

Homogenization of the variational inequality
for the p – Laplacian in the perforated domain with nonlinear
restrictions on the flux in the case when $p \in (1, 2)$

J. I. Diaz, D. Gomez-Castro, A. V. Podolskiy, T.A. Shaposhnikova

Works [1], [2] are concerned with the investigation of asymptotic behavior of the solution of the variational inequality for the p - Laplace operator, where $p \in [2, n)$ and ε – periodically perforated domain with the nonlinear Robin type boundary condition. In the present work we investigate a similar homogenization problem for the p – Laplacian in the case when $p \in (1, 2)$. It's a known fact (ref. [3]) that for this values of p considered problems describe motion of the non-Newtonian fluids.

Let Ω be a bounded domain in \mathbb{R}^n , $n \geq 3$, with a smooth boundary $\partial\Omega$. Denote $Y = (-1/2, 1/2)^n$, G_0 is the unit ball centered at the origin. For $\delta > 0$ we define $\delta B = \{x | \delta^{-1}x \in B\}$. We define

$$\tilde{\Omega}_\varepsilon = \{x \in \Omega | \rho(x, \partial\Omega) > 2\varepsilon\}, \quad G_\varepsilon = \bigcup_{j \in \Upsilon_\varepsilon} (a_\varepsilon G_0 + \varepsilon j) = \bigcup_{j \in \Upsilon_\varepsilon} G_\varepsilon^j,$$

where $0 < \varepsilon \ll 1$, $a_\varepsilon = C_0 \varepsilon^\alpha$, $\alpha = \frac{n}{n-p}$, $\Upsilon_\varepsilon = \{j \in \mathbb{Z}^n | (a_\varepsilon G_0 + \varepsilon j) \cap \tilde{\Omega}_\varepsilon \neq \emptyset\}$, $|\Upsilon_\varepsilon| \cong d\varepsilon^{-n}$, $d = \text{const} > 0$, and \mathbb{Z}^n is the set of vectors z with integer coordinates. Let $Y_\varepsilon^j = \varepsilon Y + \varepsilon j$, $j \in \Upsilon_\varepsilon$. Note that $\overline{G_\varepsilon^j} \subset Y_\varepsilon^j$ and the center of the ball G_ε^j coincides with the center of Y_ε^j .

We define sets

$$\Omega_\varepsilon = \Omega \setminus \overline{G_\varepsilon}, \quad S_\varepsilon = \partial G_\varepsilon, \quad \partial\Omega_\varepsilon = \partial\Omega \cup S_\varepsilon.$$

Consider following problem

$$\begin{cases} -\Delta_p u_\varepsilon = f, & x \in \Omega_\varepsilon, \\ -\partial_{\nu_p} u_\varepsilon \in \varepsilon^{-\gamma} \sigma(u_\varepsilon), & x \in S_\varepsilon, \\ u_\varepsilon = 0, & x \in \partial\Omega, \end{cases} \quad (1)$$

where $p \in (1, 2)$, $\Delta_p u \equiv \text{div}(|\nabla u|^{p-2} \nabla u)$, $\partial_{\nu_p} u \equiv |\nabla u|^{p-2} (\nabla u, \nu)$, ν is the outward unit normal to S_ε , $\gamma = \alpha(p-1)$. It is assumed that $f \in L^{p'}(\Omega)$,

$p' = \frac{p}{p-1}$. Let

$$\sigma(\lambda) = \begin{cases} \sigma_0(\lambda), & \lambda > 0, \\ (-\infty, 0], & \lambda = 0, \\ -\infty, & \lambda < 0, \end{cases} \quad (2)$$

where $\sigma_0 \in C^1(\mathbb{R})$, $\sigma_0(0) = 0$, $\sigma_0'(\lambda) \geq k_1 = \text{const} > 0$.

We note that boundary value problem (1) with such function as $\sigma(\lambda)$ in the boundary condition relates to the problem with the one-sided restrictions, i.e. Signorini problem

$$\begin{cases} u_\varepsilon \geq 0, \partial_{\nu_p} u_\varepsilon + \varepsilon^{-\gamma} \sigma_0(u_\varepsilon) \geq 0, & \text{on } S_\varepsilon, \\ u_\varepsilon(\partial_{\nu_p} u_\varepsilon + \varepsilon^{-\gamma} \sigma_0(u_\varepsilon)) = 0, & \text{on } S_\varepsilon. \end{cases}$$

Define function

$$\psi(\lambda) = \begin{cases} \int_0^\lambda \sigma_0(s) ds, & \lambda \geq 0, \\ +\infty, & \lambda < 0. \end{cases}$$

The weak solution of the problem (1) is defined as a function $u_\varepsilon \in K_\varepsilon = \{g \in W^{1,p}(\Omega_\varepsilon, \partial\Omega) : g \geq 0 \text{ a.e. on } S_\varepsilon\}$, satisfying integral inequality

$$\int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon \nabla (\phi - u_\varepsilon) dx + \varepsilon^{-\gamma} \int_{S_\varepsilon} (\hat{\psi}(\phi) - \hat{\psi}(u_\varepsilon)) ds \geq \int_{\Omega_\varepsilon} f(\phi - u_\varepsilon) dx \quad (3)$$

for an arbitrary function $\phi \in K_\varepsilon$, $\hat{\psi}(\lambda) \equiv \int_0^\lambda \sigma_0(\tau) d\tau$.

Let $H(\lambda)$ be a solution of the functional inclusion

$$B_0 |H|^{p-2} H \in \sigma(\lambda - H), \quad (4)$$

where $B_0 = \text{const} > 0$.

Inclusion (4) has a unique solution

$$H(\lambda) = \begin{cases} H_0(\lambda), & \lambda > 0, \\ \lambda, & \lambda \leq 0, \end{cases} \quad (5)$$

where $H_0(\lambda)$ is the solution of the functional equation

$$B_0 |H_0|^{p-2} H_0 = \sigma_0(\lambda - H_0), \quad (6).$$

Note that

$$H(u) = H_0(u^+) + u^-,$$

and

$$|H(u)|^{p-2}H(u) = |H_0(u^+)|^{p-2}H_0(u^+) + |u^-|^{p-2}u^-,$$

where u^+ is the positive part of u , u^- is its negative part, $H_0(0) = 0$.

Let $\tilde{u}_\varepsilon \in W_0^{1,p}(\Omega)$ be a $W^{1,p}$ – extension of u_ε , that satisfies inequalities

$$\|\tilde{u}_\varepsilon\|_{W^{1,p}(\Omega)} \leq K\|u_\varepsilon\|_{W^{1,p}(\Omega_\varepsilon)}, \quad \|\nabla \tilde{u}_\varepsilon\|_{L^p(\Omega)} \leq K\|\nabla u_\varepsilon\|_{L^p(\Omega_\varepsilon)}, \quad (7)$$

In view of estimates (3), we have

$$\|\nabla u_\varepsilon\|_{L^p(\Omega_\varepsilon)} \leq K.$$

Hence, using this inequality and estimations (7) we conclude that there exists a subsequence (denote as the original sequence), such that as $\varepsilon \rightarrow 0$

$$\tilde{u}_\varepsilon \rightarrow u \text{ weakly in } W_0^{1,p}(\Omega). \quad (8)$$

The following theorem gives us the description of function u .

Theorem 1. *Let $\alpha = \frac{n}{n-p}$, $\gamma = \alpha(p-1)$, $p \in (1, 2)$, $n \geq 3$. Suppose that $u_\varepsilon \in W^{1,p}(\Omega_\varepsilon, \partial\Omega)$ is the weak solution of the problem (1), where $\sigma(\lambda)$ is given by formula (2), $\tilde{u}_\varepsilon \in W_0^{1,p}(\Omega)$ is the $W^{1,p}$ – extension of u_ε . Then the function u defined in (8) is a weak solution of the following problem*

$$\begin{cases} -\Delta_p u + \mathcal{A}(n, p)|H(u)|^{p-2}H(u) = f, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \quad (9)$$

where function $H(\lambda)$ is given by formula (5), where function $H_0(\lambda)$ is a solution of the equation (6) with the constant $B_0 = \left(\frac{n-p}{p-1}\right)^{p-1} C_0^{1-p}$, $\mathcal{A}(n, p) = \left(\frac{n-p}{p-1}\right)^{p-1} C_0^{n-p} \omega_n$, ω_n is the surface area of the unit sphere in \mathbb{R}^n .

Proof. Taking into account that $\hat{\psi}(\lambda) - \hat{\psi}(\eta) \leq \sigma_0(\lambda)(\lambda - \eta)$ for all $\lambda, \eta \in \mathbb{R}$ and using monotonicity of function $|\lambda|^{p-2}\lambda$ for $p > 1$, from inequality (3) we derive that u_ε satisfies following inequality

$$\int_{\Omega_\varepsilon} |\nabla \phi|^{p-2} \nabla \phi \nabla (\phi - u_\varepsilon) dx + \varepsilon^{-\gamma} \int_{S_\varepsilon} \sigma_0(\phi) (\phi - u_\varepsilon) ds \geq \int_{\Omega_\varepsilon} f (\phi - u_\varepsilon) dx, \quad (10)$$

for an arbitrary function $\phi \in K_\varepsilon$.

Assume that (17) $\phi = v - W_\varepsilon H(v)$, where $H(\lambda)$ is defined by (5). We define W_ε as follows

$$W_\varepsilon = \begin{cases} w_\varepsilon^j, & x \in T_\varepsilon^j \setminus \overline{G_\varepsilon^j}, j \in \Upsilon_\varepsilon, \\ 1, & x \in G_\varepsilon, \\ 0, & x \in \mathbb{R}^n \setminus \bigcup_{j \in \Upsilon_\varepsilon} T_\varepsilon^j, \end{cases}$$

where w_ε^j is the solution of the following value problem

$$\begin{cases} \Delta_p w_\varepsilon^j = 0, & x \in T_\varepsilon^j \setminus \overline{G_\varepsilon^j}, \\ w_\varepsilon^j = 1, & x \in \partial G_\varepsilon^j, \\ w_\varepsilon^j = 0, & x \in \partial T_\varepsilon^j, \end{cases}$$

Let T_ε^j denote the ball of radius $\varepsilon/4$ which center coincides with the center of cube Y_ε^j .

We have

$$W_\varepsilon \rightharpoonup 0 \text{ weakly in } W_0^{1,p}(\Omega), \text{ as } \varepsilon \rightarrow 0, \quad (11)$$

and

$$\int_{\Omega_\varepsilon} |\nabla W_\varepsilon|^q dx \leq K \varepsilon^{n(p-q)/(n-p)}, \quad (12)$$

where $1 \leq q \leq p$.

Note that $\phi|_{S_\varepsilon} = v^+ - H_0(v^+) \geq 0$.

Consider integral

$$I_\varepsilon \equiv \int_{\Omega_\varepsilon} |\nabla(v - W_\varepsilon H(v))|^{p-2} \nabla(v - W_\varepsilon H(v)) \nabla(v - W_\varepsilon H(v) - u_\varepsilon) dx.$$

Lets it transform to the following form

$$\begin{aligned} I_\varepsilon &= \int_{\Omega_\varepsilon} \left[|\nabla(v - W_\varepsilon H(v))|^{p-2} \nabla(v - W_\varepsilon H(v)) - |\nabla v|^{p-2} \nabla v + \right. \\ &\quad \left. + |\nabla(W_\varepsilon H(v))|^{p-2} \nabla(W_\varepsilon H(v)) \right] \nabla(v - W_\varepsilon H(v) - u_\varepsilon) dx + \\ &\quad + \int_{\Omega_\varepsilon} |\nabla v|^{p-2} \nabla v \nabla(v - W_\varepsilon H(v) - u_\varepsilon) dx - \\ &\quad - \int_{\Omega_\varepsilon} |\nabla(W_\varepsilon H(v))|^{p-2} \nabla(W_\varepsilon H(v)) \nabla(v - W_\varepsilon H(v) - u_\varepsilon) dx \equiv \\ &\quad \equiv J_\varepsilon^1 + J_\varepsilon^2 + J_\varepsilon^3. \end{aligned}$$

Further we will use following lemma

Lemma 1. *Let $p \in (1, 2)$, $n \geq 2$. Then there exists constant $C = C(n, p)$ such that for all $a, b \in \mathbb{R}^n$ following inequality is valid*

$$\left| |a - b|^{p-2}(a - b) - (|a|^{p-2}a - |b|^{p-2}b) \right| \leq C \left(|a||b| \right)^{\frac{p-1}{2}}.$$

Proof. Without loss of generality we can assume that $|a| \geq |b| > 0$. Let $u = \frac{a}{|a|}$, $v = \frac{b}{|b|}$, $|u| = |v| = 1$, $\xi = u \cdot v$, $\xi \in [-1, 1]$, $k = \frac{|a|}{|b|} \geq 1$.

The desired inequality in new variables takes the following form

$$||ku - v|^{p-2}(ku - v) - (k^{p-1}u - v)| \leq Ck^{(p-1)/2}.$$

By squaring this inequality we get

$$\begin{aligned} \mathfrak{R}(k, \xi) &\equiv (k^2 - 2k\xi + 1)^{p-1} + k^{2(p-1)} + 1 - \\ &- 2k^{p-1}\xi - 2(k^2 - 2k\xi + 1)^{(p-2)/2}(k^p + 1 - k\xi - k^{p-1}\xi) \leq C^2k^{p-1}. \end{aligned}$$

Consider function

$$\begin{aligned} f(k, \xi) &= \mathfrak{R}(k, \xi)/k^{p-1} = k^{p-1} \left(1 - \frac{2\xi}{k} + \frac{1}{k^2} \right)^{p-1} + k^{p-1} + k^{1-p} - 2\xi - \\ &- 2 \left(1 - \frac{2\xi}{k} + \frac{1}{k^2} \right)^{(p-2)/2} (k^{p-1} - \xi - k^{p-2}\xi + k^{-1}). \end{aligned}$$

Decomposing functions $(1 - 2\xi/k + 1/k^2)^\beta$ for $\beta = p - 1, (p - 2)/2$ in Taylor series as $k \rightarrow \infty$, $k > 1 + \sqrt{2}$, and equating the coefficients of corresponding degrees, we obtain

$$f(k, \xi) = \alpha k^{1-p} + \beta k^{p-2} + o\left(\frac{1}{k}\right),$$

where α and β depend only on p and ξ .

Hence, $f(k, \xi) \rightarrow 0$ as $k \rightarrow \infty$. Thus there exists $k_1 > 1 + \sqrt{2}$ such that $f(k, \xi) < 1$ for all $k > k_1$, $|\xi| \leq 1$. It's easy to show that function $f(k, \xi)$ is continuous on the set $D = \{(k, \xi) | 1 \leq k \leq k_1, |\xi| \leq 1\}$. So there exists positive constant M that depends on p such that $\max_{(k, \xi) \in D} |f(k, \xi)| \leq M$.

Hence, function $|f|$ is bounded by $\max(M, 1)$ for all permissible k and ξ .

Q.E.D. \square

Using Lemma 1, for J_ε^1 we have

$$\begin{aligned}
|J_\varepsilon^1| &\leq K \int_{\Omega_\varepsilon} |\nabla v|^{\frac{p-1}{2}} |\nabla(W_\varepsilon H(v))|^{\frac{p-1}{2}} \left(|\nabla(W_\varepsilon H(v))| + |\nabla v| + |\nabla u_\varepsilon| \right) dx \leq \\
&\leq K \int_{\Omega_\varepsilon} (|\nabla v|^{\frac{p-1}{2}} |\nabla(W_\varepsilon H(v))|^{\frac{p+1}{2}} + |\nabla v|^{\frac{p+1}{2}} |\nabla(W_\varepsilon H(v))|^{\frac{p-1}{2}} + \\
&\quad + |\nabla v|^{\frac{p-1}{2}} |\nabla(W_\varepsilon H(v))|^{\frac{p-1}{2}} |\nabla u_\varepsilon|) dx \leq \\
&\leq K \int_{\Omega_\varepsilon} \left(|\nabla W_\varepsilon|^{\frac{p+1}{2}} + |\nabla u_\varepsilon| |\nabla W_\varepsilon|^{\frac{p-1}{2}} \right) dx \leq \\
&\leq K \left\{ \|\nabla W_\varepsilon\|_{L^{\frac{p+1}{2}}(\Omega)}^{\frac{2}{p+1}} + \|\nabla u_\varepsilon\|_{L^p(\Omega)} \|\nabla W_\varepsilon\|_{L^{p/2}(\Omega)}^{\frac{p-1}{2}} \right\}.
\end{aligned}$$

In view of $\frac{p+1}{2} < p$ and estimation (12) we derive

$$\lim_{\varepsilon \rightarrow 0} J_\varepsilon^1 = 0. \quad (14)$$

Moreover convergence (8) implies

$$\lim_{\varepsilon \rightarrow 0} J_\varepsilon^2 = \int_{\Omega} |\nabla v|^{p-2} \nabla v \nabla(v - u) dx. \quad (15)$$

Consider J_ε^3 . We have

$$\begin{aligned}
J_\varepsilon^3 &\equiv - \int_{\Omega_\varepsilon} |\nabla(W_\varepsilon H_0(v^+)) + \nabla(W_\varepsilon v^-)|^{p-2} \nabla(W_\varepsilon H_0(v^+) + W_\varepsilon v^-) \\
&\quad \nabla(v - W_\varepsilon H_0(v^+) - W_\varepsilon v^- - u_\varepsilon) dx = \\
&= - \int_{\Omega_\varepsilon} |\nabla(W_\varepsilon H_0(v^+))|^{p-2} \nabla(W_\varepsilon H_0(v^+)) \nabla(v - W_\varepsilon H_0(v^+) - W_\varepsilon v^- - u_\varepsilon) dx - \\
&\quad - \int_{\Omega_\varepsilon} |\nabla(W_\varepsilon v^-)|^{p-2} \nabla(W_\varepsilon v^-) \nabla(v - W_\varepsilon H_0(v^+) - W_\varepsilon v^- - u_\varepsilon) dx = \\
&= - \int_{\Omega_\varepsilon} |\nabla W_\varepsilon|^{p-2} \nabla W_\varepsilon \nabla[|H_0(v^+)|^{p-2} H_0(v^+)(v - W_\varepsilon H_0(v^+) - W_\varepsilon v^- - u_\varepsilon)] dx - \\
&\quad - \int_{\Omega_\varepsilon} |\nabla W_\varepsilon|^{p-2} \nabla W_\varepsilon \nabla(|v^-|^{p-2} v^-(v - W_\varepsilon H_0(v^+) - W_\varepsilon v^- - u_\varepsilon)) dx + \alpha_\varepsilon,
\end{aligned}$$

where $\alpha_\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$. From the Green's formula we derive

$$\begin{aligned}
J_\varepsilon^3 = & - \sum_{j \in \Upsilon_\varepsilon} \int_{\partial G_\varepsilon^j} \partial_{\nu_p} w_\varepsilon^j |H_0(v^+)|^{p-2} H_0(v^+) (v^+ - H_0(v^+) - u_\varepsilon) ds - \\
& - \sum_{j \in \Upsilon_\varepsilon} \int_{\partial G_\varepsilon^j} \partial_{\nu_p} w_\varepsilon^j |v^-|^{p-2} v^- (v^+ - H_0(v^+) - u_\varepsilon) ds - \\
& - \sum_{j \in \Upsilon_\varepsilon} \int_{\partial T_\varepsilon^j} \partial_{\nu_p} w_\varepsilon^j |H_0(v^+)|^{p-2} H_0(v^+) (v - u_\varepsilon) ds - \\
& - \sum_{j \in \Upsilon_\varepsilon} \int_{\partial T_\varepsilon^j} \partial_{\nu_p} w_\varepsilon^j |v^-|^{p-2} v^- (v - u_\varepsilon) ds + \alpha_\varepsilon,
\end{aligned} \tag{16}$$

Taking into account that

$$\partial_{\nu_p} w_\varepsilon^j \Big|_{\partial G_\varepsilon^j} = \frac{(n-p)\varepsilon^{-\frac{n}{n-p}}}{(p-1)C_0(1-\kappa_\varepsilon)}, \tag{17}$$

$$\partial_{\nu_p} w_\varepsilon^j \Big|_{\partial T_\varepsilon^j} = - \frac{(n-p)2^{2(n-1)/(p-1)}C_0^{(n-p)/(p-1)}\varepsilon^{1/(p-1)}}{(p-1)(1-\kappa_\varepsilon)}, \tag{18}$$

where $\kappa_\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$.

Formula (17), (18) imply that

$$\begin{aligned}
& - \sum_{j \in \Upsilon_\varepsilon} \int_{\partial G_\varepsilon^j} \partial_{\nu_p} w_\varepsilon^j |v^-|^{p-2} v^- (v^+ - H_0(v^+) - u_\varepsilon) ds = \\
& = \sum_{j \in \Upsilon_\varepsilon} \int_{\partial G_\varepsilon^j} \partial_{\nu_p} w_\varepsilon^j |v^-|^{p-2} v^- u_\varepsilon ds \leq 0.
\end{aligned} \tag{19}$$

Taking into account that $\gamma = \alpha(p-1)$ we obtain

$$\begin{aligned}
& J_\varepsilon^3 + \varepsilon^{-\gamma} \int_{S_\varepsilon} \sigma_0(v^+ - H_0(v^+)) (v^+ - H_0(v^+) - u_\varepsilon) ds \leq \\
& \leq \varepsilon^{-\gamma} \int_{S_\varepsilon} \left[\sigma_0(v^+ - H_0(v^+)) - B_0 |H_0(v^+)|^{p-2} H_0(v^+) \right] (v^+ - H_0(v^+) - u_\varepsilon) ds - \\
& - \sum_{j \in \Upsilon_\varepsilon} \int_{\partial T_\varepsilon^j} \partial_{\nu_p} w_\varepsilon^j \left[|H_0(v^+)|^{p-2} H_0(v^+) + |v^-|^{p-2} v^- \right] (v - u_\varepsilon) ds + \kappa_\varepsilon,
\end{aligned} \tag{20}$$

where $\kappa_\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$.

We will use next lemma to pass to the limit in the last integral (ref. [4]).

Lemma 2. *Let $u_\varepsilon \in H_0^1(\Omega)$ and $u_\varepsilon \rightharpoonup u_0$ as $\varepsilon \rightarrow 0$ in $H_0^1(\Omega)$, then*

$$\left| 2^{2(n-1)} \varepsilon \sum_{j=1}^{N_\varepsilon} \int_{\partial T_{\varepsilon/4}^j} u_\varepsilon dS - \omega_n \int_{\Omega} u_0 dx \right| \rightarrow 0, \quad \varepsilon \rightarrow 0,$$

where ω_n is the surface area of the unit sphere in \mathbb{R}^n .

In respect that H_0 is the solution of the equation (6) and using Lemma 2 we get

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} J_\varepsilon^3 + \varepsilon^{-\gamma} \int_{S_\varepsilon} \sigma_0(v^+ - H_0(v^+))(v^+ - H_0(v^+) - u_\varepsilon) ds &\leq \\ &\leq \mathcal{A}(n, p) \int_{\Omega} \{|H_0(v^+)|^{p-2} H_0(v^+) + |v^-|^{p-2} v^-\} (v - u) dx, \end{aligned} \quad (21)$$

where $\mathcal{A}(n, p) = \left(\frac{n-p}{p-1}\right)^{p-1} C_0^{n-p} \omega_n$.

From (10)-(21) we derive that u satisfies following inequality

$$\begin{aligned} &\int_{\Omega} |\nabla u|^{p-2} \nabla v \nabla (v - u) dx + \\ &+ \mathcal{A}(n, p) \int_{\Omega} \{|H_0(v^+)|^{p-2} H_0(v^+) + |v^-|^{p-2} v^-\} (v - u) dx \geq \\ &\geq \int_{\Omega} f(v - u) dx. \end{aligned} \quad (22)$$

This inequality implies that u is a weak solution of the problem (9). \square

In the next theorem we will prove convergence in the norm of space $W^{1,p}(\Omega_\varepsilon)$ of the solution of the problem (1) with a corrector to the solution of the homogenized problem.

Theorem 2. *Let $\alpha = \frac{n}{n-p}$, $\gamma = \alpha(p-1)$, $p \in (1, 2)$, $n \geq 3$. Suppose that $u_\varepsilon \in W^{1,p}(\Omega_\varepsilon)$ is a weak solution of the problem (1), u is a weak solution of the problem (9) possessing an additional smoothness, $u \in W^{1,\infty}(\Omega)$. Then*

$$\|\nabla(u_\varepsilon - u + W_\varepsilon H(u))\|_{L^p(\Omega_\varepsilon)} \rightarrow 0, \quad \varepsilon \rightarrow 0.$$

Proof. Inequality (3) implies that

$$\begin{aligned} \int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon \nabla (\phi - u_\varepsilon) dx + \varepsilon^{-\gamma} \int_{S_\varepsilon} \sigma_0(\phi) (\phi - u_\varepsilon) ds &\geq \\ &\geq \int_{\Omega_\varepsilon} f(\phi - u_\varepsilon) dx. \end{aligned} \quad (23)$$

In inequality (23) we substitute $\phi = u - W_\varepsilon H(u)$ and in integral identity of the problem (9) we take $v = u - W_\varepsilon H(u) - \tilde{u}_\varepsilon$ as a test function, where \tilde{u}_ε is a $W^{1,p}$ -extension u_ε on Ω . By subtracting from inequality (23) integral identity we get

$$\begin{aligned} &\int_{\Omega_\varepsilon} (|\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon - |\nabla(u - W_\varepsilon H(u))|^{p-2} \nabla(u - W_\varepsilon H(u))) \nabla \Psi dx + \\ &\quad \int_{\Omega_\varepsilon} (|\nabla(u - W_\varepsilon H(u))|^{p-2} \nabla(u - W_\varepsilon H(u)) - |\nabla u|^{p-2} \nabla u + \\ &\quad + |\nabla(W_\varepsilon H(u))|^{p-2} \nabla(W_\varepsilon H(u))) \nabla \Psi dx + \varepsilon^{-\gamma} \int_{S_\varepsilon} \sigma_0(u^+ - H_0(u^+)) \Psi ds - \\ &\quad - \int_{\Omega_\varepsilon} |\nabla(W_\varepsilon H(u))|^{p-2} \nabla(W_\varepsilon H(u)) \nabla \Psi dx - \int_{G_\varepsilon} |\nabla u|^{p-2} \nabla u \nabla(u - H(u) - \tilde{u}_\varepsilon) dx + \\ &\quad - \mathcal{A}(n, p) \int_{\Omega} |H(u)|^{p-2} H(u) (u - W_\varepsilon H(u) - \tilde{u}_\varepsilon) dx \geq - \int_{G_\varepsilon} f(u - H(u) - \tilde{u}_\varepsilon) dx, \end{aligned} \quad (24)$$

where $\Psi = u - W_\varepsilon H(u) - u_\varepsilon$.

We will use inequality that is valid for all $1 < p < 2$ and for an arbitrary $\xi, \eta \in \mathbb{R}^n$ [5]

$$C \frac{|\xi - \eta|^2}{|\xi|^{2-p} + |\eta|^{2-p}} \leq (|\xi|^{p-2} \xi - |\eta|^{p-2} \eta) (\xi - \eta). \quad (25)$$

Estimation (25) implies

$$\begin{aligned} I_{1,\varepsilon} &\equiv C \int_{\Omega_\varepsilon} \frac{|\nabla(u_\varepsilon - u + W_\varepsilon H(u))|^2}{|\nabla u_\varepsilon|^{2-p} + |\nabla(u - W_\varepsilon H(u))|^{2-p}} \leq \\ &\leq \int_{\Omega_\varepsilon} (|\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon - |\nabla(u - W_\varepsilon H(u))|^{p-2} \nabla(u - W_\varepsilon H(u))) \end{aligned}$$

$$\times \nabla(u_\varepsilon - u + W_\varepsilon H(u))dx. \quad (26)$$

Consider integral

$$I_{2,\varepsilon} \equiv \int_{\Omega_\varepsilon} \left(|\nabla(u - W_\varepsilon H(u))|^{p-2} \nabla(u - W_\varepsilon H(u)) - \right. \\ \left. - (|\nabla u|^{p-2} \nabla u - |\nabla(W_\varepsilon H(u))|^{p-2} \nabla(W_\varepsilon H(u))) \nabla(u - W_\varepsilon H(u) - u_\varepsilon) \right) dx. \quad (27)$$

Using Lemma 1 we get

$$|I_{2,\varepsilon}| \leq C \int_{\Omega_\varepsilon} |\nabla u|^{(p-1)/2} |\nabla(W_\varepsilon H(u))|^{(p-1)/2} |\nabla(u - W_\varepsilon H(u) - u_\varepsilon)| dx. \quad (28)$$

In respect of estimation (12) we derive

$$\lim_{\varepsilon \rightarrow 0} I_{2,\varepsilon} = 0. \quad (29)$$

From (24), (26), (29) and condition $u \in W^{1,\infty}(\Omega)$ we obtain

$$I_{1,\varepsilon} \leq - \int_{\Omega_\varepsilon} |\nabla(W_\varepsilon H(u))|^{p-2} \nabla(W_\varepsilon H(u)) \nabla(u - W_\varepsilon H(u) - u_\varepsilon) dx + \\ + \varepsilon^{-\gamma} \int_{\tilde{S}_\varepsilon} \sigma_0(u^+ - H_0(u^+))(u^+ - H_0(u^+) - u_\varepsilon) ds - \\ - \mathcal{A}(n, p) \int_{\Omega} |H(u)|^{p-2} H(u) (u - W_\varepsilon H(u) - \tilde{u}_\varepsilon) dx + \kappa_\varepsilon. \quad (30)$$

Note that $H(u) = H_0(u^+) + u^-$ and $|H(u)|^{p-2} H(u) = |H_0(u^+)|^{p-2} H_0(u^+) + |u^-|^{p-2} u^-$. We have

$$I_{1,\varepsilon} \leq \\ - \int_{\Omega_\varepsilon} |\nabla W_\varepsilon|^{p-2} \nabla W_\varepsilon \nabla[|H_0(u^+)|^{p-2} H_0(u^+) (u - W_\varepsilon H_0(u^+) - W_\varepsilon u^- - u_\varepsilon)] dx - \\ - \int_{\Omega_\varepsilon} |\nabla W_\varepsilon|^{p-2} \nabla W_\varepsilon \nabla(|u^-|^{p-2} u^- (u - W_\varepsilon H_0(u^+) - W_\varepsilon u^- - u_\varepsilon)) dx + \\ + \varepsilon^{-\gamma} \int_{\tilde{S}_\varepsilon} \sigma_0(u^+ - H_0(u^+))(u^+ - H_0(u^+) - u_\varepsilon) ds - \\ - \mathcal{A}(n, p) \int_{\Omega} |H(u)|^{p-2} H(u) (u - W_\varepsilon H(u) - \tilde{u}_\varepsilon) dx + \kappa_\varepsilon, \quad (31)$$

where $\kappa_\varepsilon \rightarrow 0$, $\varepsilon \rightarrow 0$.

Taking into account the definition of the function W_ε we derive that first two integrals in the right hand side of inequality (39) are equal to

$$\begin{aligned}
& - \sum_{j \in \Upsilon_\varepsilon} \int_{\partial G_\varepsilon^j} \partial_{\nu_p} w_\varepsilon^j |H_0(u^+)|^{p-2} H_0(u^+) (u^+ - H_0(u^+) - u_\varepsilon) ds - \\
& - \sum_{j \in \Upsilon_\varepsilon} \int_{\partial G_\varepsilon^j} \partial_{\nu_p} w_\varepsilon^j |u^-|^{p-2} u^- (u^+ - H_0(u^+) - u_\varepsilon) ds - \\
& - \sum_{j \in \Upsilon_\varepsilon} \int_{\partial T_\varepsilon^j} \partial_{\nu_p} w_\varepsilon^j |H_0(u^+)|^{p-2} H_0(u^+) (u - u_\varepsilon) ds - \\
& - \sum_{j \in \Upsilon_\varepsilon} \int_{\partial T_\varepsilon^j} \partial_{\nu_p} w_\varepsilon^j |u^-|^{p-2} u^- (u - u_\varepsilon) ds. \tag{32}
\end{aligned}$$

Using formula (17), (18) and inequality (19), in which function v is replaced by u , we get

$$\begin{aligned}
& I_{1,\varepsilon} \leq \\
& \leq - \sum_{j \in \Upsilon_\varepsilon} \int_{\partial G_\varepsilon^j} \partial_{\nu_p} w_\varepsilon^j |H_0(u^+)|^{p-2} H_0(u^+) (u^+ - H_0(u^+) - u_\varepsilon) ds - \\
& - \sum_{j \in \Upsilon_\varepsilon} \int_{\partial T_\varepsilon^j} \partial_{\nu_p} w_\varepsilon^j |H_0(u^+)|^{p-2} H_0(u^+) (u - u_\varepsilon) ds - \\
& - \sum_{j \in \Upsilon_\varepsilon} \int_{\partial T_\varepsilon^j} \partial_{\nu_p} w_\varepsilon^j |u^-|^{p-2} u^- (u - u_\varepsilon) ds + \tag{33} \\
& + \varepsilon^{-\gamma} \int_{S_\varepsilon} \sigma_0(u^+ - H_0(u^+)) (u^+ - H_0(u^+) - u_\varepsilon) ds - \\
& - \mathcal{A}(n, p) \int_{\Omega} |H(u)|^{p-2} H(u) (u - W_\varepsilon H(u) - \tilde{u}_\varepsilon) dx + \kappa_\varepsilon.
\end{aligned}$$

In respect that $H_0(\lambda)$ is a solution of the equation (6) and since

$$\begin{aligned}
& \lim_{\varepsilon \rightarrow 0} - \left\{ \sum_{j \in \Upsilon_\varepsilon} \int_{\partial T_\varepsilon^j} \partial_{\nu_p} w_\varepsilon^j |H_0(u^+)|^{p-2} H_0(u^+) (u - u_\varepsilon) ds + \right. \\
& \left. + \sum_{j \in \Upsilon_\varepsilon} \int_{\partial T_\varepsilon^j} \partial_{\nu_p} w_\varepsilon^j |u^-|^{p-2} u^- (u - u_\varepsilon) ds \right\} =
\end{aligned}$$

$$= \lim_{\varepsilon \rightarrow 0} \mathcal{A}(n, p) \int_{\Omega} |H(u)|^{p-2} H(u) (u - W_{\varepsilon} H(u) - \tilde{u}_{\varepsilon}) dx.$$

we derive that

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega_{\varepsilon}} \frac{|\nabla(u_{\varepsilon} - (u - W_{\varepsilon} H(u)))|^2}{|\nabla u_{\varepsilon}|^{2-p} + |\nabla(u - W_{\varepsilon} H(u))|^{2-p}} dx = 0. \quad (34)$$

Using Holder's inequality from (34) we have that

$$\|\nabla(u_{\varepsilon} - (u - W_{\varepsilon} H(u)))\|_{L^p(\Omega_{\varepsilon})} \rightarrow 0, \text{ as } \varepsilon \rightarrow 0.$$

□

REFERENCES

- [1] Jaeger W., Neuss-Radu M., Shaposhnikova T.A. Homogenization of a variational inequality for the Laplace operator with nonlinear restriction for the flux on the interior boundary of a perforated domain. // Nonlinear Analysis: Real World Applications, 2014, vol.15, pp. 367-380.
- [2] Gomez D., Perez E.M., Podolskiy A.V., Shaposhnikova T.A. Homogenization for the p-Laplace Operator in Perforated Media with Nonlinear Restrictions on the Boundary of the Perforations: A Critical Case. Dan, 2015, V. 92, N. 1, pp. 433-438.
- [3] G. Bogner, E. Rozgonyi, The local analytic solution to some nonlinear diffusion-reaction problems. WSEAS TRANSACTIONS on MATHEMATICS, 2008, issue 6, v. 7, pp. 382-395.
- [4] M.N.Zubova, T.A.Shaposhnikova Homogenization of Boundary Value Problems in Perforated Domains with the Third Boundary Condition and the Resulting Change in the Character of the Nonlinearity in the Problem// Differential Equations, pp. 78-90, 2011.
- [5] J.I. Diaz, Nonlinear Partial Diff. Eq. and free boundaries, Vol. 1, Elliptic Eq. Research Notes in Math., Pitman, London, 1985.