DPHIL DISSERTATION

Aggregation-diffusion equations in biology with a gradient flow structure

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Preamble

This thesis is concerned with the analysis of non-linear partial differential equations arising naturally from biological models. These models include non-linearities, long-range interactions, or local effects which pose a challenge that require new PDE techniques. In this thesis, the main tools that we need are Otto calculus, Wasserstein gradient flows, viscosity solutions, C_0-semigroup theory, and finite volumes.

The models we are interested in show a dichotomy between diffusion and aggregation. Therefore, one of the main question is to understand the long-time dynamics and to check wether diffusion, aggregation, or a mix of both dominates the behaviour of the equation. Hence, this work contains various parabolic PDEs of aggregation-diffusion type for which we analyse different properties. For example existence, uniqueness, long-time behaviour, steady states, or minimisers of the associated free energy functional, among others.

Chapter 1 is an introduction, presenting the mathematical context, motivations and necessary tools for the chapters to follow. Chapter 2 to 4 each correspond to a manuscript. Chapter 5 presents new outcomes and perspectives.

List of works contained in this thesis

In Chapter 2 we study well-posedness and long-time behaviour of aggregation–diffusion equations of the form $\frac{\partial \rho}{\partial t} = \Delta \rho^m + \mathrm{div}(\rho(\nabla V + \nabla W * \rho))$ in the fast-diffusion range, $0 < m < 1$, and V and W regular enough. We develop a well-posedness theory, first in a ball and then in $\mathbb{R}^d$, and characterise the long-time asymptotics in the space $W^{-1,1}$ for radial initial data. In the radial setting and for the mass equation, viscosity solutions are used to prove partial mass concentration asymptotically as $t \to \infty$, i.e. the limit as $t \to \infty$ is of the form $\alpha\delta_0 + \widehat{\rho}\, dx$ with $\alpha \geq 0$ and $\widehat{\rho} \in L^1$ where δ_0 is the Dirac at $x = 0$. Finally, we give instances of $W \neq 0$ showing that partial mass concentration does happen in infinite time, i.e. $\alpha > 0$. The results of this chapter have been already published in *Journal of Differential Equations*; [93].
Key words: *Nonlinear parabolic equations, nonlinear diffusion, Dirac delta formation, blow-up in infinite time, viscosity solutions*.

In Chapter 3 we focus on a family of nonlinear continuity equations for the evolution of a non-negative density ρ with a continuous and compactly-supported nonlinear mobility $\mathrm{m}(\rho)$ not necessarily concave. The velocity field is the negative gradient of the variation of a free energy including internal and confinement energy terms. Problems with compactly supported mobility are often called saturation problems since the values of the density are constrained below a maximal value. Taking advantage of a family of approximating problems, we show the existence of C_0-semigroups of L^1 contractions. We study the ω-limit of the problem, its most relevant properties, and the appearance of free boundaries in the long-time behaviour. This problem has a formal gradient-flow structure, and we discuss the local/global minimisers of the corresponding free energy in the natural topology related to the set of initial data for the L^∞-constrained gradient flow of probability densities. Furthermore, we analyse a structure-preserving implicit finite-volume scheme and discuss its convergence and long-time behaviour. The results of this chapter are contained in a preprint that is currently under peer review; [94].
Key words: *Saturation, nonlinear parabolic equations, long-time behaviour, C_0-semigroup, free boundary, Euler-Lagrange condition, implicit finite-volume scheme*.

In Chapter 4 we give sharp conditions for global-in-time existence of gradient-flow solutions to a Cahn-Hilliard-type equation, with backwards second-order degenerate diffusion, in any dimension and for general initial data. Our equation is the 2-Wasserstein gradient flow of a free energy with two competing terms: the

Dirichlet energy and the power-law internal energy. Homogeneity of the functionals reveals critical regimes that we analyse. Sharp conditions for global in time solutions, constructed via the minimising movement scheme, also known as JKO scheme, are obtained. Furthermore, we study a system of two Cahn-Hilliard-type equations exhibiting an analogous gradient flow structure. The results of this chapter have been already published in the *Proceedings of the London Mathematical Society*; [88].
Key words: *Cahn-Hilliard, aggregation-diffusion, variational problems, Wasserstein gradient flows*.

In Chapter 5 we explore perspectives and new outcomes to continue the research presented in this thesis in the near future. In particular, we give a small summary on two on-going projects. In Section 5.1 we suggest a deterministic particle approximation to the fourth order aggregation-diffusion equation introduced in Chapter 4. The result of this project is an on-going work with Charles Elbar. In Section 5.2 we present a new approach to study the Keller–Segel system. We suggest a Li–Yau and an Aronson–Bénilan type estimate that entail new L^∞ estimates on the density depending on its initial mass, up to the critical mass. As a consequence, we also show global existence of smooth solutions under fewer regularity assumptions on the initial data compared to previous literature. The result of this project is an on-going work with Charles Elbar and Filippo Santambrogio.
Key words Section 5.1: *Cahn-Hilliard, deterministic particle approximation, Wasserstein gradient flows, non-local Cahn-Hilliard equations, aggregation-diffusion*.
Key words Section 5.2: *Li–Yau, Aronson–Bénilan, Lipschitz estimates, Keller–Segel equation, Liouville equation, Lane–Emden equation*.

How to read this thesis

Each chapter is written to be self-contained. In Chapter 1 we provide an overview of the problem, presenting Chapter 2 to 4. We comment the main results and novelties of this thesis. Furthermore, we introduce the techniques that we use during the rest of the thesis.

Chapter 2 and 3 are tackling similar problems since both study second order aggregation-diffusion problems. Chapter 4 focuses on a fourth-order PDE that models cell-cell adhesion. We present the existence theory for the single and two-species case, and we explore their gradient flow structure. Chapter 5 presents new outcomes and perspectives to continue this line of research. We introduce two on-going projects. In Section 5.1, building from the theory presented in Chapter 4, we introduce a deterministic particle approximation for the single species case. Finally, Section 5.2 introduce a new perspective towards the Keller–Segel equation via a Li–Yau and an Aronson–Bénilan type estimate.

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Statement of originality

I hereby declare that my dissertation entitled "Aggregation-diffusion equations in biology with a gradient flow structure" is not substantially the same as any that I have submitted, or, is being concurrently submitted for a degree or diploma or other qualification at the University of Oxford or any other University or similar institution. I further state that no substantial part of my dissertation has already been submitted, or, is being concurrently submitted for any such degree, diploma or other qualification at the University of Oxford or any other University or similar institution, except as declared in this text. This dissertation is the result of my own work and includes nothing which is the outcome of work done in collaboration except where specifically indicated in this text.

Chapter 1 motivates and the research questions studied in this thesis, gives an overview to the mathematical methods and techniques that are relevant for the following chapters, Chapter 2–4.

The original research problem that led to the results in Chapter 2 and Chapter 3 was suggested by my thesis supervisors Professor José Antonio Carrillo[1] and Doctor David Gómez-Castro[2]. They are original research work produced in collaboration with Professor José Antonio Carrillo and Doctor David Gómez-Castro. The topic on Chapter 4 was brought to me by Professor José Antonio Carrillo and Professor Antonio Esposito[3]. Chapter 4 is original research work produced in collaboration with Professor José Antonio Carrillo, Professor Antonio Esposito and Carles Falcó[1].

Finally, Chapter 5 discusses perspectives and new outcomes for the near future. We introduce two on-going projects and comment on the main expected results. Section 5.1 is on-going original research work that it is being produced in collaboration with Doctor Charles Elbar[4]. Section 5.2 is on-going original research work that it is being produced in collaboration with Doctor Charles Elbar[4] and Professor Filippo Santambrogio[4].

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Contents

1 Introduction

Contents

Haciendo y deshaciendo se va aprendiendo. – Spanish proverb

Over the last few decades the mathematical community has been interested in modelling different biological phenomena with systems of partial differential equations (PDE). For instance, there exist PDE models that explain the formation of cells by meiosis [68], embrio-genesis or angio-genesis [114, 240], the Balo disease [224], bio-convection [123], cell-cell adhesion [287, 262, 107, 186], and several other phenomena.

The growing interest on these biological models justifies the intensive mathematical study of this type of PDE systems. A particular example to which the community has devoted a huge effort over the last 25 years is the Keller-Segel system, which is by now well understood in its classical setting. The Keller-Segel system was derived by Keller, and Segel in [223], and by Patlak in [271], and it models a phenomenon known as *chemotaxis* that governs the behaviour of a family of amoeba, the *Dictyostelium discoideum*. This behaviour is summarised as a competition between aggregation and diffusion and it is capsulised in the system[1]

$$\begin{cases} \dfrac{\partial \rho}{\partial t} = \Delta\rho - \operatorname{div}(\rho\nabla c), \\ -\Delta c = \rho, \\ \rho(\cdot, t) = \rho_0 \geq 0. \end{cases}$$

The classical Keller-Segel equation in dimension 2 reads as follows,

$$\begin{cases} \dfrac{\partial \rho}{\partial t} = \operatorname{div}(\rho\nabla(\log(\rho) - \mathcal{K} * \rho)) & \text{in } (0,+\infty)\times\mathbb{R}^2, \\ \mathcal{K}(x) := -\dfrac{1}{2\pi}\log|x| & \text{in } \mathbb{R}^2, \end{cases} \tag{1.1}$$

where $\mathcal{K}$ is the Poisson kernel. It has a very rich mathematical structure, depending on the initial mass, we find in the literature: Global existence for subcritical mass [213, 50, 46]; blow-up (Dirac delta formation) at infinite time for critical mass [49, 151, 144]; or blow-up at finite time (*chemotactic collapse*) [203, 278, 126, 127].

Since the model developed by Keller, Segel and Patlak was introduced in the 1970s the community has proposed further more general aggregation-diffusion equations to explain various phenomena appearing in Biology [207, 17]. The diffusion term describes population pressure effects, whereby groups of cells naturally spread away from areas of high concentration. The aggregation term is used to describe an attractor towards high-density population areas, due to a physical or chemical pressure that produces an increasing gradient towards cells direct their movements.

[1] Here we present a simplification by Nanjundiah [263] of the original model.

Figure 1.1: Aggregation phenomenon observed in *Dictostelium discoideum* to create a more complex structure. Source: *Wikipedia* https://en.wikipedia.org/wiki/Dictyostelium_discoideum.

This thesis focuses on three particular problems from the family of aggregation-diffusion equations. The first problem that we discuss in this work concerns the competition between fast diffusion and aggregation. By fast diffusion we refer to $\Delta\rho^m$ for $0 < m < 1$, in contraposition to the case $m > 1$ which is often regarded as *slow diffusion*. Notice that when $0 < \rho < 1$, the diffusion is faster for the case $0 < m < 1$ since ρ^m is bigger. However for larger values of ρ the situation is opposed and the case $m > 1$ actually diffuses faster. The fast diffusion equation appears in plasma physics (the case $m = \frac{1}{2}$ is known as the Okuda-Dawson law [267, 36]). It is also used to model diffusion of impurities in silicon [229]. In Chapter 2, we study in more detail the competition between fast diffusion and different types of aggregation, focusing on the long-time dynamics.

The second project of this thesis (Chapter 3) contains a thoughtful study of the competition between diffusion and drift with a saturation effect that prevents "overcrowding". Aggregation-diffusion equations with saturation appear naturally in mathematical biology, for instance in order to explain chemotaxis models with prevention of overcrowding [112, 206, 107]. Moreover, it has further applications in mathematical physics. It describes the relaxation of gas of fermions [217, 218], phase segregation [291, 314], or thin liquid films [245, 252] among others.

Finally, on Chapter 4 we focus on fourth-order aggregation-diffusion equation with local effects. In particular, we study a model that was proposed for tissue growth and patterning due to cell-cell adhesion [186] (see also [187] and the references therein). Nevertheless, the family of PDEs that we consider also have applications on lubrication theory [208, 39, 201].

Beyond the biology and the few mathematical physics applications that we just mentioned, the family of aggregation-diffusion equations also have meaningful applications on the field of global optimisation and machine learning. Currently, many researchers are exploring these ideas in order to perform gradient-free optimisation [275, 100, 101], gradient-free sampling [231, 160, 110], neural network calibration [260, 124, 238], or Bayesian parameter estimation [51, 199, 91].

By now, the literature concerning aggregation-diffusion problems is extensive (see [17, 200] and the references therein). There exists several techniques developed by the community in order to understand this type of problems. In our case, we take advantage of the 2-Wasserstien gradient flow structure of the problems using Otto calculus, the notion of viscosity solutions, or semigroup theory, among other tools. In the the remaining of this chapter we will first make a brief introduction of Chapter 2 to 4 and afterwards we will make a short review of the techniques mentioned above.

1.1 Outline of this thesis

The mathematical results of this thesis are structured in three chapters. In the following, we provide a short detailed description in the Section 1.1.1 to 1.1.3 presenting the main results.

1.1.1 Singularity formation for fast-diffusions with aggregation, Chapter 2

In this chapter we study the nonlinear aggregation-diffusion equation

$$\frac{\partial \rho}{\partial t} = \Delta \rho^m + \operatorname{div}(\rho \nabla V) + \operatorname{div}(\rho \nabla W * \rho)$$

where $0 < m < 1$ (fast-diffusion range). In order to get a better overall picture we focus on the problem posed in a ball B_R. We take a more general kernel $K(x,y) = K(y,x)$ and consider

$$\begin{cases} \dfrac{\partial \rho}{\partial t} = \Delta \rho^m + \operatorname{div}(\rho \nabla V) + \operatorname{div}\left(\rho \nabla \displaystyle\int_{B_R} K(\cdot, y)\rho(y)\mathrm{d}y\right) & \text{in } (0,\infty) \times B_R, \\ \rho(0,x) = \rho_0(x) & \text{for } x \in B_R, \\ \left(\nabla \rho^m + \rho \nabla V + \rho \nabla \displaystyle\int_{B_R} K(\cdot, y)\rho(y)\mathrm{d}y\right) \cdot \nu(x) = 0 & \text{on } (0,\infty) \times \partial B_R \end{cases} \tag{1.2}$$

where we have included a no-flux boundary condition on the boundary of the ball. As a convenient assumption, we also require that V does not produce any flux across the boundary $\nabla V(x) \cdot \nu(x) = 0$, on ∂B_R. We are in particular interested in the interaction kernel $K(x,y) = \eta(x) W(x-y) \eta(y)$ where η is any function in $C_c^\infty(B_R)$. Here, K does not produce any flux across the boundary. In this chapter we will study the case in which $V \in W^{2,\infty}(B_R)$, $K \in W_c^{2,\infty}(B_R \times B_R)$ and $V, K \geq 0$ as a convenient assumption.

The main goal of this project is to understand the long-time dynamics of (1.2) for the case in the ball B_R and to extend it to $\mathbb{R}^d$. In this part of the thesis we are inspired on the result by Carrillo, Gómez-Castro and Vázquez [96] for the case $W = 0$. In this result, they only cover the case with the local interaction drift V and they are able to prove partial mass concentration at time infinity for certain choices of initial datum and drifts. The authors base their analysis on a combination of techniques coming from classical PDE analysis, gradient flows (see Section 1.2), and viscosity solutions (see Section 1.3). Building up on this previous result we want to extend their techniques in order to also cover (1.2).

Uniqueness and local in time existence follows from classical (but technical) PDE analysis theory. Problem (1.2) is a 2-Wasserstein gradient flow of the free energy functional

$$\mathcal{F}[\rho] = \frac{1}{m-1}\int_{B_R} \rho(x)^m \mathrm{d}x + \int_{B_R} V(x)\rho(x)\mathrm{d}x + \frac{1}{2}\iint_{B_R \times B_R} K(x,y)\rho(y)\rho(x)\mathrm{d}y\mathrm{d}x. \tag{1.3}$$

We are able to show that solutions to (1.2) are such that for every $0 \leq t_1 < t_2 < \infty$

$$0 \leq \int_{t_1}^{t_2}\int_{B_R} \rho \left|\nabla\left(\frac{m}{m-1}\rho^{m-1} + V + \int_{B_R} K(\cdot,y)\rho(y)\mathrm{d}y\right)\right|^2 \mathrm{d}x\mathrm{d}t = \mathcal{F}[\rho_{t_1}] - \mathcal{F}[\rho_{t_2}].$$

Since, the free energy is also bounded from below we recover equicontinuity in time to $W^{-1,1}(B_R)$. We can use it in order to construct a subsequence (in time) that converges to a time-independent measure $\widehat{\mu}$ in the space $W^{-1,1}(B_R)$. Nevertheless, the properties of the negative Sobolev space are insufficient to study any further properties on $\widehat{\mu}$. Thus, we realise that a more natural setting to characterise the time limit is the one of the mass equation and viscosity solutions. We assume the initial datum ρ_0 and the potentials V and K are radially symmetric. We consider the spatial variable $v = |x|^d|B_1|$ and mass function

$$M(t,v) = \int_{\widetilde{B_v}} \rho_t(x)\mathrm{d}x$$

where $\widetilde{B_v}$ is the ball centered at 0 and with volume $|\widetilde{B_v}| = v$. Here, we use the notation R_v for $R_v = |B_R|$. On this new variable, equation (1.2) reads like

$$\partial_t M = \kappa(v)^2 \partial_v \left(\partial_v M\right)^m + \kappa(v)^2 \partial_v M \partial_v \left(V + W * \partial_v M\right), \tag{1.4}$$

where $\kappa(v) = d\omega_d^{\frac{1}{d}} v^{\frac{d-1}{d}}$ (ω_d refers to the volume of the d-dimensional sphere). Now, we consider viscosity solutions (see Section 1.3) as the natural notion of solutions and we study (1.2) under this newer framework and the further symmetry assumptions specified above. In particular, we show that there exists $\widehat{M}$ such that $\frac{\partial \widehat{M}}{\partial v} = \widehat{\mu}$ in $\mathcal{M}([0, R_v])$ up to an isometry and that $\widehat{M}$ is also a time-independent viscosity solution of (1.4).

Therefore, we just need to show that $\widehat{M}$ is locally Lipschitz in $(0, R_v]$ in order to show that $\widehat{\mu} = \alpha\delta_0 + \widehat{\rho}\,\mathrm{d}x$ with $\widehat{\rho} \in L^1(B_R)$ (see Remark 2.8 for a rigorous justification of this statement). This reasoning leads to the main result of Chapter 2.

Main Result 1 (Chapter 2, Theorem 2.10). *Under suitable conditions on the initial datum ρ_0 and the potentials V and K, it follows that $\frac{\partial \widehat{M}}{\partial v}(R_v) > 0$ and for a.e. $v \in (0, R_v)$*

$$\frac{\partial \widehat{M}}{\partial v}(v) = \left(\left(\frac{\partial \widehat{M}}{\partial v}(R_v) \right)^{m-1} + \frac{1-m}{m} \left(V(x) + \int_{B_R} K(x,y)\mathrm{d}\widehat{\rho}(y) \right)\Bigg|_{v=|B_1||x|^d} \right)^{-\frac{1}{1-m}}.$$

In particular $\widehat{M} \in W^{2,\infty}_{loc}((0, R_v])$, i.e. $W^{2,\infty}$ for every compact set in $(0, R_v]$.

This result also characterises $\widehat{\rho}$, the absolutely continuous part of the asymptotic state $\widehat{\mu}$, as a solution to

$$\widehat{\rho}(x) = \left(h + \frac{1-m}{m} \left(V(x) + \int_{B_R} K(x,y)\mathrm{d}\widehat{\rho}(y) \right) \right)^{-\frac{1}{1-m}}$$

for some $h > 0$. The proof is based on a careful analysis using the notion of inf- and sup-convolutions for viscosity solutions and a result concerning the equivalence between viscosity solutions and weak solutions, in the spirit of [211, 254, 288].

Finally, under further assumptions we extend the theory to $\mathbb{R}^d$ and we construct a family of examples of potentials V and W where we can observe the presence of a Dirac delta at the origin (both, B_R and $\mathbb{R}^d$).

As we mention above the main novelty of this work consist on the extension of [96] to the case of non-local aggregation. Moreover, we also manage to improve the existing viscosity solution theory in order to study the dynamics of aggregation-diffusion problems, managing to obtain a deeper understanding of the techniques involved.

Open problems. There exists several directions in which we can continue this project. A first natural questions is to extend the result to the cases in which V and W are more singular, e.g. the Riesz kernel. We also wonder what happens in the case of non-radial problems. Another open question is what happens if $\nabla V \cdot \nu \neq 0$ or if the support of K is non-compact.

Furthermore, in the context of viscosity solutions we wonder if we can use the theory of viscosity solutions developed during this chapter in order to study further aggregation-diffusion problems and their long-time dynamics.

Another interesting question is to study the problem in $\mathbb{R}^d$ for more general kernels. Chapter 2 covers the case of $\mathbb{R}^d$ but only when you assume that the growth of V and W is no faster than the quadratic. In [81] the authors discuss the appearance of blow up for the case $V = 0$ and $W(x) = \frac{|x|^4}{4}$. They also perform some numerical experiments for other cases of W that suggest blow up as well depending on the exponent of the fast diffusion and the dimension. Thereby, an interesting question is to study the problem in $\mathbb{R}^d$ for more general potentials V and W.

1.1.2 Drift-diffusion equations with saturation, Chapter 3

In this chapter we study the equation

$$\partial_t \rho = \operatorname{div}\left(\mathrm{m}(\rho)\nabla(U'(\rho) + V)\right), \quad \text{in } (0,\infty)\times\Omega \tag{1.5}$$

for $\Omega \subseteq \mathbb{R}^d$ bounded and no-flux boundary condition. Here, $\mathrm{m}(s)$ represents a continuous and compactly supported nonlinear mobility (*saturation*) not necessarily concave; U corresponds to the diffusive potential and it includes all the porous medium cases, i.e. $U(s) = \frac{1}{m-1}s^m$ for $m > 0$ or $U(s) = s\log(s)$ if $m = 1$; V corresponds to the attractive potential and it is such that $V \geq 0$, $V \in W^{2,\infty}(\Omega)$.

In a previous work by Bailo, Carrillo, and Hu [19], they construct an implicit finite volume scheme for (1.5) when $\mathrm{m}(\rho) = \rho\sigma(\rho)$ with σ decreasing. Their numerical experiments suggest the appearance of a *freezing* behaviour, i.e. free boundaries at the saturation level, see Figures 3.5 and 3.6. Thus, the main question that we want to solve in this chapter is whether we can show analytically the appearance of this phenomenon. In order to do that, we first study existence of the problem (1.5) using a suitable regularised approximation of the problem. We show that the problem admits an L^1-contractive C_0-semigroup (see Section 1.4). Later, we characterise the L^1-local minimisers of the associated free-energy functional in the corresponding class of measures. Moreover, we understand the long-time behaviour of the constructed solutions in view of its gradient flow structure. All of this can be summarised in the following three main results.

Main Result 2 (Existence, Section 3.2.2)**.** *Consider $\mathcal{A} := \{\rho \in L^1(\Omega) : 0 \leq \rho \leq \alpha\}$ where α is the saturation level. Then, the problem* (1.5) *admits a free-energy dissipating semigroup $S_t : \mathcal{A} \to \mathcal{A}$, in the sense of Definition 3.2.*

Main Result 3 (L^1-local minimisers, Section 3.2.3)**.** *Consider the corresponding free energy*

$$\mathcal{F}[\rho] = \int U(\rho) + \int V\rho \tag{1.6}$$

and the set $\mathcal{A}_M := \{\rho \in \mathcal{A} : \|\rho\|_{L^1(\Omega)} = M\}$ with $M \in (0, \alpha|\Omega|)$. Then, under suitable conditions on U, we have that

$$\widehat{\rho}(x) = \min\{\alpha, (U')^{-1}(C - V(x))\}_+ \tag{1.7}$$

is the local and global minimiser of $\mathcal{F}$ on $\mathcal{A}_M$; where C is unique and only depends on the mass M.

Main Result 4 (Long-time behaviour, Section 3.2.4)**.** *There exists a time-limit operator $S_\infty : \mathcal{A} \to \mathcal{A}$, in the sense of Definition 3.7, such that for any $\rho_0 \in \mathcal{A}$ we have $S_t\rho_0 \to S_\infty\rho_0$ strongly in $L^1(\Omega)$ as $t \to \infty$. Furthermore, S_∞ is an L^1-contraction. We present situations where $S_\infty\rho_0 = \widehat{\rho}$ for all $\rho_0 \in \mathcal{A}_M$ and some where this does not hold.*

With regard the L^1-local and global minimiser in (1.7), it follows that it is such that $S_t\widehat{\rho} = \widehat{\rho}$ for any $t > 0$. However, $\widehat{\rho}$ is not necessarily the unique steady state and we provide some examples for which (1.5) has infinitely many steady states different from $\widehat{\rho}$ and they are limits $t \to \infty$ for some initial datum other than themselves. Finally, we consider a variation of the implicit Finite-Volume scheme introduced in [19]. We study further properties of the method and we connect the numerical results with the ones obtained during the mathematical analysis stage.

The main novelty of this chapter is to provide a unified theory based on C_0-semigroup theory (see Section 1.4) where we do not restrict to the one-dimensional case or concave mobilities, usual assumptions on previous literature. Even though drift-diffusion phenomenon with prevention of overcrowding are ubiquitous in nature, before our result presented in Chapter 3, the mathematical study dedicated to models like (1.5) was still scarce and it focused mainly on the 1-dimensional case or concave mobilities.

Open problems. As we just mention, the literature about problems with saturation is still scarce. Thus, there exist many interesting questions that are worth studying. First, let us notice that our existence proof is based on the construction of a suitable C_0-semigroup. Thereby, an interesting question is whether we can show uniqueness. In particular, we expect it might be possible to prove uniqueness using the notion of entropy solutions [72, 219]. A further natural question would be if we can study an aggregation-diffusion problem, i.e. if we can replace V by $V + W * \rho$. This problem is completely open. The results presented in this thesis could be extended in order to prove existence of a C_0-semigroup of solutions. However, it is to prove that L^1 contraction or comparison principle do not hold for a general kernel and the long-time behaviour is a difficult problem as it has already been explored in [19].

All the theory developed in Chapter 3 is for the case of a bounded domain $\Omega \subseteq \mathbb{R}^d$. Thus, another interesting research question is to study the theory of this problem for $\mathbb{R}^d$ and compare the results with those from Chapter 3. This question is currently work-in-progress.

Finally, let us mention that after our preprint there has been further development of the theory. In particular, in [2] the author studies the C^α regularity of the problem (1.5).

1.1.3 A family of fourth-order aggregation-diffusion equations, Chapter 4

In the fourth chapter of this thesis we are interested into understanding cell-cell adhesion, a phenomenon responsible of tissue growth and patterning formation, see Figure 1.2. In [186], Falcó, Baker and Carrillo introduce the system

$$\begin{cases} \partial_t \rho &= -\mathrm{div}\left(\rho \nabla \left(\kappa \Delta \rho + \alpha \Delta \eta + \beta \rho + \omega \eta\right)\right), \\ \partial_t \eta &= -\mathrm{div}\left(\eta \nabla \left(\alpha \Delta \rho + \Delta \eta + \omega \rho + \eta\right)\right) \end{cases} \tag{1.8}$$

in order to understand this cell-sorting phenomena (see [187] and the references therein). The parameters in the model are such that $\beta, \omega \in \mathbb{R}$ and the diffusion matrix

$$A = \begin{pmatrix} \kappa & \alpha \\ \alpha & 1 \end{pmatrix}$$

is positive definite.

As the first step of the project we move to the corresponding one-species equation (with a more general local pressure) given by

$$\partial_t \rho = -\mathrm{div}(\rho \nabla(\Delta \rho)) - \chi \Delta \rho^m, \quad \text{in } (0, \infty) \times \mathbb{R}^d \tag{1.9}$$

where $m \geq 1$ and $\chi > 0$. The main goal of this chapter is to understand the 2-Wasserstein gradient flow structure of (1.9) and (1.8) (see Section 1.2) and to show existence when possible. A first analysis shows that the problem (1.9) is the gradient flow of the free energy given by

$$\mathcal{F}_m[\rho] = \begin{cases} \frac{1}{2} \displaystyle\int_{\mathbb{R}^d} |\nabla \rho(x)|^2 \, \mathrm{d}x - \chi \mathcal{E}_m[\rho], & \rho \in \mathcal{P}^a(\mathbb{R}^d), \ \nabla \rho \in L^2(\mathbb{R}^d), \\ +\infty, & \text{otherwise}, \end{cases}$$

where

$$\mathcal{E}_m[\rho] = \begin{cases} \displaystyle\int_{\mathbb{R}^d} \rho(x) \log \rho(x) \, \mathrm{d}x, & m = 1, \\ \frac{1}{m-1} \displaystyle\int_{\mathbb{R}^d} \rho^m \mathrm{d}x, & m > 1. \end{cases}$$

The approach used in this chapter is based on the Gagliardo-Nirenberg inequality. Using this functional inequality, we find that there exists a critical exponent $m_c := 2 + \frac{2}{d}$ governing the qualitative properties of the PDE. We

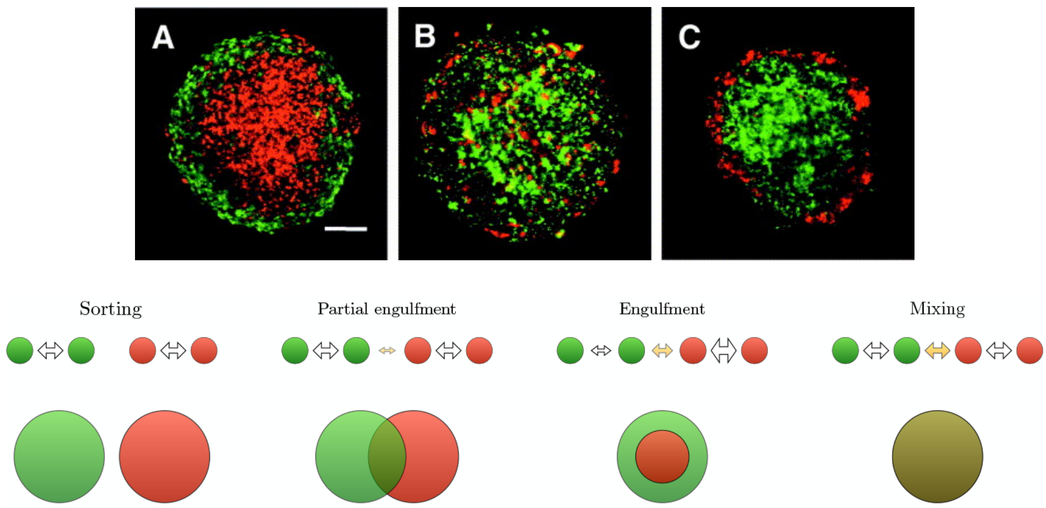

Figure 1.2: Figure representing the cell-cell adhesion behaviour of two populations of cells. In the picture above we can observe an experiment by Duguay, Foty, Steinberg, [166]. In the picture below we show the results of the PDE mathematical model (1.8) for different set of values of the constants $\kappa, \alpha, \beta, \omega$. These constants represent the strength of attraction and diffusion between cells. By Falcó, Baker, Carrillo, [186]. See also the non-local model for cell-cell adhesion introduced by Carrillo, Murakawa, Sato, Togashi, Trush in [107].

are able to distinguish between three different cases (see Theorem 4.1). If $1 \leq m < m_c$ (subcritical regime), the free energy is bounded from below and we are able to recover H^1 *a priori* estimates. If $m > m_c$ (supercritical regime), the free energy is unbounded. If $m = m_c$ we are on the critical regime. In this case, there exists a parameter χ_c such that for $0 < \chi \leq \chi_c$ the free-energy is bounded from below in the set of probability measures and if $0 < \chi < \chi_c$ we can obtain H^1 *a priori* estimates again. However, if $\chi > \chi_c$ the free-energy is unbounded. The critical parameter χ_c is identified by the sharp constant of a suitable Gagliardo-Nirenberg inequality [246]. This discussion leads to the first main result of Chapter 4.

Main Result 5 (Chapter 4, Theorem 4.3)**.** *Under suitable conditions on the initial datum ρ_0. If $1 \leq m < 2+2/d$ or $m = 2 + 2/d$ with $0 < \chi < \chi_c$, then there exists a weak solution to* (1.9).

Taking advantage of the study on the free energy discussed above we show existence via the JKO scheme (see Section 1.2.1). Using similar techniques, we also show existence of solutions for the system (1.8), which leads to the second main result of Chapter 4.

Main Result 6 (Chapter 4, Theorem 4.5)**.** *Under suitable conditions on the initial datum (ρ_0, η_0) there exists a weak solution to* (1.8).

This result is also generalised to a wider class of systems that also allows more variety on the nonlinear self-diffusion. The main novelty of this chapter is to provide a theory for (1.9) distinguishing three different regimes depending on the value of the exponent m. We do that by taking a careful analysis on the free energy. Moreover, we are able to provide existence for one of them. Another novelty refers to how to study a two species system as an extension of the theory for one single species.

Open problems. Since the problem is motivated from a biological phenomena (see Figure 1.2) the first natural question arising is whether we can provide a particle approximation for our model. We intend to solve this question in the near future. We provide further details in Section 5.1. Another very interesting question is to see if we can understand the long-time dynamics. For instance, in [171] the authors study the long-time behaviour of a second order non-local Cahn-Hilliard with some similarities to (1.9). Can we extend the theory introduced in [171] in order to cover a local fourth-order equation?

1.2 Gradient flows, Otto calculus and Wasserstein distance

Otto calculus, introduced in [269], and perfected afterwards, is a notion that allows to perform calculus on spaces of probability measures with bounded p-moment, $\mathcal{P}_p(\mathbb{R}^d)$ (which are not vector spaces). The advantage of Otto calculus is that it allows us to give a geometric intuition to some aggregation-diffusion problems. Let us introduce the Benamou-Brenier formulation for the optimal transport problem with quadratic cost.

Let us consider $\rho_0 = \frac{1}{N}\sum_{i=1}^N \delta_{x_i}$ and $\rho_1 = \frac{1}{N}\sum_{i=1}^N \delta_{y_i}$. Then, for $0 \le t \le 1$ we can think of $\rho_t = \frac{1}{N}\sum_{i=1}^N \delta_{x_i(t)}$ for particle trajectories $x_i(0) = x_i$, $x_i(1) = y_i$. Our goal is to move the particles at the position ρ_0 at time $t = 0$ to the position ρ_1 at time $t = 1$ using the smallest possible kinetic energy (see Figure 1.3 for an example). Roughly speaking what we want is to minimise the action

$$A_t \coloneqq \frac{1}{N}\sum_i |\dot{x}_i(t)|^2.$$

Let us define $v_t(x_i(t)) \coloneqq \dot{x}_i(t)$ over supp ρ_t. Hence,

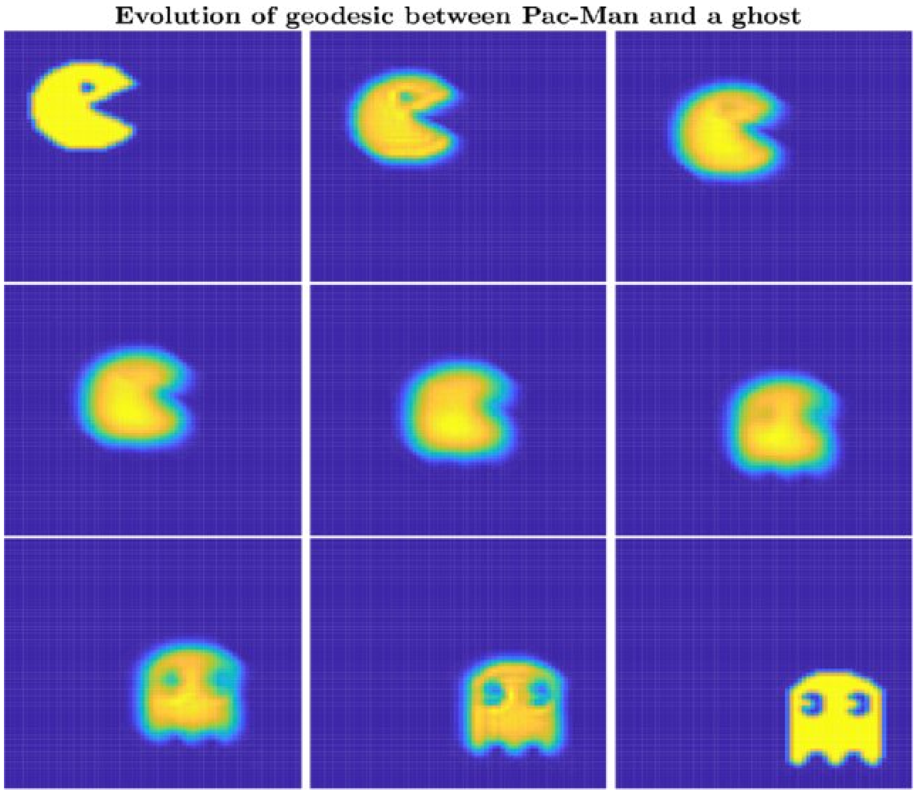

Figure 1.3: Computation of an interpolation measure by the Benamou-Brenier formula between Pac-Man and the Ghost characteristic sets suitably normalized. This figure has been taken from the manuscript [77] by Carrillo, Craig, Wang and Wei. The interested reader can also find a video for this simulation at https://figshare.com/projects/Primal_dual_methods_for_Wasserstein_gradient_flows/59474 as supplementary material of [77].

$$\partial_t \rho_t + \operatorname{div}(v_t \rho_t) = 0 \quad \text{in the distributional sense.} \tag{1.10}$$

Then, we have that

$$A_t = \int_{\mathbb{R}^d} |v_t(x)|^2 \mathrm{d}\rho_t(x).$$

Thereby, the total cost can be written

$$\int_0^1 A_t \mathrm{d}t = \int_0^1 \int_{\mathbb{R}^d} |v_t(x)|^2 \mathrm{d}\rho_t(x).$$

Hence, our problem consists in minimising a kinetic energy,

$$I := \inf_{\rho,v} \left\{ \int_0^1 \int_{\mathbb{R}^d} |v_t(x)|^2 \,\mathrm{d}x\,\mathrm{d}\rho_t(x) \ : \quad \partial_t \rho_t = -\mathrm{div}(\rho_t v_t) \right\}, \tag{1.11}$$

where the infimum is taken over all the curves of probability measures and where ρ_0 and ρ_1 correspond to the times $t = 0$, and $t = 1$. Let us define the push-forward of μ by T, which we denote $T_{\#}\mu \in \mathcal{P}(\Omega)$, by

$$T_{\#}\mu(B) := \mu(T^{-1}(B)) \quad \text{for all } B \text{ Borel measurable sets.}$$

Hence, one can state the following rigorous result that relates the dynamic formulation and the famous Monge problem.

Theorem 1.1 (Benamou-Brenier Formula, [30]). *Assume that ρ_0, ρ_1 are absolutely continuous probability measures in $\mathbb{R}^d$ with compact support. Let I be the infimum value in (1.11), then,*

$$I = \inf_{T_{\#}\rho_0 = \rho_1} \int |x - T(x)|^2 \rho_0(x)\,\mathrm{d}x. \tag{1.12}$$

In particular, (1.12) defines a distance known as the 2-Wasserstein distance (see [7, Section 7]), which is defined by

$$\mathcal{W}_2(\rho_0, \rho_1)^2 := \inf_{T_{\#}\rho_0 = \rho_1} \int |x - T(x)|^2 \rho_0(x)\mathrm{d}x.$$

Furthermore, Theorem 1.1 allows us to characterise the geodesic curves as

$$\rho_t = (\mathrm{id} + t\nabla\varphi)_{\#}\rho_0.$$

In particular, ρ_t is a distributional solution of the continuity equation (1.10) where $v_t = \nabla\varphi \circ (\mathrm{id} + t\nabla\varphi)^{-1}$ (see e.g. [6, Lecture 16]). Therefore, at least formally, the tangent space $T_{\rho_0}\mathcal{P}_2(\mathbb{R}^d)$ can be defined by the elements

$$s := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \rho_t = -\,\mathrm{div}(\rho_t v_t).$$

Then, this suggests a natural way to define the *Wasserstein norm* of the derivative $\partial_t \rho_t$ at ρ_t as

$$\|\partial_t \rho_t\|_{\rho_t}^2 := \inf_{v} \left\{ \int_{\mathbb{R}^d} \rho_t(x)|v_t|^2 \mathrm{d}x \ : \ \partial_t \rho_t = -\,\mathrm{div}(\rho_t v_t) \right\}. \tag{1.13}$$

Hence, equation (1.10) gives, at each time t, a constraint on the divergence form $\rho_t v_t$, and we get the formula

$$\mathcal{W}_2^2(\rho_0, \rho_1) = \inf_{\rho_t} \left\{ \int_0^1 \|\partial_t \rho_t\|_{\rho_t}^2 \mathrm{d}t \ : \ \rho(0,\cdot) = \rho_0,\ \rho(1,\cdot) = \rho_1 \right\}.$$

Let us now take (1.10) such that $v_t = \nabla\psi_t$. Thus, the definition in (1.13) can be rewritten as follows

$$\|\partial_t \rho_t\|_{\rho_t}^2 = \int_{\mathbb{R}^d} \rho_t |\nabla\psi_t|^2 \mathrm{d}x.$$

In particular, we can consider a more general version of this definition. If we have $\rho \in \mathcal{P}_2(\mathbb{R}^d)$ and $h : \mathbb{R}^d \to \mathbb{R}$ such that $\int_{\mathbb{R}^d} h = 0$ we can define

$$\|h\|_{\rho}^2 := \int_{\mathbb{R}^d} \rho |\nabla\psi|^2 \mathrm{d}x, \quad \text{where } \mathrm{div}(\rho\nabla\psi) = -h.$$

Furthermore, this definition can be extended in order to canonically construct the scalar product.

Definition 1.2 (Wasserstein scalar product at ρ)**.** *Given two functions $h_1, h_2 : \mathbb{R}^d \to \mathbb{R}$ with $\int_{\mathbb{R}^d} h_1 = \int_{\mathbb{R}^d} h_2 = 0$, one can define their Wasserstein scalar product at ρ as*

$$\langle h_1, h_2 \rangle_\rho := \int_{\mathbb{R}^d} \nabla\psi_1 \cdot \nabla\psi_2 \rho \,\mathrm{d}x, \quad \textit{where } \operatorname{div}(\rho\nabla\psi_i) = -h_i.$$

Taking advantage of the scalar product we can also define the gradient of a functional in the Wasserstein space.

Definition 1.3 (Gradient with respect to the Wasserstein scalar product at $\overline{\rho}$)**.** *Given a functional $\mathcal{F} : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$, its gradient with respect to the Wasserstein scalar product at $\overline{\rho} \in \mathcal{P}_2(\mathbb{R}^d)$ is the unique function $\mathrm{grad}_{\mathcal{W}_2}\mathcal{F}[\overline{\rho}]$ (if it exists) such that*

$$\left\langle \mathrm{grad}_{\mathcal{W}_2}\mathcal{F}[\overline{\rho}], \frac{\partial \rho_\varepsilon}{\partial \varepsilon} \right\rangle_{\overline{\rho}} = \left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} \mathcal{F}[\rho_\varepsilon]$$

for any smooth curve $\rho_\varepsilon : (-\varepsilon_0, \varepsilon_0) \to \mathcal{P}(\mathbb{R}^d)$ with $\rho_0 = \overline{\rho}$.

Let us now denote the first L^2–variation $\frac{\delta\mathcal{F}[\overline{\rho}]}{\delta\rho}$ which is the function such that

$$\left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} \mathcal{F}[\rho_\varepsilon] = \int_{\mathbb{R}^d} \left.\frac{\partial \rho_\varepsilon(x)}{\partial \varepsilon}\right|_{\varepsilon=0} \frac{\delta\mathcal{F}[\overline{\rho}]}{\delta\rho}(x)\,\mathrm{d}x$$

for any smooth curve $\rho : (-\varepsilon_0, \varepsilon_0) \to \mathcal{P}_2(\mathbb{R}^d)$ such that $\rho_0 = \overline{\rho}$. Therefore, taking avantage of the continuity equation (1.10) and Definition 1.3 we obtain that

$$\left\langle \mathrm{grad}_{\mathcal{W}_2}\mathcal{F}[\overline{\rho}], \frac{\partial \rho_\varepsilon}{\partial \varepsilon} \right\rangle_{\overline{\rho}} = \int_{\mathbb{R}^d} \left.\frac{\partial \rho_\varepsilon}{\partial \varepsilon}\right|_{\varepsilon=0} \frac{\delta\mathcal{F}[\overline{\rho}]}{\delta\rho}\,\mathrm{d}x = -\int_{\mathbb{R}^d} \operatorname{div}(\overline{\rho}\nabla\psi)\frac{\delta\mathcal{F}[\overline{\rho}]}{\delta\rho}\,\mathrm{d}x = \int_{\mathbb{R}^d} \overline{\rho}\nabla\frac{\delta\mathcal{F}[\overline{\rho}]}{\delta\rho} \cdot \nabla\psi\,\mathrm{d}x.$$

Thus, from the definition of the Wasserstein scalar product, one deduces that whenever the L^2-variation exists, then

$$\mathrm{grad}_{\mathcal{W}_2}\, \mathcal{F}[\rho] = -\mathrm{div}\left(\rho\nabla\frac{\delta\mathcal{F}}{\delta\rho}\right).$$

As a consequence, problems of the form

$$\frac{\partial\rho}{\partial t} = \mathrm{div}\left(\rho\nabla\frac{\delta\mathcal{F}}{\delta\rho}\right) \tag{1.14}$$

can be formally (and sometimes rigorously) gradient flows of the free-energy $\mathcal{F}$ with respect to the Otto differential structure. In particular, this notion of gradient flow structure can be made rigorous through the so-called JKO-scheme [215] which we develop further below. This interpretation of the problem is very useful to study short and long-time properties of equations of the form (1.14). We will take advantage of this tool through the different chapters of the thesis.

Here we have presented a brief introduction of Otto Calculus, we suggest reading the lecture notes [6, 191] and the very detailed books [7, 304]. Furthermore, [283] contains a very nice review with emphasis on the examples.

1.2.1 The JKO scheme

As we mention earlier, the JKO scheme refers to a notion of 2-Wasserstein gradient flow. It has been shown to be a very useful tool that can be used to show existence of solutions for various different types of aggregation-diffusion equations. For example, the heat equation [191], the Fokker-Planck equation [215, 190, 285], Keller-Segel [44, 46, 108, 158, 225, 169], nonlocal equations [85, 153, 59, 90], cross-diffusion systems [92, 226, 165], or Cahn-Hilliard [245, 232], among others.

Let us now recall some basic facts about discretization of gradient flows. We start by considering the gradient flow problem

$$\begin{cases} \dfrac{\mathrm{d}x(t)}{\mathrm{d}t} = -\nabla F(x(t)), \\ x(0) = x_0 \end{cases} \tag{1.15}$$

where $F : \mathbb{R}^d \longrightarrow \mathbb{R}$ smooth enough and $x_0 \in \mathbb{R}^d$. One way of constructing a solution is with a time discretization via an implicit Euler scheme. Let us fix a time step $\tau > 0$. Then, the implicit Euler scheme goes as follows,

$$\begin{cases} x_\tau^{n+1} = x_\tau^n - \tau \nabla F(x_\tau^{n+1}), \\ x_\tau^0 = x_0, \end{cases}$$

which might have multiple solutions. Let us also present the sequence

$$\begin{cases} x_\tau^{n+1} \in \underset{x}{\operatorname{argmin}} \left\{ \dfrac{|x - x_\tau^n|^2}{2\tau} + F(x) \right\}, \\ x_\tau^0 = x_0. \end{cases}$$

We notice that if F is lower semicontinuous and bounded from below we have at least a solution for τ small enough (we can pick any). Furthermore, if F smooth enough we also have that

$$x_\tau^{n+1} \in \underset{x}{\operatorname{argmin}}\, F(x) + \frac{|x - x_\tau^n|^2}{2\tau} \quad \text{implies that} \quad \nabla F(x_\tau^{n+1}) + \frac{x_\tau^{n+1} - x_\tau^n}{\tau} = 0,$$

and if it converges we recover a discrete in time implicit Euler scheme for $x' = -\nabla F(x)$. Then, when $\tau \to 0$, a suitable interpolation of our sequence converges to the solution of the problem.

Using De Giorgi minimising movements [149], we can generalise this strategy to a metric space (X, d). We consider the minimising problem

$$\begin{cases} x_\tau^{n+1} \in \underset{x}{\operatorname{argmin}} \left\{ \dfrac{d(x, x_\tau^n)^2}{2\tau} + F(x) \right\}, \\ x_\tau^0 = x_0. \end{cases}$$

If X is compact and F is lower semicontinuous and bounded from below, then there exists a sequence x_τ^n. We define the piecewise interpolation $x_\tau(t) = x_\tau^n$ for $t \in ((n-1)\tau, n\tau]$ and $x_\tau(0) = x_0$. Let us notice that since x_τ^{n+1} is a minimiser in particular

$$\frac{d(x_\tau^{n+1}, x_\tau^n)^2}{2\tau} + F(x_\tau^{n+1}) \le F(x_\tau^n) \quad \text{implies that} \quad F(x_\tau^{n+1}) \le F(x_\tau^n)$$

and hence we recover that

$$\sup_n F(x_\tau^n) \le F(x_0).$$

Let us take $0 \le t < s \le T$ such that $t \in ((k-1)\tau, k\tau]$ and $s \in ((l-1)\tau, l\tau]$. We notice that

$$\sum_{n=k}^{l} d(x_\tau^{n+1}, x_\tau^n)^2 \le 2\tau \left(F(x_\tau^k) - F(x_\tau^{l+1}) \right) \le C\tau.$$

Thereby,

$$\begin{aligned} d(x_\tau(t), x_\tau(s)) &\le \sum_{n=k}^{l-1} d(x_\tau^{n+1}, x_\tau^n) \le \left(\sum_{n=k}^{l-1} d(x_\tau^{n+1}, x_\tau^n)^2 \right)^{\frac{1}{2}} |l-k|^{\frac{1}{2}} \\ &\le C(x_0, T) \left(\sqrt{|s-t|} + \sqrt{\tau} \right). \end{aligned}$$

Since X is compact, from the Ascoli-Arzelà Theorem we recover $x_\tau(t) \to x(t)$ uniformly in time. Formally, the limit satisfies

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\nabla_d x.$$

Let us take

$$\partial_t \rho = -\nabla_{\mathcal{W}_2}(\mathcal{F}[\rho]) = \operatorname{div}\left(\rho \nabla \frac{\delta \mathcal{F}}{\delta \rho}\right) \tag{1.16}$$

with $\mathcal{F} : \mathcal{P}(\mathbb{R}^d) \longrightarrow (-\infty, +\infty]$. Let us point out that $\mathcal{P}_2(\mathbb{R}^d)$ is not compact. Then, we can understand the problem as a gradient flow in the 2-Wasserstein space $(\mathcal{P}_2(\mathbb{R}^d), \mathcal{W}_2)$, [215, 268]. This allows us to define rigorously the $\mathcal{W}_2$-gradient flow via the JKO scheme. The JKO scheme was introduced for Fokker-Planck, where the authors prove that *de Giorgi minimising movements in* $\mathcal{W}_2$ converge. In particular, the JKO scheme looks like

$$\begin{cases} \rho_\tau^{n+1} \in \underset{\rho \in \mathcal{P}_2(\mathbb{R}^d)}{\operatorname{argmin}} \left\{ \dfrac{\mathcal{W}_2(\rho, \rho_\tau^n)^2}{2\tau} + \mathcal{F}[\rho] \right\}, \\ \rho_\tau^0 = \rho_0. \end{cases}$$

For further details we recommend the interested reader to see also [7], and [283, Chapter 8].

1.2.2 A notion of convexity

The notions of convexity and λ-convexity are relevant and it can be useful in order to obtain relevant properties of our gradient flow problem such as uniqueness or long-time behaviour via contractivity. Let us focus first on the Eulerian setting (1.15). F is λ-convex if and only if $F(x) + \frac{\lambda}{2}|x|^2$ is convex for all $x \in \mathbb{R}^d$. Let us assume x_1, x_2 are both solutions to (1.15). Then, thanks to its gradient flow structure and the λ-convexity of F we have that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}|x_1(t) - x_2(t)|^2 &= (x_1(t) - x_2(t)) \cdot (x_1'(t) - x_2'(t)) = -(x_1(t) - x_2(t)) \cdot (\nabla F(x_1(t)) - \nabla F(x_2(t))) \\ &\leq -\lambda |x_1(t) - x_2(t)|^2. \end{aligned}$$

Thus, from Grönwall's inequality it follows that

$$|x_1(t) - x_2(t)| \leq e^{-\lambda t}|x_1(0) - x_2(0)|$$

which, for instance, implies uniqueness of solutions. This idea can be extended to the case $(\mathcal{P}_2(\mathbb{R}^d), \mathcal{W}_2)$ using the notion of gradient flow for the $\mathcal{W}_2$ distance, the JKO scheme (Section 1.2.1). In order to do that we need to introduce the notion of λ-convexity along constant speed geodesic curves, also known as displacement convexity [253]. We say that a curve $\gamma : [0, 1] \longrightarrow \mathcal{P}(\mathbb{R}^d)$ is a *constant speed curve* if

$$\mathcal{W}_2(\gamma_s, \gamma_t) = (t - s)\mathcal{W}_2(\gamma_0, \gamma_1) \quad \text{for all } 0 \leq s \leq t \leq 1.$$

Therefore, we can introduce the definition of λ-geodesically convex functionals.

Definition 1.4. *For $\lambda \in \mathbb{R}$ (positive or negative), we say $\mathcal{F}$ is λ-geodesically convex if for any $\nu_0, \nu_1 \in \mathrm{Dom}(\mathcal{F})$ there exists a constant speed geodesic γ with $\gamma_0 = \nu_0$, $\gamma_1 = \nu_1$ such that*

$$\mathcal{F}[\gamma_t] \leq (1-t)\mathcal{F}[\gamma_0] + t\mathcal{F}[\gamma_1] - \frac{\lambda}{2}t(1-t)\mathcal{W}_2(\gamma_0, \gamma_1)^2 \quad \textit{for all } t \in [0, 1].$$

Then, using this notion of convexity one can extend the Eulerian theory and apply it to the problem (1.16). In particular, if ρ_1, ρ_2 are both solutions to (1.16) and $\mathcal{F}$ is λ-geodesically convex we have that

$$\mathcal{W}_2(\rho_1(t), \rho_2(t)) \leq e^{-\lambda t}\mathcal{W}_2(\rho_1(0), \rho_2(0)),$$

see [7, Theorem 11.2.1]. Let us remark that the case $\lambda < 0$ still provides continuous dependence and in particular uniqueness. Such a result is very fruitful since it helps to understand uniqueness, stability, or the long-time dynamics of a PDE among other applications. Finally, for the interested reader we recommend to go to [7, Chapter 11]. For $\lambda > 0$ the ultracontractivity leads to uniqueness of steady states.

Connection to this thesis. Every chapter in this thesis has a connection to Wasserstein gradient flows and hence, also to optimal transport.

In Chapter 2 we can rewrite (1.2) as the 2-Wasserstein gradient flow of the free-energy functional (1.3). In particular, for a suitable class of admissible measures we show that the functional is bounded from below and we are also able to show that the solutions of our problem are such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}[\rho_t] = -\int \rho \left|\nabla \frac{\delta \mathcal{F}}{\delta \rho}[\rho]\right|^2 \mathrm{d}x.$$

The combination of both results is enough to recover equicontinuity in time of our solutions in the $W^{-1,1}$ sense. This is a key ingredient in order to show convergence in time to a limit, up to a subsequence, $t_n \to \infty$. Once we have constructed a time limit, these two ingredients are also key in order to show that the limit does not depend on time.

In Chapter 3 we can also take advantage of a similar formal gradient-flow structure known as the generalised Wasserstein distance, [161, 104] in order to study (1.5). This time, we consider the free-energy functional (1.6). Once more, for a suitable class of admissible measures the functional is bounded from below. Moreover, we also show that the solutions of the problem satisfy

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}[\rho_t] = -\int \mathrm{m}(\rho) \left|\nabla \frac{\delta \mathcal{F}}{\delta \rho}[\rho]\right|^2 \mathrm{d}x.$$

Thus, analogously to the previous case we can prove convergence to a stationary case (as $t \to \infty$ on this case).

In Chapter 4, problems (1.8) and (1.9) can also be expressed as a 2-Wasserstein gradient flow. Nevertheless, on this project we exploit the gradient flow structure and we show existence of weak solutions through a JKO scheme. We present further properties of the same problem in Section 5.1. Here, we take advantage of the notion of the 2-Wasserstein λ-convexity introduced above. λ-convexity provides the stability we need in order to get a deterministic particle approximation of (1.9).

1.3 Viscosity solutions

Viscosity solutions form a general theory of "weak" solutions (not necessarily differentiable, but only C^0) which applies to certain fully nonlinear elliptic and parabolic solutions of Partial Differential Equations of 1st and 2nd order.

Following [221, Chapter 2], let us first define the notion of viscosity solutions for a degenerate PDE of the general form

$$F(x, \rho, D\rho, D^2\rho) = 0 \tag{1.17}$$

that makes sense when $\rho \in C^0(\Omega)$ for $\Omega \subseteq \mathbb{R}^d$. Here, the "coefficients" $F = F(y, p, q, X)$ is a continuous and possibly nonlinear function defined on $\Omega \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}(d)$. For this problem we also need to impose the degenerate ellipticity condition

$$X \le Y \text{ in } \mathbb{S}(d) \quad \Rightarrow \quad F(y, p, q, X) \le F(y, p, q, Y).$$

We are now ready to introduce the notion of viscosity solution for (1.17).

Definition 1.5 (Viscosity solution elliptic problem)**.** *Let $\rho \in C^0(\Omega)$, $\Omega \subseteq \mathbb{R}^d$. We say ρ is a viscosity subsolution of the problem (1.17) if for every point $x_0 \in \Omega$ we have that for all the test functions $\varphi \in C^2(B_r(x_0))$ on a ball $B_r(x_0) \subseteq \Omega$ such that $\rho \le \varphi$, $\rho(x_0) = \varphi(x_0)$ and $\frac{\partial \varphi}{\partial x}(x) \neq 0$ when $x \in B_r(x_0)\backslash\{x_0\}$ it holds*

$$F(x_0, \varphi(x_0), D\varphi(x_0), D^2\varphi(x_0)) \ge 0.$$

Analogously, if we reverse all the inequalities, we say that ρ is a viscosity supersolution. Finally, we say that ρ is a viscosity solution if it is both, a viscosity subsolution and supersolution.

This definition can be extended in order to cover nonlinear degenerate parabolic equations, see for example [315]. In this case we consider the problem

$$\partial_t \rho + F(t, x, \rho, D\rho, D^2\rho) = 0, \tag{1.18}$$

that makes sense when $\rho \in C([0,T]; C^0(\Omega))$, $F = F(s, y, p, q, X)$ is defined on $[0,T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}(d)$. Once more, we also impose the condition

$$X \le Y \text{ in } \mathbb{S}(d) \quad \Rightarrow \quad F(s, y, p, q, X) \le F(s, y, p, q, Y).$$

Hence, we can also define the notion of viscosity solution also for a parabolic problem (1.18).

Definition 1.6 (Viscosity solution parabolic problem)**.** *Let $\rho \in C([0,T]; C^0(\Omega))$, $\Omega \subseteq \mathbb{R}^d$. We say ρ is a viscosity solution of the problem* (1.18) *if for every point $(t_0, x_0) \in [0,T] \times \Omega$ we have that for all test functions $\varphi \in C^2((t_0 - \varepsilon, t_0 + \varepsilon) \times B_r(x_0))$ on a ball $B_r(x_0) \subseteq \Omega$ such that $\rho \le \varphi$, $\rho(t_0, x_0) = \varphi(t_0, x_0)$ and $\frac{\partial \varphi}{\partial x}(x) \neq 0$ when $x \in B_r(x_0) \backslash \{x_0\}$ it holds*

$$\partial_t \varphi(t_0, x_0) + F((t_0, x_0), \varphi(t_0, x_0), D\varphi(t_0, x_0), D^2\varphi(t_0, x_0)) \ge 0.$$

Analogously, if we reverse all the inequalities, we say that ρ is a viscosity supersolution. Finally, we say that ρ is a viscosity solution if it is both, a viscosity subsolution and supersolution.

The definition of viscosity solution means that when we touch from above (respectively, below) by a smooth test function at a point, the test function is a supersolution of the PDE law (respectively, subsolution) at that point. The key idea behind the notion of viscosity solutions is to use the Maximum Principle in order to pass derivatives to smooth test functions without the more standard notion of duality. Hence, it forms a weak derivative nonlinear theory useful even for linear PDEs.

Viscosity solutions were introduced by Crandall and Lions [135] as a uniqueness criterion for first order PDEs. The essential idea regarding passage to the limit in the vanishing viscosity sense was first noticed by Evans in [180, 181]. Afterwards, this notion was extended by Crandall, Evans and Lions [132]. Jensen proved a comparison principle for this notion of solution [214]. Later on, Ishii and Lions extended the theory of viscosity solutions to second order elliptic PDEs [210, 212]. Finally, we mention that for further regularity estimates of viscosity solutions to fully nonlinear equations one can consult the book of Cabré and Caffarelli [61] and for a more detailed introduction to viscosity solutions we recommend the book of Katzourakis [221]. On the following, we give further details on the notion of a maximum principle for viscosity solutions, Section 1.3.1, and in Section 1.3.2 we explain some stability properties.

1.3.1 A maximum principle for viscosity solutions

There are many version of maximum principle for viscosity solutions. In this thesis we have used the following. In [222], Kawohl and Kutev study the subsolutions of the general elliptic equation

$$F(x, u, Du, D^2u) = 0 \quad \text{in}\, \Omega, \tag{1.19}$$

monotone with respect to D^2u, where Ω is a connected domain and F is a Caratheodory function. They assume,

1. For every $(x, p, X) \in \Omega \backslash N \times \mathbb{R}^d \times (\mathbb{R}^d)^2$, where $N \subseteq \Omega$ denotes a nowhere dense set, F satisfies,

$$r \ge s \quad \Rightarrow \quad 0 \le F(x, r, p, X) - F(x, s, p, X). \tag{1.20}$$

2. Equation (1.19) is strictly elliptic with modulus of ellipticity ω_1

$$X \ge Y \quad \Rightarrow \quad \omega_1(\lambda_K \operatorname{trace}(X - Y)) \le F(x, r, p, Y) - F(x, r, p, X), \tag{1.21}$$

for every $x \in \Omega \backslash N$, and $|r|$, $|p|$, $\|X\|$, $\|Y\| \le K$, and for some positive constant λ_K depending on K. Furthermore, as a modulus of continuity, the real function $s \mapsto \omega_1(s)$ satisfies the usual conditions,

$$\omega(s) \in C[0, \infty), \quad \omega(s) > \omega(t) \text{ for } s > t > 0 \quad \text{and} \quad \omega(0) = 0. \tag{1.22}$$

3. For every $x \in \Omega \backslash N$, and $|r|, |p|, |q|, \|X\|, \|Y\| \leq K$ there exists some positive constants A_K and B_K depending on K such that
$$|F(x,r,p,X) - F(x,r,q,Y)| \leq \omega_2(A_K\|X-Y\| + B_K|p-q|) \tag{1.23}$$
for ω_2 some modulus of ellipticity satisfying (1.22).

4. Ω is connected and there exists a positive s_0 such that
$$\omega_1(s) \geq \omega_2(s) \quad \text{for all } s \in [0, s_0]. \tag{1.24}$$

5. F is such that
$$F(x,0,0,0) \geq 0 \quad \text{for every } x \in \Omega \backslash N. \tag{1.25}$$

Then, we have the following strong interior maximum principle.

Theorem 1.7 (Kawohl-Kutev, [222])**.** *Assume $F(x,z,p,X)$ satisfies the hypothesis (1.20)-(1.25). If $u \in USC(\Omega)$ is a viscosity supersolution of (1.19) and if u has a maximum Θ at some interior point $x_0 \in \Omega$, then $u(x)$ must be constant in Ω, $F(x,0,0,0) \equiv 0$ everywhere in $\Omega \backslash N$, and $F(x,z,0,0)$ must be independent of z for $z \in [0,\Theta]$ and $x \in \Omega \backslash N$.*

1.3.2 Stability properties

We want to describe limiting operations of viscosity solutions. Let us remark that viscosity solutions are very stable under passage through various types of limits. Hence, most frequently, in order to prove existence of viscosity solutions for a PDE, it suffices to construct an adequate sequence of approximate PDEs whose solutions are locally bounded and equicontinuous, without the necessity of having any control on the derivatives, since we have passed all the derivatives to test functions. This remark can be summarised in the following theorem on [221, Chapter 3 Theorem 2] that we state here.

Theorem 1.8 (Stability under C^0 perturbations)**.** *Let $\Omega \subseteq \mathbb{R}^d$ and let also $\{F_k\}_{k=1}^{\infty}$, F be in $C^0(\Omega \times \mathbb{R} \times \mathbb{R}^d \times (\mathbb{R}^d)^2)$ such that*
$$F_k \to F \quad \text{locally uniformly as } k \to \infty.$$
Assume that $\{\rho_k\}_{k=1}^{\infty} \subseteq C^0(\Omega)$ is a sequence of viscosity solutions of
$$F_k(\cdot, \rho_k, D\rho_k, D^2\rho_k) = 0, \quad \text{on } \Omega.$$
If $\rho_k \to \rho$ locally uniformly to some $\rho \in C^0(\Omega)$ as $k \to \infty$, then the limit function ρ is a viscosity solution of the limit PDE
$$F(\cdot, \rho, D\rho, D^2\rho) = 0, \quad \text{on } \Omega.$$

Connection to this thesis. In Chapter 2, we focus on understanding the long-time dynamics of the competition between fast-diffusion and aggregation, i.e.
$$\partial_t \rho = \Delta \rho^m + \operatorname{div}\left(\rho \nabla (V + W * \rho)\right), \quad \text{in } (0,\infty) \times B_R$$
on the range $0 < m < 1$. We take the potentials such that $V, W \geq 0$ in $W^{2,\infty}(B_R)$ and we show convergence of the solution to a time-independent measure $\widehat{\mu} \in W^{-1,1}(B_R)$ as time $t \to \infty$. Afterwards, under further symmetry assumptions, we take advantage of the notion of viscosity solutions in order to characterise the time limit. We consider that the initial datum, V and W are radially symmetric, we take $M = \int_{\widetilde{B_v}} \rho$ where $\widetilde{B_v}$ is the ball with volume v and we move to the mass equation
$$\partial_t M = \kappa(v)^2 \partial_v \left(\partial_v M\right)^m + \kappa(v)^2 \partial_v M \partial_v \left(V + W * \partial_v M\right),$$
where $\kappa(v) = d\omega_d^{\frac{1}{d}} v^{\frac{d-1}{d}}$ (ω_d refers to the volume of the d-dimensional sphere). Then, using viscosity solutions and their properties we show that the time limit is of the form
$$\widehat{\mu} = \left(\|\rho_0\|_{L^1(B_R)} - \|\overline{\rho}\|_{L^1(B_R)}\right) \delta_0 + \overline{\rho}\, \mathrm{d}x.$$
Here, δ_0 is a Dirac delta at 0 and $\overline{\rho}$ is the density of the absolutely continuous part of the measure.

1.4 C_0-semigroup theory

The basic notion underlying "semigroup theory" is to understand the equation as an autonomous infinite dimensional dynamical system and to try to use ODE theory, i.e. finite-dimensional dynamical systems, in the PDE setting. This viewpoint has several applications in the analysis of PDEs including the development of well-posedness theories, the analysis of nonlinear smoothing effects, and the local stability and global dynamics of nonlinear PDEs. The origin of the theory dates back from the 1930's with the work of Stone [297] on one-parameter groups of unitary operators in a Hilbert space. This theory was further developed, see the work from Hille [205] and the references therein. In particular, since a very early stage of the theory it was clear that it had several applications concerning PDEs of parabolic type, as it can be seen in the work of Feller [189], Hille [204], Yosida [312, 313], Crandall and Liggett [134], Brezis [54] and Komura [230]. Of course, since the first half of the XXth century this theory has evolved and it now includes many more cases. For the interested readers, they can find a deeper discussion in [53, 177, 142, 111, 143, 272, 6].

In the following we will first introduce the theory of semigroup for the heat equation in Section 1.4.1. Afterwards, we cover the theory for more general cases, Section 1.4.2. Finally, in Section 1.4.3 we focus on the specific case of interest of this thesis. We focus on drift-diffusion problems in divergence form and we explain its L^1-contractivity property.

1.4.1 Semigroup solutions of heat equation

We now proceed to explain the ideas behind linear C_0-semigroup theory taking the heat equation as an example. First, let us introduce the definition of semigroup.

Definition 1.9. *Let X be a Banach space. A one-parameter semigroup on X is a family of operators $\{T(t)\}_{t\geq 0} \subseteq \mathcal{L}(X)$ such that it fulfills the following properties*
i) $T(0) = I$.
ii) $T(t)T(s) = T(t+s)$ for all $s, t \geq 0$.
Furthermore, we say that it is strongly continuous if
iii) for all $f \in X$, $T(h)f \to f$ strongly in X as $h \to 0^+$.
We call this a C_0-semigroup.

Remark 1.10. *The "semi" is required because the heat equation can be solved forward in time, but not backwards, t and s are required to be non-negative.*

We motivate this definition. In order to do this, first, we briefly review what we know from finite dimensional dynamical systems.

Let us consider first $A \in \mathcal{L}(\mathbb{R}^d; \mathbb{R}^d)$, i.e. a square matrix. Then, the solution to the problem

$$\begin{cases} \partial_t u = Au, & t \in \mathbb{R}, \\ u(0) = u_0 \in \mathbb{R}^d \end{cases} \tag{1.26}$$

is given by the exponential matrix $u(t) = e^{At}u_0$, which just follows from the linear ODE theory. Let us pay attention to the following remark. For each fixed $u_0 \in \mathbb{R}^d$ thanks to the triangular inequality we have that

$$\left|e^{At}u_0 - u_0\right| = \left|\sum_{j=1}^{\infty} \frac{t^j}{j!} A^j u_0\right| \leq \left(\sum_{j=1}^{\infty} \frac{|t|^j}{j!} \|A\|^j_{\mathcal{L}(\mathbb{R}^d)}\right) |u_0| = \left(e^{t\|A\|_{\mathcal{L}(\mathbb{R}^d)}} - 1\right) |u_0|$$

and in particular,

$$\left\|e^{At} - I\right\|_{\mathcal{L}(\mathbb{R}^d)} \leq e^{t\|A\|_{\mathcal{L}(\mathbb{R}^d)}} - 1.$$

Thereby, it follows in this case that the operator e^{At} converges uniformly in $\mathcal{L}(\mathbb{R}^d)$ to the identity operator as $t \to 0$. Hence, since

$$e^{A(t+h)} - e^{At} = e^{At}\left(e^{Ah} - I\right),$$

it follows that for any fixed $t \in \mathbb{R}$ we have that $e^{A(t+h)} \to e^{At}$ uniformly as $h \to 0$.

Now let us extrapolate this idea to the heat equation. Let us consider the problem

$$\begin{cases} \partial_t u = \Delta u, & x \in \Omega, t > 0, \\ u = 0 & \text{on } \partial\Omega, \\ u(0, \cdot) = g \end{cases} \tag{1.27}$$

where $g \in H^2(\Omega) \cap H_0^1(\Omega)$. In particular, it has a unique solution given by

$$u(t, x) = \sum_{k=1}^{\infty} a_k e^{-\lambda_k t} \phi_k(x)$$

where λ_k, ϕ_k are the eigenvalues and eigenfunctions for $-\Delta$ on $H_0^1(\Omega)$ and with ϕ_k chosen to be orthonormal in $L^2(\Omega)$ and $a_k = \langle \phi_k, g \rangle_{L^2(\mathbb{R}^d)}$. The above defines in a natural way a one-parameter family of operators

$$\begin{aligned} T(t)\colon L^2(\mathbb{R}^d) &\longrightarrow L^2(\mathbb{R}^d) \\ g(x) \quad &\mapsto \sum_{k=1}^{\infty} a_k e^{-\lambda_k t} \phi_k(x) \end{aligned}$$

defined for all $t \geq 0$ such that for each $g \in H^2(\Omega) \cap H_0^1(\Omega)$ the solution to (1.27) is given by $u(t, \cdot) = T(t)g(\cdot)$. Let us also note that by construction we clearly have $T(0)g = g$ for all $g \in L^2(\Omega)$. For a fixed $g \in L^2(\Omega)$ and for all $s, t \geq 0$ we have that $T(t)T(s)g(x) = T(t+s)g(x)$ so that $T(s)T(t) = T(s+t)$. In particular, it follows that the family of operators $\{T(t)\}_{t \geq 0}$ forms a *semigroup* of operators, Definition 1.9. Here, we recall that a semigroup is an algebraic structure consisting of a set and an associative binary operator. Semigroups not necessarily need to have multiplicative inverse or identity elements. Therefore, the family $\{T(t)\}_{t \geq 0}$ forms a semigroup with identity, although we will not make this distinction.

Finally, notice that for all $g \in L^2(\Omega)$ and $h > 0$ it follows that

$$\|T(h)g - g\|_{L^2(\Omega)}^2 = \left\| \sum_{k=1}^{\infty} \langle g, \phi_k \rangle_{L^2(\Omega)} \left(e^{-\lambda_k t} - 1 \right) \phi_k \right\|_{L^2(\Omega)}^2 = \sum_{k=1}^{\infty} |\langle g, \phi_k \rangle|^2 \left(e^{-\lambda_k h} - 1 \right)^2,$$

where the last equality is a consequence of the orthonormality of $\{\phi_k\}_{k=1}^{\infty}$. Thereby, for each $g \in L^2(\Omega)$ we have that $T(h)g \to g$ in $L^2(\Omega)$ as $h \to 0^+$. See [111] for further details on the semigroup for the heat equation.

Thus, the family $\{T(t)\}_{t \geq 0}$ forms a semigroup of operators which is strongly continuous, Definition 1.9. Such families of operators are ubiquitous in the study of PDEs and a great tool in order to understand evolution in time problems [177, 272].

1.4.2 Extension of the theory

The theory of semigroups can be extended to cover more general problems. Thanks to the Hille-Yosida and the Crandall-Liggett Theorem we can study the evolutionary equation (1.26) for further cases.

First, let us consider a Hilbert space H and $A : D(A) \subseteq H \to H$ a *maximal monotone operator* (see [54, Section 7.1] for a definition and elementary properties of maximal monotone operators). Then, the evolution in time problem (1.26) has a unique solution. This can be summarised in the celebrated Hille-Yosida Theorem [54, Theorem 7.4].

Theorem 1.11 (Hille-Yosida). *Let A be a maximal monotone operator. Then, given any $u_0 \in D(A)$ there exists a unique function $u \in C^1([0, +\infty); H) \cap C([0, +\infty); D(A))$ satisfying (1.26). Moreover,*

$$|u(t)| \leq |u_0| \quad \textit{and} \quad \left| \frac{\mathrm{d}u}{\mathrm{d}t}(t) \right| = |Au(t)| \leq |Au_0| \quad \forall t \geq 0.$$

In particular, the Hille-Yosida allows us to define a family of operators $\{T(t)\}_{t\geq 0}$ from the Hilbert space H into itself. Furthermore, this family of operators is a C_0-semigroup in the sense of Definition 1.9 that solves (1.26).

There is another case that we can cover. Let us now consider a Banach space X and let $A : D(A) \subseteq X \to X$ be *m-accretive*. The notion of m-accretive operators were introduced and studied intensively in the late 1960s and early 1970s [220, 134, 136, 25, 57]. Moreover, using this notion we can construct a solution for (1.26) in the case in which A is m-accretive. This can be summarised with the Crandall-Liggett Generation Theorem.

Theorem 1.12 (Crandall-Liggett, [134])**.** *Let A be a m-accretive operator. Then, for any $t > 0$ the sequence*

$$u_n = \left(I + \frac{t}{n}A\right)^{-n} u_0 \tag{1.28}$$

converges for every $u_0 \in D(A)$. Moreover, the rate of convergence is at least sublinear.

Thereby, thanks to the Crandall-Liggett Generation Theorem we can use (1.28) in order to define the one-parameter family of operators

$$T(t)u_0 := \lim_{n\to\infty} \left(I + \frac{t}{n}A\right)^{-n} u_0.$$

This family of operators describes a nonlinear semigroup for (1.26) where A is m-accretive (and nonlinear).

1.4.3 L^1 contraction for equations in divergence form

The semigroup that appears in this thesis, the free-energy dissipating semigroup described in Definition 3.2, is in particular an L^1-contraction, i.e. for any $\rho, \eta \in D(A)\cap L^1$ the semigroup $T(t)$ is such that $\|T(t)\rho - T(t)\eta\|_{L^1} \leq \|\rho - \eta\|_{L^1}$. We use semigroup theory in order to study the problem (1.5). The fact that the evolution equation (1.5) is L^1-contractive is not very surprising since there exists previous literature covering similar problems. In [219], Karlsen and Risebro cover the problem

$$\partial_t \rho = \Delta\Phi(\rho) + \operatorname{div}\left(k(x)f(\rho)\right), \quad \text{in } (0,T)\times\mathbb{R}^d. \tag{1.29}$$

Let us first notice that classical solutions may not exist in general for this problem. The authors assume that the diffusion $\Phi : \mathbb{R} \to \mathbb{R}$ satisfies

$$\Phi \in \mathrm{Lip}_{loc}(\mathbb{R}) \text{ and } \Phi(\cdot) \text{ is nondecreasing with } \Phi(0) = 0.$$

For the flux given by $k(x)f(\rho)$ they consider $k : \mathbb{R}^d \to \mathbb{R}^d$, $f : \mathbb{R} \to \mathbb{R}$ with $f(0) = 0$ and such that

$$k \in W^{1,1}_{loc}(\mathbb{R}^d) \cap C(\mathbb{R}^d); \quad k, \operatorname{div} k \in L^\infty(\mathbb{R}^d); \quad f \in \mathrm{Lip}_{loc}(\mathbb{R}).$$

In Lemma 3.26, the reader can see a proof of an L^1 contraction result for a regularised version of the problem of interest from Chapter 3, a particular but representative case of (1.29). However, since weak solutions may not be unique and classical solutions may not exist for this family of equations in general, the authors use the notion of *entropy solutions* in order to show that they are L^1-contractive. We say that a function $\rho \in L^1 \cap L^\infty$ is an entropy solution of the initial value problem (1.29) if

$$\begin{cases} \partial_t|\rho - c| + \operatorname{div}\left[\operatorname{sign}(\rho - c)k(x)\left(f(\rho) - f(c)\right)\right] - \Delta|\Phi(\rho) - \Phi(c)| \\ \qquad\qquad +\operatorname{sign}(\rho - c)\operatorname{div}(k(x)f(c)) \leq 0 \quad \text{in } \mathcal{D}' \text{ for all } c \in \mathbb{R}, \\ \nabla\Phi(\rho) \text{ belongs to } L^2. \end{cases}$$

Hence, with this notion of solution for (1.29), in [219, Theorem 1.2] they show that:

Theorem 1.13. *Let $\rho, \eta \in L^\infty(0,T;BV(\mathbb{R}^d))$ be entropy solutions of* (1.29) *with initial data $\rho_0, \eta_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \cap BV(\mathbb{R}^d)$. Then, for almost all $t \in (0,T)$,*

$$\|\rho(t,\cdot) - \eta(t,\cdot)\|_{L^1(\mathbb{R}^d)} \le \|\rho_0 - \eta_0\|_{L^1(\mathbb{R}^d)}.$$

In particular, there exists at most one entropy solution of the initial value problem (1.29).

The notion of entropy solution was introduced independently by Kružkov [233] and Volpert [305] in order to cover the case in which $\Phi' \equiv 0$ (first order problems) and k is smooth. Here, the notion of entropy solutions is much simpler. Afterwards, Volpert and Hudjaev [306] studied the notion of entropy solution when $\Phi' \ge 0$, they were the first to study strongly degenerate parabolic equations and they showed existence of a BV entropy solution using the viscosity method. Further results in the one-dimensional case were obtained by Bénilan and Touré [33]. An important step forward in the general case of $\Phi(\cdot)$ being just non-decreasing was made by Carrillo [72]. He showed uniqueness of entropy solutions using an elegant extension of the *doubling of variables* technique introduced by Kružkov [233]. Furthermore, in [72] the author also showed existence of an entropy solution using the semigroup method.

Let us also mention that after the fundamental paper from Carrillo, in which he extends the doubling of variables method *à la Kružkov*, his work was extended to cover more cases. First, the already mentioned work by Karlsen and Risebro [219] to study degenerate parabolic equations. Further cases covered are anisotropic diffusion [31, 32], kinetic solutions suitable for quasilinear diffusion operator [117, 273], Leray–Lions type diffusion [73], triply nonlinear degenerate problems [8, 9], or the Stefan problem [11] among others. For the interested reader we also recommend the expository paper [10].

Connection to this thesis. In Chapter 3 we construct our theory for drift-diffusion equations with saturation using C_0-semigroups. We show that our family of PDEs admits a C_0-semigroup $S_t : \mathcal{A} \to \mathcal{A}$ where $\mathcal{A}$ is a suitable set of measures. We use semigroup theory in order to study existence. Moreover, for our case we prove that the C_0-semigroup is also L^1 contractive. Taking advantage of this property, we are also able to explore its long-time behaviour. In particular, we show that there exists a time-limit operator S_∞ such that $S_t\rho_0 \to S_\infty\rho_0$ strongly in L^1 as $t \to \infty$ for every $\rho_0 \in \mathcal{A}$. We also show that S_∞ is an L^1-contraction which implies that our problem has singletons.

2 Partial mass concentration for fast-diffusions with non-local aggregation terms

This chapter is taken from the article "Partial mass concentration for fast-diffusions with non-local aggregation terms" written in collaboration with José Antonio Carrillo[1] and David Gómez-Castro[2], and published in *Journal of Differential Equations*, Volume 409, pp. 700-773 (2024); [93].

Contents

It does not matter how slowly you go as long as you do not stop. – Confucius

2.1 Introduction and problem setting

Nonlinear aggregation-diffusion equations of the form

$$\frac{\partial \rho}{\partial t} = \Delta \rho^m + \operatorname{div}(\rho \nabla V) + \operatorname{div}(\rho \nabla W * \rho), \tag{2.1}$$

are frequent in continuous descriptions of density populations. Here, ρ_t corresponds to a time-dependent probability measure over the whole space $\mathbb{R}^d$. The constant $m > 0$ is the diffusion exponent leading to three cases: slow diffusion if $m > 1$, linear diffusion if $m = 1$, or fast diffusion if $0 < m < 1$. The nonlinear diffusion equation corresponding to $V = W = 0$ is well-known, see [302]. The potentials $V(x)$ and $W(x)$ respectively model the confinement behaviour and the interaction kernel describing attraction/repulsion between the individuals of the population. In this work, V and W are assumed to be bounded below, so we can restrict without lost of generality to $V, W \geq 0$.

This family of equations with linear diffusion first appeared to explain a biological phenomenon known as chemotaxis cell movement [223, 271]. Chemotaxis refers to the natural phenomenon under which cells diffuse in space while secreting some chemical substances attracting other/same cells. The well-known Keller-Segel equation models an aggregation-diffusion behaviour of cell populations. Nonlinear slow diffusions were introduced

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to cope with overcrowding effects [65, 207]. Surveys about the classical Keller-Segel model and more general aggregation-diffusion equations can be found in [209, 78, 200]. A lot of work has been devoted to this family of equations since the 90s to classify the different cases in which diffusion dominates over aggregation or viceversa [213, 203, 50, 49, 48, 69, 74, 99, 98]. Apart from applications in mathematical biology [300, 207], this family of equations also has applications in gravitational collapse or statistical mechanics among others [289, 290].

The time-dependent equation (2.1) is the (formal) 2-Wasserstein gradient flow of a free-energy functional [269, 102, 105, 7, 283] defined for probability densities $\rho \in L^1 \cap \mathcal{P}(\mathbb{R}^d)$ by

$$\mathcal{F}[\rho] := -\tfrac{1}{1-m} \int_{\mathbb{R}^d} \rho(x)^m \,\mathrm{d}x + \int_{\mathbb{R}^d} V(x)\rho(x)\mathrm{d}x + \frac{1}{2} \int\int_{\mathbb{R}^d\times\mathbb{R}^d} W(x-y)\,\rho(x)\rho(y)\,\mathrm{d}x\,\mathrm{d}y\,, \tag{2.2}$$

for $m \in (0,1)$ with W assumed to be symmetric, $W(x) = W(-x)$. Notice that the suitable extension to probability measures $\mu \in \mathcal{P}(\mathbb{R}^d)$ is done by taking

$$\tilde{\mathcal{F}}[\mu] := -\tfrac{1}{1-m} \int_{\mathbb{R}^d} \mu_{\mathrm{ac}}(x)^m \,\mathrm{d}x + \int_{\mathbb{R}^d} V(x)\mathrm{d}\mu(x) + \frac{1}{2} \int\int_{\mathbb{R}^d\times\mathbb{R}^d} W(x-y)\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y)$$

where μ_{ac} is the absolutely continuous part of μ with respect to the Lebesgue measure [152].

Nonlinear equations of the form (2.1) show challenging behaviours with regard to their long-time behaviour and the properties of their steady states. By now, the case of the porous medium diffusion $m > 1$ is well understood [102, 48, 228, 74, 99, 98], and so is the case of linear diffusion $m = 1$ [12, 251, 270, 163, 49, 47, 197]. However, for the case $0 < m < 1$ much less work has been done. Recently, a work by Carrillo, Hittmeir, Volzone, and Yao [98] studies the equation

$$\frac{\partial \rho}{\partial t} = \Delta \rho^m + \mathrm{div}(\rho \nabla W * \rho) \tag{2.3}$$

and show that all its stationary solutions, with no restriction on $m > 0$, are radially decreasing up to translation. Another work by Carrillo, Delgadino, Dolbeault, Frank and Hoffmann [80] shows that, under certain conditions on the potential W, the energy minimiser of the corresponding energy functional over probability measures with zero center of mass is likewise split as

$$\widehat{\mu} = \left(1 - \|\widehat{\rho}\|_{L^1(\mathbb{R}^d)}\right)\delta_0 + \widehat{\rho}\,\mathrm{d}x,$$

where δ_0 denotes the Dirac delta at 0 and $\widehat{\rho}(x) > 0$ is the density of the absolutely continuous part of the measure. Furthermore, the presence of the concentrated point measure is known for specific choices of W, see [81]. To the best of our knowledge there are no results in the literature showing that solutions of the parabolic problem (2.3) actually converge to these type of minimisers with partial mass concentrated at a Dirac delta at the origin.

This work shows that partial mass concentration occurs in the long time asymptotics for the case $0 < m < 1$. More precisely, we find conditions on the initial data ρ_0 and the potentials V and W so that

- a notion of solution of the Cauchy problem is provided leading to global in time solutions.
- as $t \to \infty$, the solution experiments a one-point blow up of the form,

$$\widehat{\mu} = \left(\|\rho_0\|_{L^1(\mathbb{R}^d)} - \|\widehat{\rho}\|_{L^1(\mathbb{R}^d)}\right)\delta_0 + \widehat{\rho}\,\mathrm{d}x,$$

  with $\widehat{\rho} \in L^1(\mathbb{R}^d)$.

A previous study by Carrillo, Gómez-Castro, and Vázquez in [96] shows a similar asymptotic behavior for the easier case of no interaction $W = 0$ and V regular enough. In this work, we will further refine the techniques from [96] in order to improve the results given there and to include the case $W \geq 0$. In contrast to [96], our main results regarding partial mass concentration for long times hold for general radial $L^1 \cap L^\infty$ initial data. Notice the authors in [96] can only prove partial mass concentration for a particular choice of initial data, and any other initial datum above it.

The fast-diffusion case involves new difficulties. It is known that Dirac measures are invariant by the fast-diffusion equation $\frac{\partial u}{\partial t} = \Delta u^m$ when $0 < m < \frac{d-2}{d}$ but they are not generated from L^1 initial data [55]. However, if we add a drift term, the aggregation caused by it can be strong enough to overcome the fast-diffusion repulsion and produce infinite-time concentration [96]. The range $\frac{d-2}{d} < m < 1$ is better understood. For example, for the quadratic confinement potential its long-time asymptotics are given by integrable stationary solutions (even if the initial data is given by a Dirac delta), see for example [45] and its references. In order to analyse the Cauchy problem for (2.1), we will take advantage from the *a priori* estimates of ρ that comes from its structure and the formal interpretation of (2.1) as the 2-Wasserstein gradient flow associated to the free energy (2.2) leading to a control of the dissipation of the free energy.

One of the key ingredients to analyse partial mass concentration in these models will be to study the equation (2.1) in mass variables for radial solutions under the assumptions of radial symmetry of V and W. Let us introduce the spatial variable $v = |x|^d |B_1|$ and consider the mass function

$$M(t, v) = \int_{\widetilde{B_v}} \rho_t(x) \, \mathrm{d}x,$$

where $\widetilde{B}_v$ is the ball centred at 0 such that $|\widetilde{B_v}| = v$. The choice of the volume variable is so that $\rho = \frac{\partial M}{\partial v}$. For convenience let us define the radius $R_v = R^d |B_1|$. We will prove that M satisfies the following nonlinear aggregation-diffusion equation in the viscosity sense

$$\frac{\partial M}{\partial t} = \kappa(v)^2 \frac{\partial}{\partial v} \left(\frac{\partial M}{\partial v} \right)^m + \kappa(v)^2 \frac{\partial M}{\partial v} \frac{\partial}{\partial v} \left(V + W * \frac{\partial M}{\partial v} \right), \tag{2.4}$$

where $\kappa(v) = d \omega_d^{\frac{1}{d}} v^{\frac{d-1}{d}}$. Equation (2.4) is a Hamilton-Jacobi type problem amenable to be treated by means of the notion of viscosity solution [133]. Notice that in (2.4) the diffusion is non-linear of p-Laplace type with $p = m + 1$. We will demonstrate that the notion of viscosity solution is the natural way to study the formation of a Dirac delta comprising part of the mass of the initial data. Linking the fact that (2.1) has a gradient flow structure with the notion of viscosity solution in mass variable will be crucial in order to obtain our main result of partial mass concentration for long times for generic initial data.

In the radial setting, the formation of a Dirac delta at 0 is equivalent to the loss of the Dirichlet boundary condition $M(t, 0) = 0$. There are very few results of loss of the Dirichlet boundary condition in finite or infinite time in the literature of parabolic equations. One example of this is the previous work [96], in which with similar techniques the authors prove concentration in infinite time. There are some other examples like [15, 294], where they study equations of the type $u_t = u_{xx} + |u_x|^p$, or [26, 276, 277, 258] for a family of viscous Hamilton-Jacobi equations.

2.2 Main results

The aim of this section is to present the main results of this chapter, the key ideas of the proof, and the structure of the rest of the chapter. In the sequel, we denote the dependence on time of ρ with a subindex, referring to it like ρ_t. It should not be confused with the time derivative that we denote by $\frac{\partial \rho}{\partial t}$ or $\frac{\partial \rho_t}{\partial t}$ if we want to emphasise the time dependence. Notice that (2.1) is in divergence form and one expects conservation of mass under suitable assumptions. We will carry on the dependence of the mass from the initial data instead of normalizing it to one as it is customary in the gradient flow literature.

2.2.1 The case of the ball of radius R

The overall picture we have of the problem is better when we focus on the problem posed in a ball B_R. We take a more general kernel K with $K(x,y) = K(y,x)$ and consider the aggregation-diffusion problem

$$\frac{\partial \rho}{\partial t} = \Delta \rho^m + \operatorname{div}(\rho \nabla V) + \operatorname{div}\left(\rho \nabla \int_{B_R} K(\cdot, y)\rho(y)\,\mathrm{d}y\right) \qquad \text{in } (0,\infty)\times B_R, \tag{2.5a}$$

$$\rho(0,x) = \rho_0(x), \qquad \text{for } x \in B_R, \tag{2.5b}$$

with no-flux condition on the boundary of the ball

$$\left(\nabla \rho^m + \rho \nabla V + \rho \nabla \int_{B_R} K(\cdot, y)\rho(y)\,\mathrm{d}y\right)\cdot \nu(x) = 0 \qquad \text{on } (0,\infty)\times \partial B_R\,. \tag{2.5c}$$

As a convenient assumption, we require that V does not produce flux across the boundary

$$\nabla V(x)\cdot \nu(x) = 0 \quad \text{on } \partial B_R. \tag{2.6}$$

In order to pass to the limit as $R\to\infty$ we are particular interested in kernels of the form

$$K(x,y) = \eta(x)W(x-y)\eta(y),$$

where η is any function in $C_c^\infty(B_R)$. In this case, K does not produce any flux across the boundary. This problem is the formal 2-Wasserstein gradient-flow of the free energy

$$\mathcal{F}_R[\rho] = \frac{1}{m-1}\int_{B_R}\rho(x)^m\,\mathrm{d}x + \int_{B_R} V(x)\rho(x)\,\mathrm{d}x + \frac{1}{2}\int_{B_R}\int_{B_R} K(x,y)\rho(y)\rho(x)\,\mathrm{d}y\,\mathrm{d}x. \tag{2.7}$$

The corresponding equation for the mass is

$$\frac{\partial M}{\partial t} = \kappa(v)^2 \frac{\partial}{\partial v}\left(\frac{\partial M}{\partial v}\right)^m + \kappa(v)^2\frac{\partial M}{\partial v}\mathfrak{E}[\rho], \quad \kappa(v) = d\omega_d^{\frac{1}{d}} v^{\frac{d-1}{d}}, \quad \text{in } (0,\infty)\times(0,R_v), \tag{2.8}$$

where $R_v = |B_R|$, and slightly abusing the notation, we define

$$\mathfrak{E}[\mu] = \frac{\partial V}{\partial v} + \frac{\partial}{\partial v}\left(\int_{B_R} K(\cdot,y)\mathrm{d}\mu(y)\right).$$

The precise statement will be given below in subsection 2.5.1.

We devote section 2.3 to the well-posedness theory. The aim is to prove a well-posedness result for (2.5) for the following notion of solution.

Definition 2.1. *We say that ρ is a weak L^1 solution of the problem (2.5) in $(0,T)\times B_R$ if $\rho \in C([0,T];L^1(B_R))$, $\rho^m \in L^1(0,T;W^{1,1}(B_R))$ and*

$$\int_{B_R}\rho_0\varphi(0) + \int_0^T\int_{B_R}\rho_t\frac{\partial\varphi}{\partial t} = \int_0^T\int_{B_R}\nabla\rho^m\nabla\varphi + \int_0^T\int_{B_R}\rho_t\left(\nabla V + \nabla\int_{B_R}K(\cdot,y)\rho(y)\mathrm{d}y\right)\nabla\varphi.$$

for all $\varphi \in C^\infty([0,T]\times B_R)$ with $\varphi(T,x)=0$. Furthermore, we say that it is a strong solution if it is a weak solution and

1. *$\rho \in L^\infty((0,T)\times B_R)$,*
2. *$\rho^m \in L^2(0,T;H^2(B_R))$,*
3. *$\frac{\partial\rho}{\partial t} \in L^2((0,T)\times B_R)$.*

The well-posedness result is the following.

Theorem 2.2. *Assume the basic hypotheses on the initial data ρ_0 and the potentials V and K*

$$\begin{aligned} &\rho_0 \in L^\infty(B_R),\ \rho_0 \geq 0, \\ &V \in W^{2,\infty}(B_R),\ V \geq 0,\ \nabla V(x)\cdot \nu(x) = 0 \text{ on } \partial B_R, \\ &K \in W^{2,\infty}(B_R \times B_R),\ K(x,y) = K(y,x) \geq 0 \text{ for all } x,y \in B_R, \\ &\qquad \nabla_x K(x,y)\cdot\nu(x) = 0 \text{ for all } (x,y) \in (\partial B_R \times B_R). \end{aligned} \tag{H0}$$

Then, there exists a unique strong solution for the problem (2.5) *for all $T > 0$.*

The proof is constructive, in subsection 2.3.1 we replace the fast diffusion by a uniformly elliptic operator and fixed time-dependent drift. Using the classical theory [234] we prove well-posedness of classical solutions for this problem. In subsection 2.3.2 we replace the uniformly elliptic operator by the fast diffusion and with compactness arguments we prove the existence of strong solutions. Uniqueness follows from an L^1 comparison principle. Finally, using a fixed-point argument on the drift we show well-posedness for the problem (2.5). The proof can be found in subsection 2.3.3 using the previous sections.

Once we have shown well-posedness for the problem, we prove the so-called free-energy dissipation formula.

Proposition 2.3. *Assume the basic hypotheses* (H0) *on the initial data ρ_0 and the potentials V and K, then the strong solution ρ of the problem* (2.5) *satisfies the identity*

$$\int_{t_1}^{t_2}\int_{B_R} \rho \left| \nabla \left(U'(\rho) + V + \int_{B_R} K(\cdot,y)\rho(y)\,\mathrm{d}y \right) \right|^2 \mathrm{d}x\,\mathrm{d}t = \mathcal{F}_R[\rho_{t_1}] - \mathcal{F}_R[\rho_{t_2}], \tag{2.9}$$

where $U(\rho) = \frac{\rho^m}{m-1}$. In particular, $\mathcal{F}_R[\rho_t]$ is a non-increasing function of time.

We prove this result in subsection 2.4.1. We use this estimate to prove long-time asymptotics up to a subsequence in the following sense. Let us take $\{t_n\}$ a sequence of times such that $t_n \to \infty$. Let ρ be a weak solution of the problem (2.5), so we define the sequence of functions

$$\begin{aligned} \rho^{[n]} \colon [0,1]\times B_R &\longrightarrow \mathbb{R} \\ (t,x) &\mapsto \rho(t+t_n,x). \end{aligned} \tag{2.10}$$

In section 2.4 we prove the following:

Theorem 2.4. *Assume the basic hypotheses* (H0) *on the initial data ρ_0 and the potentials V and K. Let be the sequence $\rho^{[n]}$ be defined as in* (2.10)*. Then, there exists $\widehat{\mu} \in W^{-1,1}(B_R)$ such that, up to a subsequence, $\rho^{[n]} \to \widehat{\mu}$ in $C([0,1];W^{-1,1}(B_R))$.*

A key difficult point is to show that the limit $\widehat{\mu}$ in the previous theorem does not depend on time. We will show later that it is a solution of the stationary problem in the sense of mass. In [96] the authors can only show that $\widehat{\mu}$ does not depend on time for a specific family of explicit initial data (which are singular at 0), for which the mass function is point-wise increasing with respect to t. We can make the characterisation for general initial data in $L^1 \cap L^\infty$ with radial symmetry.

Our method relies on taking advantage of the negative Sobolev space $W^{-1,1}(B_R)$ defined as the distributions with finite norm

$$\|\mu\|_{W^{-1,1}(B_R)} := \inf_{\mu = \operatorname{div}(F)} \|F\|_{L^1(B_R)}. \tag{2.11}$$

The key reason to use $W^{-1,1}(B_R)$ is that the sequence $\rho^{[n]}$ is equicontinuous in time in this space. If $t,s \in [0,1]$, from (2.47), it follows that

$$\|\rho_t^{[n]} - \rho_s^{[n]}\|_{W^{-1,1}(B_R)} \leq \int_s^t \left\| \frac{\partial \rho_\sigma^{[n]}}{\partial \sigma} \right\|_{W^{-1,1}(B_R)} \mathrm{d}\sigma \leq \|\rho_0\|_{L^1(B_R)}^{\frac12} \left(\mathcal{F}_R[\rho_{t_n}] - \mathcal{F}_R[\rho_{t_n+1}]\right)^{\frac12} |t-s|^{\frac12}. \tag{2.12}$$

We have shown that $\mathcal{F}_R[\rho_t]$ is non-increasing in time, and we will also show that the free energy is bounded below. Hence, there is a limit as $t \to \infty$ for $\mathcal{F}_R[\rho_t] \to L$.

We prove Theorem 2.4 using the estimate (2.12). To prove the convergence claim of a subsequence of $\rho^{[n]}$ we use the Ascoli-Arzelá theorem. Passing to the limit in (2.12) we also show that $\|\widehat{\mu}_t - \widehat{\mu}_s\|_{W^{-1,1}(B_R)} = 0$, so $\widehat{\mu}$ does not depend on time. The details of the proof are given in subsection 2.4.2.

Remark 2.5. *The Benamou-Brenier formula [283] and the 2-Wasserstein gradient flow structure of the problem can be used to prove that the convergence of the full sequence $\rho^{[n]}$ also holds in $C([0,1];\mathcal{P}(B_R))$ endowed with the 2-Wasserstein distance. We will not include details of this proof since the asymptotic result is essentially analogous.*

We present the mass equation and some of its properties in subsection 2.5.1. Once we have presented the problem, in subsection 2.5.2 we prove through the construction of ρ that the mass M is a viscosity solution of (2.8) for any finite time in the precise sense given in Definition 2.35.

After that we will study the asymptotic in time of M and link it to the asymptotic in time of the solution of the problem (2.5). In the same sense as for ρ in (2.10) we define the sequence in $[0,1]\times[0,R_v]$ given by

$$M^{[n]}(t,v) = \int_{\widetilde{B_v}} \rho_t^{[n]}(x)\,\mathrm{d}x, \tag{2.13}$$

which are the solutions of the mass equation (2.49) for the time interval $[t_n, t_n+1]$ after a translation to $[0,1]$.

Theorem 2.6. *Assume the basic hypotheses (H0), and that the initial datum ρ_0 and potentials V and K are radially symmetric, and let the sequence $M^{[n]}$ be defined as in (2.13). Then, there exists $\widehat{M} \in C((0,R_v])$ such that, if we define $\widehat{M}(0) = 0$, then the following properties hold:*

- *The function $\widehat{M}$ is non-decreasing in v,*
- *up to a subsequence, $M^{[n]} \to \widehat{M}$ in $C_{loc}([0,1]\times(0,R_v])$ and point-wise in $[0,1]\times[0,R_v]$,*
- *$\frac{\partial \widehat{M}}{\partial v} = \widehat{\mu}$ belongs to $\mathcal{M}([0,R_v])$ in the precise sense given in Remark 2.39,*
- *$\widehat{M}$ is a stationary viscosity solution of (2.4) in the sense given in Definition 2.35.*

We present its proof in subsection 2.5.3.

Remark 2.7. *Notice that the convergence happens point-wise in $[0,R_v]$ so that, due to the no-flux condition on ρ, we have that $\widehat{M}(R_v) = M(t,R_v) = \|\rho_0\|_{L^1(B_R)}$.*

Remark 2.8. *Let us now justify that, if $\widehat{M}$ is locally Lipschitz in $(0,R_v]$, then*

$$\widehat{\mu} = \alpha\delta_0 + \widehat{\rho}\,\mathrm{d}x$$

with $\widehat{\rho} \in L^1(B_R)$. Since $\widehat{M}$ is Lipschitz on $[\kappa,R_v]$ for $\kappa > 0$, its weak derivative is a classical derivative a.e. due to Rademacher's differentiation theorem, and it is given by a function in $L^\infty(\kappa,R_v)$. Since they will match in an overlapping set, we denote it simply by $\widehat{M}'$. The extension towards $v=0$ builds a function $\widehat{M}' \in L^\infty_{loc}((0,R_v])$. In radial coordinates this is $\widehat{\rho} \in L^\infty_{loc}(\overline{B}_R\backslash\{0\})$. By construction, it satisfies the fundamental theorem of calculus outside of $v=0$

$$\widehat{M}(v_1) - \widehat{M}(v_0) = \int_{v_0}^{v_1} \widehat{M}', \qquad \textit{for all } 0 < v_0 \le v_1 \le R_v.$$

Hence we recover

$$\int_0^{R_v} \widehat{M}' = \widehat{M}(R_v) - \widehat{M}(0^+) < \infty,$$

since $\widehat{M}$ is non-decreasing, and thus we have shown $\widehat{M}' \in L^1(0, R_v)$. Notice that we do not know anything about the continuity of $\widehat{M}$ at $v = 0$. Using the same argument, we have that

$$\widehat{M}(v) = \widehat{M}(0^+) + \int_0^v \widehat{M}', \qquad \forall v \in (0, R_v].$$

Since we are fixing $\widehat{M}(0) = 0$ in terms of density functions we have

$$\widehat{\mu} = \frac{\partial \widehat{M}}{\partial v} = \widehat{M}(0^+)\delta_0 + \widehat{M}' \, \mathrm{d}x,$$

proving the statement.

There are two examples where we can prove this regularity. We devote subsection 2.5.4 to this goal. The first example is

Proposition 2.9. *Assume the basic hypotheses (H0), and that the initial datum ρ_0 and potentials V and K are radially symmetric. Let the sequence $M^{[n]}$ be defined as in (2.13) and $\widehat{M}$ its limit obtained in Theorem 2.6. Assume, furthermore, that for all $v \in [0, R_v]$*

$$\frac{\partial}{\partial v} \left(V(x) + \int_{B_R} K(x,y) \, \mathrm{d}\widehat{\rho}(y) \right) \bigg|_{v = |B_1||x|^d} \geq 0.$$

Then, $\widehat{M}$ is $C^2((0, R_v))$.

We prove this result in subsection 2.5.4 and follows directly from the regularity theory of Caffarelli and Cabré [61]. A much more difficult result is the following

Theorem 2.10. *Assume the basic hypotheses (H0), that the initial datum ρ_0 and potentials V and K are radially symmetric, and furthermore, that V and K have compact support. Then, $\widehat{M}$ is linear in an interval $[R_v - b, R_v]$ for some $b > 0$, $\frac{\partial \widehat{M}}{\partial v}(R_v) > 0$, and for a.e. $v \in (0, R_v)$*

$$\frac{\partial \widehat{M}}{\partial v}(v) = \left(\left(\frac{\partial \widehat{M}}{\partial v}(R_v) \right)^{m-1} + \tfrac{1-m}{m} \left(V(x) + \int_{B_R} K(x,y) \, \mathrm{d}\widehat{\rho}(y) \right) \bigg|_{v = |B_1||x|^d} \right)^{-\frac{1}{1-m}}. \tag{2.14}$$

In particular, $\widehat{M} \in W^{2,\infty}_{loc}((0, R_v])$.

The proof can be found in subsection 2.5.4 and uses inf- and sup-convolutions. Hence, we have characterised $\widehat{\rho}$, the absolutely continuous part of the asymptotic state $\widehat{\mu}$, as

$$\widehat{\rho}(x) = \left(h + \tfrac{1-m}{m} \left(V(x) + \int_{B_R} K(x,y) \, \mathrm{d}\widehat{\rho}(y) \right) \right)^{-\frac{1}{1-m}}$$

for some $h > 0$.

2.2.2 The problem in $\mathbb{R}^d$

Under certain assumptions, we extend the results in the ball B_R to the whole space $\mathbb{R}^d$ to study the problem (2.1). In order to do this, we define

$$\mathcal{F}_{R,\eta}[\rho] := \frac{1}{m-1} \int_{B_R} \rho(x)^m \, \mathrm{d}x + \int_{B_R} V_R(x)\rho(x) \, \mathrm{d}x + \frac{1}{2} \int_{B_R} \int_{B_R} \eta(x) W(x-y) \eta(y) \rho(y) \rho(x) \, \mathrm{d}y \, \mathrm{d}x,$$

the free energy associated to the problem (2.5) when $K(x,y) = \eta(x)W(x-y)\eta(y)$. Whenever we want to refer to the free energy associated to the problem in the whole space, we will write $\mathcal{F}_{\infty,\eta}$. We keep this notation for the free energy in section 2.6.

Notice that $\mathcal{F}_{R,\eta}$ is a generalisation of the free energy studied in [96], which corresponds to $W = 0$ (i.e. $\mathcal{F}_{\infty,0}$) whereas the free energy for (2.1) corresponds to $\mathcal{F}_{\infty,1}$.

First we extend the theory from section 2.3 and section 2.4 for ρ. For the treatment of $\mathbb{R}^d$ we rely heavily on the control of the free energy. For this we assume that, even when $W = 0$ the free energy is bounded below

$$\underline{\mathcal{F}_0} := \inf_{\rho \in \mathcal{P}_{ac}(\mathbb{R}^d)} \mathcal{F}_{\infty,0}[\rho] > -\infty. \tag{H1}$$

Some examples for which this assumption is satisfied are provided in [80], or [96, Section 7.2]. This allows us to control the second-order moment. We also require the control of a higher order moment, and we are only able to work in the range

$$m > \frac{1 + d - \sqrt{2d+1}}{d}. \tag{H2}$$

Remark 2.11. *Assumption (H2) is the sharp value of m for the Carlson-Levin inequality [80, Lemma 5] for the p-moment $p = \frac{2}{1-m}$. Carlson type inequalities appear in the literature after [71] and its sharp form is established by Levin in [239]. We need to take into account the sharp value of m at the Step 4.a of the proof of Theorem 2.12 in order to control the tail of the diffusive part of the free energy.*

We require that the initial datum has controlled moment of certain order higher than 2

$$\int_{\mathbb{R}^d} |x|^{\frac{2}{1-m}} \rho_0 < \infty \tag{H3}$$

We will also assume that the growth of V and W is no faster than the quadratic in the sense that

$$|\nabla V(x)| \leq C(1 + |x|) \qquad \text{for all } x \in \mathbb{R}^d, \tag{H4}$$

$$|\nabla W(x)| \leq C(1 + |x|) \qquad \text{for all } x \in \mathbb{R}^d, \tag{H5}$$

$$\Delta V, \Delta W \in L^\infty(\mathbb{R}^d). \tag{H6}$$

The assumptions on the potentials imply that the growth at infinity cannot be more than quadratic, this is quite restrictive in view of the minimization results [80, 81]. In this chapter, we also do not allow for singular at the origin attractive interaction potentials since they are qualitatively less interesting. For singular attractive potentials, analogously to the classical Keller-Segel model, we expect that there are solutions that blow-up in finite time. Furthermore, the corresponding free energy is not bounded below. Nevertheless, showing a similar result to the present work for fast-growing at infinity potentials or allowing for singular attractive potentials are challenging open problems.

We consider the solution for the problem (2.5) in the bounded domain $(0,T) \times B_R$ and for the kernel K of the form, $K(x,y) = \eta(x)W(x-y)\eta(y)$ with $\eta \in C_c^\infty(\mathbb{R}^d)$ a cut-off function. Due to Theorem 2.26 we know there exists a unique strong solution to this problem. We denote this solution by $\rho^{R,\eta}$ and it is such that

$$\rho^{R,\eta} \in L^\infty(0,T; H^1(B_R)).$$

We construct the extension by zero of $\rho^{R,\eta}$ to the whole space $\mathbb{R}^d$, i.e. $\rho^{R,\eta,*} = \rho^{R,\eta}\chi_{B_R}$.

In order to pass to the limit we build a sequence η_j as follows. We take a smooth radially-symmetric and non-increasing function η_1 such that

$$\eta_1(x) = \begin{cases} 1 & \text{if } |x| \leq \frac{1}{2}, \\ 0 & \text{if } |x| \geq 1, \end{cases}$$

and define $\eta_j(x) = \eta_1(x/j)$. In this way, η_j is such that $\eta_j \nearrow 1$ when $j \to \infty$. We are able to prove the following result.

Theorem 2.12. *Assume $\rho_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, and the following technical assumptions (H1)-(H6). For every cut-off function η_j with j fixed, there exists a sequence of radius R_i^j, with $R_i^j \to \infty$ as $i \to \infty$, and $\rho^{\infty,\eta_j} \in C_{loc}([0,\infty); L^2_{loc}(\mathbb{R}^d))$ such that*

$$\rho^{R_i^j,\eta_j,*} \to \rho^{\infty,\eta_j} \quad \text{in } C_{loc}([0,\infty); L^2_{loc}(\mathbb{R}^d)) \text{ as } i \to \infty, \tag{2.15}$$

$$\mathcal{F}_{R_i^j,\eta_j}[\rho_t^{R_i^j,\eta_j}] \to \mathcal{F}_{\infty,\eta_j}[\rho_t^{\infty,\eta_j}] \quad \text{for every } t \in [0,\infty) \text{ as } i \to \infty. \tag{2.16}$$

Furthermore, ρ^{∞,η_j} is a locally-strong solution of

$$\frac{\partial \rho}{\partial t} = \Delta \rho^m + \operatorname{div}(\rho \nabla V) + \operatorname{div}\left(\rho \nabla \int_{\mathbb{R}^d} K_{\eta_j}(\cdot, y)\rho(y)\,\mathrm{d}y\right) \qquad \text{in } (0,\infty)\times\mathbb{R}^d$$

where $K_{\eta_j}(x,y) = \eta_j(x)W(x-y)\eta_j(y)$ in the sense of Definition 2.49. Moreover, there exists $\rho^{\infty,1} \in C_{loc}([0,\infty); L^2_{loc}(\mathbb{R}^d))$ such that

$$\rho^{\infty,\eta_j} \to \rho^{\infty,1} \quad \text{in } C_{loc}([0,\infty); L^2_{loc}(\mathbb{R}^d)) \text{ as } j \to \infty, \tag{2.17}$$

$$\mathcal{F}_{\infty,\eta_j}[\rho_t^{\infty,\eta_j}] \to \mathcal{F}_{\infty,1}[\rho_t^{\infty,1}] \quad \text{for every } t \in [0,\infty) \text{ as } j \to \infty, \tag{2.18}$$

and $\rho^{\infty,1}$ is a locally-strong solution of the problem (2.1) in the sense of Definition 2.49.

We prove the result in subsection 2.6.1. In Proposition 2.58 we show that these short-time limits also hold for M in the sense of viscosity solutions. We require the additional assumptions on the tails

$$\lim_{\sigma\to\infty} \sup_{|x|>\sigma} \frac{|\nabla W(x)|}{V(x)} = 0. \tag{H7}$$

and the following uniformity on the tails of V: for every $K \Subset \mathbb{R}^d$ we have that

$$C(K) = \sup_{\substack{x\in K\\ y\in\mathbb{R}^d}} \frac{V(y-x)}{1+V(y)} < \infty. \tag{H8}$$

In order to prove the long-time asymptotics we take again $\{t_n\}$, a sequence of times such that $t_n \to \infty$. Then, we define the sequence of functions

$$\begin{aligned} \rho^{\infty,1,[n]} \colon [0,1]\times B_R &\longrightarrow \mathbb{R} \\ (t,x) &\mapsto \rho^{\infty,1}(t+t_n,x). \end{aligned} \tag{2.19}$$

In subsection 2.6.2 we prove:

Theorem 2.13. *Assume all the hypothesis from Theorem 2.12. Let be the sequence $\rho^{\infty,1,[n]}$ defined as in (2.19). Then, there exists $\widehat{\mu} \in W^{-1,1}_{loc}(\mathbb{R}^d)$ and a subsequence such that*

$$\rho^{\infty,1,[n]} \to \widehat{\mu} \quad \text{in } C([0,1]; W^{-1,1}_{loc}(\mathbb{R}^d)) \qquad \text{and} \qquad \rho^{\infty,1,[n]} \rightharpoonup \widehat{\mu} \quad \text{weak}-* \text{ in } L^\infty(0,1;\mathcal{M}(\mathbb{R}^d)).$$

Furthermore, if we also assume (2.92) and that $\inf_{x\in\mathbb{R}^d\setminus B_\sigma} V(x) \to \infty$ when $\sigma\to\infty$, $\widehat{\mu}$ is such that $\widehat{\mu}(\mathbb{R}^d) = \|\rho_0\|_{L^1(\mathbb{R}^d)}$.

Remark 2.14. *An analogous improvement to Remark 2.5 using the Benamou-Brenier formula and the 2-Wasserstein gradient flow structure of the problem can also be done in $\mathbb{R}^d$ to show convergence of the full sequence in $C([0,1];\mathcal{P}_2(\mathbb{R}^d))$ endowed with the 2-Wasserstein distance.*

In Proposition 2.59 we prove that the long-time asymptotics also hold for the mass M in the sense of viscosity solutions. Once we can link $\widehat{\mu}$ to $\widehat{M}$, the asymptotic in time of M, we are able to characterise $\widehat{\mu}$.

Remark 2.15. *If $\widehat{M} \in W^{1,\infty}_{loc}((0,\infty))$ and the mass is not escaping through infinity (i.e. $\|\rho_t\|_{L^1} = \|\rho_0\|_{L^1}$), then there exists $\widehat{\rho} \in L^1(\mathbb{R}^d)$ such that*

$$\widehat{\mu} = \alpha\delta_0 + \widehat{\rho}\,\mathrm{d}x,$$

with $\widehat{\rho}$ integrable and $\alpha = \|\rho_0\|_{L^1(\mathbb{R}^d)} - \|\widehat{\rho}\|_{L^1(\mathbb{R}^d)} \geq 0$.

In $\mathbb{R}^d$ we are only able to make the extension of the simpler result Proposition 2.9 from the B_R case, which we state below as Proposition 2.60.

We conclude the chapter with section 2.7 where we construct a family of examples of V and W where the presence of a Dirac delta at the origin for $\widehat{\mu}$ can be shown. We provide an appendix with technical results that are needed in the proofs. In section 2.A we show *a priori* estimates for the $\mathbb{R}^d$ case.

2.3 Well-posedness for the aggregation-diffusion equation in B_R

We restrict ourselves to the bounded domain B_R, adding a no-flux boundary condition. Through this section we will work with a fixed time-dependent drift E. First, in subsection 2.3.1 we review the classical theory for uniformly elliptic diffusion Φ and we obtain some *a priori* estimates that will be useful to us on different parts of this work. After that, in subsection 2.3.2 we consider the fast diffusion case, $\Phi(s) = s^m$, $0 < m < 1$. We obtain uniqueness by an L^1 continuous dependence argument and existence follows from a convergence argument, using the solutions from the previous subsection.

2.3.1 Uniformly elliptic diffusion with time-dependent drift in B_R

In this subsection we consider the problem

$$\begin{cases} \frac{\partial \rho}{\partial t} = \Delta\Phi(\rho) + \operatorname{div}(\rho E) & \text{in } (0,\infty)\times B_R \\ (\nabla\Phi(\rho) + \rho E)\cdot\nu(x) = 0 & \text{on } (0,\infty)\times\partial B_R, \\ \rho(0,x) = \rho_0(x). \end{cases} \tag{2.20}$$

where $E = E_t(x)$ is assumed to be smooth, more precisely $E \in C^\infty([0,\infty)\times\overline{B_R})$, and $\Phi \in C^2$ is uniformly elliptic, in the sense that there exist constants such that

$$0 < c_1 \leq \Phi'(\rho) \leq c_2 < \infty. \tag{2.21}$$

Furthermore, we assume,

$$E_t(x)\cdot\nu(x) = 0, \quad \text{on } \partial B_R,\ \forall t \in (0,\infty). \tag{2.22}$$

As it is explained in [96], under these assumptions, existence, uniqueness and the maximum principle hold from the classical theory [234, 196, 4, 250, 311]. Classical solutions to (2.20) with initial data $\rho_0 \in W^{s,p}(\Omega)$ with $p \geq 2$, $p > d$ and $1 \leq s < \min\{1+\frac{1}{p}, 2-\frac{d}{p}\}$ satisfy

$$\rho \in C^1\left((0,T);C(\overline{B_R})\right)\cap C\left((0,T);C^2(\overline{B_R})\right)\cap C\left([0,T);W^{s,p}(\Omega)\right)$$

where T is the maximal existence time. If $T < \infty$ then $\|\rho_t\|_{W^{s,p}} \to \infty$ as $t \nearrow T$. Due to the linear growth of $\operatorname{div}(\rho E)$, we can use the results in [234, 4] to show the solution exists until the blow-up of the L^∞ norm.

Let us now discuss further properties of the solution of (2.20). First, we will present some *a priori* estimates that were already introduced in [96], where E did not depend on time t. However, some results in [96] also hold for time-dependent E. In particular, we can state the following a priori estimates.

Proposition 2.16 (L^p estimates, [96])**.** *Assume (2.21) and $E_t(x)\cdot\nu(x) \geq 0$. Then, the unique classical solution of (2.20) satisfies that*

$$\|(\rho_t)_\pm\|_{L^p(B_R)} \leq e^{\frac{p-1}{p}\int_0^T \|\operatorname{div} E_t\|_{L^\infty(B_R)}}\|(\rho_0)_\pm\|_{L^p(B_R)}. \tag{2.23}$$

Remark 2.17. *Let us prove the preservation of non-negativity by freezing coefficients. It follows that $u = \rho$ is a solution to the problem*

$$\frac{\partial u}{\partial t} = \Phi'(\rho)\Delta u + \big(\Phi''(\rho)\nabla\rho + E\big)\cdot\nabla u + \mathrm{div}(E)u \quad \textit{in } (0,\infty)\times B_R.$$

Since ρ is a classical solution and Φ is C^2, all the coefficients are continuous. With the change of variable $v = e^{-\lambda t}u$ for any $\lambda \in \mathbb{R}$ we can replace the zero order term coefficient by a non-negative quantity. Then, if we assume $\rho_0(x) \geq 0$, the maximum principle [182, Section 7 - Theorem 12] implies that $\rho_t(x) \geq 0$.

Furthermore, due to the no-flux boundary condition, for $p = 1$ we also have

$$\|\rho_t\|_{L^1(B_R)} = \|\rho_0\|_{L^1(B_R)}, \tag{2.24}$$

because, along solutions of (2.20) *we have*

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\rho_t\|_{L^1(B_R)} = \int_{B_R}\mathrm{div}(\nabla\Phi(\rho_t) + \rho_t E_t) = \int_{\partial B_R}(\nabla\Phi(\rho_t) + \rho_t E_t)\cdot\nu(x) = 0.$$

Next, in order to formulate the *a priori* estimates on $\nabla\Phi(\rho)$ and $\nabla\rho$ we define,

$$\Psi(s) = 2\int_0^s \Phi(\sigma)\,\mathrm{d}\sigma. \tag{2.25}$$

Lemma 2.18 (*a priori* estimates on $\nabla\Phi(\rho)$, [96])**.** *Assume* (2.21) *and* (2.22)*. Then, the unique classical solution of* (2.20) *satisfies that*

$$\int_0^T\int_{B_R}|\nabla\Phi(\rho_t)|^2 \leq \int_{B_R}\Psi(\rho_0) + \int_0^T \|E_t\|^2_{L^\infty(B_R)}\|\rho_t\|^2_{L^2(B_R)}. \tag{2.26}$$

From this point we will include the proofs for the following *a priori* estimates because the time dependence becomes relevant and, therefore, there are significant differences with respect to the results shown in [96].

In order to obtain an *a priori* estimate for $\nabla\rho$ we define an auxiliary function G_Φ, given by the conditions

$$G''_\Phi(s) = \frac{1}{\Phi'(s)}, \qquad G'_\Phi(0) = G_\Phi(0) = 0. \tag{2.27}$$

Due to (2.21) and the regularity of Φ, G_Φ is well defined and $C^2((0,\infty))$.

Lemma 2.19 (*a priori* estimates on $\nabla\rho$)**.** *Assume* (2.21) *and* (2.22)*. Then, the unique classical solution of* (2.20) *satisfies that*

$$\int_{B_R}G_\Phi(\rho_T) + \frac{1}{2}\int_0^T\int_{B_R}|\nabla\rho_t|^2 \leq \int_{B_R}G_\Phi(\rho_0) + \frac{1}{2}\int_0^T\left\|\frac{\rho_t}{\Phi'(\rho_t)}\right\|^2_{L^\infty(B_R)}\|E_t\|^2_{L^2(B_R)}. \tag{2.28}$$

Proof. We can compute that

$$\frac{\partial}{\partial t}\int_{B_R}G_\Phi(\rho) = -\int_{B_R}\nabla G'_\Phi(\rho)\cdot(\nabla\Phi(\rho) + \rho E_t) = -\int_{B_R}|\nabla\rho|^2 - \int_{B_R}\frac{\rho}{\Phi'(\rho)}\nabla\rho\cdot E_t.$$

Then, we have

$$\int_{B_R}\frac{\partial}{\partial t}G_\Phi(\rho) + \int_{B_R}|\nabla\rho|^2 \leq \left\|\frac{\rho}{\Phi'(\rho)}\right\|_{L^\infty(B_R)}\|\nabla\rho\|_{L^2(B_R)}\|E_t\|_{L^2(B_R)}.$$

Applying Young's inequality we obtain

$$\int_{B_R}\frac{\partial}{\partial t}G_\Phi(\rho) + \frac{1}{2}\int_{B_R}|\nabla\rho|^2 \leq \frac{1}{2}\left\|\frac{\rho}{\Phi'(\rho)}\right\|^2_{L^\infty(B_R)}\|E_t\|^2_{L^2(B_R)},$$

obtaining the desired result if we integrate in time. □

We apply a similar argument to obtain *a priori* estimates for the time derivative.

Lemma 2.20 (*a priori* estimates on $\frac{\partial \rho}{\partial t}$)**.** *Assume* (2.21), (2.22)*, and* $\Phi(\rho_0) \in H^1(B_R)$*. Then, the unique classical solution of* (2.20) *satisfies that*

$$\begin{aligned}\frac{1}{2}\int_0^T \int_{B_R} \Phi'(\rho_t) \left|\frac{\partial \rho_t}{\partial t}\right|^2 + \frac{1}{2}\int_{B_R} |\nabla \Phi(\rho_T)|^2 \le \frac{1}{2}\int_{B_R} |\nabla \Phi(\rho_0)|^2 + \frac{1}{2}\int_0^T \int_{B_R} \Phi'(\rho_t)|\nabla \rho_t|^2 |E_t|^2 \\ + \int_0^T \int_{B_R} \rho_t \nabla \Phi(\rho_t) \cdot \frac{\partial E_t}{\partial t} \\ - \int_{B_R} \zeta(\rho_0) \operatorname{div} E_0 + \int_{B_R} \zeta(\rho_T) \operatorname{div} E_T,\end{aligned} \tag{2.29}$$

where $\zeta : [0, \infty) \to \mathbb{R}$ *is a function such that* $\zeta'(s) = s\Phi'(s)$*.*

Proof. We will use the notation $w = \Phi(\rho)$. When ρ is smooth, we can take $\frac{\partial w}{\partial t}$ as a test function and integrate in B_R. Notice that $\frac{\partial w_t}{\partial t} = \Phi'(\rho_t)\frac{\partial \rho_t}{\partial t}$, so

$$\int_{B_R} \Phi'(\rho_t) \left|\frac{\partial \rho_t}{\partial t}\right|^2 = \int_{B_R} \frac{\partial w_t}{\partial t} \cdot (\Delta w_t + \operatorname{div}(\rho_t E_t)).$$

We can integrate by parts to recover

$$\int_{B_R} \Phi'(\rho_t) \left|\frac{\partial \rho_t}{\partial t}\right|^2 = -\frac{1}{2}\frac{d}{dt}\int_{B_R} |\nabla w_t|^2 - \int_{B_R} \rho_t \nabla \frac{\partial w_t}{\partial t} \cdot E_t.$$

Integrating in $[0, T]$ we have

$$\int_0^T \int_{B_R} \Phi'(\rho_t) \left|\frac{\partial \rho_t}{\partial t}\right|^2 + \frac{1}{2}\int_{B_R} |\nabla w_T|^2 = \frac{1}{2}\int_{B_R} |\nabla w_0|^2 - \int_0^T \int_{B_R} \rho_t \nabla \left(\frac{\partial w_t}{\partial t}\right) \cdot E_t.$$

Integrating by parts in time the second integral on the RHS,

$$\begin{aligned}\int_0^T \int_{B_R} \Phi'(\rho_t) \left|\frac{\partial \rho_t}{\partial t}\right|^2 + \frac{1}{2}\int_{B_R} |\nabla w_T|^2 = \frac{1}{2}\int_{B_R} |\nabla w_0|^2 + \int_0^T \int_{B_R} \nabla w_t \cdot \left[\frac{\partial \rho_t}{\partial t} E_t + \rho_t \frac{\partial E_t}{\partial t}\right] \\ + \int_{B_R} \rho_0 \nabla w_0 \cdot E_0 - \int_{B_R} \rho_T \nabla w_T \cdot E_T.\end{aligned}$$

At this point we introduce ζ. Then, integrating by parts we get

$$\begin{aligned}\int_0^T \int_{B_R} \Phi'(\rho_t) \left|\frac{\partial \rho_t}{\partial t}\right|^2 + \frac{1}{2}\int_{B_R} |\nabla w_T|^2 = \frac{1}{2}\int_{B_R} |\nabla w_0|^2 + \int_0^T \int_{B_R} \nabla w_t \cdot \left[\frac{\partial \rho_t}{\partial t} E_t + \rho_t \frac{\partial E_t}{\partial t}\right] \\ - \int_{B_R} \zeta(\rho_0) \operatorname{div} E_0 + \int_{B_R} \zeta(\rho_T) \operatorname{div} E_T.\end{aligned}$$

Notice that,

$$\frac{\partial \rho_t}{\partial t} \nabla w_t = \Phi'(\rho_t)^{\frac{1}{2}} \frac{\partial \rho_t}{\partial t} \Phi'(\rho_t)^{\frac{1}{2}} \nabla \rho_t.$$

Therefore, applying Young's inequality, we recover,

$$\begin{aligned}\frac{1}{2}\int_0^T \int_{B_R} \Phi'(\rho_t) \left|\frac{\partial \rho_t}{\partial t}\right|^2 + \frac{1}{2}\int_{B_R} |\nabla w_T|^2 \le \frac{1}{2}\int_{B_R} |\nabla w_0|^2 + \frac{1}{2}\int_0^T \int_{B_R} \Phi'(\rho_t)|\nabla \rho_t|^2 |E_t|^2 \\ + \int_0^T \int_{B_R} \rho_t \nabla w_t \cdot \frac{\partial E_t}{\partial t}\end{aligned}$$

$$- \int_{B_R} \zeta(\rho_0) \operatorname{div} E_0 + \int_{B_R} \zeta(\rho_T) \operatorname{div} E_T,$$

where all the terms on the RHS are bounded. This is because $E \in W^{1,\infty}([0,T] \times \overline{B_R})$, Φ is uniformly elliptic, the L^p estimate (2.23), the *a priori* estimate (2.26) on $\|\nabla\Phi(\rho)\|_{L^2((0,T)\times B_R)}$ and the *a priori* estimate (2.28) on $\|\nabla\rho\|_{L^2((0,T)\times B_R)}$. □

Remark 2.21. *Let us observe that, in particular, because Φ' is non-negative, we have the inequality,*

$$\begin{aligned} \frac{1}{2}\left(\min_{\substack{t\in(0,T)\\ x\in B_R}} \Phi'(\rho_t)\right) \int_0^T \int_{B_R} \left|\frac{\partial \rho_t}{\partial t}\right|^2 &\leq \frac{1}{2}\int_{B_R} |\nabla\Phi(\rho_0)|^2 - \frac{1}{2}\int_{B_R} |\nabla\Phi(\rho_T)|^2 \\ &\quad + \frac{1}{2}\int_0^T \int_{B_R} \Phi'(\rho_t)|\nabla\rho_t|^2|E_t|^2 + \int_0^T \int_{B_R} \rho_t \nabla\Phi(\rho_t)\cdot\frac{\partial E_t}{\partial t} \\ &\quad - \int_{B_R} \zeta(\rho_0) \operatorname{div} E_0 + \int_{B_R} \zeta(\rho_T) \operatorname{div} E_T. \end{aligned} \tag{2.30}$$

2.3.2 Fast Diffusion with time-dependent drift in B_R

In this subsection, we will focus in the Fast Diffusion problem with known transport,

$$\begin{cases} \frac{\partial\rho}{\partial t} = \Delta\rho^m + \operatorname{div}(\rho E) & \text{in } (0,\infty)\times B_R \\ (\nabla\rho^m + \rho E)\cdot\nu(x) = 0 & \text{on } (0,\infty)\times\partial B_R, \\ \rho(0,x) = \rho_0(x). \end{cases} \tag{2.31}$$

For convenience in this section, we will denote $\Phi(s) = s^m$. Let us define our notions of solutions for the problem (2.31).

Definition 2.22 (Weak solution). *ρ is said to be a weak solution of the problem* (2.31) *in $(0,T)\times B_R$ if it is $C([0,T];L^1(B_R)), \Phi(\rho) \in L^1(0,T;W^{1,1}(B_R))$ and for all $\varphi \in X = \{\varphi \in C^\infty([0,T]\times\overline{B_R}) : \varphi(T) = 0\}$, it satisfies,*

$$\int_{B_R} \rho_0\varphi(0) + \int_0^T \int_{B_R} \rho_t \frac{\partial\varphi}{\partial t} = \int_0^T \int_{B_R} \nabla\Phi(\rho)\nabla\varphi + \int_0^T \int_{B_R} \rho_t E_t \nabla\varphi.$$

Definition 2.23 (Strong solution). *ρ is said to be a strong solution of the problem* (2.31) *in $(0,T)\times B_R$ if it is a weak solution such that*

1. *$\Phi(\rho) \in L^1(0,T;H^2(B_R))$;*
2. *$\frac{\partial\rho}{\partial t} \in L^2((0,T)\times B_R)$.*

Remark 2.24. *Since $\frac{\partial\rho}{\partial t} \in L^2((0,T)\times B_R)$ we have that,*

$$\int_{B_R} |\rho_{s_2} - \rho_{s_1}| \leq \int_{s_1}^{s_2} \int_{B_R} \left|\frac{\partial\rho_s(x)}{\partial t}\right| dx\, ds \leq |s_2 - s_1|^{\frac{1}{2}} |B_R|^{\frac{1}{2}} \left\|\frac{\partial\rho}{\partial t}\right\|_{L^2((0,T)\times B_R)}$$

and ρ is such that $\rho \in C^{\frac{1}{2}}([0,T];L^1(B_R))$.

Uniqueness

The uniqueness result can be obtained by continuous dependence in L^1 with respect ρ_0 and E. Given two strong solutions ρ and $\overline{\rho}$ to the problem (2.31) with given drifts E and $\overline{E}$ respectively, we follow an argument similar to [96, Theorem 2.10].

Theorem 2.25 (L^1 continuous dependence)**.** *Let ρ and $\overline{\rho}$ be two strong solution of (2.31) with given smooth drifts E and $\overline{E}$ respectively. If $E = \overline{E}$ then we have the L^1 contraction principle*

$$\int_{B_R} [\rho_T - \overline{\rho}_T]_+ \leq \int_{B_R} [\rho_0 - \overline{\rho}_0]_+.$$

If $E \not\equiv \overline{E}$ and $\rho \in L^\infty((0,T) \times B_R)$ then we have that

$$\int_{B_R} [\rho_T - \overline{\rho}_T]_+ \leq \int_{B_R} [\rho_0 - \overline{\rho}_0]_+ + \|\rho_0\|_{L^1(B_R)} \int_0^T \|\operatorname{div}(E_t - \overline{E}_t)\|_{L^\infty(B_R)} + C_1 \left(\int_0^T \|E_t - \overline{E}_t\|^2_{L^2(B_R)} \right)^{1/2}, \tag{2.32}$$

where,

$$C_1 = \left(\int_{B_R} G_\Phi(\rho_0) + \frac{1}{2} \int_0^T \left\| \frac{\rho_t}{\Phi'(\rho_t)} \right\|^2_{L^\infty(B_R)} \|E_t\|^2_{L^2(B_R)} \right)^{1/2},$$

and G_Φ is defined in (2.27).

Proof. Denote $w_t = \Phi(\rho_t) - \Phi(\overline{\rho}_t)$. Let j be convex and denote $p = j'$. We have

$$\begin{aligned}\int_{B_R} \frac{\partial}{\partial t}(\rho_t - \overline{\rho}_t) p(w_t) &= \int_{B_R} p(w_t) \operatorname{div}\left(\nabla w_t + (\rho_t E_t - \overline{\rho}_t \bar{E}_t)\right) \\ &= - \int_{B_R} p'(w_t) |\nabla w|^2 - \int_{B_R} \nabla p(w_t) \cdot (\rho_t E_t - \overline{\rho}_t \overline{E}_t).\end{aligned}$$

Then,

$$\begin{aligned}\int_{B_R} \frac{\partial}{\partial t}(\rho_t - \overline{\rho}_t) p(w_t) &\leq - \int_{B_R} \nabla p(w_t) \cdot (\rho_t E_t - \overline{\rho}_t \overline{E}_t) \\ &= - \int_{B_R} \nabla p(w_t) \cdot \rho_t (E_t - \overline{E}_t) - \int_{B_R} \nabla p(w_t) \cdot \overline{E}_t (\rho_t - \overline{\rho}_t) \\ &= \int_{B_R} p(w_t) \left(\nabla \rho_t \cdot (E_t - \overline{E}_t) + \rho_t \operatorname{div}(E_t - \overline{E}_t) \right) \\ &\quad + \int_{B_R} p(w_t) \left(\nabla(\rho_t - \overline{\rho}_t) \cdot \overline{E}_t + (\rho_t - \overline{\rho}_t) \operatorname{div} \overline{E}_t \right).\end{aligned}$$

Now, we take the limit $p(s) \to \operatorname{sign}_0^+(s)$, the positive part function such that $\operatorname{sign}_0^+(0) = 0$, and get the bound

$$\begin{aligned}\int_{B_R} [\rho_T - \overline{\rho}_T]_+ &\leq \int_{B_R} [\rho_0 - \overline{\rho}_0]_+ + \int_0^T \int_{B_R} \operatorname{sign}_0^+(w_t) \left(\nabla \rho_t \cdot (E_t - \overline{E}_t) + \rho_t \operatorname{div}(E_t - \overline{E}_t) \right) \\ &\quad + \int_0^T \int_{B_R} \nabla(\rho_t - \overline{\rho}_t)_+ \cdot \overline{E}_t + (\rho_t - \overline{\rho}_t)_+ \operatorname{div} \overline{E}_t \\ &\leq \int_{B_R} [\rho_0 - \overline{\rho}_0]_+ \\ &\quad + \|\nabla \rho_t\|_{L^2((0,T)\times B_R)} \|E_t - \overline{E}_t\|_{L^2((0,T)\times B_R)} + \|\rho_0\|_{L^1(B_R)} \int_0^T \|\operatorname{div}(E_t - \overline{E}_t)\|_{L^\infty(B_R)} \\ &\quad + \int_0^T \int_{B_R} \operatorname{div}\left([\rho_t - \overline{\rho}_t]_+ \overline{E}_t \right),\end{aligned}$$

due to a direct application of the Dominated Convergence Theorem using the properties of the strong solutions. The last term vanishes since $\overline{E}_t(x) \cdot \nu(x) = 0$ on ∂B_R. If $E = \overline{E}$ then, the L^1 contraction is obvious. If $E \neq \overline{E}$, considering the *a priori* estimate (2.28), we recover the desired inequality. □

Existence

We proceed to prove existence for the problem (2.31), by constructing solutions for regular Φ. We consider the following sequence Φ_k of functions satisfying (2.21) given by $\Phi_k(0) = 0$ and

$$\Phi_k'(s) \sim \begin{cases} mk^{m-1} & s > k, \\ ms^{m-1} & s \in [k^{-1}, k], \\ mk^{1-m} & s < k^{-1}, \end{cases} \tag{2.33}$$

up to a smoothing. The sequence Φ_k is such that, $\Phi_k(s) \to \Phi(s)$ for all $s \geq 0$. We take the sequence $\rho^{(k)}$ given as the unique solutions of the problems

$$\begin{cases} \dfrac{\partial \rho^{(k)}}{\partial t} = \Delta \Phi_k(\rho^{(k)}) + \operatorname{div}(\rho^{(k)} E^{(k)}) & \text{in } (0,\infty) \times B_R, \\ \left(\nabla \Phi_k(\rho^{(k)}) + \rho^{(k)} E^{(k)}\right) \cdot \nu(x) = 0 & \text{on } (0,\infty) \times \partial B_R, \\ \rho^{(k)}(0,x) = \rho_0^{(k)}(x) & x \in B_R, \end{cases} \tag{2.34}$$

where $\rho_0^{(k)} \in C^2(\overline{B_R})$ such that $\rho_0^{(k)} \to \rho_0$ in $L^\infty(B_R)$ and $E^{(k)}$ smooth such that $E^{(k)} \to E$ in $W^{1,\infty}([0,T] \times B_R)$.

We will prove existence of solutions for the problem (2.31) using a compactness-type argument.

Theorem 2.26 (Existence)**.** *Assume $\rho_0 \in L^\infty(B_R)$, $\rho_0 \geq 0$, $E \in W^{1,\infty}([0,T] \times B_R)$, and $\rho^{(k)}$ the strong solutions of* (2.34)*. Then, $\rho^{(k)}$ converges weakly in $H^1((0,T) \times B_R)$, strongly in $L^2((0,T) \times B_R)$ and in $C([0,T]; L^1(B_R))$ to ρ, the unique strong solution of* (2.31)*.*

Proof. We divide the proof in several steps.

Step 1: $\rho^{(k)}$ is uniformly bounded in $L^2(0,T; H^1(B_R))$. Due to the L^p estimate (2.23) and the control on $E^{(k)}$, there exists a uniform constant Λ_T such that

$$\|\rho^{(k)}\|_{L^\infty((0,T)\times B_R)} \leq \exp\left(\int_0^T \|\operatorname{div} E_t^{(k)}\|_{L^\infty(B_R)}\right) \|\rho_0^{(k)}\|_{L^\infty(B_R)} \leq \Lambda_T < \infty, \tag{2.35}$$

since $\rho_0^{(k)} \to \rho_0$ in $L^\infty(B_R)$. Because of the *a priori* estimate (2.28),

$$\int_0^T \int_{B_R} |\nabla \rho_t^{(k)}|^2 \leq \int_{B_R} G_{\Phi_k}(\rho_0^{(k)}) + \frac{1}{2}\int_0^T \left\| \frac{\rho_t^{(k)}}{\Phi_k'(\rho_t^{(k)})} \right\|_{L^\infty(B_R)} \|E_t^{(k)}\|_{L^2(B_R)}, \tag{2.36}$$

where G_{Φ_k} is defined in (2.27). Since G_{Φ_k} are non-decreasing functions we have that

$$0 = G_{\Phi_k}(0) \leq G_{\Phi_k}(\rho_0^{(k)}(x)) \leq G_{\Phi_k}(\|\rho_0^{(k)}\|_{L^\infty(B_R)}).$$

Moreover, $G_{\Phi_k}(\sigma) \to \frac{\sigma^{3-m}}{m(2-m)(3-m)}$ for all $\sigma \geq 0$. Therefore, for k big enough we have that

$$G_{\Phi_k}(\|\rho_0^{(k)}\|_{L^\infty(B_R)}) \leq 2\frac{\|\rho_0^{(k)}\|^{3-m}_{L^\infty(B_R)}}{m(2-m)(3-m)}.$$

Then, $G_{\Phi_k}(\rho_0^{(k)}(x))$ is uniformly bounded. The second term of the RHS of the estimate (2.36) is also uniformly bounded,

$$\left| \frac{\rho_t^{(k)}}{\Phi_k'(\rho_t^{(k)})} \right| \leq 2 \begin{cases} \dfrac{1}{m}|\rho_t^{(k)}|^{2-m} & \text{if } |\rho_t^{(k)}| \geq k^{-1}, \\ \dfrac{1}{m}k^{m-2} & \text{if } |\rho_t^{(k)}| < k^{-1}. \end{cases}$$

where the constant 2 accounts for the regularisation of Φ_k' on the corners. Since $\|\rho^k\|_{L^\infty((0,T)\times B_R)}$ is uniformly bounded, so is this quantity. $\|E_t^{(k)}\|_{L^2(B_R)}$ is uniformly bounded by $C\|E_t\|_{L^2(B_R)}$ because of the convergence $E^{(k)} \to E$ in $W^{1,\infty}([0,T]\times B_R)$. In particular,

$$\sup_k \int_0^T \int_{B_R} |\nabla \rho_t^{(k)}|^2 < \infty.$$

Therefore, $\rho^{(k)}$ is uniformly bounded in $L^2(0,T;H^1(B_R))$.

Step 2: $\nabla\Phi_k(\rho^{(k)})$ is uniformly bounded in $L^2((0,T)\times B_R)$. Due to (2.26), it follows that

$$\int_0^T \int_{B_R} |\nabla \Phi_k(\rho_t^{(k)})|^2 \le \int_{B_R} \Psi_k(\rho_0^{(k)}) + \int_0^T \|E_t^{(k)}\|^2_{L^\infty(B_R)} \|\rho_t\|^2_{L^2(B_R)},$$

where $\Psi_k(s) = 2\int_0^s \Phi_k(\sigma)\,d\sigma$. Hence, for $k > \Lambda_T$, $\Phi_k(\rho^{(k)}) \le \Phi(\rho^{(k)})$, so $\Psi_k(\rho^{(k)}) \le \Psi(\rho^{(k)})$ we have,

$$\begin{aligned}\int_0^T \|\nabla\Phi_k(\rho_t^{(k)})\|^2_{L^2(B_R)} &\le \int_{B_R} \Psi(\rho_0^{(k)}) + \int_0^T C\|E_t\|^2_{L^\infty(B_R)}\|\rho_t\|^2_{L^2(B_R)} \\ &\le \frac{2}{m+1}|B_R|\|\rho_0\|^{m+1}_{L^\infty(B_R)} + C\int_0^T \|E_t\|^2_{L^\infty(B_R)}\|\rho_t\|^2_{L^2(B_R)}.\end{aligned}$$

Step 3: $\frac{\partial\rho^{(k)}}{\partial t}$ is uniformly bounded in $L^2((0,T)\times B_R)$. From (2.30) we get that

$$\begin{aligned}\frac{1}{2}\int_0^T\int_{B_R} \Phi_k'(\rho_t^{(k)})\left|\frac{\partial \rho_t^{(k)}}{\partial t}\right|^2 \le{}& \frac12\int_{B_R}|\nabla\Phi_k(\rho_0^{(k)})|^2 + \frac12\int_0^T\int_{B_R}\nabla\Phi_k(\rho_t^{(k)})\nabla\rho_t^{(k)}|E_t^{(k)}|^2 \\ &+ \int_0^T\int_{B_R}\rho_t\nabla\Phi_k(\rho_t^{(k)})\cdot\frac{\partial E_t^{(k)}}{\partial t} \\ &- \int_{B_R}\zeta_k(\rho_0^{(k)})\operatorname{div} E_0^{(k)} + \int_{B_R}\zeta_k(\rho_T^{(k)})\operatorname{div} E_T^{(k)}.\end{aligned}$$

Furthermore, because Φ_k' is non-increasing we also have that

$$\begin{aligned}\frac12\int_0^T\int_{B_R}\left|\frac{\partial\rho_t^{(k)}}{\partial t}\right|^2 \le{}& \frac{1}{\Phi_k'(\|\rho_t^{(k)}\|_{L^\infty((0,T)\times B_R)})}\Bigg(\frac12\int_{B_R}|\nabla\Phi_k(\rho_0)|^2 + \frac12\int_0^T\int_{B_R}\nabla\Phi_k(\rho_t^{(k)})\nabla\rho_t^{(k)}|E_t^{(k)}|^2 \\ &+ \int_0^T\int_{B_R}\rho_t\nabla\Phi_k(\rho_t^{(k)})\cdot\frac{\partial E_t^{(k)}}{\partial t} - \int_{B_R}\zeta_k(\rho_0^{(k)})\operatorname{div}E_0^{(k)} + \int_{B_R}\zeta_k(\rho_T^{(k)})\operatorname{div}E_T^{(k)}\Bigg).\end{aligned}$$

Then, $\Phi_k'(\|\rho_t^{(k)}\|_{L^\infty}) \ge \Phi_k'(\Lambda_T) \ge \Gamma_T > 0$ since $\Phi_k'(\Lambda_T) \to \Phi'(\Lambda_T) > 0$. Let us now apply Hölder and Young inequalities,

$$\begin{aligned}\frac12\int_0^T\int_{B_R}\left|\frac{\partial\rho_t^{(k)}}{\partial t}\right|^2 \le{}& \frac{1}{\Gamma_T}\Bigg(\frac12\int_{B_R}|\nabla\Phi_k(\rho_0^{(k)})|^2 + \frac12\int_0^T\|\nabla\Phi_k(\rho_t^{(k)})\|_{L^2(B_R)}\|\nabla\rho_t^{(k)}\|_{L^2(B_R)}\|E_t^{(k)}\|^2_{L^\infty(B_R)} \\ &+ \int_0^T\|\nabla\Phi_k(\rho_t^{(k)})\|_{L^2(B_R)}\|\rho_t\|_{L^2(B_R)}\left\|\frac{\partial E_t^{(k)}}{\partial t}\right\|_{L^\infty(B_R)} \\ &- \int_{B_R}\zeta_k(\rho_0^{(k)})\operatorname{div}E_0^{(k)} + \int_{B_R}\zeta_k(\rho_T^{(k)})\operatorname{div}E_T^{(k)}\Bigg) \\ \le{}& \frac{1}{2\Gamma_T}\int_{B_R}|\nabla\Phi_k(\rho_0^{(k)})|^2\end{aligned}$$

$$
\begin{aligned}
&+ \frac{C}{4\Gamma_T}\|E\|^2_{L^\infty((0,T)\times B_R)} \int_0^T \left(\|\nabla\Phi_k(\rho_t^{(k)})\|^2_{L^2(B_R)} + \|\nabla\rho_t^{(k)}\|^2_{L^2(B_R)}\right) \\
&+ \frac{C}{2\Gamma_T}\left\|\frac{\partial E}{\partial t}\right\|_{L^\infty((0,T)\times B_R)} \int_0^T \left(\|\nabla\Phi_k(\rho_t^{(k)})\|^2_{L^2(B_R)} + \|\rho_t\|^2_{L^2(B_R)}\right) \\
&- \frac{C}{\Gamma_T}\int_{B_R} \zeta_k(\rho_0^{(k)}) \operatorname{div} E_0 + C\int_{B_R} \zeta_k(\rho_T^{(k)}) \operatorname{div} E_T.
\end{aligned}
$$

From Step 1 we already know $\rho^{(k)}$ is uniformly bounded in $L^2(0,T;H^1(B_R))$, from Step 2 $\nabla\Phi_k(\rho^{(k)})$ is uniformly bounded in $L^2((0,T)\times B_R)$ and from the assumptions, $E \in W^{1,\infty}([0,T]\times\overline{B_R})$. The function ζ_k is such that $\zeta_k(\rho_t^{(k)}) = \int_{B_R} \rho_t^{(k)} \nabla\Phi_k(\rho_t^{(k)})$, then the last two terms of the RHS are uniformly bounded from a combination of all the previously discussed above. Therefore, the sequence $\frac{\partial\rho^{(k)}}{\partial t}$ is uniformly bounded in $L^2((0,T)\times B_R)$.

Step 4: A subsequence converges to the strong solution. We divide this step in subsequent steps.

Step 4.a: Convergence by compactness. The sequence $\rho^{(k)}$ is uniformly bounded in $H^1((0,T)\times B_R)$ by Steps 1 and 3. Therefore, by the Sobolev compactness embedding Theorems, there exists a subsequence $\rho^{(k)}$ such that

$$
\rho^{(k)} \to \rho \quad \text{in } L^2((0,T)\times B_R) \quad \text{and} \quad \rho^{(k)} \to \rho \quad \text{in } C([0,T];L^1(B_R)). \tag{2.37}
$$

This also implies that, up to further a subsequence, the weak convergence in $H^1((0,T)\times B_R)$ and

$$
\rho^{(k)} \to \rho \quad \text{a.e.} \tag{2.38}
$$

Step 4.b: $\rho \in L^2(0,T;H^1(B_R))$. $\nabla\rho^{(k)}$ is uniformly bounded on $L^2((0,T)\times B_R)$, then, taking into account (2.37)

$$
\nabla\rho^{(k)} \rightharpoonup \nabla\rho \quad \text{weakly in } L^2((0,T)\times B_R),
$$

and $\rho \in L^2(0,T;H^1(B_R))$.

Step 4.c: $\frac{\partial\rho}{\partial t} \in L^2((0,T)\times B_R)$. The sequence $\frac{\partial\rho^{(k)}}{\partial t}$ is uniformly bounded in $L^2((0,T)\times B_R)$. Then, up to a subsequence, $\frac{\partial\rho^{(k)}}{\partial t} \rightharpoonup \xi$ weakly in $L^2((0,T)\times B_R)$. From (2.37) we recover $\xi = \frac{\partial\rho}{\partial t}$ and $\frac{\partial\rho}{\partial t} \in L^2((0,T)\times B_R)$.

Step 4.d: $\Phi(\rho) \in L^2(0,T;H^1(B_R))$. From Step 2 we know that the sequence $\nabla\Phi_k(\rho^{(k)})$ is uniformly bounded on $L^2((0,T)\times B_R)$. Then, up to a subsequence,

$$
\nabla\Phi_k(\rho^{(k)}) \rightharpoonup \xi \quad \text{weakly in } L^2((0,T)\times B_R).
$$

Recall that by (2.37), $\rho^{(k)} \to \rho$ in $L^2((0,T)\times B_R)$. Furthermore, $\Phi_k \to \Phi$ uniformly in $[0,\Lambda_T]$. Then,

$$
\begin{aligned}
\|\Phi_k(\rho^{(k)}) - \Phi(\rho)\|_{L^2((0,T)\times B_R)} &\le \|\Phi_k(\rho^{(k)}) - \Phi(\rho^{(k)})\|_{L^2((0,T)\times B_R)} + \|\Phi(\rho^{(k)}) - \Phi(\rho)\|_{L^2((0,T)\times B_R)} \\
&\le \|\Phi_k - \Phi\|_{L^\infty(0,\Lambda_T)}|B_R|^{1/2} + [\Phi]_{C^m}\||\rho^{(k)} - \rho|^m\|_{L^2((0,T)\times B_R)} \to 0,
\end{aligned}
$$

where $[\Phi]_{C^m}$ is the m-Hölder semi-norm of Φ. The convergence to 0 follows from the Dominated Convergence Theorem: we have almost everywhere convergence and they are bounded functions. Therefore, this implies that

$$
\Phi_k(\rho^{(k)}) \to \Phi(\rho) \quad \text{strongly in } L^2((0,T)\times B_R).
$$

Then, we also obtain $\nabla\Phi_k(\rho^{(k)}) \rightharpoonup \nabla\Phi(\rho)$ weakly in $L^2((0,T)\times B_R)$, and $\Phi(\rho) \in L^2(0,T;H^1(B_R))$.

Step 4.e: $\Delta\Phi(\rho) \in L^2((0,T)\times B_R)$. Due to the smoothness of $E^{(k)}$, $\rho^{(k)}$ are classical solutions. Then, from *a posteriori* computation we get that

$$
\|\Delta\Phi_k(\rho^{(k)})\|_{L^2((0,T)\times B_R)} \le \left\|\frac{\partial\rho^{(k)}}{\partial t} - \operatorname{div}\left(\rho^{(k)}E^{(k)}\right)\right\|_{L^2((0,T)\times B_R)}
$$

$$\leq \left\| \frac{\partial \rho^{(k)}}{\partial t} \right\|_{L^2((0,T)\times B_R)} + C\|\rho\|_{L^2(0,T;H^1(B_R))}\|E\|_{L^\infty(0,T;W^{1,\infty}(B_R))}.$$

Therefore, we conclude that $\Delta\Phi_k(\rho^{(k)}) \rightharpoonup \Delta\Phi(\rho)$ weakly in $L^2((0,T)\times B_R)$.

Step 4.f: Distributional solution. We have now all the ingredients to see that for every test function $\varphi \in X = \{\varphi \in C^\infty([0,T]\times \overline{B_R}) \,:\, \varphi(T) = 0\}$,

$$\int_{B_R} \rho_0^{(k)}\varphi(0) + \int_0^T \int_{B_R} \rho_t^{(k)} \frac{\partial \varphi}{\partial t} = \int_0^T \int_{B_R} \nabla\Phi_k(\rho_t^{(k)})\nabla\varphi + \int_0^T \int_{B_R} \rho_t^{(k)} E_t \nabla\varphi$$

converges to

$$\int_{B_R} \rho_0\varphi(0) + \int_0^T \int_{B_R} \rho_t \frac{\partial \varphi}{\partial t} = \int_0^T \int_{B_R} \nabla\Phi(\rho)\nabla\varphi + \int_0^T \int_{B_R} \rho_t E_t \nabla\varphi.$$

Therefore, ρ is a strong solution of (2.31).

Step 5: Convergence of the whole sequence. Since there is a unique strong solution of the limit problem, every subsequence has a further subsequence converging to the same limit, and hence the whole sequence converges. □

Before finishing this section we want to remark that ρ, the strong solution obtained in Theorem 2.26, preserves all the *a priori* estimates discussed during subsection 2.3.1, which are, the L^p estimate (2.23) and the *a priori* estimates of the $L^2((0,T)\times B_R)$ norm of $\nabla\Phi(\rho)$, $\nabla\rho$ and $\frac{\partial\rho}{\partial t}$ which are presented in (2.26), (2.28), and (2.29) respectively.

2.3.3 Proof of Theorem 2.2

We divide the proof in several steps. Let us fix an arbitrary time T:

Step 1: Define the contraction mapping. The goal is to prove existence and uniqueness of the solution using Banach fixed point Theorem to an operator $\mathcal{K}$ defined as follows. We consider $T_0 \leq T$. First we define

$$\mathcal{T} :\, X \longrightarrow C([0,T_0]; L^1(B_R)),$$

as the functional that maps the vector field E with the solution of (2.31) for initial data ρ_0 and vector field E, where due to Theorem 2.26 we can take the domain as

$$X = \{E \in W^{1,\infty}([0,T_0]\times B_R) \text{ that satisfies (2.22)}\}.$$

As a second step of the construction of our fixed-point operator, we define the map from ρ to E by

$$\mathcal{E} : C([0,T_0]; L^1(B_R)) \longrightarrow X, \qquad \mathcal{E}[\rho](x) = \nabla V(x) + \nabla \int_{B_R} K(x,y)\cdot\rho(y)\,\mathrm{d}y.$$

Finally, we define the operator we will use for a fixed point argument as

$$\mathcal{K} = \mathcal{T}\circ\mathcal{E} : C([0,T_0]; L^1(B_R)) \longrightarrow C([0,T_0]; L^1(B_R)). \tag{2.39}$$

It is easy to see that the ball

$$A := \{u \in C([0,T_0]; L^1(B_R)) : \sup_{t\in(0,T_0)} \|u_t\|_{L^1(B_R)} \leq \|\rho_0\|_{L^1(B_R)}\}$$

is such that $\mathcal{K}(A) \subseteq A$ because of the L^p estimate (2.23) for $p = 1$, independently of T.

Step 2: Prove that our mapping is contractive. Let us now prove that the operator $\mathcal{K} : A \to A$ is contracting for small enough T_0. We look at the Lipschitz constant of $\mathcal{T}$ and $\mathcal{E}$. We know that, due to the estimate (2.23) for $p = \infty$, we have the bound

$$\|\mathcal{T}[E]\|_{L^\infty((0,T_0)\times B_R)} \leq \|\mathcal{T}[E]\|_{L^\infty((0,T)\times B_R)} \leq \exp\left(\int_0^T \|\operatorname{div} E_t\|_{L^\infty((0,T)\times} B_R)\right) \|\rho_0\|_{L^\infty(B_R)}.$$

Notice that the time integral has been extended from $[0, T_0]$ to $[0, T]$. Then, due to the L^1 continuous dependence result from Theorem 2.25 we have that

$$\|\mathcal{T}[E]_t - \mathcal{T}[\overline{E}]_t\|_{L^1(B_R)} \leq C_2 \left(T_0^{1/2} \sup_{s\in(0,T_0)} \|E_s - \overline{E}_s\|_{L^2(B_R)} + T_0 \sup_{s\in(0,T_0)} \|\operatorname{div}(E_s - \overline{E}_s)\|_{L^\infty(B_R)}\right),$$

for all $t \in [0, T_0]$, where C_2 is equal to the sum of the constant C_1 from Theorem 2.25 and $\|\rho_0\|_{L^1(B_R)}$, i.e.

$$C_2 = \|\rho_0\|_{L^1(B_R)} + \left(\int_{B_R} G_\Phi(\rho_0) + \frac{1}{2}\int_0^T \|\mathcal{T}[E]_t\|^{2(2-m)}_{L^\infty(B_R)} \|E_t\|^2_{L^\infty(B_R)}\right)^{1/2},$$

with G_Φ defined in (2.27). Let us plug in the potential $\mathcal{E}[\rho]$. We start by realising that if we replace E by $\mathcal{E}[\rho]$, everything is bounded. This is because the terms $\|\mathcal{E}[\rho]_t\|_{L^\infty(B_R)}$ and $\|\operatorname{div}\mathcal{E}[\rho]_t\|_{L^\infty(B_R)}$ can be bounded as follows,

$$\begin{aligned}
\|\mathcal{E}[\rho]_t\|_{L^\infty(B_R)} &\leq \|\nabla V\|_{L^\infty(B_R)} + \|\nabla_x K\|_{L^\infty(B_R\times B_R)}\|\rho_t\|_{L^1(B_R)}\\
&= \|\nabla V\|_{L^\infty(B_R)} + \|\nabla_x K\|_{L^\infty(B_R\times B_R)}\|\rho_0\|_{L^1(B_R)}\\
&= C_3(\|\nabla V\|_{L^\infty(B_R)}, \|\nabla_x K\|_{L^\infty(B_R\times B_R)}, \|\rho_0\|_{L^1}),\\
\|\operatorname{div}\mathcal{E}[\rho]_t\|_{L^\infty(B_R)} &\leq \|\Delta V\|_{L^\infty(B_R)} + \|\Delta_x K\|_{L^\infty(B_R\times B_R)}\|\rho_0\|_{L^1(B_R)},\\
&= C_4(\|\Delta V\|_{L^\infty(B_R)}, \|\Delta_x K\|_{L^\infty(B_R\times B_R)}, \|\rho_0\|_{L^1}).
\end{aligned}$$

In particular, when we replace E by $\mathcal{E}[\rho]$

$$C_2 \leq C_5 = \|\rho_0\|_{L^1(B_R)} + \left(\int_{B_R} G_\Phi(\rho_0) + \frac{T}{2} C_3^2 \exp\left(2(2-m)C_4\right)\right)^{1/2}.$$

Notice that

$$\|\mathcal{E}[\rho]_t - \mathcal{E}[\overline{\rho}]_t\|_{L^2(B_R)} \leq \|\nabla_x K\|_{L^2(B_R\times B_R)}\|\rho_t - \overline{\rho}_t\|_{L^1(B_R)} = C_6(\|\nabla_x K\|_{L^2(B_R\times B_R)})\|\rho_t - \overline{\rho}_t\|_{L^1(B_R)},$$
$$\|\operatorname{div}(\mathcal{E}[\rho]_t - \mathcal{E}[\overline{\rho}]_t)\|_{L^\infty(B_R)} \leq \|\Delta_x K\|_{L^\infty(B_R\times B_R)}\|\rho_t - \overline{\rho}_t\|_{L^1(B_R)} = C_7(\|\Delta_x K\|_{L^2(B_R\times B_R)})\|\rho_t - \overline{\rho}_t\|_{L^1(B_R)}.$$

Finally, getting everything together, we have for all $t \in [0, T_0]$ that

$$\begin{aligned}
\|\mathcal{K}[\rho]_t - \mathcal{K}[\overline{\rho}]_t\|_{L^1(B_R)} &\leq C_5 \left(T_0^{1/2} \sup_{s\in(0,T_0}) \|\mathcal{E}[\rho]_s - \mathcal{E}[\overline{\rho}]_s\|_{L^2(B_R)} + T_0 \sup_{s\in(0,T_0)} \|\operatorname{div}(\mathcal{E}[\rho]_s - \mathcal{E}[\overline{\rho}]_s)\|_{L^\infty(B_R)}\right)\\
&\leq C_5(C_6 + C_7)(T_0^{1/2} + T_0) \sup_{s\in(0,T_0)} \|\rho_s - \overline{\rho}_s\|_{L^1(B_R)}.
\end{aligned} \tag{2.40}$$

The constant $C(T) := C_5(C_6 + C_7)$ is bounded and independent of T_0. Taking T_0 small enough so that $C(T)(T_0^{1/2} + T_0) < 1$ and applying Banach fixed point Theorem we get existence and uniqueness of solutions for the problem (2.5) in the time interval $[0, T_0]$.

Step 3: Extension to time T. We repeat the argument from Step 2 choosing ρ_{T_0} as the initial data. Due to the non-flux boundary condition, we have that $\|\rho_{T_0}\|_{L^1(B_R)} = \|\rho_0\|_{L^1(B_R)}$. Furthermore, we bound uniformly $\|G_\Phi(\rho_{T_0})\|_{L^1(B_R)}$. From the definition (2.27) of G_Φ and the L^p estimate (2.23), we have that for all $t \in [0,T]$

$$\int_{B_R} G_\Phi(\rho_t) = \frac{\|\rho_t\|^{3-m}_{L^{3-m}(B_R)}}{(3-m)(2-m)m} \leq \frac{\exp\big((2-m)\|\operatorname{div}\mathcal{E}[\rho]_t\|_{L^\infty(B_R)}T\big)}{(3-m)(2-m)m}\|\rho_0\|^{3-m}_{L^{3-m}(B_R)}$$
$$\leq \frac{\exp\left((2-m)C_4T\right)}{(3-m)(2-m)m}\|\rho_0\|^{3-m}_{L^{3-m}(B_R)} = C_8(\|\nabla V\|_{L^\infty(B_R)}, \|\nabla_x K\|_{L^\infty(B_R\times B_R)}, \|\rho_0\|_{L^{3-m}(B_R)}).$$

Hence, if we consider the constant,

$$C_9 = \|\rho_0\|_{L^1(B_R)} + \left(C_8 + \frac{T}{2}C_3^2\exp\left(2(2-m)C_4\right)\right)^{1/2},$$

we get the estimate,

$$\|\mathcal{K}[\rho]_t - \mathcal{K}[\overline{\rho}]_t\|_{L^1(B_R)} \leq C_9(C_6+C_7)(T_0^{1/2}+T_0)\sup_{s\in(T_0,2T_0)}\|\rho_s - \overline{\rho}_s\|_{L^1(B_R)},$$

for all $t \in [T_0, 2T_0]$. The constant $\overline{C}(T) := C_9(C_6+C_7)$ is independent of any time step. If we choose the time step T_0 even smaller than in Step 2, satisfying $\overline{C}(T)(T_0^{1/2}+T_0) < 1$, we can apply Banach fixed point Theorem again to obtain existence and uniqueness in the time interval $[T_0, 2T_0]$. We keep repeating successively this argument until we reach the value T since now the time step T_0 only depends on ρ_0. Since this construction works for an arbitrary time T, by the uniqueness of the solution, we have existence and uniqueness for every time. □

2.4 Long time asymptotic behaviour

In this Section we want to study the asymptotic behaviour of the solutions of the problem (2.5) when $t \to \infty$. In order to be able do that, we will need some *a priori* estimates that come from the dissipation of the free energy. We obtain them in subsection 2.4.1. After that, in subsection 2.4.2 we study the asymptotic behaviour of the solution using these estimates.

2.4.1 Free energy and its dissipation

We can derive a variational interpretation of the problem (2.5) that leads to additional *a priori* estimates using its gradient flow structure. The equation (2.5) can be rewritten as

$$\frac{\partial\rho}{\partial t} = \operatorname{div}\left(\rho\nabla\left\{U'(\rho) + V + \int_{B_R} K(\cdot,y)\rho(y)\,\mathrm{d}y\right\}\right), \tag{2.41}$$

where, $U(\rho) = \frac{\rho^m}{m-1}$. Formulation (2.41) shows that the equation (2.5) is formally the gradient flow of the free energy

$$\mathcal{F}_R[\mu] := \int_{B_R} U(\mu(x))\,\mathrm{d}x + \int_{B_R} V(x)\,\mathrm{d}\mu(x) + \frac{1}{2}\int_{B_R}\int_{B_R} K(x,y)\,\mathrm{d}\mu(y)\mathrm{d}\mu(x).$$

Thanks to the free energy we can get some useful *a priori* estimates. We can prove Proposition 2.3.

Proof of Proposition 2.3. First we point that $U''(s) = \frac{\Phi'(s)}{s}$ so this is a smooth function for $s > 0$, and $U(s) = \frac{s^m}{m-1}$. We can pick for $\varepsilon \in (0,1)$ the approximation given by

$$U''_\varepsilon(s) = \begin{cases} U''(\varepsilon) & s \leq \varepsilon, \\ U''(s) & s > \varepsilon, \end{cases} \qquad U'_\varepsilon(\varepsilon) = U'(\varepsilon), \qquad U_\varepsilon(\varepsilon) = U(\varepsilon).$$

Notice $U_\varepsilon \in C^2([0,\infty))$, $U_\varepsilon(s) = U(s)$ for $s \geq \varepsilon$, $0 \leq U_\varepsilon''(s) \leq U''(s)$ for $s > 0$, and $U_\varepsilon \to U$ uniformly over compacts of $[0,\infty)$. We define the free energy,

$$\mathcal{F}_{R,\varepsilon}[\rho] = \int_{B_R} U_\varepsilon(\rho(x))\,\mathrm{d}x + \int_{B_R} V(x)\rho(x)\,\mathrm{d}x + \frac{1}{2}\int_{B_R}\int_{B_R} K(x,y)\rho(y)\rho(x)\,\mathrm{d}y\,\mathrm{d}x.$$

Because $\frac{\partial\rho}{\partial t} \in L^2((0,T)\times B_R)$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_{R,\varepsilon}[\rho_t] = -\int_{B_R} \rho_t \left[U''(\rho_t)\nabla\rho_t + E\right]\cdot\left[U_\varepsilon''(\rho_t)\nabla\rho_t + E\right], \qquad \text{for a.e. } t\in(0,T)$$

and

$$\int_{t_1}^{t_2}\int_{B_R} \rho\left[U''(\rho)\nabla\rho + E\right]\cdot\left[U_\varepsilon''(\rho)\nabla\rho + E\right] = \mathcal{F}_{R,\varepsilon}[\rho_{t_1}] - \mathcal{F}_{R,\varepsilon}[\rho_{t_2}], \tag{2.42}$$

where E denotes $\nabla V + \nabla\int_{B_R} K(\cdot,y)\rho(y)\,dy$. By construction we know $U_\varepsilon \to U$ uniformly over compact sets and hence $\mathcal{F}_{R,\varepsilon}[\rho_t] \to \mathcal{F}_R[\rho_t]$ for every $t \geq 0$. We can expand (2.42) as

$$\begin{aligned}\int_{t_1}^{t_2}\int_{B_R} \rho U''(\rho)U_\varepsilon''(\rho)|\nabla\rho|^2 &= \mathcal{F}_{R,\varepsilon}[\rho_{t_1}] - \mathcal{F}_{R,\varepsilon}[\rho_{t_2}] \\ &\quad - \int_{t_1}^{t_2}\int_{B_R}\nabla\Phi(\rho)\cdot E - \int_{t_1}^{t_2}\int_{B_R}\rho U_\varepsilon''(\rho)\nabla\rho\cdot E - \int_{t_1}^{t_2}\int_{B_R}\rho|E|^2.\end{aligned} \tag{2.43}$$

Notice that $\nabla\Phi(\rho) \in L^2$ and this guarantees that the third-to-last term of the RHS is finite. For the second-to-last term we do

$$\left|\rho U_\varepsilon''(\rho)\nabla\rho\cdot E\right| \leq \rho U_\varepsilon''(\rho)|\nabla\rho||E| \leq \rho U''(\rho)|\nabla\rho||E| = |\nabla\Phi(\rho)||E|.$$

The last term is bounded due to the L^p estimate (2.23). Now, since $\rho, U_\varepsilon'', U'' \geq 0$ we can use Fatou's lemma to deduce from (2.43)

$$\int_{t_1}^{t_2}\int_{B_R} \rho U''(\rho)^2|\nabla\rho|^2 < \infty. \tag{2.44}$$

With this estimate, we control the RHS of (2.43) by

$$\rho U''(\rho)U_\varepsilon''(\rho)|\nabla\rho|^2 \leq \rho U''(\rho)U''(\rho)|\nabla\rho|^2$$

With the controls already proved, we can apply the Dominated Convergence Theorem in all terms of (2.43), and the proof is complete. □

Corollary 2.27. *Assume the basic hypotheses* (H0) *on the initial data* ρ_0 *and the potentials* V *and* K*, then* $\mathcal{F}_R[\rho_t]$ *is non-increasing, bounded below, and therefore has a finite limit as* $t \to \infty$.

Proof. Proposition 2.3 gives us that $\mathcal{F}_R[\rho_t]$ is non-increasing along solutions of (2.5). Furthermore, we can prove that $\mathcal{F}_R[\rho_t]$ is bounded below for solutions of (2.5) because

$$\mathcal{F}_R[\rho] \geq -\frac{1}{1-m}\|\rho\|_{L^1(B_R)}|B_R|^{1-m} - \|V_-\|_{L^\infty(B_R)}\|\rho\|_{L^1(B_R)} - \frac{1}{2}\|K_-\|_{L^\infty(B_R\times B_R)}\|\rho\|^2_{L^1(B_R)}, \tag{2.45}$$

so $\mathcal{F}_R[\rho_t]$ is uniformly bounded in time and it has a finite limit as $t \to \infty$. □

We are also able to deduce using these energy estimates an L^1 bound of $\nabla\rho^m$. We will do it on a bounded subdomain Ω to profit from it when $R \to \infty$.

Corollary 2.28. *Assume the basic hypotheses* (H0) *on the initial data* ρ_0 *and the potentials* V *and* K*, then we have that*

$$\int_{t_1}^{t_2}\int_\Omega |\nabla\rho_t^m| \leq \|\rho_0\|_{L^1(B_R)}\left(\mathcal{F}_R[\rho_{t_1}] - \mathcal{F}_R[\rho_{t_2}] + \int_{t_1}^{t_2}\int_\Omega \rho_t\left|\nabla V + \nabla\int_{B_R} K(\cdot,y)\rho_t(y)\,\mathrm{d}y\right|^2\right)^{\frac{1}{2}}, \forall\Omega \subseteq \overline{B_R}.$$

Proof. We have that

$$\int_{t_1}^{t_2}\int_\Omega |\nabla \rho_t^m| = \int_{t_1}^{t_2}\int_\Omega \rho \left|\frac{m}{m-1}\nabla \rho^{m-1}\right| \leq \int_{t_1}^{t_2} \|\rho_t\|_{L^1(B_R)} \left(\int_\Omega \rho_t \left|\nabla \frac{m}{m-1}\rho^{m-1}\right|^2\right)^{\frac{1}{2}} \mathrm{d}t. \tag{2.46}$$

Hence, we can conclude the result using Proposition 2.3, Jensen's inequalities and the conservation of the L^1 norm. □

2.4.2 Asymptotics in time. Proof of Theorem 2.4

Proof of Theorem 2.4. Due to the Sobolev embedding, for $p < \infty$ large enough $W_0^{1,p}(\Omega) \subset C(\overline{\Omega})$. Therefore, by standard duality arguments

$$\mathcal{M}(B_R) \subset (C(\overline{\Omega}))' \subset (W_0^{1,p}(\Omega))' = W^{-1,p'}(\Omega) \subset W^{-1,1}(\Omega).$$

The key reason to use this space is the following *a priori* estimate

$$\begin{aligned}
\int_0^1 \left\| \frac{\partial \rho_t^{[n]}}{\partial t}\right\|^2_{W^{-1,1}(B_R)} &= \int_0^1 \left\| \operatorname{div}\left(\rho_t^{[n]} \nabla \left(\frac{m}{m-1}\left(\rho_t^{[n]}\right)^{m-1} + V + \int_{B_R} K(\cdot,y)\rho_t^{[n]}(y)\,\mathrm{d}y\right)\right)\right\|^2_{W^{-1,1}(B_R)} \\
&\leq \int_0^1 \left\| \rho_t^{[n]} \nabla \left(\frac{m}{m-1}\left(\rho_t^{[n]}\right)^{m-1} + V + \int_{B_R} K(\cdot,y)\rho_t^{[n]}(y)\,\mathrm{d}y\right)\right\|^2_{L^1(B_R)} \\
&\leq \int_{t_n}^{t_n+1} \left\|\rho_t^{\frac{1}{2}}\right\|^2_{L^2(B_R)} \left\| \rho_t^{\frac{1}{2}} \nabla \left(\frac{m}{m-1}\rho_t^{m-1} + V + \int_{B_R} K(\cdot,y)\rho_t(y)\,\mathrm{d}y\right)\right\|^2_{L^2(B_R)} \\
&\leq \|\rho_0\|_{L^1(B_R)} \left(\mathcal{F}_R[\rho_{t_n}] - \mathcal{F}_R[\rho_{t_n+1}]\right),
\end{aligned}$$

which involves only the energy. Hence, it follows that

$$\|\rho_t^{[n]} - \rho_s^{[n]}\|_{W^{-1,1}(B_R)} \leq \int_s^t \left\|\frac{\partial \rho_\sigma^{[n]}}{\partial \sigma}\right\|_{W^{-1,1}(B_R)} \mathrm{d}\sigma \leq \|\rho_0\|^{\frac{1}{2}}_{L^1(B_R)} \left(\mathcal{F}_R[\rho_{t_n}] - \mathcal{F}_R[\rho_{t_n+1}]\right)^{\frac{1}{2}} |t-s|^{\frac{1}{2}}. \tag{2.47}$$

Since $\rho_0 \in L^\infty$, $\mathcal{F}_R[\rho_0] < \infty$. From Corollary 2.27 the free energy is uniformly bounded. The free energy is also decreasing in time along solutions of (2.5) from Proposition 2.3. Therefore, $\mathcal{F}_R[\rho_{t_n}] - \mathcal{F}_R[\rho_{t_n+1}]$ is uniformly bounded by a constant C. By properties of the Bochner integral we can take advantage of (2.47). In particular, this last result implies that the sequence $\rho^{[n]}$ is equicontinuous. Next, we realised that $\|\rho_t^{[n]}\|_{L^1(B_R)} = \|\rho_0\|_{L^1(B_R)}$, and, in particular, the sequence $\rho_t^{[n]}$ is uniformly bounded on $L^1(B_R)$. Furthermore, because $L^1(B_R)$ is compactly embedded in $W^{-1,1}(B_R)$ we know by Ascoli-Arzela Theorem that, up to a subsequence, we have the convergence

$$\rho^{[n_k]} \to \widehat{\mu} \quad \text{in } C([0,1]; W^{-1,1}(B_R)).$$

For any $t, s \in [0,1]$, using (2.47) we also have that the distribution $\widehat{\mu}_t$ does not depend on time because

$$\begin{aligned}
\|\widehat{\mu}_t - \widehat{\mu}_s\|_{W^{-1,1}(B_R)} &= \lim_{k\to\infty} \|\rho_t^{[n_k]} - \rho_s^{[n_k]}\|_{W^{-1,1}(B_R)} \\
&\leq \|\rho_0\|^{\frac{1}{2}}_{L^1(B_R)} |t-s|^{\frac{1}{2}} \lim_{k\to\infty}\left(\mathcal{F}_R[\rho_{t_{n_k}}] - \mathcal{F}_R[\rho_{t_{n_k}+1}]\right)^{\frac{1}{2}} = 0,
\end{aligned}$$

where the last step is again due to Corollary 2.27. □

Notice that in the setting of strong solutions we cannot characterise $\widehat{\mu}$ as a stationary solution of the problem (2.5) because the $W^{-1,1}$ convergence is not sufficient to pass to the limit under the non-linearity of the fast diffusion term. Therefore, we aim to characterise this limit in the mass equation.

2.5 An equation for the mass

The aim of this section is to develop the well-posedness theory for the mass equation (2.49) in order to characterise the stationary state. We will show that the natural notion of solution in this setting is the notion of viscosity solution.

In order to study the asymptotic behaviour we will rely on the construction of $\widehat{\mu}$ in subsection 2.4.2 as a limit of the densities $\rho^{[n]}$ defined in (2.10) and on the results from DiBenedetto [159, Chapter III].

2.5.1 Mass equation for the regularised problem

Assuming further to the basic hypotheses (H0) on the initial data ρ_0 and the potentials V and K and that they are radially symmetric, it suffices to study the mass variable

$$M(t,v) = \int_{\widetilde{B_v}} \rho_t(x)\,\mathrm{d}x. \tag{2.48}$$

We will denote by r the radial variable $r = |x|$, by $v = |B_1| r^d$, the volumetric variable and by $\widetilde{B_v}$, the ball centred in the origin and with radius $r = (v|B_1|^{-1})^{1/d}$. It was shown in [96] that M satisfies a Hamilton-Jacobi type equation which is better written in the volume variable v as

$$\frac{\partial M}{\partial t} = \kappa(v)^2 \frac{\partial}{\partial v}\Phi\left(\frac{\partial M}{\partial v}\right) + \kappa(v)^2 \frac{\partial M}{\partial v}\mathfrak{E}[\rho], \quad \kappa(v) = d\omega_d^{\frac{1}{d}} v^{\frac{d-1}{d}}, \quad \text{in } (0,\infty)\times(0,R_v), \tag{2.49}$$

where $R_v = |B_R|$, and $\mathfrak{E}$ is an operator written in volumetric coordinates that substitutes the previous operator E that we have in earlier sections. We also recall that $\Phi(s) = s^m$. We point out that $\rho \geq 0$, and for that reason we do not need to take the signed power. It makes sense to write this operator in volumetric coordinates because if η and μ are radially symmetric, so it is $(\eta\mu)$, and, since V and W are also radially symmetric we also have radial symmetry for $V + \int_{B_R} K(\cdot,y)\mu(y)\,dy$ (the problem in which we are the most interested). We are denoting by $\mathcal{M}_{\mathrm{rad}}(\overline{B_R})$ the measures invariant by rotation with respect to the origin, and similarly for $L^p_{\mathrm{rad}}(B_R)$ and $W^{-1,1}_{\mathrm{rad}}(B_R)$. Therefore, we can define $\mathfrak{E}$ like:

$$\begin{aligned} \mathfrak{E} : \mathcal{M}_{\mathrm{rad}}(\overline{B_R}) &\longrightarrow C([0,R_v]) \\ \mu &\longmapsto \frac{\partial V}{\partial v} + \frac{\partial}{\partial v}\left(\int_{B_R} K(\cdot,y)\,\mathrm{d}\mu(y)\right), \end{aligned} \tag{2.50}$$

which is such that $\mathfrak{E}[\mu]$ is the volumetric derivative of $\Upsilon[\mu]$, where Υ is the operator:

$$\begin{aligned} \Upsilon : \mathcal{M}_{\mathrm{rad}}(\overline{B_R}) &\longrightarrow C([0,R_v]) \\ \mu &\longmapsto V + \int_{B_R} K(\cdot,y)\,\mathrm{d}\mu(y). \end{aligned} \tag{2.51}$$

We now need to show that we can extend $\mathfrak{E}$ in a suitable way such that

$$\mathfrak{E} : W^{-1,1}_{\mathrm{rad}}(B_R) \longrightarrow C([0,R_v])$$

is continuous. Here $W^{-1,1}_{\mathrm{rad}}(B_R)$ is the closure of $L^1_{\mathrm{rad}}(B_R)$ in $W^{-1,1}(B_R)$.

In the following results we first point out that the change to volumetric variable is precisely mapping L^p to L^p with no weights and after that we explain how to extend the map $\mathfrak{E}$ from $\mathcal{M}_{\mathrm{rad}}(\overline{B_R})$ to $W^{-1,1}_{\mathrm{rad}}(B_R)$.

Lemma 2.29. *The change of variable map*

$$\begin{aligned} \mathcal{I} : L^p_{\mathrm{rad}}(B_R) &\longrightarrow L^p(0,R_v), && \text{where } R_v = |B_1|R^d \\ f &\longmapsto \mathcal{I}[f] := g, && \text{where } g(w) = f((w/|B_1|)^{\frac{1}{d}} e_1). \end{aligned}$$

is an L^p isometry. The L^1 isometry is extended uniquely to an isometry in the total variation distance

$$\mathcal{I} : \mathcal{M}_{\mathrm{rad}}\left(\overline{B_R}\right) \to \mathcal{M}([0,R_v]), \qquad \text{where } R_v = |B_1|R^d.$$

Remark 2.30. *Let Σ be the Borel σ-algebra in $\Omega \subseteq B_R$. The standard form to define the total variation norm (i.e., $\|\nu\|_{TV} = |\nu|(\Omega)$) is equivalent to the definition*

$$|||\nu|||_{TV} := \sup_{A\in\Sigma} |\nu(A)|.$$

This is a consequence of the Hahn's Decomposition Theorem. In order to prove Lemma 2.29 we use this second notion.

Proof of Lemma 2.29. We divide the proof in two steps.

Step 1: Extension. The map $\mathcal{I}$ is defined such that for every $\varphi \in C([0, R_v])$,

$$\int_{B_R} \varphi(\omega_d |x|^d)\, \mathrm{d}\mu(x) = \int_0^{R_v} \varphi(\sigma)\, \mathrm{d}\mathcal{I}[\mu](\sigma).$$

In particular, $\mathcal{I}[\mu]((a,b)) = \mu(\widetilde{B_b}\backslash\overline{\widetilde{B_a}})$ (and analogously for the intervals which are close), which implies surjectivity of the mapping $\mathcal{I}$ from $\mathcal{M}_{\mathrm{rad}}(\overline{B_R})$ to $\mathcal{M}([0, R_v])$. Furthermore, given $\mu \in \mathcal{M}(\overline{B_R})$, the following statement holds

$$\int_{B_R} \varphi\, \mathrm{d}\mu = 0 \quad \text{for all } \varphi \in C(\overline{B_R}) \qquad \Longleftrightarrow \qquad \mu \equiv 0.$$

If μ is radially symmetric, then we can reduce this set to radial test functions $x \mapsto \varphi(\omega_d|x|^d)$ with $\varphi \in C([0, R_v])$. From this identity we can easily check that $\mathcal{I}$ is injective by showing its kernel is trivial. If $\mathcal{I}[\mu] = 0$ then

$$\int_{B_R} \varphi(\omega_d|x|^d)\, \mathrm{d}\mu(x) = \int_0^{R_v} \varphi(\sigma)\, \mathrm{d}\mathcal{I}[\mu](\sigma) = 0 \quad \text{for all } \varphi \in C([0, R_v]),$$

so $\mu = 0$.

Step 2: Isometry in the total variation distance. We denote by Σ_{rad} the σ-algebra generated by open annuli, and by Σ the σ-algebra generated by open intervals. Let us take $\mu_1, \mu_2 \in \mathcal{M}_{\mathrm{rad}}(\overline{B_R})$. The map $\mathcal{I}$ is such that

$$|\mu_1(\widetilde{B_b}\backslash\overline{\widetilde{B_a}}) - \mu_2(\widetilde{B_b}\backslash\overline{\widetilde{B_a}})| = |\mathcal{I}[\mu_1]((a,b)) - \mathcal{I}[\mu_2]((a,b))|.$$

Therefore, we get that

$$|||\mu_1 - \mu_2|||_{TV} = \sup_{A\in\Sigma_{\mathrm{rad}}} |\mu_1(A) - \mu_2(A)| = \sup_{B\in\Sigma} |\mathcal{I}[\mu_1](B) - \mathcal{I}[\mu_2](B)| = |||\mathcal{I}[\mu_1] - \mathcal{I}[\mu_2]|||_{TV}.$$

This completes the proof. □

We now prove a lemma of differentiation in volumetric variables.

Lemma 2.31. *Assume the basic hypotheses (H0), and that the initial datum ρ_0 and potentials V and K are radially symmetric. Then, the map*

$$\mathfrak{E}[\mu](v) = \frac{v^{\frac{d-1}{d}}}{d|B_1|^{\frac{1}{d}}} e_1 \cdot \left((\nabla V)(re_1) + \left\langle (\nabla_x K)(re_1, \cdot), \mu \right\rangle_{W_0^{1,\infty}(B_R)\times W^{-1,1}(B_R)} \right),$$

where $r = (v|B_1|^{-1})^{1/d}$, extends (2.50) and it is continuous

$$\mathfrak{E} : W^{-1,1}_{\mathrm{rad}}(B_R) \to C([0, R_v]).$$

Proof. We have that $V(re_1) + \int_{B_R} K(re_1, y)\mu(y)\,dy$ is radially symmetric. We can compute its derivative as

$$\begin{aligned}\mathfrak{E}[\mu] &= \left(\frac{v}{|B_1|}\right)^{\frac{1}{d}} e_1 \cdot \left((\nabla V)\left(\left(\frac{v}{|B_1|}\right)^{\frac{1}{d}} e_1\right) + \left\langle (\nabla_x K)\,(re_1,\cdot)\,,\mu\right\rangle_{W_0^{1,\infty}(B_R)\times W^{-1,1}(B_R)}\right)\frac{1}{vd} \\ &= \frac{v^{\frac{d-1}{d}}}{d|B_1|^{\frac{1}{d}}} e_1 \cdot \left((\nabla V)(re_1) + \left\langle (\nabla_x K)\,(re_1,\cdot)\,,\mu\right\rangle_{W_0^{1,\infty}(B_R)\times W^{-1,1}(B_R)}\right)\end{aligned}$$

where $r = (v|B_1|^{-1})^{1/d}$. The details on the change of variables can be found, for example, in [23, 24]. Finally, since $\nabla_x K \in W_0^{1,\infty}(B_R \times B_R)$ and $\mathfrak{E}[\mu]$ is affine on μ the continuity follows. □

Once we have introduced the notation we recall some *a priori* estimates for the solutions of equation (2.49) using Lemma 2.18. Their proofs can be found in [96].

Lemma 2.32. *If $\rho_t \in L^q(B_R)$ for some $q \in [1,\infty)$ then*

$$[M(t,\cdot)]_{C^{\frac{q-1}{q}}([0,R_v])} \le \|\rho_t\|_{L^q(B_R)}. \tag{2.52}$$

If $q = \infty$ the same holds in $W^{1,\infty}(0,R_v)$.

Lemma 2.33. *There exists a constant $C > 0$, independent of ρ or Φ, such that*

$$\int_0^T \int_0^{R_v} \left|\frac{\partial M}{\partial t}\right|^2 \mathrm{d}v\,\mathrm{d}t \le C\left(\int_{B_R} \Psi(\rho_0) + \|E\|^2_{L^\infty((0,T)\times B_R)} \int_0^T \int_{B_R} \rho_t(x)^2\,\mathrm{d}x\,\mathrm{d}t\right). \tag{2.53}$$

In particular, if $\rho_0 \in L^2(B_R)$ and $\Psi(\rho_0) \in L^1(B_R)$ then $M \in C^{\frac{1}{2}}(0,T;L^1(0,R_v))$.

Similarly to the argument in [96, Appendix A], we can adapt DiBenedetto's theory [159] so that we get the following a priori estimate

Theorem 2.34. *Assume the basic hypotheses (H0), and that the initial datum ρ_0 and potentials V and K are radially symmetric. Let ρ be the strong solution of (2.5) and M its mass. Then, for every $\varepsilon \in (0,R_v)$ we have the following interior regularity estimate: for any $T_1 > 0$ and any $\varepsilon \in (0,R_v)$ there exists $\gamma > 0$ and $\alpha \in (0,1)$ depending only on d, m, $\|\rho_0\|_{L^\infty(B_R)}, \|\nabla V\|_{L^\infty(B_R)}, \|\nabla_x K\|_{L^\infty(B_R\times B_R)}, \varepsilon, T_1$, such that*

$$|M(t_1,v_1) - M(t_2,v_2)| \le \gamma \left(\frac{|v_1 - v_2| + \|\rho_0\|_{L^1(B_R)}^{\frac{m-1}{m+1}} |t_1 - t_2|^{\frac{1}{m+1}}}{\varepsilon + \|\rho_0\|_{L^1(B_R)}^{\frac{m-1}{m+1}} T_1^{\frac{1}{m+1}}}\right)^\alpha \tag{2.54}$$

for all $(t_i,v_i) \in [T_1,+\infty) \times [\varepsilon,R_v]$.

2.5.2 Viscosity solutions

We start by providing a self-contained definition of (2.49). Since we cannot guarantee that $\partial M/\partial v$ is positive, it is better to simplify (2.49) expanding the second derivative and dividing by $\Phi'(\partial M/\partial v)$. Let us state what is a viscosity solution for our problem (2.49).

Definition 2.35 (Viscosity solution)**.** *A function $M \in C([0,T];C((0,R_V)) \cap BV([0,R_v]))$ is a viscosity supersolution of (2.49) if, for every $t_0 > 0$, $v_0 \in (0,R_v)$ and for every $\varphi \in C^2((t_0-\varepsilon,t_0+\varepsilon)\times(v_0-\varepsilon,v_0+\varepsilon))$ such that $M \ge \varphi$, $M(v_0) = \varphi(v_0)$ and $\frac{\partial\varphi}{\partial v}(v) \ne 0$ for all $v \ne v_0$ it holds that*

$$\frac{1}{\Phi'\left(\frac{\partial\varphi}{\partial v}(t_0,v_0)\right)}\frac{\partial\varphi}{\partial t}(t_0,v_0) \ge \kappa(v_0)^2\frac{\partial^2\varphi}{\partial v^2}(t_0,v_0) + \kappa(v_0)^2\frac{\frac{\partial\varphi}{\partial v}(t_0,v_0)}{\Phi'\left(\frac{\partial\varphi}{\partial v}(t_0,v_0)\right)}\mathfrak{E}\left[\mathcal{I}^{-1}\left[\frac{\partial M}{\partial v}\right]\right](t_0,v_0).$$

The corresponding definition of subsolution is made by inverting the inequalities. A viscosity solution is a function that is a viscosity sub and supersolution.

Remark 2.36. *Notice that, in the viscosity formulation, we replace $\partial M/\partial v$ by the test function in the non-linear term, but we preserve it in the non-local one.*

In the following we prove that M is a viscosity solution for the problem (2.49).

Theorem 2.37. *Assume the basic hypotheses (H0), and that the initial datum ρ_0 and potentials V and K are radially symmetric. Let ρ be the strong solution of (2.5) and take $0 < T < \infty$. Then, M defined in (2.48) is a viscosity solution of (2.49) for $\Phi(s) = s^m$, $0 < m < 1$, and the drift $\mathfrak{E}\left[\mathcal{I}^{-1}\left[\frac{\partial M}{\partial v}\right]\right]$.*

Proof. We proceed analogously to the constructive argument used for the existence of strong solutions to (2.5).

Step 1: Uniformly elliptic diffusion with fixed time-dependent drift. Given $E \in W^{1,\infty}([0,T]\times B_R)$ a fixed time-dependent drift. We take a sequence of smooth drifts $E^{(k)}$ such that $E^{(k)} \to E$ in $W^{1,\infty}([0,T] \times B_R)$ as in Theorem 2.26. Let us take $\rho^{(k)}$ the strong solution of (2.34) with the drift $E^{(k)}$. From the regularity theory we know that $\rho^{(k)} \in C^1\left((0,T); C(\overline{B_R})\right) \cap C\left((0,\infty); C^2(\overline{B_R})\right) \cap C\left([0,\infty) \times \overline{B_R}\right)$. In particular,

$$M^{(k)}(t,v) = \int_{\widetilde{B_v}} \rho_t^{(k)}(x)\,\mathrm{d}x$$

is a classical solution of (2.49) and, hence, a viscosity solution thanks to the regularity of $\rho^{(k)}$.

Step 2: Fast diffusion with time-dependent drift. If we take the sequence Φ_k as in (2.33), we have by Theorem 2.26 that

$$\rho^{(k)} \to \rho \quad \text{in}\, C([0,T]; L^1(B_R)) \tag{2.55}$$

with ρ the unique strong solution of (2.31) for a fixed potential E. We define

$$M(t,v) = \int_{\widetilde{B_v}} \rho_t(x)\,\mathrm{d}x.$$

Then, we have

$$M^{(k)} \to M \quad \text{in}\, C([0,T] \times [0,R_v]). \tag{2.56}$$

Indeed, it follows from (2.55) combined with

$$|M^{(k)}(t,v) - M(t,v)| \le \|\rho_t^{(k)} - \rho_t\|_{L^1(\widetilde{B_v})} \le \|\rho_t^{(k)} - \rho_t\|_{L^1(B_R)}, \quad \text{for all } 0 < t \le T,$$

which implies,

$$\|M^{(k)} - M\|_{C([0,T]\times[0,R_v])} \le \|\rho_t^{(k)} - \rho_t\|_{L^\infty(0,T;L^1(B_R))}.$$

Next, we use the stability result Theorem 1.8. The sequence $M^{(k)}$ satisfy the assumptions of the Theorem since we have (2.56) and $1/\Phi_k' \to 1/\Phi'$ locally uniformly in $[0,Q]$ for every $Q > 0$, which is easy to check using (2.33). Therefore, due to the stability of viscosity solutions, M is also a viscosity solution of (2.49) for $\Phi(s) = s^m$, $0 < m < 1$, and the drift $E \in W^{1,\infty}([0,T] \times B_R)$.

Step 3: Fast diffusion with confinement and aggregation. Let ρ be the strong solution of the problem (2.5). From Theorem 2.2, ρ is the unique fixed point of the mapping (2.39). Then, there exists a sequence $\rho^{\{k\}}$ by Banach fixed-point iteration such that

$$\rho^{\{k\}} \to \rho \quad \text{strongly in}\, C([0,T]; L^1(B_R)). \tag{2.57}$$

Define

$$M(t,v) = \int_{\widetilde{B_v}} \rho_t(x)\,dx.$$

Similarly as in Step 2, one can show

$$M^{\{k\}} \to M \quad \text{in}\, C([0,T] \times [0,R_v]). \tag{2.58}$$

Since $\mathcal{I}[\rho_t^{\{k\}}] = \frac{\partial M^{\{k\}}(t,\cdot)}{\partial v}$ and $\mathcal{I}$ is an isometry from $L^1_{\mathrm{rad}}(B_R)$ to $L^1(0,R_v)$, considering (2.57) and (2.58) we also get,

$$\frac{\partial M^{\{k\}}}{\partial v} \to \frac{\partial M}{\partial v} \quad \text{in}\, C([0,T];L^1(0,R_v)),$$

and that $\mathcal{I}[\rho_t] = \frac{\partial M(t,\cdot)}{\partial v}$. Taking into account (2.57) and Lemma 2.31, one obtains

$$\mathfrak{E}[\rho_t^{\{k\}}] \to \mathfrak{E}[\rho] \quad \text{in}\, C([0,T]\times[0,R_v]). \tag{2.59}$$

Once more, from (2.58) and (2.59), combined with the stability result Theorem 1.8 we have that M is a viscosity solution of the problem (2.49) for $\Phi(s) = s^m$, $0<m<1$, and the drift $\mathfrak{E}\left[\mathcal{I}^{-1}\left[\frac{\partial M}{\partial v}\right]\right]$. □

Remark 2.38. *In the following, every time we refer to M as a viscosity solution of (2.49) we will assume that we are taking $\Phi(s) = s^m$, $0<m<1$, and the drift $\mathfrak{E}\left[\mathcal{I}^{-1}\left[\frac{\partial M}{\partial v}\right]\right]$ unless it is stated otherwise.*

2.5.3 Convergence of the mass solution as $t\to\infty$. Proof of Theorem 2.6

We already know that the asymptotic in time of $\rho^{[n]}$ is a distribution $\widehat{\mu} \in W^{-1,1}(B_R)$. In this subsection, we intend to link this distribution with the asymptotic of $M^{[n]}$, defined in (2.13), as $n\to\infty$. In fact, we can study the limit $M^{[n]} \to \widehat{M}$ and discuss its relationship with the limit $\widehat{\mu}$ obtained in Theorem 2.4.

We will start by proving that $M^{[n]} \to \widehat{M}$ converges point-wise in $[0,1]\times[0,R_v]$ uniformly over compact subsets on $[0,1]\times(0,R_v]$. We also identify $\frac{\partial \widehat{M}}{\partial v}$ with $\widehat{\mu}$.

Remark 2.39. *We identify $\frac{\partial \widehat{M}}{\partial v}$ with $\widehat{\mu}$ in the sense of the isometry $\mathcal{I}$ that we discussed in Lemma 2.29. In this way, we prove that $\frac{\partial \widehat{M}}{\partial v} = \mathcal{I}[\widehat{\mu}]$ in $L^\infty(0,1;\mathcal{M}([0,R_v]))$.*

We are now able to prove Theorem 2.6. We divide the proof in several steps.

Step 1: Ascoli-Arzela over compacts of $[0,1]\times(0,R_v]$. From (2.54) we know that for all $v_1,v_2 \in [\frac{1}{k},R_v]$ and $t_1,t_2\in[T,\infty)$, we have the estimate,

$$|M(t_1,v_1) - M(t_2,v_2)| \le C_k\left(|v_1-v_2| + |t_1-t_2|^{\frac{1}{m+1}}\right)^\alpha.$$

In particular, for n large enough the sequence $M^{[n]}$ is such that for all $v_1,v_2\in[\frac{1}{k},R_v]$ and $t_1,t_2\in[0,1]$

$$\left|M^{[n]}(t_1,v_1) - M^{[n]}(t_2,v_2)\right| \le C_k\left(|v_1-v_2| + |t_1-t_2|^{\frac{1}{m+1}}\right)^\alpha, \tag{2.60}$$

where C_k does not depend on n. Taking successive subsequences $n(k,j)$ we get that for every $k\in\mathbb{Z}_{>0}$

$$M^{[n(k,j)]} \to \widehat{M} \quad \text{in}\, C\left([0,1]\times\left[\tfrac{1}{k},R_v\right]\right) \text{ as } j\to\infty.$$

The diagonal satisfies

$$M^{[n(k,k)]} \to \widehat{M} \quad \text{in}\, C_{loc}([0,1]\times(0,R_v]). \tag{2.61}$$

From here on we re-label this sequence as $M^{[n]}$.

Step 2: Extension of $\widehat{M}$ and some properties. Based on the fact that $M^{[n]}(t,0)=0$ for all n, the extension of $\widehat{M}$ so that $\widehat{M}(t,0)=0$ is natural. Notice here that the limit of $\widehat{M}(t,v)$ as $v\to0^+$ does not need to be 0. In any case, due to (2.61), we have that

$$M^{[n]}(t,v) \to \widehat{M}(t,v) \quad \text{point-wise in } [0,1]\times[0,R_v].$$

Furthermore, because $M^{[n]}$ are non-decreasing functions on v, $M^{[n]}(t,0)=0$ and $M^{[n]}(t,R_v) = \|\rho_0\|_{L^1(B_R)}$, thus we get that $M^{[n]}$ are functions with bounded variation given by $\|\rho_0\|_{L^1(B_R)}$. As a conclusion their derivatives $\frac{\partial M^{[n]}}{\partial v} \in \mathcal{M}([0,R_v])$ have total mass $\|\rho_0\|_{L^1(B_R)}$. Then, from Banach-Alaoglu Theorem and (2.61), up to a subsequence,

$$\frac{\partial M^{[n]}}{\partial v} \rightharpoonup \frac{\partial \widehat{M}}{\partial v} \quad \text{weak}-*\text{ in } L^\infty(0,1;\mathcal{M}([0,R_v])). \tag{2.62}$$

Step 3: Identification between $\frac{\partial \widehat{M}}{\partial v}$ *and* $\mathcal{I}[\widehat{\mu}]$. From Theorem 2.4, since $\rho_t^{[n]}$ are measures with total mass $\|\rho_0\|_{L^1(B_R)}$, we can use Banach-Alaouglu Theorem to deduce that, up to a subsequence

$$\rho_t^{[n]} \rightharpoonup \widehat{\mu} \quad \text{weak} - * \text{ in } L^\infty(0,1;\mathcal{M}(\overline{B_R})).$$

Since $\mathcal{I}$ is a linear isometry it follows that,

$$\mathcal{I}[\rho_t^{[n]}] \rightharpoonup \mathcal{I}[\widehat{\mu}] \quad \text{weak} - * \text{ in } L^\infty(0,1;\mathcal{M}([0,R_v])). \tag{2.63}$$

Furthermore, because we have that $\mathcal{I}[\rho_t^{[n]}] = \frac{\partial M^{[n]}}{\partial v}(t,\cdot)$, combining (2.62) and (2.63) we obtain that $\frac{\partial \widehat{M}}{\partial v} = \mathcal{I}[\widehat{\mu}]$ in $L^\infty(\mathcal{M}([0,R_v]))$ and in particular $\widehat{M}$ does not depend on time. Taking into account Theorem 2.4 and Lemma 2.31 we get that

$$\mathfrak{E}[\rho^{[n]}] \to \mathfrak{E}[\widehat{\mu}] \quad \text{in } C([0,1]\times[0,R_v]). \tag{2.64}$$

Finally, from (2.60) and (2.64), combined with the stability result Theorem 1.8 we have that $\widehat{M}$ is a viscosity solution of (2.49). □

2.5.4 Some examples where $\widehat{M}$ is regular

When the flux $\mathfrak{E}$ has non-negative sign. Proof of Proposition 2.9

We proceed similarly to the Step 4 of the proof of [96, Theorem 5.4]. We consider the notion of viscosity solution for $\widehat{M}$. At a point of contact $v_0 \in (0,R_v)$ of a viscosity test function touching from below, we deduce

$$-\frac{\partial^2 \varphi}{\partial v^2}(v_0) \geq \frac{1}{m}\left(\frac{\partial \varphi}{\partial v}(v_0)\right)^{2-m} \mathfrak{E}[\widehat{\mu}](v_0) \geq 0,$$

which implies that $\widehat{M}$ is a viscosity super-solution of $-\Delta M = 0$. Due to [211], $\widehat{M}$ is a distributional super-solution of $-\Delta M = 0$ as well. In particular, distributional super-solutions are concave which implies $\widehat{M} \in W^{1,\infty}([\overline{\varepsilon}, R_v - \overline{\varepsilon}])$ for all $\overline{\varepsilon} > 0$. $\widehat{M}$ is a supersolution of the problem

$$-\frac{\partial^2 \widehat{M}}{\partial v^2}(v_0) \geq \frac{1}{m}\left(\frac{\partial \widehat{M}}{\partial v}(v_0)\right)^{2-m} \mathfrak{E}[\widehat{\mu}](v_0) \geq 0,$$

in the distributional sense. We can think of the right-hand side as a datum

$$f := \frac{1}{m}\left(\frac{\partial \widehat{M}}{\partial v}(v_0)\right)^{2-m} \mathfrak{E}[\widehat{\mu}](v_0) \in L^\infty(\overline{\varepsilon}, R_v - \overline{\varepsilon}).$$

Using the regularisation results from [61] we get that $\widehat{M} \in C^{1,\alpha}(2\overline{\varepsilon}, R_v - 2\overline{\varepsilon})$. Furthermore, since $\mathfrak{E}[\widehat{\mu}] \in W^{1,\infty}([0,R_v])$, $f \in C^{0,\beta}(2\overline{\varepsilon}, R_v - 2\overline{\varepsilon})$ for some $\beta > 0$, which guarantees $\widehat{M} \in C^{2,\beta}(4\overline{\varepsilon}, R_v - 4\overline{\varepsilon})$. □

The regularity of $\widehat{M}$ when V, K are compactly supported

In this subsection, we will focus on discussing about the regularity of $\widehat{M}$. Here, we assume the basic hypotheses (H0), that the initial data ρ_0 and the potentials V and K are radially symmetric, and furthermore, that V and K have compact support. In order to do that, let us fix $a > 0$ and define the infimal convolution of a function M as

$$M_\varepsilon(v) := \inf_{\overline{v}\in(a,R_v)} \left(M(\overline{v}) + \frac{|v-\overline{v}|^2}{2\varepsilon}\right),$$

where $\varepsilon > 0$, and we define the supremal convolution as

$$M^\varepsilon(v) := \sup_{\overline{v}\in(a,R_v)} \left(M(\overline{v}) - \frac{|v-\overline{v}|^2}{2\varepsilon}\right).$$

It is not difficult to show that there exists $r(\varepsilon) \to 0$ such that is equivalent to:

$$M_\varepsilon(v) = \inf_{\bar{v} \in (a,R_v) \cap B_{r(\varepsilon)}(v)} \left(M(\bar{v}) + \frac{|v-\bar{v}|^2}{2\varepsilon} \right), \tag{2.65}$$

$$M^\varepsilon(v) = \sup_{\bar{v} \in (a,R_v) \cap B_{r(\varepsilon)}(v)} \left(M(\bar{v}) - \frac{|v-\bar{v}|^2}{2\varepsilon} \right). \tag{2.66}$$

We start recalling some useful properties of M_ε. Let $M : (0, R_v) \to \mathbb{R}$ be bounded and lower semicontinuous in $(0, R_v)$. It is well known that M_ε is an increasing sequence of semiconcave functions in $(0, R_v)$ which converges pointwise to M. From the fact that all the M_ε are semiconcave, we obtain several well-known properties that we discuss in Remark 2.47.

For these, and some further properties of the infimal convolution, see [133, 254] and [216, Lemma A.1]. The next lemma is the counterpart of [216, Lemma A.1 (iii)] for our setting.

Lemma 2.40. *Assume that $M : (a, R_v) \to \mathbb{R}$ is bounded and lower semicontinuous in (a, R_v). If M is a viscosity solution in the sense*

$$-\frac{\partial^2 M}{\partial v^2} \geq f\left(v, M, \frac{\partial M}{\partial v}\right)$$

in (a, R_v), then M_ε is a viscosity solution to

$$-\frac{\partial^2 M_\varepsilon}{\partial v^2} \geq f_\varepsilon\left(v, M_\varepsilon, \frac{\partial M_\varepsilon}{\partial v}\right)$$

in $(a + r(\varepsilon), R_v - r(\varepsilon))$, where

$$f_\varepsilon(v, s, q) := \inf_{\bar{v} \in B_{r(\varepsilon)}(v)} f(\bar{v}, s, q).$$

Respectively, if M bounded and upper semicontinuous in (a, R_v) is a viscosity subsolution, M^ε is a viscosity solution in the sense,

$$-\frac{\partial^2 M^\varepsilon}{\partial v^2} \leq f^\varepsilon\left(v, M_\varepsilon, \frac{\partial M_\varepsilon}{\partial v}\right)$$

in $(a + r(\varepsilon), R_v - r(\varepsilon))$, where

$$f^\varepsilon(v, s, q) := \sup_{\bar{v} \in B_{r(\varepsilon)}} f(\bar{v}, s, q).$$

Remark 2.41. *From Theorem 2.6 we know that $\widehat{M}$ is continuous in $(0, R_v]$ and, in particular, $\widehat{M}$ is lower and upper semicontinuous in $[a, R_v]$ for every $a > 0$.*

Another well-known result, see for example [221, Chapter 4 Theorem 7(f)], gives us convergence of the infimal and supremal convolution to the original solution.

Lemma 2.42. *Assume $M \in C^0([a, R_v])$, then we have that*

$$M_\varepsilon \nearrow M \quad \text{in}\, C^0([a, R_v]) \quad \text{as}\, \varepsilon \to 0, \tag{2.67}$$

and, respectively

$$M^\varepsilon \searrow M \quad \text{in}\, C^0([a, R_v]) \quad \text{as}\, \varepsilon \to 0. \tag{2.68}$$

Focusing on our problem, observe that $\widehat{M}$ is a viscosity solution of the equation

$$-\frac{\partial^2 \widehat{M}}{\partial v^2}(v_0) = \frac{1}{m} \mathfrak{E}[\widehat{\mu}](v_0) \left(\frac{\partial \widehat{M}}{\partial v}(v_0) \right)^{2-m}. \tag{2.69}$$

Therefore, from Lemma 2.40 we have that $\widehat{M_\varepsilon}$ is a viscosity solution in the sense

$$-\frac{\partial^2 \widehat{M_\varepsilon}}{\partial v^2}(v_0) \geq \frac{1}{m}\mathfrak{E}_\varepsilon(v_0)\left(\frac{\partial \widehat{M_\varepsilon}}{\partial v}(v_0)\right)^{2-m} \tag{2.70}$$

in $(a + r(\varepsilon), R_v - r(\varepsilon))$, where $r(\varepsilon) \to 0$ when $\varepsilon \to 0$ and with

$$\mathfrak{E}_\varepsilon(v_0) = \min_{v \in \overline{B_{r(\varepsilon)}(v_0)}} \mathfrak{E}[\widehat{\mu}](v). \tag{2.71}$$

And respectively, $\widehat{M^\varepsilon}$ is a viscosity solution in the sense

$$-\frac{\partial^2 \widehat{M^\varepsilon}}{\partial v^2}(v_0) \leq \frac{1}{m}\mathfrak{E}^\varepsilon(v_0)\left(\frac{\partial \widehat{M^\varepsilon}}{\partial v}(v_0)\right)^{2-m} \tag{2.72}$$

in $(a + r(\varepsilon), R_v - r(\varepsilon))$ and with

$$\mathfrak{E}^\varepsilon(v_0) = \max_{v \in \overline{B_{r(\varepsilon)}(v_0)}} \mathfrak{E}[\widehat{\mu}](v). \tag{2.73}$$

Notice that $\mathfrak{E}_\varepsilon$ and $\mathfrak{E}^\varepsilon$ converge uniformly to $\mathfrak{E}[\widehat{\mu}]$.

Remark 2.43. *Let $\widetilde{\omega}$ be the modulus of continuity of $\mathfrak{E}[\widehat{\mu}]$. Since $\mathfrak{E}_\varepsilon(v_0) = \mathfrak{E}[\widehat{\mu}](v_\varepsilon)$ where $|v_0 - v_\varepsilon| \leq r(\varepsilon)$, then*

$$|\mathfrak{E}_\varepsilon(v_0) - \mathfrak{E}[\widehat{\mu}](v_0)| \leq \tilde{\omega}(r(\varepsilon)). \tag{2.74}$$

Notice that $\omega(\varepsilon) = \widetilde{\omega}(r(\varepsilon))$ is a modulus of continuity. Respectively for $\mathfrak{E}^\varepsilon$.

Furthermore, since we are assuming V and K compactly supported and non-negative, then the function

$$\Upsilon[\widehat{\mu}](v) = -\int_v^{R_v} \mathfrak{E}[\widehat{\mu}](s)\mathrm{d}s = \Big(V(x) + \int_{B_R} K(x,y)\,\mathrm{d}\widehat{\mu}(y)\Big)\Big|_{v=|B_1||x|^d}.$$

is also compactly supported and non-negative in $[0, R_v)$. In particular, there must exist $b > 0$ such that

$$\begin{aligned} \Upsilon[\widehat{\mu}] &\equiv 0 \quad \text{in } [R_v - b, R_v), \\ \Upsilon[\widehat{\mu}] &\geq 0 \quad \text{in } (0, R_v - b). \end{aligned} \tag{2.75}$$

We present some extra results that follows from the remark we have just noticed.

Lemma 2.44. *Let $\widehat{M}$ be a viscosity solution of (2.69). Assume V and K compactly supported so that $\Upsilon[\widehat{\mu}]$ satisfies (2.75), then $\widehat{M}$ is linear in the interval $[R_v - b, R_v]$. Moreover, if $\widehat{M}(0^+) < \widehat{M}(R_v)$, $\widehat{M}$ is non-constant in that interval.*

Proof. The result follows as a consequence of the maximum principle for semicontinuous viscosity solutions of nonlinear elliptic partial differential equations derived by Kawohl and Kutev in [222], see Section 1.3.1, where we state this version of the maximum principle. Let us define

$$\begin{aligned} F^\infty \colon \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^{d\times d} &\longrightarrow \mathbb{R} \\ (z, r, p, X) \qquad &\mapsto \ -X - \frac{1}{m}\mathfrak{E}[\widehat{\mu}](z)\, p^{2-m}, \end{aligned}$$

We are interested on studying the fully nonlinear equation

$$F^\infty(v, \widehat{M}, D\widehat{M}, D^2\widehat{M}) = 0.$$

Since $\mathfrak{E}[\widehat{\mu}] \equiv 0$ in $(R_v - b, R_v)$ it follows that $\widehat{M}$ is linear on that interval. Let us now prove the assumptions (1.20)-(1.25) for F^∞ to see that $\widehat{M}$ is also non-constant. (1.20) and (1.25) follow immediately for F^∞. Furthermore,

$$F^\infty(z, r, p, Y) - F^\infty(z, r, p, X) = -(Y - X),$$

and (1.21) also follows for the modulus of ellipticity $\omega_1(s) = s$. Finally,

$$\begin{aligned} |F^\infty(z, r, p, X) - F^\infty(z, r, q, Y)| &= \left| Y - X + \frac{1}{m}\mathfrak{E}[\widehat{\mu}](z)\left(p^{2-m} - q^{2-m}\right)\right| \\ &\leq |Y - X| + \frac{1}{m}\left\|\mathfrak{E}[\widehat{\mu}]\right\|_{L^\infty(0,R_v)}(2-m)\,|\xi|^{1-m}|p-q|, \end{aligned}$$

for some $\xi \in (0, K)$ between p and q. Then, (1.23) is satisfied for

$$A_K = 1, \qquad B_K = \frac{2-m}{m}\left\|\mathfrak{E}[\widehat{\mu}]\right\|_{L^\infty(0,R_v)} K^{1-m},$$

with ω_2 the identity once more. Therefore, because $\omega_1 \equiv \omega_2$, (1.24) is also trivially satisfied. As a consequence, we can apply Theorem 1.7 to F^∞ and the result follows immediately from it. □

Remark 2.45. *If M is non-decreasing, then M_ε and M^ε are both non-decreasing. Furthermore, if we combine this with the maximum principle stated on Theorem 1.7 it implies that if M is non-constant, then M_ε and M^ε are also strictly increasing.*

Because, $\widehat{M}$ is increasing and linear at the end of its domain we can take advantage of the following result.

Lemma 2.46. *Assume M is linear on an interval, $M(v) = Av + B$ on (α, β). Then, the infimal convolution is $M_\varepsilon(v) = Av + B - \frac{\varepsilon}{2}A^2$ on $(\alpha + \varepsilon A, \beta - \varepsilon A)$ (resp. $M^\varepsilon(v) = Av + B + \frac{\varepsilon}{2}A^2$).*

Proof. We are doing the proof for the infimal convolution (analogue for the supremal case). It is easy to compute that if $v \in (\alpha + \varepsilon A, \beta - \varepsilon A)$, then, due to the convexity of $M(\bar{v}) + \frac{|v-\bar{v}|^2}{2\varepsilon}$ the infimum is achieved at the point $\bar{v} = v - \varepsilon A$. Therefore,

$$M_\varepsilon(v) = Av + B - \frac{\varepsilon}{2}A^2.$$ □

Finally, before presenting the regularity result we need to see that we can integrate both, the infimal and supremal convolution. The theory we are exposing now is devoted to that goal and follows from the fact that the infimal and supremal convolution are semiconcave and semiconvex respectively. We will only discuss the case of the infimal convolution, the supremal convolution is analogous.

Remark 2.47. *Semiconcavity implies that $\widehat{M}_\varepsilon$ is locally Lipschitz in $(a + r(\varepsilon), R_v - r(\varepsilon))$ [183, Theorem 6.7]. Hence, by Rademacher's Theorem [183, Theorem 6.6], $D\widehat{M}_\varepsilon$ is an a.e. defined derivative in $(a+r(\varepsilon), R_v - r(\varepsilon))$ which coincides with the distributional derivative. Alexandrov's Theorem states that $\widehat{M}_\varepsilon$ is twice differentiable a.e., i.e. for a.e. $v_0 \in (a + r(\varepsilon), R_v - r(\varepsilon))$, there exists $\widehat{X}_\varepsilon(v_0)$ such that*

$$\widehat{M}_\varepsilon(v_0 + w) = \widehat{M}_\varepsilon(v_0) + D\widehat{M}_\varepsilon(v_0)w + \frac{1}{2}\widehat{X}_\varepsilon(v_0)w^2 + \mathfrak{R}_{v_0}(|w|^2),$$

where

$$|\mathfrak{R}_{v_0}(\gamma)| \leq \omega_{v_0}(\gamma)\gamma^2.$$

For a proof see [304, Theorem 14.1]. To avoid confusion with the distributional and measure derivative we avoid the notation $D^2\widehat{M}_\varepsilon$ and we mantain the notation $\widehat{X}_\varepsilon$. Furthermore, there exists a sequence $\widehat{M}_{\varepsilon,j}$ regular such that

$$(\widehat{M}_{\varepsilon,j}, D\widehat{M}_{\varepsilon,j}, D^2\widehat{M}_{\varepsilon,j}) \to (\widehat{M}_\varepsilon, D\widehat{M}_\varepsilon, \widehat{X}_\varepsilon) \quad \text{a.e. in } (a + r(\varepsilon), R_v - r(\varepsilon)). \tag{2.76}$$

We can also prove the following result,

Lemma 2.48. *Let $\widehat{M}_\varepsilon$ be a viscosity solution in the sense (2.70). Then,*

$$-\widehat{X}_\varepsilon(v_0) \geq \frac{1}{m}\mathfrak{E}_\varepsilon(v_0)\left(\frac{\partial \widehat{M}_\varepsilon}{\partial v}(v_0)\right)^{2-m} \qquad \text{a.e. in } (a+r(\varepsilon), R_v - r(\varepsilon)).$$

The result is analogous for $\widehat{M}^\varepsilon$, a viscosity solution in the sense (2.72).

Proof. Let us take $v_0 \in (a+r(\varepsilon), R_v - r(\varepsilon))$ and take a test function φ,

$$\varphi(v) = \widehat{M}_\varepsilon(v_0) + D\widehat{M}_\varepsilon(v_0)(v-v_0) + \frac{1}{2}(\widehat{X}_\varepsilon(v_0) - \gamma)(v-v_0)^2.$$

For $|v-v_0|$ small enough

$$\frac{1}{2}\gamma|v-v_0|^2 \geq |\mathfrak{R}_{v_0}(|v-v_0|^2)|,$$

and hence, in a small enough neighbourhood of v_0, we have that $\varphi(v) \leq \widehat{M}_\varepsilon(v)$. Thus, φ touches $\widehat{M}_\varepsilon$ from below at v_0. Since $\widehat{M}_\varepsilon$ is a viscosity solution in the sense (2.70)

$$-D^2\varphi(v_0) \geq \frac{1}{m}\mathfrak{E}_\varepsilon(v_0)\left(D\widehat{M}_\varepsilon(v_0)\right)^{2-m}.$$

Computing precisely $D^2\varphi(v_0)$ we deduce

$$-(\widehat{X}_\varepsilon(v_0) - \gamma) \geq \frac{1}{m}\mathfrak{E}_\varepsilon(v_0)\left(D\widehat{M}_\varepsilon(v_0)\right)^{2-m}.$$

Since this holds for any $\gamma > 0$ and a.e. $v_0 \in (a+r(\varepsilon), R_v - r(\varepsilon))$, the result is proven. □

Once we have presented the problem, we proceed to state the main result of the subsection about the regularity of $\widehat{M}$. We prove $\widehat{M} \in W^{2,\infty}_{loc}((0,R_v])$ using the infimal and the supremal convolutions and all the properties we have discussed through this subsection. The first step of the proof of the Theorem where we show that $\widehat{M}_\varepsilon$ is also a distributional supersolution is based on [288, Lemma 4.1].

We are now able to prove Theorem 2.10. Notice that (2.14) is written in our notation as

$$\frac{\partial \widehat{M}}{\partial v}(v) = \left(\left(\frac{\partial \widehat{M}}{\partial v}(R_v)\right)^{m-1} - \frac{1-m}{m}\int_v^{R_v} \mathfrak{E}(\sigma)\,\mathrm{d}\sigma\right)^{-\frac{1}{1-m}}$$

Proof of Theorem 2.10. We divide the proof in several steps.

Step 1: Weak formulation. Let us take $\Theta : \mathbb{R} \to \mathbb{R}$ smooth and non-decreasing. Let us take the regular sequence $\widehat{M}_{\varepsilon,j}$ like in (2.76). Let us now take a non-negative test function $\varphi \in C_c^\infty((a+r(\varepsilon), R_v - r(\varepsilon)))$. Then, we can integrate by parts in order to get

$$\int_{a+r(\varepsilon)}^{R_v - r(\varepsilon)} \left(-\frac{\partial^2 \widehat{M}_{\varepsilon,j}}{\partial v^2}(\sigma) - \frac{1}{m}\mathfrak{E}_\varepsilon(\sigma)\left(\frac{\partial \widehat{M}_{\varepsilon,j}}{\partial v}(\sigma)\right)^{2-m}\right)\Theta'\left(\frac{\partial \widehat{M}_{\varepsilon,j}}{\partial v}\right)\varphi\,\mathrm{d}\sigma$$
$$= \int_{a+r(\varepsilon)}^{R_v - r(\varepsilon)} \left(\Theta\left(\frac{\partial \widehat{M}_{\varepsilon,j}}{\partial v}\right)\frac{\mathrm{d}\varphi}{\mathrm{d}v} - \frac{1}{m}\mathfrak{E}_\varepsilon(\sigma)\left(\frac{\partial \widehat{M}_{\varepsilon,j}}{\partial v}(\sigma)\right)^{2-m}\Theta'\left(\frac{\partial \widehat{M}_{\varepsilon,j}}{\partial v}\right)\varphi\right)\mathrm{d}\sigma.$$

From Remark 2.47 we have that $\left|\frac{\partial \widehat{M}_{\varepsilon,j}}{\partial v}\right|$ is uniformly bounded in the support of φ. Hence, by the Dominated Convergence Theorem

$$\lim_{j\to\infty}\int_{a+r(\varepsilon)}^{R_v - r(\varepsilon)} \Theta\left(\frac{\partial \widehat{M}_{\varepsilon,j}}{\partial v}\right)\frac{\mathrm{d}\varphi}{\mathrm{d}v} - \frac{1}{m}\mathfrak{E}_\varepsilon(\sigma)\left(\frac{\partial \widehat{M}_{\varepsilon,j}}{\partial v}(\sigma)\right)^{2-m}\Theta'\left(\frac{\partial \widehat{M}_{\varepsilon,j}}{\partial v}\right)\varphi\,\mathrm{d}\sigma$$

$$= \int_{a+r(\varepsilon)}^{R_v - r(\varepsilon)} \Theta\left(\frac{\partial \widehat{M}_\varepsilon}{\partial v}\right)\frac{\mathrm{d}\varphi}{\mathrm{d}v} - \frac{1}{m}\mathfrak{E}_\varepsilon(\sigma)\left(\frac{\partial \widehat{M}_\varepsilon}{\partial v}(\sigma)\right)^{2-m}\Theta'\left(\frac{\partial \widehat{M}_\varepsilon}{\partial v}\right)\varphi\,\mathrm{d}\sigma.$$

Notice that $\frac{\partial^2 \widehat{M}_{\varepsilon,j}}{\partial v^2} \geq C > -\infty$, $\left|\frac{\partial \widehat{M}_{\varepsilon,j}}{\partial v}\right|$ is uniformly bounded in the support of φ and the first and second derivatives converge a.e. Then, we use Fatou's lemma and the Dominated Convergence Theorem to deduce

$$\begin{aligned}
&\liminf_{j\to\infty}\int_{a+r(\varepsilon)}^{R_v - r(\varepsilon)}\left(-\frac{\partial^2 \widehat{M}_{\varepsilon,j}}{\partial v^2}(\sigma) - \frac{1}{m}\mathfrak{E}_\varepsilon(\sigma)\left(\frac{\partial \widehat{M}_{\varepsilon,j}}{\partial v}(\sigma)\right)^{2-m}\right)\Theta'\left(\frac{\partial \widehat{M}_{\varepsilon,j}}{\partial v}\right)\varphi\,\mathrm{d}\sigma \\
&\quad\geq \int_{a+r(\varepsilon)}^{R_v - r(\varepsilon)}\lim_{j\to\infty}\left(-\frac{\partial^2 \widehat{M}_{\varepsilon,j}}{\partial v^2}(\sigma) - \frac{1}{m}\mathfrak{E}_\varepsilon(\sigma)\left(\frac{\partial \widehat{M}_{\varepsilon,j}}{\partial v}(\sigma)\right)^{2-m}\right)\Theta'\left(\frac{\partial \widehat{M}_{\varepsilon,j}}{\partial v}\right)\varphi\,\mathrm{d}\sigma \\
&\quad= \int_{a+r(\varepsilon)}^{R_v - r(\varepsilon)}\left(-\widehat{X}_\varepsilon(\sigma) - \frac{1}{m}\mathfrak{E}_\varepsilon(\sigma)\left(\frac{\partial \widehat{M}_\varepsilon}{\partial v}(\sigma)\right)^{2-m}\right)\Theta'\left(\frac{\partial \widehat{M}_\varepsilon}{\partial v}\right)\varphi\,\mathrm{d}\sigma \geq 0.
\end{aligned}\tag{2.77}$$

The final inequality follows from Lemma 2.48. We have deduced that

$$\int_{a+r(\varepsilon)}^{R_v - r(\varepsilon)}\left(\Theta\left(\frac{\partial \widehat{M}_\varepsilon}{\partial v}(\sigma)\right)\frac{\mathrm{d}\psi}{\mathrm{d}v} - \frac{1}{m}\mathfrak{E}_\varepsilon(\sigma)\left(\frac{\partial \widehat{M}_\varepsilon}{\partial v}(\sigma)\right)^{2-m}\Theta'\left(\frac{\partial \widehat{M}_\varepsilon}{\partial v}(\sigma)\right)\psi\right)\mathrm{d}\sigma \geq 0.$$

Step 2: A point-wise estimate. Since $\Upsilon[\widehat{\mu}]$ satisfies (2.75), we are on the assumptions of Lemma 2.44 and therefore $\widehat{M}$ is linear and non-constant on the interval $[R_v - b, R_v]$. Then, by Lemma 2.46, $\widehat{M}_\varepsilon$ is also linear on $[R_v - b + r(\varepsilon), R_v - r(\varepsilon)]$ and it is such that

$$\frac{\partial \widehat{M}_\varepsilon}{\partial v}(v) = \frac{\partial \widehat{M}}{\partial v}(R_v) \qquad \text{on } [R_v - b + r(\varepsilon), R_v - r(\varepsilon)],$$

and $\mathfrak{E}_\varepsilon = 0$ on $[R_v - b + r(\varepsilon), R_v - r(\varepsilon)]$. Due to the Lebesgue differentiation Theorem, for almost every $v \in (a + r(\varepsilon), R_v - 2r(\varepsilon))$ if we take $\psi \to \chi_{[v, R_v - 2r(\varepsilon)]}$ we have,

$$\Theta\left(\frac{\partial \widehat{M}_\varepsilon}{\partial v}(v)\right) - \Theta\left(\frac{\partial \widehat{M}_\varepsilon}{\partial v}(R_v)\right) \geq \int_v^{R_v}\frac{1}{m}\mathfrak{E}_\varepsilon(\sigma)\left(\frac{\partial \widehat{M}_\varepsilon}{\partial v}(\sigma)\right)^{2-m}\Theta'\left(\frac{\partial \widehat{M}_\varepsilon}{\partial v}(\sigma)\right)\mathrm{d}\sigma.$$

Moreover, for Θ non-decreasing, we have that

$$\frac{\partial \widehat{M}_\varepsilon}{\partial v}(v) \geq \Theta^{-1}\left(\Theta\left(\frac{\partial \widehat{M}_\varepsilon}{\partial v}(R_v)\right) + \int_v^{R_v}\frac{1}{m}\mathfrak{E}_\varepsilon(\sigma)\left(\frac{\partial \widehat{M}_\varepsilon}{\partial v}(\sigma)\right)^{2-m}\Theta'\left(\frac{\partial \widehat{M}_\varepsilon}{\partial v}(\sigma)\right)\mathrm{d}\sigma\right).$$

We take a sequence Θ_m such that $s^{2-m}\Theta_m'(s) \to 1 - m$ uniformly over compacts and $\Theta_m(1) = -1$. This corresponds to $\Theta(s) = -s^{m-1}$ so $\Theta^{-1}(s) = (-s)^{-\frac{1}{1-m}}$. Therefore, we get the estimate

$$\frac{\partial \widehat{M}_\varepsilon}{\partial v}(v) \geq \left(\left(\frac{\partial \widehat{M}}{\partial v}(R_v)\right)^{m-1} - \frac{1-m}{m}\int_v^{R_v}\mathfrak{E}_\varepsilon(\sigma)\,d\sigma\right)^{-\frac{1}{1-m}}. \tag{2.78}$$

Proceeding analogously for $\widehat{M}^\varepsilon$ we deduce the converse formula

$$\frac{\partial \widehat{M}^\varepsilon}{\partial v}(v) \leq \left(\left(\frac{\partial \widehat{M}}{\partial v}(R_v)\right)^{m-1} - \frac{1-m}{m}\int_v^{R_v}\mathfrak{E}^\varepsilon(\sigma)\,\mathrm{d}\sigma\right)^{-\frac{1}{1-m}}. \tag{2.79}$$

Step 3: Formula for $\widehat{M}$. Let $a < v_1 \le v_2 < R_v$. Then, due to (2.78) we have that for ε small enough

$$\widehat{M}_\varepsilon(v_2) - \widehat{M}_\varepsilon(v_1) \ge \int_{v_1}^{v_2} \left(\left(\frac{\partial \widehat{M}}{\partial v}(R_v) \right)^{m-1} - \frac{1-m}{m} \int_v^{R_v} \mathfrak{E}_\varepsilon(\sigma) \, \mathrm{d}\sigma \right)^{-\frac{1}{1-m}} \mathrm{d}v$$

Due to the uniform convergence of $\widehat{M}_\varepsilon$ and $\mathfrak{E}_\varepsilon$ we deduce that

$$\widehat{M}(v_2) - \widehat{M}(v_1) \ge \int_{v_1}^{v_2} \left(\left(\frac{\partial \widehat{M}}{\partial v}(R_v) \right)^{m-1} - \frac{1-m}{m} \int_v^{R_v} \mathfrak{E}(\sigma) \, \mathrm{d}\sigma \right)^{-\frac{1}{1-m}} \mathrm{d}v$$

Arguing conversely for (2.79) we deduce the other inequality. Due to the bounds and known regularity of $\mathfrak{E}$, it follows that $\widehat{M} \in W^{2,\infty}_{loc}((0, R_v])$ and

$$\frac{\partial \widehat{M}}{\partial v}(v) = \left(\left(\frac{\partial \widehat{M}}{\partial v}(R_v) \right)^{m-1} - \frac{1-m}{m} \int_v^{R_v} \mathfrak{E}(\sigma) \, \mathrm{d}\sigma \right)^{-\frac{1}{1-m}}.$$

This concludes the proof. □

2.6 The problem in $\mathbb{R}^d$

In this section, we will focus on the problem (2.1). The strategy we follow to study (2.1) consists in extending the results of the problem in the ball B_R considering the kernel $K_\eta(x, y) = \eta(x)W(x - y)\eta(y)$, with $\eta \in C_c^\infty(\mathbb{R}^d)$ a cut-off.

We recall that in order to to be able to use the results we have obtained for the ball B_R, the potential V needs to fulfil the hypothesis (2.6). Therefore, for each one of the different problems on the ball we need to modify slightly the potential V. We take the following modification of V. Let

$$g(x) = \begin{cases} 1, & \text{if } |x| \le \frac{1}{2}, \\ 0, & \text{if } |x| \ge \frac{3}{4}, \end{cases}$$

and define

$$V_R(x) = V(x) g(\tfrac{x}{R}).$$

With this construction we have that

$$\begin{aligned} & |V_R| \le C|V|, |\nabla V_R| \le C(1 + |\nabla V|), \text{ and } |D^2 V_R| \le C(1 + |D^2 V|) \text{ in } B_R, \\ & \nabla V_R \cdot \nu = 0 \text{ on } \partial B_R, \\ & V_R = V \text{ in } B_{\frac{R}{2}} \end{aligned} \tag{2.80}$$

for some constant C that does not depend on R.

2.6.1 Short-time well-posedness. Proof of Theorem 2.12

First we are going to construct locally-strong solutions in the whole space and study some of their properties. Let us start defining the notion of locally-strong solution.

Definition 2.49 (Locally-strong solution)**.** *A measure ρ is said to be a locally-strong solution of the problem (2.1) in $(0, T) \times \mathbb{R}^d$ if it is a distributional solution such that*

1. $\rho^m \in L^2(0, T; H^2_{loc}(\mathbb{R}^d))$;

2. $\partial_t \rho \in L^2(0,T; L^2_{loc}(\mathbb{R}^d))$.

Next, we make some remarks about the problem.

Remark 2.50. *From assumption* (H4) *one gets immediately by the mean value theorem that*

$$V(x) \le C(1+|x|^2). \tag{2.81}$$

From assumption (H5) *we also get in a similar way that*

$$W(x-y) \le C(1+|x|^2)(1+|y|^2) \quad \text{for every } x,y \in \mathbb{R}^d. \tag{2.82}$$

Before starting with the proof of Theorem 2.12 we need to show some *a priori* estimates. First we deal with the p-th order moment

$$\mathbf{m_p}(\rho) := \int_{\mathbb{R}^d} |x|^p \rho$$

Remark 2.51. *From assumption* (H3) *one gets immediately that* $\mathbf{m_2}(\rho_0)$ *is bounded. Using Hölder inequality we obtain that*

$$\mathbf{m_2}(\rho_0) \le \left(\int_{\mathbb{R}^d} |x|^{\frac{2}{1-m}} \rho_0 \right)^{1-m} \|\rho_0\|^m_{L^1(\mathbb{R}^d)}.$$

Lemma 2.52 (The second moment is bounded for finite time)**.** *Assume* (H1) *and* (H3)*. Then, the second moment is uniformly bounded in the sense that*

$$\mathbf{m_2}(\rho_t^{R,\eta,*}) \le \left(\mathbf{m_2}(\rho_0) + \sup_R \mathcal{F}_{R,1}[\rho_0] - \underline{\mathcal{F}_0} \right) e^t, \tag{2.83}$$

where $\rho^{R,\eta,*}$ *is the extension by zero of* $\rho^{R,\eta}$ *to the whole space* $\mathbb{R}^d$*, i.e.* $\rho^{R,\eta,*} = \rho^{R,\eta}\chi_{B_R}$.

Proof. Let us take the time derivative. We apply integration by parts, Holder and Young's inequality successively in order to get,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{m_2}(\rho_t^{R,\eta,*}) &= \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} |x|^2 \rho_t^{R,\eta,*} = \frac{\mathrm{d}}{\mathrm{d}t}\int_{B_R} |x|^2 \rho_t^{R,\eta} = \int_{B_R} |x|^2 \partial_t \rho_t^{R,\eta} \\
&= \int_{B_R} |x|^2 \operatorname{div}\left(\rho_t^{R,\eta} \left[\nabla \left(\frac{m}{m-1} (\rho_t^{R,\eta})^{m-1} + V_R + \int_{B_R} K_\eta(x,y) \rho_t^{R,\eta}(y)\,\mathrm{d}y \right) \right] \right) \mathrm{d}x \\
&= -2 \int_{B_R} x \rho_t^{R,\eta} \left[\nabla \left(\frac{m}{m-1} (\rho_t^{R,\eta})^{m-1} + V_R + \int_{B_R} K_\eta(x,y) \rho_t^{R,\eta}(y)\,\mathrm{d}y \right) \right] \mathrm{d}x \\
&\le 2 \left(\int_{B_R} |x|^2 \rho_t^{R,\eta} \right)^{\frac{1}{2}} \\
&\quad \times \left(\int_{B_R} \rho_t^{R,\eta} \left| \nabla \left(\frac{m}{m-1} (\rho_t^{R,\eta})^{m-1} + V_R + \int_{B_R} K_\eta(x,y) \rho_t^{R,\eta}(y)\,\mathrm{d}y \right) \right|^2 \mathrm{d}x \right)^{\frac{1}{2}} \\
&\le \int_{B_R} |x|^2 \rho_t^{R,\eta} \\
&\quad + \left(\int_{B_R} \rho_t^{R,\eta} \left| \nabla \left(\frac{m}{m-1} (\rho_t^{R,\eta})^{m-1} + V_R + \int_{B_R} K_\eta(x,y) \rho_t^{R,\eta}(y)\,\mathrm{d}y \right) \right|^2 \mathrm{d}x \right).
\end{aligned}$$

Integrating in time between 0 and t and taking into account (2.9) we have that

$$\mathbf{m_2}(\rho_t^{R,\eta,*}) - \mathbf{m_2}(\rho_0^{R,\eta,*}) \le \int_0^t \mathbf{m_2}(\rho_s^{R,\eta,*})\,ds + \mathcal{F}_{R,\eta}[\rho_0^{R,\eta,*}] - \mathcal{F}_{R,\eta}[\rho_t^{R,\eta,*}],$$

which gives us the bound,

$$\mathbf{m_2}(\rho_t^{R,\eta,*}) \le \int_0^t \mathbf{m_2}(\rho_s^{R,\eta,*})\, ds + \left(\mathbf{m_2}(\rho_0) + \sup_\eta \sup_R \mathcal{F}_{R,\eta}[\rho_0] - \underline{\mathcal{F}_0}\right).$$

Finally, applying Grönwall's inequality we recover (2.83). □

The previous Lemma only ensures boundedness of $\mathbf{m_2}(\rho)$. However, taking advantage of this result we can also get an *a priori* estimate for the p-order moment for $p = \frac{2}{1-m} > 2$.

Lemma 2.53. *Assume (H1) and (H3)-(H5). Then, the p-moment for $p = \frac{2}{1-m}$ is uniformly bounded from above if we pick the sequence $\eta = \eta_j$*

$$\mathbf{m_p}(\rho_t^{R,\eta_j,*}) \le A \exp(Bt), \qquad \forall t \in [0,T] \tag{2.84}$$

where A and B are constants that depend only on m, T, the constants in (H1)-(H5), $\|\rho_0\|_{L^1(\mathbb{R}^d)}$, $\|\rho_0\|_{L^{\frac{1}{1-m}}(\mathbb{R}^d)}$, $\mathbf{m_p}(\rho_0)$, $\|\nabla\eta_1\|_{L^\infty(\mathbb{R}^d)}$, and the right-hand side of (2.83).

The proof of this result is on Section 2.A. Once we have an *a priori* estimate for a $p > 2$ order moment we are ready to prove Theorem 2.12.

Proof of Theorem 2.12. We divide the proof of the Theorem into 5 steps.

Step 1: Compactness in R. If we use the L^p estimate (2.23), we get that,

$$\|\rho_t^{R,\eta_j}\|_{L^p(\mathbb{R}^d)} \le e^{C(p,T)}\|\rho_0\|_{L^p(\mathbb{R}^d)}, \qquad \forall t \in [0,T]. \tag{2.85}$$

where

$$C(p,T) = \sup_{R,j} \frac{p-1}{p} \int_0^T \left(\|\Delta V_R\|_{L^\infty(\mathbb{R}^d)} + \left\|\Delta \int_{\mathbb{R}^d} K_{\eta_j}(\cdot,y)\rho_t^{R,\eta_j}(y)\,\mathrm{d}y\right\|_{L^\infty(\mathbb{R}^d)}\right) \mathrm{d}t.$$

The term $\|\Delta V_R\|_{L^\infty(\mathbb{R}^d)}$ is uniformly bounded in R by the assumption (H6) and (2.80). Furthermore, since

$$|\nabla\eta_j(x)| \le j^{-1}\|\nabla\eta_1\|_{L^\infty(\mathbb{R}^d)}\chi_{B_j}(x) \qquad \text{and} \qquad |\Delta\eta_j(x)| \le j^{-2}\|\Delta\eta_1\|_{L^\infty(\mathbb{R}^d)}\chi_{B_j}(x),$$

we can bound $\|\Delta \int_{\mathbb{R}^d} K_{\eta_j}(\cdot,y)\rho^{R,\eta_j}(y)\,\mathrm{d}y\|_{L^\infty(\mathbb{R}^d)}$ uniformly in j.

$$\begin{aligned}
&\left|\Delta \int_{\mathbb{R}^d} K_{\eta_j}(x,y)\rho_t^{R,\eta_j}(y)\,\mathrm{d}y\right| \\
&\quad \le \left|\int_{\mathbb{R}^d} \left(\Delta\eta_j(x)W(x-y) + 2\nabla\eta_j(x)\cdot\nabla W(x-y) + \eta_j(x)\Delta W(x-y)\right)\rho_t^{R,\eta_j}(y)\,\mathrm{d}y\right| \\
&\quad \le C\Big(\int_{\mathbb{R}^d} \|\Delta\eta_1\|_{L^\infty(\mathbb{R}^d)}(j^{-2}+1)(1+|y|^2)\rho_t^{R,\eta_j}(y)\,\mathrm{d}y \\
&\qquad + \int_{\mathbb{R}^d} \|\nabla\eta_1\|_{L^\infty(\mathbb{R}^d)}(j^{-1}+1)\rho_t^{R,\eta_j}(y)\,\mathrm{d}y + \int_{\mathbb{R}^d} \|\eta_1\|_{L^\infty(\mathbb{R}^d)}\rho_t^{R,\eta_j}(y)\,\mathrm{d}y\Big) \\
&\quad \le C(\|\eta_1\|_{W^{2,\infty}(\mathbb{R}^d)})\left(\|\rho_0\|_{L^1(\mathbb{R}^d)} + \mathbf{m_2}(\rho_t^{R,\eta_j})\right).
\end{aligned}$$

Then, since the second moment is uniformly bounded due to (2.83), we have that the sequence $\rho^{R,\eta_j,*}$ is uniformly bounded in $L^p((0,T)\times\mathbb{R}^d)$ for every $p \ge 1$.

Due to the uniform bound in $L^\infty(0,T;L^p(\mathbb{R}^d))$ we can do a diagonal argument in time to prove that for every j fixed there exists a sequence $R_i^j \nearrow \infty$ as $i \to \infty$ such that

$$\rho^{R_i^j,\eta_j,*} \rightharpoonup \rho^{\infty,\eta_j} \quad \text{in } L^p_{loc}(0,\infty;L^p(\mathbb{R}^d)).$$

In Section 2.A we prove uniform *a priori* estimates. For every $\omega > 0$ and i such that $R_i^j > \omega, j$, we deduce:

- from (2.107), $\nabla \rho^{R_i^j,\eta_j}$ is uniformly bounded in $L^2((0,T)\times B_\omega)$,
- from (2.108), $(\rho^{R_i^j,\eta_j})^m$ is uniformly bounded in $L^2((0,T);H^1(B_\omega))$,
- from (2.109), $\partial_t \rho^{R_i^j,\eta_j}$ is uniformly bounded in $L^2((0,T)\times B_\omega)$, and
- from (2.110), $\Delta(\rho^{R_i^j,\eta})^m$ is uniformly bounded in $L^2((0,T)\times B_\omega)$.

Step 2: A subsequence converges to a locally-strong solution for a fixed cut-off η_j. Thanks to all the compactness results obtained in Step 1 we can prove convergence to a locally-strong solution.

Step 2.a: Convergence by compactness. The sequence $\rho^{R_i^j,\eta_j}$ is uniformly bounded in $H^1((0,T)\times B_\omega)$ by Step 1. Since $H^1((0,T)\times B_\omega)$ is compactly embedded in $C^{\frac{1}{2}}(0,T;H^1(B_\omega))$, using Ascoli-Arzela Theorem and a diagonal argument in T and ω, for j fixed up to a further subsequence R_i^j, the convergence is such that

$$\rho^{R_i^j,\eta_j,*} \to \rho^{\infty,\eta_j} \quad \text{in } C_{loc}([0,\infty);L^2_{loc}(\mathbb{R}^d)) \qquad \text{as } i\to\infty, \text{ and for each } t \text{ locally } a.e. \text{ in } \mathbb{R}^d. \tag{2.86}$$

In particular, we also have $\rho^{\infty,\eta_j}\in H^1_{loc}((0,\infty)\times\mathbb{R}^d)$. Moreover, by the Dominated Convergence Theorem we have

$$(\rho^{R_i^j,\eta_j,*})^m \to (\rho^{\infty,\eta_j})^m \quad \text{strongly in } C_{loc}((0,\infty);L^2_{loc}(\mathbb{R}^d)).$$

Furthermore, since $(\rho^{R_i^j,\eta_j})^m$ is uniformly bounded in $L^2(0,T;H^2(B_\omega))$ for any T and ω, due to up to a further subsequence in i on R_i^j; by Banach-Alaoglu Theorem and the previous characterisation of the limit

$$((\rho^{R_i^j,\eta_j,*})^m) \rightharpoonup ((\rho^{\infty,\eta_j})^m) \quad \text{weakly in } L^2_{loc}(0,\infty;H^2_{loc}(\mathbb{R}^d)).$$

Therefore, $(\rho^{\infty,\eta_j})^m\in L^2_{loc}(0,\infty;H^2_{loc}(\mathbb{R}^d))$.

Step 2.b: Distributional solution. We now have all the ingredients to see that for every test function $\phi\in X=\left\{\phi\in C_c^\infty([0,T]\times\mathbb{R}^d):\phi(T)=0\right\}$,

$$\int_{\mathbb{R}^d}\rho_0\phi(0)+\int_0^T\int_{\mathbb{R}^d}\rho_t^{R_i^j,\eta_j}\frac{\partial\varphi}{\partial t}=\int_0^T\int_{\mathbb{R}^d}\nabla((\rho_t^{R_i^j,\eta_j})^m)\nabla\phi+\int_0^T\int_{\mathbb{R}^d}\rho_t^{R_i^j,\eta_j}E_t^{R_i^j,\eta_j}\nabla\phi$$

converges to

$$\int_{\mathbb{R}^d}\rho_0\phi(0)+\int_0^T\int_{\mathbb{R}^d}\rho_t^{\infty,\eta_j}\frac{\partial\phi}{\partial t}=\int_0^T\int_{\mathbb{R}^d}\nabla((\rho^{\infty,\eta_j})^m)\nabla\phi+\int_0^T\int_{\mathbb{R}^d}\rho_t^{\infty,\eta_j}E_t^{\infty,\eta_j}\nabla\phi.$$

Step 3: Extension $\eta_j\to 1$. All the bounds discussed in the Step 1 are uniform on $\mathrm{supp}\eta_j$. Therefore, in the same way we have just done in Step 2, taking successive subsequences η_j we get that there exists $\rho^{\infty,1}\in L^\infty(0,T;L^p(\mathbb{R}^d))$ such that,

$$\rho^{\infty,\eta_j}\to\rho^{\infty,1} \quad \text{in } C_{loc}([0,\infty);L^2_{loc}(\mathbb{R}^d)). \tag{2.87}$$

Additionally, $\rho^{\infty,1}$ is a locally-strong solution of the problem (2.1) in $(0,T)\times\mathbb{R}^d$ in the case $\eta\equiv 1$ for every $T>0$.

Step 4: Convergence of the free energy with a fixed cut-off η_j. In this step we claim that we have the convergence stated at (2.16). Our strategy to prove that the limit holds is to control the tails of the three terms.

Step 4.a: Convergence of the diffusive term of the energy. First we control the tails. Let $\sigma>0$. If $m>\frac{1+d-\sqrt{2d+1}}{d}$, we can use Hölder inequality to obtain,

$$\int_{\mathbb{R}^d\setminus B_\sigma}\left(\rho_t^{R_i^j,\eta_j,*}\right)^m\le\left(\int_{\mathbb{R}^d\setminus B_\sigma}|x|^{\frac{2}{1-m}}\rho_t^{R_i^j,\eta_j,*}\right)^m\left(\int_{\mathbb{R}^d\setminus B_\sigma}\frac{1}{|x|^{\frac{2m}{(1-m)^2}}}\right)^{1-m}$$

$$\leq C_m \mathbf{m}_{\frac{\mathbf{2}}{\mathbf{1-m}}} \left(\rho_t^{R_i^j,\eta_j,*}\right)^m \sigma^{-\frac{2m}{(1-m)}+d(1-m)}.$$

The $\frac{2}{1-m}$-th moment is bounded due to (2.84) and the exponent on σ is negative with our choice of m (notice also this range is connect with the reverse Hardy-Littlewood inequality Remark 2.11) Thus, by the Fatou's lemma, due to local-a.e. convergence, for every $\sigma < \kappa < \infty$ and $t \in [0,T]$ the limit also satisfies,

$$\int_{B_\kappa \setminus B_\sigma} \left(\rho_t^{\infty,\eta_j}\right)^m \leq \omega_{T,m}(\sigma^{-1}) \to 0.$$

Now letting $\kappa \to \infty$ we obtain that the formula also holds replacing B_κ by $\mathbb{R}^d$. Eventually we recover that, for each $\sigma > 0$ we have that

$$\begin{aligned}&\limsup_{i\to\infty} \sup_{t\in[0,T]} \left| \int_{\mathbb{R}^d} \left(\rho_t^{R_i^j,\eta_j,*}\right)^m - \int_{\mathbb{R}^d} \left(\rho_t^{\infty,\eta_j}\right)^m \right| \\ &\qquad \leq \limsup_{i\to\infty} \sup_{t\in[0,T]} \left| \int_{B_\sigma} \left(\rho_t^{R_i^j,\eta_j,*}\right)^m - \int_{B_\sigma} \left(\rho_t^{\infty,\eta_j}\right)^m \right| + 2\omega_{T,m}(\sigma^{-1}).\end{aligned}$$

The first term is 0 because we have proven that the limit $i \to \infty$ is such that

$$\left(\rho^{R_i^j,\eta_j,*}\right)^m \to \left(\rho^{\infty,\eta_j}\right)^m \quad \text{in } L^2_{loc}(0,\infty; H^1_{loc}(\mathbb{R}^d)).$$

The second term vanishes as $\sigma \to \infty$ we deduce that the $\limsup$ is actually a limit, and its value is 0.

Step 4.b: Convergence of the confinement term of the energy. Analogously as before we write

$$\begin{aligned}&\limsup_{i\to\infty} \sup_{t\in[0,T]} \left| \int_{\mathbb{R}^d} V_{R_i^j} \rho_t^{R_i^j,\eta_j,*} - \int_{\mathbb{R}^d} V \rho^{\infty,\eta_j} \right| \\ &\qquad \leq \limsup_{i\to\infty} \sup_{t\in[0,T]} \left| \int_{B_\sigma} V_{R_i^j} \rho_t^{R_i^j,\eta_j,*} - \int_{B_\sigma} V \rho^{\infty,\eta_j} \right| \\ &\qquad\quad + \limsup_{i\to\infty} \sup_{t\in[0,T]} \int_{\mathbb{R}^d\setminus B_\sigma} V_{R_i^j} \rho_t^{R_i^j,\eta_j,*} + \sup_{t\in[0,T]} \int_{\mathbb{R}^d\setminus B_\sigma} V \rho_t^{\infty,\eta_j}.\end{aligned} \tag{2.88}$$

The first term is 0 since we know that when we take a sequence $i \to \infty$ we have that

$$\rho^{R_i^j,\eta_j,*} \to \rho^{\infty,\eta_j} \quad \text{in } L^2(0,T; H^1_{loc}(\mathbb{R}^d)).$$

From (2.81) we also know that $V \in L^\infty_{loc}(\mathbb{R}^d)$. From (2.81), for σ big enough, $V_{R_i^j}$ is such that $V_{R_i^j}(x) \leq C(1+|x|^2)$, and we get that,

$$\int_{\mathbb{R}^d\setminus B_\sigma} V_{R_i^j} \rho_t^{R_i^j,\eta_j,*} \leq \frac{C}{\sigma^{p-2}} \left(\|\rho_0\|_{L^1} + \mathbf{m_p}(\rho_t^{R_i^j,\eta_j,*}) \right).$$

Here we use $p = \frac{2}{1-m} > 2$, where we can control the moment. Using Fatou this uniform estimate for $t \in [0,T]$ also holds for the last term in (2.88). Letting $\sigma \to \infty$ we deduce that the first $\limsup$ in (2.88) vanishes.

Step 4.c: Convergence of the aggregation term of the free energy. Due to the local convergence (2.86), since the function $\eta_j(x)W(x-y)\eta_j(y)$ is supported on a compact set in $\mathbb{R}^d \times \mathbb{R}^d$ we observe that

$$\begin{aligned}\lim_{i\to\infty} \sup_{t\in[0,T]} \Bigg| &\int_{\mathbb{R}^d} \rho_t^{R_i^j,\eta_j,*}(x)\eta_j(x) \int_{\mathbb{R}^d} W(x-y)\eta_j(y)\rho_t^{R_i^j,\eta_j,*}(y)\,\mathrm{d}y\,\mathrm{d}x \\ &- \int_{\mathbb{R}^d} \rho_t^{\infty,\eta_j}(x)\eta_j(x) \int_{\mathbb{R}^d} W(x-y)\eta_j(y)\rho_t^{\infty,\eta_j}(y)\,\mathrm{d}y\,\mathrm{d}x \Bigg| = 0.\end{aligned}$$

Step 5: Convergence of the free energy when $\eta_j \to 1$. We now devote this step to prove (2.18). Steps 4a and 4b are analogous and therefore, up to a subsequence in j,

$$\lim_{j\to\infty}\int_{\mathbb{R}^d}\left(\rho_t^{\infty,\eta_j}\right)^m=\int_{\mathbb{R}^d}\left(\rho_t^{\infty,1}\right)^m \qquad \text{and} \qquad \lim_{j\to\infty}\int_{\mathbb{R}^d} V\rho_t^{\infty,\eta_j}=\int_{\mathbb{R}^d} V\rho^{\infty,1}.$$

To study the convergence of the aggregation term of the free energy, we define the symmetric bilinear form

$$\mathfrak{W}(u,v)=\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} u(x)W(x-y)v(y)\,\mathrm{d}y\,\mathrm{d}x.$$

Notice that the aggregation term on $\mathcal{F}_{\infty,\eta}[\rho]$ is $\frac{1}{2}\mathfrak{W}(\rho\eta_j,\rho\eta_j)$ (even for $\eta=1$). Now, we would like to prove that the limit

$$\lim_{j\to\infty}\mathfrak{W}(\rho_t^{\infty,\eta_j}\eta_j,\rho_t^{\infty,\eta_j}\eta_j)=\mathfrak{W}(\rho_t^{\infty,1},\rho_t^{\infty,1}) \tag{2.89}$$

holds. Let us proceed to check the claim. We have that

$$\left|\mathfrak{W}(\rho_t^{\infty,\eta_j}\eta_j,\rho_t^{\infty,\eta_j}\eta_j)-\mathfrak{W}(\rho_t^{\infty,1},\rho_t^{\infty,1})\right|\le\left|\mathfrak{W}(\rho_t^{\infty,\eta_j}\eta_j,\rho_t^{\infty,\eta_j}\eta_j-\rho_t^{\infty,1})\right|+\left|\mathfrak{W}(\rho_t^{\infty,\eta_j}\eta_j-\rho_t^{\infty,1},\rho_t^{\infty,1})\right|.$$

We focus on the first term of the RHS since the computations for the second one work in an analogous way. From (2.82) and taking $p=\frac{2}{1-m}$, we have that

$$\begin{aligned}&\left|\mathfrak{W}\left(\rho_t^{\infty,\eta_j}\eta_j,\rho_t^{\infty,\eta_j}\eta_j-\rho_t^{\infty,1}\right)\right|\\&\qquad\le C\left|\int_{\mathbb{R}^d}(1+|x|^2)\rho_t^{\infty,\eta_j}(x)\eta_j(x)\mathrm{d}x\right|\left|\int_{\mathbb{R}^d}(1+|y|^2)\left[\rho_t^{\infty,\eta_j}(y)\eta_j(y)-\rho_t^{\infty,1}(y)\right]\,\mathrm{d}y\right|.\end{aligned}$$

The first integral on the RHS is bounded due to (2.83). With a similar argument as before, we will prove that the second integral tends to zero as $j\to\infty$. For any $\sigma>0$ we have that

$$\begin{aligned}&\limsup_{j\to\infty}\sup_{t\in[0,T]}\left|\int_{\mathbb{R}^d}(1+|y|^2)\left[\rho_t^{\infty,\eta_j}(y)\eta_j(y)-\rho_t^{\infty,1}(y)\right]\,\mathrm{d}y\right|\\&\quad\le\limsup_{j\to\infty}\sup_{t\in[0,T]}\int_{B_\sigma}(1+|y|^2)\left|\rho_t^{\infty,\eta_j}(y)\eta_j(y)-\rho_t^{\infty,1}(y)\right|\,\mathrm{d}y\\&\qquad+\limsup_{j\to\infty}\sup_{t\in[0,T]}\int_{\mathbb{R}^d\setminus B_\sigma}(1+|y|^2)\rho_t^{\infty,\eta_j}(y)\eta_j(y)+\sup_{t\in[0,T]}\int_{\mathbb{R}^d\setminus B_\sigma}(1+|y|^2)\rho_t^{\infty,1}(y)\,\mathrm{d}y.\end{aligned} \tag{2.90}$$

The first $\limsup$ on the RHS is 0 due to the local convergences. For the second term we bound

$$\int_{\mathbb{R}^d\setminus B_\sigma}(1+|y|^2)\rho_t^{\infty,\eta_j}(y)\eta_j(y)\,\mathrm{d}y\le\frac{1+\sigma^2}{1+\sigma^p}\left(\|\rho_0\|_{L^1}+\mathbf{m_p}(\rho_t^{\infty,\eta_j})\right),$$

where we use $p=\frac{2}{1-m}>2$ because this moment is uniformly bounded. The last term of the RHS of (2.90) can also be bounded in a similar way by Fatou. Letting $\sigma\to\infty$ we recover that the RHS of (2.90) is 0. With this we conclude with the proof of (2.89). □

2.6.2 Long-time asymptotics. Proof of Theorem 2.13

Let us now take t_n a sequence of times such that $t_n\to\infty$. We define the sequence of functions

$$\begin{aligned}\rho^{\infty,1,[n]}\colon [0,1]\times\mathbb{R}^d&\longrightarrow\mathbb{R}\\(t,x)\quad&\mapsto\ \rho^{\infty,1}(t+t_n,x).\end{aligned} \tag{2.91}$$

In this subsection, we are going to study the limit when $n\to\infty$ of this sequence. Furthermore, we would also like to prove that there is no mass escaping through infinity when we take the limit in time. In $\mathbb{R}^d$, the free-energy of the FDE, $\partial_t u = \Delta u^m$ in the range $0 < m < 1$, is not bounded from below and is such that $u(t) \to 0$ as $t \to \infty$. This implies the necessity of requiring further assumptions. In fact, it suffices V to not be critical in the sense of constants, i.e. for some $\varepsilon \in (0,1)$

$$\begin{aligned} &\inf_{\substack{\rho\in\mathcal{M}_{ac}(\mathbb{R}^d),\\ \|\rho\|_{L^1(\mathbb{R}^d)}=\|\rho_0\|_{L^1(\mathbb{R}^d)}}} \left(\frac{1}{m-1}\int_{\mathbb{R}^d}\rho^m + \int_{\mathbb{R}^d} V\rho + \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K(x,y)\rho(y)\rho(x)\,\mathrm{d}y\,\mathrm{d}x\right)\\ &\geq \inf_{\substack{\rho\in\mathcal{M}_{ac}(\mathbb{R}^d),\\ \|\rho\|_{L^1(\mathbb{R}^d)}=\|\rho_0\|_{L^1(\mathbb{R}^d)}}} \left(\frac{1}{m-1}\int_{\mathbb{R}^d}\rho^m + (1-\varepsilon)\int_{\mathbb{R}^d} V\rho\right) > -\infty. \end{aligned} \tag{2.92}$$

From this assumption we can deduce the following result stated in [96],

Lemma 2.54 (Boundedness of $\|V\rho\|_{L^1(\mathbb{R}^d)}$, [96])**.** *Assume* (2.92)*. Then, there exists a constant* $C > 0$ *such that*

$$\int_{\mathbb{R}^d}\rho^m + \int_{\mathbb{R}^d} V\rho \leq C(1+\mathcal{F}[\rho]), \tag{2.93}$$

Moreover, by an argument from [79] extended in [96], we give a sufficient condition on V so that (2.92) holds.

Theorem 2.55 (Family of admissible potentials, [79, 96])**.** *Assume that, for some* $\alpha \in (0,m)$ *we have that*

$$\chi_V = \sum_{j=1}^{\infty} 2^{jn} V(2^j)^{-\frac{\alpha}{1-m}} < \infty. \tag{2.94}$$

Then, (2.92) *holds for any* $\varepsilon \in (0,1)$*.*

Then, we can prove Theorem 2.13 about the asymptotic in time of $\rho^{\infty,1}$.

Proof of Theorem 2.13. We divide the proof in several steps.

Step 1: Convergence to the limit. Let us take the sequence of functions defined in (2.91). Fix R_i^j and η_j and take $\Omega \subseteq B_{R_i^j}$ bounded. From the definition of the space $W^{-1,1}$ we have the *a priori* estimate,

$$\|u\|_{W^{-1,1}(\Omega)} = \inf_{\substack{u|_\Omega = \operatorname{div} F\\ F\in L^1(\Omega)}} \|F\|_{L^1(\Omega)} = \inf_{\substack{u|_\Omega = \operatorname{div} F\\ F\in L^1(\Omega)}} \|F\|_{L^1(\Omega)} \leq \inf_{\substack{u = \operatorname{div} F\\ F\in L^1(B_{R_i^j})}} \|F\|_{L^1(B_{R_i^j})} \leq \|u\|_{W^{-1,1}(B_{R_i^j})}.$$

Therefore, using the linearity of the extension and a similar strategy as in the proof of Theorem 2.4, we have that

$$\begin{aligned} \|\rho_t^{R_i^j,\eta_j,[n],*} - \rho_s^{R_i^j,\eta_j,[n],*}\|_{W^{-1,1}(\Omega)} &\leq \|\rho_t^{R_i^j,\eta_j,[n]} - \rho_s^{R_i^j,\eta_j,[n]}\|_{W^{-1,1}(B_{R_i^j})} \leq \int_s^t \left\|\frac{\partial\rho_\sigma^{R_i^j,\eta_j,[n]}}{\partial\sigma}\right\|_{W^{-1,1}(B_{R_i^j})} \mathrm{d}\sigma\\ &\leq \|\rho_0\|^{\frac{1}{2}}_{L^1(B_{R_i^j})}\left(\mathcal{F}_{R_i^j,\eta_j}[\rho_{t_n}^{R_i^j,\eta_j}] - \mathcal{F}_{R_i^j,\eta_j}[\rho_{t_n+1}^{R_i^j,\eta_j}]\right)^{\frac{1}{2}} |t-s|^{\frac{1}{2}}. \end{aligned}$$

Furthermore, from (2.16) taking the limit $i \to \infty$ we get that

$$\|\rho_t^{\infty,\eta_j,[n]} - \rho_s^{\infty,\eta_j,[n]}\|_{W^{-1,1}(\Omega)} \leq \|\rho_0\|^{\frac{1}{2}}_{L^1(B_{R_i^j})}\left(\mathcal{F}_{\infty,\eta_j}[\rho_{t_n}^{\infty,\eta_j}] - \mathcal{F}_{\infty,\eta_j}[\rho_{t_n+1}^{\infty,\eta_j}]\right)^{\frac{1}{2}} |t-s|^{\frac{1}{2}},$$

and, from the further convergence (2.18), taking $j \to \infty$ we obtain,

$$\|\rho_t^{\infty,1,[n]} - \rho_s^{\infty,1,[n]}\|_{W^{-1,1}(\Omega)} \leq \|\rho_0\|^{\frac{1}{2}}_{L^1(B_{R_i^j})}\left(\mathcal{F}_{\infty,1}[\rho_{t_n}^{\infty,1}] - \mathcal{F}_{\infty,1}[\rho_{t_n+1}^{\infty,1}]\right)^{\frac{1}{2}} |t-s|^{\frac{1}{2}}.$$

Take $\Omega = B_1$. By the Ascoli-Arzelá Theorem and the L^p bounds, there exists a subsequence in n such that $\rho^{\infty,1,[n]} \to \widehat{\mu}^{\{1\}}$ in $C([0,1]; W^{-1,1}(\Omega))$. Hence,

$$\|\widehat{\mu}_t^{\{1\}} - \widehat{\mu}_s^{\{1\}}\|_{W^{-1,1}(\Omega)} \leq \liminf \|\rho_t^{\infty,1,[n]} - \rho_s^{\infty,1,[n]}\|_{W^{-1,1}(\Omega)} = 0,$$

and $\widehat{\mu}^{\{1\}}$ does not depend on time in Ω.

Similarly, we construct a further subsequence that converges in B_2, and so on $\widehat{\mu}^{\{2\}} \in W^{-1,1}(B_2)$ (i.e. time independent). Using test functions in $W_0^{1,\infty}(B_1)$ we observe that $\widehat{\mu}^{\{2\}}|_{B_1} = \widehat{\mu}^{\{1\}}$. We proceed inductively for all B_k. Lastly, by a diagonal argument we get a distribution $\widehat{\mu}$ and a final subsequence $\rho^{\infty,1,[n]}$ that converges to it in $C([0,1]; W_{loc}^{-1,1}(\mathbb{R}^d))$.

Step 2: The limit is also weak-$*$ in the sense of measures. Since, $\rho_t^{\infty,1,[n]}$ are measures with total mass $\|\rho_0\|_{L^1(\mathbb{R}^d)}$, from Banach-Alaoglu Theorem, up to a subsequence, we have that

$$\rho_t^{\infty,1,[n]} \rightharpoonup \widehat{\mu} \quad \text{weak} - * \text{ in } L^\infty(0,1; \mathcal{M}(\mathbb{R}^d)).$$

Step 3: There is no mass escaping through infinity. Let us compute. Taking advantage of (2.93) we have that

$$\|\rho_0\|_{L^1(\mathbb{R}^d)} - \int_{B_R} \widehat{\mu}(x)\,\mathrm{d}x = \int_{\mathbb{R}^d \setminus B_R} \mathrm{d}\widehat{\mu} \leq \frac{1}{\inf_{x \in \mathbb{R}^d \setminus B_R} V(x)} \int_{\mathbb{R}^d} V\,\mathrm{d}\widehat{\mu} \leq \frac{C}{\inf_{x \in \mathbb{R}^d \setminus B_R} V(x)} = \omega(R^{-1}) \to 0$$

since $\inf_{x \in \mathbb{R}^d \setminus B_\sigma} V(x) \to \infty$ when $\sigma \to \infty$. So we conclude that the limit is also a probability measure. □

2.6.3 Viscosity solutions

The goal of this subsection is to study the mass problem in the whole space $\mathbb{R}^d$ taking advantage from viscosity solutions and all the theory developed on section 2.5. In order to do that, we define analogously $\mathfrak{E}_{\infty,\eta}$, $\Upsilon_{\infty,\eta}$ and $\mathcal{I}_\infty$ for the whole space $\mathbb{R}^d$ and we change the notation to $\mathfrak{E}_{R,\eta}$, $\Upsilon_{R,\eta}$ and $\mathcal{I}_R$ for the case we studied on section 2.5, where we extend by 0 if we need to. Furthermore, in the same way we did on section 2.5, we assume V, W and η radially symmetric. We study the problem,

$$\begin{cases} \dfrac{\partial M}{\partial t} = \kappa(v)^2 \dfrac{\partial}{\partial v}\left(\dfrac{\partial M}{\partial v}\right)^m + \kappa(v)^2 \dfrac{\partial M}{\partial v} \mathfrak{E}_{\infty,\eta}\left[\mathcal{I}_\infty^{-1}\left[\dfrac{\partial M}{\partial v}\right]\right](t,v), & t, v > 0, \\ M(t,0) = 0, & t > 0 \\ M(0,v) = \displaystyle\int_{\widetilde{B_v}} \rho_0(x)\,\mathrm{d}x, & v > 0 \end{cases} \tag{2.95}$$

for which we need a notion of viscosity solution. Since at this point we cannot guarantee that $\partial M/\partial v$ is positive, it is better to simplify (2.95) expanding the second derivative and multiplying by $\frac{1}{m}\left(\frac{\partial M}{\partial v}\right)^{1-m}$.

Definition 2.56. *A function $M \in C([0,T]; C((0,\infty)) \cap BV([0,\infty))$ is a viscosity supersolution of (2.95) if, for every $t_0 > 0$, $v_0 \in (0,\infty)$ and for every $\varphi \in C^2((t_0 - \varepsilon, t_0 + \varepsilon) \times (v_0 - \varepsilon, v_0 + \varepsilon))$ such that $M \geq \varphi$, $M(v_0) = \varphi(v_0)$ and $\frac{\partial \varphi}{\partial v}(v) \neq 0$ for all $v \neq v_0$ it holds that*

$$\frac{1}{m}\left(\frac{\partial \varphi}{\partial v}(t_0,v_0)\right)^{1-m} \frac{\partial \varphi}{\partial t}(t_0,v_0) \geq \kappa(v_0)^2 \left(\frac{\partial^2 \varphi}{\partial v^2}(t_0,v_0) + \frac{1}{m}\left(\frac{\partial \varphi}{\partial v}(t_0,v_0)\right)^{2-m} \mathfrak{E}_{\infty,\eta}\left[\mathcal{I}_\infty^{-1}\left[\frac{\partial M}{\partial v}\right]\right](t_0,v_0)\right).$$

The corresponding definition of subsolution is made by inverting the inequalities. A viscosity solution is a function that is a viscosity sub and supersolution.

Remark 2.57. *Notice that, in the viscosity formulation we do not replace $\partial M/\partial v$ by the test function in the non-local term.*

In order to obtain viscosity solutions of the problem (2.95), we base our strategy on extending the solutions of the problem in the ball (2.49). We define,

$$M^{R,\eta}(t,v) = \int_{\widetilde{B_v}} \rho_t^{R,\eta},$$

which we now extend to the whole space in the natural way,

$$M^{R,\eta,*}(t,v) = \int_{\widetilde{B_v}} \rho_t^{R,\eta,*} = \begin{cases} M^{R,\eta}(t,v) & \text{if } v \le R_v, \\ \|\rho_0\|_{L^1(B_R)} & \text{if } v > R_v. \end{cases} \tag{2.96}$$

Once we have presented the problem we are ready to state the main result of this subsection.

Proposition 2.58. *Assume all the hypothesis from Theorem 2.12. Assume furthermore that ρ_0, V, and W are radially symmetric and (2.92). Thus, for j fixed there exists a sequence $R_i^j \to \infty$ and $M^{\infty,\eta_j} \in C([0,\infty) \times (0,\infty))$ such that:*

- $\mathfrak{E}_{R_i^j,\eta_j}[\rho^{R_i^j,\eta_j,*}] \to \mathfrak{E}_{\infty,\eta_j}[\rho^{\infty,\eta_j}]$ *in* $C_{loc}([0,\infty) \times (0,\infty))$,
- $\frac{\partial M^{\infty,\eta_j}}{\partial v} = \mathcal{I}_\infty[\rho^{\infty,\eta_j}]$ *belongs to* $C_{loc}([0,\infty); L^2_{loc}[0,\infty))$,
- $\sup_{[0,T]\times[0,\infty)} |M^{R_i^j,\eta_j,*}(t,v) - M^{\infty,\eta_j}(t,v)| \to 0$,
- M^{∞,η_j} *is a viscosity solution of* (2.95).

Then, if we let $\eta_j \nearrow 1$ there exists $M^{\infty,1} \in C([0,\infty) \times (0,\infty))$ such that:

- $\mathfrak{E}_{\infty,\eta_j}[\rho^{\infty,\eta_j}] \to \mathfrak{E}_{\infty,1}[\rho^{\infty,1}]$ *in* $C_{loc}([0,\infty) \times (0,\infty))$,
- $\frac{\partial M^{\infty,1}}{\partial v} = \mathcal{I}_\infty[\rho^{\infty,1}]$ *belongs to* $C_{loc}([0,\infty); L^2_{loc}[0,\infty))$,
- $\sup_{[0,T]\times[0,\infty)} |M^{\infty,\eta_j}(t,v) - M^{\infty,1}(t,v)| \to 0$,
- $M^{\infty,1}$ *is a viscosity solution of* (2.95) *for* $\eta \equiv 1$.

Proof. We divide the proof in several steps.

Step 1: Convergence on R. Let us fix η_j. Due to the C^α regularity (2.54) and the uniform boundedness of $\|M^{R_i^j,\eta_j,*}\|_{L^\infty([0,1]\times\mathbb{R})}$ we know that, up to a further subsequence on i, by the Ascoli-Arzela Theorem, for any $T > 0$ and $v_1, v_2 > 0$,

$$M^{R_i^j,\eta_j,*} \to M^{\infty,\eta_j} \quad \text{in } C([0,T] \times [v_1,v_2]). \tag{2.97}$$

On Theorem 2.12 we show that, up to a subsequence in i, we have that

$$\rho^{R_i^j,\eta_j,*} \to \rho^{\infty,\eta_j} \quad \text{in } C_{loc}([0,\infty); L^2_{loc}(\mathbb{R}^d)).$$

From here it follows immediately that

$$\mathfrak{E}_{R_i^j,\eta_j}[\rho^{R_i^j,\eta_j,*}] \to \mathfrak{E}_{\infty,\eta_j}[\rho^{\infty,\eta_j}] \quad \text{in } C_{loc}([0,\infty) \times \mathbb{R}^d). \tag{2.98}$$

From (2.97) and (2.98), applying the stability result Theorem 1.8 we recover that M^{∞,η_j} is a viscosity solution of the mass equation (2.95) for every cut-off $\eta_j \in C_c^\infty(\mathbb{R}^d)$.

Furthermore, from Theorem 2.12 we know that

$$\rho^{R_i^j,\eta_j,*} \to \rho^{\infty,\eta_j} \quad \text{in } C_{loc}([0,\infty); L^2_{loc}(\mathbb{R}^d)).$$

Then, since $\mathcal{I}_\infty[\rho_t^{R_i^j,\eta_j,*}] = \frac{\partial M^{R_i^j,\eta_j,*}(t,\cdot)}{\partial v}$ and $\mathcal{I}_\infty$ is an isometry from $L^p_{rad}(\mathbb{R}^d)$ to $L^p(0,\infty)$ we also get that

$$\frac{\partial M^{R_i^j,\eta_j,*}(t,\cdot)}{\partial v} \to \frac{\partial M^{\infty,\eta_j}(t,\cdot)}{\partial v} \quad \text{in } C_{loc}([0,\infty); L^2_{loc}(\mathbb{R}^d)),$$

and that $\mathcal{I}_\infty[\rho_t^{\infty,\eta_j}] = \frac{\partial M^{\infty,\eta_j}(t,\cdot)}{\partial v}$ in $C_{loc}([0,\infty); L^2_{loc}[0,\infty))$.

Step 2: Asymptotics in space. With all the theory we have already developed we can study the asymptotic behaviour in space. We compute the following,

$$\begin{aligned}\sup_{[0,T]\times[0,\infty)} |M^{R_i^j,\eta_j,*} - M^{\infty,1}| &= \sup_{[0,T]\times[0,\infty)} \left| \int_{\widetilde{B_v}} \rho_t^{R_i^j,\eta_j,*} - \rho^{\infty,\eta_j} \right| \\ &\le \sup_{[0,T]} \int_{B_\sigma} \left| \rho_t^{R_i^j,\eta_j,*} - \rho^{\infty,\eta_j} \right| + \frac{1}{\sigma^2} \int_{\mathbb{R}^d \setminus B_\sigma} |x|^2 \rho_t^{R_i^j,\eta_j,*} + \frac{1}{\sigma^2} \int_{\mathbb{R}^d \setminus B_\sigma} |x|^2 \rho^{\infty,\eta_j}.\end{aligned}$$

Due to the (2.15), the first term in the RHS converges to 0 for every B_σ when we take the limit in i. For the second term we are taking advantage of (2.83). Since $\mathbf{m_2}(\rho^{R_i^j,\eta_j,*})$ is uniformly bounded the second term converges to 0 when $\sigma \to \infty$. For the third term we use again the same argument combined with Fatou's Lemma.

Step 3: Extension $\eta \to 1$. All the bounds discussed in the previous step are uniform on $\operatorname{supp} \eta_j$. Then, in the same way we have just done, there exists $M^{\infty,1}$ such that, up to a subsequence on j,

$$M^{\infty,\eta_j} \to M^{\infty,1} \quad \text{in } C_{loc}([0,T] \times (0,\infty)).$$

We now claim that

$$\mathfrak{E}_{\infty,\eta_j}[\rho^{\infty,\eta_j}] \to \mathfrak{E}_{\infty,1}[\rho^{\infty,1}] \quad \text{in } C_{loc}([0,T] \times [0,\infty)).$$

In order to prove this, it is enough to show that for a fixed $r > 0$, when $j \to \infty$, we have that

$$\eta_j(re_1) \frac{\partial}{\partial r} \int_{\mathbb{R}^d} \rho_t^{\infty,\eta_j}(y) W(re_1 - y) \eta_j(y)\, \mathrm{d}y \longrightarrow \frac{\partial}{\partial r} \int_{\mathbb{R}^d} \rho_t^{\infty,1}(y) W(re_1 - y)\, \mathrm{d}y.$$

Let us take j big enough so that $\eta_j(re_1) = 1$ and compute,

$$\begin{aligned}&\left| \int_{\mathbb{R}^d} \nabla W(re_1 - y) \left[\rho_t^{\infty,1}(y) - \rho_t^{\infty,\eta_j}(y) \eta_j(y) \right] \mathrm{d}y \right| \\ &\quad \le \left| \int_{\mathbb{R}^d} \nabla W(re_1 - y) \rho_t^{\infty,\eta_j}(y) \left[1 - \eta_j(y)\right] \mathrm{d}y \right| + \left| \int_{\mathbb{R}^d} \nabla W(re_1 - y) \left[\rho_t^{\infty,1}(y) - \rho_t^{\infty,\eta_j}(y) \right] \mathrm{d}y \right|.\end{aligned}$$

Let us control the first term,

$$\begin{aligned}\left| \int_{\mathbb{R}^d} \nabla W(re_1 - y) \rho_t^{\infty,\eta_j}(y) \left[1 - \eta_j(y)\right] \mathrm{d}y \right| &\le \left| \int_{\mathbb{R}^d \setminus B_{\frac{j}{2}}} \nabla W(re_1 - y) \rho_t^{\infty,\eta_j}(y)\, \mathrm{d}y \right| \\ &\le (1+r) \int_{\mathbb{R}^d \setminus B_{\frac{j}{2}}} (1+|y|) \rho_t^{\infty,\eta_j}(y)\, \mathrm{d}y \\ &\le (1+r) \frac{1 + j/2}{1 + (j/2)^2} \int_{\mathbb{R}^d \setminus B_{\frac{j}{2}}} (1+|y|^2) \rho_t^{\infty,\eta_j}(y)\, \mathrm{d}y.\end{aligned}$$

For the second one, we again introduce an intermediate ball of radius B_σ and we obtain that

$$\limsup_{j\to\infty} \sup_{t\in[0,T]} \left| \int_{\mathbb{R}^d} \nabla W(re_1 - y) \left[\rho_t^{\infty,1}(y) - \rho_t^{\infty,\eta_j}(y) \right] \mathrm{d}y \right|$$

$$\leq (1+r) \limsup_{j\to\infty} \sup_{t\in[0,T]} \Big(\int_{B_\sigma} (1+|y|) \left|\rho_t^{\infty,1}(y) - \rho_t^{\infty,\eta_j}(y)\right| \mathrm{d}y$$
$$+ \frac{1+\sigma}{1+\sigma^2} \int_{\mathbb{R}^d\setminus B_\sigma} |y|^2 \rho_t^{\infty,1}(y)\,\mathrm{d}y + \frac{1+\sigma}{1+\sigma^2} \int_{\mathbb{R}^d\setminus B_\sigma} (1+|y|^2)\rho_t^{\infty,\eta_j}(y)\,\mathrm{d}y \Big)$$
$$\leq (1+r) \limsup_{j\to\infty} \sup_{t\in[0,T]} \int_{B_\sigma} (1+|y|) \left|\rho_t^{\infty,1}(y) - \rho_t^{\infty,\eta_j}(y)\right| \mathrm{d}y + 2\omega_T(\sigma^{-1}) \to 0$$

as $\sigma \to \infty$. The first term is zero by local convergence (the argument works analogously to the computation above), and the second one converges to zero as $\sigma \to \infty$.

Then, due to the stability result Theorem 1.8, for any $T > 0$ we have that $M^{\infty,1}$ is a viscosity solution of the problem (2.95) in $[0,T]\times\mathbb{R}^d$ for $\eta \equiv 1$.

We also know from the previous step that $\mathcal{I}_\infty[\rho_t^{\infty,\eta_j}] = \frac{\partial M^{\infty,\eta_j}(t,\cdot)}{\partial v}$. Then, since $\mathcal{I}_\infty$ is an isometry, from (2.17) we also get that,

$$\frac{\partial M^{\infty,\eta_j}(t,\cdot)}{\partial v} \to \frac{\partial M^{\infty,1}(t,\cdot)}{\partial v} \quad \text{in } C_{loc}([0,\infty); L^2_{loc}(0,\infty)),$$

and that $\mathcal{I}_\infty[\rho_t^{\infty,1}] = \frac{\partial M^{\infty,1}(t,\cdot)}{\partial v}$ in $C_{loc}([0,\infty); L^2_{loc}([0,\infty))$. Since $M(0,v) = M_0(v)$, with a diagonal argument we can show that, up to a further subsequence, the limit $M^{\infty,1}$ is such that $M^{\infty,1} \in C_{loc}([0,\infty)\times[0,\infty))$.

Furthermore, through the control of the tails we have uniform convergence in v since, introducing balls of radius B_σ, we have that as $\sigma \to \infty$.

$$\limsup_j \sup_{[0,T]\times[0,\infty)} |M^{\infty,\eta_j} - M^{\infty,1}| = \sup_{[0,T]\times[0,\infty)} \left| \int_{\widetilde{B_v}} \rho_t^{\infty,\eta_j} - \rho^{\infty,1} \right|$$
$$\leq \limsup_j \sup_{[0,T]} \int_{B_\sigma} \left|\rho_t^{\infty,\eta_j} - \rho^{\infty,1}\right| + \frac{1}{\sigma^2} \int_{\mathbb{R}^d\setminus B_\sigma} |x|^2 \rho_t^{\infty,\eta_j} + \frac{1}{\sigma^2} \int_{\mathbb{R}^d\setminus B_\sigma} |x|^2 \rho^{\infty,1} \to 0$$

The result follows from the local convergence and the boundedness of the second moment in $[0,T]$. □

2.6.4 Convergence of the mass equation as $t \to \infty$ in the whole space

Let us recall $\rho^{\infty,1,[n]}$ defined in (2.91). Now we define,

$$M^{[n]}(t,v) = \int_{\widetilde{B_v}} \rho_t^{\infty,1,[n]}(x)\,\mathrm{d}x, \tag{2.99}$$

which are solutions to the mass equation (2.95) for the time interval $[t_n, t_n+1]$ after a translation to $[0,1]$.

On the assumptions of Theorem 2.12 we already know that the asymptotic in time of $\rho^{\infty,1,[n]}$ is a distribution $\widehat{\mu} \in W^{-1,1}_{loc}(\mathbb{R}^d)$. We can study the limit $M^{[n]} \to \widehat{M}$ and discuss its relationship with the limit $\widehat{\mu}$.

Proposition 2.59. *Assume all the hypothesis from Theorem 2.12. Assume furthermore that ρ_0, V, and W are radially symmetric, (2.92), (H7), and (H8). Let the sequence $M^{[n]}$ be defined as in (2.99), then there exists $\widehat{M} \in C([0,1]\times(0,\infty))$, such that, if we define $\widehat{M}(t,0) = 0$ for all $t \in [0,1]$, then the following properties hold:*

- *The function $\widehat{M}$ is non-decreasing,*
- *up to a subsequence, $M^{[n]} \to \widehat{M}$ in $C_{loc}([0,1]\times(0,\infty))$ and point-wise in $[0,1]\times[0,\infty)$,*
- *$\widehat{M}$ does not depend on time,*
- *$\frac{\partial \widehat{M}}{\partial v} = \mathcal{I}_\infty[\widehat{\mu}]$ in $L^\infty(\mathcal{M}([0,\infty)))$,*
- *$\mathfrak{E}_{\infty,1}[\rho^{\infty,1,[n]}] \to \mathfrak{E}_{\infty,1}[\widehat{\mu}]$ in $C_{loc}([0,1]\times[0,\infty))$,*

- *$\widehat{M}$ is a viscosity solution of (2.95) for $\eta \equiv 1$.*

Furthermore, if we also assume that $\inf_{x\in\mathbb{R}^d\setminus B_\sigma} V(x) \to \infty$ when $\sigma \to \infty$. Then,

$$\|\rho_0\|_{L^1(\mathbb{R}^d)} - \widehat{M}(t,v) = \int_{\mathbb{R}^d\setminus \widetilde{B_v}} \widehat{\mu} = \omega(v^{-1}) \to 0.$$

The proof is very similar to the one presented in Theorem 2.6 for the case of the ball B_R.

Proof. We divide the proof in several steps.
Step 1: Ascoli-Arzela over compacts of $[0,1]\times(0,\infty)$. For any $k \in \mathbb{Z}_{>0}$ we know from (2.54) that for all $v_1, v_2 \in [\frac{1}{k}, k]$ and $t_1, t_2 \in [0,1]$ we have the estimate,

$$\left| M^{[n]}(t_1,v_1) - M^{[n]}(t_2,v_2) \right| \le C_k \left(|v_1 - v_2| + |t_1-t_2|^{\frac{1}{m+1}} \right)^\alpha. \tag{2.100}$$

Furthermore, taking successive subsequences,

$$M^{[n(k,j)]} \to \widehat{M} \quad \text{in } C\left([0,1]\times\left[\tfrac{1}{k},k\right]\right) \text{ as } j\to\infty,$$

and the diagonal satisfies

$$M^{[n(k,k)]} \to \widehat{M} \quad \text{in } C_{loc}([0,1]\times(0,\infty)). \tag{2.101}$$

From here on we re-label this sequence again as $M^{[n]}$.
Step 2: Extension of $\widehat{M}$ and some properties. Since $M^{[n]}(t,0) = 0$ for all n, the extension of $\widehat{M}$ so that $\widehat{M}(t,0) = 0$ is natural. Therefore, due to (2.100) we have that

$$M^{[n]}(t,v) \to \widehat{M}(t,v) \quad \text{point-wise in } [0,1]\times[0,\infty).$$

Since $M^{[n]}$ are non-decreasing functions bounded from below by 0 and from above by $\|\rho_0\|_{L^1(\mathbb{R}^d)}$, they all have bounded variation $\|\rho_0\|_{L^1(\mathbb{R}^d)}$. From Banach-Alaoglu Theorem and (2.100), up to a subsequence, we have that,

$$\frac{\partial M^{[n]}}{\partial v} \rightharpoonup \frac{\partial \widehat{M}}{\partial v} \quad \text{weak}-* \text{ in } L^\infty(0,1;\mathcal{M}([0,\infty))). \tag{2.102}$$

Step 3: Identification between $\frac{\partial \widehat{M}}{\partial v}$ and $\mathcal{I}_\infty[\widehat{\mu}]$. From Theorem 2.13 we have that

$$\rho_t^{\infty,1,[n]} \rightharpoonup \widehat{\mu} \quad \text{weak}-* \text{ in } L^\infty(0,1;\mathcal{M}(\mathbb{R}^d)).$$

Since $\mathcal{I}_\infty$ is a linear isometry we also obtain that

$$\mathcal{I}_\infty[\rho_t^{\infty,1,[n]}] \rightharpoonup \mathcal{I}_\infty[\widehat{\mu}] \quad \text{weak}-* \text{ in } L^\infty(0,1;\mathcal{M}([0,\infty))). \tag{2.103}$$

Furthermore, since $\mathcal{I}_\infty[\rho_t^{\infty,1,[n]}] = \frac{\partial M^{[n]}}{\partial v}(t,\cdot)$, if we combine (2.102) and (2.103) we obtain that $\frac{\partial \widehat{M}}{\partial v} = \mathcal{I}_\infty[\widehat{\mu}]$ in $L^\infty(\mathcal{M}([0,\infty)))$. We claim that

$$\mathfrak{E}_{\infty,1}[\rho^{\infty,1,[n]}] \to \mathfrak{E}_{\infty,1}[\widehat{\mu}] \quad \text{in } C_{loc}([0,1]\times[0,\infty)).$$

Let us proceed to prove it. Let us take a cut-off function $\psi \in C_c^\infty(\mathbb{R}^d)$ such that $\psi(x) = 1$ if $x \in B_\sigma$ and $\psi(x) = 0$ if $x \in \mathbb{R}^d\setminus B_{2\sigma}$.

$$\begin{aligned}
&\limsup_{n\to\infty} \left| \int_{\mathbb{R}^d} \nabla W(re_1 - y) \left[\rho_t^{\infty,1,[n]}(y) - \widehat{\mu}(y) \right] \,\mathrm{d}y \right| \\
&\quad \le \limsup_{n\to\infty} \left| \int_{\mathbb{R}^d} \psi(y) \nabla W(re_1 - y) \left[\rho_t^{\infty,1,[n]}(y) - \widehat{\mu}(y) \right] \,\mathrm{d}y \right|
\end{aligned}$$

$$+ \limsup_{n\to\infty} \left| \int_{\mathbb{R}^d} (1-\psi(y))\nabla W(re_1 - y) \left[\rho_t^{\infty,1,[n]}(y) - \widehat{\mu}(y)\right] \mathrm{d}y \right|.$$

The first term of the RHS converges to zero since $\psi(\cdot)\nabla W(re_1 - \cdot) \in W_0^{1,\infty}(B_{2\sigma})$ and $\rho^{\infty,1,[n]} \to \widehat{\mu}$ in $C([0,1]; W_{loc}^{-1,1}(\mathbb{R}^d))$ due to Theorem 2.13. We focus now on the second term,

$$\begin{aligned}
&\limsup_{n\to\infty} \left| \int_{\mathbb{R}^d} (1-\psi(y))\nabla W(re_1 - y) \left[\rho_t^{\infty,1,[n]}(y) - \widehat{\mu}(y)\right] \mathrm{d}y \right| \\
&\quad \le \limsup_{n\to\infty} \int_{\mathbb{R}^d\setminus B_\sigma} |\nabla W(re_1 - y)| \left[\rho_t^{\infty,1,[n]}(y) + \widehat{\mu}(y)\right] \mathrm{d}y \\
&\quad = \limsup_{n\to\infty} \int_{\mathbb{R}^d\setminus B_\sigma} \frac{|\nabla W(re_1 - y)|}{V(re_1 - y)} \frac{V(re_1 - y)}{1+V(y)} (1+V(y)) \left[\rho_t^{\infty,1,[n]}(y) + \widehat{\mu}(y)\right] \mathrm{d}y \\
&\quad \le C \left\| \frac{|\nabla W|}{V} \right\|_{L^\infty(\mathbb{R}^d\setminus re_1 - B_\sigma)} \left(\limsup_{n\to\infty} \int_{\mathbb{R}^d\setminus B_\sigma} (1+V(y))\rho_t^{\infty,1,[n]}(y)\,\mathrm{d}y + \int_{\mathbb{R}^d\setminus B_\sigma} (1+V(y))\widehat{\mu}(y)\,\mathrm{d}y \right) \\
&\quad \to 0,
\end{aligned}$$

as $\sigma \to \infty$ due to (H7), (H8) and (2.93).

Thus, using the stability results Theorem 1.8 we get that $\widehat{M}$ is a viscosity solution of (2.95). □

Let us state a case in which Remark 2.15 holds and we can characterize $\widehat{\mu}$.

Proposition 2.60. *Assume all the hypothesis from Theorem 2.12. Assume furthermore that ρ_0, V and W are radially symmetric and the technical assumptions (2.92), (H7) and (H8). If $\mathfrak{E}_{\infty,1}[\widehat{\mu}]$ is such that*

$$\mathfrak{E}_{\infty,1}[\widehat{\mu}](v) \ge 0.$$

Then $\widehat{M}$, the limit in time obtained in Proposition 2.59, is $C^2((0,\infty))$.

The proof is analogous to the one from Proposition 2.9.

Let us assume that V is a radially increasing and symmetric potential that satisfies the hypothesis presented on Theorem 2.12, (2.92), (H7) and (H8). If we take $V(x) \ge 2|x|^2 + V_0(x)$, with $V(0) = 0$ and we choose $W(x) = \frac{|x|^2}{2}$. Then $\mathfrak{E}_{\infty,1}[\widehat{\mu}] \ge 0$ for every $\widehat{\mu}$ positive and radially symmetric. On section 2.7, we show that there exists concentration for this example.

2.7 Examples of concentration phenomena

In this section we prove that if ρ solves the Euler-Lagrange equation

$$-\frac{m}{1-m}\rho^{m-1}(x) + V(x) + \int_{B_R} \eta(x)W(x-y)\eta(y)\rho(y)\,\mathrm{d}y = -h, \tag{2.104}$$

then, for certain choices of V and W, we have concentration phenomena (i.e., a Dirac delta formation) for both: The problem in B_R, and the problem in the whole space $\mathbb{R}^d$ with $\eta \equiv 1$. Through this section, we will assume $\eta(x) = 1$ if $|x| \le \frac{R}{\sqrt{2}}$ whenever we are studying the problem on the ball B_R.

Theorem 2.61. *Assume ρ solves the Euler-Lagrange equation (2.104), ρ is radial and $\int \rho(x)\,\mathrm{d}x \le 1$. Let us assume that V is a radially increasing and symmetric potential that satisfies the hypothesis presented on Theorem 2.12, (2.92), (H7) and (H8). If $V(x) \ge 2|x|^2 + V_0(x)$, $V(0) = 0$ and $W(x) = \frac{|x|^2}{2}$, then*

$$\rho(x) \le \left(\frac{1-m}{m}V_0(x)\right)^{-\frac{1}{1-m}}. \tag{2.105}$$

Remark 2.62. *If V_0 is such that $I_0 := \int_{B_R} \left(\frac{1-m}{m} V_0(x)\right)^{-\frac{1}{1-m}} < 1$, then, estimate (2.105), implies that the solution ρ of the Euler-Lagrange equation (2.104) has mass $\|\rho\|_{L^1(B_R)} \leq I_0 < 1$. Let us restrict to the class of probability measures, $\mu \in \mathcal{P}(B_R)$. Theorem 2.61 implies that ρ, the absolutely continuous part of any $\mu \in \mathcal{P}(B_R)$, solving (2.104) has mass strictly less than 1. Indeed, integrating in (2.105) gives $\|\rho\|_{L^1(B_R)} = \|\mu_{\mathrm{ac}}\|_{L^1(B_R)} \leq I_0 < 1 = \mu(B_R)$, i.e. $\mu_{\mathrm{s}}(B_R) > 0$, where μ_{s} denotes the singular part of μ.*

Proof of Theorem 2.61. Since $W(x) = \frac{|x|^2}{2}$, we can rewrite (2.104) like,

$$-\frac{m}{1-m}\rho^{m-1} + V + \frac{\eta(x)}{2}|x|^2 A = -h - \frac{\eta(x)}{2} B,$$

where

$$A = \int_{B_R} \eta(y)\rho(y)\,\mathrm{d}y$$

and

$$B = \int_{B_R} \eta(y)|y|^2\rho(y)\,\mathrm{d}y.$$

Since $V(0) = 0$, if we evaluate on $x = 0$ we get that

$$-\frac{m}{1-m}\rho^{m-1}(0) = -h - \frac{B}{2},$$

from where we deduce that $h + \frac{B}{2} \geq 0$. Then, we get that ρ is of the form,

$$\rho(x) = \left[\frac{1-m}{m}\left(h + \frac{\eta(x)}{2}B + \frac{\eta(x)}{2}|x|^2 A + V(x)\right)\right]^{-\frac{1}{1-m}}.$$

We distinguish between two cases. First, if $|x| \leq \frac{R}{\sqrt{2}}$, we have that $\eta(x) = 1$ and $h + \frac{\eta(x)}{2}B \geq 0$. Therefore,

$$\rho(x) \leq \left(\frac{1-m}{m}V(x)\right)^{-\frac{1}{1-m}} \leq \left(\frac{1-m}{m}V_0(x)\right)^{-\frac{1}{1-m}}$$

and we recover (2.105). For the second case, if $|x| > \frac{R}{\sqrt{2}}$,

$$0 \leq B = \int_{B_R} \eta(y)|y|^2\rho(y)\,\mathrm{d}y \leq R^2 \int_{B_R} \rho(y)\,\mathrm{d}y \leq R^2.$$

Thus, $h \geq -\frac{B}{2} \geq -\frac{R^2}{2}$, and, in particular,

$$\rho(x) \leq \left[\frac{1-m}{m}\left(-\frac{R^2}{2} + V(x)\right)\right]^{-\frac{1}{1-m}}.$$

$V(x) \geq |x|^2 + V_0(x) \geq \frac{R^2}{2} + V_0(x)$ when $|x| \geq \frac{R}{\sqrt{2}}$. Then,

$$\rho(x) \leq \left(\frac{1-m}{m}V_0(x)\right)^{-\frac{1}{1-m}},$$

and we recover (2.105) again, finishing with the proof. □

Remark 2.63. *The proof works in the same way when we are in the whole space $\mathbb{R}^d$ and $\eta \equiv 1$. In this case, we just need $V(x) \geq V_0(x)$. In the same way as before, we also obtain examples where aggregation happens.*

2.A Uniform estimates for ρ as $R \to \infty$

In this appendix we include some of the *a priori* estimates that we need in section 2.6 to prove Theorem 2.12.

First, we realise that using the estimate for the second order moment (2.83) we are able to obtain the *a priori* estimate for a p-order moment with $p = \frac{2}{1-m} > 2$.

Proof of Lemma 2.53. In the same way we did before for the second moment, we take the time derivative,

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\mathbf{m_p}(\rho_t^{R,\eta_j,*}) &= -\int_{\mathbb{R}^d} |x|^{p-2}x \cdot \nabla(\rho_t^{R,\eta_j,*})^m - \int_{\mathbb{R}^d} |x|^{p-2}x \cdot \rho^{R,\eta_j,*}\nabla\left(V_R + \int_{\mathbb{R}^d} K_\eta(x,y)\rho_t^{R,\eta_j,*}(y)\,\mathrm{d}y\right)\\ &\le (p-1)\mathbf{m_{p-2}}((\rho_t^{R,\eta_j,*})^m)\\ &\quad + \left\|\frac{\nabla V_R + \nabla\int_{\mathbb{R}^d} K_\eta(\cdot,y)\rho_t^{R,\eta_j,*}(y)\,\mathrm{d}y}{1+|\cdot|}\right\|_{L^\infty(\mathbb{R}^d)} \int_{\mathbb{R}^d} |x|^{p-1}(1+|x|)\rho_t^{R,\eta_j,*}\,\mathrm{d}x.\end{aligned} \tag{2.106}$$

Now we bound each of the terms. For the first term we have that

$$\mathbf{m_{p-2}}\left((\rho_t^{R,\eta_j,*})^m\right) \le \left(\mathbf{m_{\frac{p-2}{m}}}(\rho_t^{R,\eta_j,*})\right)^m \|\rho_t^{R,\eta_j,*}\|_{L^{\frac{1}{1-m}}} \le m\,\mathbf{m_p}(\rho_t^{R,\eta_j,*}) + (1-m)\|\rho_t^{R,\eta_j,*}\|_{L^{\frac{1}{1-m}}}^{\frac{1}{1-m}}.$$

Notice that we have chosen p such that $p = \frac{p-2}{m}$. The second term is bounded due to the L^p estimate (2.23). The term $\nabla V_R/(1+|x|)$ is uniformly bounded by hypothesis. We expand the derivative

$$\begin{aligned}\nabla\int_{\mathbb{R}^d} K(x,y)\rho_t^{R,\eta_j,*}(y)\,\mathrm{d}y &= \int_{B_R} \eta_j(x)\nabla W(x-y)\eta_j(y)\rho_t^{R,\eta_j,*}(y)\,\mathrm{d}y\\ &\quad + \int_{\mathbb{R}^d} \nabla\eta_j(x)W(x-y)\eta_j(y)\rho_t^{R,\eta_j,*}(y)\,\mathrm{d}y.\end{aligned}$$

Since we assume $|\nabla W(x)| \le C(1+|x|)$ from (H5) and from there we also deduce $W(x-y) \le C(1+|x|^2)(1+|y|^2)$ at (2.82), we obtain that

$$\begin{aligned}\left|\int_{B_R} \eta_j(x)\nabla W(x-y)\eta_j(y)\rho_t^{R,\eta_j,*}(y)\,\mathrm{d}y\right| &\le C(W)\int_{\mathbb{R}^d} \eta_j(x)(1+|x|+|y|)\eta_j(y)\,\rho_t^{R,\eta_j,*}(y)\,\mathrm{d}y\\ &\le C(W)\left((1+|x|)\|\rho_t^{R,\eta_j,*}\|_{L^1(\mathbb{R}^d)} + \mathbf{m_1}(\rho_t^{R,\eta_j,*})\right)\\ &\le C(W)\left((1+|x|)\|\rho_0\|_{L^1(\mathbb{R}^d)} + \|\rho_0\|_{L^1(\mathbb{R}^d)}^{\frac{1}{2}} + \mathbf{m_2}(\rho_t^{R,\eta_j,*})^{\frac{1}{2}}\right)\end{aligned}$$

For the next term, we take into account that $|\nabla\eta_j(x)| \le j^{-1}\|\nabla\eta_1\|_{L^\infty(\mathbb{R}^d)}\chi_{B_j}(x)$, and we can therefore estimate

$$\begin{aligned}\left|\int_{\mathbb{R}^d} \nabla\eta_j(x)W(x-y)\eta_j(y)\rho_t^{R,\eta_j,*}(y)\,\mathrm{d}y\right| &\le C(W)|\nabla\eta_j(x)|(1+|x|^2)\left(\|\rho_0\|_{L^1(\mathbb{R}^d)} + \mathbf{m_2}(\rho_t^{R,\eta_j,*})\right)\\ &\le C(W)\|\nabla\eta_1\|_{L^\infty(\mathbb{R}^d)}(j^{-1}+|x|)\left(\|\rho_0\|_{L^1(\mathbb{R}^d)} + \mathbf{m_2}(\rho_t^{R,\eta_j,*})\right)\end{aligned}$$

where $C(W)$ is the combination of the constants appearing in (H5) and (2.82), and we control $\mathbf{m_2}$ due to (2.83). Lastly, using Hölder inequality on the $p-1$ moment and Young's inequality afterwards we get,

$$\begin{aligned}\int |x|^{p-1}(1+|x|)\rho_t^{R,\eta_j,*} &= \mathbf{m_{p-1}}(\rho_t^{R,\eta,*}) + \mathbf{m_p}(\rho_t^{R,\eta_j,*}) \le \|\rho_t^{R,\eta_j,*}\|_{L^1(\mathbb{R}^d)}^{\frac{1}{p}}\mathbf{m_p}(\rho_t^{R,\eta_j,*})^{\frac{p-1}{p}} + \mathbf{m_p}(\rho_t^{R,\eta_j,*})\\ &\le \frac{\|\rho_t^{R,\eta_j,*}\|_{L^1(\mathbb{R}^d)}}{p} + \frac{2p-1}{p}\mathbf{m_p}(\rho_t^{R,\eta_j,*}).\end{aligned}$$

Integrating (2.106) in time from 0 to t and using the previous computations we get that

$$\mathbf{m_p}(\rho_t^{R,\eta_j,*}) - \mathbf{m_p}(\rho_0^{R,\eta_j,*}) \leq C\left(1 + \int_0^t \mathbf{m_p}(\rho_s^{R,\eta_j,*})\,\mathrm{d}s\right)$$

where C is a constant with the same dependencies as A and B in the statement. Therefore, by Gronwall's inequality, we recover (2.84). □

Once we have obtain an estimate for the $p = \frac{2}{1-m}$ order moment we focus on obtaining a collection of *a priori* results. In order to do that, let us take $\varphi \in C_c^\infty(\mathbb{R}^d)$ such that $\varphi(x) = 1$ for all $x \in B_{\omega_1}$ and $\varphi(x) = 0$ for all $x \in \mathbb{R}^d \backslash B_{\omega_2}$. We fix a cut-off function ς such that $\varsigma(x) = \varphi^2(x)$. Take R large enough so that $\omega_1 < \omega_2 < R$. We present here the following collection of *a priori* results.

Lemma 2.64 (*a priori* estimates on $\nabla\rho^{R,\eta}$)**.** *Let $\rho^{R,\eta}$ be the unique strong solution of* (2.5) *in the bounded domain $(0,T)\times B_R$ for the kernel $K(x,y) = \eta(x)W(x-y)\eta(y)$. We have that*

$$\begin{aligned}&\int_0^T \int_{B_R} |\nabla\rho_t^{R,\eta}|^2\varsigma \\ &\quad\leq C\left(m, \|\rho^{R,\eta}\|_{L^{3-m}((0,T)\times\mathbb{R}^d)}, \|\rho^{R,\eta}\|_{L^\infty((0,T)\times\mathbb{R}^d)}, \|\varsigma^{\frac{1}{2}}\|_{W^{1,\infty}(\mathbb{R}^d)}, \|E^{R,\eta}\|_{L^2((0,T)\times B_{\omega_2})}\right).\end{aligned} \tag{2.107}$$

Proof. We use a similar strategy to the one followed for the *a priori* estimate (2.28). In this case, the function G_Φ appearing in (2.28) corresponds to $G(s) = \frac{1}{m(2-m)(3-m)}s^{3-m}$, where we define G using (2.27) for the function $\Phi(s) = s^m$. For R large enough we get that,

$$\begin{aligned}\partial_t \int_{B_R} (G(\rho_t^{R,\eta}))\varsigma &= -\int_{B_R} \nabla(G'(\rho_t^{R,\eta})\varsigma)\cdot\left(\nabla(\rho_t^{R,\eta})^m + \rho_t^{R,\eta}E_t^{R,\eta}\right) \\ &= -\int_{B_R} |\nabla\rho_t^{R,\eta}|^2\varsigma - \int_{B_R}\frac{1}{m}(\rho_t^{R,\eta})^{2-m}\nabla\rho_t^{R,\eta}\cdot E_t^{R,\eta}\varsigma \\ &\quad - \int_{B_R}\frac{1}{m(2-m)}(\rho_t^{R,\eta})^{2-m}\left(\nabla(\rho_t^{R,\eta})^m + \rho_t^{R,\eta}E_t^{R,\eta}\right)\nabla\varsigma.\end{aligned}$$

Thus, using Hölder, Young's inequality and integrating in time we get (2.107). □

Lemma 2.65 (*a priori* estimates on $\nabla(\rho^{R,\eta})^m$)**.** *Let $\rho^{R,\eta}$ be the unique strong solution of* (2.5) *in the bounded domain $(0,T)\times B_R$ for the kernel $K(x,y) = \eta(x)W(x-y)\eta(y)$. We have that*

$$\begin{aligned}&\int_0^T \int_{B_R} |\nabla(\rho_t^{R,\eta})^m|^2\varsigma \\ &\leq C\left(m, \|\rho^{R,\eta}\|_{L^{m+1}((0,T)\times\mathbb{R}^d)}, \|\rho^{R,\eta}\|_{L^\infty((0,T)\times\mathbb{R}^d)}, \|\varsigma^{\frac{1}{2}}\|_{W^{1,\infty}(\mathbb{R}^d)}, \|E^{R,\eta}\|_{L^\infty((0,T)\times B_{\omega_2})}\right).\end{aligned} \tag{2.108}$$

Proof. With a similar strategy to the one followed for the *a priori* estimate (2.26) we obtain that,

$$\begin{aligned}\partial_t \int_{B_R}\frac{1}{m+1}(\rho_t^{R,\eta})^{m+1}\varsigma &= \int_{B_R}(\rho_t^{R,\eta})^m\partial_t\rho_t^{R,\eta}\varsigma \\ &= \int_{B_R}(\rho_t^{R,\eta})^m \operatorname{div}(\nabla(\rho_t^{R,\eta})^m + \rho_t^{R,\eta}E_t^{R,\eta})\varsigma \\ &= -\int_{B_R}|\nabla(\rho_t^{R,\eta})^m|^2\varsigma - \int_{B_R}\rho_t^{R,\eta}\nabla(\rho_t^{R,\eta})^m E_t^{R,\eta}\varsigma \\ &\quad - \int_{B_R}(\rho_t^{R,\eta})^m\nabla(\rho_t^{R,\eta})^m\nabla\varsigma - \int_{B_R}(\rho_t^{R,\eta})^{m+1}E_t^{R,\eta}\nabla\varsigma.\end{aligned}$$

Thus, using Hölder, Young's inequality and integrating in time we get (2.108). □

Lemma 2.66 (*a priori* estimates on $\partial_t\rho^{R,\eta}$). *Let $\rho^{R,\eta}$ be the unique strong solution of (2.5) in the bounded domain $(0,T)\times B_R$ for the kernel $K(x,y)=\eta(x)W(x-y)\eta(y)$. We have that*

$$\int_0^T\int_{B_R}\left|\frac{\partial\rho_t^{R,\eta}}{\partial t}\right|^2\varsigma \le C\left(m,\|\rho^{R,\eta}\|_{L^\infty((0,T)\times\mathbb{R}^d)},\|(\rho^{R,\eta})^m\|_{L^\infty((0,T);H^1(B_{\omega_2}))}\|\varsigma^{\frac12}\|_{W^{1,\infty}(\mathbb{R}^d)},\|E^{R,\eta}\|_{W^{1,2}((0,T);L^2(B_{\omega_2})}\right). \tag{2.109}$$

Proof. This time we use a similar strategy to the one from the *a priori* estimate (2.29). Applying Young's inequality we have that

$$\begin{aligned}\int_{B_R} m(\rho_t^{R,\eta})^{m-1}\left|\frac{\partial\rho_t^{R,\eta}}{\partial t}\right|^2\varsigma &= \int_{B_R}\frac{\partial}{\partial t}(\rho_t^{R,\eta})^m\cdot\left(\Delta(\rho_t^{R,\eta})^m+\operatorname{div}(\rho_t^{R,\eta}E_t^{R,\eta})\right)\varsigma\\ &= -\int_{B_R}\nabla\left(\frac{\partial}{\partial t}(\rho_t^{R,\eta})^m\right)\nabla(\rho_t^{R,\eta})^m\varsigma-\int_{B_R}\frac{\partial}{\partial t}(\rho_t^{R,\eta})^m\nabla(\rho_t^{R,\eta})^m\nabla\varsigma\\ &\quad -\int_{B_R}\rho_t^{R,\eta}E_t^{R,\eta}\nabla\left(\frac{\partial}{\partial t}(\rho_t^{R,\eta})^m\right)\varsigma-\int_{B_R}\rho_t^{R,\eta}E_t^{R,\eta}\frac{\partial}{\partial t}(\rho_t^{R,\eta})^m\nabla\varsigma.\end{aligned}$$

Integrating in time from 0 to T we get that

$$\begin{aligned}\int_0^T\int_{B_R} m(\rho_t^{R,\eta})^{m-1}\left|\frac{\partial\rho_t^{R,\eta}}{\partial t}\right|^2\varsigma-\frac12\int_{B_R}|\nabla\rho_0^m|^2\varsigma+\frac12\int_{B_R}|\nabla(\rho_T^{R,\eta})^m|^2\varsigma\\ =\int_0^T\int_{B_R}\left(\nabla(\rho_t^{R,\eta})^m+(\rho_t^{R,\eta})^m\right)\left(E_t^{R,\eta}\frac{\partial\rho_t^{R,\eta}}{\partial t}+\rho_t^{R,\eta}\frac{\partial E_t^{R,\eta}}{\partial t}\right)\varsigma\\ -\int_0^T\int_{B_R}\nabla(\rho_t^{R,\eta})^m\frac{\partial}{\partial t}(\rho_t^{R,\eta})^m\nabla\varsigma.\end{aligned}$$

Let us bound each one of the terms on the RHS. We obtain the following bound for the first term,

$$\begin{aligned}&\int_0^T\int_{B_R}\left(\nabla(\rho_t^{R,\eta})^m+(\rho_t^{R,\eta})^m\right)\left(E_t^{R,\eta}\frac{\partial\rho_t^{R,\eta}}{\partial t}\right)\varsigma\\ &\le\frac{\|\rho_t^{R,\eta}\|^{1-m}_{L^\infty((0,T)\times\mathbb{R}^d)}}{m}\int_0^T\|(\rho_t^{R,\eta})^m\|^2_{H^1(B_{\omega_1})}\|E_t^{R,\eta}\varsigma^{\frac12}\|_{L^2(B_{\omega_1})}\\ &\quad+\frac{m}{4\|\rho_t^{R,\eta}\|^{1-m}_{L^\infty((0,T)\times\mathbb{R}^d)}}\int_0^T\int_{B_R}\left|\frac{\partial}{\partial t}\rho_t^{R,\eta}\right|^2\varsigma.\end{aligned}$$

For the second one we get the bound,

$$\begin{aligned}&\int_0^T\int_{B_R}\left(\nabla(\rho_t^{R,\eta})^m+(\rho_t^{R,\eta})^m\right)\left(\rho_t^{R,\eta}\frac{\partial E_t^{R,\eta}}{\partial t}\right)\varsigma\\ &\le\int_0^T\|(\rho_t^{R,\eta})^m\|_{H^1(B_{\omega_1})}\|\rho_t^{R,\eta}\|_{L^\infty(B_{\omega_1})}\left\|\frac{\partial E_t^{R,\eta}}{\partial t}\varsigma\right\|_{L^2(B_R)}.\end{aligned}$$

And for the last we have the bound,

$$-\int_0^T\int_{B_R}\nabla(\rho_t^{R,\eta})^m\frac{\partial}{\partial t}(\rho_t^{R,\eta})^m\nabla\varsigma\le\int_0^T\|\nabla(\rho_t^{R,\eta})^m\|_{L^2(B_{\omega_2})}\|\varsigma^{-\frac12}\nabla\varsigma\|_{L^\infty(B_{\omega_2})}\left\|\frac{\partial(\rho_t^{R,\eta})^m}{\partial t}\varsigma^{\frac12}\right\|_{L^2(B_{\omega_2})}$$

$$+\frac{1}{m}\int_0^T \|\nabla(\rho_t^{R,\eta_j})^m\|^2_{L^2(B_{\omega_2})}\|\varsigma^{-\frac{1}{2}}\nabla\varsigma\|^2_{L^\infty(B_{\omega_2})} + \frac{m}{4\|\rho_t^{R,\eta}\|^{1-m}_{L^\infty((0,T)\times\mathbb{R}^d)}}\int_0^T\int_{B_R}\left|\frac{\partial}{\partial t}\rho_t^{R,\eta}\right|^2\varsigma.$$

Thus, putting everything together, we obtain (2.109). □

Lemma 2.67 (*a priori* estimates on $\Delta(\rho^{R,\eta})^m$)**.** *Let $\rho^{R,\eta}$ be the unique strong solution of* (2.5) *in the bounded domain $(0,T)\times B_R$ for the kernel $K(x,y)=\eta(x)W(x-y)\eta(y)$. We have that*

$$\int_0^T\int_{B_R}\left|\Delta((\rho_t^{R,\eta})^m)\right|^2\varsigma \le C, \tag{2.110}$$

where C depend on the constants (2.107) *and* (2.109)*.*

Proof. We have that

$$\begin{aligned}\int_0^T\int_{B_R}\left|\Delta((\rho_t^{R,\eta})^m)\right|^2\varsigma &= \int_0^T\int_{B_R}\left|\frac{\partial\rho_t^{R,\eta}}{\partial t}\right|^2\varsigma + \int_0^T\int_{B_R}\left|\operatorname{div}\left(\rho_t^{R,\eta}E_t^{R,\eta}\right)\right|^2\varsigma \\ &\quad - 2\int_0^T\int_{B_R}\frac{\partial\rho_t^{R,\eta}}{\partial t}\operatorname{div}\left(\rho_t^{R,\eta}E_t^{R,\eta}\right)\varsigma.\end{aligned}$$

From here, we use the results from Lemma 2.64 and Lemma 2.66 combined with Hölder inequality. □

3 Drift-diffusion equations with saturation

This chapter is taken from the article "Drift-diffusion equations with saturation" written in collaboration with José Antonio Carrillo[1] and David Gómez-Castro[2], and accepted for publication in *SIAM Journal on Mathematical Analysis*. *Preprint* arXiv:2410.10040 (2024); [94].

Contents

Mathematics is the art of giving the same name to different things. – Henri Poincaré

3.1 Introduction and problem setting

Aggregation and drift-diffusion equations are frequent in continuous descriptions of density populations since they are natural macroscopic models associated to microscopic particle dynamics, see for instance [107] and the references therein. Some models contain a more general nonlinear mobility, usually called of saturation type, preventing overcrowding. This family of partial differential equations include models of the form

$$\frac{\partial \rho}{\partial t} = \operatorname{div}\left(\mathrm{m}(\rho)\nabla\left(U'(\rho) + V\right)\right). \tag{3.1a}$$

Here, we consider U convex and V a given potential regular enough. Furthermore, we work on a bounded domain Ω, where we set the natural no-flux condition

$$\mathrm{m}(\rho)\nabla(U'(\rho) + V)\cdot\nu(x) = 0 \qquad \text{for all } t > 0, x \in \partial\Omega\,, \tag{3.1b}$$

leading to conservation of the total mass. The case of linear mobility $\mathrm{m}(\rho) = \rho$ is well understood, see [78, 17] and the references therein. For this family of nonlinear parabolic equations (3.1), the well-posedness theory, their long-time behaviour, and the main qualitative properties of the solutions, self-similar solutions and their steady states have been fairly well-analysed [102, 302, 80, 98, 96, 93]. Moreover, the family of Cauchy problems of the form (3.1) with linear mobility are 2-Wasserstein gradient flows [269, 105, 106, 7, 283, 284] of the free-energy functional

$$\mathcal{F}[\rho] = \int_\Omega U(\rho(x))\mathrm{d}x + \int_\Omega V(x)\rho(x)\mathrm{d}x. \tag{3.2}$$

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When $\mathrm{m}(\rho)$ is a non-linear mobility (not necessarily bounded) there is also an extensive literature. A suitable notion of generalised Wasserstein distance was introduced in [161] by extending the Benamou-Brenier formulation. This approach only produces well-defined distances if $\mathrm{m}(\rho)$ is concave. The corresponding Otto calculus yields that the formal gradient flow of the free energy (3.10) in these non-linear mobility Wasserstein-type distances corresponds to the family of PDEs

$$\frac{\partial \rho}{\partial t} = \operatorname{div}\left(\mathrm{m}(\rho)\nabla \frac{\delta \mathcal{F}}{\delta \rho}[\rho]\right).$$

We will take advantage of the free-energy dissipation structure of this formulation. The result in [161] has been extended to cover more cases, including more general families of non-linear mobilities [244, 104, 162, 157]. Several aggregation-diffusion related equations with non-linear mobility have been also analysed by different methods in PDE theory for instance: Newtonian interaction potentials (i.e., $U = 0$) [97, 95], porous medium equations with non-local pressure [63, 295, 296], Cahn-Hilliard type equations [40, 245, 173], or interaction systems on graphs [202], among others.

Our work focuses on a non-linear mobility $\mathrm{m}(\rho)$ of *saturation*-type, i.e., the support of the mobility is a finite interval. More precisely, the mobility $\mathrm{m}(\rho)$ satisfies the following assumptions:

(H_1) There is some $\alpha \in (0, \infty)$ such that $\mathrm{m}(0) = \mathrm{m}(\alpha) = 0$, and $\mathrm{m} > 0$ in $(0, \alpha)$. We assume that $\mathrm{m} \in C([0, \alpha]) \cap C^1((0, \alpha))$.

(H_2) We deal with initial data ρ_0 in the admissible class of densities $\mathcal{A} := \{\rho \in L^1(\Omega) : 0 \le \rho \le \alpha\}$.

(H_3) V is of class $C^2(\overline{\Omega})$. Without loss of generality we assume $V \ge 0$.

(H_4) $U \in W^{1,1}((0, \alpha)) \cap C^2((0, \alpha))$ and convex $U'' \ge 0$. We assume that U is not trivial, i.e.,

$$\text{there exists } s_0 \in (0, \alpha) \text{ such that } U''(s_0) > 0. \tag{3.3}$$

We will make a further technical assumption (H_5), that is postponed to the next section. Notice that, unlike in previous literature, we do not assume that $\nabla V \cdot \nu = 0$ on $\partial\Omega$. Thus, $\rho \equiv 0$ and $\rho \equiv \alpha$ are constants solutions.

These aggregation-diffusion equations with mobility of saturation-type appear naturally in mathematical biology, in order to explain chemotaxis models with prevention of overcrowding [112, 206, 107], in mathematical physics, to describe the relaxation of gas of fermions [217, 218], in phase segregation [291, 314], or in thin liquid films [245, 252] among others. Despite the interest in applications of models with mobility of saturation-type, the literature devoted to problem (3.1) with the saturation-type mobility satisfying assumptions (H_1)-(H_4), up to our knowledge, is scarce. In [58, 156] the authors consider the Keller-Segel model with prevention of overcrowding, which they obtain by choosing $\mathrm{m}(\rho) = \rho(1-\rho)$. They study the competition between the chemotaxis term with a saturation effect and a linear diffusion term. More recently, in [154], the authors obtain a rigorous limit from discrete distributions to a family of one-dimensional non-local interaction equations with saturation. Furthermore, this result is extended to a family of one-dimensional aggregation–diffusion equation in [184]. In both cases, the authors only cover the case $\mathrm{m}(\rho) = \rho\phi(\rho)$ where ϕ is decreasing and positive only in a finite interval. In [185], the authors prove well-posedness of entropy solutions for a wide class of one-dimensional non-local transport equations with a general saturation-type mobility.

To our knowledge, there is no literature analysing the family of Cauchy problems (3.1) with saturation-type mobility in higher dimensions. Furthermore, the only work dealing with the long-time behaviour is from the numerical analysis viewpoint by implicit finite-volume schemes introduced in [19]. They show the existence of certain weak stationary solutions with kinks depending on the mass of the solution and their numerical experiments suggest the appearance of kinks in the long-time behaviour for certain initial data. Therefore, the main goal of this chapter is to provide a unified theory for the Cauchy problems (3.1) with a general saturation-type mobility satisfying (H_1)-(H_5), including the existence theory, minimisation of the free energy, and their long-time behaviour. Furthermore, we complement our mathematical analysis results with numerical analysis by showing the convergence of suitable implicit finite volume schemes related to [19] and clarifying their long time behaviour.

Main analytical results. Our main analytical results concern the existence of certain solutions to (3.1) with a general saturation-type mobility satisfying (H_1)-(H_5) via approximation arguments and semigroup theory, the characterization of L^1-local minimisers of the associated free-energy functional in the admissible set of bounded integrable densities $\mathcal{A}$ defined in (H_2), and the long-time behaviour of the constructed solutions in view of its gradient flow structure. Moreover, we study these aspects with the greatest generality on both the saturation-type non-linear mobility $\mathrm{m}(\rho)$ and the diffusion potential $U(\rho)$ which, in particular, includes the classical porous medium/fast diffusion non-linearities at zero density.

To tackle the existence of certain solutions to (3.1), we proceed by stability arguments within the family of problems of the form (3.1) with a general saturation-type mobility satisfying (H_1)-(H_5). More precisely, we construct suitable approximating problems of the form (3.1) which admit classical solutions while keeping the assumptions (H_1)-(H_5), see Theorem 3.4. Passing to the limit in these approximating problems, we are able to construct a C_0-semigroup, denoted by $\{S_t\}_{t\geq 0}$, of weak solutions defined for any initial datum $\rho_0 \in \mathcal{A}$, see Theorem 3.5. This semigroup $\{S_t\}_{t\geq 0}$, referred as free-energy dissipating semigroup in the sequel, enjoys mass conservation, comparison principle, L^1-contraction, and free-energy dissipation, see Definition 3.2. This notion of semigroup allows us next to study the long-time behaviour, leading to the first global-in-time existence result for this family of equations in higher dimensions allowing for free boundaries both at zero density and saturated density value α.

The second goal of our analysis is to study the minimisation of the free-energy functional $\mathcal{F}$ in the class of admissible densities $\mathcal{A}$. We obtain the Euler-Lagrange conditions for the L^1-local minimisation, see Theorem 3.6. When U is strictly convex, we show that the unique local minimiser for a fixed mass is explicit,

$$\widehat{\rho}(x) = \min\left\{\alpha, \Big((U')^{-1}(C_0 - V(x))\Big)_+\right\}$$

where the constant C_0 comes from the mass constraint. Notice that this is a truncation by α of the usual family of minimisers for the linear-mobility case. The Euler-Lagrange conditions are already well-understood for the linear mobility case, but this seems to be new in the literature for minimisation in the set $\mathcal{A}$ although related to constrained minimisation problems as in [131].

We next focus on the long-time behaviour of the constructed solutions showing that there exists a time-limit operator $S_\infty : \mathcal{A} \to \mathcal{A}$, see Definition 3.7, such that for any $\rho_0 \in \mathcal{A}$, we have asymptotic time convergence of the constructed semigroup $\{S_t\}_{t\geq 0}$, that is

$$S_t\rho_0 \to S_\infty\rho_0, \qquad \text{in } L^1(\Omega) \text{ as } t \to \infty.$$

We show that S_∞ is still an L^1-contraction. Hence, the ω-limit set $\{S_\infty\rho_0 : \rho_0 \in \mathcal{A}\}$ is an L^1-continuous subset of $\mathcal{A}$, see Theorem 3.8.

We further analyse the structure of the ω-limit set. The classical solutions for the approximating problems have a unique element in the ω-limit set, i.e., the global attractor, corresponding to the unique constant-in-time solution and the unique global (and L^1-local) minimiser of the free energy, see Theorem 3.9. Under certain convexity assumption for the nonlinear diffusion, we can characterize fully the ω-limit set again given by the unique constant-in-time solution and the unique global (and L^1-local) minimiser of the free energy, see Theorem 3.10. On the other hand, we construct examples of degenerate non-linearities where $S_\infty\rho_0$ is not an L^1-local minimiser of $\mathcal{F}$, but only saddle points of the free energy, see Figure 3.4.

Finally, we are able to justify mathematically the behaviour numerically observed in [19]: the appearance of kinks in the long-time behaviour and the complicated structure of the ω-limit set when non-linearities are degenerate combined with a saturated-type mobility.

Numerical analysis. The design of numerical schemes for aggregation and drift-diffusion equations is a crucial tool to understand the dynamics of this family of equations. In particular, we need to develop methods that keep the structural properties of the gradient flow of densities: the non-negativity of the solution, the dissipation property, and a corresponding set of stationary states which capture the long-time asymptotics.

Finite-volume methods allow us to obtain schemes with these properties. In [41], the authors propose first and second-order-accurate finite-volume schemes treating non-linear diffusion equations as a non-linear continuity equation. Another method is proposed in [75] for aggregation-diffusion equations. Moreover, a generalisation for high-order approximations is proposed in [299]. In [3], the authors propose several fully discrete, implicit-in-time discretizations for the Keller-Segel model in one dimension. This work is generalised in [18], where the authors introduce a fully discrete (in both space and time) implicit finite-volume scheme for the aggregation-diffusion equation with linear mobility $\mathrm{m}(s) = s$. Furthermore, as it is shown in [21], this method converges under suitable assumptions on the diffusion functions and potentials involved and assumptions on the boundary conditions. In [19], the authors extend this scheme to cover non-linear mobilities of saturation-type.

Here, we focus on a variation of the implicit finite-volume scheme introduced and analysed in [19], where we study the case $\mathrm{m}(s) = \mathrm{m}^{(1)}(s)\mathrm{m}^{(2)}(s)$ where $\mathrm{m}^{(1)}$ is non-decreasing and $\mathrm{m}^{(2)}$ is non-increasing. Our main results show that the proposed implicit finite volume scheme is well-defined, convergent and structure preserving together with a characterisation of the long-time behaviour of the fully discrete scheme. Moreover, we show that the long-time asymptotics of the numerical scheme capture the long-time behaviour of the constructed solutions to (3.1). More precisely, we prove well-posedness, free-energy dissipation, mass conservation, a discrete L^1 contraction property, and a comparison principle for the already mentioned method and an approximating version of it, see Theorem 3.13. Furthermore, to keep the analogy with the continuous case and the C_0-semigroup theory, we show that our method admits a free-energy dissipating numerical scheme, see Definition 3.12. We also show that under high regularity of the solution the scheme converges, see Theorem 3.14.

We finally conclude by discussing the long-time behaviour of the numerical scheme, and its rate of convergence to the long-time behaviour of the continuous problem (3.1). For the approximating problem, we also prove that the long-time behaviour coincides with the unique constant-in-time solution and the global attractor, see Theorem 3.15, analogously to the continuous problem. We also analyse the existence of a time-limit operator for the numerical scheme reproducing the theory studied at the continuous level, see Theorem 3.16. Moreover, we also show examples with complicated long-time asymptotics leading to free boundaries, infinitely many steady states with large basin of attraction, and saturation effects leading to "freezing" behaviour, i.e., free boundaries at the saturation level α.

Open problems. Showing uniqueness of the constructed free-energy dissipating solutions is an interesting open problem. We may expect uniqueness of enhanced notions of solution as entropy solutions (see, e.g., [72, 219]), but we do not deal with this question in this work. We only prove convergence of the numerical scheme as the mesh is refined in the case where the solution to the continuous problem is very regular. It would be interesting to have a proof of the convergence of the numerical scheme that does not use information of the continuous solution or reduces the regularity needed. We do not discuss higher regularity of the solutions. The study of C^α regularity is an interesting open problem, specially in the cases with free boundary or freezing behaviour. The problem with V replaced by the aggregation term $V + W * \rho$ is completely open. Our results can be used to prove existence of a semigroup of solutions. However, there is no L^1 contraction or comparison principle for general W and their long-time behaviour is a difficult problem as numerically investigated in [19].

Structure of the chapter. In Section 3.2, we introduce the hypotheses, the notion of solutions and the numerical scheme and present our main Theorems. In Section 3.3, we study the existence of weak solutions, proving Theorems 3.4 and 3.5. In Section 3.4, we analyse local minimisers of the free energy, proving Theorem 3.6. In Section 3.5, we focus on the long-time behaviour for both the regularised and the original problem, i.e. Theorem 3.8, and, subsequently, in Section 3.6, we deal with its ω-limit, Theorems 3.9 and 3.10. Finally, we devote Section 3.7 to the numerical analysis of the implicit finite volume scheme, and, in particular, we prove Theorems 3.13 to 3.16. The main goals of this work are schematically described in the diagrams (D_1) and (D_2), which we present in Section 3.2.

3.2 Main Results

The aim of this section is to present our main results and the key ideas of their proof. Our focus is the initial value problem

$$\begin{cases} \dfrac{\partial \rho}{\partial t} = \operatorname{div}\left(\mathrm{m}(\rho)\nabla\left(U'(\rho)+V\right)\right) & \text{in } (0,\infty)\times\Omega, \\ \mathrm{m}(\rho)\nabla(U'(\rho)+V)\cdot\nu(x) = 0 & \text{on } (0,\infty)\times\partial\Omega, \\ \rho(0,x)=\rho_0(x) & x\in\Omega, \end{cases} \tag{P}$$

where $\Omega \subseteq \mathbb{R}^d$ is a bounded, connected, and smooth domain. We make an additional technical assumption on the non-linearities.

(H_5) We assume that the diffusion is continuous, in the sense that

$$\mathrm{m}U'' \in L^1(0,\alpha), \tag{3.4a}$$

and we define

$$\Phi(s) := \int_{s_0}^{s} \mathrm{m}(\tau)U''(\tau)\mathrm{d}\tau.$$

Furthermore, we also assume that Φ is strictly increasing at 0 and α, i.e.,

$$\Phi(0) < \Phi(s) < \Phi(\alpha) \quad \text{for all } s \in (0,\alpha). \tag{3.4b}$$

Lastly, we impose a technical regularity condition which will be suitable for compactness estimates

$$\sup_{s\in[0,\alpha]} \left|\frac{(\Phi(\alpha)-\Phi(s))\Phi(s)}{\Phi'(s)}\right| + \left|\frac{(\Phi(s)-\Phi(0))\Phi(s)}{\Phi'(s)}\right| < \infty. \tag{3.4c}$$

In order to study the steady states of (P) we will sometimes assume some of the following strict convexity to different degrees

$$U''(s) > 0, \qquad \text{for a.e. } s \in (0,\alpha). \tag{SC_U}$$

$$\inf_{s\in(0,\alpha)} U''(s) > 0. \tag{USC_U}$$

For certain statements on numerical schemes, we will assume

$$U \in C^1([0,\alpha]). \tag{3.5}$$

Remarks. *Notice that, since* $\mathrm{m} \in C^1((0,\alpha))$ *and* $U \in C^3((0,\alpha))$, *then* $\Phi \in C^2((0,\alpha))$.

These main hypothesis (H) and (SC_U) *are satisfied for the Porous Medium / Fast Diffusion cases, for example, if* Φ *is* $C^2((0,\alpha))$ *and*

$$\Phi'(s) = \begin{cases} a(s)s^{m_1-1} & \text{if } s\in(0,s_1), \\ -b(s)(\alpha-s)^{m_2-1} & \text{if } s\in(s_2,\alpha). \end{cases}$$

For $m_1, m_2 > 0$ *and* $0 < \underline{a} \le a(s) \le \overline{a}, 0 < \underline{b} \le b(s) \le \overline{b}$. *The hypothesis* ($\mathrm{USC}_U$) *only holds if* $m_1, m_2 < 1$. *The hypothesis* (3.5) *only holds if* $m_1, m_2 > 1$. *With respect to assumption* (H_5), *if* $m_1, m_2 > 0$ *then* Φ *fulfills* (H_5). *However, if either* m_1 *or* m_2 *is strictly smaller than* -1, *i.e. Ultrafast Diffusion, then the assumption 3.4c does not hold. Furthermore, let us remark that the mobility* m *does not need to be concave in order to fulfill* (H_5). *For example, the case* $\mathrm{m}(s) = s^2(\alpha - s^2)$ *and* $U(s) = s^m$ *with* $m > 0$.

Our first aim is to construct a family of approximating problem (P_ε) with a well-posedness theory in the classical sense. We next use these approximating problems (P_ε) to obtain existence of (P) by compactness arguments. We study the long-time behaviour $t \to \infty$ for both problems (P_ε) and (P). Furthermore, we also discuss whether the limits $t \to \infty$ and $\varepsilon \to 0$ commute. The following diagram describes the different questions we analyse in the analytical part of this work.

$$\begin{array}{ccc} \rho_t^{(\varepsilon)} \text{ solution to } (P_\varepsilon) & \xrightarrow{t \to \infty} & \widehat{\rho}^{(\varepsilon)}(x) = (U_\varepsilon')^{-1}\Big(C_\varepsilon - V(x)\Big) \\ \Big\downarrow {\scriptstyle \varepsilon \to 0} & & \Big\downarrow {\scriptstyle \varepsilon \to 0} \\ \rho_t^{(0)} \text{ solution to } (P) & \overset{?}{\underset{t \to \infty}{\dashrightarrow}} & \widehat{\rho}^{(0)}(x) = T_{0,\alpha} \circ (U')^{-1}\Big(C_0 - V(x)\Big). \end{array} \tag{D_1}$$

Notice that we provide a counterexample for the commutativity of the diagram in section 3.6.3. In fact, the regularisation (P_ε) chooses always the L^1-local minimiser as $\varepsilon \to 0$, see Theorems 3.9 and 3.10 (taking first $t \to \infty$ and then $\varepsilon \to 0$ in (D_1)). Nevertheless there exists some cases for which the diagram is commutative, for instance $\mathrm{m}(s) = s(\alpha - s)$, $U(s) = s\log(s)$ and $V(x) = |x|^2$. A more complete version of this diagram, including the numerical results, is provided at the end of this section in (D_2). Here, we use the truncation function defined as

$$T_{0,\alpha}(s) = \begin{cases} \alpha & \text{if } s > \alpha, \\ s & \text{if } s \in [0,\alpha], \\ 0 & \text{if } s < 0. \end{cases}$$

We use the notation $T_{0,\alpha} \circ (U')^{-1}$ in a generalised sense. Recall that $U' : (0,\alpha) \to \mathbb{R}$ is non-decreasing. We define

$$\underline{\zeta} := U'(0^+) \quad \text{and} \quad \overline{\zeta} := U'(\alpha^-). \tag{3.6}$$

Either of these values can be infinite. With this definition, we define

$$T_{0,\alpha} \circ (U')^{-1}(\zeta) := \begin{cases} \alpha & \text{if } \zeta \geq \overline{\zeta}, \\ (U')^{-1}(\zeta) & \text{if } \zeta \in (\underline{\zeta}, \overline{\zeta}), \\ 0 & \text{if } \zeta \leq \underline{\zeta}. \end{cases}$$

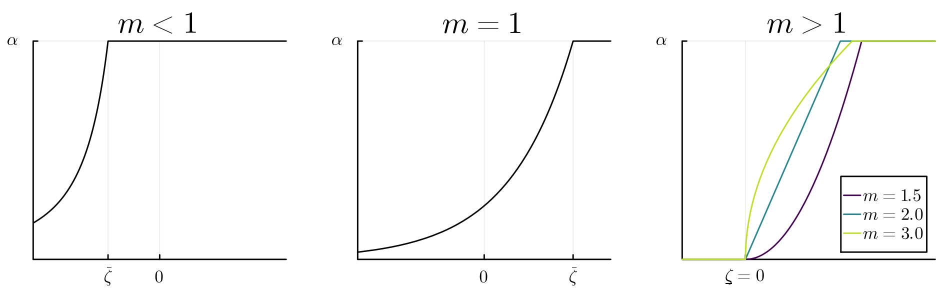


Figure 3.1: $T_{0,\alpha} \circ (U')^{-1}(\zeta)$ for $U(s) = \frac{s^m}{m-1}$ and different choices of the exponent m.

3.2.1 Notions of solution

Throughout this chapter we use the following notion of weak solution.

Definition 3.1 (Weak solution)**.** *We say ρ is a weak solution of the problem (P) in $(0,T)\times\Omega$ if $\rho \in L^1((0,T)\times\Omega)$, $\Phi(\rho) \in L^2(0,T;H^1(\Omega))$, and*

$$\int_\Omega \rho_0 \varphi(0) + \int_0^T \int_\Omega \rho_t \frac{\partial \varphi}{\partial t} = \int_0^T \int_\Omega \mathrm{m}(\rho) \nabla (U'(\rho) + V) \cdot \nabla \varphi,$$

for all $\varphi \in C^\infty([0,T]\times\Omega)$ such that $\varphi(T,\cdot)=0$. Let us recall that $\nabla\Phi(\rho) = \mathrm{m}(\rho)\nabla U'(\rho)$. Respectively, for $\varepsilon>0$ and the problem (P_ε), we consider the analogous version of weak solution.

Beside $\mathcal{A}$, we consider the following sets of initial data

$$\mathcal{A}_+ \coloneqq \left\{\rho \in L^1(\Omega) : \exists \delta > 0 \text{ s.t. } \delta \le \rho \le \alpha - \delta\right\},$$
$$\mathcal{A}_M \coloneqq \left\{\rho \in \mathcal{A} : \|\rho\|_{L^1(\Omega)} = M\right\}.$$

We will work with the notion of semigroup of solutions (possibly non-unique) for (P) as follows.

Definition 3.2. *We say that $S_t : \mathcal{A}\to\mathcal{A}$ is a free-energy dissipating semigroup of solutions for (P) if*

i) For $\rho_0\in\mathcal{A}$, $\rho_t = S_t\rho_0$ is a weak solution to (P).

ii) S_t is a C_0-semigroup in L^1, i.e., for $t,h>0$ we have

$$S_{t+h} = S_t S_h, \qquad \lim_{t\to 0^+} \|S_t\rho_0 - \rho_0\|_{L^1(\Omega)} = 0 \text{ for all } \rho_0 \in \mathcal{A}.$$

iii) $S_t:\mathcal{A}\to\mathcal{A}$ is an L^1-contraction, i.e., for any $\rho_0,\eta_0\in\mathcal{A}$ we have that $\|S_t\rho_0 - S_t\eta_0\|_{L^1(\Omega)} \le \|\rho_0 - \eta_0\|_{L^1(\Omega)}$.

iv) Free-energy dissipation and $C^{\frac12}_{loc}([0,\infty),W^{-1,1}(\Omega))$ continuity: If $\rho_0\in\mathcal{A}_+$ then calling $\rho_t = S_t\rho_0$ we have

For all $0<t_1<t_2$ we get

$$\int_{t_1}^{t_2}\int_\Omega \mathrm{m}(\rho_\sigma)|\nabla(U'(\rho_\sigma)+V)|^2 \le \mathcal{F}[\rho_{t_1}] - \mathcal{F}[\rho_{t_2}], \tag{3.7}$$

$$\|\rho_{t_2} - \rho_{t_1}\|_{W^{-1,1}(\Omega)} \le \|\mathrm{m}\|^{\frac12}_{L^\infty(0,\alpha)}|\Omega|^{\frac12}\left(\mathcal{F}[\rho_{t_1}] - \mathcal{F}[\rho_{t_2}]\right)^{\frac12}|t_2-t_1|^{\frac12}. \tag{3.8}$$

In particular, $t\mapsto \mathcal{F}[S_t\rho_0]$ is non-increasing.

Remarks. *Notice that for a C_0-semigroup it follows $t\mapsto S_t\rho_0$ in $C([0,T];L^1(\Omega))$. Since $\rho\ge 0$, weak solutions are mass preserving by using the test function $\varphi(t,x)=1$, i.e., $\|S_t\rho_0\|_{L^1(\Omega)} = \|\rho_0\|_{L^1(\Omega)}$. For $\rho\in\mathcal{A}$ we have $0\le\|\rho\|_{L^1}\le\alpha|\Omega|$, mass conservation implies that $S_t0=0$ and $S_t\alpha=\alpha$. Using that $s_+ = \frac{s+|s|}{2}$, mass conservation and the L^1-contraction directly imply the L^1 comparison principle*

$$\|(S_t\rho_0 - S_t\eta_0)_+\|_{L^1(\Omega)} \le \|(\rho_0-\eta_0)_+\|_{L^1(\Omega)}, \qquad \text{for all } \rho_0,\eta_0\in\mathcal{A}.$$

Let us finally point out that we recall the definition of the negative Sobolev space and some properties at the beginning of Section 3.3.

3.2.2 Existence of solutions for (P) by approximation

Consider $\varepsilon\in(0,1]$. We will work on approximating problems of the form

$$\begin{cases} \dfrac{\partial\rho}{\partial t} = \Delta\Phi_\varepsilon(\rho) + \operatorname{div}(\mathrm{m}_\varepsilon(\rho)\nabla V) & \text{in } (0,\infty)\times\Omega,\\ \left(\mathrm{m}_\varepsilon(\rho)\nabla(U'_\varepsilon(\rho)+V)\right)\cdot\nu(x) = 0 & \text{on } (0,\infty)\times\partial\Omega,\\ \rho(0,x) = \rho_0(x) & x\in\Omega. \end{cases} \tag{P_ε}$$

Here we regularise the mobility and the non-linear diffusion. We also make the problem uniformly elliptic by assuming $\Phi_\varepsilon\in C^3([0,\alpha])$ such that

$$\begin{gathered} \underline{\Phi'_\varepsilon}(s) \le \Phi'_\varepsilon(s) \le (1+\varepsilon)\underline{\Phi'_\varepsilon}(s) \quad \text{where} \quad \underline{\Phi'_\varepsilon}(s) = \min(\Phi'(s),\kappa(\varepsilon)^{-1}) + \varepsilon,\\ \underline{\Phi_\varepsilon}(s_0) = \Phi_\varepsilon(s_0) = 0 \text{ for } s_0 \text{ given by (3.3)},\\ \Phi_\varepsilon \to \Phi \text{ in } C^2_{loc}((0,\alpha)), \end{gathered} \tag{3.9}$$

where $\kappa(\varepsilon) \to 0$ as $\varepsilon \to 0$. Given any family $\kappa(\varepsilon) \to 0$ and due to (3.3), it is easy to construct such family of Φ_ε.

We also construct a suitable $\mathrm{m}_\varepsilon \in C^1([0,\alpha])$ such that, for $\varepsilon > 0$,

$$\mathrm{m}_\varepsilon(0) = \mathrm{m}_\varepsilon(\alpha) = 0, \qquad \mathrm{m}_\varepsilon > 0 \text{ in } (0,\alpha), \qquad \text{and } |\mathrm{m}_\varepsilon'| > 1 \text{ near } 0 \text{ and } \alpha. \tag{M$_\varepsilon$}$$

To connect this problem with (P) we consider the approximation of U given by the conditions $U_\varepsilon \in C^{3,1}((0,\alpha))$, $U_\varepsilon(\frac{\alpha}{2}) = U(\frac{\alpha}{2}), U_\varepsilon'(\frac{\alpha}{2}) = U'(\frac{\alpha}{2})$, and we define $U_\varepsilon''(s) := \Phi_\varepsilon'(s)/\mathrm{m}_\varepsilon(s)$. By construction, we already have $U_\varepsilon'' \to U''$ point-wise in $(0,\alpha)$. We make a few more assumptions for the convergence as $\varepsilon \to 0$, namely

$$\begin{gathered}\mathrm{m}_\varepsilon \to \mathrm{m} \text{ pointwise as } \varepsilon \to 0, \qquad \sup_{\varepsilon\in(0,1)} \|U_\varepsilon'\|_{L^1((0,\alpha))} < \infty \text{ and} \\ (\varepsilon, s) \mapsto \mathrm{m}_\varepsilon(s) \text{ is } C([0,1]\times[0,\alpha]) \cap C^\infty_{loc}((0,1]\times[0,\alpha]).\end{gathered} \tag{M$_0$}$$

In section 3.3.1 we construct such regularised mobilities.

Lemma 3.3. *There exists $\kappa(\varepsilon) \to 0$ and m_ε such that (M$_\varepsilon$) and (M$_0$) hold.*

Moreover, if we consider the free energy of the regularised problem

$$\mathcal{F}_\varepsilon[\rho] = \int_\Omega U_\varepsilon(\rho(x))\,\mathrm{d}x + \int_\Omega V(x)\rho(x)\,\mathrm{d}x, \tag{3.10}$$

we can rewrite (P$_\varepsilon$) again as a formal generalized mobility Wasserstein-type gradient flow.

First, we will prove a well-posedness result for (P$_\varepsilon$).

Theorem 3.4 (Well-posedness of (P$_\varepsilon$))**.** *Let $\varepsilon > 0$ be fixed and assume (M$_\varepsilon$) and (3.9). It follows that:*

i) If $\rho_0 \in \mathcal{A}_+ \cap C^2(\overline{\Omega})$, then problem (P$_\varepsilon$) has a unique classical solution.

ii) These classical solutions can be uniquely extended to a free-energy dissipating semigroup for (P$_\varepsilon$), denoted by $S_t^{(\varepsilon)}$.

iii) If $\rho_0 \in \mathcal{A} \setminus \{0,\alpha\}$ then $0 < S_t^{(\varepsilon)}\rho < \alpha$ in Ω for $t > 0$.

iv) $S_t^{(\varepsilon)} : \mathcal{A}_+ \to \mathcal{A}_+$.

We show Theorem 3.4–Item i in Section 3.3.1 and discuss the remaining items in section 3.3.3. We can now prove the left side of the diagram (D$_1$).

Theorem 3.5 (Existence for (P))**.** *There exists a sequence $\varepsilon_k \to 0$ and $S_t : \mathcal{A} \to \mathcal{A}$ a free-energy dissipating semigroup for (P) such that*

$$S^{(\varepsilon_k)}\rho_0 \to S\rho_0 \qquad \textit{in } C_{loc}([0,\infty); L^1(\Omega)) \textit{ for all } \rho_0 \in \mathcal{A}.$$

We prove the existence result on section 3.3.4.

Remarks. *Notice in (M$_\varepsilon$) and (M$_0$) that since $\mathrm{m}_\varepsilon \to \mathrm{m}$ pointwise in $[0,\alpha]$ and the fact that the map $(\varepsilon,s) \mapsto \mathrm{m}_\varepsilon(s)$ is C^1 then it holds the convergence*

$$\mathrm{m}_\varepsilon \to \mathrm{m} \quad \textit{in } C([0,\alpha]) \cap C^1_{loc}((0,\alpha)). \tag{3.11}$$

The semigroup can also be constructed using the theory of m-accretive operators in $L^1(\Omega)$, see e.g., [52]. With the construction we have made $U_\varepsilon'' \geq \varepsilon/\mathrm{m}_\varepsilon > 0$, $\mathrm{m}_\varepsilon(0) = \mathrm{m}_\varepsilon(\alpha) = 0$, and $|\mathrm{m}_\varepsilon'| > 1$ at $0, \alpha$, so

$$U_\varepsilon' : (0,\alpha) \to (-\infty,\infty) \textit{ is a strictly-increasing bijection with continuous inverse}.$$

Therefore, for each $C \in \mathbb{R}$ we have

$$\rho_t^{(\varepsilon)} = (U_\varepsilon')^{-1}(C - V) \text{ is a constant-in-time classical solution to } (\mathrm{P}_\varepsilon).$$

Notice that they $\rho_t^{(\varepsilon)}$ tend uniformly to 0 as $C \to -\infty$ and to α as $C \to \infty$. We point out that if $\rho_0 \in \mathcal{A}_+$ then

$$(U_\varepsilon')^{-1}(C_1 - V) \le \rho_0 \le (U_\varepsilon')^{-1}(C_2 - V) \text{ for some } C_1, C_2 \in \mathbb{R}. \tag{3.12}$$

Theorem 3.4–Item iv follows by using these bounds and the comparison principle can also be used for $S_t^{(\varepsilon)}\rho_0$. Notice that

$$0 \le \Phi_\varepsilon' \le (1+\varepsilon)\left(\Phi' + \varepsilon\right)$$

Since $\Phi_\varepsilon' \to \Phi'$ uniformly over compacts of $(0,\alpha)$ we have also almost everywhere convergence. Since $\Phi' \in L^1(0,\alpha)$ we can use the Dominated Convergence Theorem to show that

$$\Phi_\varepsilon' \to \Phi' \qquad \text{in } L^1(0,\alpha). \tag{3.13}$$

This, in particular, implies that $\Phi_\varepsilon \to \Phi$ uniformly in $[0,\alpha]$.

3.2.3 L^1-local minimisers of the free energy

In Section 3.4 we characterise the L^1-local minimisers, by deducing (and solving in some cases) the corresponding Euler-Lagrange condition. Since (P_ε) are particular examples of the family (P), we state the following result only for the later, more general, class.

Theorem 3.6 (Euler-Lagrange condition)**.** *If $\widehat{\rho}$ is a local minimiser of $\mathcal{F}$ on $\mathcal{A}_M$ with the L^1 topology, then there exists $C \in \mathbb{R}$ such that*

$$\begin{aligned} U'(\widehat{\rho}(x)) + V(x) &\ge C, \qquad \text{for a.e. } x \text{ such that } 0 \le \widehat{\rho}(x) < \alpha. \\ U'(\widehat{\rho}(x)) + V(x) &\le C, \qquad \text{for a.e. } x \text{ such that } 0 < \widehat{\rho}(x) \le \alpha. \end{aligned} \tag{3.14}$$

Furthermore, if U' is invertible

$$\widehat{\rho}(x) = T_{0,\alpha} \circ (U')^{-1}(C - V(x)) \quad \text{a.e. in } \Omega. \tag{3.15}$$

Lastly, if we assume (SC_U) and $M \in (0, \alpha|\Omega|)$, there exists a unique C such that (3.15) has mass M.

Remark. *The Euler-Lagrange condition for the case of $\mathrm{m}(\rho) = \rho$ is well-understood [22, 82, 83]. Here, we adapt these techniques in order to study the case with saturation.*

3.2.4 Long-time behaviour. Relation to free-energy minimisers

We begin this section by giving an interpretation of long-time behaviour in terms of semigroups.

Definition 3.7. *We say that a semigroup S for a problem (P) has a time-limit operator $S_\infty : \mathcal{A} \to \mathcal{A}$ if all the following are satisfied:*

i) For any $\rho_0 \in \mathcal{A}$ there exists a limit in time

$$S_t\rho_0 \to S_\infty\rho_0 \qquad \text{strongly in } L^1(\Omega) \text{ as } t \to \infty.$$

ii) S_∞ is stationary for the semigroup, i.e., $S_t S_\infty = S_\infty$.

iii) For any $\rho_0 \in \mathcal{A}$, $S_\infty\rho_0$ is a constant-in-time weak solution to (P).

With this definition we can now proceed to study the other three sides of the diagram (D_1). First, in Section 3.5 we construct a time-limit operator for the problems (P_ε) and (P).

Theorem 3.8 (Long-time behaviour for (P) and (P_ε))**.** *We have that:*

i) For $\varepsilon > 0$ the free-energy dissipating semigroup $S^{(\varepsilon)}$ for (P_ε) has a time-limit operator, which we denote $S^{(\varepsilon)}_\infty$.

ii) Any free-energy dissipating semigroup S for (P) has a time-limit operator, which we denote S_∞.

Both $S^{(\varepsilon)}_\infty$ and S_∞ are L^1-contractions.

In section 3.6.1 we study the time-limit operator of problem (P_ε).

Theorem 3.9 (The global attractors of (P_ε))**.** *Let $\varepsilon > 0$ be fixed, $M \in (0, \alpha|\Omega|)$ and define the corresponding form of (3.15), i.e.,*

$$\widehat{\rho}^{(\varepsilon)}(x) := (U'_\varepsilon)^{-1} \left(C_\varepsilon - V(x) \right), \quad \text{in } \Omega, \tag{3.16}$$

where C_ε is uniquely determined by the mass condition $\int_\Omega (U'_\varepsilon)^{-1} (C_\varepsilon - V) = M$. Then:

i) $\widehat{\rho}^{(\varepsilon)}$ is the unique fixed-point of the semigroup in $\mathcal{A}_M$, i.e., $\widehat{\rho} \in \mathcal{A}_M$ such that $S^{(\varepsilon)}_t \widehat{\rho} = \widehat{\rho}$ for all $t > 0$. In particular, it is the global attractor, i.e., for any $\rho_0 \in \mathcal{A}_M$ we have $S^{(\varepsilon)}_\infty \rho_0 = \widehat{\rho}^{(\varepsilon)}$.

ii) $\widehat{\rho}^{(\varepsilon)}$ is the unique constant-in-time weak solution of (P_ε) in $\mathcal{A}_M \cap \mathcal{A}_+$.

iii) $\widehat{\rho}^{(\varepsilon)}$ is the unique L^1-local minimiser of the free energy $\mathcal{F}_\varepsilon$ over $\mathcal{A}_M$. It is also the unique global minimiser of the free energy $\mathcal{F}_\varepsilon$ over $\mathcal{A}_M$.

We next focus on studying the ω-limit of (P) in more detail. First, in section 3.6.2 we obtain the following result.

Theorem 3.10 (On steady states for (P))**.** *Assume (SC_U) and that $M \in (0, \alpha|\Omega|)$. Then we can define*

$$\widehat{\rho}^{(0)}(x) := T_{0,\alpha} \circ (U')^{-1}(C_0 - V(x)), \quad \text{in } \Omega,$$

where C_0 is uniquely determined by the mass condition $\int_\Omega T_{0,\alpha} \circ (U')^{-1}(C_0 - V) = M$. We have that:

i) $S_t \widehat{\rho}^{(0)} = \widehat{\rho}^{(0)}$ for any $t > 0$.

ii) $\widehat{\rho}^{(0)}$ is the unique L^1-local minimiser of the free energy (3.2) over $\mathcal{A}_M$. It is the unique global minimiser over $\mathcal{A}_M$.

iii) If we also assume (USC_U), then $\widehat{\rho}^{(0)}$ is the limit as $\varepsilon \to 0$ of the constant-in-time weak solutions of (P_ε) given by (3.16)

$$\widehat{\rho}^{(\varepsilon)} \to \widehat{\rho}^{(0)} \quad \text{in } L^1(\Omega) \text{ as } \varepsilon \to 0.$$

We finally show that the double limits in diagram (D_1) do not commute by giving a counterexample. In section 3.6.3, we provide a non-linearity U, and a potential V, such that the problem (P) has infinitely many steady states different from $\widehat{\rho}^{(0)}$. Moreover, each of them have a large basin of attraction of initial data.

Remarks. *The long-time behaviour result holds even if $\mathcal{F}[\rho_t]$ never becomes finite. This is a powerful consequence of the L^1-contraction theory. The Wasserstein-type gradient-flow theory is usually not able to deal with these cases. In section 3.6.3 we construct m, U, V such that the global minimiser of $\mathcal{F}$ is not the global attractor, i.e., there exists ρ_0 such that $S_\infty \rho_0 \neq \widehat{\rho}^{(0)}$. Furthermore, we construct a curve of stationary weak solutions such that each of them attracts some initial data (see Figure 3.3 and Figure 3.4). For certain choices of V (e.g., V convex) it is easy to show that there exists a global attractor for (P). See, e.g., [102, Section 3] and [228].*

3.2.5 Numerical analysis

We now study the implicit Finite-Volume scheme proposed by Bailo, Carrillo, and Hu in [19]. Here, we consider a small generalisation of the method in [19] that also fit to the regularised problem (P_ε). For the sake of clarity, in this chapter we only cover the 1-dimensional case. We devote Section 3.7 to the numerical analysis. Let us fix $\varepsilon \geq 0$. We prove a first result which we will need for the structure of our paper. We will assume that

$$\mathrm{m}(s) = \mathrm{m}^{(1)}(s)\mathrm{m}^{(2)}(s), \tag{M^Δ}$$

where $\mathrm{m}^{(j)}$ are Lipschitz continuous, $\mathrm{m}^{(1)}$ is non-decreasing, and $\mathrm{m}^{(2)}$ is non-increasing. This is required to perform the up-winding below. This assumption is not too restrictive, since we show the following result in section 3.7.1.

Lemma 3.11. *Assume (H_1). Then, there exists $\mathrm{m}^{(1)} \in C^1((0,\alpha))$ non-decreasing and $\mathrm{m}^{(2)} \in C^1((0,\alpha))$ non-increasing such that (M^Δ) holds. Furthermore, if m is Lipschitz in $[0,\alpha]$, $\mathrm{m}' \geq 0$ in a neighbourhood of 0 and $\mathrm{m}' \leq 0$ in a neighbourhood of α, then $\mathrm{m}^{(1)}$ and $\mathrm{m}^{(2)}$ are Lipschitz.*

Without loss of generality, we can restrict to $x \in (0,1)$, we pick $I = \{1, \cdots, N\}$ for some $N \in \mathbb{N}$, and let $\Delta x = 1/N$. We now present (a small generalisation of) the Finite-Volumes scheme constructed in [19]. The method is given as

$$\begin{aligned}
\frac{\rho_i^{n+1} - \rho_i^n}{\Delta t} &= -\frac{F_{i+\frac12}(\rho^{n+1}) - F_{i-\frac12}(\rho^{n+1})}{\Delta x}, \qquad i \in I, n \in \mathbb{N},\\
F_{i+\frac12}(\rho) &= \mathrm{m}^{(1)}(\rho_i)\mathrm{m}^{(2)}(\rho_{i+1})(v_{i+\frac12}(\rho))^+ + \mathrm{m}^{(1)}(\rho_{i+1})\mathrm{m}^{(2)}(\rho_i)(v_{i+\frac12}(\rho))^-,\\
v_{i+\frac12}(\rho) &= -\frac{\xi_{i+1}(\rho) - \xi_i(\rho)}{\Delta x},\\
\xi_i(\rho) &= U'(\rho_i) + V(x_i),\\
F_{\frac12}(\rho) &= F_{N+\frac12}(\rho) = 0.
\end{aligned} \tag{P^Δ}$$

Here, we are using the notation $u = u^+ + u^-$. We consider the initial condition

$$\rho_i^0 = \frac{1}{\Delta x}\int_{x_{i-\frac12}}^{x_{i+\frac12}} \rho_0(x)\,\mathrm{d}x. \tag{3.17}$$

In [19, Theorem 2.4] the authors prove the decay of a discrete energy, defined for $\varepsilon \geq 0$ as

$$E^\Delta[\rho] = \Delta x \sum_{i\in I} \left(U(\rho_i) + V(x_i)\rho_i\right). \tag{3.18}$$

We introduce the following discrete version of (P_ε):

(P^Δ) when we replace $\mathrm{m}^{(i)}$ and U by $\mathrm{m}_\varepsilon^{(i)}$ and U_ε.
We will use the notation $\rho^{\varepsilon,n}, F^\varepsilon, v^\varepsilon, \xi^\varepsilon$. ($\mathrm{P}_\varepsilon^\Delta$)
The corresponding energy is denoted E_ε^Δ.

Remarks. *For the approximating problems, due to ellipticity, $U_\varepsilon'(0) = -\infty, U_\varepsilon'(\alpha) = \infty$. We can only deal with solutions in $0 < \rho_i^0 < \alpha$. This is also the case in general if $U \notin C^1([0,\alpha])$. We will separate the case of $U \in C^1([0,1])$. Even if $U \in C^1([0,\alpha])$, from the computational point of view the approximating problem ($\mathrm{P}_\varepsilon^\Delta$) is usually better than (P^Δ) since typically we will be able to use Newton iteration to compute the next step (we can guarantee the Jacobian is invertible). The scheme introduced in [19] is done for $\mathrm{m}^{(1)}(s) = s$ and $\xi_i = U'(\rho) + V(x_i) + W * \rho$. This problem is significantly more difficult. For example, it does not have a comparison principle.*

Finite-time properties of the numerical schemes

We introduce

$$\mathcal{A}_\Delta = \{\rho \in \mathbb{R}^{|I|} : 0 \le \rho_i \le \alpha\}, \qquad \mathcal{A}_{\Delta,+} = \{\rho \in \mathbb{R}^{|I|} : 0 < \rho_i < \alpha\}, \qquad \|\rho\|_{L^1_\Delta} = \sum_{i\in I} |\rho_i|.$$

We will use the notation $\Delta = (\Delta t, \Delta x)$. In the same fashion we worked with semigroups, we can write

$$J^\Delta : \rho^0 \in \mathcal{A}_\Delta \mapsto \rho^1, \qquad J^\Delta_\varepsilon : \rho^0 \in \mathcal{A}_\Delta \mapsto \rho^{\varepsilon,1}.$$

Formally, $\rho^n := (J^\Delta)^n \rho^0$. For $\rho, \eta \in \mathcal{A}_{\Delta,+}$ we say that $\rho \le \eta$ if $\rho_i \le \eta_i$ for all $i \in I$. We use the following definition.

Definition 3.12. *Fix Δ. We say that (P^Δ) is a free-energy dissipating numerical scheme over $\mathcal{B} \subset \mathcal{A}_\Delta$ if*

i) For $\rho^0 \in \mathcal{B}$, there exists a unique $\rho \in \mathcal{B}$ that solves (P^Δ). We call this unique solution ρ^1.

ii) The solution map $J^\Delta : \rho_0 \in \mathcal{B} \to \rho^1 \in \mathcal{B}$ is an L^1_Δ-contraction. There is mass conservation.

iii) There is free-energy dissipation, i.e., for $\rho^0 \in \mathcal{B}$ we have $E^\Delta[J^\Delta \rho^0] \le E^\Delta[\rho^0]$.

In [19] the authors do not prove that this implicit scheme admits a solution, or whether it is unique.

Theorem 3.13 (Well-posedness theory)**.** *We have that*

i) (P^Δ) and ($\mathrm{P}^\Delta_\varepsilon$) for $\varepsilon > 0$ are free-energy dissipating numerical schemes in $\mathcal{A}_{\Delta,+}$. If $\rho^0 \in \mathcal{A}_{\Delta,+}$ then $J^\Delta_\varepsilon \rho^0 \to J^\Delta \rho^0$ as $\varepsilon \to 0$.

ii) If we also assume $U \in C^1([0,\alpha])$ then (P^Δ) is a free-energy dissipating numerical scheme in $\mathcal{A}_\Delta$.

This scheme is convergent at least under high regularity of the solution. We only include a small remark below about the regularity result, we will not discuss it any further and just focus on the convergence.

Theorem 3.14 (Convergence as $\Delta \to 0$)**.** *Let $\rho_0 \in \mathcal{A}_+$ be fixed, ρ a solution to (P), and $R := \rho([0,T] \times \overline{\Omega}) \subset [0,\alpha]$, $U \in C^{3+\gamma}(R)$, $\mathrm{m}_\varepsilon^{(j)} \in C^{1+\gamma}(R)$ for $j = 1, 2$, $V \in C^{2+\gamma}(\overline{\Omega})$, and $\rho \in C_t^{1+\beta}, C_x^{2+\gamma}$. Then, if ρ_i^n is the solution to (P^Δ) it is such that*

$$\sup_{0 \le n \le \frac{T}{\Delta t}} \Delta x \sum_{i\in I} |\rho_i^n - \rho(t_n, x_i)| \le C((\Delta t)^\beta + (\Delta x)^\gamma).$$

Hence, the numerical solution converges to the solution of the continuous problem as $\Delta \to 0$.

Notice that this theorem can also be applied to (P_ε) if the solution of the continuous problem had the regularity needed. The proof of these results can be found in section 3.7.1.

Remarks. *The comparison principle holds, i.e., for $\underline{\rho}^0, \overline{\rho}^0 \in \mathcal{B}$ if $\underline{\rho}^0 \le \overline{\rho}^0$ then $J^\Delta \underline{\rho}_i^0 \le J^\Delta \overline{\rho}_i^0$. In order to obtain the convergence result presented in Theorem 3.14, we rely on consistency and stability. Thus, it requires for the solution of the problem to be smooth. We expect this to happen in (P_ε) with $\varepsilon > 0$ due to uniform ellipticity. For* (P) *we can construct examples where it does not hold. Theorem 3.14 implies uniqueness of smooth solutions of* (P).

Long-time behaviour for the numerical problems

In section 3.7.2 we focus on the long-time analysis of the numerical method. First, we study the limit in the time step of the solutions of ($\mathrm{P}^\Delta_\varepsilon$).

Theorem 3.15 (Asymptotics for ($\mathrm{P}^\Delta_\varepsilon$))**.** *Let $\varepsilon > 0$ be fixed. Assume (M^Δ), $M \in (0, \alpha|\Omega|)$ and let*

$$\rho_i^{\varepsilon,\infty} = \left(U'_\varepsilon\right)^{-1}\left(C^\Delta_\varepsilon - V(x_i)\right),$$

where C^Δ_ε is uniquely determined by the mass condition $\Delta x \sum_i \left(U'_\varepsilon\right)^{-1}\left(C^\Delta_\varepsilon - V(x_i)\right) = M$. Then:

i) $\rho^{\varepsilon,\infty}$ *is a unique constant-in-time solution of* (P^{Δ}_{ε}) *in* $\mathcal{A}_{\Delta,+}$ *of mass* M.

ii) *It is the global attractor, i.e., for any* $\rho_0 \in \mathcal{A}_{\Delta,+}$ *we have* $(J^{\Delta}_{\varepsilon})^n \rho^0 \to \rho^{\varepsilon,\infty}$ *in* $\mathbb{R}^{|I|}$ *as* $n \to \infty$.

iii) *Consider* $\widehat{\rho}^{(\varepsilon)}$ *obtained in Theorem 3.9. Then* $|C^{\Delta}_{\varepsilon} - C_{\varepsilon}| \le \mathsf{C}(\varepsilon)\Delta x$.

For (P^{Δ}) we can prove existence of a steady state, but not its uniqueness. Furthermore, we study some properties of one of the steady states.

Theorem 3.16 (Asymptotics for (P^{Δ}))**.** *Assume* (3.5). *Then:*

i) *For every* $\rho^0 \in \mathcal{A}$ *there exists* $J^{\Delta,\infty}\rho^0 := \lim_n (J^{\Delta})^n \rho^0$. *This limit is a fixed point of* J^{Δ} *(i.e., a constant-in-time solution).*

ii) *The operator* $J^{\Delta,\infty}$ *is an* L^1_{Δ}*-contraction.*

Assume, furthermore, that U *satisfies* (SC_U). *Let* $M \in (0, \alpha|\Omega|)$ *and define*

$$\rho_i^{0,\infty} = T_{0,\alpha} \circ (U')^{-1}(C^{\Delta}_0 - V(x_i)), \quad \text{in } \mathbb{R}^{|I|},$$

where C^{Δ}_0 *is determined by the mass condition* $\Delta x \sum_i T_{0,\alpha} \circ (U')^{-1}(C^{\Delta}_0 - V(x_i)) = M$. *Then:*

iii) $\rho^{0,\infty}$ *is a constant-in-time solution to* (P^{Δ}), *i.e.,* $J^{\Delta}\rho^{0,\infty} = \rho^{0,\infty}$. *In particular,* $J^{\Delta,\infty}\rho^{0,\infty} = \rho^{0,\infty}$.

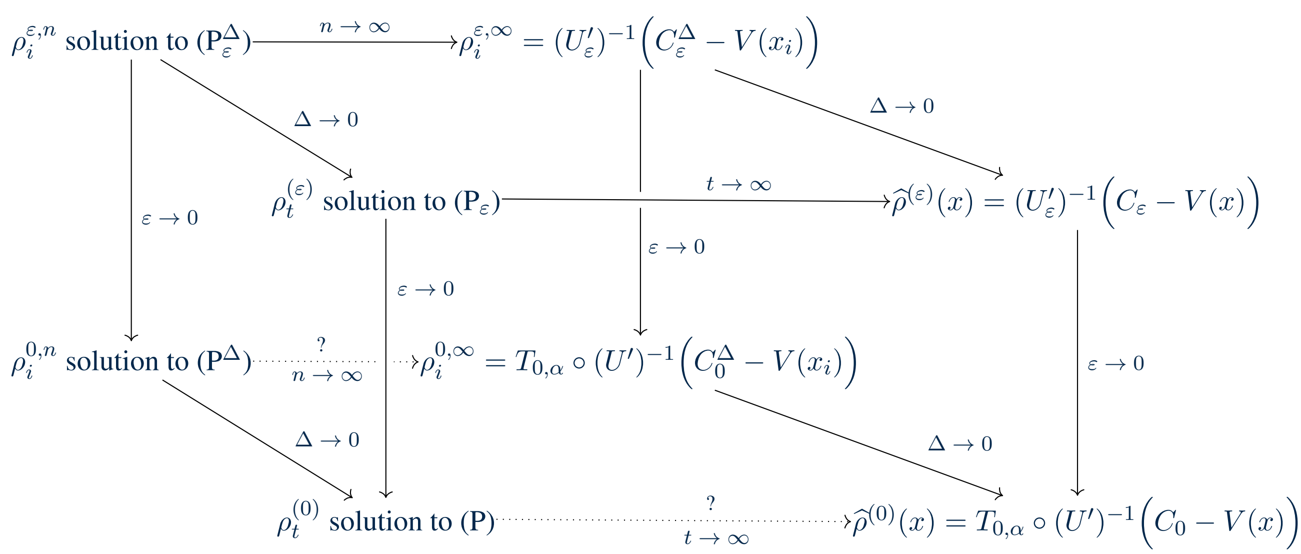
iv) *For the same mass,* $\rho^{\varepsilon,\infty} \to \rho^{0,\infty}$ *as* $\varepsilon \to 0$.

v) *Let* $\widehat{\rho}^{(0)}$ *be as obtained in Theorem 3.10. Then,* $C^{\Delta}_0 \to C_0$ *as* $\Delta x \to 0$. *If* (USC_U) *and* $(U')^{-1} \in C^{\gamma}([0,\alpha])$, *then* $|C^{\Delta}_0 - C_0| \le \mathsf{C}(\Delta x)^{\gamma}$.

As we mention above for (P^{Δ}) we cannot prove uniqueness of a steady state. In section 3.7.3 we reproduce the example from section 3.6.3. Thus, analogously to the continuous case, in the discrete problem there exists a steady state different from $\rho^{0,\infty}$ that attracts a large class of initial data. Finally, in section 3.7.4 we show some numerical experiments.

3.2.6 A complete convergence diagram

As a summary of the results in the previous sections, we present the diagram (D_2) superseding the diagram (D_1). This diagram summarises the most important results of this work.



(D_2)

Notice again that a similar counterexample for the commutativity of the diagram (D_1) can be given here to the commutativity of the backward face of (D_2) in section 3.7.3.

3.3 Well-posedness theory

We divide this section as follows. In section 3.3.1 we construct the regularised mobility m_ε, we obtain some auxiliary results, we discuss well-posedness, and we obtain a Strong Maximum Principle using classical theory. In section 3.3.2 we obtain some *a priori* estimates for the sequence ρ that come from the structure of the equation. Then, in section 3.3.3 we understand the problem through a free-energy dissipating semigroup. Finally, we devote section 3.3.4 to prove existence of weak solutions for the problem (P) as a limit of classical solutions of the problem (P_ε).

Before we begin, we recall some functional analysis results. We will take advantage of the negative Sobolev space

$$W^{-1,1}(\Omega) \coloneqq \left\{\rho \in \mathcal{D}'(\Omega) : \exists F \in L^1(\Omega) \text{ s.t. } \rho = \operatorname{div}(F)\right\} \qquad \text{with } \|\rho\|_{W^{-1,1}(\Omega)} \coloneqq \inf_{\rho=\operatorname{div}(F)} \|F\|_{L^1(\Omega)}.$$

This space will be very helpful, since it allows us to use the free-energy dissipation to prove that if ρ solves (P) then

$$\begin{aligned}
\int_{t_1}^{t_2} \left\|\frac{\partial\rho}{\partial t}\right\|_{W^{-1,1}(\Omega)}^2 &= \int_{t_1}^{t_2} \left\|\operatorname{div}\left(\mathrm{m}(\rho)\nabla\left(U'(\rho)+V\right)\right)\right\|_{W^{-1,1}(\Omega)}^2 \\
&\le \int_{t_1}^{t_2} \left\|\mathrm{m}(\rho)\nabla\left(U'(\rho)+V\right)\right\|_{L^1(\Omega)}^2 \\
&\le \|\mathrm{m}\|_{L^\infty([0,\alpha])}|\Omega| \int_{t_1}^{t_2} \left\|\mathrm{m}(\rho)^{\frac{1}{2}}\nabla\left(U'(\rho)+V\right)\right\|_{L^2(\Omega)}^2 \\
&\le \|\mathrm{m}\|_{L^\infty([0,\alpha])}|\Omega| \left(\mathcal{F}[\rho_{t_1}] - \mathcal{F}[\rho_{t_2}]\right).
\end{aligned} \tag{3.19}$$

The decay of the free energy allows to control the right-hand side uniformly.

Lemma 3.17. *Let $\rho^{[n]} \in C([t_1,t_2]; W^{-1,1}(\Omega))$ such that*

$$\int_{t_1}^{t_2} \left\|\frac{\partial\rho^{[n]}}{\partial t}\right\|_{W^{-1,1}(\Omega)}^2 \le C$$

and uniformly bounded in $L^1((t_1,t_2)\times\Omega)$. Then, up to a subsequence,

$$\rho^{[n]} \to \rho^\infty \text{ in } C([t_1,t_2]; W^{-1,1}(\Omega)) \text{ as } n \to \infty.$$

Proof. By properties of the Bochner integral (see, e.g., [256]), we get that a family of $\rho^{[n]}$ with such uniform bound are equicontinuous as $C([t_1,t_2]; W^{-1,1}(\Omega))$ functions. We recall that $L^1((t_1,t_2)\times\Omega) = L^1(t_1,t_2; L^1(\Omega))$ and that $L^1(\Omega)$ is compactly embedded in $W^{-1,1}(\Omega)$. Thus, the result follows from the Ascoli-Arzelà theorem. □

3.3.1 Well-posedness and interior bounds for (P_ε)

We start by showing some auxiliary results.

Proof of Lemma 3.3. First, we take $0 \le \kappa < \alpha/2$. Since $\Phi'_\varepsilon \le (1+\varepsilon)(\Phi'+\varepsilon)$ we observe that

$$\begin{aligned}
\int_0^\alpha \left| \int_{\frac{\alpha}{2}}^s \frac{\Phi'_\varepsilon(\sigma)}{\mathrm{m}(\sigma)} \chi_{[\kappa,\alpha-\kappa]}\,\mathrm{d}\sigma \right| \mathrm{d}s \le (1+\varepsilon) \int_0^\alpha \left| \int_{\frac{\alpha}{2}}^s \frac{\Phi'(\sigma)}{\mathrm{m}(\sigma)}\,\mathrm{d}\sigma \right| \mathrm{d}s \\
+ (1+\varepsilon)\varepsilon \left(\int_0^{\frac{\alpha}{2}} \int_s^{\frac{\alpha}{2}} \frac{\chi_{[\kappa,\alpha-\kappa]}(\sigma)}{\mathrm{m}(\sigma)} \mathrm{d}\sigma\mathrm{d}s + \int_{\frac{\alpha}{2}}^\alpha \int_{\frac{\alpha}{2}}^s \frac{\chi_{[\kappa,\alpha-\kappa]}(\sigma)}{\mathrm{m}(\sigma)}\,\mathrm{d}\sigma\mathrm{d}s \right).
\end{aligned}$$

The first term is finite since $U' \in L^1((0,\alpha))$. We pick

$$\kappa(\varepsilon) = \varepsilon\frac{\alpha}{2} + (1-\varepsilon)\inf\left\{\kappa \in (0,\tfrac{\alpha}{2}] : \int_0^{\frac{\alpha}{2}}\int_s^{\frac{\alpha}{2}} \frac{\chi_{[\kappa,\alpha-\kappa]}(\sigma)}{\mathrm{m}(\sigma)}\,\mathrm{d}\sigma\mathrm{d}s + \int_{\frac{\alpha}{2}}^{\alpha}\int_{\frac{\alpha}{2}}^{s} \frac{\chi_{[\kappa,\alpha-\kappa]}(\sigma)}{\mathrm{m}(\sigma)}\,\mathrm{d}\sigma\mathrm{d}s \le \varepsilon^{-\frac{1}{2}}\right\}.$$

The set is non-empty, since the function inside infimum is continuous with respect to κ, and takes value 0 at $\kappa = \frac{\alpha}{2}$. Since the function inside the infimum is non-negative and non-increasing, we have that $\kappa(0^+) = 0$.

We will use it to bound the L^1 norm of $U_\varepsilon'(s) = U'(\frac{\alpha}{2}) + \int_{\frac{\alpha}{2}}^{s} \frac{\Phi_\varepsilon'(\sigma)}{\mathrm{m}_\varepsilon(\sigma)}\,\mathrm{d}\sigma$. Thus, it suffices to estimate

$$\int_0^\alpha \left|\int_{\frac{\alpha}{2}}^{s} \frac{\Phi_\varepsilon'(\sigma)}{\mathrm{m}_\varepsilon(\sigma)}\,\mathrm{d}\sigma\right| \mathrm{d}s = \int_0^{\frac{\alpha}{2}}\int_s^{\frac{\alpha}{2}} \frac{\Phi_\varepsilon'(\sigma)}{\mathrm{m}_\varepsilon(\sigma)}\,\mathrm{d}\sigma\mathrm{d}s + \int_{\frac{\alpha}{2}}^{\alpha}\int_{\frac{\alpha}{2}}^{s} \frac{\Phi_\varepsilon'(\sigma)}{\mathrm{m}_\varepsilon(\sigma)}\,\mathrm{d}\sigma\mathrm{d}s.$$

We split

$$\frac{\Phi_\varepsilon'(\sigma)}{\mathrm{m}_\varepsilon(\sigma)} = \frac{\Phi_\varepsilon'(\sigma)}{\mathrm{m}_\varepsilon(\sigma)}\chi_{[0,\kappa(\varepsilon))} + \frac{\Phi_\varepsilon'(\sigma)}{\mathrm{m}_\varepsilon(\sigma)}\chi_{[\kappa(\varepsilon),\alpha-\kappa(\varepsilon)]} + \frac{\Phi_\varepsilon'(\sigma)}{\mathrm{m}_\varepsilon(\sigma)}\chi_{(\alpha,\alpha-\kappa(\varepsilon)]}.$$

First we build a lower bound with a mobility of the form

$$\underline{\mathrm{m}}(\varepsilon,s) = \begin{cases} s & \text{if } 0 \le s < \kappa(\varepsilon),\\ \max\{\kappa(\varepsilon), \mathrm{m}(\kappa(\varepsilon))\} & \text{if } s = \kappa(\varepsilon),\\ \mathrm{m}(s) & \text{if } \kappa(\varepsilon) < s < \alpha-\kappa(\varepsilon),\\ \max\{\kappa(\varepsilon), \mathrm{m}(\alpha-\kappa(\varepsilon))\} & \text{if } s = \alpha-\kappa(\varepsilon),\\ \alpha - s & \text{if } \alpha-\kappa(\varepsilon) < s \le \alpha. \end{cases}$$

Since we know that $\Phi_\varepsilon' \le (1+\varepsilon)(\kappa(\varepsilon)^{-1}+\varepsilon)$ and if $\mathrm{m}_\varepsilon \ge \underline{\mathrm{m}}(\varepsilon,\cdot)$, it follows that

$$\begin{aligned}\int_0^\alpha \left|\int_{\frac{\alpha}{2}}^{s} \frac{\Phi_\varepsilon'(\sigma)}{\mathrm{m}_\varepsilon(\sigma)}\chi_{[0,\kappa(\varepsilon)]}\mathrm{d}\sigma\right| \mathrm{d}s &\le (1+\varepsilon)\int_0^\alpha \left|\int_{\frac{\alpha}{2}}^{s} \frac{\kappa(\varepsilon)^{-1}+\varepsilon}{\underline{\mathrm{m}}(\varepsilon,\sigma)}\chi_{[0,\kappa(\varepsilon)]}\mathrm{d}\sigma\right|\mathrm{d}s\\ &= (1+\varepsilon)(\kappa(\varepsilon)^{-1}+\varepsilon)\int_0^{\kappa(\varepsilon)}\int_s^{\kappa(\varepsilon)} \frac{d\sigma}{\sigma}\mathrm{d}s = (1+\varepsilon)(\kappa(\varepsilon)^{-1}+\varepsilon)\kappa(\varepsilon),\end{aligned}$$

so it is bounded, and similarly for the terms $\chi_{(\alpha-\kappa(\varepsilon),\alpha]}$. We also build the following upper bound

$$\overline{\mathrm{m}}(\varepsilon,s) = \begin{cases} B_\varepsilon s & \text{if } 0 \le s \le \kappa(\varepsilon),\\ \min\{B_\varepsilon\kappa(\varepsilon), (1+\varepsilon)\mathrm{m}(\kappa(\varepsilon))\} & \text{if } s = \kappa(\varepsilon),\\ (1+\varepsilon)\mathrm{m}(s) & \text{if } \kappa(\varepsilon) < s < \alpha-\kappa(\varepsilon),\\ \min\{B_\varepsilon\kappa(\varepsilon), (1+\varepsilon)\mathrm{m}(\alpha-\kappa(\varepsilon))\} & \text{if } s = \kappa(\varepsilon),\\ B_\varepsilon(\alpha - s) & \text{if } \alpha-\kappa(\varepsilon) \le s \le \alpha, \end{cases}$$

where

$$B_\varepsilon = \max\left\{1+\varepsilon, \frac{(1+\varepsilon)\mathrm{m}(\kappa(\varepsilon))}{\kappa(\varepsilon)}, \frac{(1+\varepsilon)\mathrm{m}(\alpha-\kappa(\varepsilon))}{\kappa(\varepsilon)}\right\}.$$

With this construction it is easy to see that in $(0,1]\times[0,\alpha]$ we have $\overline{\mathrm{m}} \ge \underline{\mathrm{m}}$. Furthermore, $\overline{\mathrm{m}}$ is lower semi-continuous, and $\underline{\mathrm{m}}$ is upper semi-continuous. The same holds when we divide by $s(\alpha-s)$. Due to the Katětov–Tong insertion theorem there exists $v_0 \in C((0,1]\times[0,\alpha])$ such that $\frac{\underline{\mathrm{m}}}{s(\alpha-s)} \le v_0 \le \frac{\overline{\mathrm{m}}}{s(\alpha-s)}$.

We first build an auxiliary sequence v_n. Let $K_n \coloneqq [\varepsilon_n, 1]\times[0,\alpha]$ where $\varepsilon_n = \frac{1}{n+1}$ for $n \ge 0$. We are going to inductively construct a sequence of smooth functions u_n for $n \ge 1$ such that

$$u_n = u_{n+1} \text{ in } K_{n-1} \quad \text{and} \quad \underline{\mathrm{m}} \le u_{n+1} \le \overline{\mathrm{m}} + \varepsilon s(\alpha-s) \text{ in } K_{n+1}.$$

There exists $v_1 \in C^\infty(K_1)$ with $\|\frac{\varepsilon}{2}+v_0-v_1\|_{L^\infty(K_1)} \le \frac{\varepsilon_1}{2}$, which in particular implies $\underline{\mathrm{m}}(\varepsilon,s) \le v_1(\varepsilon,s)s(\alpha-s) \le \overline{\mathrm{m}}(\varepsilon,s)+\varepsilon s(\alpha-s)$ for all $(\varepsilon,s)\in K_1$. We now set $u_1 = s(\alpha-s)v_1$. Again, there exists $v_{n+1}\in C^\infty(K_{n+1})$ such that $\|\frac{\varepsilon}{2}+v_0-v_{n+1}\|_{L^\infty(K_{n+1})} \le \frac{\varepsilon_{n+1}}{2}$. Once more, this bound implies that $\underline{\mathrm{m}}(\varepsilon,s) \le v_{n+1}(\varepsilon,s)s(\alpha-s) \le \overline{\mathrm{m}}(\varepsilon,s)+\varepsilon s(\alpha-s)$ for all $(\varepsilon,s)\in K_{n+1}$.

We now construct u_{n+1} from u_n and v_{n+1}. Consider a non-decreasing function $\psi \in C^\infty([0,1])$ such that $\psi(0)=\psi^{(k)}(0)=\psi^{(k)}(1)=0$ and $\psi(1)=1$ for all $k\ge 1$. Let $\theta_{n+1}(\varepsilon)=\psi(\frac{\varepsilon_{n-1}-\varepsilon}{\varepsilon_{n-1}-\varepsilon_n})$. With this construction, we define

$$u_{n+1}(\varepsilon,s)=\begin{cases} u_n(\varepsilon,s) & \text{in } K_{n-1},\\ u_n(\varepsilon,s)(1-\theta_{n+1}(\varepsilon))+v_{n+1}(\varepsilon,s)s(\alpha-s)\theta_{n+1}(\varepsilon) & \text{in } K_n\setminus K_{n-1},\\ v_{n+1}(\varepsilon,s)s(\alpha-s) & \text{in } K_{n+1}\setminus K_n,\end{cases}$$

which also satisfies the desired bounds. Since we are pasting C^∞ functions with C^∞ contact conditions, we have $u_{n+1}\in C^\infty(K_{n+1})$.

Let us now define the limit function $u\in C^\infty((0,1]\times[0,\alpha])$ so that $u=u_n$ in K_{n-1} for all $n\ge 1$. Notice that $u(\varepsilon,0)=u(\varepsilon,\alpha)=0$ for $\varepsilon>0$. Furthermore, due to the lower bound we get $\frac{\partial u}{\partial s}(\varepsilon,0), -\frac{\partial u}{\partial s}(\varepsilon,\alpha)\ge 1$ for any $\varepsilon>0$. Since $\underline{\mathrm{m}}(0,s)=\overline{\mathrm{m}}(0,s)=\mathrm{m}(s)$ we have an extension $u\in C([0,1]\times[0,\alpha])$ where $u(0,s)=\mathrm{m}(s)$. Finally, we define $\mathrm{m}_\varepsilon(s)=u(\varepsilon,s)$. Since $\mathrm{m}_\varepsilon \ge \underline{\mathrm{m}}(\varepsilon,\cdot)$ the corresponding U_ε satisfies the desired properties. □

Lemma 3.18. *If* (3.4c) *is satisfied, then it follows that*

$$\sup_{(\varepsilon,s)\in(0,1]\times[0,\alpha]} \left|\frac{(\Phi_\varepsilon(\alpha)-\Phi_\varepsilon(s))\Phi_\varepsilon(s)}{\Phi'_\varepsilon(s)}\right| + \left|\frac{(\Phi_\varepsilon(s)-\Phi_\varepsilon(0))\Phi_\varepsilon(s)}{\Phi'_\varepsilon(s)}\right| < \infty. \tag{3.20}$$

Proof. We work first on the term $\Phi(s)-\Phi(0)$. Using (3.9), we point out that

$$\Phi_\varepsilon(s)=\int_{s_0}^s \Phi'_\varepsilon(\sigma)\,\mathrm{d}\sigma,\quad \Phi_\varepsilon(s)-\Phi_\varepsilon(0)=\int_0^s \Phi'_\varepsilon(\sigma)\,\mathrm{d}\sigma \ge 0,\quad |\underline{\Phi_\varepsilon}(s)|\le|\Phi_\varepsilon(s)|\le(1+\varepsilon)|\underline{\Phi_\varepsilon}(s)|,$$
$$\text{and}\quad \underline{\Phi_\varepsilon}(s)-\underline{\Phi_\varepsilon}(0)\le\Phi_\varepsilon(s)-\Phi_\varepsilon(0)\le(1+\varepsilon)(\underline{\Phi_\varepsilon}(s)-\underline{\Phi_\varepsilon}(0)).$$

Therefore, we have the estimates

$$0\le\frac{\underline{\Phi_\varepsilon}(s)-\underline{\Phi_\varepsilon}(0)}{(1+\varepsilon)\underline{\Phi'_\varepsilon}(s)}|\underline{\Phi_\varepsilon}(s)|\le\frac{\Phi_\varepsilon(s)-\Phi_\varepsilon(0)}{\Phi'_\varepsilon(s)}|\Phi_\varepsilon(s)|\le(1+\varepsilon)^2\frac{\underline{\Phi_\varepsilon}(s)-\underline{\Phi_\varepsilon}(0)}{\underline{\Phi'_\varepsilon}(s)}|\underline{\Phi_\varepsilon}(s)|.$$

We distinguish two cases:

Case I: s such that $\Phi'(s)\le\kappa(\varepsilon)^{-1}$. In this case, we have $\underline{\Phi_\varepsilon}'(s)=\Phi'(s)+\varepsilon$, and we point out that

$$\underline{\Phi_\varepsilon}(s)-\underline{\Phi_\varepsilon}(0)=\int_0^s\underline{\Phi_\varepsilon}'(\sigma)\,\mathrm{d}\sigma\le\int_0^s(\Phi'(s)+\varepsilon)\,\mathrm{d}s=\Phi(s)-\Phi(0)+\varepsilon s.$$

Therefore, we can estimate

$$\frac{\underline{\Phi_\varepsilon}(s)-\underline{\Phi_\varepsilon}(0)}{\underline{\Phi'_\varepsilon}(s)}\le\frac{\Phi(s)-\Phi(0)+\varepsilon s}{\Phi'(s)+\varepsilon}\le\frac{\Phi(s)-\Phi(0)}{\Phi'(s)}+s,$$

and therefore

$$\frac{\underline{\Phi_\varepsilon}(s)-\underline{\Phi_\varepsilon}(0)}{\underline{\Phi'_\varepsilon}(s)}|\underline{\Phi_\varepsilon}(s)|\le\left(\frac{\Phi(s)-\Phi(0)}{\Phi'(s)}+s\right)(|\Phi(s)|+\varepsilon)<\infty.$$

Case II: s such that $\Phi'(s)>\kappa(\varepsilon)^{-1}$. In this case we have that $\underline{\Phi'_\varepsilon}(s)=\kappa(\varepsilon)^{-1}+\varepsilon\ge\kappa(\varepsilon)^{-1}$, and hence

$$\frac{\underline{\Phi_\varepsilon}(s)-\underline{\Phi_\varepsilon}(0)}{\underline{\Phi'_\varepsilon}(s)}\le\kappa(\varepsilon)\left(\Phi(s)-\Phi(0)+\varepsilon s\right).$$

In particular, we recover that

$$\frac{\underline{\Phi_\varepsilon}(s)-\underline{\Phi_\varepsilon}(0)}{\underline{\Phi'_\varepsilon}(s)}|\underline{\Phi_\varepsilon}(s)| \le \kappa(\varepsilon)\left(\Phi(s)-\Phi(0)+\varepsilon s\right)\left(|\Phi(s)|+\varepsilon s\right) < \infty.$$

We can proceed similarly for $\Phi(\alpha)-\Phi(s)$. □

From (3.9), our problem has uniformly elliptic diffusion, and from our set of basic assumptions, V is regular enough. Then, the problem (P_ε) is uniformly parabolic and existence, uniqueness, and the maximum principle hold from the classical theory [234, 196, 4, 311], further details can also be seen in [96]. Furthermore, from [4], we know that classical solutions to (P_ε) are as regular as we want depending on the regularity of the initial data ρ_0. This corresponds to Theorem 3.4–Item i.

Afterwards, we continue by proving an interior bounds result for the classical solutions.

Lemma 3.19 (Strong Maximum Principle)**.** *Assume* (M_ε) *and* $\rho_0 \in \mathcal{A}\cap C^2(\overline{\Omega})$*. Then, the unique classical solution of* (P_ε) *satisfies either* $\rho^{(\varepsilon)} \equiv 0$*,* $\rho^{(\varepsilon)} \equiv \alpha$*, or* $0 < \rho_t^{(\varepsilon)} < \alpha$ *in* Ω *for all* $t > 0$*.*

Proof. Assume $\rho^{(\varepsilon)}$ is not constantly 0 or α. Let us freeze the coefficients to study strict positivity. Consider the extension

$$c(s) = \begin{cases} \frac{\mathrm{m}_\varepsilon(s)}{s} & \text{if } s\in(0,\alpha],\\ \mathrm{m}'_\varepsilon(0) & \text{if } s=0.\end{cases}$$

Then $u=\rho^{(\varepsilon)}$ is a solution of the following problem

$$\frac{\partial u}{\partial t} = \Phi'_\varepsilon(\rho^{(\varepsilon)})\Delta u + \Big(\Phi''_\varepsilon(\rho^{(\varepsilon)})\nabla\rho^{(\varepsilon)} + \mathrm{m}'_\varepsilon(\rho^{(\varepsilon)})\nabla V\Big)\cdot\nabla u + c(\rho^{(\varepsilon)})(\Delta V)u \quad \text{in } (0,\infty)\times\Omega.$$

Since $\rho^{(\varepsilon)}$ is a classical solution, all the coefficients are continuous. Furthermore, from (M_ε) it follows that $c(\rho^{(\varepsilon)})\Delta V$ is bounded. With the change of variable $v=e^{-\lambda t}u$ for any $\lambda\in\mathbb{R}$ we can replace the zero order coefficient by a non-negative quantity. Thus, we can apply the Strong Maximum Principle [182, Section 7 - Theorem 12] to deduce that the solution $u=\rho^{(\varepsilon)}$ is strictly positive for all $t>0$.

Analogously, $u=\alpha-\rho^{(\varepsilon)}$ solves the problem,

$$\begin{cases} \frac{\partial u}{\partial t} = \operatorname{div}\Big(\Phi'_\varepsilon(\rho^{(\varepsilon)})\nabla u\Big) + \mathrm{m}'_\varepsilon(\rho^{(\varepsilon)})\nabla u\cdot\nabla V - \frac{\mathrm{m}_\varepsilon(\rho^{(\varepsilon)})}{\alpha-\rho^{(\varepsilon)}}(\Delta V)u & \text{in } (0,\infty)\times\Omega,\\ u=\alpha-\rho^{(\varepsilon)}\ge 0 & \text{on } (0,\infty)\times\partial\Omega,\\ u=\alpha-\rho_0\ge 0 & \text{on } \{t=0\}\times\Omega.\end{cases}$$

Since $0\le\frac{\mathrm{m}_\varepsilon(\rho^{(\varepsilon)})}{\alpha-\rho^{(\varepsilon)}}\le\varepsilon^{-1}$ due to (M_ε), once again, from the Strong Maximum Principle it follows that $u>0$ in $(0,\infty)\times\Omega$ and, in particular, $\rho^{(\varepsilon)}<\alpha$ in Ω for all $t>0$. □

3.3.2 A priori estimates for (P_ε)

Let us first comment on some properties of Φ_ε and $\rho^{(\varepsilon)}$. We point out that for every $0\le t_1<t_2<\infty$ and $p\in[1,\infty]$, the unique classical solution of (P_ε) is such that,

$$\|\rho^{(\varepsilon)}\|_{L^p((t_1,t_2)\times\Omega)} \le C(p,t_1,t_2) := \alpha(t_2-t_1)^{\frac{1}{p}}|\Omega|^{\frac{1}{p}}. \tag{3.21}$$

Spatial regularity

In order to obtain an *a priori* estimate on $\nabla\rho^{(\varepsilon)}$ we need to define the following auxiliary flow,

$$G''_\varepsilon(s) = \frac{1}{\Phi'_\varepsilon(s)}, \quad G'_\varepsilon(0) = G_\varepsilon(0) = 0. \tag{3.22}$$

Lemma 3.20 (*A priori* estimates on $\nabla\rho^{(\varepsilon)}$)**.** *Assume* ($\mathrm{M}_\varepsilon$) *and let* $0 < t_1 < t_2 < \infty$. *Then, the unique classical solution* $\rho^{(\varepsilon)}$ *of* (P_ε) *is such that*

$$\|\nabla\rho^{(\varepsilon)}\|^2_{L^2((t_1,t_2)\times\Omega)} \leq 2\int_\Omega G_\varepsilon(\rho^{(\varepsilon)}_{t_1}) + (t_2-t_1)\left\|\frac{\mathrm{m}_\varepsilon(\rho^{(\varepsilon)})}{\Phi'_\varepsilon(\rho^{(\varepsilon)})}\right\|_{L^\infty(\Omega)}\|\nabla V\|^2_{L^2(\Omega)}. \tag{3.23}$$

Proof. We compute in order to obtain that

$$\begin{aligned}\frac{\partial}{\partial t}\int_\Omega G_\varepsilon(\rho^{(\varepsilon)}) &= -\int_\Omega \nabla G'_\varepsilon(\rho^{(\varepsilon)})\cdot\Big(\nabla\Phi_\varepsilon(\rho^{(\varepsilon)}) + \mathrm{m}_\varepsilon(\rho^{(\varepsilon)})\nabla V\Big)\\ &= -\int_\Omega|\nabla\rho^{(\varepsilon)}|^2 - \int_\Omega\nabla\rho^{(\varepsilon)}G''_\varepsilon(\rho^{(\varepsilon)})\mathrm{m}_\varepsilon(\rho^{(\varepsilon)})\cdot\nabla V.\end{aligned}$$

Therefore, applying Young's inequality and integrating in time from t_1 to t_2 we get,

$$\int_{t_1}^{t_2}\|\nabla\rho^{(\varepsilon)}\|^2_{L^2(\Omega)} \leq \int_\Omega G_\varepsilon(\rho^{(\varepsilon)}_{t_1}) - \int_\Omega G_\varepsilon(\rho^{(\varepsilon)}_{t_2}) + \frac12\int_{t_1}^{t_2}\|\nabla\rho^{(\varepsilon)}\|^2_{L^2(\Omega)} + \frac12\int_{t_1}^{t_2}\left\|\frac{\mathrm{m}_\varepsilon(\rho^{(\varepsilon)})}{\Phi'_\varepsilon(\rho^{(\varepsilon)})}\right\|_{L^\infty(\Omega)}\|\nabla V\|^2_{L^2(\Omega)}.$$

Since $G_\varepsilon(s)\geq 0$ for all $s\geq 0$ we recover (3.23). □

Finally, we would also like to obtain an *a priori* estimate on $\nabla\Phi_\varepsilon(\rho^{(\varepsilon)})$.

Lemma 3.21 (*a priori* estimates on $\Phi_\varepsilon(\rho^{(\varepsilon)})$)**.** *Assume* ($\mathrm{M}_\varepsilon$), $\varepsilon\in(0,1]$ *and let* $0<t_1<t_2<\infty$. *Then, the unique classical solution* $\rho^{(\varepsilon)}$ *of* (P_ε) *is such that*

$$\|\nabla\Phi_\varepsilon(\rho^{(\varepsilon)})\|^2_{L^2((t_1,t_2)\times\Omega)} \leq 4\alpha|\Omega|\|\Phi_\varepsilon\|_{L^\infty(0,\alpha)} + (t_2-t_1)\|\mathrm{m}_\varepsilon\|_{L^\infty(0,\alpha)}\|\nabla V\|^2_{L^2(\Omega)}. \tag{3.24}$$

Proof. We define $\vartheta_\varepsilon(s) \coloneqq \int_0^s\Phi_\varepsilon(\sigma)\,\mathrm{d}\sigma$. We can compute that,

$$\frac{\partial}{\partial t}\int_\Omega\vartheta_\varepsilon(\rho^{(\varepsilon)}) = -\int_\Omega|\nabla\Phi_\varepsilon(\rho^{(\varepsilon)})|^2 - \int_\Omega\nabla\Phi_\varepsilon(\rho^{(\varepsilon)})\cdot\mathrm{m}_\varepsilon(\rho^{(\varepsilon)})\nabla V.$$

Therefore, applying Young's inequality and integrating in time from t_1 to t_2 we get that

$$\begin{aligned}\int_{t_1}^{t_2}\|\nabla\Phi_\varepsilon(\rho^{(\varepsilon)})\|^2_{L^2(\Omega)} \leq \int_\Omega\vartheta_\varepsilon(\rho^{(\varepsilon)}_{t_1}) - \int_\Omega\vartheta_\varepsilon(\rho^{(\varepsilon)}_{t_2}) + \frac12\int_{t_1}^{t_2}\|\nabla\Phi_\varepsilon(\rho^{(\varepsilon)})\|^2_{L^2(\Omega)}\\ + \frac12\int_{t_1}^{t_2}\left\|\mathrm{m}_\varepsilon(\rho^{(\varepsilon)})\right\|_{L^\infty(\Omega)}\|\nabla V\|^2_{L^2(\Omega)}.\end{aligned}$$

Finally, we point out that $|\vartheta_\varepsilon(\rho)|\leq\alpha\|\Phi_\varepsilon\|_{L^\infty(0,\alpha)}$ for any $\rho\in\mathcal{A}_+$. Hence, we recover the desired inequality. □

Free energy and its dissipation

In order to obtain an *a priori* estimate on the time derivative and be able to discuss the long time asymptotic behaviour of the problem, we need to take advantage of the gradient flow structure of the problem.

Proposition 3.22. *Assume* (M_ε) *and let* $\rho_0\in\mathcal{A}_+\cap C^2(\overline{\Omega})$. *Then, the unique classical solution* $\rho^{(\varepsilon)}$ *of the problem* (P_ε) *is such that,*

$$\int_{t_1}^{t_2}\int_\Omega\mathrm{m}_\varepsilon(\rho^{(\varepsilon)})\left|\nabla\Big(U'_\varepsilon(\rho^{(\varepsilon)})+V\Big)\right|^2\mathrm{d}x\,\mathrm{d}t = \mathcal{F}_\varepsilon[\rho^{(\varepsilon)}_{t_1}] - \mathcal{F}_\varepsilon[\rho^{(\varepsilon)}_{t_2}] \qquad \text{for all } 0\leq t_1<t_2. \tag{3.25}$$

In particular, $\mathcal{F}_\varepsilon[\rho^{(\varepsilon)}_t]$ *is a non-increasing sequence in t.*

Proof. Combining the sufficient regularity of $\rho^{(\varepsilon)}$, the regularity of U_ε in $(0,\alpha)$ and (3.12), we can rigorously take the derivative

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_\varepsilon[\rho^{(\varepsilon)}] = -\int_\Omega \mathrm{m}_\varepsilon(\rho^{(\varepsilon)}) \left|\nabla \left(U_\varepsilon'(\rho^{(\varepsilon)}) + V\right)\right|^2,$$

Integrating in (t_1, t_2) yields the result. □

For the set $\mathcal{A}$, we can prove the following auxiliary result about the free energy.

Lemma 3.23. *Let $\varepsilon \geq 0$. Then $\mathcal{F}_\varepsilon$ has a global minimiser in $\mathcal{A}$ and is lower semicontinuous in $\mathcal{A}$ with respect to the weak L^1-topology.*

Proof. Using that $0 \le \rho \le \alpha$, the positivity of V, and the continuity of U_ε, it is trivial to see that $\mathcal{F}$ is bounded from below.

Let us prove the continuity claim. The classical theorems for lower semi-continuity are written for functions $U : \mathbb{R} \to \mathbb{R}$ (e.g., [198]). However, the result in our setting has a very simple proof. First, we prove strong L^1 continuity. If $u_n \to u$ strongly in $L^1(\Omega)$, there exists a subsequence, u_{n_k}, converging a.e. in Ω. Since $U \in C([0,\alpha])$, V is continuous and $0 \le u_{n_k} \le \alpha$, by the Dominated Convergence Theorem we get $\mathcal{F}_\varepsilon[u_{n_k}] \to \mathcal{F}_\varepsilon[u]$. Since every sequence has a convergent subsequence, and they all share a limit, $\mathcal{F}_\varepsilon[u_n] \to \mathcal{F}_\varepsilon[u]$. The strong L^1 continuity is proven.

We now prove weak-L^1 lower-semicontinuity. Let $u_n \rightharpoonup u$ weakly in $L^1(\Omega)$ and $L = \liminf_n \mathcal{F}_\varepsilon[u_n]$. Notice that L is finite since $\mathcal{F}$ is bounded below. Pick a subsequence $v_i = u_{n_i}$ such that $\mathcal{F}_\varepsilon[v_i] \to L$. Due to Mazur's lemma, there exist convex combinations

$$y_j = \sum_{i=j}^{N_j} \theta_i^{(j)} v_i, \quad \text{where } \theta_i^{(j)} \in [0,1] \text{ and } \sum_{i=j}^{N_j} \theta_i^{(j)} = 1,$$

such that $y_j \to u$ strongly in $L^1(\Omega)$. Let $\delta > 0$. There exists $\mathcal{I}(\delta)$ such that for $i \ge \mathcal{I}(\delta)$ we get $\mathcal{F}_\varepsilon[v_i] \le \delta + L$. Since U_ε is convex, so is $\mathcal{F}_\varepsilon$, and for $j \ge \mathcal{I}(\delta)$

$$\mathcal{F}_\varepsilon[y_j] \le \sum_{i=j}^{N_j} \theta_i^{(j)} \mathcal{F}_\varepsilon[v_i] \le \sum_{i=j}^{N_j} \theta_i^{(j)} (\delta + L) = \delta + L.$$

Using the strong-L^1 lower semi-continuity and this estimate we deduce that $\mathcal{F}[u] \le L + \delta$ for any $\delta > 0$. Hence, we deduce that $\mathcal{F}[u] \le L$.

Lastly, we prove the existence of a global minimiser. Taking a minimising sequence $\rho_n \in \mathcal{A}$, we have that it is bounded in $L^\infty(\Omega)$, so up to a subsequence, $\rho_{n_k} \rightharpoonup \rho^\star$ weakly-$\star$ in $L^\infty(\Omega)$. Due to the lower semi-continuous in the weak-L^1 topology we notice that the infimum is achieved at $\rho^\star$. □

Stability

We devote this subsection to show some stability result that will appear several times through the manuscript.

Lemma 3.24 (L^1-stability of weak solutions)**.** *Let $\mathrm{m}^{\{k\}} \to \mathrm{m}$ and $\Phi^{\{k\}} \to \Phi$ in $C([0,\alpha])$, $\rho^{\{k\}} \to \rho$ in $L^1((0,T)\times\Omega)$ and assume $\Phi^{\{k\}}(\rho^{\{k\}})$ are uniformly bounded in $L^2(0,T;H^1(\Omega))$. If $\rho^{\{k\}}$ are weak solutions of* (P) *with $\mathrm{m}^{\{k\}}$ and $\Phi^{\{k\}}$, then ρ is a solution of* (P) *with m and Φ.*

Proof. We take a subsequence that converges a.e. in $(0,T)\times\Omega$. Since $\Phi^{\{k\}}$ converges uniformly and $0 \le \rho^{\{k\}} \le \alpha$, by the Dominated Convergence Theorem we recover

$$\Phi^{\{k\}}(\rho^{\{k\}}) \to \Phi(\rho) \text{ in } L^p((0,T)\times\Omega) \text{ for all } p \in [1,\infty).$$

Similarly, $m^{\{k\}}(\rho^{\{k\}}) \to m(\rho)$ strongly in $L^p((0,T)\times\Omega)$. Due to the boundedness in $L^2(0,T;H^1(\Omega))$, up to a further subsequence

$$\Phi^{\{k\}}(\rho^{\{k\}}) \rightharpoonup \Phi(\rho) \quad \text{weakly in } L^2(0,T;H^1(\Omega)).$$

This fact and the $L^1(\Omega)$ convergence for $\rho^{\{k\}}$ are sufficient to pass to the limit in the weak formulation. □

Using standard arguments of Calculus of Variations (see e.g., [279, Chapter 2]) we prove suitable convergence of the free energy dissipation terms for the approximating problems.

Lemma 3.25 (Stability of the dissipation term)**.** *Assume that* $\Phi^{\{k\}}, \Phi \in C^1((0,\alpha))$, $\mathrm{m}^{\{k\}} \in C([0,\alpha])$. *Let* $\mathrm{m}^{\{k\}} \to \mathrm{m}$ *and* $\Phi^{\{k\}} \to \Phi$ *in* $C([0,\alpha])$ *and* $\rho^{\{k\}} \to \rho$ *in* $L^1((t_1,t_2)\times\Omega)$. *Let us consider the associated* $U^{\{k\}}, U$ *(i.e.,* $(U^{\{k\}})'' = (\Phi^{\{k\}})'/\mathrm{m}^{\{k\}}$, $U'' = \Phi'/\mathrm{m}$ *with prescribed values of* $U^{\{k\}}(\frac{\alpha}{2}) = U(\frac{\alpha}{2})$ *and* $(U^{\{k\}})'(\frac{\alpha}{2}) = U'(\frac{\alpha}{2})$*) and the energy dissipation terms*

$$\mathcal{D}[\rho] := \int_{t_1}^{t_2}\int_\Omega \mathrm{m}(\rho)|\nabla(U'(\rho)+V)|^2, \qquad \mathcal{D}_k[\rho] := \int_{t_1}^{t_2}\int_\Omega \mathrm{m}^{\{k\}}(\rho)|\nabla((U^{\{k\}})'(\rho)+V)|^2.$$

Then,

$$\mathcal{D}[\rho] \le \liminf_{k\to\infty} \mathcal{D}_k[\rho^{\{k\}}].$$

Proof. For the second part of the statement, we assume that this $\liminf$ is finite, otherwise there is nothing to prove. There exists a subsequence $k_j \to \infty$ such that

$$\lim_{j\to\infty} \mathcal{D}_{k_j}[\rho^{\{k_j\}}] = \liminf_{k\to\infty} \mathcal{D}_k[\rho^{\{k\}}].$$

We pick a subsequence, not relabelled, with convergence a.e. in $(t_1,t_2)\times\Omega$. Because of uniform boundedness, we can use the Dominated Convergence Theorem to prove L^p convergence of $\mathrm{m}^{\{k_j\}}(\rho^{\{k_j\}})$ and $\Phi^{\{k_j\}}(\rho^{\{k_j\}})$ for any $p<\infty$. Define

$$\xi_j := \frac{1}{\mathrm{m}^{\{k_j\}}(\rho^{\{k_j\}})^{\frac12}} \nabla\Phi^{\{k_j\}}(\rho^{\{k_j\}}) + \mathrm{m}^{\{k_j\}}(\rho^{\{k_j\}})^{\frac12}\nabla V.$$

This is an $L^2((t_1,t_2)\times\Omega)^d$ bounded sequence. Thus, from the Banach-Alaoglu Theorem, it has a weakly convergent subsequence (which we will not relabel). Let $\xi \in L^2((t_1,t_2)\times\Omega)^d$ be its limit.

We know strong L^1 convergence of all terms except $\nabla\Phi^{\{k_j\}}(\rho^{\{k_j\}})$. But we can write the weak L^p convergence for $p\in(1,2)$

$$\nabla\Phi^{\{k_j\}}(\rho^{\{k_j\}}) = \mathrm{m}^{\{k_j\}}(\rho^{\{k_j\}})^{\frac12}\Big(\xi_j - \mathrm{m}^{\{k_j\}}(\rho^{\{k_j\}})^{\frac12}\nabla V\Big) \rightharpoonup \mathrm{m}^{\{k_j\}}(\rho^{\{k_j\}})\Big(\xi - \mathrm{m}^{\{k_j\}}(\rho^{\{k_j\}})^{\frac12}\nabla V\Big)$$

Thus $\Phi^{\{k_j\}}(\rho^{\{k_j\}})$ converges weakly in $L^p(t_1,t_2;W^{1,p}(\Omega))$ for $p\in[1,2)$ and we have

$$\nabla\Phi(\rho) = \mathrm{m}(\rho)^{\frac12}\Big(\xi - \mathrm{m}(\rho)^{\frac12}\nabla V\Big).$$

Now we can apply Mazur's lemma similarly to the proof of Lemma 3.23 using the convexity of the map $\xi \mapsto |\xi|^2$. □

L^1 contraction and continuous dependence

We study the L^1 contraction and continuous dependence result for classical solutions of the regularised problem (P_ε).

Lemma 3.26. *Let* $\varepsilon>0$. *Consider* $\rho^{(\varepsilon)}$ *and* $\eta^{(\varepsilon)}$ *two classical solutions of the problem* (P_ε) *corresponding to the initial data* $\rho_0^{(\varepsilon)}$ *and* $\eta_0^{(\varepsilon)}$ *respectively. Then, it follows that for all* $t>0$ *the problem satisfies the* L^1*-contraction estimate*

$$\|\rho_t^{(\varepsilon)} - \eta_t^{(\varepsilon)}\|_{L^1(\Omega)} \le \|\rho_0^{(\varepsilon)} - \eta_0^{(\varepsilon)}\|_{L^1(\Omega)}, \tag{3.26}$$

and the L^1*-comparison principle*

$$\int_\Omega (\rho_t^{(\varepsilon)} - \eta_t^{(\varepsilon)})^+ \le \int_\Omega (\rho_0^{(\varepsilon)} - \eta_0^{(\varepsilon)})^+. \tag{3.27}$$

Proof. We divide the proof in several steps.

Step 1: Auxiliary function. Let us fix $\delta > 0$. We choose an auxiliary function j_δ such that $j_\delta''(s) = \frac{1}{2\delta}\chi_{(-\delta,\delta)}(s)$ and $j_\delta(s) \to |s|$ as $\delta \to 0$. We also define ψ_δ such that $\psi_\delta'(s) = s j_\delta''(s) = \frac{s}{2\delta}\chi_{(-\delta,\delta)}(s)$ and $\psi_\delta(0) = 0$. Thus, $|\psi_\delta'(s)| \le \frac{1}{2}\chi_{(-\delta,\delta)}(s)$, and $|\psi_\delta(s)| \le \delta$. Hence,

$$\frac{\partial}{\partial t}\int_\Omega j_\delta(\rho_\tau^{(\varepsilon)} - \eta_\tau^{(\varepsilon)}) = \int_\Omega j_\delta'(\rho_\tau^{(\varepsilon)} - \eta_\tau^{(\varepsilon)}) \operatorname{div}\left(\nabla\left(\Phi_\varepsilon(\rho_\tau^{(\varepsilon)}) - \Phi_\varepsilon(\eta_\tau^{(\varepsilon)})\right)\right) + \int_\Omega j_\delta'(\rho_\tau^{(\varepsilon)} - \eta_\tau^{(\varepsilon)}) \operatorname{div}\left((\mathrm{m}_\varepsilon(\rho_\tau^{(\varepsilon)}) - \mathrm{m}_\varepsilon(\eta_\tau^{(\varepsilon)}))\nabla V\right) =: \mathcal{I} + \mathcal{J}.$$

Step 2: Diffusion. Let us start studying the diffusive terms. We compute to obtain,

$$\begin{aligned}
\mathcal{I} &= -\int_\Omega j_\delta''(\rho_\tau^{(\varepsilon)} - \eta_\tau^{(\varepsilon)})\nabla(\rho_\tau^{(\varepsilon)} - \eta_\tau^{(\varepsilon)}) \cdot \nabla\left(\Phi_\varepsilon(\rho_\tau^{(\varepsilon)}) - \Phi_\varepsilon(\eta_\tau^{(\varepsilon)})\right) \\
&= -\int_\Omega j_\delta''(\rho_\tau^{(\varepsilon)} - \eta_\tau^{(\varepsilon)})\Phi_\varepsilon'(\rho_\tau^{(\varepsilon)})|\nabla(\rho_\tau^{(\varepsilon)} - \eta_\tau^{(\varepsilon)})|^2 \\
&\quad - \int_\Omega j_\delta''(\rho_\tau^{(\varepsilon)} - \eta_\tau^{(\varepsilon)})\left(\Phi_\varepsilon'(\rho_\tau^{(\varepsilon)}) - \Phi_\varepsilon'(\eta_\tau^{(\varepsilon)})\right)\nabla\eta_\tau^{(\varepsilon)} \cdot \nabla(\rho_\tau^{(\varepsilon)} - \eta_\tau^{(\varepsilon)}) \\
&\le -\int_\Omega \frac{\Phi_\varepsilon'(\rho_\tau^{(\varepsilon)}) - \Phi_\varepsilon'(\eta_\tau^{(\varepsilon)})}{\rho_\tau^{(\varepsilon)} - \eta_\tau^{(\varepsilon)}}\nabla\psi_\delta(\rho_\tau^{(\varepsilon)} - \eta_\tau^{(\varepsilon)}) \cdot \nabla\eta_\tau^{(\varepsilon)} \\
&= -\int_\Omega \left(\frac{\Phi_\varepsilon'(\rho_\tau^{(\varepsilon)}) - \Phi_\varepsilon'(\eta_\tau^{(\varepsilon)})}{\rho_\tau^{(\varepsilon)} - \eta_\tau^{(\varepsilon)}} - \Phi_\varepsilon''(\eta_\tau^{(\varepsilon)})\right)\nabla\psi_\delta(\rho_\tau^{(\varepsilon)} - \eta_\tau^{(\varepsilon)}) \cdot \nabla\eta_\tau^{(\varepsilon)} \\
&\quad - \int_\Omega \Phi_\varepsilon''(\eta_\tau^{(\varepsilon)})\nabla\psi_\delta(\rho_\tau^{(\varepsilon)} - \eta_\tau^{(\varepsilon)}) \cdot \nabla\eta_\tau^{(\varepsilon)} =: \mathcal{I}_1 + \mathcal{I}_2.
\end{aligned}$$

In order to study $\mathcal{I}_1$ we take advantage of the following remark. Using a Taylor expansion we have that

$$\left|\frac{\Phi_\varepsilon'(\rho_\tau^{(\varepsilon)}) - \Phi_\varepsilon'(\eta_\tau^{(\varepsilon)})}{\rho_\tau^{(\varepsilon)} - \eta_\tau^{(\varepsilon)}} - \Phi_\varepsilon''(\eta_\tau^{(\varepsilon)})\right| \le \frac{1}{2}\|\Phi_\varepsilon'''\|_{L^\infty(0,\alpha)}|\rho_\tau^{(\varepsilon)} - \eta_\tau^{(\varepsilon)}|.$$

Therefore,

$$\begin{aligned}
\mathcal{I}_1 &\le \frac{1}{2}\|\Phi_\varepsilon'''\|_{L^\infty(0,\alpha)}\int_\Omega |\rho_\tau^{(\varepsilon)} - \eta_\tau^{(\varepsilon)}||\psi_\delta'(\rho_\tau^{(\varepsilon)} - \eta_\tau^{(\varepsilon)})\nabla(\rho_\tau^{(\varepsilon)} - \eta_\tau^{(\varepsilon)}) \cdot \nabla\eta_\tau^{(\varepsilon)}| \\
&= \frac{1}{2}\|\Phi_\varepsilon'''\|_{L^\infty(0,\alpha)}\int_{|\rho_\tau^{(\varepsilon)} - \eta_\tau^{(\varepsilon)}| < \delta} |\rho_\tau^{(\varepsilon)} - \eta_\tau^{(\varepsilon)}||\psi_\delta'(\rho_\tau^{(\varepsilon)} - \eta_\tau^{(\varepsilon)})\nabla(\rho_\tau^{(\varepsilon)} - \eta_\tau^{(\varepsilon)}) \cdot \nabla\eta_\tau^{(\varepsilon)}| \\
&\le \frac{\delta}{2}\|\Phi_\varepsilon'''\|_{L^\infty(0,\alpha)}\|\nabla(\rho_\tau^{(\varepsilon)} - \eta_\tau^{(\varepsilon)})\|_{L^2(\Omega)}\|\nabla\eta_\tau^{(\varepsilon)}\|_{L^2(\Omega)}.
\end{aligned}$$

Next, we use properties of ψ_δ in order to deal with $\mathcal{I}_2$. We integrate by parts in order to obtain that

$$\mathcal{I}_2 = \int_\Omega \psi_\delta(\rho_\tau^{(\varepsilon)} - \eta_\tau^{(\varepsilon)}) \operatorname{div}\left(\Phi_\varepsilon''(\eta_\tau^{(\varepsilon)})\nabla\eta_\tau^{(\varepsilon)}\right) - \int_{\partial\Omega} \psi_\delta(\rho_\tau^{(\varepsilon)} - \eta_\tau^{(\varepsilon)})\Phi_\varepsilon''(\eta_\tau^{(\varepsilon)})\nabla\eta_\tau^{(\varepsilon)} \cdot n,$$

and $|\mathcal{I}_2| \le C\delta$.

Step 3: Drift. Let us focus now on the drift term. Integrating by parts we obtain that

$$\begin{aligned}
\mathcal{J} &= -\int_\Omega j_\delta''(\rho_\tau^{(\varepsilon)} - \eta_\tau^{(\varepsilon)})\nabla(\rho_\tau^{(\varepsilon)} - \eta_\tau^{(\varepsilon)})(\mathrm{m}_\varepsilon(\rho_\tau^{(\varepsilon)}) - \mathrm{m}_\varepsilon(\eta_\tau^{(\varepsilon)})) \cdot \nabla V \\
&= -\int_\Omega \frac{\mathrm{m}_\varepsilon(\rho_\tau^{(\varepsilon)}) - \mathrm{m}_\varepsilon(\eta_\tau^{(\varepsilon)})}{\rho_\tau^{(\varepsilon)} - \eta_\tau^{(\varepsilon)}}\nabla\psi_\delta(\rho_\tau^{(\varepsilon)} - \eta_\tau^{(\varepsilon)}) \cdot \nabla V
\end{aligned}$$

$$= -\int_\Omega \left(\frac{\mathrm{m}_\varepsilon(\rho_\tau^{(\varepsilon)}) - \mathrm{m}_\varepsilon(\eta_\tau^{(\varepsilon)})}{\rho_\tau^{(\varepsilon)} - \eta_\tau^{(\varepsilon)}} - \mathrm{m}_\varepsilon'(\rho_\tau^{(\varepsilon)}) \right) \nabla \psi_\delta(\rho_\tau^{(\varepsilon)} - \eta_\tau^{(\varepsilon)}) \cdot \nabla V$$
$$- \int_\Omega \mathrm{m}_\varepsilon'(\rho_\tau^{(\varepsilon)}) \nabla \psi_\delta(\rho_\tau^{(\varepsilon)} - \eta_\tau^{(\varepsilon)}) \cdot \nabla V =: \mathcal{J}_1 + \mathcal{J}_2.$$

We study $\mathcal{J}_1$ in the same way we have studied $\mathcal{I}_1$. Using a Taylor expansion we have that

$$\left| \frac{\mathrm{m}_\varepsilon(\rho_\tau^{(\varepsilon)}) - \mathrm{m}_\varepsilon(\eta_\tau^{(\varepsilon)})}{\rho_\tau^{(\varepsilon)} - \eta_\tau^{(\varepsilon)}} - \mathrm{m}_\varepsilon'(\rho_\tau^{(\varepsilon)}) \right| \le \frac{1}{2} \|\mathrm{m}_\varepsilon''\|_{L^\infty} |\rho_\tau^{(\varepsilon)} - \eta_\tau^{(\varepsilon)}|.$$

Hence, we can estimate

$$|\mathcal{J}_1| \le \frac{1}{2} \|\mathrm{m}_\varepsilon''\|_{L^\infty} \int_\Omega |\rho_\tau^{(\varepsilon)} - \eta_\tau^{(\varepsilon)}| \left| \psi_\delta'(\rho_\tau^{(\varepsilon)} - \eta_\tau^{(\varepsilon)}) \nabla (\rho_\tau^{(\varepsilon)} - \eta_\tau^{(\varepsilon)}) \cdot \nabla V \right|$$
$$= \frac{1}{2} \|\mathrm{m}_\varepsilon''\|_{L^\infty} \int_{|\rho_\tau^{(\varepsilon)} - \eta_\tau^{(\varepsilon)}| < \delta} |\rho_\tau^{(\varepsilon)} - \eta_\tau^{(\varepsilon)}| \left| \psi_\delta'(\rho_\tau^{(\varepsilon)} - \eta_\tau^{(\varepsilon)}) \nabla (\rho_\tau^{(\varepsilon)} - \eta_\tau^{(\varepsilon)}) \cdot \nabla V \right|$$
$$\le \frac{\delta}{2} \|\mathrm{m}_\varepsilon''\|_{L^\infty} \|\nabla (\rho_\tau^{(\varepsilon)} - \eta_\tau^{(\varepsilon)})\|_{L^2(\Omega)} \|\nabla V\|_{L^2(\Omega)}.$$

Integrating by parts and using similar ideas, and the properties of ψ_δ we can deal with $\mathcal{J}_2$,

$$\mathcal{J}_2 = \int_\Omega \psi_\delta(\rho_\tau^{(\varepsilon)} - \eta_\tau^{(\varepsilon)}) \operatorname{div} \left(\mathrm{m}_\varepsilon(\rho_\tau^{(\varepsilon)}) \nabla V \right) - \int_{\partial\Omega} \psi_\delta(\rho_\tau^{(\varepsilon)} - \eta_\tau^{(\varepsilon)}) \mathrm{m}_\varepsilon(\rho_\tau^{(\varepsilon)}) \nabla V \cdot n,$$

and therefore $|\mathcal{J}_2| \le C\delta$.

Step 4: Limit $\delta \to 0$. Hence, combining everything and integrating in time from 0 to t we recover that,

$$\int_\Omega j_\delta(\rho_t^{(\varepsilon)} - \eta_t^{(\varepsilon)}) \le \int_\Omega j_\delta(\rho_0^{(\varepsilon)} - \eta_0^{(\varepsilon)}) + C\delta.$$

Thus, if we take the limit $\delta \to 0$ we obtain the L^1 contraction,

$$\|\rho_t^{(\varepsilon)} - \eta_t^{(\varepsilon)}\|_{L^1(\Omega)} \le \|\rho_0^{(\varepsilon)} - \eta_0^{(\varepsilon)}\|_{L^1(\Omega)}.$$

Now, we choose j_δ such that $j_\delta''(s) = \frac{1}{\delta}\chi_{(0,\delta)}(s)$ and $j_\delta(s) \to s^+$ as $\delta \to 0$. Analogously to the previous case we recover,

$$\int_\Omega (\rho_t^{(\varepsilon)} - \eta_t^{(\varepsilon)})^+ \le \int_\Omega (\rho_0^{(\varepsilon)} - \eta_0^{(\varepsilon)})^+. \qquad \square$$

Remark 3.27. *This argument is valid only for classical solutions. For strong solutions, the classical argument for L^1-contraction can be found in the book from Vázquez [302], but it seems difficult to adapt it directly to the case of non-linear mobility. A more suitable notion of solution to deal with L^1 contractions is that of entropy solutions, but we will not discuss them here. We point the reader to [72, 219].*

3.3.3 Semigroup theory for the problem (P_ε). Proof of Theorem 3.4

In this subsection we intend to develop the semigroup theory for the problem (P_ε). In order to do that, we take advantage of the classical theory constructed in the previous subsections, where we consider initial data $\rho_0 \in \mathcal{A}_+ \cap C^2(\overline{\Omega})$. We have already discussed well-posedness, i.e., Theorem 3.4–Item i, in section 3.3.1. We now proceed to prove the remaining claims.

Proof of Theorem 3.4–Item ii. $S^{(\varepsilon)}$ is a free-energy dissipating semigroup. We divide the proof in several steps.

Step 1: Extension to $\mathcal{A}$. First we point out that, so far, we have defined $S_t^{(\varepsilon)}$ in $\mathcal{A}_+ \cap C^2(\overline{\Omega})$ in Theorem 3.4–Item i. Since $S_t^{(\varepsilon)}$ is an L^1-contraction in $\mathcal{A}_+ \cap C^2(\overline{\Omega})$ (i.e., 1-Lipschitz) and $\mathcal{A}_+ \cap C^2(\overline{\Omega})$ is L^1-dense in $\mathcal{A}$, $S_t^{(\varepsilon)}$ admits a unique continuous extension to $\mathcal{A}$. We prove the four properties of a free-energy dissipating semigroup given in Definition 3.2.

Step 2: Mass preserving and L^1-contraction property. They follow from the conservation of mass for classical solutions and Lemma 3.26 combined with the density of $\mathcal{A}_+ \cap C^2(\overline{\Omega})$ in $\mathcal{A}$.

Step 3: C_0-semigroup. Take $\eta_0 \in \mathcal{A}_+ \cap C^2(\overline{\Omega})$. Then

$$\begin{aligned}\|S_h\rho_0 - \rho_0\|_{L^1(\Omega)} &\le \|S_h\rho_0 - S_h\eta_0\|_{L^1(\Omega)} + \|S_h\eta_0 - \eta_0\|_{L^1(\Omega)} + \|\eta_0 - \rho_0\|_{L^1(\Omega)} \\ &\le \|S_h\eta_0 - \eta_0\|_{L^1(\Omega)} + 2\|\eta_0 - \rho_0\|_{L^1(\Omega)}\end{aligned}$$

Letting $h \to 0$, and using the properties of classical solutions

$$\limsup_{h\to 0} \|S_h\rho_0 - \rho_0\|_{L^1(\Omega)} \le 2\|\eta_0 - \rho_0\|_{L^1(\Omega)}.$$

Now we can take the infimum over $\eta_0 \in \mathcal{A}_+ \cap C^2(\overline{\Omega})$.

Step 4: Energy dissipation and $C([0,T]; W^{-1,1}(\Omega))$. For data in $\rho_0 \in \mathcal{A}_+ \cap C^2(\overline{\Omega})$ we recall Proposition 3.22. Hence, we can apply (3.19). In particular,

$$\|\rho_{t_2} - \rho_{t_1}\|_{W^{-1,1}(\Omega)} \le \int_{t_1}^{t_2} \left\| \frac{\partial \rho_\sigma}{\partial t} \right\|_{W^{-1,1}(\Omega)} \mathrm{d}\sigma \le \|\mathrm{m}_\varepsilon\|_{L^\infty([0,\alpha])}^{\frac{1}{2}} |\Omega|^{\frac{1}{2}} \left(\mathcal{F}_\varepsilon[\rho_{t_1}] - \mathcal{F}_\varepsilon[\rho_{t_2}]\right)^{\frac{1}{2}} |t_2 - t_1|^{\frac{1}{2}}.$$

Let $\rho_0 \in \mathcal{A}_+$. Due to the upper and lower bound (3.12), there exists an approximating sequence $\rho_0^{\{k\}} \in \mathcal{A}_+ \cap C^2(\overline{\Omega})$ such that

$$0 < c_1 \le (U_\varepsilon')^{-1}(C_1 - V) \le \rho_0^{\{k\}} \le (U_\varepsilon')^{-1}(C_2 - V) \le c_2 < \alpha$$

uniformly. Since we have a comparison principle, these bounds are satisfied by $S_t^{(\varepsilon)}\rho_0^{(k)}$ for all positive times. Due to Lemma 3.25 and the continuity of U_ε, we can pass to the limit in the estimates for $S_t^{(\varepsilon)}\rho_0^{\{k\}}$.

Step 5: Weak solution. It follows from the stability result in Lemma 3.24.

Proof of Theorem 3.4–Item iii. Strong Maximum Principle. Take $\rho_0 \in \mathcal{A}$ and $\rho_0 \not\equiv 0, \alpha$. We prove first that $\rho > 0$. Since $\rho_0 \ge 0$, there exists a $c > 0$ and small ball $B_r(x)$ such that $\rho_0 \ge c$ in $B_r(x)$. Taking a smooth function $0 \le \eta_0 \le c$ that is 0 outside $B_r(x)$ and positive in $B_{r/2}(x)$. By the comparison principle and Lemma 3.19 we have $S_t^{(\varepsilon)}\rho_0 \ge S_t^{(\varepsilon)}\eta_0 > 0$ in Ω. The claim $\rho < \alpha$ follows similarly.

Proof of Theorem 3.4–Item iv. Strict positivity. As already mentioned, by using (3.12) we have $0 < (U_\varepsilon')^{-1}(C_1 - V) \le S_t^{(\varepsilon)}\rho_0 \le (U_\varepsilon')^{-1}(C_2 - V) < \alpha$. □

3.3.4 Existence for (P). Proof of Theorem 3.5

We divide the proof of Theorem 3.5 in several Lemmas.

Lemma 3.28. *Under the assumption (H$_4$) the regularised diffusion U_ε from* (3.9), (M$_0$), *and* (M$_\varepsilon$) *is such that $U_\varepsilon \to U$ uniformly in $[0,\alpha]$ and $U_\varepsilon' \to U'$, $U_\varepsilon'' \to U''$ uniformly over compacts of $(0,\alpha)$.*

Proof. Recall that $U_\varepsilon'' = \frac{\Phi_\varepsilon'}{\mathrm{m}_\varepsilon}$ and (3.9) includes that $\Phi_\varepsilon \to \Phi$ in $C^2_{loc}((0,\alpha))$. From (3.11), the regularised mobility m_ε converges uniformly to m, a continuous function that is positive in $(0,\alpha)$. Thus, $U_\varepsilon'' \to U''$ uniformly over compacts of $(0,\alpha)$. Since $U_\varepsilon'(\frac{\alpha}{2}) = U'(\frac{\alpha}{2})$ we also get $U_\varepsilon' \to U'$ uniformly over compacts of $(0,\alpha)$. Similarly, since $U_\varepsilon(\frac{\alpha}{2}) = U(\frac{\alpha}{2})$ we get convergence of $U_\varepsilon \to U$ uniformly over compacts of $(0,\alpha)$. Lastly, since $\|U_\varepsilon'\|_{L^1((0,\alpha))}$ is bounded, we have that U_ε are uniformly continuous in $[0,\alpha]$, and the uniform convergence over $[0,\alpha]$ follows. □

Using a sequence of classical solutions of the approximating problems (P_ε) we are able to prove existence of a free-energy dissipating semigroup for (P). First, we prove the convergence of the semigroup in the weaker space $W^{-1,1}$.

Lemma 3.29 ($W^{-1,1}$ precompactness)**.** *There exists a sequence $\varepsilon_k \to 0$ and a semigroup $S : [0,\infty) \times \mathcal{A} \to W^{-1,1}(\Omega)$ such that, for each $\rho_0 \in \mathcal{A}$,*

$$S^{(\varepsilon_k)}\rho_0 \to S\rho_0, \qquad \text{in } C_{loc}([0,\infty); W^{-1,1}(\Omega)). \tag{3.28}$$

Proof. Our proof relies on the Ascoli-Arzelà theorem on $C(X,Y)$ with the topology of compact convergence (i.e., uniform convergence over compact sets), where $X := [0,\infty) \times \mathcal{A}$ and $Y := W^{-1,1}(\Omega)$. We observe that X is a topological space and Y is a metric space (in particular a uniform space). We want to show that $H := \{S^{(\varepsilon)} : \varepsilon \in (0,1)\}$ is precompact in $C(X,Y)$ with the topology of compact convergence. It suffices to prove that (i) H are equicontinuous and (ii) that for each $x = (t,\rho_0) \in X_\ell$ the set $H(x) := \{S_t^{(\varepsilon)}\rho_0 : \varepsilon \in (0,1)\}$ is pre-compact in Y.

The proof of (ii) is simple. Given $x = (t,\rho_0) \in X$ we have that $\|S_t^{(\varepsilon)}\rho_0\|_{L^1} = \|\rho_0\|_{L^1}$. Due to the compact embedding of L^1 in $W^{-1,1}$ we observe that the set $H(x)$ is precompact in $W^{-1,1}(\Omega)$.

Now let us prove (i). We point that the energy is controlled in $\mathcal{A}$ since

$$|\mathcal{F}_\varepsilon[\rho]| \le \int_\Omega |U_\varepsilon(\rho)| + \int_\Omega V\rho \le \max_{\substack{\varepsilon\in[0,1]\\ s\in[0,\alpha]}} |U_\varepsilon(s)||\Omega| + \alpha\|V\|_{L^1(\Omega)} =: \mathsf{C}.$$

Thus, we can control the time continuity using the gradient-flow structure

$$\|S_{t+h}^{(\varepsilon)}\rho_0 - S_t^{(\varepsilon)}\rho_0\|_{W^{-1,1}(\Omega)} \le C(\mathcal{F}_\varepsilon[S_t^{(\varepsilon)}\rho_0] - \mathcal{F}_\varepsilon[S_{t+h}^{(\varepsilon)}\rho_0])^{\frac12} h^{\frac12} \le \mathsf{C}h^{\frac12}.$$

Using the continuous embedding of $L^1(\Omega)$ in $W^{-1,1}(\Omega)$, and the L^1-contraction property of the semigroup, we recover that

$$\begin{aligned}
\|S_{t+h}^{(\varepsilon)}\rho_0 - S_t^{(\varepsilon)}\eta_0\|_{W^{-1,1}(\Omega)} &\le \|S_{t+h}^{(\varepsilon)}\rho_0 - S_t^{(\varepsilon)}\rho_0\|_{W^{-1,1}(\Omega)} + \|S_t^{(\varepsilon)}\rho_0 - S_t^{(\varepsilon)}\eta_0\|_{W^{-1,1}(\Omega)} \\
&\le \|S_{t+h}^{(\varepsilon)}\rho_0 - S_t^{(\varepsilon)}\rho_0\|_{W^{-1,1}(\Omega)} + C(\Omega)\|S_t^{(\varepsilon)}\rho_0 - S_t^{(\varepsilon)}\eta_0\|_{L^1(\Omega)} \\
&\le \mathsf{C}h^{\frac12} + C(\Omega)\|\rho_0 - \eta_0\|_{L^1(\Omega)}.
\end{aligned}$$

This shows that the family H are equicontinuous.

Therefore, by the Ascoli-Arzelà theorem, there exists $\varepsilon_k \to 0$ such that $S^{(\varepsilon_k)}$ converges to S uniformly over compacts of X. □

We also show that the convergence happens in L^p for any ρ_0 fixed, although it may not be uniform in ρ_0.

Lemma 3.30 (L^p pre-compactness)**.** *Given a sequence $\varepsilon_k \to 0$, $\rho_0 \in \mathcal{A}$ and $T > 0$ fixed, there exists a sub-sequence and $u \in L^\infty((0,T)\times\Omega)$ such that*

$$S^{(\varepsilon_k)}\rho_0 \to u \quad \text{in } L^p((0,T)\times\Omega) \text{ for all } p \in [1,\infty).$$

Proof. Through the proof we consider the notation $\rho^{(\varepsilon)} = S^{(\varepsilon)}\rho_0$. Let us recall that from our assumptions we know that Φ and Φ_ε are non-decreasing and there exists $s_0 \in (0,\alpha)$ such that $\Phi'(s_0) > 0$ and $\Phi_\varepsilon(s_0) = \Phi(s_0) = 0$. Notice that this implies that $\Phi_\varepsilon, \Phi < 0$ in $(0, s_0)$ and $\Phi_\varepsilon, \Phi > 0$ in (s_0, α). We consider

$$\Psi_1^{(\varepsilon)}(s) = \int_{s_0}^{\max\{s,s_0\}} [\Phi_\varepsilon(\alpha) - \Phi_\varepsilon(\sigma)]\Phi_\varepsilon(\sigma)\,\mathrm{d}\sigma \quad \text{and} \quad \Psi_2^{(\varepsilon)}(s) = \int_0^{\min\{s,s_0\}} [\Phi_\varepsilon(\sigma) - \Phi_\varepsilon(0)]\Phi_\varepsilon(\sigma)\,\mathrm{d}\sigma.$$

Notice that we can write $(\Psi_1^{(\varepsilon)})'(s) = [\Phi_\varepsilon(\alpha) - \Phi_\varepsilon(s)][\Phi_\varepsilon(s)]^+$ and $(\Psi_2^{(\varepsilon)})'(s) = [\Phi_\varepsilon(s) - \Phi_\varepsilon(0)][\Phi_\varepsilon(s)]^-$. We will prove convergence of $\Psi_1^{(\varepsilon)}(\rho^{(\varepsilon)})$ and $\Psi_2^{(\varepsilon)}(\rho^{(\varepsilon)})$ and combine them to show the limit of $\rho^{(\varepsilon)}$. We divide the proof in several steps.

Step 1: Application of Aubin-Lions Lemma. We define

$$F_\varepsilon = \mathrm{m}_\varepsilon(\rho^{(\varepsilon)})\nabla\left(U_\varepsilon'(\rho^{(\varepsilon)}) + V\right),$$

which is uniformly bounded in ε in $L^2((0,T)\times\Omega)$ due to (3.25) and the uniform bound on m_ε. Furthermore, we get

$$\frac{\partial}{\partial t}\left(\Psi_1^{(\varepsilon)}(\rho^{(\varepsilon)})\right) = \operatorname{div}\left([\Phi_\varepsilon(\alpha) - \Phi_\varepsilon(\rho^{(\varepsilon)})][\Phi_\varepsilon(\rho^{(\varepsilon)})]^+ F_\varepsilon\right) - \nabla([\Phi_\varepsilon(\alpha) - \Phi_\varepsilon(\rho^{(\varepsilon)})][\Phi_\varepsilon(\rho^{(\varepsilon)})]^+)\cdot F_\varepsilon. \tag{3.29}$$

Let us show that this is bounded in $L^1(0,T;H^{-1}(\Omega))$. We observe that

$$\|[\Phi_\varepsilon(\alpha) - \Phi_\varepsilon(\rho^{(\varepsilon)})][\Phi_\varepsilon(\rho^{(\varepsilon)})]^+ F_\varepsilon\|_{L^2((0,T)\times\Omega)} \le 2\|\Phi_\varepsilon\|^2_{L^\infty(0,\alpha)}\|F_\varepsilon\|_{L^2((0,T)\times\Omega)},$$

and this is uniformly bounded in ε. Therefore, the first term on the right-hand side of (3.29) is controlled in $L^2(0,T;H^{-1}(\Omega))$. The second term on the right-hand side of (3.29) is bounded in $L^1((0,T)\times\Omega)$ due to (3.24) and (3.25). Hence, $\frac{\partial}{\partial t}\left(\Psi_1^{(\varepsilon)}(\rho^{(\varepsilon)})\right)$ is uniformly bounded in $L^1(0,T;H^{-1}(\Omega))$. Moreover, from Lemma 3.18, we have that

$$\left|\nabla\Psi_1^{(\varepsilon)}(\rho^{(\varepsilon)})\right| = \left|\frac{[\Phi_\varepsilon(\alpha) - \Phi_\varepsilon(\rho^{(\varepsilon)})][\Phi_\varepsilon(\rho^{(\varepsilon)})]^+}{\Phi_\varepsilon'(\rho^{(\varepsilon)})}\nabla\Phi_\varepsilon(\rho^{(\varepsilon)})\right| \le C\left|\nabla\Phi_\varepsilon(\rho^{(\varepsilon)})\right|.$$

The elements $\Phi_\varepsilon(\rho^{(\varepsilon)})$ are integrable and uniformly bounded, for ε small enough. This, in addition to (3.24), implies that $\Psi_1^{(\varepsilon)}(\rho^{(\varepsilon)})$ is uniformly bounded in $L^2(0,T;H^1(\Omega))$. Let us take the given sequence ε_k. Due to the Aubin-Lions Lemma there exists $\varkappa_1 \in L^2((0,T)\times\Omega)$ and a subsequence such that

$$\Psi_1^{(\varepsilon_k)}(\rho^{(\varepsilon_k)}) \to \varkappa_1 \quad \text{strongly in } L^2((0,T)\times\Omega) \text{ and a.e.} \tag{3.30}$$

Working analogously, we can prove that there exists $\varkappa_2 \in L^2((0,T)\times\Omega)$ such that, up to a further subsequence, $\Psi_2^{(\varepsilon_k)}(\rho^{(\varepsilon_k)}) \to \varkappa_2$ strongly in $L^2((0,T)\times\Omega)$ and a.e..

Step 2: Characterisation of the limit. By construction $\Phi_\varepsilon(s_0)$ is non-decreasing, $\Phi_\varepsilon(s_0) = 0$, and $\Phi_\varepsilon'(s_0) > 0$. Therefore, we have that $\Phi_\varepsilon(s) > 0$ for all $\varepsilon \in [0,1]$ and $s > s_0$. Joining this fact with (3.4b) we recover that $(\Psi_1^{(\varepsilon)})'(s) > 0$ for all $s \in (s_0,\alpha)$. Clearly, $\Psi_1^{(\varepsilon)}(s) = 0$ for $s < s_0$. Likewise, $(\Psi_2^{(\varepsilon)})'(s) = [\Phi_\varepsilon(s) - \Phi_\varepsilon(0)]\,[\Phi_\varepsilon(s)]^- \le 0$ and is strictly negative in $(0, s_0)$.

We can invert the functions $s \mapsto \Psi_i^{(\varepsilon)}(s)/\Psi_i^{(\varepsilon)}(\alpha)$. For $\varkappa \in (0,1)$ by Bolzano's theorem there exists $s(\varkappa) \in [0,\alpha]$ such that $\Psi_i^{(\varepsilon)}(s(\varkappa)) = \varkappa\Psi_i^{(\varepsilon)}(\alpha)$. Because of the construction of $\Phi_i^{(\varepsilon)}$, we know that $s(\varkappa) \in (0,\alpha)$. We define

$$\varpi_1^{(\varepsilon)} : [0,1] \to [s_0,\alpha], \qquad \varpi_1^{(\varepsilon)}(\varkappa) = \begin{cases} s_0 & \text{if } \varkappa = 0, \\ s(\varkappa) & \text{if } \varkappa \in (0,1), \\ \alpha & \text{if } \varkappa = 1. \end{cases}$$

This is a non-decreasing function. We can make a similar construction for $\Psi_2^{(\varepsilon)}$, which we denote $\varpi_2^{(\varepsilon)}$. For $\varepsilon \geq 0$, it holds that

$$\varpi_1^{(\varepsilon)}\left(\frac{\Psi_1^{(\varepsilon)}(s)}{\Psi_1^{(\varepsilon)}(\alpha)}\right) = \begin{cases} s_0 & \text{if } s \in [0, s_0], \\ s & \text{if } s \in (s_0, \alpha), \end{cases} \quad \text{and} \quad \varpi_2^{(\varepsilon)}\left(\frac{\Psi_2^{(\varepsilon)}(s)}{\Psi_2^{(\varepsilon)}(\alpha)}\right) = \begin{cases} s & \text{if } s \in [0, s_0], \\ s_0 & \text{if } s \in (s_0, \alpha]. \end{cases} \tag{3.31}$$

We can therefore write

$$\rho^{(\varepsilon)} = \varpi_1^{(\varepsilon)}\left(\frac{\Psi_1^{(\varepsilon)}(\rho^{(\varepsilon)})}{\Psi_1^{(\varepsilon)}(\alpha)}\right) + \varpi_2^{(\varepsilon)}\left(\frac{\Psi_2^{(\varepsilon)}(\rho^{(\varepsilon)}))}{\Psi_2^{(\varepsilon)}(\alpha)}\right) - s_0. \tag{3.32}$$

To show the convergence along the sequence ε_k of each term, we write the triangular inequality,

$$\begin{aligned} &\left|\varpi_1^{(\varepsilon)}\left(\frac{\Psi_1^{(\varepsilon)}(\rho^{(\varepsilon)})}{\Psi_1^{(\varepsilon)}(\alpha)}\right) - \varpi_1^{(0)}\left(\frac{\varkappa_1(t,x)}{\Psi_1^{(0)}(\alpha)}\right)\right| \\ &\qquad \leq \left|\varpi_1^{(\varepsilon)}\left(\frac{\Psi_1^{(\varepsilon)}(\rho^{(\varepsilon)})}{\Psi_1^{(\varepsilon)}(\alpha)}\right) - \varpi_1^{(0)}\left(\frac{\Psi_1^{(\varepsilon)}(\rho^{(\varepsilon)})}{\Psi_1^{(\varepsilon)}(\alpha)}\right)\right| + \left|\varpi_1^{(0)}\left(\frac{\Psi_1^{(\varepsilon)}(\rho^{(\varepsilon)})}{\Psi_1^{(\varepsilon)}(\alpha)}\right) - \varpi_1^{(0)}\left(\frac{\varkappa_1(t,x))}{\Psi_1^{(0)}(\alpha)}\right)\right|. \end{aligned}$$

Due to (3.30), the uniform convergence of $\Phi_\varepsilon \to \Phi$, and the continuity of $\varpi_1^{(0)}$ we have that

$$\varpi_1^{(0)}\left(\frac{\Psi_1^{(\varepsilon_k)}(\rho^{(\varepsilon_k)})}{\Psi_1^{(\varepsilon_k)}(\alpha)}\right) \to \varpi_1^{(0)}\left(\frac{\varkappa_1}{\Psi_1^{(0)}(\alpha)}\right) \quad \text{a.e. } (0,T) \times \Omega \tag{3.33}$$

and similarly for $\Psi_2^{(\varepsilon_k)}$.

Hence, it suffices to prove uniform convergence of $\varpi_1^{(\varepsilon_k)}$ to $\varpi_1^{(0)}$. Let us take $0 < \varkappa_* \leq \varkappa^* < 1$ arbitrarily. For any $\varkappa \in [\varkappa_*, \varkappa^*]$

$$0 \leq (\varpi_1^{(\varepsilon)})'(\varkappa) = \frac{\Psi_1^{(\varepsilon)}(\alpha)}{(\Psi_1^{(\varepsilon)})'(s(\varkappa))} \leq \frac{\Psi_1^{(\varepsilon)}(\alpha)}{[\Phi_\varepsilon(\alpha) - \Phi_\varepsilon(s(\varkappa^*))]\Phi_\varepsilon(s(\varkappa_*))} \leq C(\varkappa_*, \varkappa^*)$$

independent of ε. Due to (3.13) we have that $\Phi_\varepsilon \to \Phi$ and $\Psi_1^{(\varepsilon)} \to \Psi_1$ uniformly in $[0, \alpha]$ as $\varepsilon \to 0$ and hence $C_\varepsilon(\varkappa_*, \varkappa^*)$ converges as $\varepsilon \to 0$. We conclude that $\varpi_1^{(\varepsilon)}$ are uniformly Lipschitz over compact subsets of $(0, 1)$. We can use the Ascoli-Arzelà to prove that, up to a further subsequence of ε_k, they converge uniformly over compacts of $(0, 1)$. Furthermore, due to (3.31) we can establish that the limit is $\varpi_1^{(0)}$. Also, we observe that $\varpi_1^{(\varepsilon_k)}(0) = s_0 = \varpi_1^{(0)}(0)$, $\varpi_1^{(\varepsilon_k)}(1) = \alpha = \varpi_1^{(0)}(1)$. Since $\varpi_1^{(\varepsilon_k)}$ are non-decreasing functions and $\varpi_1^{(0)}$ is continuous, then the convergence is uniform in $[0, 1]$. Up to a further subsequence the same reasoning holds for $\varpi_2^{(\varepsilon_k)}$. Taking into account (3.32) and (3.33), it follows that

$$\rho^{(\varepsilon_k)} \to u := \varpi_1^{(0)}\left(\frac{\varkappa_1}{\Psi_1^{(0)}(\alpha)}\right) + \varpi_2^{(0)}\left(\frac{\varkappa_2}{\Psi_2(\alpha)}\right) - s_0 \quad \text{a.e. } (0,T) \times \Omega.$$

By the Dominated Convergence Theorem the convergence also holds in $L^p((0,T) \times \Omega)$ for $p \in [1, \infty)$. □

We are now ready to show the main result of this subsection.

Proof of Theorem 3.5. We divide the proof in several steps.

Step 1: For $\rho_0 \in \mathcal{A}$ the $C([0,T]; W^{-1,1})$ limit also happens in $L^1((0,1) \times \Omega)$. In Lemma 3.29 we describe the $C([0,T]; W^{-1,1}(\Omega))$ limit. Due to Lemma 3.30, for any $\rho_0 \in \mathcal{A}$ and any $T > 0$ we have that, up to a subsequence ε_{k_ℓ}, $S^{(\varepsilon_{k_\ell})}\rho_0 \to u$ strongly in $L^1((0,T) \times \Omega)$. Due to the uniqueness of the limit, $u = S\rho_0$. By the stability Lemma 3.24, we conclude that $S\rho_0$ is a weak solution of (P).

Step 2: Uniform time continuity for good initial data. Let us fix $\varepsilon > 0$. For $\rho_0 \in \mathcal{A}_+ \cap C^2(\overline{\Omega})$ we have the L^1 limit

$$\lim_{h\to 0} \frac{S_h^{(\varepsilon)}\rho_0 - \rho_0}{h} = \frac{\partial}{\partial t}\rho(0,x) = \Delta\Phi_\varepsilon(\rho_0) + \operatorname{div}(\mathrm{m}_\varepsilon(\rho_0)\nabla V).$$

Notice that if $\Delta\Phi_\varepsilon(\rho_0) + \operatorname{div}(\mathrm{m}_\varepsilon(\rho_0)\nabla V) = 0$ then ρ_0 is a stationary strong solution, so $S_h^{(\varepsilon)}\rho_0 = \rho_0$. There exists $h_0(\varepsilon,\delta) > 0$ such that

$$\|S_h^{(\varepsilon)}\rho_0 - \rho_0\|_{L^1(\Omega)} \le (1+\delta)h\|\Delta\Phi_\varepsilon(\rho_0) + \operatorname{div}(\mathrm{m}_\varepsilon(\rho_0)\nabla V)\|_{L^1(\Omega)}, \qquad \forall h \le h_0.$$

Let us take $t = hk$ for some $h \le h_0$. Then, using the triangular inequality and the L^1-contraction property it follows that

$$\begin{aligned}
\|S_h^{(\varepsilon)}\rho_0 - \rho_0\|_{L^1(\Omega)} &\le \sum_{j=1}^{k} \|S_{jk}^{(\varepsilon)}\rho_0 - S_{(j-1)k}^{(\varepsilon)}\rho_0\|_{L^1(\Omega)} \le k\|S_h^{(\varepsilon)}\rho_0 - \rho_0\|_{L^1(\Omega)} \\
&\le (1+\delta)hk\|\Delta\Phi_\varepsilon(\rho_0) + \operatorname{div}(\mathrm{m}_\varepsilon(\rho_0)\nabla V)\|_{L^1(\Omega)} \\
&\le (1+\delta)t\|\Delta\Phi_\varepsilon(\rho_0) + \operatorname{div}(\mathrm{m}_\varepsilon(\rho_0)\nabla V)\|_{L^1(\Omega)}.
\end{aligned}$$

We now take $\delta \to 0$ to recover that

$$\|S_t^{(\varepsilon)}\rho_0 - \rho_0\|_{L^1(\Omega)} \le t\|\Delta\Phi_\varepsilon(\rho_0) + \operatorname{div}(\mathrm{m}_\varepsilon(\rho_0)\nabla V)\|_{L^1(\Omega)}.$$

Therefore, due to the semigroup property and the L^1 contraction, for any $t, s \ge 0$ it follows that

$$\|S_t^{(\varepsilon)}\rho_0 - S_s^{(\varepsilon)}\rho_0\|_{L^1(\Omega)} \le |t-s|\|\Delta\Phi_\varepsilon(\rho_0) + \operatorname{div}(\mathrm{m}_\varepsilon(\rho_0)\nabla V)\|_{L^1(\Omega)}, \tag{3.34}$$

which is uniformly bounded since $\rho_0 \in \mathcal{A}_+ \cap C^2(\overline{\Omega})$.

Step 3: Convergence $C([0,T];L^1(\Omega))$ for good initial data. Take ε_{k_ℓ} any subsequence of ε_k. We recall that $L^1((0,T)\times\Omega) = L^1(0,T;L^1(\Omega))$, where the latter L^1 space is understood in the Bochner sense. As it happens for the usual L^1 spaces, we have a.e. convergence in time (see e.g., [256, Theorem 9.2]), i.e., there exists a further sub-sequence, still denote ε_{k_ℓ}, such that $S_t^{(\varepsilon_{k_\ell})}\rho_0 \to S_t\rho_0$ in $L^1(\Omega)$ for a.e. $t \in (0,T)$. Notice that (3.34) gives a uniform-Lipschitz time continuity. Let us call

$$L \coloneqq \sup_{\varepsilon\in(0,1]} \|\Delta\Phi_\varepsilon(\rho_0) + \operatorname{div}(\mathrm{m}_\varepsilon(\rho_0)\nabla V)\|_{L^1(\Omega)}$$

which is finite since $\rho_0 \in \mathcal{A}_+ \cap C^2(\overline{\Omega})$, $\Phi_\varepsilon \to \Phi$ in $C^2_{loc}((0,\alpha))$ due to (3.9) and $\mathrm{m}_\varepsilon \to \mathrm{m}$ in $C([0,\alpha]) \cap C^1_{loc}((0,\alpha))$ due to (3.11). Thus, $S\rho_0 \in C([0,T];L^1(\Omega))$. Let $\Lambda = \{t \in [0,T] : S_t^{(\varepsilon_{k_\ell})}\rho_0 \to S_t\rho_0\}$. Take $\delta > 0$. Since $[0,T]\setminus\Lambda$ has measure 0, the covering $\{(t-\delta,t+\delta)\}_{t\in\Lambda}$ admits a finite sub-cover given by $t_1,\dots,t_N$. For $t \in [0,T]$ there exists t_i such that $|t-t_i| < \delta$. We estimate

$$\begin{aligned}
\|S_t^{(\varepsilon_{k_\ell})}\rho_0 - S_t\rho_0\|_{L^1(\Omega)} &\le \|S_t^{(\varepsilon_{k_\ell})}\rho_0 - S_{t_i}^{(\varepsilon_{k_\ell})}\rho_0\|_{L^1(\Omega)} + \|S_{t_i}^{(\varepsilon_{k_\ell})}\rho_0 - S_{t_i}\rho_0\|_{L^1(\Omega)} + \|S_{t_i}\rho_0 - S_t\rho_0\|_{L^1(\Omega)} \\
&\le 2L\delta + \|S_{t_i}^{(\varepsilon_{k_\ell})}\rho_0 - S_{t_i}\rho_0\|_{L^1(\Omega)}.
\end{aligned}$$

Since there are finitely many t_i and we have pointwise convergence over all of them, we can write

$$\limsup_{\ell\to\infty} \sup_{t\in[0,T]} \|S_t^{(\varepsilon_{k_\ell})}\rho_0 - S_t\rho_0\|_{L^1(\Omega)} \le 2L\delta.$$

Since any sub-sequence of ε_k has a further sub-sequence that converges in $C([0,T];L^1(\Omega))$ and they all do so to the same limit, the whole sequence converges in $C([0,T];L^1(\Omega))$.

Step 4: L^1-contraction for good initial data. Take $\rho_0, \eta_0 \in \mathcal{A}_+ \cap C^2(\overline{\Omega})$. Then, combining Step 3 and using (3.34), the L^1-contraction property of $S^{(\varepsilon)}$, we have that

$$\begin{aligned}\|S_t\rho_0 - S_t\eta_0\|_{L^1(\Omega)} &\le \|S_t\rho_0 - S_t^{(\varepsilon_k)}\rho_0\|_{L^1(\Omega)} + \|S_t^{(\varepsilon_k)}\eta_0 - S_t\eta_0\|_{L^1(\Omega)} + \|S_t^{(\varepsilon_k)}\rho_0 - S_t^{(\varepsilon_k)}\eta_0\|_{L^1(\Omega)} \\ &\le \|S_t\rho_0 - S_t^{(\varepsilon_k)}\rho_0\|_{L^1(\Omega)} + \|S_t^{(\varepsilon_k)}\eta_0 - S_t\eta_0\|_{L^1(\Omega)} + \|\rho_0 - \eta_0\|_{L^1(\Omega)}.\end{aligned}$$

If we take the limit $k \to \infty$ it follows $\|S_t\rho_0 - S_t\eta_0\|_{L^1(\Omega)} \le \|\rho_0 - \eta_0\|_{L^1(\Omega)}$. Furthermore, by density, we can extend this result to $\rho_0, \eta_0 \in \mathcal{A}$.

Step 5: Convergence in $C([0,T];L^1)$ for $\rho_0 \in \mathcal{A}$. Let us take $\eta_0 \in \mathcal{A}_+ \cap C^2(\overline{\Omega})$. From the L^1-contraction property (Step 2) it follows that

$$\begin{aligned}\|S_t^{(\varepsilon_k)}\rho_0 - S_t\rho_0\|_{L^1(\Omega)} &\le \|S_t^{(\varepsilon_k)}\rho_0 - S_t^{(\varepsilon_k)}\eta_0\|_{L^1(\Omega)} + \|S_t^{(\varepsilon_k)}\eta_0 - S_t\eta_0\|_{L^1(\Omega)} + \|S_t\eta_0 - S_t\rho_0\|_{L^1(\Omega)} \\ &\le 2\|\rho_0 - \eta_0\|_{L^1(\Omega)} + \|S_t^{(\varepsilon_k)}\eta_0 - S_t\eta_0\|_{L^1(\Omega)}.\end{aligned}$$

Hence, we have that

$$\limsup_{k\to\infty} \sup_{t\in[0,T]} \|S_t^{(\varepsilon_k)}\rho_0 - S_t\rho_0\|_{L^1(\Omega)} \le 2\|\rho_0 - \eta_0\|_{L^1(\Omega)}.$$

Therefore, if we take the infimum over $\eta_0 \in \mathcal{A}_+ \cap C^2(\overline{\Omega})$ we conclude the proof.

Step 6: S is a free-energy dissipating semigroup over $\mathcal{A}$. From Step 4, S is an L^1-contraction for $\rho_0, \eta_0 \in \mathcal{A}$. From Lemmas 3.21 and 3.24, we know that $S\rho_0$ are weak solutions to (P).

Let us now show that S_t is a C_0-semigroup. The semigroup property is a direct consequence of Step 3. Due to the point-wise convergence $S_0\rho_0 = \rho_0$, and we know $t \mapsto S_t\rho_0$ is continuous.

Furthermore, for $\rho_0 \in \mathcal{A}$, if we use the notation $\rho_t^{(\varepsilon)} = S_t^{(\varepsilon)}\rho_0$ we know

$$\int_{t_1}^{t_2}\int_\Omega \mathrm{m}_\varepsilon(\rho^{(\varepsilon)})|\nabla(U'(\rho^{(\varepsilon)}) + V)|^2 = \mathcal{F}_\varepsilon[\rho_{t_1}^{(\varepsilon)}] - \mathcal{F}_\varepsilon[\rho_{t_2}^{(\varepsilon)}].$$

We recall that $U_\varepsilon \to U$ uniformly (Lemma 3.28) and $\rho^{(\varepsilon_k)} \to \rho$ strongly in $C([0,T];L^1(\Omega))$. Hence, it follows that $\mathcal{F}_\varepsilon[\rho_t^{(\varepsilon)}] \to \mathcal{F}[\rho_t]$. We can pass to the limit due to Lemma 3.25. We can also pass to the limit in the $C([0,T];W^{-1,1}(\Omega))$ estimate.

Step 7: $S\rho_0 \in \mathcal{A}$. The semigroup $S^{(\varepsilon_k)}$ is such that $0 \le S_t^{(\varepsilon_k)}\rho_0 \le \alpha$ for every k and every $t \ge 0$. Therefore, Step 3 implies that $0 \le S_t\rho_0 \le \alpha$ for every $t \ge 0$. □

3.4 Local minimisers of the free energy. Proof of Theorem 3.6

When V is not radially increasing we could build several constant-in-time weak solutions by "pasting" constant-in-time weak solutions of the problem (P) of the form $T_{0,\alpha} \circ (U')^{-1}(C - V(x))$ (Theorem 3.10–Item i) for different values of the constants C, see Figure 3.2. However, as stated in Theorem 3.6, there exists a unique L^1-local minimiser, which corresponds to the case in which there is only one constant involved.

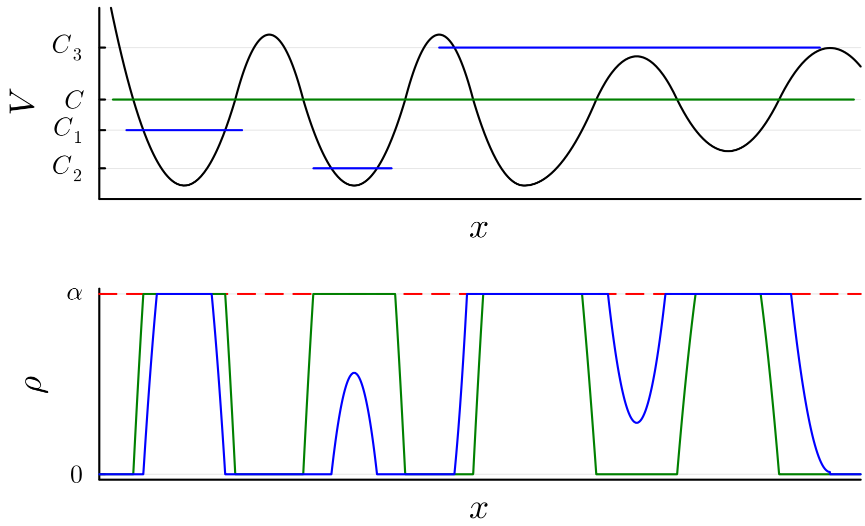


Figure 3.2: Example of steady states for $U(s) = s^2$ and V, the potential above, not radially increasing. The blue steady state is not an L^1-local minimiser of the free energy (3.2). On the other hand, the green steady state is an L^1-local minimiser, since there is only one constant involved, C; see Theorem 3.6.

The phenomenon exposed in Figure 3.2 is known in the no-saturation case $\mathrm{m}(\rho) = \rho$. We now proceed to show it in the case with saturation. Before the proof, we construct some auxiliary scaffolding.

Proof of Theorem 3.6. We split the proof in several steps.

Step 1: Euler-Lagrange condition (3.14). We distinguish two cases

Step 1.a: Euler-Lagrange condition when $0 \le \widehat{\rho} < \alpha$. We take sets $A_\lambda = \{x \in \Omega : \widehat{\rho}(x) < \alpha - \lambda\}$. If $|A_0| = 0$, the statement does claim anything and the proof is complete. For the rest of the proof, we assume that $|A_\lambda| > 0$ for all λ small enough.

Now, for $\delta > 0$ small, we consider the variation $\rho_\delta(x) = \widehat{\rho}(x) + \delta\varphi(x)$ for $0 \le \psi \in C^\infty(\Omega)$ and

$$\varphi(x) = \chi_{A_\lambda}(x) \left(\psi(x) - \widehat{\rho}(x)(\alpha - \widehat{\rho}(x)) \frac{\int_{A_\lambda} \psi(y) \mathrm{d}y}{\int_{A_\lambda} \widehat{\rho}(y)(\alpha - \widehat{\rho}(y)) \mathrm{d}y} \right). \tag{3.35}$$

By construction $\int_\Omega \varphi = 0$. In order to prove $\rho_\delta \ge 0$, we consider

$$0 < \delta < \frac{\int_{A_\lambda} \widehat{\rho}(\alpha - \widehat{\rho})}{\lambda \int_{A_\lambda} \psi}.$$

Lastly we check that $\rho_\delta \le \alpha$. If $\widehat{\rho}(x) \ge \alpha - \lambda$ then $\rho_\delta(x) = \widehat{\rho}(x) \le \alpha$. If $\widehat{\rho}(x) < \alpha - \lambda$ then it suffices that $0 < \delta < \frac{1}{\|\psi\|_{L^\infty}}$. We have concluded that $\rho_\delta \in \mathcal{A}_M$. Computing the first variation

$$0 \le \lim_{\delta \to 0^+} \frac{\mathcal{F}[\rho_\delta] - \mathcal{F}[\widehat{\rho}]}{\delta} = \int_{A_\lambda} \big(U'(\widehat{\rho}) + V\big) \varphi = \int_{A_\lambda} \big(U'(\widehat{\rho}(x)) + V(x) - C_\lambda\big) \psi(x) \, \mathrm{d}x,$$

where

$$C_\lambda = \frac{\int_{A_\lambda} (U'(\widehat{\rho}) + V) \widehat{\rho}(\alpha - \widehat{\rho})}{\int_{A_\lambda} \widehat{\rho}(\alpha - \widehat{\rho})}.$$

Since this holds for any $\psi \in C^\infty(\Omega)$ non-negative, we conclude

$$U'(\widehat{\rho}) + V \ge C_\lambda, \qquad \text{a.e. in } A_\lambda.$$

Since these sets A_λ are monotonically increasing to A_0, we can let $\lambda \to 0$ to deduce the claim with

$$C = \frac{\int_{\widehat{\rho}<\alpha}(U'(\widehat{\rho}) + V)\widehat{\rho}(\alpha - \widehat{\rho})}{\int_{\widehat{\rho}<\alpha}\widehat{\rho}(\alpha - \widehat{\rho})} = \frac{\int_{\Omega}(U'(\widehat{\rho}) + V)\widehat{\rho}(\alpha - \widehat{\rho})}{\int_{\Omega}\widehat{\rho}(\alpha - \widehat{\rho})}.$$

Step 1.b: Euler-Lagrange condition when $0 < \widehat{\rho} \le \alpha$. The proof is similar. Let us now take the sets $A^\lambda = \{x \in \Omega : \widehat{\rho}(x) > \lambda\}$. As we explain before, we assume that $|A^\lambda| > 0$ for all λ small enough. For $\delta > 0$ small, we choose the variation $\rho^\delta(x) = \widehat{\rho}(x) - \delta\varphi(x)$ for $0 \le \psi \in C^\infty(\Omega)$ and

$$\varphi(x) = \chi_{A^\lambda}(x)\left(\psi(x) - \widehat{\rho}(x)(\alpha - \widehat{\rho}(x))\frac{\int_{A^\lambda}\psi(y)\,\mathrm{d}y}{\int_{A^\lambda}\widehat{\rho}(y)(\alpha - \widehat{\rho}(y))\,\mathrm{d}y}\right).$$

With the same assumptions on δ that we have considered earlier is easy to prove that $\rho^\delta \in \mathcal{A}$. Computing the first variation

$$0 \le \lim_{\delta\to 0^+}\frac{\mathcal{F}[\rho^\delta] - \mathcal{F}[\widehat{\rho}]}{\delta} = -\int_\Omega \left(U'(\widehat{\rho}) + V\right)\varphi = -\int_{A^\lambda}\left(U'(\widehat{\rho}(x)) + V(x) - C^\lambda\right)\psi(x)\,\mathrm{d}x,$$

where

$$C^\lambda = \frac{\int_{A^\lambda}(U'(\widehat{\rho}) + V)\widehat{\rho}(\alpha - \widehat{\rho})}{\int_{A^\lambda}\widehat{\rho}(\alpha - \widehat{\rho})}.$$

Analogously to what we remarked earlier, it follows that

$$U'(\widehat{\rho}) + V \le C^\lambda, \qquad \text{a.e. in } A^\lambda,$$

and

$$C^\lambda \to C = \frac{\int_{\widehat{\rho}>0}(U'(\widehat{\rho}) + V)\widehat{\rho}(\alpha - \widehat{\rho})}{\int_{\widehat{\rho}>0}\widehat{\rho}(\alpha - \widehat{\rho})} = \frac{\int_{\Omega}(U'(\widehat{\rho}) + V)\widehat{\rho}(\alpha - \widehat{\rho})}{\int_{\Omega}\widehat{\rho}(\alpha - \widehat{\rho})}.$$

Step 2: Proof of formula (3.15). We re-write the Euler-Lagrange conditions using that U' is non-decreasing as

$$\begin{aligned} \widehat{\rho}(x) &\ge (U')^{-1}(C - V(x)), &\qquad &\text{if } 0 \le \widehat{\rho}(x) < \alpha, \\ \widehat{\rho}(x) &\le (U')^{-1}(C - V(x)), &\qquad &\text{if } 0 < \widehat{\rho}(x) \le \alpha. \end{aligned}$$

We distinguish the three possible cases

- Let $x \in \Omega$ be such that $(U')^{-1}(C - V(x)) \in (0, \alpha)$. If $\widehat{\rho}(x) = 0$ we get to contradiction with the first condition. If $\widehat{\rho}(x) = \alpha$ we get into contradiction with the second condition. Thus, we conclude

$$\widehat{\rho}(x) = (U')^{-1}(C - V(x)).$$

- Let $x \in \Omega$ be such that $(U')^{-1}(C - V(x)) \le 0$. If $\widehat{\rho}(x) > 0$ we get a contradiction with the second condition. Since we know $\widehat{\rho} \ge 0$ in Ω, we conclude $\widehat{\rho}(x) = 0$.
- Let $x \in \Omega$ be such that $(U')^{-1}(C - V(x)) \ge \alpha$. If $\widehat{\rho}(x) < \alpha$ we get a contradiction with the first condition. Therefore, we conclude that $\widehat{\rho}(x) = \alpha$.

Step 3: Uniqueness of the constant. Due to (SC_U), we have that U' is invertible with continuous inverse. Furthermore, the function

$$F(C) \coloneqq \int_\Omega T_{0,\alpha} \circ (U')^{-1}(C - V(x))$$

is non-decreasing and continuous. Let us recall the definition of $\underline{\zeta}$ and $\overline{\zeta}$ in (3.6). Then, we distinguish three cases of C:

1. $C \geq \max V + \overline{\zeta}$. Then $F(C) = \alpha|\Omega|$.
2. $C \leq \min V + \underline{\zeta}$. Then $F(C) = 0$.
3. $\min V + \underline{\zeta} < C < \max V + \overline{\zeta}$ in Ω. Then, by the continuity of V there exists $\delta > 0$ and $A \subset \Omega$ of positive measure, such that for all $c \in \mathbb{R}$ satisfying $|c - C| < \delta$ and $x \in A$, then $\underline{\zeta} < c - V(x) < \overline{\zeta}$. Hence, we can write
$$F(C) = \int_A (U')^{-1}(C - V(x)) + \int_{\Omega \setminus A} T_{0,\alpha} \circ (U')^{-1}(C - V(x)).$$
The second term of the sum is still non-decreasing. Due to (SC_U), the first term of the sum is strictly increasing.

Thus, if $M \in (0, \alpha|\Omega|)$ there is a unique C such that $F(C) = M$. □

Remark 3.31. *Notice that we are taking perturbations $\widehat{\rho} + \varepsilon\varphi$ with φ given by* (3.35) *(and the corresponding modification at α). This means that if $\widehat{\rho} + \varepsilon\varphi$ converges to $\widehat{\rho}$ in a topology $\mathcal{T}$, then any local minimiser in the topology $\mathcal{T}$ satisfies the Euler-Lagrange conditions and it is therefore* (3.15)*. In particular, $\mathcal{T}$ can be any L^p topology, C^k topology, or even 2-Wasserstein. Notice, however, that $\widehat{\rho} + \varepsilon\varphi$ does not converge to $\widehat{\rho}$ in the ∞-Wasserstein topology. In fact, there can be many more ∞-Wasserstein local minimisers (see Remark 3.39 below). For necessary condition for ∞-Wasserstein minimisation we point the reader to [22, 83].*

3.5 Existence of long-time behaviour. Proof of Theorem 3.8

We study the long-time behaviour of the problem using semigroup theory. The proof of this result relies on an Aubin-Lions compactness argument, and the application of L^1 contraction techniques, we devote section 3.5.1 to this goal. Afterwards, in the next two subsections we discuss the two examples and for each one of them, we construct a time-limit operator. First, in section 3.5.2 we describe the time-limit operator of (P), which corresponds to Theorem 3.8–Item ii. Afterwards, in section 3.5.3 we do the same for the problem (P_ε), which corresponds to Theorem 3.8–Item i.

3.5.1 From compactness to convergence

We present the following auxiliary lemma for the existence of a time-limit operator.

Lemma 3.32. *Let S be a free-energy dissipating semigroup for* (P)*. Assume that, for each $\rho_0 \in \mathcal{A}_+$, there exists $t_n \to \infty$ and $\rho^\infty \in \mathcal{A}$ such that*
$$S_{\bullet + t_n}\rho_0 \to \rho^\infty \text{ in } L^1((0,1) \times \Omega).$$
Then, there exists a time-limit operator for S_∞ and it is an L^1-contraction.

Proof. We divide the proof in several steps.

Step 1: Analysis for $\rho_0 \in \mathcal{A}_+$.

Step 1.a: Convergence in $C([0,1]; L^1(\Omega))$. Consider a sub-sequence in time t_{n_k}. Let us recall again that $L^1((0,1) \times \Omega) = L^1(0,1; L^1(\Omega))$ where the latter L^1 space is in the Bochner sense (see [256, Theorem 9.2]). Then, up to a further subsequence still denoted t_{n_k}, we have that
$$S_{s + t_{n_k}}\rho_0 \to \rho^\infty \qquad \text{in } L^1(\Omega) \text{ for a.e. } s \in [0,1].$$
Let Λ be the set of $s \in [0,1]$ where the convergence happens. Taking $\delta > 0$, then $\{(s - \delta, s + \delta)\}_{s \in \Lambda}$ is a cover of $[0,1]$, and there exists a finite subcover $\{(s_i - \delta, s_i + \delta)\}_{1 \leq i \leq N}$. For $s \in [0,T]$ and $\sigma \in \Lambda$ there exists $1 \leq i \leq N$ such that
$$\|S_{s+t_{n_k}}\rho_0 - \rho^\infty\|_{L^1(\Omega)} \leq \|S_{s+t_{n_k}}\rho_0 - S_{s_i + t_{n_k}}\rho_0\|_{L^1(\Omega)} + \|S_{s_i + t_{n_k}}\rho_0 - \rho^\infty\|_{L^1(\Omega)}$$

$$\leq \|S_{|s-s_i|}\rho_0 - \rho_0\|_{L^1(\Omega)} + \|S_{s_i+t_{n_k}}\rho_0 - \rho^\infty\|_{L^1(\Omega)}.$$

Since there is a finite number of s_i, we recover

$$\limsup_{k\to\infty} \sup_{s\in[0,1]} \|S_{s+t_{n_k}}\rho_0 - \rho^\infty\|_{L^1(\Omega)} \leq \sup_{\tau\in[0,\delta]} \|S_\tau \rho_0 - \rho_0\|_{L^1(\Omega)}.$$

Letting $\delta \to 0$ we have shown the convergence of this subsequence. Since any sub-sequence of t_n has a further subsequence converging in $C([0,1];L^1(\Omega))$, and they all do so to ρ^∞, then the whole sequence $S_{\bullet+t_n}\rho_0$ converges in $C([0,1];L^1(\Omega))$ to ρ^∞.

Step 1.b: ρ^∞ is stationary for S, i.e., $S_t\rho^\infty = \rho^\infty$ for all $t \geq 0$. Due to the semigroup property for $a \in [0,1]$

$$S_a S_{t_n}\rho = S_{a+t_n}\rho.$$

Since S_a is 1-Lipschitz in L^1 we can pass to the limit on both sides to recover $S_a\rho^\infty = \rho^\infty$. Once more, due to the semigroup property, $S_t\rho^\infty = \rho^\infty$ for all $t > 0$.

Step 1.c: Convergence of the whole sequence. Let us define

$$F(t) \coloneqq \|S_t\rho_0 - \rho^\infty\|_{L^1(\Omega)} = \|S_t\rho_0 - S_t\rho^\infty\|_{L^1(\Omega)}.$$

This function is clearly non-negative. Due to the L^1 contraction, it is non-increasing. We have assumed that $F(t_n) \to 0$. Thus, $F(t) \to 0$ as $t \to \infty$.

Step 2: General $\rho_0 \in \mathcal{A}$. From the previous step, there exists $S_\infty : \mathcal{A}_+ \to \mathcal{A}$ that is 1-Lipschitz. There is a unique continuous extension $S_\infty : \mathcal{A} \to \mathcal{A}$, and it is also 1-Lipschitz.

For $\rho_0 \in \mathcal{A}$ we can now simply take $\eta_0 \in \mathcal{A}_+$ and write

$$\begin{aligned} \|S_t\rho_0 - S_\infty\rho_0\|_{L^1(\Omega)} &\leq \|S_t\rho_0 - S_t\eta_0\|_{L^1(\Omega)} + \|S_t\eta_0 - S_\infty\eta_0\|_{L^1(\Omega)} + \|S_\infty\eta_0 - S_\infty\rho\|_{L^1(\Omega)} \\ &\leq \|S_t\eta_0 - S_\infty\eta_0\|_{L^1(\Omega)} + 2\|\eta_0 - \rho_0\|_{L^1(\Omega)}. \end{aligned}$$

Letting $t \to \infty$ we arrive at

$$\limsup_{t\to\infty} \|S_t\rho_0 - S_\infty\rho_0\|_{L^1(\Omega)} \leq 2\|\eta_0 - \rho_0\|_{L^1(\Omega)}.$$

Taking infimum over $\eta_0 \in \mathcal{A}_+$ we get the result.

Step 3: The limit is a constant-in-time weak solution. Since the free energy is bounded from below and the energy $\mathcal{F}[\rho_t]$ decays with time, it has a limit as $t \to \infty$. Hence, due to the stability of the disperssion term in Lemma 3.25, it follows that $\mathrm{m}(\rho^\infty)^{\frac{1}{2}}\nabla(U'(\rho^\infty) + V) = 0$ almost everywhere Ω. Multiplying once again by $\mathrm{m}(\rho^\infty)^{\frac{1}{2}}$ we get $\nabla\Phi(\rho^\infty) + \mathrm{m}(\rho^\infty)\nabla V = 0$. Given a test function $\varphi \in H^1(\Omega)$, we can multiply by $\nabla\varphi$ and integrate in Ω to the deduce the weak formulation of the stationary problem. □

3.5.2 For problem (P)

Lemma 3.33. *Let $\rho_0 \in \mathcal{A}_+$. Then, there exists $\rho^\infty \in \mathcal{A}$ and $t_n \to \infty$ such that $S_{\bullet+t_n}\rho_0 \to \rho^\infty$ in $L^1((0,1)\times\Omega)$ as $n \to \infty$.*

Proof. We use the lighter notation $\rho_s^{[n]} = S_{s+t_n}\rho_0$ for $s \in [0,1]$ and $\rho_t = S_t\rho_0$ for $t \geq 0$. First, we prove convergence by compactness. Analogously to the Step 1 of the proof of Lemma 3.30, up to a subsequence, we obtain that for all $p \in [1,\infty)$,

$$\rho^{[n]} \to \rho^\infty \quad \text{strongly in } L^p([0,1]\times\Omega). \tag{3.36}$$

Using Lemma 3.17 we have $\rho^{[n]} \to \rho^\infty$ in $C([0,1];W^{-1,1}(\Omega))$.

Lastly, we show that the limit is stationary. From Theorem 3.5 it follows that S is a free-energy dissipating semigroup. Hence, due to (3.8), $\mathcal{F}[\rho_t]$ is non-increasing and bounded below, therefore it admits a limit which we denote $\underline{\mathcal{F}}$, the limit does not depend on time. □

3.5.3 For problem (P_ε)

The proof of Lemma 3.33 can be applied also to (P_ε). However, using that approach some of the technical become too complicated. We include now an elementary proof that works for $\varepsilon > 0$.

Lemma 3.34. *Let $\varepsilon > 0$ and $\rho_0 \in \mathcal{A}_+$. Then, there exists $\rho^{(\varepsilon),\infty} \in \mathcal{A}$ and $t_n \to \infty$ such that $S^{(\varepsilon)}_{\bullet+t_n}\rho_0 \to \rho^{(\varepsilon),\infty}$ in $L^1((0,1)\times\Omega)$ as $n \to \infty$.*

Proof. Once more, we use the lighter notation $\rho_s^{(\varepsilon),[n]} = S^{(\varepsilon)}_{s+t_n}\rho_0$ for $s \in [0,1]$ and $\rho_t^{(\varepsilon)} = S_t^{(\varepsilon)}\rho_0$ for $t \geq 0$. We prove convergence by compactness arguments. Using Lemma 3.17 we have that, up to a subsequence,

$$\rho^{(\varepsilon),[n]} \to \rho^{(\varepsilon),\infty} \quad \text{in } C([0,1]; W^{-1,1}(\Omega)).$$

For the point-wise convergence we aim to use the Aubin-Lions Lemma with the spaces $H^1(\Omega) \subset L^2(\Omega) \subset W^{-1,1}(\Omega)$. From (3.23) we recover

$$\|\nabla\rho^{(\varepsilon),[n]}\|^2_{L^2((0,1)\times\Omega)} \leq 2\int_\Omega G_\varepsilon(\rho^{(\varepsilon)}_{t_n}) + C\|\nabla V\|^2_{L^2(\Omega)},$$

where G_ε is defined in (3.22). Next, we show that the first term in the RHS is uniformly bounded for each $\varepsilon > 0$ fixed. Taking advantage of the definition of G_ε, (3.9), and (3.21) it follows that

$$\int_\Omega G_\varepsilon(\rho^{(\varepsilon)}_{t_n}) = \int_\Omega \int_{\frac{\alpha}{2}}^{\rho^{(\varepsilon)}_{t_n}(x)} \int_{\frac{\alpha}{2}}^{\sigma} G''_\varepsilon(s)\mathrm{d}s\, d\sigma \mathrm{d}x \leq C(\varepsilon,\alpha)\left(|\Omega| + \|\rho^{(\varepsilon)}_{t_n}\|_{L^1(\Omega)} + \|\rho^{(\varepsilon)}_{t_n}\|^2_{L^2(\Omega)}\right) \leq C(\varepsilon,\alpha,|\Omega|).$$

Hence, for each $\varepsilon > 0$ fixed, the sequence $\rho^{(\varepsilon),[n]}$ is uniformly bounded in $L^2(0,1;H^1(\Omega))$. Therefore, we can apply the Aubin-Lions Lemma, up to a subsequence, we have that

$$\rho^{(\varepsilon),[n]} \to \rho^{(\varepsilon),\infty} \quad \text{in } L^2((0,1)\times\Omega).$$

Due to (3.8), $\mathcal{F}_\varepsilon[\rho_t]$ is non-increasing and bounded below, therefore it admits a limit which we denote $\underline{\mathcal{F}_\varepsilon}$. Hence, the limit does not depend on time. □

Theorem 3.8 follows from the combination of Lemmas 3.32 to 3.34.

3.6 Analysis of the long-time limit

In this Section we understand the long-time behaviour. First, in section 3.6.1 we focus on the global attractor of (P_ε) for $\varepsilon > 0$. Afterwards, in section 3.6.2 we study some properties of the ω-limit of (P). Finally, in section 3.6.3 we construct an example in order to show that uniqueness of constant-in-time solutions of (P) is not necessarily true, and that the extra constant-in-time solutions also attract a large class of initial data.

3.6.1 The global attractors for (P_ε). Proof of Theorem 3.9

We first show an auxiliary result.

Lemma 3.35 (A generalisation of Theorem 3.9–Item ii)**.** *If ρ is a constant-in-time weak solution of (P_ε) then exactly one of the following holds: $\rho \equiv 0$ in Ω, $\rho \equiv \alpha$ in Ω, or $\rho = \rho^{(\varepsilon),\infty}$ given by*

$$\rho^{(\varepsilon),\infty}(x) = (U'_\varepsilon)^{-1}(C_\varepsilon - V(x)),$$

where C_ε is uniquely determined by the mass of ρ.

Proof. We divide the proof in several steps.

Step 1: $\rho \equiv 0$, $\rho \equiv \alpha$, or $0 < \rho < \alpha$. We write

$$-\Delta \Phi_\varepsilon(\rho) = \operatorname{div}\left(\mathrm{m}_\varepsilon(\rho)\nabla V\right)$$

in the weak sense. Let us now pick $w = \Phi_\varepsilon(\rho)$. Since ρ is a weak solution of (P_ε) and Φ_ε fulfils the assumption (3.9) we know that $w \in H^1(\Omega) \cap L^\infty(\Omega)$. Therefore, w satisfies the equation

$$-\Delta w = \operatorname{div}\left(f_\varepsilon(w)\nabla V\right), \qquad f_\varepsilon(w) = \mathrm{m}_\varepsilon(\Phi_\varepsilon^{-1}(w))$$

with no-flux boundary condition. The right-hand side is in $L^2(\Omega)$, and so $w \in H^2_{loc}(\Omega)$. Notice that $f_\varepsilon(0) = 0$. By a bootstrap argument $w \in W^{2,\infty}_{loc}(\Omega)$ (see e.g., [61]). So $\rho \in W^{2,\infty}_{loc}(\Omega)$, and it is a classical solution of the interior equation.

Assume there is a set of positive measure such that $\rho > 0$. Let $A \subset \Omega$ be the largest set where this is satisfied. Since ρ is continuous, then A is an open set. Assume, towards contradiction, that $A \subsetneq \Omega$. Take a point in $x_0 \in \Omega \cap \partial A$. Let r be small enough that $B(x_0, 4r) \subset \Omega$. Notice that $u = \rho$ satisfies the elliptic problem

$$-\operatorname{div}(a(x)\nabla u + b(x)u) = 0, \qquad \text{in } \Omega,$$

where $a(x) = \Phi_\varepsilon'(\rho(x)) \geq c(\varepsilon)$ and bounded, and

$$b(x) = \begin{cases} \dfrac{\mathrm{m}_\varepsilon(\rho)}{\rho}\nabla V & \text{if } \rho(x) > 0, \\ \mathrm{m}_\varepsilon'(0)\nabla V & \text{if } \rho(x) = 0 \end{cases}$$

is continuous. Hence, we can use Harnack's inequality (see [196]). Then $\sup_{B(x_0,r)} \rho \leq C \inf_{B(x_0,r)} \rho = 0$. This contradicts the definition of x_0. Similarly, taking $u = \alpha - \rho$ we deduce $\rho \equiv \alpha$ in Ω or $\rho < \alpha$ in Ω.

Step 2: Characterisation of the case $0 < \rho < \alpha$. Let us take $\delta \in (0,1)$. Notice that $U_\varepsilon'' = \Phi_\varepsilon'/\mathrm{m}_\varepsilon$ is singular at 0 and α. For this proof we consider a smoothing of U_ε and we define $U_{\varepsilon,\delta}$ such that $U_{\varepsilon,\delta}(\delta) = U_\varepsilon(\delta)$, $U_{\varepsilon,\delta}'(\delta) = U_\varepsilon'(\delta)$ and $U_{\varepsilon,\delta}''(s) = \min\{U_\varepsilon''(s), \delta^{-1}\}$. Using $\varphi = U_{\varepsilon,\delta}'(\rho) + V$ as a test function, we have that

$$\begin{aligned} 0 &= \int_\Omega \mathrm{m}_\varepsilon(\rho)\nabla\left(U_\varepsilon'(\rho) + V\right)\cdot\nabla\left(U_{\varepsilon,\delta}'(\rho) + V\right) \\ &= \int_\Omega \mathrm{m}_\varepsilon(\rho)U_\varepsilon''(\rho)U_{\varepsilon,\delta}''(\rho)|\nabla\rho|^2 + \int_\Omega \mathrm{m}_\varepsilon(\rho)U_{\varepsilon,\delta}''(\rho)\nabla\rho\cdot\nabla V + \int_\Omega \mathrm{m}_\varepsilon(\rho)U_\varepsilon''(\rho)\nabla\rho\cdot\nabla V + \int_\Omega \mathrm{m}_\varepsilon(\rho)|\nabla V|^2. \end{aligned} \tag{3.37}$$

Only the first two terms vary with δ. From this equality we can estimate the first term on the right-hand side. Using that $0 \leq U_{\varepsilon,\delta}'' \leq U_\varepsilon''$ we have

$$0 \leq \mathrm{m}_\varepsilon(\rho)U_{\varepsilon,\delta}''(\rho)|\nabla\rho| \leq \mathrm{m}_\varepsilon(\rho)U_\varepsilon(\rho)|\nabla\rho| = |\nabla\Phi_\varepsilon(\rho)|.$$

Hence, this quantity is in $L^2(\Omega)$ due to our notion of weak solution. Hence, we can bound

$$\int_\Omega \mathrm{m}_\varepsilon(\rho)U_\varepsilon''(\rho)U_{\varepsilon,\delta}''(\rho)|\nabla\rho|^2 \leq 2\int_\Omega |\nabla\Phi_\varepsilon(\rho)||\nabla V| - \int_\Omega \mathrm{m}_\varepsilon(\rho)|\nabla V|^2 \leq 2\int_\Omega |\nabla\Phi_\varepsilon(\rho)||\nabla V|.$$

We observe that as $\delta \to 0$ we have pointwise that

$$\mathrm{m}_\varepsilon(\rho)U_\varepsilon''(\rho)U_{\varepsilon,\delta}''(\rho)|\nabla\rho|^2 \nearrow \mathrm{m}_\varepsilon(\rho)U_\varepsilon''(\rho)^2|\nabla\rho|^2.$$

Therefore, by the monotone convergence theorem $\mathrm{m}_\varepsilon(\rho)|\nabla U_\varepsilon'(\rho)|^2 \in L^1$ as we have convergence as $\delta \to 0$ of the first term in (3.37). Since $\mathrm{m}_\varepsilon(\rho)U_{\varepsilon,\delta}''(\rho)|\nabla\rho|$ is uniformly bounded in $L^2(\Omega)$, a subsequence admits a weak-L^2 limit. By point-wise convergence we deduce that, up to a subsequence, $\mathrm{m}_\varepsilon(\rho)U_{\varepsilon,\delta}''(\rho)\nabla\rho \rightharpoonup \mathrm{m}_\varepsilon(\rho)U_\varepsilon''(\rho)\nabla\rho$ weakly in $L^2(\Omega)$ as $\delta \to 0$. We can therefore pass to the limit in (3.37) to deduce

$$\int_\Omega \mathrm{m}_\varepsilon(\rho)\left|\nabla(U_\varepsilon'(\rho) + V)\right|^2 = 0.$$

Since $0 < \rho < \alpha$ a.e. in Ω, we deduce $U_\varepsilon'(\rho) + V = C_\varepsilon$ and, in particular, $\rho = (U_\varepsilon')^{-1}(C_\varepsilon - V(x))$. Using Theorem 3.6 the constant is uniquely defined. And the proof is complete. □

Any constant-in-time weak solution of the problem (P_ε) is described in Lemma 3.35. The combination of this result with Theorem 3.8 yields that the time-limit operator S_∞ for the problem (P_ε) is such that $S_\infty^{(\varepsilon)}\rho_0 = 0$, $S_\infty^{(\varepsilon)}\rho_0 = \alpha$ or $S_\infty^{(\varepsilon)}\rho_0(x) = (U_\varepsilon')^{-1}(C_\varepsilon - V(x))$, where the constant C_ε depends only on the mass of ρ_0, i.e., the only possible constant-in-time weak solutions. Hence, as we point out in the statement, Theorem 3.9–Item ii follows as a consequence of Lemma 3.35. Using this result, we are now ready to study further properties of $\widehat{\rho}^{(\varepsilon)}$.

Proof of Theorem 3.9–Item i and Theorem 3.9–Item iii. The fact that it is the unique L^1-local minimiser follows directly from Theorem 3.6. Combining this with Lemma 3.23 we obtain that it is also the unique global minimiser. Furthermore, from Theorem 3.8 and Lemma 3.35 we have that for any $\rho_0 \in \mathcal{A}$

$$S_t^{(\varepsilon)}\rho_0 \to S_\infty^{(\varepsilon)}\rho_0 = \rho^{(\varepsilon),\infty} \quad \text{strongly in } L^1(\Omega) \text{ as } t \to \infty,$$

finishing the proof. □

3.6.2 The ω-limit of (P). Proof of Theorem 3.10

We proceed to study the limit of the constant-in-time weak solutions of (P_ε) as $\varepsilon \to 0$. In order to do that and study its properties we present some auxiliary results.

Lemma 3.36. *Assume* (USC_U). *Consider the approximation U_ε defined with the condition* (3.9). *Then, the functions $(U_\varepsilon')^{-1}$ are such that $(U_\varepsilon')^{-1} \to T_{0,\alpha} \circ (U')^{-1}$ pointwise in $\mathbb{R}$ and uniformly over compacts of $\mathbb{R}\setminus\{\underline{\zeta}, \overline{\zeta}\}$ as $\varepsilon \searrow 0$, where $\underline{\zeta}, \overline{\zeta}$ are defined in* (3.6).

Proof. We recall that Lemma 3.28 ensures $U_\varepsilon' \to U'$ pointwise in $(0, \alpha)$. We will use the notations (3.6). In order to show the convergence, we divide the proof into the sets $(\underline{\zeta}, \overline{\zeta})$, $\{\underline{\zeta}, \overline{\zeta}\}$, and $\mathbb{R}\setminus[\underline{\zeta}, \overline{\zeta}]$.

Step 1: Convergence in $(\underline{\zeta}, \overline{\zeta})$. We first prove that $(U_\varepsilon')^{-1}$ are uniformly Lipschitz over compacts. We know that it is non-decreasing. So the derivative is bounded below. Take any $\zeta_1, \zeta_2 \in (\underline{\zeta}, \overline{\zeta})$ with $\zeta_1 < \zeta_2$. By construction $0 < U'(\zeta_1) \le U'(\zeta_2) < \alpha$. Let us define

$$s_1 := \frac{(U')^{-1}(\zeta_1)}{2} \in \big(0, U'(\zeta_1)\big) \quad \text{and} \quad s_2 := \frac{(U')^{-1}(\zeta_2) + \alpha}{2} \in \big(U'(\zeta_2), \alpha\big).$$

Due to uniform convergence over compacts (see Lemma 3.28), for ε small enough, $(U_\varepsilon')^{-1}([\zeta_1, \zeta_2]) \subseteq [s_1, s_2]$. We pick $\zeta \in [\zeta_1, \zeta_2]$. Recalling (3.9), we observe that

$$\left((U_\varepsilon')^{-1}\right)'(\zeta) = \frac{1}{U_\varepsilon''\left((U_\varepsilon')^{-1}(\zeta)\right)} = \frac{\mathrm{m}_\varepsilon\left((U_\varepsilon')^{-1}(\zeta)\right)}{\Phi_\varepsilon'\left((U_\varepsilon')^{-1}(\zeta)\right)} \le \frac{\mathrm{m}_\varepsilon\left((U_\varepsilon')^{-1}(\zeta)\right)}{\underline{\Phi_\varepsilon'}\left((U_\varepsilon')^{-1}(\zeta)\right)}.$$

We distinguish two cases:

Step 1.a: $\underline{\Phi_\varepsilon'}((U_\varepsilon')^{-1}(\zeta)) \ge \kappa(\varepsilon)^{-1}$. In this case we simply estimate

$$\left((U_\varepsilon')^{-1}\right)'(\zeta) \le \frac{\mathrm{m}_\varepsilon\left((U_\varepsilon')^{-1}(\zeta)\right)}{\kappa(\varepsilon)^{-1} + \varepsilon} \le \kappa(\varepsilon) \max_{\varepsilon \in [0,1]} \|\mathrm{m}_\varepsilon\|_{L^\infty(0,\alpha)}.$$

Step 1.b: $\underline{\Phi_\varepsilon'}((U_\varepsilon')^{-1}(\zeta)) < \kappa(\varepsilon)^{-1}$. Then we have that

$$\left((U_\varepsilon')^{-1}\right)'(\zeta) \le \frac{\mathrm{m}_\varepsilon\left((U_\varepsilon')^{-1}(\zeta)\right)}{\mathrm{m}\left((U_\varepsilon')^{-1}(\zeta)\right) U''\left((U_\varepsilon')^{-1}(\zeta)\right) + \varepsilon} \le \frac{\max_{\varepsilon \in [0,1]} \|\mathrm{m}_\varepsilon\|_{L^\infty(0,\alpha)}}{\min_{s \in [s_1, s_2]} \mathrm{m}(s)\, U''(s)}.$$

The denominator is positive due to (H_1) and (USC_U).

Thus, we have show that $(U'_\varepsilon)^{-1}$ are uniformly Lipschitz over compacts of $(\underline{\zeta}, \overline{\zeta})$. Now we prove point-wise convergence. Let $\zeta \in (\underline{\zeta}, \overline{\zeta})$ and $\delta > 0$ small enough so that $\zeta \in (\underline{\zeta} + \delta, \overline{\zeta} - \delta)$. Due to continuity, $s := T_{0,\alpha} \circ (U')^{-1}(\zeta) \in (0, \alpha)$. Let $\zeta_\varepsilon := U'_\varepsilon(s)$. By Lemma 3.28, there exists $\varepsilon_0 > 0$ such that for $\varepsilon < \varepsilon_0$ we have $\zeta_\varepsilon \in (\underline{\zeta} + \delta, \overline{\zeta} - \delta)$. In the previous step we have shown that $(U'_\varepsilon)^{-1}$ are uniformly Lipschitz in $(\underline{\zeta} + \delta, \overline{\zeta} - \delta)$. Letting $L(\delta)$ be the continuity constant we have

$$|(U'_\varepsilon)^{-1}(\zeta) - T_{0,\alpha} \circ (U')^{-1}(\zeta)| = |(U'_\varepsilon)^{-1}(\zeta) - (U'_\varepsilon)^{-1}(\zeta_\varepsilon)| \le L(\delta)|\zeta - \zeta_\varepsilon| \to 0 \quad \text{as } \varepsilon \to 0. \tag{3.38}$$

Since this holds for any $\zeta \in (\underline{\zeta}, \overline{\zeta})$, point-wise convergence holds. Joining this fact with the uniformly Lipschitz continuity over compacts of $(\underline{\zeta}, \overline{\zeta})$, we obtain uniform convergence over compact sets of $(\underline{\zeta}, \overline{\zeta})$.

Step 2: Point-wise convergence at $\{\underline{\zeta}, \overline{\zeta}\}$. Let us now assume $0 < \lambda < \overline{\zeta} - \underline{\zeta}$. Then, due to monotonicity it follows that

$$0 \le \limsup_{\varepsilon \to 0^+} (U'_\varepsilon)^{-1}(\underline{\zeta}) \le \limsup_{\varepsilon \to 0^+} (U'_\varepsilon)^{-1}(\underline{\zeta} + \lambda) = T_{0,\alpha} \circ (U')^{-1}(\underline{\zeta} + \lambda).$$

Letting then $\lambda \to 0$ we get $(U'_\varepsilon)^{-1}(\underline{\zeta}) \to 0 = T_{0,\alpha} \circ (U')^{-1}(\underline{\zeta})$. Analogously, $(U'_\varepsilon)^{-1}(\overline{\zeta}) \to \alpha = T_{0,\alpha} \circ (U')^{-1}(\overline{\zeta})$.

Step 3: Uniform convergence on $(-\infty, \underline{\zeta})$ *and* $(\overline{\zeta}, \infty)$. Finally, for the set $(-\infty, \underline{\zeta})$ we can compute, using that $U'_\varepsilon(\zeta) \in (0, \alpha)$ for all $\zeta \in \mathbb{R}$ and $T_{0,\alpha} \circ (U')^{-1}(\zeta) = 0$ for $\zeta \le \underline{\zeta}$ we have that

$$\sup_{\zeta < \underline{\zeta}} \left|(U'_\varepsilon)^{-1}(\zeta) - T_{0,\alpha} \circ (U')^{-1}(\zeta)\right| = \sup_{\zeta < \underline{\zeta}} (U'_\varepsilon)^{-1}(\zeta) \le (U'_\varepsilon)^{-1}(\underline{\zeta}) \to 0 \text{ as } \varepsilon \to 0.$$

Analogously for the set $(\overline{\zeta}, \infty)$. $\square$

Now we are ready to prove convergence of $\widehat{\rho}^{(\varepsilon)}$ as $\varepsilon \to 0$ to a stationary weak solution of (P).

Lemma 3.37. *Assume* $M \in (0, \alpha|\Omega|)$ *and* (USC_U). *Let* C_ε *and* C_0 *be given by Theorem 3.6. Then, as* $\varepsilon \to 0$ $C_\varepsilon \to C_0$ *and*

$$\widehat{\rho}^{(\varepsilon)} = (U'_\varepsilon)^{-1}(C_\varepsilon - V) \to \widehat{\rho}^{(0)} := T_{0,\alpha} \circ (U')^{-1}(C_0 - V) \quad \text{in } L^1(\Omega).$$

Furthermore, $\widehat{\rho}^{(0)}$ *is a constant-in-time weak solution to* (P)*. If* S *is the free-energy dissipating semigroup coming from Theorem 3.5, then* $S_t \widehat{\rho}^{(0)} = \widehat{\rho}^{(0)}$.

Proof. We divide the proof into several steps.

Step 1: The set $\{C_\varepsilon : 0 < \varepsilon < 1\}$ *is bounded.* Let us argue by contradiction. Assume there exists a sequence C_{ε_k} such that $C_{\varepsilon_k} \searrow -\infty$. Then, for every $\zeta \in \mathbb{R}$ there exists a constant $N_0(\zeta)$ such that $C_{\varepsilon_k} - V(x) < \zeta$ for all $x \in \overline{\Omega}$, $k \ge N_0(\zeta)$. Hence, for $k \ge N_0(\zeta)$,

$$M = \int_\Omega \widehat{\rho}^{(\varepsilon_k)} \le |\Omega| \left(U'_{\varepsilon_k}\right)^{-1}(\zeta).$$

Due to Lemma 3.36 we pass to the limit in $k \to \infty$ to recover $M \le |\Omega|\, T_{[0,\alpha]} \circ (U')^{-1}(\lambda)$. As $\lambda \to -\infty$ we recover $M = 0$, a contradiction. Therefore, the set $\{C_\varepsilon : 0 < \varepsilon < 1\}$ is bounded from below.

Similarly, using that $\lim_{\zeta \to \infty} \left(U'_{\varepsilon_k}\right)^{-1}(\zeta) = \alpha$, and arguing by contradiction, we are able to prove that $\{C_\varepsilon : 0 < \varepsilon < 1\}$ is bounded from above.

Step 2: Convergence. Since the set $\{C_\varepsilon : 0 < \varepsilon < 1\} \subset \mathbb{R}$ is bounded there is a sequence C_{ε_k} and a constant $\widetilde{C}$ such that $C_{\varepsilon_k} \to \widetilde{C}$. In the following we will prove that $\widehat{\rho}^{(\varepsilon_k)} \to T_{0,\alpha} \circ (U')^{-1}(\widetilde{C} - V)$ in $L^1(\Omega)$.

First, we prove point-wise convergence, we separate the domain into two subsets

$$A_1 := \left\{x : \widetilde{C} - V(x) \ne \underline{\zeta}, \overline{\zeta}\right\}, \qquad A_2 := \Omega \setminus A_1.$$

For $x \in A_1$, Lemma 3.36 implies uniform convergence in a neighbourhood of $\widetilde{C} - V(x)$, and it implies

$$\widehat{\rho}^{(\varepsilon_k)}(x) \to T_{0,\alpha} \circ (U')^{-1}(\widetilde{C} - V(x)).$$

For $x \in A_2$ we use a monotonicity argument. Let us take any $\lambda > 0$ such that $\tilde{C} - V(x) \pm \lambda \neq \underline{\zeta}, \overline{\zeta}$. Then, there exist $\lambda_\varepsilon \to \lambda$ such that $C_\varepsilon - V(x) \pm \lambda_\varepsilon \neq \underline{\zeta}, \overline{\zeta}$. We observe

$$(U'_{\varepsilon_k})^{-1}(C_{\varepsilon_k} - V(x) - \lambda_{\varepsilon_k}) \leq \widehat{\rho}^{(\varepsilon_k)}(x) \leq (U'_{\varepsilon_k})^{-1}(C_{\varepsilon_k} - V(x) + \lambda_{\varepsilon_k}).$$

Now, let $k \to \infty$ We recover that

$$T_{0,\alpha} \circ (U')^{-1}(\widetilde{C} - V(x) - \lambda) \leq \liminf_{k\to\infty} \widehat{\rho}^{(\varepsilon_k)}(x) \leq \limsup_{k\to\infty} \widehat{\rho}^{(\varepsilon_k)}(x) \leq T_{0,\alpha} \circ (U')^{-1}(\widetilde{C} - V(x) + \lambda).$$

The function $T_{0,\alpha} \circ (U')^{-1}$ is continuous. Therefore, as $\lambda \to 0^+$, we have that

$$\lim_{k\to\infty} \widehat{\rho}^{(\varepsilon_k)}(x) = T_{0,\alpha} \circ (U')^{-1}(\widetilde{C} - V(x)),$$

from where it follows pointwise convergence in Ω. Furthermore, since $0 \leq \widehat{\rho}^{(\varepsilon_k)} \leq \alpha$, by the Dominated Convergence Theorem it follows that

$$\widehat{\rho}^{(\varepsilon_k)} \to T_{0,\alpha} \circ (U')^{-1}(\widetilde{C} - V) \quad \text{in } L^1(\Omega).$$

Due to conservation of mass and Theorem 3.6, we have that $\widetilde{C} = C_0(M)$.

Step 3: Convergence of the whole sequence. Every sequence C_{ε_k} has a convergent subsequence and the limit is unique, so $C_\varepsilon \to C_0$ and

$$\widehat{\rho}^{(\varepsilon)} \to T_{0,\alpha} \circ (U')^{-1}(C_0 - V) \quad \text{in } L^1(\Omega).$$

Step 4: $\widehat{\rho}^{(0)}$ is a constant-in-time weak solution. Due to Lemma 3.24, $\widehat{\rho}^{(0)}$ is a weak solution of the problem (P) and it does not depend on time.

Step 5: $\widehat{\rho}^{(0)}$ is stationary for the semigroup. Given that $S_t^{(\varepsilon_k)} \widehat{\rho}^{(\varepsilon_k)} = \widehat{\rho}^{(\varepsilon_k)}$, due to the $L^1(\Omega)$ convergence, as take $k \to \infty$ we get $S_t \widehat{\rho}^{(0)} = \widehat{\rho}^{(0)}$. □

We conclude this section with the proof of the main theorem.

Proof of Theorem 3.10. Theorem 3.10–Item i and Theorem 3.10–Item iii follow from Lemma 3.37. To prove Theorem 3.10–Item ii, from Theorem 3.6 it follows that $\widehat{\rho}^{(0)}$ is the unique L^1-local minimiser of the free energy (3.2). If we combine this result with Lemma 3.23 we also have that $\widehat{\rho}^{(0)}$ is the unique global minimiser. □

3.6.3 Examples with several steady states with non-trivial basin of attraction

During this subsection we prove that the diagram (D_1) presented in Section 3.2 is not commutative. In order to do that, we construct a counterexample. We take $U(s) = \frac{1}{m-1}s^m$ with $m > 1$ and V a double well potential. First, we explain the counterexample for the case with no saturation and linear mobility, and then we generalise it to the problem with saturation.

Linear mobility

Let us discuss first the following example. For $m > 1$ we consider the famous Barenblatt solution of mass M

$$\mathfrak{B}(x, M) = \left(\frac{m-1}{m} \left(C - \frac{|x|^2}{2} \right) \right)_+^{\frac{1}{m-1}}, \qquad \text{where } C \text{ is s.t.} \int_{\mathbb{R}^d} \mathfrak{B}(x, M) dx = M.$$

Since $m > 1$, $\operatorname{supp}\mathfrak{B}(\cdot, 2) = B_{\widehat{R}}$ for some $\widehat{R} > 0$. Take $x_1, x_2 \in \mathbb{R}^d$ with $|x_1 - x_2| > 2\widehat{R}$ and consider

$$\overline{\rho}(x) = \mathfrak{B}(x - x_1, 2) + \mathfrak{B}(x - x_2, 2).$$

We also consider

$$V(x) = \begin{cases} \frac{|x-x_i|^2}{2} & \text{if } |x - x_i| \le \widehat{R} \\ \frac{|x|^2}{2} & \text{if } |x| \gg 1 \\ \text{smooth} & \text{in the intermediate regions.} \end{cases}$$

For $0 \le \rho_0 \le \overline{\rho}$, by the comparison principle the semigroup solution $S_t\rho_0$ is the unique weak solution to

$$\frac{\partial \rho}{\partial t} = \Delta \rho^m + \operatorname{div}(\rho \nabla V) \qquad \text{in } (0, \infty) \times \mathbb{R}^d. \tag{3.39}$$

Due to the comparison principle we get $0 \le \rho_t \le \overline{\rho}$. The ρ_t is also a solution to $\frac{\partial \rho}{\partial t} = \Delta \rho^m + \operatorname{div}(\rho(x - x_i))$ in each ball $B(x_i, \widehat{R})$ with Dirichlet and no-flux boundary condition $(\nabla \rho^m + \rho \nabla V) \cdot \nu = 0$ in any set Ω such that $\overline{\Omega} \supset B(x_1, \widehat{R}) \cup B(x_2, \widehat{R})$. We notice that there is no mass exchange between these two balls $B(x_i, \widehat{R})$, $i = 1, 2$. Consider $M_1, M_2 \le 1$ and $0 \le \rho_0 \le \overline{\rho}$ such that

$$\int_{x_1 + B_{\widehat{R}}} \rho_0 = M_1, \qquad \int_{x_2 + B_{\widehat{R}}} \rho_0 = M_2. \tag{3.40}$$

We show an example of such initial datum in Figure 3.3. Then, from [109] it follows that

$$S_\infty \rho_0 = \mathfrak{B}(x - x_1, M_1) + \mathfrak{B}(x - x_2, M_2). \tag{3.41}$$

See Figure 3.3. Notice that, naturally, $0 \le S_\infty\rho_0 \le \overline{\rho}$. We show a numerical example in Figure 3.7. In particular, if $M_1 \ne M_2$ then $S_\infty\rho_0$ satisfies the Euler-Lagrange condition (3.14) with different constants in each component of its support. Thus, $S_\infty\rho_0$ is not an L^1-local minimiser of the free energy. However, it attracts some initial data.

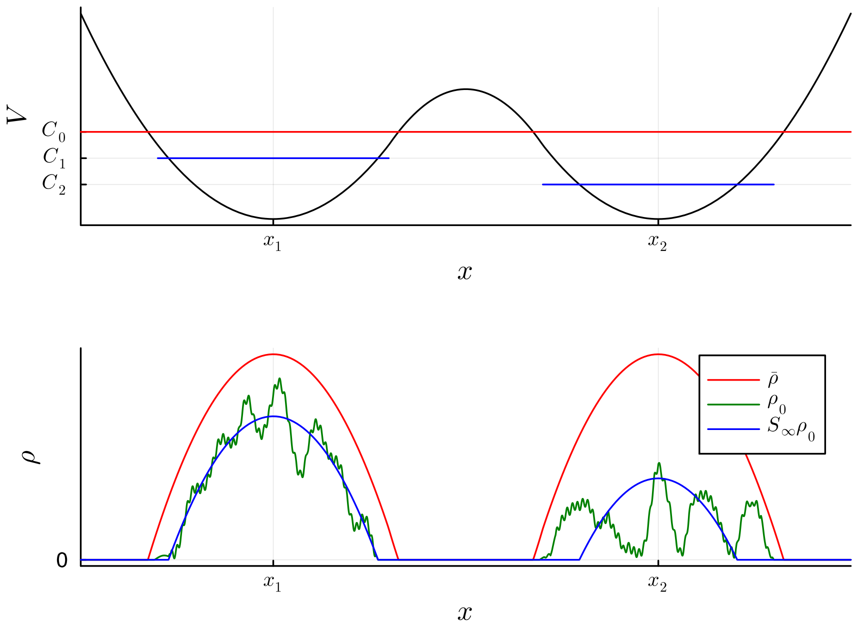


Figure 3.3: Double-well potential and the levels of energy. Since the initial data ρ_0 (green) is such that $0 \le \rho_0 \le \overline{\rho}$ (red), it converges to $S_\infty\rho_0$ (blue).

Remark 3.38. *This example can be generalised to U such that U' invertible with $U'(0) = 0$, and potentials V that are uniformly convex, in the sense that*

$$D^2V \geq \lambda I$$

for some $\lambda > 0$, in a region $B(x_1, \widehat{R}) \cup B(x_2, \widehat{R}) \subset \overline{\Omega}$. Then, from [105], we have that

$$S_t\rho_0 \to S_\infty\rho_0, \qquad \text{in } \mathcal{W}_2(\Omega) \text{ as } t \to \infty.$$

Using this convergence result and the 2-Wasserstein distance, we can reconstruct the previous example for more choices of the potential V.

Remark 3.39 (Local minimiser in p-Wasserstein spaces)**.** *Let us make $M_1 + M_2 = 1$ and $M_1 \neq M_2$. Even though $\widehat{\rho}$ is not a L^1 local minimiser, or even a 2-Wasserstein local minimiser, it is a local minimiser in the ∞-Wasserstein sense, see e.g., [83]. The p-Wasserstein cost of moving a mass $\mathfrak{m}$ from $B(x_1, \widehat{R})$ to $B(x_2, \widehat{R})$ can be estimated by*

$$\mathfrak{m}^{\frac{1}{p}} \min_{\substack{x \in B(x_1, \widehat{R}) \\ y \in B(x_2, \widehat{R})}} |x - y| \leq cost_p \leq \mathfrak{m}^{\frac{1}{p}} \max_{\substack{x \in B(x_1, \widehat{R}) \\ y \in B(x_2, \widehat{R})}} |x - y|.$$

Hence, for M fixed the curve of steady states

$$M_1 \in (0, M) \mapsto \mathfrak{B}(x - x_1, M_1) + \mathfrak{B}(x - x_2, M - M_1) \tag{3.42}$$

is continuous in p-Wasserstein distance for $p \in [1, \infty)$. As $p \to \infty$ we get an infinite cost to pass any mass, so (3.42) is not continuous in ∞-Wasserstein distance. We show this curve and the effect on the free energy in Figure 3.4. The minimum is achieved for the case $M_1 = \frac{M}{2}$, i.e., $C_1 = C_2$. Figure 3.4 also shows that (for these examples) the cases $M_1 \neq M_2$ (i.e., $C_1 \neq C_2$) are saddle points of the free energy (3.2) with respect to the L^1 topology.

Furthermore, even though $\widehat{\rho}$ is not a critical point of the 2-Wasserstein gradient flow it is indeed stationary, and in fact a saddle point. We show that it has a large basin of attraction.

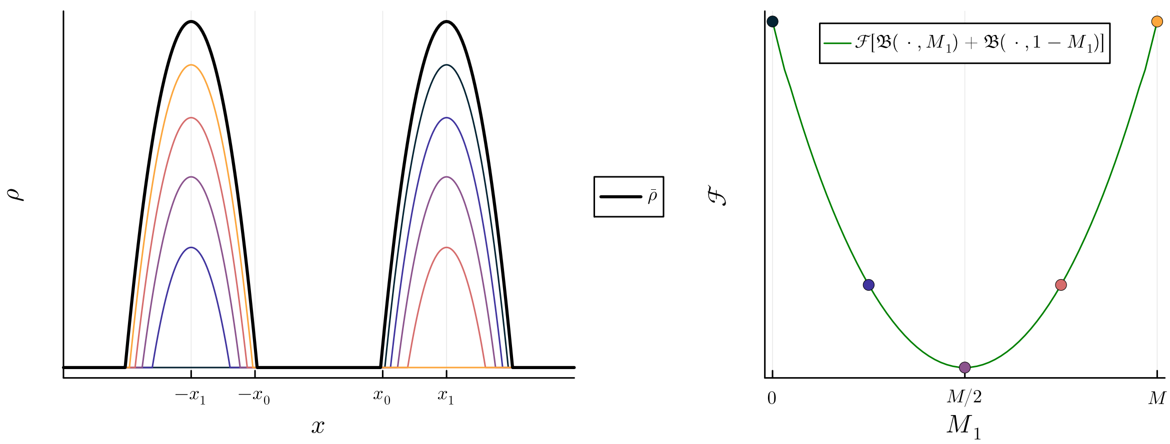


Figure 3.4: Free energy of (3.41) for different values of M_1.

Extension to the saturation case

Using the linear mobility case, we extend the counterexample, and we show that (D_1) is not a commutative diagram. We keep the notation from the previous subsection. For the construction of this counter-example let us set $\alpha = 4\sup_{x\in\overline{\Omega}} \overline{\rho}$, and the mobility $\mathrm{m}(s) = \mathrm{m}^{(1)}(s)\mathrm{m}^{(2)}(s)$ with,

$$\mathrm{m}^{(1)}(s) = \begin{cases} s & \text{if} \quad 0 \leq s \leq \dfrac{\alpha}{4}, \\ \alpha & \text{if} \quad \dfrac{3\alpha}{4} \leq s \leq \alpha, \end{cases} \qquad \mathrm{m}^{(2)}(s) = \begin{cases} 1 & \text{if} \quad 0 \leq s \leq \dfrac{\alpha}{4}, \\ 1 - \dfrac{s}{\alpha} & \text{if} \quad \dfrac{3\alpha}{4} \leq s \leq \alpha, \end{cases} \tag{3.43}$$

and regular for $s \in \left(\frac{\alpha}{4}, \frac{3\alpha}{4}\right)$. Notice that with the construction, if $\rho \leq \overline{\rho}$ then $\mathrm{m}(\rho) = \rho$ and thus, solving (3.39) is equivalent to solving

$$\partial_t \rho = \operatorname{div}\left(\mathrm{m}(\rho)\nabla\left(\frac{m}{m-1}\rho^{m-1} + V\right)\right). \tag{3.44}$$

Taken V as in the previous section, $\overline{\rho}$ and $S_\infty\rho_0$ are also steady solutions of (3.44). In fact, $S_\infty\rho_0$ is still an attractor for all initial data $0 \leq \rho_0 \leq \overline{\rho}$ satisfying the mass condition (3.40).

Finally, let us remark that we can also study this effect at $\rho = \alpha$. Let us consider the same construction of the non-linear mobility $\mathrm{m}(s)$. If ρ is a solution of (3.44) then $u = \alpha - \rho$ solves

$$\partial_t u = \operatorname{div}\left(\mathrm{m}(u)\nabla(U'(u) - V)\right), \tag{3.45}$$

where $U'(s) = \frac{-m}{m-1}(\alpha - s)^{m-1}$. The comparison principle for this problem is inherited through the change of variables. We can use the sub-solution $\underline{u} = \alpha - \overline{\rho}$ to be a sub-solution of the problem (3.45). If $\alpha \geq u_0 \geq \underline{u}$ with correct mass M_1 in $B(x_1, \widehat{R})$ and mass M_2 in $B(x_2, \widehat{R})$ then its limit is $u_\infty = \alpha - S_\infty\rho_0$. Once more, there are different constants in each part of the support, so it is not an L^1-local minimiser.

Moreover, these examples can be generalised to larger families of potentials U and V.

3.7 An upwind numerical scheme

In this Section we study the numerical method (P^Δ). For simplicity, we restrict ourselves to the case of dimension 1. However, these results can be extended to $\Omega \subset \mathbb{R}^d$ with $d > 1$, see e.g., [70].

We consider a variation of the numerical method from [19] that can be also adapted to the regularised problem, i.e., $\varepsilon > 0$. In section 3.7.1 we recall some of the basic properties of the method from [19]. Here we also study well-posedness, convergence of the regularised method ($\mathrm{P}_\varepsilon^\Delta$) to ($\mathrm{P}^\Delta$) as $\varepsilon \to 0$, and convergence from the discrete to continuous solutions as $\Delta \to 0$.

Afterwards, in section 3.7.2 we study the long-time behaviour. We find the global attractors of ($\mathrm{P}_\varepsilon^\Delta$), and we study their limit as $\varepsilon \to 0$. Nevertheless, analogous to the phenomenon observed in the continuous case, this last limit might not be the unique constant-in-time solution (P^Δ). In section 3.7.3, applying the same strategy from section 3.6.3, we construct examples of a different constant-in-time solution of (P^Δ) with a large basin of attraction. Therefore, the back face of the diagram (D_2) is not commutative for the numerical method (P^Δ) and its regularisation ($\mathrm{P}_\varepsilon^\Delta$) either. Finally, in section 3.7.4 we perform some numerical experiments.

3.7.1 The numerical method. Presentation and analysis

In [19] the authors propose the scheme (P^Δ) with $\mathrm{m}^{(1)}(s) = s$, and U locally bounded. In this subsection we present and recall some of the properties of the scheme. For a variation of the method that suits our case better, we obtain existence, uniqueness, and a comparison principle. Later, we also show convergence of solutions of ($\mathrm{P}_\varepsilon^\Delta$) to a solution of ($\mathrm{P}^\Delta$) as $\varepsilon \to 0$. This is all included in Theorem 3.13. Furthermore, under high regularity of the continuous solutions, we show that the discrete solution of ($\mathrm{P}_\varepsilon^\Delta$) and ($\mathrm{P}^\Delta$) converges to the one of the continuous case (P_ε) and (P) respectively as $\Delta \to 0$, Theorem 3.14. Therefore, this section contains the analysis of the left face of the diagram (D_2), i.e., the results concerning the numerical scheme at finite time. Let us first prove the lemma on the decomposition of m.

Proof of Lemma 3.11. Our problem is equivalent to decomposing $\log \mathrm{m}$ as a the sum of a non-decreasing and non-increasing function. Since $\frac{\mathrm{d}}{\mathrm{d}s}\log \mathrm{m}(s) = \frac{\mathrm{m}'(s)}{\mathrm{m}(s)}$ we can expand the Fundamental Theorem of Calculus

$$\log \mathrm{m}(s) = \log \mathrm{m}(\tfrac{\alpha}{2}) + \int_{\frac{\alpha}{2}}^{s} \frac{(\mathrm{m}'(\sigma))_+}{\mathrm{m}(\sigma)}\mathrm{d}\sigma + \int_{\frac{\alpha}{2}}^{s} \frac{(\mathrm{m}'(\sigma))_-}{\mathrm{m}(\sigma)}\mathrm{d}\sigma$$

Therefore, we can write

$$\mathrm{m}^{(1)}(s) = \mathrm{m}(\tfrac{\alpha}{2})\exp\left(\int_{\frac{\alpha}{2}}^{s} \frac{(\mathrm{m}'(\sigma))_+}{\mathrm{m}(\sigma)}\mathrm{d}\sigma\right), \qquad \text{and} \qquad \mathrm{m}^{(2)}(s) = \exp\left(\int_{\frac{\alpha}{2}}^{s} \frac{(\mathrm{m}'(\sigma))_-}{\mathrm{m}(\sigma)}\mathrm{d}\sigma\right).$$

Under the additional assumption, let the neighbourhoods be $(0, \delta)$ and $(\alpha - \delta, \alpha)$. Due to the assumption $\frac{\mathrm{d}m^{(1)}}{\mathrm{d}s} = 0$ in $(\alpha - \delta, \alpha)$ and $\frac{\mathrm{d}m^{(2)}}{\mathrm{d}s} = 0$ is constant in $(0, \delta)$. We can write

$$\frac{\mathrm{d}m^{(1)}}{\mathrm{d}s}(s) = \mathrm{m}^{(1)}(s)\frac{(\mathrm{m}'(s))_+}{\mathrm{m}(s)} = \frac{(\mathrm{m}'(s))_+}{\mathrm{m}^{(2)}(s)}$$

In $(0, \delta)$ we have $\frac{\mathrm{dm}^{(1)}}{\mathrm{d}s}(s) = \mathrm{m}'(s)/\mathrm{m}^{(2)}(0^+)$, so it is bounded. Similarly for $\mathrm{m}^{(2)}$. □

In [19, Proposition 2.2], the authors prove that the scheme preserves boundedness and non-negativity, i.e., $0 \le \rho_i^n \le \alpha$. We show this also holds for our scheme.

We write the system (P^Δ) as $H(\rho^{n+1}) = \rho^n$ where

$$H_i(\rho) = \rho_i + \Delta t \frac{F_{i+\frac{1}{2}}(\rho) - F_{i-\frac{1}{2}}(\rho)}{\Delta x} \tag{3.46}$$

We show first a continuous dependence result and comparison principle.

Lemma 3.40 (Continuous dependence). *Let $\underline{\rho}, \overline{\rho} \in \mathcal{A}_\Delta$, and that for some $H : Q \to \mathbb{R}^{|I|}$ where*

$$Q = \left\{ \rho \in \mathbb{R}^{|I|} : \min\{\underline{\rho}_i, \overline{\rho}_i\} \le \rho_i \le \max\{\underline{\rho}_i, \overline{\rho}_i\} \right\}$$

and H is such that it satisfies:

i) We have mass conservation in the sense that $\sum_i H_i(\rho) = \sum_i \rho_i$ for all $\rho \in Q$.

ii) In the weak sense we have the monotonicity condition: $\frac{\partial H_i}{\partial s_j}(\rho) \le 0$ for all $j \ne i$ and $\rho \in Q$.

Assume that $\underline{\rho}, \overline{\rho}$ are almost solutions in the sense that

$$|H_i(\underline{\rho}) - \underline{f}_i| \le \underline{g}_i, \qquad |H_i(\overline{\rho}) - \overline{f}_i| \le \overline{g}_i.$$

Then, we have that

$$\sum_i |\underline{\rho}_i - \overline{\rho}_i| \le \sum_i |\underline{f}_i - \overline{f}_i| + 2 \sum_i \max\{\underline{g}_i, \overline{g}_i\}, \tag{3.47}$$

and

$$\sum_i (\underline{\rho}_i - \overline{\rho}_i)^+ \le \sum_i (\underline{f}_i - \overline{f}_i)^+ + 2 \sum_i \max\{\underline{g}_i, \overline{g}_i\}.$$

Proof. Due to the conservation of mass we have $\sum_i \frac{\partial H_i}{\partial s_j} = 1$, so $\frac{\partial H_j}{\partial s_j} \ge 0$. Abusing slightly the notation, we write $H_i(\rho) = H_i(\rho_i, (\rho_j)_{j \ne i})$. With the monotonicity indicated

$$H_i\Big(\underline{\rho}_i, (\min\{\underline{\rho}_j, \overline{\rho}_j\})_{j\ne i}\Big) \ge H_i(\underline{\rho}) \ge \underline{f}_i - \underline{g}_i, \qquad H_i\Big(\overline{\rho}_i, (\min\{\underline{\rho}_j, \overline{\rho}_j\})_{j\ne i}\Big) \ge H_i(\overline{\rho}) \ge \overline{f}_i - \overline{g}_i.$$

We have that

$$H_i\Big((\min\{\underline{\rho}_j, \overline{\rho}_j\})_{j\in I}\Big) = \begin{cases} H_i\Big(\underline{\rho}_i, (\min\{\underline{\rho}_j, \overline{\rho}_j\})_{j\ne i}\Big), & \text{if } \underline{\rho}_i \le \overline{\rho}_i, \\ H_i\Big(\overline{\rho}_i, (\min\{\underline{\rho}_j, \overline{\rho}_j\})_{j\ne i}\Big), & \text{if } \overline{\rho}_i \le \underline{\rho}_i. \end{cases}$$

In either case $H_i(\min\{\underline{\rho}, \overline{\rho}\}) \ge \min\{\underline{f}_i, \overline{f}_i\} - \max\{\underline{g}_i, \overline{g}_i\}$. Similarly, we get $H_i(\max\{\underline{\rho}, \overline{\rho}\}) \le \max\{\underline{f}_i, \overline{f}_i\} + \max\{\underline{g}_i, \overline{g}_i\}$. Then, we write

$$\begin{aligned} \sum_{i\in I} \Big(\max\{\underline{\rho}_i, \overline{\rho}_i\} - \min\{\underline{\rho}_i, \overline{\rho}_i\}\Big) &= \sum_{i\in I} \Big(H_i(\max\{\underline{\rho}, \overline{\rho}\}) - H_i(\min\{\underline{\rho}, \overline{\rho}\})\Big) \\ &\le \sum_{i\in I} (\max\{\underline{f}_i, \overline{f}_i\} - \min\{\underline{f}_i, \overline{f}_i\} + 2\max\{\underline{g}_i, \overline{g}_i\}). \end{aligned}$$

Taking into account that $|a-b| = \max\{a,b\} - \min\{a,b\}$ we recover

$$\sum_{i\in I} |\underline{\rho}_i - \overline{\rho}_i| \le \sum_{i\in I} |\underline{f}_i - \overline{f}_i| + 2\sum_{i\in I} \max\{\underline{g}_i, \overline{g}_i\}.$$

Similarly to before, we also have due to mass conservation that

$$\sum_{i\in I}(\underline{\rho}_i - \overline{\rho}_i) = \sum_{i\in I}(H_i(\underline{\rho}) - H_i(\overline{\rho}_i)) \le \sum_{i\in I}(\underline{f}_i - \overline{f}_i) + \sum_{i\in I}(\underline{g}_i + \overline{g}_i).$$

Using that $a^+ = \frac{|a|+a}{2}$ we deduce that

$$\sum_{i\in I}(\underline{\rho}_i - \overline{\rho}_i)^+ \le \sum_{i\in I}(\underline{f}_i - \overline{f}_i)^+ + 2\sum_{i\in I} \max\{\underline{g}_i, \overline{g}_i\}.$$

□

Thanks to this result, we recover the uniqueness and the comparison principle.

Lemma 3.41 (Comparison principle)**.** *In the hypotheses of Lemma 3.40 if $H_i(\underline{\rho}) \le H_i(\overline{\rho})$ for all i, then $\underline{\rho}_i \le \overline{\rho}_i$ for all i.*

Proof. Define $\underline{f}_i := H_i(\underline{\rho})$ and $\overline{f}_i := H_i(\overline{\rho})$. By assumption $\underline{f}_i \le \overline{f}_i$. Therefore, we deduce that $(\underline{f}_i - \overline{f}_i)^+ = 0$. Hence, we recover

$$\sum_i (\underline{\rho}_i - \overline{\rho}_i)^+ \le 0,$$

and we deduce $\overline{\rho}_i \ge \underline{\rho}_i$. □

In particular, the up-winding scheme that we have chosen is such that H satisfies mass conservation, conservation of non-negativity, and it is monotone.

Lemma 3.42. *If we take F from (P^Δ), then the corresponding H defined in*

$$H_i(\lambda,\rho) = \rho_i + \lambda\Delta t \frac{F_{i+\frac12}(\rho) - F_{i-\frac12}(\rho)}{\Delta x}, \tag{3.48}$$

satisfies the hypothesis of Lemma 3.40 for $\lambda \ge 0$.

This lemma applies also for ($\mathrm{P}_\varepsilon^\Delta$).

Proof. The mass conservation follows almost immediately from the telescopic sum and the no-flux condition

$$\sum_{i\in I} H_i(\rho) = \sum_{i\in I}\rho_i + \lambda\frac{\Delta t}{\Delta x}\sum_{i\in I}\Big(F_{i+\frac12}(\rho) - F_{i-\frac12}(\rho)\Big) = \sum_{i\in I}\rho_i + \lambda\frac{\Delta t}{\Delta x}\Big(F_{N+\frac12}(\rho) - F_{\frac12}(\rho)\Big) = \sum_{i\in I}\rho_i.$$

Let us now prove the monotonicity of H, by computing the derivative $\frac{\partial H_i}{\partial \rho_j}$. We proceed by taking derivatives in each term of the composition. First, we compute

$$\frac{\partial \xi_i}{\partial \rho_j} = \begin{cases} U''(\rho_i) & \text{if } j = i, \\ 0 & \text{otherwise} \end{cases} \qquad \frac{\partial v_{i+\frac12}}{\partial \rho_j} = \begin{cases} -\dfrac{U''(\rho_{i+1})}{\Delta x} & \text{if } j = i+1, \\ \dfrac{U''(\rho_i)}{\Delta x} & \text{if } j = i, \\ 0 & \text{otherwise.} \end{cases}$$

The function $F_{i+\frac12}$ is Lipschitz (but not smoother to the presence of the positive and negative part), but we can suitably differentiate it to obtain (using the Kronecker delta notation)

$$\frac{\partial F_{i+\frac12}}{\partial \rho_j} = (\mathrm{m}^{(1)})'(\rho_i)\delta_{ij}\mathrm{m}^{(2)}(\rho_{i+1})(v_{i+\frac12}(\rho))^+ + \mathrm{m}^{(1)}(\rho_i)(\mathrm{m}^{(2)})'(\rho_{i+1})\delta_{i+1,j}(v_{i+\frac12}(\rho))^+$$

$$+ \mathrm{m}^{(1)}(\rho_i)\mathrm{m}^{(2)}(\rho_{i+1})\mathrm{sign}^+(v_{i+\frac12}(\rho))\frac{\partial v_{i+\frac12}}{\partial \rho_j} + (\mathrm{m}^{(1)})'(\rho_{i+1})\delta_{i+1,j}\mathrm{m}^{(2)}(\rho_i)(v_{i+\frac12}(\rho))_-$$
$$+ \mathrm{m}^{(1)}(\rho_{i+1})(\mathrm{m}^{(2)})'(\rho_i)\delta_{ij}(v_{i+\frac12}(\rho))_- + \mathrm{m}^{(1)}(\rho_{i+1})\mathrm{m}^{(2)}(\rho_i)\mathrm{sign}_-(v_{i+\frac12}(\rho))\frac{\partial v_{i+\frac12}}{\partial \rho_j}.$$

Therefore, using $(\mathrm{m}^{(1)})' \geq 0$ and $(\mathrm{m}^{(2)})' \leq 0$ we have that

$$\frac{\partial F_{i+\frac12}}{\partial \rho_i} = \begin{cases} (\mathrm{m}^{(1)})'(\rho_i)\mathrm{m}^{(2)}(\rho_{i+1})v_{i+\frac12}(\rho) + \mathrm{m}^{(1)}(\rho_i)\mathrm{m}^{(2)}(\rho_{i+1})\dfrac{\partial v_{i+\frac12}}{\partial \rho_i} \geq 0 & \text{if } v_{i+\frac12} \geq 0, \\ \mathrm{m}^{(1)}(\rho_{i+1})(\mathrm{m}^{(2)})'(\rho_i)v_{i+\frac12}(\rho) + \mathrm{m}^{(1)}(\rho_{i+1})\mathrm{m}^{(2)}(\rho_i)\dfrac{\partial v_{i+\frac12}}{\partial \rho_i} \geq 0 & \text{if } v_{i+\frac12} < 0. \end{cases}$$

Similarly, $\frac{\partial F_{i+\frac12}}{\partial \rho_{i+1}} \leq 0$. It is trivial to see that $\frac{\partial F_{i+\frac12}}{\partial \rho_j} = 0$ if $j \neq i, i+1$. Lastly, we can compute

$$\frac{\partial H_i}{\partial \rho_j} = \begin{cases} \lambda \dfrac{\Delta t}{\Delta x}\dfrac{\partial F_{i+\frac12}}{\partial \rho_{i+1}} \leq 0 & \text{if } j = i+1, \\ -\lambda \dfrac{\Delta t}{\Delta x}\dfrac{\partial F_{i-\frac12}}{\partial \rho_{i-1}} \leq 0 & \text{if } j = i-1, \\ 0 & \text{if } j \neq i-1, i, i+1. \end{cases}$$

This concludes the proof. □

Lemma 3.43 (Constant sub and super solution)**.** *Assume $\varepsilon \in [0,1]$. Let F^ε be given as in ($\mathrm{P}^\Delta_\varepsilon$). We define*

$$H_i^\varepsilon(\lambda, \rho) = \rho_i + \lambda \Delta t \frac{F^\varepsilon_{i+\frac12}(\rho) - F^\varepsilon_{i-\frac12}(\rho)}{\Delta x}, \tag{3.49}$$

and we take $c^n \in (0, \alpha)$. Then, there exist $\underline{c}^{n+1}, \overline{c}^{n+1} \in (0,\alpha)$ such that for $\underline{\rho} = (\underline{c}^{n+1}, \cdots, \underline{c}^{n+1})$ and $\overline{\rho} = (\overline{c}^{n+1}, \cdots, \overline{c}^{n+1}) \in \mathbb{R}^{|I|}$ satisfy

$$H_i^\varepsilon(\lambda, \underline{\rho}) \leq c^n \leq H_i^\varepsilon(\lambda, \overline{\rho})$$

for any $i \in I$, $\lambda \in [0,1]$.

Proof. Consider $\varepsilon \in (0,1]$. Notice that if $\rho_i = \underline{c}^{n+1}$ for all $i \in I$ then $v^\varepsilon_{i+\frac12} = -\frac{V(x_{i+1}) - V(x_i)}{\Delta x}$. Therefore, must only satisfy

$$\underline{c}^{n+1} + \lambda(\Delta t)\mathrm{m}_\varepsilon(\underline{c}^{n+1})\frac{v^\varepsilon_{i+\frac12} - v^\varepsilon_{i-\frac12}}{\Delta x} \leq c^n, \quad \text{for } 2 \leq i \leq N-1,$$
$$\underline{c}^{n+1} + \lambda(\Delta t)\mathrm{m}_\varepsilon(\underline{c}^{n+1})\frac{v^\varepsilon_{1+\frac12}}{\Delta x} \leq c^n, \qquad \underline{c}^{n+1} - \lambda(\Delta t)\mathrm{m}_\varepsilon(\underline{c}^{n+1})\frac{v^\varepsilon_{N-\frac12}}{\Delta x} \leq c^n.$$

Since we are in the discrete setting, we can simply set the equation

$$\underline{c}^{n+1} + 2(\Delta t)\frac{\|\nabla V\|_{L^\infty}}{\Delta x} \sup_{\varepsilon \in (0,1]} \mathrm{m}_\varepsilon(\underline{c}^{n+1}) \leq c^n.$$

We are trying to solve a problem of the form $F(\underline{c}^{n+1}) \leq c^n$. Since the family $(\varepsilon, s) \mapsto \mathrm{m}_\varepsilon(s)$ is $C([0,1]\times[0,\alpha])$, F is a continuous function. We have $F(0) = 0$ and $F(0) \geq 0$ for $s \in [0,\alpha]$. By Bolzano's Theorem, there exists $\underline{c}^{n+1} > 0$ such that $F(c^{n+1}) \leq c^n$. Thus, the result also holds for $\varepsilon = 0$.

For the super-solution we apply the same argument to

$$\alpha - \overline{c}^{n+1} + \Delta t \, 2\frac{\|\nabla V\|_{L^\infty}}{\Delta x} \sup_{\varepsilon \in (0,1]} \mathrm{m}_\varepsilon(\overline{c}^{n+1}) \leq \alpha - c^n.$$ □

Remark 3.44. *Notice that this sub and super solutions do not pass to the limit when $\Delta \to 0$. The key problem is the behaviour at the endpoints. If we assume that $V \in C^3(\overline{\Omega})$ and $\nabla V \cdot n = 0$ then one can write a better equation for the sub and super solutions, depending on ΔV and $D^3\rho$, that will, in fact, converge to the constant sub and super solutions*

$$\frac{d\underline{c}}{dt} = -\mathrm{m}(\underline{c})\|\Delta V\|_{L^\infty}, \qquad \frac{d\overline{c}}{dt} = \mathrm{m}(\overline{c})\|\Delta V\|_{L^\infty}.$$

Notice that if m is not Lipschitz at 0 or α, there can be finite-time extinction to 0 or α.

We can also show that the problem has energy dissipation, a simplified version of [19, Theorem 2.4] for the case $W = 0$ which we include for the convenience of the reader.

Lemma 3.45 (Energy dissipation)**.** *Let $\varepsilon \geq 0$ and $\lambda \geq 0$. Assume $\rho \in \mathbb{R}^{|I|}$ with $E_\varepsilon^\Delta[\rho] < \infty$ satisfies $H^\varepsilon(\lambda, \rho) = \rho^{\varepsilon,n}$, then $E_\varepsilon^\Delta[\rho] \leq E_\varepsilon^\Delta[\rho^{\varepsilon,n}]$.*

Proof. Since U_ε is convex

$$\begin{aligned}
E_\varepsilon^\Delta(\rho) - E_\varepsilon^\Delta(\rho^{\varepsilon,n}) &= \Delta x \sum_i (U_\varepsilon(\rho_i) - U_\varepsilon(\rho_i^{\varepsilon,n})) + \sum_i V(x_i)(\rho_i - \rho_i^{\varepsilon,n}) \\
&\leq \Delta x \sum_i (U_\varepsilon'(\rho_i) + V(x_i)) \frac{\rho_i - \rho_i^{\varepsilon,n}}{\Delta t} = -\lambda \Delta x \sum_i \xi_i^\varepsilon(\rho) \frac{F_{i+\frac12}^\varepsilon(\rho) - F_{i-\frac12}^\varepsilon(\rho)}{\Delta x} \\
&= \lambda \Delta x \sum_i \frac{\xi_{i+1}^\varepsilon(\rho) - \xi_i^\varepsilon(\rho)}{\Delta x} F_{i+\frac12}^\varepsilon(\rho) \\
&= -\lambda \Delta x \sum_i \left(\mathrm{m}_\varepsilon^{(1)}(\rho_i)\mathrm{m}_\varepsilon^{(2)}(\rho_{i+1})((v_{i+\frac12}(\rho))^+)^2 + \mathrm{m}_\varepsilon^{(1)}(\rho_{i+1})\mathrm{m}_\varepsilon^{(2)}(\rho_i)((v_{i+\frac12}(\rho))^-)^2 \right) \\
&\leq 0.
\end{aligned}$$

□

In the next Lemma we discuss the existence of a solution of the numerical scheme. We recall the notion of topological degree in $\mathbb{R}^{|I|}$, see e.g., [150]. The topological degree is a function $\deg : X \to \mathbb{Z}$ where

$$X = \{(f, D, y) : D \subset \mathbb{R}^d \text{ is open and bounded, } f : \overline{D} \to \mathbb{R}^d \text{ continuous }, y \in \mathbb{R}^d \setminus f(\partial D)\}.$$

We use three key properties. The first one is that $\deg(\mathrm{id}, D, y) = 1$. The second one is that if $h : [0,1] \times \overline{D} \to \mathbb{R}^d$ is continuous and $y \notin h(\lambda, \partial D)$ then $\deg(h(\lambda, \cdot), D, y)$ is constant in λ. Lastly, we use the fact that

Theorem 3.46 ([150, Theorem 3.1])**.** *If $\deg(f, D, y) \neq 0$, then there exists $x \in D$ such that $f(x) = y$.*

Using this tool we prove the following

Lemma 3.47 (Existence for (P^Δ))**.** *Assume that $U \in C^1([0,\alpha])$, and that $\rho^n \in \mathcal{A}_\Delta$. Then, there exists $\rho \in \mathcal{A}_\Delta$ that solves (P^Δ).*

Proof. First, we consider the extension

$$\widetilde{U'}(\rho) = \begin{cases} U'(0) & \text{if } s < 0 \\ U'(s) & \text{if } s \in [0,\alpha] \\ U'(\alpha) & \text{if } s > \alpha \end{cases} \qquad \widetilde{\mathrm{m}}^{(i)}(\rho) = \begin{cases} 0 & \text{if } s < 0 \\ \mathrm{m}^{(i)} & \text{if } s \in [0,\alpha] \\ 0 & \text{if } s > \alpha \end{cases}$$

We can extend $\widetilde{F}_{i+\frac12}$ with these definitions. We define

$$\widetilde{H}_i(\lambda, \rho) = \rho_i + \lambda \Delta t \frac{\widetilde{F}_{i+\frac12}(\rho) - \widetilde{F}_{i-\frac12}(\rho)}{\Delta x} \qquad \text{for } \lambda \in [0,1] \text{ and } \rho \in \mathbb{R}^{|I|}.$$

Now we need to pick D such that $\widetilde{H}(\lambda, \partial D) \not\ni y$ for any $\lambda \in [0,1]$. We look at the one-parameter family of open sets

$$D_R = \left\{ \rho \in \mathbb{R}^{|I|} : \sum_{i \in I} |\rho_i| < R \right\}.$$

Let $\rho^n = \widetilde{H}(\lambda, \rho)$ with $\rho \in \partial D_R$. Because of how we have constructed the extension, we can apply Lemma 3.40 to $\widetilde{H}(\lambda, \cdot)$. Since $\underline{\rho} = (0, \cdots, 0)$ and $\overline{\rho} = (\alpha, \cdots, \alpha)$ satisfy $\rho = \widetilde{H}(\lambda, \rho)$ and $0 \le \rho_i^n \le \alpha$ we conclude that $0 \le \rho_i \le \alpha$ for all $i \in I$. We observe

$$\sum_i |\rho_i| = \sum_i \rho_i = \sum_i \widetilde{H}_i(\lambda, \rho) = \sum_i \rho_i^n.$$

Hence, for $R > \sum_{i \in I} \rho_i^n$ it is clear that $\rho_i^n \notin \widetilde{H}(\lambda, \partial D_R)$ for any $\lambda > 0$. Therefore, we can state that

$$\deg(\widetilde{H}(\lambda, \cdot), D_R, \rho^n) = \deg(\widetilde{H}(0, \cdot), D_R, \rho^n) = \deg(\mathrm{id}, D_R, \rho^n) = 1 \neq 0.$$

Hence, there exists at least one solution $\rho \in \mathbb{R}^{|I|}$ of the extended problem. By the comparison principle $\rho \in \mathcal{A}_\Delta$. Due to the construction of the extension, ρ solves (P^Δ). □

Lemma 3.48 (Existence under general assumptions)**.** *Let $\rho^n \in \mathcal{A}_{\Delta,+}$. Then, there exists $\rho \in \mathcal{A}_{\Delta,+}$ that solves* (P^Δ).

Notice that this lemma applies also for ($\mathrm{P}^\Delta_\varepsilon$).

Proof. We make some adaptations on the proof of Lemma 3.47 to be able to work in the more general case $U \in C^1((0, \alpha))$. Consider $\delta > 0$ such that $\delta \le \rho_i^n \le \alpha - \delta$. We use Lemma 3.43 to show the existence of $\delta_1, \delta_2 > 0$ such that

$$H_i(\lambda, (\delta_1, \cdots, \delta_1)) \le \rho_i^n \le H_i(\lambda, (\alpha - \delta_2, \cdots, \alpha - \delta_2)) \text{ for all } \lambda \in [0,1]. \tag{3.50}$$

We can now apply the same reasoning as in Lemma 3.47 using the open set

$$D = \{\rho \in \mathbb{R}^{|I|} : \rho_i \in (\tfrac{\delta_1}{2}, \alpha - \tfrac{\delta_2}{2})\}.$$

If $\rho^n = H(\lambda, \rho)$ because of the sub and super solution and Lemma 3.41 we know $\rho \notin \partial D$. We recover a solution $\rho \in D$. □

We now use the combination of all these Lemmas to prove the main result.

Proof of Theorem 3.13. If $\rho^0 \in \mathcal{A}_{\Delta,+}$ existence for (P^Δ) and ($\mathrm{P}^\Delta_\varepsilon$) follows as in Lemma 3.48 and uniqueness by (3.41). We due to Lemma 3.43 we can also build uniform sub and super solutions that they are uniform in $\varepsilon \in [0,1]$. Due to compactness, any $\varepsilon_k \to 0$ has a subsequence so that $J^\Delta_{\varepsilon_k} \rho^0 \to u$. We now we use the uniform convergence over compacts set of $U'_\varepsilon \to U'$ (Lemma 3.28) to show it is the unique solution to (P^Δ), so $u = J^\Delta \rho^0$. Since every sequence has a convergent sub-sequence converging, and they all share the same limit, then the whole sequence $J^\Delta_\varepsilon \rho^0$ converges to $J^\Delta \rho^0$ as $\varepsilon \to 0$. If $\rho_0 \in \mathcal{A}_\Delta$ and $U' \in C^1([0,1])$ existence follows by Lemma 3.47 and uniqueness by Lemma 3.41. In both cases we can use Lemma 3.40 to prove L^1_Δ contraction and Lemma 3.45 to show that there is free-energy dissipation. □

Remark 3.49. *In [19] the authors have a more general setting where $\xi_i = U'(\rho_i) + V(x_i) + \sum_j W(x_i - x_j)\rho_j$. For bounded domains, however, it is more natural to write $\xi_i = U'(\rho_i) + V(x_i) + \sum_j K(x_i, x_j)\rho_j$ where $K(x_i, x_j) = K(x_j, x_i)$. The proof of existence is still valid. Our proof of uniqueness uses strongly the monotonicity of the problem (i.e., the existence of a comparison principle), which does not hold for general W. Hence, when $K \neq 0$ but it is smooth, existence and uniqueness can be obtained using* (3.47) *to prove continuous dependence with respect to ∇V, and arguing by fixed-point for Δt small.*

We now discuss convergence of discrete solutions of the scheme (P^Δ) to solutions of the continuous problem (P) (including both, $\varepsilon > 0$ and $\varepsilon = 0$) under high regularity of the solution. In particular, this completes the analysis of the left face of the diagram (D_2).

Proof of Theorem 3.14. We will simply check that $u_i^n = \rho(t_n, x_i)$ has the correct consistency rate. To avoid confusion, we denote the velocity and the flux of the exact solutions by

$$\mathfrak{v}(t,x) = -\nabla(U(\rho)+V), \qquad \mathfrak{F} = \mathrm{m}^{(1)}(\rho)\mathrm{m}^{(2)}(\rho)\mathfrak{v}.$$

It is easy to see that

$$|v^n_{i+\frac12} - \mathfrak{v}(t_n, x_{i+\frac12})| \le C([U'']_{C^\gamma}, \|\nabla\rho\|_{L^\infty}, [\nabla V]_{C^\gamma})(\Delta x)^\gamma. \tag{3.51}$$

Similarly, we can show that

$$\left| \frac{v^n_{i+\frac12} - v^n_{i-\frac12}}{\Delta x} - \operatorname{div}\mathfrak{v}(t_n, x_i)\right| \le C([U''']_{C^\gamma}, [\nabla\rho]_{C^{1,\gamma}}, [\nabla V]_{C^\gamma})(\Delta x)^\gamma. \tag{3.52}$$

We write the decomposition

$$\operatorname{div}\mathfrak{F} = (\mathrm{m}^{(1)})'(\rho)\mathrm{m}^{(2)}(\rho)\nabla\rho\cdot\mathfrak{v} + \mathrm{m}^{(1)}(\rho)(\mathrm{m}^{(2)})'(\rho)\nabla\rho\cdot\mathfrak{v} + \mathrm{m}^{(1)}(\rho)\mathrm{m}^{(2)}(\rho)\operatorname{div}\mathfrak{v}.$$

We separate four cases. First, let us consider $v^n_{i+\frac12}, v^n_{i-\frac12} \ge 0$. Then, we have

$$\begin{aligned}\frac{F^n_{i+\frac12} - F^n_{i-\frac12}}{\Delta x} &= \frac{1}{\Delta x}\Big(\mathrm{m}^{(1)}(u_i^n)\mathrm{m}^{(2)}(u^n_{i+1})v^n_{i+\frac12} - \mathrm{m}^{(1)}(u^n_{i-1})\mathrm{m}^{(2)}(u^n_i)v^n_{i-\frac12}\Big)\\ &= \frac{\mathrm{m}^{(1)}(u_i^n) - \mathrm{m}^{(1)}(u^n_{i-1})}{\Delta x}\mathrm{m}^{(2)}(u^n_{i+1})v^n_{i+\frac12} + \mathrm{m}^{(1)}(u^n_{i-1})\frac{\mathrm{m}^{(2)}(u^n_{i+1}) - \mathrm{m}^{(2)}(u^n_i)}{\Delta x}v^{\varepsilon,n}_{i+\frac12}\\ &\quad + \mathrm{m}^{(1)}(u^n_{i-1})\mathrm{m}^{(2)}(u^n_i)\frac{v^n_{i+\frac12} - v^n_{i-\frac12}}{\Delta x}.\end{aligned}$$

Therefore, using the regularity and the previous estimates (3.51) and (3.52) we obtain that

$$\left|\frac{F^n_{i+\frac12} - F^n_{i-\frac12}}{\Delta x} - \operatorname{div}\mathfrak{F}(t_n, x_i)\right| \le C(\Delta x)^\gamma. \tag{3.53}$$

Similarly, if $v^n_{i+\frac12}, v^n_{i-\frac12} \le 0$. The third case is $v^n_{i+\frac12} \ge 0 \ge v^n_{i-\frac12}$. In this setting,

$$\begin{aligned}\frac{F^n_{i+\frac12} - F^n_{i-\frac12}}{\Delta x} &= \frac{1}{\Delta x}\Big(\mathrm{m}^{(1)}(u_i^n)\mathrm{m}^{(2)}(u^n_{i+1})v^n_{i+\frac12} - \mathrm{m}^{(1)}(u^n_i)\mathrm{m}^{(2)}(u^n_{i-1})v^n_{i-\frac12}\Big)\\ &= \mathrm{m}^{(1)}(u_i^n)\frac{\mathrm{m}^{(2)}(u^n_{i+1}) - \mathrm{m}^{(2)}(u^n_{i-1})}{\Delta x}v^n_{i+\frac12} + \mathrm{m}^{(1)}(u^n_i)\mathrm{m}^{(2)}(u^n_{i-1})\frac{v^n_{i+\frac12} - v^n_{i-\frac12}}{\Delta x}.\end{aligned}$$

Due to the change of sign and the similarity to $\mathfrak{v}(t_n, x_{i+\frac12})$ we deduce $|\mathfrak{v}(t_n, x_{i+\frac12})|, |v^n_{i+\frac12}| \le C(\Delta x)^\gamma$. Using this estimate,

$$\left|\mathrm{m}^{(1)}(u_i^n)\frac{\mathrm{m}^{(2)}(u^n_{i+1}) - \mathrm{m}^{(2)}(u^n_{i-1})}{\Delta x}v^n_{i+\frac12}\right| \le \mathrm{m}^{(1)}(\alpha)\|(\mathrm{m}^{(2)})'\|_{L^\infty}\|\nabla\rho\|_{L^\infty}C(\Delta x)^\gamma.$$

The same happens in the corresponding terms of $\operatorname{div}\mathfrak{F}$. Hence, we have (3.53). The final case is $v^n_{i+\frac12} \le 0 \le v^n_{i-\frac12}$, which is analogous to the third. Thus, we have that

$$\begin{aligned}\left|\frac{u_i^{n+1} - u_i^n}{\Delta t} + \frac{F^n_{i+\frac12} - F^n_{i-\frac12}}{\Delta x}\right| &\le \left|\frac{u_i^{n+1} - u_i^n}{\Delta t} - \frac{\partial\rho}{\partial t}\right| + \left|\frac{\partial\rho}{\partial t} - \operatorname{div}\mathfrak{F}\right| + \left|\frac{F^n_{i+\frac12} - F^n_{i-\frac12}}{\Delta x} - \operatorname{div}\mathfrak{F}^{(\varepsilon)}(t_n, x_i)\right|\\ &\le C((\Delta t)^\beta + (\Delta x)^\gamma).\end{aligned}$$

We deduce the estimate in the statement from the stability of the numerical scheme. $\square$

3.7.2 Long-time behaviour for the numerical method

In this subsection we study the long-time behaviour of the numerical method. First we focus on the regularised problem ($\mathrm{P}_\varepsilon^\Delta$) for $\varepsilon > 0$. The long-time behaviour analysis we study here is based on the gradient flow structure of the problem. We explain this in more detail in the following remark.

Remark 3.50. *Notice that we can write*

$$\begin{aligned} F_{i+\frac{1}{2}}^{\varepsilon,n+1} &= \Theta_{i+\frac{1}{2}}^{\varepsilon,n+1} v_{i+\frac{1}{2}}^{\varepsilon,n+1}, \\ \Theta_{i+\frac{1}{2}}^{\varepsilon,n+1} &= \mathrm{m}_\varepsilon^{(1)}(\rho_i^{\varepsilon,n+1})\mathrm{m}_\varepsilon^{(2)}(\rho_{i+1}^{\varepsilon,n+1})\mathrm{sign}^+(v_{i+\frac{1}{2}}^{\varepsilon,n+1}) - \mathrm{m}_\varepsilon^{(1)}(\rho_{i+1}^{\varepsilon,n+1})\mathrm{m}_\varepsilon^{(2)}(\rho_i^{\varepsilon,n+1})\mathrm{sign}^-(v_{i+\frac{1}{2}}^{\varepsilon,n+1}). \end{aligned}$$

From the result on energy dissipation obtained at Lemma 3.45 and the notation that we use for the problem ($\mathrm{P}_\varepsilon^\Delta$), it follows that

$$0 \le \Delta t \Delta x \sum_{m=n}^{n+k-1} \sum_i \Theta_{i+\frac{1}{2}}^{\varepsilon,m+1} \left| v_{i+\frac{1}{2}}^{\varepsilon,m+1} \right|^2 \le E_\varepsilon^\Delta[\rho^{\varepsilon,n}] - E_\varepsilon^\Delta[\rho^{\varepsilon,n+k}]. \tag{3.54}$$

Proof of Theorem 3.15–Item i. If $H^\varepsilon(\rho) = \rho$, the free energy dissipation states that $F_{i+\frac{1}{2}}^\varepsilon(\rho) = 0$. Since $0 < \rho_i < \alpha$ for all $i \in I$, we get $v_{i+\frac{1}{2}}^\varepsilon(\rho) = 0$ for all $i \in I$, and thus $\xi_i(\rho)$ is constant. This concludes the proof. □

Lemma 3.51. *For $\rho \in \mathbb{R}^{|I|}$, the discrete version of the $W^{-1,1}$ norm, i.e.,*

$$\|\rho\|_{W_\Delta^{-1,1}(0,1)} = \inf \left\{ \Delta x \sum_{i=0}^{N} |F_{i+\frac{1}{2}}| : \text{for each } i \in I \text{ we have that } \rho_i = \frac{F_{i+\frac{1}{2}} - F_{i-\frac{1}{2}}}{\Delta x} \right\}, \tag{3.55}$$

defines a norm.

Proof. The homogeneity with constants and the triangle inequality are obvious. Lastly, if $\|u\|_{W_\Delta^{-1,1}} = 0$ then $F = 0$ so $u = 0$. Therefore, $\|\cdot\|_{W_\Delta^{-1,1}}$ is a norm. □

All norms in $\mathbb{R}^{|I|}$ are equivalent, which is a great advantage of the discrete setting. Notice that, in the definition (3.55), we have not specified that $F_{\frac{1}{2}}$ or $F_{N+\frac{1}{2}}$ vanish, but this is allowed. Lemma 3.51 is a key step in the proof of the next result.

Proof of Theorem 3.15–Item ii. We divide the proof in several steps.

Step 1: The constant is determined uniquely by $\sum_i \rho^{\varepsilon,\infty}$. Due to the no-flux boundary condition it follows immediately that the mass is preserved, i.e.,

$$\Delta x \sum_{i=1}^{|I|} \rho_i^{\varepsilon,n+1} = \Delta x \sum_{i=1}^{|I|} \rho_i^{\varepsilon,n} = \int_0^1 \rho_0 \quad \text{for every } n.$$

Due to the convergence in $\mathbb{R}^{|I|}$, $\rho^{\varepsilon,\infty}$ also satisfies the mass condition. Define the function

$$P(C) = \Delta x \sum_{i=1}^{|I|} (U_\varepsilon')^{-1} \left(C - V(x_i) \right).$$

It is easy to see that it is the strictly monotone, continuous, with $P(-\infty) = -\infty$, and $P(\infty) = \infty$. Therefore, the mass condition

$$\Delta x \sum_{i=1}^{|I|} (U_\varepsilon')^{-1} \left(C_\varepsilon^\Delta - V(x_i) \right) = \int_0^1 \rho_0 \, \mathrm{d}x$$

determines C_ε^Δ uniquely.

Step 2: Convergence in time. Due to the no-flux assumption of our problem

$$\sum_i |\rho_i^{\varepsilon,n}| = \sum_i |\rho_i^0| \quad \text{for all } n.$$

Therefore, up to a subsequence n_j, there exists $u \in \mathbb{R}^{|I|}$ such that $\rho^{\varepsilon,n_j} \to u \quad$ in $\mathbb{R}^{|I|}$. From the construction of the norm $W_\Delta^{-1,1}$, the sequence $\rho^{\varepsilon,n}$ satisfies,

$$\begin{aligned}\left\| \frac{\rho^{\varepsilon,n+1}-\rho^{\varepsilon,n}}{\Delta t} \right\|^2_{W_\Delta^{-1,1}(0,1)} &\le \left(\Delta x \sum_{i\in I} |F^{\varepsilon,n+1}_{i+\frac12}| \right)^2 \le (\Delta x)|I|(\Delta x) \sum_{i\in I} |F^{\varepsilon,n+1}_{i+\frac12}|^2 \\ &= \Delta x \sum_{i\in I} (\Theta^{\varepsilon,n+1}_{i+\frac12})^2 |v^{\varepsilon,n+1}_{i+\frac12}|^2.\end{aligned}$$

Hence, using (3.54) and the fact that $|\Theta^{\varepsilon,n}_{i+\frac12}| \le \mathrm{m}_\varepsilon^{(1)}(\alpha)\mathrm{m}_\varepsilon^{(2)}(0)$, we recover

$$\|\rho^{\varepsilon,n+1}-\rho^{\varepsilon,n}\|_{W_\Delta^{-1,1}(0,L)} \le C(\Delta t)^{\frac12} \left(\Delta t \Delta x \sum_{i\in I} \Theta^{\varepsilon,n+1}_{i+\frac12} |v^{\varepsilon,n+1}_{i+\frac12}|^2 \right)^{\frac12} \le C(\Delta t)^{\frac12} (E_\varepsilon^\Delta[\rho^{\varepsilon,n}] - E_\varepsilon^\Delta[\rho^{\varepsilon,n+1}])^{\frac12}.$$

We recall that $E_\varepsilon^\Delta[\rho^{\varepsilon,n}]$ is a non-increasing sequence due to Lemma 3.45. Furthermore, it is also bounded from below by $\min_{s\in[0,\alpha]} U_\varepsilon(s) > -\infty$. Hence, there exists $E_\varepsilon^{\Delta,\infty} \in \mathbb{R}$ such that as $n \to \infty$

$$E_\varepsilon^\Delta[\rho^{\varepsilon,n}] \searrow E_\varepsilon^{\Delta,\infty}.$$

Hence, it follows that

$$\lim_{k\to\infty} \|\rho^{\varepsilon,n_k} - \rho^{\varepsilon,n_k+1}\|_{W_\Delta^{-1,1}(0,L)} \le C(\Delta t)^{\frac12} \lim_{k\to\infty} (E_\varepsilon^\Delta[\rho^{\varepsilon,n_k}] - E_\varepsilon^\Delta[\rho^{\varepsilon,n_k+1}])^{\frac12} = 0.$$

Therefore, we obtain that $\rho^{\varepsilon,n_k+1} \to u$ in $\mathbb{R}^{|I|}$.

Step 3: Stationary sub and super solution. Since $\rho^0 \in \mathcal{A}_{\Delta,+}$ we can pick $C_1, C_2 \in \mathbb{R}$ such that

$$(U_\varepsilon')^{-1}(C_1 - V(x_i)) \le \rho_i^0 \le (U_\varepsilon')^{-1}(C_2 - V(x_i)), \qquad \forall i \in I.$$

These upper and lower bounds are stationary solutions, and we have a comparison principle, they preserved for all time. Thus, we have that $\rho_i^{\varepsilon,n} \in [\delta, \alpha-\delta]$ for some $\delta > 0$ and all n, i.

Step 4: Solution of the scheme ($\mathrm{P}_\varepsilon^\Delta$)*.* Due the convergence and the continuity of the mobility $\mathrm{m}_\varepsilon^{(j)}$ (with $j = 1,2$) and U_ε' in $[\delta, \alpha-\delta]$, it follows that

$$\mathrm{m}_\varepsilon^{(j)}(\rho^{\varepsilon,n_k+1}) \to \mathrm{m}_\varepsilon^{(j)}(u), \quad U_\varepsilon'(\rho^{\varepsilon,n_k+1}) \to U_\varepsilon'(u) \quad \text{in } \mathbb{R}^{|I|}.$$

Since $H^\varepsilon(\rho^{\varepsilon,n_k+1}) = \rho^{\varepsilon,n_k}$, we have that u satisfies $H^\varepsilon(u) = u$. Since $0 < u_i < \alpha$ for all $i \in I$ we conclude that for all $i \in I$ we have $u_i = (U_\varepsilon')^{-1}(C - V(x_i))$ for some C. By conservation of mass we precisely characterise $u = \rho^{\varepsilon,\infty}$.

Step 5: The whole sequence converges. Since every sequence has a convergent subsequence, and they all share the same limit. □

We connect the discrete and continuous stationary states, and we provide a rate of convergence on Δx. We start with the case $\varepsilon > 0$ fixed.

Proof of Theorem 3.15–Item iii. Let us remark that,

$$\Delta x \sum_i \left(U_\varepsilon'\right)^{-1}\left(C_\varepsilon^\Delta - V(x_i)\right) = M = \int_0^1 \left(U_\varepsilon'\right)^{-1}\left(C_\varepsilon - V(x)\right) \mathrm{d}x. \tag{3.56}$$

Since we know that $0 < \widehat{\rho}^{(\varepsilon)} < \alpha$, in particular

$$\left| \left(\left(U_\varepsilon'\right)^{-1}\right)' \left(C_\varepsilon - V(x_i)\right)\right| \leq C \quad \forall\, i \in I.$$

Since $\widehat{\rho}^{(\varepsilon)} \in \mathcal{A}_+$, we can perform a first order Taylor expansion to obtain that for $h \in [0, \Delta x)$,

$$\widehat{\rho}^{(\varepsilon)}(x_i + h) = \left(U_\varepsilon'\right)^{-1}\left(C_\varepsilon - V(x_i)\right) + O(h).$$

Then, in particular,

$$\int_0^1 \left(U_\varepsilon'\right)^{-1}\left(C_\varepsilon - V(x)\right) \mathrm{d}x = \Delta x \sum_i \Big(\left(U_\varepsilon'\right)^{-1}\left(C_\varepsilon - V(x_i)\right) + O(\Delta x)\Big). \tag{3.57}$$

Combining (3.56) and (3.57) it follows that,

$$O(\Delta x) = \Delta x \sum_i \Big(\left(U_\varepsilon'\right)^{-1}\left(C_\varepsilon^\Delta - V(x_i)\right) - \left(U_\varepsilon'\right)^{-1}\left(C_\varepsilon - V(x_i)\right)\Big) = \Delta x \sum_i \left(\left(U_\varepsilon'\right)^{-1}\right)'(\zeta_i)(C_\varepsilon^\Delta - C_\varepsilon),$$

with $\zeta_i \in (C_\varepsilon^\Delta - V(x_i), C_\varepsilon - V(x_i))$. It satisfies, $0 < \underline{C} \leq \left((U_\varepsilon')^{-1}\right)'(\zeta_i) \leq \overline{C} < \infty$, for all $i \in I$. From here, it follows,

$$\left|C_\varepsilon^\Delta - C_\varepsilon\right| = \left| \frac{O(\Delta x)}{\Delta x \sum_i \left((U_\varepsilon')^{-1}\right)'(\zeta_i)} \right| \leq \left| \frac{O(\Delta x)}{\Delta x \sum_i \underline{C}} \right| = O(\Delta x). \qquad \square$$

Once we have understood the long-time behaviour of the regularised numerical method ($\mathrm{P}_\varepsilon^\Delta$), we now proceed to study (P^Δ) and the properties of its long-time behaviour.

Proof of Theorem 3.16. Theorem 3.16–Item i follows analogously to Theorem 3.15–Item ii. Since J^Δ is an L^1_Δ contraction, so is $(J^\Delta)^n$, and hence $J^{\Delta,\infty}$. Theorem 3.16–Item ii follows as $n \to \infty$.

Theorem 3.16–Item iii. Constant-in-time solution to (P^Δ). The first part of the proof is analogous to the one in the continuous case done for Theorem 3.10. In order to finish the proof, we show that $\rho_i^{0,\infty} = T_{0,\alpha} \circ (U')^{-1}(C_0^\Delta - V(x_i))$ is a stationary state at every point x_i for $i \in I$. The argument that we consider is the following: $\rho^{0,\infty}$ is a stationary state if and only if $F_{i\pm\frac{1}{2}} = 0$ for all $i \in I$.

Let us assume $\rho_i^{0,\infty} = 0$ and $0 < \rho_{i+1}^{0,\infty} < \alpha$. This means that $C_0^\Delta - V(x_i) \leq \underline{\zeta} = U'(0^+) < C_0^\Delta - V(x_{i+1}) < \overline{\zeta} = U'(\alpha^-)$. Hence, we can compute that

$$v_{i+\frac{1}{2}}(\rho^{0,\infty}) = -\frac{U'(\rho_{i+1}^{0,\infty}) + V(x_{i+1}) - (U'(\rho_i^{0,\infty}) + V(x_i))}{\Delta x} = \frac{U'(0) + V(x_i) - C_0^\Delta}{\Delta x} \geq 0,$$

where, in the last inequality, we are using that U' is increasing. Since $\mathrm{m}_\varepsilon^{(1)}(0) = 0$, we get that

$$F_{i+\frac{1}{2}}(\rho^{0,\infty}) = \mathrm{m}_\varepsilon^{(1)}(\rho_i^{0,\infty})\mathrm{m}_\varepsilon^{(2)}(\rho_{i+1}^{0,\infty})(v_{i+\frac{1}{2}}(\rho^{0,\infty}))^+ = 0,$$

and $\rho^{0,\infty}$ is a stationary state at the point x_i. Arguing analogously we can prove the same result for all the different combinations of $\rho_i^{0,\infty}$ and $\rho_{i+1}^{0,\infty}$ taking values 0, α, or in $(0, \alpha)$.

Theorem 3.16–Item iv. Convergence of numerical steady states as $\varepsilon \to 0$. We argue analogously to Lemma 3.37 to show that there exists ε_k such that $C^{\Delta}_{\varepsilon_k} \to \widetilde{C}$ as $k \to \infty$ and

$$(U'_{\varepsilon_k})^{-1}(C^{\Delta}_{\varepsilon_k} - V(x)) \to T_{0,\alpha} \circ (U')^{-1}(\widetilde{C} - V(x)) \quad \text{pointwise in } [0,\alpha]. \tag{3.58}$$

Due to the mass condition $\widetilde{C} = C^{\Delta}_0$. As usual, we realise that every sequence ε_k has a subsequence where (3.58) holds, and the limit is shared amongst convergent sequences. This proves the convergence as $\varepsilon \to 0$.

Theorem 3.16–Item v. Convergence of the numerical solution steady state as $\Delta \to 0$. Convergence without rates follows as in the previous step. Let us now prove rates of convergence. Since $(U')^{-1}$ is Hölder, then $T_{0,\alpha} \circ (U')^{-1}$ is also Hölder. Let us also assume we are in the case $0 < M < \alpha$. This second step of the proof works analogously to the proof of Theorem 3.15–Item iii. We just need to adapt the Taylor expansion to its Hölder version. If we do that we arrive at

$$\begin{aligned} O((\Delta x)^{\gamma}) &= \int_0^1 (T_{0,\alpha} \circ (U')^{-1}(C^{\Delta}_0 - V(x)) - T_{0,\alpha} \circ (U')^{-1}(C_0 - V(x))) \\ \underline{\zeta} &= (C^{\Delta}_0 - C_0) \int_0^1 (T_{0,\alpha} \circ (U')^{-1})'(\widehat{\zeta}(x))\,\mathrm{d}x, \end{aligned} \tag{3.59}$$

where $\widehat{\zeta}(x)$ lies between $C^{\Delta}_0 - V(x), C_0 - V(x)$. Since $0 < M < \alpha$, there exists $z \in (0,1)$ such that $U'(0^+) =: \underline{\zeta} < C_0 - V(z) < \overline{\zeta} := U'(\alpha^-)$. Let us define

$$\ell := \min\left\{(C_0 - V(z)) - \underline{\zeta}, \overline{\zeta} - (C_0 - V(z))\right\}.$$

From the previous step, we know that there exists $\delta > 0$ small enough such that for every $\Delta x < \delta$ we have that $|C^{\Delta}_0 - C_0| \le \frac{\ell}{4}$. From the continuity of V, there exists an interval $K \subseteq (0,1)$ small enough such that for every $x \in K$ we have that $|V(z) - V(x)| \le \frac{\ell}{4}$. Therefore, it follows that $|\widehat{\zeta}(x) - \underline{\zeta}|, |\overline{\zeta} - \widehat{\zeta}(x)| \ge \frac{\ell}{2} > 0$ for all $x \in K$. In particular, $(U')^{-1}\left(\left\{\widehat{\zeta}(x) : x \in K\right\}\right) \subseteq [s_1, s_2]$, a compact subset of $(0,\alpha)$. Then, we obtain that

$$\begin{aligned} \int_0^1 (T_{0,\alpha} \circ (U')^{-1})'(\widehat{\zeta}(x))\,\mathrm{d}x &\ge \int_K ((U')^{-1})'(\widehat{\zeta}(x))\,\mathrm{d}x = \int_K \frac{1}{U''((U')^{-1}(\widehat{\zeta}(x)))}\,\mathrm{d}x \\ &\ge \frac{|K|}{\max_{s \in [s_1,s_2]} U''(s)} > 0. \end{aligned}$$

Thus, combining this bound with (3.59) we have that $|C^{\Delta}_0 - C_0| = O((\Delta x)^{\gamma})$. □

3.7.3 Time asymptotics for $\varepsilon = 0$. The minimiser might not be an attractor

We can reproduce the example at section 3.6.3 to prove that the back face of the diagram (D_2) is not commutative for the numerical scheme (P^{Δ}_{ε}) either. Analogously to the continuous case we construct a counterexample using $U(s) = \frac{1}{m-1}s^m$ and V a double-well potential. For $m > 1$ we consider the discrete Barenblatt

$$\mathcal{B}_i = \left(\frac{m-1}{m}\left(C^{\Delta}_0 - \frac{|x_i|^2}{2}\right)\right)_+^{\frac{1}{m-1}}.$$

We select C^{Δ}_0 such that $\Delta x \sum_{i=1}^{M} \mathcal{B}_i > 1$. Working analogously to the proof of Theorem 3.16 we can show that $\mathcal{B}_i$ is a discrete stationary state. Let us remind the reader that the method (P^{Δ}_{ε}) applies only to dimension 1. Since $m > 1$, $\operatorname{supp} \mathcal{B} \subset \{j : |x_j| < \widehat{R}\}$. Take $L, K \in I$ with $|x_L - x_K| > 2\widehat{R}$, and consider $\overline{\rho}_i = \mathcal{B}_{i-L} + \mathcal{B}_{i-K}$. We choose again the potential,

$$V(x) = \begin{cases} \frac{|x-x_L|^2}{2} & \text{if } |x - x_L| \le \widehat{R} \\ \frac{|x-x_K|^2}{2} & \text{if } |x - x_K| \le \widehat{R} \\ \frac{|x|^2}{2} & \text{if } |x| \gg 1 \\ \text{smooth} & \text{in the intermediate regions} \end{cases}$$

For each $0 \le \rho_i^0 \le \overline{\rho}_i$ we solve (P_ε^Δ) with linear mobility $\mathrm{m}(s) = s$. We are solving (P_ε^Δ) in each interval $(x_j - \widehat{R}, x_j + \widehat{R})$, $j = L, K$. From the Comparison Principle in Theorem 3.13, we get $0 \le \rho_i^n \le \overline{\rho}_i$. Consider

$$\widehat{\rho}_i = \left(\frac{m-1}{m}\left(C_L^\Delta - \frac{|x_i - x_L|^2}{2}\right)\right)_+^{\frac{1}{m-1}} + \left(\frac{m-1}{m}\left(C_K^\Delta - \frac{|x_i - x_K|^2}{2}\right)\right)_+^{\frac{1}{m-1}},$$

and select $C_l^\Delta, C_k^\Delta \le C_0^\Delta$ such that

$$\Delta x \sum_{|x_i - x_K| \le \widehat{R}} \widehat{\rho}_i = \Delta x \sum_{|x_i - x_K| \le \widehat{R}} \rho_i^0, \qquad \Delta x \sum_{|x_i - x_L| \le \widehat{R}} \widehat{\rho}_i = \Delta x \sum_{|x_i - x_L| \le \widehat{R}} \rho_i^0.$$

In particular, $\widehat{\rho}_i$ is such that, $0 \le \widehat{\rho}_i \le \overline{\rho}_i$.

Due to compactness, the scheme ρ_i^n is such that, up to a subsequence, $\rho_i^n \to \xi_i \quad \text{in } \mathbb{R}^{|I|}$. From the Comparison Principle, the only candidate for ξ_i is $\widehat{\rho}_i$ and therefore, up to a subsequence, $\rho_i^n \to \widehat{\rho}_i \quad \text{in } \mathbb{R}^{|I|}$. When

$$\Delta x \sum_{|x_i - x_L| \le \widehat{R}} \rho_i^0 \ne \Delta x \sum_{|x_i - x_K| \le \widehat{R}} \rho_i^0,$$

then $C_L^\Delta \ne C_K^\Delta$.

Remark 3.52. *Following section 3.6.3 we can extend this case to further examples with saturation.*

3.7.4 Numerical experiments

We implement the scheme (P^Δ) using the `julia` language [43]. The fixed point problem for the implicit time-stepping is solved using a Newton method through the `NLSolve.jl` package (see [259]) using automatic differentiation via the `ForwardDiff.jl` package.

Convex potential We exemplify the behaviour for the nonlinear diffusion ρ^2 with a convex potential and free boundaries at levels $\rho = 0, \alpha$ in Figure 3.5.

In the left plot we show the profiles at different times illustrating the formation of the free boundary and the kinks in the upper constraint. In the right plot we display the speed of convergence to the steady state. Numerical experiments indicate that it is exponential as suggested by the linear character of the log plot of the error towards the exact steady solution. For t large the error becomes so small that its log is computationally $-\inf$.

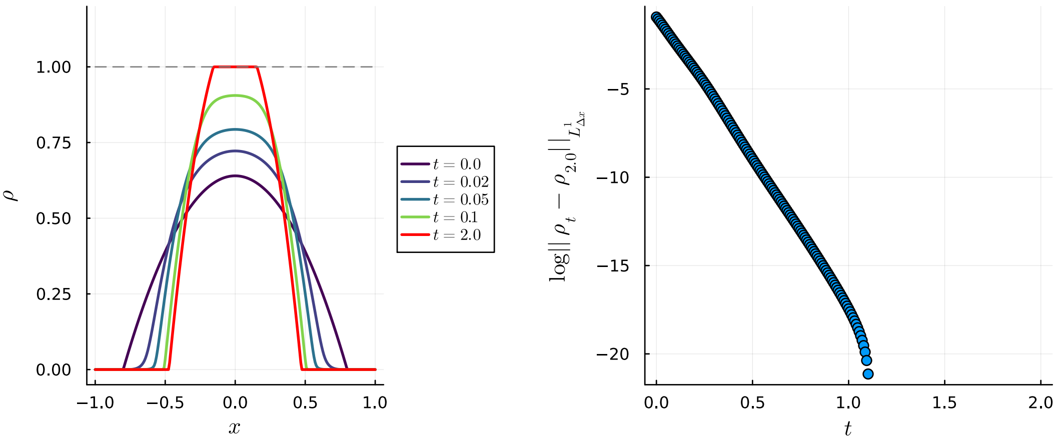


Figure 3.5: $\mathrm{m}(\rho) = \rho(1-\rho)$, $U(\rho) = \rho^2$ and $V(x) = 10x^2$. $\Delta t = \Delta x = 2^{-7}$. Left: profiles at different times. Right: distance from ρ_t to ρ_2.

Double well potential. Formation of Barenblatt from above We exemplify the behaviour for a double well potential where a Barenblatt profile appears from level $\rho = \alpha$ in Figure 3.6. This shows the formation of a gap in the upper free boundary of the problem.

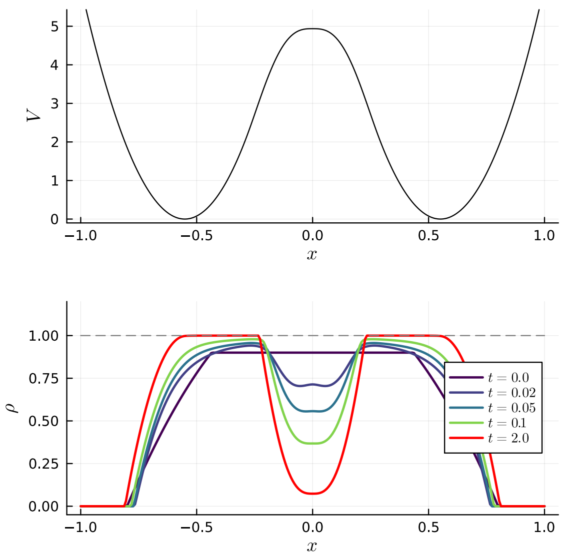


Figure 3.6: $\mathrm{m}(\rho) = \rho(1-\rho)$, $U(\rho) = \rho^2$. $\Delta t = \Delta x = 2^{-7}$

Non-minimising steady states Finally, we show in Figure 3.7 a numerical experiment in which the asymptotic state is a non-minimising linear combination of two Barenblatt profiles with disjoint supports. It illustrates our theoretical findings in Section 3.4 and Theorem 3.6, observing that the global minimizer of the free energy does not always attract all initial data and that these non-minimising steady states have a large basin of attraction.

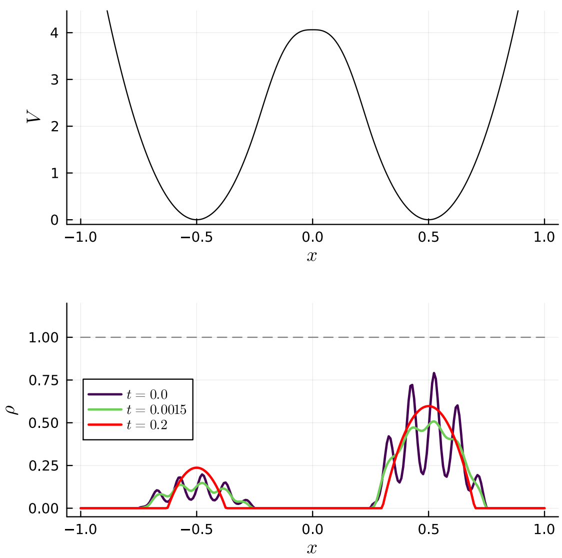


Figure 3.7: $\mathrm{m}(\rho) = \rho(1-\rho)$, $U(\rho) = \rho^2$. $\Delta t = 2^{-12}$, $\Delta x = 2^{-7}$.

4 Competing effects in fourth-order aggregation-diffusion equations

This chapter is taken from the article "Competing effects in fourth-order aggregation-diffusion equations" written in collaboration with José Antonio Carrillo[1], Antonio Esposito[2] and Carles Falcó[1], and published in *Proceedings of the London Mathematical Society*, Volume 129 (2), e12623 (2024); [88].

Contents

The beauty of mathematics only shows itself to more patient followers. – Maryam Mirzakhani

4.1 Introduction and problem setting

In this chapter we are interested in the mathematical analysis of the equation

$$\partial_t \rho = -\operatorname{div}(\rho \nabla(\Delta\rho)) - \chi \Delta \rho^m, \tag{4.1}$$

where $m \geq 1$, and its extension to systems. We look for solutions of (4.1) in the set of probability densities, $\rho \in L^1_+(\mathbb{R}^d) := \{\rho \in L^1(\mathbb{R}^d) : \rho \geq 0\}$, thus setting the mass to one in the sequel without loss of generality. The parameter $\chi > 0$ measures the relative balance between aggregation, modelled by backwards degenerate diffusion, and repulsion, modelled by fourth-order diffusion. The case of general masses can be reduced to (4.1) with a suitable parameter χ upon a standard time rescaling and mass normalisation, cf. Remark 4.12.

Equation (4.1) is related to the classical thin-film equations from lubrication theory, cf. [208, 39, 268, 38, 137, 201] and the references therein. Starting from a conjecture of Hocherman and Rosenau, [208], the authors in [40] study well-posedness and finite-time singularities of Cahn-Hilliard-type equations, in one spatial dimension on bounded interval with periodic boundary conditions. More precisely, they analyse the family of equations of the form

$$\partial_t \rho = -(\rho^n \rho_{xxx})_x - (\rho^{m-1}\rho_x)_x, \tag{4.2}$$

proving that for nonnegative (weak) solutions, blow-up can only occur for $m \geq n+3$. The results in [208, 40] hold for general degenerate mobilities, as in [40, Conjectures 1 and 2]. Afterwards, several contributions to

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the analysis of the one dimensional problem have been made. Linear in-/stability of steady states for the one-dimensional periodic problem was analysed in [235, 292]. Using the dissipation of a suitable energy functional, the authors of [237] were able to further characterise the energy landscape distinguishing between local minima and saddles among periodic steady states. Stability of droplets steady states with a fixed contact angle for the one-dimensional periodic problem was further studied in [236].

The critical case $m = n + 3$ in one dimension is analysed in [308], where blow-up in finite time can only happen above a certain critical mass identified thanks to a sharp Sz.-Nagy inequality, cf. [301, 257]. Existence of selfsimilar blow-up solutions of (4.2) is explored in [293] for the critical case $m = n + 3$. In particular, for $n = 1$, there exists a family of blowing-up symmetric selfsimilar solutions with zero contact angle. Further analysis of one-dimensional self-similar solutions, both expanding and blowing-up, for the critical cases of (4.2) has been done in [179, 178, 292].

The nonlinear Cahn-Hilliard-type equations (4.1) have also been recently proposed as approximations of nonlocal aggregation-diffusion models of the form

$$\frac{\partial \rho}{\partial t} = \Delta \rho^s + \operatorname{div}(\rho \nabla (W * \rho)), \quad s \geq 1, \tag{4.3}$$

by truncation of the Fourier expansion of the interaction potential W, see [35]. This approximation has been rendered rigorous under certain assumptions on the interaction potential W in [172].

The connection between aggregation-diffusion and Cahn-Hilliard equations has also been generalised to systems of aggregation-diffusion equations modelling tissue growth and patterning due to cell-cell adhesion [107]. The authors in [186] show that cell-sorting phenomena are kept for the resulting system of equations:

$$\partial_t \rho = -\operatorname{div}\left(\rho \nabla \left(\kappa \Delta \rho + \alpha \Delta \eta + \beta \rho + \omega \eta\right)\right), \tag{4.4a}$$

$$\partial_t \eta = -\operatorname{div}\left(\eta \nabla \left(\alpha \Delta \rho + \Delta \eta + \omega \rho + \eta\right)\right). \tag{4.4b}$$

The parameters in the model are such that $\beta, \omega \in \mathbb{R}$ and the matrix

$$A = \begin{pmatrix} \kappa & \alpha \\ \alpha & 1 \end{pmatrix},$$

is positive definite. We extend the theory developed for the one-species case (4.1) to construct solutions to the systems of equations (4.4). The nonlocal-to-local limit in the context of systems has also been studied rigorously in [87]. We also mention that different multi-species Cahn-Hilliard equations are considered in [176, 175, 168] and references therein.

Equation (4.1) can be interpreted as 2-Wasserstein gradient flow of the (extended) energy functional

$$\mathcal{F}_m[\rho] = \begin{cases} \frac{1}{2} \int_{\mathbb{R}^d} |\nabla \rho(x)|^2 \, \mathrm{d}x - \chi \mathcal{E}_m[\rho], & \rho \in \mathcal{P}^a(\mathbb{R}^d), \ \nabla \rho \in L^2(\mathbb{R}^d), \\ +\infty, & \text{otherwise}, \end{cases} \tag{4.5}$$

as already noted in [292], being

$$\mathcal{E}_m[\rho] = \begin{cases} \int_{\mathbb{R}^d} \rho(x) \log \rho(x) \, \mathrm{d}x, & m = 1, \\ \frac{1}{m-1} \int_{\mathbb{R}^d} \rho^m \mathrm{d}x, & m > 1. \end{cases} \tag{4.6}$$

This gradient flow structure was made rigorous for related Cahn-Hilliard equations in [252, 245]. However, the former does not include the second-order backwards diffusion term in (4.1), while the latter is concerned with more general, density-dependent, mobilities.

As for the multi-species case, by defining the free energy functional as

$$\tilde{\mathcal{F}}[\rho, \eta] = \int_{\mathbb{R}^d} \left(\frac{\kappa}{2} |\nabla \rho|^2 + \frac{1}{2} |\nabla \eta|^2 + \alpha \nabla \rho \cdot \nabla \eta - \frac{\beta}{2} \rho^2 - \frac{1}{2} \eta^2 - \omega \rho \eta \right) \mathrm{d}x,$$

system (4.4) can be written as a 2-Wasserstein gradient flow with respect to the (extended) free energy functional

$$\boldsymbol{\mathcal{F}}[\rho,\eta]=\begin{cases}\tilde{\boldsymbol{\mathcal{F}}}[\rho,\eta] & \text{if } (\rho,\eta)\in\mathcal{P}^a(\mathbb{R}^d)^2,\ (\nabla\rho,\nabla\eta)\in L^2(\mathbb{R}^d)^2\\ +\infty & \text{otherwise.}\end{cases} \tag{4.7}$$

Our main goal is to show global existence of weak solutions of (4.1) for $m < m_c := 2+\frac{2}{d}$ (subcritical case) and for $m = m_c$ (critical case) for *subcritical mass*, $0<\chi<\chi_c$, by leveraging the aforementioned gradient flow structure. The critical parameter χ_c is identified by the sharp constant of a suitable functional inequality [246]. The critical exponent m_c is determined by scaling arguments using mass-preserving dilations of densities in the energy functional (4.5). Moreover, we also obtain global existence of weak solutions for the system (4.4) by an analogous approach. In fact, we employ the (by now) classical variational minimising movement scheme, or JKO scheme, [215, 7] to obtain an approximation of a candidate solution. A crucial step will be to use the flow interchange technique, developed in [252, 245], to gain suitable regularity. Afterwards, we check that limits of the variational scheme are indeed weak solutions in any dimension.

Our main result provides sharp conditions on the exponent of backwards diffusion in (4.1) to ensure global existence of solutions in the natural class of initial data for any dimension compared to previous literature [308, 292, 245, 247].

The key ingredient to take advantage of the gradient flow structure of (4.1) and system (4.4) is to have uniform bounds on the competing terms in the free energies (4.5) and (4.7), respectively. Interestingly, this is reminiscent of similar arguments developed for generalisations of the Patlak-Keller-Segel equation for chemotactic cell movement [48, 78, 103]. Actually, we can draw a nice parallelism with this well studied problem. Generalised Patlak-Keller-Segel equations are of the form (4.3). In particular, let us focus on the power-law kernel

$$W_k(x)=\begin{cases}\dfrac{|x|^k}{k} & \text{if } k\neq 0,\\ \log|x| & \text{if } k=0.\end{cases}$$

We find an immediate connection with the problem (4.1). Analogously to the case we are studying in this work, there exists a critical exponent, $s_c = 1-\frac{k}{d}$, also found via mass-preserving dilations on the corresponding free energy functional which characterises the behaviour of (4.3).

The case $s > s_c$ is the diffusion dominated regime and global well-posedness for (4.3) is expected, see for instance [65, 298, 48, 66, 67, 99]. This is analogous to the case $1 \le m < m_c$ for (4.1).

As for the range $1 \le s < s_c$, aggregation-dominated regime for Eq. (4.3) — analogous to the case $m > m_c$ for (4.1) — coexistence of blow-up and global existence depending on the initial data is expected, see [249, 29, 120] for instance.

In the fair competition regime $s = s_c$ — analogous to our critical exponent $m = m_c$ — there exists a dichotomy between aggregation and diffusion in terms of the *initial mass*: M, analogous to our parameter χ. Sharp constants of variants of Hardy-Littlewood-Sobolev type inequalities determine the critical value of the mass M_c for (4.3), analogously to our critical parameter χ_c. We note that for our fourth-order Cahn-Hilliard type equation, the crucial functional inequality was established in [246], see a limiting case in [248]. In the supercritical mass case, $M > M_c$, there exists solutions that blow-up in finite time, see for instance [48, 27, 29, 66, 67]. In the subcritical mass case, $M < M_c$, global existence of solutions is shown and spreading self-similar solutions are expected to attract the long time dynamics, see for instance [163, 50, 29, 27, 66, 67].

In the critical case $M = M_c$, there are infinitely many stationary states given by the optimisers of the variants of the HLS inequalities, solutions are globally well-posed blowing-up at infinite time for bounded second moment initial data if $m = 1$, and local stability of stationary solutions is expected, see [163, 48, 47, 309, 66, 67].

We shall perform a parallel study to nonlinear Keller-Segel equations (4.3) for our family of Cahn-Hilliard equations (4.1), depending on the critical exponent case m_c and parameter χ.

Finally, we want to emphasize that our work sets the path to many other interesting open questions. Uniqueness is widely open being the functionals not convex, even in subsets, in any obvious manner. Existence of minimisers in the subcritical case in the whole space is not clear since we do not know at present how to bound uniformly in time the second moment or any other quantity controlling escape of mass at infinity. Long time asymptotics are, in turn, widely open in all global existence cases. Free boundary problem techniques could help understand if the evolution leads to compactly supported solutions corresponding to compactly supported initial data. This conjecture is corroborated by numerical experiments being this another challenging problem. In the two-species case, we can identify other interesting issues such as sharp segregation for specific parameter values between the two species not only at steady states but along their evolutions. This information is important for the applications in mathematical biology [107, 186].

Structure of the chapter. We structure the chapter as follows. Section 4.2 is devoted to the precise statements of the main theorems together with some preliminary material used in the sequel. We will analyse the existence of global minimisers of the energy (4.5) following the strategies in [163, 48, 66] in Section 4.3. In Section 4.4 we deal with the core main result of global existence of weak solutions to the single equation (4.1) in any dimension for generic initial data. Finally, Section 4.5 focuses on the generalisation of this approach to the case of systems of the form (4.4).

4.2 Main results and preliminaries

We begin by listing the main results covered in this chapter. First we study some properties of the free energy functional $\mathcal{F}_m$ and its minimisers. The following theorem summarises the results proven in Section 4.3.

Theorem 4.1. *Let $\mathcal{F}_m$ be as in (4.5). The following holds:*

(1) *If $1 \leq m < 2 + 2/d$, then $\mathcal{F}_m$ is bounded from below.*

(2) *If $m = 2 + 2/d$, then, for the subcritical and critical mass regimes, $\mathcal{F}_m$ is bounded from below. Furthermore, for the critical mass, the infimum is achieved. In the supercritical mass regime, $\mathcal{F}_m$ is unbounded from below.*

(3) *If $m > 2 + 2/d$, then $\mathcal{F}_m$ is unbounded from below.*

Case (1) is proven in Proposition 4.11. Case (2) is a combination of two results. In Proposition 4.13 we show that for $\chi \leq \chi_c$ the free energy is bounded from below. In Proposition 4.15, we prove that the infimum is achieved for critical mass and that the free energy is unbounded if $\chi > \chi_c$. Finally, in Proposition 4.17 we show case (3).

Throughout the chapter, we denote by $\mathcal{P}(\mathbb{R}^d)$ the set of probability measures on $\mathbb{R}^d$, for $d \in \mathbb{N}$, and by $\mathcal{P}_2(\mathbb{R}^d) := \{\rho \in \mathcal{P}(\mathbb{R}^d) : \mathrm{m}_2(\rho) < +\infty\}$, being $\mathrm{m}_2(\rho) := \int_{\mathbb{R}^d} |x|^2 \,\mathrm{d}\rho(x)$ the 2^{nd}-order moment of ρ. We shall use $\mathcal{P}^a(\mathbb{R}^d)$ and $\mathcal{P}_2^a(\mathbb{R}^d)$ for elements in $\mathcal{P}(\mathbb{R}^d)$ and $\mathcal{P}_2(\mathbb{R}^d)$ which are absolutely continuous with respect to the Lebesgue measure. In order to deal with L^p-regularity, we set

$$2^* := \begin{cases} +\infty & \text{if } d = 1, 2, \\ 2^* = \frac{2d}{d-2} & \text{if } d \geq 3. \end{cases}$$

The second result we prove is the existence of weak solutions to (4.1), in the following sense:

Definition 4.2 (Weak solution)**.** *A weak solution to (4.1) on the time interval $[0, T]$, with initial datum $\rho_0 \in \mathcal{P}_2^a(\mathbb{R}^d)$ such that $\nabla\rho_0 \in L^2(\mathbb{R}^d)$, is a narrowly continuous curve $\rho : [0, T] \to \mathcal{P}_2(\mathbb{R}^d)$ satisfying the following properties:*

i) *$\rho \in L^\infty([0, T]; L^p(\mathbb{R}^d)) \cap L^\infty([0, T]; H^1(\mathbb{R}^d)) \cap L^2([0, T]; H^2(\mathbb{R}^d))$, for any $p \in [1, 2^*]$, $d \neq 2$, and for any $p \in [1, 2^*)$ when $d = 2$;*

ii) for every $\varphi \in C_c^2(\mathbb{R}^d)$ and every $0 \le s_1 < s_2 \le T$ it holds

$$\int_{\mathbb{R}^d} \varphi(x)\rho(s_2,x)\,\mathrm{d}x = \int_{\mathbb{R}^d} \varphi(x)\rho(s_1,x)\,\mathrm{d}x - \int_{s_1}^{s_2}\int_{\mathbb{R}^d} \left(\rho\Delta\rho\Delta\varphi + \Delta\rho\nabla\rho\cdot\nabla\varphi\right)\,\mathrm{d}x\,\mathrm{d}t$$
$$- \chi \int_{s_1}^{s_2}\int_{\mathbb{R}^d} \rho^m \Delta\varphi\,\mathrm{d}x\,\mathrm{d}t.$$

Theorem 4.3. *Assume $1 \le m < 2 + 2/d$ or $m = 2 + 2/d$ with subcritical mass $\chi < \chi_c$ and let $\rho_0 \in \mathcal{P}_2^a(\mathbb{R}^d)$ be an initial datum such that $\mathcal{F}_m[\rho_0] < +\infty$. Then there exists a weak solution to* (4.1).

We extend our results from the one species case to construct weak solutions to system (4.4), in the following sense.

Definition 4.4 (Weak solution for the system)**.** *A weak solution to* (4.4) *on the time interval $[0,T]$, with initial datum $\sigma_0 = (\rho_0, \eta_0) \in \mathcal{P}_2^a(\mathbb{R}^d)^2$ such that $\nabla\rho_0, \nabla\eta_0 \in L^2(\mathbb{R}^d)$, consists of a pair of narrowly continuous curves $\rho, \eta : [0,T] \to \mathcal{P}_2(\mathbb{R}^d)$ satisfying the following properties:*

i) $\rho, \eta \in L^\infty([0,T]; L^p(\mathbb{R}^d)) \cap L^\infty([0,T]; H^1(\mathbb{R}^d)) \cap L^2([0,T]; H^2(\mathbb{R}^d))$, for any $p \in [1, 2^*]$, $d \ne 2$, and for any $p \in [1, 2^*)$ when $d = 2$;

ii) for every $\varphi, \psi \in C_c^2(\mathbb{R}^d)$ and every $0 \le s_1 < s_2 \le T$ it holds

$$\int_{\mathbb{R}^d} \varphi(x)\rho(s_2,x)\,\mathrm{d}x = \int_{\mathbb{R}^d} \varphi(x)\rho(s_1,x)\,\mathrm{d}x - \kappa\int_{s_1}^{s_2}\int_{\mathbb{R}^d} \rho\Delta\rho\Delta\varphi + \nabla\rho\cdot\nabla\varphi\Delta\rho\,\mathrm{d}x\,\mathrm{d}t$$
$$- \alpha\int_{s_1}^{s_2}\int_{\mathbb{R}^d} \rho\Delta\eta\Delta\varphi + \nabla\rho\cdot\nabla\varphi\Delta\eta\,\mathrm{d}x\,\mathrm{d}t$$
$$- \frac{\beta}{2}\int_{s_1}^{s_2}\int_{\mathbb{R}^d} \rho^2\Delta\varphi\,\mathrm{d}x\,\mathrm{d}t + \omega\int_{s_1}^{s_2}\int_{\mathbb{R}^d} \rho\nabla\eta\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t,$$
$$\int_{\mathbb{R}^d} \psi(x)\eta(s_2,x)\,\mathrm{d}x = \int_{\mathbb{R}^d} \psi(x)\eta(s_1,x)\,\mathrm{d}x - \int_{s_1}^{s_2}\int_{\mathbb{R}^d} \eta\Delta\eta\Delta\psi + \nabla\eta\cdot\nabla\psi\Delta\eta\,\mathrm{d}x\,\mathrm{d}t$$
$$- \alpha\int_{s_1}^{s_2}\int_{\mathbb{R}^d} \eta\Delta\rho\Delta\psi + \nabla\eta\cdot\nabla\psi\Delta\rho\,\mathrm{d}x\,\mathrm{d}t$$
$$- \frac{1}{2}\int_{s_1}^{s_2}\int_{\mathbb{R}^d} \eta^2\Delta\psi\,\mathrm{d}x\,\mathrm{d}t + \omega\int_{s_1}^{s_2}\int_{\mathbb{R}^d} \eta\nabla\rho\cdot\nabla\psi\,\mathrm{d}x\,\mathrm{d}t.$$

Theorem 4.5. *Let $(\rho_0, \eta_0) \in \mathcal{P}_2^a(\mathbb{R}^d) \times \mathcal{P}_2^a(\mathbb{R}^d)$ be an initial datum such that $\boldsymbol{\mathcal{F}}[\rho_0, \eta_0] < +\infty$. Then there exists a weak solution to* (4.4).

The last result is generalised to a wider class of systems allowing for nonlinear self-diffusion terms.

4.2.1 Preliminaries

We present the notation and we collect some *a priori* results that we will use throughout the chapter.

A key tool for the analysis is the Wasserstein metric, that is a distance function in the space of probability measures with finite second order moments.

Definition 4.6 (2-Wasserstein distance)**.** *For μ, $\nu \in \mathcal{P}_2(\mathbb{R}^d)$, we define the 2-Wasserstein distance, $\mathcal{W}_2(\mu,\nu)$, between μ and ν as*

$$\mathcal{W}_2(\mu,\nu) := \min_{\gamma\in\Gamma(\mu,\nu)} \left\{ \int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^2\,\mathrm{d}\gamma(x,y) \right\}^{\frac{1}{2}},$$

where $\Gamma(\mu,\nu)$ is the set of transport plans between μ and ν,

$$\Gamma(\mu,\nu) = \left\{ \gamma \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d) \,:\, (\pi_x)_{\#}\gamma = \mu,\ (\pi_y)_{\#}\gamma = \nu \right\},$$

and π_x and π_y are the projections onto the first and the second variables respectively.

In the expression above, marginals are the push-forward of γ through π_i. For a measure $\rho \in \mathcal{P}(\mathbb{R}^d)$ and a Borel map $T : \mathbb{R}^d \to \mathbb{R}^n$, $n \in \mathbb{N}$, the push-forward of ρ through T is defined by

$$\int_{\mathbb{R}^n} f(y)\,\mathrm{d}T_{\#}\rho(y) = \int_{\mathbb{R}^d} f(T(x))\,\mathrm{d}\rho(x) \qquad \text{for all } f \text{ Borel functions on } \mathbb{R}^d.$$

We refer the reader to [7, 283, 304] for further details on optimal transport theory and Wasserstein spaces.

In order to obtain strong convergence of ρ in $L^2([0,T];L^2(\mathbb{R}^d))$ we take advantage of a refined version of the Aubin-Lions Lemma for compactness in measures, due to Rossi and Savaré. It relies on two conditions: tightness and weak integral equi-continuity.

Proposition 4.7 ([280], Theorem 2)**.** *Let X be a separable Banach space and consider*

- *a lower semicontinuous functional $\mathcal{I} : X \to [0,+\infty]$ with relatively compact sublevels in X,*
- *a pseudo-distance $g : X \times X \to [0,+\infty]$, i.e. g is lower semicontinuous and such that $g(\rho,\eta) = 0$ for any $\rho,\eta \in X$ with $\mathcal{I}[\rho] < \infty$ and $\mathcal{I}[\eta] < \infty$ implies $\rho = \eta$.*

Let U be a set of measurable functions $u : (0,T) \to X$, with a fixed $T > 0$. Assume further that U is tight with respect to $\mathcal{I}$

$$\sup_{u\in U} \int_0^T \mathcal{I}[u(t)]\,\mathrm{d}t < \infty\,, \tag{4.8}$$

and satisfies the weak integral equi-continuity condition

$$\lim_{h\downarrow 0} \sup_{u\in U} \int_0^{T-h} g\left(u(t+h),u(t)\right)\mathrm{d}t = 0\,. \tag{4.9}$$

Then U contains an infinite sequence $(u_n)_{n\in\mathbb{N}}$ convergent in measure, with respect to $t \in (0,T)$, to a measurable $\tilde{u} : (0,T) \to X$, i.e.

$$\lim_{n\to\infty} |\{t \in (0,T) :\ \|u_n(t) - \tilde{u}(t)\|_X \geq \delta\}| = 0, \quad \forall \delta \geq 0\,. \tag{4.10}$$

In addition to the strong convergence given by Proposition 4.7, we will need an L^2 bound on $\Delta\rho$ to obtain suitable compactness in time and space for $\nabla\rho$ and $\Delta\rho$. We employ the *flow interchange* technique, developed by Matthes, McCann and Savaré in [252] and previously used in [268] — we also refer the reader to [59, 84, 153, 155] for further details. The idea of the flow interchange consists in considering the dissipation of the free energy $\mathcal{F}_m$ along a solution of an auxiliary gradient flow, and using the Evolution Variational Inequality (EVI) afterwards to obtain the desired estimate. For the reader's convenience we recall the definition of λ-flow for a general functional $\mathcal{G}$, which is connected to the EVI.

Definition 4.8 (λ-flow)**.** *A semigroup $S_{\mathcal{G}} : [0,+\infty] \times \mathcal{P}_2(\mathbb{R}^d) \to \mathcal{P}_2(\mathbb{R}^d)$ is a λ-flow for a functional $\mathcal{G} : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R} \cup \{\infty\}$ with respect to the distance $\mathcal{W}_2$ if, for an arbitrary $\rho \in \mathcal{P}_2(\mathbb{R}^d)$, we have that the curve $t \mapsto S^t_{\mathcal{G}}\rho$ is absolutely continuous on $[0,\infty)$ and it satisfies the EVI*

$$\frac{1}{2}\frac{\mathrm{d}^+}{\mathrm{d}t}\mathcal{W}_2^2(S^t_{\mathcal{G}}\rho,\mu) + \frac{\lambda}{2}\mathcal{W}_2^2(S^t_{\mathcal{G}}\rho,\mu) \leq \mathcal{G}[\mu] - \mathcal{G}[S^t_{\mathcal{G}}\rho], \tag{EVI}$$

for all $t > 0$, with respect to every reference measure $\mu \in \mathcal{P}_2(\mathbb{R}^d)$ such that $\mathcal{G}[\mu] < \infty$.

As shown in the seminal work by Jordan, Kinderlehrer and Otto [215], the heat equation can be regarded as a 2-Wasserstein steepest descent of the Boltzmann entropy

$$\mathcal{E}[\rho] = \begin{cases} \int_{\mathbb{R}^d} \rho(x)\log\rho(x)\,\mathrm{d}x, & \rho\log\rho \in L^1(\mathbb{R}^d), \\ +\infty & \text{otherwise} \end{cases} \tag{4.11}$$

We mention [7, 283] and the recent [191, Chapter 3.3] for further details. The functional $\mathcal{E}$ is 0-convex along geodesics and it possesses a unique 0-flow, which we denote $S_{\mathcal{E}}$, given by the heat semigroup, see for example

[7, 139, 155]. We will use the heat equation as the auxiliary flow and the free energy (4.11) as the auxiliary functional.

In order to illustrate the method, let us calculate the dissipation of the Boltzmann entropy along solutions of our equation, (4.1). For simplicity, we consider $m = 2$, although the method generalises to other exponents. In this case, a formal computation yields

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d}\rho\log\rho\,\mathrm{d}x &= -\int_{\mathbb{R}^d}\log\rho\,\mathrm{div}\,(\rho\nabla(\Delta\rho))\,\mathrm{d}x - 2\chi\int_{\mathbb{R}^d}\log\rho\,\mathrm{div}\,(\rho\nabla\rho)\,\mathrm{d}x.\\ &= \int_{\mathbb{R}^d}\nabla\rho\cdot\nabla(\Delta\rho)\,\mathrm{d}x + 2\chi\int_{\mathbb{R}^d}|\nabla\rho|^2\,\mathrm{d}x.\\ &\le -\int_{\mathbb{R}^d}(\Delta\rho)^2\,\mathrm{d}x + 2C,\end{aligned}$$

where the constant $C > 0$ is given in Proposition 4.11. By integrating in time, we obtain

$$\|\Delta\rho\|^2_{L^2((0,T)\times\mathbb{R}^d)} \le \mathcal{E}[\rho_0] - \mathcal{E}[\rho_T] + 2CT \le \|\rho_0\|^2_{L^2(\mathbb{R}^d)} - \mathcal{E}[\rho_T] + 2CT.$$

It remains to notice that $\mathcal{E}[\rho]$ can be bounded from below by the second moment of ρ, $\mathrm{m}_2(\rho)$, which gives the desired L^2 bound on $\Delta\rho$. Although this formal computation requires further regularity, it illustrates how we may use an auxiliary flow to obtain H^2 estimates for our equation. In Lemma 4.24, we shall make this calculation fully rigorous by considering, instead, the dissipation of our energy functional $\mathcal{F}_m[\rho]$, (4.5), along solutions of the heat equation with suitable initial data.

Remark 4.9. *We remind the reader of the following bound for the Boltzmann entropy functional* $\mathcal{E}$*,*

$$\mathcal{E}[\rho] = \int_{\mathbb{R}^d}\rho\log\rho \ge -C(1+\mathrm{m}_2(\rho)).$$

To prove this, let $M(x) := (2\pi)^{-d/2}\exp\left(-|x|^2/2\right)$ *, and consider the relative entropy*

$$\mathcal{E}(\rho|M) := \int_{\mathbb{R}^d}\rho\log\frac{\rho}{M}\,\mathrm{d}x\,.$$

Jensen's inequality implies that

$$\mathcal{E}(\rho|M) \ge \log\left(\int_{\mathbb{R}^d}\frac{\rho}{M}M\mathrm{d}x\right)\int_{\mathbb{R}^d}\frac{\rho}{M}\,\mathrm{d}x = 0\,,$$

and thus, we conclude that

$$0 \le \mathcal{E}(\rho|M) = \int_{\mathbb{R}^d}\rho\log\rho\,\mathrm{d}x + \frac{d}{2}\log(2\pi) + \frac{1}{2}\int_{\mathbb{R}^d}|x|^2\rho(x)\,\mathrm{d}x\,,$$

which implies

$$\int_{\mathbb{R}^d}\rho\log\rho \ge \frac{d}{2}\log(2\pi) - \frac{1}{2}\mathrm{m}_2(\rho)\,.$$

4.3 Properties of the energy functional

The energy $\mathcal{F}_m$ plays a crucial role in the analysis of (4.1), as it provides uniform bounds we hinge on for the construction of weak solutions. Furthermore, in the theory of gradient flows, the dynamical problem is usually related to energy minimisers via stationary states. This is, indeed, a valuable advantage of studying Eq. (4.1) in the Wasserstein space $(\mathcal{P}_2(\mathbb{R}^d), \mathcal{W}_2)$. As we shall see below, the Gagliardo–Nirenberg inequality is essential for a thorough study of our problem as it reveals critical regimes. For the reader's convenience we recall it in the lemma below, cf. for instance [56, 264].

Lemma 4.10 (Gagliardo–Nirenberg interpolation inequality)**.** *Let $\theta \in [0,1]$, $1 \le p, q \le +\infty$, and $1 \le r < \infty$ such that $\frac{1}{p} = \theta\left(\frac{1}{r} - \frac{1}{d}\right) + \frac{1-\theta}{q}$. Then, it holds*

$$\|f\|_{L^p(\mathbb{R}^d)} \le C\|\nabla f\|^{\theta}_{L^r(\mathbb{R}^d)}\|f\|^{1-\theta}_{L^q(\mathbb{R}^d)},$$

where C denotes a positive constant depending on p, q, r, θ, but not on f. In the case $d = 2$, $\theta \in [0,1)$.

In the proposition below we find a range of exponents for which the free energy $\mathcal{F}_m$ is bounded from below, thus proving Theorem 4.1, case (1). In turn, this implies further regularity for the density ρ and provides the critical exponent, $m_c = 2 + 2/d$.

Proposition 4.11 (Lower bound for the free energy and induced regularity)**.** *Assume $\rho \in L^1_+(\mathbb{R}^d)$ and let $1 \le m < 2 + \frac{2}{d}$. Set $\alpha := 1 + \frac{\frac{2}{d}(m-1)}{2+\frac{2}{d}-m}$, for $m > 1$, and $\alpha := 2$, for $m = 1$. The following properties hold.*

(1) *Lower bound for the free energy: let $\nabla\rho \in L^2(\mathbb{R}^d)$, then $\mathcal{F}_m[\rho]$ is bounded from below as*

$$\mathcal{F}_m[\rho] \ge -C\|\rho\|^{\alpha}_{L^1(\mathbb{R}^d)}, \tag{4.12}$$

where $C = C(m, d, \chi)$.

(2) *H^1-bound: assume $\mathcal{F}_m[\rho] < +\infty$, then the following bound holds*

$$\|\nabla\rho\|^2_{L^2(\mathbb{R}^d)} \le C\left(\mathcal{F}_m[\rho] + \|\rho\|^{\alpha}_{L^1(\mathbb{R}^d)}\right), \tag{4.13}$$

where $C = C(m, d, \chi)$.

(3) *L^p-regularity: assume $\mathcal{F}_m[\rho] < +\infty$, then $\rho \in L^p(\mathbb{R}^d)$ for any $p \in [1, 2^*]$, $d \ne 2$, and for any $p \in [1, 2^*)$ when $d = 2$. In particular, there exists a constant $C = C(m, p, d, \rho, \chi) > 0$ such that*

$$\|\rho\|_{L^p(\mathbb{R}^d)} \le C < +\infty. \tag{4.14}$$

Proof. We divide the proof in several steps.

Step 1: Lower bound for the free energy. Let $1 < m < 2 + \frac{2}{d}$. By applying Gagliardo–Nirenberg inequality to $\|\rho\|_{L^m(\mathbb{R}^d)}$ we find

$$\|\rho\|_{L^m(\mathbb{R}^d)} \le C\|\nabla\rho\|^{\theta}_{L^2(\mathbb{R}^d)}\|\rho\|^{1-\theta}_{L^1(\mathbb{R}^d)},$$

where $\theta = \frac{2d}{d+2}\frac{m-1}{m} \in (0,1)$. By applying Young's inequality with $p = \frac{2}{m\theta} = \frac{1+\frac{2}{d}}{m-1} > 1$ and p' its conjugate, we have

$$\|\rho\|^m_{L^m(\mathbb{R}^d)} \le \frac{\varepsilon^p\|\nabla\rho\|^2_{L^2(\mathbb{R}^d)}}{p} + \frac{C^{mp'}}{\varepsilon^{p'}}\frac{\|\rho\|^{m(1-\theta)p'}_{L^1(\mathbb{R}^d)}}{p'}.$$

Therefore, taking any $0 < \varepsilon < (p(m-1)/2\chi)^{1/p}$ we obtain the bound

$$\begin{aligned}\mathcal{F}_m[\rho] &= \frac{1}{2}\|\nabla\rho\|^2_{L^2(\mathbb{R}^d)} - \frac{\chi}{m-1}\|\rho\|^m_{L^m(\mathbb{R}^d)} \\ &\ge \left(\frac{1}{2} - \frac{\chi\varepsilon^p}{p(m-1)}\right)\|\nabla\rho\|^2_{L^2(\mathbb{R}^d)} - \frac{\chi C^{mp'}}{p'(m-1)\varepsilon^{p'}}\|\rho\|^{m(1-\theta)p'}_{L^1(\mathbb{R}^d)} \\ &\ge -C\|\rho\|^{\alpha}_{L^1(\mathbb{R}^d)},\end{aligned} \tag{4.15}$$

where $C = C(m, d, \chi)$, and $\alpha = m(1-\theta)p' = 1 + \frac{\frac{2}{d}(m-1)}{2+\frac{2}{d}-m}$.

Note that in case of linear diffusion $m = 1$, i.e. $\mathcal{F}_1[\rho]$ as functional, we can argue similarly by using that

$$\mathcal{F}_1[\rho] \geq \mathcal{F}_2[\rho] \geq -C\|\rho\|^2_{L^1(\mathbb{R}^d)}.$$

Note that the first inequality holds because $\mathcal{E}_1[\rho] \leq \mathcal{E}_2[\rho]$, since $x \log x \leq x^2$, for $x > 0$.

Step 2: H^1-bound. The result follows from (4.15) by noting that $\mathcal{F}_m[\rho] < +\infty$ and choosing again $0 < \varepsilon < (p(m-1)/2\chi)^{1/p}$.

Step 3: L^p-regularity. From the previous case, we have $\nabla\rho \in L^2(\mathbb{R}^d)$, and thus we can apply Gagliardo–Nirenberg inequality to obtain

$$\|\rho\|_{L^p(\mathbb{R}^d)} \leq C\|\nabla\rho\|^{\theta}_{L^2(\mathbb{R}^d)}\|\rho\|^{1-\theta}_{L^1(\mathbb{R}^d)} \leq C < \infty,$$

with $\theta = \frac{2d}{d+2}\frac{p-1}{p} \in [0,1]$ and $p \in [1, 2^*]$ for $d = 1$ and $d \geq 3$. Note that for $d \geq 3$ and $p = 2^*$ we have $\theta = 1$. In the case $d = 2$, we need to impose $\theta < 1$, and $p \in [1, 2^*)$. □

In the critical exponent case, $m_c = 2+\frac{2}{d}$, deriving energy bounds and induced regularity as in Proposition 4.11 reveals the critical mass

$$\chi_c := \frac{m_c - 1}{2C_{GN}}, \tag{4.16}$$

where C_{GN} stands for the sharp constant from the Gagliardo–Nirenberg inequality, for $m = m_c$ given by

$$\|\rho\|^{m_c}_{L^{m_c}(\mathbb{R}^d)} \leq C_{GN}\|\nabla\rho\|^2_{L^2(\mathbb{R}^d)}\|\rho\|^{2/d}_{L^1(\mathbb{R}^d)} \tag{4.17}$$

Remark 4.12. *(Critical mass and the parameter χ). The critical mass in* (4.16) *is obtained for the sharp Gagliardo–Nirenberg constant, C_{GN}. This value is found in [246], extending to general dimension [308]. Note that we refer to χ_c as the critical mass since we assume that all densities are probability measures with unit mass. However, upon rescaling* (4.1) *using the change of variables*

$$\tau = t/\chi^{\frac{1}{m-2}} \quad \text{and} \quad \tilde{\rho} = \rho\chi^{\frac{1}{m-2}},$$

Eq. (4.1) *becomes $\partial_\tau\tilde{\rho} = -\mathrm{div}(\tilde{\rho}\,\nabla(\Delta\tilde{\rho})) - \Delta\tilde{\rho}^{\,m}$. Therefore, one can distinguish between subcritical, critical, and supercritical regimes, in terms of the usual mass $\|\tilde{\rho}\|_{L^1(\mathbb{R}^d)}$. More precisely, by denoting $M := \|\tilde{\rho}\|_{L^1}$, the critical mass is*

$$M_c := \left(\frac{m_c - 1}{2C_{GN}}\right)^{\frac{d}{2}}.$$

We show that for $\chi \leq \chi_c$ and $m = m_c$ the free energy is bounded from below, which covers Theorem 4.1, case (2).

Proposition 4.13. *Let $m = m_c$, $\chi \leq \chi_c$, and assume $\rho \in \mathcal{P}^a(\mathbb{R}^d)$, $\nabla\rho \in L^2(\mathbb{R}^d)$. The free energy* (4.5) *satisfies the bound*

$$\mathcal{F}_{m_c}[\rho] \geq \|\nabla\rho\|^2_{L^2(\mathbb{R}^d)}\left(\frac{1}{2} - \frac{\chi C_{GN}}{m_c - 1}\|\rho\|^{\frac{2}{d}}_{L^1(\mathbb{R}^d)}\right) \geq 0.$$

Moreover, if $\chi < \chi_c$ and $\mathcal{F}_{m_c}[\rho] < +\infty$, then

$$\|\rho\|_{L^p(\mathbb{R}^d)}, \|\nabla\rho\|_{L^2(\mathbb{R}^d)} < C,$$

where $C = C(m, d, \chi)$ and $p \in [1, 2^*]$, $d \neq 2$, or $p \in [1, 2^*)$ when $d = 2$. Furthermore, for $\chi = \chi_c$, the optimisers of the Gagliardo–Nirenberg inequality (4.17) *are the set of global minimisers of the free energy.*

Proof. From the Gagliardo–Nirenberg inequality (4.17), we can deduce that

$$\mathcal{F}_{m_c}[\rho] = \frac{1}{2}\|\nabla\rho\|^2_{L^2(\mathbb{R}^d)} - \frac{\chi}{m_c-1}\|\rho\|^{m_c}_{L^{m_c}(\mathbb{R}^d)} \geq \|\nabla\rho\|^2_{L^2(\mathbb{R}^d)}\left(\frac{1}{2} - \frac{\chi C_{GN}}{m_c-1}\|\rho\|^{\frac{2}{d}}_{L^1(\mathbb{R}^d)}\right).$$

In particular, since $\chi \leq \chi_c$ and $\|\rho\|_{L^1(\mathbb{R}^d)} = 1$, we obtain

$$\mathcal{F}_{m_c}[\rho] \geq 0.$$

The last properties are simple consequences of the Gagliardo–Nirenberg inequality (4.17), the definition of the free energy and $\chi < \chi_c$. □

Summarising the previous two propositions, using the Gagliardo–Nirenberg inequality we showed that the free energy $\mathcal{F}_{m_c}[\rho]$ is uniformly bounded from below when the exponent m is subcritical, $1 \leq m < m_c$, or when $m = m_c$ and we have subcritical or critical mass, $\chi \leq \chi_c$. Moreover, this induces further regularity in the subcritical exponent and critical exponent with subcritical mass cases. In section 4.4, we use this information to prove existence of weak solutions to (4.1).

In order to gain further intuitions on the remaining cases, $m = m_c$ with $\chi \geq \chi_c$ and $m > m_c$, we study energy minimisers distinguishing between the two cases.

4.3.1 Critical exponent case

First, we focus on the critical case given by $m = m_c = 2 + \frac{2}{d}$, and study properties of the free energy (4.5), following ideas from [48]. This highlights an interesting connection with Patlak-Keller-Segel sytems [78], and more broadly with aggregation-diffusion equations, as mentioned in the introduction.

A crucial observation concerns the homogeneity of the energy funcional $\mathcal{F}_{m_c}$: mass-preservation dilation implies that, in this critical case, the aggregation and diffusion terms in the energy functional (4.5) have the same homogeneity.

Lemma 4.14 (Scaling properties of the free energy)**.** *Assume $\rho \in L^{m_c}(\mathbb{R}^d)$ such that $\nabla\rho \in L^2(\mathbb{R}^d)$. Let $\rho_\lambda(x) := \lambda^d\rho(\lambda x)$, for any $x \in \mathbb{R}^d$, then*

$$\|\rho_\lambda\|^{m_c}_{L^{m_c}(\mathbb{R}^d)} = \lambda^{d+2}\|\rho\|^{m_c}_{L^{m_c}(\mathbb{R}^d)}, \quad \|\nabla\rho_\lambda\|^2_{L^2(\mathbb{R}^d)} = \lambda^{d+2}\|\nabla\rho\|^2_{L^2(\mathbb{R}^d)},$$

for all $\lambda \in (0, +\infty)$. In particular,

$$\mathcal{F}_{m_c}[\rho_\lambda] = \lambda^{d+2}\mathcal{F}_{m_c}[\rho].$$

Proof. We have

$$\begin{aligned}\mathcal{F}_{m_c}[\rho_\lambda] &= \frac{\lambda^{2d}}{2}\int_{\mathbb{R}^d}|\nabla\rho(\lambda x)|^2\,\mathrm{d}x - \frac{\chi\lambda^{dm_c}}{m_c-1}\int_{\mathbb{R}^d}\rho^{m_c}(\lambda x)\,\mathrm{d}x \\ &= \frac{\lambda^{d+2}}{2}\int_{\mathbb{R}^d}|\nabla\rho(x)|^2\,\mathrm{d}x - \frac{\chi\lambda^{d(m_c-1)}}{m_c-1}\int_{\mathbb{R}^d}\rho^{m_c}(x)\,\mathrm{d}x = \lambda^{d+2}\mathcal{F}_{m_c}[\rho]\,,\end{aligned}$$

since $d(m_c - 1) = d + 2$. □

Next, we study the infimum of the free energy $\mathcal{F}_{m_c}$. Let us define $\mu_\chi := \inf_{\rho\in\mathcal{Y}}\mathcal{F}_{m_c}[\rho]$, where

$$\mathcal{Y} = \{\rho \in \mathcal{P}^a(\mathbb{R}^d) \cap L^{m_c}(\mathbb{R}^d) : \nabla\rho \in L^2(\mathbb{R}^d)\}.$$

The next result completes Theorem 4.1, case (2).

Proposition 4.15 (Infimum of the free energy). *We have*

$$\mu_\chi = \begin{cases} 0 & \text{if } \chi \in (0, \chi_c]\,, \\ -\infty & \text{if } \chi > \chi_c\,. \end{cases}$$

Moreover, for $\rho \in \mathcal{Y}$,

$$(\chi_c - \chi)\, \|\nabla\rho\|^2_{L^2(\mathbb{R}^d)} \le \frac{(m_c-1)\mathcal{F}_{m_c}[\rho]}{C_{GN}} \le (\chi_c + \chi)\, \|\nabla\rho\|^2_{L^2(\mathbb{R}^d)}, \tag{4.18}$$

where the critical mass χ_c *is defined in* (4.16) *and* C_{GN} *is the sharp constant in the Gagliardo–Nirenberg inequality* (4.17). *In particular, the infimum is not achieved for* $\chi < \chi_c$*, and there exists a minimiser in* $\mathcal{Y}$ *for* $\chi = \chi_c$.

Proof. Let $\rho \in \mathcal{Y}$. By Gagliardo-Nirenberg inequality (4.17),

$$\mathcal{F}_{m_c}[\rho] \ge \|\nabla\rho\|^2_{L^2(\mathbb{R}^d)} \left(\frac{1}{2} - \frac{\chi C_{GN}}{m_c - 1}\|\rho\|^{2/d}_{L^1(\mathbb{R}^d)}\right) = \|\nabla\rho\|^2_{L^2(\mathbb{R}^d)}\,(\chi_c - \chi)\,\frac{C_{GN}}{m_c-1},$$

and also

$$\mathcal{F}_{m_c}[\rho] \le \frac{1}{2}\|\nabla\rho\|^2_{L^2(\mathbb{R}^d)} + \frac{\chi}{m_c-1}\|\rho\|^{m_c}_{L^{m_c}(\mathbb{R}^d)} \le \|\nabla\rho\|^2_{L^2(\mathbb{R}^d)}\,(\chi_c + \chi)\,\frac{C_{GN}}{m_c-1},$$

which gives (4.18).

Case I: $\chi \le \chi_c$. We first show $\mu_\chi = 0$. From (4.18) we see that $\mu_\chi \ge 0$. Let $u_\varepsilon(x) = \varepsilon^d u(\varepsilon x)$, where $u \in \mathcal{Y}$. Then, $u_\varepsilon \in \mathcal{Y}$ and by Lemma 4.14, we have $\|\nabla u_\varepsilon\|_{L^2(\mathbb{R}^d)} = O(\varepsilon^{d/2+1})$. Hence, by sending $\varepsilon \downarrow 0$ in (4.18) we obtain $\mu_\chi = 0$.

Next note that if $\chi < \chi_c$ the inequality in (4.18) is strict and the infimum cannot be achieved. When the mass is critical, $\chi = \chi_c$, we exploit [246], where equality in the Gagliardo–Nirenberg inequality is proven for a non-negative radial symmetric function that can be chosen in $\mathcal{Y}$. In particular, we have a minimiser for $\mathcal{F}_{m_c}$. Moreover, all minimisers coincide with scalings of this fixed profile, that is, the set of global minimisers is given by the optimisers of the Gagliardo–Nirenberg inequality (4.17).

Case II: $\chi > \chi_c$. The arguments presented here are inspired by [307]. Fix $\delta \in (\chi_c/\chi, 1)$. Due to the Gagliardo–Nirenberg inequality, there exists a nonzero function $\rho^* \in L^{m_c}(\mathbb{R}^d)$ with $\nabla\rho^* \in L^2(\mathbb{R}^d)$ such that

$$C_{GN}\delta \le \frac{\|\rho^*\|^{m_c}_{L^{m_c}(\mathbb{R}^d)}}{\|\nabla\rho^*\|^2_{L^2(\mathbb{R}^d)}\|\rho^*\|^{2/d}_{L^1(\mathbb{R}^d)}} \le C_{GN}\,; \tag{4.19}$$

for instance ρ^* could be chosen as the optimizer of the Gagliardo–Nirenberg inequality (4.17) for the critical exponent m_c. Now let $\lambda > 0$, and consider the function $\rho_\lambda(x) = \lambda^d \rho^*\left(\lambda\|\rho^*\|^{1/d}_{L^1(\mathbb{R}^d)}x\right)$. It is easy to check $\rho_\lambda \in \mathcal{Y}$. From Lemma 4.14, (4.19), and the definition of the critical mass (4.16), we obtain

$$\begin{aligned}
\mathcal{F}_{m_c}[\rho_\lambda] &= \frac{\lambda^{d+2}}{\|\rho^*\|_{L^1(\mathbb{R}^d)}}\left[\frac{\|\nabla\rho^*\|^2_{L^2(\mathbb{R}^d)}\|\rho^*\|^{2/d}_{L^1(\mathbb{R}^d)}}{2} - \frac{\chi\|\rho^*\|^{m_c}_{L^{m_c}(\mathbb{R}^d)}}{m_c-1}\right] \\
&= \frac{\lambda^{d+2}}{2}\|\rho^*\|^{2/d-1}_{L^1(\mathbb{R}^d)}\|\nabla\rho^*\|^2_{L^2(\mathbb{R}^d)}\left[1 - \frac{2\chi}{m_c-1}\frac{\|\rho^*\|^{m_c}_{L^{m_c}(\mathbb{R}^d)}}{\|\rho^*\|^{2/d}_{L^1(\mathbb{R}^d)}\|\nabla\rho^*\|^2_{L^2(\mathbb{R}^d)}}\right] \\
&\le \frac{\lambda^{d+2}}{2}\|\rho^*\|^{2/d-1}_{L^1(\mathbb{R}^d)}\|\nabla\rho^*\|^2_{L^2(\mathbb{R}^d)}\left(1 - \frac{\chi}{\chi_c}\delta\right).
\end{aligned}$$

Owing to the choice of δ and taking the limit $\lambda \to +\infty$ we obtain $\mu_\chi = -\infty$.

□

Self-similarity

In the critical case $m = m_c = 2 + \frac{2}{d}$ we may assume the self-similar ansatz

$$\rho(x,t) = t^{-a} u(xt^{-b})\,. \tag{4.20}$$

Mass conservation gives the usual relation between the exponents $a = bd$. Moreover, assuming (4.20) is a solution of (4.1), we obtain

$$au + b\nabla u(z)\cdot z = \text{div}\,(u\nabla(\Delta u(z))) + \chi \Delta u(z)^{m_c}$$

with $b = \frac{1}{d+4}$, $a = bd$. In particular, we obtain the equation

$$\text{div}\,(u\nabla(\Delta u(z))) + \chi \Delta u(z)^{m_c} - b\,\text{div}(zu) = 0,$$

which is the equation for steady states of the corresponding evolution problem

$$\partial_t u = -\text{div}\,(u\nabla(\Delta u(z))) - \chi \Delta u(z)^{m_c} + b\,\text{div}(zu). \tag{4.21}$$

The above evolution PDE is (at least formally) a 2-Wasserstein gradient flow of the energy

$$\mathcal{L}[u] = \mathcal{F}_{m_c}[u] + \frac{b}{2}\int |z|^2 u(z)\,\mathrm{d}z.$$

For this energy, one can prove existence of minimisers using the direct method of Calculus of Variations. The main advantage with respect to the minimisation of $\mathcal{F}_m$ is the presence of the additional term, fundamental for the compactness of the minimising sequence, as we shall see also in Proposition 4.18. As the proof of the latter proposition applies to a wider range of exponents, including $m = m_c$, we postpone this proof to Section 4.4, below that of Proposition 4.18.

Proposition 4.16 (Existence of minimisers for $\mathcal{L}$)**.** *Given $\chi < \chi_c$, the functional $\mathcal{L} : \mathcal{P}^a(\mathbb{R}^d) \to [-\infty, +\infty]$ admits minimisers in the set $\{u \in \mathcal{P}^a(\mathbb{R}^d) : \nabla u \in L^2(\mathbb{R}^d),\, \mathrm{m}_2(u) < \infty\}$.*

A natural question to ask is whether one can characterise energy minimisers, in the spirit of [48, 74, 67], and check if these are steady states of Eq. (4.21). In turn, one would be able to characterise self-similar profiles for (4.1).

As mentioned in Remark 4.28, Eq. (4.21) admits weak solutions arguing as in Section 4.4. Studying the long-time behaviour of solutions to (4.21) is also another interesting open problem we leave to future investigation, as well as a thorough study of energy minimisers for the subcritical case $m < m_c$.

4.3.2 Supercritical exponent case

We study the infimum of the free energy $\mathcal{F}_m$ when $m > m_c$, i.e. it is supercritical. As before, we define the set

$$\mathcal{Y} = \{\rho \in \mathcal{P}^a(\mathbb{R}^d) \cap L^m(\mathbb{R}^d) : \nabla\rho \in L^2(\mathbb{R}^d)\},$$

and we prove that the free energy is not bounded from below. This is, indeed, Theorem 4.1, case (3).

Proposition 4.17 (Infimum of the free energy)**.** *Assume $m > m_c$. Then* $\inf_{\rho\in\mathcal{Y}} \mathcal{F}_m[\rho] = -\infty$.

Proof. Given $\rho \in \mathcal{Y}$, we define $\rho_\lambda(x) := \lambda^d \rho(\lambda x)$, for any $x \in \mathbb{R}^d$. Note that $\rho_\lambda \in \mathcal{Y}$. Then, we have

$$\mathcal{F}_m[\rho_\lambda] = \frac{\lambda^{d+2}}{2}\|\nabla\rho\|^2_{L^2(\mathbb{R}^d)} - \frac{\chi\lambda^{d(m-1)}}{m-1}\|\rho\|^m_{L^m(\mathbb{R}^d)} = \lambda^{d+2}\left[\mathcal{F}_m[\rho] - \frac{\chi\lambda^{d(m-m_c)}-1}{m-1}\|\rho\|^m_{L^m(\mathbb{R}^d)}\right].$$

Let us note that for any λ big enough

$$\mathcal{F}_m[\rho] - \frac{\chi\lambda^{d(m-m_c)}-1}{m-1}\|\rho\|^m_{L^m(\mathbb{R}^d)} < 0\,.$$

Therefore, by letting $\lambda \to +\infty$, $\mathcal{F}_m[\rho_\lambda] \to -\infty$, obtaining the desired result. □

Finally, we briefly discuss on finite-time blow-up of classical solutions for the supercritical regimes. This shows that our main global in time existence results in Theorem 4.3 for (4.1) are sharp. Our arguments are based on the computation for the evolution of the second-order moment $\mathrm{m}_2(\rho)$ as classically done in Keller-Segel models [163, 50, 48, 67, 246]. We assume the solutions are classical solutions such that the following computations using integration by parts are allowed. More precisely, one can find that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{m}_2(\rho) &= 2\int_{\mathbb{R}^d} x\cdot(\rho\nabla(\Delta\rho)) + \chi\nabla\rho^m)\,\mathrm{d}x = -2d\int_{\mathbb{R}^d}\rho\Delta\rho\,\mathrm{d}x - 2\int_{\mathbb{R}^d}(x\cdot\nabla\rho)\Delta\rho\,\mathrm{d}x - 2d\chi\int_{\mathbb{R}^d}\rho^m\,\mathrm{d}x\\ &= (d+2)\int_{\mathbb{R}^d}|\nabla\rho|^2\,\mathrm{d}x - 2d\chi\int_{\mathbb{R}^d}\rho^m\,\mathrm{d}x\\ &= 2(d+2)\left[\mathcal{F}_m[\rho] - \chi\left(\frac{1}{m_c-1} - \frac{1}{m-1}\right)\|\rho\|^m_{L^m(\mathbb{R}^d)}\right],\end{aligned} \tag{4.22}$$

where we used that

$$\begin{aligned}\int_{\mathbb{R}^d}(x\cdot\nabla\rho)\Delta\rho\,\mathrm{d}x &= -\int_{\mathbb{R}^d}\nabla(x\cdot\nabla\rho)\cdot\nabla\rho\,\mathrm{d}x = -\int_{\mathbb{R}^d}|\nabla\rho|^2\,\mathrm{d}x - \int_{\mathbb{R}^d}x\cdot D^2\rho\nabla\rho\,\mathrm{d}x\\ &= -\int_{\mathbb{R}^d}|\nabla\rho|^2\,\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^d}x\cdot\nabla|\nabla\rho|^2\,\mathrm{d}x = \left(\frac{d}{2}-1\right)\int_{\mathbb{R}^d}|\nabla\rho|^2\,\mathrm{d}x\,.\end{aligned}$$

We observe that this computation could be made rigorous for the solutions constructed in Theorem 4.3 by using the flow interchange technique with a suitable auxiliary flow [252], in the same spirit as in Proposition 4.24. A short-time existence of solutions in the super critical exponent is expected as in [108] but it is not a trivial question for the variational scheme below.

Note that for the critical case $m = m_c$, (4.22) reduces to

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{m}_2(\rho) = 2(d+2)\mathcal{F}_{m_c}[\rho]\,.$$

In particular, by using Proposition 4.15 we obtain that the second moment is non-decreasing in time for the subcritical and critical mass regimes, $\chi \leq \chi_c$. In the supercritical mass regime, by using the above equation and that free energy $\mathcal{F}_{m_c}[\rho]$ is unbounded from below, see Proposition 4.15, the authors of [246] are able to show that any solution to (4.1) with an initial datum ρ_0 satisfying $\mathcal{F}[\rho_0] < 0$, has a finite-time blow-up in the L^{m_c}-norm.

A similar argument also works in the supercritical exponent case. If $m > m_c$ then

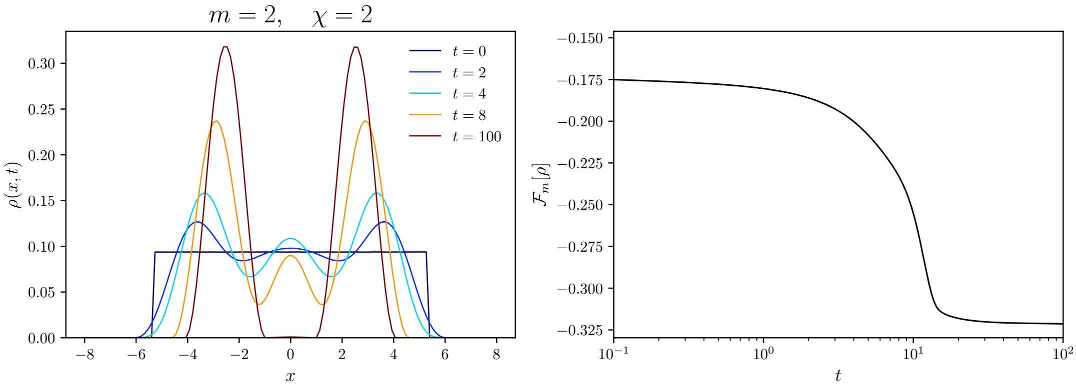
$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{m}_2(\rho) \leq 2(d+2)\mathcal{F}_m[\rho] \leq 2(d+2)\mathcal{F}_m[\rho_0] < 0\,,$$

for an initial datum with $\mathcal{F}_m[\rho_0] < 0$, which can be chosen by Proposition 4.17. If the initial second moment is finite, then there exists some time $t^* > 0$ such that $\mathrm{m}_2(\rho(t^*)) = 0$, implying that such solutions can only exist locally in time.

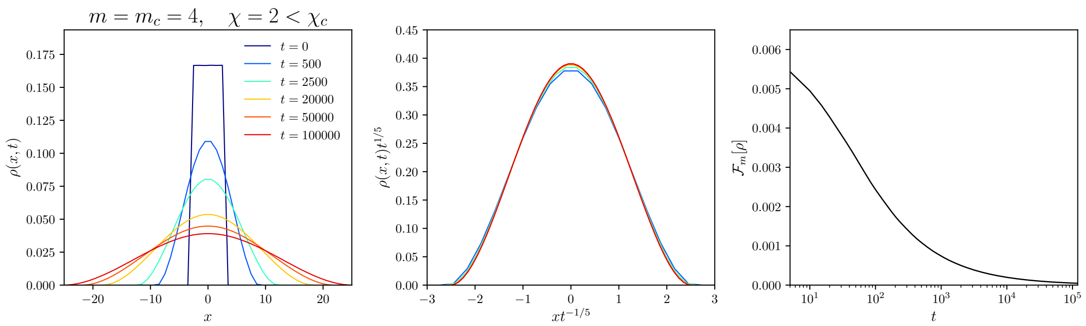
Our main results are also summarised in Figure 4.1, where we plot numerical solutions to (4.1) in one spatial dimension and for different values of the exponent m and the mass parameter χ. These are based on the finite-volume scheme presented in [20]. In particular, we observe that for subcritical exponent, solutions evolve towards a compactly supported steady state while the free energy stays bounded from below. For critical exponent with subcritical mass, we also notice that the free energy is bounded by zero from below, but in this case solutions tend to the self-similar profile mentioned in the previous sections. By plotting the solution in self-similar variables, this scaling is numerically verified. Finally, we observe finite-time blow-up for critical exponent with supercritical mass, and for supercritical exponent. In both cases, the free energy is unbounded from below.

4.4 Existence of weak solutions via the JKO scheme

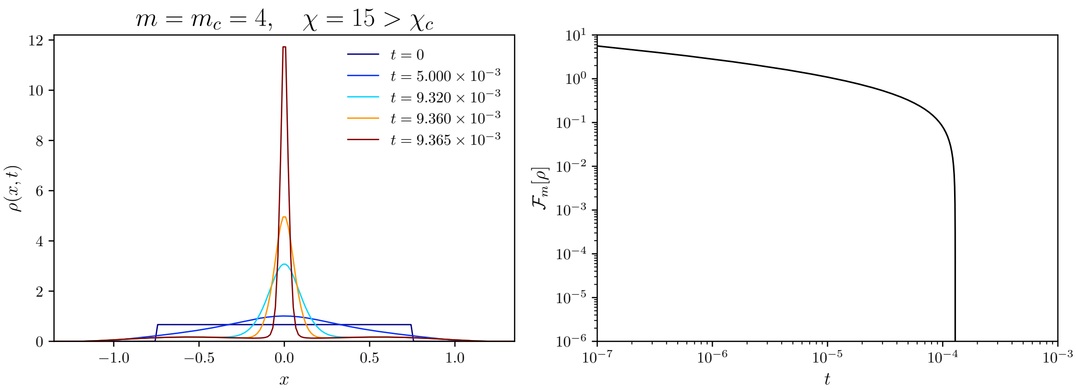
Once we understood the properties of the free energy (4.5) we study existence of weak solutions of (4.1). The variational structure of Eq. (4.1) allows to construct a candidate approximating solutions by means of the so-called JKO scheme or minimising movement, cf. [215, 7]. For a fixed $\tau > 0$, we define the following sequence



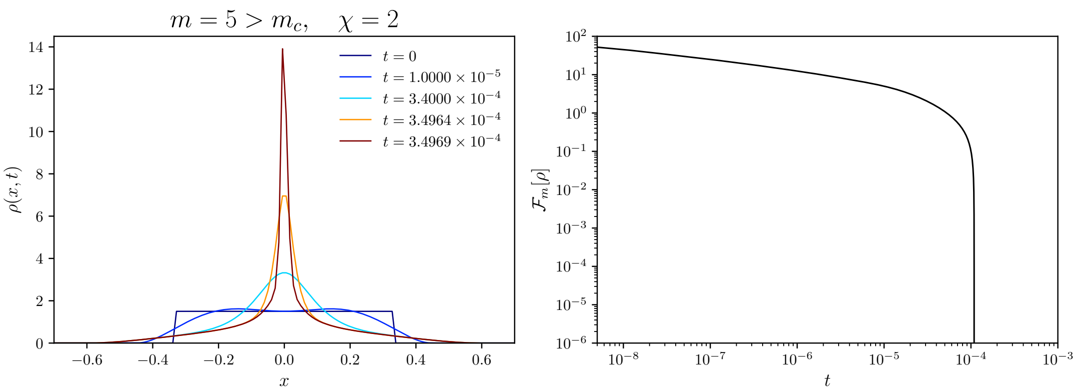
(a) Subcritical exponent $m < m_c$.



(b) Critical exponent $m = m_c$, subcritical mass $\chi < \chi_c$.



(c) Critical exponent $m = m_c$, supercritical mass $\chi > \chi_c$.



(d) Supercritical exponent $m > m_c$.

Figure 4.1: Numerical solutions to (4.1) in one spatial dimension for different values of m and χ, and decay of the free energy $\mathcal{F}_m[\rho]$ as a function of time.

recursively

$$\begin{aligned} \rho_\tau^0 &:= \rho_0, \\ \rho_\tau^{k+1} &\in \underset{\rho\in\mathcal{P}(\mathbb{R}^d)}{\operatorname{argmin}} \left\{ \frac{\mathcal{W}_2^2(\rho,\rho_\tau^k)}{2\tau} + \mathcal{F}_m[\rho] \right\}, \text{ given } \rho_\tau^k,\ k\geq 0. \end{aligned} \tag{4.23}$$

First, we prove the above scheme is well-defined, which is not immediate due to the negative component in the energy functional, or destabilising term. Let us fix $\overline{\rho}\in\mathcal{P}_2^a(\mathbb{R}^d)$ and define the functional

$$\begin{aligned} \mathcal{A}_m\colon \mathcal{P}(\mathbb{R}^d) &\longrightarrow \overline{\mathbb{R}} \\ \rho \quad &\mapsto \frac{\mathcal{W}_2^2(\rho,\overline{\rho})}{2\tau} + \mathcal{F}_m[\rho]. \end{aligned}$$

Proposition 4.18. *Let $\overline{\rho}\in\mathcal{P}_2^a(\mathbb{R}^d)$ and $1\leq m<2+\frac{2}{d}$ or $m=2+\frac{2}{d}$ with subcritical mass, i.e. $\chi<\chi_c$. The functional $\mathcal{A}_m$ admits minimisers in the set $\{\rho\in\mathcal{P}^a(\mathbb{R}^d) : \nabla\rho\in L^2(\mathbb{R}^d)\}$. Moreover, $\rho\in L^p(\mathbb{R}^d)$ for any $p\in[1,2^*]$, $d\neq 2$, and for any $p\in[1,2^*)$ when $d=2$;.*

Existence of minimisers is based on the *direct method of calculus of variations*, as we prove below.

Remark 4.19. *Note that $\mathcal{W}_2^2(\rho,\overline{\rho})$ and $\|\nabla\rho\|^2_{L^2(\mathbb{R}^d)}$ are lower semicontinuous with respect to weak convergence in $\mathcal{P}(\mathbb{R}^d)$ and $L^2(\mathbb{R}^d)$, respectively. However, the negative terms in the free energy,*

$$-\frac{1}{m-1}\|\rho\|^m_{L^m(\mathbb{R}^d)} \qquad \textit{and} \qquad -\int_{\mathbb{R}^d}\rho\log\rho\,dx,$$

are both upper (and not lower) semicontinuous with respect to the weak convergence in $L^m(\mathbb{R}^d)$. In particular, our functional $\mathcal{A}_m$ cannot be weakly lower semicontinuous.

Proof Proposition 4.18. We split the proof into three parts.

Step 1: Boundedness from below and minimising sequence. Taking into account the definition of the free energy functional (4.5), we look for minimisers $\rho\in\mathcal{P}^a(\mathbb{R}^d)$ such that $\nabla\rho\in L^2(\mathbb{R}^d)$, otherwise the functional is infinite. Due to (4.12) we have the bound from below

$$\mathcal{F}_m[\rho]\geq C, \tag{4.24}$$

which implies $\mathcal{A}_m[\rho]\geq C$. Boundedness from below ensures we can consider a minimising sequence, ρ_n, for which we also know $\mathcal{A}_m\leq C$. Since the functional $\mathcal{F}_m$ is bounded from below, we obtain the bound for the second order moment

$$\mathrm{m}_2(\rho_n)\leq 2\mathcal{W}_2^2(\rho_n,\overline{\rho})+2\mathrm{m}_2(\overline{\rho}) = 4\tau\mathcal{A}_m[\rho_n]-4\tau\mathcal{F}_m[\rho_n]+2\mathrm{m}_2(\overline{\rho})\leq CT\left(1+\mathrm{m}_2(\overline{\rho})\right), \tag{4.25}$$

for a different constant C.

Step 2: Lower semicontinuity and compactness. First we comment on the lower semicontinuity of $\mathcal{A}_m$ with respect to a suitable convergence, i.e. $\liminf_{n\to\infty}\mathcal{A}_m[\rho_n]\geq\mathcal{A}_m[\rho]$. From Remark 4.19 we infer we cannot have lower semicontinuity with respect to weak convergence in all terms, but we have it with respect to the convergence

$$\nabla\rho_n \rightharpoonup \nabla\rho \quad \text{in } L^2(\mathbb{R}^d),$$

$$\begin{cases} \rho_n\to\rho \quad \text{in } L^m(\mathbb{R}^d) & \text{if } 1<m\leq 2+\frac{2}{d}, \\ \rho_n\log\rho_n\to\rho\log\rho \quad \text{in } L^1(\mathbb{R}^d) & \text{if } m=1. \end{cases}$$

Let us note that (4.13) combined with (4.24) implies that ρ_n and $\nabla\rho_n$ are uniformly bounded on $L^m(\mathbb{R}^d)$ and $L^2(\mathbb{R}^d)$, respectively, as ρ_n is a minimising sequence.

Step 2.a: Strong L^m convergence of ρ_n. If $1 < m < 2 + \frac{2}{d}$ or $m = 2 + \frac{2}{d}$ with $\chi < \chi_c$, since

$$\|\rho_n\|_{L^m(\mathbb{R}^d)} \leq C, \tag{4.26}$$

by Banach-Alaoglu Theorem, up to pass to a subsequence,

$$\rho_n \rightharpoonup \rho \quad \text{weakly in } L^m(\mathbb{R}^d). \tag{4.27}$$

Taking into account (4.24), (4.25), and (4.26), we can restrict to the set

$$\mathcal{H}_m := \left\{ f \in \mathcal{P}^a(\mathbb{R}^d) : \mathrm{m}_2(f),\ \|\nabla f\|_{L^2(\mathbb{R}^d)},\ |\mathcal{F}_m[f]| \leq C \right\}.$$

Furthermore, from (4.14), if $f \in \mathcal{H}_m$, then

$$\|f\|_{L^p(\mathbb{R}^d)} \leq C, \tag{4.28}$$

for all $p \in [1, 2^*]$, $d \neq 2$, and for any $p \in [1, 2^*)$ when $d = 2$;.

Next, we prove that $\mathcal{H}_m$ is relatively compact in $L^m(\mathbb{R}^d)$ by means of Kolmogorov–Riesz–Fréchet Theorem [54, Corollary 4.27]. In particular, we first show the uniform continuity estimate: $\|f(\cdot + h) - f(\cdot)\|_{L^m(\mathbb{R}^d)} \to 0$ as $|h| \to 0^+$. We distinguish two cases: $m = 2$ and $m \neq 2$.

Case I: $m = 2$. Let us take

$$\begin{aligned}\int_{\mathbb{R}^d} |f(x+h) - f(x)|^2 \mathrm{d}x &= \int_{\mathbb{R}^d} \left| \int_0^1 \frac{\mathrm{d}}{\mathrm{d}s} \left(f(x+sh)\right) \mathrm{d}s \right|^2 \mathrm{d}x = |h|^2 \int_{\mathbb{R}^d} \left| \int_0^1 \nabla f(x+sh)\,\mathrm{d}s \right|^2 \mathrm{d}x \\ &\leq |h|^2 \int_{\mathbb{R}^d} \int_0^1 |\nabla f(x+sh)|^2 \,\mathrm{d}s\,\mathrm{d}x \to 0,\end{aligned}$$

since $\|\nabla f\|_{L^2(\mathbb{R}^d)} \leq C$ for every $f \in \mathcal{H}_m$.

Case II: $m \neq 2$. We use L^p interpolation and apply *Case I* afterwards. If $1 < m < 2$,

$$\begin{aligned}\|f(\cdot + h) - f(\cdot)\|_{L^m(\mathbb{R}^d)} &\leq \|f(\cdot + h) - f(\cdot)\|_{L^1(\mathbb{R}^d)}^{\frac{2-m}{m}} \|f(\cdot + h) - f(\cdot)\|_{L^2(\mathbb{R}^d)}^{\frac{2m-2}{m}} \\ &\leq \left(2\|f\|_{L^1(\mathbb{R}^d)}\right)^{\frac{2-m}{m}} \|f(\cdot + h) - f(\cdot)\|_{L^2(\mathbb{R}^d)}^{\frac{2m-2}{m}} \to 0.\end{aligned}$$

If $2 < m < 2 + \frac{2}{d}$ or $m = 2 + \frac{2}{d}$ with subcritical mass, by using a different L^p interpolation we obtain

$$\begin{aligned}\|f(\cdot + h) - f(\cdot)\|_{L^m(\mathbb{R}^d)} &\leq \|f(\cdot + h) - f(\cdot)\|_{L^{2(m-1)}(\mathbb{R}^d)}^{\frac{m-1}{m}} \|f(\cdot + h) - f(\cdot)\|_{L^2(\mathbb{R}^d)}^{\frac{1}{m}} \\ &\leq \left(2\|f\|_{L^{2(m-1)}(\mathbb{R}^d)}\right)^{\frac{m-1}{m}} \|f(\cdot + h) - f(\cdot)\|_{L^2(\mathbb{R}^d)}^{\frac{1}{m}} \to 0,\end{aligned}$$

and the convergence follows from (4.14) given that $2(m-1) < 2^*$.

In order to prove uniform integrability at infinity we first use Holder's inequality to show that

$$\int_{\mathbb{R}^d \setminus B_R} f(x)^m \,\mathrm{d}x \leq \frac{1}{R^{2\delta}} \left(\int_{\mathbb{R}^d} |x|^2 f(x)\,\mathrm{d}x \right)^{\delta} \left(\int_{\mathbb{R}^d} f(x)^{\frac{m-\delta}{1-\delta}} \,\mathrm{d}x \right)^{1-\delta}.$$

Now $\delta \in (0,1)$ can be chosen so that the exponent $p := \frac{m-\delta}{1-\delta}$ satisfies $p \in [1, 2^*]$, $d \neq 2$, or $p \in [1, 2^*)$ when $d = 2$. Hence, by (4.14), $\|f\|_{L^p(\mathbb{R}^d)}$ is uniformly bounded. In particular, by taking the $R \to +\infty$ limit, and using that $f \in \mathcal{H}_m$ has uniformly bounded second moments we obtain the uniform integrability at infinity.

Then, $\mathcal{H}_m$ is relatively compact in $L^m(\mathbb{R}^d)$ and combining it with (4.27) we obtain,

$$\rho_n \to \rho \quad \text{in } L^m(\mathbb{R}^d). \tag{4.29}$$

If $m = 1$, since $\mathcal{F}_1[\rho] \geq \mathcal{F}_2[\rho]$ we have that $\mathcal{H}_1 \subseteq \mathcal{H}_2$. From here, we recover (4.28), and (4.29) for $m = 2$. We show $\rho_n \log \rho_n \to \rho \log \rho$ in $L^1(\mathbb{R}^d)$ via an extended version of Lebesgue's Dominated Convergence Theorem [281, Chapter 4, Theorem 17]. Note that strong convergence in $L^2(\mathbb{R}^d)$ implies that, up to a subsequence,

$$\rho_n \log \rho_n \to \rho \log \rho \quad \text{a.e. in } x \in \mathbb{R}^d.$$

Furthermore, it is easy to check the majorant $|\rho_n(x) \log \rho_n(x)| \leq \rho_n^2(x) + \rho_n^{\frac{1}{2}}(x)$, for any $x \in \mathbb{R}^d$. We claim that $\rho_n^2 + \rho_n^{\frac{1}{2}} \to \rho^2 + \rho^{\frac{1}{2}}$ strongly in $L^1(\mathbb{R}^d)$. Since $\mathcal{H}_1 \subseteq \mathcal{H}_2$ it is enough to show $\rho_n^{\frac{1}{2}} \to \rho^{\frac{1}{2}}$ strongly in $L^1(\mathbb{R}^d)$. Applying Jensen's inequality for concave functions we have $\rho_n^{\frac{1}{2}} \in L^1(\mathbb{R}^d)$, while continuity of the square root function ensures

$$\rho_n^{\frac{1}{2}}(x) \to \rho^{\frac{1}{2}}(x) \quad \text{a.e. in } x \in \mathbb{R}^d.$$

By applying Fatou's Lemma,

$$\liminf_{n\to\infty} \int_{\mathbb{R}^d} \rho_n^{\frac{1}{2}}\, dx \geq \int_{\mathbb{R}^d} \rho^{\frac{1}{2}}\, \mathrm{d}x\,, \tag{4.30}$$

and concavity implies,

$$\limsup_{n\to\infty} \int_{\mathbb{R}^d} \rho_n^{\frac{1}{2}}\, \mathrm{d}x \leq \int_{\mathbb{R}^d} \rho^{\frac{1}{2}}\, \mathrm{d}x. \tag{4.31}$$

Combining (4.30) and (4.31) we infer

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} \rho_n^{\frac{1}{2}}\, dx = \int_{\mathbb{R}^d} \rho^{\frac{1}{2}}\, \mathrm{d}x.$$

Applying the extended Dominated Convergence Theorem we obtain

$$\rho_n \log \rho_n \to \rho \log \rho \quad \text{in } L^1(\mathbb{R}^d).$$

Step 2.b: Weak L^2 convergence of $\nabla\rho_n$. Given that $\nabla\rho$ is bounded in $L^2(\mathbb{R}^d)$, from Banach-Alaoglu Theorem we obtain that up to a subsequence,

$$\nabla\rho_n \rightharpoonup \nabla\rho \quad \text{weakly in } L^2(\mathbb{R}^d).$$

Note that the limit is $\nabla\rho$, which can be checked by testing $\nabla\rho$ against a smooth and compactly supported test function, and using the convergence $\rho_n \to \rho$ in $L^m(\mathbb{R}^d)$ that we proved in the previous step.

Step 3: Existence of minimisers. Due to the Weierstrass criterion for the existence of minimisers, cf. e.g. [283, Box 1.1], $\mathcal{A}_m$ has at least one minimiser in $\mathcal{H}_m$. □

As mentioned in Section 4.3.1, the proof of Proposition 4.16 can be obtained by adapting the previous one to the functional $\mathcal{L} : \mathcal{P}^a(\mathbb{R}^d) \to \mathbb{R}$ given by

$$\mathcal{L}[u] = \mathcal{F}_{m_c}[u] + \frac{b}{2} \int |z|^2 u(z)\, \mathrm{d}z.$$

Proof of Proposition 4.16. Boundedness from below follows from Gagliardo–Nirenberg inequality and non-negativity of the additional term in $\mathcal{L}[u]$, as noted in Proposition 4.13. For a minimising sequence $\{u_n\}_{n\in\mathbb{N}}$, since $\chi < \chi_c$ we derive the following bounds, again as a consequence of Gagliardo–Nirenberg, cf. Proposition 4.13:

$$\|u_n\|_{L^p(\mathbb{R}^d)} \leq C, \quad \|\nabla u_n\|_{L^2(\mathbb{R}^d)} \leq C, \quad \mathrm{m}_2(u_n) \leq C,$$

for any $p \in [1, 2^*]$, $d \neq 2$, and for any $p \in [1, 2^*)$ when $d = 2$; the constant $C = C(m_c, p, d, \chi) > 0$. Kolmogorov–Riesz–Fréchet Theorem provides relatively compactness in $L^{m_c}(\mathbb{R}^d)$ for $\{f \in \mathcal{P}^a(\mathbb{R}^d) : \mathrm{m}_2(f), \|\nabla f\|_{L^2(\mathbb{R}^d)}, |\mathcal{L}[\rho]| \leq C\}$, arguing as in Proposition 4.18. For the sake of completeness, we point out the additional term is lower semicontinuous with respect to the weak-L^2 convergence by applying a cut-off and the monotone convergence Theorem. Choosing $p = 2$ we infer weak-L^2 convergence of u_n from the above uniform bounds. Proceeding as in Proposition 4.18, and for $\chi < \chi_c$, we can show existence of minimisers in $\{u \in \mathcal{P}^a(\mathbb{R}^d) : \nabla u \in L^2(\mathbb{R}^d), \mathrm{m}_2(u) < +\infty\}$. □

Proposition 4.18 guarantees the sequence is well-defined, as we can solve the minimisation problem in (4.23). Next, we set up the approximating solution to (4.1). Let $T > 0$, and consider $N := \left[\frac{T}{\tau}\right]$. We define the curve $\rho_\tau : [0,T] \to \mathcal{P}(\mathbb{R}^d)$ as the piecewise constant interpolation

$$\rho_\tau(t) := \rho_\tau^k, \quad t \in ((k-1)\tau, k\tau]\,, \tag{4.32}$$

where ρ_τ^k is defined in (4.23). We can prove convergence of this piecewise interpolation to a continuous curve with respect to the 2-Wasserstein distance.

Lemma 4.20 (Narrow convergence & discrete uniform estimates)**.** *Let $\rho_0 \in \mathcal{P}_2^a(\mathbb{R}^d)$ such that $\mathcal{F}_m[\rho_0] < +\infty$ and $1 \le m < 2 + \frac{2}{d}$ or $m = 2 + \frac{2}{d}$ with subcritical mass $\chi < \chi_c$. There exists an absolutely continuous curve $\tilde{\rho} : [0,T] \to \mathcal{P}_2(\mathbb{R}^d)$ such that, up to a subsequence, $\rho_\tau(t)$ narrowly converges to $\tilde{\rho}(t)$, uniformly in $t \in [0,T]$. Moreover, we obtain the following discrete uniform bounds:*

$$\sup_k \|\nabla \rho_\tau^k\|_{L^2(\mathbb{R}^d)} \le C_1 \left(\mathcal{F}_m[\rho_0] + \|\rho_0\|_{L^1(\mathbb{R}^d)}^\alpha\right)^{1/2} < +\infty; \tag{4.33}$$

$$\sup_k \|\rho_\tau^k\|_{L^p(\mathbb{R}^d)} \le C_2 < +\infty; \tag{4.34}$$

$$\mathrm{m}_2(\rho_\tau^k) \le 2\mathrm{m}_2(\rho_0) + 4T\left(\mathcal{F}_m[\rho_0] + C\right), \tag{4.35}$$

for constants $C_1 = C_1(m,d,\chi) > 0$ and $C_2 = C_2(m,p,d,\rho_0,\chi) > 0$, for any $p \in [1, 2^*]$, $d \ne 2$, and any $p \in [1, 2^*)$ when $d = 2$.

Proof. By construction of the sequence we have

$$\mathcal{F}_m[\rho_\tau^k] \le \frac{\mathcal{W}_2^2(\rho_\tau^k, \rho_\tau^{k-1})}{2\tau} + \mathcal{F}_m[\rho_\tau^k] \le \mathcal{F}_m[\rho_\tau^{k-1}]. \tag{4.36}$$

In particular, this gives

$$\sup_k \mathcal{F}_m[\rho_\tau^k] \le \mathcal{F}_m[\rho_0] < +\infty,$$

which together with (4.13) and (4.14) implies that $\|\nabla \rho_\tau^k\|_{L^2(\mathbb{R}^d)}$ and $\|\rho_\tau^k\|_{L^p(\mathbb{R}^d)}$ are uniformly bounded in k and τ for $p \in [1, 2^*]$, $d \ne 2$, or $p \in [1, 2^*)$ when $d = 2$. Hence we obtain (4.33) and (4.34).

Next, by summing up over k in (4.36) and using that the free energy is bounded from below, (4.12), we deduce

$$\sum_{k=i+1}^{j} \frac{\mathcal{W}_2^2(\rho_\tau^k, \rho_\tau^{k-1})}{2\tau} \le \mathcal{F}_m[\rho_\tau^i] - \mathcal{F}_m[\rho_\tau^j] \le \mathcal{F}_m[\rho_0] + C. \tag{4.37}$$

Therefore the 2-Wasserstein distance between ρ_0 and $\rho_\tau(t)$ is uniformly bounded. Indeed, for $t \in ((j-1)\tau, j\tau]$,

$$\mathcal{W}_2^2(\rho_0, \rho_\tau(t)) \le j \sum_{k=1}^{j} \mathcal{W}_2^2(\rho_\tau^k, \rho_\tau^{k-1}) \le 2j\tau(\mathcal{F}_m[\rho_0] + C) \le 2T(\mathcal{F}_m[\rho_0] + C).$$

Furthermore, we obtain second order moments are uniformly bounded on compact time intervals $[0,T]$ since

$$\mathrm{m}_2\left(\rho_\tau(t)\right) \le 2\mathrm{m}_2(\rho_0) + 2\mathcal{W}_2^2(\rho_0, \rho_\tau(t)) \le 2\mathrm{m}_2(\rho_0) + 4T\left(\mathcal{F}_m[\rho_0] + C\right).$$

Let us now prove equi-continuity. Consider $0 \le s < t$ such that $s \in ((i-1)\tau, i\tau]$ and $t \in ((j-1)\tau, j\tau]$. Using Cauchy–Schwarz inequality and (4.37) we have

$$\begin{aligned} \mathcal{W}_2(\rho_\tau(s), \rho_\tau(t)) &\le \sum_{k=i+1}^{j} \mathcal{W}_2(\rho_\tau^k, \rho_\tau^{k-1}) \le \left(\sum_{k=i+1}^{j} \mathcal{W}_2^2(\rho_\tau^k, \rho_\tau^{k-1})\right)^{\frac{1}{2}} |j-i|^{\frac{1}{2}} \\ &\le \left(2(\mathcal{F}_m[\rho_0] + C)\right)^{\frac{1}{2}} \left(\sqrt{|t-s|} + \sqrt{\tau}\right). \end{aligned} \tag{4.38}$$

Thus, ρ_τ is $\frac{1}{2}$-Hölder equi-continuous up to a negligible error of order $\sqrt{\tau}$. Therefore, by a refined version of the Ascoli-Arzelà Theorem [7, Proposition 3.3.1], we obtain that ρ_τ admits a subsequence narrowly converging to a limit $\tilde{\rho}$ as $\tau \to 0^+$, uniformly on $[0,T]$. Moreover, using the uniform bound (4.35) and that $|\cdot|^2$ is lower semicontinuous and bounded from below, we obtain that the limiting curve $\tilde{\rho}$ has bounded second order moments,

$$\mathrm{m}_2(\tilde{\rho}(t)) \leq \liminf_{\tau\downarrow 0} \mathrm{m}_2(\rho_\tau(t)), \quad \forall t \in [0,T].$$

□

The bounds (4.34) and (4.33) imply weak convergence of the interpolation ρ_τ to a probability density $\tilde{\rho}$ with regularity provided below.

Proposition 4.21 (Weak convergence). *Let $\rho_0 \in \mathcal{P}_2^a(\mathbb{R}^d)$ such that $\mathcal{F}_m[\rho_0] < +\infty$ and $1 \leq m < 2 + \frac{2}{d}$ or $m = 2 + \frac{2}{d}$ with subcritical mass $\chi < \chi_c$. The piecewise interpolation ρ_τ in (4.32) is such that $\rho_\tau \in L^\infty([0,T]; L^p(\mathbb{R}^d)) \cap L^\infty([0,T]; H^1(\mathbb{R}^d))$, for any $p \in [1, 2^*]$, $d \neq 2$, and $p \in [1, 2^*)$ when $d = 2$. In particular, the limit $\tilde{\rho}$ belongs to $L^\infty([0,T]; L^p(\mathbb{R}^d)) \cap L^\infty([0,T]; H^1(\mathbb{R}^d))$ and*

$$\rho_\tau \rightharpoonup \tilde{\rho} \quad \text{in } L^2([0,T]; H^1(\mathbb{R}^d)).$$

Proof. From (4.34) in Lemma 4.20 we have

$$\|\rho_\tau\|_{L^\infty([0,T];L^p(\mathbb{R}^d))} = \sup_{t\in(0,T)} \|\rho_\tau(t)\|_{L^p(\mathbb{R}^d)} = \sup_k \|\rho_\tau^k\|_{L^p(\mathbb{R}^d)} < +\infty,$$

for $p \in [1, 2^*]$, $d \neq 2$, and $p \in [1, 2^*)$ when $d = 2$. Analogously, from (4.33) we obtain $\nabla\rho_\tau \in L^\infty([0,T]; L^2(\mathbb{R}^d))$. In particular, for any compact time interval $[0,T]$ with $T > 0$, we have $\|\rho_\tau\|_{L^2([0,T];H^1(\mathbb{R}^d))} \leq C$ uniformly in τ and the weak convergence follows from Banach-Alaoglu Theorem. Regularity of the limit follows from standard arguments. □

The uniform-in-τ $L^\infty([0,T]; H^1(\mathbb{R}^d))$ estimate allows us to obtain strong convergence of ρ_τ via a refined version of the Aubin-Lions Lemma due to Rossi and Savaré — cf. Proposition 4.7.

Proposition 4.22 (Strong convergence of ρ_τ). *Let $\rho_0 \in \mathcal{P}_2^a(\mathbb{R}^d)$ such that $\mathcal{F}_m[\rho_0] < +\infty$ and $1 \leq m < 2 + \frac{2}{d}$ or $m = 2 + \frac{2}{d}$ with subcritical mass $\chi < \chi_c$. The sequence ρ_τ converges, up to a subsequence, strongly to the curve $\tilde{\rho}$ in $L^2([0,T]; L^2(\mathbb{R}^d))$ for every $T > 0$.*

Proof. We apply Proposition 4.7 to a subset $U = \{\rho_\tau\}_{\tau\geq 0}$ for $X = L^2(\mathbb{R}^d)$ and $g := \mathcal{W}_2$, the 2-Wasserstein distance. Further, we consider the functional $\mathcal{I} : L^2(\mathbb{R}^d) \to [0, +\infty]$ defined by

$$\mathcal{I}[\rho] = \begin{cases} \|\rho\|^2_{H^1(\mathbb{R}^d)} + \int_{\mathbb{R}^d} |x|^2 \rho(x)\,\mathrm{d}x & \rho \in \mathcal{P}_2(\mathbb{R}^d) \cap H^1(\mathbb{R}^d), \\ +\infty & \text{otherwise.} \end{cases}$$

Note that $\mathcal{W}_2$ is a distance on the proper domain of $\mathcal{I}$. Indeed, if $\mathcal{I}[\rho] < \infty$ then $\rho \in \mathcal{P}_2(\mathbb{R}^d)$. Lower semicontinuity of $\mathcal{I}$ follows from standard arguments — see for instance [59].

Next, let $B_c = \{\rho \in L^2(\mathbb{R}^d) : \mathcal{I}[\rho] \leq c\}$ be a sublevel of $\mathcal{I}$. We notice that $B_c \subset \mathcal{P}_2(\mathbb{R}^d) \cap H^1(\mathbb{R}^d)$ and thus we can apply Kolmogorov-Riesz-Fréchet Theorem [54, Corollary 4.27] as in the proof of Proposition 4.18 to obtain that B_c is relatively compact. Hence we have $\mathcal{I}$ is an admissible functional.

The tightness condition (4.8) follows from the uniform-in-τ second order moment and $L^\infty([0,T]; H^1(\mathbb{R}^d))$ bounds for ρ_τ given in (4.35) and Proposition 4.21. The integral equi-continuity condition (4.9) can be seen from the Hölder equi-continuity of ρ_τ, proved in Lemma 4.20. More precisely, for $h > \tau$ we have

$$\int_0^{T-h} \mathcal{W}_2\left(\rho_\tau(t+h), \rho_\tau(t)\right) \mathrm{d}t \leq \int_0^{T-h} C\left(\sqrt{h} + \sqrt{\tau}\right) \mathrm{d}t \leq 2CT\sqrt{h},$$

for a constant $C > 0$ independent of τ and h. If instead, $h < \tau$, we can write

$$\int_0^{T-h} \mathcal{W}_2\left(\rho_\tau(t+h), \rho_\tau(t)\right) \mathrm{d}t \leq h \sum_{k=0}^{N-1} \mathcal{W}_2(\rho_\tau^{k+1}, \rho_\tau^k) \leq h\sqrt{N} \sum_{k=0}^{N-1} \mathcal{W}_2^2(\rho_\tau^{k+1}, \rho_\tau^k) \leq Ch\sqrt{T},$$

where $C > 0$ is the constant defined in (4.37).

Hence we can apply Proposition 4.7 to obtain that there exists a subsequence, that we label by $\tau \downarrow 0$, such that ρ_τ converges in measure to $\tilde{\rho}$, as in (4.10), where $X := L^2(\mathbb{R}^d)$. Let us denote by $A_\delta(\tau) := \{t \in (0,T) : \|\rho_\tau(t) - \tilde{\rho}(t)\|_X \geq \delta\}$, which vanishes as $\tau \to 0$. Owing to (4.34) and Proposition 4.21 we can prove (see e.g. [90, Proposition 4.3])

$$\limsup_{\tau \to 0} \|\rho_\tau - \tilde{\rho}\|^2_{L^2([0,T];L^2(\mathbb{R}^d))} \leq \delta T^{1/2},$$

hence strong convergence in $L^2([0,T];L^2(\mathbb{R}^d))$ since δ is arbitrarily small. □

4.4.1 Flow interchange

The strong convergence of the sequence ρ_τ obtained in Proposition 4.22 is not enough to pass to the limit in the Euler-Lagrange equation associated to (4.23) and arrive to a weak formulation of our equation. We use the heat equation as auxiliary flow to obtain uniform bounds on the Hessian of the sequence $\{\rho_\tau\}_\tau$, cf. section 4.2. More precisely, we exploit that the heat equation is a 2-Wasserstein gradient flow of the entropy functional $\mathcal{E}[\rho] = \int \rho \log \rho \, \mathrm{d}x$.

In the following, for $\mu \in \mathcal{P}_2(\mathbb{R}^d)$ such that $\mathcal{E}[\mu] < \infty$, we denote by $S^t_{\mathcal{E}}\mu$ the solution at time t of the heat equation for an initial value μ at $t = 0$. Furthermore, we also define the dissipation of $\mathcal{F}_m$ along $S_{\mathcal{E}}$ by

$$D_{\mathcal{E}}\mathcal{F}_m[\rho] := \limsup_{s \downarrow 0} \left\{ \frac{\mathcal{F}_m[\rho] - \mathcal{F}_m[S^s_{\mathcal{E}}\rho]}{s} \right\}.$$

Remark 4.23. *Given some initial datum $\mu_0 \in \mathcal{P}(\mathbb{R}^d)$ the solution of the heat equation, $S^t_{\mathcal{E}}\mu_0$, can be written as the convolution of the heat kernel G_t with the initial condition, i.e.*

$$S^t_{\mathcal{E}}\mu_0 = G_t * \mu_0 = (4\pi t)^{-d/2} \int_{\mathbb{R}^d} e^{-|x-y|^2/4t} \mathrm{d}\mu_0(y).$$

As a consequence, $S^t_{\mathcal{E}}\mu_0 \in C^\infty\left((0,+\infty) \times \mathbb{R}^d\right)$. Moreover, for solutions of the heat equation we can integrate by parts to obtain the well-known equality

$$\int_{\mathbb{R}^d} |\Delta S^t_{\mathcal{E}}\mu_0|^2 \, \mathrm{d}x = \int_{\mathbb{R}^d} |D^2 S^t_{\mathcal{E}}\mu_0|^2 \, \mathrm{d}x \,. \tag{4.39}$$

We are now ready to prove an H^2 bound for ρ^τ.

Lemma 4.24 (H^2 uniform bound)**.** *Let ρ_0 such that $\mathcal{F}_m[\rho_0] < +\infty$, and $1 \leq m < 2 + \frac{2}{d}$ or $m = 2 + \frac{2}{d}$ with sub-critical mass $\chi < \chi_c$. The piecewise interpolation ρ_τ constructed in (4.32) is such that $\rho_\tau \in L^2([0,T];H^2(\mathbb{R}^d))$. In particular, we obtain the uniform-in-τ bound*

$$\|\Delta\rho_\tau\|^2_{L^2([0,T];L^2(\mathbb{R}^d))} \leq d\|D^2\rho_\tau\|^2_{L^2([0,T];L^2(\mathbb{R}^d))} \leq C,$$

where $C = C(m,d,\rho_0,T) > 0$.

Proof. For all $s > 0$, we consider $S^s_{\mathcal{E}}\rho^{k+1}_\tau$. Then, by the definition of the scheme (4.23) and of ρ^{k+1}_τ, we have the inequality

$$\frac{1}{2\tau}\mathcal{W}_2^2(\rho^k_\tau, \rho^{k+1}_\tau) + \mathcal{F}_m[\rho^{k+1}_\tau] \leq \frac{1}{2\tau}\mathcal{W}_2^2(\rho^k_\tau, S^s_{\mathcal{E}}\rho^{k+1}_\tau) + \mathcal{F}_m[S^s_{\mathcal{E}}\rho^{k+1}_\tau],$$

from which we obtain,

$$\tau\frac{\mathcal{F}_m[\rho^{k+1}_\tau] - \mathcal{F}_m[S^s_{\mathcal{E}}\rho^{k+1}_\tau]}{s} \leq \frac{1}{2}\frac{\mathcal{W}_2^2(\rho^k_\tau, S^s_{\mathcal{E}}\rho^{k+1}_\tau) - \mathcal{W}_2^2(\rho^k_\tau, \rho^{k+1}_\tau)}{s}.$$

By taking the $\limsup$ as $s \downarrow 0$ we obtain

$$\tau D_{\mathcal{E}}\mathcal{F}_m[\rho^{k+1}_\tau] \leq \frac{1}{2}\left.\frac{\mathrm{d}^+}{\mathrm{d}t}\right|_{t=0} \mathcal{W}_2^2(\rho^k_\tau, S^t_{\mathcal{E}}\rho^{k+1}_\tau) \leq \mathcal{E}[\rho^k_\tau] - \mathcal{E}[\rho^{k+1}_\tau], \tag{4.40}$$

where in the last inequality we use the (EVI), as $S_\mathcal{E}$ is a 0-flow, cf. Definition 4.8. Note that

$$D_\mathcal{E}\mathcal{F}_m[\rho_\tau^{k+1}] = \limsup_{s\downarrow 0}\left\{\frac{\mathcal{F}_m[\rho_\tau^{k+1}]-\mathcal{F}_m[S_\mathcal{E}^s\rho_\tau^{k+1}]}{s}\right\} = \limsup_{s\downarrow 0}\int_0^1\left(-\frac{\mathrm{d}}{\mathrm{d}z}\bigg|_{z=st}\mathcal{F}_m[S_\mathcal{E}^z\rho_\tau^{k+1}]\right)\mathrm{d}t. \quad (4.41)$$

From this point of the proof, we distinguish between two cases.

Case I: $1<m<2+\frac{2}{d}$ *or* $m=2+\frac{2}{d}$ *with subcritical mass* $\chi<\chi_c$. Let us compute the time derivative:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_m[S_\mathcal{E}^t\rho_\tau^{k+1}] = -\int_{\mathbb{R}^d}|\Delta S_\mathcal{E}^t\rho_\tau^{k+1}|^2\mathrm{d}x - \frac{\chi m}{m-1}\int_{\mathbb{R}^d}(S_\mathcal{E}^t\rho_\tau^{k+1})^{m-1}\Delta S_\mathcal{E}^t\rho_\tau^{k+1}\mathrm{d}x. \quad (4.42)$$

Therefore, combining (4.40), (4.41) and (4.42) we obtain

$$\tau\limsup_{s\downarrow 0}\int_0^1\left(\int_{\mathbb{R}^d}|\Delta S_\mathcal{E}^{st}\rho_\tau^{k+1}|^2\,\mathrm{d}x + \frac{\chi m}{m-1}\int_{\mathbb{R}^d}(S_\mathcal{E}^{st}\rho_\tau^{k+1})^{m-1}\Delta S_\mathcal{E}^{st}\rho_\tau^{k+1}\,\mathrm{d}x\right)\mathrm{d}t \le \mathcal{E}[\rho_\tau^k]-\mathcal{E}[\rho_\tau^{k+1}].$$

By applying Young's inequality we have

$$\begin{aligned}
&\int_{\mathbb{R}^d}|\Delta S_\mathcal{E}^{st}\rho_\tau^{k+1}|^2\,\mathrm{d}x + \frac{\chi m}{m-1}\int_{\mathbb{R}^d}(S_\mathcal{E}^{st}\rho_\tau^{k+1})^{m-1}\Delta S_\mathcal{E}^{st}\rho_\tau^{k+1}\,\mathrm{d}x\\
&\quad\ge \int_{\mathbb{R}^d}|\Delta S_\mathcal{E}^{st}\rho_\tau^{k+1}|^2\,\mathrm{d}x - \frac{\chi m}{m-1}\int_{\mathbb{R}^d}|(S_\mathcal{E}^{st}\rho_\tau^{k+1})^{m-1}||\Delta S_\mathcal{E}^{st}\rho_\tau^{k+1}|\,\mathrm{d}x\\
&\quad\ge \frac12\int_{\mathbb{R}^d}|\Delta S_\mathcal{E}^{st}\rho_\tau^{k+1}|^2\,\mathrm{d}x - \frac{\chi^2m^2}{2(m-1)^2}\int_{\mathbb{R}^d}|S_\mathcal{E}^{st}\rho_\tau^{k+1}|^{2(m-1)}\,\mathrm{d}x\,,
\end{aligned}$$

which gives

$$\begin{aligned}
&\frac{\tau}{2}\liminf_{s\downarrow 0}\int_0^1\int_{\mathbb{R}^d}|\Delta S_\mathcal{E}^{st}\rho_\tau^{k+1}|^2\,\mathrm{d}x\,\mathrm{d}t\\
&\quad\le \mathcal{E}[\rho_\tau^k]-\mathcal{E}[\rho_\tau^{k+1}] + \tau\frac{\chi^2m^2}{2(m-1)^2}\limsup_{s\downarrow 0}\int_0^1\int_{\mathbb{R}^d}|S_\mathcal{E}^{st}\rho_\tau^{k+1}|^{2(m-1)}\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

In order to take the $s\downarrow 0$ limit in the above expression, first we note that, in view of Remark 4.23, we can write $\|\Delta S_\mathcal{E}^{st}\rho_\tau^{k+1}\|_{L^2(\mathbb{R}^d)} = \|D^2S_\mathcal{E}^{st}\rho_\tau^{k+1}\|_{L^2(\mathbb{R}^d)}$. Since the auxiliary flow is the heat equation with initial datum $\rho_\tau^{k+1}\in H^1(\mathbb{R}^d)$, we have $S_\mathcal{E}^t\rho_\tau^{k+1}\to\rho_\tau^{k+1}$ in $L^2(\mathbb{R}^d)$ as well as $\nabla S_\mathcal{E}^t\rho_\tau^{k+1}\to\nabla\rho_\tau^{k+1}$ in $L^2(\mathbb{R}^d)$ as $t\downarrow 0$ — by noticing that $\nabla S_\mathcal{E}^t\rho_\tau^{k+1}$ is a solution to the heat equation with initial datum $\nabla\rho_\tau^{k+1}\in L^2(\mathbb{R}^d)$. By the weak lower-semicontinuity of the H^1 seminorm we have

$$\liminf_{s\downarrow 0}\int_0^1\int_{\mathbb{R}^d}|D^2S_\mathcal{E}^{st}\rho_\tau^{k+1}|^2\,\mathrm{d}x\,\mathrm{d}t \ge \int_{\mathbb{R}^d}|D^2\rho_\tau^{k+1}|^2\,\mathrm{d}x\,. \quad (4.43)$$

Next, we focus on the term involving $\|S_\mathcal{E}^{st}\rho_\tau^{k+1}\|_{L^{2(m-1)}(\mathbb{R}^d)}$ and distinguish between two cases, depending on the value of m. We apply Young's convolution inequality to $S_\mathcal{E}^{st}\rho_\tau^{k+1} = G_{st}*\rho_\tau^{k+1}$, as noticed in Remark 4.23.

If $\frac32\le m<2+\frac{2}{d}$ or $m=2+\frac{2}{d}$ with subcritical mass, then $1\le 2(m-1)<2^*$ and, by (4.14), $\rho_\tau^{k+1}\in L^{2(m-1)}(\mathbb{R}^d)$. Furthermore, we have

$$\|S_\mathcal{E}^{st}\rho_\tau^{k+1}\|_{L^{2(m-1)}(\mathbb{R}^d)} \le \|G_{st}\|_{L^1(\mathbb{R}^d)}\|\rho_\tau^{k+1}\|_{L^{2(m-1)}(\mathbb{R}^d)} = \|\rho_\tau^{k+1}\|_{L^{2(m-1)}(\mathbb{R}^d)},$$

In particular, we obtain

$$\limsup_{s\downarrow 0}\int_0^1\int_{\mathbb{R}^d}|S_\mathcal{E}^{st}\rho_\tau^{k+1}|^{2(m-1)}\,\mathrm{d}x\,\mathrm{d}t \le \int_{\mathbb{R}^d}|\rho_\tau^{k+1}|^{2(m-1)}\,\mathrm{d}x\,.$$

If $1 < m < \frac{3}{2}$, we use that the function $|\cdot|^{2(m-1)}$ is concave and apply Jensen's inequality to find

$$\int_{\mathbb{R}^d} |S_{\mathcal{E}}^{st}\rho_\tau^{k+1}|^{2(m-1)}\,\mathrm{d}x \le \left|\int_{\mathbb{R}^d} S_{\mathcal{E}}^{st}\rho_\tau^{k+1}\,\mathrm{d}x\right|^{2(m-1)} = \|G_{st} * \rho_\tau^{k+1}\|_{L^1(\mathbb{R}^d)}^{2(m-1)}$$
$$\le \|G_{st}\|_{L^1(\mathbb{R}^d)}^{2(m-1)} \|\rho_\tau^{k+1}\|_{L^1(\mathbb{R}^d)}^{2(m-1)} = \|\rho_\tau^{k+1}\|_{L^1(\mathbb{R}^d)}^{2(m-1)},$$

whence

$$\limsup_{s\downarrow 0} \int_0^1 \int_{\mathbb{R}^d} |S_{\mathcal{E}}^{st}\rho_\tau^{k+1}|^{2(m-1)}\,\mathrm{d}x\,\mathrm{d}t \le \left|\int_{\mathbb{R}^d} \rho_\tau^{k+1}\,\mathrm{d}x\right|^{2(m-1)}.$$

As a consequence,

$$\frac{\tau}{2}\int_{\mathbb{R}^d} |D^2\rho_\tau^{k+1}|^2\,\mathrm{d}x \le \mathcal{E}[\rho_\tau^k] - \mathcal{E}[\rho_\tau^{k+1}] + \tau\frac{\chi^2 m^2}{2(m-1)^2}\|\rho_\tau^{k+1}\|_{L^q(\mathbb{R}^d)}^{2(m-1)},$$

with $q = 2(m-1)$ for $\frac{3}{2} \le m < 2 + \frac{2}{d}$ or $m = 2 + \frac{2}{d}$ with subcritical mass, and $q = 1$ for $1 < m < \frac{3}{2}$. By summing up over k from 0 to $N-1$, considering that $x\log x \le x^2$ and Remark 4.9, we recover, further using Jensen's inequality for concave functions for $q = 1$,

$$\frac{1}{2}\|D^2\rho_\tau\|_{L^2([0,T];L^2(\mathbb{R}^d))}^2 \le \mathcal{E}[\rho_0] - \mathcal{E}[\rho_\tau^N] + \frac{\chi^2 m^2}{2(m-1)^2}\sum_{k=0}^{N-1}\tau\|\rho_\tau^{k+1}\|_{L^q(\mathbb{R}^d)}^{2(m-1)}$$
$$\le \|\rho_0\|_{L^2(\mathbb{R}^d)}^2 + C(1 + \mathrm{m}_2(\rho_\tau^N)) + \frac{\chi^2 m^2}{2(m-1)^2 T^{2(m-1)-1}}\|\rho_\tau\|_{L^q([0,T];L^q(\mathbb{R}^d))}^{2(m-1)},$$

which is uniformly bounded, due to Lemma 4.20. In particular, we also obtain

$$\|\Delta\rho_\tau\|_{L^2([0,T];L^2(\mathbb{R}^d))} \le \sqrt{d}\|D^2\rho_\tau\|_{L^2([0,T];L^2(\mathbb{R}^d))} \le C(m,d,\rho_0,\chi,T).$$

Case II: $m = 1$. Let us compute the time derivative,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_1[S_{\mathcal{E}}^t\rho_\tau^{k+1}] = -\int_{\mathbb{R}^d}|\Delta S_{\mathcal{E}}^t\rho_\tau^{k+1}|^2\,\mathrm{d}x - \chi\int_{\mathbb{R}^d}\Delta S_{\mathcal{E}}^t\rho_\tau^{k+1}(1+\log S_{\mathcal{E}}^t\rho_\tau^{k+1})\,\mathrm{d}x$$
$$= -\int_{\mathbb{R}^d}|\Delta S_{\mathcal{E}}^t\rho_\tau^{k+1}|^2\,\mathrm{d}x + \chi\int_{\mathbb{R}^d}\nabla S_{\mathcal{E}}^t\rho_\tau^{k+1}\cdot\nabla\log S_{\mathcal{E}}^t\rho_\tau^{k+1}\,\mathrm{d}x\,. \tag{4.44}$$

By combining (4.40), (4.41) and (4.44), we obtain

$$\tau\limsup_{s\downarrow 0}\int_0^1\left(\int_{\mathbb{R}^d}|\Delta S_{\mathcal{E}}^{st}\rho_\tau^{k+1}|^2\,\mathrm{d}x - \chi\int_{\mathbb{R}^d}\nabla S_{\mathcal{E}}^{st}\rho_\tau^{k+1}\cdot\nabla\log S_{\mathcal{E}}^{st}\rho_\tau^{k+1}\,\mathrm{d}x\right)\mathrm{d}t \le \mathcal{E}[\rho_\tau^k] - \mathcal{E}[\rho_\tau^{k+1}]\,.$$

Similarly to the previous case, we obtain

$$\tau\liminf_{s\downarrow 0}\int_0^1\int_{\mathbb{R}^d}|\Delta S_{\mathcal{E}}^{st}\rho_\tau^{k+1}|^2\,\mathrm{d}x\,\mathrm{d}t$$
$$\le \mathcal{E}[\rho_\tau^k] - \mathcal{E}[\rho_\tau^{k+1}] + \chi\tau\limsup_{s\downarrow 0}\int_0^1\int_{\mathbb{R}^d}\nabla S_{\mathcal{E}}^{st}\rho_\tau^{k+1}\cdot\nabla\log S_{\mathcal{E}}^{st}\rho_\tau^{k+1}\,\mathrm{d}x\,\mathrm{d}t$$
$$= \mathcal{E}[\rho_\tau^k] - \mathcal{E}[\rho_\tau^{k+1}] + \chi\tau\limsup_{s\downarrow 0}\left(\mathcal{E}[\rho_\tau^{k+1}] - \mathcal{E}[S_{\mathcal{E}}^s\rho_\tau^{k+1}]\right),$$

where we recognised the third term as the Fisher information functional for solutions of the heat equation. Next, using well-known properties of the heat equation and the estimates in Lemma 4.20 we have

$$\limsup_{s\downarrow 0}\left(\mathcal{E}[\rho_\tau^{k+1}] - \mathcal{E}[S_{\mathcal{E}}^s\rho_\tau^{k+1}]\right) \le C,$$

for a constant C independent of k. By summing up over k from 0 to $N-1$, and using (4.39) and (4.43) again we obtain

$$\|D^2\rho_\tau\|^2_{L^2([0,T];L^2(\mathbb{R}^d))} \le \mathcal{E}[\rho_0] - \mathcal{E}[\rho_\tau^N] + \tau N C,$$

and in particular, $\Delta\rho_\tau$ is uniformly bounded in $L^2([0,T];L^2(\mathbb{R}^d))$. □

Proposition 4.25 (Strong convergence of $\nabla\rho_\tau$)**.** *Let ρ_0 be such that $\mathcal{F}_m[\rho_0] < +\infty$, and $1 \le m < 2+\frac{2}{d}$ or $m = 2+\frac{2}{d}$ with subcritical mass $\chi < \chi_c$. Up to a subsequence, the sequence $\rho_\tau : [0,T] \to \mathcal{P}_2(\mathbb{R}^d)$ converges strongly to the curve $\tilde\rho$ in $L^2([0,T];H^1(\mathbb{R}^d))$.*

Proof. First note that due to Lemma 4.24, $D^2\rho_\tau \rightharpoonup D^2\tilde\rho$ in $L^2([0,T];L^2(\mathbb{R}^d))$. The limit can be uniquely identified by integrating against a smooth and compactly supported test function and using the convergence $\nabla\rho_\tau \rightharpoonup \nabla\tilde\rho$ in $L^2([0,T];L^2(\mathbb{R}^d))$, cf. Proposition 4.21. Next, we claim strong convergence of ρ_τ in $L^2([0,T];H^1(\mathbb{R}^d))$ follows from the strong convergence in $L^2([0,T];L^2(\mathbb{R}^d))$, cf. Proposition 4.22, and the fact that the term $\|\rho_\tau\|_{L^2([0,T];H^2(\mathbb{R}^d))}$ is uniformly bounded in τ, as given in Lemma 4.24. More precisely, using Gagliardo–Nirenberg (for the gradient) and Cauchy–Schwarz inequalities, we obtain

$$\begin{aligned}\int_0^T \|\nabla\rho_\tau(t) - \nabla\tilde\rho(t)\|^2_{L^2(\mathbb{R}^d)}\,\mathrm{d}t &\le C\int_0^T \|D^2\rho_\tau(t) - D^2\tilde\rho(t)\|_{L^2(\mathbb{R}^d)}\|\rho_\tau(t)-\tilde\rho(t)\|_{L^2(\mathbb{R}^d)}\,\mathrm{d}t\\ &\le C\|D^2\rho_\tau - D^2\tilde\rho\|_{L^2([0,T];L^2(\mathbb{R}^d))}\|\rho_\tau - \tilde\rho\|_{L^2([0,T];L^2(\mathbb{R}^d))}.\end{aligned}$$

The result is obtained by using that the norms $\|D^2\rho_\tau\|_{L^2([0,T];L^2(\mathbb{R}^d))}$ and $\|D^2\tilde\rho\|_{L^2([0,T];L^2(\mathbb{R}^d))}$ are uniformly bounded in τ — Lemma 4.24. □

The strong convergence of $\nabla\rho_\tau$ allows us to improve the result of ρ_τ given by Proposition 4.22 via interpolation inequalities. In particular, we obtain the integrability exponent needed to pass to the limit $\tau \to 0$ in the weak formulation.

Corollary 4.26 (Higher integrability)**.** *Assume $1 < m < 2+\frac{2}{d}$ or $m = 2+\frac{2}{d}$ with subcritical mass $\chi < \chi_c$. Then, the sequence $\rho_\tau : [0,T] \to \mathcal{P}_2(\mathbb{R}^d)$ converges strongly, up to subsequence, to the curve $\tilde\rho$ in $L^m([0,T];L^m(\mathbb{R}^d))$ for every $T > 0$.*

Proof. The proof is based on that of Proposition 4.11. For $1 < m < 2+\frac{2}{d}$, by applying Gagliardo–Nirenberg and Hölder inequalities we obtain

$$\begin{aligned}&\int_0^T \|\rho_\tau(t) - \tilde\rho(t)\|^m_{L^m(\mathbb{R}^d)}\,\mathrm{d}t\\ &\quad\le C\int_0^T \|\nabla\rho_\tau(t) - \nabla\tilde\rho(t)\|^{m\theta}_{L^2(\mathbb{R}^d)}\|\rho_\tau(t)-\tilde\rho(t)\|^{m(1-\theta)}_{L^1(\mathbb{R}^d)}\,\mathrm{d}t \qquad (4.45)\\ &\quad\le C\|\nabla\rho_\tau - \nabla\tilde\rho\|^{m\theta}_{L^2([0,T];L^2(\mathbb{R}^d))}\left(\int_0^T \|\rho_\tau(t)-\tilde\rho(t)\|^{\alpha}_{L^1(\mathbb{R}^d)}\,\mathrm{d}t\right)^{\frac{m(1-\theta)}{\alpha}},\end{aligned}$$

where $\theta = \frac{2d}{d+2}\frac{m-1}{m} \in (0,1)$ and $\alpha = 1 + \frac{\frac{2}{d}(m-1)}{2+\frac{2}{d}-m}$. The result follows from the strong convergence of $\nabla\rho_\tau$ and by noting that the second term is uniformly bounded in τ due to the narrow convergence of ρ_τ given in Lemma 4.20, being ρ_t and ρ probability densities.

In the critical case $m = m_c = 2+\frac{2}{d}$, (4.45) gives

$$\begin{aligned}\int_0^T \|\rho_\tau(t) - \tilde\rho(t)\|^m_{L^m(\mathbb{R}^d)}\,\mathrm{d}t &\le C\int_0^T \|\nabla\rho_\tau(t) - \nabla\tilde\rho(t)\|^2_{L^2(\mathbb{R}^d)}\|\rho_\tau(t)-\tilde\rho(t)\|^{\frac{2}{d}}_{L^1(\mathbb{R}^d)}\,\mathrm{d}t\,,\\ &\le C\|\nabla\rho_\tau - \nabla\tilde\rho\|^2_{L^2([0,T];L^2(\mathbb{R}^d))}\|\rho_\tau(t)-\tilde\rho(t)\|^{\frac{2}{d}}_{L^\infty([0,T];L^1(\mathbb{R}^d))},\end{aligned}$$

where the second term is uniformly bounded in τ by Lemma 4.20 and Proposition 4.21. Again, the result follows from the strong convergence of $\nabla\rho_\tau$. □

4.4.2 Consistency of the scheme

The results from the previous subsection ensure we can prove that $\tilde{\rho}$ is a weak solution of (4.1) in the sense of Definition 4.2. This subsection completes the proof of Theorem 4.3.

Proof of Theorem 4.3. We prove the theorem by showing that the sequence $\rho_\tau : [0,T] \to \mathcal{P}_2(\mathbb{R}^d)$ converges, up to a subsequence, to a weak solution $\tilde{\rho}$ of (4.1). Let us focus on two consecutive steps in the JKO scheme, ρ_τ^k and ρ_τ^{k+1}, and consider the perturbation $\rho^\varepsilon = P^\varepsilon_{\#}\rho_\tau^{k+1}$ given by $P^\varepsilon = \mathrm{id} + \varepsilon\zeta$, where ζ is a vector field $\zeta \in C_c^\infty(\mathbb{R}^d;\mathbb{R}^d)$ and $\varepsilon \geq 0$. From the definition of the scheme we have

$$\frac{1}{2\tau}\left(\frac{\mathcal{W}_2^2(\rho_\tau^k,\rho^\varepsilon) - \mathcal{W}_2^2(\rho_\tau^k,\rho_\tau^{k+1})}{\varepsilon}\right) + \frac{\mathcal{F}_m[\rho^\varepsilon] - \mathcal{F}_m[\rho_\tau^{k+1}]}{\varepsilon} \geq 0. \tag{4.46}$$

As we want to let $\varepsilon \to 0$ and recover the Euler–Lagrange equation of the minimisation problem (4.23), we examine each term in (4.46).

Step 1: Wasserstein distance terms. We consider, in view of Brenier's Theorem, the optimal map $\mathcal{T}$ between ρ_τ^k and ρ_τ^{k+1} (see, e.g., [303, 304, 283]), so that

$$\mathcal{W}_2^2(\rho_\tau^k,\rho_\tau^{k+1}) = \int_{\mathbb{R}^d} |x - \mathcal{T}(x)|^2 \rho_\tau^k(x)\,\mathrm{d}x\,.$$

Moreover, from the definition of the Wasserstein distance, we also have

$$\begin{aligned}\mathcal{W}_2^2(\rho_\tau^k,\rho^\varepsilon) &\leq \int_{\mathbb{R}^d} |x - P^\varepsilon(\mathcal{T}(x))|^2 \rho_\tau^k(x)\,\mathrm{d}x = \int_{\mathbb{R}^d} |x - \mathcal{T}(x) - \varepsilon\zeta(\mathcal{T}(x))|^2 \rho_\tau^k(x)\,\mathrm{d}x \\ &= \mathcal{W}_2^2(\rho_\tau^k,\rho_\tau^{k+1}) - 2\varepsilon \int_{\mathbb{R}^d} (x - \mathcal{T}(x))\cdot\zeta(\mathcal{T}(x))\rho_\tau^k(x)\,\mathrm{d}x + O(\varepsilon^2)\,.\end{aligned}$$

Consequently,

$$\frac{\mathcal{W}_2^2(\rho_\tau^k,\rho^\varepsilon) - \mathcal{W}_2^2(\rho_\tau^k,\rho_\tau^{k+1})}{2\tau\varepsilon} \leq -\frac{1}{\tau}\int_{\mathbb{R}^d} (x - \mathcal{T}(x))\cdot\zeta(\mathcal{T}(x))\rho_\tau^k(x)\,\mathrm{d}x + O(\varepsilon)\,. \tag{4.47}$$

Step 2: Aggregation terms. We use the area formula [7, Section 5.5] and that $\det\nabla P^\varepsilon(x) = 1 + \varepsilon\mathrm{div}\,\zeta(x) + O(\varepsilon^2)$. For the case $1 < m < 2 + \frac{2}{d}$ or $m = 2 + \frac{2}{d}$ with subcritical mass, we obtain

$$\int_{\mathbb{R}^d} (\rho^\varepsilon)^m\,\mathrm{d}x = \int_{\mathbb{R}^d} \left(\frac{\rho_\tau^{k+1}}{\det\nabla P^\varepsilon}\right)^m \det\nabla P^\varepsilon\,\mathrm{d}x = \int_{\mathbb{R}^d} (\rho_\tau^{k+1})^m \left(1 - \varepsilon(m-1)(\mathrm{div}\,\zeta) + O(\varepsilon^2)\right)\,\mathrm{d}x\,.$$

Thus, we find

$$-\frac{1}{m-1}\int_{\mathbb{R}^d} \frac{(\rho^\varepsilon)^m - (\rho_\tau^{k+1})^m}{\varepsilon}\,\mathrm{d}x = \int_{\mathbb{R}^d} (\rho_\tau^{k+1})^m(\mathrm{div}\,\zeta)\,\mathrm{d}x + O(\varepsilon)\,.$$

For the case $m = 1$ we have that

$$\begin{aligned}\int_{\mathbb{R}^d} \rho^\varepsilon \log(\rho^\varepsilon)\,\mathrm{d}x &= \int_{\mathbb{R}^d} \frac{\rho_\tau^{k+1}}{\det\nabla P^\varepsilon} \log\left(\frac{\rho_\tau^{k+1}}{\det\nabla P^\varepsilon}\right)\det\nabla P^\varepsilon\,\mathrm{d}x \\ &= \int_{\mathbb{R}^d} \rho_\tau^{k+1}\log\rho_\tau^{k+1} - \rho_\tau^{k+1}\log\left(1 + \varepsilon\mathrm{div}\,\zeta + O(\varepsilon^2)\right)\,\mathrm{d}x.\end{aligned}$$

Therefore,

$$-\int_{\mathbb{R}^d} \frac{\rho^\varepsilon\log\rho^\varepsilon - \rho_\tau^{k+1}\log\rho_\tau^{k+1}}{\varepsilon}\,\mathrm{d}x = \int_{\mathbb{R}^d} \frac{\rho_\tau^{k+1}\log\left(1 + \varepsilon\mathrm{div}\,\zeta + O(\varepsilon^2)\right)}{\varepsilon}\,\mathrm{d}x,$$

and taking the limit in ε we obtain,

$$-\lim_{\varepsilon\to 0}\int_{\mathbb{R}^d}\frac{\rho^\varepsilon\log\rho^\varepsilon-\rho_\tau^{k+1}\log\rho_\tau^{k+1}}{\varepsilon}\,\mathrm{d}x=\int_{\mathbb{R}^d}\rho_\tau^{k+1}(\operatorname{div}\zeta)\,\mathrm{d}x.$$

In particular,

$$-\lim_{\varepsilon\to 0}\frac{\mathcal{E}_m[\rho^\varepsilon]-\mathcal{E}_m[\rho_\tau^{k+1}]}{\varepsilon}=\int_{\mathbb{R}^d}(\rho_\tau^{k+1})^m(\operatorname{div}\zeta)\,\mathrm{d}x, \tag{4.48}$$

holds for every $1\le m<2+\frac{2}{d}$ or $m=2+\frac{2}{d}$ with subcritical mass.

Step 3: Diffusion terms. We use the definition of push-forward and the area formula to obtain

$$\begin{aligned}\int_{\mathbb{R}^d}|\nabla P_\#\rho_\tau^{k+1}(x)|^2\,\mathrm{d}x&=\int_{\mathbb{R}^d}\left|\nabla\left(\frac{\rho_\tau^{k+1}}{\det\nabla P^\varepsilon}\circ(P^\varepsilon)^{-1}\right)(x)\right|^2\,\mathrm{d}x\\&=\int_{\mathbb{R}^d}\left|\nabla(P^\varepsilon)^{-1}(x)\nabla\left(\frac{\rho_\tau^{k+1}}{\det\nabla P^\varepsilon}\right)((P^\varepsilon)^{-1}(x))\right|^2\,\mathrm{d}x\\&=\int_{\mathbb{R}^d}\left|\nabla(P^\varepsilon)^{-1}(P^\varepsilon(x))\nabla\left(\frac{\rho_\tau^{k+1}}{\det\nabla P^\varepsilon}\right)(x)\right|^2|\det\nabla P^\varepsilon(x)|\,\mathrm{d}x\\&=\int_{\mathbb{R}^d}\left|(\nabla P^\varepsilon(x))^{-1}\nabla\left(\frac{\rho_\tau^{k+1}}{\det\nabla P^\varepsilon}\right)(x)\right|^2|\det\nabla P^\varepsilon(x)|\,\mathrm{d}x\,.\end{aligned}$$

Next, we observe that $(\nabla P^\varepsilon)^{-1}=\mathrm{I}_d-\varepsilon\nabla\zeta+O(\varepsilon^2)$, with I_d the identity matrix. Hence, we have

$$\int_{\mathbb{R}^d}|\nabla P_\#\rho_\tau^{k+1}|^2\,\mathrm{d}x=\int_{\mathbb{R}^d}\left|\nabla\rho_\tau^{k+1}-\varepsilon(\rho_\tau^{k+1}\nabla(\operatorname{div}\zeta)+\nabla\zeta\nabla\rho_\tau^{k+1}+\frac{1}{2}(\operatorname{div}\zeta)\nabla\rho_\tau^{k+1})\right|^2\,\mathrm{d}x+O(\varepsilon^2),$$

and, in particular,

$$\begin{aligned}&\frac{1}{2}\int_{\mathbb{R}^d}\frac{|\nabla P_\#\rho_\tau^{k+1}|^2-|\nabla\rho_\tau^{k+1}|^2}{\varepsilon}\,\mathrm{d}x\\&=-\int_{\mathbb{R}^d}\left(\rho_\tau^{k+1}\nabla(\operatorname{div}\zeta)\cdot\nabla\rho_\tau^{k+1}+(\nabla\zeta\nabla\rho_\tau^{k+1})\cdot\nabla\rho_\tau^{k+1}+\frac{1}{2}\operatorname{div}\zeta|\nabla\rho_\tau^{k+1}|^2\right)\mathrm{d}x+O(\varepsilon).\end{aligned} \tag{4.49}$$

Step 4: Letting $\varepsilon\to 0$. Let us perform again the same computation for $\varepsilon\le 0$. Then, we consider $\zeta=\nabla\varphi$ and compute the limit $\varepsilon\to 0$. By taking into account (4.47), (4.48), and (4.49), we have that,

$$\begin{aligned}&\frac{1}{\tau}\int_{\mathbb{R}^d}(x-\mathcal{T}(x))\cdot\nabla\varphi(\mathcal{T}(x))\rho_\tau^k(x)\,\mathrm{d}x\\&=-\int_{\mathbb{R}^d}\left(\rho_\tau^{k+1}\nabla(\Delta\varphi)\cdot\nabla\rho_\tau^{k+1}+(D^2\varphi\nabla\rho_\tau^{k+1})\cdot\nabla\rho_\tau^{k+1}+\frac{1}{2}\Delta\varphi|\nabla\rho_\tau^{k+1}|^2\right)\mathrm{d}x\\&\quad+\chi\int_{\mathbb{R}^d}(\rho_\tau^{k+1})^m\Delta\varphi\,\mathrm{d}x\,.\end{aligned} \tag{4.50}$$

Next, we rewrite the left-hand side of (4.50) by considering a Taylor expansion of φ on $\mathcal{T}(x)$. Since ρ_τ is Holder continuous, (4.38), we have

$$\int_{\mathbb{R}^d}(x-\mathcal{T}(x))\cdot\nabla\varphi(\mathcal{T}(x))\rho_\tau^k(x)\,\mathrm{d}x=\int_{\mathbb{R}^d}\varphi(x)\left[\rho_\tau^k(x)-\rho_\tau^{k+1}(x)\right]\,\mathrm{d}x+O(\tau)\,.$$

Let $0\le s_1<s_2\le T$ be fixed with,

$$h_1=\left[\frac{s_1}{\tau}\right]+1\quad\text{and}\quad h_2=\left[\frac{s_2}{\tau}\right].$$

By summing with respect to k in (4.50), we obtain,

$$\begin{aligned}
&\int_{\mathbb{R}^d} \varphi(x)\rho_\tau^{h_2+1}(x)\,\mathrm{d}x - \int_{\mathbb{R}^d} \varphi(x)\rho_\tau^{h_1}(x)\,\mathrm{d}x + O(\tau)\\
&= \sum_{j=h_1}^{h_2} \tau \int_{\mathbb{R}^d}\Big(\rho_\tau^{j+1}\nabla(\Delta\varphi)\cdot\nabla\rho_\tau^{j+1} + (D^2\varphi\nabla\rho_\tau^{j+1})\cdot\nabla\rho_\tau^{j+1} + \frac{1}{2}\Delta\varphi|\nabla\rho_\tau^{j+1}|^2\Big)\mathrm{d}x\\
&\quad - \chi\sum_{j=h_1}^{h_2}\tau\int_{\mathbb{R}^d}(\rho_\tau^{j+1})^m\Delta\varphi\,\mathrm{d}x\,.
\end{aligned}$$

Using the definition of the piecewise constant interpolation ρ_τ and integration by parts, cf. Remark 4.27, this is equivalent to

$$\begin{aligned}
&\int_{\mathbb{R}^d} \varphi(x)\rho_\tau(s_2,x)\,\mathrm{d}x - \int_{\mathbb{R}^d} \varphi(x)\rho_\tau(s_1,x)\,\mathrm{d}x + O(\tau)\\
&= \int_{s_1}^{s_2}\int_{\mathbb{R}^d}\Big(\rho_\tau\nabla(\Delta\varphi)\cdot\nabla\rho_\tau + (D^2\varphi\nabla\rho_\tau)\cdot\nabla\rho_\tau + \frac{1}{2}\Delta\varphi|\nabla\rho_\tau|^2\Big)\,\mathrm{d}x\,\mathrm{d}t - \chi\int_{s_1}^{s_2}\int_{\mathbb{R}^d}\rho_\tau^m\Delta\varphi\,\mathrm{d}x\,\mathrm{d}t \qquad (4.51)\\
&= -\int_{s_1}^{s_2}\int_{\mathbb{R}^d}(\rho_\tau\Delta\rho_\tau\Delta\varphi + \Delta\rho_\tau\nabla\rho_\tau\cdot\nabla\varphi)\,\mathrm{d}x\,\mathrm{d}t - \chi\int_{s_1}^{s_2}\int_{\mathbb{R}^d}\rho_\tau^m\Delta\varphi\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

By combining Lemma 4.24, Proposition 4.22, Proposition 4.25, and Corollary 4.26 we can pass to the limit in (4.51) as $\tau \to 0^+$, and recover a weak solution. □

Remark 4.27. *Assume $\rho \in H^2(\mathbb{R}^d)$ and $\varphi \in C_0^3(\mathbb{R}^d)$ — this is indeed not a restriction as $\zeta \in C_c^\infty(\mathbb{R}^d;\mathbb{R}^d)$. Using integration by parts several times, we have*

$$\begin{aligned}
&\int_{\mathbb{R}^d}\Big(\rho\nabla\rho\cdot\nabla(\Delta\varphi) + \nabla\rho\cdot(D^2\varphi\nabla\rho) + \frac{1}{2}\Delta\varphi|\nabla\rho|^2\Big)\,\mathrm{d}x\\
&= -\int_{\mathbb{R}^d}\rho\Delta\rho\Delta\varphi\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^d}\left(2\nabla\rho\cdot(D^2\varphi\nabla\rho) - \Delta\varphi|\nabla\rho|^2\right)\,\mathrm{d}x\\
&= -\int_{\mathbb{R}^d}\rho\Delta\rho\Delta\varphi\,\mathrm{d}x + \int_{\mathbb{R}^d}\left(\nabla\rho\cdot(D^2\varphi\nabla\rho) + \nabla\varphi\cdot(D^2\rho\nabla\rho)\right)\,\mathrm{d}x\\
&= -\int_{\mathbb{R}^d}(\rho\Delta\rho\Delta\varphi + \Delta\rho\nabla\rho\cdot\nabla\varphi)\,\mathrm{d}x\,.
\end{aligned}$$

Remark 4.28. *We observe that the addition of an external potential to the energy $\mathcal{F}_m$, thus to (4.1), even nonlocal, does not bring further difficulties to our strategy under minimal regularity assumptions. Indeed, the above proof can be integrated with previous results, e.g. [215, 252].*

4.5 Extension to systems of two interacting species

In this section, we extend the one-species theory to study system (4.4) and prove existence of weak solutions. First, we obtain some basic properties of the free energy functional, defined in (4.7), we recall here for the reader's convenience:

$$\boldsymbol{\mathcal{F}}[\rho,\eta] = \begin{cases} \tilde{\boldsymbol{\mathcal{F}}}[\rho,\eta] & \text{if } (\rho,\eta)\in\mathcal{P}^a(\mathbb{R}^d)^2,\ (\nabla\rho,\nabla\eta)\in L^2(\mathbb{R}^d)^2,\\ +\infty & \text{otherwise,}\end{cases}$$

being

$$\tilde{\boldsymbol{\mathcal{F}}}[\rho,\eta] = \int_{\mathbb{R}^d}\left(\frac{\kappa}{2}|\nabla\rho|^2 + \frac{1}{2}|\nabla\eta|^2 + \alpha\nabla\rho\cdot\nabla\eta - \frac{\beta}{2}\rho^2 - \frac{1}{2}\eta^2 - \omega\rho\eta\right)\,\mathrm{d}x.$$

We remind the reader the parameters in the model are such that $\beta, \omega \in \mathbb{R}$ and the matrix

$$A = \begin{pmatrix} \kappa & \alpha \\ \alpha & 1 \end{pmatrix}$$

is assumed to be positive definite.

Remark 4.29. *Throughout this section we restrict to the case ρ, η have both mass equal to 1. Our result holds true when $\int \rho \, dx = \int \eta \, dx = M \neq 1$ up to changing variables as*

$$\tau = Mt, \quad \tilde{\rho} = \rho/M, \quad \tilde{\eta} = \eta/M.$$

If the masses are different we consider the Wasserstein distance between measures with given mass for each species and the corresponding distance on the product space.

Proposition 4.30 (Lower bound for the free energy and induced regularity)**.** *Assume $(\rho, \eta) \in \mathcal{P}^a(\mathbb{R}^d)^2$. The following properties hold.*

(1) *Lower bound for the free energy: let $\nabla\rho, \nabla\eta \in L^2(\mathbb{R}^d)$, then $\boldsymbol{\mathcal{F}}[\rho, \eta]$ is bounded from below as*

$$\boldsymbol{\mathcal{F}}[\rho, \eta] \geq -C \left(\|\rho\|^2_{L^1(\mathbb{R}^d)} + \|\eta\|^2_{L^1(\mathbb{R}^d)} \right), \tag{4.52}$$

where $C = C(\kappa, \alpha, \beta, \omega, d) > 0$.

(2) *H^1-bound: assume $\boldsymbol{\mathcal{F}}[\rho, \eta] < +\infty$, then the following bound holds*

$$\|\nabla f\|^2_{L^2(\mathbb{R}^d)} \leq C \left(\mathcal{F}_2[f] + \|f\|^2_{L^1(\mathbb{R}^d)} \right), \quad \textit{for } f \in \{\rho, \eta\} \tag{4.53}$$

where $C = C(d) > 0$.

(3) *L^p-regularity: assume $\boldsymbol{\mathcal{F}}[\rho, \eta] < +\infty$, then $\rho, \eta \in L^p(\mathbb{R}^d)$ for any $p \in [1, 2^*]$, $d \neq 2$, and for any $p \in [1, 2^*)$ when $d = 2$. In particular, there exists a constant $C = C(p, d, f) > 0$ such that*

$$\|f\|_{L^p(\mathbb{R}^d)} \leq C < +\infty, \quad \textit{for } f \in \{\rho, \eta\}. \tag{4.54}$$

Proof. We divide the proof into two steps.

Step 1: Lower bound for the free energy. By using Cauchy–Schwarz and Young inequalities we obtain

$$\begin{aligned}
\boldsymbol{\mathcal{F}}[\rho, \eta] &= \int_{\mathbb{R}^d} \left(\frac{\kappa}{2}|\nabla\rho|^2 + \frac{1}{2}|\nabla\eta|^2 + \alpha\nabla\rho \cdot \nabla\eta - \frac{\beta}{2}\rho^2 - \frac{1}{2}\eta^2 - \omega\rho\eta \right) \mathrm{d}x \\
&\geq \int_{\mathbb{R}^d} \left(\frac{\kappa}{2}|\nabla\rho|^2 + \frac{1}{2}|\nabla\eta|^2 - |\alpha||\nabla\rho||\nabla\eta| - \frac{\beta + |\omega|}{2}\rho^2 - \frac{1 + |\omega|}{2}\eta^2 \right) \mathrm{d}x \\
&\geq \int_{\mathbb{R}^d} \left(\frac{\kappa - |\alpha|\varepsilon}{2}|\nabla\rho|^2 + \frac{1 - |\alpha|\varepsilon^{-1}}{2}|\nabla\eta|^2 - \frac{\beta + |\omega|}{2}\rho^2 - \frac{1 + |\omega|}{2}\eta^2 \right) \mathrm{d}x.
\end{aligned}$$

Since the matrix A is positive definite, $\kappa - \alpha^2 > 0$ we can choose $\varepsilon \in (|\alpha|, \frac{\kappa}{|\alpha|})$ so that $1 - |\alpha|\varepsilon^{-1} > 0$ and $\kappa - |\alpha|\varepsilon > 0$. Hence, we obtain

$$\boldsymbol{\mathcal{F}}[\rho, \eta] \geq (\kappa - |\alpha|\varepsilon)\mathcal{F}_2[\rho] + (1 - |\alpha|\varepsilon^{-1})\mathcal{F}_2[\eta], \tag{4.55}$$

where we implicitly have two different values of χ in the two energies, depending on the parameters of the system. This is not an issue as we are in the subcritical exponent case, $m = 2$. The energy is, therefore, bounded from below, and the result follows from the one-species case (4.12).

Step 2: H^1-bound and L^p-regularity. Given $\boldsymbol{\mathcal{F}}[\rho, \eta] < +\infty$, then (4.55) implies $\mathcal{F}_2[\rho], \mathcal{F}_2[\eta] < +\infty$. The results follow from the one-species case (4.13), (4.14). □

4.5.1 The JKO scheme

Analogously to the problem for the one-species case, we can use the JKO scheme to construct an approximation to a candidate of a solution.

Remark 4.31. *For the sake of completeness we specify the notation for the 2-Wasserstein distance in the product space. Let $\sigma_1 = (\rho_1, \eta_1) \in \mathcal{P}_2(\mathbb{R}^d)^2$ and $\sigma_2 = (\rho_2, \eta_2) \in \mathcal{P}_2(\mathbb{R}^d)^2$. The 2-Wasserstein distance between σ_1 and σ_2 is denoted as*

$$\mathrm{d}_W^2(\sigma_1, \sigma_2) = \mathcal{W}_2^2(\rho_1, \rho_2) + \mathcal{W}_2^2(\eta_1, \eta_2)\,. \tag{4.56}$$

Furthermore, note that for $\sigma = (\rho, \eta) \in \mathcal{P}_2(\mathbb{R}^d)^2$, $\mathrm{m}_2(\sigma) = \mathrm{m}_2(\rho) + \mathrm{m}_2(\eta)$.

As in the one-species case, we consider the following recursive scheme, for $\sigma_0 \in \mathcal{P}_2(\mathbb{R}^d)^2$.

- Let $\tau > 0$ and set $\sigma_\tau^0 := \sigma_0 = (\rho_0, \eta_0)$.
- Given $\sigma_\tau^k = (\rho_\tau^k, \eta_\tau^k) \in \mathcal{P}(\mathbb{R}^d)^2$ for $k \geq 0$, choose

$$\sigma_\tau^{k+1} = (\rho_\tau^{k+1}, \eta_\tau^{k+1}) \in \underset{\sigma \in \mathcal{P}(\mathbb{R}^d)^2}{\operatorname{argmin}} \left\{ \frac{\mathrm{d}_W^2(\sigma, \sigma_\tau^k)}{2\tau} + \boldsymbol{\mathcal{F}}[\sigma] \right\}. \tag{4.57}$$

We start checking that the scheme (4.57) is well-defined. Let us fix $\bar{\sigma} = (\bar{\rho}, \bar{\eta}) \in \mathcal{P}_2^a(\mathbb{R}^d)^2$ and define the functional

$$\begin{aligned} \boldsymbol{\mathcal{A}} \colon \mathcal{P}(\mathbb{R}^d)^2 &\longrightarrow \overline{\mathbb{R}} \\ \sigma \quad &\longmapsto \frac{\mathrm{d}_W^2(\sigma, \overline{\sigma})}{2\tau} + \boldsymbol{\mathcal{F}}[\sigma]. \end{aligned}$$

Proposition 4.32. *Let $\overline{\sigma} \in \mathcal{P}_2^a(\mathbb{R}^d)^2$. The functional $\boldsymbol{\mathcal{A}}$ admits a minimiser in the set*

$$\left\{ \sigma = (\rho, \eta) \in \mathcal{P}^a(\mathbb{R}^d)^2 \, : \, \nabla\rho, \nabla\eta \in L^2(\mathbb{R}^d) \right\}.$$

Again, we employ the direct method of calculus of variations and the results from the one-species case, cf. Proposition 4.18.

Proof. We divide the proof in several steps.

Step 1: Boundedness from below. Analogously to Proposition 4.18 we note that $\boldsymbol{\mathcal{A}}[\sigma] \geq C$. This ensures that we can consider a minimising sequence $\{\sigma_n\}_n$, where $\sigma_n = (\rho_n, \eta_n)$, satisfying:

$$\mathrm{m}_2(\rho_n) + \mathrm{m}_2(\eta_n) \leq CT(1 + \mathrm{m}_2(\overline{\rho}) + \mathrm{m}_2(\overline{\eta})).$$

Step 2: $\boldsymbol{\mathcal{A}}$ is lower semicontinuous. Repeating the argument in Proposition 4.18 we know that, up to a subsequence,

$$\nabla\rho_n \rightharpoonup \nabla\rho \quad \text{and} \quad \nabla\eta_n \rightharpoonup \nabla\eta \qquad \text{in } L^2(\mathbb{R}^d), \tag{4.58a}$$

$$\rho_n \to \rho \quad \text{and} \quad \eta_n \to \eta \qquad \text{in } L^2(\mathbb{R}^d). \tag{4.58b}$$

Next, we write

$$\nabla\rho_n \cdot \nabla\eta_n = \frac{\alpha}{2}\left|\nabla(\rho_n + \alpha^{-1}\eta_n)\right|^2 - \frac{\alpha}{2}|\nabla\rho_n|^2 - \frac{1}{2\alpha}|\nabla\eta_n|^2.$$

Note that $\rho_n + \alpha^{-1}\eta_n \to \rho + \alpha^{-1}\eta$ and also $\nabla(\rho_n + \alpha^{-1}\eta_n) \rightharpoonup \nabla\rho + \alpha^{-1}\nabla\eta$ in $L^2(\mathbb{R}^d)$. By using the lower semicontinuity of the H^1 seminorm and that $\kappa - \alpha^2 > 0$, we obtain

$$\liminf_{n \to +\infty} \int_{\mathbb{R}^d} \left(\frac{\kappa}{2}|\nabla\rho_n|^2 + \frac{1}{2}|\nabla\eta_n|^2 + \alpha\nabla\rho_n \cdot \nabla\eta_n \right) \mathrm{d}x$$

$$
\begin{aligned}
&= \liminf_{n\to+\infty} \int_{\mathbb{R}^d} \left(\frac{\kappa-\alpha^2}{2} |\nabla \rho_n|^2 + \frac{\alpha^2}{2} \left|\nabla(\rho_n + \alpha^{-1}\eta_n)\right|^2 \right) \mathrm{d}x \\
&\geq \int_{\mathbb{R}^d} \left(\frac{\kappa-\alpha^2}{2} |\nabla \rho|^2 + \frac{\alpha^2}{2} \left|\nabla(\rho + \alpha^{-1}\eta)\right|^2 \right) \mathrm{d}x \\
&= \int_{\mathbb{R}^d} \left(\frac{\kappa}{2} |\nabla \rho|^2 + \frac{1}{2} |\nabla \eta|^2 + \alpha \nabla\rho\cdot\nabla\eta \right) \mathrm{d}x \,.
\end{aligned}
$$

In order to deal with the other terms involved in the free energy, the quadratic terms follow from the convergence (4.58). In order to deal with the last term, we now claim that

$$
\rho_n\eta_n \to \rho\eta \quad \text{in } L^1(\mathbb{R}^d).
$$

This follows from

$$
\begin{aligned}
\|\rho_n\eta_n - \rho\eta\|_{L^1(\mathbb{R}^d)} &\leq \|\eta_n(\rho-\rho_n)\|_{L^1(\mathbb{R}^d)} + \|\rho(\eta-\eta_n)\|_{L^1(\mathbb{R}^d)} \\
&\leq \|\eta_n\|_{L^2(\mathbb{R}^d)}\|\rho-\rho_n\|_{L^2(\mathbb{R}^d)} + \|\rho\|_{L^2(\mathbb{R}^d)}\|\eta-\eta_n\|_{L^2(\mathbb{R}^d)} \\
&\to 0\,.
\end{aligned}
$$

Step 3. Existence of minimisers follows then from the Weierstrass criterion, cf. e.g. [283, Box 1.1]. □

Let $T>0$, and consider $N := \left[\frac{T}{\tau}\right]$. We define the curve $\sigma_\tau : [0,T] \to \mathcal{P}(\mathbb{R}^d)^2$ as the piecewise constant interpolation

$$
\sigma_\tau(t) = \sigma_\tau^k, \quad t\in((k-1)\tau, k\tau]\,, \tag{4.59}
$$

where $\sigma_\tau^k = (\rho_\tau^k, \eta_\tau^k)$ is defined in (4.57). In the following, we prove the two-species analogous of Lemma 4.20, Proposition 4.21, and Proposition 4.22.

Lemma 4.33 (Narrow convergence & discrete uniform estimates)**.** *Let $\sigma_0 \in \mathcal{P}_2^a(\mathbb{R}^d)^2$ such that $\boldsymbol{\mathcal{F}}[\sigma_0] < +\infty$. There exists an absolutely continuous curve $\tilde\sigma : [0,T] \to \mathcal{P}_2(\mathbb{R}^d)^2$ such that, up to a subsequence, $\sigma_\tau(t)$ narrowly converges to $\tilde\sigma(t)$, uniformly in $t\in[0,T]$.*
Moreover, we obtain the following discrete uniform bounds:

$$
\sup_k \|\nabla\rho_\tau^k\|_{L^2(\mathbb{R}^d)} + \sup_k \|\nabla\eta_\tau^k\|_{L^2(\mathbb{R}^d)} \leq C_1 < +\infty\,, \tag{4.60}
$$

$$
\sup_k \|\rho_\tau^k\|_{L^p(\mathbb{R}^d)} + \sup_k \|\eta_\tau^k\|_{L^p(\mathbb{R}^d)} \leq C_2 < +\infty\,, \tag{4.61}
$$

$$
\mathrm{m}_2(\sigma_\tau(t)) \leq 2\mathrm{m}_2(\sigma_0) + 4T\left(\boldsymbol{\mathcal{F}}[\sigma_0] + C\right), \tag{4.62}
$$

for $p\in[1,2^*]$, $d\neq 2$, and $p\in[1,2^*)$ when $d=2$. The constants $C_1>0$ and $C_2>0$ are independent of k and τ.

Proof. The proof works analogously to the one from Lemma 4.20. By construction of the sequence we obtain that,

$$
\boldsymbol{\mathcal{F}}[\sigma_\tau^k] \leq \frac{d_W^2(\sigma_\tau^k, \sigma_\tau^{k-1})}{2\tau} + \boldsymbol{\mathcal{F}}[\sigma_\tau^k] \leq \boldsymbol{\mathcal{F}}[\sigma_\tau^{k-1}], \tag{4.63}
$$

and, in particular,

$$
\sup_k \boldsymbol{\mathcal{F}}[\sigma_\tau^k] \leq \boldsymbol{\mathcal{F}}[\sigma_0] < +\infty.
$$

This combined with (4.53) and (4.54) implies that $\|\nabla\rho_\tau^k\|_{L^2(\mathbb{R}^d)}$, $\|\nabla\eta_\tau^k\|_{L^2(\mathbb{R}^d)}$, and $\|\rho_\tau^k\|_{L^p(\mathbb{R}^d)}$, $\|\eta_\tau^k\|_{L^p(\mathbb{R}^d)}$ are uniformly bounded in k and τ for $p\in[1,2^*]$, $d\neq2$, and $p\in[1,2^*)$ when $d=2$. From here we recover (4.60) and (4.61).

Summing up over k in (4.63), we obtain that

$$
\sum_{k=i+1}^{j} \frac{d_W^2(\sigma_\tau^k, \sigma_\tau^{k-1})}{2\tau} \leq \boldsymbol{\mathcal{F}}[\sigma_\tau^i] - \boldsymbol{\mathcal{F}}[\sigma_\tau^j] \leq \boldsymbol{\mathcal{F}}[\sigma_0] + C, \tag{4.64}
$$

where the last inequality holds because the free energy is bounded from below from (4.52). Therefore, the distance d_W between σ_0 and $\sigma_\tau(t)$ is uniformly bounded, as for $t \in ((j-1)\tau, j\tau]$,

$$d_W^2(\sigma_0, \sigma_\tau(t)) \le j \sum_{k=1}^{j} d_W^2(\sigma_\tau^k, \sigma_\tau^{k-1}) \le 2j\tau(\boldsymbol{\mathcal{F}}[\sigma_0] + C) \le 2T(\boldsymbol{\mathcal{F}}[\sigma_0] + C).$$

Furthermore, this last inequality combined with the triangular inequality for the 2-Wasserstein distance gives us that second order moments are uniformly bounded on compact time intervals $[0, T]$:

$$\mathrm{m}_2(\sigma_\tau(t)) \le 2\mathrm{m}_2(\sigma_0) + 2d_W^2(\sigma_0, \sigma_\tau(t)) \le 2\mathrm{m}_2(\sigma_0) + 4T\left(\boldsymbol{\mathcal{F}}[\sigma_0] + C\right).$$

We can now prove equicontinuity. Consider $0 \le s < t \le T$ such that $s \in ((i-1)\tau, i\tau]$ and $t \in ((j-1)\tau, j\tau]$. Then, combining Cauchy–Schwarz inequality with (4.64) we have,

$$\begin{aligned} d_W(\sigma_\tau(s), \sigma_\tau(t)) &\le \sum_{k=i+1}^{j} d_W(\sigma_\tau^k, \sigma_\tau^{k-1}) \le \left(\sum_{k=i+1}^{j} d_W^2(\sigma_\tau^k, \sigma_\tau^{k-1})\right)^{\frac{1}{2}} |j-i|^{\frac{1}{2}} \\ &\le \left(2(\boldsymbol{\mathcal{F}}[\sigma_0] + C)\right)^{\frac{1}{2}} \left(\sqrt{|t-s|} + \sqrt{\tau}\right). \end{aligned} \tag{4.65}$$

From here we obtain that σ_τ is $\frac{1}{2}$-Holder equi-continuous up to a negligible error of order $\sqrt{\tau}$. Thus, using the refined version of the Ascoli-Arzelà Theorem [7, Proposition 3.3.1], it follows that σ_τ admits a subsequence narrowly converging to a limit $\tilde{\sigma} = (\tilde{\rho}, \tilde{\eta}) \in \mathcal{P}(\mathbb{R}^d)^2$ as $\tau \to 0^+$, uniformly on $[0, T]$. Furthermore, using that $|\cdot|^2$ is lower semicontinuous and the uniform bound (4.62), we obtain that the limiting curve $\tilde{\sigma}$ is such that,

$$\mathrm{m}_2(\tilde{\sigma}(t)) \le \liminf_{\tau \downarrow 0} \mathrm{m}_2(\sigma_\tau(t)) \le C. \qquad \square$$

Proposition 4.34 (Weak convergence). *Let $\sigma_0 \in \mathcal{P}_2^a(\mathbb{R}^d)^2$ such that $\boldsymbol{\mathcal{F}}[\sigma_0] < +\infty$. The piecewise interpolation σ_τ constructed in (4.59) is such that $\sigma_\tau \in L^\infty([0,T]; H^1(\mathbb{R}^d))^2$. In particular, the limit $\tilde{\sigma}$ belongs to $L^\infty([0,T]; H^1(\mathbb{R}^d))^2$ and*

$$\sigma_\tau \rightharpoonup \tilde{\sigma} \quad \text{in } L^2([0,T]; H^1(\mathbb{R}^d))^2.$$

Proof. From (4.60) in Lemma 4.33 we have

$$\|\rho_\tau\|_{L^\infty([0,T];H^1(\mathbb{R}^d))} = \sup_{t \in (0,T)} \|\rho_\tau(t)\|_{L^2(\mathbb{R}^d)} = \sup_k \|\rho_\tau^k\|_{H^1(\mathbb{R}^d)} < +\infty,$$

and analogously for η_τ. In particular, for any compact time interval $[0, T]$ with $T > 0$, we have

$$\|\rho_\tau\|_{L^2([0,T];H^1(\mathbb{R}^d))} + \|\eta_\tau\|_{L^2([0,T];H^1(\mathbb{R}^d))} \le C$$

uniformly in τ and the weak convergence follows from Banach-Alaoglu Theorem. Regularity of the limit follows from standard arguments. $\square$

Proposition 4.35 (Strong convergence of σ_τ). *Let $\sigma_0 \in \mathcal{P}_2^a(\mathbb{R}^d)$ such that $\boldsymbol{\mathcal{F}}[\sigma_0] < +\infty$. The sequence $\sigma_\tau : [0,T] \to \mathcal{P}_2(\mathbb{R}^d)^2$ converges, up to a subsequence, strongly to the curve $\tilde{\sigma}$ in $L^2([0,T]; L^2(\mathbb{R}^d))^2$ for every $T > 0$.*

Proof. We apply Proposition 4.7 to a subset $U = \{\sigma_\tau\}_{\tau \ge 0}$ for $X = L^2(\mathbb{R}^d)^2$ ang $g := d_W$ defined in (4.56). Similarly to the one-species case, we consider the functional $\boldsymbol{\mathcal{I}} : L^2(\mathbb{R}^d)^2 \to [0, +\infty]$ defined by

$$\boldsymbol{\mathcal{I}}[\rho, \eta] = \begin{cases} \|\rho\|_{H^1(\mathbb{R}^d)}^2 + \|\eta\|_{H^1(\mathbb{R}^d)}^2 + \mathrm{m}_2(\rho) + \mathrm{m}_2(\eta) & \rho, \eta \in \mathcal{P}_2(\mathbb{R}^d) \cap H^1(\mathbb{R}^d), \\ +\infty & \text{otherwise.} \end{cases}$$

Note that d_W is a distance on the proper domain of $\mathcal{I}$. Indeed, given $\sigma = (\rho, \eta)$, if $\mathcal{I}[\sigma] < +\infty$ then $\sigma \in \mathcal{P}_2(\mathbb{R}^d)^2$. As in Proposition 4.22, the functional $\mathcal{I}$ is lower semicontinuous from standard arguments [59] and has relatively compact subsets from Kolmogorov-Riesz-Fréchet Theorem [54, Corollary 4.27].

Proving that $\mathcal{I}$ and d_W satisfy the tightness and integral equicontinuity conditions in Proposition 4.7 can be done as in the one-species case by using arguments analogous to those in Proposition 4.22. Tightness follows from the uniform-in-τ second order moment and $L^\infty([0,T]; H^1(\mathbb{R}^d))$ bounds for σ_τ^k given in Lemma 4.33. Equi-continuity is a consequence from the Hölder equi-continuity of σ_τ proved in Lemma 4.33. □

4.5.2 Flow interchange

As in the one-species case we can obtain H^2 bounds for ρ and η using the flow interchange technique. In order to do so, we consider the decoupled system of heat equations as an auxiliary flow,

$$\begin{cases} \partial_t \mu_1 = \Delta \mu_1 \,, \\ \partial_t \mu_2 = \Delta \mu_2 \,, \end{cases} \tag{4.66}$$

and the auxiliary functional,

$$\mathcal{E}[\mu_1, \mu_2] = \begin{cases} \displaystyle\int_{\mathbb{R}^d} [\mu_1 \log \mu_1 + \mu_2 \log \mu_2] \,\mathrm{d}x, & \mu_1 \log \mu_1,\ \mu_2 \log \mu_2 \in L^1(\mathbb{R}^d); \\ +\infty & \text{otherwise.} \end{cases}$$

For any $\mu = (\mu_1, \mu_2) \in \mathcal{P}_2(\mathbb{R}^d)^2$ such that $\mathcal{E}[\mu] < \infty$, we denote by $S_{\mathcal{E}}^t \mu := (S_{\mathcal{E}}^t \mu_1, S_{\mathcal{E}}^t \mu_2)$ the solution at time $t > 0$ to system (4.66) for an initial value μ. Furthermore, we define the dissipation of $\mathcal{F}$ along the flow $S_{\mathcal{E}}$ as

$$D_{\mathcal{E}} \mathcal{F}[\sigma] := \limsup_{s \downarrow 0} \left\{ \frac{\mathcal{F}[\sigma] - \mathcal{F}[S_{\mathcal{E}}^s \sigma]}{s} \right\},$$

where σ denotes $\sigma := (\rho, \eta) \in \mathcal{P}(\mathbb{R}^d)^2$.

Lemma 4.36 (H^2 uniform bound)**.** *Let σ_0 such that $\mathcal{F}[\sigma_0] < +\infty$. The piecewise interpolation σ_τ in (4.59) is such that $\sigma_\tau \in L^2([0,T]; H^2(\mathbb{R}^d))^2$. In particular, we obtain the uniform bound*

$$\|D^2 \rho_\tau\|^2_{L^2([0,T]; L^2(\mathbb{R}^d))} + \|D^2 \eta_\tau\|^2_{L^2([0,T]; L^2(\mathbb{R}^d))} \le C,$$

where $C > 0$ is independent of τ.

Proof. We proceed analogously to the one-species case. Note that $\sigma_\tau \in L^2([0,T]; H^1(\mathbb{R}^d))^2$ by Proposition 4.34. For all $s > 0$, we consider $S_{\mathcal{E}}^s \sigma_\tau^{k+1} = (S_{\mathcal{E}}^s \rho_\tau^{k+1}, S_{\mathcal{E}}^s \eta_\tau^{k+1})$. Then, by the definition of the scheme (4.23) and of σ_τ^{k+1}, we have the inequality

$$\frac{1}{2\tau} d_W^2(\sigma_\tau^k, \sigma_\tau^{k+1}) + \mathcal{F}[\sigma_\tau^{k+1}] \le \frac{1}{2\tau} d_W^2(\sigma_\tau^k, S_{\mathcal{E}}^s \sigma_\tau^{k+1}) + \mathcal{F}[S_{\mathcal{E}}^s \sigma_\tau^{k+1}],$$

from which we obtain,

$$\tau \frac{\mathcal{F}[\sigma_\tau^{k+1}] - \mathcal{F}[S_{\mathcal{E}}^s \sigma_\tau^{k+1}]}{s} \le \frac{1}{2} \frac{d_W^2(\sigma_\tau^k, S_{\mathcal{E}}^s \sigma_\tau^{k+1}) - d_W^2(\sigma_\tau^k, \sigma_\tau^{k+1})}{s}.$$

By taking the $\limsup$ as $s \downarrow 0$ and considering the definition of the distance d_W, we obtain

$$\tau D_{\mathcal{E}} \mathcal{F}[\sigma_\tau^{k+1}] \le \frac{1}{2} \frac{\mathrm{d}^+}{\mathrm{d}t}\bigg|_{t=0} d_W^2(\sigma_\tau^k, S_{\mathcal{E}}^t \sigma_\tau^{k+1}) \le \mathcal{E}[\sigma_\tau^k] - \mathcal{E}[\sigma_\tau^{k+1}], \tag{4.67}$$

where in the last inequality we use the (EVI), as $S_{\mathcal{E}}$ is a 0-flow, cf. Definition 4.8.

The dissipation of $\boldsymbol{\mathcal{F}}$ along the flow $S_{\boldsymbol{\mathcal{E}}}$ can be written as

$$D_{\boldsymbol{\mathcal{E}}}\boldsymbol{\mathcal{F}}[\sigma_\tau^{k+1}] = \limsup_{s\downarrow 0}\left\{\frac{\boldsymbol{\mathcal{F}}[\sigma_\tau^{k+1}]-\boldsymbol{\mathcal{F}}[S_{\boldsymbol{\mathcal{E}}}^s\sigma_\tau^{k+1}]}{s}\right\} = \limsup_{s\downarrow 0}\int_0^1\left(-\left.\frac{\mathrm{d}}{\mathrm{d}z}\right|_{z=st}\boldsymbol{\mathcal{F}}[S_{\boldsymbol{\mathcal{E}}}^z\sigma_\tau^{k+1}]\right)\mathrm{d}t. \quad (4.68)$$

Let us calculate the time derivative:

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\mathcal{F}}[S_{\boldsymbol{\mathcal{E}}}^t\sigma_\tau^{k+1}]=-\int\left(\kappa|\Delta S_{\boldsymbol{\mathcal{E}}}^t\rho_\tau^{k+1}|^2+|\Delta S_{\boldsymbol{\mathcal{E}}}^t\eta_\tau^{k+1}|^2+2\alpha\Delta S_{\boldsymbol{\mathcal{E}}}^t\rho_\tau^{k+1}\Delta S_{\boldsymbol{\mathcal{E}}}^t\eta_\tau^{k+1}\right)\mathrm{d}x$$
$$+\int\left(\beta|\nabla S_{\boldsymbol{\mathcal{E}}}^t\rho_\tau^{k+1}|^2+|\nabla S_{\boldsymbol{\mathcal{E}}}^t\eta_\tau^{k+1}|^2+2\omega\nabla S_{\boldsymbol{\mathcal{E}}}^t\rho_\tau^{k+1}\cdot\nabla S_{\boldsymbol{\mathcal{E}}}^t\eta_\tau^{k+1}\right)\mathrm{d}x.$$

By applying Young's inequality, we obtain

$$\begin{aligned}-\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\mathcal{F}}[S_{\boldsymbol{\mathcal{E}}}^t\sigma_\tau^{k+1}] \geq &\int_{\mathbb{R}^d}\left((\kappa-|\alpha|\varepsilon)|\Delta S_{\boldsymbol{\mathcal{E}}}^t\rho_\tau^{k+1}|^2+(1-|\alpha|\varepsilon^{-1})|\Delta S_{\boldsymbol{\mathcal{E}}}^t\eta_\tau^{k+1}|^2\right)\mathrm{d}x\\ &-\int_{\mathbb{R}^d}\left((\beta+|\omega|)|\nabla S_{\boldsymbol{\mathcal{E}}}^t\rho_\tau^{k+1}|^2+(1+|\omega|)|\nabla S_{\boldsymbol{\mathcal{E}}}^t\eta_\tau^{k+1}|^2\right)\mathrm{d}x,\end{aligned} \quad (4.69)$$

where ε can be chosen such that $\kappa-|\alpha|\varepsilon>0$ and $1-|\alpha|\varepsilon^{-1}>0$. Therefore, combining (4.67), (4.68) and (4.69) we obtain

$$\tau\liminf_{s\downarrow 0}\int_0^1\int_{\mathbb{R}^d}\left((\kappa-|\alpha|\varepsilon)|\Delta S_{\boldsymbol{\mathcal{E}}}^{st}\rho_\tau^{k+1}|^2+(1-|\alpha|\varepsilon^{-1})|\Delta S_{\boldsymbol{\mathcal{E}}}^{st}\eta_\tau^{k+1}|^2\right)\mathrm{d}x\,\mathrm{d}t$$
$$\leq \tau\limsup_{s\downarrow 0}\int_0^1\int_{\mathbb{R}^d}\left((\beta+|\omega|)|\nabla S_{\boldsymbol{\mathcal{E}}}^{st}\rho_\tau^{k+1}|^2+(1+|\omega|)|\nabla S_{\boldsymbol{\mathcal{E}}}^{st}\eta_\tau^{k+1}|^2\right)\mathrm{d}x\mathrm{d}t+\boldsymbol{\mathcal{E}}[\sigma_\tau^k]-\boldsymbol{\mathcal{E}}[\sigma_\tau^{k+1}].$$

Next, we recognize $\nabla S_{\boldsymbol{\mathcal{E}}}^{st}\sigma_\tau^{k+1}$ as the solution of the system of heat equations with initial data $\nabla\sigma_\tau^{k+1}\in L^2(\mathbb{R}^d)^2$. Hence, $\nabla S_{\boldsymbol{\mathcal{E}}}^{st}\sigma_\tau^{k+1}\to\nabla\sigma_\tau^{k+1}$ in $L^2(\mathbb{R}^d)^2$ as $s\downarrow 0$. In particular,

$$\limsup_{s\downarrow 0}\int_0^1\int_{\mathbb{R}^d}|\nabla S_{\boldsymbol{\mathcal{E}}}^{st}\sigma_\tau^{k+1}|^2\,\mathrm{d}x\,\mathrm{d}t=\int_{\mathbb{R}^d}|\nabla\sigma_\tau^{k+1}|^2\,\mathrm{d}x$$

Moreover, by well-known properties of the heat equation and the weak lower semicontinuity of the H^1 seminorm we have

$$\begin{aligned}\liminf_{s\downarrow 0}\int_0^1\int_{\mathbb{R}^d}|\Delta S_{\boldsymbol{\mathcal{E}}}^{st}\rho_\tau^{k+1}|^2+|\Delta S_{\boldsymbol{\mathcal{E}}}^{st}\eta_\tau^{k+1}|^2\,\mathrm{d}x\,\mathrm{d}t &= \liminf_{s\downarrow 0}\int_0^1\int_{\mathbb{R}^d}|D^2S_{\boldsymbol{\mathcal{E}}}^{st}\rho_\tau^{k+1}|^2+|D^2S_{\boldsymbol{\mathcal{E}}}^{st}\eta_\tau^{k+1}|^2\,\mathrm{d}x\,\mathrm{d}t\\ &\geq\int_{\mathbb{R}^d}|D^2\rho_\tau^{k+1}|^2+|D^2\eta_\tau^{k+1}|^2\,\mathrm{d}x\,.\end{aligned}$$

Thus we have found

$$\tau\|D^2\rho_\tau^{k+1}\|_{L^2(\mathbb{R}^d)}^2+\tau\|D^2\eta_\tau^{k+1}\|_{L^2(\mathbb{R}^d)}^2\leq C\left(\boldsymbol{\mathcal{E}}[\sigma_\tau^k]-\boldsymbol{\mathcal{E}}[\sigma_\tau^{k+1}]\right)+C\left(\tau\|\nabla\rho_\tau^{k+1}\|_{L^2(\mathbb{R}^d)}^2+\tau\|\nabla\eta_\tau^{k+1}\|_{L^2(\mathbb{R}^d)}^2\right),$$

for a constant $C=C(\kappa,\alpha,\beta,\omega)$ independent of τ. By summing up over k from 0 to $N-1$ we obtain the desired H^2 bound by using Lemma 4.33 since we have:

$$\begin{aligned}&\|D^2\rho_\tau\|_{L^2([0,T];L^2(\mathbb{R}^d))}^2+\|D^2\eta_\tau\|_{L^2([0,T];L^2(\mathbb{R}^d))}^2\\ &\quad\leq C\left(\boldsymbol{\mathcal{E}}[\sigma_0]-\boldsymbol{\mathcal{E}}[\sigma_\tau^N]\right)+C\left(\|\nabla\rho_\tau\|_{L^2([0,T];L^2(\mathbb{R}^d))}^2+\|\nabla\eta_\tau\|_{L^2([0,T];L^2(\mathbb{R}^d))}^2\right).\end{aligned}$$

□

The obtained H^2 bound allows us to obtain a two-species analogous of Proposition 4.25.

Proposition 4.37 (Strong convergence of $\nabla\sigma_\tau$)**.** *Let σ_0 such that $\boldsymbol{\mathcal{F}}[\sigma_0]<+\infty$. Up to a subsequence, the sequence $\sigma_\tau:[0,T]\to\mathcal{P}_2(\mathbb{R}^d)^2$ converges strongly to the curve $\tilde\sigma$ in $L^2([0,T];H^1(\mathbb{R}^d))^2$.*

Proof. The result follows by applying Proposition 4.25 to $\nabla\rho_\tau$ and $\nabla\eta_\tau$ together with the uniform H^2 bound derived in Lemma 4.36. □

4.5.3 Consistency of the scheme

Now we are ready to prove that $\tilde{\sigma} = (\tilde{\rho}, \tilde{\eta})$ is a weak solution of the problem (4.4) in the sense of Definition 4.4. This subsection completes the proof of Theorem 4.5.

Proof of Theorem 4.5. We prove the theorem by showing that the sequence $\sigma_\tau : [0,T] \to \mathcal{P}_2(\mathbb{R}^d)^2$ converges, up to a subsequence, to a weak solution $\tilde{\sigma}$ of (4.4). We will only prove the consistency for the first equation (4.4a). The case (4.4b) will work analogously. Let us fix two consecutive steps in the JKO scheme $\sigma_\tau^k = (\rho_\tau^k, \eta_\tau^k)$, $\sigma_\tau^{k+1} = (\rho_\tau^{k+1}, \eta_\tau^{k+1})$, and consider the perturbation $\sigma^\varepsilon = (\rho^\varepsilon, \eta_\tau^{k+1})$ where $\rho^\varepsilon = P^\varepsilon_{\#}\rho_\tau^{k+1}$ given by $P^\varepsilon = \mathrm{id} + \varepsilon\zeta$, where ζ is a vector field $\zeta \in C_c^\infty(\mathbb{R}^d;\mathbb{R}^d)$, and $\varepsilon \geq 0$. By applying the definition of the scheme we obtain,

$$\frac{1}{2\tau}\left(\frac{d_W^2(\sigma_\tau^k,\sigma^\varepsilon) - d_W^2(\sigma_\tau^k,\sigma_\tau^{k+1})}{\varepsilon}\right) + \frac{\boldsymbol{\mathcal{F}}[\sigma^\varepsilon] - \boldsymbol{\mathcal{F}}[\sigma_\tau^{k+1}]}{\varepsilon} \geq 0. \tag{4.70}$$

We proceed now to analyse each one of the terms in (4.70).

Step 1: Wasserstein distance terms. We first realise that

$$\frac{d_W^2(\sigma_\tau^k,\sigma^\varepsilon) - d_W^2(\sigma_\tau^k,\sigma_\tau^{k+1})}{2\tau\varepsilon} = \frac{\mathcal{W}_2^2(\rho_\tau^k,\rho^\varepsilon) - \mathcal{W}_2^2(\rho_\tau^k,\rho_\tau^{k+1})}{2\tau\varepsilon}. \tag{4.71}$$

Therefore, Step 1 of the proof of Theorem 4.3 applies to this case. Let $\mathcal{T}$ be the optimal map between ρ_τ^k and ρ_τ^{k+1}, then

$$\frac{d_W^2(\sigma_\tau^k,\sigma^\varepsilon) - d_W^2(\sigma_\tau^k,\sigma_\tau^{k+1})}{2\tau\varepsilon} \leq -\frac{1}{\tau}\int_{\mathbb{R}^d}(x-\mathcal{T}(x))\cdot\zeta(\mathcal{T}(x))\rho_\tau^k(x)\mathrm{d}x + O(\varepsilon).$$

Step 2: Self-aggregation and self-diffusion terms. As in the one-species case; cf. Theorem 4.3, we have

$$-\int_{\mathbb{R}^d}\frac{(\rho^\varepsilon)^2 - (\rho_\tau^{k+1})^2}{\varepsilon} = \int_{\mathbb{R}^d}(\rho_\tau^{k+1})^2(\operatorname{div}\zeta)\,\mathrm{d}x + O(\varepsilon) \tag{4.72}$$

and

$$\begin{aligned}\frac{1}{2}\int_{\mathbb{R}^d}\frac{|\nabla\rho^\varepsilon|^2 - |\nabla\rho_\tau^{k+1}|^2}{\varepsilon}\,\mathrm{d}x = -\int_{\mathbb{R}^d}\Big(&\rho_\tau^{k+1}\nabla(\operatorname{div}\zeta)\cdot\nabla\rho_\tau^{k+1} + (\nabla\zeta\nabla\rho_\tau^{k+1})\cdot\nabla\rho_\tau^{k+1}\\ &+\frac{1}{2}\operatorname{div}\zeta|\nabla\rho_\tau^{k+1}|^2\Big)\,\mathrm{d}x + O(\varepsilon).\end{aligned} \tag{4.73}$$

Step 3: Cross-interaction terms. For the second-order term we use the area formula to obtain,

$$\begin{aligned}\int_{\mathbb{R}^d}\frac{\rho^\varepsilon(x) - \rho_\tau^{k+1}(x)}{\varepsilon}\eta_\tau^{k+1}(x)\,\mathrm{d}x &= \int_{\mathbb{R}^d}\rho_\tau^{k+1}(x)\frac{\eta_\tau^{k+1}(P^\varepsilon(x)) - \eta_\tau^{k+1}(x)}{\varepsilon}\,\mathrm{d}x\\ &= \int_{\mathbb{R}^d}\rho_\tau^{k+1}(x)\nabla\eta_\tau^{k+1}(x)\cdot\zeta(x)\,\mathrm{d}x + O(\varepsilon).\end{aligned} \tag{4.74}$$

Similarly, for the fourth-order term, we use the fact that

$$\nabla\eta_\tau^{k+1}(P^\varepsilon(x)) = \nabla\eta_\tau^{k+1}(x) + \varepsilon D^2\eta_\tau^{k+1}(x)\zeta(x) + O(\varepsilon^2),$$

and argue as in the one-species case to obtain

$$\begin{aligned}\int_{\mathbb{R}^d}\frac{\nabla\rho^\varepsilon - \nabla\rho_\tau^{k+1}}{\varepsilon}\cdot\nabla\eta_\tau^{k+1}\,\mathrm{d}x = -\int_{\mathbb{R}^d}\Big(&\rho_\tau^{k+1}\nabla(\operatorname{div}\zeta)\cdot\nabla\eta_\tau^{k+1} + \nabla\rho_\tau^{k+1}\cdot(\nabla\zeta\nabla\eta_\tau^{k+1})\\ &-\nabla\rho_\tau^{k+1}\cdot(D^2\eta_\tau^{k+1}\zeta)\Big)\,\mathrm{d}x + O(\varepsilon).\end{aligned} \tag{4.75}$$

Step 4: Taking the limit $\varepsilon \to 0$. Analogously to the one species case we perform the same computation for $\varepsilon \leq 0$ and we take again $\zeta = \nabla\varphi$. If we consider $\varepsilon \to 0$, and thanks to (4.71), (4.72), (4.73), (4.74), and (4.75), we have,

$$\begin{aligned}
&\frac{1}{\tau}\int_{\mathbb{R}^d}(x-\mathcal{T}(x))\cdot\nabla\varphi(\mathcal{T}(x))\rho_\tau^k(x)\,\mathrm{d}x \\
&= -\kappa\int\left(\rho_\tau^{k+1}\nabla(\Delta\varphi)\cdot\nabla\rho_\tau^{k+1} + (D^2\varphi\nabla\rho_\tau^{k+1})\cdot\nabla\rho_\tau^{k+1} + \frac{1}{2}\Delta\varphi|\nabla\rho_\tau^{k+1}|^2\right)\mathrm{d}x \\
&\quad -\alpha\int\left(\rho_\tau^{k+1}\nabla(\Delta\varphi)\cdot\nabla\eta_\tau^{k+1} + \nabla\rho_\tau^{k+1}\cdot(D^2\varphi\nabla\eta_\tau^{k+1}) - \nabla\rho_\tau^{k+1}\cdot(D^2\eta_\tau^{k+1}\nabla\varphi)\right)\mathrm{d}x \\
&\quad + \frac{\beta}{2}\int(\rho_\tau^{k+1})^2\Delta\varphi\,\mathrm{d}x - \omega\int\rho_\tau^{k+1}\nabla\eta_\tau^{k+1}\cdot\nabla\varphi\,\mathrm{d}x.
\end{aligned} \tag{4.76}$$

As in the one-species case, and using the Holder continuity of ρ_τ, (4.65), we have

$$\int_{\mathbb{R}^d}(x-\mathcal{T}(x))\cdot\nabla\varphi(\mathcal{T}(x))\rho_\tau^k(x)\,\mathrm{d}x = \int_{\mathbb{R}^d}\varphi(x)\left[\rho_\tau^k(x) - \rho_\tau^{k+1}(x)\right]\,\mathrm{d}x + O(\tau)\,.$$

Let $0 \leq s_1 < s_2 \leq T$ be fixed with,

$$h_1 = \left[\frac{s_1}{\tau}\right] + 1 \quad \text{and} \quad h_2 = \left[\frac{s_2}{\tau}\right].$$

By summing on (4.76) and using the definition of piecewise interpolation, we obtain,

$$\begin{aligned}
&\int_{\mathbb{R}^d}\varphi(x)\rho_\tau(s_2,x)\,\mathrm{d}x - \int_{\mathbb{R}^d}\varphi(x)\rho_\tau(s_1,x)\,\mathrm{d}x + O(\tau) \\
&= \kappa\int_{s_1}^{s_2}\int_{\mathbb{R}^d}\left(\rho_\tau\nabla(\Delta\varphi)\cdot\nabla\rho_\tau + (D^2\varphi\nabla\rho_\tau)\cdot\nabla\rho_\tau + \frac{1}{2}\Delta\varphi|\nabla\rho_\tau|^2\right)\mathrm{d}x\,\mathrm{d}t \\
&\quad + \alpha\int_{s_1}^{s_2}\int_{\mathbb{R}^d}\left(\rho_\tau\nabla(\Delta\varphi)\cdot\nabla\eta_\tau + \nabla\rho_\tau\cdot(D^2\varphi\nabla\eta_\tau) - \nabla\rho_\tau\cdot(D^2\eta_\tau\nabla\varphi)\right)\mathrm{d}x\mathrm{d}t \\
&\quad - \frac{\beta}{2}\int_{s_1}^{s_2}\int_{\mathbb{R}^d}\rho_\tau^2\Delta\varphi\,\mathrm{d}x\,\mathrm{d}t + \omega\int_{s_1}^{s_2}\int_{\mathbb{R}^d}\rho_\tau\nabla\eta_\tau\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned} \tag{4.77}$$

Integrating by parts in the first two terms after the equality, as in Remarks 4.27 and 4.38, we obtain

$$\begin{aligned}
\int_{\mathbb{R}^d}\varphi(x)\rho_\tau(s_2,x)\,\mathrm{d}x = &\int_{\mathbb{R}^d}\varphi(x)\rho_\tau(s_1,x)\,\mathrm{d}x + O(\tau) \\
&- \kappa\int_{s_1}^{s_2}\int_{\mathbb{R}^d}\left(\rho_\tau\Delta\rho_\tau\Delta\varphi + \Delta\rho_\tau\nabla\rho_\tau\cdot\nabla\varphi\right)\mathrm{d}x\,\mathrm{d}t \\
&- \alpha\int_{s_1}^{s_2}\int_{\mathbb{R}^d}\left(\rho_\tau\Delta\eta_\tau\Delta\varphi + \Delta\eta_\tau\nabla\rho_\tau\cdot\nabla\varphi\right)\mathrm{d}x\,\mathrm{d}t \\
&- \frac{\beta}{2}\int_{s_1}^{s_2}\int_{\mathbb{R}^d}\rho_\tau^2\Delta\varphi\mathrm{d}x\mathrm{d}t + \omega\int_{s_1}^{s_2}\int_{\mathbb{R}^d}\rho_\tau\nabla\eta_\tau\cdot\nabla\varphi\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

By combining Proposition 4.35, Lemma 4.36, and Proposition 4.37 we can pass to the limit as $\tau \to 0^+$, and, in this way, recover a weak solution. As aforementioned, an analogous argument for the species η can be repeated to obtain (4.4b). □

Remark 4.38. *Assume $\rho, \eta \in H^2(\mathbb{R}^d)$ and $\varphi \in C_0^3(\mathbb{R}^d)$. Using integration by parts, we have*

$$\begin{aligned}
&\int_{\mathbb{R}^d}\left(\rho\nabla\Delta\varphi\cdot\nabla\eta + \nabla\rho\cdot(D^2\varphi\nabla\eta) - \nabla\rho\cdot(D^2\eta\nabla\varphi)\right)\,\mathrm{d}x \\
&= -\int_{\mathbb{R}^d}\rho\Delta\eta\Delta\varphi\,\mathrm{d}x + \int_{\mathbb{R}^d}\left(\nabla\rho\cdot(D^2\varphi\nabla\eta) - \nabla\rho\cdot(D^2\eta\nabla\varphi) - \Delta\varphi\nabla\rho\cdot\nabla\eta\right)\,\mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&= -\int_{\mathbb{R}^d} \rho \Delta\eta \Delta\varphi \, \mathrm{d}x + \int_{\mathbb{R}^d} \left(\nabla\rho \cdot (D^2\varphi \nabla\eta) + \nabla\varphi \cdot (D^2\rho\nabla\eta) \right) \, \mathrm{d}x \\
&= -\int_{\mathbb{R}^d} \left(\rho\Delta\eta\Delta\varphi + \Delta\eta \nabla\rho \cdot \nabla\varphi \right) \, \mathrm{d}x \, .
\end{aligned}
$$

4.5.4 Extension to generalised self-diffusion systems

In this subsection we remark that, taking advantage of the one and two species cases, we can generalise the existence theory to the following system with nonlinear self-diffusion terms

$$
\partial_t \rho = -\mathrm{div}\left(\rho \nabla \left(\kappa\Delta\rho + \alpha\Delta\eta + \frac{\beta}{m_1 - 1}\rho^{m_1 - 1} + \omega\eta \right) \right), \tag{4.78a}
$$

$$
\partial_t \eta = -\mathrm{div}\left(\eta \nabla \left(\alpha\Delta\rho + \Delta\eta + \omega\rho + \frac{1}{m_2 - 1}\eta^{m_2 - 1} \right) \right), \tag{4.78b}
$$

where $1 \le m_1, m_2 < 2 + \frac{2}{d}$. As before, the parameters in the model are such that $\beta, \omega \in \mathbb{R}$ and the matrix

$$
A = \begin{pmatrix} \kappa & \alpha \\ \alpha & 1 \end{pmatrix},
$$

is positive definite. In this case, we consider

$$
\tilde{\boldsymbol{\mathcal{F}}}_{m_1,m_2}[\rho,\eta] = \int_{\mathbb{R}^d} \left(\frac{\kappa}{2}|\nabla\rho|^2 + \frac{1}{2}|\nabla\eta|^2 + \alpha\nabla\rho\cdot\nabla\eta - \omega\rho\eta \right) \mathrm{d}x - \frac{\beta}{m_1}\mathcal{E}_{m_1}[\rho] - \frac{1}{m_2}\mathcal{E}_{m_2}[\eta],
$$

where $\mathcal{E}_m$ is the entropy defined in (4.6). The system of equations above can be written as a 2-Wasserstein gradient flow with respect to the (extended) free energy functional

$$
\boldsymbol{\mathcal{F}}_{m_1,m_2}[\rho,\eta] = \begin{cases} \tilde{\boldsymbol{\mathcal{F}}}_{m_1,m_2}[\rho,\eta] & \text{if } (\rho,\eta) \in \mathcal{P}^a(\mathbb{R}^d)^2, \ (\nabla\rho,\nabla\eta) \in L^2(\mathbb{R}^d)^2, \\ +\infty & \text{otherwise.} \end{cases}
$$

We can obtain the following lower bound for the free energy:

$$
\begin{aligned}
\boldsymbol{\mathcal{F}}_{m_1,m_2}[\rho,\eta] &= \int_{\mathbb{R}^d} \left(\frac{\kappa}{2}|\nabla\rho|^2 + \frac{1}{2}|\nabla\eta|^2 + \alpha\nabla\rho\cdot\nabla\eta - \omega\rho\eta \right) \mathrm{d}x - \frac{\beta}{m_1}\mathcal{E}_{m_1}[\rho] - \frac{1}{m_2}\mathcal{E}_{m_2}[\eta] \\
&\ge \int_{\mathbb{R}^d} \left(\frac{\kappa}{2}|\nabla\rho|^2 + \frac{1}{2}|\nabla\eta|^2 - |\alpha||\nabla\rho||\nabla\eta| - \frac{|\omega|}{2}\rho^2 - \frac{|\omega|}{2}\eta^2 \right) \mathrm{d}x - \frac{\beta}{m_1}\mathcal{E}_{m_1}[\rho] - \frac{1}{m_2}\mathcal{E}_{m_2}[\eta] \\
&\ge \int_{\mathbb{R}^d} \left(\frac{\kappa - |\alpha|\varepsilon}{4}|\nabla\rho|^2 - \frac{|\omega|}{2}\rho^2 \right) \mathrm{d}x + \int_{\mathbb{R}^d} \left(\frac{1 - |\alpha|\varepsilon^{-1}}{4}|\nabla\eta|^2 - \frac{|\omega|}{2}\eta^2 \right) \mathrm{d}x \\
&\quad + \int_{\mathbb{R}^d} \frac{\kappa - |\alpha|\varepsilon}{4}|\nabla\rho|^2 \, \mathrm{d}x - \frac{\beta}{m_1}\mathcal{E}_{m_1}[\rho] + \int_{\mathbb{R}^d} \frac{1 - |\alpha|\varepsilon^{-1}}{4}|\nabla\eta|^2 \, \mathrm{d}x - \frac{1}{m_2}\mathcal{E}_{m_2}[\eta].
\end{aligned}
$$

Therefore, since we can take ε such that $\kappa - |\alpha|\varepsilon$, $1 - |\alpha|\varepsilon^{-1} > 0$, it follows that

$$
\boldsymbol{\mathcal{F}}_{m_1,m_2}[\rho,\eta] \ge C\left(\mathcal{F}_2[\rho] + \mathcal{F}_2[\eta] + \mathcal{F}_{m_1}[\rho] + \mathcal{F}_{m_2}[\eta] \right). \tag{4.79}
$$

In particular, for $1 \le m_1, m_2 < 2 + \frac{2}{d}$, the free energy is bounded from below. Furthermore, (4.79) gives the basic estimates that we used for the existence of the one and two species cases. Since the cross-interacting terms are kept as in (4.4) and the new terms with exponents m_1 and m_2 have already been treated on the one species case, our previous results can be easily generalised to obtain existence for the problem (4.78).

In addition to that, using a scaling argument, we can show that the free energy is unbounded from below if $m_1 > 2 + \frac{2}{d}$, or equally $m_2 > 2 + \frac{2}{d}$. Without loss of generality we state the result for m_1. A thorough analysis of more general systems, as well as the other cases for the exponents, will be object of further investigation, as it is beyond the purpose of the current manuscript.

Proposition 4.39. *Assume $m_1 > m_c$ and denote*

$$\mathcal{Y} := \left\{(\rho,\eta) \in \mathcal{P}^a(\mathbb{R}^d)^2 \cap L^{m_1}(\mathbb{R}^d) \times L^{m_2}(\mathbb{R}^d) : \nabla\rho, \nabla\eta \in L^2(\mathbb{R}^d)\right\}.$$

Then

$$\inf_{(\rho,\eta)\in\mathcal{Y}} \boldsymbol{\mathcal{F}}_{m_1,m_2}[\rho,\eta] = -\infty,$$

Proof. Given $(\rho,\eta) \in \mathcal{Y}$ we define $\rho_\lambda(x) := \lambda^d \rho(\lambda x)$, for any $x \in \mathbb{R}^d$ and any $\lambda \in (0,+\infty)$. Note that $(\rho_\lambda,\eta) \in \mathcal{Y}$. Then, we have,

$$\begin{aligned}
\boldsymbol{\mathcal{F}}_{m_1,m_2}[\rho_\lambda,\eta] &= \frac{\kappa}{2}\lambda^{d+2}\|\nabla\rho\|^2_{L^2(\mathbb{R}^d)} - \lambda^{d(m_1-1)}\frac{\beta}{m_1(m_1-1)}\|\rho\|^{m_1}_{L^{m_1}(\mathbb{R}^d)} \\
&\quad + \frac{1}{2}\|\nabla\eta\|^2_{L^2(\mathbb{R}^d)} - \frac{1}{m_2}\mathcal{E}_{m_2}[\eta] + \int_{\mathbb{R}^d} \alpha\lambda^d \nabla\rho(\lambda x)\cdot\nabla\eta(x)\,\mathrm{d}x - \int_{\mathbb{R}^d} \omega\lambda^d\rho(\lambda x)\eta(x)\,\mathrm{d}x \\
&= \lambda^{d+2}\left(\frac{\kappa}{2}\|\nabla\rho\|^2_{L^2(\mathbb{R}^d)} - \lambda^{d(m_1-m_c)}\frac{\beta}{m_1(m_1-1)}\|\rho\|^{m_1}_{L^{m_1}(\mathbb{R}^d)}\right) \\
&\quad + \frac{1}{2}\|\nabla\eta\|^2_{L^2(\mathbb{R}^d)} - \frac{1}{m_2}\mathcal{E}_{m_2}[\eta] + \int_{\mathbb{R}^d} \alpha\lambda^d \nabla\rho(\lambda x)\cdot\nabla\eta(x)\,\mathrm{d}x - \int_{\mathbb{R}^d} \omega\lambda^d\rho(\lambda x)\eta(x)\,\mathrm{d}x.
\end{aligned}$$

Therefore, if we take ρ and η such that $\lambda\times\mathrm{supp}(\rho)\cap\mathrm{supp}(\eta) = \emptyset$ for big enough λ it follows that $\boldsymbol{\mathcal{F}}_{m_1,m_2}[\rho_\lambda,\eta] \to -\infty$ when $\lambda \to +\infty$; for instance we could consider the support of ρ to be an annulus and that of η to be a ball. □

5 Perspectives and new outcomes

Contents

Un grano no hace granero, pero ayuda al compañero. – Spanish proverb

5.1 A deterministic particle approximation for a fourth-order aggregation-diffusion equation

This section presents a work in preparation together with Charles Elbar[1]. "A deterministic particle approximation for a fourth-order aggregation-diffusion equation", currently *preprint in preparation*.

This project is a continuation of Chapter 4. The main goal is to obtain a deterministic particle approximation (DPA) of the equation

$$\partial_t\rho + \mathrm{div}(\rho\nabla\Delta\rho) + \Delta\rho^m = 0 \tag{5.1}$$

where $m > 1$. In Chapter 4 we understand the problem as a 2-Wasserstein gradient flow. We describe a subcritical, critical and supercritical regime depending on the value of the exponent m based on an analysis of the corresponding free-energy functional. There exists a critical value $m_c := 2 + \frac{2}{d}$ and we say we are in the subcritical regime if $m < m_c$, that we are in the supercritical if $m > m_c$ and that we are in the critical regime if $m = m_c$. After this classification, using a JKO scheme, we show existence for the subcritical regime and certain cases of the critical regime. More recently, in [60] the authors study the stability for $m = 2$ in dimension $d = 1$. Moreover, there exists further literature focused on the mathematical analysis of some related equations, mostly of Cahn-Hilliard and thin-film type, c.f. [174, 308, 292, 245, 246].

Regarding applications, as already mentioned in Chapter 4, the problem (5.1) is related to the classical thin-film equations from lubrication theory, cf. [208, 39, 268, 38, 137, 201] and the references therein. In addition, this family of equations also models tissue growth and patterning due to cell-cell adhesion. In particular, in [186], Falcó, Baker and Carrillo show that a 2 species system version of (5.1) keeps cell-sorting phenomena.

One of the aims of this project is to show that the local fourth-order competing effects problem (5.1) can be asymptotically computed as a limit from a class of nonlocal partial differential equations. As a consequence of this analysis we will be able to recover a qualitative particle approximation, i.e. a DPA.

Obtaining a DPA for a problem is very relevant since it provides a microscopic description of the equation. For ease of presentation, let us focus on the more general aggregation equation (which is a nonlocal equation)

$$\partial_t\rho - \mathrm{div}(\rho\nabla W_\varepsilon * \rho) = 0, \tag{5.2}$$

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where W_ε is an interaction kernel depending on a small parameter ε accounting for the range of interaction between the particles or cells. More precisely, this formulation hinges on deterministic approaches since *particles are solutions*, i.e. the empirical measure

$$\rho^N(t) := \frac{1}{N}\sum_{i=1}^{N}\delta_{X_i(t)}$$

is a weak solution of (5.1), where $X_i(t)$ gives the position of the particles. For any $i = 1, \dots, N$, the positions X_i solve the system of ODEs

$$\dot{X}_i(t) = -\frac{1}{N}\sum_{j=1}^{N}\nabla W_\varepsilon(X_i(t) - X_j(t)).$$

Moreover, if $W_\varepsilon \to \delta_0$ as $\varepsilon \to 0$ we recover the porous medium equation

$$\partial_t\rho - \frac{1}{2}\Delta\rho^2 = 0$$

as a limit of (5.2) when $\varepsilon \to 0$. Choosing appropriately the interaction kernel $W_\varepsilon = \omega_\varepsilon * \omega_\varepsilon$ one can make this argument rigorous in order to provide a DPA. The result we explain here is due to Lions and Mas-Gallic in [242] who studied the problem for the first time. Recently, there has been several further results in this direction for more general non-linear diffusion evolution in time equations [59, 90, 164, 130, 89, 5, 86] and for certain cases of aggregation-diffusion equations [76, 138].

With respect to fourth-order differential equations, a first non-local Cahn-Hilliard equation was obtained by Giacomin and Lebowitz [193, 194] looking at a microscopic description. Furthermore, in [37] Bertini, Landim and Olla derived a constant mobility Cahn-Hilliard model as a hydrodynamic limit from a stochastic Ginzburg-Landau model. Since then, there has also been several results regarding non-local to local convergence for various fourth-order models. The case of Cahn-Hilliard type equations with constant mobility is covered in [255, 146, 147, 148, 1]. Degenerate mobility with aggregation given by a fixed potential is studied in [172, 173, 170]. This result is then extended to its corresponding cross-diffusion system in [87]. Finally in [145] the authors cover non-local to local convergence for a Cahn-Hilliard type cross-diffusion system with non-linear mobility. However, even though the non-local to local convergence problem for Cahn-Hilliard is well-studied, up to our knowledge, there are no results in the literature providing a deterministic particles approximation for any fourth-order partial differential equation. In order to derive this DPA let us make the following observation. From a Taylor expansion we have that

$$B_\varepsilon[\rho_\varepsilon] := \frac{\rho - \rho * \omega_\varepsilon}{\varepsilon^2} = -\Delta\rho + O(\varepsilon)$$

for ω_ε symmetric and an aproximation of δ_0 as $\varepsilon \to 0$. Hence, we can approximate (5.1) by

$$\partial_t\rho - \operatorname{div}\left(\rho\nabla\left(\frac{\rho - \rho * \omega_\varepsilon}{\varepsilon^2}\right)\right) + \Delta\rho^m = 0$$

as it is suggested in [172]. Nevertheless, this approximation contains the terms $\Delta\rho^2$ and $\Delta\rho^m$ which diffuses particle initial data to a continuous density, i.e., the empirical measure does not remain as a sum of Dirac deltas due to the smoothing effect of the porous medium equation. In order to overcome this difficulty we suggest the following equation

$$\partial_t\rho - \operatorname{div}\left(\rho\nabla\left(\frac{\rho * \widetilde{\omega}_\varepsilon * \widetilde{\omega}_\varepsilon - \rho * \omega_\varepsilon * \widetilde{\omega}_\varepsilon * \widetilde{\omega}_\varepsilon}{\varepsilon^2}\right)\right) + \frac{m}{m-1}\operatorname{div}(\rho\nabla\widetilde{\omega}_\varepsilon * (\rho * \widetilde{\omega}_\varepsilon)^{m-1}) = 0 \tag{5.3}$$

where $\widetilde{\omega}_\varepsilon = \omega_{\widetilde{\varepsilon}}$ corresponds to another kernel converging to δ_0 at the same time as ω_ε but slower, $\varepsilon \ll \widetilde{\varepsilon}$. This formulation has the advantage that an empirical measure is a weak solution where the equation of motion for the particles is given by

$$\dot{X}_i(t) = -\frac{1}{N}\sum_{j=1}^{N}\nabla W_\varepsilon(X_i(t) - X_j(t)) + m\sum_{j=1}^{N}\nabla\widetilde{\omega}_\varepsilon(X_i(t) - X_j(t))\left(\frac{1}{N}\sum_{k=1}^{N}\widetilde{\omega}_\varepsilon(X_j(t) - X_k(t))\right)^{m-1}$$

where
$$W_\varepsilon = \frac{\widetilde{\omega}_\varepsilon * \widetilde{\omega}_\varepsilon - \omega_\varepsilon * \widetilde{\omega}_\varepsilon * \widetilde{\omega}_\varepsilon}{\varepsilon^2}.$$

At this point we realise that both (5.1) and (5.3) have a (formal) gradient flow structure with respect to the 2-Wasserstein distance $\mathcal{W}_2$ [304, 283, 284]. In particular, for (5.1) we consider the (extended) energy functional

$$\mathcal{F}[\rho] := \begin{cases} \frac{1}{2}\int_{\mathbb{R}^d} |\nabla\rho(x)|^2 \,\mathrm{d}x - \mathcal{E}_m[\rho], & \rho \in \mathcal{P}^a(\mathbb{R}^d),\ \nabla\rho \in L^2(\mathbb{R}^d), \\ +\infty, & \text{otherwise}, \end{cases} \tag{5.4}$$

where $\mathcal{P}^a$ refers to the absolutely continuous probability measures. For (5.3) we take into account the regularised functional
$$\mathcal{F}_\varepsilon[\rho] := \frac{1}{4}\mathcal{D}_\varepsilon[\rho * \widetilde{\omega}_\varepsilon] - \mathcal{E}_m[\rho * \widetilde{\omega}_\varepsilon]$$

where
$$\mathcal{D}_\varepsilon[\rho] = \iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{\omega_\varepsilon(y)}{\varepsilon^2}|\rho(x) - \rho(x-y)|^2 \mathrm{d}x\mathrm{d}y \quad \text{and} \quad \mathcal{E}_m[\rho] = \frac{1}{m-1}\int_{\mathbb{R}^d} \rho^m(x)\mathrm{d}x.$$

Thanks to this structure, we show stability of the gradient flows in $\varepsilon \to 0$. This stability is established in the framework introduced by Sandier and Serfaty [282, 286] and the concept of λ-gradient flows developed in [7]. There are several stability results of this nature in the literature covering various cases of non-linear diffusion [242, 90, 89]. There are also results in this direction for aggregation-diffusion equations, such as [76, 129] where the authors also introduce a deterministic particle approximation through a blob method. From this approach and taking advantage of the λ-convexity of the regularised free-energy functional we are able to achieve a rigorous particle approximation that follows from the λ-stability (or contractivity) of Wasserstein gradient flows [7]. However, one can only achieve a qualitative result since the initial datum needs to be approximated enough [129, Theorem 1.4]. Thereby, quantitative results are left as an open problem.

Further related to particle methods, we refer the reader to Chertock's comprehensive review on deterministic particle methods [122]. We also mention the seminal paper by Oelschäger [265], where a stochastic particle approximation is proven for classical and positive solutions of the quadratic porous medium equation in $\mathbb{R}^d$ and for weak solutions in the one dimensional case. We also mention the much more recent results in [119, 118] covering systems. In [274], the author derive strong L^1-solutions of the quadratic porous medium equation from a stochastic mean field interacting particle system with the addition of a vanishing Brownian motion. Finally, in [266, 261, 192] they study the viscous porous medium as a limit of a sequence of distributions of the solutions to nonlinear stochastic differential equations.

Outline and expected main results

This project has three main goals. We want to study well-posedness of solutions for the non-local problem. Then we will discuss convergence of the solutions of the non-local problem to the solutions of the local one. Finally, we intend to develop a blob method [128] for this family of equations in order to obtain a deterministic particle approximation.

First of all, in order to obtain our goals we need to define our mollifying sequence. We want to work on the moderate regime and hence a natural choice is the sequence generated by $\omega_\varepsilon(x) = \varepsilon^{-d}\omega_1(x/\varepsilon)$ for $\varepsilon > 0$. We assume that the function ω_1 satisfies

$$\begin{gathered} \omega_1 \text{ is compactly supported in the unit ball.} \\ \int_{\mathbb{R}^d} \omega_1(y)\,dy = 1, \ \int_{\mathbb{R}^d} y\omega_1(y)\,dy = 0, \ \int_{\mathbb{R}^d} y_i y_j \omega_1(y)\,dy = \delta_{i,j}\frac{2}{d}. \end{gathered} \tag{H$_\omega$}$$

In particular, for $\widetilde{\omega}_\varepsilon$ we make the choice $\widetilde{\omega}_\varepsilon = \omega_{\widetilde{\varepsilon}}$ where $\varepsilon \ll \widetilde{\varepsilon}$. Furthermore, in order to show convergence from non-local to local we need to introduce a viscosity term $\widetilde{\varepsilon}\Delta\rho^2$ to recover H^1 compactness. However, in order to preserve the DPA we take the non-local approximation of the viscosity term and we consider the problem

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{v}_{\varepsilon,\alpha}[\rho]) = 0, \\ \mathbf{v}_{\varepsilon,\alpha}[\rho] = \nabla \left(-B_\varepsilon[\rho * \widetilde{\omega}_\varepsilon * \widetilde{\omega}_\varepsilon] + \dfrac{m}{m-1} \widetilde{\omega}_\varepsilon * (\rho * \widetilde{\omega}_\varepsilon)^{m-1} - \widetilde{\varepsilon} \rho * \omega_\alpha * \omega_\alpha \right), \end{cases} \tag{5.5}$$

in $(0,T) \times \mathbb{R}^d$ with initial condition $\rho(0,\cdot) = \rho_0$. The non-local problem (5.5) contains three different mollifiers ω_α, $\widetilde{\omega}_\varepsilon$ and ω_ε. We take the limit $\alpha \to 0$ first in order to recover the H^1 compactness we need for the non-local to local convergence. Afterwards, we take the limits $\varepsilon, \widetilde{\varepsilon} \to 0$ at the same pace but different speed $\varepsilon \ll \widetilde{\varepsilon}$. Furthermore, (5.5) corresponds (formally) to the 2-Wasserstein gradient flow of the free-energy functional

$$\mathcal{F}_{\varepsilon,\alpha}[\rho] \coloneqq \frac{1}{4} \mathcal{D}_\varepsilon[\rho * \widetilde{\omega}_\varepsilon] - \mathcal{E}_m[\rho * \widetilde{\omega}_\varepsilon] + \widetilde{\varepsilon} \mathcal{E}_2[\rho * \omega_\alpha]. \tag{5.6}$$

Following the same notation we introduce the local version of (5.5)

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{v}[\rho]) = 0, \\ \mathbf{v}[\rho] = \nabla \left(\Delta \rho + \dfrac{m}{m-1} \rho^{m-1} \right), \end{cases} \tag{5.7}$$

set also on $(0,T) \times \mathbb{R}^d$ with initial condition $\rho(0,\cdot) = \rho_0$. Thereby, the first main goal of this project is to show well-posedness for the non-local problem (5.5). We fix $\varepsilon, \alpha > 0$. In this case, at least formally, we are able to show existence of weak solutions when $m \in (1, +\infty)$ and uniqueness when $m \in [2, +\infty)$. Once we finish the discussion on well-posedness we focus on the non-local to local convergence. Here, for $1 < m < m_c = 2 + \frac{2}{d}$, we want to show convergence of the solutions of (5.5) to the solution of (5.7). We intend to do that by taking advantage of its gradient flow structure studied in Chapter 4.

Thus, it only remains to study the deterministic particle approximation via the blob method. A key first ingredient to apply the blob method is to realise that the regularised free-energy functional (5.6) is λ-geodesically convex for some $\lambda = \lambda(\varepsilon, \alpha)$, see Section 1.2.2. In order to continue with the analysis we can take advantage of the non-local to local convergence result. In particular, for any $t \in [0,T]$, $N \in \mathbb{N}$, the empirical measure

$$\rho^N_{\varepsilon,\alpha}(t) = \frac{1}{N} \sum_{i=1}^N \delta_{X^i_{\varepsilon,\alpha}(t)} \tag{5.8}$$

is a weak solution to (5.5) provided the particles satisfy the ODE system given by

$$\begin{aligned} \dot{X}_i(t) = &-\frac{1}{N} \sum_{j=1}^N \nabla W_\varepsilon(X_i - X_j) + m \sum_{j=1}^N \nabla \widetilde{\omega}_\varepsilon(X_i - X_j) \left(\frac{1}{N} \sum_{k=1}^N \widetilde{\omega}_\varepsilon(X_j - X_k) \right)^{m-1} \\ &- 2\widetilde{\varepsilon} \sum_{j=1}^N \nabla \omega_\alpha(X_i - X_j) \frac{1}{N} \sum_{k=1}^N \omega_\alpha(X_j - X_k) \end{aligned} \tag{5.9}$$

where

$$W_\varepsilon = \frac{1}{\varepsilon^2} \left(\widetilde{\omega}_\varepsilon * \widetilde{\omega}_\varepsilon - \omega_\varepsilon * \widetilde{\omega}_\varepsilon * \widetilde{\omega}_\varepsilon \right).$$

Therefore, (at least formally) combining the non-local to local convergence result with the λ-convexity of the functional we expect to get a result like the following one.

Main Expected Result 1 (Continuous dependence)**.** *Assume $m \in (1, m_c)$ and that we have suitable conditions on the initial datum ρ_0. Then, the solution $\rho_{\varepsilon,\alpha}$ of (5.5) and the empirical measure $\rho^N_{\varepsilon,\alpha}$ in (5.8) satisfy that*

$$\mathcal{W}_2(\rho^N_{\varepsilon,\alpha}(t), \rho_{\varepsilon,\alpha}(t)) \le e^{-\lambda(\varepsilon,\alpha)t} \mathcal{W}_2(\rho^N_{\varepsilon,\alpha}(0), \rho_{\varepsilon,\alpha}(0)), \quad t \in [0,T],$$

where $\mathcal{W}_2$ denotes the 2-Wasserstein distance. In particular, (5.8) converges narrowly to a weak solution of the problem (5.7) when $\varepsilon, \alpha \to 0$.

Then, in view of this main result and [129], if the initial distribution of particles X_i is cleverly chosen so that $\mathcal{W}_2(\rho^N_{\varepsilon,\alpha}(0), \rho_{\varepsilon,\alpha}(0))$, then one can take $N = o\left(e^{-\lambda(\varepsilon,\alpha)}\right)$ to fulfill the hypothesis on the initial condition. Furthermore, we also want to remark that, up to our knowledge, this result would introduce the first deterministic particle approximation for a fourth-order partial differential equation.

Open problems. A natural question that arises is whether we can replicate this result to the two species case and recover a deterministic particle approximation for the problem (1.8). Moreover, let us notice that the particle approximation that we present here is only qualitative. The main result only addresses a qualitative deterministic particle approximation (logarithmic rates). Thus, an interesting second question is to try to get algebraic rates for the particle approximation.

5.2 A Li-Yau and Aronson-Bénilan approach for the Keller-Segel system with critical exponent

This section presents a work in preparation together with Charles Elbar[1] and Filippo Santambrogio[1]. "A Li-Yau and Aronson-Bénilan approach for the Keller-Segel system with critical exponent", currently *preprint in preparation*.

In this project we want to study the Keller–Segel system in $\mathbb{R}^d$

$$\begin{cases} \partial_t \rho = \Delta \rho^m - \operatorname{div}(\rho \nabla u) & \text{in } (0,\infty)\times\mathbb{R}^d, \\ -\Delta u = \rho & \text{in } (0,\infty)\times\mathbb{R}^d, \end{cases} \tag{5.10}$$

where ρ denotes the density of cells or bacteria and u is a chemoattractant that they produce and move toward. Furthermore, we take $\rho_0 \geq 0$ the initial condition, i.e. $\rho(0,\cdot) = \rho_0$. We cover the case in which the dimension is $d \geq 2$, and $m = 2 - \frac{2}{d}$ is the critical exponent which balances the nonlinear diffusion given by $\Delta\rho^m$ with the chemotactic attraction $\operatorname{div}(\rho\nabla u)$. Moreover, by the fundamental property of the Newtonian potential $\mathcal{N}$ in $\mathbb{R}^d$, we know that the chemoattractant u is given by

$$u(t,x) = -(\mathcal{N} * \rho)(t,x),$$

where

$$\mathcal{N}(x) = \begin{cases} \dfrac{1}{(d-2)\,\omega_d}\,|x|^{2-d} & \text{if } d \geq 3, \\ \dfrac{1}{2\pi}\,\log|x| & \text{if } d = 2, \end{cases}$$

and ω_d is the surface measure of the unit sphere $\mathbb{S}^{d-1} \subset \mathbb{R}^d$.

The model was proposed for the first time by Keller and Segel in [223] and Patlak in [271] to explain a phenomenon known as *chemotaxis*, [167]. In addition to that, the Keller-Segel system has further applications for different biological phenomenon connected to the evolution of unicellular organism to come up with more complex structures. For instance, it explains pattern formation of cells through meiosis [68]; embrio-genesis and angio-genesis [114, 240]; the Balo disease [224]; and bio-convection [123]. In physics, the Keller-Segel system also models the movement of self-gravitating Brownian particles [115, 116].

With respect to the mathematical analysis, the community has devoted a lot work to this problem over the last few decades [209, 78]. Keller-Segel presents a dichotomy between diffusion and aggregation. For instance, for the 2-dimensional case this dichotomy depends on the evolution of the second moment. In particular, in [50] Blanchet, Dolbeault and Perthame prove that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2} |x|^2 \rho(t,x)\mathrm{d}x = 4M\left(1 - \frac{M}{8\pi}\right)$$

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showing that $M = 8\pi$ is the critical mass. If $M < 8\pi$ and the initial datum has finite free energy (5.12), then there exists global in time solutions [213, 50, 46]. If $M > 8\pi$ there exists blow-up in finite time. In [203], Herrero and Velázquez construct an example of a radial solution with finite time blow-up. Afterwards, in [278] Raphaël and Schweyer show the existence of singularities for the radial case with mass close to 8π. Finally, in [126, 127], Collot, Ghould, Masmoudi and Nguyen are able to drop the mass assumption from the previous work and covered the case $M > 8\pi$. If the total mass is $M = 8\pi$ and the initial free energy (5.12) is finite solutions exists globally in time but they blow-up at time infinity [49, 144].

In the case of higher dimension there exists also a dichotomy depending on the initial mass. There exists a critical mass M_c such that if $M < M_c$ the solutions remain bounded globally in time and there exists self-similar solutions decaying in time with the porous medium equation [48, 27]. If $M > M_c$, in [28] Bedrossian and Kim show that all radial solutions blow up in finite time. If $M = M_c$ Blanchet, Carrillo and Laurençot show that the solutions are globally well-posed and their L^∞ norm is globally bounded in time [48]. Furthermore, Yao showed that radial solutions with compact support converge to some stationary solution in this family [309]. Furthermore, the critical mass M_c is characterised in the following way.

Definition 5.1 (Critical mass). *We define the critical mass as*

$$M_c = \begin{cases} 8\pi, & d = 2, \\ \left(\dfrac{2}{C_*(m-1)}\right)^{\frac{1}{2-m}}, & d \geq 3, \end{cases}$$

where C_* is the smallest constant in the inequality

$$\int_{\mathbb{R}^d} \rho(x)u(x)\mathrm{d}x \leq C_* \|\rho\|_{L^m}^m \|\rho\|_{L^1}^{2-m}, \tag{5.11}$$

for $m = 2 - \frac{2}{d}$ and $-\Delta u = \rho$. This is valid for all nonnegative $\rho \in L^1(\mathbb{R}^d) \cap L^m(\mathbb{R}^d)$. The existence the constant C_* is obtained from the Hardy–Littlewood–Sobolev (HLS) inequality combined with Hölder's inequality, see for instance [48, 66]. For convenience, we will loosely refer to (5.11) as "the HLS inequality", or the Lane–Emden inequality although it is slightly adapted to our Keller–Segel setting.

In this project, we intend to give a new perspective to the study of the Keller–Segel system. Thus, our main goal is to study the classical Li–Yau and Aronson–Bénilan estimates when you add to the diffusive equation the chemotactic aggregative term from the Keller–Segel system. Let us present these two classical estimates.

- For the heat equation $\partial_t \rho = \Delta \rho$, Li and Yau [241, 310] showed

$$\Delta \left(\log \rho(t,\cdot)\right) \geq -\frac{d}{2t}.$$

- For the porous medium equation $\partial_t \rho = \Delta \rho^m$, Aronson and Bénilan [13, 14] derived

$$\Delta \left(\frac{m}{m-1}\rho^{m-1}(t,\cdot)\right) \geq -\frac{\alpha}{t}, \quad \alpha = \frac{d}{d(m-1)+2}.$$

Hence, as mentioned above our main goal is to discuss whether these pointwise inequalities still hold, up to some constants. This implies robustness of Li–Yau and Aronson–Bénilan estimates even with a nonlocal term. As a result, we can derive L^∞ bounds for ρ, proving global existence, even if the initial data has a low regularity.

Moreover, with this new approach we are able to follow the dichotomy of the mass introduced above. Furthermore, we are able to recover some of the classical results of global existence and more important, we also improve them in two different ways. First, this strategy will lead us to recover an explicit L^∞ estimate on the solutions, this estimate is decreasing in time. In addition to this, with our approach we can also consider as a initial data a measure with mass small enough. If the mass is bigger but smaller than the critical mass, we only require the free energy of the initial data to be bounded.

Outline and expected main results

Let us now present the main goals of this project. However, let us first remark that formally, (5.10) can be viewed as the gradient flow in the Wasserstein metric of the free-energy functional

$$\mathcal{F}[\rho] = \begin{cases} \displaystyle\int_{\mathbb{R}^2} \rho(x) \log \rho(x)\, \mathrm{d}x - \frac{1}{2} \int_{\mathbb{R}^2} \rho(x) u(x) \mathrm{d}x, & d = 2, \\ \displaystyle\int_{\mathbb{R}^d} \frac{\rho(x)^m}{m-1}\, \mathrm{d}x - \frac{1}{2} \int_{\mathbb{R}^d} \rho(x)\, u(x) \mathrm{d}x, & d \geq 3. \end{cases} \tag{5.12}$$

For some of the main results we will need to assume $\mathcal{F}[\rho_0] < \infty$.

Now, we are ready to discuss our results (at least formally). First, for the case of dimension 2 we expect to get a Li–Yau type differential inequality.

Main Expected Result 2 (Li–Yau estimate for Keller-Segel in dimension $d = 2$)**.** *Let $\rho_0 \in \mathcal{M}_+(\mathbb{R}^2)$, a non-negative measure, such that $\mathcal{F}[\rho_0], \mathrm{m}_2[\rho_0] < +\infty$. Let us consider $M = \|\rho_0\|_{L^1(\mathbb{R}^2)} \in (0, 8\pi)$ the mass of the initial condition. Then, the Keller–Segel system* (5.10) *has a global solution ρ and there exists a constant $C > 0$ depending on T such that*

$$\Delta[\log \rho(t, x) - u(t, x)] \geq -C(T) \left(1 + \frac{1}{t}\right) \quad \textit{for a.e. } (t, x) \in (0, T) \times \mathbb{R}^2.$$

Moreover,

$$\|\rho(t, \cdot)\|_{L^\infty} \lesssim C(T) \left(1 + \frac{1}{t}\right) \quad \textit{for a.e. } t \in (0, T).$$

Furthermore, at least formally, we expect to extend this result in order to cover the case of critical mass $M = 8\pi$. In addition to this, under a smallness assumption on the mass, we think that there exists a Li–Yau type estimate like

$$\Delta[\log \rho(t, x) - u(t, x)] \geq -\frac{C}{t} \quad \text{for a.e. } (t, x) \in (0, T) \times \mathbb{R}^2$$

where the only assumption required is that $\rho \in \mathcal{M}_+(\mathbb{R}^2)$, i.e. $\mathcal{F}[\rho_0], \mathrm{m}_2[\rho_0] < \infty$ do not necessarily hold. This type of result is interesting because, for example, it would allow us to consider a small mass Dirac delta as an initial datum.

In an analogous way to the 2-dimensional case, we think we can cover the subcritical mass regime through an Aronson–Bénilan type differential inequality.

Main Expected Result 3 (Aronson–Bénilan estimate for Keller-Segel in dimension $d \geq 3$)**.** *Let $\rho_0 \in \mathcal{M}_+(\mathbb{R}^d)$, a non-negative measure. Assume furthermore $\mathcal{F}[\rho_0] < \infty$ and that the mass $M = \|\rho_0\|_{L^1(\mathbb{R}^d)}$ is small enough. Then, the Keller–Segel system* (5.10) *has a global solution ρ and there exists a constant $C > 0$ such that*

$$\Delta \left[\frac{m}{m-1} \rho^{m-1}(t, x) - u(t, x)\right] \geq -C(T) \left(1 + \frac{1}{t}\right) \quad \textit{for a.e. } (t, x) \in (0, T) \times \mathbb{R}^d.$$

Moreover,

$$\|\rho(t, \cdot)\|_{L^\infty} \lesssim C(T) \left(1 + \frac{1}{t}\right) \quad \textit{for a.e. } t \in (0, T).$$

Under a further smallness assumption on the mass we expect to be able to drop the assumptions on the boundedness of the free energy at time $t = 0$, i.e. $\mathcal{F}[\rho_0] < +\infty$ do not necessarily hold. In this case, we think we can recover an Aronson–Bénilan estimate like

$$\Delta \left[\frac{m}{m-1} \rho^{m-1}(t, x) - u(t, x)\right] \geq -\frac{C}{t} \quad \text{for a.e. } (t, x) \in (0, T) \times \mathbb{R}^d.$$

Moreover, we wonder if we can also cover all the cases $M \in (0, M_c)$ and the critical mass case $M = M_c$.

The main results of this project just presented imply a different series of results. In the following, we highlight them and compare it with the previous literature.

Existence of global solutions. Main Expected Result 2 and 3 provide uniform L^∞ bounds which can be used to prove existence of global-in-time solutions. In order to prove such result we can take an approximation of the initial data with smoother functions. This implies an L^∞ estimate valid for short times. Afterwards, the main challenge is to extend this bound beyond short time intervals. However, both Main Expected Result 2 and 3 yield an L^∞ estimate independent of the initial approximation, and therefore by iteration on the initial condition one can construct a global in time solution.

Smoothing of the initial condition. Our result also shows that the initial data is immediately smoothed out, becoming instantly bounded in L^∞, even if it was initially just a measure in $\mathcal{M}_+(\mathbb{R}^d)$. This smoothing property holds provided the total mass is below a certain threshold, which we explicitly compute. Nevertheless we still do not know whether the threshold that we compute is sharp.

Precise L^∞ estimates and applications. Main Expected Result 2 and 3 provide an explicit decay estimate of the L^∞ norm. Such estimates can be particularly useful in applications. For instance, in [27], Bedrossian obtained a similar decay, although derived by a different method and under a different mass threshold. In the case of [27], the decay obtained was used to prove the convergence to a self-similar profile. The author remarked that this kind of decay could be obtained with the Aronson—Bénilan method but suggested that such an estimate was not yet available for the Keller—Segel system. Thus, with this project we confirm that in fact this estimate is available.

Lipschitz estimates for the porous medium equation. To justify the Aronson–Bénilan estimate, we relied on a comparison principle, which required Lipschitz regularity of the pressure. Generally, such Lipschitz estimates are not available for arbitrary initial data in the porous medium equation or the critical Keller–Segel system for dimension $d \geq 3$. Earlier, Caffarelli and Friedman [62] established Hölder continuity of solutions to the porous medium equation uniformly in space and time. This result was further refined by Caffarelli, Vázquez, and Wolanski [64], showing that the pressure becomes Lipschitz continuous after some *waiting time*. Recently, there has been further work in this direction, in [227, 42, 140, 141] it was shown that a weighted Lipschitz norm of the pressure, of the form $|p|^\alpha|\nabla p|^\beta$ for some α, β remains bounded. In this project, we are able to obtain explicit Lipschitz estimates on ∇p for initial densities that decay polynomially at infinity.

The Liouville equation and the Lane–Emden equation. These two equations play a key role on the theory we develop in here. The Liouville equation

$$\Delta h + e^h = 0 \quad \text{in } \mathbb{R}^2 \tag{5.13}$$

was first studied by Liouville in [243]. In particular, he was the first one to give a representation formula for the equation. More recently, in [121] Chen and Li used the *moving planes* technique in order to show that if $\int_{\mathbb{R}^2} e^h < \infty$ then there exists a unique representation formula for the solutions of the equation (5.13). There has been further results concerning this problem. For instance, in [125] the authors studied the problem when you drop the assumption on the finiteness of the mass. For further literature review on this problem we recommend the interested reader to consult [113]. In particular, it is a well-known result that any nontrivial solution of (5.13) is such that $\int_{\mathbb{R}^2} e^h \geq 8\pi$, see [121, Lemma 1.1]. We can adapt this result in order to show that any nonnegative, nontrivial subsolution of $\Delta h + e^h \geq 0$ also satisfies $\int_{\mathbb{R}^2} e^h \geq 8\pi$. This result will be key in our analysis since we use it in order to recover L^∞ bounds of the solutions for (5.10) on dimension $d = 2$ if and only if $\|\rho_0\|_{L^1(\mathbb{R}^2)} < 8\pi$.

On dimension $d \geq 3$ in order to perform our analysis we require the Lane–Emden equation

$$\frac{m}{m-1}\Delta h + h^q = 0 \quad \text{in } \mathbb{R}^d \tag{5.14}$$

in the case $q = \frac{d}{d-2}$. In the seminal paper [195], Gidas and Spruck assert that (5.14) has no positive solutions. Afterwards, there has been further work focusing on understanding qualitative properties of the Lane–Emden equation [16, 34, 188]. One of the goals of this project is to prove that any nonnegative, nontrivial subsolution of $\frac{m}{m-1}\Delta h + h^{\frac{d}{d-2}} \geq 0$ is such that $\int_{\mathbb{R}^d} h^{\frac{d}{d-2}} \geq M_c$, where M_c is the critical mass from Definition 5.1. Up to our knowledge, this result would be new. Our idea for the proof relies on an optimal-control argument for a radially decreasing rearrangement of the subsolutions. Furthermore, and analogously to the 2-dimensional case, this result will help us to understand better the dynamics of (5.10). It allows us to recover L^∞ estimates of the solutions for (5.10) on dimension $d \geq 3$ if and only if the mass is subcritical, Definition 5.1.

Open problems. Up to our knowledge, this project is the first one that explores Li–Yau and Aronson–Bénilan estimates for diffusion equations with a non-local aggregation term. We wonder if we can extend the theory presented here to cover further cases, for example the diffusion dominated case, i.e. $m > 2 - \frac{2}{d}$. Can we also obtain an Aronson–Bénilan type differential inequality for this case as well?

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