

A FAMILY OF FINITE ELEMENT STOKES COMPLEXES IN THREE DIMENSIONS*

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Abstract. We construct finite element Stokes complexes on tetrahedral meshes. In the lowest order case, the finite elements in the complex have 4, 18, 16, and 1 degrees of freedom on each tetrahedron, respectively. As a consequence, we obtain grad curl-conforming finite elements and inf-sup stable Stokes pairs on tetrahedral meshes which fit into complexes. We show that the new elements lead to convergent algorithms for solving a grad curl model problem as well as solving the Stokes system with precise divergence-free condition. We demonstrate the validity of the algorithms by numerical experiments.

Key words. grad curl-conforming, finite elements, Stokes complexes, divergence free, $\text{curl}\Delta\text{curl}$ problems

AMS subject classifications. 65N30, 35Q60, 65N15, 35B45

1. Introduction. The discrete de Rham complexes are now an important tool in designing finite elements and analyzing numerical schemes, c.f., [3, 4, 5, 10, 21, 29]. Motivated by problems in fluid and solid mechanics, there is an increased interest in de Rham complexes with enhanced smoothness, sometimes referred to as Stokes complexes [11, 14, 34]:

$$(1.1) \quad 0 \longrightarrow \mathbb{R} \xrightarrow{\subset} H^2(\Omega) \xrightarrow{\nabla} \mathbf{H}^1(\text{curl}; \Omega) \xrightarrow{\nabla \times} \mathbf{H}^1(\Omega) \xrightarrow{\nabla \cdot} L^2(\Omega) \longrightarrow 0,$$

where $\mathbf{H}^1(\text{curl}; \Omega) := \{\mathbf{u} \in \mathbf{H}^1(\Omega) : \text{curl } \mathbf{u} \in \mathbf{H}^1(\Omega)\}$. A slightly different version is the following [13, 29]:

$$(1.2) \quad 0 \longrightarrow \mathbb{R} \xrightarrow{\subset} H^1(\Omega) \xrightarrow{\nabla} H(\text{grad curl}; \Omega) \xrightarrow{\nabla \times} \mathbf{H}^1(\Omega) \xrightarrow{\nabla \cdot} L^2(\Omega) \longrightarrow 0.$$

Here $H(\text{grad curl}; \Omega) := \{\mathbf{u} \in \mathbf{L}^2(\Omega) : \text{curl } \mathbf{u} \in \mathbf{H}^1(\Omega)\}$ is larger than $\mathbf{H}^1(\text{curl}; \Omega)$ in (1.1), whereas the last two spaces stay the same.

Neilan [29] constructed the first discrete finite element subcomplex of (1.1) on tetrahedral meshes, which involves supersmoothness on lower-dimensional simplices of the mesh. As a result, the construction in [29] also requires high order polynomials, with degree 9, 8, 7, and 6, respectively, for the finite elements in the sequence. To reduce the polynomial degree, two discrete complexes are constructed on Alfeld splits [16] and Worsey-Farin splits [19], respectively. As a summary, these constructions involve either a large number of degrees of freedom (DOFs) or an extensive use of macroelement structures.

In this paper we construct a simple discrete subcomplex of (1.2):

$$(1.3) \quad 0 \longrightarrow \mathbb{R} \xrightarrow{\subset} \Sigma_h \xrightarrow{\nabla} V_h \xrightarrow{\nabla \times} \Sigma_h^+ \xrightarrow{\nabla \cdot} W_h \longrightarrow 0.$$

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In the lowest order case, the spaces in (1.3) have 4, 18, 16, and 1 DOFs on each element, respectively. The DOFs are those of the Whitney forms (e.g., the Nédélec element and the Raviart-Thomas element), plus vertex evaluation for V_h and Σ_h^+ . See Figure 1 below. Our construction is inspired by the modified Bernardi-Raugel bubble functions by Guzmán and Neilan [20]. For the velocity space Σ_h^+ , we extend the low order construction in [20] by enriching the vector-valued Lagrange finite elements with the modified Bernardi-Raugel bubbles and/or suitable interior bubbles in high order cases. Then we construct the entire complex using the Poincaré operators. The restriction of (1.3) to each face coincides with the 2D sequences in [23].

Applications of (1.3) include the discretization of incompressible flows, where Σ_h^+ and W_h are the finite element spaces for the velocity and pressure, respectively. Stokes complexes provide a solution to the important problem of preserving the divergence-free condition (incompressibility) precisely in the discretization of the Navier-Stokes equations [14, 24]. In this direction, the last two spaces Σ_h^+ - W_h in (1.3) extend the low order construction in Guzmán and Neilan [20] to an arbitrary order, avoiding supersmoothness or an extensive use of macroelement structures. Moreover, obtaining the entire complex (1.3) has other benefits. For example, a discrete subcomplex provides an explicit characterization for the kernel of differential operators, which is crucial for the construction of robust preconditioners in the framework of the subspace correction methods and auxiliary space preconditioning technology [22, 26, 31, 35]. With an explicit characterization of the kernel spaces in a discrete complex, one may construct parameter robust preconditioners for solving the Navier-Stokes equations, c.f., [15].

Another application of (1.3), though less addressed in the literature, is on the high order curl problems in electromagnetism and continuum mechanics [9, 27, 30]. Conforming discretization of $H(\text{grad curl}; \Omega)$ can be viewed as a natural candidate for solving these problems. In this paper, we show that V_h in (1.3) lead to a convergence scheme for a high order curl problem. Our construction thus extends the results in two space dimensions (2D) [23, 36] and simplifies the 3D $H(\text{grad curl})$ -conforming element [37] by two of the authors, which has at least 315 DOFs on each tetrahedron. In fact, solving high order curl problems is a subtle issue. Similar to the discretization of the Maxwell equations, notorious spurious numerical solutions may appear on non-convex domains if the smoothness of the finite elements are higher than necessary. We leave detailed discussions to future work, but only mention that V_h and the entire complex (1.3) provide the structures that guarantee the convergence of these problems.

The remaining part of the paper is organized as follows. In Section 2, we present preliminaries. In Section 3, we construct local shape function spaces using various versions of bubble functions and the Poincaré operators. In Section 4, we define DOFs to construct global discrete Stokes complexes. In Section 5, we prove properties of the discrete complexes, including the exactness and the approximation properties. Numerical experiments are presented in Section 6 to validate the grad curl-conforming elements for solving a high order curl problem. Finally, we summarize our results and give possible extensions in Section 7.

2. Preliminaries. Unless otherwise specified, we assume that $\Omega \in \mathbb{R}^3$ is a contractible Lipschitz domain throughout the paper. We adopt conventional notations for Sobolev spaces such as $H^m(D)$ or $H_0^m(D)$ on a sub-domain $D \subset \Omega$ furnished with the norm $\|\cdot\|_{m,D}$ and the semi-norm $|\cdot|_{m,D}$. In the case of $m = 0$, the space $H^0(D)$ coincides with $L^2(D)$ which is equipped with the inner product $(\cdot, \cdot)_D$ and the norm $\|\cdot\|_D$. When $D = \Omega$, we drop the subscript D . We use $\tilde{L}^2(D)$ to denote L^2 functions

with vanishing mean:

$$\dot{L}^2(D) = \left\{ q \in L^2(D) : \int_D q dV = 0 \right\}.$$

We also use $\mathbf{H}^m(D)$, $\mathbf{H}_0^m(D)$, and $\mathbf{L}^2(D)$ to denote the vector-valued Sobolev spaces $[\mathbf{H}^m(D)]^3$, $[\mathbf{H}_0^m(D)]^3$, and $[\mathbf{L}^2(D)]^3$.

In addition to the standard Sobolev spaces, we also define

$$\begin{aligned} H(\text{curl}; \Omega) &:= \{ \mathbf{u} \in \mathbf{L}^2(\Omega) : \nabla \times \mathbf{u} \in \mathbf{L}^2(\Omega) \}, \\ H(\text{grad curl}; \Omega) &:= \{ \mathbf{u} \in \mathbf{L}^2(\Omega) : \nabla \times \mathbf{u} \in \mathbf{H}^1(\Omega) \}, \\ H(\text{curl}^2; \Omega) &:= \{ \mathbf{u} \in H(\text{curl}; \Omega) : \nabla \times \mathbf{u} \in H(\text{curl}; \Omega) \}. \end{aligned}$$

In general $H(\text{grad curl}; \Omega) \subseteq H(\text{curl}^2; \Omega)$. If Ω is convex or has $C^{1,1}$ boundary, then for any function $\mathbf{u} \in H(\text{curl}^2; \Omega)$ with certain boundary conditions, e.g., $\mathbf{u} \times \mathbf{n} = 0$ on $\partial\Omega$, we have $\nabla \times \mathbf{u} \in \mathbf{H}^1$ since $\nabla \times (\nabla \times \mathbf{u}) \in \mathbf{L}^2$ and $\nabla \cdot (\nabla \times \mathbf{u}) = 0$ [17]. This implies that for these domains we actually have $H(\text{grad curl}; \Omega) \cap H_0(\text{curl}; \Omega) = H(\text{curl}^2; \Omega) \cap H_0(\text{curl}; \Omega)$, where $H_0(\text{curl}; \Omega) := \{ \mathbf{u} \in H(\text{curl}; \Omega) : \mathbf{u} \times \mathbf{n} = 0 \text{ on } \partial\Omega \}$. In particular, any $H(\text{curl}^2; \Omega)$ -conforming finite element is automatically $H(\text{grad curl}; \Omega)$ -conforming. Therefore in this paper we will focus on the construction of grad curl-conforming finite elements, which naturally fit in the Stokes complex (1.2).

For a subdomain D , we use $P_k(D)$, or simply P_k when there is no possible confusion, to denote the space of polynomials with degree at most k on D . We also denote by $\mathcal{H}_k(D)$ the space of homogeneous polynomials of degree k on D . Let $\mathbf{P}_k = [P_k]^3$ and $\mathcal{H}_k(D) = [\mathcal{H}_k(D)]^3$ be the corresponding spaces of vector-valued polynomials.

Let $\mathcal{T}_h$ be a partition of the domain Ω consisting of shape-regular tetrahedra. We denote h_K as the diameter of an element $K \in \mathcal{T}_h$ and h as the mesh size of $\mathcal{T}_h$. Denote by $\mathcal{V}_h(K)$, $\mathcal{E}_h(K)$, and $\mathcal{F}_h(K)$ the sets of vertices, edges, and faces of $K \in \mathcal{T}_h$. With the affine mapping

$$(2.1) \quad F_K(\hat{\mathbf{x}}) = B_K \hat{\mathbf{x}} + \mathbf{b}_K,$$

we can map the reference element $\hat{K}$ (the tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$) to the element K . We use the notation $\hat{\cdot}$ to denote the variables relating to $\hat{K}$.

For each $K \in \mathcal{T}_h$, let x_K be the barycenter of K . We denote K^r as the partition of K by adjoining the vertices of K with the new vertex x_K , known as the Alfeld split of K [1]. We also denote

$$\begin{aligned} \mathbf{P}_k^c(K^r) &= \{ \mathbf{v} \in \mathbf{H}^1(K) : \mathbf{v}|_T \in \mathbf{P}_k(T) \text{ for all } T \in K^r \}, \\ \dot{\mathbf{P}}_k^c(K^r) &= \{ \mathbf{v} \in \mathbf{H}_0^1(K) : \mathbf{v}|_T \in \mathbf{P}_k(T) \text{ for all } T \in K^r \}, \\ \dot{P}_k(K^r) &= \{ q \in \dot{L}^2(K) : q|_T \in P_k(T) \text{ for all } T \in K^r \}. \end{aligned}$$

We use C to denote a generic positive h -independent constant.

3. Local shape function spaces.

3.1. Modified bubble functions. Let $x_1, \dots, x_4$ be the four vertices of the element K , and $x_0 = x_K$. Let λ_0 be the continuous, piecewise linear function satisfying

$\lambda_0(x_j) = \delta_{0j}$ for $0 \leq j \leq 4$. Denote

$$\mathbf{P}_l^\perp(K) = \left\{ \mathbf{v} \in \mathbf{P}_l(K) : \int_K \mathbf{v} \cdot \boldsymbol{\kappa} dV = 0 \text{ for all } \boldsymbol{\kappa} \in \mathcal{R}_{l-1} \right\}$$

with $\mathcal{R}_l = \mathbf{P}_{l-1}(K) \oplus \{\mathbf{p} \in \mathcal{H}_l(K) : \mathbf{x} \cdot \mathbf{p} = 0\}$ for $l \geq 1$ and $\mathcal{R}_l = 0$ for $l = 0, -1$. We will enrich the velocity space with bubble functions from the following space:

$$(3.1) \quad \mathbf{M}_k(K^r) = \{\mathbf{v} \in \mathring{\mathbf{P}}_k^c(K^r) : \mathbf{v} = \sum_{j=1}^k \lambda_0^j \mathbf{w}_{k-j} \text{ with } \mathbf{w}_{k-j} \in \mathbf{P}_{k-j}^\perp(K)\}.$$

We recall the following property [20, Theorem 3.3].

LEMMA 3.1. *Let $k \geq 1$. For any $K \in \mathcal{T}_h$ and for any $p \in \mathring{P}_{k-1}(K^r)$, there exists a unique $\mathbf{v} \in \mathbf{M}_k(K^r)$ satisfying*

$$\operatorname{div} \mathbf{v} = p \text{ on } K.$$

Let $\lambda_i (i = 1, 2, 3, 4)$ be the barycentric coordinates of K , i.e., $\lambda_i(x_j) = \delta_{ij}$. We define the scalar face bubbles

$$B_i = \prod_{j=1, j \neq i}^4 \lambda_j \text{ for } 1 \leq i \leq 4$$

and the scalar interior bubble

$$B_0 = \prod_{j=1}^4 \lambda_j.$$

The Bernardi-Raugel face bubbles are given as

$$\mathbf{b}_i^f = B_i \mathbf{n}_i \text{ for } 1 \leq i \leq 4,$$

where $\mathbf{n}_i$ is the outward unit normal to $f_i \in \mathcal{F}_h(K)$.

According to [20, Proposition 4.2], we can modify the Bernardi-Raugel face bubbles such that they have constant divergence.

LEMMA 3.2. *There exists $\boldsymbol{\beta}_i^f \in \mathbf{P}_3^c(K^r)$ such that*

$$(3.2) \quad \boldsymbol{\beta}_i^f|_{\partial K} = \mathbf{b}_i^f|_{\partial K}, \quad \nabla \cdot \boldsymbol{\beta}_i^f \in P_0(K).$$

We refer to the functions $\boldsymbol{\beta}_i^f \in \mathbf{P}_3^c(K^r)$, $i = 1, 2, 3, 4$ which satisfy (3.2) as the *modified Bernardi-Raugel bubbles* on a tetrahedron K (c.f., [20]). Denote

$$B^1 := \operatorname{span}\{\boldsymbol{\beta}_i^f, i = 1, 2, 3, 4\},$$

To construct high order elements, we will use certain interior bubbles. Denote

$$\mathcal{S}_k(K) = \begin{cases} \mathcal{H}_k(K), & k = 1, \\ \mathcal{H}_k(K) \oplus \mathcal{H}_{k-1}(K), & k \geq 2, \end{cases}$$

and

$$\mathring{\mathcal{S}}_k(K) := \{u - \frac{1}{|K|} \int_K u dV : u \in \mathcal{S}_k(K)\}.$$

According to Lemma 3.1, there exists a unique subspace $\mathring{B}^{k+1} \subset \mathbf{M}_{k+1}(K^r)$ such that $\nabla \cdot \mathring{B}^{k+1} = \mathring{\mathcal{S}}_k(K)$, and $\dim \mathring{B}^{k+1} = \dim \mathring{\mathcal{S}}_k(K)$.

Remark 3.3. With the constructive proof of Lemma 3.1 (c.f., [20, Theorem 3.3]), we can obtain explicit forms of the interior bubbles in the implementation.

LEMMA 3.4. For $k \geq 1$, a function $\mathbf{v} \in \mathring{B}^{k+1}$ is uniquely determined by

$$(3.3) \quad \int_K \mathbf{v} \cdot \nabla q dV \text{ for all } q \in \mathring{S}_k(K).$$

Proof. From the construction, $\dim \mathring{B}^{k+1} = \dim \mathring{S}_k(K)$. Suppose that the functionals in (3.3) vanish on $\mathbf{v}$. It suffices to show $\mathbf{v} = 0$. Indeed, we have from integration by parts

$$0 = \int_K \mathbf{v} \cdot \nabla q dV = \int_K \nabla \cdot \mathbf{v} q dV.$$

Taking $q = \nabla \cdot \mathbf{v}$, we obtain $\nabla \cdot \mathbf{v} = 0$ and therefore $\mathbf{v} = 0$ since $\text{div} : \mathring{B}^{k+1} \rightarrow \mathring{S}_k(K)$ is bijective by the construction of $\mathring{B}^{k+1}$. $\square$

3.2. Poincaré operators. For any complex

$$(3.4) \quad \dots \longrightarrow V^{k-1} \xrightarrow{d^{k-1}} V^k \xrightarrow{d^k} V^{k+1} \longrightarrow \dots,$$

where $V^\bullet$ are linear vector spaces and $d^\bullet$ are linear operator, we call a graded operator $\mathbf{p}^k : V^k \rightarrow V^{k-1}$ Poincaré operators if it satisfies

- the null-homotopy property:

$$(3.5) \quad d^{k-1} \mathbf{p}^k + \mathbf{p}^{k+1} d^k = \text{id}_{V^k};$$

- the complex property:

$$\mathbf{p}^{k-1} \circ \mathbf{p}^k = 0.$$

LEMMA 3.5. If there exist Poincaré operators $\mathbf{p}^\bullet$ for (3.4), then (3.4) is exact.

Proof. Assume that $d^k u = 0$ for $u \in V^k$. From the null-homotopy identity, $u = d^{k-1}(\mathbf{p}^k u)$. This implies the exactness of (3.4) at V^k . $\square$

For the 3D de Rham complex,

$$(3.6) \quad 0 \longrightarrow \mathbb{R} \xrightarrow{\subset} C^\infty \xrightarrow{d^0} [C^\infty]^3 \xrightarrow{d^1} [C^\infty]^3 \xrightarrow{d^2} C^\infty \longrightarrow 0,$$

there exist Poincaré operators and they have the explicit form [11, 21, 25]:

$$(3.7) \quad \mathbf{p}^1 \mathbf{u} = \int_0^1 \mathbf{u}(W + t(\mathbf{x} - W)) \cdot (\mathbf{x} - W) dt,$$

$$(3.8) \quad \mathbf{p}^2 \mathbf{u} = \int_0^1 \mathbf{u}(W + t(\mathbf{x} - W)) \times t(\mathbf{x} - W) dt,$$

$$(3.9) \quad \mathbf{p}^3 \mathbf{u} = \int_0^1 t^2 \mathbf{u}(W + t(\mathbf{x} - W)) (\mathbf{x} - W) dt,$$

where W is a base point. In addition to the complex property and the null-homotopy identity, these operators further satisfy

- the polynomial preserving property: if u is a polynomial of degree r , then $\mathbf{p}u$ is a polynomial of degree at most $r + 1$.

3.3. Local shape function spaces. On each $K \in \mathcal{T}_h$, we construct the local shape function spaces of (1.3) as follows:

(3.10)

$$0 \longrightarrow \mathbb{R} \xrightarrow{\subset} \Sigma_h^r(K) \xrightarrow{\nabla} V_h^{r-1,k+1}(K) \xrightarrow{\nabla \times} \Sigma_h^{k,+}(K) \xrightarrow{\nabla \cdot} W_h^{k-1}(K) \longrightarrow 0.$$

Different choices of r and k will lead to various versions of the complex (3.10).

We choose $\Sigma_h^r(K) := P_r(K)$, $W_h^{k-1}(K) := P_{k-1}(K)$, and set $\Sigma_h^{k,+}(K) = P_k(K) \oplus B$, where

$$B = \begin{cases} B^1, & k = 1, \\ B^1 \oplus \mathring{B}^2, & k = 2, \\ \mathring{B}^k, & k \geq 3. \end{cases}$$

Note that for $k = 1$, we only supply $P_1(K)$ with the modified Bernardi-Raugel face bubbles; for $k = 2$, we supply $P_2(K)$ with both face and interior bubbles, while for $k \geq 3$ we only need to supply $P_k(K)$ with interior bubbles. It is easy to see the face bubbles $\{\beta_i^f\}_{i=1}^4$ and $P_2(K)$ are linearly independent, and hence, $P_2(K) \oplus B^1$ and $P_1(K) \oplus B^1$ are direct sums. From the explicit form (3.1) of the functions in $M_k(K^r)$, we see that $M_k(K^r) \oplus P_k(K)$ is a direct sum, and hence, $\mathring{B}^k \oplus P_k(K)$, is also a direct sum.

Remark 3.6. The idea of enriching with modified bubbles is inspired by [20], where the case of $k = 1$ is defined and used to construct a stable Stokes finite element pair. Here we extend it to high order cases.

Define

$$(3.11) \quad V_h^{r-1,k+1}(K) = \nabla \Sigma_h^r(K) \oplus \mathfrak{p}^2 \Sigma_h^{k,+}(K).$$

The right-hand side of (3.11) is a direct sum. In fact, if $\mathbf{u} \in \nabla \Sigma_h^r(K) \cap \mathfrak{p}^2 \Sigma_h^{k,+}(K)$, then $\nabla \times \mathbf{u} = 0$ and $\mathfrak{p}^1 \mathbf{u} = 0$. By the null-homotopy identity (3.5), $\mathbf{u} = \mathfrak{p}^2 \nabla \times \mathbf{u} + \nabla \mathfrak{p}^1 \mathbf{u} = 0$.

Remark 3.7. For the bubble functions in $\Sigma_h^{k,+}(K)$, we choose the barycenter x_K as the base point W , c.f., [11]. For other functions, we choose $W = 0$ to be the origin.

Remark 3.8. The Koszul operator $\kappa \mathbf{u} := \mathbf{u} \times \mathbf{x}$ exerting on homogeneous polynomials has similar properties as the Poincaré operator $\mathfrak{p}^2$ [3, 4]. For polynomial bases in $\Sigma_h^{k,+}(K)$ other than the bubbles, we can replace the Poincaré operator $\mathfrak{p}^2$ by the Koszul operator. However, to get the complex property it seems necessary to use the Poincaré operators for the bubbles.

By the definition of the shape function spaces, it is easy to show that the sequence (3.10) is a complex. By the properties of the Poincaré operators, we can verify that the sequence

$$(3.12) \quad 0 \longleftarrow \Sigma_h^r(K) \xleftarrow{\mathfrak{p}^1} V_h^{r-1,k+1}(K) \xleftarrow{\mathfrak{p}^2} \Sigma_h^{k,+}(K) \xleftarrow{\mathfrak{p}^3} W_h^{k-1}(K) \longleftarrow 0$$

is also a complex with the Poincaré operators in (3.7)-(3.9). From Lemma 3.5, we obtain the exactness.

LEMMA 3.9. *The complex (3.10) is exact.*

From the definition, we see that $V_h^{r-1,k+1}(K)$ has two parts: one from the gradient on $\Sigma_h^r(K)$ and the other from the Poincaré operator on $\Sigma_h^{k,+}$. The first part is easy

to implement: we may remove the constant (kernel of gradient) from the bases of Σ_h^r and apply gradient to the rest. The $\mathbf{p}^2 \Sigma_h^{k,+}(K)$ part calls for more explanation as we cannot obtain a basis by applying the Poincaré operator to a basis of $\Sigma_h^{k,+}$ (as the results are not linearly independent). Now we show how to obtain a basis for the $\mathbf{p}^2 \Sigma_h^{k,+}(K)$ part to implement $V_h^{r-1,k+1}(K)$.

We first claim $\mathbf{P}_k(K) = \nabla \times \mathbf{P}_{k+1}(K) \oplus \mathbf{p}^3 P_{k-1}(K)$. In fact, for all $\mathbf{u} \in \mathbf{P}_k(K)$, the null-homotopy identity (3.5) leads to $\mathbf{u} = \nabla \times \mathbf{p}^2 \mathbf{u} + \mathbf{p}^3 \nabla \cdot \mathbf{u} \in \nabla \times \mathbf{P}_{k+1}(K) + \mathbf{p}^3 P_{k-1}(K)$. Moreover, if $\mathbf{u} \in \nabla \times \mathbf{P}_{k+1}(K) \cap \mathbf{p}^3 P_{k-1}(K)$, then $\nabla \cdot \mathbf{u} = 0$ and $\mathbf{p}^2 \mathbf{u} = 0$, which follows from (3.5) again that $\mathbf{u} = 0$.

We then have the decomposition $\Sigma_h^{k,+}(K) = \nabla \times \mathbf{P}_{k+1}(K) \oplus \mathbf{p}^3 P_{k-1}(K) \oplus B$, which leads to

$$(3.13) \quad \mathbf{p}^2 \Sigma_h^{k,+}(K) = \mathbf{p}^2 \nabla \times \mathbf{P}_{k+1}(K) + \mathbf{p}^2 B + \mathbf{p}^2 \mathbf{p}^3 P_{k-1}(K) = \mathbf{p}^2 \nabla \times \mathbf{P}_{k+1} + \mathbf{p}^2 B,$$

where we used $\mathbf{p}^2 \mathbf{p}^3 = 0$.

From the exactness and the decomposition of $\Sigma_h^{k,+}(K)$, we obtain

$$\begin{aligned} \dim \mathbf{p}^2 \Sigma_h^{k,+} &= \dim V_h^{r-1,k+1}(K) - \dim \nabla \Sigma_h^r(K) \\ &= \dim \Sigma_h^{k,+}(K) - \dim W_h^{k-1}(K) \\ &= \dim \nabla \times \mathbf{P}_{k+1}(K) + \dim \mathbf{p}^3 P_{k-1}(K) + \dim B - \dim W_h^{k-1}(K) \\ &= \dim \nabla \times \mathbf{P}_{k+1}(K) + \dim B \\ &\geq \dim \mathbf{p}^2 \nabla \times \mathbf{P}_{k+1}(K) + \dim \mathbf{p}^2 B, \end{aligned}$$

which together with (3.13) leads to

$$\mathbf{p}^2 \Sigma_h^{k,+}(K) = \mathbf{p}^2 \nabla \times \mathbf{P}_{k+1}(K) \oplus \mathbf{p}^2 B.$$

Therefore, to implement $\mathbf{p}^2 V_h^{r-1,k+1}(K)$, we take the bases of B and the bases of $\nabla \times \mathbf{P}_{k+1}(K)$, and apply the Poincaré operator on them.

To show the approximation property of the finite element space $V_h^{r-1,k+1}$, we demonstrate that $V_h^{r-1,k+1}(K)$ contains polynomials of certain degree.

LEMMA 3.10. *The inclusion $\mathbf{P}_s(K) \subseteq V_h^{r-1,k+1}(K)$ holds, where $s = \min\{r-1, k+1\}$.*

Proof. From the null-homotopy property, $\mathbf{P}_s(K) = \text{grad } \mathbf{p}^1 \mathbf{P}_s(K) + \mathbf{p}^2 \text{curl } \mathbf{P}_s(K)$. By definition, $V_h^{r-1,k+1}(K) = \text{grad } \Sigma_h^r(K) + \mathbf{p}^2 \Sigma_h^{k,+}(K)$. For $s = \min\{r-1, k+1\}$, we have $\mathbf{p}^1 \mathbf{P}_s(K) \subseteq P_r(K) = \Sigma_h^r(K)$ and $\text{curl } \mathbf{P}_s(K) \subseteq \mathbf{P}_k(K) \subseteq \Sigma_h^{k,+}(K)$. Therefore the desired inclusion holds. $\square$

4. Degrees of freedom and global finite element spaces. In this section, we construct grad curl-conforming finite elements and discrete Stokes complexes on tetrahedra. The discrete complex with global finite element spaces is given by

$$(4.1) \quad 0 \longrightarrow \mathbb{R} \xrightarrow{\subset} \Sigma_h^r \xrightarrow{\nabla} V_h^{r-1,k+1} \xrightarrow{\nabla \times} \Sigma_h^{k,+} \xrightarrow{\nabla \cdot} W_h^{k-1} \longrightarrow 0.$$

Taking $r = k, k+1$, and $k+2$ in (4.1) yields three versions of grad curl-conforming element spaces $V_h^{k-1,k+1}$, $V_h^{k,k+1}$, and $V_h^{k+1,k+1}$. Fig. 1 demonstrates the complex (4.1) for the case $k = 1$.

We define DOFs for each space in (4.1).

The DOFs for the Lagrange element $\Sigma_h^r(K)$ can be given as follows.

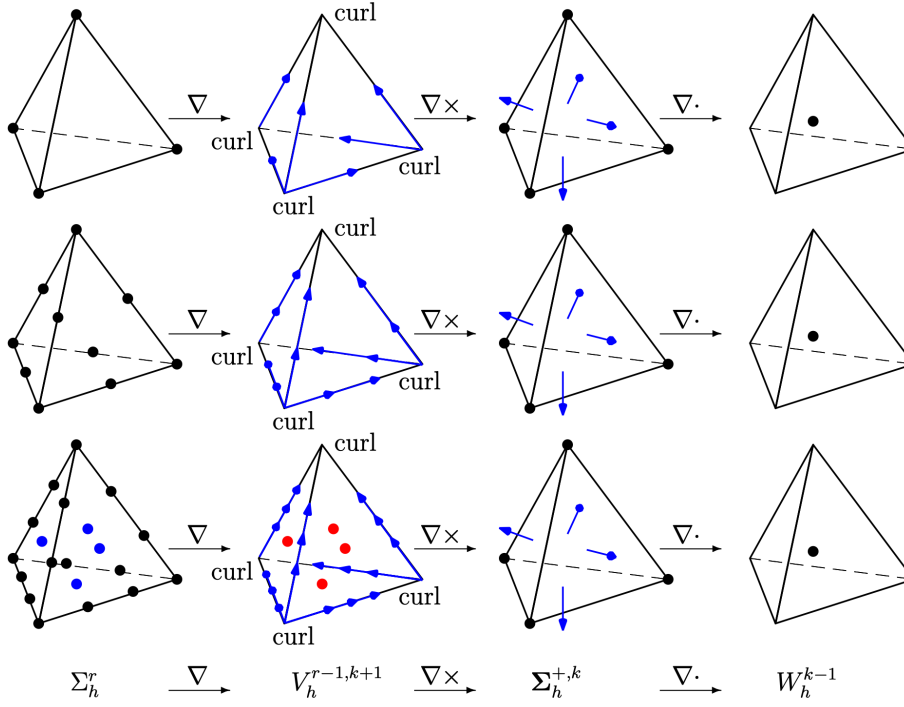


Fig. 1: The lowest-order ($k = 1$) finite element complex (4.1) on tetrahedra with $r = k$ in the first row, $r = k + 1$ in the second row, and $r = k + 2$ in the third row.

- Vertex DOFs $M_v(u)$ at all the vertices $v_i \in \mathcal{V}_h(K)$:

$$M_v(u) = \{u(v_i)\}.$$

- Edge DOFs $M_e(u)$ on all the edges $e_i \in \mathcal{E}_h(K)$:

$$M_e(u) = \left\{ \int_{e_i} u v ds \text{ for all } v \in P_{r-2}(e_i) \right\}.$$

- Face DOFs $M_f(u)$ on all the faces $f_i \in \mathcal{F}_h(K)$:

$$M_f(u) = \left\{ \int_{f_i} u v dA \text{ for all } v \in P_{r-3}(f_i) \right\}.$$

- Interior DOFs $M_K(u)$ in the element K :

$$M_K(u) = \left\{ \int_K u v dV \text{ for all } v \in P_{r-4}(K) \right\}.$$

We now equip the space $V_h^{r-1,k+1}(K)$ with the following DOFs:

- Vertex DOFs $\mathbf{M}_v(\mathbf{u})$ at all vertices $v_i \in \mathcal{V}_h(K)$:

$$(4.2) \quad \mathbf{M}_v(\mathbf{u}) = \{(\nabla \times \mathbf{u})(v_i)\}.$$

- Edge DOFs $\mathbf{M}_e(\mathbf{u})$ on all edges $e_i \in \mathcal{E}_h(K)$:

$$\begin{aligned} \mathbf{M}_e(\mathbf{u}) = & \left\{ \int_{e_i} \mathbf{u} \cdot \boldsymbol{\tau}_i q \, ds \text{ for all } q \in P_{r-1}(e_i) \right\} \\ & \cup \left\{ \int_{e_i} \nabla \times \mathbf{u} \cdot \mathbf{q} \, ds \text{ for all } \mathbf{q} \circ F_K \in \mathbf{P}_{k-2}(\hat{e}_i) \right\}. \end{aligned} \quad (4.3)$$

- Face DOFs $\mathbf{M}_f(\mathbf{u})$ at all faces $f_i \in \mathcal{F}_h(K)$ (with two mutually orthogonal unit vector $\boldsymbol{\tau}_i^1$ and $\boldsymbol{\tau}_i^2$ in the face f_i and the unit normal vector $\mathbf{n}_i$):

$$\begin{aligned} \mathbf{M}_f(\mathbf{u}) = & \left\{ \int_{f_i} \nabla \times \mathbf{u} \cdot \mathbf{n}_i q \, dA \text{ for all } q \in P_{k-3}(f_i)/\mathbb{R} \right\} \\ & \cup \left\{ \int_{f_i} \nabla \times \mathbf{u} \cdot \boldsymbol{\tau}_i^1 q \, dA \text{ for all } q \in P_{k-3}(f_i) \right\} \\ & \cup \left\{ \int_{f_i} \nabla \times \mathbf{u} \cdot \boldsymbol{\tau}_i^2 q \, dA \text{ for all } q \in P_{k-3}(f_i) \right\} \\ & \cup \left\{ \int_{f_i} \mathbf{u} \cdot \mathbf{q} \, dA \text{ for all } \mathbf{q} \circ F_K = B_K \hat{\mathbf{q}}, \hat{\mathbf{q}} \in P_{r-3}(\hat{f}_i) \hat{\mathbf{x}}_{\hat{f}_i} \right\}, \end{aligned} \quad (4.4)$$

where $\hat{\mathbf{x}}_{\hat{f}_i} = [\hat{\mathbf{x}} - (\hat{\mathbf{x}} \cdot \hat{\mathbf{n}}_i) \hat{\mathbf{n}}_i]|_{\hat{f}_i}$.

- Interior DOFs $\mathbf{M}_K(\mathbf{u})$ for the element K :

$$\begin{aligned} \mathbf{M}_K(\mathbf{u}) = & \left\{ \int_K \mathbf{u} \cdot \mathbf{q} \, dV \text{ for all } \mathbf{q} \circ F_K = B_K \hat{\mathbf{q}}, \hat{\mathbf{q}} \in P_{r-4}(\hat{K}) \hat{\mathbf{x}} \right\} \\ & \cup \left\{ \int_K \nabla \times \mathbf{u} \cdot \mathbf{q} \, dV \text{ for all } \mathbf{q} \circ F_K = B_K^{-T} \hat{\mathbf{q}}, \hat{\mathbf{q}} \in \hat{\nabla} \times \hat{V}_h^{r-1, k+1}(\hat{K}) \right\}, \end{aligned} \quad (4.5)$$

where $\hat{V}_h^{r-1, k+1}(K) = \{\mathbf{u} \in V_h^{r-1, k+1}(K) : \text{DOFs (4.2)–(4.4) vanish on } \mathbf{u}\}$.

The DOFs for $\boldsymbol{\Sigma}_h^{k,+}(K)$ can be given similarly to $\boldsymbol{\Sigma}_h^r(K)$ with some additional face or interior integration DOFs to take care of the bubble functions (see Lemma 3.2 and Lemma 3.4).

- Vertex DOFs $\mathbf{M}_v(\mathbf{u})$ at all the vertices $v_i \in \mathcal{V}_h(K)$:

$$\mathbf{M}_v(\mathbf{u}) = \{\mathbf{u}(v_i)\}. \quad (4.6)$$

- Edge DOFs $\mathbf{M}_e(\mathbf{u})$ on all the edges $e_i \in \mathcal{E}_h(K)$:

$$\mathbf{M}_e(\mathbf{u}) = \left\{ \int_{e_i} \mathbf{u} \cdot \mathbf{v} \, ds \text{ for all } \mathbf{v} \in \mathbf{P}_{k-2}(e_i) \right\}. \quad (4.7)$$

- Face DOFs $\mathbf{M}_f(\mathbf{u})$ on all the faces $f_i \in \mathcal{F}_h(K)$:

$$\begin{aligned} \mathbf{M}_f(\mathbf{u}) = & \left\{ \int_{f_i} \mathbf{u} \cdot \mathbf{v} \, dA \text{ for all } \mathbf{v} \in \mathbf{P}_{k-3}(f_i) \right\} \\ & \cup \left\{ \int_{f_i} \mathbf{u} \cdot \mathbf{n}_i \, dA \text{ when } k = 1, 2 \right\}. \end{aligned} \quad (4.8)$$

- Interior DOFs $\mathbf{M}_K(\mathbf{u})$ in the element K :

$$\begin{aligned} \mathbf{M}_K(\mathbf{u}) = & \left\{ \int_K \mathbf{u} \cdot \mathbf{v} \, dV \text{ for all } \mathbf{v} = B_K^{-T} \hat{\mathbf{v}}, \hat{\mathbf{v}} \in \hat{\nabla} \times \hat{V}_h^{r-1, k+1}(\hat{K}) \right\} \\ & \cup \left\{ \int_K \mathbf{u} \cdot \nabla v \, dV \text{ for all } v \in P_{k-1}(K)/\mathbb{R} \right\}. \end{aligned} \quad (4.9)$$

The DOFs for $W_h^{k-1}(K)$ can be given as follows.

- Interior DOFs $M_K(u)$ in the element K :

$$M_K(u) = \left\{ \int_K u \cdot v dV \text{ for all } v \in P_{k-1}(K) \right\}$$

LEMMA 4.1. *The DOFs for $\Sigma_h^{k,+}(K)$ are unisolvent.*

Proof. The case of $k = 1$ is proved in [20, Lemma 4.3], and the case of $k = 2$ can be proved similarly. We only prove the lemma for $k \geq 3$. For $\mathbf{u} \in \Sigma_h^{k,+}(K)$, rewrite $\mathbf{u} = \mathbf{w} + \sum_{i=1}^{N_k-1} b_i \hat{\beta}_i$ with $b_i \in \mathbb{R}$, $\mathbf{w} \in \mathbf{P}_k(K)$, and $\hat{\beta}_i \in \hat{B}^k$. Suppose that the DOFs (4.6)–(4.9) vanish on $\mathbf{u}$. We must show that $\mathbf{u} = 0$. Since $\hat{\beta}_i$ vanish on ∂K , $\mathbf{w}$ vanishes on ∂K by the DOFs in (4.6)–(4.8). The DOFs in the second set of (4.9) leads to $\nabla \cdot \mathbf{u} = 0$ since $\nabla \cdot \mathbf{u} \in P_{k-1}(K)/\mathbb{R}$. Therefore $\mathbf{u} = \nabla \times \mathbf{v}$ with $\mathbf{v} \in \dot{V}_h^{r-1,k+1}(K)$. Using the DOFs in the first set of (4.9), we obtain $\mathbf{u} = 0$. $\square$

LEMMA 4.2. *The DOFs for $V_h^{r-1,k+1}(K)$ are unisolvent.*

Proof. Since the complex (3.10) is exact, we have

$$(4.10) \quad \dim V_h^{r-1,k+1}(K) = \dim \Sigma_h^{k,+}(K) + \dim \Sigma_h^r(K) - \dim W_h^{k-1}(K) - 1.$$

We can check that the space of the DOFs has the same dimension. Then it suffices to show that if all the DOFs vanish on $\mathbf{u} \in V_h^{r-1,k+1}(K)$, then $\mathbf{u} = 0$. To see this, we first show that $\nabla \times \mathbf{u} = 0$. Using the properties of the Poincaré operators, we have $\nabla \times \mathbf{p}^2 \Sigma_h^{k,+}(K) \subset \Sigma_h^{k,+}(K)$. By integration by parts, the following DOFs for $\Sigma_h^{k,+}(K)$ vanish on $\nabla \times \mathbf{u}$:

$$\int_{f_i} \nabla \times \mathbf{u} \cdot \mathbf{n}_i dA = \int_{\partial f_i} \mathbf{u} \cdot \boldsymbol{\tau}_{\partial f_i} ds = 0,$$

and

$$\int_K \nabla \times \mathbf{u} \cdot \nabla v dV = \int_{\partial K} \nabla \times \mathbf{u} \cdot \mathbf{n}_{\partial K} v dA = 0 \text{ for any } v \in P_{k-1}(K).$$

By the unisolvence of the DOFs for $\Sigma_h^{k,+}(K)$, we get $\nabla \times \mathbf{u} = 0$ in K .

Therefore on each f_i , there exists a $\phi_i \in P_r(f_i)$ such that $\mathbf{n}_i \times \mathbf{u}|_{f_i} \times \mathbf{n}_i = \nabla_{f_i} \phi_i$. Here ∇_{f_i} is the face gradient on f_i . By the edge DOFs of $V_h^{r-1,k+1}(K)$, we get $\mathbf{u} \cdot \boldsymbol{\tau}_i = 0$ on the edge $\mathbf{e}_i$. Therefore ϕ_i is a constant on all the edges of f_i . Without loss of generality, we can choose this constant to be zero. Then ϕ_i has the form $\phi_i = B_i|_{f_i} \psi_i$ with $\psi_i \in P_{r-3}(f_i)$. By the property of Koszul operators in 2D [3, Theorem 7.1], for any function $\psi_i \in P_{r-3}(f_i)$, there exists $\mathbf{q}_i \in P_{r-3}(\hat{f}_i) B_K \hat{\mathbf{x}}_{\hat{f}_i}$ satisfying $\mathbf{q}_i \perp \mathbf{n}_i$ and $\nabla_{f_i} \cdot \mathbf{q}_i = \psi_i$. By the DOFs in (4.4), we have

$$0 = (\mathbf{u}, \mathbf{q}_i)_{f_i} = -(\phi_i, \nabla_{f_i} \cdot \mathbf{q}_i)_{f_i} = -(B_i|_{f_i} \psi_i, \psi_i)_{f_i}.$$

This implies that $\psi_i = 0$, i.e., $\mathbf{u} \times \mathbf{n}_i = 0$ on f_i .

Since $\nabla \times \mathbf{u} = 0$ and $\mathbf{u} \times \mathbf{n}_i = 0$ on f_i , there exists $\phi = B_0 \psi$ with $\psi \in P_{r-4}(K)$ such that $\mathbf{u} = \nabla \phi$. We choose $\mathbf{q} \in P_{r-4}(K) B_K \hat{\mathbf{x}}$ such that $\nabla \cdot \mathbf{q} = \psi$. Then

$$0 = (\mathbf{u}, \mathbf{q}) = (\nabla \phi, \mathbf{q}) = -(\phi, \nabla \cdot \mathbf{q}) = -(B_0 \psi, \psi). \quad \square$$

This implies that $\psi = 0$ and hence $\phi = 0$ and $\mathbf{u} = 0$.

Equipping the local spaces with the above DOFs, we obtain the global finite element spaces Σ_h^r , $V_h^{r-1,k+1}$, $\Sigma_h^{k,+}$, and W_h^{k-1} .

LEMMA 4.3. *The following conformity holds:*

$$V_h^{r-1,k+1} \subset H(\text{grad curl}; \Omega).$$

Proof. If we can verify $V_h^{r-1,k+1} \subset H(\text{curl}; \Omega)$, then the conformity follows from $\nabla \times V_h^{r-1,k+1} \subseteq \Sigma_h^{k,+} \subset \mathbf{H}^1(\Omega)$. To this end, we must show $\mathbf{u} \times \mathbf{n}_i = 0$ for all $f_i \in \mathcal{F}_h(K)$ if the DOFs (4.2)-(4.4) vanish on $\mathbf{u} \in V_h^{r-1,k+1}(K)$. From the vanishing DOFs involving $\nabla \times \mathbf{u}$ and $\int_{f_i} \nabla \times \mathbf{u} \cdot \mathbf{n}_i dA = \int_{\partial f_i} \mathbf{u} \cdot \boldsymbol{\tau}_{\partial f_i} ds = 0$, we have $\nabla \times \mathbf{u} = 0$ on ∂K . Proceeding as in the proof of Lemma 4.2, we can show that $\mathbf{u} \times \mathbf{n}_i = 0$ on each f_i .

5. Global Finite element complexes. We now present properties of the complex (4.1) with the global finite element spaces. The first property we will show is the surjectivity of $\nabla : \Sigma_h^{k,+} \rightarrow W_h^{k-1}$. To this end, we need the following property for the local complex.

LEMMA 5.1. *For any $q \in W_h^{k-1}(K) \cap \dot{L}^2(K)$, there exists $\mathbf{v} \in \Sigma_h^{k,+}(K) \cap \mathbf{H}_0^1(K)$ such that $\nabla \cdot \mathbf{v} = q$ and $\|\mathbf{v}\|_{1,K} \leq C\|q\|_K$.*

Proof. For a fixed $q \in W_h^{k-1}(K) \cap \dot{L}^2(K)$, there exists $\mathbf{w} \in \mathbf{H}_0^1(K)$ such that (see e.g. [17, Corollary 2.4])

$$\nabla \cdot \mathbf{w} = q \text{ in } \Omega.$$

Let $\mathbf{v} \in \Sigma_h^{k,+}(K)$ be the unique function that satisfies

$$\int_K \mathbf{v} \cdot \nabla p dV = \int_K \mathbf{w} \cdot \nabla p dV, \quad \forall p \in P_{k-1}(K),$$

with the remaining DOFs in (4.6)-(4.9) vanishing on $\mathbf{v}$. Then $\mathbf{v} \in \Sigma_h^{k,+}(K) \cap \mathbf{H}_0^1(K)$. Moreover, integrating by parts, we have

$$\begin{aligned} (\nabla \cdot \mathbf{v}, p) &= (\mathbf{v}, \nabla p) = (\mathbf{w}, \nabla p) = (\nabla \cdot \mathbf{w}, p) = (q, p), \quad \forall p \in P_{k-1}(K)/\mathbb{R}, \\ (\nabla \cdot \mathbf{v}, 1) &= \langle \mathbf{v} \cdot \mathbf{n}, 1 \rangle = 0 = (q, 1). \end{aligned}$$

This implies $\nabla \cdot \mathbf{v} - q = 0$ since $\nabla \cdot \mathbf{v} - q \in P_{k-1}(K)$.

We now prove $\|\mathbf{v}\|_{1,K} \leq C\|q\|_K$ by a scaling argument. Denote

$$n_{k-1} = \dim P_{k-1}(K),$$

we can express $\mathbf{v}$ as

$$\mathbf{v} = \sum_{i=2}^{n_{k-1}} (\mathbf{w}, \nabla p_i) \mathbf{N}_i,$$

where $\{p_i\}_{i=2}^{n_{k-1}}$ is a set of basis functions of $P_{k-1}(K)/\mathbb{R}$ and $\mathbf{N}_i$ is the dual basis of p_i with respect to the DOFs $(\mathbf{w}, \nabla p_i)$, i.e., $(\mathbf{N}_i, \nabla p_j) = \delta_{ij}$. Setting $\hat{\mathbf{v}} = \det(B_K) B_K^{-1} \mathbf{v} \circ F_K$ and $\hat{p} = p \circ F_K$ with B_K and F_K defined in (2.1), we obtain

$$\begin{aligned} \|\mathbf{v}\|_{1,K}^2 &\leq Ch_K^{-3} \|\hat{\mathbf{v}}\|_{1,\hat{K}}^2 \leq Ch_K^{-3} \sup_{2 \leq i \leq n_{k-1}} |(\hat{\mathbf{w}}, \hat{\nabla} \hat{p}_i)|^2 \\ &= Ch_K^{-3} \sup_{2 \leq i \leq n_{k-1}} |(\hat{\nabla} \cdot \hat{\mathbf{w}}, \hat{p}_i)|^2 \leq Ch_K^{-3} \|\hat{\nabla} \cdot \hat{\mathbf{w}}\|_{\hat{K}}^2 \leq C \|\nabla \cdot \mathbf{w}\|_K^2 = C\|q\|_K^2. \quad \square \end{aligned}$$

LEMMA 5.2. For any $q \in W_h^{k-1}$, there exists $\mathbf{v} \in \Sigma_h^{k,+}$ such that $\nabla \cdot \mathbf{v} = q$ and $\|\mathbf{v}\|_1 \leq C\|q\|$.

Proof. Given $q \in W_h^{k-1} \subset L^2(\Omega)$, according to [6, Theorem 2], there exists $\mathbf{w} \in \mathbf{H}^1(\Omega)$ satisfying $\nabla \cdot \mathbf{w} = q$ and $\|\mathbf{w}\|_1 \leq C\|q\|$. Let $\mathbf{I}_h \mathbf{w} \in \Sigma_h^k \subset \Sigma_h^{k,+}$ denote the Scott-Zhang interpolation of $\mathbf{w}$ (see [32, (2.13)] for its definition), where Σ_h^k is the vector-valued Lagrange finite element space of degree k . We also let $\mathbf{v}_1 \in \Sigma_h^{k,+}$ be the unique function that satisfies

$$\int_{f_i} \mathbf{v}_1 \cdot \mathbf{n}_i dA = \int_{f_i} (\mathbf{w} - \mathbf{I}_h \mathbf{w}) \cdot \mathbf{n}_i dA, \quad \forall f_i \in \mathcal{F}_h,$$

with other DOFs in (4.6)-(4.9) vanishing on $\mathbf{v}_1$. Then we have, for any $K \in \mathcal{T}_h$,

$$(\nabla \cdot \mathbf{v}_1 + \nabla \cdot \mathbf{I}_h \mathbf{w}, 1)_K = \langle \mathbf{v}_1 \cdot \mathbf{n} + \mathbf{I}_h \mathbf{w} \cdot \mathbf{n}, 1 \rangle_{\partial K} = \langle \mathbf{w} \cdot \mathbf{n}, 1 \rangle_{\partial K} = (\nabla \cdot \mathbf{w}, 1)_K = (q, 1)_K,$$

which means $(q - \nabla \cdot \mathbf{v}_1 - \nabla \cdot \mathbf{I}_h \mathbf{w})|_K \in W_h^{k-1}(K) \cap \dot{L}^2(K)$. By Lemma 5.1, there exists $\mathbf{v}_{2,K} \in \Sigma_h^{k,+}(K) \cap \mathbf{H}_0^1(K)$ such that

$$\nabla \cdot \mathbf{v}_{2,K} = (q - \nabla \cdot \mathbf{v}_1 - \nabla \cdot \mathbf{I}_h \mathbf{w})|_K, \quad \forall K \in \mathcal{T}_h$$

and

$$\|\mathbf{v}_{2,K}\|_{1,K} \leq C(\|\mathbf{v}_1\|_{1,K} + \|\mathbf{I}_h \mathbf{w}\|_{1,K} + \|q\|_K).$$

Define $\mathbf{v}_2 \in \mathbf{H}_0^1(\Omega) \cap \Sigma_h^{k,+}$ by $\mathbf{v}_2|_K = \mathbf{v}_{2,K}$. Setting $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{I}_h \mathbf{w}$, we have

$$\nabla \cdot \mathbf{v} = \nabla \cdot (\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{I}_h \mathbf{w}) = q \text{ and } \|\mathbf{v}\|_1 \leq C(\|\mathbf{v}_1\|_1 + \|\mathbf{I}_h \mathbf{w}\|_1 + \|q\|).$$

We apply the same scaling argument as used in Lemma 5.1 and the approximation property of the Scott-Zhang interpolation $\mathbf{I}_h \mathbf{w}$ [32, (4.1)] to obtain

$$\begin{aligned} \|\mathbf{v}_1\|_{1,K}^2 &\leq Ch_K^{-3} \|\hat{\mathbf{v}}_1\|_{1,\hat{K}}^2 \leq Ch_K^{-3} |\langle (\mathbf{w} - \mathbf{I}_h \mathbf{w}) \cdot \mathbf{n}_i, 1 \rangle_{\partial K}|^2 \leq Ch_K^{-1} \|\mathbf{w} - \mathbf{I}_h \mathbf{w}\|_{\partial K}^2 \\ &\leq C(h_K^{-2} \|\mathbf{w} - \mathbf{I}_h \mathbf{w}\|_K^2 + \|\mathbf{w} - \mathbf{I}_h \mathbf{w}\|_{1,K}^2) \leq C\|\mathbf{w}\|_{1,\omega(K)}^2 \end{aligned}$$

with $\omega(K) = \text{Int} \{ \bar{K}_i | \bar{K}_i \cap \bar{K} \neq \emptyset, K_i \in \mathcal{T}_h \}$. Summing over $K \in \mathcal{T}_h$, we obtain

$$\|\mathbf{v}_1\|_1 \leq C\|\mathbf{w}\|_1,$$

which together with $\|\mathbf{I}_h \mathbf{w}\|_1 \leq C\|\mathbf{w}\|_1$ [32, (4.5)] and $\|\mathbf{w}\|_1 \leq C\|q\|$ leads to

$$\|\mathbf{v}\|_1 \leq C\|q\|. \quad \square$$

COROLLARY 5.3. The inf-sup condition for the Stokes problem holds, i.e., there exists a positive constant $\alpha > 0$ not depending on h , such that

$$\sup_{0 \neq \mathbf{v} \in \Sigma_h^{k,+}} \frac{(\nabla \cdot \mathbf{v}, q)}{\|\mathbf{v}\|_1} \geq \alpha\|q\|, \quad \forall q \in W_h^{k-1}.$$

Corollary 5.3 implies that $\Sigma_h^{k,+} - W_h^{k-1}$ leads to convergent algorithms for solving the Stokes problem with a precise divergence-free condition.

THEOREM 5.4. The complex (4.1) is exact on contractible domains.

Proof. The exactness at Σ_h^r and $V_h^{r-1,k+1}$ follows from the exactness of the standard finite element differential forms (e.g., [3]). The exactness at W_h^{k-1} , i.e., the surjectivity of $\nabla \cdot : \Sigma_h^{k,+} \rightarrow W_h^{k-1}$ is verified in Lemma 5.2.

Finally, the exactness at $\Sigma_h^{k,+}$ follows from a dimension count. Let $\mathcal{V}$, $\mathcal{E}$, $\mathcal{F}$, and $\mathcal{K}$ denote the number of vertices, edges, faces, and 3D cells, respectively. Then we have

$$\dim \Sigma_h^r = \mathcal{V} + (r-1)\mathcal{E} + \frac{1}{2}(r-2)(r-1)\mathcal{F} + \frac{1}{6}(r-3)(r-2)(r-1)\mathcal{K},$$

$$\dim W_h^{k-1} = \frac{k(k+1)(k+2)}{6}\mathcal{K}.$$

From the DOFs (4.2) -(4.5),

$$\begin{aligned} \dim V_h^{r-1,k+1} - \dim \Sigma_h^{k,+} &= r\mathcal{E} + \frac{1}{2}(r-2)(r-1)\mathcal{F} \\ &\quad - \mathcal{F} + \frac{1}{6}[(r-3)(r-2)(r-1) - k(k+1)(k+2) + 6]\mathcal{K}. \end{aligned}$$

From the above dimension count, we have

$$-1 + \dim \Sigma_h^r - \dim V_h^{r-1,k+1} + \dim \Sigma_h^{k,+} - \dim W_h^{k-1} = 0,$$

where we have used Euler's formula $\mathcal{V} - \mathcal{E} + \mathcal{F} - \mathcal{K} = 1$. This completes the proof. $\square$

Remark 5.5. The finite element spaces with vanishing boundary conditions also form an exact complex on contractible domains:

$$(5.1) \quad 0 \xrightarrow{\subset} \mathring{\Sigma}_h^r \xrightarrow{\nabla} \mathring{V}_h^{r-1,k+1} \xrightarrow{\nabla \times} \mathring{\Sigma}_h^{k,+} \xrightarrow{\nabla \cdot} \mathring{W}_h^{k-1} \longrightarrow 0,$$

where $\mathring{\Sigma}_h^r = \Sigma_h^r \cap H_0^1(\Omega)$, $\mathring{V}_h^{r-1,k+1} = V_h^{r-1,k+1} \cap H_0(\text{grad curl}; \Omega)$, $\mathring{\Sigma}_h^{k,+} = \Sigma_h^{k,+} \cap \mathbf{H}_0^1(\Omega)$, $\mathring{W}_h^{k-1} = W_h^{k-1} \cap \mathbf{L}^2(\Omega)$ with $H_0(\text{grad curl}; \Omega) = \{\mathbf{u} \in H(\text{grad curl}; \Omega) : \mathbf{u} \times \mathbf{n} = 0 \text{ and } \nabla \times \mathbf{u} = 0 \text{ on } \partial\Omega\}$.

For $\delta > 0$, denote $\Sigma = H^{3/2+\delta}(\Omega)$ and $V = \{\mathbf{u} \in \mathbf{H}^{1/2+\delta}(\Omega) : \nabla \times \mathbf{u} \in H^{3/2+\delta}(\Omega)\}$. We use $\pi_h : \Sigma \rightarrow \Sigma_h^r$, $\tilde{\pi}_h : \Sigma \rightarrow \Sigma_h^{k,+}$, $\mathbf{r}_h : V \rightarrow V_h^{r-1,k+1}$, and $i_h : L^2(\Omega) \rightarrow W_h^{k-1}$ to denote the interpolation operators defined by the DOFs for Σ_h^r , $\Sigma_h^{k,+}$, $V_h^{r-1,k+1}$, and W_h^{k-1} , respectively.

We summarize the interpolations defined above in the following diagram:

$$(5.2) \quad \begin{array}{ccccccc} \mathbb{R} & \xrightarrow{\subset} & H^1(\Omega) & \xrightarrow{\nabla} & H(\text{grad curl}; \Omega) & \xrightarrow{\nabla \times} & \mathbf{H}^1(\Omega) \xrightarrow{\nabla \cdot} L^2(\Omega) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ \mathbb{R} & \xrightarrow{\subset} & \Sigma & \xrightarrow{\nabla} & V & \xrightarrow{\nabla \times} & \Sigma \xrightarrow{\nabla \cdot} L^2(\Omega) \longrightarrow 0 \\ & & \downarrow \pi_h & & \downarrow \mathbf{r}_h & & \downarrow \tilde{\pi}_h \\ \mathbb{R} & \xrightarrow{\subset} & \Sigma_h^r & \xrightarrow{\nabla} & V_h^{r-1,k+1} & \xrightarrow{\nabla \times} & \Sigma_h^{k,+} \xrightarrow{\nabla \cdot} W_h^{k-1} \longrightarrow 0 \\ & & & & & & \downarrow i_h \end{array}$$

By a similar argument as in [28, Theorem 5.49], the interpolations in (5.2) commute with the differential operators.

LEMMA 5.6. *The last two rows of the complex (5.2) are a commuting diagram, i.e.,*

$$(5.3) \quad \nabla \pi_h u = \mathbf{r}_h \nabla u \text{ for all } u \in \Sigma,$$

$$(5.4) \quad \nabla \times \mathbf{r}_h \mathbf{u} = \tilde{\pi}_h \nabla \times \mathbf{u} \text{ for all } \mathbf{u} \in V,$$

$$(5.5) \quad \nabla \cdot \tilde{\pi}_h \mathbf{u} = i_h \nabla \cdot \mathbf{u} \text{ for all } \mathbf{u} \in \Sigma.$$

We adopt the following Piola mapping to transform the finite element function $\mathbf{u}$ on a general element K to a function $\hat{\mathbf{u}}$ on the reference element $\hat{K}$:

$$(5.6) \quad \mathbf{u} \circ F_K = B_K^{-T} \hat{\mathbf{u}}.$$

By a simple calculation, we have

$$(5.7) \quad (\nabla \times \mathbf{u}) \circ F_K = \frac{B_K}{\det(B_K)} \hat{\nabla} \times \hat{\mathbf{u}},$$

$$(5.8) \quad \mathbf{n}_i \circ F_K = \frac{B_K^{-T} \hat{\mathbf{n}}_i}{|B_K^{-T} \hat{\mathbf{n}}_i|},$$

$$(5.9) \quad \boldsymbol{\tau}_i \circ F_K = \frac{B_K \hat{\boldsymbol{\tau}}_i}{|B_K \hat{\boldsymbol{\tau}}_i|}.$$

The following lemma relates the interpolation on K to that on $\hat{K}$.

LEMMA 5.7. *For $\mathbf{u} \in W$, we have $\widehat{\mathbf{r}_K \mathbf{u}} = \mathbf{r}_{\hat{K}} \hat{\mathbf{u}}$ with the transformation (5.6).*

Proof. Following [7, Proposition 3.4.7], we only need to show the DOFs for defining $\widehat{\mathbf{r}_K \mathbf{u}}$ are linear combinations of those for defining $\mathbf{r}_{\hat{K}} \hat{\mathbf{u}}$.

By the transformations (5.6), (5.7), (5.8), and (5.9), we have that all the DOFs in (4.2)–(4.5) are linear combinations of those for $\hat{\mathbf{u}}$ on $\hat{K}$. For instance,

$$\begin{aligned} \int_{f_i} \nabla \times \mathbf{u} \cdot \boldsymbol{\tau}_i^1 dA &= \frac{1}{\det(B_K)} \int_{\hat{f}_i} \hat{\nabla} \times \hat{\mathbf{u}} \cdot B_K^T \boldsymbol{\tau}_i^1 d\hat{A} \\ &= \frac{1}{\det(B_K)} \int_{\hat{f}_i} \hat{\nabla} \times \hat{\mathbf{u}} \cdot ((B_K^T \boldsymbol{\tau}_i^1 \cdot \hat{\boldsymbol{\tau}}_i^1) \hat{\boldsymbol{\tau}}_i^1 + (B_K^T \boldsymbol{\tau}_i^1 \cdot \hat{\boldsymbol{\tau}}_i^2) \hat{\boldsymbol{\tau}}_i^2 + (B_K^T \boldsymbol{\tau}_i^1 \cdot \hat{\mathbf{n}}_i) \hat{\mathbf{n}}_i) d\hat{A} \\ &= \frac{1}{\det(B_K)} \int_{\hat{f}_i} \hat{\nabla} \times \hat{\mathbf{u}} \cdot ((B_K^T \boldsymbol{\tau}_i^1 \cdot \hat{\boldsymbol{\tau}}_i^1) \hat{\boldsymbol{\tau}}_i^1 + (B_K^T \boldsymbol{\tau}_i^1 \cdot \hat{\boldsymbol{\tau}}_i^2) \hat{\boldsymbol{\tau}}_i^2) d\hat{A} \\ &\quad + \frac{(B_K^T \boldsymbol{\tau}_i^1 \cdot \hat{\mathbf{n}}_i)}{\det(B_K)} \int_{\partial \hat{f}_i} \hat{\mathbf{u}} \cdot \hat{\boldsymbol{\tau}}_{\partial \hat{f}_i} d\hat{s}. \end{aligned}$$

This completes the proof. $\square$

Next, we establish the approximation property of the interpolation operators.

THEOREM 5.8. *Assume that $\mathbf{u} \in \mathbf{H}^{s+(r-k-1)}(\Omega)$ and $\nabla \times \mathbf{u} \in \mathbf{H}^s(\Omega)$, $s \geq 3/2 + \delta$ with $\delta > 0$, and $r = k, k+1$, or $k+2$. Then we have the following error estimates for the interpolation $\mathbf{r}_h$,*

$$(5.10) \quad \|\mathbf{u} - \mathbf{r}_h \mathbf{u}\| \leq Ch^{\min\{s+(r-k-1), r\}} (\|\mathbf{u}\|_{s+(r-k-1)} + \|\nabla \times \mathbf{u}\|_s),$$

$$(5.11) \quad \|\nabla \times (\mathbf{u} - \mathbf{r}_h \mathbf{u})\| \leq Ch^{\min\{s, k+1\}} \|\nabla \times \mathbf{u}\|_s,$$

$$(5.12) \quad |\nabla \times (\mathbf{u} - \mathbf{r}_h \mathbf{u})|_1 \leq Ch^{\min\{s-1, k\}} \|\nabla \times \mathbf{u}\|_s.$$

Proof. By the identity $\widehat{\mathbf{r}_K \mathbf{u}} = \mathbf{r}_{\hat{K}} \hat{\mathbf{u}}$ and the inclusion $\mathbf{P}_{r-1}(K) \subseteq V_h^{r-1, k+1}(K)$ (Lemma 5.7 and Lemma 3.10), the proof is standard, c.f., [28, Theorem 5.41]. Here we have used Lemma 5.6 to prove (5.11) and (5.12). $\square$

6. Applications to $-\text{curl } \Delta \text{curl}$ problems. In this section, we use the three grad curl-conforming finite element families to solve a problem with curl Δ curl operator: for $\mathbf{f} \in H(\text{div}^0; \Omega)$, find $\mathbf{u}$, such that

$$\begin{aligned} -\nabla \times \Delta(\nabla \times \mathbf{u}) + \mathbf{u} &= \mathbf{f} \quad \text{in } \Omega, \\ \nabla \cdot \mathbf{u} &= 0 \quad \text{in } \Omega, \\ \mathbf{u} \times \mathbf{n} &= 0 \quad \text{on } \partial\Omega, \\ \nabla \times \mathbf{u} &= 0 \quad \text{on } \partial\Omega. \end{aligned} \quad (6.1)$$

Here $\mathbf{n}$ is the unit outward normal vector on $\partial\Omega$, and $H(\text{div}^0; \Omega)$ is the space of $\mathbf{L}^2(\Omega)$ functions with vanishing divergence, i.e.,

$$H(\text{div}^0; \Omega) := \{\mathbf{u} \in \mathbf{L}^2(\Omega) : \nabla \cdot \mathbf{u} = 0\}.$$

Taking divergence on both sides of the first equation of (6.1), we see that $\nabla \cdot \mathbf{u} = 0$ automatically holds with $\mathbf{f} \in H(\text{div}^0; \Omega)$.

The variational formulation reads: find $\mathbf{u} \in H_0(\text{grad curl}; \Omega)$, such that

$$a(\mathbf{u}, \mathbf{v}) = (\mathbf{f}, \mathbf{v}) \quad \forall \mathbf{v} \in H_0(\text{grad curl}; \Omega), \quad (6.2)$$

with $a(\mathbf{u}, \mathbf{v}) := (\nabla \nabla \times \mathbf{u}, \nabla \nabla \times \mathbf{v}) + (\mathbf{u}, \mathbf{v})$. The weak form (6.2) can be regarded as a model problem for the high order problems in MHD, e.g., [9, (1)] and continuum mechanics with size effects, e.g., [27, (3.27)], [30, (35)].

Remark 6.1. The grad curl operator appears in the following complex (referred to as the grad curl complex)

$$0 \longrightarrow H^q \xrightarrow{\text{grad}} H^{q-1} \otimes \mathbb{V} \xrightarrow{\text{grad curl}} H^{q-3} \otimes \mathbb{T}^{\text{curl}} \xrightarrow{\text{div}} H^{q-4} \otimes \mathbb{M} \xrightarrow{\text{div}} H^{q-5} \otimes \mathbb{V} \longrightarrow 0, \quad (6.3)$$

which can be derived from de Rham complexes [6, (46)]. Thus (6.2) is closely related to one of the Hodge-Laplacian problems associated to the grad curl complex.

Remark 6.2. With the given boundary conditions and the identity for vector Laplacian $-\Delta \mathbf{u} = -\nabla \nabla \cdot \mathbf{u} + \nabla \times \nabla \times \mathbf{u}$, the above weak form is equivalent to the quad-curl problem, i.e., $(\nabla \nabla \times \mathbf{u}, \nabla \nabla \times \mathbf{v}) = (\nabla \times \nabla \times \mathbf{u}, \nabla \times \nabla \times \mathbf{v})$.

The grad curl-conforming finite element method for (6.2) reads: seek $\mathbf{u}_h \in \hat{V}_h^{r-1, k+1}$, such that

$$a(\mathbf{u}_h, \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h) \quad \forall \mathbf{v}_h \in \hat{V}_h^{r-1, k+1}. \quad (6.4)$$

THEOREM 6.3. *We assume that Ω is a simply-connected Lipschitz polyhedral domain with a connected boundary. There exists a constant $\alpha > 1/2$ such that the solution $\mathbf{u}$ of (6.1) satisfies*

$$\mathbf{u} \in \mathbf{H}^\alpha(\Omega), \quad \nabla \times \mathbf{u} \in \mathbf{H}^{1+\alpha}(\Omega),$$

and it holds

$$\|\mathbf{u}\|_\alpha + \|\nabla \times \mathbf{u}\|_{1+\alpha} \leq C \|\mathbf{f}\|.$$

Proof. The claim that $\mathbf{u} \in \mathbf{H}^\alpha(\Omega)$ follows from the embedding $H_0(\text{curl}; \Omega) \cap H(\text{div}; \Omega) \hookrightarrow \mathbf{H}^\alpha(\Omega)$ with $\alpha > 1/2$ [2], and it holds

$$\|\mathbf{u}\|_\alpha \leq C (\|\mathbf{u}\| + \|\nabla \cdot \mathbf{u}\| + \|\nabla \times \mathbf{u}\|) = C (\|\mathbf{u}\| + \|\nabla \times \mathbf{u}\|).$$

Furthermore, by Poincaré inequality, we have

$$\|\mathbf{u}\|_\alpha \leq C(\|\mathbf{u}\| + \|\nabla \nabla \times \mathbf{u}\|) \leq C\|\mathbf{f}\|.$$

If $-\Delta(\nabla \times \mathbf{u})$ belongs to $\mathbf{L}^2(\Omega)$, then from the boundary condition $\nabla \times \mathbf{u} = 0$ and the regularity of the Laplace problem [28, Theorem 3.18], we can obtain $\nabla \times \mathbf{u} \in \mathbf{H}^{1+\alpha}(\Omega)$ with $\alpha > 1/2$, and

$$\|\nabla \times \mathbf{u}\|_{1+\alpha} \leq C\|\Delta(\nabla \times \mathbf{u})\|.$$

It suffices to show that $(\nabla \times)^3 \mathbf{u} \in \mathbf{L}^2(\Omega)$ and $\|(\nabla \times)^3 \mathbf{u}\| \leq C\|\mathbf{f}\|$ since $-\Delta(\nabla \times \mathbf{u}) = -\nabla \nabla \cdot \nabla \times \mathbf{u} + (\nabla \times)^3 \mathbf{u} = (\nabla \times)^3 \mathbf{u}$. If we can prove

$$(6.5) \quad g(\mathbf{v}) := ((\nabla \times)^2 \mathbf{u}, \nabla \times \mathbf{v}) \leq C_0 \|\mathbf{v}\|, \text{ for all } \mathbf{v} \in H_0(\text{curl}; \Omega),$$

where $H_0(\text{curl}; \Omega) = \{\mathbf{u} \in H(\text{curl}; \Omega) : \mathbf{u} \times \mathbf{n} = 0 \text{ on } \partial\Omega\}$, then, by Hahn Banach theorem, there is a unique extension of the map $g(\mathbf{v})$ to a bounded linear functional from all of $\mathbf{L}^2(\Omega)$ to $\mathbb{R}$ with the bound C_0 . Moreover, by Riesz representation theorem, there exists a unique element $\boldsymbol{\phi} \in \mathbf{L}^2(\Omega)$ such that

$$g(\mathbf{v}) = ((\nabla \times)^2 \mathbf{u}, \nabla \times \mathbf{v}) = (\boldsymbol{\phi}, \mathbf{v}), \text{ for } \mathbf{v} \in H_0(\text{curl}; \Omega).$$

From the definition of the adjoint of $\nabla \times$, we have $(\nabla \times)^3 \mathbf{u} = \boldsymbol{\phi} \in \mathbf{L}^2(\Omega)$ and $\|(\nabla \times)^3 \mathbf{u}\| = \|g\|_{\mathcal{L}(\mathbf{L}^2(\Omega), \mathbb{R})} \leq C_0$.

To prove (6.5), we first seek $q \in H_0^1(\Omega)$ such that

$$-\Delta q = \nabla \cdot \mathbf{v} \in H^{-1}(\Omega).$$

Then it holds $\|\nabla q\| \leq \|\mathbf{v}\|$. Applying [17, Theorem 3.6] to $\mathbf{v} - \nabla q$, there exists a divergence-free vector potential $\mathbf{w} \in H_0(\text{curl}; \Omega)$ satisfying

$$(6.6) \quad \mathbf{v} - \nabla q = \nabla \times \mathbf{w}.$$

Since $\mathbf{v} - \nabla q \in H_0(\text{curl}; \Omega)$, then $\mathbf{w} \in H_0(\text{curl curl}; \Omega)$. From (6.6) and the Friedrichs inequality [28, Corollary 3.51], we have

$$\begin{aligned} ((\nabla \times)^2 \mathbf{u}, \nabla \times \mathbf{v}) &= ((\nabla \times)^2 \mathbf{u}, (\nabla \times)^2 \mathbf{w}) \\ &= (\mathbf{f} - \mathbf{u}, \mathbf{w}) \leq \|\mathbf{f} - \mathbf{u}\| \|\mathbf{w}\| \leq C\|\mathbf{f} - \mathbf{u}\| \|\nabla \times \mathbf{w}\| \\ &\leq C\|\mathbf{f} - \mathbf{u}\| (\|\mathbf{v}\| + \|\nabla q\|) \leq C\|\mathbf{f} - \mathbf{u}\| \|\mathbf{v}\| \leq C\|\mathbf{f}\| \|\mathbf{v}\|, \end{aligned}$$

which leads to (6.5) with $C_0 = C\|\mathbf{f}\|$. $\square$

To estimate the error in the sense of $H(\text{curl})$ -norm, we introduce the following auxiliary problem. Find $\mathbf{w}$ such that

$$\begin{aligned} -\nabla \times \Delta(\nabla \times \mathbf{w}) + \mathbf{w} &= (\nabla \times)^2(\mathbf{u} - \mathbf{u}_h) \text{ in } \Omega, \\ \nabla \cdot \mathbf{w} &= 0 \text{ in } \Omega, \\ \mathbf{w} \times \mathbf{n} &= 0 \text{ on } \partial\Omega, \\ \nabla \times \mathbf{w} &= 0 \text{ on } \partial\Omega. \end{aligned}$$

Due to the special form of the right-hand side in the auxiliary problem, we can have a better regularity estimate by a suitable modification to the proof of Theorem 6.3. This result will play an important role in the dual argument in the approximation analysis below.

THEOREM 6.4. *We assume that Ω is a simply-connected Lipschitz polyhedral domain with a connected boundary. The solution $\mathbf{w}$ of (6.7) satisfies*

$$\|\mathbf{w}\|_\alpha + \|\nabla \times \mathbf{w}\|_{1+\alpha} \leq C \|\nabla \times (\mathbf{u} - \mathbf{u}_h)\|.$$

REMARK 6.5. Furthermore, if Ω is convex, then the constant α in Theorem 6.3 and Theorem 6.4 can be 1.

THEOREM 6.6. *For $r = k$, $r = k + 1$, or $r = k + 2$, if $\mathbf{u} \in \mathbf{H}^{s+(r-k-1)}(\Omega)$ and $\nabla \times \mathbf{u} \in H^s(\Omega)$, $s \geq 1 + \alpha$, we have the following error estimates for the numerical solution $\mathbf{u}_h$:*

$$(6.8) \quad \|\mathbf{u} - \mathbf{u}_h\|_{H(\text{grad curl}; \Omega)} \leq Ch^{\min\{s-1, k\}} (\|\mathbf{u}\|_{s-1} + \|\nabla \times \mathbf{u}\|_s),$$

$$(6.9) \quad \|\nabla \times (\mathbf{u} - \mathbf{u}_h)\| \leq Ch^{\min\{s, k+1, 2\alpha\}} (\|\mathbf{u}\|_{s-1} + \|\nabla \times \mathbf{u}\|_s),$$

$$(6.10) \quad \|\mathbf{u} - \mathbf{u}_h\| \leq Ch^{\min\{s, k+1, 2\alpha\}} (\|\mathbf{u}\|_s + \|\nabla \times \mathbf{u}\|_s) \text{ when } r = k + 1, k + 2.$$

PROOF. The estimates (6.8) and (6.9) follow immediately from Céa's lemma, the dual argument, and Theorem 5.8. Proceeding as in the proof of [33, Theorem 6], we can show that (6.10) holds. $\square$

REMARK 6.7. The estimate for $\|\mathbf{u} - \mathbf{u}_h\|$ is not optimal for the family $r = k + 2$.

We now carry out several numerical tests to validate our new elements. We consider the problem (6.1) on a unit cube $\Omega = (0, 1) \times (0, 1) \times (0, 1)$ with an exact solution

$$\mathbf{u} = \begin{pmatrix} \sin(\pi x_1)^3 \sin(\pi x_2)^2 \sin(\pi x_3)^2 \cos(\pi x_2) \cos(\pi x_3) \\ \sin(\pi x_2)^3 \sin(\pi x_3)^2 \sin(\pi x_1)^2 \cos(\pi x_3) \cos(\pi x_1) \\ -2 \sin(\pi x_3)^3 \sin(\pi x_1)^2 \sin(\pi x_2)^2 \cos(\pi x_1) \cos(\pi x_2) \end{pmatrix}.$$

Then, by a simple calculation, we can obtain the source term $\mathbf{f}$. We denote the finite element solution as $\mathbf{u}_h$. To measure the error between the exact solution and the finite element solution, we denote

$$\mathbf{e}_h = \mathbf{u} - \mathbf{u}_h.$$

For the mesh, we partition the unit cube into N^3 small cubes and then partition each small cube into 6 congruent tetrahedra.

We first use the lowest-order ($k = 1$) elements in the families $r = k$ and $r = k + 1$ to solve the problem (6.1) on the uniform tetrahedral mesh. Tables 1 and 2 illustrate errors and convergence rates for the two families. We observe that the numerical solution converges to the exact one at rate h for the case $r = k = 1$, and at rate h^2 for $r = k + 1 = 2$ in the sense of the L^2 -norm. In addition, the two families have the same convergence rate h^2 in the $H(\text{curl})$ -norm and h in the $H(\text{grad curl})$ -norm, respectively. All results agree with Theorem 6.6.

We now test the third-order element ($k = 3$). Tables 3 demonstrates numerical data for the family $r = k$, again they are consistent with Theorem 6.6.

Our code for the basis functions of the grad curl-conforming elements when $k = 1, 3$ is available at

<https://github.com/QianZhangMath/3D-gradcurl-conforming-FE-18DOF>.

7. Concluding remarks. In this paper we constructed 3D finite element Stokes complexes on tetrahedral meshes. Generalizing the modified Bernardi-Raugel bubbles in [20] to an arbitrary order and utilizing the Poincaré operators for the de Rham

Table 1: Numerical results by the grad curl-conforming element with $r = k$ and $k = 1$

N	$\ e_h\ $	rates	$\ \nabla \times e_h\ $	rates	$\ \nabla \nabla \times e_h\ $	rates
45	8.642113e-03		7.620755e-02		2.862735e+00	
50	7.401715e-03	1.4705	6.317760e-02	1.7797	2.601358e+00	0.9087
55	6.443660e-03	1.4544	5.314638e-02	1.8141	2.382186e+00	0.9235
60	5.687783e-03	1.4340	4.527838e-02	1.8414	2.196043e+00	0.9351

Table 2: Numerical results by the grad curl-conforming element with $r = k + 1$ and $k = 1$

N	$\ e_h\ $	rates	$\ \nabla \times e_h\ $	rates	$\ \nabla \nabla \times e_h\ $	rates
30	1.334051e-02		1.453615e-01		4.055510e+00	
35	1.033747e-02	1.6544	1.135563e-01	1.6018	3.567777e+00	0.8312
40	8.212073e-03	1.7237	9.077071e-02	1.6772	3.178759e+00	0.8646
45	6.662599e-03	1.7753	7.399883e-02	1.7344	2.862553e+00	0.8896

Table 3: Numerical results by the grad curl-conforming element with $r = k$ and $k = 3$

N	$\ e_h\ $	rates	$\ \nabla \times e_h\ $	rates	$\ \nabla \nabla \times e_h\ $	rates
10	3.047288e-04		2.974941e-03		2.909078e-01	
12	1.719285e-04	3.1392	1.403569e-03	4.1202	1.779005e-01	2.6973
14	1.070064e-04	3.0761	7.353798e-04	4.1932	1.162168e-01	2.7620
16	7.125639e-05	3.0450	4.174453e-04	4.2405	7.986321e-02	2.8094

complexes, we obtain simple finite element spaces with canonical DOFs. The newly obtained finite elements allow further applications in mass-conservative approximation of fluid mechanics and high order models in continuum mechanics and electromagnetism.

Since the entire discrete de Rham complexes are obtained, one may further investigate robust solvers in the framework of subspace correction [26, 31]. These results also show promising directions for elasticity as in [8, 12, 18].

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