| 9th | | | | 549 | 1 | | 17 |
| 12th | | | | | | | 2 |
| 15th | | | | | | | 2 |
| 18th | | | | | | | 1 |

*Notes*. The language shown is Sumerian (a linguistic isolate).[913] According to the data, ordinal numbers first appeared in the ED IIIb period; in Sumerian, they took the form #(diš)-kam (i.e., 1(diš)-kam = 'first', 2(diš)-kam = 'second', etc.). *Key*: ED = Early Dynastic IIIb period, 2500–2340 BC, sample comprised of 100.0% administrative texts; OA = Old Akkadian period, 2340–

[913] Data were obtained from the CDLI database on 5 October 2015.

2200 BC, 96.0% administrative texts and 4.0% royal/monumental inscriptions; Lagash II period, 2200–2100 BC, 100.0% administrative texts; Ur III period, 2100–2000 BC, 99.8% administrative texts and 0.1% school texts, with negligible amounts of legal texts, school texts, letters, royal/monumental inscriptions, genre uncertain, or not specified; EOB = Early Old Babylonian period, 2000–1900 BC, 99.0% administrative texts and 1.0% royal/monumental inscriptions; OldAssyr. = Old Assyrian period, 1950–1850 BC, 9.6% administrative texts, 4.1% legal texts, and 3.7% letters (the genre(s) of the remaining 82.7% was not specified in the source data); OB = Old Babylonian period, 1900–1600 BC, 46.2% literary texts, 28.7% administrative texts, 17.5% letters, 2.2% mathematical texts, 2.0% royal/monumental inscriptions, 1.8% lexical texts, 0.6% legal texts, 0.6% school texts, and 0.2% of uncertain genre. The few high OB ordinals suggest a possible influence of administrative cycles (i.e., ordinals pertaining to the twelfth, fifteenth, and eighteenth days of the month). Since modern ordinal frequency is unrelated to the development of the associated number system or the general level of numeracy in the community of speakers, Sumerian cannot be presumed to have had restricted numbers solely on the basis of the availability of ordinal numbers.

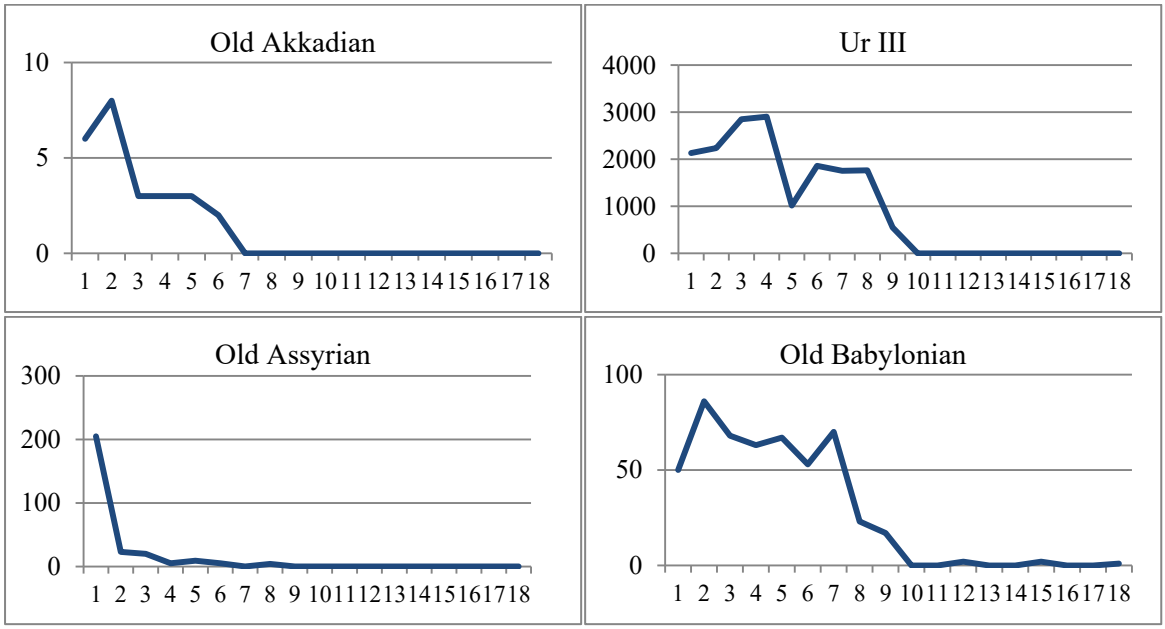



Figure B.2. Graphical display of ordinal number word frequency in Sumerian (2500–1600 BC). The top two graphs (Old Akkadian and Ur III) represent periods occurring before the attainment of a writing system with high fidelity to language[914]; as a consequence, these corpora are composed almost exclusively of administrative texts. In comparison, the bottom two graphs (Old Assyrian and Old Babylonian) represent time periods occurring after the writing system has developed high fidelity to language; relatedly, these corpora include a variety of textual genres. The distribution for the Old Assyrian period appears highly similar to those for modern languages (see Table B.1 and Figure B.1), as well as that of literary texts (Figure B.3). However, because 82.7% of the texts included are currently unassigned to a genre, caution is warranted before drawing any firm conclusions about their similarities. In contrast, the texts of the Old Babylonian period span a variety of genres, with very few unassigned texts. The distribution shows an unusual higher use frequency for 'second' than 'first', and significant use frequency for the ordinals 'third' to 'seventh'. These features may be artifices of the corpora data (i.e., few exemplars, the development of literacy, etc.) or an influence from administrative texts (compare to the distribution for administrative texts shown in Figure B.3).

[914] Overmann 2016a.

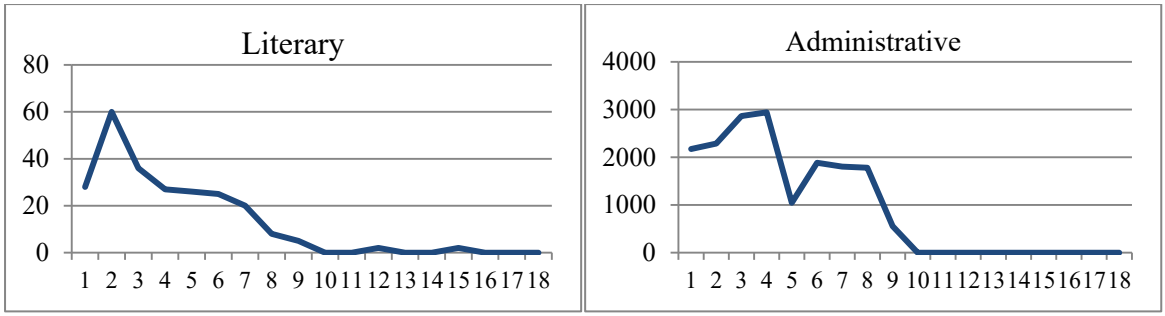



*Figure B.3*. Ordinal frequency in Sumerian literary and administrative texts. The distribution of literary texts includes the Old Babylonian (1900–1600 BC, *n* = 232, 98.3%), Middle Babylonian (1400–1100 BC, *n* = 3, 1.3%), and Neo-Babylonian periods (626–539 BC, *n* = 1, 0.4%). The distribution of administrative texts includes ED IIIb (2500–2340 BC, *n* = 2, 0.01%), Old Akkadian (2340–2200 BC, *n* = 24, 0.1%), Lagash II (2200–2100, *n* = 10, 0.1%), Ur III (2100–2000 BC, *n* = 17,026, 98.2%), Early Old Babylonian (2000–1900, *n* = 98, 0.6%), Old Assyrian (1950–1850 BC, *n* = 26, 0.2%), Old Babylonian (*n* = 144, 0.8%), and Neo-Assyrian periods (911–612 BC, *n* = 2, 0.01%). The administrative text distribution is nearly identical to that of the Ur III period (Figure B.2). Both distributions are characterized by a higher frequency of word use for 'second' relative to 'first' and elevated usage (relative to modern corpora) between 'fourth' and 'seventh' (literary) or 'eighth' (administrative). Interestingly, the literary distribution might be said to resemble the one for the Arabic corpus (though again, caution in interpreting the results is warranted because of the restrictions of the Sumerian sample).

# APPENDIX C

# COMPARISONS OF TOKENS AND NUMERICAL NOTATIONS (*CA*. 8500–3500 BC)

Table C.1

*Comparison of Numerical Impressions on Bullae and Shapes of Associated Tokens*

| Provenience | CDLI no | Type | $N_{64}$ | $N_{45}$ | $N_{34}$ | $N_{14}$ | $N_1$ | $N_{39}$ | $N_{41}$ | $N_3$ | Other |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Uruk (Iraq) | P274833 | Bulla | | 2 ($N_{45}$) | | 2 ($N_{14}$) | | 2 ($N_{39}$) | | | |
| | | Token(s) | | 2 SPHERES | | 2 spheres | | 2 disks | | | |
| | P274835 | Bulla | | | | 5 ($N_{14}$) | | | | | |
| | | Token(s) | | | | 5 spheres | | | | | |
| | P274843 | Bulla | | | | | | | | 7 ($N_3$) | |
| | | Token(s) | | | | | | | | 7 ovoid | |
| | P285630 | Bulla | 5 ($N_{64}$) | | | | | 2 ($N_{39}$) | | | |
| | | Token(s) | 5 TETRA | | | | | 2 disks | | | |
| | P285629 | Bulla | | | | 1 ($N_{14}$) | | 1 ($N_{39}$) | | | |
| | | Token(s) | | | | 1 sphere | | 1 disk | | | 1 square |
| | P285631 | Bulla | | | | | | 4 ($N_{39}$) | | | |
| | | Token(s) | | | | | | 4 disks | | | |
| | P285632 | Bulla | | | | | | 5 ($N_{39}$) | | | |
| | | Token(s) | | | | | | 5 disks | | | |

(Continued)

| Provenience | CDLI no. | Type | $N_{64}$ | $N_{45}$ | $N_{34}$ | $N_{14}$ | $N_1$ | $N_{39}$ | $N_{41}$ | $N_3$ | Other |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Uruk (Iraq) | P285633 | Bulla<br>Token(s) | | | | 4 ($N_{14}$)<br>4 spheres | | | | | |
| | P285634 | Bulla<br>Token(s) | | | | 4 ($N_{14}$)<br>4 spheres | | 1 ($N_{39}$)<br>1 disk | | | |
| | P285635 | Bulla<br>Token(s) | | | 1 ($N_{34}$)<br>1 CONE | 4 ($N_{14}$)<br>4 spheres | 1 ($N_1$)<br>1 cylinder | 3 ($N_{39}$)<br>1 disk | | | 2 lens |
| | P285636 | Bulla<br>Token(s) | | 2 SPHERES | 2 ($N_{34}$) | 1 sphere | 1 ($N_1$) | 1 ($N_{39}$)<br>1 disk | | | |
| | P285637 | Bulla<br>Token(s) | | 1 ($N_{45}$)<br>1 SPHERE | | | | 1 ($N_{39}$)<br>3 disks | | | |
| | P285638 | Bulla<br>Token(s) | | | | | | 3 ($N_{39}$)<br>3 disks | | | |
| Umma (Iraq) | P235737 | Bulla<br>Token(s) | | 1 ($N_{45}$)<br>1 cone | | 4 ($N_{14}$)<br>1 sphere | 2 ($N_1$)<br>1 cylinder | | 4 ($N_{41}$)<br>4 tetra | | 4 triangle |
| | P235738 | Bulla<br>Token(s) | | 1 ($N_{45}$)<br>1 SPHERE | 5 ($N_{34}$)<br>5 cones | 7 ($N_{14}$)<br>7 spheres | 3 ($N_1$)<br>3 cylinders | | | | |
| | P235739 | Bulla<br>Token(s) | | | | 5 ($N_{14}$)<br>1 spheres | | | 4 tetra | | |
| | P235740 | Bulla<br>Token(s) | | 1 ($N_{45}$)<br>1 SPHERE | | 4 ($N_{14}$)<br>4 spheres | 3 ($N_1$)<br>3 cylinders | | | | |
| | P235741 | Bulla<br>Token(s) | | | | 7 ($N_{14}$)<br>7? 1LG 6sm | 8 ($N_1$)<br>3 cylinders | 3 ($N_{39}$)<br>2 disks | 6 tetra | | |
| | P235742 | Bulla<br>Token(s) | | | | 6 ($N_{14}$)<br>6 spheres | 2 ($N_1$)<br>2 cylinders | | | | |
| | P235743 | Bulla<br>Token(s) | | 1 ($N_{45}$)<br>1? LG | | | | | | | |
| | P235744 | Bulla<br>Token(s) | | 1 ($N_{45}$)<br>2? LG | | | | | | | 4? sm |
| | P235745 | Bulla<br>Token(s) | | 1 ($N_{45}$)<br>1 SPHERE | | | | | | | |

(Continued)

| Provenience | CDLI no | Type | $N_{64}$ | $N_{45}$ | $N_{34}$ | $N_{14}$ | $N_1$ | $N_{39}$ | $N_{41}$ | $N_3$ | Other |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Susa (Iran) | P274841 | Bulla | | | | 4 ($N_{14}$) | 3 ($N_1$) | | | | |
| | | Token(s) | 1 TETRA | | | 2 spheres | 2 cylinders | | | | 2 bow ties |
| | P281696 | Bulla | | 3 ($N_{45}$) | | 3 ($N_{14}$) | 1 ($N_1$) | | | | |
| | | Token(s) | | 3 cones | | 3 spheres | | | | | 1 CONE |
| | P285645 | Bulla | | | | 3 ($N_{14}$) | 3 ($N_1$) | | | | |
| | | Token(s) | | | | 3 spheres | 3 cylinders | | | | |
| | P285654 | Bulla | | | | | 6 ($N_1$) | | | | |
| | | Token(s) | | | | | 6 cylinders | | | | |
| | P285655 | Bulla | | | | | 7 ($N_1$) | | | | |
| | | Token(s) | | | | | 7 cylinders | | | | |
| | P285657 | Bulla | | 1 ($N_{45}$) | | | 2 ($N_1$) | | | | |
| | | Token(s) | | | | | 2 cylinders | | | | |
| | P285658 | Bulla | | | | 3 ($N_{14}$) | 5 ($N_1$) | | | | |
| | | Token(s) | | | | 3 spheres | 5 cylinders | | | | |
| | P285659 | Bulla | | | | 4 ($N_{14}$) | 1 ($N_1$) | | | | |
| | | Token(s) | | | | 4 spheres | 1 cylinder | | | | 1 CONE |
| Tepe Yahya (Iran) | P285661 | Bulla | | | 1 ($N_{34}$) | 2 ($N_{14}$) | | | | | |
| | | Token(s) | | | 1 cone | 2 spheres | | | | | |
| Habuba Kabira (Syria) | P283935 | Bulla | | | | | | | | 6 ($N_3$) | |
| | | Token(s) | | | | | | | | 6 ovoids | |
| | P285660 | Bulla | | | | | 11 ($N_1$) | | | | |
| | | Token(s) | | | | | | | | 11 ovoids | |

*Notes*. Correspondences between tokens and numerical impressions from the cities of Uruk ($n$ = 13) and Umma ($n$ = 9), modern Iraq; Susa ($n$ = 8) and Tepe Yahya ($n$ = 1), modern Iran; and Habuba Kabira, modern Syria ($n$ = 2). The data were sourced from Amiet 1972; Damerow et al. 1995; Le Brun et al. 1978; Schmandt-Besserat 1992a; and the CDLI database. *Key*: The number of impressions (on bullae surfaces) and tokens (found either inside of or in proximity with the associated bulla) is indicated, along with the interpreted numerical impression and token shape (e.g., $N_1$ and cylinder). Large tokens are indicated in capital letters (e.g., CONE), small ones with small letters (e.g., sphere). Question marks indicate tokens of undetermined shape; their size is indicated as large or small (i.e., LG or sm). Tokens other than the plain shapes thought to have represented quantity are listed in the column 'Other'. Analysis of the data is presented in Chapter 8. Although Damerow (2012, 159) noted that the "combinations of differently shaped tokens in the [bullae] do not show regularities that would indicate the representation of standardized numerical systems," there is an overall similarity in the token–impression correspondences when considered as a group, though their representation of number also varied by location (and probably time).

# APPENDIX D

# NEWLY CATALOGUED TOKENS

Table D.1

*Newly Catalogued Tokens by Site, Reference(s), Shape, and Quantity*

| Site/References | Type | Quantity | Site/References | Type | Quantity | Site/References | Type | Quantity |
|---|---|---|---|---|---|---|---|---|
| Chagha Sefid, Iran (Hole 1977) | Cone | 10 | Chogha Bonut, Iran (Alizadeh 2003) | Cone | 7 | Chogha Mish, Iran (Delougaz et al. 1996) | Cone | 10 |
| | Sphere | 19 | | Sphere | 14 | | Sphere | 13 |
| | Disk | 15 | | Disk | 6 | | Disk | 17 |
| | Cylinder | 0 | | Cylinder | 0 | | Cylinder | 1 |
| | Tetrahedra | 0 | | Tetrahedra | 2 | | Tetrahedra | 10 |
| | Ovoid | 2 | | Ovoid | 5 | | Ovoid | 4 |
| | Rectangle | 0 | | Rectangle | 0 | | Rectangle | 1 |
| | Triangle | 0 | | Triangle | 1 | | Triangle | 8 |
| | Biconoid | 0 | | Biconoid | 0 | | Biconoid | 1 |
| | Paraboloid | 0 | | Paraboloid | 1 | | Paraboloid | 3 |
| | Oval-Rhomboid | 0 | | Oval-Rhomboid | 1 | | Oval-Rhomboid | 0 |
| | Animal | 0 | | Animal | 0 | | Animal | 2 |
| | Miscellaneous | 1 | | Miscellaneous | 1 | | Miscellaneous | 0 |
| | Total | 47 | | Total | 38 | | Total | 70 |

(Continued)

| Site/References | Type | Quantity |
|---|---|---|
| Ganj-Dareh, Iran (Morales et al. 1990) | Cone | 644 |
| | Sphere | 0 |
| | Disk | 0 |
| | Cylinder | 0 |
| | Tetrahedra | 0 |
| | Ovoid | 0 |
| | Rectangle | 0 |
| | Triangle | 0 |
| | Biconoid | 0 |
| | Paraboloid | 0 |
| | Oval-Rhomboid | 0 |
| | Animal | 0 |
| | Miscellaneous | 0 |
| | Total | 644 |
| Tepe Abdul Hosein, Iran (Pullar 1990) | Cone | 20 |
| | Sphere | 10 |
| | Disk | 0 |
| | Cylinder | 12 |
| | Tetrahedra | 0 |
| | Ovoid | 0 |
| | Rectangle | 0 |
| | Triangle | 0 |
| | Biconoid | 0 |
| | Paraboloid | 0 |
| | Oval-Rhomboid | 0 |
| | Animal | 0 |
| | Miscellaneous | 0 |
| | Total | 42 |
| Sheikh-e Abad, Iran (Cole et al. 2013) | Cone | 1 |
| | Sphere | 3 |
| | Disk | 2 |
| | Cylinder | 5 |
| | Tetrahedra | 0 |
| | Ovoid | 0 |
| | Rectangle | 0 |
| | Triangle | 0 |
| | Biconoid | 0 |
| | Paraboloid | 0 |
| | Oval-Rhomboid | 0 |
| | Animal | 0 |
| | Miscellaneous | 0 |
| | Total | 11 |
| Zagheh, Iran (Moghimi et al. 2014) | Cone | 94 |
| | Sphere | 84 |
| | Disk | 31 |
| | Cylinder | 0 |
| | Tetrahedra | 2 |
| | Ovoid | 14 |
| | Rectangle | 9 |
| | Triangle | 0 |
| | Biconoid | 0 |
| | Paraboloid | 0 |
| | Oval-Rhomboid | 0 |
| | Animal | 0 |
| | Miscellaneous | 4 |
| | Total | 238 |
| Tall-i-Bakun, Iran (Alizadeh 2006; Langsdorff et al. 1942) | Cone | 17 |
| | Sphere | 6 |
| | Disk | 16 |
| | Cylinder | 2 |
| | Tetrahedra | 1 |
| | Ovoid | 0 |
| | Rectangle | 1 |
| | Triangle | 2 |
| | Biconoid | 0 |
| | Paraboloid | 5 |
| | Oval-Rhomboid | 0 |
| | Animal | 0 |
| | Miscellaneous | 3 |
| | Total | 53 |
| Abu Salabikh, Iraq (Baker et al. 1993) | Cone | 7 |
| | Sphere | 4 |
| | Disk | 32 |
| | Cylinder | 8 |
| | Tetrahedra | 1 |
| | Ovoid | 5 |
| | Rectangle | 4 |
| | Triangle | 4 |
| | Biconoid | 0 |
| | Paraboloid | 0 |
| | Oval-Rhomboid | 0 |
| | Animal | 0 |
| | Miscellaneous | 9 |
| | Total | 74 |

(Continued)

| Site/References | Type | Quantity |
|---|---|---|
| Assur, Iraq (Schmitt 2012) | Cone | 7 |
| | Sphere | 5 |
| | Disk | 0 |
| | Cylinder | 0 |
| | Tetrahedra | 4 |
| | Ovoid | 0 |
| | Rectangle | 0 |
| | Triangle | 0 |
| | Biconoid | 0 |
| | Paraboloid | 0 |
| | Oval-Rhomboid | 0 |
| | Animal | 0 |
| | Miscellaneous | 0 |
| | Total | 16 |
| Tell Abada, Iraq (Jasim 1985; Jasim et al. 1986) | Cone | 64 |
| | Sphere | 84 |
| | Disk | 15 |
| | Cylinder | 9 |
| | Tetrahedra | 0 |
| | Ovoid | 0 |
| | Rectangle | 0 |
| | Triangle | 0 |
| | Biconoid | 0 |
| | Paraboloid | 0 |
| | Oval-Rhomboid | 0 |
| | Animal | 0 |
| | Miscellaneous | 8 |
| | Total | 180 |
| Jemdet Nasr, Iraq (MacKay 1931; Matthews 2002) | Cone | 40 |
| | Sphere | 12 |
| | Disk | 4 |
| | Cylinder | 0 |
| | Tetrahedra | 2 |
| | Ovoid | 0 |
| | Rectangle | 0 |
| | Triangle | 0 |
| | Biconoid | 0 |
| | Paraboloid | 0 |
| | Oval-Rhomboid | 0 |
| | Animal | 1 |
| | Miscellaneous | 2 |
| | Total | 61 |
| Tell Madhhur, Iraq (Roaf 1989) | Cone | 2 |
| | Sphere | 0 |
| | Disk | 0 |
| | Cylinder | 0 |
| | Tetrahedra | 0 |
| | Ovoid | 0 |
| | Rectangle | 0 |
| | Triangle | 0 |
| | Biconoid | 0 |
| | Paraboloid | 0 |
| | Oval-Rhomboid | 0 |
| | Animal | 0 |
| | Miscellaneous | 0 |
| | Total | 2 |
| Qal'at 'Ana, Iraq (Northedge et al. 1998) | Cone | 1 |
| | Sphere | 0 |
| | Disk | 0 |
| | Cylinder | 0 |
| | Tetrahedra | 0 |
| | Ovoid | 0 |
| | Rectangle | 0 |
| | Triangle | 0 |
| | Biconoid | 0 |
| | Paraboloid | 0 |
| | Oval-Rhomboid | 0 |
| | Animal | 0 |
| | Miscellaneous | 0 |
| | Total | 1 |
| Tell Oueili, Iraq (Breniquet 1987) | Cone | 0 |
| | Sphere | 0 |
| | Disk | 4 |
| | Cylinder | 0 |
| | Tetrahedra | 0 |
| | Ovoid | 0 |
| | Rectangle | 0 |
| | Triangle | 0 |
| | Biconoid | 0 |
| | Paraboloid | 0 |
| | Oval-Rhomboid | 0 |
| | Animal | 0 |
| | Miscellaneous | 0 |
| | Total | 4 |

(Continued)

| Site/References | Type | Quantity |
|---|---|---|
| Es-Sifya, Jordan (Mahasneh et al. 1998) | Cone | 36 |
| | Sphere | 38 |
| | Disk | 1 |
| | Cylinder | 0 |
| | Tetrahedra | 0 |
| | Ovoid | 0 |
| | Rectangle | 0 |
| | Triangle | 0 |
| | Biconoid | 0 |
| | Paraboloid | 0 |
| | Oval-Rhomboid | 0 |
| | Animal | 0 |
| | Miscellaneous | 3 |
| | Total | 78 |
| Amuq, Syria (Yener et al. 2000) | Cone | 3 |
| | Sphere | 5 |
| | Disk | 5 |
| | Cylinder | 0 |
| | Tetrahedra | 0 |
| | Ovoid | 0 |
| | Rectangle | 0 |
| | Triangle | 0 |
| | Biconoid | 0 |
| | Paraboloid | 0 |
| | Oval-Rhomboid | 0 |
| | Animal | 0 |
| | Miscellaneous | 0 |
| | Total | 13 |
| ’Ain Ghazal, Palestine (Iceland n.d.) | Cone | 22 |
| | Sphere | 95 |
| | Disk | 14 |
| | Cylinder | 1 |
| | Tetrahedra | 0 |
| | Ovoid | 4 |
| | Rectangle | 0 |
| | Triangle | 0 |
| | Biconoid | 0 |
| | Paraboloid | 0 |
| | Oval-Rhomboid | 0 |
| | Animal | 0 |
| | Miscellaneous | 1 |
| | Total | 137 |
| Tell ’Atij, Syria (Weiss 1991) | Cone | 8 |
| | Sphere | 3 |
| | Disk | 5 |
| | Cylinder | 1 |
| | Tetrahedra | 0 |
| | Ovoid | 0 |
| | Rectangle | 0 |
| | Triangle | 0 |
| | Biconoid | 0 |
| | Paraboloid | 0 |
| | Oval-Rhomboid | 1 |
| | Animal | 0 |
| | Miscellaneous | 3 |
| | Total | 21 |
| Abu Hureyra, Syria (Moore et al. 2000) | Cone | 0 |
| | Sphere | 8 |
| | Disk | 5 |
| | Cylinder | 0 |
| | Tetrahedra | 0 |
| | Ovoid | 1 |
| | Rectangle | 0 |
| | Triangle | 0 |
| | Biconoid | 0 |
| | Paraboloid | 0 |
| | Oval-Rhomboid | 0 |
| | Animal | 0 |
| | Miscellaneous | 3 |
| | Total | 17 |
| Tell Brak, Syria (Oates et al. 2001) | Cone | 8 |
| | Sphere | 20 |
| | Disk | 51 |
| | Cylinder | 0 |
| | Tetrahedra | 0 |
| | Ovoid | 0 |
| | Rectangle | 1 |
| | Triangle | 0 |
| | Biconoid | 0 |
| | Paraboloid | 0 |
| | Oval-Rhomboid | 0 |
| | Animal | 0 |
| | Miscellaneous | 6 |
| | Total | 86 |

(Continued)

| Site/References | Type | Quantity |
|---|---|---|
| Amuq, Tell Kurdu, Turkey (Yener et al. 2000) | Cone | 2 |
| | Sphere | 0 |
| | Disk | 2 |
| | Cylinder | 0 |
| | Tetrahedra | 0 |
| | Ovoid | 0 |
| | Rectangle | 0 |
| | Triangle | 0 |
| | Biconoid | 0 |
| | Paraboloid | 0 |
| | Oval-Rhomboid | 0 |
| | Animal | 0 |
| | Miscellaneous | 0 |
| | Total | 4 |
| Hacınebi, Turkey (Stein 2002; Stein et al. 1996) | Cone | 0 |
| | Sphere | 10 |
| | Disk | 2 |
| | Cylinder | 0 |
| | Tetrahedra | 0 |
| | Ovoid | 0 |
| | Rectangle | 0 |
| | Triangle | 0 |
| | Biconoid | 0 |
| | Paraboloid | 0 |
| | Oval-Rhomboid | 0 |
| | Animal | 0 |
| | Miscellaneous | 0 |
| | Total | 12 |
| Cafer Höyük, Turkey (Cauvin et al. 1999) | Cone | 1 |
| | Sphere | 1 |
| | Disk | 0 |
| | Cylinder | 0 |
| | Tetrahedra | 0 |
| | Ovoid | 0 |
| | Rectangle | 0 |
| | Triangle | 0 |
| | Biconoid | 0 |
| | Paraboloid | 0 |
| | Oval-Rhomboid | 0 |
| | Animal | 0 |
| | Miscellaneous | 0 |
| | Total | 2 |
| Ziyaret Tepe, Turkey (MacGinnis et al. 2014; Ziyaret Tepe Project database) | Cone | 137 |
| | Sphere | 33 |
| | Disk | 73 |
| | Cylinder | 81 |
| | Tetrahedra | 142 |
| | Ovoid | 0 |
| | Rectangle | 19 |
| | Triangle | 0 |
| | Biconoid | 0 |
| | Paraboloid | 0 |
| | Oval-Rhomboid | 0 |
| | Animal | 0 |
| | Miscellaneous | 5 |
| | Total | 490 |
| Çatalhöyük, Turkey (Hamilton et al. 1996) | Cone | 0 |
| | Sphere | 9 |
| | Disk | 2 |
| | Cylinder | 0 |
| | Tetrahedra | 0 |
| | Ovoid | 0 |
| | Rectangle | 0 |
| | Triangle | 0 |
| | Biconoid | 0 |
| | Paraboloid | 0 |
| | Oval-Rhomboid | 0 |
| | Animal | 0 |
| | Miscellaneous | 0 |
| | Total | 11 |

*Notes*. The newly catalogued tokens are listed by provenience, modern country in which the site is located, and the source references where the data were obtained. Overall, the new tokens occurred within 13 of Schmandt-Besserat's 16 categories[915]; most were the types thought to have been used in accounting (e.g., cones, spheres, disks, etc.). Also see the discussion in Chapter 8.

[915] Schmandt-Besserat 1992a.

# APPENDIX E

# ANALYSIS OF NUMERICAL IMPRESSIONS FROM THE URUK V PERIOD (*CA.* 3500–3350 BC)

Table E.1

*Uruk V Numerical Impressions by Source, Sign, Quantity, and System*

| Source | CDLI no. | Face | $N_{45}$ | $N_{48}$ | $N_{34}$ | $N_{15}$ | $N_{14}$ | $N_1$ | $N_{39}$ | $N_{24}$ | $N_{22}$ | $N_{23}$ | $N_{51}$ | Other | System |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Uruk | P000753 | Obverse Col 1 | | | | | 1 | 9 | | | | | | | S, B, E, U |
| | | Obverse Col 2 | | | | | | 5 | | | | | | | |
| | | Obverse Col 3 | | | | | | 3 | | | | | | | |
| | | Bottom Col 1 | | | | | | 3 | | | | | | | |
| | | Bottom Col 2 | | | | | | 1 | | | | | | | |
| | | Reverse Col 1 | | | | | | 6 | | | | | | | |
| | | Reverse Col 2 | | | | | | 5 | | | | | | | |
| | P000754 | Obverse Col 1 | | | | | | 2 | | | | | | | S, B, G, E, U, Š |
| | | Obverse Col 2 | | | | | | 2 | | | | | | | |
| | | Bottom Col 1 | | | | | | 2 | | | | | | | |
| | | Reverse Col 1 | | | | | | 1 | | | | | | | |
| | P000755 | Obverse Col 1 | | | | | 4 | 4 | | | | | | | S, B, G, E, U, Š |
| | P000793 | Obverse Col 1 | | | | | | 4 | | | | | | | S, B, G, E, U, Š |
| | | Obverse Col 2 | | | | | | 1 | | | | | | | |

(Continued)

| Source | CDLI no. | Face | $N_{45}$ | $N_{48}$ | $N_{34}$ | $N_{15}$ | $N_{14}$ | $N_{1}$ | $N_{39}$ | $N_{24}$ | $N_{22}$ | $N_{23}$ | $N_{51}$ | Other | System |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Uruk | P000794 | Obverse Col 1 | | | | | | 3 | | | | | | | S, B, G, E, U, Š |
| | | Obverse Col 2 | | | | | | 2 | | | | | | | |
| | | Bottom Col 1 | | | | | | 1 | | | | | | | |
| | P000795 | Obverse Col 1 | | | | | | 3 | | | | | | | S, B, G, E, U, Š |
| | P000796 | Obverse Col 1 | | | | | | 5 | | | | | | | S, B, G, E, U, Š |
| | | Obverse Col 2 | | | | | | 5 | | | | | | | |
| | | Reverse Col 1 | | | | | | 1 | | | | | | | |
| | | Reverse Col 2 | | | | | | 2 | | | | | | | |
| | P000797 | Obverse Col 1 | | | | | 2 | | | | | | | | S, B, G, E, U, Š |
| | | Reverse Col 1 | | | | | 3 | | | | | | | | |
| | P000798 | Obverse Col 2 | | | | | | 2 | | | | | | | S, B, G, E, U, Š |
| | P000799 | Obverse Col 1 | | | | | | 2 | | | | | | | S, B, G, E, U, Š |
| | P000800 | Obverse Col 1 | | | | | | 2 | | | | | | | S, B, G, E, U, Š |
| | | Obverse Col 2 | | | | | | 1 | | | | | | | |
| | P000801 | Obverse Col 1 | | | | | | 4 | | | | | | | S, B, G, E, U, Š |
| | P000803 | Obverse Col 1 | | | | | | 2 | | | | | | | S, B, G, E, U, Š |
| | | Obverse Col 2 | | | | | | 3 | | | | | | | |
| | P000842 | Obverse Col 1 | | | | | 2 | 5 | | | | | | DUG~b | S, B, G, E, U, Š |
| | P000843 | Obverse Col 1 | | | | | | 3 | | | | | | | S, B, G, E, U, Š |
| | P000845 | Obverse Col 1 | | | | | | 2 | | | | | | | S, B, G, E, U, Š |
| | P000847 | Obverse Col 1 | | | | | | 3 | | | | | | | S, B, G, E, U, Š |
| | | Obverse Col 2 | | | | | | 2 | | | | | | | |
| | P000852 | Obverse Col 1 | | | 1 | | 1 | 4 | | | | | | UDU~a | S, B |
| | | Reverse Col 1 | | | | | | 9 | | | | | | UDU~a | |
| | P000899 | Obverse Col 1 | | | 1 | | 4 | 3 | | | | | | | S, Š, B |
| | P000901 | Obverse Col 1 | | | | | | 2 | | | | | | | S, B, G, E, U, Š |
| | P000902 | Obverse Col 1 | | | 2 | | | 5 | | | | | | | S, Š, B |
| | P000904 | Obverse Col 1 | | | | | | 2 | | | | | | | S, B, G, E, U, Š |
| | P000905 | Obverse Col 1 | | | 1 | | 2 | 2 | | | | | | | S, Š, B |
| | P000906 | Obverse Col 1 | | | 2 | | 2 | | | | | | | | S, Š, B |
| | | Reverse Col 1 | | | | | | 3 | | | | | | | |

(Continued)

| Source | CDLI no. | Face | $N_{45}$ | $N_{48}$ | $N_{34}$ | $N_{15}$ | $N_{14}$ | $N_{1}$ | $N_{39}$ | $N_{24}$ | $N_{22}$ | $N_{23}$ | $N_{51}$ | Other | System |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Uruk | P000913 | Obverse Col 1 | | | | | 1 | 2 | | | | | | | S, B, G, E, U, Š |
| | P000918 | Obverse Col 1 | | | | | 1 | 2 | | | | | | | Š |
| | | Reverse Col 1 | | | | | 3 | | 2 | | | | | | |
| | P000929 | Obverse Col 1 | | | | | | 1 | | | | | | | S, B, G, E, U, Š |
| | | Obverse Col 2 | | | | | | 7 | | | | | | | |
| | P000930 | Obverse Col 1 | | | | | | 1 | | | | | | | S, B, G, E, U, Š |
| | P000932 | Obverse Col 1 | | | | | | 4 | | | | | | | S, B, G, E, U, Š |
| | P000935 | Obverse Col 1 | | | | | 1 | | | | | | | | S, B, G, E, U, Š |
| | P000940 | Obverse Col 1 | | | | | 2 | | | | | | | | S, B, G, E, U, Š |
| | P000941 | Obverse Col 1 | | | | | 1 | 3 | | | | | | | S, B, G, E, U, Š |
| | P000942 | Obverse Col 1 | | | | | | 5 | | | | | | | S, B, G, E, U, Š |
| | | Bottom Col 1 | | | | | | 2 | | | | | | | |
| | P000944 | Obverse Col 1 | | | | | | 3 | | | | | | | S, B, G, E, U, Š |
| | | Obverse Col 2 | | | | | | 2 | | | | | | | |
| | P000949 | Obverse Col 1 | | | | | 1 | 2 | | | | | | | S, B, G, E, U, Š |
| | P000952 | Obverse Col 1 | | | | | 1 | 3 | | | | | | | S, B, G, E, U, Š |
| | | Obverse Col 2 | | | | | | 4 | | | | | | | |
| | | Bottom Col 1 | | | | | | 1 | | | | | | | |
| | P000954 | Obverse Col 1 | | | | | 1 | 2 | | | | | | | S, B, G, E, U, Š |
| | P000957 | Bottom Col 1 | | | | | | | | | | 5 | | | Not assigned to a system. |
| | P000970 | Obverse Col 1 | | | | | | 4 | | | | | | | S, B, G, E, U, Š |
| | | Obverse Col 2 | | | | | | 4 | | | | | | | |
| | P000971 | Obverse Col 1 | | | | | | 3 | | | | | | | S, B, G, E, U, Š |
| | | Obverse Col 1b | | | | | | 2 | | | | | | | |
| | | Obverse Col 2 | | | | | | 2 | | | | | | | |
| | | Obverse Col 3 | | | | | | 2 | | | | | | | |
| | | Reverse Col 1 | | | | | | 4 | | | | | | | |

(Continued)

| Source | CDLI no. | Face | $N_{45}$ | $N_{48}$ | $N_{34}$ | $N_{15}$ | $N_{14}$ | $N_{1}$ | $N_{39}$ | $N_{24}$ | $N_{22}$ | $N_{23}$ | $N_{51}$ | Other | System |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Uruk | P000972 | Obverse Col 1 | | | | | | 4 | | | | | | | S, B, G, E, U, Š |
| | | Obverse Col 1b | | | | | | 2 | | | | | | | |
| | | Obverse Col 2 | | | | | | 7 | | | | | | | |
| | | ObverseCol2b | | | | | | 2 | | | | | | | |
| | | Bottom Col 1 | | | | | | 2 | | | | | | | |
| | | Bottom Col 2 | | | | | | 2 | | | | | | | |
| | P000973 | Obverse Col 1 | | | | | | 4 | | | | | | | S, B, G, E, U, Š |
| | | Obverse Col 2 | | | | | | 5 | | | | | | | |
| | P000974 | Obverse Col 1 | | | | | 1 | 5 | | | | | | | S, B, G, E, U, Š |
| | P000975 | Obverse Col 1 | | | | | | 3 | | | | | | | S, B, G, E, U, Š |
| | | Obverse Col 2 | | | | | | 2 | | | | | | | |
| | | Obverse Col 3 | | | | | | 1 | | | | | | | |
| | P000976 | Obverse Col 1 | | | | | | 2 | | | | | | | S, B, G, E, U, Š |
| | P000987 | Obverse Col 1 | | | | | | 8 | | | | | | | S, B, G, E, U, Š |
| | | Bottom Col 1 | | | | | 2 | | | | | | | | |
| | | Bottom Col 2 | | | | | | 4 | | | | | | | |
| | | Bottom Col 3 | | | | | | 1 | | | | | | | |
| | | Bottom Col 3b | | | | | | 2 | | | | | | | |
| | | Reverse Col 1 | | | | | | 6 | | | | | | | |
| | | Reverse Col 2 | | | | | 2 | 2 | | | | | | | |
| | P000992 | Obverse Col 1 | | | | | 5 | | | | | | | | S, B, G, E, U, Š |
| | P001013 | Obverse Col 2 | | | | | | 1 | | | | | | | S, B, G, E, U, Š |
| | P001014 | Obverse Col 1 | | | | | | 1 | | | | | | | S, B, G, E, U, Š |
| | P001225 | Obverse Col 1 | | | | | | 6 | | | | | | | S, B, G, E, U, Š |
| | | Obverse Col 2 | | | | | | 7 | | | | | | | |
| | | Bottom Col 1 | | | | | | 5 | | | | | | | |
| | | Reverse Col 2 | | | | | | 2 | | | | | | | |
| | P001311 | Obverse Col 1 | | | | | | 3 | | | | | | | S, B, G, E, U, Š |
| | P001545 | Obverse Col 1 | | | | | 1 | | | | | | | | S, B, G, E, U, Š |
| | P001546 | Obverse Col 1 | | | | | 1 | 6 | | | | | | | S, B, E, U |
| | P001547 | Obverse Col 1 | | | | | 1 | | | | | | | | S, B, G, E, U, Š |
| | | Reverse Col 1 | | | | | | | | | | | | X | |
| | P001548 | Obverse Col 1 | | | | | | 2 | | | | | | | S, B, G, E, U, Š |

(Continued)

| Source | CDLI no. | Face | $N_{45}$ | $N_{48}$ | $N_{34}$ | $N_{15}$ | $N_{14}$ | $N_{1}$ | $N_{39}$ | $N_{24}$ | $N_{22}$ | $N_{23}$ | $N_{51}$ | Other | System |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Uruk | P001690 | Obverse Col 1 | | | | | | 3 | | | | | | | S, B, G, E, U, Š |
| | P004402 | Obverse Col 1 | | | | | 4 | | | | | | | | S, B, G, E, U, Š |
| | P005067 | Obverse Col 1 | | | | | 3 | 3 | | | | | | | S, B, G, E, U, Š |
| Tell Jokha | P200007 | Obverse Col 1 | | | | | 3 | | | | | | | | S, B, G, E, U, Š |
| | P200008 | Obverse Col 1 | | | | | 3 | 4 | | | | | | | S, B, G, E, U, Š |
| | P200009 | Obverse Col 1<br>Obverse Col 2 | | | | | 6<br>1 | | | | | | | | G, Š |
| | P200010 | Obverse Col 1 | | | | | **12** | | | | | | | | G (Note 2) |
| | P006269 | Obverse Col 1 | | | | | 7 | 2 | | | 2 | | | | G |
| Tutub | P235768 | Obverse Col 1 | | | | | 1 | 5 | | | | | | | S, B, G, E, U, Š |
| Nineveh | P235769 | Obverse Col 1 | | | | | 3 | | | | | | | | S, B, G, E, U, Š |
| Jemdet Nasr | P005303 | Obverse Col 1 | | | | | | 8 | | | | | | | S, B, G, E, U, Š |
| | P005304 | Obverse Col 1<br>Obverse Col 2 | | | | | 5<br>3 | | | | | | | | G (Note 2) |
| Susa | P008209 | Reverse Col 1 | | | | | | 6 | | | | | | | S, B, G, E, U, Š |
| | P008218 | Obverse Col 1 | 2 | | | | 8 | 5 | | | | | | | S, Š, G |
| | P008226 | Obverse Col 1 | 2 | | | | 2 | | | | | | | | S, Š, G |
| | P008256 | Obverse Col 1<br>Obverse Col 2 | | | | | | 6<br>7 | | | | | | | S, B, G, E, U, Š |
| | P008268 | Obverse Col 1 | | | | | | 8 | | | | | | | S, B, G, E, U, Š |
| | P008280 | Obverse Col 1 | | | 9 | | 1 | | | | | | | | S, Š |
| | P008304 | Obverse Col 1 | | | | | 2 | 3 | | | | | | | S, B, G, E, U, Š |
| | P008306 | Obverse Col 1 | | | | | 4 | | | | | | | | S, B, G, E, U, Š |
| | P008336 | Obverse Col 1 | | | | | | 4 | | | | | | | S, B, G, E, U, Š |
| | P008358 | Obverse Col 1<br>Obverse Col 2<br>Obverse Col 3 | | | | | | 4<br>4<br>4 | | | | | | X | S, B, G, E, U, Š |
| | P008363 | Obverse Col 1<br>Reverse Col 1<br>Top | | 1 | 1 | | 4<br>4<br>1 | | | | | | | | Š |
| | P008364 | Obverse Col 1 | | | | | 1 | 8 | | | | | | | S, B, E, U |
| | P008367 | Obverse Col 1<br>Reverse Col 1 | | | | | | 3 | | | | | | M072 | S, B, G, E, U, Š |

(Continued)

| Source | CDLI no. | Face | $N_{45}$ | $N_{48}$ | $N_{34}$ | $N_{15}$ | $N_{14}$ | $N_{1}$ | $N_{39}$ | $N_{24}$ | $N_{22}$ | $N_{23}$ | $N_{51}$ | Other | System |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Susa | P008374 | Obverse Col 1 | | | | | | 5 | | | | | | | S, B, G, E, U, Š |
| | | Reverse Col 1 | | | | | 1 | | | | | | | | |
| | P008400 | Obverse Col 2 | | | | | | | | | | | **7** | | B |
| | | Obverse Col 3 | | | | | | 2 | | | | | | | |
| | P008459 | Obverse Col 1 | | | | | 1 | | | | | | | | S, B, G, E, U, Š |
| | | Obverse Col 2 | | | | | | 4 | | | | | | | |
| | P008462 | Obverse Col 1 | | | | | 5 | | | | | | | | G (Note 1) |
| | | Obverse Col 2 | | | | | 4 | | | | | | | | |
| | P008488 | Obverse Col 1 | | | 2 | | | | | | | | | | S, Š, B |
| | P008553 | Obverse Col 1 | | | | | 4 | | | | | | | | G (Note 1) |
| | | Obverse Col 2 | | | | | 3 | | | | | | | | |
| | | Obverse Col 3 | | | | | **14** | | | | | | | | |
| | | Obverse Col 3b | | | | | **14** | | | | | | | | |
| | P008560 | Obverse Col 1 | | | | | 6 | | | | | | | | G (Note 1) |
| | | Obverse Col 2 | | | | | 4 | | | | | | | | |
| | | Obverse Col 3 | | | | | 6 | | | | | | | | |
| | | Obverse Col 3b | | | | | 6 | | | | | | | | |
| | P008632 | Obverse Col 1 | | | | | 6 | 3 | | | | | | | G, Š |
| | P008673 | Obverse Col 1 | | | | | 2 | | | | | | | | G (Note 1) |
| | | Obverse Col 2 | | | | | 5 | | | | | | | | |
| | | Obverse Col 3 | | | | | 9 | | | | | | | | |
| | | Obverse Col 3b | | | | | 9 | | | | | | | | |
| | P008682 | Obverse Col 1 | | | | | 2 | 5 | | | | | | | S, B, G, E, U, Š |
| | | Obverse Col 2 | | | | | | 1 | | | | | | | |
| | | Reverse Col 1 | | | | | 1 | 4 | | | | | | | |
| | P009501 | Obverse Col 1 | | | | | 2 | 3 | | | | | | | G (Note 2) |
| | | Obverse Col 2 | | | | | 5 | | | | | | | | |
| | P008962 | Obverse Col 2 | | | 2 | | 4 | 4 | | | | | | $\lvert M_{351}+1(N_{14})\rvert$ | S, Š, B |
| | P009093 | Obverse Col 1 | 2 | | 3 | | 1 | 3 | | | | | | M017 | Š |
| | P009170 | Obverse Col 1 | | | | | 1 | | | | | | | | S, B, G, E, U, Š |
| | P009171 | Obverse Col 1 | | | | | | 1 | | | | | | | S, B, G, E, U, Š |
| | P009173 | Obverse Col 1 | | | | | 2 | | | | | | | | S, B, G, E, U, Š |
| | P009212 | Obverse Col 1 | 4 | | | | | | | | | | | | S, Š, G |

(Continued)

| Source | CDLI no. | Face | $N_{45}$ | $N_{48}$ | $N_{34}$ | $N_{15}$ | $N_{14}$ | $N_{1}$ | $N_{39}$ | $N_{24}$ | $N_{22}$ | $N_{23}$ | $N_{51}$ | Other | System |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Susa | P009215 | Obverse Col 1 | | | | | 5 | 2 | | | | | | | G (Note 2) |
| | | Obverse Col 2 | | | | | 1 | 4 | | | | | | | |
| | | Obverse Col 3 | | | | | 1 | | | | | | | | |
| | | Obverse Col 3b | | | | | 4 | 5 | | | | | | | |
| | P009218 | Obverse Col 1 | | | | | 5 | | | | | | | | S, B, G, E, U, Š |
| | P009323 | Obverse Col 1 | 2 | | | | 8 | 3 | | | 2 | | | M157~a | G |
| | P009387 | Obverse Col 1 | | | | | 5 | | | | | | | | S, B, G, E, U, Š |
| | | Reverse Col 1 | | | | | | | | | | | | X | |
| | P009405 | Obverse Col 1 | | | | | | 3 | | | | | | | S, B, G, E, U, Š |
| | P009406 | Obverse Col 1 | | | | | 1 | 1 | | | | | | | S, B, G, E, U, Š |
| | P009407 | Obverse Col 1 | | | | | | 3 | | | | | | | S, Š, B |
| | | Reverse Col 1 | | | 2 | | | | | | | | | | |
| | P009408 | Obverse Col 1 | | | | | | 4 | | | | | | | S, B, G, E, U, Š |
| | P009409 | Obverse Col 1 | | | | | 1 | 1 | | | | | | | S, B, G, E, U, Š |
| | P009410 | Obverse Col 1 | | | | | 1 | 6 | | | | | | | S, B, G, E, U, Š |
| | P009426 | Obverse Col 1 | | | | | 2 | 2 | | | | | | | S, B, G, E, U, Š |
| | P009427 | Obverse Col 1 | | | | | 1 | 1 | | | | | | | S, B, G, E, U, Š |
| | P009428 | Obverse Col 1 | | | | | | 5 | | | | | | | S, B, G, E, U, Š |
| | P009429 | Obverse Col 1 | | | | | | 6 | | | | | | | S, B, G, E, U, Š |
| | P009430 | Obverse Col 1 | | | | | | 2 | | | | | | | S, B, G, E, U, Š |
| | P009431 | Obverse Col 1 | | | | | 1 | 3 | | | | | | | S, B, G, E, U, Š |
| | P235750 | Obverse Col 1 | | | | | 1 | | | | | | | GADA~a@g | S, B, G, E, U, Š |
| | P235751 | Obverse Col 1 | | | | | 1 | 4 | | | | | | ZATU644 | S, B, G, E, U, Š |
| | P281697 | Obverse Col 1 | | | | | 1 | | | | | | | | S, B, G, E, U, Š |
| | P281699 | Obverse Col 1 | | | | | 3 | | | | | | | | S, B, G, E, U, Š |
| | P281700 | Obverse Col 1 | | | | | | 1 | | | | | | | S, B, G, E, U, Š |
| | P281703 | Obverse Col 1 | | | | | 2 | | | | | | | | S, B, G, E, U, Š |
| | P281708 | Obverse Col 1 | | | | | | 2 | | | | | | | S, B, G, E, U, Š |
| | P281709 | Obverse Col 1 | | | | | | 9 | | | | | | | S, B, G, E, U, Š |
| | P281710 | Obverse Col 1 | 1 | | | | 4 | | | | | | | | S, Š, G |
| | P281711 | Obverse Col 1 | | | | | 6 | 2 | | | | | | | G, Š |
| | P281713 | Obverse Col 1 | 2 | | 2 | | 3 | | | | | | | | Š |
| | P281715 | Obverse Col 1 | | | | | 3 | | | | | | | | S, B, G, E, U, Š |

(Continued)

| Source | CDLI no. | Face | $N_{45}$ | $N_{48}$ | $N_{34}$ | $N_{15}$ | $N_{14}$ | $N_{1}$ | $N_{39}$ | $N_{24}$ | $N_{22}$ | $N_{23}$ | $N_{51}$ | Other | System |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Susa | P281717 | Obverse Col 1 | | | | | 1 | | | | | | | | S, B, G, E, U, Š |
| | P281718 | Obverse Col 1 | | | | | | 4 | | | | | | | S, B, G, E, U, Š |
| | P281719 | Obverse Col 1 | | | | | 4 | | | | | | | | S, B, G, E, U, Š |
| | P281720 | Obverse Col 1 | | | | | 1 | 2 | | | | | | | S, B, G, E, U, Š |
| | P281721 | Obverse Col 1 | | | | | | 3 | | | | | | | S, B, G, E, U, Š |
| | P281722 | Obverse Col 1 | | | | | 4 | | | | | | | | S, B, G, E, U, Š |
| | P281723 | Obverse Col 1 | | | | 2 | | | 4 | | | | | | $N_{15}$ = S'; $N_{39}$ = Š |
| | P281724 | Obverse Col 1 | 2 | | | | 4 | 1 | | | | | | | S, Š, G |
| | P281725 | Obverse Col 1 | | | | | | 3 | | | | | | | S, B, G, E, U, Š |
| | P281726 | Obverse Col 1 | | | | | 1 | | | | | | | | S, B, G, E, U, Š |
| | P281728 | Obverse Col 1 | | | | | 2 | 1 | | | | | | | S, B, G, E, U, Š |
| | P281729 | Obverse Col 1 | | | | | | 3 | | | | | | | S, B, G, E, U, Š |
| | P281730 | Obverse Col 1 | 1 | | | | | | | | | | | | S, Š, G |
| | P281731 | Obverse Col 1 | 1 | | 3 | | 4 | | | | | | | | Š |
| | P283936 | Obverse Col 1 | | | | | 6 | | | | | | | | G, Š |
| | | Obverse Col 2 | | | | | 5 | | | | | | | | |
| | | Obverse Col 3 | | | | | 6 | | | | | | | | |
| | | Obverse Col 3b | | | | | 5 | | | | | | | | |
| | P333432 | Obverse Col 1 | | | | | 5 | | | | | | | | S, B, G, E, U, Š |
| | P333451 | Obverse Col 1 | | | | | 1 | | | | | | | | S, B, G, E, U, Š |
| Chogha Mish | P009401 | Obverse Col 1 | | | | | 1 | 3 | | | | | | | S, B, G, E, U, Š |
| | | Obverse Col 2 | | | | | | 2 | | | | | | | |
| Godin Tepe | P009447 | Obverse Col 1 | | | | | 2 | 3 | | | | | | | S, B, G, E, U, Š |
| | P009448 | Obverse Col 1 | | | | | 3 | 3 | | | | | | | S, B, G, E, U, Š |
| | P009449 | Obverse Col 1 | | | | | 4 | 4 | | | | | | | S, B, G, E, U, Š |
| | P009450 | Obverse Col 1 | | | | | 5 | 3 | | | | | | | S, B, G, E, U, Š |
| | P009451 | Obverse Col 1 | | | | | | 4 | | | | | | | Š |
| | | Obverse Col 2 | | | | | 1 | | 4 | 1 | | | | | |
| | P009452 | Obverse Col 1 | | | | | 1 | 4 | | | | | | | S, B, G, E, U, Š |
| | | Reverse Col 1 | | | | | 1 | 3 | | | | | | | |
| | P009453 | Obverse Col 1 | | | | | 2 | 3 | | | | | | | G, Š |
| | | Reverse Col 1 | | | | | 9 | | | | | | | | |
| | P009455 | Obverse Col 1 | | | | | | 4 | | | | | | M009 | S, B, G, E, U, Š |

(Continued)

| Source | CDLI no. | Face | $N_{45}$ | $N_{48}$ | $N_{34}$ | $N_{15}$ | $N_{14}$ | $N_{1}$ | $N_{39}$ | $N_{24}$ | $N_{22}$ | $N_{23}$ | $N_{51}$ | Other | System |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Godin Tepe | P283933 | Obverse Col 1 | | | | | | 5 | 2 | 1 | | | | | Š |
| | P333406 | Obverse Col 1 | | | 2 | | 4 | 6 | | | | | | | S, Š, B |
| Tepe Sialk | P009507 | Obverse Col 1 | 2 | | | | 5 | | | | | | | | S, Š, G |
| | P009508 | Obverse Col 1 | | | | | 1 | | | | | | | | S, B, G, E, U, Š |
| | | Reverse Col 1 | | | | | 2 | | | | | | | | |
| | P009511 | Obverse Col 1 | | | | | 1 | | | | | | | X | S, B, G, E, U, Š |
| | | Reverse Col 1 | | | | | 1 | | | | | | | M387 | |
| | P009512 | Obverse Col 1 | | | | | 1 | | | | | | | | S, B, G, E, U, Š |
| | | Obverse Col 2 | | | | | 1 | | | | | | | | |
| | P009515 | Obverse Col 1 | | | | | 6 | 3 | | | | | | | G, Š |
| | P009517 | Obverse Col 1 | 2 | | | | 3 | 2 | 2 | | | | | | Š |
| | | Reverse Col 1 | | | | | 1 | | | | | | | | |
| | P009518 | Obverse Col 1 | | | | | 2 | 1 | 2 | | | | | | Š |
| | | Reverse Col 1 | | | | | 1 | 1 | | | | | | | |
| | P009520 | Obverse Col 1 | | | | | **23** | | | | | | | | S, B, G, E, U, Š |
| | | Obverse Col 2 | | | | | | | | | | | | M072 | |
| | P009522 | Obverse Col 1 | | | | | 1 | 6 | | | | | | | S, B, G, E, U, Š |
| | | Reverse Col 1 | | | | | 1 | | | | | | | | |
| | P009554 | Reverse Col 1 | | | | | 1 | | | | | | | | S, B, G, E, U, Š |
| Jebel Aruda | P235754 | Obverse Col 1 | | | 2 | | 7 | 5 | | | | | | | S, Š, B |
| | P235755 | Obverse Col 1 | | | 1 | | **10** | 3 | | | | | | | S, Š, B |
| | P235756 | Obverse Col 1 | 2 | | | | 4 | | | | | | | | S, Š, G |
| | P235757 | Obverse Col 1 | 1 | | 2 | | 7 | 2 | | | | | | | Š |
| | P235759 | Obverse Col 1 | | | 3 | | **22** | 5 | | | | | | | S, Š, B |
| | P235760 | Obverse Col 1 | | | | | **10** | | | | | | | | S, B, G, E, U, Š |
| | P235761 | Obverse Col 1 | | | | | 5 | 5 | | | | | | | S, B, G, E, U, Š |
| | P235762 | Obverse Col 1 | | | | | | 5 | | | | | | | S, B, G, E, U, Š |
| | P235763 | Obverse Col 1 | 1 | | | | | 4 | | | | | | | S, Š, G |
| | P235765 | Obverse Col 1 | | | | | | 1 | | | | | | | S, B, G, E, U, Š |
| | P249252 | Obverse Col 1 | | | | | 5 | | | | | | | | S, B, G, E, U, Š |
| Mari | P235767 | Obverse Col 1 | | | | | **11** | 3 | | | | | | | S, B, G, E, U, Š |

(Continued)

| Source | CDLI no. | Face | $N_{45}$ | $N_{48}$ | $N_{34}$ | $N_{15}$ | $N_{14}$ | $N_{1}$ | $N_{39}$ | $N_{24}$ | $N_{22}$ | $N_{23}$ | $N_{51}$ | Other | System |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Habuba Kabira | P235770 | Obverse Col 1 | | | | | 1 | | | | | | | | G (Note 2) |
| | | Obverse Col 2 | | | | | | 4 | | | | | | | |
| | | Obverse Col 3 | | | | | | 3 | | | | | | | |
| | P283931 | Obverse Col 1 | 1 | | 3 | | 4 | | | | | | | | S, Š, G |
| | | Obverse Col 1 | 1 | | | | 1 | | | | | | | | |
| | | Obverse Col 2 | | | | | 4 | | | | | | | | |
| | P283932 | Obverse Col 1 | 2 | | | | | | | | | | | | S, Š, G |
| Tell Brak | P235766 | Obverse Col 1 | | | 1 | | 1 | 3 | | | | | | | S, Š, B |
| Uncertain | P200038 | Obverse Col 1 | 2 | | | | 2 | 3 | | | | | | | S, Š, G |
| | | Obverse Col 2 | 1 | | | | 4 | | | | | | | | |
| | P252149 | Obverse Col 1 | | | | | | 8 | | | | | | | G (Note 1) |
| | | Obverse Col 2 | | | | | | 9 | | | | | | | |
| | P252150 | Obverse Col 1 | | | | | 6 | 2 | | | | | | | S, B, G, E, U, Š |
| | | Obverse Col 2 | | | | | 2 | 1 | | | | | | | |
| | P252151 | Obverse Col 1 | | | | | 5 | | | | | | | | S, B, G, E, U, Š |
| | P252152 | Obverse Col 1 | | | | | 4 | 5 | | | | | | | S, B, G, E, U, Š |
| | P252153 | Obverse Col 1 | | | | | 2 | 4 | | | | | | | S, B, G, E, U, Š |
| | P252154 | Obverse Col 1 | | | | | 4 | 4 | | | | | | | S, B, G, E, U, Š |
| | P252155 | Obverse Col 1 | | | | | 1 | 9 | | | | | | | S, B, G, E, U, Š |
| | P252156 | Obverse Col 1 | 2 | 2 | 1 | | 3 | 3 | | | | | | | S, Š |
| | P252157 | Obverse Col 1 | 1 | | 6 | | 3 | 5 | | | | | | [grain] | Š |
| | P252158 | Obverse Col 1 | | | | | 2 | 5 | | | | | | [grain] | Š |
| | P252159 | Obverse Col 1 | | | | | 4 | 4 | | | | | | | S, B, G, E, U, Š |
| | P325364 | Obverse Col 1 | | | | | 2 | 1 | | | | | | | S, B, G, E, U, Š |
| | | Obverse Col 2 | | | | | | 7 | | | | | | | |
| | P387697 | Obverse Col 1 | | | | | 3 | 4 | | | | | | | S, B, G, E, U, Š |

*Notes*. Analysis of 193 numerical tablets from Uruk ($n$ = 58), Tell Jokha ($n$ = 5), Tutub ($n$ = 1), Nineveh ($n$ = 1), and Jemdet Nasr ($n$ = 2), modern Iraq; Susa ($n$ = 75), Chogha Mish ($n$ = 1), Godin Tepe ($n$ = 10), and Tepe Sialk ($n$ = 10), modern Iran; Jebel Aruda ($n$ = 11), Mari ($n$ = 1), Habuba Kabira ($n$ = 3), and Tell Brak ($n$ = 1), modern Syria; and 14 whose provenience is uncertain. These data represent all the artifacts from the Uruk V period with interpretations of the numerical signs contained on them, as listed in the CDLI database on February 10, 2016. The CDLI identifier for each artifact is provided, along with the quantities of numerical signs on each face. For analytical purposes, other information provided in the CDLI, such as sign variants, were omitted. Numerical signs were assigned to possible counting systems based on sign quantities, sign order, and bundling relations as provided by Nissen et al. 1993. In most cases, the available data was consistent with the use of multiple systems, though in a few cases the presence of certain signs (e.g., $N_{34}$), relations (the relative order

of $N_{34}$ and $N_{45}$), or arrangements (along the sides of the obverse face) narrowed the possible system(s) represented by the signs. An analysis of the data is presented in Chapter 9.

*Note 1*. Five artifacts (P008673, P008553, P008560, P008462, and P252149) were assigned to System G because the tablet contained area calculations with amounts corresponding to the quantities of the numerical impressions. Numbers of impressions highlighted in bold font exceeded the bundling relations of any system listed by Nissen et al. 1993.

*Note 2*. Five additional artifacts (P005304, P009215, P009501, P200010, and P235770) were assigned to System G because the arrangement of numerical notations was similar to that of tablets interpreted as containing area calculations (i.e., the numerical impressions were arranged along the sides of the obverse face).