

# DEFECTIVE DP-COLORINGS OF SPARSE SIMPLE GRAPHS

YIFAN JING, ALEXANDR KOSTOCHKA, FUHONG MA, AND JINGWEI XU

**ABSTRACT.** DP-coloring (also known as correspondence coloring) is a generalization of list coloring developed recently by Dvořák and Postle. We introduce and study  $(i, j)$ -defective DP-colorings of simple graphs. Let  $g_{DP}(i, j, n)$  be the minimum number of edges in an  $n$ -vertex DP- $(i, j)$ -critical graph. In this paper we determine sharp bound on  $g_{DP}(i, j, n)$  for each  $i \geq 3$  and  $j \geq 2i + 1$  for infinitely many  $n$ .

*Mathematics Subject Classification:* 05C15, 05C35.

*Key words and phrases:* Defective Coloring, List Coloring, DP-coloring.

## 1. INTRODUCTION

**1.1. Defective Coloring.** A *proper  $k$ -coloring* of a graph  $G$  is a partition of  $V(G)$  into  $k$  independent sets  $V_1, \dots, V_k$ . A  $(d_1, \dots, d_k)$ -*defective coloring* (or simply  $(d_1, \dots, d_k)$ -*coloring*) of a graph  $G$  is a partition of  $V(G)$  into sets  $V_1, V_2, \dots, V_k$  such that for every  $i \in [k]$ , every vertex in  $V_i$  has at most  $d_i$  neighbors in  $V_i$ . The ordinary proper  $k$ -coloring is a partial case of such coloring, namely it is a  $(0, 0, \dots, 0)$ -defective coloring. A significant amount of interesting papers were devoted to defective colorings of graphs, see e.g. [1, 9, 11, 13, 15, 17, 21, 23, 24, 28].

For every  $(i, j) \neq (0, 0)$ , it is an NP-complete problem to decide whether a graph  $G$  has an  $(i, j)$ -coloring. Even the problem of checking whether a given planar graph of girth 9 has a  $(0, 1)$ -coloring is NP-complete; this was showed by Esperet, Montassier, Ochem, and Pinlou [15]. Since the parameter is NP-complete, a number of papers considered how sparse can be graphs with no  $(i, j)$ -coloring for given  $i$  and  $j$ ; the reader may look at [3, 4, 5, 6, 7, 8, 19, 20]. Among the measures of how “sparse” is a graph, one of the most used is the *maximum average degree*,  $mad(G) = \max_{G' \subseteq G} \frac{2|E(G')|}{|V(G')|}$ . In this paper we restrict ourselves to coloring with 2 colors. A very useful notion in the studies of defective colorings with two colors is that of  $(i, j)$ -*critical graphs* which are the graphs that do not have  $(i, j)$ -coloring but every proper subgraph of which has such a coloring. Let  $f(i, j, n)$  denote the minimum number of edges in an  $(i, j)$ -critical  $n$ -vertex graph. One simple example is that  $f(0, 0, n) = n$  for odd  $n \geq 3$ : the  $n$ -cycle is not bipartite, but every graph with fewer than  $n$  edges has a vertex of degree at most 1 and hence cannot be  $(0, 0)$ -critical. The papers cited above

---

A.K. was partially supported by NSF grant DMS1600592, by grants 18-01-00353A and 19-01-00682 of the Russian Foundation for Basic Research and by Arnold O. Beckman Campus Research Board Award RB20003 of the University of Illinois at Urbana-Champaign.

F.M. is corresponding author.

J.X. was partially supported by Arnold O. Beckman Campus Research Board Award RB20003 of the University of Illinois at Urbana-Champaign.

showed several interesting bounds on  $f(i, j, n)$ . For example, they contain lower bounds that are exact for infinitely many  $n$  in the cases when  $j \geq 2i + 2$  and when  $(i, j) \in \{(0, 1), (1, 1)\}$ .

**1.2. Defective List Coloring.** Recall that a *list* for a graph  $G$  is a function  $L : V(G) \rightarrow \mathcal{P}(\mathbb{N})$  that assigns to each  $v \in V(G)$  a set  $L(v)$ . A list  $L$  is an  $\ell$ -*list* if  $|L(v)| = \ell$  for every  $v \in V(G)$ . An  $L$ -*coloring* of  $G$  is a function  $\phi : V(G) \rightarrow \bigcup_{v \in V(G)} L(v)$  such that  $\phi(v) \in L(v)$  for every  $v \in V(G)$  and  $\phi(u) \neq \phi(v)$  whenever  $uv \in E(G)$ . A graph  $G$  is  $k$ -*choosable* if  $G$  has an  $L$ -coloring for every  $k$ -list assignment  $L$ . The following notion was introduced in [14, 26] and studied in [27, 30, 16, 17]: A  $d$ -*defective list  $L$ -coloring* of  $G$  is a function  $\phi : V(G) \rightarrow \bigcup_{v \in V(G)} L(v)$  such that  $\phi(v) \in L(v)$  for every  $v \in V(G)$  and every vertex has at most  $d$  neighbors of the same color. If  $G$  has a  $d$ -defective list  $L$ -coloring from every  $k$ -list assignment  $L$ , then it is called  $d$ -*defective  $k$ -choosable*. As in the case of ordinary coloring, a direction of study is showing that “sparse” graphs are  $d$ -defective  $k$ -choosable. As mentioned before, in this paper we consider only  $k = 2$ . The best known bounds on maximum average degree that guarantee that a graph is  $d$ -defective 2-choosable are due to Havet and Sereni [16] (a new proof of the lower bound is due to Hendrey and Wood [17]):

**Theorem A** ([16]). *For every  $d \geq 0$ , if  $\text{mad}(G) < \frac{4d+4}{d+2}$ , then  $G$  is  $d$ -defective 2-choosable. On the other hand, for every  $\epsilon > 0$ , there is a graph  $G_\epsilon$  with  $\text{mad}(G_\epsilon) < 4 + \epsilon - \frac{2d+4}{d^2+2d+2}$  that is not  $(d, d)$ -colorable.*

**1.3. Defective DP-Coloring.** Dvořák and Postle [12] introduced and studied DP-coloring which generalizes list coloring. This notion was extended to multigraphs by Bernshteyn, Kostochka and Pron [2].

**Definition 1.** Let  $G$  be a multigraph. A *cover* of  $G$  is a pair  $\mathcal{H} = (L, H)$ , consisting of a graph  $H$  (called the *cover graph* of  $G$ ) and a function  $L : V(G) \rightarrow 2^{V(H)}$ , satisfying the following requirements:

- (1) the family of sets  $\{L(u) : u \in V(G)\}$  forms a partition of  $V(H)$ ;
- (2) for every  $u \in V(G)$ , the graph  $H[L(u)]$  is complete;
- (3) if  $E(H[L(u), L(v)]) \neq \emptyset$ , then either  $u = v$  or  $uv \in E(G)$ ;
- (4) if the multiplicity of an edge  $uv \in E(G)$  is  $k$ , then  $H[L(u), L(v)]$  is the union of at most  $k$  matchings connecting  $L(u)$  with  $L(v)$ .

A cover  $(L, H)$  of  $G$  is  $k$ -*fold* if  $|L(u)| = k$  for every  $u \in V(G)$ .

Throughout this paper, we consider only 2-fold covers.

**Definition 2.** Let  $G$  be a multigraph and  $\mathcal{H} = (L, H)$  be a cover of  $G$ . An  $\mathcal{H}$ -*map* is an injection  $\phi : V(G) \rightarrow V(H)$ , such that  $\phi(v) \in L(v)$  for every  $v \in V(G)$ . The subgraph of  $H$  induced by  $\phi(V(G))$  is called the  $\phi$ -*induced cover graph*, denoted by  $H_\phi$ .

**Definition 3.** Let  $\mathcal{H} = (L, H)$  be a cover of  $G$ . For  $u \in V(G)$ , let  $L(u) = \{p(u), r(u)\}$ , where  $p(u)$  and  $r(u)$  are called the *poor* and the *rich* vertices, respectively. Given  $i, j \geq 0$  and  $i \leq j$ . An  $\mathcal{H}$ -map  $\phi$  is an  $(i, j)$ -*defective- $\mathcal{H}$ -coloring* of  $G$  if the degree of every poor vertex in  $H_\phi$  is at most  $i$ , and the degree of every rich vertex in  $H_\phi$  is at most  $j$ .

**Definition 4.** A multigraph  $G$  is  $(i, j)$ -defective-DP-colorable if for every 2-fold cover  $\mathcal{H} = (L, H)$  of  $G$ , there exists an  $(i, j)$ -defective- $\mathcal{H}$ -coloring. We say  $G$  is  $(i, j)$ -defective-DP-critical, if  $G$  is not  $(i, j)$ -defective-DP-colorable, but every proper subgraph of  $G$  is.

If  $uv \in E(G)$  and in a 2-fold cover  $\mathcal{H} = (L, H)$  of  $G$  some vertex  $\alpha \in L(u)$  has no neighbors in  $L(v)$ , then also some  $\beta \in L(v)$  has no neighbors in  $L(u)$ . In this case, adding  $\alpha\beta$  to  $H$  makes it only harder to find an  $(i, j)$ -defective- $\mathcal{H}$ -coloring of  $G$ . Thus, below we consider only *full* 2-fold covers, i.e. the covers  $\mathcal{H} = (L, H)$  of  $G$  such that for every edge  $e$  connecting  $u$  with  $v$  in  $G$ , the matching in  $\mathcal{H} = (L, H)$  corresponding to  $e$  consists of two edges.

For brevity, in the rest of the paper, we call an  $(i, j)$ -defective- $\mathcal{H}$ -coloring simply by an  $(i, j, \mathcal{H})$ -coloring (or ‘ $\mathcal{H}$ -coloring’, if  $i$  and  $j$  are clear from the context). Similarly, instead of “ $(i, j)$ -defective-DP-colorable” and “ $(i, j)$ -defective-DP-critical” we will say “ $(i, j)$ -colorable” and “ $(i, j)$ -critical”.

Denote the minimum number of edges in an  $n$ -vertex  $(i, j)$ -critical multigraph by  $f_{DP}(i, j, n)$ , and the minimum number of edges in an  $n$ -vertex  $(i, j)$ -critical simple graph by  $g_{DP}(i, j, n)$ . By definition,  $f_{DP}(i, j, n) \leq g_{DP}(i, j, n)$ . Recently [18], linear lower bounds on  $f_{DP}(i, j, n)$  were proved that are exact for infinitely many  $n$  for every choice of  $i \leq j$ .

**Theorem B** ([18]).

- (1) If  $i = 0$  and  $j \geq 1$ , then  $f_{DP}(0, j, n) \geq n + j$ . This is sharp for every  $j \geq 1$  and every  $n \geq 2j + 2$ .
- (2) If  $i \geq 1$  and  $j \geq 2i + 1$ , then  $f_{DP}(i, j, n) \geq \frac{(2i+1)n - (2i-j)}{i+1}$ . This is sharp for each such pair  $(i, j)$  for infinitely many  $n$ .
- (3) If  $i \geq 1$  and  $i + 2 \leq j \leq 2i$ , then  $f_{DP}(i, j, n) \geq \frac{2jn+2}{j+1}$ . This is sharp for each such pair  $(i, j)$  for infinitely many  $n$ .
- (4) If  $i \geq 1$ , then  $f_{DP}(i, i+1, n) \geq \frac{(2i^2+4i+1)n+1}{i^2+3i+1}$ . This is sharp for each  $i \geq 1$  for infinitely many  $n$ .
- (5) If  $i \geq 1$ , then  $f_{DP}(i, i, n) \geq \frac{(2i+2)n}{i+2}$ . This is sharp for each  $i \geq 1$  for infinitely many  $n$ .

The bound in Part (1) is also sharp for simple graphs.

For  $i > 0$  we do not know simple graphs for which the bounds of Theorem B are sharp. In fact, we think that  $g_{DP}(i, j, n) > f_{DP}(i, j, n)$  for  $i > 0$ . It follows from [22] that  $g_{DP}(1, 1, n) > f_{DP}(1, 1, n)$  and  $g_{DP}(2, 2, n) > f_{DP}(2, 2, n)$ . The goal of this paper is to find a lower bound on  $g_{DP}(i, j, n)$  for  $i \geq 3$  and  $j \geq 2j + 1$  that is exact for infinitely many  $n$  for each such pair  $(i, j)$ . It differs from the bound of Theorem B(2) but only by 1.

## 2. RESULTS

The goal of this paper is to prove the following extremal result.

**Theorem 2.1.** Let  $i \geq 3$ ,  $j \geq 2i + 1$  be positive integers, and let  $G$  be an  $(i, j)$ -critical simple graph. Then

$$g_{DP}(i, j, n) \geq \frac{(2i+1)n + j - i + 1}{i+1}.$$

This is sharp for each such pair  $(i, j)$  for infinitely many  $n$ .

Since every non- $(i, j)$ -colorable graph contains an  $(i, j)$ -critical subgraph, Theorem 2.1 yields the following.

**Corollary 2.2.** *Let  $G$  be a simple graph. If  $i \geq 3$  and  $j \geq 2i + 1$  and for every subgraph  $H$  of  $G$ ,  $|E(H)| \leq \frac{(2i+1)|V(H)|+j-i+1}{i+1}$ , then  $G$  is  $(i, j)$ -colorable. This is sharp.*

In the next section we introduce a more general framework to prove the lower bound of Theorem 2.1. The lower bound of Theorem 2.1 will be proved in Section 4. In the last section, we present constructions showing that our bounds are sharp for each  $i \leq j$  for infinitely many  $n$ .

### 3. A MORE GENERAL SETTING

We need the following more general framework. Instead of  $(i, j)$ -colorings of a cover  $(L, H)$  of a graph  $G$ , we will consider  $(L, H)$ -maps  $\phi$  with variable restrictions on the ‘allowed’ degrees of the vertices in  $H_\phi$ .

**Definition 5** (Capacity). A *capacity function* on  $G$  is a map  $\mathbf{c} : V(G) \rightarrow \{-1, 0, \dots, i\} \times \{-1, 0, \dots, j\}$ . For  $u \in V(G)$ , denote  $\mathbf{c}(u)$  by  $(\mathbf{c}_1(u), \mathbf{c}_2(u))$ . We call such pair  $(G, \mathbf{c})$  a *weighted pair*.

Below, let  $(G, \mathbf{c})$  be a weighted pair, and  $\mathcal{H} = (L, H)$  be a cover of  $G$ .

**Definition 6** (A  $\mathbf{c}$ -coloring). A  $(\mathbf{c}, \mathcal{H})$ -coloring of  $G$  is an  $\mathcal{H}$ -map  $\phi$  such that for each  $u \in V(G)$ , the degree of  $p(u)$  in  $H_\phi$  is at most  $\mathbf{c}_1(u)$ , and that of  $r(u)$  is at most  $\mathbf{c}_2(u)$ . We call  $\mathbf{c}_1(u)$  the *capacity* of  $p(u)$  and  $\mathbf{c}_2(u)$  the *capacity* of  $r(u)$ . If the capacity of some  $v$  in  $V(H)$  is  $-1$ , then  $v$  is not allowed in the image of any  $(\mathbf{c}, \mathcal{H})$ -coloring of  $G$ . If for every cover  $\mathcal{H}$  of  $G$ , there is a  $(\mathbf{c}, \mathcal{H})$ -coloring, we say that  $G$  is  $\mathbf{c}$ -colorable.

If  $\mathbf{c}(v) = (i, j)$  for all  $v \in V(G)$ , then any  $(\mathbf{c}, \mathcal{H})$ -coloring of  $G$  is an  $(i, j, \mathcal{H})$ -coloring in the sense of Definition 3. So, Definition 6 is a refinement of Definition 3. Similarly, we say that  $G$  is  $\mathbf{c}$ -critical if  $G$  is not  $\mathbf{c}$ -colorable, but every proper subgraph of  $G$  is. For every vertex  $x$  in the cover graph, we slightly abuse the notation of  $\mathbf{c}$  and denote the capacity of  $x$  by  $\mathbf{c}(x)$ .

**Definition 7.** For a vertex  $u \in V(G)$ , the  $(i, j, \mathbf{c})$ -potential of  $u$  is

$$\rho_{\mathbf{c}}(u) := i - j + 1 + \mathbf{c}_1(u) + \mathbf{c}_2(u).$$

The  $(i, j, \mathbf{c})$ -potential of a subgraph  $G'$  of  $G$  is

$$(1) \quad \rho_{G, \mathbf{c}}(G') := \sum_{u \in V(G')} \rho_{\mathbf{c}}(u) - (i + 1)|E(G')|.$$

For a subset  $S \subseteq V(G)$ , the  $(i, j, \mathbf{c})$ -potential of  $S$ ,  $\rho_{G, \mathbf{c}}(S)$ , is the  $(i, j, \mathbf{c})$ -potential of  $G[S]$ . The  $(i, j, \mathbf{c})$ -potential of  $(G, \mathbf{c})$  is defined by  $\rho(G, \mathbf{c}) := \min_{S \subseteq V(G)} \rho_{G, \mathbf{c}}(S)$ .

When clear from the text, we call the  $(i, j, \mathbf{c})$ -potential simply by potential.

Observe that the potential function is submodular:

**Lemma 3.1.** For all  $A, B \subseteq V(G)$ ,

$$(2) \quad \rho_{G,\mathbf{c}}(A) + \rho_{G,\mathbf{c}}(B) = \rho_{G,\mathbf{c}}(A \cup B) + \rho_{G,\mathbf{c}}(A \cap B) + (i+1)|E(A \setminus B, B \setminus A)|.$$

*Proof.* Since  $G$  and  $\mathbf{c}$  are fixed, we omit the subscripts in the proof. By definition,

$$\begin{aligned} \rho(A) &= \rho(A \setminus B) + \rho(A \cap B) - (i+1)|E(A \setminus B, A \cap B)|, \\ \rho(B) &= \rho(B \setminus A) + \rho(A \cap B) - (i+1)|E(B \setminus A, A \cap B)|. \end{aligned}$$

Hence

$$\rho(A) + \rho(B) = \rho(A \cap B) + \rho(A \cup B) + (i+1)|E(A \setminus B, B \setminus A)| \geq \rho(A \cap B) + \rho(A \cup B).$$

□

The following theorem implies the lower bound in Theorem 2.1.

**Theorem 3.2.** Let  $i \geq 3$ ,  $j \geq 2i+1$  be positive integers, and  $(G, \mathbf{c})$  be a weighted pair such that  $G$  is  $\mathbf{c}$ -critical, then  $\rho(G, \mathbf{c}) \leq i - j - 1$ .

To deduce the lower bound in Theorem 2.1, simply set  $\mathbf{c}(v) = (i, j)$  for every  $v \in V(G)$ . We will prove Theorem 3.2 in the next section.

#### 4. PROOF OF THEOREM 3.2

Suppose there exists a  $\mathbf{c}$ -critical graph  $G$  with  $\rho(G, \mathbf{c}) \geq i - j$ . Choose such  $(G, \mathbf{c})$  with  $|V(G)| + |E(G)|$  minimum. We say that  $G'$  is *smaller* than  $G$  if  $|V(G')| + |E(G')| < |V(G)| + |E(G)|$ . Let  $\mathcal{H} = (L, H)$  be an arbitrary cover of  $G$ .

For a subgraph  $G'$  of  $G$ , let  $\mathcal{H}_{G'} = (L_{G'}, H_{G'})$  denote the subcover *induced* by  $G'$ , i.e.,

- (1)  $L_{G'} = L|_{V(G')}$ , where ' $f|_S$ ' is the restriction of function  $f$  to subdomain  $S$ ;
- (2)  $V(H_{G'}) = L(V(G'))$  and  $L_{G'}(v) = L(v)$  for every  $v \in V(G')$ ;
- (3)  $H_{G'}[L(u) \cup L(v)] = H[L(u) \cup L(v)]$  for every  $uv \in E(G')$ , and for  $x, y$  such that  $xy \notin E(G')$ , there is no edge between  $L_{G'}(x)$  and  $L_{G'}(y)$ .

For a subset  $S$  of  $V(G)$ , let  $\mathcal{H}_S = (L_S, H_S)$  denote the subcover induced by  $G[S]$ . If a capacity function is the restriction of  $\mathbf{c}$  to some  $S \subseteq V(G)$ , we denote this capacity function by  $\mathbf{c}$  instead of  $\mathbf{c}|_S$ , for simplicity.

For two vertices  $x, y$ , we use  $x \sim y$  to indicate that  $x$  is adjacent to  $y$ , and  $x \not\sim y$  to indicate that  $x$  is not adjacent to  $y$ .

**Lemma 4.1.** Let  $S$  be a proper subset of  $V(G)$ . If  $\rho_{G,\mathbf{c}}(S) \leq i - j$ , then  $S = \{x\}$  for some  $x \in V(G)$  with  $\rho_{\mathbf{c}}(x) = i - j$ .

*Proof.* Suppose the lemma fails. Let  $S$  be a maximal proper subset of  $V(G)$  such that  $\rho_{G,\mathbf{c}}(S) \leq i - j$  and  $|S| \geq 2$ . If  $|N(v) \cap S| \geq 2$  for some  $v \in V(G) \setminus S$ , then

$$\rho_{G,\mathbf{c}}(S \cup \{v\}) \leq i - j + 2i + 1 - 2(i+1) = i - j - 1.$$

If  $S \cup \{v\} \neq V(G)$ , this contradicts the maximality of  $S$ , otherwise this contradicts the choice of  $G$ . Thus

$$(3) \quad |N(v) \cap S| \leq 1 \text{ for every } v \in V(G) \setminus S.$$

Since  $G$  is  $\mathbf{c}$ -critical,  $G[S]$  admits an  $(\mathbf{c}, \mathcal{H}_S)$ -coloring  $\phi$ .

Construct  $G'$  from  $G - S$  by adding a new vertex  $v^*$  adjacent to every  $u \in V(G) - S$  that was adjacent to a vertex in  $S$ . Define a capacity function  $\mathbf{c}'$  by letting  $\mathbf{c}'(v^*) = (-1, 0)$  and  $\mathbf{c}'(u) = \mathbf{c}(u)$  for  $u \in V(G' - v^*)$ .

By (3),  $G'$  is simple. Suppose  $\rho_{G', \mathbf{c}'}(A) \leq i - j - 1$  for some  $A \subseteq V(G')$ . Since  $G' - v^* \subseteq G$  and  $\mathbf{c}'(u) = \mathbf{c}(u)$  for  $u \in V(G' - v^*)$ ,  $v^* \in A$ . Then using (2) and  $\rho_{G, \mathbf{c}}(S) \leq i - j = \rho_{G', \mathbf{c}'}(v^*)$ ,

$$\begin{aligned} \rho_{G, \mathbf{c}}(S \cup (A - v^*)) &= \rho_{G, \mathbf{c}}(S) + \rho_{G, \mathbf{c}}(A - v^*) - (i + 1)|E_G(S, A - v^*)| \\ &\leq \rho_{G', \mathbf{c}'}(v^*) + \rho_{G', \mathbf{c}'}(A - v^*) - (i + 1)|E_{G'}(v^*, A - v^*)| = \rho_{G', \mathbf{c}'}(A) \leq i - j - 1. \end{aligned}$$

Again, this contradicts either the maximality of  $S$  or the choice of  $G$ . This yields

$$(4) \quad \rho(G', \mathbf{c}') \geq i - j.$$

For every  $x \in S$  and  $y \in N(x) \setminus S$ , denote the neighbor of  $\phi(x)$  in  $L(y)$  by  $y_\phi$ . Let  $\mathcal{H}' = (L', H')$  be the cover of  $G'$  defined as follows:

- 1)  $L'(v^*) = \{p(v^*), r(v^*)\}$ , and  $L'(u) = L(u)$  for every  $u \in V(G) \setminus S$ ;
- 2)  $y_\phi \sim r(v^*)$  for every  $y \in N(S)$ , and  $H'[\{u, w\}] = H[\{u, w\}]$  for every  $u, w \in V(G' - v^*)$ .

By (4) and the minimality of  $G$ ,  $G'$  has a  $(\mathbf{c}', \mathcal{H}')$ -coloring  $\psi$ . Since  $\mathbf{c}'(v^*) = (-1, 0)$ ,

$$(5) \quad \psi(v^*) = r(v^*) \text{ and } \psi(y) \neq y_\phi \text{ for every } y \in N(S).$$

Let  $\theta$  be an  $\mathcal{H}$ -map such that  $\theta|_S = \phi$  and  $\theta|_{V(G) \setminus S} = \psi|_{V(G' - v^*)}$ . By (5),  $\theta$  is a  $(\mathbf{c}, \mathcal{H})$ -coloring of  $G$ , a contradiction.  $\square$

Lemma 4.1 implies that

$$(6) \quad \text{for every } F \subseteq V(G), \rho_{G, \mathbf{c}}(F) \geq i - j.$$

**Lemma 4.2.** *For every  $u \in V(G)$ , the following statements hold:*

- (i)  $\mathbf{c}_1(u), \mathbf{c}_2(u) \geq 0$ ;
- (ii)  $d_G(u) \geq 2$ ;
- (iii)  $\rho_{\mathbf{c}}(u) \geq i - j + 1$ .

*Proof.* We prove (i) by contradiction. Suppose there is a vertex  $u \in V(G)$  with  $L(u) = \{\alpha(u), \beta(u)\}$ , where  $\mathbf{c}(\alpha(u)) = -1$ . Let  $v \in N_G(u)$  and  $L(v) = \{\alpha(v), \beta(v)\}$ , where  $\alpha(v)\alpha(u), \beta(v)\beta(u) \in E(H)$ .

**Case 1.**  $\mathbf{c}(\beta(u)) = 0$ . If  $\min\{\mathbf{c}_1(v), \mathbf{c}_2(v)\} = -1$ , then

$$\rho_{G, \mathbf{c}}(\{u, v\}) = \rho_{\mathbf{c}}(u) + \rho_{\mathbf{c}}(v) - (i + 1) \leq (i - j) + (i - j + 1 - 1 + j) - (i + 1) = i - j - 1,$$

a contradiction to (6). Thus  $\mathbf{c}_1(v), \mathbf{c}_2(v) \geq 0$ .

For every  $w \in N_G(v)$ , let  $L(w) = \{\alpha(w), \beta(w)\}$ , so that  $\alpha(w) \sim \alpha(v), \beta(w) \sim \beta(v)$ . Since  $G$  is  $\mathbf{c}$ -critical, graph  $G - v$  has a  $(\mathbf{c}, \mathcal{H}_{G-v})$ -coloring  $\phi$ .

**Case 1.1:**  $\mathbf{c}(\alpha(w)) = -1$  for all  $w \in N(v)$  (in particular, this happens if  $d(v) = 1$ ). Then  $\phi(w) = \beta(w)$  for every  $w \in N(v)$ . Extend  $\phi$  to  $v$  by  $\phi(v) = \alpha(v)$ . This  $\phi$  is a  $(\mathbf{c}, \mathcal{H})$ -coloring of  $G$ , a contradiction.

**Case 1.2:** There is  $w \in N(v)$  such that  $\mathbf{c}(\alpha(w)) \geq 0$ . Then  $\rho_{\mathbf{c}}(w) \geq i - j + 1$  since otherwise  $\rho_{G, \mathbf{c}}(\{u, v, w\}) \leq 2(i - j) - 1 < i - j - 1$ . Define  $(G', \mathbf{c}')$  as follows:

- 1)  $G' = G - vw$  and  $\mathcal{H}' = (L, H')$  is the sub-cover of  $\mathcal{H}$  induced by  $G'$ ;
- 2)  $\mathbf{c}'$  differs from  $\mathbf{c}$  only for  $\alpha(v)$  and  $\alpha(w)$ :  $\mathbf{c}'(\alpha(v)) = \mathbf{c}(\alpha(v)) - 1$  and  $\mathbf{c}'(\alpha(w)) = \mathbf{c}(\alpha(w)) - 1$ .

By the minimality of  $G$ , if  $G'$  is not colorable, then there is  $F \subseteq V(G')$  with  $\rho_{G, \mathbf{c}'}(F) \leq i - j - 1$ . By (6),  $F \cap \{v, w\} \neq \emptyset$ . If  $v, w \in F$ , then

$$\rho_{G, \mathbf{c}}(F) = \rho_{G, \mathbf{c}'}(F) + 2 - (i + 1) < \rho_{G, \mathbf{c}'}(F) < i - j.$$

If  $v \in F$  and  $w \notin F$ , then  $\rho_{G,\mathbf{c}}(V(F)) \leq \rho_{G,\mathbf{c}'}(F) + 1 \leq i - j$ . Since  $\rho_{\mathbf{c}}(v) \geq i - j + 1$ , this contradicts Lemma 4.1. Since  $\rho_{\mathbf{c}}(w) \geq i - j + 1$ , the case when  $w \in F, v \notin F$  is impossible for the same reason. Thus,  $G'$  admits a  $(\mathbf{c}', \mathcal{H}')$ -coloring  $\phi$ . Since  $\mathbf{c}(\alpha(u)) = -1$  and  $\mathbf{c}(\beta(u)) = 0$ , we have  $\phi(u) = \beta(u)$  and  $\beta(u)$  has no neighbors in  $H'_\phi$ . Then  $\phi(v) = \alpha(v)$  and by the construction of  $G'$ , independently of the color of  $w$ ,  $\phi$  is a  $(\mathbf{c}, \mathcal{H})$ -coloring of  $G$ .

**Case 2.**  $\mathbf{c}(\beta(u)) \geq 1$ . Then  $\rho_{\mathbf{c}}(u) \geq i - j + 1$ . Since  $G$  is  $\mathbf{c}$ -critical,  $G - uv$  has a  $(\mathbf{c}, \mathcal{H}_{G-uv})$ -coloring  $\phi$ . If  $\mathbf{c}(\beta(v)) = -1$ , then  $\phi(u) = \beta(u)$  and  $\phi(v) = \alpha(v)$ . So,  $\phi$  is also a  $(\mathbf{c}, \mathcal{H})$ -coloring of  $G$ , a contradiction. Hence  $\mathbf{c}(\beta(v)) \geq 0$ . Also by Lemma 4.1,

$$i - j + 1 \leq \rho_{G,\mathbf{c}}(\{u, v\}) = \rho_{\mathbf{c}}(v) + \rho_{\mathbf{c}}(u) - (i + 1) \leq \rho_{\mathbf{c}}(v) + i - j + 1 - 1 + j - i - 1,$$

thus  $\rho_{\mathbf{c}}(v) \geq i - j + 2$ .

Define  $(G', \mathbf{c}')$  as follows:

- 1)  $G' = G - vu$  and  $\mathcal{H}' = (L, H')$  is the sub-cover of  $\mathcal{H}$  induced by  $G'$ ;
- 2)  $\mathbf{c}'$  differs from  $\mathbf{c}$  only for  $\beta(v)$  and  $\beta(u)$ :  $\mathbf{c}'(\beta(v)) = \mathbf{c}(\beta(v)) - 1$  and  $\mathbf{c}'(\beta(u)) = \mathbf{c}(\beta(u)) - 1$ .

Repeating the argument of Case 1.2, we prove that  $G'$  has a  $(\mathbf{c}', \mathcal{H}_{G'})$ -coloring  $\psi$ , which is also a  $(\mathbf{c}, \mathcal{H})$ -coloring of  $G$ , a contradiction. This proves (i).

For (ii), suppose there is a vertex  $u$  with  $N(u) = \{v\}$ . By (i),  $\mathbf{c}_1(u), \mathbf{c}_2(u) \geq 0$ . Since  $G$  is  $\mathbf{c}$ -critical,  $G - u$  has a  $(\mathbf{c}, \mathcal{H}_{G-u})$ -coloring  $\phi$ . Now choosing  $\phi(u) \in L(u)$  not adjacent to  $\phi(v)$  we obtain a  $(\mathbf{c}, \mathcal{H})$ -coloring of  $G$ , a contradiction. This proves (ii), and (iii) follows immediately from (i).  $\square$

Lemmas 4.1 and 4.2 (iii) imply that

$$(7) \quad \text{for every } \emptyset \neq F \subsetneq V(G), \rho_{G,\mathbf{c}}(F) \geq i - j + 1.$$

We say a vertex  $v \in V(G)$  is a  $(d; c_1, c_2)$ -vertex if  $d(v) = d$ ,  $\mathbf{c}_1(v) = c_1$ , and  $\mathbf{c}_2(v) = c_2$ . The following lemma is a crucial ingredient of our argument.

**Lemma 4.3.** *Let  $\emptyset \neq F \subsetneq V(G)$ . If  $\rho_{G,\mathbf{c}}(F) \leq i - j + 1$ , then  $V(G) \setminus F = \{x\}$ , where  $x$  is a  $(2; i, j)$ -vertex.*

*Proof.* Suppose the lemma fails. Then there is a maximal  $\emptyset \neq F \subsetneq V(G)$  such that  $\rho_{G,\mathbf{c}}(F) \leq i - j + 1$  and  $V(G) - F$  is not a single  $(2; i, j)$ -vertex.

If  $V(G) - F = \{v\}$ , then by the choice of  $F$  either  $\rho(v) < 2i + 1$  or by Lemma 4.2 (ii),  $d(v) \geq 3$ . In both cases, by (2),

$$\rho(V(G)) = \rho(F) + \rho(v) - (i + 1)d(v) < (i - j + 1) + (2i + 1) - 2(i + 1) = i - j,$$

a contradiction. Hence  $|V(G \setminus F)| \geq 2$ . Because of (7), we have

$$(8) \quad |N(v) \cap F| \leq 1, \text{ for every } v \in V(G) \setminus F.$$

Let  $Y$  be the set of all  $(2; i, j)$ -vertices in  $V(G) - F$ , and  $X = V(G) \setminus F \setminus Y$ .

**Claim 1.** *Both  $X$  and  $Y$  are independent sets.*

*Proof of Claim 1.* Suppose  $u, v \in Y$  and  $u \sim v$ . Let  $u' \in N(u) - v, v' \in N(v) - u$ . Since  $G$  is  $\mathbf{c}$ -critical,  $G - \{u, v\}$  has a  $(\mathbf{c}, \mathcal{H}_{G-\{u,v\}})$ -coloring  $\phi$ . We extend  $\phi$  to  $u$  and  $v$  by choosing  $\phi(u) \in L(u)$  with  $\phi(u) \approx \phi(u')$  and  $\phi(v) \in L(v)$  with  $\phi(v) \approx \phi(v')$ . Since  $u$  and  $v$  are  $(2; i, j)$ -vertices, the new  $\phi$  is a  $(\mathbf{c}, \mathcal{H})$ -coloring of  $G$ , a contradiction.

Now suppose  $u, v \in X$  and  $u \sim v$ . Let  $G' = G - uv$ . Define  $\mathbf{c}'$  by  $\mathbf{c}'(y) = \mathbf{c}(y)$  for  $y \notin \{u, v\}$  and  $\mathbf{c}'_k(x) = \mathbf{c}_k(x) - 1$  for  $x \in \{u, v\}$  and  $k \in \{1, 2\}$ . If  $G'$  has a  $(\mathbf{c}', \mathcal{H})$ -coloring  $\phi$ , then  $\phi$  is a  $(\mathbf{c}, \mathcal{H})$ -coloring of  $G$ , a contradiction. Thus  $G'$  has no such coloring. By the minimality of  $G$ , this yields that there is  $Q \subseteq V(G')$  with  $\rho_{G', \mathbf{c}'}(Q) < i - j$ . By the construction of  $G'$ ,  $Q \cap \{u, v\} \neq \emptyset$ . If  $\{u, v\} \subseteq Q$ , then

$$\rho_{G, \mathbf{c}}(Q) = \rho_{G', \mathbf{c}'}(Q) + 4 - (i + 1) < i - j,$$

a contradiction. Thus by the symmetry between  $u$  and  $v$  we may assume  $u \in V(Q)$  and  $v \notin V(Q)$ . In this case,

$$\rho_{G, \mathbf{c}}(Q) \leq \rho_{G', \mathbf{c}'}(Q) + 2 \leq i - j - 1 + 2 = i - j + 1.$$

Note that by maximality of  $F$ ,  $F \setminus Q \neq \emptyset$  and  $F \cap Q \neq \emptyset$ . Then by (2) and (7),

$$\rho_{G, \mathbf{c}}(F \cup Q) \leq 2(i - j + 1) - \rho_{G, \mathbf{c}}(F \cap Q) \leq i - j + 1.$$

Since  $v \notin F \cup Q$  and  $v$  is not a  $(2; i, j)$ -vertex, this contradicts the maximality of  $F$ .  $\bowtie$

Let  $X_1 := N_G(F) \cap X$ ,  $Y_1 := N_G(F) \cap Y$ ,  $X_0 := X \setminus X_1$ , and  $Y_0 := Y \setminus Y_1$ .

**Claim 2.** *Let  $u \in F \cap N_G(X \cup Y)$ . For any  $(\mathbf{c}, \mathcal{H}_F)$ -coloring  $\phi$  of  $G[F]$ , the degree of  $\phi(u)$  in the  $\phi$ -induced subgraph  $H_\phi$  is equal to  $\mathbf{c}(\phi(u))$ .*

*Proof of Claim 2.* Suppose that for some  $(\mathbf{c}, \mathcal{H}_F)$ -coloring  $\phi$  of  $G[F]$ , the degree of  $\phi(u)$  in  $H_\phi$  is at most  $\mathbf{c}(\phi(u)) - 1$ . Let  $w \in N(u) \setminus F$ . Denote  $S := X_1 \cup Y_1 \setminus \{w\}$ .

**Case 1:**  $w \in Y_1$ . Let  $w'$  be the other neighbor of  $w$ . By Claim 1 and (8),  $w' \in X$ . Construct  $G'$  from  $G - F - w$  by adding a new vertex  $v^*$  adjacent to each vertex in  $S$ . By (8),

$$(9) \quad |E_G(F, S')| = |E_{G'}(v^*, S')| \text{ for every } S' \subseteq S.$$

Define  $\mathbf{c}'$  by  $\mathbf{c}'(x) = \mathbf{c}(x)$  for all  $x \in V(G') \setminus \{v^*\}$  and  $\mathbf{c}'(v^*) = (0, -1)$ . Define  $L'$  by  $L'(v^*) = \{p(v^*), r(v^*)\}$  and  $L'(x) = L(x)$  for  $x \in V(G') - v^*$ . Let  $\mathcal{H}' = (L', H')$  be a cover of  $G'$  such that  $H'[\{x, y\}] = H[\{x, y\}]$  for  $x, y \in V(G') - v^*$  and the neighbors of  $p(v^*)$  and  $r(v^*)$  are defined as follows. For each  $v \in S$ , if  $z \in N(v) \cap F$  and  $L(v) = \{\alpha(v), \beta(v)\}$  where  $\alpha(v) \sim \phi(z)$ , then  $p(v^*) \sim \alpha(v)$  and  $r(v^*) \sim \beta(v)$ .

If there is  $Q \subset V(G')$  with  $\rho_{G', \mathbf{c}'}(Q) \leq i - j - 1$ , then  $v^* \in Q$  since  $G' - v^* \subset G$ . In this case, using (2) and (9) and remembering that  $\rho_{G', \mathbf{c}'}(v^*) = i - j$ ,

$$\begin{aligned} \rho_{G, \mathbf{c}}((Q - v^*) \cup F) &= \rho_{G, \mathbf{c}}(F) + \rho_{G, \mathbf{c}}(Q - v^*) - (i + 1)|E_G(F, Q - v^*)| \\ &\leq (i - j + 1) + \rho_{G', \mathbf{c}'}(Q - v^*) - (i + 1)|E_{G'}(v^*, Q - v^*)| = 1 + \rho_{G', \mathbf{c}'}(Q) \leq i - j. \end{aligned}$$

Since  $w \notin (Q - v^*) \cup F$ , this contradicts (7). Thus  $\rho(G', \mathbf{c}') \geq i - j$ . Hence by the minimality of  $G$ ,  $G'$  has a  $(\mathbf{c}', \mathcal{H}')$ -coloring  $\psi$ . Since  $\mathbf{c}'(v^*) = (0, -1)$ ,  $\psi(v^*) = p(v^*)$  and the degree of  $p(v^*)$  in  $\mathcal{H}'_\psi$  is zero. Hence

$$(10) \quad \psi(v) = \beta(v) \text{ for every } v \in S.$$

Define an  $\mathcal{H}$ -map  $\theta$  of  $G$  by  $\theta(x) = \phi(x)$  for  $x \in F$ ,  $\theta(v) = \psi(v)$  for  $v \in V(G) - F - w$  and choosing  $\theta(w) \in L(w)$  with  $\theta(w) \asymp \psi(w')$ . We claim that for every  $v \in V(G)$  the degree of  $\theta(v)$  in  $H_\theta$  is at most its capacity. This is true for each  $v \in F - u$  since by (10) for each neighbor  $v'$  of  $v$  in  $V(G) - F$ ,  $\psi(v') \asymp \phi(v)$ . For the same reason, this is true for each  $x \in V(G) - F - w$ . This is true for  $u$  by its choice and the fact the only possible neighbor



of  $\phi(u)$  outside of  $\phi(F)$  is  $\theta(w)$ . And this is true for  $w$ , since  $w$  is a  $(2; i, j)$ -vertex and  $\theta(w) \approx \psi(w')$ .

**Case 2:**  $w \in X_1$ . Construct  $G'$  from  $G - F$  by adding a new vertex  $v^*$  adjacent to each vertex in  $S - w$ . As in Case 1, (9) holds. Define  $\mathbf{c}''$  by  $\mathbf{c}''(x) = \mathbf{c}(x)$  for all  $x \in V(G') \setminus \{w, v^*\}$ ,  $\mathbf{c}''(v^*) = (0, -1)$  and  $\mathbf{c}''(w) = (\mathbf{c}_1(w) - 1, \mathbf{c}_2(w) - 1)$ . Define  $L''$  and  $\mathcal{H}'' = (L'', H'')$  exactly as we defined  $L'$  and  $\mathcal{H}' = (L', H')$  in Case 1.

Suppose there is  $Q \subseteq V(G')$  with  $\rho_{G', \mathbf{c}''}(Q) \leq i - j - 1$ . If  $v^* \notin Q$ , then  $w \in Q$ , for otherwise  $Q \subseteq G$ . In this case  $\rho_{G, \mathbf{c}}(Q \cup F) \leq i - j + 1 + (i - j - 1 + 2) - (i + 1) < i - j$ . So  $v^* \in Q$ . Moreover, if  $Q \neq V(G')$ , then repeating the argument of Case 1 we get a contradiction. If  $Q = V(G')$ , then since  $v^*w \notin E(G')$  and  $\rho_{G', \mathbf{c}''}(w) = \rho_{G, \mathbf{c}}(w) - 2$ ,

$$\begin{aligned} \rho_{G, \mathbf{c}}(V(G)) &= \rho_{G, \mathbf{c}}(F) + \rho_{G, \mathbf{c}}(Q - v^*) - (i + 1)|E_G(F, Q - v^*)| \leq (i - j + 1) + \rho_{G', \mathbf{c}''}(Q - v^*) \\ &\quad + 2 - (i + 1)(|E_{G'}(v^*, Q - v^*)| + 1) = 3 + \rho_{G', \mathbf{c}''}(Q) - (i + 1) < i - j, \end{aligned}$$

a contradiction. Thus in all cases  $\rho(G', \mathbf{c}'') \geq i - j$ . So by the minimality of  $G$ ,  $G'$  has a  $(\mathbf{c}'', \mathcal{H}'')$ -coloring  $\psi$ . As in Case 1,  $\psi(v^*) = p(v^*)$  and (10) holds.

Define an  $\mathcal{H}$ -map  $\theta$  of  $G$  by  $\theta(x) = \phi(x)$  for  $x \in F$  and  $\theta(v) = \psi(v)$  for  $v \in V(G) - F$ . We claim that for every  $v \in V(G)$  the degree of  $\theta(v)$  in  $H_\theta$  is at most its capacity. If  $v \neq w$ , then the proof of it is exactly as in Case 1. For  $v = w$  this follows from the fact that  $\mathbf{c}''(w) = (\mathbf{c}_1(w) - 1, \mathbf{c}_2(w) - 1)$ .  $\boxtimes$

Let  $Q$  be an auxiliary graph with  $V(Q) = X$ ,  $E(Q) = Y_0$ , where  $y \in Y_0$  has end vertices  $x_1, x_2$  in  $Q$  if  $N_G(y) = \{x_1, x_2\}$ . From now on, we fix a  $(\mathbf{c}, \mathcal{H}_F)$ -coloring  $\psi$  of  $F$ .

For every  $v \in X_1 \cup Y_1$ , let  $v_F$  be its neighbor in  $F$  and denote by  $\overline{\psi(v)}$  the vertex in  $L(v)$  such that  $\psi(v_F) \approx \overline{\psi(v)}$ . Define function  $w_\psi : X \rightarrow \mathbb{Z}$  by:  $w(x) = \mathbf{c}(\overline{\psi(x)}) - |E(x, Y_1)|$  for  $x \in X_1$ , and  $w(x) = \mathbf{c}_2(x) - |E(x, Y_1)|$  for  $x \in X_0$ . Note that  $w_\psi$  is determined by the coloring  $\psi$  on  $F$ .

**Claim 3.** There exists no  $(\mathbf{c}, \mathcal{H}_F)$ -coloring  $\psi$ , such that for every  $A \subseteq X$ ,

$$(11) \quad \sum_{x \in A} w_\psi(x) \geq |E(Q[A])|.$$

*Proof of Claim 3.* Suppose the  $(\mathbf{c}, \mathcal{H}_F)$ -coloring  $\psi$  satisfies (11). Define an auxiliary bipartite graph  $B = B(V_1, V_2)$  with partite sets  $V_1$  and  $V_2$  where  $V_1 = Y_0$  and  $V_2$  has exactly  $w(x)$  copies of  $x$  for each  $x \in X$ . For each edge  $xy \in E(G)$  with  $x \in X, y \in Y_0$ , vertex  $y$  is adjacent to each copy of  $x$  in  $B$ . For any  $S \subseteq V_1$ , by (11),  $|S| \leq |N_B(S)|$ . This means that  $B$  satisfies Hall's condition and hence has a matching  $M$  saturating  $V_1$ . An orientation  $D$  of  $Q$  can then be formed as follows: for each pair  $x_1, x_2 \in X$ , and each edge  $e$  connecting  $x_1, x_2$  in  $Q$  ( $e$  equals some  $y \in Y_0$  such that  $y \sim x_1$  and  $y \sim x_2$ ), orient  $e$  from  $x_1$  to  $x_2$  if the edge of  $M$  connects  $y$  to a copy of  $x_1$  in  $V_2$ , and from  $x_2$  to  $x_1$  otherwise. Then for every  $x \in X$ ,  $d^+(x) \leq w(x)$ .

Define an  $\mathcal{H}$ -map  $\phi$  of  $G$  as follows. For every  $v \in F$ ,  $\phi(v) = \psi(v)$ . For every  $x \in X_0$ ,  $\phi(x) = r(x)$ . For every  $u \in X_1 \cup Y_1$ ,  $\phi(u) = \overline{\psi(u)}$ . For every  $y \in Y_0$ , if  $y = x_1x_2$  in  $D$ , then choose  $\phi(y) \in L(y)$  so that  $\phi(y) \approx \phi(x_2)$ .

Let us check that for every  $v \in V(G)$ , the degree of  $\phi(v)$  in  $H_\phi$  is at most its capacity. This is true for  $v \in F$  because  $\phi(v)$  has no neighbors in  $\phi(V(G) - F)$  by the choice of colors

for vertices in  $X_1 \cup Y_1$ . This is true for each  $y \in Y$ , because each of them has two neighbors and  $\phi(y)$  has capacity at least  $i$ . This is true for each  $x \in X$  by (11) and the choice of  $w$ ,  $D$  and the colors for the vertices in  $Y_0$ . Thus  $\phi$  is a  $(\mathbf{c}, \mathcal{H})$ -coloring of  $G$ , a contradiction.  $\bowtie$

By Claim 3, there is  $A \subseteq X$  with

$$(12) \quad \sum_{x \in A} w_\psi(x) \leq |E(Q[A])| - 1.$$

Let  $Y' \subseteq Y$  consist of the vertices  $y \in Y_0$  that have both neighbors in  $A$  and the vertices  $y \in Y_1$  with a neighbor in  $A$ . Then

$$\begin{aligned} \rho_{G, \mathbf{c}}(F \cup A \cup Y') &= \rho_{G, \mathbf{c}}(F) + \sum_{x \in A} (\rho_{\mathbf{c}}(x) - |E(x, Y_1)|) - |A \cap X_1|(i+1) - |E(Q[A])| \\ &\leq i - j + 1 + \sum_{x \in A} (i - j + 1 + \mathbf{c}_1(x) + \mathbf{c}_2(x) - |E(x, Y_1)| - w_\psi(x)) - 1 - |A \cap X_1|(i+1) \\ &\leq i - j + 1 + \sum_{x \in A \cap X_0} (i - j + 1 + \mathbf{c}_1(x)) + \sum_{x \in A \cap X_1} (-j + \max\{\mathbf{c}_1(x), \mathbf{c}_2(x)\}) - 1 \leq i - j. \end{aligned}$$

By Lemma 4.1, the equality can hold only if  $A = X, Y' = Y$ . Moreover, in this case every non-strict inequality in the chain above is an equality, and (12) is an equality. The latter yields

$$(13) \quad \sum_{x \in X} w_\psi(x) = |Y_0| - 1, \text{ and (11) holds for every } A \neq X,$$

and the former yields

$$(14) \quad \overline{\psi(x)} = p(x) \text{ for every } x \in X_1.$$

We consider two cases:

**Case 1.**  $X_1 \neq \emptyset$ . Let  $v \in X_1$ , and  $v_F$  be its neighbor in  $F$ . Then  $\psi(v_F) \sim r(v)$ . Define  $\mathbf{c}'$  on  $F$  that differs from  $\mathbf{c}$  only in that  $\mathbf{c}'(\psi(v_F)) = \mathbf{c}(\psi(v_F)) - 1$ . By (7),  $\rho(G, \mathbf{c}') \geq i - j$ . By the minimality of  $G$ , we can find a  $(\mathbf{c}', \mathcal{H}_F)$ -coloring  $\psi'$  of  $G[F]$ . By Claim 2,  $\psi'(v_F) \neq \psi(v_F)$  and  $\overline{\psi'(v)} = r(v)$ . Then the above chain of inequalities with  $\psi'$  in place of  $\psi$  does not satisfy (14), a contradiction.

**Case 2.**  $X_1 = \emptyset$ . Then  $Y_1 \neq \emptyset$ . Let  $v \in Y_1$ ,  $v_F$  be its neighbor in  $F$  and  $x'$  be the other neighbor. If  $\overline{\psi(v)} \approx r(x')$ , then we can define a new function  $w'$  that differs from  $w$  only in that  $w'(x') = \mathbf{c}_2(x') - |E(x', Y_1)| + 1$ . Then by (13),  $\sum_{x \in X} w'_\psi(x) = |Y_0|$  and (11) holds with  $w'$  in place of  $w$  for every  $A \neq X$ . Repeating the proof of Claim 3, we construct an orientation  $D$  of the auxiliary graph  $Q$  such that for every  $x \in X$ ,  $d^+(x) \leq w'(x)$ . Then we define a map  $\phi$  exactly as in the proof of Claim 3 and check that for every  $u \in V(G)$ , the degree of  $\phi(u)$  in  $H_\phi$  is at most its capacity almost as in that proof with a change only for  $u = x'$ : the degree of  $r(x')$  does not exceed  $\mathbf{c}_2(x')$  because in addition to other conditions,  $\phi(v) \approx r(x')$ . Thus  $\overline{\psi(v)} \sim r(x')$ .

Define  $\mathbf{c}'$  as in Case 1. By the same argument, we can find a  $(\mathbf{c}', \mathcal{H}_F)$ -coloring  $\psi'$  such that  $\psi'(v) \neq \psi(v)$ . Now  $\overline{\psi'(v)} \approx r(x')$ . This contradicts the previous paragraph.  $\square$

Denote the set of  $(2; i, j)$ -vertices in  $G$  by  $Y$  and let  $X = V(G) \setminus Y$ . The proof of the following lemma is very similar to the proof of Claim 1 in Lemma 4.3, so we omit the details.

**Lemma 4.4.** *Both  $X$  and  $Y$  are independent sets.*

Given  $A \subseteq X$ , let  $N_2(A)$  denote the set of vertices  $v$  in  $Y$  such that  $|N(v) \cap A| = 2$ . Let  $G_A$  be the subgraph of  $G$  induced by  $A \cup N_2(A)$ . By Lemmas 4.3 and 4.4, we have

$$(15) \quad \rho_{G, \mathbf{c}}(G_{X'}) = \sum_{v \in X'} \rho_{\mathbf{c}}(v) - |N_2(X')| > i - j + 1, \text{ for every } X' \subsetneq X$$

Let  $\mathcal{G}$  be the collection of functions  $g : Y \rightarrow X$  such that  $g(y) \in N(y)$  for every  $y \in Y$ . For each  $x \in X$ , let  $\lambda(x, g) = |g^{-1}(x)|$ , and for each  $y \in Y$ , let  $x_{y,g}$  denote the vertex in  $N(y) \setminus \{g(y)\}$ .

**Lemma 4.5.** *For every  $g \in \mathcal{G}$  there is  $x \in X$  such that  $\lambda(x, g) > \mathbf{c}_2(x)$ .*

*Proof.* Suppose some  $g \in \mathcal{G}$  satisfies  $\lambda(x, g) \leq \mathbf{c}_2(x)$  for every  $x \in X$ . Define an  $\mathcal{H}$ -map  $\phi$  by:  $\phi(x) = r(x)$  for every  $x \in X$ ,  $\phi(y) \sim \phi(x_{y,g})$  for every  $y \in Y$ . Then  $\phi$  is a  $(\mathbf{c}, \mathcal{H})$ -coloring of  $G$ , a contradiction.  $\square$

Let  $\widehat{G}$  be an auxiliary multigraph, where  $V(\widehat{G}) = X$  and  $E(\widehat{G}) = Y$ , such that for every  $a, b \in V(\widehat{G})$ , each  $y \in N(a) \cap N(b)$  corresponds bijectively to an edge between  $a$  and  $b$ . Let  $\mathcal{D}$  be the collection of all digraphs obtained by orienting the edges of  $\widehat{G}$ . Define the bijection

$$\pi : \mathcal{G} \rightarrow \mathcal{D},$$

so that for every  $g \in \mathcal{G}$ , the head of each edge  $y \in E(\widehat{G})$  in  $\pi(g)$  is  $g(y)$ . For  $g \in \mathcal{G}$ , let  $S_g := \{v \in X : \lambda(v, g) > \mathbf{c}_2(v)\}$  and let  $S'_g$  be the set of the vertices  $v \in X$  such that  $\pi(g)$  has a directed path from  $v$  to  $S_g$ . By definition,  $S'_g \supseteq S_g$ , and by Lemma 4.5,  $|S_g| \geq 1$  for all  $g$ .

Now we fix  $g_0 \in \mathcal{G}$  such that

$$(16) \quad \sum_{v \in X} \max\{0, \lambda(v, g_0) - \mathbf{c}_2(v)\} = \min_{g \in \mathcal{G}} \sum_{v \in X} \max\{0, \lambda(v, g) - \mathbf{c}_2(v)\}.$$

**Lemma 4.6.**  $|S'_{g_0}| \geq 2$ .

*Proof.* Suppose  $S'_{g_0} = \{v\}$ . Then by Lemma 4.5,  $S_{g_0} = \{v\}$  and  $d_{\pi(g_0)}^-(v) = 0$ . This means  $\lambda(v, g_0) = 0$ , thus by the definition of  $S_{g_0}$ ,  $\mathbf{c}_2(v) = -1$ , a contradiction to Lemma 4.2.  $\square$

**Lemma 4.7.** *For every  $x \in S'_{g_0}$ ,  $\lambda(x, g_0) \geq \mathbf{c}_2(x)$ .*

*Proof.* Suppose there is  $v_0 \in S'_{g_0}$  such that  $\lambda(v_0, g_0) < \mathbf{c}_2(v_0)$ . Let  $P := v_0 v_1 \dots v_t$  be a  $v_0, v_t$ -path, where  $v_t \in S_{g_0}$ . Obtain  $D \in \mathcal{D}$  from  $\pi(g_0)$  by reversing all edges in  $P$ , and denote  $\pi^{-1}(D)$  by  $h$ . Then

$$\sum_{v \in X} \max\{0, \lambda(v, h) - \mathbf{c}_2(v)\} = \sum_{v \in X} \max\{0, \lambda(v, g_0) - \mathbf{c}_2(v)\} - 1,$$

contradicting (16).  $\square$

We say that a vertex  $y \in Y$  adjacent to  $u$  and  $v$  in  $G$  is *even* (with respect to  $\mathcal{H}$ ), if in  $H$ , each vertex in  $L(y)$  is adjacent either to both rich vertices in  $L(u) \cup L(v)$ , or to both poor vertices in  $L(u) \cup L(v)$ . Otherwise, we say  $y$  is *odd*.

**Lemma 4.8.** *For each  $v \in S'_{g_0}$ ,  $\lambda(v, g_0) \leq \mathbf{c}_1(v) + \mathbf{c}_2(v) + 1$ .*

*Proof.* Suppose there is  $v \in S'_{g_0}$  with  $\lambda(v, g_0) \geq \mathbf{c}_1(v) + \mathbf{c}_2(v) + 2$ . By Lemma 4.7 and the definition of  $S'_{g_0}$ ,

$$2 + \mathbf{c}_1(v) + \sum_{x \in S'_{g_0}} \mathbf{c}_2(x) \leq \sum_{x \in S'_{g_0}} \lambda(x, g_0) = |N_2(S'_{g_0})|.$$

Therefore, by Lemma 4.6,

$$\begin{aligned} \rho_{G, \mathbf{c}}(G_{S'_{g_0}}) &= (i - j + 1)|S'_{g_0}| + \sum_{x \in S'_{g_0}} \mathbf{c}_1(x) + \sum_{x \in S'_{g_0}} \mathbf{c}_2(x) - |N_2(S'_{g_0})| \\ &\leq (i - j + 1)|S'_{g_0}| + (|S'_{g_0}| - 1)i - 2 \\ &= i - j - 1 + (|S'_{g_0}| - 1)(2i - j + 1) \leq i - j - 1, \end{aligned}$$

which contradicts (15) or the choice of  $G$ .  $\square$

**Lemma 4.9.**  $|S_{g_0}| > 1$ .

*Proof.* Suppose  $S_{g_0} = \{v\}$ . Let  $\phi_1, \phi_2$  be  $\mathcal{H}$ -maps defined by:  $\phi_1(x) = \phi_2(x) = r(x)$  for  $x \in X \setminus \{v\}$ ,  $\phi_1(v) = r(v)$ ,  $\phi_2(v) = p(v)$ ; for every  $y \in Y$ ,  $\phi_1(y) \approx \phi_1(x_{y, g_0})$  and  $\phi_2(y) \approx \phi_2(x_{y, g_0})$ . If  $g_0^{-1}(v)$  contains at most  $\mathbf{c}_2(v)$  odd vertices, then  $\phi_1$  is a  $(\mathbf{c}, \mathcal{H})$ -coloring, a contradiction. Similarly, if  $g_0^{-1}(v)$  contains at most  $\mathbf{c}_1(v)$  even vertices, then  $\phi_2$  is a  $(\mathbf{c}, \mathcal{H})$ -coloring. Hence  $g_0^{-1}(v)$  contains at least  $\mathbf{c}_1(v) + 1$  even vertices and at least  $\mathbf{c}_2(v) + 1$  odd vertices. Thus  $\lambda(v, g_0) \geq \mathbf{c}_1(v) + \mathbf{c}_2(v) + 2$ . This contradicts Lemma 4.8.  $\square$

For  $v \in X$ , let  $\lambda_S(v, g) := |g^{-1}(v) \cap N_2(S_g)|$ , and  $\lambda_{\bar{S}}(v, g) := |g^{-1}(v) \cap (Y \setminus N_2(S_g))|$ . Thus  $\lambda(v, g) = \lambda_S(v, g) + \lambda_{\bar{S}}(v, g)$  for every  $v \in X$ . The proof goes by induction on  $|S_{g_0}|$ . Lemma 4.9 provides the base of induction. Now we do an induction step.

If for some  $v \in S_{g_0}$ ,  $\lambda_S(v, g_0) \geq \lambda(v, g_0) - \mathbf{c}_2(v)$ , consider the digraph  $D$  obtained from  $\pi(g_0)$  by reversing  $\lambda(v, g_0) - \mathbf{c}_2(v)$  edges in  $\pi(g_0)[S_{g_0}]$  each with head vertex  $v$ , and let  $h = \pi^{-1}(D)$ . Then  $\sum_{v \in X} \max\{0, \lambda(v, g_0) - \mathbf{c}_2(v)\} = \sum_{v \in X} \max\{0, \lambda(v, h) - \mathbf{c}_2(v)\}$  and  $S_h = S_{g_0} \setminus \{v\}$ , so  $|S_h| = |S_{g_0}| - 1$ . Hence by induction assumption, the theorem holds. Thus we may assume that

$$(17) \quad \text{for every } v \in S_{g_0}, \lambda_S(v, g_0) < \lambda(v, g_0) - \mathbf{c}_2(v).$$

Let  $Y_e$  be the set of even vertices in  $Y$ , and  $Y_o$  be the set of odd vertices in  $Y$ . For every  $g \in \mathcal{G}$  and every  $v \in X$ , let  $\lambda_{\bar{S}}^e(v, g) = |g^{-1}(v) \cap (Y_e \setminus N_2(S_g))|$  and  $\lambda_{\bar{S}}^o(v, g) = |g^{-1}(v) \cap (Y_o \setminus N_2(S_g))|$ , so for every  $v \in X$ ,  $\lambda_{\bar{S}}(v, g) = \lambda_{\bar{S}}^e(v, g) + \lambda_{\bar{S}}^o(v, g)$ . Define

$$T_{g_0} = \{v \in S_{g_0} : \lambda_{\bar{S}}^e(v, g_0) + \lambda_S(v, g_0) \geq \mathbf{c}_1(v) + 1\} \quad \text{and}$$

$$R_{g_0} = \{u \in S_{g_0} : \lambda_{\bar{S}}^o(u, g_0) + \lambda_S(u, g_0) \geq \mathbf{c}_2(u) + 1\}.$$

**Lemma 4.10.**  $R_{g_0} \cap T_{g_0} \neq \emptyset$ .

*Proof.* Suppose  $R_{g_0} \cap T_{g_0} = \emptyset$ . Let  $\phi$  be an  $\mathcal{H}$ -map such that  $\phi(v) = r(v)$  for  $v \in X \setminus R_{g_0}$ ,  $\phi(x) = p(x)$  for  $x \in R_{g_0}$ , and  $\phi(y) \approx \phi(x_{y,g_0})$  for  $y \in Y$ .

Let us check that the degree of  $\phi(v)$  in  $H_\phi$  is at most  $\mathbf{c}(\phi(v))$  for each  $v \in V(G)$ . This is true for each  $y \in Y$  because  $\phi(y)$  has at most one neighbor in  $H_\phi$ , and  $\mathbf{c}_2(y) > \mathbf{c}_1(y) = i \geq 3$ . This is true for each  $x \in X - S_{g_0}$  because  $\phi(x)$  has at most  $\lambda(x, g_0)$  neighbors in  $H_\phi$ ,  $\mathbf{c}(\phi(x)) = \mathbf{c}_2(x)$ , and  $\lambda(x, g_0) \leq \mathbf{c}_2(x)$ . Suppose  $v \in S_{g_0} \setminus R_{g_0}$ . Then  $\phi(v) = r(v)$ , so  $\mathbf{c}(\phi(v)) = \mathbf{c}_2(v)$ . If  $\phi(y)$  is a neighbor of  $\phi(v)$  in  $\mathcal{H}$ , and the neighbor  $x$  of  $y$  in  $G$  distinct from  $v$  is not in  $S_{g_0}$ , then  $\phi(x) = r(x)$ , and  $\phi(y) \approx \phi(x)$ . Hence in order  $\phi(y)$  to be a neighbor of  $\phi(v)$ , vertex  $y$  needs to be odd. The total number of such neighbors is  $\lambda_{\bar{S}}^o(v, g_0)$ . By the definition of  $R_{g_0}$ ,  $\lambda_{\bar{S}}^o(v, g_0) + \lambda_S(v, g_0) \leq \mathbf{c}_2(v)$ , thus our claim holds for  $v$ . Finally, if  $u \in R_{g_0}$ , then  $\phi(u) = p(u)$  and  $\mathbf{c}(\phi(u)) = \mathbf{c}_1(u)$ . Symmetrically to above, the total number of neighbors  $\phi(y)$  of  $\phi(u)$  in  $\mathcal{H}$  such that the neighbor  $x$  of  $y$  in  $G$  distinct from  $u$  is not in  $S_{g_0}$  is  $\lambda_{\bar{S}}^e(u, g_0)$ . Since  $u \notin T_{g_0}$ ,  $\lambda_{\bar{S}}^e(u, g_0) + \lambda_S(u, g_0) \leq \mathbf{c}_1(u)$ . Thus,  $\phi$  is a  $(\mathbf{c}, \mathcal{H})$ -coloring of  $G$ , a contradiction.  $\square$

By definition, for every  $v \in T_{g_0} \cap R_{g_0}$ ,

$$\begin{aligned} (\mathbf{c}_1(v) + 1) + (\mathbf{c}_2(v) + 1) &\leq (\lambda_{\bar{S}}^e(v, g_0) + \lambda_S(v, g_0)) + (\lambda_{\bar{S}}^o(v, g_0) + \lambda_S(v, g_0)) \\ &= \lambda_{\bar{S}}(v, g_0) + 2\lambda_S(v, g_0) = \lambda(v, g_0) + \lambda_S(v, g_0). \end{aligned}$$

By (17), this is at most  $2\lambda(v, g_0) - \mathbf{c}_2(v) - 1$ . Therefore, for every  $v \in T_{g_0} \cap R_{g_0}$ ,

$$(18) \quad \lambda(v, g_0) \geq \mathbf{c}_2(v) + \frac{\mathbf{c}_1(v)}{2} + \frac{3}{2}.$$

**Lemma 4.11.**  $|R_{g_0} \cap T_{g_0}| \leq 1$ .

*Proof.* Suppose there are distinct  $u, v \in R_{g_0} \cap T_{g_0}$ . By (18) and Lemma 4.7,

$$\begin{aligned} |N_2(S'_{g_0})| &= \sum_{x \in S'_{g_0}} \lambda(x, g_0) = \lambda(u, g_0) + \lambda(v, g_0) + \sum_{x \in S'_{g_0} \setminus S_{g_0}} \lambda(x, g_0) + \sum_{x \in S_{g_0} \setminus \{u, v\}} \lambda(x, g_0) \\ &\geq |S_{g_0}| - 2 + \sum_{x \in S'_{g_0}} \mathbf{c}_2(x) + \frac{\mathbf{c}_1(u) + \mathbf{c}_1(v)}{2} + 3. \end{aligned}$$

Therefore,

$$\begin{aligned} \rho_{G, \mathbf{c}}(S'_{g_0}) &= (i - j + 1)|S'_{g_0}| + \sum_{x \in S'_{g_0}} \mathbf{c}_1(x) + \sum_{x \in S'_{g_0}} \mathbf{c}_2(x) - |N_2(S'_{g_0})| \\ &\leq (i - j + 1)|S'_{g_0}| + (|S'_{g_0}| - 1)i - |S_{g_0}| - 1 \\ &\leq i - j - 1 + (|S'_{g_0}| - 1)(2i - j + 1) - 1 \leq i - j - 2, \end{aligned}$$

contradicting the choice of  $G$ .  $\square$

Now the only remaining case is that  $|R_{g_0} \cap T_{g_0}| = 1$ . Let  $R_{g_0} \cap T_{g_0} = \{v\}$ .

Define  $\mathcal{H}$ -maps  $\phi_1$  and  $\phi_2$  as follows:  $\phi_1(x) = \phi_2(x) = r(x)$  for every  $x \in X \setminus R_{g_0}$ ,  $\phi_1(x) = \phi_2(x) = p(x)$  for all  $x \in R_{g_0} \setminus \{v\}$ ,  $\phi_1(v) = r(v)$ ,  $\phi_2(v) = p(v)$ , and for every  $y \in Y$ ,  $\phi_1(y) \approx \phi_1(x_{y,g_0})$  and  $\phi_2(y) \approx \phi_2(x_{y,g_0})$ .

If  $g_0^{-1}(v)$  contains at most  $\mathbf{c}_1(v)$  even vertices, then repeating the proof of Lemma 4.10 we conclude that  $\phi_1$  is a  $(\mathbf{c}, \mathcal{H})$ -coloring, a contradiction. Similarly, if  $g_0^{-1}(v)$  contains at most

$\mathbf{c}_2(v)$  odd vertices, then  $\phi_2$  is a  $(\mathbf{c}, \mathcal{H})$ -coloring. So we have  $\lambda(v, g_0) \geq \mathbf{c}_1(v) + \mathbf{c}_2(v) + 2$ , this contradicts Lemma 4.8 completing the proof of the theorem.

## 5. CONSTRUCTIONS

In this section, we construct  $(i, j)$ -critical graphs with  $i \geq 3, j \geq 2i + 1$  that attain equality of the upper bound in Theorem 3.2. We first define flags, which will be used to control the capacity of the vertices.

**Definition 8** (flags). Given a vertex  $v$ , a *flag at  $v$*  is a graph containing  $i + 1$  many degree 2 vertices  $\{u_1, \dots, u_{i+1}\}$  and a degree  $(i + 2)$  vertex  $x$ , such that all of these vertices are adjacent to  $v$ , and  $x$  is adjacent to all the vertices in  $\{u_1, \dots, u_{i+1}\}$ . See Figure 1.  $x$  is called the *top vertex* in this flag,  $v$  is the *base vertex* of the flag, and  $u_1, \dots, u_{i+1}$  are *middle vertices*.

In the cover graph (we abbreviate 'the flag-induced cover graph' here by 'flag'), we say that a flag (with base vertex  $v$ , top vertex  $x$ , middle vertices  $u_1, \dots, u_{i+1}$ ) is *parallel* if  $p(x) \sim p(v), r(x) \sim r(v)$  and  $u_t$  is even for every  $t$ ; when  $p(x) \sim r(v), r(x) \sim p(v)$  and  $u_t$  is odd for every  $t$ , we call such flag a *twisted flag*.

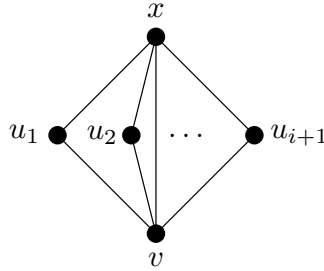


FIGURE 1. A flag at vertex  $v$ .

The following observation about flags is easy to check by hand.

**Claim 4.** Let  $\mathcal{H} = (H, L)$  a 2-fold cover of a graph  $G$  and  $F$  be a flag with base  $v$  in  $\mathcal{H}$ . Let  $\phi$  be a coloring of  $H - (V(F) - v)$ . If  $F$  is parallel and  $\phi(v) = r(v)$ , then in any extension of  $\phi$  to  $F$ ,  $\phi(v)$  will have a neighbor in  $\phi(F)$ , and there is an extension in which  $\phi(v)$  will have exactly one neighbor in  $\phi(F)$ . Similarly, if  $F$  is twisted and  $\phi(v) = p(v)$ , then in any extension of  $\phi$  to  $F$ ,  $\phi(v)$  will have a neighbor in  $\phi(F)$ , and there is an extension in which  $\phi(v)$  will have exactly one neighbor in  $\phi(F)$ . In all other cases, we can extend  $\phi$  to  $F$  so that  $\phi(v)$  will have no neighbors in  $\phi(F)$ .  $\boxtimes$

Hence, adding a parallel flag on a vertex  $v$  essentially decreases  $\mathbf{c}_2(v)$  by 1, and adding a twisted flag on  $v$  essentially decreases  $\mathbf{c}_1(v)$  by 1.

Given  $m \geq 1$ , we now construct the graph  $G_m$ . When  $m \geq 2$ , let  $G_m$  be obtained from a path  $v_1 \dots v_m$ , by adding  $i + 1$  flags to  $v_1$ , adding  $i$  flags to  $v_t$  for every  $1 < t < m$ , and adding  $i + j + 1$  flags to  $v_m$ . When  $m = 1$ , we define  $G_1$  as a single base vertex with  $i + j + 2$  flags. See Figure 2.

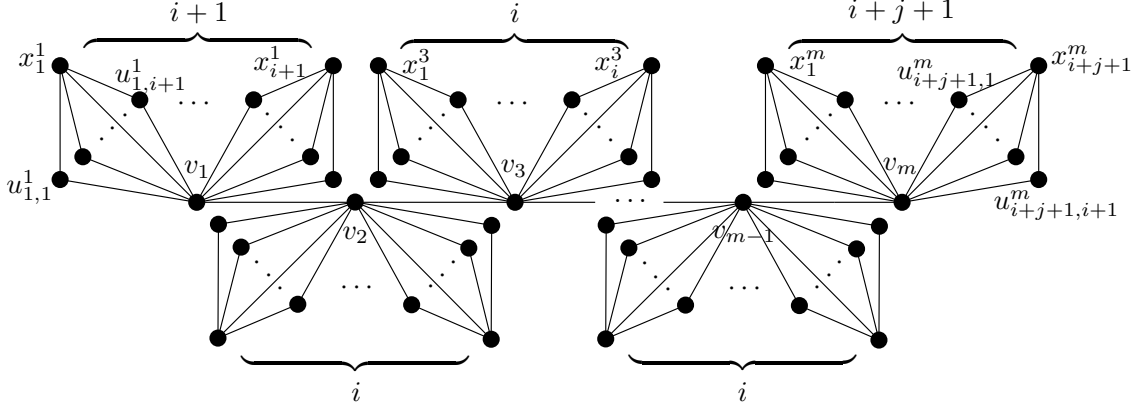


FIGURE 2. Critical graphs  $G_m$  for  $(i, j)$ -colorings.

Note that for every  $m \geq 1$ ,  $|V(G_m)| = (i+2)(mi+j+2) + m$  and  $|E(G_m)| = (2i+3)(mi+j+2) + m - 1$ , thus

$$|E(G_m)| = \frac{(2i+1)|V(G_m)| + j - i + 1}{i+1}.$$

**Proposition 5.1.** *Let  $i \geq 1$ ,  $j \geq 2i+1$  be integers. Then  $G_m$  is  $(i, j)$ -critical for every  $m$ .*

*Proof.* We first construct for each  $m$  a 2-fold cover  $\mathcal{H}_m = (L_m, H_m)$  of  $G_m$ , such that no  $\mathcal{H}_m$ -map is an  $\mathcal{H}_m$ -coloring. When  $m = 1$ , let  $i+1$  flags be twisted and the remaining  $j+1$  flags be parallel. When  $m \geq 2$ , let  $j$  flags based on  $v_m$  be parallel, and the remaining flags in  $G_m$  be twisted. For the path  $v_1 \cdots v_m$  in  $H_m$ , let  $r(v_t) \sim p(v_{t+1})$  for  $t = 1, \dots, m-2$ , and  $r(v_{m-1}) \sim r(v_m)$ . Suppose  $\phi$  is an  $\mathcal{H}_m$ -coloring of  $G_m$ . By Claim 4,  $\phi(v_1) = r(v_1)$ . Since  $p(v_2) \sim r(v_1)$ , and there are  $i$  twisted flags based  $v_2$ ,  $\phi(v_2)$  has to be  $r(v_2)$ . Similarly,  $\phi(v_t) = r(v_t)$  for all  $t = 1, \dots, m-1$ . Now since there are  $i+1$  twisted flags based on  $v_m$ , by Claim 4,  $\phi(v_m)$  cannot be  $p(v_m)$ . But then again by Claim 4,  $\phi(v_m)$  has  $j$  neighbors from the parallel flags plus  $\phi(v_{m-1}) = r(v_{m-1})$  also is its neighbor, a contradiction.

We now show that every proper subgraph of  $G_m$  is  $(i, j)$ -colorable. It suffices to show that  $G_m - e$  is  $(i, j)$ -colorable for any  $e \in E(G_m)$ .

**Claim 5.** *Let  $F$  be obtained by removing an edge  $e$  from a flag with base  $v$ . Let  $\mathcal{H} = (L, H)$  be a 2-fold cover of  $F$ . Then for each of the choices  $\phi(v) = p(v)$  and  $\phi(v) = r(v)$ , there is an  $\mathcal{H}$ -map  $\phi$ , such that the degree of  $\phi(v)$  in  $H_\phi$  is 0.*

*Proof of Claim 5.* Denote the top vertex by  $x$ . If  $x \sim v$ , define  $\phi(x) = r(x)$  and for each middle vertex  $u$ , define  $\phi(u)$  so that  $\phi(u) \sim \phi(v)$ . Then  $\phi$  is a desired  $\mathcal{H}$ -map. Now assume  $x \sim v$ . Choose  $\phi(x) \sim \phi(v)$ . If  $e = xu_t$  for some middle vertex  $u_t$ , then let  $\phi(u_t) \sim \phi(v)$ ; if  $e = u_tv$ , let  $\phi(u_t) \sim \phi(x)$ . In either case,  $\phi(u_t)$  is adjacent to neither  $\phi(x)$  or  $\phi(v)$ . For the remaining middle vertices, choose  $\phi(u_k) \sim \phi(v)$ ,  $k \neq t$ . There are only  $i$  such  $u_k$ 's, thus  $\phi$  is a desired  $\mathcal{H}$ -map.  $\boxtimes$

Claim 5 essentially says that removing an edge from a flag is ‘equivalent’ (with respect to coloring) to removing the whole flag. Hence  $G_1 - e$  contains either at most  $i$  twisted flags, or at most  $j$  parallel flags. In either case  $G_1 - e$  is colorable.

Let  $m \geq 2$  and  $\mathcal{H} = (L, H)$  be a 2-fold cover of  $G_m - e$ . We will construct an  $\mathcal{H}$ -map  $\phi$ . If for a cover  $\mathcal{H}' = (L', H')$  of  $G_m$ , there are at most  $i$  twisted flags on  $v_m$  in  $H'$ , then we can define an  $\mathcal{H}'$ -map  $\phi'$  of  $G_m$  by  $\phi'(v_m) = p(v_m)$  and  $\phi'(v_k) = r(v_k)$  for all  $k \neq m$ . Since all the possible neighbors of  $\phi'(v_m)$  in  $H'_{\phi'}$  will be from the twisted flags based on  $v_m$ , the degree of  $\phi'(v_m)$  will not exceed  $i$ . For  $k = 1, \dots, m-1$ , since  $j > i+1$ , the degree of  $\phi'(v_k)$  will not exceed  $j$ . Hence

(19) *we consider only covers of  $G_m - e$  with at least  $i+1$  twisted flags on  $v_m$ .*

**Case 1:**  $e$  belongs to some flag  $F$ . If  $F$  is based on  $v_m$ , then by (19) there are at most  $j-1$  parallel flags on  $v_m$ . Let  $\phi(v_k) = r(v_k)$  for each  $k$ . Then by Claims 4 and 5, we can extend  $\phi$  to each of the flags so that the degree of  $\phi(v_k)$  in  $H'_{\phi}$  will be at most  $j$  for each  $k$ .

If  $F$  is based on  $v_t$  for some  $t \neq m$ , then there are at most  $i-1$  twisted or parallel flags based on  $v_t$  when  $t > 1$  and at most  $i$  such flags based on  $v_t$  when  $t = 1$ . Let  $\phi(v_k) = r(v_k)$  for  $k \in \{1, \dots, t-1, m\}$ , and  $\phi(v_l) = p(v_l)$  for  $l = t, \dots, m-1$ . Again by Claims 4 and 5, we can extend  $\phi$  to each of the flags so that for all  $k$ , the degree of  $\phi(v_k)$  in  $H'_{\phi}$  is at most  $i+1 < j$ , for each  $k = 1, \dots, t-1$ , and the degree of  $\phi(v_k)$  in  $H'_{\phi}$  is at most  $i$  for each  $k = t, \dots, m-1$ . Moreover, by (19) we can provide that the degree of  $\phi(v_m)$  in  $H'_{\phi}$  is at most  $j$ . Thus in all cases,  $\phi$  can be extended to an  $\mathcal{H}$ -coloring.

**Case 2:**  $e = v_t v_{t+1}$  for some  $t \in \{1, \dots, m-1\}$ . Let  $\phi(v_k) = r(v_k)$ , for each  $k \in \{1, \dots, t, m\}$ , and  $\phi(v_k) = p(v_k)$  for each  $k = t+1, \dots, m-1$ . Similarly to Case 1,  $\phi$  again can be extended to an  $\mathcal{H}$ -coloring of  $G_m - e$ .  $\square$

## REFERENCES

- [1] D. Archdeacon, A note on defective colorings of graphs in surfaces. *J. Graph Theory* 11 (1987), 517–519.
- [2] A. Bernshteyn, A. Kostochka and S. Pron, On DP-coloring of graphs and multigraphs, *Sib Math J.* 58 (2017), No 1, 28–36.
- [3] O. V. Borodin, A. O. Ivanova, M. Montassier, P. Ochem and A. Raspaud, Vertex decompositions of sparse graphs into an edgeless subgraph and a subgraph of maximum degree at most  $k$ . *J. Graph Theory* 65 (2010), 83–93.
- [4] O. V. Borodin, A. O. Ivanova, M. Montassier, and A. Raspaud,  $(k, j)$ -coloring of sparse graphs. *Discrete Appl. Math.* 159 (2011), 1947–1953.
- [5] O. V. Borodin, A. O. Ivanova, M. Montassier, and A. Raspaud,  $(k, 1)$ -coloring of sparse graphs. *Discrete Math.* 312 (2012), 1128–1135.
- [6] O. V. Borodin and A. V. Kostochka, Vertex decompositions of sparse graphs into an independent set and a subgraph of maximum degree at most 1. *Sibirsk. Mat. Zh.* 52 (2011), 1004–1010.
- [7] O. V. Borodin and A. V. Kostochka, Defective 2-colorings of sparse graphs. *J. Combin. Theory Ser. B* 104 (2014), 72–80.
- [8] O. V. Borodin, A. V. Kostochka, and M. Yancey, On 1-improper 2-coloring of sparse graphs. *Discrete Math.* 313 (2013), 2638–2649.
- [9] L. J. Cowen, R. Cowen, and D. R. Woodall, Defective colorings of graphs in surfaces: partitions into subgraphs of bounded valency, *J. Graph Theory* 10 (1986), 187–195.
- [10] W. Cushing and H. A. Kierstead, Planar graphs are 1-relaxed 4-choosable. *European J. Combin.* 31 (2010), 1385–1397.
- [11] P. Dorbec, T. Kaiser, M. Montassier, and A. Raspaud, Limits of near-coloring of sparse graphs. *J. Graph Theory* 75 (2014), 191–202.
- [12] Z. Dvořák, and L. Postle, Correspondence coloring and its application to list-coloring planar graphs without cycles of lengths 4 to 8. *J. Combin. Theory Ser. B* 129 (2018), 38–54.



- [13] K. Edwards, D. Y. Kang, J. Kim, S.-i. Oum, and P. Seymour, A relative of Hadwigers conjecture. *SIAM J. Discrete Math.* 29 (2015), 2385–2388.
- [14] N. Eaton and T. Hull, Defective list colorings of planar graphs. *Bull. Inst. Combin. Appl.* 25 (1999), 79–87.
- [15] L. Esperet, M. Montassier, P. Ochem, and A. Pinlou, A complexity dichotomy for the coloring of sparse graphs. *J. Graph Theory* 73 (2013), 85–102.
- [16] F. Havet, and J.-S. Sereni, Improper choosability of graphs and maximum average degree. *J. Graph Theory* 52 (2006), 181–199.
- [17] K. Hendrey, and D. Wood. Defective and clustered choosability of sparse graphs. *arXiv:1806.07040* preprint, 2018.
- [18] Y. Jing, A. Kostochka, F. Ma, P. Sittitrai, and J. Xu, Defective DP-colorings for sparse multigraphs. *arXiv:1912.03421* preprint, 2019.
- [19] J. Kim, A. V. Kostochka, and X. Zhu, Improper coloring of sparse graphs with a given girth, I:  $(0, 1)$ -colorings of triangle-free graphs. *Eur. J. Comb.* 42 (2014), 26–48.
- [20] J. Kim, A. V. Kostochka, and X. Zhu, Improper coloring of sparse graphs with a given girth, II: Constructions. *J. Graph Theory* 81 (2015), 403–413.
- [21] M. Kopreski and G. Yu, Maximum average degree and relaxed coloring. *Discrete Math.* 340 (2017), 2528–2530.
- [22] A. V. Kostochka and J. Xu, On 2-defective DP-colorings of sparse graphs, to appear in *Eur. J. Comb.*
- [23] L. Lovász, On decomposition of graphs. *Studia Sci. Math. Hungar.* 1 (1966), 237–238.
- [24] P. Ossona de Mendez, S.-I. Oum, and D. R. Wood, Defective colouring of graphs excluding a subgraph or minor, to appear in *Combinatorica*.
- [25] P. Sittitrai, and K. Nakprasit, Analogue of DP-coloring on variable degeneracy and its applications on list vertex-arboricity and DP-coloring. *arXiv:1807.00815* preprint, 2018.
- [26] R. Škrekovski, List improper colourings of planar graphs. *Combin. Probab. Comput.* 8 (1999), 293–299.
- [27] R. Škrekovski, List improper colorings of planar graphs with prescribed girth. *Discrete Math.* 214 (2000), 221–233.
- [28] J. Van den Heuvel and D. R. Wood, Improper colourings inspired by Hadwiger’s conjecture. *J. London Math. Soc.* 98 (2018), 129–148.
- [29] D. R. Wood, Defective and clustered graph colouring. *Electron. J. Combin.* #DS23, 2018.
- [30] D. R. Woodall. Defective choosability of graphs in surfaces. *Discuss. Math. Graph Theory* 31 (2011), 441–459.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN, URBANA, IL 61801, USA.

*E-mail address:* yifanjing17@gmail.com.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN, URBANA, IL 61801, USA, AND SOBOLEV INSTITUTE OF MATHEMATICS, NOVOSIBIRSK 630090, RUSSIA.

*E-mail address:* kostochk@math.uiuc.edu.

SCHOOL OF MATHEMATICS, SHANDONG UNIVERSITY, JINAN 250100, CHINA.

*E-mail address:* mafuhongsdnu@163.com.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN, URBANA, IL 61801, USA.

*E-mail address:* jx6@illinois.edu.