

A NEW CONSTRUCTION OF COMPACT TORSION-FREE G_2 -MANIFOLDS BY GLUING FAMILIES OF EGUCHI–HANSON SPACES

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Abstract

We give a new construction of compact Riemannian 7-manifolds with holonomy G_2 . Let M be a torsion-free G_2 -manifold (which can have holonomy a proper subgroup of G_2) such that M admits an involution ι preserving the G_2 -structure. Then $M/\langle\iota\rangle$ is a G_2 -orbifold, with singular set L an associative submanifold of M , where the singularities are locally of the form $\mathbb{R}^3 \times (\mathbb{R}^4/\{\pm 1\})$. We resolve this orbifold by gluing in a family of Eguchi–Hanson spaces, parametrized by a nonvanishing closed and coclosed 1-form λ on L .

Much of the analytic difficulty lies in constructing appropriate closed G_2 -structures with sufficiently small torsion to be able to apply the general existence theorem of the first author. In particular, the construction involves solving a family of elliptic equations on the noncompact Eguchi–Hanson space, parametrized by the singular set L . We also present two generalizations of the main theorem, and we discuss several methods of producing examples from this construction.

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1. Introduction

Compact G_2 holonomy manifolds have much in common with the better understood compact Calabi–Yau manifolds. In particular, they are both Ricci-flat compact Riemannian manifolds, and they both admit

parallel spinors, making them candidates for *supersymmetric compactification spaces* in physics [1]. Both Calabi–Yau metrics, which are of holonomy $SU(n)$, and holonomy G_2 metrics on compact manifolds, are in some sense *transcendental* objects, in the sense that explicit formulas for such metrics are not expected to exist. Instead, one proves existence of such metrics by establishing that some nonlinear elliptic partial differential equation on the manifold has a smooth solution.

However, Calabi–Yau manifolds can be, at least partially, studied fruitfully using methods of algebraic geometry. Such tools are not available for general torsion-free G_2 -manifolds, as they are 7-dimensional objects. Moreover, there is no analogue of a Calabi–Yau Theorem in G_2 geometry. That is, we are far from knowing useful sufficient conditions for a compact smooth oriented spin 7-manifold to admit G_2 holonomy metrics. Because of this, there are far fewer known examples of compact G_2 holonomy manifolds.

The first construction of compact G_2 holonomy manifolds was due to the first author, originally presented in [18, 19] and extended in [20]. The construction involves the resolution of orbifold singularities obtained by taking the quotient of the flat 7-torus T^7 by a finite group action preserving the canonical flat G_2 -structure. One then very carefully defines a closed G_2 -structure φ on the resolved manifold, with sufficiently small torsion, and invokes a theorem [18, Th. 11.6.1] of the first author (this is Theorem 2.7 below) to conclude that there exists a torsion-free G_2 -structure $\tilde{\varphi}$ in the same cohomology class as φ . This generalizes the familiar Kummer-type construction of the Calabi–Yau metric on a $K3$ surface [37], and is an example of a geometric gluing construction. Several hundred distinct topological types, distinguishable by their second and third Betti numbers b^2 and b^3 , were produced from this construction.

Somewhat later, a second construction of compact G_2 holonomy manifolds was given by Kovalev [27], based on an idea of Donaldson [20, Method 3, page 303]. Kovalev’s construction involves the *twisted connect sum* of two asymptotically cylindrical G_2 -manifolds built from asymptotically cylindrical Calabi–Yau manifolds. This construction is also a geometric gluing. The main difficulty of the construction is in finding suitable building blocks that satisfy the correct matching conditions to be able to carry out the gluing. More recently, Corti–Haskins–Nordström–Pacini [6, 7] generalized the Kovalev construction, producing tens of thousands of topologically distinct compact G_2 holonomy manifolds. Moreover, further work by Crowley–Goette–Nordström in [9] then resulted in the first examples of inequivalent torsion-free G_2 -structures on the same underlying smooth manifold.

In the present paper we present a new construction of smooth compact G_2 holonomy manifolds, based on an idea due to the first author

[20, Method 2, page 303]. This construction is also essentially a geometric gluing, however, it differs from the two earlier constructions in a significant way:

- In the new construction of the present paper, there are three pieces being glued together, as opposed to two, and two of the three pieces being glued in *do not come initially equipped* with torsion-free G_2 -structures. This does not happen in the earlier constructions of the first author or of Kovalev and Corti–Haskins–Nordström–Pacini. In those cases, the pieces being glued together came already equipped with torsion-free G_2 -structures, and the main analytic difficulty was in controlling the size of the torsion in the gluing annulus. In the present construction we need to work hard to construct closed G_2 -structures with sufficiently small torsion on two of the three pieces to be able to apply Theorem 2.7.

In fact, our construction actually produces a smooth compact torsion-free G_2 -manifold N , starting from a smooth compact torsion-free G_2 -manifold M which admits a G_2 -involution ι , defined in §2.7. In practice, the initial manifold M would be of the form $M = \mathcal{S}^1 \times Y$ for a Calabi–Yau 3-fold Y . An antiholomorphic isometric involution on Y induces a G_2 -involution on M . This is explained in §2.7.

Our construction can be very roughly summarized as follows. We take the quotient $M/\langle\iota\rangle$ of M by the action of ι . The fixed point set L of ι is a smooth compact 3-dimensional totally geodesic associative submanifold of M . The space $M/\langle\iota\rangle$ is a compact orbifold, with singular set L . Locally, the orbifold singularities are of the form $\mathbb{R}^3 \times (\mathbb{R}^4/\{\pm 1\})$. We then cut out a tubular neighbourhood of L in $M/\langle\iota\rangle$, and glue in a smooth family of Eguchi–Hanson spaces, smoothly parametrized by L , to obtain a smooth compact 7-manifold N . We show that N admits a 1-parameter family of G_2 -structures (φ_t^N, g_t^N) that are closed and, for small t , have torsion small enough in a precise sense to be able to invoke Theorem 2.7 to establish the existence of a torsion-free G_2 -structure on N . When the fundamental group of N is finite, this will be a holonomy G_2 metric on N .

As mentioned above, the main technical difficulties arise in constructing closed G_2 -structures with sufficiently small torsion on two of the three pieces being glued together. These three pieces are roughly described as follows:

- (i) the interior of the complement of a neighbourhood of L in $M/\langle\iota\rangle$;
- (ii) an annulus in the quotient $\nu/\{\pm 1\}$ of the normal bundle ν of L in M by the action of $\{\pm 1\}$ on fibres;
- (iii) the product of L with a neighbourhood of the “bolt” \mathcal{S}^2 in the Eguchi–Hanson space $T^*\mathcal{S}^2$.

The pieces (ii) and (iii) do not come equipped with torsion-free G_2 -structures. On both pieces, there is a natural way to define a G_2 -structure, but that is not even closed. In both cases, one can then naturally correct these G_2 -structures to closed versions. But in both cases the torsion is still too large to be able to use these closed G_2 -structures to construct a G_2 -structure on N that satisfies the hypotheses of Theorem 2.7. Thus, in both cases we must perform a *further correction*. For (ii), this correction involves modifying both the exponential map of L in M and the connection on ν , both of which are ingredients used in constructing the initial G_2 -structure on (ii). For (iii), this correction involves solving a family of elliptic equations on the noncompact Eguchi–Hanson space, smoothly parametrized by the points in L .

A key point is that in order to perform our construction, we require the existence of a nonvanishing harmonic 1-form λ on L . This is a very strong assumption. (An explanation for the necessity of this assumption is given in Remark 6.5.) In the case when $M = \mathcal{S}^1 \times Y$, the submanifold L is the disjoint union of two copies of a single special Lagrangian submanifold of Y . Its metric is induced from the Calabi–Yau metric on Y , which exists but cannot be made explicit. We discuss this point in §7, where we present several applications of the theorem to situations that could produce new examples. Some of these examples are obtained by using generalizations of our theorem described in §6.5–§6.6.

Here is our main result, which is Theorem 6.4 below. The various objects and terminology used in the theorem are made precise in the paper.

Theorem 1.1. *Let (M, φ, g) be a compact torsion-free G_2 -manifold, and let $\iota : M \rightarrow M$ be a nontrivial involution preserving (φ, g) , so that the fixed locus L of ι is a compact associative 3-fold in M , as described in §2.7 below.*

Suppose L is nonempty, and suppose there exists a closed, coclosed, nonvanishing 1-form λ on L . That is, $\lambda \in \Omega^1(L)$ with $d\lambda = 0$ and $d^\lambda = 0$, where d^* is defined using $g|_L$, and $\lambda|_x \neq 0$ in T_x^*L for all $x \in L$.*

Then there exists a compact 7-manifold N defined as a resolution of singularities $\pi : N \rightarrow M/\langle \iota \rangle$ of the 7-orbifold $M/\langle \iota \rangle$ along its singular locus $L \subset M/\langle \iota \rangle$, by gluing in a bundle $\sigma : P \rightarrow L$ along L , with fibre the Eguchi–Hanson space X described in §2.5, where P is constructed using λ . The preimage $\pi^{-1}(L)$ is a 5-submanifold Q of N , and $\pi|_Q : Q \rightarrow L$ is a smooth bundle with fibre S^2 . The fundamental group of N satisfies $\pi_1(N) \cong \pi_1(M/\langle \iota \rangle)$, and the Betti numbers are

$$b^k(N) = b^k(M/\langle \iota \rangle) + b^{k-2}(L).$$

There exists a smooth family $(\tilde{\varphi}_t^N, \tilde{g}_t^N)$ of torsion-free G_2 -structures on N for $t \in (0, \epsilon]$, with $\epsilon > 0$ small, such that $(\tilde{\varphi}_t^N, \tilde{g}_t^N) \rightarrow \pi^(\varphi, g)$ in*

C^0 away from Q as $t \rightarrow 0$, and for each $x \in L$ the fibre $\pi^{-1}(x) \cong \mathcal{S}^2$ with metric $\tilde{g}_t^N|_{\pi^{-1}(x)}$ approximates a small round 2-sphere with area $\pi t^2 |\lambda|_x|$ for small t . The metrics \tilde{g}_t^N on N have holonomy G_2 if and only if $M/\langle \iota \rangle$ has finite fundamental group.

In §6.5–§6.6 we generalize Theorem 1.1, replacing $M/\langle \iota \rangle$ by a more general G_2 -orbifold, and twisting the 1-form λ by a principal \mathbb{Z}_2 -bundle on L .

The paper is organized as follows. In §2 we discuss some background material. The material in §3–§5 is developed to prove our main theorem. Specifically, in §3 we study G_2 -structures and correction forms on the normal bundle ν of L in M , in §4 we construct G_2 -structures and correction forms on the resolution P of $\nu/\{\pm 1\}$, and in §5 we state and prove a theorem for the further correction on piece (iii) mentioned above. We then use the material from §3–§5 to prove our main theorem in §6. Section 7 applies our theorem to several situations where new examples can be obtained, including examples that would arise as consequences of the Strominger-Yau-Zaslow (SYZ) conjecture. Finally, in §8 we briefly outline some directions for future study. An appendix presents two calculations that are needed in §3.4.

Notation. All manifolds and tensors are assumed to be smooth unless explicitly stated otherwise. Similarly, all bundles are at least smooth fibre bundles, although some of them will be vector bundles. The term *nonvanishing* means the same thing as *nowhere zero* or *nowhere vanishing*. If E is a vector bundle over M , then $\Gamma^\infty(E)$ denotes the space of smooth sections of E .

We use $v \cdot \beta$ to denote the interior product of a vector v with a form β . The symbol \cdot is also used to denote various natural bilinear pairings throughout, and occasionally in §6 to denote ordinary multiplication in some particularly complicated expressions.

Convention. There are two sign conventions in G_2 geometry. The convention we choose to use in the present paper is the one used by Bryant [3] and the first author [20], but differs from the convention used by Bryant–Salamon [4] or Harvey–Lawson [17]. A detailed discussion of sign conventions and orientations in G_2 geometry can be found in [24].

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2. Background material

2.1. G_2 -structures on 7-manifolds. We now introduce the holonomy group G_2 , following [20, §10.1].

Definition 2.1. Let (x_1, \dots, x_7) be the standard coordinates on \mathbb{R}^7 . Define a 3-form φ_0 on \mathbb{R}^7 by

$$\begin{aligned} \varphi_0 = & dx_1 \wedge dx_2 \wedge dx_3 - dx_1 \wedge dx_4 \wedge dx_5 - dx_1 \wedge dx_6 \wedge dx_7 - dx_2 \wedge dx_4 \wedge dx_6 \\ (2.1) \quad & + dx_2 \wedge dx_5 \wedge dx_7 - dx_3 \wedge dx_4 \wedge dx_7 - dx_3 \wedge dx_5 \wedge dx_6. \end{aligned}$$

The subgroup of $GL(7, \mathbb{R})$ preserving φ_0 is the *exceptional Lie group* G_2 . It is a compact, semisimple, 14-dimensional Lie group, a subgroup of $SO(7)$. It also preserves the Euclidean metric

$$(2.2) \quad g_0 = (dx_1)^2 + \dots + (dx_7)^2$$

on \mathbb{R}^7 , the orientation on \mathbb{R}^7 , and the Hodge dual 4-form

$$\begin{aligned} * \varphi_0 = & dx_4 \wedge dx_5 \wedge dx_6 \wedge dx_7 \\ & - dx_2 \wedge dx_3 \wedge dx_6 \wedge dx_7 - dx_2 \wedge dx_3 \wedge dx_4 \wedge dx_5 \\ & - dx_1 \wedge dx_3 \wedge dx_5 \wedge dx_7 + dx_1 \wedge dx_3 \wedge dx_4 \wedge dx_6 \\ (2.3) \quad & - dx_1 \wedge dx_2 \wedge dx_5 \wedge dx_6 - dx_1 \wedge dx_2 \wedge dx_4 \wedge dx_7. \end{aligned}$$

A G_2 -structure on a 7-manifold M is a principal subbundle Q of the frame bundle of M , with structure group G_2 . Each G_2 -structure Q gives rise to a 3-form φ , a metric g , a 4-form $*\varphi$, and an orientation on M , such that every tangent space of M admits an isomorphism with \mathbb{R}^7 identifying $\varphi, g, *\varphi$ and the orientation with $\varphi_0, g_0, *\varphi_0$ and the standard orientation on \mathbb{R}^7 respectively. Here $*\varphi$ comes from φ by the Hodge star determined by g and the orientation on M . Conversely, φ determines Q uniquely. By an abuse of notation, we will refer to the pair (φ, g) as a G_2 -structure. We call the triple (M, φ, g) a G_2 -manifold. (In contrast to [20, Ch. 10], we do not require (φ, g) to be torsion-free in a G_2 -manifold (M, φ, g) , in the sense below.)

Sometimes we write $g, *\varphi$ as $g_\varphi, *_\varphi \varphi$ to emphasize that they are determined by φ . Note that $g_\varphi, *_\varphi \varphi$ are *nonlinear* functions of φ , since although $*_\varphi$ is linear, it depends on g_φ , and so on φ . Here is [20, Prop.s 10.1.3 & 10.1.5]:

Theorem 2.2. *Let (M, φ, g) be a G_2 -manifold. The following are equivalent:*

- (i) $\text{Hol}(g) \subseteq G_2$, and φ is the induced 3-form,
- (ii) $\nabla \varphi = 0$ on M , where ∇ is the Levi-Civita connection of g , and
- (iii) $d\varphi = 0$ and $d(*_\varphi \varphi) = 0$ on M .

If these hold then g is Ricci-flat.

We call $\nabla\varphi$ the *torsion* of the G_2 -structure (φ, g) . If (M, φ, g) is a G_2 -manifold with (φ, g) torsion-free, we call (M, φ, g) a *torsion-free G_2 -manifold*. Here is [20, Prop. 10.2.2 & Th. 10.4.4]:

Theorem 2.3. *Let (M, φ, g) be a compact torsion-free G_2 -manifold. Then $\text{Hol}(g) = G_2$ if and only if $\pi_1(M)$ is finite. In this case the moduli space of metrics with holonomy G_2 on M , up to diffeomorphisms isotopic to the identity, is a smooth manifold of dimension $b^3(M)$.*

2.2. Exterior forms on G_2 -manifolds. Exterior forms on a G_2 -manifold have a natural decomposition:

Proposition 2.4. *Let (M, φ, g) be a G_2 -manifold. Then $\Lambda^k T^*M$ has a natural orthogonal splitting into components as follows, where Λ_l^k is a vector subbundle of rank l corresponding to an irreducible representation of G_2 :*

$$\begin{array}{ll} \text{(i)} & \Lambda^1 T^*M = \Lambda_7^1, \\ \text{(iii)} & \Lambda^3 T^*M = \Lambda_1^3 \oplus \Lambda_7^3 \oplus \Lambda_{27}^3, \\ \text{(v)} & \Lambda^5 T^*M = \Lambda_7^5 \oplus \Lambda_{14}^5, \end{array} \quad \begin{array}{ll} \text{(ii)} & \Lambda^2 T^*M = \Lambda_7^2 \oplus \Lambda_{14}^2, \\ \text{(iv)} & \Lambda^4 T^*M = \Lambda_1^4 \oplus \Lambda_7^4 \oplus \Lambda_{27}^4, \\ \text{(vi)} & \Lambda^6 T^*M = \Lambda_7^6. \end{array}$$

The Hodge star $*$ of g gives an isometry between Λ_l^k and Λ_l^{7-k} . Here

$$\begin{aligned} \Lambda_{14}^2 &= \text{Ker}((*\varphi \wedge \cdot) : \Lambda^2 T^*M \rightarrow \Lambda^6 T^*M), & \Lambda_1^3 &= \langle \varphi \rangle, & \Lambda_1^4 &= \langle *\varphi \rangle, \\ \Lambda_7^4 &= \text{Im}((\varphi \wedge \cdot) : T^*M \rightarrow \Lambda^4 T^*M), \\ \Lambda_7^5 &= \text{Im}((*\varphi \wedge \cdot) : T^*M \rightarrow \Lambda^5 T^*M). \end{aligned}$$

Write $\Omega^k = \Gamma^\infty(\Lambda^k T^*M)$ and $\Omega_l^k = \Gamma^\infty(\Lambda_l^k)$ for the vector spaces of smooth sections, so that we have splittings $\Omega^2 = \Omega_7^2 \oplus \Omega_{14}^2$, $\Omega^3 = \Omega_1^3 \oplus \Omega_7^3 \oplus \Omega_{27}^3$, and so on. Write $\pi_l : \Omega^k \rightarrow \Omega_l^k$ for the projections to the factors in these splittings. So, for instance, if $\xi \in \Omega^2$ is a 2-form on M then $\xi = \pi_7(\xi) + \pi_{14}(\xi)$.

These splittings are crucial in understanding the linearization of the map Θ that takes a G_2 -structure φ to the dual 4-form $*_\varphi\varphi$, which we now discuss.

Definition 2.5. Let M be a 7-manifold. We call a 3-form φ on M a *positive 3-form* if for every $p \in M$, there exists an isomorphism between $T_p M$ and \mathbb{R}^7 that identifies $\varphi|_p$ and the 3-form φ_0 of (2.1). Each positive 3-form φ determines a unique G_2 -structure (φ, g) on M . Similarly, call a 4-form ψ on M a *positive 4-form* if for every $p \in M$, there exists an isomorphism between $T_p M$ and \mathbb{R}^7 that identifies $\psi|_p$ and the 4-form $*\varphi_0$ of (2.3).

Write points of $\Lambda^k T^*M$ as (x, α) for $x \in M$ and $\alpha \in \Lambda^k T_x^*M$. Write $\Lambda_+^3 T^*M$ and $\Lambda_+^4 T^*M$ for the subsets of (x, α) in $\Lambda^3 T^*M$, $\Lambda^4 T^*M$ with α positive (that is, α is identified with $\varphi_0, *\varphi_0$ in (2.1), (2.3) by some isomorphism $T_x M \cong \mathbb{R}^7$). Then $\Lambda_+^3 T^*M \rightarrow M$, $\Lambda_+^4 T^*M \rightarrow M$ are fibre

bundles (but not vector bundles), whose sections are positive 3- and 4-forms.

As the stabilizer group of φ_0 in $GL(7, \mathbb{R})$ is G_2 , we see that $\Lambda_+^3 T^*M \rightarrow M$ is a bundle with fibre $GL(7, \mathbb{R})/G_2$, which has dimension $49 - 14 = 35$. Since $\Lambda^3 T^*M$ has rank $\binom{7}{3} = 35$, we see that $\Lambda_+^3 T^*M$ is an open submanifold in $\Lambda^3 T^*M$ (note that it is not a vector subbundle). The stabilizer group of $*\varphi_0$ in $GL(7, \mathbb{R})$ is $G_2 \times \{\pm 1\}$, so again $\Lambda_+^4 T^*M$ is open in $\Lambda^4 T^*M$.

Define $\Theta : \Gamma^\infty(\Lambda_+^3 T^*M) \rightarrow \Gamma^\infty(\Lambda_+^4 T^*M)$ by $\Theta(\varphi) = *_\varphi \varphi$, where $*_\varphi \varphi$ is the Hodge dual form defined using the metric g_φ and orientation induced by the G_2 -structure (φ, g_φ) associated to φ . Then Θ is a smooth, nonlinear map.

Here is [20, Prop. 10.3.5], slightly rewritten.

Proposition 2.6. *There exist universal constants $\epsilon, C > 0$ such that the following holds. Let (M, φ, g) be a G_2 -manifold, and ξ be a 3-form on M with $\|\xi\|_{C^0} \leq \epsilon$. Then $\varphi + \xi$ is a positive 3-form, and*

$$(2.4) \quad \Theta(\varphi + \xi) = *_\varphi \varphi + *_\varphi \left(\frac{4}{3} \pi_1(\xi) + \pi_7(\xi) - \pi_{27}(\xi) \right) + F_\varphi(\xi),$$

where the nonlinear function $F_\varphi : \{\xi \in \Omega^3 : \|\xi\|_{C^0} \leq \epsilon\} \rightarrow \Omega^4$ satisfies

$$(2.5) \quad F_\varphi(0) = 0, \quad |F_\varphi(\xi)| \leq C|\xi|^2, \quad |\nabla F_\varphi(\xi)| \leq C(|\xi|^2 |\nabla \varphi| + |\xi| |\nabla \xi|).$$

Here all norms and the covariant derivatives are taken with respect to g .

Equation (2.4) computes the derivative $D_\varphi \Theta : \Omega^3 \rightarrow \Omega^4$ of Θ at φ : we have

$$(2.6) \quad D_\varphi \Theta = *_\varphi \circ \left(\frac{4}{3} \pi_1 + \pi_7 - \pi_{27} \right).$$

2.3. An existence theorem for torsion-free G_2 -manifolds. An important technical tool that we will require is the following theorem of the first author, which says that if one can find a G_2 -structure (φ, g) with $d\varphi = 0$ on a compact 7-manifold M , whose torsion is sufficiently small in a certain sense, then there exists a *torsion-free* G_2 -structure $(\tilde{\varphi}, \tilde{g})$ on M which is close to (φ, g) , and in the same de Rham cohomology class. It was first used in [18, 19] to construct the first compact examples of manifolds with G_2 -holonomy.

Theorem 2.7 (Joyce [20, Th. 11.6.1]). *Let α, K_1, K_2 , and K_3 be any positive constants. Then there exist $\epsilon \in (0, 1]$ and $K_4 > 0$, such that whenever $0 < t \leq \epsilon$, the following holds.*

Let M be a compact 7-manifold, with a G_2 -structure (φ, g) satisfying $d\varphi = 0$. Suppose there is a closed 4-form ψ on M such that:

- (i) $\|\Theta(\varphi) - \psi\|_{C^0} \leq K_1 t^\alpha$, $\|\Theta(\varphi) - \psi\|_{L^2} \leq K_1 t^{\frac{7}{2} + \alpha}$, and
- $\|d(\Theta(\varphi) - \psi)\|_{L^{14}} \leq K_1 t^{-\frac{1}{2} + \alpha}.$

- (ii) the **injectivity radius** inj of g satisfies $\text{inj} \geq K_2 t$.
- (iii) the **Riemann curvature tensor** Rm of g satisfies $\|\text{Rm}\|_{C^0} \leq K_3 t^{-2}$.

Then there exists a smooth, **torsion-free** G_2 -structure $(\tilde{\varphi}, \tilde{g})$ on M such that $\|\tilde{\varphi} - \varphi\|_{C^0} \leq K_4 t^\alpha$ and $[\tilde{\varphi}] = [\varphi]$ in $H^3(M, \mathbb{R})$. Here all norms are computed using the original metric g .

Remark 2.8. We have rewritten the theorem slightly: [20, Th. 11.6.1] is expressed in terms of the 3-form $\chi = \varphi - *_g \psi$ rather than the 4-form $\Theta(\varphi) - \psi$, and also [20, Th. 11.6.1] fixes $\alpha = \frac{1}{2}$, which is sufficient for the applications in [20], but [20, bottom of p. 296] explains how to generalize to all $\alpha > 0$.

2.4. G_2 -manifolds and hyperKähler 4-manifolds. Let us identify \mathbb{R}^7 with coordinates (x_1, \dots, x_7) with the space $\mathbb{R}^3 \oplus \mathbb{H}$ with coordinates $((x_1, x_2, x_3), (y_1, \dots, y_4))$, where $\mathbb{H} = \mathbb{R}^4$ is the quaternions, by

$$\begin{aligned} (x_1, \dots, x_7) &\cong ((x_1, x_2, x_3), (y_1, y_2, y_3, y_4)) \\ &= ((x_1, x_2, x_3), (x_4, x_5, x_6, x_7)). \end{aligned}$$

Then in equations (2.1)–(2.3) we have

$$\begin{aligned} (2.7) \quad \varphi_0 &= dx_1 \wedge dx_2 \wedge dx_3 - dx_1 \wedge \omega_0^I - dx_2 \wedge \omega_0^J - dx_3 \wedge \omega_0^K, \\ g_0 &= (dx_1)^2 + (dx_2)^2 + (dx_3)^2 + h_0, \\ * \varphi_0 &= \text{vol}_{\mathbb{H}} - dx_2 \wedge dx_3 \wedge \omega_0^I - dx_3 \wedge dx_1 \wedge \omega_0^J - dx_1 \wedge dx_2 \wedge \omega_0^K, \end{aligned}$$

where

$$\begin{aligned} \omega_0^I &= dy_1 \wedge dy_2 + dy_3 \wedge dy_4, & \omega_0^J &= dy_1 \wedge dy_3 + dy_4 \wedge dy_2, \\ \omega_0^K &= dy_1 \wedge dy_4 + dy_2 \wedge dy_3, & h_0 &= (dy_1)^2 + (dy_2)^2 + (dy_3)^2 + (dy_4)^2, \end{aligned} \quad (2.8)$$

$$\text{vol}_{\mathbb{H}} = dy_1 \wedge dy_2 \wedge dy_3 \wedge dy_4 = \frac{1}{2} \omega_0^I \wedge \omega_0^I = \frac{1}{2} \omega_0^J \wedge \omega_0^J = \frac{1}{2} \omega_0^K \wedge \omega_0^K.$$

Here h_0 is the Euclidean metric on \mathbb{H} , with volume form $\text{vol}_{\mathbb{H}}$. As in [20, Ch. 7], the metric h_0 is *hyperKähler*, so it is Kähler with respect to three different complex structures I, J, K on \mathbb{H} . Here $\omega_0^I, \omega_0^J, \omega_0^K$ are the Kähler forms of I, J, K for h_0 . Explicitly, we have $\omega_0^I(v, w) = h_0(Iv, w)$ and similarly for J, K . It is clear that the three Kähler forms are all *self-dual* with respect to h_0 and the standard orientation. The three complex structures satisfy the quaternion multiplication relations when acting on tangent vector fields:

$$\begin{aligned} (2.9) \quad I^2 &= J^2 = K^2 = -\mathbf{1}, & IJ &= -JI = K, \\ JK &= -KJ = I, & KI &= -IK = J. \end{aligned}$$

In fact, \mathbb{H} has an entire \mathcal{S}^2 family of complex structures, with respect to each of which h_0 is a Kähler metric. Explicitly, $J_{\mathbf{u}} = u_1 I + u_2 J + u_3 K$

is such a complex structure, for any $\mathbf{u} = (u_1, u_2, u_3) \in \mathbb{R}^3$ satisfying $u_1^2 + u_2^2 + u_3^2 = 1$, with Kähler form $\omega_0^{\mathbf{u}} = u_1\omega_0^I + u_2\omega_0^J + u_3\omega_0^K$.

Given a complex structure I acting on tangent vectors, we define its dual map acting on 1-forms, also denoted I , by $(I\alpha)(v) = \alpha(Iv)$. In particular, with this definition it follows that when acting on 1-forms, we have:

$$(2.10) \quad \begin{aligned} I^2 = J^2 = K^2 &= -\mathbf{1}, & IJ &= -JI = -K, \\ JK &= -KJ = -I, & KI &= -IK = -J. \end{aligned}$$

Explicitly, here we have

$$(2.11) \quad \begin{aligned} I(dy_1) &= -dy_2, & I(dy_2) &= dy_1, & I(dy_3) &= -dy_4, & I(dy_4) &= dy_3, \\ J(dy_1) &= -dy_3, & J(dy_2) &= dy_4, & J(dy_3) &= dy_1, & J(dy_4) &= -dy_2, \\ K(dy_1) &= -dy_4, & K(dy_2) &= -dy_3, & K(dy_3) &= dy_2, & K(dy_4) &= dy_1. \end{aligned}$$

We note here for future use that, from the orthogonality of I , the self-duality $*\omega_I = \omega_I$, and the relation $\omega_0^I(v, w) = h_0(Iv, w)$ it follows that

$$(2.12) \quad *(\alpha \wedge \omega_I) = -I\alpha \quad \text{for any 1-form } \alpha,$$

and similarly for J, K . Equation (2.12) is a purely linear algebraic fact that holds on any hyperKähler 4-manifold.

The Lie subgroup of $\mathrm{GL}(4, \mathbb{R})$ acting on $\mathbb{H} = \mathbb{R}^4$ preserving $\omega_0^I, \omega_0^J, \omega_0^K$ is $\mathrm{SU}(2) \cong \mathrm{Sp}(1)$. So the action of $\mathrm{SU}(2)$ on $\mathbb{R}^7 \cong \mathbb{R}^3 \oplus \mathbb{H}$ which is trivial on \mathbb{R}^3 and left multiplication on \mathbb{H} preserves $\varphi_0, *\varphi_0$ by (2.7)–(2.8). This induces an embedding of Lie subgroups $\mathrm{SU}(2) \hookrightarrow G_2$, since G_2 is the subgroup of $\mathrm{GL}(7, \mathbb{R})$ fixing φ_0 .

Riemannian 4-manifolds with holonomy $\mathrm{SU}(2)$ are known as *Calabi–Yau 2-folds*, or *hyperKähler 4-manifolds*, as in [20, Ch.s 6 & 7] for instance. By the general theory of Riemannian holonomy, the inclusion $\mathrm{SU}(2) \hookrightarrow G_2$ above means that if X is a hyperKähler 4-manifold, with metric h , volume form vol_X , complex structures I, J, K , and Kähler form $\omega^I, \omega^J, \omega^K$, then the product 7-manifold $M = \mathbb{R}^3 \times X$ is a torsion-free G_2 -manifold, with 3-form φ , metric g and 4-form ψ given by

$$(2.13) \quad \begin{aligned} \varphi &= dx_1 \wedge dx_2 \wedge dx_3 - dx_1 \wedge \omega^I - dx_2 \wedge \omega^J - dx_3 \wedge \omega^K, \\ g &= (dx_1)^2 + (dx_2)^2 + (dx_3)^2 + h, \\ \psi &= \mathrm{vol}_X - dx_2 \wedge dx_3 \wedge \omega^I - dx_3 \wedge dx_1 \wedge \omega^J - dx_1 \wedge dx_2 \wedge \omega^K, \end{aligned}$$

as in (2.7). The G_2 -manifolds of this type have holonomy $\mathrm{SU}(2) \subset G_2$.

2.5. The Eguchi–Hanson space. Following our discussion of hyperKähler 4-manifolds in §2.4, we now describe a particular example of a hyperKähler 4-manifold that plays a crucial role in our construction. The Eguchi–Hanson space was originally discovered in [14, 15] but it can be described from several points of view. These descriptions include:

the resolution (blow-up) of $\mathbb{C}^2/\mathbb{Z}_2$ at the origin, which describes it as a complex manifold; the simplest case of both the Calabi construction [5] of hyperKähler metrics on $T^*\mathbb{CP}^n$ and the Stenzel construction [35] of Calabi–Yau metrics on T^*S^n ; and the hyperKähler quotient construction of Eguchi–Hanson space. A good survey of hyperKähler manifolds in general and the Eguchi–Hanson space in particular, discussing various different approaches, can be found in [10].

Write the Eguchi–Hanson space as X , with blow-up map $B : X \rightarrow \mathbb{C}^2/\{\pm 1\}$, and complex structure I on X , \mathbb{C}^2 , and $\mathbb{C}^2/\{\pm 1\}$. The exceptional divisor of the blow-up is written $Y = B^{-1}(0)$, where $Y \cong \mathbb{CP}^1$. Write $r : \mathbb{C}^2 \rightarrow [0, \infty)$ for the radius function $r(z_1, z_2) = (|z_1|^2 + |z_2|^2)^{1/2}$. Also write $r : \mathbb{C}^2/\{\pm 1\} \rightarrow [0, \infty)$ for the function descending from $r : \mathbb{C}^2 \rightarrow [0, \infty)$, and write $r : X \rightarrow [0, \infty)$ in place of $r \circ B$, so that $Y = r^{-1}(0) \subset X$.

Remark 2.9. The map $B : X \rightarrow \mathbb{C}^2/\{\pm 1\}$ is *not a smooth map* in the sense of orbifolds. This means that given some smooth function or tensor T on \mathbb{C}^2 which is invariant under $\{\pm 1\}$, then T descends to a smooth function or tensor on $\mathbb{C}^2/\{\pm 1\}$ in the orbifold sense, but the pullback $B^*(T)$ *may not be smooth (or defined) on the exceptional divisor* $Y = B^{-1}(0)$.

However, since B is defined using complex geometry, holomorphic objects on $\mathbb{C}^2/\{\pm 1\}$ do pull back to holomorphic (and hence smooth) objects on X . For example, the holomorphic functions z_1^2 , z_2^2 , and $z_1 z_2$ on $\mathbb{C}^2/\{\pm 1\}$ pull back to holomorphic functions on X . Although $r^2 : \mathbb{C}^2/\{\pm 1\} \rightarrow \mathbb{R}$ is smooth in the orbifold sense, its pullback $r^2 : X \rightarrow \mathbb{R}$ is not smooth on X . But $r^4 : X \rightarrow \mathbb{R}$ is smooth, since

$$r^4 = |z_1^2|^2 + |z_2^2|^2 + 2|z_1 z_2|^2,$$

where z_1^2 , z_2^2 , and $z_1 z_2$ are holomorphic, and hence smooth, on X . Near Y , we can think of r^4 as the squared distance to Y in X .

In §4 we will work with a bundle $\sigma : P \rightarrow L$ with Eguchi–Hanson space fibres X_x for $x \in L$. We will have to be careful that the metrics and forms we define on P extend smoothly over the exceptional divisors Y_x in the fibres X_x .

On $\mathbb{R}^4 = \mathbb{C}^2$ use real coordinates (y_1, y_2, y_3, y_4) and complex co-ordinates (z_1, z_2) with $z_1 = y_1 + iy_2$, $z_2 = y_3 + iy_4$, so that $r^2 = y_1^2 + y_2^2 + y_3^2 + y_4^2 = |z_1|^2 + |z_2|^2$. We have a Euclidean metric h_0 , complex structures I, J, K where z_1, z_2 are holomorphic with respect to I , and Kähler forms $\omega_0^I, \omega_0^J, \omega_0^K$, where

$$\begin{aligned} h_0 &= dy_1^2 + dy_2^2 + dy_3^2 + dy_4^2 = |dz_1|^2 + |dz_2|^2, \\ \omega_0^I &= dy_1 \wedge dy_2 + dy_3 \wedge dy_4 = \frac{i}{2}(dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2), \end{aligned}$$

and

$$\begin{aligned}\omega_0^J &= dy_1 \wedge dy_3 + dy_4 \wedge dy_2, & \omega_0^K &= dy_1 \wedge dy_4 + dy_2 \wedge dy_3, \\ \omega_0^J + i\omega_0^K &= dz_1 \wedge dz_2.\end{aligned}$$

Since $I(dz_1) = idz_1$ and $I(dz_2) = idz_2$, we have

$$Idy_1 = -dy_2, \quad Idy_2 = dy_1, \quad Idy_3 = -dy_4, \quad Idy_4 = dy_3.$$

Then we may also write

$$(2.14) \quad \omega_0^I = -\frac{1}{4}d[I(d(r^2))],$$

where $r^2 = y_1^2 + y_2^2 + y_3^2 + y_4^2 : \mathbb{C}^2 \rightarrow \mathbb{R}$ is the Kähler potential of ω_0 on (\mathbb{C}^2, I) .

For $a > 0$, define $f_a : (0, \infty) \rightarrow \mathbb{R}$ by

$$\begin{aligned}(2.15) \quad f_a(r) &= \sqrt{r^4 + a^2} - a \log \left(\frac{\sqrt{r^4 + a^2} + a}{r^2} \right) \\ &= \sqrt{r^4 + a^2} + 2a \log r - a \log \left(\sqrt{r^4 + a^2} + a \right).\end{aligned}$$

Then $f_a(r) = af_1(a^{-1/2}r)$. There is a unique Ricci-flat Kähler metric h_a on (X, I) with Kähler form ω_a^I , where analogous to (2.14), on $X \setminus Y$ we have

$$(2.16) \quad \omega_a^I|_{X \setminus Y} = -\frac{1}{4}d[I(df_a(r))].$$

The function r^2 is the Kähler potential of the Euclidean metric h_0 on $\mathbb{C}^2/\{\pm 1\}$. We need to understand the difference between the metrics h_a and h_0 near infinity. Define a smooth function $G_a : (0, \infty) \rightarrow \mathbb{R}$ by

$$(2.17) \quad G_a(r) = f_a(r) - r^2.$$

Then from the first line of (2.15) we can show that for large r we have

$$(2.18) \quad \nabla^k G_a(r) = O(r^{-2-k}) \text{ as } r \rightarrow \infty, \text{ for all } k \geq 0,$$

where ∇ is the Levi-Civita connection of h_0 . Equation (2.18) when $k = 2$ implies that $h_a = h_0 + O(r^{-4})$ as $r \rightarrow \infty$. That is, h_a is *Asymptotically Locally Euclidean* (ALE). See [20, Chapters 7–8] for more background on ALE metrics.

The expression (2.16) defines ω_a^I on $X \setminus Y$. We will show how it in fact extends smoothly over Y in a unique way. From the second line of (2.15), write

$$(2.19) \quad f_a(r) = 2a \log r + H_a(r^4),$$

where $H_a : (-a^2, \infty) \rightarrow \mathbb{R}$ is the smooth function given by

$$H_a(u) = \sqrt{u + a^2} - a \log \left(\sqrt{u + a^2} + a \right).$$

Note that as $r^4 : X \rightarrow \mathbb{R}$ is smooth by Remark 2.9 and takes values in $[0, \infty) \subset (-a^2, \infty)$, this implies that $H_a(r^4)$ is a smooth function on all of X , including Y .

Define 2-forms ω^J, ω^K on X by $\omega^J + i\omega^K = \pi^*(\omega_0^J + i\omega_0^K)$. These are smooth by Remark 2.9, because $\omega_0^J + i\omega_0^K$ is holomorphic on $\mathbb{C}^2/\{\pm 1\}$. There are unique complex structures J_a, K_a on X such that h_a is Kähler with respect to J_a, K_a with Kähler forms ω^J, ω^K . Then (X, h_a) is hyperKähler, with complex structures (I, J_a, K_a) and Kähler forms $(\omega_a^I, \omega^J, \omega^K)$. Note that $h_a, J_a, K_a, \omega_a^I$ depend on $a \in (0, \infty)$, but I, ω^J, ω^K do not. As $a \rightarrow 0$, these $h_a, (I, J_a, K_a), (\omega_a^I, \omega^J, \omega^K)$ approach the Euclidean hyperKähler structure on $\mathbb{C}^2/\{\pm 1\}$.

There is a natural holomorphic map $\pi : X \rightarrow Y = \mathbb{CP}^1$, acting by $\pi(x) = [z_1, z_2]$ if $B(x) = (z_1, z_2)$ for $x \in X \setminus Y$, which realizes X as the total space of the holomorphic line bundle $\mathcal{O}(-2) = T^*\mathbb{CP}^1$ over \mathbb{CP}^1 . On $X \setminus Y$ we have

$$(2.20) \quad -\frac{1}{4}d[I(d(2\log r))] = \pi^*(\omega_{\mathbb{CP}^1})|_{X \setminus Y},$$

where $\omega_{\mathbb{CP}^1}$ is the Kähler form of the usual Fubini–Study metric on \mathbb{CP}^1 . Combining (2.16), (2.19) and (2.20) shows that

$$(2.21) \quad \omega_a^I = a \pi^*(\omega_{\mathbb{CP}^1}) - \frac{1}{4}d[I(dH_a(r^4))].$$

Note that although (2.16) only makes sense on $X \setminus Y$ since $f_a(r(x)) \rightarrow -\infty$ as $x \rightarrow y \in Y$ in X , equation (2.21) is valid on all of X , because both $\pi^*(\omega_{\mathbb{CP}^1})$ and $H_a(r^4)$ are smooth on X .

Remark 2.10. The submanifold Y of X , corresponding to the zero section $\mathbb{CP}^1 \cong \mathcal{S}^2$ in $X = T^*\mathbb{CP}^1$, is sometimes called the *bolt* of X . It is well known that the Fubini–Study metric on \mathbb{CP}^1 is isometric to the round metric on S^2 of radius $\frac{1}{2}$. It follows from (2.21) that the area of the bolt is πa .

By the material of §2.4, we see that we have a torsion-free G_2 -structure (φ_a, g_a) on $\mathbb{R}^3 \times X$ for $a > 0$, with 4-form ψ_a , given by

$$\begin{aligned} \varphi_a &= dx_1 \wedge dx_2 \wedge dx_3 - dx_1 \wedge \omega_a^I - dx_2 \wedge \omega^J - dx_3 \wedge \omega^K, \\ g_a &= (dx_1)^2 + (dx_2)^2 + (dx_3)^2 + h_a, \\ \psi_a &= \text{vol}_X - dx_2 \wedge dx_3 \wedge \omega_a^I - dx_3 \wedge dx_1 \wedge \omega^J - dx_1 \wedge dx_2 \wedge \omega^K. \end{aligned}$$

We can generalize this by rescaling distances in X by $t > 0$, replacing $h, \omega_a^I, \omega^J, \omega^K, \text{vol}_{\mathbb{H}}$ by $t^2h, t^2\omega_a^I, t^2\omega^J, t^2\omega^K, t^4\text{vol}_{\mathbb{H}}$, respectively, giving a family of torsion-free G_2 -structures $(\varphi_{a,t}, g_{a,t})$ on $\mathbb{R}^3 \times X$ for $a, t > 0$

with 4-forms $\psi_{a,t}$:

$$\begin{aligned}
 \varphi_{a,t} &= dx_1 \wedge dx_2 \wedge dx_3 - t^2 dx_1 \wedge \omega_a^I - t^2 dx_2 \wedge \omega^J - t^2 dx_3 \wedge \omega^K, \\
 (2.22) \quad g_{a,t} &= (dx_1)^2 + (dx_2)^2 + (dx_3)^2 + t^2 h_a, \\
 \psi_{a,t} &= t^4 \text{vol}_X - t^2 dx_2 \wedge dx_3 \wedge \omega_a^I - t^2 dx_3 \wedge dx_1 \wedge \omega^J \\
 &\quad - t^2 dx_1 \wedge dx_2 \wedge \omega^K.
 \end{aligned}$$

These $(\varphi_{a,t}, g_{a,t}), \psi_{a,t}$ will be our local models in §4.

2.6. G_2 -manifolds and Calabi–Yau 3-folds. Let us identify \mathbb{R}^7 with coordinates (x_1, \dots, x_7) with $\mathbb{R} \oplus \mathbb{C}^3$ with coordinates $(x, (z_1, z_2, z_3))$ by

$$(x_1, \dots, x_7) \cong (x, (z_1, z_2, z_3)) = (x_4, (x_1 + ix_5, x_2 + ix_6, x_3 + ix_7)).$$

Then in equations (2.1)–(2.3) we have

$$\begin{aligned}
 \varphi_0 &= dx \wedge \omega_0 + \text{Re } \Omega_0, \quad g = dx^2 + h_0, \\
 * \varphi_0 &= -dx \wedge \text{Im } \Omega_0 + \frac{1}{2} \omega_0 \wedge \omega_0, \\
 \text{where } \omega_0 &= \frac{i}{2} (dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2 + dz_3 \wedge d\bar{z}_3), \\
 (2.23) \quad \Omega_0 &= dz_1 \wedge dz_2 \wedge dz_3, \quad \text{and} \quad h_0 = |dz_1|^2 + |dz_2|^2 + |dz_3|^2,
 \end{aligned}$$

so that ω_0 is the standard Kähler form, Ω_0 the holomorphic volume form, and h_0 the Kähler metric on \mathbb{C}^3 , normalized by the equation $\text{vol}_{\mathbb{C}^3} = \frac{1}{6} \omega_0^3 = \frac{i}{8} \Omega_0 \wedge \bar{\Omega}_0$. The Lie subgroup of $\text{GL}(6, \mathbb{R})$ acting on $\mathbb{R}^6 \cong \mathbb{C}^3$ preserving both ω_0 and Ω_0 is $\text{SU}(3)$. So the action of $\text{SU}(3)$ on $\mathbb{R}^7 \cong \mathbb{R} \oplus \mathbb{C}^3$ which is trivial on \mathbb{R} and as usual on \mathbb{C}^3 preserves $\varphi_0, * \varphi_0$ by (2.23). This induces an embedding of Lie subgroups $\text{SU}(3) \hookrightarrow G_2$, since G_2 is the subgroup of $\text{GL}(7, \mathbb{R})$ fixing φ_0 .

Riemannian 6-manifolds with holonomy $\text{SU}(3)$ are known as *Calabi–Yau 3-folds*, as in [20, Ch. 6] for instance. By the general theory of Riemannian holonomy, the inclusion $\text{SU}(3) \hookrightarrow G_2$ above means that if Y is a Calabi–Yau 3-fold, with Ricci-flat Kähler metric h , complex structure J , Kähler form ω , and holomorphic $(3, 0)$ -form Ω , normalized by $\text{vol}_Y = \frac{1}{6} \omega^3 = \frac{i}{8} \Omega \wedge \bar{\Omega}$ and

$$h(v, w) = \omega(v, Jw)$$

for vector fields v and w on Y , and $\mathcal{S}^1 = \mathbb{R}/\mathbb{Z}$ is the circle with coordinate $x \in \mathbb{R}/\mathbb{Z}$ and metric dx^2 , then the product 7-manifold $M = \mathcal{S}^1 \times Y$ is a torsion-free G_2 -manifold, with 3-form φ , metric g and 4-form ψ given by

$$(2.24) \quad \varphi = \text{Re } \Omega + dx \wedge \omega, \quad g = dx^2 + h, \quad \psi = -dx \wedge \text{Im } \Omega + \frac{1}{2} \omega \wedge \omega,$$

as in (2.23). The G_2 -manifolds of this type have holonomy $\text{SU}(3) \subset G_2$.

2.7. G_2 -involutions and their fixed point sets. We now define what we mean by a G_2 -involution ι , which we need to construct the orbifold $M/\langle\iota\rangle$.

Definition 2.11. Let (M, φ) be a G_2 -manifold, and $\iota : M \rightarrow M$ be a diffeomorphism of M . We call ι a G_2 -*involution* if ι is an involution which preserves the G_2 -structure. This means $\iota^2 = \mathbf{1}$, where $\mathbf{1}$ is the identity map of M , and $\iota^*(\varphi) = \varphi$. Such a map ι also clearly satisfies $\iota^*(g) = g$ and $\iota^*(\psi) = \psi$.

Example 2.12. Suppose that $M = \mathcal{S}^1 \times Y$, with Y a Calabi–Yau 3-fold, as described in §2.6, and $\mathcal{S}^1 = \mathbb{R}/\mathbb{Z}$. An *anti-holomorphic isometric involution* τ of Y is a diffeomorphism satisfying $\tau^2 = \mathbf{1}$ (involution), $\tau^*(g) = g$ (isometry), and $\tau^*(J) = -J$ (anti-holomorphic). It follows that $\tau^*(\omega) = -\omega$ and $\tau^*(\Omega) = \bar{\Omega}$. Define an involution on $M = \mathcal{S}^1 \times Y$ by $\iota(x, y) = (-x, \tau(y))$. From equation (2.24) we see that

$$\begin{aligned} \iota^*(\varphi) &= \operatorname{Re}(\tau^*(\Omega)) + (-dx) \wedge \tau^*(\omega) = \operatorname{Re}(\bar{\Omega}) + (-dx) \wedge (-\omega) \\ &= \operatorname{Re}(\Omega) + dx \wedge \omega = \varphi, \end{aligned}$$

so that such an ι is a G_2 -involution.

We will need to understand the fixed point set of a G_2 -involution. The following is [20, Prop. 10.8.1], except for L totally geodesic, which is obvious.

Proposition 2.13. *Let (M, φ) be a connected, compact G_2 -manifold, and let ι be a G_2 -involution of M . Let L be the fixed point set of ι . That is, $L = \{p \in M : \iota(p) = p\}$. Suppose that ι is not the identity ($L \neq M$), and that ι has at least one fixed point ($L \neq \emptyset$). Then L is a smooth, orientable 3-dimensional compact submanifold of M which is totally geodesic, and, with respect to a canonical orientation, is **associative**.*

Remark 2.14. An associative submanifold is calibrated with respect to φ , in the sense of Harvey–Lawson [17]. In particular, we note here for future reference the important fact that the normal bundle ν of an associative submanifold L of (M, φ) is always topologically trivial. This fact is well-known, but hard to find in the literature, so we outline the argument here. We use the algebraic properties of the octonions (see [17]). The fibre of the normal bundle ν_p over a point $p \in L$ is a coassociative 4-plane in $T_p M$, and thus the cross product $s_p \times X_p$ of a normal vector $s_p \in \nu_p$ with a tangent vector $X_p \in T_p L$ always lies in the normal space ν_p . Let $\{e_1, e_2, e_3\}$ be a global frame for L , which exists since all 3-manifolds are parallelizable. Let s be a non-vanishing section of ν , which exists because ν is an \mathbb{R}^4 -bundle over a 3-manifold. Then the set $\{s, s \times e_1, s \times e_2, s \times e_3\}$ is a global frame for ν , so ν is topologically trivial.

Example 2.15. Consider the G_2 -involution on $\mathcal{S}^1 \times Y$ of Example 2.12. If (x, y) is a fixed point of ι , then $x = -x$ in \mathbb{R}/\mathbb{Z} and $\tau(y) = y$. This means $x = 0$ or $x = \frac{1}{2}$, and y is a fixed point of τ . Since $\tau^*(\omega) = -\omega$ and $\tau^*(\text{Im}(\Omega)) = -\text{Im}(\Omega)$, we see that on the fixed point set L of τ , which by Proposition 2.13 is a 3-dimensional submanifold of Y , the forms ω and $\text{Im}(\Omega)$ both vanish. Thus L is a special Lagrangian submanifold of Y , and the fixed point set of ι in M is two copies of L , namely $\{0\} \times L$ and $\{\frac{1}{2}\} \times L$.

3. G_2 -structures on the normal bundle ν of L in M

From here until the end of §6.4 we will work in the following situation.

Assumption 3.1. Suppose we are given a compact, torsion-free G_2 -manifold (M, φ, g) , and a nontrivial involution $\iota : M \rightarrow M$ with $\iota^*(\varphi) = \varphi$, so that $\iota^*(g) = g$. Write L for the fixed locus of ι in M . Then L is an associative 3-fold in (M, φ, g) by Proposition 2.13, and is compact as M is.

We suppose L is nonempty, but not necessarily connected. Suppose we are given a closed, coclosed, nonvanishing 1-form λ on L . That is, $\lambda \in \Omega^1(L)$ with $d\lambda = d^*\lambda = 0$, where d^* is defined using $g|_L$, and $\lambda|_x \neq 0$ in T_x^*L for all $x \in L$. (See Remark 6.5 for a justification of the necessity of this assumption.)

3.1. Choosing a tubular neighbourhood of L . Let $(M, \varphi, g), \iota, L$ be as in Assumption 3.1. Write $\nu \rightarrow L$ for the normal bundle of L in M , a rank 4 real vector bundle in the exact sequence

$$(3.1) \quad 0 \longrightarrow TL \longrightarrow TM|_L \longrightarrow \nu \longrightarrow 0.$$

There is a unique vector bundle isomorphism

$$(3.2) \quad TM|_L \cong \nu \oplus TL$$

compatible with (3.1) such that the subbundles ν, TL in $TM|_L$ are orthogonal with respect to g . Write $g_L \in \Gamma^\infty(S^2T^*L)$ for the restriction of g to L , and $h_\nu \in \Gamma^\infty(S^2\nu^*)$ for the restriction of g to the factor ν in (3.2), so that

$$g|_L = h_\nu \oplus g_L \quad \text{in } \Gamma^\infty(S^2T^*M|_L).$$

Write ∇^g for the Levi-Civita connection of g , and ∇^ν for the connection on $\nu \rightarrow L$ induced from the restriction of ∇^g to $TM|_L$ using (3.2), and ∇^{g_L} for the Levi-Civita connection of g_L . Recall by Proposition 2.13 that L is totally geodesic in (M, g) , hence the second fundamental form of L in M vanishes. Consequently, suppose $v, w \in \Gamma^\infty(TL)$, and $\alpha \in \Gamma^\infty(\nu)$. Choose vector fields $\tilde{v}, \tilde{w} \in \Gamma^\infty(TM)$ with $\tilde{v}|_L \cong 0 \oplus v$, $\tilde{w}|_L \cong \alpha \oplus w$ under the isomorphism (3.2). Then under (3.2) we have

$$(3.3) \quad (\nabla_{\tilde{v}}^g \tilde{w})|_L \cong (\nabla_v^\nu \alpha) \oplus (\nabla_v^{g_L} w).$$

We will choose data $U_R \subset \nu$ and $\Upsilon : U_R \rightarrow M$ as in the next definition:

Definition 3.2. Write points of ν as (x, α) for $x \in L$ and $\alpha \in \nu_x$. Fix $R > 0$, and define

$$(3.4) \quad U_R = \{(x, \alpha) \in \nu : |\alpha|_{h_\nu} < R\},$$

so that U_R is a tubular neighbourhood of the zero section in ν . Write $\pi : U_R \rightarrow L$ for the projection $\pi : (x, \alpha) \mapsto x$. We will choose a smooth map $\Upsilon : U_R \rightarrow M$ satisfying the following conditions:

- (i) Υ is a diffeomorphism with an open neighbourhood of L in M .
- (ii) $\Upsilon(x, 0) = x$ for $x \in L$.
- (iii) $\Upsilon(x, -\alpha) = \iota \circ \Upsilon(x, \alpha)$ for all $(x, \alpha) \in U_R$. (Compatibility with the involution.)
- (iv) Identify L with the zero section $\{(x, 0) : x \in L\}$ in $U_R \subset \nu$. Then there is a natural identification $TU_R|_L \cong \nu \oplus TL$. Also (i), (ii) imply that $\Upsilon_*|_L : TU_R|_L \rightarrow TM|_L$ is a vector bundle isomorphism. The composition of these isomorphisms $\nu \oplus TL \cong TU_R|_L \cong TM|_L$ must agree with (3.2). Another way to say this is that the induced pushforward $\Upsilon_* : TU_R \rightarrow TM$ restricted to the zero section of TU_R is the identity map on $T_x L \oplus \nu_x$.

One way to define such a map Υ is using *exponential normal coordinates* along L : for each (x, α) in U_R there is a unique geodesic $\gamma : [0, 1] \rightarrow M$ in (M, g) with $\gamma(0) = x$, $\dot{\gamma}(0) = \alpha$ and length $|\alpha|_{h_\nu}$, and we set $\Upsilon(x, \alpha) = \gamma(1)$. Then provided $R > 0$ is small enough, conditions (i)–(iv) hold. However, in Proposition 3.8 we will choose U_R, Υ to satisfy an extra condition, which exponential normal coordinates may not satisfy.

3.2. Power series decomposition of $\varphi, * \varphi, g$ on ν . Use the notation of §3.1. Let V denote the vertical subbundle of $T\nu$. We have $V \cong \pi^* \nu$ as a bundle over ν , where $\pi : \nu \rightarrow L$ is the projection. The total space of ν admits a 1-parameter family of diffeomorphisms given by dilation in the fibres. Explicitly, given $t \in \mathbb{R}$, the dilation map $t : \nu \rightarrow \nu$ given by $t(x, \alpha) = (x, t\alpha)$ is a diffeomorphism if $t \neq 0$. The vector field δ on ν whose flow is this family is called the *dilation vector field* on ν . It is a vertical vector field, and with respect to the isomorphism $V \cong \pi^* \nu$ we have $\delta|_{(x, \alpha)} \cong \alpha$ at each $(x, \alpha) \in \nu$.

Define smooth maps Υ_t for $t \in \mathbb{R}$ by

$$(3.5) \quad \begin{aligned} \Upsilon_t : U_{|t|^{-1}R} &\rightarrow M, & \Upsilon_t : (x, \alpha) &\mapsto \Upsilon(x, t\alpha), & 0 \neq t \in \mathbb{R}, \\ \Upsilon_0 : \nu &\rightarrow M, & \Upsilon_0 : (x, \alpha) &\mapsto \Upsilon(x, 0\alpha) = x, & t = 0, \end{aligned}$$

where $U_{|t|^{-1}R}$ is as in (3.4) with $|t|^{-1}R$ in place of R , so that $\Upsilon_1 = \Upsilon$. Note that $\Upsilon_t = \Upsilon \circ t$ is the composition of Υ with dilation by t .

Consider the pullbacks $\Upsilon_t^*(\varphi)$, $\Upsilon_t^*(\ast\varphi)$, $\Upsilon_t^*(g)$. These are defined on ν for $t = 0$ and on $U_{|t|^{-1}R}$ for $t \neq 0$, where $\lim_{t \rightarrow 0} U_{|t|^{-1}R} = \nu$. Since Υ_t depends smoothly on $t \in \mathbb{R}$, so do $\Upsilon_t^*(\varphi)$, $\Upsilon_t^*(\ast\varphi)$, $\Upsilon_t^*(g)$. Thus we can consider the Taylor series of $\Upsilon_t^*(\varphi)$, $\Upsilon_t^*(\ast\varphi)$, $\Upsilon_t^*(g)$ in t at $t = 0$.

From Definition 3.1(iii) and $\iota^*(\varphi) = \varphi$ we see that $\Upsilon_{-t}^*(\varphi) = \Upsilon_t^*(\varphi)$, and similarly for $\ast\varphi, g$, so there are no odd powers of t in the Taylor series. Hence we may write

$$(3.6) \quad \Upsilon_t^*(\varphi) \sim \sum_{n=0}^{\infty} t^{2n} \varphi^{2n},$$

$$(3.7) \quad \Upsilon_t^*(\ast\varphi) \sim \sum_{n=0}^{\infty} t^{2n} \psi^{2n},$$

$$(3.8) \quad \Upsilon_t^*(g) \sim \sum_{n=0}^{\infty} t^{2n} g^{2n}.$$

Here $\varphi^{2n}, \psi^{2n}, g^{2n}$ are defined and smooth on all of ν , and ‘ \sim ’ in (3.6) means that on any compact subset $S \subseteq \nu$, so that $S \subseteq U_{|t|^{-1}R}$ for sufficiently small t , we have

$$\sup_S \left| \Upsilon_t^*(\varphi) - \sum_{n=0}^k t^{2n} \varphi^{2n} \right| = o(t^{2k}) \quad \text{as } t \rightarrow 0 \text{ for all } k = 0, 1, \dots,$$

and similarly for (3.7)–(3.8).

Remark 3.3. In fact as φ satisfies an elliptic equation, there is a unique real analytic structure on M such that $\varphi, \ast\varphi, g$ are real analytic. If we choose Υ also to be real analytic, then on any compact subset $S \subseteq \nu$, the sums in (3.6)–(3.8) converge absolutely for small enough t , and the equations hold exactly. But we will not need this.

Note that $\varphi^{2n}, \psi^{2n}, g^{2n}$ depend on the choice of map Υ .

Since $d\varphi = 0$ and $d(\ast\varphi) = 0$ we see from (3.6)–(3.7) that

$$(3.9) \quad d\varphi^{2n} = 0 \quad \text{and} \quad d\psi^{2n} = 0 \quad \text{for all } n = 0, 1, \dots$$

Let $s \neq 0$ and consider the dilation $s : \nu \rightarrow \nu$. Then $\Upsilon_{ts} = \Upsilon_t \circ s$, so $s^* \Upsilon_t^* = \Upsilon_{st}^*$. It follows that

$$\sum_{n=0}^{\infty} t^{2n} s^* \varphi^{2n} \sim s^* \Upsilon_t^*(\varphi) = \Upsilon_{ts}^*(\varphi) \sim \sum_{n=0}^{\infty} t^{2n} s^{2n} \varphi^{2n}$$

and similarly for $\ast\varphi$ and g . Thus $\varphi^{2n}, \psi^{2n}, g^{2n}$ are homogeneous of order $2n$ under dilations in ν , so that

$$\mathcal{L}_\delta \varphi^{2n} = 2n \varphi^{2n}, \quad \mathcal{L}_\delta \psi^{2n} = 2n \psi^{2n}, \quad \mathcal{L}_\delta g^{2n} = 2n g^{2n},$$

where \mathcal{L}_δ is the Lie derivative. Using Cartan’s formula $\mathcal{L}_v \beta = d(v \cdot \beta) + v \cdot (d\beta)$ for a vector field v and exterior form β , and equation (3.9), we see that

$$(3.10) \quad \varphi^{2n} = \frac{1}{2n} d(\delta \cdot \varphi^{2n}), \quad \psi^{2n} = \frac{1}{2n} d(\delta \cdot \psi^{2n}) \quad \text{for } n = 1, 2, \dots,$$

so the φ^{2n} and ψ^{2n} are exact for $n > 0$.

Now let $\tilde{\nabla}^\nu$ be a connection on ν . This could be the connection ∇^ν defined in §3.1, but later we will want the freedom to choose a different connection. On ν , the connection $\tilde{\nabla}^\nu$ induces a splitting of vector bundles

$$(3.11) \quad T\nu = V \oplus H, \quad \text{where } V \cong \pi^*(\nu) \text{ and } H \cong \pi^*(TL).$$

Here V and H are the vertical and horizontal subbundles of the connection, respectively. The exact sequence

$$0 \longrightarrow \pi^*(\nu) \cong V \longrightarrow T\nu \longrightarrow \pi^*(TL) \cong H \longrightarrow 0$$

is independent of choices, but the embedding $H \subset T\nu$ splitting this exact sequence depends on the choice of $\tilde{\nabla}^\nu$. Dual to (3.11) we have a splitting $T^*\nu = V^* \oplus H^*$, and hence a splitting

$$(3.12) \quad \Lambda^k T^*\nu = \bigoplus_{i+j=k, 0 \leq i \leq 4, 0 \leq j \leq 3} \Lambda^i V^* \otimes \Lambda^j H^*.$$

Write $\varphi_{i,j}^{2n}, \psi_{i,j}^{2n}$ for the components of φ^{2n}, ψ^{2n} in $\Lambda^i V^* \otimes \Lambda^j H^*$ in the splitting (3.12). We will call such forms of type (i, j) . Hence we have

$$(3.13) \quad \varphi^{2n} = \varphi_{0,3}^{2n} + \varphi_{1,2}^{2n} + \varphi_{2,1}^{2n} + \varphi_{3,0}^{2n}, \quad \psi^{2n} = \psi_{1,3}^{2n} + \psi_{2,2}^{2n} + \psi_{3,1}^{2n} + \psi_{4,0}^{2n}.$$

Note that the components $\varphi_{i,j}^{2n}, \psi_{i,j}^{2n}$ depend on the choice of connection $\tilde{\nabla}^\nu$, although φ^{2n}, ψ^{2n} do not.

Similarly we decompose $g^{2n} = g_{0,2}^{2n} + g_{1,1}^{2n} + g_{2,0}^{2n}$, with $g_{0,2}^{2n} \in S^2 H^*$, $g_{1,1}^{2n} \in V^* \otimes H^*$ and $g_{2,0}^{2n} \in S^2 V^*$.

In fact, for $n > 0$ we can decompose φ^{2n}, ψ^{2n} further. As the dilation vector field δ lies in the V factor in (3.11), we see that $\delta \cdot \varphi_{i,j}^{2n}$ is of type $(i-1, j)$, and similarly for $\delta \cdot \psi_{i,j}^{2n}$. Since closed 1-forms on L pull back to closed sections of H^* , we can show that if $\beta = \beta_{i,j}$ is a form of type (i, j) then $d\beta = (d\beta)_{i+1,j} + (d\beta)_{i,j+1} + (d\beta)_{i-1,j+2}$ is the sum of terms of type $(i+1, j)$ and $(i, j+1)$ and $(i-1, j+2)$ only. That is, by taking the exterior derivative, the number of horizontal components can never decrease, and can never increase by more than two. Define forms $\dot{\varphi}_{i,j}^{2n}, \ddot{\varphi}_{i,j}^{2n}, \ddot{\ddot{\varphi}}_{i,j}^{2n}$, and $\dot{\psi}_{i,j}^{2n}, \ddot{\psi}_{i,j}^{2n}, \ddot{\ddot{\psi}}_{i,j}^{2n}$ of type (i, j) for $n > 0$ by

$$(3.14) \quad \begin{aligned} \frac{1}{2n} d(\delta \cdot \varphi_{i,j}^{2n}) &= \dot{\varphi}_{i,j}^{2n} + \ddot{\varphi}_{i-1,j+1}^{2n} + \ddot{\ddot{\varphi}}_{i-2,j+2}^{2n}, \\ \frac{1}{2n} d(\delta \cdot \psi_{i,j}^{2n}) &= \dot{\psi}_{i,j}^{2n} + \ddot{\psi}_{i-1,j+1}^{2n} + \ddot{\ddot{\psi}}_{i-2,j+2}^{2n}. \end{aligned}$$

Then (3.10) implies that

$$(3.15) \quad \varphi_{i,j}^{2n} = \dot{\varphi}_{i,j}^{2n} + \ddot{\varphi}_{i,j}^{2n} + \ddot{\ddot{\varphi}}_{i,j}^{2n}, \quad \psi_{i,j}^{2n} = \dot{\psi}_{i,j}^{2n} + \ddot{\psi}_{i,j}^{2n} + \ddot{\ddot{\psi}}_{i,j}^{2n} \quad \text{for } n > 0.$$

By considering equation (3.14) for various values of (i, j) , we derive the following conditions:

$$\begin{aligned}
 \ddot{\varphi}_{3,0}^{2n} &= \ddot{\varphi}_{2,1}^{2n} = \ddot{\varphi}_{3,0}^{2n} = 0, \\
 \ddot{\psi}_{4,0}^{2n} &= \ddot{\psi}_{3,1}^{2n} = \ddot{\psi}_{4,0}^{2n} = 0 \quad \text{for } n > 0 \text{ } (j = -1 \text{ or } -2), \\
 \dot{\varphi}_{0,3}^{2n} &= 0 \quad \text{for } n > 0 \text{ } (i = 0, j = 3).
 \end{aligned}
 \tag{3.16}$$

3.3. Understanding the leading terms in the power series. On the fibres of the rank 7 vector bundle $TM|_L$ over L we have a G_2 -structure $(\varphi, g)|_L$. Pulling back by $\pi : \nu \rightarrow L$ gives a G_2 -structure $\pi^*((\varphi, g)|_L)$ on $\pi^*(TM|_L) \rightarrow \nu$. But (3.2) gives an isomorphism

$$\pi^*(TM|_L) \cong \pi^*(\nu) \oplus \pi^*(TL),$$

and (3.11) an isomorphism

$$T\nu = V \oplus H \cong \pi^*(\nu) \oplus \pi^*(TL).$$

Combining these gives an isomorphism $T\nu \cong \pi^*(TM|_L)$. Let (φ^ν, g^ν) be the G_2 -structure on the fibres of $T\nu$ (that is, the G_2 -structure on ν) identified with $\pi^*((\varphi, g)|_L)$ by this isomorphism, and ψ^ν the corresponding 4-form $*_{g^\nu}(\varphi^\nu)$. One can think of (φ^ν, g^ν) as a G_2 -structure on ν that is “constant” on the fibres, since at each point $(x, \alpha) \in \nu$ it corresponds to the G_2 -structure $\varphi(x)$ at $(x, 0) \in \nu$ via the canonical isomorphism $T_{(x, \alpha)}(\nu_x) \cong \nu_x$ between the tangent space of a vector space and the vector space itself.

Thus $(\nu, \varphi^\nu, g^\nu)$ is a G_2 -manifold, which is generally not torsion-free, as we will see shortly. We will use the metric g^ν to measure the size of tensors on ν .

Write $\varphi_{i,j}^\nu, \psi_{i,j}^\nu, g_{i,j}^\nu$ for the components of $\varphi^\nu, \psi^\nu, g^\nu$ of type (i, j) . As $\varphi^\nu, \psi^\nu, g^\nu$ are unchanged under $-1 : \nu \rightarrow \nu$, the components with i odd are zero, giving

$$\varphi^\nu = \varphi_{0,3}^\nu + \varphi_{2,1}^\nu, \quad \psi^\nu = \psi_{2,2}^\nu + \psi_{4,0}^\nu, \quad g^\nu = g_{0,2}^\nu + g_{2,0}^\nu.$$

The fact that $g_{1,1}^\nu = 0$ says that H is orthogonal to V . Since dilations rescale the V factor in (3.11) but fix the H factor, the effect of Lie derivative by the dilation vector field δ is

$$(3.17) \quad \mathcal{L}_\delta \varphi_{i,j}^\nu = i \varphi_{i,j}^\nu, \quad \mathcal{L}_\delta \psi_{i,j}^\nu = i \psi_{i,j}^\nu, \quad \mathcal{L}_\delta g_{i,j}^\nu = i g_{i,j}^\nu.$$

Return now to $\varphi^{2n}, \psi^{2n}, g^{2n}$ defined in §3.2. Observe that $|\varphi_{i,j}^{2n}|_{g^\nu}, |\psi_{i,j}^{2n}|_{g^\nu}, |g_{i,j}^{2n}|_{g^\nu}$ are continuous functions on ν which are homogeneous of some degree under dilations. Explicitly, since $t^* \varphi_{i,j}^{2n} = t^{2n} \varphi_{i,j}^{2n}$, the function $|\varphi_{i,j}^{2n}|_{g^\nu}$ is homogeneous of order $2n - i$ with respect to dilations, because the g^ν -norm on vertical k -forms is homogeneous of degree $-k$ by (3.17). Similarly $|\psi_{i,j}^{2n}|_{g^\nu}$ and $|g_{i,j}^{2n}|_{g^\nu}$ are also homogeneous of degree $2n - i$.

Write $r : \nu \rightarrow [0, \infty)$ for the radius function, where $r(x, \alpha) = |\alpha|_{g^\nu}$. When restricted to the unit sphere bundle of ν , which is compact, the continuous functions $|\varphi_{i,j}^{2n}|_{g^\nu}$, $|\psi_{i,j}^{2n}|_{g^\nu}$, $|g_{i,j}^{2n}|_{g^\nu}$ are bounded. Thus, by their homogeneity, we deduce that

$$(3.18) \quad |\varphi_{i,j}^{2n}|_{g^\nu} = O(r^{2n-i}), \quad |\psi_{i,j}^{2n}|_{g^\nu} = O(r^{2n-i}), \quad |g_{i,j}^{2n}|_{g^\nu} = O(r^{2n-i}).$$

As the degree of homogeneity cannot be negative for $\varphi_{i,j}^{2n}, \psi_{i,j}^{2n}, g_{i,j}^{2n} \neq 0$, by continuity at $r = 0$, we have

$$(3.19) \quad \varphi_{i,j}^{2n} = 0, \quad \psi_{i,j}^{2n} = 0, \quad g_{i,j}^{2n} = 0 \quad \text{if } i > 2n.$$

An immediate consequence of combining (3.19) and (3.14) is

$$(3.20) \quad \begin{aligned} \dot{\varphi}_{i,j}^{2n} &= \ddot{\varphi}_{i-1,j+1}^{2n} = \ddot{\ddot{\varphi}}_{i-2,j+2}^{2n} = 0, \quad \text{for } n > 0, (i > 2n), \\ \dot{\psi}_{i,j}^{2n} &= \ddot{\psi}_{i-1,j+1}^{2n} = \ddot{\ddot{\psi}}_{i-2,j+2}^{2n} = 0, \quad \text{for } n > 0, (i > 2n). \end{aligned}$$

The terms in (3.18) which are homogeneous of $O(r^0)$ are those which may be nonzero at the zero section $L \subset \nu$. But at the zero section, $\varphi^\nu, \psi^\nu, g^\nu$ agree with $\Upsilon^*(\varphi), \Upsilon^*(\psi), \Upsilon^*(g)$, that is, with $\Upsilon_1^*(\varphi), \Upsilon_1^*(\psi), \Upsilon_1^*(g)$. So (3.6)–(3.8) yield

$$\varphi_{i,j}^\nu = \varphi_{i,j}^i, \quad \psi_{i,j}^\nu = \psi_{i,j}^i, \quad g_{i,j}^\nu = g_{i,j}^i, \quad (\text{no sum over } i).$$

If $i > 0$ then $\varphi_{i,j}^i = \dot{\varphi}_{i,j}^i + \ddot{\varphi}_{i,j}^i + \ddot{\ddot{\varphi}}_{i,j}^i$, but $\dot{\varphi}_{i,j}^i, \ddot{\varphi}_{i,j}^i$ are respectively components of $\frac{1}{i}d(\delta \cdot \varphi_{i+1,j-1}^i)$ and $\frac{1}{i}d(\delta \cdot \varphi_{i+2,j-2}^i)$, which are zero by (3.19). So $\ddot{\ddot{\varphi}}_{i,j}^i = \ddot{\ddot{\psi}}_{i,j}^i = 0$, and similarly $\ddot{\ddot{\psi}}_{i,j}^i = \ddot{\ddot{\psi}}_{i,j}^i = 0$. Hence we have

$$(3.21) \quad \varphi^\nu = \varphi_{0,3}^0 + \dot{\varphi}_{2,1}^2, \quad \psi^\nu = \dot{\psi}_{2,2}^2 + \dot{\psi}_{4,0}^4.$$

Here $\varphi_{0,3}^0 = \pi^*(\text{vol}_L)$ is the pullback to ν of the volume form vol_L on L , and is a closed 3-form on ν .

We also find it useful to define $\varphi_t^\nu, \psi_t^\nu, g_t^\nu$ on ν to be the pullbacks of $\varphi^\nu, \psi^\nu, g^\nu$ under $t : \nu \rightarrow \nu$ for $t > 0$. This scales each component by the powers of t in (3.6)–(3.8), so that by (3.21) we have

$$(3.22) \quad \varphi_t^\nu = \varphi_{0,3}^0 + t^2 \dot{\varphi}_{2,1}^2, \quad \psi_t^\nu = t^2 \dot{\psi}_{2,2}^2 + t^4 \dot{\psi}_{4,0}^4, \quad g_t^\nu = g_{0,2}^0 + t^2 g_{2,0}^2.$$

Then (φ_t^ν, g_t^ν) is a G_2 -structure on ν , with 4-form $\psi_t^\nu = *_{g_t^\nu}(\varphi_t^\nu) = \Theta(\varphi_t^\nu)$, and $\varphi_t^\nu, \psi_t^\nu, g_t^\nu$ are the leading order approximations to $\Upsilon_t^*(\varphi), \Upsilon_t^*(\psi), \Upsilon_t^*(g)$ near the zero section $L \subset \nu$, just as $\varphi^\nu, \psi^\nu, g^\nu$ approximate $\Upsilon^*(\varphi), \Upsilon^*(\psi), \Upsilon^*(g)$. If we estimate the size of tensors using g_t^ν rather than g^ν , the answers differ by powers of t . For example, if $\beta_{i,j}$ is a form on ν of type (i, j) then

$$(3.23) \quad |\beta_{i,j}|_{g_t^\nu} = t^{-i} |\beta_{i,j}|_{g^\nu}.$$

We may summarize all the work above, also using (3.20), with the following pair of equations:

$$\begin{aligned}
 \Upsilon_t^*(\varphi) \sim & \\
 & \left. \begin{aligned} & \varphi_{0,3}^0 + t^2 \dot{\varphi}_{2,1}^2 \\ & + t^2 \ddot{\varphi}_{1,2}^2 + t^2 \dot{\varphi}_{1,2}^2 + t^4 \dot{\varphi}_{3,0}^4 \\ & + t^2 \ddot{\varphi}_{0,3}^2 + t^2 \ddot{\varphi}_{0,3}^2 + t^4 \ddot{\varphi}_{2,1}^4 + t^2 \dot{\varphi}_{0,3}^2 + t^4 \dot{\varphi}_{2,1}^4 \\ & \qquad \qquad \qquad + t^4 \ddot{\varphi}_{1,2}^4 \qquad \qquad \qquad + t^4 \ddot{\varphi}_{1,2}^4 + t^4 \dot{\varphi}_{1,2}^4 + t^6 \dot{\varphi}_{3,0}^6 \end{aligned} \right] \begin{aligned} & \varphi_t^\nu = O(1) \\ & O(tr) \\ & O(t^2 r^2) \\ & O(t^3 r^3) \end{aligned} \\
 (3.24) \qquad & + \dots,
 \end{aligned}$$

$$\begin{aligned}
 \Upsilon_t^*(\psi) \sim & \\
 & \left. \begin{aligned} & t^2 \dot{\psi}_{2,2}^2 + t^4 \dot{\psi}_{4,0}^4 + \\ & t^2 \ddot{\psi}_{1,3}^2 + t^4 \ddot{\psi}_{3,1}^4 + t^2 \dot{\psi}_{1,3}^2 + t^4 \dot{\psi}_{3,1}^4 \\ & \qquad \qquad \qquad + t^4 \ddot{\psi}_{2,2}^4 \qquad \qquad \qquad + t^4 \ddot{\psi}_{2,2}^4 + t^4 \dot{\psi}_{2,2}^4 + t^6 \dot{\psi}_{4,0}^6 \\ & \qquad \qquad \qquad + t^4 \ddot{\psi}_{1,3}^4 + t^4 \ddot{\psi}_{1,3}^4 + t^6 \ddot{\psi}_{3,1}^6 + t^4 \dot{\psi}_{1,3}^4 + t^6 \dot{\psi}_{3,1}^6 \end{aligned} \right] \begin{aligned} & \psi_t^\nu = O(1) \\ & O(tr) \\ & O(t^2 r^2) \\ & O(t^3 r^3) \end{aligned} \\
 (3.25) \qquad & + \dots,
 \end{aligned}$$

where the $O(\dots)$ are measured using g_t^ν . Explicitly, using (3.18) and (3.23) we have

$$|t^a \alpha_{i,j}^{2n}|_{g_t^\nu} = t^a |\alpha_{i,j}^{2n}|_{g_t^\nu} = t^{a-i} |\alpha_{i,j}^{2n}|_{g^\nu} = t^{a-i} O(r^{2n-i}),$$

so the terms aligned horizontally in (3.24) and (3.25) have the same order $O(t^k r^k)$ when measured using $|\cdot|_{g_t^\nu}$. The terms aligned vertically sum to a closed form, by (3.14).

3.4. Choosing Υ and $\tilde{\nabla}^\nu$ to eliminate the $O(tr)$ error terms. Taking $t = 1$, it will be important later to eliminate the $O(tr)$ terms in (3.24)–(3.25), as they would cause too large an error to apply Theorem 2.7.

Recall that φ^{2n}, ψ^{2n} depend on the choice of the map Υ , and the further decomposition into types $\varphi_{i,j}^{2n}$ and $\psi_{i,j}^{2n}$ depend on the choice of connection $\tilde{\nabla}^\nu$ on ν . In this section we will modify Υ and $\tilde{\nabla}^\nu$ to eliminate the $O(tr)$ terms in (3.24)–(3.25).

In this section we will need several times to make use of a one-to-one correspondence between tensors on the total space ν that are homogeneous of a certain degree and sections of certain bundles over L . This correspondence is more general but we will only describe two particular cases we will need. Recall that we have canonical isomorphisms $V \cong \pi^*(\nu)$ and $H^* \cong \pi^*(T^*L)$ and, given a connection $\tilde{\nabla}^\nu$ on ν , we also have isomorphisms $H \cong \pi^*(TL)$ and $V^* \cong \pi^*(\nu^*)$. Let $S^k(\nu^*)$ be the k^{th} symmetric power of ν^* , so sections of $S^k(\nu^*)$ are functions on the total space ν that are homogeneous of degree k under dilations.

Case 1: Let $\bar{\alpha}$ be a section of the bundle $S^{2n-i}(\nu^*) \otimes \Lambda^i(\nu^*) \otimes TL$. Then under these identifications $\bar{\alpha}$ corresponds to a section $v_{\bar{\alpha}}$ of $\Lambda^i(V^*) \otimes H$ which is homogeneous of degree $2n - i$. This correspondence can be seen explicitly as follows. Let v be a vector field on L and let f_1, \dots, f_i be sections of ν^* . Then

$$\bar{\alpha} = h \otimes (f_1 \wedge \dots \wedge f_i) \otimes v \longleftrightarrow v_{\bar{\alpha}} = h(\pi^* f_1) \wedge \dots \wedge (\pi^* f_i) \otimes \pi^*(v)$$

for $h \in S^{2n-i}(\nu^*)$.

Case 2: Let $\bar{\alpha}_{i,j}^{2n}$ be a section of the bundle $S^{2n-i}(\nu^*) \otimes \Lambda^i(\nu^*) \otimes \Lambda^j(T^*L)$. Then under these identifications $\bar{\alpha}_{i,j}^{2n}$ corresponds to a section of $\Lambda^i(V^*) \otimes \Lambda^j(H^*)$, that is an (i, j) form on ν , which is homogeneous of degree $2n - i$. Explicitly, if e_1, \dots, e_j are 1-forms on L and f_1, \dots, f_i are sections of ν^* , then

$$\begin{aligned} \bar{\alpha}_{i,j}^{2n} &= h \otimes (f_1 \wedge \dots \wedge f_i) \otimes (e_1 \wedge \dots \wedge e_j) \\ &\quad \updownarrow \\ \alpha_{i,j}^{2n} &= h(\pi^* f_1) \wedge \dots \wedge (\pi^* f_i) \otimes (\pi^* e_1) \wedge \dots \wedge (\pi^* e_j) \end{aligned}$$

for $h \in S^{2n-i}(\nu^*)$. Let $d^v = d^{(1,0)}$ be the vertical derivative, the part of d that takes an (i, j) form to an $(i+1, j)$ form. In this case we also need to understand the correspondence between d^v on the right hand side and a canonical operation on the left hand side. The easiest way to see this is in local coordinates. Let x_1, \dots, x_3 be local coordinates on L and let y_1, \dots, y_4 be fibre coordinates for ν^* . Locally, a decomposable (i, j) form that is homogeneous of degree $2n - i$ will be of the form $h dy_{a_1} \wedge \dots \wedge dy_{a_i} \wedge dx_{b_1} \wedge \dots \wedge dx_{b_j}$ where $h(x, y)$ is homogeneous of degree $2n - i$ in y_1, \dots, y_4 . Noting that $d^v x_b = 0$ and $d^v y_a = dy_a$, it is easy to see that d^v corresponds under the identification to the *partial antisymmetrization* map

$$\wedge : S^{2n-i}(\nu^*) \otimes \Lambda^i(\nu^*) \otimes \Lambda^j(T^*L) \rightarrow S^{2n-i-1}(\nu^*) \otimes \Lambda^{i+1}(\nu^*) \otimes \Lambda^j(T^*L).$$

We write this map explicitly in a local frame when $2n - i = 1$ or $2n - i = 2$. Let β be a section of $\Lambda^i(\nu^*)$ and γ a j -form on L . Then we have

$$\begin{aligned} f_a \otimes \beta \otimes \gamma &\mapsto (f_a \wedge \beta) \otimes \gamma, & 2n - i = 1, \\ (f_a f_b) \otimes \beta \otimes \gamma &\mapsto [f_a \otimes (f_b \wedge \beta) + f_b \otimes (f_a \wedge \beta)] \otimes \gamma, & 2n - i = 2. \end{aligned}$$

With these preparations out of the way, we begin with the following lemma.

Lemma 3.4. *In the situation above, we have*

$$(3.26) \quad (D_{\varphi^\nu} \Theta)(\ddot{\varphi}_{1,2}^2 + \dot{\varphi}_{1,2}^2 + \dot{\varphi}_{3,0}^4) = \ddot{\psi}_{1,3}^2 + \ddot{\psi}_{3,1}^4 + \dot{\psi}_{1,3}^2 + \dot{\psi}_{3,1}^4,$$

where $D_{\varphi^\nu} \Theta$ is as in equation (2.6) in §2.2.

Proof. We have $\Theta(\varphi^\nu) = \psi^\nu$ and $\Theta(\Upsilon^*(\varphi)) = \Upsilon^*(\varphi)$, since $\Theta(\varphi) = *\varphi$, and Θ is an intrinsic object on any 7-manifold, and so commutes with pullback by the local diffeomorphism Υ . Equation (3.26) now follows from (3.24)–(3.25) for $t = 1$ and small r , since $D_{\varphi^\nu}\Theta$ must map the leading term in $\Upsilon^*(\varphi) - \varphi^\nu$ to the leading term in $\Theta(\Upsilon^*(\varphi)) - \Theta(\varphi^\nu)$.
q.e.d.

In §3.1 we chose $\Upsilon : U_R \rightarrow M$. If we expand Υ in a power series around $L \subset U_R$ in terms of order $O(r^k)$ for $k = 0, 1, \dots$, then the $O(r^0)$ and $O(r^1)$ terms are determined by Definition 3.1(ii), (iv), but the $O(r^2)$ and higher terms are essentially arbitrary, apart from the \mathbb{Z}_2 -equivariance in Definition 3.1(iii).

The next lemma shows that by choosing the $O(r^2)$ terms in the expansion of Υ correctly, we can ensure the vanishing of the fourth column in (3.24). Note that the $O(r^2)$ terms in the expansion of Υ determine the $O(r)$ terms in the expansion of $\Upsilon^*(\varphi)$, since the pullback Υ^* involves one derivative of Υ .

Lemma 3.5. *We can choose the map $\Upsilon : U_R \rightarrow M$ in §3.1 so that $\dot{\varphi}_{3,0}^4 = \ddot{\varphi}_{2,1}^4 = \ddot{\varphi}_{1,2}^4 = 0$.*

Proof. Let R' and $\Upsilon' : U_{R'} \rightarrow M$ be some fixed choice of R, Υ in §3.1, and $\dot{\varphi}_{3,0}^4$ the corresponding value for $\dot{\varphi}_{3,0}^4$. Let $\bar{\alpha}$ be a smooth section of the rank 30 vector bundle $S^2\nu^* \otimes TL \rightarrow L$, to be chosen later. Under the identification (Case 1) described above with $n = 1$ and $i = 0$, the section $\bar{\alpha}$ corresponds to a vector field $v_{\bar{\alpha}}$ in $H \subset T\nu$ on ν which is homogeneous quadratic in the fibre directions. That is, $v_{\bar{\alpha}}$ is a horizontal vector field on ν such that the function $|v_{\bar{\alpha}}|_{g^\nu}$ is homogeneous of degree 2 under dilations.

For $0 < R \leq R'$ small, define $\Upsilon : U_R \rightarrow M$ by $\Upsilon = \Upsilon' \circ \exp(v_{\bar{\alpha}})$, where $\exp(v_{\bar{\alpha}}) : U_R \rightarrow \nu$ is the flow of the vector field $v_{\bar{\alpha}}$, and it maps U_R diffeomorphically to an open subset of $U_{R'}$ provided R is sufficiently small. As $v_{\bar{\alpha}} = O(r^2)$, we see that Υ and Υ' agree up to $O(r^2)$ near $L \subset U_R$, so Definition 3.1(ii), (iv) for Υ' imply (ii), (iv) for Υ . Also $v_{\bar{\alpha}}$ is invariant under $-1 : U_R \rightarrow U_R$, so Definition 3.1(iii) for Υ' implies (iii) for Υ . Hence R, Υ satisfy Definition 3.1.

We now have a 3-form $\dot{\varphi}_{3,0}^4$ on ν from the new Υ , so we can compare $\dot{\varphi}_{3,0}^4$ and $\dot{\varphi}_{3,0}^4$. They are precisely the (3,0) components that are homogeneous of degree 1 of $\Upsilon^*(\varphi)$ and $\Upsilon'^*(\varphi)$, respectively. Because $\Upsilon^*(\varphi) = \exp(v_{\bar{\alpha}})^*(\Upsilon'^*(\varphi))$ with $v_{\bar{\alpha}} = O(r^2)$, we find that

$$\begin{aligned} \Upsilon^*(\varphi) &= \Upsilon'^*(\varphi) + \mathcal{L}_{v_{\bar{\alpha}}}(\Upsilon'^*(\varphi)) + O(r^2) \\ &= \Upsilon'^*(\varphi) + d[v_{\bar{\alpha}} \cdot \Upsilon'^*(\varphi)] + O(r^2) \end{aligned}$$

since $\Upsilon'^*(\varphi)$ is closed. We take the type (3,0) components of both sides of the above equation and keep those terms that are homogeneous of

degree 1. From (3.24) with $t = 1$ the only component of $\Upsilon'^*(\varphi)$ for which $d[v_{\bar{\alpha}} \cdot \Upsilon'^*(\varphi)]$ will have a $(3, 0)$ component is $\dot{\varphi}_{2,1}^2$, and indeed $[d(v_{\bar{\alpha}} \cdot \dot{\varphi}_{2,1}^2)]_{3,0}$ is homogeneous of degree 1. Thus, we have

$$(3.27) \quad \dot{\varphi}_{3,0}^4 = \dot{\varphi}_{3,0}^{\prime 4} + [d(v_{\bar{\alpha}} \cdot \dot{\varphi}_{2,1}^2)]_{3,0} = \dot{\varphi}_{3,0}^{\prime 4} + d^v(v_{\bar{\alpha}} \cdot \dot{\varphi}_{2,1}^2).$$

Because $\dot{\varphi}_{3,0}^4, \dot{\varphi}_{3,0}^{\prime 4}$ are sections of $\Lambda^3 V^* \rightarrow \nu$ which are linear in the fibre directions, by the identification (*Case 2*) above we can identify $\dot{\varphi}_{3,0}^4, \dot{\varphi}_{3,0}^{\prime 4}$ with sections $\bar{\varphi}_{3,0}^4, \bar{\varphi}_{3,0}^{\prime 4}$ of the rank 16 vector bundle $\nu^* \otimes \Lambda^3 \nu^* \rightarrow L$. These sections $\bar{\varphi}_{3,0}^4, \bar{\varphi}_{3,0}^{\prime 4}$ are not arbitrary, however. The fourth column of (3.24) is closed, and the only $(4, 0)$ component of its exterior derivative arises from the vertical (fibre) derivative of the $(3, 0)$ term. That is, $\dot{\varphi}_{3,0}^4$ and $\dot{\varphi}_{3,0}^{\prime 4}$ are closed in the fibre directions. As explained above, under the identification this corresponds to $\bar{\varphi}_{3,0}^4$ and $\bar{\varphi}_{3,0}^{\prime 4}$ being in the kernel of $\wedge : \nu^* \otimes \Lambda^3 \nu^* \rightarrow \Lambda^4 \nu^*$, which is a rank 15 vector bundle on L .

Translating via the identifications, equation (3.27) becomes

$$\bar{\varphi}_{3,0}^4 = \bar{\varphi}_{3,0}^{\prime 4} + \bar{\alpha} \cdot \bar{\varphi}_{2,1}^2,$$

where $\bar{\varphi}_{2,1}^2$ is a section of $\Lambda^2 \nu^* \otimes T^* L \rightarrow L$, and \cdot is a natural bilinear product

$$(S^2 \nu^* \otimes TL) \times (\Lambda^2 \nu^* \otimes T^* L) \longrightarrow \nu^* \otimes \Lambda^3 \nu^*,$$

defined by combining the dual pairing of $TL, T^* L$ with the partial antisymmetrization map $S^2 \nu^* \otimes \Lambda^2 \nu^* \rightarrow \nu^* \otimes \Lambda^3 \nu^*$. A calculation shows that the map $\bar{\alpha} \mapsto \bar{\alpha} \cdot \bar{\varphi}_{2,1}^2$ is surjective onto the kernel of $\wedge : \nu^* \otimes \Lambda^3 \nu^* \rightarrow \Lambda^4 \nu^*$. (The details are given in Proposition A.1 of Appendix A.) Therefore we can choose $\bar{\alpha}$ such that $\bar{\alpha} \cdot \bar{\varphi}_{2,1}^2 = -\bar{\varphi}_{3,0}^{\prime 4}$, so that $\bar{\varphi}_{3,0}^4 = 0$, and hence $\dot{\varphi}_{3,0}^4 = 0$, as desired. Finally, since $\ddot{\varphi}_{3,0}^4 = \ddot{\varphi}_{3,0}^{\prime 4} = 0$ by (3.16) we have $\varphi_{3,0}^4 = 0$ by (3.15). Hence by (3.14) we deduce that $\ddot{\varphi}_{2,1}^4 = \ddot{\varphi}_{1,2}^4 = 0$ as well. q.e.d.

Remark 3.6. Note that in the proof above $\bar{\alpha}$ lives in a rank 30 vector bundle, but $\dot{\varphi}_{3,0}^4$ lives in a rank 15 vector bundle, so we have used only half of the freedom in $\bar{\alpha}$.

From now on, fix the map $\Upsilon : U_R \rightarrow M$ satisfying Lemma 3.5. This fixes the decomposition of $\Upsilon^*(\varphi)$ into components φ^{2n} in (3.6). In §3.2 we chose a connection $\tilde{\nabla}^\nu$ on $\nu \rightarrow L$, which was used to define the splitting $T\nu = V \oplus H$, and hence the decompositions of φ^{2n} into components $\varphi_{i,j}^{2n}$ and $\dot{\varphi}_{i,j}^{2n}, \ddot{\varphi}_{i,j}^{2n}$ in (3.13) and (3.15). The next lemma explores the effect of changing the choice of $\tilde{\nabla}^\nu$. Note that changing $\tilde{\nabla}^\nu$ does not change the fact that $\dot{\varphi}_{3,0}^4 = \ddot{\varphi}_{2,1}^4 = \ddot{\varphi}_{1,2}^4 = 0$ in Lemma 3.5, since we see from (3.24) with $t = 1$ that $\dot{\varphi}_{3,0}^4 = 0$ if and only if $|\varphi^4|_{g^\nu} = O(r^2)$, where this second condition is independent of $\tilde{\nabla}^\nu$, and $\dot{\varphi}_{3,0}^4 = 0$ implies $\dot{\varphi}_{3,0}^4 = \ddot{\varphi}_{2,1}^4 = \ddot{\varphi}_{1,2}^4 = 0$ as above.

Lemma 3.7. *In §3.2 there is a unique choice of the connection $\tilde{\nabla}^\nu$ on ν used to define the splitting $T\nu = V \oplus H$, such that $\tilde{\varphi}_{1,2}^2 = \dot{\varphi}_{1,2}^2 = \ddot{\varphi}_{0,3}^2 = 0$.*

Proof. Observe that $\ddot{\varphi}_{1,2}^2 + \dot{\varphi}_{1,2}^2$ is a $(1, 2)$ form on ν which is linear in the fibre directions. By the identification (Case 2) above we can identify $\ddot{\varphi}_{1,2}^2 + \dot{\varphi}_{1,2}^2$ with a section $\bar{\varphi}_{1,2}^2 + \bar{\varphi}_{1,2}^2$ of $\nu^* \otimes \nu^* \otimes \Lambda^2 T^*L$ over L , where the first factor of ν^* is linear functions in the fibres of $\nu \rightarrow L$, and the second factor is 1-forms on the fibres of $\nu \rightarrow L$.

Suppose $\tilde{\nabla}^\nu$ and $\tilde{\nabla}'^\nu$ are two choices of $\tilde{\nabla}^\nu$, yielding $\ddot{\varphi}_{1,2}^2 + \dot{\varphi}_{1,2}^2$ and $\ddot{\varphi}'_{1,2}^2 + \dot{\varphi}'_{1,2}^2$ above, and corresponding sections $\bar{\varphi}_{1,2}^2 + \bar{\varphi}_{1,2}^2$ and $\bar{\varphi}'_{1,2}^2 + \bar{\varphi}'_{1,2}^2$ of $\nu^* \otimes \nu^* \otimes \Lambda^2 T^*L$. Then we have $\tilde{\nabla}^\nu = \tilde{\nabla}'^\nu + \Gamma$, for Γ a smooth section of $\nu \otimes \nu^* \otimes T^*L$ over L . In terms of local frames, if e_1, \dots, e_3 is a local orthonormal frame for T^*L and f_1, \dots, f_4 is a local frame for ν^* , then $\pi^*e_1, \dots, \pi^*e_3$ is a local frame for $H^* = H'^*$, while the two connections $\tilde{\nabla}^\nu$ and $\tilde{\nabla}'^\nu$ induce different vertical frames, since $V^* \neq V'^*$. We denote these frames by $\pi^*f_1, \dots, \pi^*f_4$ and $\pi'^*f_1, \dots, \pi'^*f_4$, respectively, which is an abuse of notation as they are not actually pullbacks. The relation between these frames is

$$(\pi'^*f^p)|_{(x,\alpha)} = (\pi^*f^p)|_{(x,\alpha)} + \sum_{k=1}^3 \Gamma_{qk}^p(x) \alpha^q e_k$$

at the point $(x, \alpha^p f_p) \in \nu$. From (3.24), we see that $\ddot{\varphi}_{1,2}^2 + \dot{\varphi}_{1,2}^2$ is precisely the $(1, 2)$ part of φ^2 , and $\ddot{\varphi}_{2,1}^2$ is the $(2, 1)$ part of φ^2 . We expand $\varphi^2 = \varphi_{2,1}^2 + \varphi_{1,2}^2 + \varphi_{0,3}^2 = \varphi'_{2,1}^2 + \varphi'_{1,2}^2 + \varphi'_{0,3}^2$ in terms of these frames, and apply the identification between forms on ν and sections of bundles over L . A short calculation gives

$$(3.28) \quad \bar{\varphi}_{1,2}^2 + \bar{\varphi}_{1,2}^2 = \bar{\varphi}'_{1,2}^2 + \bar{\varphi}'_{1,2}^2 + \Gamma \cdot \bar{\varphi}_{2,1}^2,$$

where $\bar{\varphi}_{2,1}^2$ is the section of $\Lambda^2 \nu^* \otimes T^*L$ corresponding to $\ddot{\varphi}_{2,1}^2$ under the identification of $\Lambda^2 V^* \otimes H^*$ with $\Lambda^2 \nu^* \otimes T^*L$. Here \cdot is the canonical bilinear pairing

$$(\nu \otimes \nu^* \otimes T^*L) \times (\Lambda^2 \nu^* \otimes T^*L) \rightarrow \nu^* \otimes \nu^* \otimes \Lambda^2 T^*L$$

given by pairing the ν factor with the $\Lambda^2 \nu^*$ factor, and wedging the two T^*L factors together.

A calculation shows that the map $\Gamma \mapsto \Gamma \cdot \bar{\varphi}_{2,1}^2$ is an isomorphism of vector bundles on L . (The details are given in Proposition A.3 of Appendix A.) Hence if we fix a reference connection $\tilde{\nabla}'^\nu$ on ν , giving some $\bar{\varphi}'_{1,2}^2 + \bar{\varphi}'_{1,2}^2$, there is a unique Γ such that (3.28) gives $\bar{\varphi}'_{1,2}^2 + \bar{\varphi}'_{1,2}^2 = 0$, and then $\tilde{\nabla}^\nu = \tilde{\nabla}'^\nu + \Gamma$ is the unique choice of connection on ν with $\ddot{\varphi}_{1,2}^2 + \dot{\varphi}_{1,2}^2 = 0$.

By (3.19), we have $\varphi_{3,0}^2 = 0$. Thus from (3.14) we get $\ddot{\varphi}_{1,2}^2 = 0$. Therefore by (3.15) we find that $\varphi_{1,2}^2 = \ddot{\varphi}_{1,2}^2 + \dot{\varphi}_{1,2}^2$. Hence we conclude that in fact $\varphi_{1,2}^2 = 0$, which by (3.14) again implies that $\dot{\varphi}_{1,2}^2 = \ddot{\varphi}_{0,3}^2 = 0$. Thus $\ddot{\varphi}_{1,2}^2 + \dot{\varphi}_{1,2}^2 = 0$ yields $\ddot{\varphi}_{1,2}^2 = 0$, and the proof is complete. q.e.d.

We now combine Lemmas 3.4, 3.5, and 3.7:

Proposition 3.8. *We can choose the data $\Upsilon : U_R \rightarrow M$ and $\tilde{\nabla}^\nu$ in §3.1–§3.2 such that*

$$(3.29) \quad \begin{aligned} \ddot{\varphi}_{1,2}^2 &= \dot{\varphi}_{1,2}^2 = \dot{\varphi}_{3,0}^4 = \ddot{\varphi}_{0,3}^2 = \ddot{\varphi}_{2,1}^4 = \ddot{\varphi}_{1,2}^4 = 0, \\ \ddot{\psi}_{1,3}^2 &= \ddot{\psi}_{3,1}^4 = \dot{\psi}_{1,3}^2 = \dot{\psi}_{3,1}^4 = \ddot{\psi}_{2,2}^4 = \ddot{\psi}_{1,3}^4 = 0, \\ (D_{\varphi^\nu} \Theta)(\ddot{\varphi}_{0,3}^2 + \dot{\varphi}_{0,3}^2 + \dot{\varphi}_{2,1}^4) &= \ddot{\psi}_{2,2}^4 + \dot{\psi}_{2,2}^4 + \dot{\psi}_{4,0}^6. \end{aligned}$$

Proof. Lemmas 3.5 and 3.7 prove that we may choose $\Upsilon, \tilde{\nabla}^\nu$ such that the first line of (3.29) holds. Hence Lemma 3.4 shows that

$$(3.30) \quad \ddot{\psi}_{1,3}^2 + \ddot{\psi}_{3,1}^4 + \dot{\psi}_{1,3}^2 + \dot{\psi}_{3,1}^4 = 0.$$

Since $\psi_{3,1}^2 = 0$ by (3.19), we get from (3.14) that $\ddot{\psi}_{1,3}^2 = 0$. We also have $\ddot{\psi}_{3,1}^4 = 0$ by (3.16). Thus, taking the (1, 3) and (3, 1) components of (3.30) and using (3.15) we find

$$(3.31) \quad \psi_{1,3}^2 = \ddot{\psi}_{1,3}^2 + \dot{\psi}_{1,3}^2 = 0, \quad \psi_{3,1}^4 = \ddot{\psi}_{3,1}^4 + \dot{\psi}_{3,1}^4 = 0.$$

Applying (3.14) to $\psi_{1,3}^2 = 0$ and $\psi_{3,1}^4 = 0$ gives $\dot{\psi}_{1,3}^2 = 0$ and $\dot{\psi}_{3,1}^4 = \ddot{\psi}_{2,2}^4 = \ddot{\psi}_{1,3}^4 = 0$, and feeding these back into (3.31) then gives $\ddot{\psi}_{1,3}^2 = 0$ and $\ddot{\psi}_{3,1}^4 = 0$. We have thus proved the second line of (3.29).

Now in (3.24)–(3.25) with $t = 1$, all the $O(r)$ terms are zero, so the leading error terms are the $O(r^2)$ terms. Some of these $O(r^2)$ terms vanish by the first two lines of (3.29), and the rest are on the left and right hand sides of the third line of (3.29). Therefore the argument of Lemma 3.4 applied to the leading $O(r^2)$ errors proves the third line.

q.e.d.

From now on we fix the data $\Upsilon, \tilde{\nabla}^\nu$ as in Proposition 3.8. The proposition, combined with equations (3.21), (3.22), and (3.24)–(3.25) implies:

Corollary 3.9. *In the situation above, $\tilde{\varphi}^\nu = \varphi^\nu + \ddot{\varphi}_{0,3}^2$ and $\tilde{\psi}^\nu = \psi^\nu + \ddot{\psi}_{2,2}^4$ are closed forms on ν , where $|\ddot{\varphi}_{0,3}^2|_{g^\nu} = O(r^2)$ and $|\ddot{\psi}_{2,2}^4|_{g^\nu} = O(r^2)$.*

Similarly, $\tilde{\varphi}_t^\nu = \varphi_t^\nu + t^2 \ddot{\varphi}_{0,3}^2$ and $\tilde{\psi}_t^\nu = \psi_t^\nu + t^4 \ddot{\psi}_{2,2}^4$ are closed forms on ν , where $|t^2 \ddot{\varphi}_{0,3}^2|_{g_t^\nu} = O(t^2 r^2)$ and $|t^4 \ddot{\psi}_{2,2}^4|_{g_t^\nu} = O(t^2 r^2)$.

Here is the point of all this. We have constructed a G_2 -structure (φ_t^ν, g_t^ν) on ν with 4-form $\psi_t^\nu = \Theta(\varphi_t^\nu) = *_{g_t^\nu} \varphi_t^\nu$. To apply Theorem

2.7 in §6, we will need (φ_t^ν, g_t^ν) to have *small torsion*, in an appropriate sense. If $d\varphi_t^\nu = 0$ and $d\psi_t^\nu = 0$ then (φ_t^ν, g_t^ν) would be torsion-free, by Theorem 2.2. In general $d\varphi_t^\nu \neq 0$ and $d\psi_t^\nu \neq 0$, but as $d\tilde{\varphi}_t^\nu = 0$ and $d\tilde{\psi}_t^\nu = 0$ with $\tilde{\varphi}_t^\nu = \varphi_t^\nu + t^2\ddot{\varphi}_{0,3}^2$ and $\tilde{\psi}_t^\nu = \psi_t^\nu + t^4\ddot{\psi}_{2,2}^4$, we can regard $|t^2\ddot{\varphi}_{0,3}^2|_{g_t^\nu}$ and $|t^4\ddot{\psi}_{2,2}^4|_{g_t^\nu}$ as measuring the torsion of (φ_t^ν, g_t^ν) , as if $t^2\ddot{\varphi}_{0,3}^2 = t^4\ddot{\psi}_{2,2}^4 = 0$ then (φ_t^ν, g_t^ν) would be torsion-free. Thus, Corollary 3.9 basically says that the torsion of (φ_t^ν, g_t^ν) is $O(t^2r^2)$.

Remark 3.10. It would be interesting to determine an invariant geometric interpretation for the choices of Υ in Lemma 3.5 and of $\tilde{\nabla}^\nu$ in Lemma 3.7.

3.5. Comparing $\Upsilon^*(\varphi)$, $\Upsilon^*(\ast\varphi)$ and $\tilde{\varphi}^\nu, \tilde{\psi}^\nu$. We now construct forms η, ζ on U_R which we will use in §6 to interpolate between $\Upsilon^*(\varphi)$, $\Upsilon^*(\ast\varphi)$ and $\tilde{\varphi}^\nu, \tilde{\psi}^\nu$, respectively. Define a 2-form η and a 3-form ζ on U_R by

$$(3.32) \quad \begin{aligned} \eta &= \int_0^1 t^{-1}(\delta \cdot \Upsilon_t^*(\varphi))dt - \frac{1}{2}\delta \cdot \varphi_{2,1}^2, \\ \zeta &= \int_0^1 t^{-1}(\delta \cdot \Upsilon_t^*(\ast\varphi))dt - \frac{1}{2}\delta \cdot \psi_{2,2}^2 - \frac{1}{4}\delta \cdot \psi_{4,0}^4. \end{aligned}$$

Here the integrals are well defined because $\delta \cdot \Upsilon_0^*(\varphi) = 0$ and $\delta \cdot \Upsilon_0^*(\ast\varphi) = 0$, so $\delta \cdot \Upsilon_t^*(\varphi)$ and $\delta \cdot \Upsilon_t^*(\ast\varphi)$ are both $O(t)$.

Let $F_s : \nu \rightarrow \nu$ be the flow of the dilation vector field δ . Then $(\Upsilon \circ F_s)(x, \alpha) = \Upsilon(x, e^s\alpha) = \Upsilon_{e^s}(x, \alpha)$. Thus $\Upsilon_t = \Upsilon \circ F_{\log t}$, so we have

$$\frac{d}{dt}\Upsilon_t^*(\varphi) = \frac{d}{dt}(\Upsilon \circ F_{\log t})^*(\varphi) = t^{-1}\mathcal{L}_\delta(\Upsilon_t^*(\varphi))$$

for $t > 0$. Using this we compute

$$(3.33) \quad \begin{aligned} d\left[\int_0^1 t^{-1}(\delta \cdot \Upsilon_t^*(\varphi))dt\right] &= \int_0^1 t^{-1}\mathcal{L}_\delta(\Upsilon_t^*(\varphi))dt = \int_0^1 \frac{d}{dt}(\Upsilon_t^*(\varphi))dt \\ &= \Upsilon_1^*(\varphi) - \Upsilon_0^*(\varphi). \end{aligned}$$

Substituting (3.32) into (3.33), using both (3.14) and the first line of (3.29), and noting that $\Upsilon_1 = \Upsilon$, $\Upsilon_0^*(\varphi) = \varphi_{0,3}^0$, we obtain

$$\begin{aligned} d\eta &= \Upsilon_1^*(\varphi) - \Upsilon_0^*(\varphi) - \dot{\varphi}_{2,1}^2 - \ddot{\varphi}_{1,2}^2 - \ddot{\varphi}_{0,3}^2 \\ &= \Upsilon^*(\varphi) - \varphi_{0,3}^0 - \dot{\varphi}_{2,1}^2 - \ddot{\varphi}_{0,3}^2. \end{aligned}$$

In the same way, this time using the second line of (3.29) and $\Upsilon_0^*(\psi) = 0$, we obtain

$$\begin{aligned} d\zeta &= \Upsilon_1^*(\psi) - \Upsilon_0^*(\psi) - \dot{\psi}_{2,2}^2 - \ddot{\psi}_{1,3}^2 - \dot{\psi}_{4,0}^4 - \ddot{\psi}_{3,1}^4 - \ddot{\psi}_{2,2}^4 \\ &= \Upsilon^*(\psi) - \dot{\psi}_{2,2}^2 - \dot{\psi}_{4,0}^4 - \ddot{\psi}_{2,2}^4. \end{aligned}$$

Using (3.21) and $\tilde{\varphi}^\nu = \varphi^\nu + \ddot{\varphi}_{0,3}^2$ and $\tilde{\psi}^\nu = \psi^\nu + \ddot{\psi}_{2,2}^4$ from Corollary 3.9 we conclude that

$$(3.34) \quad d\eta = \Upsilon^*(\varphi) - \tilde{\varphi}^\nu|_{U_R}, \quad \text{and similarly} \quad d\zeta = \Upsilon^*(\psi) - \tilde{\psi}^\nu|_{U_R}.$$

Substituting (3.6) into (3.32) and integrating term by term yields

$$(3.35) \quad \begin{aligned} \eta &\sim \int_0^1 \left(\sum_{n=1}^{\infty} t^{2n-1} \delta \cdot \varphi^{2n} \right) dt - \frac{1}{2} \delta \cdot \varphi_{2,1}^2 \\ &= \sum_{n=1}^{\infty} \frac{1}{2n} \delta \cdot \varphi^{2n} - \frac{1}{2} \delta \cdot \varphi_{2,1}^2 = \frac{1}{4} \delta \cdot \varphi_{2,1}^4 + \frac{1}{4} \delta \cdot \varphi_{1,2}^4 + \sum_{n=3}^{\infty} \frac{1}{2n} \delta \cdot \varphi^{2n}, \end{aligned}$$

interpreted as a power series around the zero section L in U_R , where we have used $\varphi_{3,0}^2 = \varphi_{1,2}^2 = \varphi_{3,0}^4 = 0$ and $\delta \cdot \varphi_{0,3}^2 = \delta \cdot \varphi_{0,3}^4 = 0$. From (3.18), (3.35) and $|\delta| = O(r)$ we see that

$$(3.36) \quad |\eta|_{g^\nu} = O(r^3) \quad \text{and} \quad |d\eta|_{g^\nu} = |\Upsilon^*(\varphi) - \tilde{\varphi}^\nu|_{U_R}|_{g^\nu} = O(r^2)$$

for small r . Similarly using $\psi_{4,0}^2 = \psi_{3,1}^2 = \psi_{1,3}^2 = \psi_{3,1}^4 = 0$ we find that

$$\begin{aligned} \zeta &\sim \int_0^1 \left(\sum_{n=1}^{\infty} t^{2n-1} \delta \cdot \psi^{2n} \right) dt - \frac{1}{2} \delta \cdot \psi_{2,2}^2 - \frac{1}{4} \delta \cdot \psi_{4,0}^4 \\ &= \sum_{n=1}^{\infty} \frac{1}{2n} \delta \cdot \psi^{2n} - \frac{1}{2} \delta \cdot \psi_{2,2}^2 - \frac{1}{4} \delta \cdot \psi_{4,0}^4 \\ &= \frac{1}{4} \delta \cdot \psi_{2,2}^4 + \frac{1}{4} \delta \cdot \psi_{1,3}^4 + \sum_{n=3}^{\infty} \frac{1}{2n} \delta \cdot \psi^{2n}, \end{aligned}$$

and thus analogously that

$$(3.37) \quad |\zeta|_{g^\nu} = O(r^3) \quad \text{and} \quad |d\zeta|_{g^\nu} = |\Upsilon^*(\psi) - \tilde{\psi}^\nu|_{U_R}|_{g^\nu} = O(r^2).$$

4. G₂-structures on the resolution P of $\nu/\{\pm 1\}$

In this section we define a resolution $\rho : P \rightarrow \nu/\{\pm 1\}$ of the orbifold $\nu/\{\pm 1\}$, and a family of G₂-structures (φ_t^P, g_t^P) on P for $t > 0$, which are asymptotic at infinity in P to the G₂-structures $(\varphi_t^{\nu/\{\pm 1\}}, g_t^{\nu/\{\pm 1\}})$ on $\nu/\{\pm 1\}$ induced by the G₂-structure (φ_t^ν, g_t^ν) on ν defined in (3.22) of §3.3. We then modify the pair $\varphi_t^P, \psi_t^P = \Theta(\varphi_t^P)$ to a pair of closed forms $\tilde{\varphi}_t^P, \tilde{\psi}_t^P$. However, the torsion $\tilde{\psi}_t^P - \Theta(\tilde{\varphi}_t^P)$ will still be too large, and we will correct it further in §5.

We continue in the situation of §3, with $\Upsilon, \tilde{\nabla}^\nu$ chosen as in Proposition 3.8. This section will use only the following data from §3:

- The compact 3-manifold L , and rank 4 real vector bundle $\pi : \nu \rightarrow L$.
- The fibre metrics h_ν and g_L on the bundles ν and TL over L , respectively, and the radius function $r : \nu \rightarrow [0, \infty)$ defined using h_ν .

- The connection $\tilde{\nabla}^\nu$ on ν , which induces a splitting $T\nu = V \oplus H$ with $V \cong \pi^*(\nu)$ and $H \cong \pi^*(TL)$, and a notion of type (i, j) forms on ν .
- The data $\varphi_t^\nu = \varphi_{0,3}^0 + t^2 \dot{\varphi}_{2,1}^2$, $\psi_t^\nu = t^2 \dot{\psi}_{2,2}^2 + t^4 \dot{\psi}_{4,0}^4$, $g_t^\nu = g_{0,2}^\nu + t^2 g_{2,0}^\nu$ on ν from (3.22) for $t > 0$, giving a G_2 -structure (φ_t^ν, g_t^ν) on ν with 4-form $\psi_t^\nu = \Theta(\varphi_t^\nu)$.
- The forms $\ddot{\varphi}_{0,3}^2$ and $\ddot{\psi}_{2,2}^4$ on ν , where $\tilde{\varphi}_t^\nu = \varphi_t^\nu + t^2 \ddot{\varphi}_{0,3}^2$ and $\tilde{\psi}_t^\nu = \psi_t^\nu + t^4 \ddot{\psi}_{2,2}^4$ are closed forms on ν .
- The closed, coclosed, nonvanishing 1-form λ on L from Assumption 3.1.

4.1. Defining the resolution P of $\nu/\{\pm 1\}$. Define a smooth function $A : L \rightarrow (0, \infty)$ by $A = |\lambda|_{g_L}$, noting that λ is nowhere vanishing, and define a 1-form $e_1 = A^{-1}\lambda$ on L , so that e_1 is of unit length with respect to the metric g_L on L , and $\lambda = Ae_1$. It will be convenient for us to extend e_1 to an oriented orthonormal basis (e_1, e_2, e_3) of 1-forms on L , but this may not be possible globally on L , as the rank 2 subbundle $\langle e_1 \rangle^\perp \subset T^*L$ may not be trivial. So choose an open subset $L' \subseteq L$ on which $\langle e_1 \rangle^\perp$ is trivial, and choose 1-forms e_2, e_3 on L' such that (e_1, e_2, e_3) is orthonormal with respect to g_L , and oriented with respect to the orientation on L induced by φ , so that since L is associative in (M, φ, g) , we have $\varphi|_{L'} = e_1 \wedge e_2 \wedge e_3$. Then $*\lambda = Ae_2 \wedge e_3$. Later we will show that the important structures we define are independent of the choice of e_2, e_3 , and so are well defined over all of L , not just over $L' \subseteq L$.

Since L is associative in (M, φ, g) , at each point $x \in L'$ there exists an isomorphism $T_x M \cong \mathbb{R}^7$ identifying $\varphi|_x, g|_x, *\varphi|_x$ with $\varphi_0, g_0, *\varphi_0$ in (2.1)–(2.3) and identifying $T_x L$ with $\{(x_1, x_2, x_3, 0, 0, 0) : x_j \in \mathbb{R}\} \subset \mathbb{R}^7$. We can also choose this to identify $e_1|_x, e_2|_x, e_3|_x$ with dx_1, dx_2, dx_3 . Therefore from equations (2.7) and (2.13) in §2.4, we see that there are unique smooth sections $\hat{\omega}^I, \hat{\omega}^J, \hat{\omega}^K$ of $\Lambda^2 \nu^* \rightarrow L'$ such that

$$\begin{aligned} \varphi|_{L'} &= e_1 \wedge e_2 \wedge e_3 - e_1 \wedge \hat{\omega}^I - e_2 \wedge \hat{\omega}^J - e_3 \wedge \hat{\omega}^K & \text{in } \Gamma^\infty(\Lambda^3 T^* M|_{L'}), \\ *\varphi|_{L'} &= \text{vol}_\nu - e_2 \wedge e_3 \wedge \hat{\omega}^I - e_3 \wedge e_1 \wedge \hat{\omega}^J - e_1 \wedge e_2 \wedge \hat{\omega}^K & \text{in } \Gamma^\infty(\Lambda^4 T^* M|_{L'}), \end{aligned}$$

where $\hat{\omega}^I, \hat{\omega}^J, \hat{\omega}^K$ are the Hermitian forms with respect to h_ν of almost complex structures $I, J, K : \nu|_{L'} \rightarrow \nu|_{L'}$ on the fibres of ν satisfying the quaternion relations (2.9). That is, at each $x \in L'$ there exist linear coordinates (y_1, y_2, y_3, y_4) on $\nu|_{L'}$ such that $\hat{\omega}^I|_x, \hat{\omega}^J|_x, \hat{\omega}^K|_x, h_\nu|_x, I|_x, J|_x, K|_x$ are given by equations (2.7)–(2.11) for $\omega_0^I, \omega_0^J, \omega_0^K, h_0, I, J, K$.

If e'_2, e'_3 are alternative choices for e_2, e_3 yielding $\hat{\omega}'^I, \hat{\omega}'^J, \hat{\omega}'^K, I', J', K'$ then we may write $e'_2 = \cos \phi e_2 + \sin \phi e_3$ and $e'_3 = -\sin \phi e_2 + \cos \phi e_3$ for some smooth $\phi : L' \rightarrow \mathbb{R}/2\pi\mathbb{Z}$, and comparing (4.1) for e_1, e_2, e_3 and

e_1, e'_2, e'_3 we see that
(4.2)
 $\hat{\omega}'^I = \hat{\omega}^I, \quad \hat{\omega}'^J = \cos \phi \hat{\omega}^J + \sin \phi \hat{\omega}^K, \quad \hat{\omega}'^K = -\sin \phi \hat{\omega}^J + \cos \phi \hat{\omega}^K,$
 $I' = I, \quad J' = \cos \phi J + \sin \phi K, \quad K' = -\sin \phi J + \cos \phi K.$

Hence $\hat{\omega}^I, I$ are independent of the choice of e_2, e_3 in the orthonormal basis of sections (e_1, e_2, e_3) on L' . So by covering L by open $L' \subseteq L$ on which such a basis (e_1, e_2, e_3) exists, we construct global $\hat{\omega}^I \in \Gamma^\infty(\Lambda^2 \nu^*)$ and $I : \nu \rightarrow \nu$ satisfying (4.1) on each L' in the cover.

Now $I : \nu \rightarrow \nu$ with $I^2 = -1$ is a complex structure on the fibres of $\nu \rightarrow L$, making ν into a rank 2 complex vector bundle, with fibre \mathbb{C}^2 .

Remark 4.1. The complex structure I on ν , identifying the \mathbb{R}^4 fibres with \mathbb{C}^2 , can also be understood as follows. The unit 1-form e_1 on L is metric dual to a unit vector field, still denoted e_1 , on L . Because L is associative in M , it follows that *cross product* with e_1 takes normal vectors to normal vectors, and squares to minus the identity. Thus $I = e_1 \times (\cdot)$. See also Remark 2.14.

Form the quotient $\nu/\{\pm 1\}$, as a 7-dimensional orbifold, and write $\varpi : \nu \rightarrow \nu/\{\pm 1\}$ and $\pi : \nu/\{\pm 1\} \rightarrow L$ for the obvious projections and $0 : L \rightarrow \nu/\{\pm 1\}$ for the zero section. Then $\pi : \nu/\{\pm 1\} \rightarrow L$ is a fibre bundle with fibre $\mathbb{C}^2/\{\pm 1\}$.

As in §2.5, the Eguchi–Hanson space X is the blow-up $B : X \rightarrow \mathbb{C}^2/\{\pm 1\}$ of $\mathbb{C}^2/\{\pm 1\}$ at 0, with exceptional divisor $Y = B^{-1}(0)$ where $Y \cong \mathbb{CP}^1$. We can do this blow-up construction fibrewise over L . So let $\rho : P \rightarrow \nu/\{\pm 1\}$ be the bundle blow-up of $\nu/\{\pm 1\}$ along the zero section $0(L) \subset \nu/\{\pm 1\}$, using the complex structure I on the fibres $\mathbb{C}^2/\{\pm 1\}$ to define the blow-up. Write $\sigma = \pi \circ \rho : P \rightarrow L$. Then P is a smooth 7-manifold, and $\sigma : P \rightarrow L$ a smooth map which is a bundle with fibre the Eguchi–Hanson space X .

Write $Q = \rho^{-1}(0(L)) \subset P$ for the preimage of the zero section in P , and write $\text{inc} : Q \hookrightarrow P$ for the inclusion, and $\Pi = \sigma|_Q : Q \rightarrow L$. Then Q is a smooth 5-manifold, an embedded submanifold of P with embedding $\text{inc} : Q \hookrightarrow P$, and $\Pi : Q \rightarrow L$ is smooth and a fibre bundle with fibre \mathbb{CP}^1 the exceptional divisor $Y \subset X$ from §2.5. For each $x \in L$, write $P_x = \sigma^{-1}(x) \subset P$ and $Q_x = \Pi^{-1}(x) \subset Q$ for the fibres of $\sigma : P \rightarrow L$ and $\Pi : Q \rightarrow L$ over x , respectively. Then P_x is diffeomorphic to the Eguchi–Hanson space X , and $Q_x \subset P_x$ to the exceptional divisor $Y \subset X$.

As in §2.5, we have $Y \cong \mathbb{CP}^1 = \mathbb{P}(\mathbb{C}^2)$, and there is a natural projection $\pi : X \rightarrow Y$ which realizes X as the total space of the complex line bundle $T^*Y \rightarrow Y$. Similarly, there is a natural projection $\pi : P \rightarrow Q$ with $\Pi \circ \pi = \sigma : P \rightarrow L$, such that $\pi_x : P_x \rightarrow Q_x$ is identified with $\pi : X \rightarrow Y$ on the fibres over each $x \in L$. We summarize our work so

far in the diagram

$$\begin{array}{ccccc}
 P & \xrightarrow{\rho} & \nu/\{\pm 1\} & \xleftarrow{\varpi} & \nu \\
 \pi \downarrow \uparrow \text{inc} & & \pi \downarrow \uparrow 0 & & \pi \downarrow \uparrow 0 \\
 Q = \mathbb{P}_{\mathbb{C}}(\nu) & \xrightarrow{\Pi} & L & \xlongequal{\quad} & L.
 \end{array}$$

Note that $P, Q, \rho, \sigma, \pi, \Pi$ on the left hand side depend on the complex structure I on the fibres of ν used to define the blow-up.

Remark 4.2. There is an equivalent way to define P, Q using principal bundles. Corresponding to the rank 2 complex vector bundle $(\nu, I) \rightarrow L$ there is a natural $\mathrm{GL}(2, \mathbb{C})$ -principal bundle $F_{\mathrm{GL}(2, \mathbb{C})} \rightarrow L$, where points of $F_{\mathrm{GL}(2, \mathbb{C})}$ over $x \in L$ are isomorphisms $(\nu|_x, I|_x) \cong (\mathbb{C}^2, i)$. Then we may write

$$(4.3) \quad P \cong (F_{\mathrm{GL}(2, \mathbb{C})} \times X) / \mathrm{GL}(2, \mathbb{C}), \quad Q \cong (F_{\mathrm{GL}(2, \mathbb{C})} \times Y) / \mathrm{GL}(2, \mathbb{C}),$$

where $\mathrm{GL}(2, \mathbb{C})$ acts on X and Y in the natural ways, and on $F_{\mathrm{GL}(2, \mathbb{C})}$ by the principal bundle action. Equation (4.3) makes it clear that P, Q are bundles over L with fibres X, Y , and that the bundle blow-up procedure is well defined.

If we also use the Hermitian metric h_ν on the fibres of (ν, I) we can instead define a natural $\mathrm{U}(2)$ -principal bundle $F_{\mathrm{U}(2)} \rightarrow L$, where points of $F_{\mathrm{U}(2)}$ over $x \in L$ are isomorphisms $(\nu|_x, I|_x, h_\nu|_x) \cong (\mathbb{C}^2, i, h_0)$. Then as in (4.3) we have natural diffeomorphisms

$$(4.4) \quad P \cong (F_{\mathrm{U}(2)} \times X) / \mathrm{U}(2), \quad Q \cong (F_{\mathrm{U}(2)} \times Y) / \mathrm{U}(2).$$

This means that $\mathrm{U}(2)$ -invariant tensors on X , including the Eguchi–Hanson metrics h_a for $a > 0$ in §2.5, can be carried over to P .

Remark 4.3. Note that $\rho|_{P \setminus Q} : P \setminus Q \rightarrow (\nu/\{\pm 1\}) \setminus 0(L)$ is a diffeomorphism, since $B|_{X \setminus Y} : X \setminus Y \rightarrow (\mathbb{C}^2/\{\pm 1\}) \setminus \{0\}$ is a diffeomorphism. Thus, if T is any tensor on the total space of ν that is invariant under $\{\pm 1\}$, it descends to T on $\nu/\{\pm 1\}$, and then pulls back to a smooth tensor $\rho|_{P \setminus Q}^*(T)$ on $P \setminus Q$.

However, as explained in Remark 2.9 for $B : X \rightarrow \mathbb{C}^2/\{\pm 1\}$, it is *not* necessary that $\rho|_{P \setminus Q}^*(T)$ extends smoothly to a tensor on all P , although if it does extend, the extension is unique, as $P \setminus Q$ is open and dense in P .

For example, the radius function $r : \nu \rightarrow [0, \infty)$ defined using h_ν is $\{\pm 1\}$ -invariant, and so descends to $r : \nu/\{\pm 1\} \rightarrow [0, \infty)$. We write

$$\tilde{r} = r \circ \rho : P \rightarrow [0, \infty)$$

for the radius function on P , so that $Q = \tilde{r}^{-1}(0)$. In Remark 2.9 we explained that although $r^2 : \mathbb{C}^2 \rightarrow \mathbb{R}$ and $r^2 : \mathbb{C}^2/\{\pm 1\} \rightarrow \mathbb{R}$ are smooth (in the orbifold sense), $r^2 : X \rightarrow \mathbb{R}$ is *not* smooth, but $r^4 : X \rightarrow \mathbb{R}$ is.

As $r^4 : X \rightarrow \mathbb{R}$ is $U(2)$ -invariant, it follows from (4.4) that $\check{r}^2 : P \rightarrow \mathbb{R}$ is not smooth, but $\check{r}^4 : P \rightarrow \mathbb{R}$ is smooth.

In what follows, we often define metrics, exterior forms, and tensors on P by pullback along $\rho|_{P \setminus Q} : P \setminus Q \rightarrow (\nu/\{\pm 1\}) \setminus 0(L)$, as above, but we must be careful to justify that these extend smoothly from $P \setminus Q$ to P .

4.2. Constructing a splitting $TP \cong \check{V} \oplus \check{H}$. In §3.2 we chose a connection $\check{\nabla}^\nu$ on $\nu \rightarrow L$ and defined a splitting $T\nu = V \oplus H$ in (2.16), where $V \cong \pi^*(\nu)$ and $H \cong \pi^*(TL)$. Similarly, we will need a splitting $TP \cong \check{V} \oplus \check{H}$ for the fibre bundle $\sigma : P \rightarrow L$, where the vertical subbundle $\check{V} = \text{Ker}(d\sigma : TP \rightarrow \sigma^*(TL))$ is the subbundle of TP whose fibre at a point p is the tangent space at $p \in P_{\sigma(p)}$ of the Eguchi–Hanson fibre $P_{\sigma(p)}$ of P over $\sigma(p) \in L$, and the horizontal subbundle \check{H} has $\check{H} \cong \sigma^*(TL)$.

The splitting $T\nu = V \oplus H$ is $\{\pm 1\}$ -invariant, so it descends to $T(\nu/\{\pm 1\}) = V \oplus H$. For compatibility with §3 we would like $\rho : P \rightarrow \nu/\{\pm 1\}$ to map \check{V}, \check{H} on P to V, H on $\nu/\{\pm 1\}$ near infinity in P . This works for \check{V}, V because ρ is fibre-preserving and \check{V}, V are canonical. However, we cannot simply define \check{H} by pulling back H by $\rho : P \rightarrow \nu/\{\pm 1\}$, since as in Remark 4.3, the resulting \check{H} might not be smooth at $Q \subset P$.

As $\rho : P \rightarrow \nu/\{\pm 1\}$ was defined using the complex structure I on the fibres of ν in §4.1, it is the case that the pullback of H on $\nu/\{\pm 1\}$ to $P \setminus Q$ extends smoothly over Q if $\check{\nabla}^\nu I = 0$. This is a consequence of the description of P in (4.4). But $\check{\nabla}^\nu$ was determined uniquely in Lemma 3.7, so we are *not* free to choose $\check{\nabla}^\nu$ with $\check{\nabla}^\nu I = 0$.

Choose another connection $\check{\nabla}^\nu$ on $\nu \rightarrow L$ compatible with h_ν and I , so that $\check{\nabla}^\nu h_\nu = \check{\nabla}^\nu I = 0$. This can always be done by general principal bundle theory, and is not unique. Write $\check{\nabla}^\nu = \check{\nabla}^\nu + \Gamma$ for $\Gamma \in \Gamma^\infty(\text{End}(\nu) \otimes T^*L)$. Then $\check{\nabla}^\nu + s\Gamma$ for $s \in [0, 1]$ is a 1-parameter family of connections interpolating between $\check{\nabla}^\nu$ at $s = 0$, and $\check{\nabla}^\nu$ at $s = 1$. Choose a smooth function $a : \mathbb{R} \rightarrow \mathbb{R}$ with $a(r) = 1$ for $r \leq \frac{1}{2}$, and $a(r) \in (0, 1)$ for $r \in (\frac{1}{2}, 1)$, and $a(r) = 0$ for $r \geq 1$. Define a splitting $TP \cong \check{V} \oplus \check{H}$ by $\check{V} = \text{Ker}(d\sigma : TP \rightarrow \sigma^*(TL))$, and for $p \in P \setminus Q$, define $\check{H}|_p$ to be identified by $d_p\rho : T_pP \rightarrow T_{\rho(p)}(\nu/\{\pm 1\})$ with the horizontal subspace of the connection $\check{\nabla}^\nu + (a \circ \check{r}(p))\Gamma$ on ν at $\rho(p)$ in $\nu/\{\pm 1\}$.

Then $\check{H}|_p$ depends smoothly on $p \in P \setminus Q$, since $a \circ \check{r} : P \rightarrow \mathbb{R}$ is smooth. Where $\check{r} \geq 1$ in P we have $a \circ \check{r} = 0$, so \check{H} is defined using $\check{\nabla}^\nu$, and coincides with the pullback of H in §3.2 as we want. Where $\check{r} \leq \frac{1}{2}$ in P we have $a \circ \check{r} = 1$, so \check{H} is defined using $\check{\nabla}^\nu$, and as $\check{\nabla}^\nu I = 0$ we see that \check{H} extends smoothly over Q . Thus, we have defined a smooth

splitting of vector bundles on P :

$$TP = \check{V} \oplus \check{H}, \text{ where } \check{V}|_p = T_p P_{\sigma(p)} \subset T_p P, p \in P, \text{ and } \check{H} \cong \sigma^*(TL).$$

As in (3.12), this induces a splitting

$$(4.5) \quad \Lambda^k T^*P = \bigoplus_{i+j=k, 0 \leq i \leq 4, 0 \leq j \leq 3} \Lambda^i \check{V}^* \otimes \Lambda^j \check{H}^*,$$

and we call a k -form on P of type (i, j) if it lies in the factor $\Lambda^i \check{V}^* \otimes \Lambda^j \check{H}^*$ in (4.5). We write $[\alpha]_{i,j}$ for the type (i, j) component of a k -form α , so $\alpha = \sum_{i+j=k} [\alpha]_{i,j}$.

For each $x \in L$, the fibre P_x of $\sigma : P \rightarrow L$ over x is isomorphic to the Eguchi–Hanson space X , which is a complex surface, and $\check{V}_p = T_p P_x$ for $p \in P_x$. The complex structures on each P_x induce a unique vector bundle isomorphism $I : \check{V} \rightarrow \check{V}$ on P with $I^2 = -1$.

Remark 4.4. We address here a subtle point. Recall that in $TP = \check{V} \oplus \check{H}$, the vertical subbundle $\check{V} = \text{Ker}(d\sigma : TP \rightarrow \sigma^*(TL))$ is canonical, but the horizontal subbundle \check{H} depends on an arbitrary choice. However, in $T^*P = \check{V}^* \oplus \check{H}^*$, the factor \check{H}^* is canonical, because it is the annihilator $(\check{V})^\circ$, and in addition \check{H}^* is the pullback $\sigma^*(T^*L)$ of the cotangent bundle of L . But \check{V}^* depends on a choice, as it is $(\check{H})^\circ$.

Let $x \in L$, so that $P_x \subset P$ is the fibre of $\sigma : P \rightarrow L$ over x , a 4-submanifold of P . Then $(0, 1)$ -forms on P restrict to zero on P_x , so (i, j) -forms for $j > 0$ restrict to zero on P_x . If α is a k -form on P , then $\alpha = \sum_{i+j=k} [\alpha]_{i,j}$ implies that

$$(4.6) \quad [\alpha]_{k,0}|_{P_x} = \alpha|_{P_x} \quad \text{in } k\text{-forms on } P_x.$$

Here is the subtle point. Although $[\alpha]_{k,0}$ depends on the choice of \check{H} , its restriction $[\alpha]_{k,0}|_{P_x}$ does not. In fact, $[\alpha]_{k,0}$ is independent of \check{H} when considered as a section of the vector bundle $\Lambda^k \check{V}^*$ associated to $\check{V} = \text{Ker}(d\sigma : TP \rightarrow \sigma^*(TL))$. But when considered as a k -form on P , it does depend on \check{H} , as the embedding $\Lambda^k \check{V}^* \hookrightarrow \Lambda^k T^*P$ depends on \check{H} .

4.3. Defining some forms and tensors on P . In this section we define some forms and tensors on P that will be used to construct G_2 -structures (φ_t^P, g_t^P) in §4.4. Let $f_a : (0, \infty) \rightarrow \mathbb{R}$ be the smooth function in (2.15) such that $f_a(r) : \mathbb{C}^2/\{\pm 1\} \rightarrow \mathbb{R}$ is the Kähler potential of the Eguchi–Hanson space (X, h_a) , where f_a depends smoothly on $a \in (0, \infty)$. Let $A : L \rightarrow (0, \infty)$ be the smooth function $A(x) = |\lambda_x|_{g_L}$ from §4.1, so that $A \circ \sigma : P \rightarrow (0, \infty)$ is smooth, and $\check{r}|_{P \setminus Q} : P \setminus Q \rightarrow (0, \infty)$ is smooth. Thus $f_{A \circ \sigma}(\check{r}) : P \setminus Q \rightarrow (0, \infty)$ is a smooth function mapping $p \mapsto f_{A \circ \sigma(p)}(\check{r}(p))$. Note that $f_{A \circ \sigma(p)}(\check{r}(p)) \rightarrow -\infty$ as $p \rightarrow Q$ in P , so $f_{A \circ \sigma}(\check{r})$ does not extend smoothly from $P \setminus Q$ to P .

Thus, $d(f_{A \circ \sigma}(\check{r}))$ is a closed 1-form on $P \setminus Q$, and we denote by $[d(f_{A \circ \sigma}(\check{r}))]_{1,0}$ its type $(1, 0)$ -component in $\check{V}^* \subset \check{V}^* \oplus \check{H}^* = T^*P$. Ap-

plying the complex structure $I : \check{V} \rightarrow \check{V}$ in §4.2 gives another type (1,0) 1-form $I([d(f_{A \circ \sigma}(\check{r}))]_{1,0})$ on $P \setminus Q$, so we can take d of this, and divide into components of type (2,0), (1,1), and (0,2), which we do to define 2-forms $\check{\omega}^I, \kappa_{1,1}, \kappa_{0,2}$ of types (2,0), (1,1), (0,2) on $P \setminus Q \subset P$ as follows:

$$\begin{aligned}
 \check{\omega}^I &= -\frac{1}{4} [d(I([d(f_{A \circ \sigma}(\check{r}))]_{1,0}))]_{2,0} \\
 &= -\frac{1}{4} (A \circ \sigma) [d((A \circ \sigma)^{-1} I([d(f_{A \circ \sigma}(\check{r}))]_{1,0}))]_{2,0}, \\
 \kappa_{1,1} &= -\frac{1}{4} [d((A \circ \sigma)^{-1} I([d(f_{A \circ \sigma}(\check{r}))]_{1,0}))]_{1,1}, \\
 \kappa_{0,2} &= -\frac{1}{4} [d((A \circ \sigma)^{-1} I([d(f_{A \circ \sigma}(\check{r}))]_{1,0}))]_{0,2}.
 \end{aligned}
 \tag{4.7}$$

By $(A \circ \sigma)^{-1}$ we mean the smooth function $P \rightarrow (0, \infty)$ given by

$$(A \circ \sigma)^{-1}(p) = [(A \circ \sigma)(p)]^{-1} = \frac{1}{A(\sigma(p))}.$$

We can also write this as $A^{-1} \circ \sigma$ where $A^{-1}(x) = [A(x)]^{-1}$. The first two lines of (4.7) are equal because $d((A \circ \sigma)^{-1}) = \sigma^*(d(A^{-1}))$ is a 1-form of type (0,1), and so makes no contribution to the type (2,0) component.

Remark 4.5. The motivation for defining $\check{\omega}^I$ as we did in (4.7) comes from (2.16). If one thinks of restricting to a fixed fibre P_x of P , then since we take only vertical (fibre) derivatives in the definition of $\check{\omega}^I$ it corresponds to the hyperKähler form ω_a^I on $X \cong P_x$ in the domain $X \setminus Y$. We make this more explicit at the end of this section. On the other hand, the forms $\kappa_{1,1}$ and $\kappa_{2,2}$ were defined as they were in (4.7) precisely so that

$$d[(A \circ \sigma)^{-1} \check{\omega}^I + \kappa_{1,1} + \kappa_{0,2}] = 0, \tag{4.8}$$

which is easy to see because $(A \circ \sigma)^{-1} \check{\omega}^I + \kappa_{1,1} + \kappa_{0,2}$ is exact.

We now use the crucial assumption that λ is closed and coclosed on L to derive two important relations, namely (4.9) and (4.10), which will be used to construct *closed* 3- and 4-forms on P in §4.4.

Since $\lambda = Ae_1$ is a closed 1-form on L , taking the exterior product of (4.8) with $\sigma^*(\lambda) = (A \circ \sigma)\sigma^*(e_1)$, which is of type (0,1), implies that

$$d[\sigma^*(e_1) \wedge \check{\omega}^I + \sigma^*(\lambda) \wedge \kappa_{1,1} + \sigma^*(\lambda) \wedge \kappa_{0,2}] = 0, \tag{4.9}$$

with $\sigma^*(e_1) \wedge \check{\omega}^I$, $\sigma^*(\lambda) \wedge \kappa_{1,1}$, $\sigma^*(\lambda) \wedge \kappa_{0,2}$ of types (2,1), (1,2), (0,3), respectively.

Similarly, for e_1, e_2, e_3 over $L' \subset L$ as in §4.1, so that $*\lambda = Ae_2 \wedge e_3$ is a closed 2-form on L , taking the exterior product of (4.8) with $\sigma^*(\lambda) = (A \circ \sigma)\sigma^*(e_2) \wedge \sigma^*(e_3)$, which is of type (0,2), implies that

$$d[\sigma^*(e_2) \wedge \sigma^*(e_3) \wedge \check{\omega}^I + \sigma^*(\lambda) \wedge \kappa_{1,1}] = 0, \tag{4.10}$$

on $\sigma^{-1}(L') \setminus Q$, where $\sigma^*(\lambda) \wedge \kappa_{0,2} = 0$ automatically as it is a (0,4)-form.

We want to show that $\check{\omega}^I, \kappa_{1,1}, \kappa_{0,2}$ extend uniquely to smooth forms on all of P . Combining equations (2.19) and (4.7) gives the alternative expressions

$$(4.11) \quad \begin{aligned} \check{\omega}^I &= -\frac{1}{2}(A \circ \sigma) [d(I([d(\log \check{r})]_{1,0}))]_{2,0} - \frac{1}{4} [d(I([d(H_{A \circ \sigma}(\check{r}^4))]_{1,0}))]_{2,0}, \\ \kappa_{1,1} &= -\frac{1}{2} [d(I([d(\log \check{r})]_{1,0}))]_{1,1} - \frac{1}{4} [d((A \circ \sigma)^{-1} I([d(H_{A \circ \sigma}(\check{r}^4))]_{1,0}))]_{1,1}, \\ \kappa_{0,2} &= -\frac{1}{2} [d(I([d(\log \check{r})]_{1,0}))]_{0,2} - \frac{1}{4} [d((A \circ \sigma)^{-1} I([d(H_{A \circ \sigma}(\check{r}^4))]_{1,0}))]_{0,2}. \end{aligned}$$

Recall that $\pi : P \rightarrow Q$ may be considered as a complex line bundle over Q . The connection $\check{\nabla}^\nu$ on $\nu \rightarrow L$, which is compatible with h_ν and I , induces a $U(1)$ -connection on this line bundle $\pi : P \rightarrow Q$, whose curvature is a closed 2-form F_Q on Q restricting to $-2\omega_{\mathbb{CP}^1}$ in (2.20) on each fibre $Q_x \cong \mathbb{CP}^1$ of $\Pi : Q \rightarrow L$. Calculating in a local trivialization of $(\nu, I) \rightarrow L$ we can show that

$$\pi^*(F_Q)|_{P \setminus Q} = d(I([d(\log \check{r})]_{1,0})) \quad \text{where } \check{r} \leq \frac{1}{2} \text{ on } P \setminus Q.$$

Here $\check{r} \leq \frac{1}{2}$ is needed so that \check{H} used to define $[\cdots]_{1,0}$ comes from $\check{\nabla}^\nu$.

Thus (4.11) implies that where $\check{r} \leq \frac{1}{2}$ on $P \setminus Q$ we have

$$(4.12) \quad \begin{aligned} \check{\omega}^I &= -\frac{1}{2}(A \circ \sigma) \cdot [\pi^*(F_Q)]_{2,0} - \frac{1}{4} [d(I([d(H_{A \circ \sigma}(\check{r}^4))]_{1,0}))]_{2,0}, \\ \kappa_{1,1} &= -\frac{1}{2} [\pi^*(F_Q)]_{1,1} - \frac{1}{4} [d((A \circ \sigma)^{-1} \cdot I([d(H_{A \circ \sigma}(\check{r}^4))]_{1,0}))]_{1,1}, \\ \kappa_{0,2} &= -\frac{1}{2} [\pi^*(F_Q)]_{0,2} - \frac{1}{4} [d((A \circ \sigma)^{-1} \cdot I([d(H_{A \circ \sigma}(\check{r}^4))]_{1,0}))]_{0,2}. \end{aligned}$$

Now each term on the right hand sides of (4.12) extends smoothly to P , since $\pi^*(F_Q)$ is smooth on P and $\check{r}^4 : P \rightarrow \mathbb{R}$ is smooth as in Remark 4.3, so $H_{A \circ \sigma}(\check{r}^4) : P \rightarrow \mathbb{R}$ is smooth. Therefore $\check{\omega}^I, \kappa_{1,1}, \kappa_{0,2}$ are defined as smooth 2-forms on P . Also (4.8)–(4.9) hold on all of P , by continuity, and (4.10) holds on $\sigma^{-1}(L')$.

We now make Remark 4.5 more explicit. Let $x \in L$. By restricting (4.7) to P_x , using (4.6) twice, and the fact that d commutes with pullbacks we find that

$$(4.13) \quad \check{\omega}^I|_{P_x} = -\frac{1}{4} d(I(d(f_{A(x)}(\check{r})|_{P_x}))),$$

where there are now no projections $[\cdots]_{1,0}, [\cdots]_{2,0}$, and we use $A \circ \sigma|_{P_x} \equiv A(x)$. By construction, $(P_x, I|_{P_x})$ is isomorphic as a complex surface to the Eguchi–Hanson space X . By (4.4), this isomorphism is only canonical up to the action of $U(2)$ on X . Comparing (2.16) and (4.13) shows that this isomorphism $P_x \cong X$ identifies $\check{\omega}^I|_{P_x}$ with the Kähler form $\omega_{A(x)}^I$ of the hyperKähler metric $h_{A(x)}$ on X . Therefore if we define a section $\check{g}_{2,0}$ of $S^2\check{V}^* \rightarrow P$ by

$$(4.14) \quad \check{g}_{2,0}(v, w) = \check{\omega}^I(v, Iw) \quad \text{for all } v, w \in \Gamma^\infty(\check{V})$$

then $\check{g}_{2,0}|_{P_x}$ is identified with the Eguchi–Hanson metric $h_{A(x)}$ on X by the isomorphism $P_x \cong X$. That is, $\check{g}_{2,0}$ is a family of Eguchi–Hanson metrics on the fibres P_x of $\sigma : P \rightarrow L$ at $x \in L$.

4.4. A family of G_2 -structures (φ_t^P, g_t^P) on P . Now return to our choice in §4.1 of open $L' \subseteq L$, and oriented orthonormal basis (e_1, e_2, e_3) of 1-forms on L' . Recall that in §4.1 we had defined $\lambda = Ae_1$ for smooth $A : L \rightarrow (0, \infty)$, and the data $\check{\omega}^I, \check{\omega}^J, \check{\omega}^K, I, J, K$ on L' . We will show that the important structures we define are independent of the choices of L', e_2, e_3 .

In §2.5 we noted that because $B : X \rightarrow \mathbb{C}^2/\{\pm 1\}$ is a holomorphic blow-up, and $\omega_0^J + i\omega_0^K = dz_1 \wedge dz_2$ is a holomorphic $(2, 0)$ -form on $\mathbb{C}^2/\{\pm 1\}$, the pullback $\omega^J + i\omega^K := B^*(\omega_0^J + i\omega_0^K)$ is a holomorphic, and hence smooth, complex 2-form on X , so that $\omega^J = B^*(\omega_0^J)$ and $\omega^K = B^*(\omega_0^K)$ are smooth. Similarly, there are unique smooth sections $\check{\omega}^J, \check{\omega}^K$ of $\Lambda^2 \check{V}^*$ on $\sigma^{-1}(L') \subseteq P$, such that for each $x \in L'$, under the identifications $\Lambda^2 \check{V}^*|_{P_x} \cong \Lambda^2 T^*P_x$ and $\Lambda^2 T^*(\nu_x/\{\pm 1\}) \cong \pi^*(\Lambda^2 \nu_x^*)$ we have $\check{\omega}^J|_{P_x} \cong \rho^*(\pi^*(\check{\omega}^J|_x))$ and $\check{\omega}^K|_{P_x} \cong \rho^*(\pi^*(\check{\omega}^K|_x))$.

Thus, for each $x \in L'$, we have 2-forms $\check{\omega}^I|_{P_x}, \check{\omega}^J|_{P_x}, \check{\omega}^K|_{P_x}$ on P_x which are identified with the 2-forms $\omega_{A(x)}^I, \omega_{A(x)}^J, \omega_{A(x)}^K$ on the Eguchi–Hanson space X in §2.5 by the isomorphism $P_x \cong X$. We have

$$\omega_{A(x)}^I \wedge \omega_{A(x)}^I = \omega^J \wedge \omega^J = \omega^K \wedge \omega^K$$

on X by properties of hyperKähler manifolds. Thus we deduce that

$$(4.15) \quad \check{\omega}^I \wedge \check{\omega}^I = \check{\omega}^J \wedge \check{\omega}^J = \check{\omega}^K \wedge \check{\omega}^K \quad \text{in } \Lambda^4 \check{V}^* \text{ on } \sigma^{-1}(L') \subseteq P.$$

Let δ denote the dilation vector field on $\mathbb{C}^2/\{\pm 1\}$. Since the closed form $\omega_0^J + i\omega_0^K = dz_1 \wedge dz_2$ is homogeneous of order 2 under dilations, we deduce that on $\mathbb{C}^2/\{\pm 1\}$ we have

$$(4.16) \quad \omega_0^J + i\omega_0^K = \frac{1}{2} \mathcal{L}_\delta (\omega_0^J + i\omega_0^K) = \frac{1}{2} d[\delta \cdot \omega_0^J + i\delta \cdot \omega_0^K].$$

In complex coordinates, $\delta = z_k \partial_{z_k} + \bar{z}_k \partial_{\bar{z}_k}$. Hence $\delta \cdot \omega_0^J + i\delta \cdot \omega_0^K = z_1 dz_2 - z_2 dz_1$ is a holomorphic $(1, 0)$ -form on $\mathbb{C}^2/\{\pm 1\}$, so that $B^*(\delta \cdot \omega_0^J + i\delta \cdot \omega_0^K)$ is a holomorphic (and hence smooth) 1-form on X . Thus there are unique smooth sections μ^J, μ^K of \check{V}^* on $\sigma^{-1}(L') \subseteq P$, such that for each $x \in L'$, under the identifications $\Lambda^2 \check{V}^*|_{P_x} \cong \Lambda^2 T^*P_x$ and $\Lambda^2 T^*(\nu_x/\{\pm 1\}) \cong \pi^*(\Lambda^2 \nu_x^*)$ we have $\mu^J|_{P_x} \cong \frac{1}{2} \rho^*(\delta \cdot \pi^*(\check{\omega}^J|_x))$ and $\mu^K|_{P_x} \cong \frac{1}{2} \rho^*(\delta \cdot \pi^*(\check{\omega}^K|_x))$. Pulling back (4.16) to P_x gives

$$(4.17) \quad \check{\omega}^J|_{P_x} = d(\mu^J|_{P_x}) \quad \text{and} \quad \check{\omega}^K|_{P_x} = d(\mu^K|_{P_x}).$$

Now using (4.6) and the fact that restriction of 2-forms in $\Lambda^2 \check{V}^* \subset \Lambda^2 T^*P$ to P_x is an isomorphism $\Lambda^2 \check{V}^*|_{P_x} \rightarrow \Lambda^2 T^*P_x$, we see that (4.17)

implies

$$(4.18) \quad \check{\omega}^J = [d\mu^J]_{2,0} \quad \text{and} \quad \check{\omega}^K = [d\mu^K]_{2,0} \quad \text{on } \sigma^{-1}(L') \subseteq P.$$

On $\mathbb{C}^2/\{\pm 1\}$ we have $\omega_0^J \wedge \omega_0^J = \omega_0^K \wedge \omega_0^K$, so contracting with δ implies that $(\delta \cdot \omega_0^J) \wedge \omega_0^J = (\delta \cdot \omega_0^K) \wedge \omega_0^K$. Pulling this equation back to $\sigma^{-1}(L')$ as for $\check{\omega}^J, \check{\omega}^K, \mu^J, \mu^K$ implies that

$$(4.19) \quad \mu^J \wedge \check{\omega}^J = \mu^K \wedge \check{\omega}^K.$$

Finally, on $\mathbb{C}^2/\{\pm 1\}$ we also have $d[(\delta \cdot \omega_0^J) \wedge \omega_0^J] = 2\omega_0^J \wedge \omega_0^J$, so arguing as for (4.18), using (4.15) and (4.19) we deduce that on $\sigma^{-1}(L')$ we have

$$(4.20) \quad [d(\mu^J \wedge \check{\omega}^J)]_{4,0} = [d(\mu^K \wedge \check{\omega}^K)]_{4,0} = \check{\omega}^I \wedge \check{\omega}^I = \check{\omega}^J \wedge \check{\omega}^J = \check{\omega}^K \wedge \check{\omega}^K.$$

Recall we defined $\check{g}_{2,0}$ in (4.14). Now we define a tensor $\check{g}_{0,2}$ in $S^2\check{V}^* \subset S^2T^*P$ and exterior forms $\check{\varphi}_{0,3}, \dots, \theta_{2,2}$ on $\sigma^{-1}(L') \subset P$, where subscripts $(\dots)_{i,j}$ indicate a form of type (i,j) , as follows. The motivation for the definitions will be apparent when we derive (4.22) after the definitions. We define:

$$(4.21) \quad \begin{aligned} \check{g}_{0,2} &= \sigma^*(e_1)^2 + \sigma^*(e_2)^2 + \sigma^*(e_3)^2, \\ \check{\varphi}_{0,3} &= \sigma^*(e_1) \wedge \sigma^*(e_2) \wedge \sigma^*(e_3) = \sigma^*(\text{vol}_L), \\ \check{\varphi}_{2,1} &= -\sigma^*(e_1) \wedge \check{\omega}^I - \sigma^*(e_2) \wedge \check{\omega}^J - \sigma^*(e_3) \wedge \check{\omega}^K, \\ \check{\psi}_{2,2} &= -\sigma^*(e_2) \wedge \sigma^*(e_3) \wedge \check{\omega}^I - \sigma^*(e_3) \wedge \sigma^*(e_1) \wedge \check{\omega}^J \\ &\quad - \sigma^*(e_1) \wedge \sigma^*(e_2) \wedge \check{\omega}^K, \\ \check{\psi}_{4,0} &= \frac{1}{2}\check{\omega}^I \wedge \check{\omega}^I = \frac{1}{2}\check{\omega}^J \wedge \check{\omega}^J = \frac{1}{2}\check{\omega}^K \wedge \check{\omega}^K, \\ \xi_{1,2} &= -\sigma^*(\lambda) \wedge \kappa_{1,1} + [d(\sigma^*(e_2) \wedge \mu^J + \sigma^*(e_3) \wedge \mu^K)]_{1,2}, \\ \xi_{0,3} &= -\sigma^*(\lambda) \wedge \kappa_{0,2} + [d(\sigma^*(e_2) \wedge \mu^J + \sigma^*(e_3) \wedge \mu^K)]_{0,3} \\ \chi_{1,3} &= -\sigma^*(\ast\lambda) \wedge \kappa_{1,1} - [d(\sigma^*(e_3) \wedge \sigma^*(e_1) \wedge \mu^J \\ &\quad + \sigma^*(e_1) \wedge \sigma^*(e_2) \wedge \mu^K)]_{1,3}, \\ \theta_{3,1} &= \frac{1}{2}[d(\mu^J \wedge \check{\omega}^J)]_{3,1} = \frac{1}{2}[d(\mu^K \wedge \check{\omega}^K)]_{3,1}, \\ \theta_{2,2} &= \frac{1}{2}[d(\mu^J \wedge \check{\omega}^J)]_{2,2} = \frac{1}{2}[d(\mu^K \wedge \check{\omega}^K)]_{2,2}. \end{aligned}$$

Here the alternate expressions for $\check{\psi}_{4,0}, \theta_{3,1}, \theta_{2,2}$ come from (4.15) and (4.19). Using equations (4.9), (4.10), (4.18), (4.20), (4.21) and $d\text{vol}_L = 0$ we claim that

$$(4.22) \quad \begin{aligned} d\check{\varphi}_{0,3} &= 0, \quad d[\check{\varphi}_{2,1} + \xi_{1,2} + \xi_{0,3}] = 0, \\ d[\check{\psi}_{2,2} + \chi_{1,3}] &= 0, \quad d[\check{\psi}_{4,0} + \theta_{3,1} + \theta_{2,2}] = 0. \end{aligned}$$

The first equation in (4.22) is obvious. We will derive the second equation. First, we write

$$\begin{aligned} \xi_{1,2} + \xi_{0,3} &= -\sigma^*(\lambda) \wedge (\kappa_{1,1} + \kappa_{0,2}) + d(\sigma^*(e_2) \wedge \mu^J + \sigma^*(e_3) \wedge \mu^K) \\ &\quad - [d(\sigma^*(e_2) \wedge \mu^J + \sigma^*(e_3) \wedge \mu^K)]_{2,1} \end{aligned}$$

$$\begin{aligned}
&= -\sigma^*(\lambda) \wedge (\kappa_{1,1} + \kappa_{0,2}) + d(\cdots) + \sigma^*(e_2) \wedge \tilde{\omega}^J \\
&\quad + \sigma^*(e_3) \wedge \tilde{\omega}^K,
\end{aligned}$$

using (4.18) in the last step. Thus we find

$$d(\tilde{\varphi}_{2,1} + \xi_{1,2} + \xi_{0,3}) = -d(\sigma^*(\lambda) \wedge (\kappa_{1,1} + \kappa_{0,2}) + \sigma^*(e_1) \wedge \tilde{\omega}^I) = 0$$

by (4.9). The remaining two equations in (4.22) are proved similarly.

As in §4.1, we are working with an arbitrary choice of e_2, e_3 on an open $L' \subseteq L$ with (e_1, e_2, e_3) an oriented orthonormal basis of T^*L' . If e'_2, e'_3 are alternative choices for e_2, e_3 then $e'_2 = \cos \phi e_2 + \sin \phi e_3$, $e'_3 = -\sin \phi e_2 + \cos \phi e_3$ for some smooth $\phi : L' \rightarrow \mathbb{R}/2\pi\mathbb{Z}$. Equation (4.2) expresses the corresponding $\hat{\omega}^I, \hat{\omega}^J, \hat{\omega}^K, I', J', K'$ in terms of $\hat{\omega}^I, \hat{\omega}^J, \hat{\omega}^K, I, J, K$. From the definitions of $\tilde{\omega}^J, \tilde{\omega}^K, \mu^J, \mu^K$ above, we see that they transform by

$$\begin{aligned}
\tilde{\omega}^J &= \cos \phi \tilde{\omega}^J + \sin \phi \tilde{\omega}^K, & \tilde{\omega}^K &= -\sin \phi \tilde{\omega}^J + \cos \phi \tilde{\omega}^K, \\
\mu^J &= \cos \phi \mu^J + \sin \phi \mu^K, & \mu^K &= -\sin \phi \mu^J + \cos \phi \mu^K.
\end{aligned}$$

We can now check that all of $\check{g}_{0,2}, \dots, \theta_{2,2}$ in (4.21) are independent of the choice of e_2, e_3 . Thus by covering L by open $L' \subseteq L$ on which such a basis (e_1, e_2, e_3) exists, we construct global $\check{g}_{0,2}, \dots, \theta_{2,2}$ on all of P satisfying (4.21) over each L' in the cover. Then (4.22) holds over each L' in the cover, so (4.22) holds on P .

In a similar way to (3.22), for all $t > 0$ define a 3-form φ_t^P , 4-form ψ_t^P and metric g_t^P on P by

$$(4.23) \quad \varphi_t^P = \check{\varphi}_{0,3} + t^2 \check{\varphi}_{2,1}, \quad \psi_t^P = t^2 \check{\psi}_{2,2} + t^4 \check{\psi}_{4,0}, \quad g_t^P = \check{g}_{0,2} + t^2 \check{g}_{2,0}.$$

Then over $\sigma^{-1}(L')$ as above, by (4.21) we have

$$\begin{aligned}
\varphi_t^P &= \sigma^*(e_1) \wedge \sigma^*(e_2) \wedge \sigma^*(e_3) - t^2 \sigma^*(e_1) \wedge \tilde{\omega}^I \\
&\quad - t^2 \sigma^*(e_2) \wedge \tilde{\omega}^J - t^2 \sigma^*(e_3) \wedge \tilde{\omega}^K, \\
\psi_t^P &= \frac{1}{2} t^4 \tilde{\omega}^I \wedge \tilde{\omega}^I - t^2 \sigma^*(e_2) \wedge \sigma^*(e_3) \wedge \tilde{\omega}^I \\
&\quad - t^2 \sigma^*(e_3) \wedge \sigma^*(e_1) \wedge \tilde{\omega}^J - t^2 \sigma^*(e_1) \wedge \sigma^*(e_2) \wedge \tilde{\omega}^K, \\
g_t^P &= \sigma^*(e_1)^2 + \sigma^*(e_2)^2 + \sigma^*(e_3)^2 + t^2 \check{g}_{2,0}.
\end{aligned}$$

Comparing these with (2.22) and noting that on the fibres P_x of $\sigma : P \rightarrow L$, the $\tilde{\omega}^I, \tilde{\omega}^J, \tilde{\omega}^K, \check{g}_{2,0}$ correspond to $\hat{\omega}_{A(x)}^I, \hat{\omega}^J, \hat{\omega}^K, h_{A(x)}$ on the Eguchi–Hanson space X , we see that (φ_t^P, g_t^P) is a G_2 -structure on P , with 4-form $\psi_t^P = \Theta(\varphi_t^P) = *_{{g_t^P}} \varphi_t^P$.

Finally, we define on P the following forms:

$$(4.24) \quad \tilde{\varphi}_t^P = \varphi_t^P + t^2 \xi_{1,2} + t^2 \xi_{0,3}, \quad \tilde{\psi}_t^P = \psi_t^P + t^2 \chi_{1,3} + t^4 \theta_{3,1} + t^4 \theta_{2,2}.$$

Using equations (4.22) and (4.23) we find that $\tilde{\varphi}_t^P, \tilde{\psi}_t^P$ are *closed* forms. However, we emphasize that $\Theta(\tilde{\varphi}_t^P) \neq \tilde{\psi}_t^P$.

4.5. Comparing $\varphi_t^P, \psi_t^P, \tilde{\varphi}_t^P, \tilde{\psi}_t^P$ and $\varphi_t^\nu, \psi_t^\nu, \tilde{\varphi}_t^\nu, \tilde{\psi}_t^\nu$. In §3.3–§3.4 we defined a G_2 -structure (φ_t^ν, g_t^ν) with 4-form $\psi_t^\nu = *_{g_t^\nu} \varphi_t^\nu$ on ν for $t > 0$, and closed 3- and 4-forms $\tilde{\varphi}_t^\nu = \varphi_t^\nu + t^2 \ddot{\varphi}_{0,3}^2$ and $\tilde{\psi}_t^\nu = \psi_t^\nu + t^4 \ddot{\psi}_{2,2}^4$ on ν . These are all invariant under $\{\pm 1\}$, and so they descend to $\nu/\{\pm 1\}$. In §4.4 we defined a G_2 -structure (φ_t^P, g_t^P) with 4-form $\psi_t^P = *_{g_t^P} \varphi_t^P$ on P for $t > 0$, and closed 3- and 4-forms $\tilde{\varphi}_t^P = \varphi_t^P + t^2 \xi_{1,2} + t^2 \xi_{0,3}$ and $\tilde{\psi}_t^P = \psi_t^P + t^2 \chi_{1,3} + t^4 \theta_{3,1} + t^4 \theta_{2,2}$ on P . We will now compare $\varphi_t^P, \psi_t^P, \tilde{\varphi}_t^P, \tilde{\psi}_t^P$ with the pullbacks of $\varphi_t^\nu, \psi_t^\nu, \tilde{\varphi}_t^\nu, \tilde{\psi}_t^\nu$ by $\rho : P \rightarrow \nu/\{\pm 1\}$ on the region $\tilde{r} > 1$ in P , where the connections are compatible.

By the definition of ρ in §4.1, the following map is a diffeomorphism:

$$(4.25) \quad \rho|_{\tilde{r}>1} : \{p \in P : \tilde{r}(p) > 1\} \longrightarrow \{z \in \nu/\{\pm 1\} : r(z) > 1\}.$$

We have splittings $T(\nu/\{\pm 1\}) = V \oplus H$ from §3.2 and $TP = \check{V} \oplus \check{H}$ from §4.2. Here H is defined using a connection $\tilde{\nabla}^\nu$ on ν , and \check{H} is also defined using $\tilde{\nabla}^\nu$ in the region $\tilde{r} > 1$. Therefore (4.25) identifies \check{V}, \check{H} with V, H .

As in §4.1, work with an oriented orthonormal basis (e_1, e_2, e_3) of 1-forms on $L' \subseteq L$. Then in forms on $\nu/\{\pm 1\}$ we have

$$(4.26) \quad \begin{aligned} \varphi_{0,3}^0 &= \pi^*(e_1) \wedge \pi^*(e_2) \wedge \pi^*(e_3) = \pi^*(\text{vol}_L), \\ \dot{\varphi}_{2,1}^2 &= -\pi^*(e_1) \wedge \hat{\omega}^I - \pi^*(e_2) \wedge \hat{\omega}^J - \pi^*(e_3) \wedge \hat{\omega}^K, \\ \dot{\psi}_{2,2}^2 &= -\pi^*(e_2) \wedge \pi^*(e_3) \wedge \hat{\omega}^I - \pi^*(e_3) \wedge \pi^*(e_1) \wedge \hat{\omega}^J \\ &\quad - \pi^*(e_1) \wedge \pi^*(e_2) \wedge \hat{\omega}^K, \\ \dot{\psi}_{4,0}^4 &= \tfrac{1}{2} \hat{\omega}^I \wedge \hat{\omega}^I = \tfrac{1}{2} \hat{\omega}^J \wedge \hat{\omega}^J = \tfrac{1}{2} \hat{\omega}^K \wedge \hat{\omega}^K. \end{aligned}$$

In forms on $\{p \in P : \tilde{r}(p) > 1\}$ we claim that

$$(4.27) \quad \begin{aligned} \tilde{\omega}^I &= \rho|_{\tilde{r}>1}^*(\hat{\omega}^I) - \tfrac{1}{4} [d(I([d(G_{A \circ \sigma}(\tilde{r}))])_{1,0})]_{2,0}, \\ \tilde{\omega}^J &= \rho|_{\tilde{r}>1}^*(\hat{\omega}^J), \quad \tilde{\omega}^K = \rho|_{\tilde{r}>1}^*(\hat{\omega}^K), \quad \sigma^*(e_i) = \rho|_{\tilde{r}>1}^*(\pi^*(e_i)). \end{aligned}$$

Here $G_a(r) = f_a(r) - r^2$ as in (2.17). The first equation of (4.27) follows from (2.17), (4.13), and $\hat{\omega}^I = -\frac{1}{4} [d(I([d(r^2)])_{1,0})]_{2,0}$. The second and third equations follow from the definitions of $\tilde{\omega}^J, \tilde{\omega}^K$ and $\rho^*(V^*) = \check{V}^* \subset T^*P$ where $\tilde{r} > 1$ in P . The fourth equation holds as $\sigma = \pi \circ \rho$.

Now define a 2-form $\tau_{1,1}$ of type $(1, 1)$ and a 3-form $v_{1,2}$ of type $(1, 2)$ on the open subset of P where $\tilde{r} > 1$ by

$$(4.28) \quad \begin{aligned} \tau_{1,1} &= -\tfrac{1}{4} (A \circ \sigma)^{-1} \sigma^*(\lambda) \wedge I([d(G_{A \circ \sigma}(\tilde{r}))])_{1,0}, \\ v_{1,2} &= \tfrac{1}{4} (A \circ \sigma)^{-1} \sigma^*(\lambda) \wedge I([d(G_{A \circ \sigma}(\tilde{r}))])_{1,0}. \end{aligned}$$

Combining equations (4.21), (4.26), (4.27), and (4.28) one finds that

$$\begin{aligned}
 (4.29) \quad & \check{\varphi}_{0,3}|\check{r}>1 = \rho_{|\check{r}>1}^*(\varphi_{0,3}^0), \\
 & \check{\varphi}_{2,1}|\check{r}>1 = \rho_{|\check{r}>1}^*(\dot{\varphi}_{2,1}^2) + [d\tau_{1,1}]_{2,1}, \\
 & \check{\psi}_{2,2}|\check{r}>1 = \rho_{|\check{r}>1}^*(\dot{\psi}_{2,2}^2) + [dv_{1,2}]_{2,2}, \\
 & \check{\psi}_{4,0}|\check{r}>1 = \rho_{|\check{r}>1}^*(\dot{\psi}_{4,0}^4),
 \end{aligned}$$

and consequently from (3.22) and (4.23) we obtain

$$(4.30) \quad \varphi_t^P|_{\check{r}>1} = \rho_{|\check{r}>1}^*(\varphi_t^\nu) + t^2[d\tau_{1,1}]_{2,1}, \quad \psi_t^P|_{\check{r}>1} = \rho_{|\check{r}>1}^*(\psi_t^\nu) + t^2[dv_{1,2}]_{2,2}.$$

Similarly, working with an oriented orthonormal basis (e_1, e_2, e_3) on $L' \subseteq L$ as in §4.1, from the definitions and equations above we can show that

$$\begin{aligned}
 (4.31) \quad & \check{\varphi}_t^P|_{\check{r}>1} = \check{\varphi}_{0,3} - t^2 d[\sigma^*(e_1) \wedge I[d(\tfrac{1}{4}f_{A \circ \sigma}(\check{r}^2))]_{1,0} - \sigma^*(e_2) \wedge \mu^J \\
 & \quad - \sigma^*(e_3) \wedge \mu^K], \\
 & \rho_{|\check{r}>1}^*(\check{\varphi}_t^\nu) = \check{\varphi}_{0,3} - t^2 d[\sigma^*(e_1) \wedge I[\tfrac{1}{4}d(\check{r}^2)]_{1,0} - \sigma^*(e_2) \wedge \mu^J \\
 & \quad - \sigma^*(e_3) \wedge \mu^K], \\
 & \check{\psi}_t^P|_{\check{r}>1} = d[t^4(\tfrac{1}{2}\mu^J \wedge \check{\omega}^J) + t^2(\sigma^*(e_2 \wedge e_3) \wedge I[\tfrac{1}{4}d(f_{A \circ \sigma}(\check{r}^2))]_{1,0} \\
 & \quad - \sigma^*(e_3 \wedge e_1) \wedge \mu^J - \sigma^*(e_1 \wedge e_2) \wedge \mu^K)], \\
 & \rho_{|\check{r}>1}^*(\check{\psi}_t^\nu) = d[t^4(\tfrac{1}{2}\mu^J \wedge \check{\omega}^J) + t^2(\sigma^*(e_2 \wedge e_3) \wedge I[\tfrac{1}{4}d(\check{r}^2)]_{1,0} \\
 & \quad - \sigma^*(e_3 \wedge e_1) \wedge \mu^J - \sigma^*(e_1 \wedge e_2) \wedge \mu^K)].
 \end{aligned}$$

Combining $(A \circ \sigma)^{-1}\sigma^*(\lambda) = e_1$, $(A \circ \sigma)^{-1}\sigma^*(\ast\lambda) = e_2 \wedge e_3$ and equations (2.17), (4.28), and (4.31), we see that

$$(4.32) \quad \check{\varphi}_t^P|_{\check{r}>1} = \rho_{|\check{r}>1}^*(\check{\varphi}_t^\nu) + t^2 d\tau_{1,1}, \quad \check{\psi}_t^P|_{\check{r}>1} = \rho_{|\check{r}>1}^*(\check{\psi}_t^\nu) + t^2 dv_{1,2}.$$

In the region $\check{r} > 1$, from Corollary 3.9, equations (4.24), (4.29), (4.30), (4.32), and the fact that $\rho|_{\check{r}>1}$ takes \check{V}, \check{H} to V, H , we see that

$$\begin{aligned}
 (4.33) \quad & \xi_{1,2}|\check{r}>1 = [d\tau_{1,1}]_{1,2}, \quad \xi_{0,3}|\check{r}>1 = \rho_{|\check{r}>1}^*(\ddot{\varphi}_{0,3}^2) + [d\tau_{1,1}]_{0,3}, \\
 & \chi_{1,3}|\check{r}>1 = [dv_{1,2}]_{1,3}, \quad \theta_{3,1}|\check{r}>1 = 0, \quad \theta_{2,2}|\check{r}>1 = \rho_{|\check{r}>1}^*(\ddot{\psi}_{2,2}^4).
 \end{aligned}$$

Using (2.18) and (4.28) we can show that where $\check{r} > 1$ on P we have

$$\begin{aligned}
 (4.34) \quad & |\nabla^k(t^2\tau_{1,1})|_{g_t^P} = O(t^{1-k}\check{r}^{-3-k}), \quad |\nabla^k(t^2v_{1,2})|_{g_t^P} = O(t^{1-k}\check{r}^{-3-k}), \\
 & |\nabla^k(t^2[d\tau_{1,1}]_{i,3-i})|_{g_t^P} = O(t^{2-i-k}\check{r}^{-2-i-k}), \\
 & |\nabla^k(t^2[dv_{1,2}]_{i,4-i})|_{g_t^P} = O(t^{2-i-k}\check{r}^{-2-i-k}),
 \end{aligned}$$

for all $i = 0, 1, 2$ and $k = 0, 1, \dots$, where ∇ is the Levi-Civita connection of g_t^P . Thus, (4.30) and (4.32) imply that $\varphi_t^P, \psi_t^P, \tilde{\varphi}_t^P, \tilde{\psi}_t^P$ are asymptotic to $\rho^*(\varphi_t^\nu), \rho^*(\psi_t^\nu), \rho^*(\tilde{\varphi}_t^\nu), \rho^*(\tilde{\psi}_t^\nu)$ as $\tilde{r} \rightarrow \infty$ in P .

4.6. Estimating the torsion of (φ_t^P, g_t^P) . Just as for (φ_t^ν, g_t^ν) in the discussion after Corollary 3.9, we have constructed a G_2 -structure (φ_t^P, g_t^P) on P with 4-form $\psi_t^P = *_t^P \varphi_t^P$. To apply Theorem 2.7 in §6, we will need (φ_t^P, g_t^P) to have *small torsion*, in an appropriate sense. If $d\varphi_t^P = 0$ and $d\psi_t^P = 0$ then (φ_t^P, g_t^P) would be torsion-free, by Theorem 2.2. In general $d\varphi_t^P \neq 0$ and $d\psi_t^P \neq 0$, but as $d\tilde{\varphi}_t^P = 0$ and $d\tilde{\psi}_t^P = 0$ with $\tilde{\varphi}_t^P = \varphi_t^P + t^2\xi_{1,2} + t^2\xi_{0,3}$ and $\tilde{\psi}_t^P = \psi_t^P + t^2\chi_{1,3} + t^4\theta_{3,1} + t^4\theta_{2,2}$ by (4.24), we can regard $|t^2\xi_{1,2}|_{g_t^P}, |t^2\xi_{0,3}|_{g_t^P}, |t^2\chi_{1,3}|_{g_t^P}, |t^4\theta_{3,1}|_{g_t^P}, |t^4\theta_{2,2}|_{g_t^P}$ as measuring the torsion of (φ_t^P, g_t^P) . Thus, the next proposition estimates the torsion of (φ_t^P, g_t^P) .

Proposition 4.6. *In the situation of §4.4, for all $k = 0, 1, \dots$ we have*

$$(4.35) \quad |\nabla^k(t^2\xi_{1,2})|_{g_t^P} = \begin{cases} O(t^{1-k}), & \tilde{r} \leq 1, \\ O(t^{1-k}\tilde{r}^{-3-k}), & \tilde{r} \geq 1, \end{cases}$$

$$(4.36) \quad |\nabla^k(t^2\xi_{0,3})|_{g_t^P} = \begin{cases} O(t^{2-k}), & \tilde{r} \leq 1, \\ O(t^{2-k}\tilde{r}^{2-k}), & \tilde{r} \geq 1, \end{cases}$$

$$(4.37) \quad |\nabla^k(t^2\chi_{1,3})|_{g_t^P} = \begin{cases} O(t^{1-k}), & \tilde{r} \leq 1, \\ O(t^{1-k}\tilde{r}^{-3-k}), & \tilde{r} \geq 1, \end{cases}$$

$$(4.38) \quad |\nabla^k(t^4\theta_{3,1})|_{g_t^P} = \begin{cases} O(t^{1-k}), & \tilde{r} \leq 1, \\ 0, & \tilde{r} \geq 1, \end{cases}$$

$$(4.39) \quad |\nabla^k(t^4\theta_{2,2})|_{g_t^P} = \begin{cases} O(t^{2-k}), & \tilde{r} \leq 1, \\ O(t^{2-k}\tilde{r}^{2-k}), & \tilde{r} \geq 1. \end{cases}$$

Proof. The $\tilde{r} \leq 1$ estimates hold because if $\alpha_{i,j}$ is a form or tensor of type (i, j) on P then $|\alpha_{i,j}|_{g_t^P} = O(t^{-i})$ on the compact region $\tilde{r} \leq 1$ in P , by the definition (4.23) of g_t^P . The $\tilde{r} \geq 1$ estimates follow from (4.33)–(4.34) and the estimates $|\nabla^k(t^2\ddot{\varphi}_{0,3}^2)|_{g_t^P} = O(t^{2-k}\tilde{r}^{2-k})$ and $|\nabla^k(t^4\ddot{\psi}_{2,2}^4)|_{g_t^P} = O(t^{2-k}\tilde{r}^{2-k})$, which both follow from Corollary 3.9. q.e.d.

5. Correcting for the leading-order errors on P

5.1. The correction theorem. Our goal in §6 is to glue together the G_2 -structures (φ, g) on $M/\langle \iota \rangle$, and $(\tilde{\varphi}_t^\nu, \tilde{g}_t^\nu)$ on $\nu/\{\pm 1\}$, and $(\tilde{\varphi}_t^P, \tilde{g}_t^P)$ on P , to get a closed G_2 -structure (φ_t^N, g_t^N) with small torsion on a compact

7-manifold N , and then apply Theorem 2.7 to show that (φ_t^N, g_t^N) has a small deformation to a torsion-free G_2 -structure on N for small t . To do this we must ensure that the 4-form $\Theta(\varphi_t^N) - \psi_t^N$ which is a substitute for the torsion of (φ_t^N, g_t^N) satisfies several estimates, including $\|\Theta(\varphi_t^N) - \psi_t^N\|_{L^2} \leq Kt^{\frac{7}{2}+\alpha}$ for $\alpha, K > 0$.

Now in Proposition 4.6, the bounds for $|t^2\xi_{0,3}|_{g_t^P}$, $|t^4\theta_{2,2}|_{g_t^P}$ will contribute $O(t^4)$ to $\|\Theta(\varphi_t^N) - \psi_t^N\|_{L^2}$, which is good, as $4 > \frac{7}{2}$. However, the bounds for $|t^2\xi_{1,2}|_{g_t^P}$, $|t^2\chi_{1,3}|_{g_t^P}$, $|t^4\theta_{3,1}|_{g_t^P}$ would contribute $O(t^3)$ to $\|\Theta(\varphi_t^N) - \psi_t^N\|_{L^2}$, which is bad, as the error is too large to apply Theorem 2.7. In Theorem 5.1 we construct forms on P which will be used in §6 to cancel out the leading-order error terms in $t^2\xi_{1,2}$, $t^2\chi_{1,3}$, $t^4\theta_{3,1}$, and so make $\|\Theta(\varphi_t^N) - \psi_t^N\|_{L^2}$ small enough to apply Theorem 2.7.

Theorem 5.1. *There exist 2-forms $\alpha_{0,2}, \alpha_{2,0}$ and 3-forms $\beta_{0,3}, \beta_{2,1}$ on P , where $\alpha_{i,j}, \beta_{i,j}$ are of type (i, j) , satisfying for all $t > 0$ the equation*

$$(5.1) \quad \begin{aligned} (D_{\varphi_t^P}\Theta)(t^2[d\alpha_{0,2}]_{1,2} + t^4[d\alpha_{2,0}]_{3,0} + t^2\xi_{1,2}) \\ = t^2d\beta_{0,3} + t^4[d\beta_{2,1}]_{3,1} + t^2\chi_{1,3} + t^4\theta_{3,1}. \end{aligned}$$

Note that in (5.1) we have $d\beta_{0,3} = [d\beta_{0,3}]_{1,3}$ automatically, so all the derivatives in (5.1) are in fact vertical (fibre) derivatives. Moreover, for $\gamma > 0$ sufficiently small and for all $k \geq 0$, these forms satisfy the following estimates:

$$(5.2) \quad |\nabla^k(t^2\alpha_{0,2})|_{g_t^P} = \begin{cases} O(t^{2-k}), & \check{r} \leq 1, \\ O(t^{2-k}\check{r}^{-2-k+\gamma}), & \check{r} \geq 1, \end{cases}$$

$$(5.3) \quad |\nabla^k(t^2[d\alpha_{0,2}]_{i,3-i})|_{g_t^P} = \begin{cases} O(t^{2-i-k}), & \check{r} \leq 1, \ i = 0, 1, \\ O(t^{2-i-k}\check{r}^{-2-i-k+\gamma}), & \check{r} \geq 1, \ i = 0, 1, \end{cases}$$

$$(5.4) \quad |\nabla^k(t^4\alpha_{2,0})|_{g_t^P} = \begin{cases} O(t^{2-k}), & \check{r} \leq 1, \\ O(t^{2-k}\check{r}^{-2-k+\gamma}), & \check{r} \geq 1, \end{cases}$$

$$(5.5) \quad |\nabla^k(t^4[d\alpha_{2,0}]_{i,3-i})|_{g_t^P} = \begin{cases} O(t^{4-i-k}), & \check{r} \leq 1, \ i = 1, 2, 3, \\ O(t^{4-i-k}\check{r}^{-i-k+\gamma}), & \check{r} \geq 1, \ i = 1, 2, 3, \end{cases}$$

$$(5.6) \quad |\nabla^k(t^2\beta_{0,3})|_{g_t^P} = \begin{cases} O(t^{2-k}), & \check{r} \leq 1, \\ O(t^{2-k}\check{r}^{-2-k+\gamma}), & \check{r} \geq 1, \end{cases}$$

$$(5.7) \quad |\nabla^k(t^4\beta_{2,1})|_{g_t^P} = \begin{cases} O(t^{2-k}), & \check{r} \leq 1, \\ O(t^{2-k}\check{r}^{-2-k+\gamma}), & \check{r} \geq 1, \end{cases}$$

$$(5.8) \quad |\nabla^k(t^4[d\beta_{2,1}]_{i,j})|_{g_t^P} = \begin{cases} O(t^{4-i-k}), & \check{r} \leq 1, \ i = 1, 2, 3, \\ O(t^{4-i-k}\check{r}^{-i-k+\gamma}), & \check{r} \geq 1, \ i = 1, 2, 3. \end{cases}$$

If we take $\alpha_{2,0}$ and $\beta_{2,1}$ to be self-dual in the fibre directions, that is, $\alpha_{2,0} \in \Gamma^\infty(\Lambda_+^2 V^*)$ and $\beta_{2,1} \in \Gamma^\infty(\Lambda_+^2 V^* \otimes H^*)$, then $\alpha_{0,2}, \alpha_{2,0}, \beta_{0,3}, \beta_{2,1}$ are unique.

The remainder of §5 is taken up with the proof of Theorem 5.1. In §5.2 we establish several preliminary results needed for the proof. In §5.3 we collect various facts about analysis on the Eguchi–Hanson fibres. Finally, in §5.4 we complete the proof of Theorem 5.1 in three steps.

5.2. Preliminary results needed for the correction theorem.

In this section we collect several preliminary results that will be used to prove Theorem 5.1, including several results that use properties of G_2 -structures.

As in §4.1, we make an arbitrary choice of e_2, e_3 on an open $L' \subseteq L$ with (e_1, e_2, e_3) an oriented orthonormal basis of T^*L' . All our computations in this section will be independent of this choice. We now introduce some notation that will be used only within §5, to simplify our expressions. First, we will abuse notation and write e_k for $\sigma^*(e_k)$, thinking of e_k as a horizontal 1-form on P over L' . Moreover, over L' , a vertical k -form on P can be thought of as a k -form on X smoothly parametrized by $x \in L'$.

We will use $d_X, d_X^*, *_X, g_X$, and vol_X to denote the exterior derivative, its formal adjoint, the Hodge star, the metric, and the volume form on the fibre $P_x \cong X$ over $x \in L'$. Note that $d_X = d^{1,0}$ is the $(1, 0)$ part of d . That is, d_X is the exterior derivative in the vertical (fibre) direction. We have $*_X^2 = (-1)^k$ on $\Omega^k(P_x)$ and $d_X^* = -*_X d_X *_X$ on $\Omega^\bullet(P_x)$. Finally, we will denote the hyperKähler forms $\tilde{\omega}^I, \tilde{\omega}^J, \tilde{\omega}^K$ by $\omega_1, \omega_2, \omega_3$, respectively, and their associated complex structures I, J, K , by J_1, J_2, J_3 . More precisely, the forms ω_1, ω_2 , and ω_3 are a hyperKähler triple on $P_x \cong X$ for each $x \in L'$. Recall from (2.8) that $J_1 J_2 = -J_3 = -J_2 J_1$ on 1-forms, and cyclic permutations of this.

Lemma 5.2. *Let $\alpha \in \Omega^1(P_x)$ and $\gamma \in \Omega^3(P_x)$. Then we have*

$$(5.9) \quad \omega_i \wedge (*_X d_X \alpha) = -(d_X^*(J_i \alpha)) \text{vol}_X, \quad *_X(d_X \gamma) = d_X^*(*_X \gamma).$$

Moreover, if f is any function, then $d_X^*(J_k d_X f) = 0$.

Proof. The second equation is immediate from $*_X^2 = -1$ on 3-forms and $d_X^* = -*_X d_X *_X$. For the first equation, using $*_X \omega_i = \omega_i$ and $d_X \omega_i = 0$, and (2.12) we compute

$$\begin{aligned} \omega_i \wedge (*_X d_X \alpha) &= (d_X \alpha) \wedge (*_X \omega_i) = (d_X \alpha) \wedge \omega_i = d_X(\alpha \wedge \omega_i) \\ &= -d_X(*_X *_X (\alpha \wedge \omega_i)) = d_X *_X (J_i \alpha) = *_X(*_X d_X *_X)(J_i \alpha) \\ &= -*_X(d_X^*(J_i \alpha)) = -(d_X^*(J_i \alpha)) \text{vol}_X \end{aligned}$$

as claimed. Finally, from (5.9), for any function f we have $-d_X^*(J_k d_X f) \wedge \text{vol}_X = \omega_k \wedge *_X(d_X^2 f) = 0$, so $d_X^*(J_k d_X f) = 0$. q.e.d.

Consider now the G_2 -structure (φ_t^P, g_t^P) of (4.23). In the remainder of §5 we will drop the superscript P for simplicity. Thus we write

$$(5.10) \quad \begin{aligned} \varphi_t &= e_1 \wedge e_2 \wedge e_3 - t^2(\omega_1 \wedge e_1 + \omega_2 \wedge e_2 + \omega_3 \wedge e_3), \\ \psi_t &= t^4 \text{vol}_X - t^2(\omega_1 \wedge e_2 \wedge e_3 + \omega_2 \wedge e_3 \wedge e_1 + \omega_3 \wedge e_1 \wedge e_2), \\ g_t &= (e_1)^2 + (e_2)^2 + (e_3)^2 + t^2 g_X. \end{aligned}$$

At a point in P lying over $x \in L'$, the metric g_t^P is the Riemannian product of the flat metric on $\mathbb{R}^3 \cong \text{span}\{e_1, e_2, e_3\}$ with the metric g_X . Let $*_t$ and vol_t denote the Hodge star and volume form of (φ_t, g_t) . From $\text{vol}_t = t^4 e_1 \wedge e_2 \wedge e_3 \wedge \text{vol}_X$ it is easy to deduce that

$$(5.11) \quad *_t(\alpha \wedge \beta) = (-1)^{kl} t^{4-2k} (*_X \alpha) \wedge (*_{\mathbb{R}^3} \beta) \quad \begin{array}{l} \text{when } \alpha \text{ is a vertical } k\text{-form} \\ \text{and } \beta \text{ is an } l\text{-form on } \mathbb{R}^3. \end{array}$$

Recall that in §4.4 we defined a closed 3-form $\tilde{\varphi}_t^P$ and a closed 4-form $\tilde{\psi}_t^P$ by

$$(5.12) \quad \tilde{\varphi}_t^P = \varphi_t + t^2 \xi_{1,2} + t^2 \xi_{0,3}, \quad \tilde{\psi}_t^P = \psi_t + t^2 \chi_{1,3} + t^4 \theta_{3,1} + t^4 \theta_{2,2}.$$

Consider the forms $\xi_{1,2}$, $\chi_{1,3}$, and $\theta_{3,1}$, of equation (5.12). With respect to the local frame (e_1, e_2, e_3) we write

$$(5.13) \quad \begin{aligned} \xi_{1,2} &= \xi_1 \wedge e_2 \wedge e_3 + \xi_2 \wedge e_3 \wedge e_1 + \xi_3 \wedge e_1 \wedge e_2, \\ \chi_{1,3} &= \chi \wedge e_1 \wedge e_2 \wedge e_3, \\ \theta_{3,1} &= \theta_1 \wedge e_1 + \theta_2 \wedge e_2 + \theta_3 \wedge e_3, \end{aligned}$$

where $\xi_1, \xi_2, \xi_3, \chi$ are vertical 1-forms, and $\theta_1, \theta_2, \theta_3$ are vertical 3-forms. Over the open set L' , they can thus be viewed as forms on X smoothly parametrized by $x \in L'$. In this section we will write $[(p, q)]$ for a form of type (p, q) whose explicit expression will turn out to be irrelevant for us. Taking d of the expressions in (5.13), we have

$$(5.14) \quad \begin{aligned} d\xi_{1,2} &= \underbrace{(d_X \xi_1) \wedge e_2 \wedge e_3 + (d_X \xi_2) \wedge e_3 \wedge e_1 + (d_X \xi_3) \wedge e_1 \wedge e_2}_{\text{type } (2,2)} + [(1, 3)], \\ d\chi_{1,3} &= \underbrace{(d_X \chi) \wedge e_1 \wedge e_2 \wedge e_3}_{\text{type } (2,3)}, \\ d\theta_{3,1} &= \underbrace{(d_X \theta_1) \wedge e_1 + (d_X \theta_2) \wedge e_2 + (d_X \theta_3) \wedge e_3}_{\text{type } (4,1)} + [(3, 2)] + [(2, 3)]. \end{aligned}$$

Proposition 5.3. *The forms $\xi_1, \xi_2, \xi_3, \chi, \theta_1, \theta_2, \theta_3$ satisfy the following three relations:*

$$(5.15) \quad \begin{aligned} d_X^*(*_X \theta_1) + d_X^*(J_1 \chi) &= d_X^*(J_2 \xi_3) - d_X^*(J_3 \xi_2), \\ d_X^*(*_X \theta_2) + d_X^*(J_2 \chi) &= d_X^*(J_3 \xi_1) - d_X^*(J_1 \xi_3), \\ d_X^*(*_X \theta_3) + d_X^*(J_3 \chi) &= d_X^*(J_1 \xi_2) - d_X^*(J_2 \xi_1). \end{aligned}$$

Proof. We have $\psi_t = *_t \varphi_t = \Theta(\varphi_t)$, and therefore the pair (φ_t, ψ_t) satisfies the fundamental relation

$$(5.16) \quad \varphi_t \wedge (*_t d\varphi_t) = \psi_t \wedge (*_t d\psi_t).$$

Equation (5.16) is satisfied by any pair $(\varphi, \psi) \in \Omega_+^3 \oplus \Omega_+^4$ with $\psi = \Theta(\varphi)$, and is a characterization of the fact that the components $\pi_7(d\varphi)$ and $\pi_7(d\psi)$ of the torsion that are 7-dimensional can be identified by a G_2 -equivariant isomorphism. A proof of (5.16) can be found in [22, Theorem 2.23]. (Note that even though [22] uses the opposite orientation convention, the identity (5.16) is independent of the choice of orientation convention.) We will show that (5.16) implies the three equations in (5.15).

Since $\tilde{\varphi}_t^P$ and $\tilde{\psi}_t^P$ are both closed, taking d of both expressions in (5.12) and substituting (5.14) gives

$$\begin{aligned} d\varphi_t &= -t^2 d\xi_{1,2} - t^2 d\xi_{0,3} \\ &= -t^2 \underbrace{((d_X \xi_1) \wedge e_2 \wedge e_3 + (d_X \xi_2) \wedge e_3 \wedge e_1 + (d_X \xi_3) \wedge e_1 \wedge e_2)}_{\text{type } (2,2)} \\ &\quad + t^2 [(1, 3)] \end{aligned}$$

and

$$\begin{aligned} d\psi_t &= -t^2 d\chi_{1,3} - t^4 \theta_{3,1} - t^4 \theta_{2,2} \\ &= -t^2 \underbrace{((d_X \chi) \wedge e_1 \wedge e_2 \wedge e_3)}_{\text{type } (2,3)} \\ &\quad - t^4 \underbrace{((d_X \theta_1) \wedge e_1 + (d_X \theta_2) \wedge e_2 + (d_X \theta_3) \wedge e_3)}_{\text{type } (4,1)} \\ &\quad + t^4 [(3, 2)] + t^4 [(2, 3)]. \end{aligned}$$

Applying $*_t$ to both expressions and using (5.11) yields

$$\begin{aligned} *_t(d\varphi_t) &= -t^2 \underbrace{(*_X d_X \xi_1) \wedge e_1 + (*_X d_X \xi_2) \wedge e_2 + (*_X d_X \xi_3) \wedge e_3}_{\text{type } (2,1)} \\ &\quad + t^4 [(3, 0)], \\ *_t(d\psi_t) &= - \underbrace{(*_X d_X \theta_1) \wedge e_2 \wedge e_3 + (*_X d_X \theta_2) \wedge e_3 \wedge e_1 + (*_X d_X \theta_3) \wedge e_1 \wedge e_2}_{\text{type } (0,2)} \\ &\quad - \underbrace{t^2 (*_X d_X \chi)}_{\text{type } (2,0)} + t^2 [(1, 1)] + t^4 [(2, 0)]. \end{aligned}$$

Computing $\varphi_t \wedge (*_t d\varphi_t)$ and $\psi_t \wedge (*_t d\psi_t)$ using (5.10) and keeping track of types, we find

$$\begin{aligned} \varphi_t \wedge (*_t d\varphi_t) &= t^4 \underbrace{\left(\sum_{i,j=1}^3 \omega_i \wedge (*_X d_X \xi_j) \wedge e_i \wedge e_j \right)}_{\text{type } (4,2)} + t^4[(3,3)], \\ \psi_t \wedge (*_t d\psi_t) &= -t^4 \underbrace{\left(\sum_{\substack{i,j,k \\ \text{cyclic}}} (*_X d_X \theta_k) \wedge \text{vol}_X \wedge e_i \wedge e_j \right)}_{\text{type } (4,2)} \\ &\quad + t^4 \underbrace{\left(\sum_{\substack{i,j,k \\ \text{cyclic}}} (*_X d_X \chi) \wedge \omega_k \wedge e_i \wedge e_j \right)}_{\text{type } (4,2)} + t^6[(4,2)] + t^4[(3,3)]. \end{aligned}$$

Equating the terms of the form $t^4[(4,2)]$ above, the fundamental relation (5.16) yields the three equations

$$\begin{aligned} \omega_2 \wedge (*_X d_X \xi_3) - \omega_3 \wedge (*_X d_X \xi_2) &= -(*_X d_X \theta_1) \wedge \text{vol}_X + (*_X d_X \chi) \wedge \omega_1, \\ \omega_3 \wedge (*_X d_X \xi_1) - \omega_1 \wedge (*_X d_X \xi_3) &= -(*_X d_X \theta_2) \wedge \text{vol}_X + (*_X d_X \chi) \wedge \omega_2, \\ \omega_1 \wedge (*_X d_X \xi_2) - \omega_2 \wedge (*_X d_X \xi_1) &= -(*_X d_X \theta_3) \wedge \text{vol}_X + (*_X d_X \chi) \wedge \omega_3. \end{aligned}$$

Applying (5.9) now gives (5.15) as claimed. q.e.d.

We will also need explicit formulas for $D_{\varphi_t} \Theta$ acting on forms of type (3,0) and (1,2).

Lemma 5.4. *Let π_7 be the projection $\pi_7 : \Omega^3 \rightarrow \Omega_7^3$ with respect to the G_2 -structure φ_t of (5.10). If $\gamma_{3,0} = \eta$ is a vertical 3-form, then*

$$(5.17) \quad \pi_7(\gamma_{3,0}) = \frac{1}{4}\eta - \frac{t^{-2}}{4}((J_1 *_X \eta) \wedge e_2 \wedge e_3 + (J_2 *_X \eta) \wedge e_3 \wedge e_1 + (J_3 *_X \eta) \wedge e_1 \wedge e_2).$$

If $\gamma_{1,2} = \zeta_1 \wedge e_2 \wedge e_3 + \zeta_2 \wedge e_3 \wedge e_1 + \zeta_3 \wedge e_1 \wedge e_2$ where the ζ_k 's are vertical 1-forms, then

$$(5.18) \quad \begin{aligned} \pi_7(\gamma_{1,2}) &= \\ &= -\frac{t^2}{4}((*_X J_1 \zeta_1) + (*_X J_2 \zeta_2) + (*_X J_3 \zeta_3)) + \frac{1}{4}(\zeta_1 + J_3 \zeta_2 - J_2 \zeta_3) \wedge e_2 \wedge e_3 \\ &\quad + \frac{1}{4}(-J_3 \zeta_1 + \zeta_2 + J_1 \zeta_3) \wedge e_3 \wedge e_1 + \frac{1}{4}(J_2 \zeta_1 - J_1 \zeta_2 + \zeta_3) \wedge e_1 \wedge e_2. \end{aligned}$$

Proof. The G_2 -structure φ_t satisfies

$$(5.19) \quad *_t(\varphi_t \wedge *_t(\varphi_t \wedge \alpha)) = -4\alpha \quad \text{for any 1-form } \alpha.$$

This identity in fact holds for any G_2 -structure [22, Proposition A.3]. (Note that even though [22] uses the opposite orientation convention, the identity (5.19) is independent of the choice of orientation convention.) If $\gamma \in \Omega^3$, then $\pi_7\gamma = *_t(\varphi_t \wedge \alpha)$ for some 1-form α . Since $\varphi_t \wedge \gamma = \varphi_t \wedge \pi_7\gamma$, applying equation (5.19) yields

$$\pi_7\gamma = -\frac{1}{4} *_t(\varphi_t \wedge *_t(\varphi_t \wedge \gamma)).$$

Let $\gamma_{3,0} = \eta$ be a vertical 3-form. Then (5.10) and (5.11) give

$$(5.20) \quad *_t(\varphi_t \wedge \gamma_{3,0}) = t^{-2} *_X \eta.$$

Similarly if $\gamma_{1,2} = \zeta_1 \wedge e_2 \wedge e_3 + \zeta_2 \wedge e_3 \wedge e_1 + \zeta_3 \wedge e_1 \wedge e_2$ where the ζ_k 's are vertical 1-forms, we obtain

$$(5.21) \quad *_t(\varphi_t \wedge \gamma_{1,2}) = -*_X(\omega_1 \wedge \zeta_1 + \omega_2 \wedge \zeta_2 + \omega_3 \wedge \zeta_3) = J_1\zeta_1 + J_2\zeta_2 + J_3\zeta_3,$$

where we have also used (2.12). Note that the right hand sides of both (5.20) and (5.21) are forms of type (1,0). If β is of type (1,0), then again using (5.11) we compute

$$\begin{aligned} *_t(\varphi_t \wedge \beta) &= *_t(e_1 \wedge e_2 \wedge e_3 \wedge \beta - t^2(\omega_1 \wedge e_1 + \omega_2 \wedge e_2 + \omega_3 \wedge e_3) \wedge \beta) \\ &= t^2(*_X\beta) - *_X(\omega_1 \wedge \beta) \wedge e_2 \wedge e_3 \\ &\quad - *_X(\omega_2 \wedge \beta) \wedge e_3 \wedge e_1 - *_X(\omega_3 \wedge \beta) \wedge e_1 \wedge e_2 \\ &= t^2(*_X\beta) + (J_1\beta) \wedge e_2 \wedge e_3 + (J_2\beta) \wedge e_3 \wedge e_1 \\ &\quad + (J_3\beta) \wedge e_1 \wedge e_2. \end{aligned}$$

By (5.20) and (5.19), to compute $\pi_7(\gamma_{3,0})$ we substitute $\beta = -\frac{1}{4}t^{-2} *_X \eta$ into the above expression. This yields (5.17), since $*_X^2\eta = -\eta$. Similarly, by (5.21) and (5.19), to compute $\pi_7(\gamma_{1,2})$ we substitute $\beta = -\frac{1}{4}(J_1\zeta_1 + J_2\zeta_2 + J_3\zeta_3)$ into the above expression. This yields (5.18), where we have also used (2.10). q.e.d.

Corollary 5.5. *Let $D_{\varphi_t}\Theta$ be the linearization of Θ at φ_t as in (2.6). If $\gamma_{3,0} = \eta$ is a vertical 3-form, then*

$$(5.22) \quad (D_{\varphi_t}\Theta)(\gamma_{3,0}) = -\frac{t^{-2}}{2}(*_X\eta) \wedge e_1 \wedge e_2 \wedge e_3 \\ + \frac{1}{2}((J_1\eta) \wedge e_1 + (J_2\eta) \wedge e_2 + (J_3\eta) \wedge e_3).$$

If $\gamma_{1,2} = \zeta_1 \wedge e_2 \wedge e_3 + \zeta_2 \wedge e_3 \wedge e_1 + \zeta_3 \wedge e_1 \wedge e_2$ where the ζ_k 's are vertical 1-forms, then

$$(5.23) \quad (D_{\varphi_t}\Theta)(\gamma_{1,2}) = \\ \frac{1}{2}(J_1\zeta_1 + J_2\zeta_2 + J_3\zeta_3) \wedge e_1 \wedge e_2 \wedge e_3 + \frac{t^2}{2} *_X(-\zeta_1 + J_3\zeta_2 - J_2\zeta_3) \wedge e_1 \\ + \frac{t^2}{2} *_X(-J_3\zeta_1 - \zeta_2 + J_1\zeta_3) \wedge e_2 + \frac{t^2}{2} *_X(J_2\zeta_1 - J_1\zeta_2 - \zeta_3) \wedge e_3.$$

Proof. From (2.6) we get $D_{\varphi_t}\Theta = \frac{7}{3}*_t\pi_1 + 2*_t\pi_7 - *_t$. If $\gamma \in \Omega^3$, then $\pi_1\gamma = f\varphi_t$ for some function f . Then $\gamma \wedge \psi_t = (\pi_1\gamma) \wedge \psi_t = 7f \text{vol}_t$, so $\pi_1\gamma = \frac{1}{7}*_t(\gamma \wedge \psi_t)$. Since ψ_t is of type $(4,0) + (2,2)$, it follows that $\pi_1 = 0$ on forms of type $(3,0) + (1,2)$. Hence $D_{\varphi_t}\Theta = 2*_t\pi_7 - *_t$ on forms of type $(3,0) + (1,2)$. Equations (5.22) and (5.23) now follow from (5.17) and (5.18) using (5.11) and the fact that $*_x$ commutes with J_1, J_2 , and J_3 . q.e.d.

Proposition 5.6. *Let $\alpha_{0,2}$ and $\alpha_{2,0}$ be $(0,2)$ and $(2,0)$ forms, respectively. Then the 5-form*

$$(5.24) \quad v = d_X[(D_{\varphi_t}\Theta)(t^2 d_X \alpha_{0,2} + t^4 d_X \alpha_{2,0} + t^2 \xi_{1,2}) - t^2 \chi_{1,3} - t^4 \theta_{3,1}]$$

lies in Ω_{14}^5 with respect to φ_t .

Proof. We need to show that $\psi_t \wedge (*_t v) = 0$. We write

$$\alpha_{0,2} = a_1 e_2 \wedge e_3 + a_2 e_3 \wedge e_1 + a_3 e_1 \wedge e_2, \quad \alpha_{2,0} = b,$$

where a_1, a_2, a_3 are functions and b is a vertical 2-form. Then we have

$$\begin{aligned} d_X \alpha_{0,2} &= [d\alpha_{0,2}]_{1,2} \\ &= (d_X a_1) \wedge e_2 \wedge e_3 + (d_X a_2) \wedge e_3 \wedge e_1 + (d_X a_3) \wedge e_1 \wedge e_2, \\ (5.25) \quad d_X \alpha_{2,0} &= [d\alpha_{2,0}]_{3,0} = d_X b. \end{aligned}$$

For $k = 1, 2, 3$, let

$$(5.26) \quad \zeta_k = d_X a_k + \xi_k,$$

where ξ_k was defined in (5.13). Using the expressions in (5.25) and (5.13) we compute using Corollary 5.5 that

$$\begin{aligned} &(D_{\varphi_t}\Theta)(t^2 d_X \alpha_{0,2} + t^4 d_X \alpha_{2,0} + t^2 \xi_{1,2}) \\ &= (D_{\varphi_t}\Theta) \left(t^2 \sum_{\substack{i,j,k \\ \text{cyclic}}} \zeta_k \wedge e_i \wedge e_k + t^4 (d_X b) \right) \\ &= \frac{t^2}{2} (J_1 \zeta_1 + J_2 \zeta_2 + J_3 \zeta_3) \wedge e_1 \wedge e_2 \wedge e_3 \\ &\quad + \frac{t^4}{2} \sum_{i,j,k \text{ cyclic}} *_x (-\zeta_i + J_k \zeta_j - J_j \zeta_k) \wedge e_i \\ &\quad - \frac{t^2}{2} (*_x d_X b) \wedge e_1 \wedge e_2 \wedge e_3 \\ &\quad + \frac{t^4}{2} ((J_1 d_X b) \wedge e_1 + (J_2 d_X b) \wedge e_2 + (J_3 d_X b) \wedge e_3). \end{aligned}$$

Collecting terms above and using (5.13) to express $\chi_{1,3}$ and $\theta_{3,1}$ in terms of $\chi, \theta_1, \theta_2, \theta_3$, we obtain

$$(5.27) \quad \begin{aligned} & (D_{\varphi_t} \Theta)(t^2 d_X \alpha_{0,2} + t^4 d_X \alpha_{2,0} + t^2 \xi_{1,2}) - t^2 \chi_{1,3} - t^4 \theta_{3,1} \\ &= \frac{t^2}{2} A \wedge e_1 \wedge e_2 \wedge e_3 + \frac{t^4}{2} \sum_{k=1}^3 B_k \wedge e_k, \end{aligned}$$

where

$$(5.28) \quad \begin{aligned} A &= J_1 \zeta_1 + J_2 \zeta_2 + J_3 \zeta_3 - *_X d_X b - 2\chi, \\ B_1 &= *_X (-\zeta_1 + J_3 \zeta_2 - J_2 \zeta_3) + J_1 d_X b - 2\theta_1, \\ B_2 &= *_X (-\zeta_2 + J_1 \zeta_3 - J_3 \zeta_1) + J_2 d_X b - 2\theta_2, \\ B_3 &= *_X (-\zeta_3 + J_2 \zeta_1 - J_1 \zeta_2) + J_3 d_X b - 2\theta_3. \end{aligned}$$

Note that A is a vertical 1-form and each B_k is a vertical 3-form. By (5.24), the 5-form v is d_X of (5.27). Thus we have

$$v = \frac{t^2}{2} (d_X A) \wedge e_1 \wedge e_2 \wedge e_3 + \frac{t^4}{2} \sum_{k=1}^3 (d_X B_k) \wedge e_k,$$

and therefore using (5.11) and (5.10), we find

$$\begin{aligned} *_t v &= \frac{t^2}{2} (*_X d_X A) + \frac{1}{2} (*_X d_X B_1) \wedge e_2 \wedge e_3 \\ &\quad + \frac{1}{2} (*_X d_X B_2) \wedge e_3 \wedge e_1 + \frac{1}{2} (*_X d_X B_3) \wedge e_1 \wedge e_2, \\ \psi_t \wedge (*_t v) &= \frac{t^4}{2} \sum_{\substack{i,j,k \\ \text{cyclic}}} (*_X d_X B_k) \wedge \text{vol}_X \wedge e_i \wedge e_j \\ &\quad - \frac{t^4}{2} \sum_{\substack{i,j,k \\ \text{cyclic}}} (*_X d_X A) \wedge \omega_k \wedge e_i \wedge e_j. \end{aligned}$$

Hence, using (5.9) we conclude that

$$\psi_t \wedge (*_t v) = \frac{t^4}{2} \sum_{\substack{i,j,k \\ \text{cyclic}}} (d_X^* (*_X B_k) + d_X^* (J_k A)) \text{vol}_X \wedge e_i \wedge e_j.$$

Thus, we find that v is in Ω_{14}^5 with respect to φ_t if and only if

$$(5.29) \quad d_X^* (*_X B_k + J_k A) = 0 \quad \text{for all } k = 1, 2, 3.$$

Substituting the expressions (5.28) for A, B_1, B_2, B_3 we find that

$$\begin{aligned} *_X B_1 + J_1 A &= \zeta_1 - J_3 \zeta_2 + J_2 \zeta_3 + J_1 *_X d_X b - 2 *_X \theta_1 \\ &\quad - \zeta_1 - J_3 \zeta_2 + J_2 \zeta_3 - J_1 *_X d_X b - 2 J_1 \chi \\ &= 2 J_2 \zeta_3 - 2 J_3 \zeta_2 - 2 *_X \theta_1 - 2 J_1 \chi, \end{aligned}$$

and cyclic permutations of this. Cancelling the factor of 2, the conditions (5.29) become

$$d_X^*(J_2\zeta_3 - J_3\zeta_2 - *_X\theta_1 - J_1\chi) = 0,$$

and cyclic permutations of this. Recall that $\zeta_k = d_X a_k + \xi_k$. From Lemma 5.2, we have $d_X^*(J_i d_X a_j) = 0$ for any i, j and thus the conditions above become

$$d_X^*(J_2\xi_3 - J_3\xi_2 - *_X\theta_1 - J_1\chi) = 0,$$

$$d_X^*(J_3\xi_1 - J_1\xi_3 - *_X\theta_2 - J_2\chi) = 0,$$

$$d_X^*(J_1\xi_2 - J_2\xi_1 - *_X\theta_3 - J_3\chi) = 0.$$

But these are precisely the conditions (5.15) of Proposition 5.3. Thus the proof is complete. q.e.d.

5.3. Analytic results on the Eguchi–Hanson fibre $P_x \cong X$. Before we can give the proof of Theorem 5.1, in this section we collect some analytic results about the Eguchi–Hanson fibre $P_x \cong X$. First, the Riemannian manifold (X, g_X) is an example of an *asymptotically conical* (AC) Riemannian manifold, whose asymptotic cone is the cone over \mathbb{RP}^3 . More precisely, it is an *asymptotically locally Euclidean* (ALE) Riemannian manifold, because the asymptotic cone $\mathbb{R}^4/\{\pm 1\}$ is flat. We begin by stating some general results that are valid for any AC manifold, specialized to the case of dimension 4. Then we use these facts to establish results that require the stronger ALE property, as well as the specific topology of X . Finally the last result we collect in this section uses the fact that (X, g_X) is also hyperKähler.

On an AC manifold (X, g_X) , there exist reasonably nice *Hodge-theoretic* results for forms with appropriate decay at infinity. The results we use are from Lockhart [30] and Lockhart–McOwen [31]. We will not give detailed definitions of the weighted Sobolev spaces that we use, but rather only list the results we will need. A comprehensive summary of this theory using the same notation that we use here can be found in [25, Section 4]. Although [25] is written for dimension 7, it is a simple matter to translate the results to dimension 4.

Specifically, the Lockhart–McOwen facts we will need are the following. Here all norms and covariant derivatives are with respect to the metric g_X on X . Let $L_{k,\lambda}^2(E)$ denote the *weighted Sobolev space* of sections of a tensor bundle E over X with rate λ . When $E = \Lambda^p T^*X$ is the bundle of p -forms on X , we will abbreviate our notation and write

$$\Omega_{k,\lambda}^p = L_{k,\lambda}^2(\Lambda^p T^*X).$$

Similarly we write

$$(\Omega_{\pm}^2)_{k,\lambda} = L_{k,\lambda}^2(\Lambda_{\pm}^2 T^*X)$$

for the corresponding weighted Sobolev spaces of self-dual (anti-self-dual) 2-forms on X .

- (LM1) Let r be the function on $X \setminus Y$ from §2.5. A smooth section f of E that lies in $L^2_{k,\lambda}(E)$ satisfies $|\nabla^k f| = O(r^{\lambda-k})$ as $r \rightarrow \infty$ for all $k \geq 0$. Conversely, a smooth section f of E such that $|\nabla^k f| = O(r^{\lambda-k})$ for all $k \geq 0$ as $r \rightarrow \infty$ lies in $L^2_{k,\lambda+\gamma}(E)$ for any $k \geq 0$ and any $\gamma > 0$. Moreover, we have $L^2_{k,\lambda}(E) \subseteq L^2_{k,\lambda'}(E)$ if $\lambda \leq \lambda'$.
- (LM2) For any rate λ , and any $k \geq 0$, the Laplacian Δ_X on $\Lambda^0 T^*X$ induces a continuous linear map

$$(\Delta_X)_{k+2,\lambda} : \Omega^0_{k+2,\lambda} \rightarrow \Omega^0_{k,\lambda-2},$$

and for generic (“noncritical”) rates, this map is Fredholm. At such rates, we have

$$\text{Coker}(\Delta_X)_{k+2,\lambda} \cong \text{Ker}(\Delta_X)_{k,-2-\lambda}.$$

Moreover, by elliptic regularity if $(\Delta_X)_{k+2,\lambda} f$ is smooth, then f is.

- (LM3) In dimension 4 the rate $\lambda = -2$ is always critical for the Laplacian on functions, but there are no critical rates in the interval $(-2, 0)$. Take $\gamma \in (0, 2)$ so that $\lambda = -2 + \gamma$ is noncritical. Then by (LM2), the cokernel of $(\Delta_X)_{k+2,-2+\gamma}$ is isomorphic to the kernel of $(\Delta_X)_{k,-\gamma}$. By elliptic regularity, elements in this kernel are smooth, and since by (LM1) they are $O(r^{-\gamma})$ as $r \rightarrow \infty$ with $\gamma > 0$, we conclude by the maximum principle that the kernel is trivial. Similarly $\text{Ker}(\Delta_X)_{-2+\gamma} = 0$ because $\gamma < 2$. Therefore, for $\gamma \in (0, 2)$ the map

$$(\Delta_X)_{k+2,-2+\gamma} : \Omega^0_{k+2,-2+\gamma} \rightarrow \Omega^0_{k,-4+\gamma}$$

is an isomorphism.

- (LM4) The maps d_X and d_X^* on X induce continuous linear maps

$$(d_X)_{k+1,\lambda} : \Omega^p_{k+1,\lambda} \rightarrow \Omega^{p+1}_{k,\lambda-1}, \quad (d_X^*)_{k+1,\lambda} : \Omega^{p-1}_{k+1,\lambda} \rightarrow \Omega^{p-1}_{k,\lambda-1},$$

for any rate λ and any $k \geq 0$.

- (LM5) Elements of $\Omega^p_{k,\lambda}$ for $\lambda < -2$ are in the usual L^2 space. In particular, by (LM4) the spaces $d_X(\Omega^{p-1}_{k+1,\lambda+1})$ and $d_X^*(\Omega^{p+1}_{k+1,\lambda+1})$ are L^2 -orthogonal subspaces of $\Omega^p_{k,\lambda}$ if $\lambda < -2$.
- (LM6) If $w \in \Omega^1_{k+1,\lambda+1}$ is smooth, then $w \wedge d_X w$ is a 3-form that is $O(r^{2\lambda+1})$ as $r \rightarrow \infty$. If $2\lambda+1+3 < 0$, equivalently $\lambda < -2$, then we can apply Stokes’ Theorem to the exact 4-form $d_X w \wedge d_X w$ to deduce that $\int_X (d_X w \wedge d_X w) = 0$. See, for example, the proof of [25, Lem. 4.68].
- (LM7) Define the operator $\mathcal{D}_X : \Omega^0(X) \oplus \Omega^2_+(X) \rightarrow \Omega^1(X)$ by $\mathcal{D}_X(f, \beta) = d_X f + d_X^* \beta$. This operator is elliptic, and has formal adjoint $\mathcal{D}_X^* : \Omega^1(X) \rightarrow \Omega^0(X) \oplus \Omega^2_+(X)$ given by $\mathcal{D}_X^* \alpha = (d_X^* \alpha, \pi_+ d_X \alpha)$.

For any rate λ , and any $k \geq 0$, it induces a continuous linear map

$$(\mathcal{D}_X)_{k+1,\lambda} : \Omega_{k+1,\lambda}^0 \oplus (\Omega_+^2)_{k+1,\lambda} \rightarrow \Omega_{k,\lambda-1}^1,$$

and for generic (noncritical) rates, this map is Fredholm. At such rates, we have

$$\operatorname{Coker}(\mathcal{D}_X)_{k+1,\lambda} \cong \operatorname{Ker}(\mathcal{D}_X^*)_{k,-3-\lambda}.$$

Moreover, by elliptic regularity if $(\mathcal{D}_X)_{k+1,\lambda}(f, \beta)$ is smooth, then (f, β) is smooth.

- (LM8) For the operators $(\Delta_X)_{k+2,\lambda}$, $(\mathcal{D}_X)_{k+1,\lambda}$, and $(\mathcal{D}_X^*)_{k+1,\lambda}$ of (LM2) and (LM7), the kernel or cokernel can only change as we vary λ if we cross a critical rate. The critical rates are those for which there exist nontrivial sections, which are homogeneous of order λ with respect to dilations, in the kernel of the corresponding operators on the cone.
- (LM9) Let $\mathcal{H}_\lambda^p = \{\alpha \in \Omega_{k,\lambda}^p : d_X \alpha = 0 \text{ and } d_X^* \alpha = 0\}$ which by elliptic regularity consists of smooth elements so is independent of k . Then we have

$$\mathcal{H}_{-2}^1 \cong H_{\text{cs}}^1(X), \quad \mathcal{H}_{-2}^2 \cong \operatorname{Im}(H_{\text{cs}}^2(X) \rightarrow H^2(X)), \quad \mathcal{H}_{-2}^3 \cong H^3(X),$$

where $H^p(X)$ and $H_{\text{cs}}^p(X)$ denote the p^{th} de Rham cohomology and p^{th} compactly supported de Rham cohomology of X , respectively. The map $H_{\text{cs}}^2(X) \rightarrow H^2(X)$ is the obvious one. See [32, Th. 6.5.2] for a more general statement.

Corollary 5.7. *Let (X, g_X) be an AC Riemannian manifold of dimension 4. Let $w \in \Omega_{k+1,\lambda+1}^1$ such that $d_X w \in (\Omega_-^2)_{k,\lambda}$. If $\lambda < -2$ then $d_X w = 0$.*

Proof. By hypothesis we have

$$(5.30) \qquad \qquad \qquad *_X d_X w = -d_X w.$$

By (LM5), the 2-form $d_X w$ is in L^2 . Moreover, we can apply (LM6) and (5.30) to deduce that

$$\|d_X w\|_{L^2} = \int_X d_X w \wedge *_X d_X w = - \int_X d_X w \wedge d_X w = 0$$

as claimed. q.e.d.

Lemma 5.8. *Let (X, g_X) be the Eguchi–Hanson space. There are no critical rates for the operator \mathcal{D}_X^* in the interval $[-2, 0]$.*

Proof. By (LM8), we must show that there do not exist any nonzero elements homogeneous of order $\lambda \in [-2, 0]$ in the kernel of the corre-

sponding operator on the cone. Let α be such a 1-form on the cone, homogeneous of order λ . Then $d^*\alpha = 0$ and $\pi_+d\alpha = 0$. Thus $dd^*\alpha = 0$ and $d^*d\alpha = -d^* * d\alpha = 0$. So α is a harmonic 1-form. The harmonic 1-forms on the cone $\mathbb{R}^4/\{\pm 1\}$ correspond to the $\{\pm 1\}$ -invariant harmonic 1-forms on \mathbb{R}^4 . But \mathbb{R}^4 admits a global trivialization by parallel 1-forms. So any 1-form on \mathbb{R}^4 is of the form $\alpha = \sum_{j=1}^4 f_j dx_j$ and $\Delta\alpha = \sum_{j=1}^4 (\Delta f_j) dx_j$. Hence the critical rates of \mathcal{D}_X^* are a subset of the critical rates for Δ on functions on \mathbb{R}^4 , which is also a cone over S^3 . By (LM3), there are no critical rates in $(-2, 0)$ for the Laplacian on functions, hence none for \mathcal{D}_X^* in that interval.

It remains to consider the end points $\lambda = -2, 0$. It is easy to check that the only harmonic functions on a metric cone in dimension 4 that are homogeneous of order $\lambda = -2$ or $\lambda = 0$ are of the form cr^λ for $c \in \mathbb{R}$. Therefore the only harmonic 1-forms on \mathbb{R}^4 that are homogeneous of order $\lambda = -2$ or $\lambda = 0$ will be of the form $r^\lambda v$ for some parallel 1-form v . But the nonzero parallel 1-forms on \mathbb{R}^4 are not $\{\pm 1\}$ -invariant, hence they do not descend to $\mathbb{R}^4/\{\pm 1\}$. Thus $\lambda = -2$ and $\lambda = 0$ are also not critical rates for \mathcal{D}_X^* . q.e.d.

Proposition 5.9. *Let (X, g_X) be the Eguchi–Hanson space. For any $k \geq 0$ and for $\gamma > 0$ sufficiently small, we have*

$$\begin{aligned}\Omega_{k,-3+\gamma}^1 &= d_X(\Omega_{k+1,-2+\gamma}^0) \oplus d_X^*((\Omega_+^2)_{k+1,-2+\gamma}), \\ \Omega_{k,-3+\gamma}^3 &= d_X^*(\Omega_{k+1,-2+\gamma}^4) \oplus d_X((\Omega_+^2)_{k+1,-2+\gamma}),\end{aligned}$$

where in both cases the summands are L^2 -orthogonal. Moreover, any smooth closed form in $\Omega_{k,-3+\gamma}^1$ or $\Omega_{k,-3+\gamma}^3$ is necessarily d_X of a smooth form in $\Omega_{k+1,-2+\gamma}^0$ or $(\Omega_+^2)_{k+1,-2+\gamma}$, respectively.

Proof. Consider (LM7) for rate $\lambda = -2 + \gamma$. For $\gamma > 0$ sufficiently small, this rate will be noncritical, and $\text{Coker}(\mathcal{D}_X)_{k+1,-2+\gamma} \cong \text{Ker}(\mathcal{D}_X^*)_{k,-1-\gamma}$. By Lemma 5.8 and (LM8), we have that

$$\text{Ker}(\mathcal{D}_X^*)_{k,-1-\gamma} = \text{Ker}(\mathcal{D}_X^*)_{k,-2-\epsilon}$$

for some $\epsilon > 0$. Let $\alpha \in \text{Ker}(\mathcal{D}_X^*)_{k,-2-\epsilon}$. Then $d_X^*\alpha = 0$ and $\pi_+d_X\alpha = 0$. By Corollary 5.7 we have $d_X\alpha = 0$. Since $-2 - \epsilon < -2$, we deduce that $\alpha \in \mathcal{H}_{-2}^1$. By (LM9), in this case $\mathcal{H}_{-2}^1 \cong H_{\text{cs}}^1(X) = \{0\}$, because $X \cong T^*\mathcal{S}^2$. Hence, $\text{Coker}(\mathcal{D}_X)_{k+1,-2+\gamma} = \{0\}$, and therefore

$$\Omega_{k,-3+\gamma}^1 = d_X(\Omega_{k+1,-2+\gamma}^0) + d_X^*((\Omega_+^2)_{k+1,-2+\gamma}),$$

and the sum is direct by (LM5). This establishes the first decomposition. The second decomposition is obtained by applying $*_X$ to both sides of the first one.

For either of the two decompositions, suppose that $\alpha = d_X \sigma + d_X^* \tau$ is closed. Then $d_X d_X^* \tau = 0$, and at this rate it lies in L^2 . Thus $d_X^* \tau = 0$, so $\alpha = d_X \sigma$ is exact. The final statement about smoothness follows from the elliptic regularity remark in (LM7). q.e.d.

We close this section with a useful observation. The space (X, g_X) is also hyperKähler, so it is equipped with an orthonormal triple $\omega_1, \omega_2, \omega_3$ of *parallel* self-dual 2-forms. It is a standard fact that the curvature operator is anti-self-dual, since the Ricci curvature vanishes and the bundle $\Lambda_+^2(T^*X)$ admits a parallel trivialization. This in turn implies that the curvature term in the Weitzenböck formula on 2-forms vanishes, so $\Delta_d = \nabla^* \nabla$ on 2-forms. It then follows, because any self-dual 2-form η can be written as $\eta = f_1 \omega_1 + f_2 \omega_2 + f_3 \omega_3$ for some functions f_1, f_2, f_3 and the ω_i 's are parallel, that

$$(5.31) \quad \Delta_X \eta = \Delta_X (f_1 \omega_1 + f_2 \omega_2 + f_3 \omega_3) = (\Delta_X f_1) \omega_1 + (\Delta_X f_2) \omega_2 + (\Delta_X f_3) \omega_3$$

for any self-dual 2-form η on X .

We will use this fact in the proof of Theorem 5.1 in §5.4.

5.4. Proof of the correction theorem. We are now ready to give the proof of Theorem 5.1. We break the proof into three steps.

Step One. Using the expression for g_t in (5.10), the expressions (5.13) for $\xi_{1,2}, \chi_{1,3}, \theta_{3,1}$, and the estimates (4.35), (4.37), (4.38) we find that with respect to the metric g_X on X ,

$$(5.32) \quad |\nabla^k \xi_i| = O(r^{-3-k}), \quad |\nabla^k \chi| = O(r^{-3-k}), \quad |\nabla^k \theta_i| = 0,$$

for $r \geq 1$, for all $k \geq 0$ and $i = 1, 2, 3$. In particular, by (LM1) we deduce that

$$(5.33) \quad \xi_1, \xi_2, \xi_3, \chi \in \Omega_{k,-3+\gamma}^1, \quad \theta_1, \theta_2, \theta_3 \in \Omega_{k,-3+\gamma}^3,$$

for any $\gamma > 0$ and any $k \geq 0$.

In order to simplify notation we define

$$\begin{aligned} \tilde{A} &= \frac{1}{2}(-J_1 \xi_1 - J_2 \xi_2 - J_3 \xi_3) + \chi, & \tilde{B}_1 &= \frac{1}{2} *_X (\xi_1 - J_3 \xi_2 + J_2 \xi_3) + \theta_1, \\ \tilde{B}_2 &= \frac{1}{2} *_X (\xi_2 - J_1 \xi_3 + J_3 \xi_1) + \theta_2, & \tilde{B}_3 &= \frac{1}{2} *_X (\xi_3 - J_2 \xi_1 + J_1 \xi_2) + \theta_3, \end{aligned}$$

where we have used tildes to avoid confusion with the similar but different (5.28). Note that \tilde{A} is a vertical 1-form, and the \tilde{B}_i 's are vertical 3-forms on P . Since $*_X$ and the J_i 's are isometries, it follows from (5.33) that

$$(5.34) \quad \tilde{A}, *_X \tilde{B}_1, *_X \tilde{B}_2, *_X \tilde{B}_3 \in \Omega_{k,-3+\gamma}^1,$$

for any $\gamma > 0$ and all $k \geq 0$.

Define on P the following 2-form:

$$(5.35) \quad \vartheta = - *_t d_X [(D_{\varphi_t} \Theta)(t^2 \xi_{1,2}) - t^2 \chi_{1,3} - t^4 \theta_{3,1}] + t^2 (d_X \tilde{A}).$$

We will rewrite expression (5.35) as follows. Using the expressions (5.13) for $\xi_{1,2}$, $\chi_{1,3}$, and $\theta_{3,1}$, and equation (5.23) to compute $(D_{\varphi_t} \Theta)(t^2 \xi_{1,2})$, we find that

$$\begin{aligned} & (D_{\varphi_t} \Theta)(t^2 \xi_{1,2}) - t^2 \chi_{1,3} - t^4 \theta_{3,1} \\ &= \frac{t^2}{2} (J_1 \xi_1 + J_2 \xi_2 + J_3 \xi_3 - 2\chi) \wedge e_1 \wedge e_2 \wedge e_3 \\ & \quad + \frac{t^4}{2} (*_X(-\xi_1 + J_3 \xi_2 - J_2 \xi_3) - 2\theta_1) \wedge e_1 \\ & \quad + \frac{t^4}{2} (*_X(-J_3 \xi_1 - \xi_2 + J_1 \xi_3) - 2\theta_2) \wedge e_2 \\ & \quad + \frac{t^4}{2} (*_X(J_2 \xi_1 - J_1 \xi_2 - \xi_3) - 2\theta_3) \wedge e_3. \end{aligned}$$

Thus we have

$$\begin{aligned} (D_{\varphi_t} \Theta)(t^2 \xi_{1,2}) - t^2 \chi_{1,3} - t^4 \theta_{3,1} &= -t^2 \tilde{A} \wedge e_1 \wedge e_2 \wedge e_3 \\ & \quad - t^4 (\tilde{B}_1 \wedge e_1 + \tilde{B}_2 \wedge e_2 + \tilde{B}_3 \wedge e_3) \end{aligned}$$

and hence

$$\begin{aligned} & d_X [(D_{\varphi_t} \Theta)(t^2 \xi_{1,2}) - t^2 \chi_{1,3} - t^4 \theta_{3,1}] \\ &= -t^2 (d_X \tilde{A}) \wedge e_1 \wedge e_2 \wedge e_3 \\ & \quad - t^4 ((d_X \tilde{B}_1) \wedge e_1 + (d_X \tilde{B}_2) \wedge e_2 + (d_X \tilde{B}_3) \wedge e_3). \end{aligned}$$

Applying (5.11) again, we deduce that the expression (5.35) can be written as

$$\begin{aligned} & \vartheta = - *_t d_X [(D_{\varphi_t} \Theta)(t^2 \xi_{1,2}) - t^2 \chi_{1,3} - t^4 \theta_{3,1}] + t^2 (d_X \tilde{A}) \\ (5.36) \quad &= t^2 (d_X \tilde{A} + *_X d_X \tilde{A}) + (*_X d_X \tilde{B}_1) \wedge e_2 \wedge e_3 \\ & \quad + (*_X d_X \tilde{B}_2) \wedge e_3 \wedge e_1 + (*_X d_X \tilde{B}_3) \wedge e_1 \wedge e_2. \end{aligned}$$

Now on P consider the following equation:

$$(5.37) \quad (d_X *_t d_X *_t - *_t d_X *_t d_X)(t^2 \alpha_{0,2} + t^4 \alpha_{2,0}) = \vartheta.$$

If, in equation (5.37), the d_X 's were replaced by the full exterior derivative d on P , then the left hand side would be $\Delta_t = d *_t d *_t - *_t d *_t d$, the Laplacian with respect to g_t on 2-forms. Thus, we expect (5.37) to give Laplace equations on each fibre P_x . Explicitly, as before we write

$$(5.38) \quad \alpha_{0,2} = a_1 e_2 \wedge e_3 + a_2 e_3 \wedge e_1 + a_3 e_1 \wedge e_2, \quad \alpha_{2,0} = b,$$

where a_1, a_2, a_3 are functions and b is a vertical 2-form on P . Using (5.11) repeatedly we compute

$$\begin{aligned}
d_X \alpha_{0,2} &= (d_X a_1) \wedge e_2 \wedge e_3 + (d_X a_2) \wedge e_3 \wedge e_1 + (d_X a_3) \wedge e_1 \wedge e_2, \\
d_X \alpha_{2,0} &= d_X b, \\
*_t d_X \alpha_{0,2} &= t^2 ((*_X d_X a_1) \wedge e_1 + (*_X d_X a_2) \wedge e_2 + (*_X d_X a_3) \wedge e_3), \\
*_t d_X \alpha_{2,0} &= t^{-2} (*_X d_X b) \wedge e_1 \wedge e_2 \wedge e_3, \\
d_X *_t d_X \alpha_{0,2} &= t^2 ((d_X *_X d_X a_1) \wedge e_1 + (d_X *_X d_X a_2) \wedge e_2 \\
&\quad + (d_X *_X d_X a_3) \wedge e_3), \\
d_X *_t d_X \alpha_{2,0} &= t^{-2} (d_X *_X d_X b) \wedge e_1 \wedge e_2 \wedge e_3,
\end{aligned}$$

and hence we find

$$\begin{aligned}
*_t d_X *_t d_X \alpha_{0,2} &= t^{-2} ((*_X d_X *_X d_X a_1) \wedge e_2 \wedge e_3 + (*_X d_X *_X d_X a_2) \wedge e_3 \wedge e_1 \\
&\quad + (*_X d_X *_X d_X a_3) \wedge e_1 \wedge e_2) \\
(5.39) \quad &= -t^{-2} (d_X^* d_X a_1) \wedge e_2 \wedge e_3 - t^{-2} (d_X^* d_X a_2) \wedge e_3 \wedge e_1 \\
&\quad - t^{-2} (d_X^* d_X a_3) \wedge e_1 \wedge e_2, \\
*_t d_X *_t d_X \alpha_{2,0} &= t^{-2} (*_X d_X *_X d_X b) = -t^{-2} d_X^* d_X b,
\end{aligned}$$

where we have used that $d_X^* = -*_X d_X *_X$ on forms of any degree since X is even-dimensional. Similarly we find that

$$\begin{aligned}
*_t \alpha_{0,2} &= t^4 ((*_X a_1) \wedge e_1 + (*_X a_2) \wedge e_2 + (*_X a_3) \wedge e_3), \\
*_t \alpha_{2,0} &= (*_X b) \wedge e_1 \wedge e_2 \wedge e_3, \\
d_X *_t \alpha_{0,2} &= 0, \\
d_X *_t \alpha_{2,0} &= (d_X *_X b) \wedge e_1 \wedge e_2 \wedge e_3, \\
*_t d_X *_t \alpha_{0,2} &= 0, \\
*_t d_X *_t \alpha_{2,0} &= -t^{-2} (*_X d_X *_X b),
\end{aligned}$$

and hence that

$$\begin{aligned}
d_X *_t d_X *_t \alpha_{0,2} &= 0, \\
(5.40) \quad d_X *_t d_X *_t \alpha_{2,0} &= -t^{-2} (d_X *_X d_X *_X b) = t^{-2} d_X d_X^* b.
\end{aligned}$$

The expressions in (5.39) and (5.40) yield that

$$\begin{aligned}
(5.41) \quad & (d_X *_t d_X *_t - *_t d_X *_t d_X) (t^2 \alpha_{0,2} + t^4 \alpha_{2,0}) \\
&= (\Delta_X a_1) \wedge e_2 \wedge e_3 + (\Delta_X a_2) \wedge e_3 \wedge e_1 + (\Delta_X a_3) \wedge e_1 \wedge e_2 + t^2 (\Delta_X b).
\end{aligned}$$

Equation (5.41) expresses the left hand side of (5.37) in terms of the fibre Laplacians, as expected.

Equating (5.41) and (5.36), and using that $*_x^2 = -1$ on odd forms and $d_x^* = -*_x d_x *_x$ on all forms, we deduce that equation (5.37) is in fact equivalent on each fibre P_x to the following system:

$$(5.42) \quad \Delta_x a_1 = d_x^*(*_x \tilde{B}_1), \quad \Delta_x a_2 = d_x^*(*_x \tilde{B}_2), \quad \Delta_x a_3 = d_x^*(*_x \tilde{B}_3),$$

$$(5.43) \quad \Delta_x b = (d_x \tilde{A} + *_x d_x \tilde{A}).$$

By equation (5.34) and (LM4), the right hand sides of the four equations above are all in $\Omega_{k,-4+\gamma}^0$ and $(\Omega_+^2)_{k,-4+\gamma}$ for all $k \geq 0$ and all $\gamma > 0$. Moreover, the right hand side of (5.43) is also self-dual.

Therefore by equation (5.31) and fact (LM3), for $\gamma \in (0, 2)$, on each fibre P_x of $\sigma : P \rightarrow L$ for $x \in L$ there exist unique functions a_1, a_2, a_3 , and a unique self-dual 2-form b solving (5.42) and (5.43) in $L_{k+2,-2+\gamma}^2$ on P_x for all $k \geq 0$. Here we take b self-dual to ensure that $\alpha_{2,0} \in \Gamma^\infty(\Lambda_+^2 V^*)$, as in the last part of Theorem 5.1. In particular by the elliptic regularity remark in (LM2), these solutions are smooth. Moreover, by (LM1), we in fact have

$$(5.44) \quad |\nabla^k a_i| = O(r^{-2-k+\gamma}), \quad |\nabla^k b| = O(r^{-2-k+\gamma}),$$

on P_x as $r \rightarrow \infty$, for all $k \geq 0$ and $i = 1, 2, 3$.

We can show that these solutions on P_x for $x \in L$ depend smoothly on the base point x using the Banach space implicit function theorem, by trivializing locally and observing that (5.42)–(5.43) depend smoothly on the base point x . Hence smooth a_1, a_2, a_3 and b satisfying (5.42)–(5.44) exist on all of P , and are unique provided b is self-dual on the fibres P_x . By (5.10) for $t = 1$ and (5.38), we deduce that we have found unique smooth 2-forms $\alpha_{2,0}$ and $\alpha_{0,2}$ solving (5.37) such that with respect to the metric g_1 on P ,

$$|\nabla^k \alpha_{2,0}|_{g_1^P} = O(\tilde{r}^{-2-k+\gamma}), \quad |\nabla^k \alpha_{0,2}|_{g_1^P} = O(\tilde{r}^{-2-k+\gamma})$$

for $\tilde{r} \geq 1$, and these are unique provided $\alpha_{2,0} \in \Gamma^\infty(\Lambda_+^2 V^*)$. Measuring norms using the metric g_t^P instead, the estimates (5.2)–(5.5) follow easily.

If we choose $\gamma > 0$ sufficiently small, then by (LM5) since $-3 + \gamma < -2$, we deduce by L^2 -orthogonality of the images of d_x and d_x^* that the solutions to (5.43) and in fact solve the uncoupled equations

$$d_x d_x^* b = d_x \tilde{A}, \quad d_x^* d_x b = *_x d_x \tilde{A}.$$

Retracing our steps backwards using (5.39), (5.40), and (5.35), we find that the solution to (5.37) we have produced is actually a solution to

the two uncoupled equations

$$\begin{aligned} d_X *_t d_X *_t (t^2 \alpha_{0,2} + t^4 \alpha_{2,0}) &= t^2 (d_X \tilde{A}), \\ - *_t d_X *_t d_X (t^2 \alpha_{0,2} + t^4 \alpha_{2,0}) &= - *_t d_X [(D_{\varphi_t} \Theta)(t^2 \xi_{1,2}) - t^2 \chi_{1,3} - t^4 \theta_{3,1}]. \end{aligned}$$

We throw away the first equation, and write the second equation as

$$(5.45) \quad d_X *_t d_X (t^2 \alpha_{0,2} + t^4 \alpha_{2,0}) = d_X [(D_{\varphi_t} \Theta)(t^2 \xi_{1,2}) - t^2 \chi_{1,3} - t^4 \theta_{3,1}].$$

Step Two. We have constructed $\alpha_{0,2}$ and $\alpha_{2,0}$ solving (5.45) and satisfying (5.2)–(5.5). To simplify notation, define the 3-form

$$C = t^2 d_X \alpha_{0,2} + t^4 d_X \alpha_{2,0}.$$

Note that from (5.3) and (5.5) for all $k \geq 0$ and $\gamma > 0$ sufficiently small we have

$$(5.46) \quad |\nabla^k C|_{g_t^P} = \begin{cases} O(t^{1-k}), & \check{r} \leq 1, \\ O(t^{1-k\check{r}-3-k+\gamma}), & \check{r} \geq 1. \end{cases}$$

Since C is a form of type $(1, 2) + (3, 0)$, by the proof of Corollary 5.5 we have

$$(D_{\varphi_t} \Theta)(C) = 2 *_t \pi_7 C - *_t C.$$

Thus we can rewrite (5.45) as

$$\begin{aligned} d_X *_t C &= d_X [2 *_t \pi_7 C - (D_{\varphi_t} \Theta)(C)] = d_X [(D_{\varphi_t} \Theta)(t^2 \xi_{1,2}) - t^2 \chi_{1,3} - t^4 \theta_{3,1}], \\ \text{and rearrange it as} \\ (5.47) \end{aligned}$$

$$d_X [(D_{\varphi_t} \Theta)(t^2 d_X \alpha_{0,2} + t^4 d_X \alpha_{2,0} + t^2 \xi_{1,2}) - t^2 \chi_{1,3} - t^4 \theta_{3,1}] = 2 d_X *_t \pi_7 C.$$

The form $*_t \pi_7 C$ is in Ω_7^4 with respect to φ_t , and hence can be written as

$$(5.48) \quad *_t \pi_7 C = w \wedge \varphi_t$$

for some unique 1-form w on P . Since $*_t$ is an isometry and wedge product with φ_t on 1-forms is an isometry up to a constant factor, we deduce from (5.48) and (5.46) that for all $k \geq 0$ and $\gamma > 0$ sufficiently small we have

$$|\nabla^k w|_{g_t^P} = \begin{cases} O(t^{1-k}), & \check{r} \leq 1, \\ O(t^{1-k\check{r}-3-k+\gamma}), & \check{r} \geq 1, \end{cases}$$

from which it follows from (5.10) and (LM1) that

$$(5.49) \quad w \in \Omega_{k,-3+\gamma}^1 \text{ for all } k \geq 0 \text{ and } \gamma > 0 \text{ sufficiently small.}$$

By Lemma 5.4 we know $\pi_7 C$ is type $(1, 2) + (3, 0)$, so $*_t \pi_7 C$ is type $(3, 1) + (1, 3)$, and we deduce from the fact that φ_t is type $(0, 3) + (2, 1)$ that the 1-form w is in fact purely vertical. Thus, the right hand side of (5.47) is

$$(5.50) \quad 2 d_X *_t \pi_7 C = 2 d_X (w \wedge \varphi_t) = 2 (d_X w) \wedge \varphi_t,$$

where we have used the fact that $d_X \varphi_t = 0$, which follows from (5.10) as the ω_i 's are closed in the fibre direction and the e_j 's are pulled back from the base.

Proposition 5.6 precisely says that the left hand side of (5.47) lies in Ω_{14}^5 with respect to φ_t . Hence, the Ω_7^5 component of the right hand side must vanish, so by (5.50) we conclude that $\pi_7((d_X w) \wedge \varphi_t) = 0$ and thus $\pi_7(d_X w) = 0$. But this means that $(d_X w) \wedge \psi_t = 0$, and from (5.10) and the fact that $d_X w$ is of type $(2, 0)$ we deduce that $(d_X w) \wedge \omega_k = 0$ for $k = 1, 2, 3$.

Thus $d_X w$ is anti-self-dual on the fibres. Since $-3 + \gamma < -2$, by (5.49) and Corollary 5.7 we find that $d_X w = 0$ and we conclude that both sides of (5.47) are actually zero. Thus we see from equation (5.24) that $v = 0$.

Step Three. In the previous step, we showed the 2-forms $\alpha_{0,2}$ and $\alpha_{2,0}$ constructed in Step One satisfying (5.2)–(5.5) are in fact solutions to the equation

$$(5.51) \quad d_X [(D_{\varphi_t} \Theta)(t^2 d_X \alpha_{0,2} + t^4 d_X \alpha_{2,0} + t^2 \xi_{1,2}) - t^2 \chi_{1,3} - t^4 \theta_{3,1}] = 0.$$

Note that equation (5.51) is precisely d_X of equation (5.1). Rewrite (5.1) as

$$(5.52) \quad t^2 d_X \beta_{0,3} + t^4 d_X \beta_{2,1} = (D_{\varphi_t} \Theta)(t^2 d_X \alpha_{0,2} + t^4 d_X \alpha_{2,0} + t^2 \xi_{1,2}) - t^2 \chi_{1,3} - t^4 \theta_{3,1}.$$

By (5.51) the right hand side above is a 4-form on P in the kernel of d_X . In fact, in (5.27) we determined that

$$(5.53) \quad \begin{aligned} & (D_{\varphi_t} \Theta)(t^2 d_X \alpha_{0,2} + t^4 d_X \alpha_{2,0} + t^2 \xi_{1,2}) - t^2 \chi_{1,3} - t^4 \theta_{3,1} \\ &= \frac{t^2}{2} A \wedge e_1 \wedge e_2 \wedge e_3 + \frac{t^4}{2} \sum_{k=1}^3 B_k \wedge e_k, \end{aligned}$$

with A, B_1, B_2, B_3 given by (5.28). Equation (5.51) therefore says that

$$d_X A = 0, \quad d_X B_1 = 0, \quad d_X B_2 = 0 \quad \text{and} \quad d_X B_3 = 0.$$

By (5.44) we have

$$(5.54) \quad |\nabla^k(d_X a_i)| = O(r^{-3-k+\gamma}), \quad |\nabla^k(d_X b)| = O(r^{-3-k+\gamma}),$$

for $i = 1, 2, 3$ and all $k \geq 0$. Combining (5.54) with (5.32), (5.26), and (5.28) we deduce just as in (5.34) that

$$(5.55) \quad A \in \Omega_{k,-3+\gamma}^1, \quad B_1, B_2, B_3 \in \Omega_{k,-3+\gamma}^3,$$

for any $\gamma > 0$ sufficiently small and all $k \geq 0$. Now write

$$(5.56) \quad \beta_{0,3} = h e_1 \wedge e_2 \wedge e_3, \quad \beta_{2,1} = f_1 \wedge e_1 + f_2 \wedge e_2 + f_3 \wedge e_3,$$

where h is a function and f_1, f_2, f_3 are vertical 2-forms on P . We will take f_1, f_2, f_3 to be self-dual on the fibres P_x , so that $\beta_{2,1} \in \Gamma^\infty(\Lambda_+^2 V^* \otimes H^*)$, as in the last part of Theorem 5.1. We have

$$\begin{aligned} d_X \beta_{0,3} &= (d_X h) \wedge e_1 \wedge e_2 \wedge e_3, \\ d_X \beta_{2,1} &= (d_X f_1) \wedge e_1 + (d_X f_2) \wedge e_2 + (d_X f_3) \wedge e_3. \end{aligned}$$

Using the above two expressions and equations (5.52)–(5.53), we find that (5.1) is in fact equivalent on each fibre P_x to the following system of equations:

$$(5.57) \quad d_X h = \tfrac{1}{2}A, \quad d_X f_1 = \tfrac{1}{2}B_1, \quad d_X f_2 = \tfrac{1}{2}B_2, \quad d_X f_3 = \tfrac{1}{2}B_3.$$

By (5.55) and Proposition 5.9, on each fibre P_x there exists a smooth h in $\Omega_{k,-2+\gamma}^0$ and smooth f_1, f_2, f_3 in $(\Omega_+^2)_{k,-2+\gamma}$ satisfying (5.57). Suppose h' was another solution in $\Omega_{k,-2+\gamma}^0$ to $d_X h' = \frac{1}{2}A$. Then $h - h'$ is a constant function on X , and decays to zero at infinity, so $h' = h$. If f'_j was another solution in $(\Omega_+^2)_{k,-2+\gamma}$ to $d_X f'_j = \frac{1}{2}B_j$, then $f_j - f'_j$ is a closed self-dual 2-form on X (and thus also coclosed) which decays at rate $-2 + \gamma$ at infinity. Thus by (5.31), we have $f_j - f'_j = c_1 \omega_1 + c_2 \omega_2 + c_3 \omega_3$ for some decaying harmonic functions c_1, c_2, c_3 on X , which must therefore vanish by the maximum principle. Thus $f'_j = f_j$, so h, f_1, f_2, f_3 are unique.

As for a_1, a_2, a_3, b in Step One, we can show that these h, f_1, f_2, f_3 on P_x for $x \in L$ depend smoothly on the base point x using the Banach space implicit function theorem, as equations (5.57) depend smoothly on the base point x . Thus by (5.10) for $t = 1$ and (5.56), we deduce that we have found smooth 3-forms $\beta_{0,3}$ and $\beta_{2,1}$ solving equation (5.52), and hence (5.1), such that with respect to the metric g_1 on P ,

$$|\nabla^k \beta_{0,3}|_{g_1^P} = O(\check{r}^{-2-k+\gamma}), \quad |\nabla^k \beta_{2,1}|_{g_1^P} = O(\check{r}^{-2-k+\gamma})$$

for $\check{r} \geq 1$, and these $\beta_{0,3}, \beta_{2,1}$ are unique provided $\beta_{2,1} \in \Gamma^\infty(\Lambda_+^2 V^* \otimes H^*)$. Measuring norms using the metric g_t^P instead, the estimates (5.6)–(5.8) follow easily. This completes the proof of Theorem 5.1.

6. Torsion-free G_2 -structures on the resolution N of $M/\langle \iota \rangle$

Sections 6.1–6.4 prove our main theorem, Theorem 6.4, and §6.5–§6.6 explain two generalizations of it.

6.1. The resolution N of $M/\langle \iota \rangle$. In §3, given $(M, \varphi, g), \iota, L$ as in Assumption 3.1, we wrote $\nu \rightarrow L$ for the normal bundle of L in M and $r : \nu \rightarrow [0, \infty)$ for the radius function on ν , so that $r^{-1}(0)$ is the zero section $0(L)$. We chose a tubular neighbourhood $U_R \subset \nu$ of the zero section $0(L)$ in ν and a tubular neighbourhood map $\Upsilon : U_R \rightarrow M$,

which is a diffeomorphism onto an open subset of M , where $U_R = \{(x, \alpha) \in \nu : r(x, \alpha) < R\}$ for small $R > 0$. For $t > 0$ we also defined $\Upsilon_t : U_{t^{-1}R} \rightarrow M$ by $\Upsilon_t : (x, \alpha) \mapsto \Upsilon(x, t\alpha)$, as in (3.5). These are equivariant under the actions of $\{\pm 1\}$ on ν and $\langle \iota \rangle$ on M , and so they descend to $\Upsilon_t : U_{t^{-1}R}/\{\pm 1\} \rightarrow M/\langle \iota \rangle$.

In §4 we defined a 7-manifold P , a 5-submanifold $Q \subset P$, and a proper, continuous map $\rho : P \rightarrow \nu/\{\pm 1\}$, where $\rho^{-1}(0(L)) = Q$, and $\rho|_{P \setminus Q} : P \setminus Q \rightarrow (\nu/\{\pm 1\}) \setminus 0(L)$ is smooth and a diffeomorphism. We defined $\tilde{r} = r \circ \rho : P \rightarrow [0, \infty)$ to be the pullback of the radius function, so that $Q = \tilde{r}^{-1}(0)$.

For $t > 0$, define a compact, smooth 7-manifold N by

$$(6.1) \quad N = [\rho^{-1}(U_{t^{-1}R}/\{\pm 1\}) \amalg (M \setminus L)/\langle \iota \rangle] / \approx .$$

Here $\rho^{-1}(U_{t^{-1}R}/\{\pm 1\}) \subset P$ is open, a noncompact 7-manifold, and $(M \setminus L)/\langle \iota \rangle$ is the nonsingular part of the orbifold $M/\langle \iota \rangle$, another noncompact 7-manifold. We define \approx to be the equivalence relation on the disjoint union of these two 7-manifolds which identifies $x \in \rho^{-1}(U_{t^{-1}R}/\{\pm 1\}) \setminus Q$ with $\Upsilon_t \circ \rho(x)$ in $(M \setminus L)/\langle \iota \rangle$. Since $\Upsilon_t \circ \rho$ is a diffeomorphism between the relevant open subsets of $\rho^{-1}(U_{t^{-1}R}/\{\pm 1\})$ and $(M \setminus L)/\langle \iota \rangle$, and the quotient topological space is compact and Hausdorff, this N is a smooth compact 7-manifold.

Technically N depends on $t > 0$, so it might be better to write it as N_t rather than N . But there are canonical diffeomorphisms $N_t \cong N_1$ for all $t > 0$ acting as the identity on the subsets $(M \setminus L)/\langle \iota \rangle \subset N_t$, $(M \setminus L)/\langle \iota \rangle \subset N_1$, and identifying $\rho^{-1}(U_{t^{-1}R}/\{\pm 1\}) \subset N_t$ with $\rho^{-1}(U_R/\{\pm 1\}) \subset N_1$ by rescaling by t . So by an abuse of notation we will identify the N_t for $t > 0$, and write them all as N .

By another abuse of notation, we will regard $\rho^{-1}(U_{t^{-1}R}/\{\pm 1\})$ and $(M \setminus L)/\langle \iota \rangle$ as open subsets of N , and functions, metrics, and exterior forms defined on $\rho^{-1}(U_{t^{-1}R}/\{\pm 1\}) \subset P$ or on $(M \setminus L)/\langle \iota \rangle \subset M/\langle \iota \rangle$ as being defined on the corresponding subsets of N , without changing notation.

There is an obvious continuous map $\pi : N \rightarrow M/\langle \iota \rangle$ acting by $\Upsilon_t \circ \rho$ on $\rho^{-1}(U_{t^{-1}R}/\{\pm 1\})$, and by the inclusion $(M \setminus L)/\langle \iota \rangle \hookrightarrow M/\langle \iota \rangle$ on $(M \setminus L)/\langle \iota \rangle$. Then $\pi|_{N \setminus Q} : N \setminus Q \rightarrow (M/\langle \iota \rangle) \setminus L$ is smooth, and a diffeomorphism. However, as in Remark 2.9 and §4.1, π is not a smooth map of orbifolds near Q , although $\pi|_Q : Q \rightarrow L$ is smooth and an \mathcal{S}^2 -bundle.

Readers are warned about a possible source of confusion concerning the radius functions $r : \nu/\{\pm 1\} \rightarrow [0, \infty)$ and $\tilde{r} : P \rightarrow [0, \infty)$. The map $\Upsilon : U_R/\{\pm 1\} \rightarrow M/\langle \iota \rangle$ identifies $U_R/\{\pm 1\}$ with an open neighbourhood of L in $M/\langle \iota \rangle$. We think of the radius function $r : U_R/\{\pm 1\} \rightarrow [0, R]$ as being the distance to L in $M/\langle \iota \rangle$.

However, the map $\Upsilon_t \circ \rho : \rho^{-1}(U_{t^{-1}R}/\{\pm 1\}) \rightarrow M/\langle \iota \rangle$ includes a rescaling by $t > 0$ in the fibres of $\nu/\{\pm 1\}$. So \check{r} maps $\rho^{-1}(U_{t^{-1}R}/\{\pm 1\}) \rightarrow [0, t^{-1}R)$, and $t\check{r}$ on $\rho^{-1}(U_{t^{-1}R}/\{\pm 1\})$ is identified with the function r on $\Upsilon(U_R/\{\pm 1\}) \subset M/\langle \iota \rangle$. That is, *we should identify the radius function r in most of §3 with $t\check{r}$ in §4–§5, not with \check{r}* , and we should think of $t\check{r}$ (rather than \check{r}) as being the distance to Q in $\rho^{-1}(U_{t^{-1}R}/\{\pm 1\}) \subset P$.

Next we compute the Betti numbers $b^k(N) = \dim H^k(N; \mathbb{R})$. It is enough to compute b^0, \dots, b^3 , as $b^{7-k}(N) = b^k(N)$ by Poincaré duality.

Proposition 6.1. *Let $M/\langle \iota \rangle, L$ and N be as above. Then the Betti numbers of N are given by*

(6.2)
$$b^k(N) = b^k(M/\langle \iota \rangle) + b^{k-2}(L).$$

Moreover, the fundamental groups of N and $M/\langle \iota \rangle$ are isomorphic.

Proof. We have the following commutative diagram of topological spaces, where the rows are embeddings:

(6.3)
$$\begin{array}{ccc} Q & \xhookrightarrow{\text{inc}} & N \\ \downarrow \pi|_Q & & \downarrow \pi \\ L & \xhookrightarrow{\text{inc}} & M/\langle \iota \rangle. \end{array}$$

Consider the following diagram of (relative) cohomology groups:

(6.4)
$$\begin{array}{ccccccc} & 0 & & 0 & & 0 & & 0 \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \cdots \rightarrow & H^{k-1}(L; \mathbb{R}) & \rightarrow & H^k(M/\langle \iota \rangle, L; \mathbb{R}) & \rightarrow & H^k(M/\langle \iota \rangle; \mathbb{R}) & \xrightarrow{\text{inc}^*} & H^k(L; \mathbb{R}) \rightarrow \cdots \\ & \downarrow \pi|_Q^* = \begin{pmatrix} \text{id} \\ 0 \end{pmatrix} & & \downarrow \cong \pi^* & & \downarrow \pi^* & & \downarrow \pi|_Q^* = \begin{pmatrix} \text{id} \\ 0 \end{pmatrix} \\ \cdots \rightarrow & H^{k-1}(Q; \mathbb{R}) \cong & & H^k(N, Q; \mathbb{R}) & \rightarrow & H^k(N; \mathbb{R}) & \xrightarrow{\text{inc}^*} & H^k(Q; \mathbb{R}) \cong \\ & H^{k-1}(L; \mathbb{R}) \oplus & & & & & & H^k(L; \mathbb{R}) \oplus \\ & H^{k-3}(L; \mathbb{R}) & & & & & & H^{k-2}(L; \mathbb{R}) \\ & \downarrow (0 \text{ id}) & & \downarrow & & \downarrow \vdots & & \downarrow (0 \text{ id}) \\ \cdots \rightarrow & H^{k-3}(L; \mathbb{R}) & \rightarrow & 0 & \rightarrow & H^{k-2}(L; \mathbb{R}) & \xrightarrow{\text{id}} & H^{k-2}(L; \mathbb{R}) \rightarrow \cdots \\ & \downarrow 0 & & \downarrow 0 & & \downarrow 0 & & \downarrow 0. \end{array}$$

Here the first and second rows are the relative cohomology exact sequences for the pairs $(M/\langle \iota \rangle, L)$ and (N, Q) . The squares between the first and second rows commute as (6.3) commutes. In the second column, $\pi^* : H^k(M/\langle \iota \rangle, L; \mathbb{R}) \rightarrow H^k(N, Q; \mathbb{R})$ is an isomorphism since collapsing L to a point in $M/\langle \iota \rangle$ yields the same pointed topological space as collapsing Q to a point in N . Thus the second column is exact.

In the first and fourth columns, $\pi|_Q : Q \rightarrow L$ is an \mathcal{S}^2 -bundle, which is trivial as $\nu \rightarrow L$ is a trivial \mathbb{C}^2 -bundle (see Remark 2.14). Hence

$Q \cong L \times \mathcal{S}^2$, so
(6.5)

$$H^k(Q; \mathbb{R}) \cong \bigoplus_{i+j=k} H^i(L; \mathbb{R}) \otimes H^j(\mathcal{S}^2; \mathbb{R}) \cong H^k(L; \mathbb{R}) \oplus H^{k-2}(L; \mathbb{R})$$

by the Künneth Theorem. Thus the first and fourth columns are exact. The bottom left square commutes trivially, and the third row is exact.

We now know that the rows of (6.4) are exact, the first, second and fourth columns are exact, the squares between the first and second row commute, and the left hand square between the second and third row commutes. By homological algebra, it follows that there is a unique morphism ‘ \dashrightarrow ’ as shown in (6.4) which makes the third column exact. Hence $H^k(N; \mathbb{R}) \cong H^k(M/\langle \iota \rangle; \mathbb{R}) \oplus H^{k-2}(L; \mathbb{R})$. Taking dimensions proves (6.2).

For the final part, we can show that $\pi_* : \pi_1(N) \rightarrow \pi_1(M/\langle \iota \rangle)$ is an isomorphism using the Seifert–van Kampen Theorem and the fact that $\rho_* : \pi_1(P) \rightarrow \pi_1(\nu/\{\pm 1\})$ is an isomorphism, since $\pi_1(P), \pi_1(\nu/\{\pm 1\})$ are both isomorphic to $\pi_1(L)$, as the fibres of $\sigma : P \rightarrow L$ and $\pi : \nu/\{\pm 1\} \rightarrow L$ are simply-connected. q.e.d.

6.2. The G_2 -structures (φ_t^N, g_t^N) on N . Let $a : [0, \infty) \rightarrow \mathbb{R}$ be a smooth function with $a(x) = 0$ for $x \in [0, 1]$, and $a(x) \in (0, 1)$ for $x \in (1, 2)$, and $a(x) = 1$ for $x \in [2, \infty)$. Let η and ζ be as defined in §3.5. For some sufficiently small $\epsilon > 0$ and all $t \in (0, \epsilon]$, we define a 3-form φ_t^N on N by

$$(6.6) \quad \varphi_t^N = \begin{cases} \tilde{\varphi}_t^P + d[t^2\alpha_{0,2} + t^4\alpha_{2,0}], & \text{if } \check{r} \leq t^{-1/9} \\ & \text{in } \rho^{-1}(U_{t^{-1}R}/\{\pm 1\}), \\ \tilde{\varphi}_t^P + d[t^2\alpha_{0,2} + t^4\alpha_{2,0} + a(t^{1/9}\check{r}) \cdot \Upsilon_*(\eta)], & \text{if } t^{-1/9} \leq \check{r} \leq 2t^{-1/9} \\ & \text{in } \rho^{-1}(U_{t^{-1}R}/\{\pm 1\}), \\ \tilde{\varphi}_t^P + d[t^2\alpha_{0,2} + t^4\alpha_{2,0} + \Upsilon_*(\eta)] = & \text{if } 2t^{-1/9} \leq \check{r} \leq t^{-4/5} \\ \rho^*(\tilde{\varphi}_t^\nu) + d[t^2\tau_{1,1} + t^2\alpha_{0,2} + t^4\alpha_{2,0} + \Upsilon_*(\eta)], & \text{in } \rho^{-1}(U_{t^{-1}R}/\{\pm 1\}), \\ \rho^*(\tilde{\varphi}_t^\nu) + d[(1 - a(t^{4/5}\check{r})) \cdot & \text{if } t^{-4/5} \leq \check{r} \leq 2t^{-4/5} \\ (t^2\tau_{1,1} + t^2\alpha_{0,2} + t^4\alpha_{2,0}) + \Upsilon_*(\eta)], & \text{in } \rho^{-1}(U_{t^{-1}R}/\{\pm 1\}), \\ \rho^*(\tilde{\varphi}_t^\nu) + d\Upsilon_*(\eta) = \Upsilon_*(\tilde{\varphi}^\nu + d\eta) = \varphi, & \text{if } 2t^{-4/5} \leq \check{r} < t^{-1}R \\ & \text{in } \rho^{-1}(U_{t^{-1}R}/\{\pm 1\}), \\ \varphi, & \text{in } (M \setminus \Upsilon(U_R))/\langle \iota \rangle. \end{cases}$$

Here we require $\epsilon > 0$ to be small enough that if $0 < t \leq \epsilon$ then

$$1 < t^{-1/9} < 2t^{-1/9} < t^{-4/5} < 2t^{-4/5} < t^{-1}R,$$

so that each region in (6.6) is nonempty. These particular exponents of $-\frac{1}{9}$ and $-\frac{4}{5}$ have been chosen so that the torsion estimates com-

puted in Proposition 6.2 below will be sufficiently small to produce estimates in Proposition 6.3 that will satisfy the hypotheses of Theorem 2.7.

The equality in the third case in (6.6) comes from (4.32), and the equalities in the fifth case come from (3.34) and the equivalence of $\rho^*(\tilde{\varphi}_t^\nu)$ and $\Upsilon_*(\tilde{\varphi}^\nu)$ under the identification of their domains by \approx in (6.1). Considering the transitions between regions we see that φ_t^N is smooth on N , and since $\tilde{\varphi}_t^P, \tilde{\varphi}_t^\nu, \varphi$ are closed, φ_t^N is also closed.

Similarly, we define a smooth closed 4-form ψ_t^N on N for all $t \in (0, \epsilon]$ by

$$(6.7) \quad \psi_t^N = \begin{cases} \tilde{\psi}_t^P + d[t^2\beta_{0,3} + t^4\beta_{2,1}], & \text{if } \tilde{r} \leq t^{-1/9} \\ & \text{in } \rho^{-1}(U_{t^{-1}R}/\{\pm 1\}), \\ \tilde{\psi}_t^P + d[t^2\beta_{0,3} + t^4\beta_{2,1} + a(t^{1/9}\tilde{r}) \cdot \Upsilon_*(\zeta)], & \text{if } t^{-1/9} \leq \tilde{r} \leq 2t^{-1/9} \\ & \text{in } \rho^{-1}(U_{t^{-1}R}/\{\pm 1\}), \\ \tilde{\psi}_t^P + d[t^2\beta_{0,3} + t^4\beta_{2,1} + \Upsilon_*(\zeta)] = & \text{if } 2t^{-1/9} \leq \tilde{r} \leq t^{-4/5} \\ \rho^*(\tilde{\psi}_t^\nu) + d[t^2v_{1,2} + t^2\beta_{0,3} + t^4\beta_{2,1} + \Upsilon_*(\zeta)], & \text{in } \rho^{-1}(U_{t^{-1}R}/\{\pm 1\}), \\ \rho^*(\tilde{\psi}_t^\nu) + d[(1 - a(t^{4/5}\tilde{r})) \cdot & \text{if } t^{-4/5} \leq \tilde{r} \leq 2t^{-4/5} \\ (t^2v_{1,2} + t^2\beta_{0,3} + t^4\beta_{2,1}) + \Upsilon_*(\zeta)], & \text{in } \rho^{-1}(U_{t^{-1}R}/\{\pm 1\}), \\ \rho^*(\tilde{\psi}_t^\nu) + d\Upsilon_*(\zeta) = \Upsilon_*(\tilde{\psi}^\nu + d\zeta) = *\varphi, & \text{if } 2t^{-4/5} \leq \tilde{r} < t^{-1}R \\ & \text{in } \rho^{-1}(U_{t^{-1}R}/\{\pm 1\}), \\ *\varphi, & \text{in } (M \setminus \Upsilon(U_R))/\langle \iota \rangle. \end{cases}$$

We claim that for $\epsilon > 0$ small enough, φ_t^N will be a positive 3-form on N for all $t \in (0, \epsilon]$ in the sense of §2.2. To establish this, we show that if t is small enough, then at all points in N the 3-form φ_t^N is close to a positive 3-form, because then Proposition 2.6 implies that φ_t^N will be positive when t is sufficiently small. Recall that $r = t\tilde{r}$. If we consider $\varphi_t^N - \varphi_t^P$, then this difference includes terms $t^2\xi_{0,3}$ and $d\Upsilon_*(\eta)$, which by (4.24), (3.36), and Proposition 4.6, are of size $O(r^2) = O(t^2\tilde{r}^2)$. These are too big when $r = O(1)$, which happens when $\tilde{r} = O(t^{-1})$. So we can approximate φ_t^N by φ_t^P only when \tilde{r} is not too large, and the region $\tilde{r} \leq t^{-1/2}R$ will do. On the other hand, if we consider $\varphi_t^N - \varphi$, then this difference includes a term $t^2d\tau_{1,1}$, which by (4.34) is of size $O(\tilde{r}^{-4})$. This is too big when $\tilde{r} = O(1)$, which happens when $r = O(t)$. So we can approximate φ_t^N by φ only when \tilde{r} is not too small, and the region $\tilde{r} \geq t^{-1/2}R$ will do. Together these regions cover all points of N , so φ_t^N is indeed positive for all $t \in (0, \epsilon]$ if ϵ is sufficiently small.

6.3. Estimating the torsion of (φ_t^N, g_t^N) . Since $d\varphi_t^N = 0$ and $d\psi_t^N = 0$, if $\Theta(\varphi_t^N) - \psi_t^N = 0$ then the G_2 -structure (φ_t^N, g_t^N) is torsion-free. Thus, we can regard the next two propositions, which bound $\Theta(\varphi_t^N) - \psi_t^N$, as measuring the torsion of (φ_t^N, g_t^N) .

Proposition 6.2. *In the situation above, for all $t \in (0, \epsilon]$ we have*

(6.8)

$$|\Theta(\varphi_t^N) - \psi_t^N|_{g_t^N} = \begin{cases} O(t^2), & \text{if } \check{r} \leq 1 \text{ in } \rho^{-1}(U_{t^{-1}R}/\{\pm 1\}), \\ O(t^2\check{r}^2), & \text{if } 1 \leq \check{r} \leq t^{-1/9} \text{ in } \rho^{-1}(U_{t^{-1}R}/\{\pm 1\}), \\ O(t^{16/9}), & \text{if } t^{-1/9} \leq \check{r} \leq 2t^{-1/9} \text{ in } \rho^{-1}(U_{t^{-1}R}/\{\pm 1\}), \\ O(t^2\check{r}^{-2+\gamma}), & \text{if } 2t^{-1/9} \leq \check{r} \leq t^{-4/5} \text{ in } \rho^{-1}(U_{t^{-1}R}/\{\pm 1\}), \gamma > 0, \\ O(t^{16/5}), & \text{if } t^{-4/5} \leq \check{r} \leq 2t^{-4/5} \text{ in } \rho^{-1}(U_{t^{-1}R}/\{\pm 1\}), \\ 0, & \text{if } 2t^{-4/5} \leq \check{r} < t^{-1}R \text{ in } \rho^{-1}(U_{t^{-1}R}/\{\pm 1\}), \\ 0, & \text{in } (M \setminus \Upsilon(U_R))/\langle \iota \rangle, \end{cases}$$

(6.9)

$$|d(\Theta(\varphi_t^N) - \psi_t^N)|_{g_t^N} = \begin{cases} O(t), & \text{if } \check{r} \leq 1 \text{ in } \rho^{-1}(U_{t^{-1}R}/\{\pm 1\}), \\ O(t\check{r}), & \text{if } 1 \leq \check{r} \leq t^{-1/9} \text{ in } \rho^{-1}(U_{t^{-1}R}/\{\pm 1\}), \\ O(t^{8/9}), & \text{if } t^{-1/9} \leq \check{r} \leq 2t^{-1/9} \text{ in } \rho^{-1}(U_{t^{-1}R}/\{\pm 1\}), \\ O(t\check{r}^{-3+\gamma}), & \text{if } 2t^{-1/9} \leq \check{r} \leq t^{-4/5} \text{ in } \rho^{-1}(U_{t^{-1}R}/\{\pm 1\}), \gamma > 0, \\ O(t^3), & \text{if } t^{-4/5} \leq \check{r} \leq 2t^{-4/5} \text{ in } \rho^{-1}(U_{t^{-1}R}/\{\pm 1\}), \\ 0, & \text{if } 2t^{-4/5} \leq \check{r} < t^{-1}R \text{ in } \rho^{-1}(U_{t^{-1}R}/\{\pm 1\}), \\ 0, & \text{in } (M \setminus \Upsilon(U_R))/\langle \iota \rangle. \end{cases}$$

Proof. For the first two cases in (6.8), if $\check{r} \leq t^{-1/9}$ in $\rho^{-1}(U_{t^{-1}R}/\{\pm 1\})$, we have

$$\begin{aligned} \Theta(\varphi_t^N) - \psi_t^N &= \Theta(\tilde{\varphi}_t^P + d[t^2\alpha_{0,2} + t^4\alpha_{2,0}]) - \tilde{\psi}_t^P - d[t^2\beta_{0,3} + t^4\beta_{2,1}] \\ &= \Theta(\varphi_t^P + t^2\xi_{1,2} + t^2\xi_{0,3} + t^2d\alpha_{0,2} + t^4d\alpha_{2,0}) \\ &\quad - \psi_t^P - t^2\chi_{1,3} - t^4\theta_{3,1} - t^4\theta_{2,2} - t^2d\beta_{0,3} - t^4d\beta_{2,1} \\ &= (D_{\varphi_t^P}\Theta)(t^2\xi_{1,2} + t^2\xi_{0,3} + t^2d\alpha_{0,2} + t^4d\alpha_{2,0}) \\ &\quad - t^2\chi_{1,3} - t^4\theta_{3,1} - t^4\theta_{2,2} - t^2d\beta_{0,3} - t^4d\beta_{2,1} \\ &\quad + F_{\varphi_t^P}(t^2\xi_{1,2} + t^2\xi_{0,3} + t^2d\alpha_{0,2} + t^4d\alpha_{2,0}) \\ &= (D_{\varphi_t^P}\Theta)(t^2\xi_{0,3} + t^2[d\alpha_{0,2}]_{0,3} + t^4[d\alpha_{2,0}]_{2,1} + t^4[d\alpha_{2,0}]_{1,2}) \\ &\quad - t^4\theta_{2,2} - t^4[d\beta_{2,1}]_{2,2} - t^4[d\beta_{2,1}]_{1,3} \\ &\quad + F_{\varphi_t^P}(t^2\xi_{1,2} + t^2\xi_{0,3} + t^2d\alpha_{0,2} + t^4d\alpha_{2,0}), \end{aligned} \tag{6.10}$$

using (6.6)–(6.7) in the first step; (4.24) in the second; Proposition 2.6 and $*_{\varphi_t^P}\varphi_t^P = \psi_t^P$ in the third; and $d\alpha_{0,2} = [d\alpha_{0,2}]_{1,2} + [d\alpha_{0,2}]_{0,3}$, etc., and equation (5.1) of Theorem 5.1 to cancel seven terms in the fourth.

If $\epsilon > 0$ is small enough then $|\varphi_t^N - \varphi_t^P|_{g_t^P}$ is small on $\check{r} \leq t^{-1/9}$, so (2.5) in Proposition 2.6 applies for some $C > 0$. Also (2.6) implies that $|(D_{\varphi_t^P}\Theta)(\alpha)|_{g_t^P} \leq \frac{4}{3}|\alpha|_{g_t^P}$ for any 3-form α . Thus from (6.10) we deduce that if $\check{r} \leq t^{-1/9}$ we have

$$\begin{aligned} |\Theta(\varphi_t^N) - \psi_t^N|_{g_t^P} &\leq \frac{4}{3}(|t^2\xi_{0,3}|_{g_t^P} + |t^2[d\alpha_{0,2}]_{0,3}|_{g_t^P} + |t^4[d\alpha_{2,0}]_{2,1}|_{g_t^P} \\ &\quad + |t^4[d\alpha_{2,0}]_{1,2}|_{g_t^P}) + |t^4\theta_{2,2}|_{g_t^P} + |t^4[d\beta_{2,1}]_{2,2}|_{g_t^P} + |t^4[d\beta_{2,1}]_{1,3}|_{g_t^P} \\ (6.11) \quad &+ C(|t^2\xi_{1,2}|_{g_t^P} + |t^2\xi_{0,3}|_{g_t^P} + |t^2d\alpha_{0,2}|_{g_t^P} + |t^4d\alpha_{2,0}|_{g_t^P})^2. \end{aligned}$$

All the terms on the right hand side of (6.11) are estimated in Proposition 4.6 and Theorem 5.1. The dominant terms are $|t^2\xi_{0,3}|_{g_t^P}$, $|t^4\theta_{2,2}|_{g_t^P}$, which are $O(t^2)$ if $\check{r} \leq 1$ and $O(t^2\check{r}^2)$ if $1 \leq \check{r} \leq t^{-1/9}$. Thus we see that (6.12)

$$|\Theta(\varphi_t^N) - \psi_t^N|_{g_t^P} = \begin{cases} O(t^2), & \text{if } \check{r} \leq 1 \text{ in } \rho^{-1}(U_{t^{-1}R}/\{\pm 1\}), \\ O(t^2\check{r}^2), & \text{if } 1 \leq \check{r} \leq t^{-1/9} \text{ in } \rho^{-1}(U_{t^{-1}R}/\{\pm 1\}). \end{cases}$$

This is not quite what is wanted for (6.8), as we are taking norms using g_t^P rather than g_t^N . However, g_t^P and g_t^N are C^0 -close in the region $\check{r} \leq t^{-1/9}$, so $|\cdots|_{g_t^P}$ and $|\cdots|_{g_t^N}$ differ by a bounded factor, and thus (6.12) implies the first two cases in (6.8).

For the third case in (6.8), when $t^{-1/9} \leq \check{r} \leq 2t^{-1/9}$, we proceed as in (6.10)–(6.12) but including the extra terms from (6.6)–(6.7):

$$(D_{\varphi_t^P}\Theta)(d[a(t^{1/9}\check{r}) \cdot \Upsilon_*(\eta)]) - d[a(t^{1/9}\check{r}) \cdot \Upsilon_*(\zeta)].$$

Using (3.36) and (3.37), and noting as in §6.1 that r in §3.5 corresponds to $t\check{r}$ above, we find that these terms are $O(t^{16/9})$, agreeing with $O(t^2\check{r}^2)$ in the previous case when $\check{r} = O(t^{-1/9})$. The third case in (6.8) follows.

For the fourth case, if $2t^{-1/9} \leq \check{r} \leq t^{-4/5}$ in $\rho^{-1}(U_{t^{-1}R}/\{\pm 1\})$ we have

$$\begin{aligned} &\Theta(\varphi_t^N) - \psi_t^N \\ &= \Theta(\tilde{\varphi}_t^P + d[t^2\alpha_{0,2} + t^4\alpha_{2,0} + \Upsilon_*(\eta)]) \\ &\quad - (\tilde{\psi}_t^P + d[t^2\beta_{0,3} + t^4\beta_{2,1} + \Upsilon_*(\zeta)]) - \{\Theta(\varphi) - \psi\} \\ &= \Theta(\varphi_t^P + t^2\xi_{1,2} + t^2\xi_{0,3} + t^2d\alpha_{0,2} + t^4d\alpha_{2,0} + \Upsilon_*(d\eta)) \\ &\quad - \psi_t^P - t^2\chi_{1,3} - t^4\theta_{3,1} - t^4\theta_{2,2} - t^2d\beta_{0,3} - t^4d\beta_{2,1} - \Upsilon_*(d\zeta) \\ &\quad - \{\Theta(\rho^*(\varphi_t^P + t^2\ddot{\varphi}_{0,3}^2) + \Upsilon_*(d\eta)) - (\rho^*(\psi_t^P + t^4\ddot{\psi}_{2,2}^4) + \Upsilon_*(d\zeta))\} \\ &= (D_{\varphi_t^P}\Theta)(t^2\xi_{1,2} + t^2\xi_{0,3} + t^2d\alpha_{0,2} + t^4d\alpha_{2,0} + \Upsilon_*(d\eta)) \\ &\quad - t^2\chi_{1,3} - t^4\theta_{3,1} - t^2d\beta_{0,3} - t^4d\beta_{2,1} \\ &\quad + F_{\varphi_t^P}(t^2\xi_{1,2} + t^2\xi_{0,3} + t^2d\alpha_{0,2} + t^4d\alpha_{2,0} + \Upsilon_*(d\eta)) \end{aligned}$$

$$\begin{aligned}
& - (D_{\rho^*(\varphi_t^\nu)} \Theta) (t^2 \xi_{0,3} - t^2 [d\tau_{1,1}]_{0,3} + \Upsilon_*(d\eta)) \\
& - F_{\rho^*(\varphi_t^\nu)} (t^2 \xi_{0,3} - t^2 [d\tau_{1,1}]_{0,3} + \Upsilon_*(d\eta)) \\
= & (D_{\varphi_t^P} \Theta) (t^2 [d\alpha_{0,2}]_{0,3} + t^4 [d\alpha_{2,0}]_{2,1} + t^4 [d\alpha_{2,0}]_{1,2}) \\
& + (D_{\rho^*(\varphi_t^\nu)} \Theta) (t^2 [d\tau_{1,1}]_{0,3}) \\
& + (D_{\varphi_t^P} \Theta - D_{\rho^*(\varphi_t^\nu)} \Theta) (t^2 \xi_{0,3} + \Upsilon_*(d\eta)) - t^4 [d\beta_{2,1}]_{2,2} - t^4 [d\beta_{2,1}]_{1,3} \\
& + F_{\varphi_t^P} (t^2 \xi_{1,2} + t^2 \xi_{0,3} + t^2 d\alpha_{0,2} + t^4 d\alpha_{2,0} + \Upsilon_*(d\eta)) \\
(6.13) \quad & - F_{\rho^*(\varphi_t^\nu)} (t^2 \xi_{0,3} - t^2 [d\tau_{1,1}]_{0,3} + \Upsilon_*(d\eta)).
\end{aligned}$$

Here we use (6.6)–(6.7) and $\Theta(\varphi) = \psi$ in the first step; equations (3.34), (4.24) and $\Upsilon_*(\tilde{\varphi}^\nu) = \rho^*(\tilde{\varphi}^\nu)$, $\Upsilon_*(\tilde{\psi}^\nu) = \rho^*(\tilde{\psi}^\nu)$, and Corollary 3.9 in the second; equation (4.33), Proposition 2.6 and $*_{\varphi_t^P} \varphi_t^P = \psi_t^P$, $*_{\rho^*(\varphi_t^\nu)} \rho^*(\varphi_t^\nu) = \rho^*(\psi_t^\nu)$ in the third, in particular to cancel $t^4 \theta_{2,2}$ with $t^4 \psi_{2,2}$ and to cancel two $\Upsilon_*(d\zeta)$ terms; and $d\alpha_{0,2} = [d\alpha_{0,2}]_{1,2} + [d\alpha_{0,2}]_{0,3}$, etc., and equation (5.1) of Theorem 5.1 to cancel seven terms in the fourth.

Just as we did for (6.11), we can now estimate all the terms in the final step of (6.13) when $2t^{-1/9} \leq \check{r} \leq t^{-4/5}$, using Propositions 2.6 and 4.6, Theorem 5.1, equation (4.34), and the estimate

$$(6.14) \quad |D_{\varphi_t^P} \Theta - D_{\rho^*(\varphi_t^\nu)} \Theta|_{g_t^P} = O(\check{r}^{-4}).$$

The estimate (6.14) is immediate from $|\varphi_t^P - \rho^*(\varphi_t^\nu)| = O(\check{r}^{-4})$ which itself follows from (4.30) and (4.34). In this case we find that the dominant terms come from $t^2 [d\alpha_{0,2}]_{0,3}$, $t^4 [d\alpha_{2,0}]_{2,1}$, $t^4 [d\beta_{2,1}]_{2,2}$ which contribute $O(t^2 \check{r}^{-2+\gamma})$ for $\gamma > 0$ by Theorem 5.1, and all other terms are smaller. This proves the fourth case in (6.8).

The fifth case of (6.8) is as for the fourth case with $\check{r} = O(t^{-4/5})$, but with additional terms involving

$$d[a(t^{4/5} \check{r}) \cdot (t^2 \tau_{1,1} + t^2 \alpha_{0,2} + t^4 \alpha_{2,0})], \quad d[a(t^{4/5} \check{r}) \cdot (t^2 v_{1,2} + t^2 \beta_{0,3} + t^4 \beta_{2,1})].$$

Using (4.34) and Theorem 5.1 we find that these contribute errors of size $O(t^{16/5})$ and $O(t^{17/5-4\gamma/5})$ for $\gamma > 0$. The $O(t^{16/5})$ error dominates, so the fifth case follows.

The sixth and seventh cases are immediate as $\varphi_t^N = \varphi$, $\psi_t^N = *\varphi$ with $\Theta(\varphi) = *\varphi$. This proves (6.8).

Finally, as in (4.34) and Proposition 4.6 and Theorem 5.1, the principle is that adding a derivative ∇ multiplies the estimates by t^{-1} if $\check{r} \leq 1$ and by $t^{-1} \check{r}^{-1}$ if $\check{r} \geq 1$, so (6.9) follows easily in the same way as (6.8).
q.e.d.

We can now estimate the norms of $\Theta(\varphi_t^N) - \psi_t^N$ needed in Theorem 2.7.

Proposition 6.3. *For all $t \in (0, \epsilon]$ we have*
(6.15)

$$\begin{aligned} \|\Theta(\varphi_t^N) - \psi_t^N\|_{C^0} &= O(t^{16/9}), & \|\Theta(\varphi_t^N) - \psi_t^N\|_{L^2} &= O(t^{32/9}), \\ \text{and} \quad \|\mathrm{d}(\Theta(\varphi_t^N) - \psi_t^N)\|_{L^{14}} &= O(t^{8/7}), \end{aligned}$$

where the norms are computed using the metric g_t^N on N .

Proof. Table 6.1 gives the orders of the contributions to the norms in (6.15) from each region in (6.8)–(6.9). We can compute each of these contributions using (6.8)–(6.9) and the facts, which are obvious from the construction, that the region $\check{r} \leq 1$ in (N, g_t^P) has volume $O(t^4)$, and for $1 \leq \check{r} \leq 2t^{-4/5}$, the region in (N, g_t^P) with $s \leq \check{r} \leq s + \delta s$ has volume $O(t^4 s^3 \delta s)$ for small $\delta s > 0$.

Table 6.1. Contributions of regions to norms of $\Theta(\varphi_t^N) - \psi_t^N$, any $\gamma > 0$

	$\ \Theta(\varphi_t^N) - \psi_t^N\ _{C^0}$	$\ \Theta(\varphi_t^N) - \psi_t^N\ _{L^2}$	$\ \mathrm{d}(\Theta(\varphi_t^N) - \psi_t^N)\ _{L^{14}}$
$\check{r} \leq 1$	$O(t^2)$	$O(t^4)$	$O(t^{9/7})$
$1 \leq \check{r} \leq t^{-1/9}$	$O(t^{16/9})$	$O(t^{32/9})$	$O(t^{8/7})$
$t^{-1/9} \leq \check{r} \leq 2t^{-1/9}$	$O(t^{16/9})$	$O(t^{32/9})$	$O(t^{8/7})$
$2t^{-1/9} \leq \check{r} \leq t^{-4/5}$	$O(t^{20/9-\gamma/9})$	$O(t^{4-4\gamma/5})$	$O(t^{100/63-\gamma/9})$
$t^{-4/5} \leq \check{r} \leq 2t^{-4/5}$	$O(t^{16/5})$	$O(t^{18/5})$	$O(t^{107/35})$
$r \geq 2t^{-4/5}$	0	0	0

The $O(t^4)$ volume estimate in the region $\check{r} \leq 1$ gives the first row of Table 6.1. If a quantity is $O(t^a \check{r}^b)$ for $b \neq -2$ in the region $A \leq \check{r} \leq B$, then its L^2 norm on this region is given by

$$\|O(t^a \check{r}^b)\|_{L^2} = O\left(\int_A^B (t^a s^b)^2 t^4 s^3 ds\right)^{\frac{1}{2}} = O(t^{a+2}(A^{b+2} + B^{b+2})).$$

Similarly if a quantity is $O(t^a \check{r}^b)$ for $b \neq -\frac{2}{7}$ in the region $A \leq \check{r} \leq B$, then its L^{14} norm on this region is given by

$$\|O(t^a \check{r}^b)\|_{L^{14}} = O\left(\int_A^B (t^a s^b)^{14} t^4 s^3 ds\right)^{\frac{1}{14}} = O(t^{a+\frac{2}{7}}(A^{b+\frac{2}{7}} + B^{b+\frac{2}{7}})).$$

Using these observations, one obtains the entries of Table 6.1 easily. Taking the largest term in each column in Table 6.1 for small t proves (6.15). q.e.d.

6.4. Proof of the main theorem. We can now at last prove the main theorem of this paper:

Theorem 6.4. *Let (M, φ, g) be a compact, torsion-free G_2 -manifold, and let $\iota : M \rightarrow M$ be a nontrivial involution preserving (φ, g) , so that*

the fixed locus L of ι is a compact associative 3-fold in M by Proposition 2.13.

Suppose L is nonempty, and that there exists a closed, coclosed, non-vanishing 1-form λ on L . That is, $\lambda \in \Omega^1(L)$ with $d\lambda = d^*\lambda = 0$, where d^* is defined using $g|_L$, and $\lambda|_x \neq 0$ in T_x^*L for all $x \in L$.

Then there exists a compact 7-manifold N defined as a resolution of singularities $\pi : N \rightarrow M/\langle \iota \rangle$ of the 7-orbifold $M/\langle \iota \rangle$ along its singular locus $L \subset M/\langle \iota \rangle$, by gluing in a bundle $\sigma : P \rightarrow L$ along L , with fibre the Eguchi–Hanson space X from §2.5, where P is constructed using λ . The preimage $\pi^{-1}(L)$ is a 5-submanifold Q of N , and $\pi|_Q : Q \rightarrow L$ is a smooth bundle with fibre \mathcal{S}^2 . The fundamental group of N satisfies $\pi_1(N) \cong \pi_1(M/\langle \iota \rangle)$, and the Betti numbers are

$$(6.16) \quad b^k(N) = b^k(M/\langle \iota \rangle) + b^{k-2}(L).$$

There exists a smooth family $(\tilde{\varphi}_t^N, \tilde{g}_t^N)$ of torsion-free G_2 -structures on N for $t \in (0, \epsilon]$, with $\epsilon > 0$ small, such that $(\tilde{\varphi}_t^N, \tilde{g}_t^N) \rightarrow \pi^*(\varphi, g)$ in C^0 away from Q as $t \rightarrow 0$, and for each $x \in L$ the fibre $\pi^{-1}(x) \cong \mathcal{S}^2$ with metric $\tilde{g}_t^N|_{\pi^{-1}(x)}$ approximates a small round 2-sphere with area $\pi t^2 |\lambda|_x|$ for small t . The metrics \tilde{g}_t^N on N have holonomy G_2 if and only if $M/\langle \iota \rangle$ has finite fundamental group.

Proof. The first part of the theorem, the construction of $\pi : N \rightarrow M/\langle \iota \rangle$ and Q , was done in §6.1, and the claims on fundamental group and Betti numbers are proved in Proposition 6.1. For the second part, we will apply Theorem 2.7 to the family of G_2 -structures (φ_t^N, g_t^N) and 4-forms ψ_t^N on N for $t \in (0, \epsilon]$ defined in §6.2. By Proposition 6.3, to ensure that Theorem 2.7(i) holds for (φ_t^N, g_t^N) , ψ_t^N for all $t \in (0, \epsilon]$, we need to find $\alpha > 0$ such that $\alpha < \frac{16}{9}$, $\frac{7}{2} + \alpha < \frac{32}{9}$, and $-\frac{1}{2} + \alpha < \frac{8}{7}$. In particular, these all hold for $\alpha < \frac{1}{18}$.

We can see from the construction that g_t^N satisfies Theorem 2.7(ii), (iii) for some $K_2, K_3 > 0$ and all $t \in (0, \epsilon]$, since where $\tilde{r} \leq 1$ the local injectivity radius of g_t^N is $O(t)$ and the curvature is $O(t^{-2})$, and where $1 \leq \tilde{r} \leq t^{-1}R$ the local injectivity radius of g_t^N is $O(t\tilde{r})$ and the curvature is $O(t^{-2}\tilde{r}^{-2})$, and outside $\rho^{-1}(U_{t^{-1}R}/\{\pm 1\})$ the local injectivity radius and curvature of g_t^N are both $O(1)$.

Thus Theorem 2.7 shows that after making $\epsilon > 0$ smaller if necessary, for all $t \in (0, \epsilon]$ there exists a smooth, torsion-free G_2 -structure $(\tilde{\varphi}_t^N, \tilde{g}_t^N)$ on N such that $\|\tilde{\varphi}_t^N - \varphi_t^N\|_{C^0} \leq K_4 t^{1/18}$ for some $K_4 > 0$ independent of t , computing the norm using g_t^N , and $[\tilde{\varphi}_t^N] = [\varphi_t^N]$ in $H^3(M, \mathbb{R})$. The proof of Theorem 2.7 in [20] implies that $(\tilde{\varphi}_t^N, \tilde{g}_t^N)$ depends smoothly on $t \in (0, \epsilon]$, as (φ_t^N, g_t^N) does.

The claims on convergence $(\tilde{\varphi}_t^N, \tilde{g}_t^N) \rightarrow \pi^*(\varphi, g)$ away from Q as $t \rightarrow 0$, and $(\pi^{-1}(x), \tilde{g}_t^N|_{\pi^{-1}(x)})$ approximating a small round 2-sphere with area $\pi |\lambda|_x| t^2$ for $x \in L$ and small t , hold as $\|\tilde{\varphi}_t^N - \varphi_t^N\|_{C^0} \leq K_4 t^{1/18}$ and

the analogous claims hold for (φ_t^N, g_t^N) by construction, using the fact that the area of the bolt is $\pi|\lambda|_x|t^2$ as in Remark 2.10. Since $\pi_1(N) \cong \pi_1(M/\langle\iota\rangle)$, the last part follows from Theorem 2.3. This completes the proof. q.e.d.

Remark 6.5. We explain why it is necessary that the 1-form λ on L used in the construction be closed and coclosed. Without assuming $d\lambda = d(*\lambda) = 0$, suppose that using λ we construct a 7-manifold N and a family of G_2 -structures (φ_t^N, g_t^N) for $t \in (0, \epsilon]$ as in §6.2, and suppose that as in the theorem there exist a family of torsion-free G_2 -structures $(\tilde{\varphi}_t^N, \tilde{g}_t^N)$ on N with $\|\tilde{\varphi}_t^N - \varphi_t^N\|_{g_t^N, C^0} = O(t^{1/18})$ for $t \in (0, \epsilon]$. As in §6.1 we have a 5-submanifold $Q \subset N$ with a submersion $\Pi : Q \rightarrow L$, whose fibres $\Pi^{-1}(x)$ for $x \in L$ are round 2-spheres \mathcal{S}^2 with area $\pi t^2|\lambda|_x|$ in the metric g_t^N .

Recall that if $f : X \rightarrow Y$ is a submersion of compact, oriented manifolds with $\dim X = m$, $\dim Y = n$, and α is a k -form on X for $k \geq m - n$, there is a unique pushforward $f_*(\alpha)$, a $(k - m + n)$ -form on Y , with the property that if β is an $(m - k)$ -form on Y then $\int_Y f_*(\alpha) \wedge \beta = \pm \int_X \alpha \wedge f^*(\beta)$. Such pushforwards have the property that $d(f_*(\alpha)) = f_*(d\alpha)$. Consider the equations

$$(6.17) \quad \begin{aligned} \Pi_*(\tilde{\varphi}_t^N|_Q) &= \Pi_*(\varphi_t^N) + O(t^{37/18}) = -\pi t^2 \lambda + O(t^{37/18}), \\ \Pi_*(\Theta(\tilde{\varphi}_t^N)|_Q) &= \Pi_*(\Theta(\varphi_t^N)) + O(t^{37/18}) = -\pi t^2 (*\lambda) + O(t^{37/18}). \end{aligned}$$

Here the first steps hold as $\|\tilde{\varphi}_t^N - \varphi_t^N\|_{g_t^N, C^0} = O(t^{1/18})$ and the fibres of Π have volume $O(t^2)$, and the second steps hold by definition of φ_t^N . As $\tilde{\varphi}_t^N, \Theta(\tilde{\varphi}_t^N)$ are closed, the left hand sides of (6.17) are closed. Hence, multiplying (6.17) by t^{-2} and taking the limit as $t \rightarrow 0$, we see that λ and $*\lambda$ are closed.

6.5. Replacing $M/\langle\iota\rangle$ by a G_2 -orbifold. The construction of §3–§6.4 begins with a compact torsion-free G_2 -manifold (M, φ, g) with an involution $\iota : M \rightarrow M$ with $\iota^*(\varphi) = \varphi$, where the fixed locus L of ι is a 3-submanifold in M . Then $M/\langle\iota\rangle$ is an orbifold, with orbifold stratum L , and (φ, g) descends to a torsion-free G_2 -structure on $M/\langle\iota\rangle$ in the orbifold sense, also written (φ, g) . We define a resolution $\pi : N \rightarrow M/\langle\iota\rangle$ and construct torsion-free G_2 -structures $(\tilde{\varphi}_t^N, \tilde{g}_t^N)$ on N .

In fact the whole programme also works if we start not with a quotient $M/\langle\iota\rangle$, but with a compact torsion-free G_2 -orbifold (M', φ, g) , such that the only orbifold stratum of M' is a compact 3-submanifold L such that M' near L is locally modelled on $\mathbb{R}^3 \times (\mathbb{R}^4/\{\pm 1\})$, and we are given a closed, coclosed, nonvanishing 1-form λ on $(L, g|_L)$. It is not necessary that M' should be a global quotient $M/\langle\iota\rangle$.

We chose to start from $M/\langle\iota\rangle$ in the main text for simplicity, to avoid dealing with orbifolds. But in fact the general case of G_2 -orbifolds

(M', φ, g) is essentially identical, requiring no extra work, so we will not go through the details. In §7.2–§7.4 we discuss examples involving G_2 -orbifolds (M', φ, g) in which M' cannot be written as $M/\langle \iota \rangle$ for a 7-manifold M .

6.6. Twisting λ by a principal \mathbb{Z}_2 -bundle. The construction of §3–§6.4 involves a closed, coclosed, nonvanishing 1-form λ on the fixed locus L of ι in M . This λ was used in §4 to define a resolution P of $\nu/\{\pm 1\}$, where λ is used to define a complex structure I on the fibres of $\nu \rightarrow L$, and then $\rho : P \rightarrow \nu/\{\pm 1\}$ is the bundle blow-up of $\nu/\{\pm 1\}$ along the zero section $0(L)$ using I .

Consider the effect of replacing λ by $-\lambda$ in §4. This replaces I by $-I$, but the blow-up P is unchanged. It turns out that the data $\varphi_t^P, \psi_t^P, g_t^P, \tilde{\varphi}_t^P, \tilde{\psi}_t^P, \xi_{1,2}, \xi_{0,3}, \chi_{1,3}, \theta_{3,1}, \theta_{2,2}, \dots$ defined in §4.4 is unchanged as well. This holds because $\tilde{\omega}^I, e_1, e_2 \wedge e_3$ change sign, but combinations such as $\sigma^*(e_1) \wedge \tilde{\omega}^I$ and $\sigma^*(e_2) \wedge \sigma^*(e_3) \wedge \tilde{\omega}^I$ in (4.21) are unchanged. The constructions of §5–§6.4, and the final result $N, (\tilde{\varphi}_t^N, \tilde{g}_t^N)$ in Theorem 6.4, only involve data which are unchanged by replacing λ by $-\lambda$.

In fact the construction still works if λ is only defined up to sign $\pm \lambda$ locally in L . Here is what we mean by this. Suppose $\pi : Z \rightarrow L$ is a principal \mathbb{Z}_2 -bundle over L . Then we can form the vector bundles $\Lambda^k T^* L \otimes_{\mathbb{Z}_2} Z \rightarrow L$ of k -forms on L twisted by Z . The d and d^* operators on L extend naturally to $d : \Gamma^\infty(\Lambda^k T^* L \otimes_{\mathbb{Z}_2} Z) \rightarrow \Gamma^\infty(\Lambda^{k+1} T^* L \otimes_{\mathbb{Z}_2} Z)$ and $d^* : \Gamma^\infty(\Lambda^k T^* L \otimes_{\mathbb{Z}_2} Z) \rightarrow \Gamma^\infty(\Lambda^{k-1} T^* L \otimes_{\mathbb{Z}_2} Z)$, so it makes sense for a Z -twisted k -form λ to be closed and coclosed. If locally in L we choose a trivialization $Z \cong L \times \mathbb{Z}_2$, then we have local canonical isomorphisms $\Lambda^k T^* L \otimes_{\mathbb{Z}_2} Z \cong \Lambda^k T^* L$, which identify Z -twisted k -forms with ordinary k -forms. These depend on the trivialization up to sign.

We can generalize the construction of §3–§6.4 as follows: we choose a principal \mathbb{Z}_2 -bundle $\pi : Z \rightarrow L$, and take λ to be a closed, coclosed, nonvanishing, Z -twisted 1-form on L , instead of an ordinary 1-form. Locally in L we can identify λ with an ordinary 1-form on L up to sign, and this is enough to define $P, \varphi_t^P, \psi_t^P, g_t^P, \dots, N, \tilde{\varphi}_t^N, \tilde{g}_t^N$ in §4–§6.4, since as above these are all unchanged by replacing λ by $-\lambda$, and the analogue of Theorem 6.4 holds.

Proposition 6.1 and equation (6.16) must be modified. In the Z -twisted case we no longer have $Q \cong L \times \mathcal{S}^2$, so (6.5) does not hold. Instead it turns out that

$$H^k(Q; \mathbb{R}) \cong H^k(L; \mathbb{R}) \oplus H^{k-2}(L, Z; \mathbb{R}),$$

where $H^*(L, Z; \mathbb{R})$ is the Z -twisted de Rham cohomology of L , the cohomology of the complex $(\Gamma^\infty(\Lambda^k T^* L \otimes_{\mathbb{Z}_2} Z)_{k \geq 0}, d)$. Thus in the third row of (6.4) we replace $H^*(L; \mathbb{R})$ by $H^*(L, Z; \mathbb{R})$, and (6.2) and (6.16) are replaced by

$$(6.18) \quad b^k(N) = b^k(M/\langle \iota \rangle) + b^{k-2}(L, Z),$$

where $b^k(L; Z) = \dim H^k(L, Z; \mathbb{R})$ are the Z -twisted Betti numbers of L . In Example 7.3 we construct compact G_2 -manifolds using Z -twisted 1-forms.

7. Examples

7.1. General discussion on how to produce examples. To apply Theorem 6.4, we have to find:

- (a) A compact torsion-free G_2 -manifold (M, φ, g) and an involution $\iota : M \rightarrow M$ with $\iota^*(\varphi) = \varphi$ and fixed points a 3-submanifold $L \subset M$.

Or more generally, as in §6.5, a compact torsion-free G_2 -orbifold (M', φ, g) with one orbifold stratum L locally modelled on $\mathbb{R}^3 \times (\mathbb{R}^4 / \{\pm 1\})$.

- (b) A closed, coclosed, nonvanishing 1-form λ on L .

Or more generally, as in §6.6, a principal \mathbb{Z}_2 -bundle $\pi : Z \rightarrow L$ and a closed, coclosed, nonvanishing, Z -twisted 1-form on L .

There are several obvious ways to produce the data in (a):

- (i) We could make $(M, \varphi, g), \iota$ or (M', φ, g) using one of the known constructions of compact 7-manifolds with holonomy G_2 due to the first author, Kovalev–Lee, or Corti–Haskins–Nordström–Pacini [7, 18, 19, 20, 21, 27, 28], and either leave some orbifold singularities unresolved, or do the construction \mathbb{Z}_2 -equivariantly, and divide by \mathbb{Z}_2 at the end of the construction.
- (ii) We could take M' of the form $(T^3 \times K3)/\Gamma$ for Γ a finite group (e.g. $\Gamma = \mathbb{Z}_2^2$), where T^3 carries a flat metric and $K3$ a hyperKähler metric.
- (iii) We could take $M = \mathcal{S}^1 \times Y$ for Y a Calabi–Yau 3-fold, and $\iota : \mathcal{S}^1 \times Y \rightarrow \mathcal{S}^1 \times Y$ to act by $\iota(x, y) = (-x, \tau(y))$ for $\tau : Y \rightarrow Y$ an antiholomorphic involution, with fixed locus a special Lagrangian 3-fold L' in Y . Then $L = \{0, \frac{1}{2}\} \times L'$. This situation was described in Examples 2.12 and 2.15.

However, there is a difficulty in finding a suitable 1-form λ in (b). As $(L, g|_L)$ is a compact Riemannian 3-manifold, we know by Hodge theory that the vector space \mathcal{H}^1 of closed, coclosed 1-forms λ on L is isomorphic to the de Rham cohomology group $H^1(L; \mathbb{R})$, which we can often compute. But unless we know something about the metric $g|_L$ on L , we do not know that any $\lambda \in \mathcal{H}^1$ is *nonvanishing*; conceivably every $\lambda \in \mathcal{H}^1$ might have at least one zero.

Thus, to apply Theorem 6.4 we restrict to cases in which we can describe $(L, g|_L)$ as a Riemannian 3-manifold. Note in particular that in case (iii), if the Calabi–Yau metric h on Y comes from Yau’s proof of the Calabi Conjecture, then without further information on h , we

know nothing about $h|_{L'}$, so we cannot deduce the existence of closed, coclosed, nonvanishing 1-forms λ on L' .

Here are some useful facts for constructing such 1-forms λ :

- (A) Suppose (L, g) is a 3-torus T^3 with a flat metric. Then there is a space $\mathbb{R}^3 \setminus \{0\}$ of closed, coclosed, nonvanishing 1-forms λ on L .
- (B) Suppose (L, g) is a product $(\mathcal{S}^1 \times \Sigma, dx^2 + h)$, where (Σ, h) is a compact Riemannian 2-manifold (of any genus $g \geq 0$), and $x \in \mathbb{R}/\mathbb{Z}$ is the coordinate on \mathcal{S}^1 . Then $\lambda = dx$ is closed, coclosed, and nonvanishing on L .
- (C) Suppose (L, g) is a product $(\mathcal{S}^1 \times \Sigma, dx^2 + h)$, where Σ is an oriented torus T^2 , and h is a Riemannian metric on Σ , which need not be flat.

The notion of harmonic 1-form on (Σ, h) depends only on the conformal class of h , and h is conformal to a flat metric. Thus there is an $\mathbb{R}^2 \setminus \{0\}$ space of closed, coclosed, nonvanishing 1-forms on (Σ, h) , which pull back to closed, coclosed, nonvanishing 1-forms λ on L .

- (D) In each of (A)–(C) we can replace L by L/Γ , where Γ is a finite group acting freely on L by isometries preserving λ .

More generally, we can allow Γ to preserve λ only up to sign, and then λ descends to a closed, coclosed, nonvanishing, \mathbb{Z} -twisted 1-form on L/Γ .

- (E) Let (L, g) be a compact Riemannian 3-manifold, and λ a closed, coclosed, nonvanishing 1-form on (L, g) . Suppose g' is another Riemannian metric on L . Then there exists a unique closed, coclosed 1-form λ' on (L, g') with $[\lambda'] = [\lambda] \in H_{\text{dR}}^1(L; \mathbb{R})$. One can show that if $\|g' - g\|_{C^0}$ is small then $\|\lambda' - \lambda\|_{C^0} \leq C\|g' - g\|_{C^0}$ for $C > 0$ depending only on L, g, λ . But if $\|\lambda' - \lambda\|_{C^0}$ is sufficiently small then λ' is nonvanishing, as λ is. This proves that if (L, g) admits a closed, coclosed, nonvanishing 1-form λ , and g' is any other metric on L with $\|g' - g\|_{C^0}$ sufficiently small, then (L, g') also admits a closed, coclosed, nonvanishing 1-form λ' .

We can apply this in (A)–(D), so that the metric g need only be C^0 -close to being flat, or a product $dx^2 + h$.

7.2. Resolving orbifolds T^7/Γ in several stages. In [18, 19], [20, §11–§12] the first author constructed examples of compact 7-manifolds with holonomy G_2 by resolving orbifolds T^7/Γ , where Γ is a finite group acting on T^7 preserving a flat G_2 -structure (φ_0, g_0) . We explain how to modify this construction. Suppose $(T^7/\Gamma, \varphi_0, g_0)$ is such a flat G_2 -orbifold, and the orbifold singularities of T^7/Γ split as $S \amalg T$, where S is a union of orbifold strata that can be resolved using the methods of [18, 19, 20], and T is a disjoint union of orbifold strata T^3/Δ , where Δ is a finite group acting freely on $T^3 \subset T^7$, and the singularities are locally modelled on $(T^3 \times (\mathbb{R}^4/\{\pm 1\}))/\Delta$.

Apply the methods of [18, 19, 20] (which work for partial resolutions in orbifolds, as well as for complete resolutions in manifolds) to resolve the singularities S in T^7/Γ , but leave T unresolved. This gives a 7-orbifold M' with resolution map $\pi : M' \rightarrow T^7/\Gamma$, and a family of torsion-free G_2 -structures (φ_s, g_s) on M' for $s \in (0, \epsilon]$ with $\epsilon > 0$ small. Here s is the length-scale of the resolution of the singular set S , so that near $\pi^{-1}(S)$ the curvature $R(g_s)$ is $O(s^{-2})$ and the injectivity radius $\delta(g_s)$ is $O(s)$, and away from $\pi^{-1}(S)$ we have $\|g_s - \pi^*(g_0)\|_{C^0} = O(s^\kappa)$ for some $\kappa > 0$ as in Theorem 2.7 (we can take $\kappa = \frac{1}{2}$ as in [20, Th. 11.6.1]).

Now M' has singular set T . As T is disjoint from S , we have $\|g_s|_T - \pi^*(g_0|_T)\|_{C^0} = O(s^{1/2})$, so $g_s|_T$ is C^0 -close to the flat metric $g_0|_T$ on T for small $s > 0$. Suppose each component T^3/Δ of T has $b^1(T^3/\Delta) > 0$ (or more generally, T^3/Δ has a principal \mathbb{Z}_2 -bundle Z with $H^1(T^3/\Delta, Z; \mathbb{R}) \neq 0$, as in §6.6). Then as in §7.1(A), (D) there exists a closed, coclosed, nonvanishing 1-form λ_0 on $(T, g_0|_T)$ (possibly Z -twisted as in §6.6), so by §7.1(E) there also exists a closed, coclosed, nonvanishing 1-form λ_s on $(T, g_s|_T)$ for small enough $s > 0$ (possibly Z -twisted as in §6.6). Thus we can apply Theorem 6.4, modified as in §6.5–§6.6, to construct a compact 7-manifold N resolving M' , and a family $(\varphi_{s,t}^N, g_{s,t}^N)$ of torsion-free G_2 -structures on N , where $t > 0$ must be small in terms of s , that is, $0 < t \ll s \ll 1$.

Arguably these examples are not that interesting, as we could just have applied [18, 19, 20] to get torsion-free G_2 -structures (φ_s^N, g_s^N) on the same 7-manifold N in a single step, by resolving the singularities S, T simultaneously, rather than first resolving S , then resolving T . However, the G_2 -structures $(\varphi_{s,t}^N, g_{s,t}^N)$ we have constructed are different from those of [18, 19, 20], as they involve two length-scales s, t with $0 < t \ll s \ll 1$.

In fact the authors expect that one can resolve the disjoint singular sets S, T in T^7/Γ completely independently, so that there should exist torsion-free G_2 -structures $(\varphi_{s,t}^N, g_{s,t}^N)$ on N which resolve S at length-scale $s > 0$ and T at length scale $t > 0$, whenever $0 < s, t \ll 1$. But this is not proved. The methods of [18, 19, 20] give such $(\varphi_{s,t}^N, g_{s,t}^N)$ when s, t are about the same size, that is, $0 < s, t \ll 1$ with $s \approx t$, and the modification above gives $(\varphi_{s,t}^N, g_{s,t}^N)$ when t is much smaller than s , that is, $0 < t \ll s \ll 1$, but the two constructions do not overlap.

We can also generalize all this to resolve orbifolds T^7/Γ with singular set $S \amalg T_1 \amalg \cdots \amalg T_k$, for S as above and T_1, \dots, T_k as for T above, where we iteratively resolve S, T_1, T_2, \dots, T_k in this order, at length-scales s, t_1, t_2, \dots, t_k respectively, where $0 < t_k \ll t_{k-1} \ll \cdots \ll t_1 \ll s \ll 1$.

Here is an example, taken from [18, §2.1]. The labelling has been changed to be compatible with (2.1).

Example 7.1. Let $T^7 = \mathbb{R}^7/\mathbb{Z}^7$ with standard G_2 -structure (φ_0, g_0) given in (2.1)–(2.2). Let $\Gamma = \langle \alpha, \beta, \gamma \rangle \cong \mathbb{Z}_2^3$, where α, β, γ are involutions

acting by

$$\begin{aligned}\alpha(x_1, \dots, x_7) &= (-x_1, -x_2, x_3, -x_4, x_5, x_6, -x_7), \\ \beta(x_1, \dots, x_7) &= (\tfrac{1}{2} - x_1, x_2, -x_3, -x_4, x_5, -x_6, x_7), \\ \gamma(x_1, \dots, x_7) &= (x_1, \tfrac{1}{2} - x_2, -x_3, \tfrac{1}{2} - x_4, -x_5, x_6, x_7).\end{aligned}$$

These preserve the G_2 -structure (φ_0, g_0) . The only elements of Γ with fixed points are $1, \alpha, \beta, \gamma$, where α, β, γ each fix 16 T^3 's. The quotient T^7/Γ is simply-connected, with singular set:

- (a) 4 copies of T^3 from the fixed points of α .
- (b) 4 copies of T^3 from the fixed points of β .
- (c) 4 copies of T^3 from the fixed points of γ .

These 12 T^3 's are pairwise disjoint. Near each T^3 , the space T^7/Γ is modelled on $T^3 \times (\mathbb{R}^4/\{\pm 1\})$.

Split the singular set as $S \amalg T$, where S is k of the 12 T^3 's, and T is the remaining $12 - k$ of the T^3 's, for $0 < k < 12$. Then the construction above applies. The 7-manifold N resolving T^7/Γ is described in [18, §2.2], and is simply-connected with $b^2(N) = 12$ and $b^3(N) = 43$. The construction above yields a family of torsion-free G_2 -structures $(\varphi_{s,t}^N, g_{s,t}^N)$ on N , which have holonomy G_2 .

We can do the same thing for many of the examples in [18, 19, 20].

It is also possible to do a similar two-stage (or multiple-stage) resolution process for examples of G_2 -manifolds coming from the ‘twisted connected sum’ constructions of [6, 7, 27, 28]. These work by gluing together noncompact G_2 -manifolds $\mathcal{S}^1 \times Y_1$ and $\mathcal{S}^1 \times Y_2$, where Y_1, Y_2 are Asymptotically Cylindrical Calabi–Yau 3-folds satisfying a matching condition at their infinite ends. We could instead allow Y_1, Y_2 to be Calabi–Yau orbifolds, with compact singular sets Σ_1, Σ_2 locally modelled on $\mathbb{C} \times (\mathbb{C}^2/\{\pm 1\})$.

Gluing $\mathcal{S}^1 \times Y_1$ and $\mathcal{S}^1 \times Y_2$ together using [6, 7, 27, 28] would give a family of compact G_2 -orbifolds (M', φ_s, g_s) , with singular set $(\mathcal{S}^1 \times \Sigma_1) \amalg (\mathcal{S}^1 \times \Sigma_2)$ locally modelled on $\mathbb{R}^3 \times (\mathbb{R}^4/\{\pm 1\})$. We could then resolve this using Theorem 6.4 generalized as in §6.5 and §7.1(B), (E), to get torsion-free G_2 -structures $(\varphi_{s,t}^N, g_{s,t}^N)$ on a 7-manifold N . This is the same N we would get by applying [6, 7, 27, 28] to glue $\mathcal{S}^1 \times \tilde{Y}_1$ and $\mathcal{S}^1 \times \tilde{Y}_2$, where \tilde{Y}_1, \tilde{Y}_2 are the crepant resolutions of Y_1, Y_2 , so we get new torsion-free G_2 -structures on known examples of 7-manifolds with holonomy G_2 . Carrying out this procedure for some explicit examples could be a project for future study.

7.3. Resolving orbifolds $(T^3 \times K3)/\Gamma$. Next we apply our construction to resolve G_2 -orbifolds of the form $(T^3 \times X)/\Gamma$, where X is a hyperKähler $K3$ surface.

Take $T^3 = \mathbb{R}^3/\Lambda$, where Λ is a lattice in \mathbb{R}^3 (e.g. $\Lambda = \mathbb{Z}^3$). Let X be a hyperKähler $K3$ surface with metric h , complex structures I, J, K , and hyperKähler forms $\omega^I, \omega^J, \omega^K$. Define a torsion-free G_2 -structure (φ, g) on $T^3 \times X$ as in (2.13). Suppose we can find a finite group Γ with an action on $T^3 \times X$ preserving (φ, g) , so $M' = (T^3 \times X)/\Gamma$ is a 7-orbifold with G_2 -structure (φ, g) , such that the singular set of M' can be resolved using our construction.

As Γ acts by isometries, one can show that in this case the action of Γ on $T^3 \times X$ must be a product of Γ -actions on T^3 and on X . To get orbifold singularities of the right type, it is necessary that any $1 \neq \gamma \in \Gamma$ with fixed points in $T^3 \times X$ must be of order 2. There are two possibilities for such γ :

- (a) γ acts trivially on T^3 , and fixes $2k$ points on X , where γ acts on $H^2(X; \mathbb{R})$ with $10 + k$ eigenvalues $+1$ and $12 - k$ eigenvalues -1 . Hence $\text{Fix}(\gamma)$ is $2k$ copies of T^3 in $T^3 \times X$.
- (b) γ fixes l circles \mathcal{S}^1 in T^3 , where $l = 2$ or 4 , and γ fixes a 2-submanifold Σ in X . Hence $\text{Fix}(\gamma)$ is l copies of $\mathcal{S}^1 \times \Sigma$ in $T^3 \times X$.

The corresponding orbifold stratum in $M' = (T^3 \times X)/\Gamma$ is $\text{Fix}(\gamma)/\Delta$, where $\Delta = \{\delta \in \Gamma : \delta\gamma = \gamma\delta\}/\{1, \gamma\}$.

In case (b), after applying an $SO(3)$ transformation to $\langle I, J, K \rangle_{\mathbb{R}}$ we can suppose that γ fixes I and changes the signs of J, K . Hence γ is a *nonsymplectic involution* of the complex symplectic $K3$ surface $(X, I, \omega^J + i\omega^K)$. Nonsymplectic involutions have been completely classified by Nikulin [34], see also [2] and [28, §3]. Nikulin shows [34, §4] that Σ must be either the disjoint union of two T^2 's, or the disjoint union of a genus g surface and k \mathbb{CP}^1 's, where there are 64 possibilities for (g, k) with $0 \leq g \leq 10$ and $0 \leq k \leq 9$.

A huge amount is known about $K3$ surfaces, and using this we could construct many examples of hyperKähler $K3$ surfaces X with suitable actions of a finite group Γ . Basically, everything is determined by the choice of faithful Γ -representation on the $K3$ lattice $H^2(X; \mathbb{Z}) \cong 2(-E_8) \oplus 3H$, which fixes a positive definite subspace $\langle \omega^I, \omega^J, \omega^K \rangle_{\mathbb{R}} \subset H^2(X; \mathbb{R})$ satisfying some conditions.

One possibility would be to take X to be a resolution of T^4/Δ , and the Γ -action on X to lift from a Γ -action on T^4/Δ . Then $M' = (T^3 \times X)/\Gamma$ would be a partial resolution of $(T^3 \times T^4)/(\Gamma \ltimes \Delta)$, and the G_2 -manifolds N we eventually construct would be a complete resolution of $T^7/(\Gamma \ltimes \Delta)$. In this case, the 7-manifold N we get would be the same as if we resolved $T^7/(\Gamma \ltimes \Delta)$ in a single step using [18, 19, 20], as in §7.2. However, not all examples are of this type.

Example 7.2. Let C be the nonsingular sextic curve in \mathbb{CP}^2

$$C = \{[z_0, z_1, z_2] \in \mathbb{CP}^2 : z_0^6 + z_1^6 + z_2^6 = 0\}.$$

Then C has genus 10, by the degree-genus formula. Define $\pi : X \rightarrow \mathbb{CP}^2$ to be the double cover of \mathbb{CP}^2 branched over C . Then X is a complex $K3$ surface, with complex structure I . Write $\alpha : X \rightarrow X$ to be the holomorphic involution swapping the sheets of the double cover, so that $\pi \circ \alpha = \pi$. Then α is a nonsymplectic involution, with $\text{Fix}(\alpha) = C \subset X$.

Consider the antiholomorphic involution of \mathbb{CP}^2 mapping $[z_0, z_1, z_2] \mapsto [\bar{z}_0, \bar{z}_1, \bar{z}_2]$. This lifts through π to two antiholomorphic involutions of X , which we write as β and $\alpha\beta$, where we take β to have fixed set $\text{Fix}(\beta) = \{x \in X : \pi(x) \in \mathbb{RP}^2 \subset \mathbb{CP}^2\}$, and $\alpha\beta$ to be free. Then $\text{Fix}(\beta)$ is the double cover of \mathbb{RP}^2 , that is, $\text{Fix}(\beta) \cong \mathcal{S}^2$, and $\text{Fix}(\alpha\beta) = \emptyset$, and $\Gamma = \{1, \alpha, \beta, \alpha\beta\} \cong \mathbb{Z}_2^2$ acts on X .

By the Calabi Conjecture we can extend (X, I) to a hyperKähler $K3$ surface $(X, I, J, K, h, \omega^I, \omega^J, \omega^K)$, where we can choose J, K such that α, β act by

$$\begin{aligned} \alpha : \omega^I &\mapsto \omega^I, & \alpha : \omega^J &\mapsto -\omega^J, & \alpha : \omega^K &\mapsto -\omega^K, \\ \beta : \omega^I &\mapsto -\omega^I, & \beta : \omega^J &\mapsto \omega^J, & \beta : \omega^K &\mapsto -\omega^K. \end{aligned}$$

Let $T^3 = \mathbb{R}^3/\mathbb{Z}^3$ with coordinates (x_1, x_2, x_3) , $x_i \in \mathbb{R}/\mathbb{Z}$, and let Γ act on T^3 by

$$(7.1) \quad \alpha : (x_1, x_2, x_3) \mapsto (x_1, -x_2, -x_3), \quad \beta : (x_1, x_2, x_3) \mapsto (-x_1, x_2, \tfrac{1}{2} - x_3).$$

The fixed locus of α in T^3 is four \mathcal{S}^1 's, given by $x_2, x_3 \in \{0, \frac{1}{2}\}$. Also the fixed locus of β is four \mathcal{S}^1 's, and $\alpha\beta$ acts freely on T^3 .

Define a torsion-free G_2 -structure (φ, g) on $T^3 \times X$ as in (2.13). Then Γ preserves (φ, g) , so $M' = (T^3 \times X)/\Gamma$ is a compact orbifold with torsion-free G_2 -structure (φ, g) . The fixed set of α in $T^3 \times X$ is four copies of $\mathcal{S}^1 \times C$, but the four are exchanged in pairs by β , so these contribute two copies of $\mathcal{S}^1 \times C$ to the singular set of M' . The fixed set of β in $T^3 \times X$ is four copies of $\mathcal{S}^1 \times \mathcal{S}^2$, which are exchanged in pairs by α , and contributes two copies of $\mathcal{S}^1 \times \mathcal{S}^2$ to the singular set of M' . As $\alpha\beta$ acts freely, it contributes no singular points, and the singular sets from α, β are disjoint. Hence the singular set of M' is two copies of $\mathcal{S}^1 \times C$, where C has genus 10, and two copies of $\mathcal{S}^1 \times \mathcal{S}^2$.

The de Rham cohomology $H_{\text{dR}}^k(M'; \mathbb{R})$ is the subspace of $H_{\text{dR}}^k(T^3 \times X; \mathbb{R})$ fixed by Γ . It is a general fact about $K3$ surfaces X that if $\gamma : X \rightarrow X$ is an involution and the induced action of γ on $H^2(X) \cong \mathbb{R}^{22}$ has a eigenvalues $+1$ and $22-a$ eigenvalues -1 then $\chi(\text{Fix}(\alpha)) = 2a - 20$. Since $\chi(\text{Fix}(\alpha)) = -18$, $\chi(\text{Fix}(\beta)) = 2$ and $\chi(\text{Fix}(\alpha\beta)) = 0$ from above, we see that $H^2(X) \cong \mathbb{R} \oplus \mathbb{R}^{11} \oplus \mathbb{R}^{10}$ where $\alpha = 1$ and $\beta = -1$ on \mathbb{R} , and $\alpha = -1$ and $\beta = 1$ on \mathbb{R}^{11} , and $\alpha = \beta = -1$ on \mathbb{R}^{10} . Using this and (7.1) we can compute the action of Γ on

$$H_{\text{dR}}^k(T^3 \times X; \mathbb{R}) \cong \bigoplus_{i+j=k} H_{\text{dR}}^i(T^3; \mathbb{R}) \otimes H_{\text{dR}}^j(X; \mathbb{R}).$$

By taking Γ -invariant parts, we find that $b^0(M') = 1$, $b^1(M') = b^2(M') = 0$, and $b^3(M') = 23$. Moreover, M' is simply-connected.

The singular set L is a product $\mathcal{S}^1 \times (C \amalg C \amalg \mathcal{S}^2 \amalg \mathcal{S}^2)$, and the metric $g|_L$ on L is a product metric. Hence by §7.1(B) we can apply Theorem 6.4 to get a compact 7-manifold N carrying torsion-free G_2 -structures (φ_t^N, g_t^N) . Proposition 6.1 shows that $b^0(N) = 1$, $b^1(N) = 0$, $b^2(N) = 4$ and $b^3(N) = 67$, using that $b^0(L) = 4$ and $b^1(L) = 44$ (which follows from $b^1(C) = 20$), and that N is simply-connected, since M' is. Hence $\text{Hol}(g_t^N) = G_2$ by Theorem 2.3.

Both the first author [20, §12] and Kovalev and Lee [28, Table 1] construct examples of compact 7-manifolds with holonomy G_2 and $b^2 = 4$, $b^3 = 67$. In fact our 7-manifold N can be obtained from the Kovalev–Lee construction [28] by instead using the metric $dx_1^2 + dx_2^2 + R^2 dx_3^2$ on T^3 for $R \gg 0$, so that the x_3 circle is very long. As $R \rightarrow \infty$, $N \cong N_1 \#_{\text{twist}} N_2$ splits as a ‘twisted connect sum’ into a region N_1 from $x_3 \in (-\frac{1}{4}, \frac{1}{4})$ containing the resolution of the α fixed points, and a region N_2 from $x_3 \in (\frac{1}{4}, \frac{3}{4})$ containing the resolution of the β fixed points, where $N_i \cong \mathcal{S}^1 \times Y_i$ for Y_i an Asymptotically Cylindrical Calabi–Yau 3-fold obtained as a resolution of $(X \times \mathbb{C}^*)/\mathbb{Z}_2$.

Example 7.3. Work in the situation of Example 7.2, but instead of (7.1), let Γ act on T^3 by

$$\alpha : (x_1, x_2, x_3) \mapsto (x_1, -x_2, -x_3), \quad \beta : (x_1, x_2, x_3) \mapsto (-x_1, x_2, -x_3).$$

Again the fixed set of α in $T^3 \times X$ is four copies of $\mathcal{S}^1 \times C$, but now $\langle \beta \rangle = \mathbb{Z}_2$ acts freely on each copy, so they contribute four copies of $(\mathcal{S}^1 \times C)/\mathbb{Z}_2$ to the singular set of M' . Similarly, the fixed set of β in $T^3 \times X$ is four copies of $\mathcal{S}^1 \times \mathcal{S}^2$, but $\langle \alpha \rangle = \mathbb{Z}_2$ acts freely on each copy, so they contribute four copies of $(\mathcal{S}^1 \times \mathcal{S}^2)/\mathbb{Z}_2$ to the singular set of M' . Thus, the singular set L of M' is the disjoint union of four copies of $(\mathcal{S}^1 \times C)/\mathbb{Z}_2$ and four copies of $(\mathcal{S}^1 \times \mathcal{S}^2)/\mathbb{Z}_2$.

We resolve M' using Theorem 6.4 modified as in §6.6, twisting by the unique nontrivial principal \mathbb{Z}_2 -bundles Z on $(\mathcal{S}^1 \times C)/\mathbb{Z}_2$ and $(\mathcal{S}^1 \times \mathcal{S}^2)/\mathbb{Z}_2$ induced by the \mathbb{Z}_2 -actions on $\mathcal{S}^1 \times C$ and $\mathcal{S}^1 \times \mathcal{S}^2$. Then we can take the Z -twisted 1-form to be $\lambda = dx$ for $x \in \mathbb{R}/\mathbb{Z}$ the coordinate on \mathcal{S}^1 on each component, and we can apply our construction as in §7.1(B), (D), to get a compact 7-manifold N with torsion-free G_2 -structures (φ_t^N, g_t^N) . Calculation shows that the Z -twisted Betti numbers are

$$\begin{aligned} b^0((\mathcal{S}^1 \times C)/\mathbb{Z}_2, Z) &= 0, & b^1((\mathcal{S}^1 \times C)/\mathbb{Z}_2, Z) &= 11, \\ b^0((\mathcal{S}^1 \times \mathcal{S}^2)/\mathbb{Z}_2, Z) &= 0, & b^1((\mathcal{S}^1 \times \mathcal{S}^2)/\mathbb{Z}_2, Z) &= 1. \end{aligned}$$

As before we have $b^0(M') = 1$, $b^1(M') = b^2(M') = 0$, and $b^3(M') = 23$, and M' is simply-connected, so N is simply-connected, and $\text{Hol}(g_t^N) = G_2$. Using (6.18) we see that the Betti numbers of N are $b^2(N) = 0$ and $b^3(N) = 71$.

Kovalev and Lee [28, Table 1] also construct a compact 7-manifold with holonomy G_2 and $b^2 = 0$, $b^3 = 71$. But in contrast to Example 7.2, here we cannot relate our 7-manifold N to a Kovalev–Lee example by stretching an S^1 in T^3 to separate the α and β fixed points, since these both lie over $(0, 0, 0)$ in T^3 . So this example may be new.

7.4. Describing metrics on some Calabi–Yau 3-folds. Suppose Y is a 3-dimensional projective complex orbifold with trivial canonical bundle, all of whose singularities are locally modelled on $\mathbb{C} \times (\mathbb{C}^2/\{\pm 1\})$. Write $\Sigma \subset Y$ for the singular set, so that Σ is a smooth projective curve. Using the Calabi Conjecture in the orbifold category gives a Calabi–Yau metric h on Y . As in §2.6, $M' = S^1 \times Y$ is a 7-orbifold with torsion-free G_2 -structure (φ, g) , with singular set $L = S^1 \times \Sigma$ locally modelled on $\mathbb{R}^3 \times (\mathbb{R}^4/\{\pm 1\})$.

Since $g = dx^2 + h$ restricts to a product metric on $L = S^1 \times \Sigma$, as in §7.1(A) Theorem 6.4 applies to construct torsion-free G_2 -structures (φ_t^N, g_t^N) on a resolution N of $S^1 \times Y$. In this case everything is equivariant under the obvious action of S^1 on $S^1 \times Y$, and we have $N = S^1 \times \tilde{Y}$, for \tilde{Y} the crepant resolution of Y , which is the blow up of Y along Σ , and the G_2 -structures (φ_t^N, g_t^N) come from Calabi–Yau metrics \tilde{h}_t on \tilde{Y} for small $t > 0$.

Of course, we already knew by the Calabi Conjecture that \tilde{Y} admits Calabi–Yau metrics. However, our construction gives a precise description of the Calabi–Yau metrics \tilde{h}_t on \tilde{Y} in the limit as the \mathbb{CP}^1 fibres of $\pi : \tilde{N} \rightarrow N$ over the singular set Σ become small, and we believe this description is new.

Example 7.4. Define Y to be the orbifold-smooth hypersurface

$$Y = \{[z_0, z_1, z_2, z_3, z_4] \in \mathbb{CP}_{1,1,2,2,2}^4 : z_0^8 + z_1^8 + z_2^4 + z_3^4 + z_4^4 = 0\}$$

in the weighted projective space $\mathbb{CP}_{1,1,2,2,2}^4$. Then Y is a Calabi–Yau 3-orbifold. The singularities of Y are inherited from the orbifold singularities of $\mathbb{CP}_{1,1,2,2,2}^4$, are locally modelled on $\mathbb{C} \times (\mathbb{C}^2/\{\pm 1\})$, and are the genus 6 projective curve

$$\Sigma = \{[0, 0, z_2, z_3, z_4] \in \mathbb{CP}_{1,1,2,2,2}^4 : z_2^4 + z_3^4 + z_4^4 = 0\}.$$

The blow-up \tilde{Y} of Y along Σ is a Calabi–Yau 3-fold. As above we can describe Calabi–Yau metrics \tilde{h}_t on \tilde{Y} close to orbifold Calabi–Yau metrics h on Y .

7.5. Conjectural examples from $(CY3 \times S^1)/\langle \iota \rangle$. Suppose Y is a Calabi–Yau 3-fold with metric h , and $\tau : Y \rightarrow Y$ is an antiholomorphic isometric involution with fixed locus L' , a special Lagrangian 3-fold in Y . Then as in §7.1(iii) we can take $M = S^1 \times Y$ with the induced torsion-free G_2 -structure with product metric $g = dx^2 + h$, and we can define an involution $\iota : M \rightarrow M$ by $\iota(x, y) = (-x, \tau(y))$, so that ι has

fixed locus $L = \{0, \frac{1}{2}\} \times L'$. To apply Theorem 6.4, we need there to exist a closed, coclosed, nonvanishing 1-form λ' on $(L', h|_{L'})$. As in §7.1, if the Calabi–Yau metric h comes from the Calabi Conjecture then we know basically nothing about $h|_{L'}$, so we do not know whether any of the harmonic 1-forms λ on L' are nonvanishing.

However, if the Calabi–Yau manifold (Y, h) is close to some kind of singular limit, then we may have at least a conjectural description of h , which may be enough to say something useful about $h|_{L'}$. In particular, in some special cases we may expect L' to be a disjoint union of T^3 ’s with $h|_{L'}$ C^0 -close to a flat metric, and then there exist closed, coclosed, nonvanishing 1-forms λ' on L' as in §7.1(A), (E), so we could apply Theorem 6.4 to resolve $M/\langle \iota \rangle$, giving a compact 7-manifold N with torsion-free G_2 -structures (φ_t^N, g_t^N) .

The famous ‘SYZ Conjecture’ aims to explain Mirror Symmetry of Calabi–Yau 3-folds Y, \check{Y} . The original formulation by Strominger, Yau and Zaslow [36] concerned dual special Lagrangian fibrations $\pi : Y \rightarrow B, \check{\pi} : \check{Y} \rightarrow B$, with generic fibre T^3 . Later Kontsevich and Soibelman [26, §3], Morrison [33], and others extended the conjecture to describe how the Calabi–Yau metrics h, \check{h} on Y, \check{Y} degenerate as one approaches the ‘large complex structure limit’. Here is a partial statement of this conjectural picture, focussing on one side of the mirror:

Conjecture 7.5 (Strominger–Yau–Zaslow [36], Kontsevich–Soibelman [26], and Morrison [33]). *Suppose (Y, J, Ω, h) is a Calabi–Yau 3-fold which is close to the ‘large complex structure limit’. Then there should exist a compact real 3-manifold B and a continuous map $\pi : Y \rightarrow B$, called the ‘SYZ fibration’.*

There is a closed subset $\Delta \subset B$ called the ‘discriminant’, a trivalent graph in B . Outside a small open neighbourhood of $\pi^{-1}(\Delta)$ in Y , the map $\pi : Y \rightarrow B$ is smooth, with fibres $Y_b = \pi^{-1}(b)$ for $b \in B \setminus \Delta$ special Lagrangian tori T^3 in Y , with approximately flat metrics $h|_{Y_b}$, of diameters $\text{diam}(Y_b) \ll \text{diam}(Y)$. Close to $\pi^{-1}(\Delta)$, the map π need not be smooth, and the Y_b may be singular.

The analogue for $K3$ surfaces was proved by Gross and Wilson [16].

We are particularly interested in the claim that the nonsingular SYZ fibres Y_b are tori T^3 with metrics $h|_{Y_b}$ close to flat. Suppose that in the situation of Conjecture 7.5, we can find an antiholomorphic involution $\tau : Y \rightarrow Y$ which preserves the SYZ fibration $\pi : Y \rightarrow B$, in the sense that there is an involution $\sigma : B \rightarrow B$ with $\pi \circ \tau = \sigma \circ \pi$. Suppose too that σ has only finitely many fixed points b_1, \dots, b_k in B , all of which lie in $B \setminus \Delta$, and the fixed locus L' of τ is the disjoint union of fibres $Y_{b_1} \amalg \dots \amalg Y_{b_k}$.

If all this holds, and Conjecture 7.5 is true, and h is sufficiently close to the ‘large complex structure limit’, then L' would be k copies of

T^3 with metric $h|_{L'}$ close to flat, so we could apply Theorem 6.4 as in §7.1(A), (E). But this is not rigorous, as Conjecture 7.5 is not yet proved.

Here is an example in which we expect this picture to hold:

Example 7.6. Let Y_ϵ be a hypersurface in $\mathbb{CP}^1 \times \mathbb{CP}^1 \times \mathbb{CP}^1 \times \mathbb{CP}^1$ of the form

$$Y_\epsilon = \{([w_0, w_1], [x_0, x_1], [y_0, y_1], [z_0, z_1]) \in (\mathbb{CP}^1)^4 : \\ (w_0 x_0 y_0 z_0)^2 + (w_1 x_1 y_1 z_1)^2 + \epsilon P(w_0, w_1, x_0, x_1, y_0, y_1, z_0, z_1) = 0\},$$

where $\epsilon > 0$ is small, and $P(w_0, w_1, x_0, x_1, y_0, y_1, z_0, z_1)$ is a complex polynomial which is polyhomogeneous of degree $(2, 2, 2, 2)$ in (w_0, w_1) , (x_0, x_1) , (y_0, y_1) , (z_0, z_1) , and satisfies the complex conjugate equation

$$(7.2) \quad P(\bar{w}_1, \bar{w}_0, \bar{x}_1, \bar{x}_0, \bar{y}_1, \bar{y}_0, \bar{z}_1, \bar{z}_0) = \overline{P(w_0, w_1, x_0, x_1, y_0, y_1, z_0, z_1)}$$

for all w_0, \dots, z_1 . If P is generic under these conditions then Y_ϵ is non-singular for all small $\epsilon > 0$, and is a Calabi–Yau 3-fold, which approaches the ‘large complex structure limit’ as $\epsilon \rightarrow 0$. Define an antiholomorphic involution $\tau_\epsilon : Y_\epsilon \rightarrow Y_\epsilon$ by

$$\begin{aligned} \tau_\epsilon : ([w_0, w_1], [x_0, x_1], [y_0, y_1], [z_0, z_1]) \\ \longmapsto ([\bar{w}_1, \bar{w}_0], [\bar{x}_1, \bar{x}_0], [\bar{y}_1, \bar{y}_0], [\bar{z}_1, \bar{z}_0]). \end{aligned}$$

This is well defined by (7.2). The fixed locus L'_ϵ of τ_ϵ in Y_ϵ may be written

$$L'_\epsilon = \{([1, e^{i\theta_1}], [1, e^{i\theta_2}], [1, e^{i\theta_3}], [1, e^{i\theta_4}]) : \theta_1, \dots, \theta_4 \in \mathbb{R}/2\pi\mathbb{Z}, \\ 1 + e^{2i(\theta_1 + \theta_2 + \theta_3 + \theta_4)} + \epsilon P(1, e^{i\theta_1}, 1, e^{i\theta_2}, 1, e^{i\theta_3}, 1, e^{i\theta_4}) = 0\}.$$

When ϵ is small, the second line approximates $1 + e^{2i(\theta_1 + \theta_2 + \theta_3 + \theta_4)} = 0$, so that $e^{i(\theta_1 + \theta_2 + \theta_3 + \theta_4)} = i = e^{i\pi/2}$ or $e^{i(\theta_1 + \theta_2 + \theta_3 + \theta_4)} = -i = e^{-i\pi/2}$. Hence we have

$$\begin{aligned} L'_\epsilon \approx \{([1, e^{i\theta_1}], \dots, [1, e^{i\theta_4}]) : \theta_1, \dots, \theta_4 \in \mathbb{R}/2\pi\mathbb{Z}, \theta_1 + \dots + \theta_4 = \frac{\pi}{2}\} \\ \amalg \{([1, e^{i\theta_1}], \dots, [1, e^{i\theta_4}]) : \theta_1, \dots, \theta_4 \in \mathbb{R}/2\pi\mathbb{Z}, \theta_1 + \dots + \theta_4 = -\frac{\pi}{2}\}, \end{aligned}$$

which is the disjoint union of two copies of T^3 .

The authors expect that Y_ϵ has an SYZ fibration $\pi_\epsilon : Y_\epsilon \rightarrow \mathcal{S}^3$ for small $\epsilon > 0$, as in Conjecture 7.5, and that τ_ϵ is compatible with the involution $\sigma : \mathcal{S}^3 \rightarrow \mathcal{S}^3$, $\sigma : (x_1, x_2, x_3, x_4) \mapsto (x_1, -x_2, -x_3, -x_4)$ fixing two points $(\pm 1, 0, 0, 0)$ in $\mathcal{S}^3 \setminus \Delta$, and the two T^3 ’s are the fibres of π_ϵ over $(\pm 1, 0, 0, 0)$.

Set $M = \mathcal{S}^1 \times Y_\epsilon$ for small $\epsilon > 0$, with torsion-free G_2 -structure (φ, g) as in §2.5. Define an involution $\iota : M \rightarrow M$ by $\iota(x, y) = (-x, \tau_\epsilon(y))$. Then ι preserves (φ, g) , and has fixed locus $\{0, \frac{1}{2}\} \times L'_\epsilon$, which is four copies of T^3 . We can show that Y has Betti numbers $b^0(Y) = 1$, $b^1(Y) = 0$, $b^2(Y) = 4$, $b^3(Y) = 138$, so that M has Betti numbers $b^0(M) =$

1, $b^1(M) = 1$, $b^2(M) = 4$, and $b^3(M) = 142$. We can compute the action of ι on $H^*(M; \mathbb{R})$, and so show that $M/\langle \iota \rangle$ has Betti numbers $b^0(M/\langle \iota \rangle) = 1$, $b^1(M/\langle \iota \rangle) = 0$, $b^2(M/\langle \iota \rangle) = 0$, and $b^3(M/\langle \iota \rangle) = 73$. Thus Proposition 6.1 shows that the resolution N of $M/\langle \iota \rangle$ has Betti numbers $b^0(N) = 1$, $b^1(N) = 0$, $b^2(N) = 2$, and $b^3(N) = 79$. Kovalev and Lee [28, Table 1] construct a compact 7-manifold with holonomy G_2 and $b^2 = 2$, $b^3 = 79$.

If Conjecture 7.5 could be proved, it is likely that this construction would yield many examples of compact 7-manifolds with holonomy G_2 .

8. Directions for future study

There are now several natural directions of study to pursue following our main theorem. Some of these are:

- (i) One could try to further generalize the theorem to the case of singularities of the form $\mathbb{R}^3 \times (\mathbb{C}^2/\Gamma)$ for other discrete subgroups Γ of $\mathrm{Sp}(1)$, not just \mathbb{Z}_2 . It is possible that most of the analysis can be adapted to this setting. There are ALE hyperKähler 4-manifolds that are asymptotic at infinity to \mathbb{C}^2/Γ for many discrete groups Γ . See [20, 29] for details. The difficulty in applying such a generalization of our main theorem would be in finding initial G_2 -orbifolds with the correct types of singularities.
- (ii) It is likely that a version of our main theorem can be proved in the setting of $\mathrm{Spin}(7)$ -manifolds. This is currently being investigated by a student of the second author.
- (iii) The notion of a G_2 -instanton was introduced by Donaldson–Thomas [11] and elaborated on in [12]. These are special connections on vector bundles over G_2 -manifolds, which are analogues of anti-self-dual connections on 4-manifolds. A natural problem is to prove existence of such connections on the G_2 -manifolds produced by our theorem, using a gluing construction. Precisely this problem was considered in the context of the first author’s original construction of compact G_2 -manifolds by Walpuski [38].
- (iv) New topological and analytic invariants of G_2 -structures were introduced by Crowley–Nordström [8] and Crowley–Goette–Nordström [9]. In particular these invariants can sometimes be used to distinguish distinct G_2 -structures on the same underlying manifold. It would be interesting to compute these invariants for the G_2 -manifolds produced by our theorem.
- (v) The setup for the analysis used to prove our theorem seems to be related to some recent work of Donaldson [13]. This relation deserves closer study. It is possible that such a further study could improve or replace part of the work we did in §3, 4, 5 of the present paper.

- (vi) When the closed, coclosed 1-form λ on L is *not* nonvanishing, our construction fails. Generically, λ should have finitely many isolated zeroes. We cannot resolve the space at these points. In this case there is some evidence that we are instead *almost* producing objects that have isolated conical singularities. This is because one can show that *topologically*, near an isolated zero of λ our resolved manifold looks like a cone over \mathbb{CP}^3 . There is a G_2 -cone over \mathbb{CP}^3 , due to Bryant and Salamon [4].

If we could further modify the closed forms φ_t^N, ψ_t^N near the zeroes of λ to interpolate toward the Bryant–Salamon cone, then one could hope to produce the first known compact examples of *conically singular (CS) G_2 -manifolds*. This is currently being investigated by the second author and Jason Lotay. They have proved an analogue of Theorem 2.7 in the category of CS manifolds, using weighted Sobolev spaces. However, constructing the appropriate closed 3- and 4-forms is likely to be quite difficult.

The analysis of G_2 -conifolds has been studied by the second author and Lotay [23, 25]. In particular, if they succeed in producing CS G_2 -manifolds by generalizing the main theorem of the present paper, then these CS G_2 -manifolds could then be *further resolved* to produce compact smooth G_2 -manifolds using the main result of [23].

Appendix A. Two calculations needed in §3.4

In this appendix we present the details of two calculations that were needed in §3.4, for the proofs of Lemma 3.5 and 3.7. They are actually general results about the linear algebra of a 4-dimensional real vector space V equipped with the standard hyperKähler package. In the lemmas this space V is the fibre of ν at some $x \in L$. Let V be an oriented 4-dimensional real vector space, equipped with metric g , a volume form vol , three orthogonal complex structures J_1, J_2, J_3 satisfying the quaternionic relations, and a triple $\omega_1, \omega_2, \omega_3$ of 2-forms satisfying $g(u, v) = \omega_k(J_k u, v)$. We will also need the relation (2.12) that

$$(A.1) \quad \gamma \wedge \omega_k = *(J_k \gamma) \quad \text{for any 1-form } \gamma.$$

Let f^1, f^2, f^3, f^4 be an oriented orthonormal basis for V^* and let e_1, e_2, e_3 denote the standard basis of \mathbb{R}^3 .

Proposition A.1. *Consider the linear maps*

$$A : V^* \otimes \Lambda^3 V^* \longrightarrow \Lambda^4 V^*, \quad A(\gamma \otimes \eta) = \gamma \wedge \eta,$$

and

$$B : S^2 V^* \otimes \mathbb{R}^3 \longrightarrow V^* \otimes \Lambda^3 V^*,$$

$$B : h_{pq}^k f^p f^q \otimes e_k \longmapsto h_{pq}^k [f^p \otimes (f^q \wedge \omega_k) + f^q \otimes (f^p \wedge \omega_k)].$$

The map B surjects onto the kernel of A .

Proof. Note that $V^* \cong \Lambda^3 V^*$ via the isomorphism $\gamma \mapsto *\gamma$. Thus we have $A : V^* \otimes \Lambda^3 V^* \rightarrow \Lambda^4 V^*$ with $A(s_{pm}f^p \otimes *f^m) = s_{pm}f^p \wedge *f^m = s_{pm}g^{pm} \text{ vol}$, so $s_{pm}f^p \otimes *f^m \in \text{Ker } A$ if and only if $s_{pm}g^{pm} = 0$. Using (A.1), we compute

$$(A.2) \quad \begin{aligned} B(h_{pq}^k f^p f^q \otimes e_k) &= h_{pq}^k [f^p \otimes (f^q \wedge \omega_k) + f^q \otimes (f^p \wedge \omega_k)] \\ &= h_{pq}^k [f^p \otimes *(J_k f^q) + f^q \otimes *(J_k f^p)] = 2h_{pq}^k f^p \otimes *(J_k f^q). \end{aligned}$$

Thus $A[B(h_{pq}^k f^p f^q \otimes e_k)] = 2h_{pq}^k g(f^p, J_k f^q) \text{ vol} = 2h_{pq}^k \omega_k(f^p, f^q) \text{ vol} = 0$, since ω_k is skew and h_{pq}^k is symmetric in p, q . So the image of B always lies in the kernel of A .

In terms of the basis f^1, \dots, f^4 , the kernel of A is spanned by two types of elements of $V^* \otimes \Lambda^3 V^*$: those of the form $f^{p_0} \otimes *f^{q_0}$ for $p_0 \neq q_0$ and those of the form $f^{p_0} \otimes *f^{p_0} - f^{q_0} \otimes *f^{q_0}$ for $p_0 \neq q_0$. We can see from (2.11) that there exists a unique $k_0 \in \{1, 2, 3\}$ such that $J_{k_0} f^{q_0} = \pm f^{p_0}$. In the first case, set $h_{p_0 p_0}^{k_0} = \pm \frac{1}{2}$, and all other $h_{pq}^k = 0$. It follows from (A.2) that for this choice, $B(h_{pq}^k f^p f^q \otimes e_k) = f^{p_0} \otimes *f^{q_0}$. In the second case, set $h_{p_0 q_0}^{k_0} = h_{q_0 p_0}^{k_0} = \pm \frac{1}{2}$, and all other $h_{pq}^k = 0$. Noting that $J_{k_0} f^{p_0} = \mp f^{q_0}$, it is each to check that for this choice, equation (A.2) gives $B(h_{pq}^k f^p f^q \otimes e_k) = f^{p_0} \otimes *f^{p_0} - f^{q_0} \otimes *f^{q_0}$. Thus B indeed surjects onto the kernel of A . q.e.d.

Remark A.2. Proposition A.1 is used in the proof of Lemma 3.5, where $B(\bar{\alpha}) = -\bar{\alpha} \cdot \bar{\varphi}_{2,1}^2$ and $\bar{\varphi}_{2,1}^2 = -(\omega_1 \otimes e_1 + \omega_2 \otimes e_2 + \omega_3 \otimes e_3)$.

Proposition A.3. Consider the linear map

$$\begin{aligned} C : V \otimes V^* \otimes \mathbb{R}^3 &\longrightarrow V^* \otimes V^* \otimes \Lambda^2 \mathbb{R}^3, \\ C : s_q^{pk} f_p \otimes f^q \otimes e_k &\longmapsto \sum_{j=1}^3 s_q^{pk} (f_p \cdot \omega_j) \otimes f^q \otimes (e_k \wedge e_j). \end{aligned}$$

The map C is an isomorphism.

Proof. The middle factor of V^* is just coming along for the ride, so we need to check that the map $V \otimes \mathbb{R}^3 \rightarrow V^* \otimes \Lambda^2 \mathbb{R}^3$ which on decomposable elements is given by $f_p \otimes e_k \mapsto \sum_{j=1}^3 (f_p \cdot \omega_j) \otimes (e_j \wedge e_k)$ is an isomorphism. The dimensions agree, so we need only show the kernel is trivial. Let $\sum_{k=1}^3 s_k^p f_p \otimes e_k \in V \otimes \mathbb{R}^3$ be in the kernel. Then $\sum_{j,k=1}^3 s_k^p (f_p \cdot \omega_j) \otimes (e_j \wedge e_k) = 0$. It is easy to check that in an orthonormal frame we have $f_p \cdot \omega_j = -J_j f^p = -(J_j)_q^p f^q$, so we have

$$0 = \sum_{p,q=1}^4 \sum_{j,k=1}^3 s_k^p (J_j)_q^p f^q \otimes (e_j \wedge e_k) = \sum_{p,q=1}^4 \sum_{j < k} (s_k^p (J_j)_q^p - s_j^p (J_k)_q^p) f^q \otimes (e_j \wedge e_k).$$

We deduce that for any fixed pair $j \neq k$, we have

$$0 = \sum_{p=1}^4 (s_k^p(J_j)_q^p - s_j^p(J_k)_q^p).$$

Let i be the remaining index in $\{1, 2, 3\}$ and without loss of generality order them so that i, j, k is a cyclic permutation of $1, 2, 3$. Multiplying the above equation by $(J_i)_m^q$ and summing over q , we find

$$\begin{aligned} 0 &= \sum_{p=1}^4 (s_k^p(J_i)_m^q (J_j)_q^p - s_j^p(J_i)_m^q (J_k)_q^p) \\ (A.3) \quad &= \sum_{p=1}^4 (s_k^p(J_i J_j)_m^p - s_j^p(J_i J_k)_m^p) = \sum_{p=1}^4 (s_k^p(J_k)_m^p + s_j^p(J_j)_m^p). \end{aligned}$$

For $i \in \{1, 2, 3\}$ and $m \in \{1, 2, 3, 4\}$, define $t_{im} = \sum_{p=1}^4 s_i^p(J_i)_m^p$. In (A.3) we deduced that $t_{jm} + t_{km} = 0$ if $j \neq k$. Thus $t_{1m} = -t_{2m} = t_{3m} = -t_{1m}$, so $t_{1m} = 0$ and similarly $t_{im} = 0$ for all i, m . Now multiply $t_{im} = 0$ by $(J_i)_q^m$ and sum over m . From $(J_i)^2 = -I$, we find

$$0 = (J_i)_q^m t_{im} = \sum_{p=1}^4 s_i^p(J_i)_q^m (J_i)_m^p = - \sum_{p=1}^4 s_i^p \delta_q^p = -s_i^q.$$

Thus the kernel is indeed trivial and C is an isomorphism. q.e.d.

Remark A.4. Proposition A.3 is used in the proof of Lemma 3.7, where $C(\Gamma) = -\Gamma \cdot \bar{\varphi}_{2,1}^2$ and $\bar{\varphi}_{2,1}^2 = -(\omega_1 \otimes e_1 + \omega_2 \otimes e_2 + \omega_3 \otimes e_3)$.

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