

Subdegree growth rates of infinite primitive permutation groups



Simon Smith

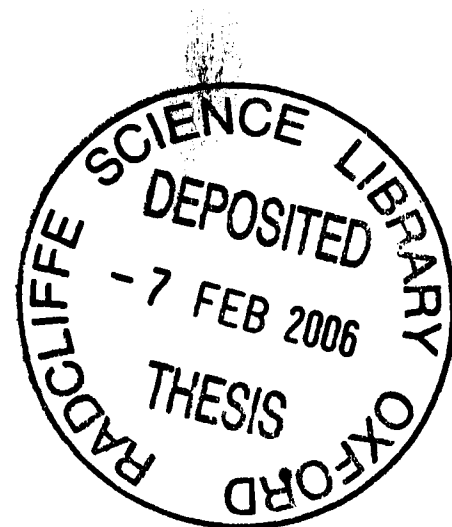
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To my parents

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Abstract

If G is a group acting on a set Ω , and $\alpha, \beta \in \Omega$, the directed graph whose vertex set is Ω and whose edge set is the orbit $(\alpha, \beta)^G$ is called an *orbital graph* of G . These graphs have many uses in permutation group theory. A graph Γ is said to be *primitive* if its automorphism group acts primitively on its vertex set, and is said to have *connectivity one* if there is a vertex α such that the graph $\Gamma \setminus \{\alpha\}$ is not connected. A *half-line* in Γ is a one-way infinite path in Γ . The *ends* of a locally finite graph Γ are equivalence classes on the set of half-lines: two half-lines lie in the same end if there exist infinitely many disjoint paths between them.

A complete characterisation of the primitive undirected graphs with connectivity one is already known. We give a complete characterisation in the directed case. This enables us to show that if G is a primitive permutation group with a locally finite orbital graph with more than one end, then G has a connectivity-one orbital graph Γ , and that this graph is essentially unique. Through the application of this result we are able to determine both the structure of G , and its action on the end space of Γ .

If $\alpha \in \Omega$, the orbits of the stabiliser G_α are called the α -*suborbits* of G . The size of an α -suborbit is called a *subdegree*. If all subdegrees of an infinite primitive group G are finite, Adeleke and Neumann claim one may enumerate them in a non-decreasing sequence (m_r) . They conjecture that the growth of the sequence (m_r) is extremal when G acts *distance transitively* on a locally finite graph; that is, for all natural numbers m the stabiliser in G of any vertex α permutes the vertices lying at distance m from α transitively. They also conjecture that for any primitive group G possessing

a finite self-paired suborbit of size m there might exist a number c which perhaps depends upon G , perhaps only on m , such that $m_r \leq c(m-2)^{r-1}$. We show their questions are poorly posed, as there exist primitive groups possessing at least two distinct subdegrees, each occurring infinitely often. The subdegrees of such groups cannot be enumerated as claimed. We give a revised definition of subdegree enumeration and growth, and show that under these new definitions their conjecture is true for groups exhibiting exponential subdegree growth above a prescribed bound.

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Chapter 1

Introduction

In this thesis we exploit the relationship between graph theory and group theory to characterise an important class of infinite directed graphs, and use this characterisation to investigate the orbital graphs of infinite primitive permutation groups. These results are applied to determine fundamental properties of infinite primitive groups whose suborbits are finite, including their structure and subdegree growth.

If G is a permutation group acting on a set Ω and $\alpha, \beta \in \Omega$, the graph $(\Omega, (\alpha, \beta)^G)$ whose vertex set is Ω and whose edge set is the orbit $(\alpha, \beta)^G$ is called an *orbital graph* of G . Note such graphs are directed. Indeed, all graphs appearing in this paper will be directed unless otherwise stated; all paths, however, will be undirected. A connected graph Γ is said to have *connectivity one* if there exists a vertex $\alpha \in V\Gamma$ such that the induced graph $\Gamma \setminus \{\alpha\}$ is not connected. Such graphs resemble trees in many respects, a property that underpins many of the results presented in this thesis.

A group G is said to act *primitively* on a set Ω if the only G -congruences are the trivial and universal equivalence relations. It is well known that for such groups every orbital graph $(\Omega, (\alpha, \beta)^G)$ is connected whenever α and β are distinct. A graph is said to be *primitive* if its automorphism group acts primitively on its vertex set, and is *locally finite* if every vertex is adjacent to only finitely many vertices.

A *half-line* is a one-way infinite, cycle-free path. An *end* of an infinite locally finite graph Γ is an equivalence class on the set of half-lines of Γ : two half-lines lie in the same end if there exist infinitely many disjoint paths in Γ joining a point in one to

a point in the other. An end is *thin* if any set of pairwise disjoint half-lines in it is finite, otherwise it is *thick*.

If G is a group acting on a set Ω and $\alpha \in \Omega$, the orbits of G_α on Ω are called the α -*suborbits* of G ; the cardinality of any α -suborbit is called a *subdegree* of G . There is a natural pairing of suborbits, described in detail in Chapter 2.

If Ω is a countable set then there is a natural topology on $\text{Sym}(\Omega)$, that of *point-wise convergence*, described in Section 2.1. Any group $G \leq \text{Sym}(\Omega)$ is said to be *closed* if it is closed in $\text{Sym}(\Omega)$ under this topology.

A group of automorphisms G of a graph Γ acts *distance-transitively* on Γ if it is vertex-transitive, and for all $k \geq 1$ the vertex-stabiliser G_α is transitive on all vertices at distance k from α in Γ . A group of permutations of a set Ω is *distance-transitive* if it acts distance-transitively on some graph Γ with $V\Gamma = \Omega$; a graph is *distance-transitive* if its automorphism group acts upon it distance-transitively. The infinite locally finite distance-transitive graphs were classified by Macpherson [17] and, independently, by Ivanov [13].

This work is divided into sections as follows. Chapter 2 is conceived as setting up notation and language, and as a collection of tools that will be used throughout this thesis.

In Chapter 3 we introduce the concept of subdegree growth. In [1, Remark 29.8], Adeleke and Neumann observe that if an infinite primitive group G possesses a finite suborbit whose pair is also finite, then every suborbit of G is finite, and one may enumerate them in a non-decreasing sequence as $m_1 \leq m_2 \leq \dots$. They observe $m_r < m_1(m_1 - 1)^{r-1}$ for all sufficiently large r , and ask whether the subdegree growth rates of infinite primitive groups that act distance-transitively on locally finite distance-transitive graphs are extremal. Furthermore, they conjecture that for any primitive group G possessing a finite self-paired suborbit of size m there might exist a number c which perhaps depends upon G , perhaps only on m , such that $m_r \leq c(m - 2)^{r-1}$ for all r . We show this question is poorly posed, as it does not consider the possible existence of a primitive group G whose suborbits are all finite, such that for some

fixed element $\alpha \in \Omega$ the group G has infinitely many α -suborbits with cardinality m , and infinitely many with cardinality m' , with m and m' distinct. For such a group, with $m < m'$, the subdegree m' does not occur in the sequence (m_r) . In this chapter we give examples of such groups, and in the process detail original methods for constructing examples of infinite primitive groups and primitive one-ended graphs. This chapter concludes with a revised definition of subdegree growth that takes into account the existence of these groups.

In Chapter 4 we turn our attention to infinite primitive graphs, and obtain a complete characterisation of the locally finite primitive connectivity-one directed graphs. This result is a generalisation of Jung and Watkins' famous classification of the locally finite primitive connectivity-one undirected graphs.

In Chapter 5 we show any primitive group possessing a locally finite orbital graph with more than one end has an orbital graph with connectivity one, and this graph is essentially unique. These results, and those of Chapter 4, underpin all subsequent arguments made in this thesis.

As an illustration of the power of the mathematics contained in Chapters 4 and 5, we apply them in Chapter 6 to examine infinite *generously transitive* permutation groups. A group G is *generously transitive* on a set Ω if, given any $\alpha, \beta \in \Omega$, there exists $g \in G$ such that $\alpha^g = \beta$ and $\beta^g = \alpha$. Finite generously transitive groups have been studied extensively by Neumann [25]. We show if G is an infinite primitive group possessing a locally finite orbital graph with more than one end then G is generously transitive if and only if G is distance-transitive.

In Chapter 7 we explicitly describe the structure of any primitive group possessing a locally finite orbital graph with more than one end.

If G is a group acting on an infinite locally finite graph Γ then there is a natural action of G on the set of ends of Γ . In [9], Halin showed one may infer details of $\text{Aut } \Gamma$ from its action on the end space of Γ . Interest in this area has continued since; more recently, in [20], Möller classified all locally finite connected graphs with infinitely many ends admitting an automorphism group that acts transitively on the end space, and in [21] investigated the structure of those locally finite connected graphs with

infinitely many ends that possess a vertex-transitive group of automorphisms that fixes an end. Chapter 8 continues in this vein as we illustrate how the results in Chapters 4 and 5 can be used to determine the action of a primitive edge-transitive group on the end space of a locally finite graph.

Finally, in Chapter 9 we examine the subdegree growth of primitive groups, and show a revised form of Adeleke and Neumann's conjecture is true for groups whose subdegree growth rate is exponential.

Chapter 2

Preliminaries

The theoretical background required to appreciate and understand this thesis is presented herein; all group theory may be found in [2] and [5], while the graph theory has been taken from [8]. In the lovely survey paper [33], Watkins builds upon the basic notions of finite graphs to produce an overview of infinite graph theory, of which the relevant concepts are also presented here. Much of what is currently known regarding groups acting on infinite graphs is summarised in the accessible survey paper [22].

2.1 Group theory

Throughout this work, G will be a group of permutations of an infinite set Ω . If $\alpha \in \Omega$ and $g \in G$, we denote the image of α under g by α^g . Following this notation, all permutations will act on the right. The set of images of α under all elements of G is called an *orbit* of G , and is denoted by α^G . There is a natural action of G on the n -element subsets and n -tuples of Ω via $\{\alpha_1, \dots, \alpha_n\}^g := \{\alpha_1^g, \dots, \alpha_n^g\}$ and $(\alpha_1, \dots, \alpha_n)^g := (\alpha_1^g, \dots, \alpha_n^g)$ respectively. If $\alpha \in \Omega$, we denote the stabiliser of α in G by G_α , and if $\Sigma \subseteq \Omega$ we denote the setwise and pointwise stabilisers of Σ in G by $G_{\{\Sigma\}}$ and $G_{(\Sigma)}$ respectively. The group G is *transitive* on Ω if G has one orbit on Ω , namely Ω itself. A transitive group G is said to act *regularly* on Ω if $G_\alpha = 1$ for each $\alpha \in \Omega$.

A G -congruence on Ω is an equivalence relation \approx on Ω satisfying

$$\alpha \approx \beta \Leftrightarrow \alpha^g \approx \beta^g,$$

for all $\alpha, \beta \in \Omega$ and $g \in G$. A transitive group G is *primitive* on Ω if the only G -congruences admitted by Ω are the trivial and universal equivalence relations. The following is well known.

Theorem 2.1. ([2, Theorem 4.7]) *If G is a transitive group of permutations on Ω , and $|\Omega| > 1$, the following are equivalent:*

(i) G is primitive on Ω ;

(ii) for every $\alpha \in \Omega$, the stabiliser G_α is a maximal subgroup of G . □

Given $\alpha, \beta \in \Omega$, the set $(\alpha, \beta)^G$ is called an *orbital* of G . It is *diagonal* if $\alpha = \beta$. An α -*suborbit* is an orbit of G_α on Ω . When the choice of α is obvious, the term α -suborbit will be abbreviated to *suborbit*. If G is transitive on Ω , the *subdegrees* of G are the cardinalities of the α -suborbits of G for some fixed $\alpha \in \Omega$.

There is a natural pairing between orbitals of G : if $\Delta := (\alpha, \beta)^G$ is an orbital, then its pair Δ^* is the orbital $(\beta, \alpha)^G$. There is also a natural correspondence between the orbital Δ and the α -suborbit $\Delta(\alpha)$, where $\Delta(\alpha) := \{\gamma \mid (\alpha, \gamma) \in \Delta\}$. For every α -suborbit Υ of G , there is an orbital Δ such that $\Upsilon = \Delta(\alpha)$, namely the orbital $\Delta := (\alpha, \beta)^G$, where β is some vertex in Υ . Thus, one may deduce that the above correspondence is bijective. The pair of the α -suborbit $\Delta(\alpha)$ is the suborbit $\Delta^*(\alpha)$. A suborbit $\Delta(\alpha)$ or orbital Δ is said to be *self-paired* if it is equal to its pair.

If Ω is a countable set, there is a natural complete topology on $\text{Sym } \Omega$ described in [3], that of *pointwise convergence*. Enumerate the set Ω as $\{\gamma_1, \gamma_2, \gamma_3, \dots\}$, then a sequence of permutations (g_n) tends to the limit g if and only if, for any $k \geq 1$, we have $\gamma_k^{g_n} = \gamma_k^g$ and $\gamma_k^{g_n^{-1}} = \gamma_k^{g^{-1}}$ for all sufficiently large n . A basis for the open sets in this topology consists of all cosets of pointwise stabilisers of finite sets. If $G \leq \text{Sym } \Omega$, the *closure* of G is the intersection of all closed subgroups of $\text{Sym } \Omega$ that contain G , and is denoted by \overline{G} .

Theorem 2.2. ([3, Proposition 2.6]) *If $G \leq \text{Sym}(\Omega)$ then G and its closure \overline{G} have the same orbits on n -tuples of Ω for all $n \geq 1$.* \square

2.2 Graph theory

Throughout this thesis, a *graph* will be a directed graph without multiple edges or loops. A graph Γ will be thought of as a pair $(V\Gamma, E\Gamma)$, where $V\Gamma$ is the set of *vertices* and $E\Gamma$ the set of *edges* of Γ . The set $E\Gamma$ consists of ordered pairs of distinct elements of $V\Gamma$. Two vertices $\alpha, \beta \in V\Gamma$ are *adjacent* if either (α, β) or (β, α) lies in $E\Gamma$. An *undirected path of length r* in Γ is a sequence of $r + 1$ vertices such that consecutive vertices are adjacent. A path $x_1 \dots x_{r+1}$ between x_1 and x_{r+1} is *directed* if $(x_i, x_{i+1}) \in E\Gamma$ for all integers i satisfying $1 \leq i \leq r$. A path $x_1 \dots x_{r+1}$ of length r is a *cycle of length r* if $x_1 = x_{r+1}$; a *loop* is a cycle of length one. A *reduced path* is a path without cycles.

Two vertices are *connected* if there exists an undirected path in Γ between them, while a graph is *connected* if any two vertices are connected. A *geodesic* between two vertices is a path of minimal length connecting one to the other. The *distance* between two connected vertices α and β in Γ is denoted by $d_\Gamma(\alpha, \beta)$, and is defined to be the length of any geodesic between them. When there can be no ambiguity as to the graph in question, $d_\Gamma(\alpha, \beta)$ will be abbreviated to $d(\alpha, \beta)$. Given a vertex $\alpha \in V\Gamma$, the sphere of radius r , denoted by $S_r(\alpha, \Gamma)$, is defined to be the set of all vertices of Γ that lie at distance r from α ; when it is obvious which graph is being considered, $S_r(\alpha)$ will often be used instead. The *r -ball* $B_r(\alpha, \Gamma)$ about α is defined by

$$B_r(\alpha, \Gamma) := \{\beta \in \Gamma \mid d_\Gamma(\alpha, \beta) \leq r\}.$$

A *tree* is a connected cycle-free graph. In a tree T , given any two vertices α and β , there is exactly one geodesic between them, which we denote by $[\alpha, \beta]_T$. If we wish to exclude α or β from the geodesic we write $(\alpha, \beta]_T$ or $[\alpha, \beta)_T$ respectively.

A graph Γ is *infinite* if $V\Gamma$ is infinite and is *locally finite* if, for all $\alpha \in V\Gamma$, the sphere $S_1(\alpha, \Gamma)$ is finite. The *in-valency* of a vertex α is equal to the number of

vertices β such that $(\beta, \alpha) \in E\Gamma$; the *out-valency* is equal to the number of vertices γ such that $(\alpha, \gamma) \in E\Gamma$. The *valency* of α is the size of the sphere $S_1(\alpha, \Gamma)$. A graph is *regular* if every vertex has the same valency.

An element $\sigma \in \text{Sym}(V\Gamma)$ is an *automorphism* of Γ if it preserves the edge-structure of Γ ; that is,

$$e \in E\Gamma \Leftrightarrow e^\sigma \in E\Gamma.$$

The set of all automorphisms of the graph Γ form a group called the *automorphism group* of Γ , denoted by $\text{Aut } \Gamma$. A graph is *primitive* if $\text{Aut } \Gamma$ is primitive on the set $V\Gamma$, and is *automorphism-regular* if $\text{Aut } \Gamma$ acts regularly on $V\Gamma$.

We use the term *undirected graph* to mean a directed graph with the property that, whenever $(\alpha, \beta) \in E\Gamma$, we have $(\beta, \alpha) \in E\Gamma$. In this case, we will replace each pair of edges (α, β) and (β, α) with the unordered pair $\{\alpha, \beta\}$. A group acting on a graph Γ is said to be *vertex-transitive* or *edge-transitive* if it acts transitively on the set of vertices or edges of Γ respectively. Similarly, a graph admitting such a group will be referred to as being *vertex-* or *edge-transitive*.

If $G \leq \text{Sym}(\Omega)$ and Δ is an orbital of G , the graph (Ω, Δ) is called an *orbital graph* of G . Note such graphs are necessarily edge-transitive. The following theorem due to D. G. Higman gives a useful test for primitivity.

Theorem 2.3. ([11]) *A transitive group of permutations of a set Ω is primitive if and only if every non-diagonal orbital graph of G is connected.* \square

If Γ is a graph and W is a subset of $V\Gamma$, the subgraph of Γ *induced* by W is the graph $(W, (W \times W) \cap E\Gamma)$. We denote the subgraph of Γ induced by the set $V\Gamma \setminus W$ as $\Gamma \setminus W$. The *connectivity* of an infinite connected graph Γ is the smallest possible size of a subset W of $V\Gamma$ for which the induced graph $\Gamma \setminus W$ is disconnected. A *block* of Γ is a connected subgraph that is maximal subject to the condition it has connectivity strictly greater than one. If Γ has connectivity one, then the vertices α for which $\Gamma \setminus \{\alpha\}$ is disconnected are called the *cut vertices* of Γ .

Consider the following construction. Let V_1 be the set of cut vertices of Γ , and let V_2 be a set in bijective correspondence with the set of blocks of Γ . We let T be

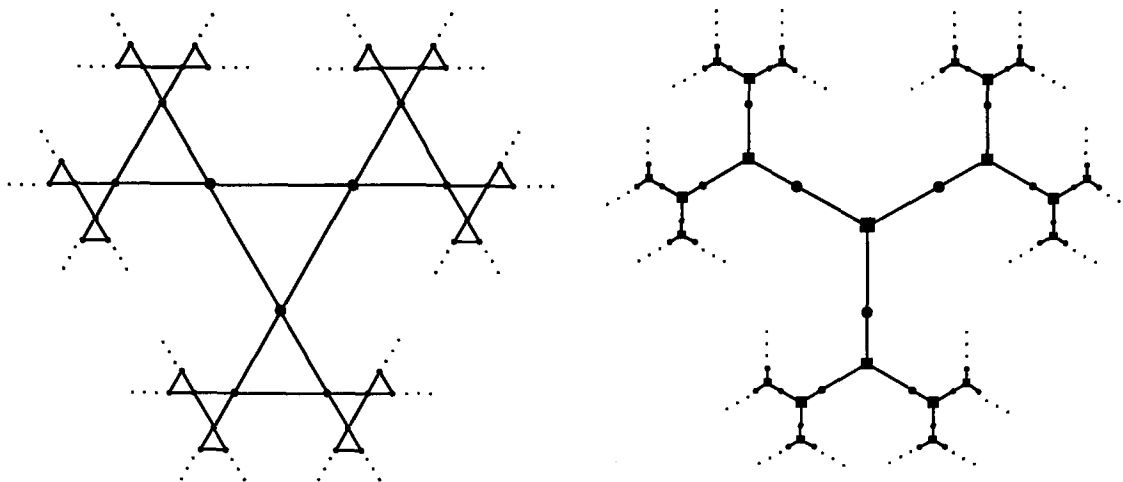


Figure 2.1: A connectivity-one graph and its block-cut-vertex tree

a bipartite graph whose parts are V_1 and V_2 . Two vertices $\alpha \in V_1$ and $x \in V_2$ are adjacent in T if and only if α is contained in the block corresponding to x ; in this case, there are two edges in T between the vertices α and x , one going in each direction. Thus, T can be considered to be an undirected graph. In fact, this construction yields a tree, which is called the *block-cut-vertex tree* of Γ . Note that if Γ has connectivity one and block-cut-vertex tree T , then any group G acting on Γ has a natural action on T . Figure 2.1 shows an undirected primitive connectivity-one graph and its associated block-cut-vertex tree.

Conversely, suppose we are given an undirected tree T . Colour one part of the natural bipartition of T violet and the other blue. If every blue vertex has valence at least three, one may construct a connectivity-one graph Γ with block-cut-vertex tree T that is not a tree as follows. Take $V\Gamma$ to be the violet vertices of T , and for each blue vertex b of T insert edges in Γ between the violet vertices to which b is adjacent in T , in such a way as to obtain a subgraph Λ_b of Γ which has connectivity strictly greater than one. Then Γ will have connectivity one, and for each blue vertex b of T the graph Λ_b will be a block of Γ . If all violet vertices of T have valency m , and the graphs Λ_b are all vertex-transitive and pairwise isomorphic, then we denote the graph Γ by $\Gamma(m, \Lambda_b)$. Figure 2.1 shows the graph $\Gamma(2, \Lambda)$, where in this case Λ is an undirected 3-cycle, along with its block-cut-vertex tree. It should be noted that, although each connectivity-one graph gives rise to a unique 2-coloured tree, each 2-coloured tree may give several connectivity-one graphs.

A concept that is fundamental to the arguments contained herein is that of an *end*. A *half-line* L of a graph Γ is a one-way infinite reduced path $x_1x_2x_3\dots$ in Γ . There is a natural equivalence relation on the set of half-lines of Γ : two half-lines L_1 and L_2 are equivalent if there exists an infinite number of pairwise-disjoint paths connecting a vertex in L_1 to a vertex in L_2 . The equivalence classes of this relation are called the *ends* of Γ . An end is called *thick* if it contains infinitely many disjoint half-lines, otherwise it is called *thin*. By convention, a finite graph has no ends.

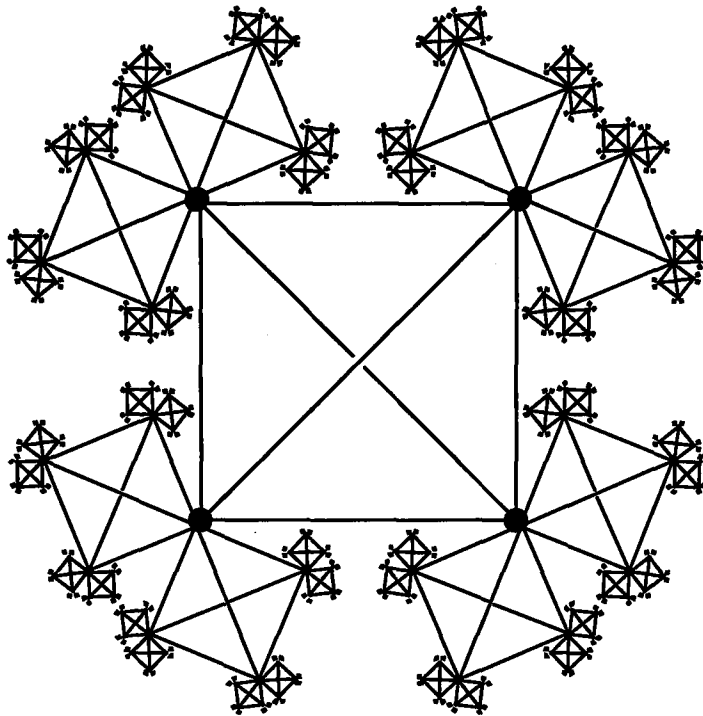
Theorem 2.4. ([33, Corollary 1.5]) *If Γ is an infinite locally finite connected vertex-transitive graph then Γ has 1, 2 or 2^{\aleph_0} ends.* \square

The image one should have in mind when picturing infinite vertex-transitive graphs with one end is that of an infinite grid, while graphs with infinitely many ends resemble trees. Vertex-transitive graphs with two ends are called *strips*; however, as we shall see in Corollary 5.2, such graphs are not primitive and familiarity with them is therefore not required.

A graph is *complete* if it is undirected, and any two distinct vertices are adjacent. The finite complete graph on t vertices will be denoted by K_t . If $t \geq 3$, the connectivity-one graph $\Gamma(m, K_t)$ in which each vertex belongs to $m \geq 2$ blocks, each of which is isomorphic to K_t , is primitive and edge-transitive with infinitely many ends. Figure 2.1 illustrates the graph $\Gamma(m, K_3)$ and its block-cut-vertex tree, while in Figure 2.2 we see the graph $\Gamma(3, K_4)$.

A group acting on a graph Γ is said to be *k-distance-transitive* if, for any four vertices $\alpha_1, \alpha_2, \beta_1, \beta_2 \in V\Gamma$ with $d(\alpha_1, \alpha_2) = d(\beta_1, \beta_2) \leq k$, there exists $g \in G$ such that $\alpha_1^g = \beta_1$ and $\alpha_2^g = \beta_2$, and is *distance-transitive* if it is *k-distance-transitive* for all positive integers k . A graph is *distance-transitive* if its automorphism group acts upon it distance-transitively. The locally finite infinite distance-transitive graphs were classified by Macpherson [17] and independently by Ivanov [13] thus.

Theorem 2.5. ([17, Theorem 1.2]) *A locally finite infinite graph is distance-transitive if and only if it is a regular combinatorial tree or is $\Gamma(m, K_t)$ for some $m \geq 2$ and $t \geq 3$, where K_t is the complete graph on t vertices.* \square

Figure 2.2: The graph $\Gamma(3, K_4)$

Interestingly, Möller shows in [23] if Γ is locally finite with more than one end, then Γ is distance-transitive if and only if Γ is 2-distance-transitive.

Two graphs Γ_1 and Γ_2 are *quasi-isometric* if there exists a map $\phi : V\Gamma_1 \rightarrow V\Gamma_2$ and there exist constants $a, b > 0$ such that for all $\alpha, \beta \in V\Gamma$ we have

$$\frac{1}{a}d_{\Gamma_1}(\alpha, \beta) - b \leq d_{\Gamma_2}(\phi(\alpha), \phi(\beta)) \leq a \cdot d_{\Gamma_1}(\alpha, \beta) + b.$$

In [21, Proposition 1], Möller shows if two locally finite connected graphs are quasi-isometric, then there is a natural bijection between their ends.

Lemma 2.6. *If G is a primitive group of permutations of an infinite set Ω and Γ_1 and Γ_2 are two orbital graphs of G then there is a natural bijection between the ends of Γ_1 and the ends of Γ_2 .*

Proof. Suppose both $\Gamma_1 = (\Omega, (\alpha_1, \beta_1)^G)$ and $\Gamma_2 = (\Omega, (\alpha_2, \beta_2)^G)$ are locally finite orbital graphs of G . Put $m_1 := d_{\Gamma_1}(\alpha_2, \beta_2)$ and $m_2 := d_{\Gamma_2}(\alpha_1, \beta_1)$, and let $a := \max\{m_1, m_2\}$. Since both Γ_1 and Γ_2 are connected, any edge in Γ_2 corresponds to a path of length m_1 in Γ_1 , and any edge in Γ_1 corresponds to a path of length m_2 in

Γ_2 . Thus, if $\alpha, \beta \in \Omega$ then $d_{\Gamma_1}(\alpha, \beta) \leq m_1 \cdot d_{\Gamma_2}(\alpha, \beta) \leq a \cdot d_{\Gamma_2}(\alpha, \beta)$, and $d_{\Gamma_2}(\alpha, \beta) \leq m_2 \cdot d_{\Gamma_1}(\alpha, \beta) \leq a \cdot d_{\Gamma_1}(\alpha, \beta)$. Therefore $(1/a)d_{\Gamma_1}(\alpha, \beta) \leq d_{\Gamma_2}(\alpha, \beta) \leq a \cdot d_{\Gamma_1}(\alpha, \beta)$, so Γ_1 and Γ_2 are quasi-isometric. \square

Henceforth, no distinction will be made between an end of Γ_1 and its corresponding end in Γ_2 . Furthermore, we define a *permutation-end* of G to be an end of an orbital graph of G . By the above lemma, the set of permutation-ends of G is equal to the set of ends of an orbital graph of G , and is independent of the orbital graph chosen.

2.3 Relations

A *relation* R of arity n on a set Ω is a subset $R \subseteq \Omega^n$. A *relational structure* is a pair $(\Omega; \mathfrak{R})$, where \mathfrak{R} is a set of relations on the set Ω .

Let Ω be a non-empty set and let \mathfrak{R} be a set of relations on Ω . Since $\text{Sym } \Omega$ acts on the set of n -tuples of Ω , there is a natural action of $\text{Sym } \Omega$ on \mathfrak{R} . An element $\sigma \in \text{Sym } \Omega$ is said to be an *automorphism* of the relational structure $(\Omega; \mathfrak{R})$ if for all $R \in \mathfrak{R}$,

$$R^\sigma = R.$$

The set of all automorphisms of the relational structure $(\Omega; \mathfrak{R})$ forms a group called the *automorphism group* of $(\Omega; \mathfrak{R})$, denoted by $\text{Aut } (\Omega; \mathfrak{R})$.

Suppose we are given two relational structures $(\Omega_1; \mathfrak{R}_1)$ and $(\Omega_2; \mathfrak{R}_2)$, and a bijective map $\phi : \mathfrak{R}_1 \rightarrow \mathfrak{R}_2$. A map $\varphi : \Omega_1 \rightarrow \Omega_2$ is said to be an *isomorphism* between the relational structures $(\Omega_1; \mathfrak{R}_1)$ and $(\Omega_2; \mathfrak{R}_2)$ if and only if $\varphi(R) = \phi(R)$ for all relations $R \in \mathfrak{R}_1$, where $\varphi(R)$ denotes the set $\{(\varphi(\alpha_1), \dots, \varphi(\alpha_n)) \mid (\alpha_1, \dots, \alpha_n) \in R\}$. Two relational structures are *isomorphic* if there exists an isomorphism between them.

Let G be a permutation group acting on a non-empty set Ω . If Θ is an orbit of G on Ω^n then it may be considered to be a relation of arity n on Ω . Following [3], we define \mathfrak{R}_G to be the set of relations consisting of all orbits of G on Ω^n where n takes every value in the set of natural numbers. The *canonical relational structure* associated with G is the relational structure $(\Omega; \mathfrak{R}_G)$.

Theorem 2.7. ([3, Theorem 2.6]) *If Ω is a countably infinite set then $G \leq \text{Sym } \Omega$ is closed if and only if G is the automorphism group of the canonical relational structure of G .* \square

Chapter 3

Subdegree enumeration and growth

3.1 Background

Interest in the subdegree growth of infinite primitive permutation groups possessing a finite suborbit whose pair is also finite stems from the following observation.

Theorem 3.1. *If G is a group acting primitively on the infinite set Ω , possessing a finite suborbit whose pair is also finite, then every suborbit of G is finite and Ω is countable.*

Proof. Fix $\alpha \in \Omega$ and let β^{G_α} be a finite α -suborbit whose pair is finite. Let Γ be the orbital graph $(\Omega, (\alpha, \beta)^G)$. By Theorem 2.3, Γ is connected. Furthermore, the out-valency of α in Γ is equal to $|\beta^{G_\alpha}|$ and the in-valency of α is equal to the size of the suborbit paired with β^{G_α} . Since both these sets are finite, the in-valency and the out-valency of α are finite, so the valency of α is finite. Because Γ is vertex-transitive, Γ is locally finite, so for all $r \geq 0$ the sphere $S_r(\alpha, \Gamma)$ is finite.

Since Γ is connected, $\Omega = \bigcup_{r=0}^{\infty} S_r(\alpha, \Gamma)$. Thus Ω is a countable union of finite sets, and is therefore countably infinite.

Since G_α fixes each sphere $S_r(\alpha, \Gamma)$ setwise, every α -suborbit of G is finite. Because G is primitive on Ω , it acts transitively on Ω , so every suborbit of G is finite. \square

Henceforth, any group whose suborbits are all finite will be described as being *locally finite*. Thus a primitive group is locally finite if and only if it has a locally finite orbital graph; of course, if it has a locally finite orbital graph then all its orbital graphs will be locally finite.

Note this theorem is not true if we merely require a primitive group G to possess a finite suborbit. Indeed, in [7] Evans constructs an infinite primitive group acting on an uncountable infinite set Ω , possessing a finite suborbit whose pair is infinite.

Adeleke and Neumann in [1, Remark 29.8] observe if G acts primitively on an infinite set Ω and has a non-trivial self-paired finite suborbit of size m then Ω is countable, all the suborbits of G are finite, $m \geq 2$, and if the suborbits are arranged in a non-decreasing sequence $1 = m_0 \leq m_1 \leq \dots$ then $m_r \leq (m - 1)m_{r-1}$ for all sufficiently large r . They then ask whether the subdegree growth rates of infinite primitive groups that act distance-transitively on locally finite distance-transitive graphs are extremal and conjecture there might exist a number c which perhaps depends upon G , perhaps only on m , such that $m_r \leq c(m - 2)^{r-1}$.

In fact, this approach is naive, as it does not consider the existence of a locally finite primitive group G , possessing at least two distinct subdegrees m and m' such that infinitely many α -suborbits have cardinality m , and infinitely many have cardinality m' . Such subdegrees will henceforth be said to occur *infinitely often*. Indeed, if G has at least two subdegrees, each occurring infinitely often, of which m is the smallest, then under the above enumeration method $m_r = m$ for all sufficiently large r . Any subdegree of G that is strictly larger than m would therefore not be present in the subdegree sequence (m_r) .

In the following section, we give examples of such groups. In subsequent sections we define comprehensive methods for enumerating the subdegrees of locally finite infinite primitive groups and measuring their subdegree growth.

3.2 Examples and constructions

3.2.1 Monster groups

In [27], Ol'shanskiĭ shows for every prime $p > 10^{10}$, there is an infinite group in which all non-trivial proper subgroups are of order p . We propose to call such groups Tarski–Ol'shanskiĭ monster groups of order p . If p is a prime number greater than 10^{10} and T_p is such a group, fix any non-trivial proper subgroup $H \leq T_p$. Let T_p act on the set of right cosets $\Omega := \{Hg \mid g \in T_p\}$ via

$$(Hg)^{g'} = Hgg'.$$

The kernel K of this action is a normal subgroup of T_p , and is therefore trivial. Indeed, suppose K is a non-trivial, proper normal subgroup of T_p . Then every non-trivial, proper subgroup of the quotient group T_p/K is of the form H'/K for some proper subgroup H' of T_p with $K < H'$. Now T_p/K is infinite, and every element has finite order, so T_p/K contains a non-trivial, proper subgroup H'/K , with $K < H' < T_p$. This is absurd, however, since both H' and K must have order p . Hence T_p is simple, and its action on Ω is faithful.

The stabiliser of the coset $H.1 \in \Omega$ is the group $H \leq T_p$. Since every subgroup of T_p has order p , this group is a maximal subgroup of T_p . Furthermore, every element of the finite group H has order p , so the orbits of H acting on $\Omega \setminus \{H\}$ must all have size p . Hence T_p acts primitively on Ω , with all non-trivial suborbits finite of size p . The group T_p contains no elements of infinite order, so, as we shall see in Theorem 7.2, every orbital graph of this group must have one end. Thus, T_p is an example of a locally finite primitive group with one permutation-end, whose suborbits are all bounded above.

3.2.2 Infinitely-ended constructions

Suppose we are given an integer $m \geq 2$, a locally finite primitive edge-transitive graph Λ with connectivity at least two, and an edge-transitive primitive non-regular group H of automorphisms of Λ . We will construct a primitive and edge-transitive group

of automorphisms G of the graph $\Gamma(m, \Lambda)$ such that the subgroup of $\text{Aut } \Lambda$ induced by $G_{\{\Lambda\}}$ is equal to \overline{H} , the closure of H in $\text{Aut } \Lambda$. This construction will provide a natural way of manufacturing large primitive groups from smaller ones, analogous to the method, discussed in Section 2.2, of creating infinitely-ended primitive graphs from primitive graphs with connectivity strictly greater than one.

We begin by describing a relational structure based on the graph $\Gamma(m, \Lambda)$. Let Γ denote this graph, and, for any graph Γ' with connectivity one, define $B(\Gamma')$ to be the set of blocks of Γ' . Let \mathfrak{R} be a set of relations on the set $V\Lambda$ such that $E\Lambda \in \mathfrak{R}$. Since all blocks of Γ are isomorphic to Λ , we may, for each block Δ of Γ , choose a graph isomorphism

$$\varphi_\Delta : \Lambda \rightarrow \Delta.$$

For each relation $R \in \mathfrak{R}$ define $R_\Delta := \varphi_\Delta(R)$, where

$$\varphi_\Delta(R) := \{(\varphi_\Delta(\alpha_1), \dots, \varphi_\Delta(\alpha_n)) \mid (\alpha_1, \dots, \alpha_n) \in R\}.$$

Now define $\mathfrak{R}_\Delta := \{R_\Delta \mid R \in \mathfrak{R}\}$ and observe the relational structure $(V\Delta; \mathfrak{R}_\Delta)$ is isomorphic to $(V\Lambda; \mathfrak{R})$.

For each vertex $\alpha \in V\Lambda$ the graph $\Gamma \setminus \{\alpha\}$ is not connected. One connected component of $\Gamma \setminus \{\alpha\}$ contains all vertices in $\Lambda \setminus \{\alpha\}$; the other connected components are disjoint from $\Lambda \setminus \{\alpha\}$. Define Γ'_α to be the subgraph of Γ consisting of all connected components of $\Gamma \setminus \{\alpha\}$ that are disjoint from $\Lambda \setminus \{\alpha\}$, and let Γ_α be the subgraph of Γ induced by $V\Gamma'_\alpha \cup \{\alpha\}$. This graph has connectivity one, and for two vertices $\alpha, \beta \in V\Lambda$ the graphs Γ_α and Γ_β are isomorphic.

Now fix $\alpha \in V\Lambda$ and let

$$\mathfrak{S}_\alpha := \bigcup_{\Delta \in B(\Gamma_\alpha)} \mathfrak{R}_\Delta.$$

This is a set of relations on the set $V\Gamma_\alpha$, so we may define Σ_α to be the relational structure $(V\Gamma_\alpha; \mathfrak{S}_\alpha)$. For each $\gamma \in V\Lambda$ choose a graph isomorphism $\phi_\gamma : \Gamma_\alpha \rightarrow \Gamma_\gamma$, with ϕ_α the identity map. Now define $\mathfrak{S}_\gamma := \phi_\gamma(\mathfrak{S}_\alpha)$, where $\phi_\gamma(\mathfrak{S}_\alpha) = \{\phi_\gamma(R) \mid R \in \mathfrak{S}_\alpha\}$, and let Σ_γ be the relational structure $(V\Gamma_\gamma; \mathfrak{S}_\gamma)$. The mapping ϕ_γ is thus an isomorphism between the relational structures Σ_α and Σ_γ . This observation will underpin the proof of our next lemma.

Now let

$$\mathfrak{S} := \bigcup_{\gamma \in V\Lambda} \mathfrak{S}_\gamma,$$

and define $\Gamma(m, \Lambda, \mathfrak{R})$ to be the relational structure $(V\Gamma; \mathfrak{S} \cup \mathfrak{R})$. This structure has many important properties, the most useful being

$$\text{Aut } \Gamma(m, \Lambda, \mathfrak{R}) \leq \text{Aut } \Gamma(m, \Lambda).$$

To see this simply observe $E\Lambda \in \mathfrak{R}$. Recall that for each block Δ of $\Gamma(m, \Lambda)$ the relational structure $(V\Delta; \mathfrak{R}_\Delta)$ is isomorphic to $(V\Lambda; \mathfrak{R})$. Henceforth, such relational structures will be referred to as *relational blocks* of $\Gamma(m, \Lambda, \mathfrak{R})$.

Recall that we have been given a primitive and non-regular group H that acts edge-transitively on Λ . The canonical set of relations \mathfrak{R}_H consists of all orbits of H on $V\Lambda^n$, where n takes every value in the set of natural numbers. Let

$$G := \text{Aut } \Gamma(m, \Lambda, \mathfrak{R}_H).$$

We will show the subgroup of $\text{Aut } \Lambda$ induced by $G_{\{\Lambda\}}$ is equal to \overline{H} , the closure of H in $\text{Aut } \Lambda$.

Lemma 3.2. *Let T be the block-cut-vertex tree of $\Gamma(m, \Lambda)$ and let x be the vertex of T corresponding to Λ . If $\alpha \in V\Lambda$ and $\beta \in V\Gamma$ lie in the same component of $T \setminus \{x\}$ then the subgroup of $\text{Aut } \Lambda$ induced by $G_{\alpha, \beta, \{\Lambda\}}$ is isomorphic to \overline{H}_α .*

Proof. Let G' be the subgroup of $\text{Aut } \Lambda$ induced by $G_{\alpha, \beta, \{\Lambda\}}$. It is clear that $G' \leq \text{Aut } (V\Lambda; \mathfrak{R}_H)$. By Theorem 2.7, $\text{Aut } (V\Lambda; \mathfrak{R}_H) = \overline{H}$. Thus,

$$G' \leq \overline{H}_\alpha.$$

Choose any automorphism $h \in \overline{H}_\alpha$. We will extend this to an automorphism σ of $\Gamma(m, \Lambda, \mathfrak{R}_H)$ that fixes β .

For each pair of vertices $\gamma_1, \gamma_2 \in V\Lambda$ the relational structures Σ_{γ_1} and Σ_{γ_2} are isomorphic, so we may choose an isomorphism

$$\phi_{(\gamma_1, \gamma_2)} : \Sigma_{\gamma_1} \rightarrow \Sigma_{\gamma_2},$$

which is equal to the identity map when $\gamma_1 = \gamma_2$. We now construct a mapping $\sigma : \Gamma(m, \Lambda, \mathfrak{R}_H) \rightarrow \Gamma(m, \Lambda, \mathfrak{R}_H)$ as follows. For each $\delta \in V\Gamma$ there exists a unique vertex $\gamma \in V\Lambda$ such that $\delta \in \Sigma_\gamma$, so set

$$\delta^\sigma := \delta^{\phi_{(\gamma, \gamma^h)}}.$$

It is simple to check this is a well-defined automorphism of $\Gamma(m, \Lambda, \mathfrak{R}_H)$. Since α and β lie in the same component of $T \setminus \{x\}$, the vertex α must be the unique element of $V\Lambda$ such that $\beta \in \Sigma_\alpha$. Hence $\beta^\sigma = \beta$, so $G' = \overline{H}_\alpha$. \square

Theorem 3.3. *Suppose H is a primitive non-regular edge-transitive group of automorphisms of a locally finite graph Λ with at least three vertices and connectivity strictly greater than one. If $m \geq 2$ then the group $\text{Aut } \Gamma(m, \Lambda, \mathfrak{R}_H)$ acts primitively and edge-transitively on the graph $\Gamma(m, \Lambda)$. Furthermore, the group induced by its action on $V\Lambda$ is \overline{H} , the closure of H in $\text{Aut } \Lambda$.*

Proof. We begin by showing the subgroup of $\text{Aut } \Lambda$ induced by $\text{Aut } \Gamma(m, \Lambda, \mathfrak{R}_H)$ on $V\Lambda$ is \overline{H} . Let $G = \text{Aut } \Gamma(m, \Lambda, \mathfrak{R}_H)$ and let G' be the subgroup of $\text{Aut } \Lambda$ induced by the action of $G_{\{\Lambda\}}$ on $V\Lambda$. By construction, $G \leq \text{Aut } \Gamma(m, \Lambda)$. Fix $\alpha \in V\Lambda$, and note that by applying Lemma 3.2 with $\beta = \alpha$ we have $G'_\alpha = \overline{H}_\alpha$. Furthermore, $G' \leq \text{Aut}(V\Lambda, \mathfrak{R}_H) = \overline{H}$, so

$$\overline{H}_\alpha \leq G' \leq \overline{H}.$$

Our choice of α was arbitrary, so we may choose a vertex $\gamma \in V\Lambda$ that is distinct from α and note that

$$\overline{H}_\gamma \leq G' \leq \overline{H}.$$

Since H acts primitively and non-regularly on $V\Lambda$, the group \overline{H}_γ does not fix α ; whence G' does not fix α . Thus

$$\overline{H}_\alpha < G' \leq \overline{H}.$$

Since H acts primitively on $V\Lambda$ the same must be true of \overline{H} ; therefore \overline{H}_α is a maximal subgroup of \overline{H} . Consequently $G' = \overline{H}$.

It remains to show G acts primitively and edge-transitively on the graph $\Gamma(m, \Lambda)$. By construction G acts transitively on the relational blocks of $\Gamma(m, \Lambda, \mathfrak{R}_H)$, and therefore acts transitively on the set of blocks of the graph $\Gamma(m, \Lambda)$.

The action of G on the vertices of Γ is transitive. Indeed, suppose δ is any vertex in $V\Gamma$. Then there exists a block Δ of Γ containing δ . Choose $g \in G$ such that $\Delta^g = \Lambda$. Since $G' = \overline{H}$, and H acts transitively on the vertices of Λ , there exists an automorphism $h \in G_{\{\Lambda\}}$ such that $\delta^{gh} = \alpha$.

Furthermore, the stabiliser G_α transitively permutes the blocks of Γ that contain α . For, suppose Λ' is a block of Γ containing α . Then there is an automorphism $g \in G$ such that $\Lambda^g = \Lambda'$. We again observe that since $G' = \overline{H}$, and H acts transitively on the vertices of Λ , there exists an automorphism $h \in G_{\{\Lambda\}}$ such that $\alpha^h = \alpha^{g^{-1}}$. Thus $\Lambda^{hg} = \Lambda'$ and $\alpha^{hg} = \alpha$.

Since H acts vertex- and edge-transitively on Λ , and G_α transitively permutes the blocks of Γ that contain α , we must have G is edge-transitive on Γ .

Finally, let ρ be a non-trivial G -congruence on $V\Gamma$. Let T be the block-cut-vertex tree of Γ . Choose $\beta \in \rho(\alpha) \setminus \{\alpha\}$ of minimal distance in T from α . We claim α and β lie in a common block of Γ . Indeed, suppose this is not the case. Let x be the vertex adjacent to β in the geodesic $[\alpha, \beta]_T$, and let γ be the vertex adjacent to x in $[\alpha, \beta]_T$. The vertex x corresponds to a block Λ' of Γ , which, since α and β lie in distinct blocks, is not equal to Λ . Because G acts transitively on the blocks of Γ , the group induced on $V\Lambda'$ by $G_{\alpha, \gamma, \{\Lambda'\}}$ is isomorphic to \overline{H}_α acting on $V\Lambda$. Since \overline{H} is primitive and non-regular on $V\Lambda$, there exists $g \in G_{\alpha, \gamma, \{\Lambda'\}}$ such that $\beta^g \neq \beta$. Thus $\beta, \beta^g \in \rho(\alpha)$ share a common block, and our claim is established. So, without loss of generality, we may assume α and β lie in Λ . Now ρ induces a \overline{H} -congruence on $V\Lambda$, so we must have $V\Lambda \subseteq \rho(\alpha)$. Furthermore, G_α acts transitively on the blocks of Γ that contain α , so ρ is the universal relation. Hence G acts primitively on Γ . \square

If H is a primitive and non-regular group of permutations of a set Ω , with $|\Omega| \geq 3$, possessing an orbital graph Λ with connectivity strictly greater than one, then the primitive group G constructed above will be called the *m-fold graph product of H* , and will be denoted by $G(m, H)$. By the above theorem, this group acts primitively

on the vertex set of $\Gamma(m, \Lambda)$. Since this graph is an orbital graph of $G(m, \Lambda)$, all orbital graphs of G will have infinitely many ends by Lemma 2.6.

Example 3.4. Let p be a prime number with $p > 10^{10}$ and let T_p be a Tarski–Ol’shanskiĭ monster group of order p . Recall this group acts primitively on its coset space Ω . Let Λ be an orbital graph of T_p . Then Λ is a one-ended primitive graph, and therefore has connectivity strictly greater than one.

By Theorem 3.3, the group $G(m, T_p)$ acts primitively on $\Gamma(m, \Lambda)$, and has infinitely many distinct subdegrees, each occurring infinitely often. In fact, for all $\alpha \in V\Gamma$, the α -subdegrees of $G(m, T_p)$ are $m(m-1)^{r-1}p^r$ for $r \geq 1$. For each positive integer r there are infinitely many α -suborbits of $G(m, T_p)$ with cardinality $m(m-1)^{r-1}p^r$.

3.2.3 One-ended constructions

Let C and D be permutation groups acting on sets Γ and Δ respectively. Define $K := C^\Delta = \{f \mid f : \Delta \rightarrow C\}$, and note this a group under the following multiplication: given $f_1, f_2 \in K$ and $\delta \in \Delta$,

$$(f_1 f_2)(\delta) := f_1(\delta) f_2(\delta).$$

The group D acts on C^Δ via

$$f^d(\delta) := f(\delta^{d^{-1}}).$$

The *wreath product* W of C by D is defined to be the semidirect product of K by D ; that is, W is the set $\{(f, d) \mid f \in K, d \in D\}$ under the multiplication

$$(f_1, d_1)(f_2, d_2) = (f_1 f_2^{d_1^{-1}}, d_1 d_2).$$

The wreath product of C by D is written $C \operatorname{Wr}_\Delta D$. When the set Δ is clear from the choice of D , it is usual to simply write $C \operatorname{Wr} D$. There is a natural action of the group W on the set $\Gamma^\Delta := \{\varphi \mid \varphi : \Delta \rightarrow \Gamma\}$, called the *product action*, defined as follows. Given $\varphi \in \Gamma^\Delta$ and $(f, d) \in W$,

$$\varphi^{(f, d)}(\delta) := \varphi(\delta^{d^{-1}}) f(\delta^{d^{-1}}),$$

for all $\delta \in \Delta$. Under certain conditions, this action is primitive.

Theorem 3.5. ([5, Lemma 2.7A]) *Let C and D be permutation groups acting on Γ and Δ respectively. If $|\Gamma|, |\Delta| > 1$ and $W := C \operatorname{Wr}_{\Delta} D$ then W acts primitively on Γ^{Δ} under the product action if and only if Δ is finite, D acts transitively on Δ and C is primitive but not regular on Γ . \square*

In the case where Δ is finite, the description of $W = C \operatorname{Wr}_{\Delta} D$ and its product action on the set Γ^{Δ} can be simplified significantly. If $\Delta = \{\delta_1, \dots, \delta_m\}$, the group C^{Δ} is isomorphic to the group C^m , and there is a bijection between Γ^{Δ} and Γ^m . In this way, an element $f \in C^{\Delta}$ can be thought of as the element $(f(\delta_1), \dots, f(\delta_m)) \in C^m$; similarly, an element $\varphi \in \Gamma^{\Delta}$ can be thought of as the element $(\varphi(\delta_1), \dots, \varphi(\delta_m)) \in \Gamma^m$. Thus, the group W is defined on the set $C^m \times D$, with multiplication

$$((c_{\delta_1}, \dots, c_{\delta_m}), d)((c'_{\delta_1}, \dots, c'_{\delta_m}), d') = (c_{\delta_1} c'_{\delta_1^{d-1}}, \dots, c_{\delta_m} c'_{\delta_m^{d-1}}, dd').$$

The product action of W on Γ^m can now be defined as follows. Given a vertex $\varphi = (\gamma_{\delta_1}, \dots, \gamma_{\delta_m}) \in \Gamma^m$ and $w = (c_{\delta_1}, \dots, c_{\delta_m})d \in W$,

$$\varphi^w := \left((\gamma_{\delta_1}^{c_{\delta_1}})^{d^{-1}}, \dots, (\gamma_{\delta_m}^{c_{\delta_m}})^{d^{-1}} \right).$$

The following result allows one to construct infinitely many examples of locally finite primitive groups with one permutation-end.

Theorem 3.6. *If G is a locally finite primitive group of permutations of an infinite set Ω and $m \geq 2$, then $G \operatorname{Wr} \operatorname{Sym}(m)$ is a locally finite primitive group of permutations of Ω^m with one permutation-end.*

This result may be deduced from the following lemmas. Fix a group G acting primitively on an infinite set Ω , possessing a finite suborbit whose pair is also finite. Let Γ be an orbital graph of G on Ω , and define $H := G \operatorname{Wr} \operatorname{Sym}(m)$. By Theorem 3.5, the group H acts primitively on the set Ω^m under the product action. Fix $\alpha, \beta \in \Omega$ such that α and β are adjacent in Γ , with $(\alpha, \beta) \in E\Gamma$. Let $\underline{\alpha} := (\alpha, \dots, \alpha) \in \Omega^m$ and $\underline{\beta} := (\beta, \alpha, \dots, \alpha) \in \Omega^m$, and define

$$\Lambda := (\Omega^m, (\underline{\alpha}, \underline{\beta})^H),$$

where H acts on Ω^m via the product action.

Lemma 3.7. *The vertex $(\gamma_1, \dots, \gamma_m)$ is adjacent to $\underline{\alpha}$ in Λ if and only if*

$$\sum_{i=1}^m d_{\Gamma}(\alpha, \gamma_i) = 1.$$

Proof. Let $\underline{\gamma} := (\gamma_1, \dots, \gamma_m) \in V\Lambda$. Since Λ is an orbital graph of H on Ω^m , it is edge transitive. Therefore $\underline{\gamma}$ lies in the sphere $S_1(\underline{\alpha}, \Lambda)$ if and only if there exists $g = (g_1, \dots, g_m)\sigma \in H$ such that $(\underline{\alpha}, \underline{\beta})^g$ is equal to $(\underline{\alpha}, \underline{\gamma})$ or $(\underline{\gamma}, \underline{\alpha})$. If $(\underline{\alpha}, \underline{\beta})^g = (\underline{\alpha}, \underline{\gamma})$ then $g \in H_{\underline{\alpha}}$ and so $g_1, \dots, g_m \in G_{\alpha}$. Thus $\underline{\gamma} = \underline{\beta}^g = (\beta^{g_1}, \alpha, \dots, \alpha)^{\sigma}$, and therefore $\sum_{i=1}^m d_{\Gamma}(\alpha, \gamma_i) = d_{\Gamma}(\alpha, \beta^{g_1}) = 1$. Otherwise, if $(\underline{\alpha}, \underline{\beta})^g = (\underline{\gamma}, \underline{\alpha})$ then $\underline{\beta}^g = \underline{\alpha}$, so $g_2, \dots, g_m \in G_{\alpha}$ but $\beta^{g_1} = \alpha$. Thus $\underline{\gamma} = \underline{\alpha}^g = (\alpha^{g_1}, \alpha, \dots, \alpha)^{\sigma}$ and $\sum_{i=1}^m d_{\Gamma}(\alpha, \gamma_i) = d_{\Gamma}(\alpha, \alpha^{g_1}) = 1$.

Conversely, suppose $\underline{\gamma} = (\gamma_1, \dots, \gamma_m) \in V\Lambda$ and $\sum_{i=1}^m d_{\Gamma}(\alpha, \gamma_i) = 1$. Then there exists a unique value of i such that $d_{\Gamma}(\alpha, \gamma_i) = 1$ and $\gamma_j = \alpha$ for all $j \neq i$. Thus, there exists $g_1 \in G$ such that $(\alpha, \beta)^{g_1}$ is equal to (α, γ_i) or (γ_i, α) . Let $\sigma \in \text{Sym}(m)$ be the 2-cycle $(1i)$ and set $g := (g_1, 1, \dots, 1)\sigma \in H$. If $(\alpha, \beta)^{g_1} = (\alpha, \gamma_i)$ then $g \in H_{\underline{\alpha}}$ and $\underline{\beta}^g = (\beta^{g_1}, \alpha, \dots, \alpha)^{\sigma} = (\gamma_i, \alpha, \dots, \alpha)^{\sigma} = \underline{\gamma}$, and therefore $d_{\Lambda}(\underline{\alpha}, \underline{\gamma}) = 1$. On the other hand, if $(\alpha, \beta)^{g_1} = (\gamma_i, \alpha)$ then $\underline{\beta}^g = \underline{\alpha}$ and $\underline{\alpha}^g = (\alpha^{g_1}, \alpha, \dots, \alpha)^{\sigma} = (\gamma_i, \alpha, \dots, \alpha)^{\sigma} = \underline{\gamma}$, so again $d_{\Lambda}(\underline{\alpha}, \underline{\gamma}) = 1$. We have thus shown $(\gamma_1, \dots, \gamma_m) \in S_1((\alpha, \dots, \alpha), \Lambda)$ if and only if $\sum_{i=1}^m d_{\Gamma}(\alpha, \gamma_i) = 1$. Since Λ is vertex-transitive the hypothesis holds for any vertex $(\delta_1, \dots, \delta_m) \in V\Lambda$. \square

Lemma 3.8. *The vertex $(\gamma_1, \dots, \gamma_m)$ lies in the sphere $S_r((\delta_1, \dots, \delta_m), \Lambda)$ if and only if*

$$\sum_{i=1}^m d_{\Gamma}(\delta_i, \gamma_i) = r.$$

Proof. We proceed by induction. Since H is vertex-transitive, the hypothesis holds when $r = 1$ by Lemma 3.7. Fix $\underline{\delta} = (\delta_1, \dots, \delta_m) \in V\Lambda$ and $k > 1$ and suppose the hypothesis is true for all $r \leq k$. Choose $\underline{\gamma} = (\gamma_1, \dots, \gamma_m) \in V\Lambda$ with $\sum_{i=1}^m d_{\Gamma}(\delta_i, \gamma_i) = k + 1$. Since $\sum_{i=1}^m d_{\Gamma}(\delta_i, \gamma_i) > 2$ there exists j for which one may choose $\gamma'_j \in S_1(\gamma_j, \Gamma)$ such that $d_{\Gamma}(\delta_j, \gamma'_j) = d_{\Gamma}(\delta_j, \gamma_j) - 1$. Let $\gamma'_i := \gamma_i$ for all $i \neq j$ and put $\underline{\gamma}' := (\gamma'_1, \dots, \gamma'_m)$. Now $\sum_{i=1}^m d_{\Gamma}(\delta_i, \gamma'_i) = k$, so by assumption, $\underline{\gamma}' \in S_k(\underline{\delta}, \Lambda)$; furthermore, $d_{\Lambda}(\underline{\gamma}, \underline{\gamma}') = 1$, so $\underline{\gamma} \in S_{k-1}(\underline{\delta}, \Lambda) \cup S_k(\underline{\delta}, \Lambda) \cup S_{k+1}(\underline{\delta}, \Lambda)$. If $\underline{\gamma}$ lies in $S_{k-1}(\underline{\delta}, \Lambda)$ or

$S_k(\underline{\delta}, \Lambda)$ then by assumption $\sum_{i=1}^m d_\Gamma(\delta_i, \gamma_i)$ is equal to $k - 1$ or k respectively; since this is not the case, we must have $\underline{\gamma} \in S_{k+1}(\underline{\delta}, \Lambda)$.

Conversely, suppose $\underline{\gamma} = (\gamma_1, \dots, \gamma_m) \in S_{k+1}(\underline{\delta}, \Lambda)$. Since $d_\Lambda(\underline{\delta}, \underline{\gamma}) > k$ we have $\sum_{i=1}^m d_\Gamma(\delta_i, \gamma_i) \geq k + 1$ by the induction hypothesis. Since Λ is connected, there exists $\underline{\gamma}' = (\gamma'_1, \dots, \gamma'_m) \in S_k(\underline{\delta}, \Lambda) \cap S_1(\underline{\gamma}, \Lambda)$. Now $\sum_{i=1}^m d_\Gamma(\delta_i, \gamma'_i) = k$ and $\sum_{i=1}^m d_\Gamma(\gamma_i, \gamma'_i) = 1$. For each i we have $d_\Gamma(\delta_i, \gamma_i) \leq d_\Gamma(\delta_i, \gamma'_i) + d_\Gamma(\gamma'_i, \gamma_i)$; whence, $\sum_{i=1}^m d_\Gamma(\delta_i, \gamma_i) \leq \sum_{i=1}^m (d_\Gamma(\delta_i, \gamma'_i) + d_\Gamma(\gamma'_i, \gamma_i)) = k + 1$. Hence $\sum_{i=1}^m d_\Gamma(\delta_i, \gamma_i) = k + 1$. \square

Lemma 3.9. *The graph Λ is infinite, primitive, locally finite and edge-transitive, with one end.*

Proof. The group H acts primitively on $V\Lambda$ by Theorem 2.3, so the graph Λ is primitive. It is infinite because $V\Lambda = \Omega^m$ is infinite, and is locally finite by Lemma 3.7; furthermore, since Λ is an orbital graph of H on Ω^m , it is edge-transitive.

It remains to prove that Λ has one end. We will show, for all $r \geq 1$, given any pair of vertices $\underline{\gamma}, \underline{\delta} \in S_{r+1}(\underline{\alpha}, \Lambda)$, there is a path connecting $\underline{\gamma}$ to $\underline{\delta}$ that is not contained in the ball $B_r(\underline{\alpha}, \Lambda)$. From this, we may deduce there is no finite subgraph of Λ that one may remove to leave at least two disjoint infinite connected components. Whence, Λ has precisely one end.

We begin by observing that, given any two vertices $(\gamma_1, \dots, \gamma_m)$ and $(\delta_1, \dots, \delta_m)$ in $V\Lambda$, for any path in Γ between γ_1 and δ_1 there exists a corresponding path in Λ between $(\gamma_1, \gamma_2, \dots, \gamma_m)$ and $(\delta_1, \gamma_2, \dots, \gamma_m)$. Indeed, by Lemma 3.7, if ξ lies on the path in Γ between γ_1 and δ_1 , then $(\xi, \gamma_2, \dots, \gamma_m)$ lies on the corresponding path in Λ between $(\gamma_1, \gamma_2, \dots, \gamma_m)$ and $(\delta_1, \gamma_2, \dots, \gamma_m)$. This observation can also be made for paths between γ_i and δ_i for all i satisfying $1 \leq i \leq m$.

Fix $r \geq 1$ and two distinct vertices $\underline{\gamma} = (\gamma_1, \dots, \gamma_m)$ and $\underline{\delta} = (\delta_1, \dots, \delta_m)$ in $S_{r+1}(\underline{\alpha}, \Lambda)$. Let $B_r := B_r(\underline{\alpha}, \Lambda)$. We will describe four vertices $\underline{\xi}_1, \dots, \underline{\xi}_4 \in V\Lambda \setminus B_r$ such that there exist paths in Λ between $\underline{\gamma}$ and $\underline{\xi}_1$; between $\underline{\delta}$ and $\underline{\xi}_4$; and between $\underline{\xi}_i$ and $\underline{\xi}_{i+1}$ for $1 \leq i < 4$ that are all disjoint from B_r , thus showing there exists a path in $\Lambda \setminus B_r$ from $\underline{\gamma}$ to $\underline{\delta}$.

Let $d_1 := d_\Gamma(\alpha_1, \delta_1) - 1$ and $d_m := d_\Gamma(\alpha_m, \gamma_m) - 1$. Choose a vertex $\gamma'_m \in$

$V\Gamma \setminus B_r(\alpha_m, \Gamma)$ such that there is a path in Γ between γ_m and γ'_m that is disjoint from $B_{d_m}(\alpha_m, \Gamma)$. Similarly, choose a vertex $\delta'_1 \in V\Gamma \setminus B_r(\alpha_1, \Gamma)$ such that there is a path in Γ between δ_1 and δ'_1 that is disjoint from $B_{d_1}(\alpha_1, \Gamma)$. Now define

$$\underline{\xi}_1 := (\gamma_1, \gamma_2, \dots, \gamma_{m-1}, \gamma'_m);$$

$$\underline{\xi}_2 := (\delta_1, \delta_2, \dots, \delta_{m-1}, \gamma'_m);$$

$$\underline{\xi}_3 := (\delta'_1, \delta_2, \dots, \delta_{m-1}, \gamma'_m);$$

$$\underline{\xi}_4 := (\delta'_1, \delta_2, \dots, \delta_{m-1}, \delta_m).$$

The path in Γ between γ_m and γ'_m that is disjoint from $B_{d_m}(\alpha_m, \Gamma)$ corresponds to a path in Λ between $(\gamma_1, \dots, \gamma_{m-1}, \gamma_m)$ and $(\gamma_1, \dots, \gamma_{m-1}, \gamma'_m)$ which, by Lemma 3.8, is disjoint from B_r . Hence, there exists a path in $\Lambda \setminus B_r$ between $\underline{\gamma}$ and $\underline{\xi}_1$. A similar argument shows there exists a path in $\Lambda \setminus B_r$ between $\underline{\delta}$ and $\underline{\xi}_4$.

Finally, observe that any vertex $\underline{\chi} = (\chi_1, \dots, \chi_m) \in V\Lambda$ satisfying $\chi_i \in V\Gamma \setminus B_r(\alpha_i, \Gamma)$ for some i does not lie in B_r by Lemma 3.8. Therefore, there exist paths in $\Lambda \setminus B_r$ between $\underline{\xi}_1$ and $\underline{\xi}_2$; between $\underline{\xi}_2$ and $\underline{\xi}_3$; and between $\underline{\xi}_3$ and $\underline{\xi}_4$. Hence, Λ has precisely one end. \square

Thus the proof of Theorem 3.6 is complete.

Example 3.10. Let p be a prime number greater than 10^{10} and let T_p be a Tarski–Ol’shanskiĭ monster group of order p . Recall this group acts primitively on its coset space Ω . Let $G := T_p \operatorname{Wr} \operatorname{Sym}(2)$. This group acts primitively on the set Ω^2 by Theorem 3.6, and has precisely three non-trivial subdegrees, $2p, p^2$ and $2p^2$, each occurring infinitely often.

Example 3.11. The orbital graphs of the group $G(m, T_p) \operatorname{Wr} \operatorname{Sym}(2)$ have precisely one end. Furthermore, this group has infinitely many distinct subdegrees, each occurring infinitely often.

3.3 Enumeration and growth

3.3.1 The upper and lower subdegree sequences

As the above examples show, the concept of subdegree growth requires re-examining. We begin by clarifying some terminology. Let G act primitively on an infinite set Ω , possessing a finite suborbit whose pair is also finite. Then every suborbit of G is finite. Fix $\alpha \in \Omega$.

If S is a set, a *multiset* chosen from S is a function μ from S into the non-negative integers. Informally, one may think of a multiset chosen from S as being a set in which each element $s \in S$ occurs with multiplicity $\mu(s)$. This definition may be extended naturally to allow elements of s to occur infinitely often in the multiset, by defining the *extended non-negative integers* to be the set comprising the non-negative integers and the first infinite cardinal number \aleph_0 , and extending μ to be a function from S to the extended non-negative numbers.

The *set of subdegrees* of G is defined to be the set whose elements are the cardinalities of its α -suborbits. The *multiset of subdegrees* of G is a function μ from the set of subdegrees to the extended non-negative integers defined as follows. If m is any subdegree of G then $\mu(m)$ is the number of α -suborbits with cardinality m . Having defined this multiset formally, we shall speak of it informally as a set of elements of the set of subdegrees, in which some elements occur with multiplicity greater than one.

For our given group G , there exists a minimal ordinal number κ such that one may enumerate all elements of the multiset of subdegrees as a monotonic increasing sequence (m_γ) for $\gamma < \kappa$. This sequence (m_γ) is called the *subdegree sequence* of G , and the ordinal number κ is called the *height* of G . Despite our informal treatment of the multiset of subdegrees, the precise definition of the subdegree sequence of G is hopefully clear: (m_γ) is a non-decreasing sequence of elements of the set of subdegrees, in which each subdegree m appears in the sequence with multiplicity $\mu(m)$.

By Theorem 3.1, the set of subdegrees of G is a finite or countably infinite set. Since each entry occurs with multiplicity one, we may enumerate all its elements

$M_1 < M_2 < \dots$. The sequence (M_r) is called the *upper subdegree sequence* of G . The *lower subdegree sequence* of G is the sequence $(m_r)_{r < \omega}$. Note that both the upper and lower subdegree sequences are subsequences of the subdegree sequence of G , and are indexed by the natural numbers.

Lemma 3.12. *If G is a locally finite primitive group with height h then*

$$\omega \leq h \leq \omega^2.$$

Proof. Let X and Y be the set and multiset of subdegrees of G respectively. Since G acts on an infinite set, and all subdegrees of G are finite, Y is infinite, so the height of G is at least ω .

Each subdegree in X occurs with multiplicity at most \aleph_0 in Y , so the subdegree sequence can be enumerated in a non-decreasing sequence of length at most ω^2 . \square

Subdegree growth is similar in many ways to the growth of connected locally finite infinite graphs, an area of research that is already well-established. Following [33], if $t_1 \leq t_2 \leq t_3 \leq \dots$ is a sequence of positive real numbers, we define the concept of *growth* as follows. If there exist positive real numbers c_1, c_2 and $d \geq 1$ such that

$$c_1 r^d \leq t_r \leq c_2 r^d,$$

we say that the sequence has *polynomial growth of degree d* . A sequence exhibiting polynomial growth of degree 0 is often called *bounded*. The growth is *subexponential* if, for all $a > 1$,

$$\liminf_{r \rightarrow \infty} \frac{t_r}{a^r} = 0;$$

it is *exponential* if there exists a constant $a > 1$ such that both $\liminf_{r \rightarrow \infty} t_r/a^r$ and $\limsup_{r \rightarrow \infty} t_r/a^r$ are non-zero and finite. The growth is said to be *super exponential* if $\liminf_{r \rightarrow \infty} t_r/a^r$ is infinite for all $a > 1$.

The *growth* of an infinite locally finite vertex-transitive connected graph Γ is the growth of the sequence $(|B_r(\alpha, \Gamma)|)$. A great deal of work has been done in this area. It was shown in [12] that it is precisely the two-ended graphs that have growth of degree one. As we will see, such graphs cannot be primitive. Furthermore, Trofimov

has shown in [32] that if a locally finite graph has polynomial growth then it is not primitive. All locally finite infinitely-ended vertex-transitive graphs have exponential sphere growth, while it is noted in [33] that one-ended graphs exhibit all possible rates of growth of degree at least 2, except super exponential.

By examining the growth of both the lower and upper subdegree sequences, one may obtain similar results pertaining to subdegree growth rates. It is natural to consider the subdegree growth of primitive groups in this way. Indeed, the intuitive but flawed approach taken by Adeleke and Neumann in [1, Remark 29.8] only considered the existence of primitive groups with height ω . For such groups, the lower subdegree sequence is equal to the subdegree sequence (although it is not necessarily equal to the upper subdegree sequence), so the two approaches are equivalent in this case.

For groups with height ω , the growth of the upper subdegree sequence is of secondary importance, as all relevant subdegree information can be found in the lower subdegree sequence. However, this is not the case for groups with height strictly greater than ω .

Lemma 3.13. *If an infinite primitive group G whose subdegrees are all finite does not have height ω , then the lower subdegree sequence is always bounded.*

Proof. Suppose G has height $h > \omega$, and let (m_γ) be the subdegree sequence of G . Then there are finite constants $c_1 := 1$ and $c_2 := m_\omega$ such that $c_1 \leq m_r \leq c_2$ for all $r < \omega$. □

The growth of the upper subdegree sequence will be used only to differentiate between the subdegree growth of groups exhibiting a bounded lower subdegree sequence; thus, when referring to the subdegree growth of a group, unless otherwise stated, it is the lower subdegree growth to which we are referring.

The constructions detailed in this chapter may be used to create myriad examples of groups with exponential, subexponential and polynomial growth, several of which are given below. The final example illustrates how they may also be used to show that the above list of possible rates of growth is not exhaustive.

Example 3.14. Fix $m \geq 2$ and $t \geq 3$, and let $G := \text{Aut } \Gamma(m, K_{t+1})$. The group G acts distance-transitively on $V\Gamma$, and has height ω . The upper and lower subdegree sequences of G are equal, with $M_r = m_r = m(m-1)^{r-1}t^r$, for all $r \in \mathbb{N}$.

Example 3.15. If p is prime, and $p > 10^{10}$, then the group $G(m, T_p)$ has a bounded lower subdegree sequence, with $m_r = mp$ for all integers $r \geq 1$. However, $M_r = m(m-1)^{r-1}p^r$ for all $r \geq 1$. Intuitively, while this group exhibits slow subdegree growth when compared to groups with non-bounded lower subdegree sequences, when instead it is compared with other groups possessing bounded lower subdegree sequences its subdegree growth is extremely fast.

Example 3.16. Given a prime integer p with $p > 10^{10}$, both T_p and $T_p \text{ Wr Sym}(2)$ have bounded upper and lower subdegree sequences. The subdegree growth of these groups is therefore, intuitively, very slow. Indeed, T_p provides a lower bound on both upper and lower subdegree growth rates.

Example 3.17. Fix integers $m, t \geq 2$ and define $\Gamma := \Gamma(m, K_{t+1})$. Let $G := \text{Aut } \Gamma$ and let $H := G \text{ Wr Sym}(2)$. This group acts primitively on the set $\Omega := V\Gamma \times V\Gamma$, with all subdegrees finite. Choose vertices α and β adjacent in Γ and let Λ be the orbital graph $(\Omega, ((\alpha, \alpha), (\alpha, \beta))^H)$. We shall denote the number of $H_{(\alpha, \alpha)}$ -orbits in the sphere $S_r((\alpha, \alpha), \Lambda)$ by n_r , and the number of $H_{(\alpha, \alpha)}$ -orbits in the ball $B_r((\alpha, \alpha), \Lambda)$ by N_r .

Since G acts distance transitively on Γ , $n_r = \lceil r/2 \rceil$, where $\lceil r/2 \rceil$ denotes the smallest integer greater than or equal to $r/2$. Furthermore, the subdegrees of H in the sphere $S_r((\alpha, \alpha), \Lambda)$ are $2m(m-1)^{r-1}t^r$ and $2m^2(m-1)^{r-2}t^r$ for $r \geq 2$, and, if r is even, $m^2(m-1)^{r-2}t^r$.

The largest suborbit in the sphere $S_r((\alpha, \alpha), \Lambda)$ has size $2m^2(m-1)^{r-2}t^r$, and for all sufficiently large integers r ,

$$2m^2(m-1)^{r-2}t^r \leq (m-1)^{2r-2}t^{2r}.$$

Thus, there exists an integer R such that, for all $r > R$, the largest suborbit in $S_r((\alpha, \alpha), \Lambda)$ has cardinality strictly less than the cardinality of every suborbit in $\Lambda \setminus B_{2r}((\alpha, \alpha), \Lambda)$.

Now consider the lower subdegree sequence (m_r) of H which, in this case, is equal to the subdegree sequence of H . Choose $r > R$ and find the largest integer $s \geq 2$ such that

$$m_s = 2m^2(m-1)^{r-2}t^r.$$

Our aim is to find the number of subdegrees that are less than or equal to m_s , and from this determine s .

Since m_s is the largest suborbit in $B_r((\alpha, \alpha), \Lambda)$ we have $s \geq N_r$. Furthermore, m_s is strictly less than the cardinality of every (α, α) -suborbit in $\Lambda \setminus B_{2r}((\alpha, \alpha), \Lambda)$, so there can be at most N_{2r} suborbits with cardinality less than or equal to m_s . Hence $s \leq N_{2r}$, and therefore $N_r \leq s \leq N_{2r}$. Since $N_r = \sum_{i=1}^r n_i$, we have $N_r \geq r(r+1)/4$ and $N_r \leq r(r+1)/2$. Thus

$$r(r+1)/4 \leq s \leq r(2r+1).$$

If the lower subdegree sequence of H exhibits polynomial growth of degree $d \geq 1$ then there exist positive real numbers c_1, c_2 such that for all $k \geq 1$,

$$c_1 k^d \leq m_k \leq c_2 k^d.$$

However, for all $r > R$ we may choose a maximal integer s with $m_s = 2m^2(m-1)^{r-2}t^r$. For all sufficiently large r we therefore have $s^d \leq r^d(2r+1)^d \leq 2m^{2d}(m-1)^{d(r-2)}t^{dr}$, so the lower subdegree growth rate of H is not polynomial.

Although the growth of the lower subdegree sequence of H is faster than polynomial growth, it is not exponential. Indeed, for all $r > R$ there exists an integer s_r with $m_{s_r} = 2m^2(m-1)^{r-2}t^r$ and $s_r \geq r(r+1)/4$. Hence, for all $a > 1$,

$$\lim_{r \rightarrow \infty} \frac{m_{s_r}}{a^{s_r}} \leq \frac{2m^2(m-1)^{r-2}t^r}{a^{r(r+1)/4}} = 0.$$

The group H is thus an example of a group exhibiting subexponential, non-polynomial growth. The existence of such a group demonstrates that the list of possible growth rates given previously is not exhaustive. On reflection, at some future date, it may prove necessary to refine the classification of possible rates of growth in light of this; for our needs, however, the large classes of exponential, subexponential and polynomial growth suffice.

3.3.2 The average subdegree sequence

Let G act primitively on an infinite set Ω , and suppose every subdegree of G is finite. A natural, but flawed, alternative method for enumerating the subdegrees of G is as follows. Fix an orbital graph Γ of G , and a vertex $\alpha \in \Omega$. Let $b_r(\alpha, \Gamma)$ be the ball-size $|B_r(\alpha, \Gamma)|$, and let $N_r(\alpha, \Gamma)$ be the number of suborbits of G_α in $B_r(\alpha, \Gamma)$. When it is clear from the context, we shall suppress the use of α and Γ . The sequence (b_r/N_r) is then the *average subdegree sequence* of G with respect to the graph Γ . The growth of this sequence can be used as a measure of the subdegree growth of G .

The usefulness of average subdegree growth as a measure of subdegree growth is limited by its dependence on the orbital graph chosen. It is possible, however, to determine the extent of this dependence.

For a real number ξ let $\lfloor \xi \rfloor$ denote the largest integer less than or equal to ξ .

Lemma 3.18. *Let G be a primitive group of permutations of an infinite set Ω , with locally finite orbital graphs Γ_1 and Γ_2 . If $(b_r(\Gamma_1)/N_r(\Gamma_1))$ and $(b_r(\Gamma_2)/N_r(\Gamma_2))$ are the average subdegree sequences with respect to Γ_1 and Γ_2 respectively, then there exists a constant $a > 0$ such that, for all integers $r \geq 1$,*

$$\frac{b_{\lfloor r/a \rfloor}(\Gamma_2)}{N_{ar}(\Gamma_2)} \leq \frac{b_r(\Gamma_1)}{N_r(\Gamma_1)} \leq \frac{b_{ar}(\Gamma_2)}{N_{\lfloor r/a \rfloor}(\Gamma_2)}.$$

Proof. In the proof of Lemma 2.6, it was shown there exists a constant $a > 0$ such that

$$(1/a)d_{\Gamma_1}(\alpha, \beta) \leq d_{\Gamma_2}(\alpha, \beta) \leq a.d_{\Gamma_1}(\alpha, \beta)$$

for all $\alpha, \beta \in \Omega$. Observe that if $\beta \in B_r(\alpha, \Gamma_1)$ then $d_{\Gamma_1}(\alpha, \beta) \leq r$, and so $d_{\Gamma_2}(\alpha, \beta) \leq ar$ and $\beta \in B_{ar}(\alpha, \Gamma_2)$.

Similarly, if $\beta \in B_{\lfloor r/a \rfloor}(\alpha, \Gamma_2)$ then $d_{\Gamma_2}(\alpha, \beta) \leq \lfloor r/a \rfloor \leq r/a$; thus $(1/a)d_{\Gamma_1}(\alpha, \beta) \leq r/a$ and $d_{\Gamma_1}(\alpha, \beta) \leq r$. Hence

$$B_{\lfloor r/a \rfloor}(\alpha, \Gamma_2) \subseteq B_r(\alpha, \Gamma_1) \subseteq B_{ar}(\alpha, \Gamma_2).$$

Thus

$$N_{\lfloor r/a \rfloor}(\Gamma_2) \subseteq N_r(\Gamma_1) \subseteq N_{ar}(\Gamma_2)$$

and

$$b_{\lfloor r/a \rfloor}(\Gamma_2) \leq b_r(\Gamma_1) \leq b_{ar}(\Gamma_2). \quad \square$$

It would be interesting to know if it is possible to improve this result. If bounds can be found that are independent of r , the average subdegree growth would be essentially independent of the orbital graph chosen.

A further, and more serious, limit on the usefulness of average subdegree growth rates is their lack of subtlety. As an illustration, suppose we are given integers $m, t \geq 2$. The groups $G := \text{Aut } \Gamma(m, K_{t+1})$ and $H := G \text{ Wr Sym}(2)$ acting on $\Omega := V\Gamma(m, K_{t+1})$ and $\Omega \times \Omega$ respectively are manifestly dissimilar; all non-diagonal orbital graphs of the former have infinitely many ends; all non-diagonal orbital graphs of the latter have just one. Their many differences are encapsulated by their lower subdegree growth rates, with G exhibiting exponential growth, and H subexponential non-polynomial growth. However, if one were instead to compare the average subdegree growth rates the groups, no such distinction is possible.

Indeed, define $\Gamma := \Gamma(m, K_{t+1})$ and $\Lambda := (\Omega^2, ((\alpha, \alpha), (\alpha, \beta))^H)$, where α and β are adjacent in Γ . The average subdegree growth rate of G with respect to Γ is exponential, with

$$\begin{aligned} \lim_{r \rightarrow \infty} \left(\frac{|B_r(\alpha, \Gamma)|}{N_r(\alpha, \Gamma)} \right)^{1/r} &\geq \left(\frac{m(m-1)^{r-1}t^r}{r} \right)^{1/r} \\ &= (m-1)t. \end{aligned}$$

Furthermore, $|B_r(\alpha, \Gamma)| \leq r|S_r(\alpha, \Gamma)|$, so the limit is equal to $(m-1)t$.

By Lemma 3.8,

$$|S_r((\alpha, \alpha), \Lambda)| = \sum_{k=0}^r |S_k(\alpha, \Gamma)| |S_{r-k}(\alpha, \Gamma)|,$$

so

$$|B_r((\alpha, \alpha), \Lambda)| \leq \frac{m(r+1)|B_r(\alpha, \Gamma)|}{(m-1)}.$$

Hence $|B_r(\alpha, \Gamma)| \leq |B_r((\alpha, \alpha), \Lambda)| \leq K(r+1)|B_r(\alpha, \Gamma)|$, where $K := m/(m-1)$.

Two vertices $(\gamma_1, \gamma_2), (\delta_1, \delta_2) \in V\Lambda$ lie in the same $H_{(\alpha, \alpha)}$ orbit if and only if the sets $\{d_\Gamma(\alpha, \gamma_1), d_\Gamma(\alpha, \gamma_2)\}$ and $\{d_\Gamma(\alpha, \delta_1), d_\Gamma(\alpha, \delta_2)\}$ are equal. Thus if n_r denotes the

number of (α, α) – *suborbits* in $S_r((\alpha, \alpha), \Lambda)$ then $1 \leq n_r \leq r$, so

$$r \leq N_r((\alpha, \alpha), \Lambda) \leq r^2.$$

Whence the average subdegree growth of H is exponential, with

$$\lim_{r \rightarrow \infty} \left(\frac{|B_r(\alpha, \Gamma)|}{N_r(\alpha, \Gamma)} \right)^{1/r} = (m-1)t.$$

In light of these problems we will henceforth focus only on the relationship between the lower and upper subdegree growth rates of a group, and its structure.

Chapter 4

A complete characterisation of the primitive directed graphs with connectivity one

The primitive undirected graphs with connectivity one were classified by Jung and Watkins in [14], wherein they show the blocks of such graphs are primitive, pairwise-isomorphic and have at least three vertices. When one considers the general case of a directed primitive graph with connectivity one, however, this result no longer holds. In this chapter we address this problem, and obtain a complete classification of the graphs in question.

4.1 Local structure

Let Γ be a primitive graph with connectivity one whose blocks have at least three vertices, and suppose G is a vertex- and edge-transitive group of automorphisms of Γ . Since Γ is a vertex-transitive graph with connectivity one, every vertex is a cut vertex. Fix some block Λ of Γ , and let H be the subgroup of the automorphism group $\text{Aut } \Lambda$ induced by the setwise stabiliser $G_{\{\Lambda\}}$ of $V\Lambda$ in G . Let T be the block-cut-vertex tree of Γ , and let x be the vertex of T that corresponds to the block Λ . Our aim in this section is to show H is primitive but not regular.

If x_1 and x_2 are distinct vertices of the tree T , we use $C(T \setminus \{x_1\}, x_2)$ to denote the connected component of $T \setminus \{x_1\}$ that contains the vertex x_2 .

Lemma 4.1. *If G acts primitively on the vertices of Γ , then the group H acts primitively on the vertices of Λ .*

Proof. Fix $\alpha \in V\Lambda$ and suppose, for a contradiction, the group H does not act primitively on $V\Lambda$. Then there is a vertex $\gamma \in V\Lambda \setminus \{\alpha\}$ such that the graph $\Lambda' := (V\Lambda, (\alpha, \gamma)^H)$ is not connected. Let $\Gamma' := (V\Gamma, (\alpha, \gamma)^G)$. We will show that this graph cannot be connected, and hence that G cannot be primitive.

Let $\{\Delta_i\}_{i \in I}$ be the set of connected components of Λ' and let

$$\mathcal{C}_i := \bigcup_{\delta \in \Delta_i} C(T \setminus \{x\}, \delta) \cap V\Gamma.$$

Suppose $\delta_i \in \mathcal{C}_i$ and $\delta_j \in \mathcal{C}_j$, with $i \neq j$. We claim δ_i and δ_j are not adjacent in Γ' . Indeed, since the distance $d_T(\alpha, \gamma)$ between α and γ in T is equal to 2, if δ_i and δ_j are to be adjacent in Γ' , it must be the case that $d_T(\delta_i, \delta_j) = 2$. If either δ_i or δ_j is not adjacent to x in T then $d_T(\delta_i, \delta_j) > 2$, so they cannot be adjacent in Γ' . If on the other hand δ_i and δ_j are adjacent to x in T , they both lie in $V\Lambda = V\Lambda'$, and therefore $\delta_i \in \Delta_i$ and $\delta_j \in \Delta_j$. In this case, if they are adjacent in Γ' then there exists $g \in G$ such that either (δ_i, δ_j) or (δ_j, δ_i) is equal to $(\alpha, \gamma)^g$. Such an automorphism must fix $V\Lambda$ setwise, and therefore lies in $G_{\{\Lambda\}}$. Thus, there exists an element $h \in H$ such that either (δ_i, δ_j) or (δ_j, δ_i) is equal to $(\alpha, \gamma)^h$, meaning that δ_i and δ_j are adjacent in Λ' , which contradicts the fact that δ_i and δ_j are in distinct components of Λ' . Hence, δ_i and δ_j are not adjacent in Γ' .

Therefore, there can be no path in Γ' between a vertex in \mathcal{C}_i and a vertex in \mathcal{C}_j whenever $i \neq j$, and so the graph Γ' is not connected. \square

Fix distinct vertices $\alpha, \beta \in V\Gamma$. Recall $[\alpha, \beta]_T$ is the T -geodesic between α and β , while $(\alpha, \beta)_T$ is the T -geodesic $[\alpha, \beta]_T$ excluding both α and β . Since $d_T(\alpha, \beta)$ is even we may choose a vertex $y \in (\alpha, \beta)_T$ that is distinct from α and β .

Lemma 4.2. *If $g \in G_\alpha$ does not fix $y \in VT$, and $\delta \notin C(T \setminus \{y\}, \alpha)$, then $\delta^g \notin C(T \setminus \{y\}, \beta)$. Similarly, if $g \in G_\beta$ does not fix y and $\delta \notin C(T \setminus \{y\}, \beta)$ then $\delta^g \notin C(T \setminus \{y\}, \alpha)$.*

Proof. If $\delta \notin C(T \setminus \{y\}, \alpha)$ and $\delta^g \in C(T \setminus \{y\}, \beta)$ then $\delta, \delta^g \notin C(T \setminus \{y\}, \alpha)$, so we must have $g \in G_{\alpha,y}$. Similarly, if $\delta \notin C(T \setminus \{y\}, \beta)$ and $\delta^g \in C(T \setminus \{y\}, \alpha)$ then $\delta, \delta^g \notin C(T \setminus \{y\}, \beta)$, so we must have $g \in G_{\beta,y}$. \square

Lemma 4.3. *If $g \in G_\alpha$ does not fix the vertex y and $\delta \notin C(T \setminus \{y\}, \alpha)$ then $d_T(y, \delta^g) > d_T(y, \delta)$. Similarly, if $g \in G_\beta$ does not fix y and $\delta \notin C(T \setminus \{y\}, \beta)$ then $d_T(y, \delta^g) > d_T(y, \delta)$.*

Proof. Let y' be the vertex adjacent to y in $[\alpha, y]_T$. If $\delta \notin C(T \setminus \{y\}, \alpha)$ then $y \in [\alpha, \delta]_T$. Since $g \in G_\alpha \setminus G_y$, both y and y' lie on the geodesic $[\delta, \delta^g]_T$, with $y' \in [\delta^g, y]_T$. Thus $d_T(\delta^g, y) = d_T(\delta^g, y') + d_T(y', y)$. Now $d_T(\delta^g, y') \geq d_T(\delta, y) + d_T(y, y')$, so $d_T(\delta^g, y) \geq d_T(\delta, y) + 1 > d_T(\delta, y)$. Interchanging α and β in the above argument completes the proof of this lemma. \square

Lemma 4.4. *Let $g_1, \dots, g_n \in G_\alpha$ and $h_1, \dots, h_n \in G_\beta$, and suppose $G_{\alpha,y} = G_{\beta,y}$. If there exists $\gamma \in VT$ such that $G_{\alpha,y} \leq G_\gamma$ then, for some $m \leq n$, there exist $g'_2, \dots, g'_m \in G_\alpha \setminus G_y$ and $g'_1 \in G_\alpha \setminus G_y \cup \{1\}$ together with $h'_1, \dots, h'_{m-1} \in G_\beta \setminus G_y$ and $h'_m \in G_\beta \setminus G_y \cup \{1\}$ such that*

$$\gamma^{g'_1 h'_1 \dots g'_m h'_m} = \gamma^{g_1 h_1 \dots g_n h_n}.$$

Proof. The proof of this lemma will be an inductive argument. Suppose there exists $\gamma \in VT$ such that $G_{\alpha,y} \leq G_\gamma$.

Let $n = 1$. When considering $h_1 \in G_\beta$ we have two cases: either $h_1 \in G_y$ or $h_1 \in G_\beta \setminus G_y$. If $h_1 \in G_y$ then $h_1 \in G_{\beta,y} = G_{\alpha,y}$, so $g_1 h_1 \in G_\alpha$. In this case, redefine $g_1 := g_1 h_1$ and set $h'_1 := 1$. Alternatively, if $h_1 \in G_\beta \setminus G_y$ then set $h'_1 := h_1$. Having found a suitable h'_1 , we will now construct g'_1 from the (possibly redefined) element $g_1 \in G_\alpha$. We again have two cases: either $g_1 \in G_y$ or $g_1 \in G_\alpha \setminus G_y$. If $g_1 \in G_y$ then $g_1 \in G_{\alpha,y}$ and so $g_1 \in G_\gamma$. In this case, we can choose $g'_1 := 1$. Otherwise, if

$g_1 \in G_\alpha \setminus G_y$, then choose $g'_1 := g_1$. In choosing g'_1 and h'_1 in this way we ensure

$$\gamma^{g_1 h_1} = \gamma^{g'_1 h'_1},$$

so the hypothesis holds when $n = 1$.

Let k be a positive integer, and suppose the hypothesis is true for all integers $n \leq k$. Fix $g_1, \dots, g_{k+1} \in G_\alpha$ and $h_1, \dots, h_{k+1} \in G_\beta$, and set $\gamma' := \gamma^{g_1 h_1 \dots g_{k+1} h_{k+1}}$. We will use induction to construct elements $g'_2, \dots, g'_m \in G_\alpha \setminus G_y$ and $g'_1 \in G_\alpha \setminus G_y \cup \{1\}$ together with $h'_1, \dots, h'_{m-1} \in G_\beta \setminus G_y$ and $h'_m \in G_\beta \setminus G_y \cup \{1\}$ such that

$$\gamma^{g'_1 h'_1 \dots g'_m h'_m} = \gamma',$$

where m is some integer less than or equal to $k + 1$.

We begin by considering $h_{k+1} \in G_\beta$. There are two cases: either $h_{k+1} \in G_y$ or $h_{k+1} \in G_\beta \setminus G_y$. If $h_{k+1} \in G_y$ then $h_{k+1} \in G_{\beta,y} = G_{\alpha,y}$, so $g_{k+1} h_{k+1} \in G_\alpha$. In this case, redefine $g_{k+1} := g_{k+1} h_{k+1}$ and set $h' := 1$. If, on the other hand, $h_{k+1} \in G_\beta \setminus G_y$, then set $h' := h_{k+1}$.

If we now consider the (possibly redefined) element $g_{k+1} \in G_\alpha$, there are again two cases: either $g_{k+1} \in G_y$, or $g_{k+1} \in G_\alpha \setminus G_y$. If $g_{k+1} \in G_y$ then $g_{k+1} \in G_{\alpha,y} = G_{\beta,y}$, so $h_k g_{k+1} h' \in G_\beta$. In this case, redefine $h_k := h_k g_{k+1} h'$; then

$$\gamma' = \gamma^{g_1 h_1 \dots g_k h_k},$$

so we can apply the induction hypothesis to $\gamma^{g_1 h_1 \dots g_k h_k}$ and we are done. If, on the other hand, $g_{k+1} \in G_\alpha \setminus G_y$, then set $g' := g_{k+1}$, and observe

$$\gamma' = \gamma^{g_1 h_1 \dots g_k h_k g' h'}.$$

By the induction hypothesis, for some $m \leq k$ there exist $g'_2, \dots, g'_m \in G_\alpha \setminus G_y$ and $g'_1 \in G_\alpha \setminus G_y \cup \{1\}$ together with $h'_1, \dots, h'_{m-1} \in G_\beta \setminus G_y$ and $h'_m \in G_\beta \setminus G_y \cup \{1\}$ such that

$$\gamma^{g_1 h_1 \dots g_k h_k} = \gamma^{g'_1 h'_1 \dots g'_m h'_m}.$$

Set $g'_{m+1} := g' \in G_\alpha \setminus G_y$ and $h'_{m+1} := h' \in G_\beta \setminus G_y \cup \{1\}$. Then

$$\gamma^{g'_1 h'_1 \dots g'_{m+1} h'_{m+1}} = \gamma',$$

so the hypothesis holds for $n = k + 1$. □

We are now in a position to present the main result of this section.

Theorem 4.5. *Let G be a vertex-transitive group of automorphisms of a connectivity-one graph Γ whose blocks have at least three vertices, and let T be the block-cut-vertex tree of Γ . If there exist distinct vertices $\alpha, \beta \in V\Gamma$ such that, for some vertices $\alpha', \beta' \in (\alpha, \beta)_T$,*

$$(i) [\alpha, \alpha']_T \cap (\beta', \beta]_T = \emptyset; \text{ and}$$

$$(ii) G_{\alpha, \alpha'} = G_{\beta, \beta'};$$

then G does not act primitively on $V\Gamma$.

Proof. Suppose G acts primitively on $V\Gamma$ and there exist distinct vertices $\alpha, \beta \in V\Gamma$ with $\alpha', \beta' \in (\alpha, \beta)_T$ such that (i) and (ii) hold. We will show the group $\langle G_\alpha, G_\beta \rangle$ generated by G_α and G_β is not transitive on $V\Gamma$. Then $G_\alpha < \langle G_\alpha, G_\beta \rangle < G$, which will contradict the assumption that G is primitive.

Choose $y \in [\alpha', \beta']_T$, and observe that by (ii) we have $G_{\alpha, y} = G_{\beta, y}$. Without loss of generality, suppose $d_T(y, \alpha) \leq d_T(y, \beta)$. As G acts primitively on $V\Gamma$ the orbit $\beta^{\langle G_\alpha, G_\beta \rangle}$ contains α . Therefore, there exist elements $g_1, \dots, g_n \in G_\alpha$ and $h_1, \dots, h_n \in G_\beta$ such that $\alpha = \beta^{g_1 h_1 \dots g_n h_n}$. By Lemma 4.4, we can find $g'_2, \dots, g'_m \in G_\alpha \setminus G_y$ and $g'_1 \in G_\alpha \setminus G_y \cup \{1\}$ together with $h'_1, \dots, h'_{m-1} \in G_\beta \setminus G_y$ and $h'_m \in G_\beta \setminus G_y \cup \{1\}$ such that

$$\alpha = \beta^{g'_1 h'_1 \dots g'_m h'_m}.$$

Suppose these automorphisms are chosen so that m is minimal.

Now either $g'_1 \in G_\alpha \setminus G_y$ or $g'_1 = 1$. If $g'_1 = 1$ then $\beta^{g'_1} = \beta$ and therefore $\beta^{g'_1 h'_1} = \beta$. Thus $\beta^{g'_2 h'_2 \dots g'_m h'_m} = \alpha$, contradicting the minimality of m . So we must have $g'_1 \in G_\alpha \setminus G_y$. Since $\beta \notin C(T \setminus \{y\}, \alpha)$, we may apply Lemma 4.2 and Lemma 4.3 to obtain $d_T(y, \beta^{g'_1}) > d_T(y, \beta)$ and $\beta^{g'_1} \notin C(T \setminus \{y\}, \beta)$.

We now observe $h'_1 \neq 1$. Indeed, if $h'_1 = 1$ then $m = 1$ and $\alpha = \beta^{g'_1}$; since $g'_1 \in G_\alpha$ this is clearly not possible.

Thus, $h'_1 \in G_\beta \setminus G_y$ and $\beta^{g'_1} \notin C(T \setminus \{y\}, \beta)$, and we can again deduce from Lemma 4.2 and Lemma 4.3 that $d_T(y, \beta^{g'_1 h'_1}) > d_T(y, \beta^{g'_1}) > d_T(y, \beta)$, and $\beta^{g'_1 h'_1} \notin C(T \setminus \{y\}, \alpha)$.

We may continue to apply Lemma 4.2 and Lemma 4.3 to obtain $\beta^{g'_1 h'_1 \dots g'_m} \notin C(T \setminus \{y\}, \beta)$ and $d_T(y, \beta^{g'_1 h'_1 \dots g'_m}) > d_T(y, \beta)$. Now either $h'_m \in G_\beta \setminus G_y$ or $h'_m = 1$. If $h'_m = 1$ then $\alpha = \beta^{g'_1 h'_1 \dots g'_m}$, so $d_T(y, \alpha) = d_T(y, \beta^{g'_1 h'_1 \dots g'_m}) > d_T(y, \beta)$. If $h'_m \in G_\beta \setminus G_y$ then, by Lemma 4.3, $d_T(y, \beta^{g'_1 h'_1 \dots g'_m h'_m}) > d_T(y, \beta)$; that is, $d_T(y, \alpha) > d_T(y, \beta)$. Thus, in both cases $d_T(y, \alpha) > d_T(y, \beta)$. This contradicts our assumption that $d_T(y, \alpha) \leq d_T(y, \beta)$. Hence $\alpha \notin \beta^{G_\alpha, G_\beta}$, and so $\langle G_\alpha, G_\beta \rangle$ cannot act transitively on the set $V\Gamma$. \square

Theorem 4.6. *Let G be a vertex-transitive group of automorphisms of a connectivity-one graph Γ whose blocks have at least three vertices. If G acts primitively on $V\Gamma$ and Λ is some block of Γ then $G_{\{\Lambda\}}$ is primitive and not regular on $V\Lambda$.*

Proof. Suppose Λ is a block of Γ and $G_{\{\Lambda\}}$ acts primitively and regularly on $V\Lambda$. If T is the block-cut-vertex tree of Γ then there exists a vertex $x \in VT$ corresponding to the block Λ . Choose distinct vertices α and β in $V\Lambda$, and observe $G_{\alpha, x} = G_{\alpha, \{\Lambda\}} \leq G_\beta$ and $G_{\beta, x} = G_{\beta, \{\Lambda\}} \leq G_\alpha$. Furthermore, $x \in (\alpha, \beta)_T$; hence G cannot act primitively on $V\Gamma$ by Theorem 4.5. \square

4.2 Global structure

In this section we will employ Theorem 4.6 to give a complete characterisation of the primitive connectivity-one directed graphs.

Lemma 4.7. *Suppose Γ is a vertex-transitive graph with connectivity one, whose blocks are vertex-transitive, have at least three vertices and are pairwise isomorphic. If Λ is a block of Γ and H is the subgroup of $\text{Aut } \Lambda$ induced by the action of $(\text{Aut } \Gamma)_{\{\Lambda\}}$ on Λ , then $H = \text{Aut } \Lambda$.*

Proof. Let T denote the block-cut-vertex tree of Γ , and let Λ be a block of Γ . We will show any automorphism of the directed graph Λ may be extended to an automorphism of Γ .

We begin by asserting that if Λ_1 and Λ_2 are blocks of Γ , and α_1 and α_2 are vertices in Λ_1 and Λ_2 respectively, then there exists an isomorphism $\rho : \Lambda_1 \rightarrow \Lambda_2$ such that $\alpha_1^\rho = \alpha_2$. Indeed, by assumption, there exists an isomorphism $\rho' : \Lambda_1 \rightarrow \Lambda_2$. Define

$\alpha'_1 := \alpha_1^{\rho'}$. Since the block Λ_2 is vertex-transitive, there exists an automorphism τ of Λ_2 such that $\alpha_1'^{\tau} = \alpha_2$. Let $\rho := \rho'\tau$. Then $\rho : \Lambda_1 \rightarrow \Lambda_2$ is an isomorphism, with $\alpha_1^{\rho} = \alpha_1^{\rho'\tau} = \alpha_1'^{\tau} = \alpha_2$.

Let x be the vertex of T that corresponds to Λ . For $k \geq 0$, define Γ_k to be the subgraph of Γ induced by the set $\{\alpha \in V\Gamma \mid d_T(x, \alpha) \leq 2k + 1\}$. We will show any automorphism $\sigma_k : \Gamma_k \rightarrow \Gamma_k$ admits an extension $\sigma_{k+1} : \Gamma_{k+1} \rightarrow \Gamma_{k+1}$. Whence, by induction, the lemma will follow.

Fix $k \geq 0$ and let $\sigma_k : \Gamma_k \rightarrow \Gamma_k$ be an automorphism. Let $\{\alpha_i\}_{i \in I}$ be the set of vertices in $V\Gamma_k \setminus V\Gamma_{k-1}$ (where $V\Gamma_{-1} := \emptyset$). Each vertex α_i belongs to a unique block Λ_i of Γ_k , and, if $k \geq 1$, the block Λ_i possesses exactly one vertex in Γ_{k-1} . Since Γ is vertex transitive, any two vertices lie in the same number of blocks of Γ , so let $\{\Lambda_{i,j}\}_{j \in J}$ be the set of blocks of Γ that contain α_i and are distinct from Λ_i . Each block $\Lambda_{i,j}$ is wholly contained in Γ_{k+1} and has exactly one vertex in Γ_k , namely α_i . If $i \in I$, set $\alpha'_i := \alpha_i^{\sigma_k}$ and $\Lambda'_i := \Lambda_i^{\sigma_k}$. Then $\Lambda'_i = \Lambda_{i'}$ for some $i' \in I$. For all $j \in J$ there exists an isomorphism $\rho_{i,j} : \Lambda_{i,j} \rightarrow \Lambda_{i',j}$ such that $\alpha_i^{\rho_{i,j}} = \alpha'_{i'}$. Thus, we may define a mapping $\sigma_{k+1} : \Gamma_{k+1} \rightarrow \Gamma_{k+1}$ with

$$\beta^{\sigma_{k+1}} := \begin{cases} \beta^{\sigma_k} & \text{if } \beta \in V\Gamma_k; \\ \beta^{\rho_{i,j}} & \text{if } \beta \in V\Lambda_{i,j}. \end{cases}$$

This is clearly a well-defined automorphism of Γ_{k+1} . □

The primitive undirected graphs with connectivity one have the following complete characterisation.

Theorem 4.8. ([14, Theorem 4.2]) *If Γ is a vertex-transitive undirected graph with connectivity one, then it is primitive if and only if the blocks of Γ are primitive, pairwise isomorphic and each has at least three vertices.* □

This useful result seems to suggest a similar characterisation may be possible for directed primitive graphs with connectivity one. This is indeed the case.

Theorem 4.9. *If Γ is a vertex-transitive directed graph with connectivity one, then it is primitive if and only if the blocks of Γ are primitive but not automorphism-regular, pairwise isomorphic and each has at least three vertices.*

Proof. Let Γ be a directed vertex-transitive graph with connectivity one. Suppose the blocks of Γ are primitive but not automorphism-regular, pairwise isomorphic and each has at least three vertices. Let \approx be a non-trivial $\text{Aut } \Gamma$ -congruence on $V\Gamma$. We will show this relation must be universal, and thus that Γ is a primitive graph. Since the relation is non-trivial, there exist distinct vertices $\alpha, \beta \in V\Gamma$ such that $\alpha \approx \beta$. Let T be the block-cut-vertex tree of Γ , let $\gamma \in V\Gamma$ be the vertex in the geodesic $[\alpha, \beta]_T$ such that $d_T(\beta, \gamma) = 2$, and let Λ be the block of Γ containing β and γ . By Lemma 4.7, the group $(\text{Aut } \Gamma)_{\{\Lambda\}}$ acts primitively but not regularly on $V\Lambda$. Thus, there exists an automorphism $g \in (\text{Aut } \Gamma)_{\gamma, \{\Lambda\}}$ that does not fix β . We are considering the full automorphism group of the connectivity-one graph Γ , so there must therefore exist an element $g' \in (\text{Aut } \Gamma)_{\alpha, \gamma, \{\Lambda\}}$ that does not fix β . Thus, β and $\beta^{g'}$ are distinct vertices in Λ . Now $\alpha \approx \beta$, so $\alpha \approx \beta^{g'}$, and therefore $\beta \approx \beta^{g'}$. Since $(\text{Aut } \Gamma)_{\{\Lambda\}}$ is primitive on $V\Lambda$ and \approx induces a non-trivial $(\text{Aut } \Gamma)_{\{\Lambda\}}$ -congruence on $V\Lambda$, this relation must be universal in Λ . By assumption, $\text{Aut } \Gamma$ acts transitively on the blocks of Γ , so if two vertices lie in the same block then they must lie in the same congruence class. Thus, if γ is any vertex of Γ , and $\alpha x_1 \alpha_1 x_2 \dots x_n \gamma$ is the geodesic in T between α and γ , then α and α_1 lie in a common block, so $\alpha \approx \alpha_1$. Similarly, $\alpha_1 \approx \alpha_2$ and $\alpha_2 \approx \alpha_3$, so $\alpha \approx \alpha_2$ and $\alpha \approx \alpha_3$. Continuing in this way we eventually obtain $\alpha \approx \gamma$. Hence, this congruence relation is universal on $V\Gamma$.

Conversely, suppose the group $\text{Aut } \Gamma$ acts primitively on $V\Gamma$. Since Γ is a directed primitive graph with connectivity one, we can obtain an undirected graph Γ' with vertex set $V\Gamma$ and edge set $\{\{\alpha, \beta\} \mid (\alpha, \beta) \in E\Gamma\}$. Two vertices are adjacent in Γ if and only if they are adjacent in Γ' . As $\text{Aut } \Gamma$ is primitive on $V\Gamma$ and $\text{Aut } \Gamma \leq \text{Aut } \Gamma'$, it follows that $\text{Aut } \Gamma'$ must be primitive on $V\Gamma$, and hence that Γ' is a primitive undirected graph. Since Γ has connectivity one, the same is true of Γ' , so we may apply Theorem 4.8 to deduce the blocks of Γ' are primitive, pairwise isomorphic and each has at least three vertices. Now, given a block Λ of Γ , there is a block Λ' of Γ' such that $V\Lambda = V\Lambda'$. Therefore, the blocks of Γ have at least three vertices, and are primitive but not automorphism-regular by Theorem 4.6. It remains to show they are pairwise isomorphic. Fix some block Λ of Γ and an edge $(\alpha, \beta) \in E\Lambda$. Let Γ_1 be

the orbital graph $(V\Gamma, (\alpha, \beta)^{\text{Aut } \Gamma})$. As $\text{Aut } \Gamma$ is primitive, this graph is a connected subgraph of Γ . Thus, every block of Γ must contain an edge in $E\Gamma_1$. Furthermore, if Λ' is a block of Γ , then any automorphism of Γ mapping the edge (α, β) to an edge in Λ' must map Λ to Λ' . Since Γ_1 is edge-transitive, the blocks of Γ must be pairwise isomorphic. \square

Chapter 5

Orbital graphs of primitive groups

To a permutation group theorist, orbital graphs offer a pictorial representation of a group's action on a set, often bestowing a vantage point from which many properties of the group may be completely determined. As such, it is natural to wish to obtain a class of graphs whose structure is well known, in which there lies an orbital graph of each locally finite infinite primitive group. When considering groups with more than one permutation-end, we will show this is indeed possible; the question as to whether this can be achieved for groups with just one permutation-end remains open.

Much of what is currently known about graphs with more than one end is underpinned by the work of Dicks and Dunwoody in their seminal text [4]. An accessible summary of their results pertaining to locally finite graphs can be found in [22].

5.1 The canonical orbital graph

In [24], Möller shows every connected locally finite primitive undirected graph with more than one end is closely related to one with connectivity one.

Theorem 5.1, ([22, Theorem 15]) *If Γ is a connected locally finite primitive undirected graph with more than one end then there exist vertices $\alpha, \beta \in V\Gamma$ such that the undirected graph $(V\Gamma, \{\alpha, \beta\}^{\text{Aut } \Gamma})$ has connectivity one and each block has at most one end.* □

Since any infinite vertex-transitive connectivity-one graph has infinitely many ends, the following is immediate from Theorem 2.4, Lemma 2.6 and Theorem 5.1.

Corollary 5.2. *If Γ is a locally finite primitive edge-transitive graph then Γ has 0, 1 or 2^{\aleph_0} ends.* \square

Recall that the set of permutation-ends of a locally finite primitive group is equal to the set of ends of any one of its orbital graphs. A consequence of the above corollary therefore, is that locally finite infinite primitive groups either have precisely one permutation-end, or they have uncountably many.

We can use Theorem 5.1 to tell us a great deal about the orbital graphs of infinite primitive permutation groups.

Theorem 5.3. *If G is a primitive group of permutations of an infinite set Ω then G has a locally finite orbital graph with more than one end if and only if G has a locally finite orbital graph with connectivity one such that each block has at most one end.*

The proof of this theorem relies on the following lemmas. Let G be a primitive group of permutations of an infinite set Ω .

Lemma 5.4. *If G has a locally finite orbital graph with more than one end then G has an orbital graph with connectivity one.*

Proof. Suppose G has a locally finite orbital graph Γ with more than one end. Let Γ' be the undirected graph whose vertex set is Ω , and whose edge set is $\{\{\alpha, \beta\} \mid (\alpha, \beta) \in E\Gamma\}$. By Theorem 5.1, we can find vertices $\alpha, \beta \in \Omega$ such that the undirected graph $(\Omega, \{\alpha, \beta\}^{\text{Aut } \Gamma})$ has connectivity one. Now G is a primitive subgroup of $\text{Aut } \Gamma$, so the undirected graph $\Gamma' := (\Omega, \{\alpha, \beta\}^G)$ is a connected subgraph of $(\Omega, \{\alpha, \beta\}^{\text{Aut } \Gamma})$, and therefore has connectivity one. Hence, the directed graph $\Gamma := (\Omega, (\alpha, \beta)^G)$ also has connectivity one. \square

The following three lemmas are generalisations of several observations made in [24]. Although Möller restricts his attention to automorphism groups of undirected graphs, for our purposes his arguments require only trivial modification.

Lemma 5.5. *If Γ is a locally finite connectivity-one orbital graph of G and $\alpha, \beta \in V\Gamma$ lie in the same block Λ of Γ , then a block of $\Lambda' := (V\Lambda, (\alpha, \beta)^{G_{\{\Lambda\}}})$ is a block of $\Gamma' := (V\Gamma, (\alpha, \beta)^G)$.*

Proof. Since Λ' is the subgraph of Γ' induced by $V\Lambda$ in Γ' , any block Θ of Λ' is a connected subgraph of Γ' with connectivity at least two, so it must be contained in some block Δ of Γ' . If $V\Delta \subseteq V\Lambda$ then Δ is a block of Λ' , so we must have $\Delta = \Theta$. Thus, it suffices to show $V\Delta \subseteq V\Lambda$.

Without loss of generality, suppose $(\alpha, \beta) \in E\Delta$. If $V\Delta = \{\alpha, \beta\}$ then $V\Delta \subseteq V\Lambda$ and we are done. So, suppose $V\Delta \neq \{\alpha, \beta\}$. Since a block has connectivity strictly greater than one, any two edges in Δ lie in a common cycle. It is therefore sufficient to show that if $\alpha_1\alpha_2\ldots\alpha_n$ is a cycle in Δ with $\alpha_1 = \alpha$ and $\alpha_2 = \beta$, then $\alpha_1, \alpha_2, \ldots, \alpha_n \in V\Lambda$.

Let T be the block-cut-vertex tree of Γ . Since α and β lie in the same block of Γ , we have $d_T(\alpha, \beta) = 2$, and so since G acts transitively on the edges of Γ , the images of any two vertices adjacent in Γ' are at distance 2 in T . Now the cycle $\alpha_1\ldots\alpha_n$ in Δ is a cycle in Γ' and therefore gives us a closed path $\alpha_1v_1\alpha_2v_2\ldots v_{n-1}\alpha_n$ in T , where each v_i is the vertex in T corresponding to the block in Γ containing the edge between α_i and α_{i+1} . Since all the α_i are distinct, we must have $v_1 = \ldots = v_{n-1}$. Furthermore, since $\alpha_1 = \alpha$ and $\alpha_2 = \beta$, we know v_1 corresponds to the block Λ , and therefore $\alpha_1, \ldots, \alpha_n \in V\Lambda$. Hence $V\Delta \subseteq V\Lambda$. \square

Let Γ be a locally finite orbital graph of G with more than one end. Fix $\alpha, \beta \in V\Gamma$ such that $\Gamma^{(1)} := (V\Gamma, (\alpha, \beta)^G)$ has connectivity one. Let Λ_1 be a block of $\Gamma^{(1)}$ containing α , and let $G^{(1)} := G_{\{\Lambda_1\}}$. This group acts primitively on $V\Lambda_1$ by Lemma 4.1. If Λ_1 has more than one end then find $\beta_2 \in \Lambda_1$ such that $\Gamma^{(2)} := (V\Lambda_1, (\alpha, \beta_2)^{G^{(1)}})$ has connectivity one. Let Λ_2 be a block of $\Gamma^{(2)}$ containing α and put $G^{(2)} := G_{\{\Lambda_2\}}$. Again we note this group acts primitively on $V\Lambda_2$ by Lemma 4.1. For $i \geq 2$, if Λ_i has more than one end, find $\beta_{i+1} \in \Lambda_i$ such that $\Gamma^{(i+1)} := (V\Lambda_i, (\alpha, \beta_{i+1})^{G^{(i)}})$ has connectivity one. Let Λ_{i+1} be a block of $\Gamma^{(i+1)}$ containing α and put $G^{(i+1)} := G_{\{\Lambda_{i+1}\}}$. By Lemma 4.1, this group acts primitively on $V\Lambda_{i+1}$.

Lemma 5.6. *There is an $n \geq 1$ such that Λ_n has at most one end.*

Proof. Let s_i be the size of the smallest orbit of $G_\alpha^{(i-1)}$ on $V\Gamma^{(i)}$ and let n_i be the number of blocks in $\Gamma^{(i)}$ that contain α . Since $G^{(i-1)}$ is vertex- and edge-transitive on $\Gamma^{(i)}$, and each edge lies in a unique block, $G^{(i-1)}$ acts transitively on the blocks of $\Gamma^{(i)}$; therefore, the stabiliser $G_\alpha^{(i-1)}$ acts transitively on the blocks of $\Gamma^{(i)}$ that contain α . Thus s_i is simply n_i multiplied by the size of the smallest orbit of $G_\alpha^{(i)}$ on Λ_i ; that is, $s_i = n_i s_{i+1}$. If there were no $n \geq 1$ such that Λ_n has at most one end then $n_i \geq 2$ for all $i \geq 1$, and therefore $s_i > s_{i+1}$ for all $i \geq 1$. Since all values of s_i are non-negative, this is a contradiction. \square

Lemma 5.7. *For all $i \geq 1$ the graph Λ_i is a block of $(V\Gamma, (\alpha, \beta_i)^G)$.*

Proof. We use an inductive argument. Suppose Λ_i is a block of the orbital graph $(V\Gamma, (\alpha, \beta_i)^G)$. Now Λ_{i+1} is a block of $(V\Lambda_i, (\alpha, \beta_{i+1})^{G_{\{\Lambda_i\}}})$, so by Lemma 5.5 it is a block of $(V\Gamma, (\alpha, \beta_{i+1})^G)$. By construction we know Λ_1 is a block of $(V\Gamma, (\alpha, \beta_1)^G)$, and so the induction hypothesis is true for all $i \geq 1$. \square

Proof of Theorem 5.3. Suppose G has a locally finite orbital graph with connectivity one such that each block has at most one end. Such a graph has more than one end.

Conversely, suppose G has a locally finite orbital graph with more than one end. By Lemma 5.4, G has a locally finite connectivity-one orbital graph Γ . Using the notation described above, if we set $\Gamma^{(1)} := \Gamma$ then, by Lemma 5.6, there exists a positive integer n such that Λ_n has at most one end. Put $\Gamma' := (V\Gamma, (\alpha, \beta_n)^G)$. By Lemma 5.7, Λ_n is a block of Γ' , and since $\Lambda_n \neq \Gamma'$, we see Γ' has connectivity one. Furthermore, G acts edge-transitively on Γ' , so every block of Γ' is isomorphic to Λ_n , and thus has at most one end. \square

Combining Theorem 4.6, Theorem 4.9 and Theorem 5.3 we obtain the following.

Theorem 5.8. *If G is a primitive group of permutations of an infinite set Ω then G has a locally finite orbital graph with more than one end if and only if G has a locally finite orbital graph Γ with connectivity one, whose blocks are primitive but not automorphism-regular, are pairwise isomorphic, have at least three vertices and at*

most one end. Furthermore, if Λ is a block of Γ , then $G_{\{\Lambda\}}$ acts primitively but not regularly on $V\Lambda$. \square

If G is a primitive group of permutations of an infinite set Ω and Γ is a locally finite orbital graph with connectivity one whose blocks have at most one end, then we shall call Γ a *canonical orbital graph of G* . If Γ is such a graph, and Λ is a block of Γ containing precisely one end ϵ , then ϵ will be called a *block-end* of Γ . If an end ϵ is not a block-end of Γ , it will be called a *tree-end*, as it corresponds to an end of the block-cut-vertex tree of Γ .

Observe that G permutes the block-ends of Γ transitively. Indeed, G acts edge-transitively on Γ , and each edge belongs to a unique block of Γ , so G permutes the blocks, and therefore the block-ends, transitively.

Since Γ has connectivity one, any thick ends of Γ must be block-ends. Thus, if Γ has a thick end then G acts transitively on the thick ends of Γ .

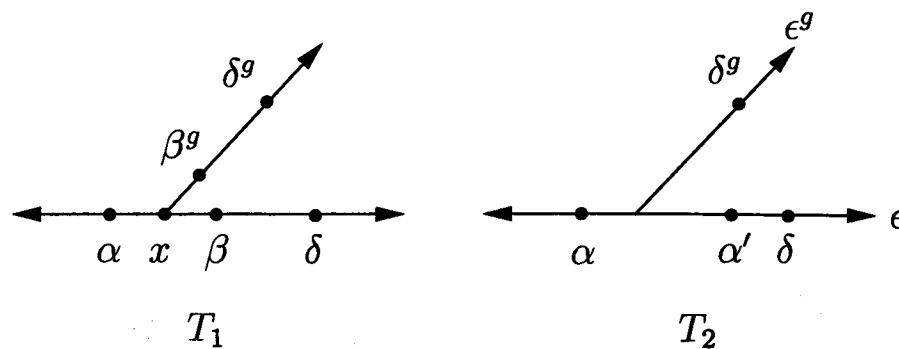
5.2 Uniqueness

Recall from Chapter 2 that if G is a primitive group and Γ_1 and Γ_2 are orbital graphs of G then there is a natural bijection between the ends of Γ_1 and the ends of Γ_2 .

If T is a tree, and $\alpha, \beta \in VT$, there is exactly one geodesic in T between α and β , which we denote by $[\alpha, \beta]_T$. If ϵ is an end of T , then there is precisely one half-line in T that lies in ϵ with initial vertex α . We denote this half-line by $[\alpha, \epsilon)_T$. A connected subgraph C of a graph Γ is said to *contain the end ϵ* if for every half-line $(\alpha_i)_{i \in \mathbb{N}}$ lying in ϵ we have $\alpha_i \in VC$ for all sufficiently large i .

Lemma 5.9. *Suppose Γ_1 and Γ_2 are canonical orbital graphs of a primitive group G . If ϵ is a block-end of Γ_1 then ϵ is a block-end of Γ_2 .*

Proof. Let T_1 and T_2 be the block-cut-vertex trees of Γ_1 and Γ_2 respectively. Suppose, for a contradiction, that there is a block-end ϵ of Γ_1 that is a tree-end of Γ_2 . Since ϵ is a tree-end of Γ_2 , it is an end of T_2 . As it is a block-end of Γ_1 , there exists a block Λ_1 of Γ_1 whose end is ϵ , and a vertex $x \in VT_1 \setminus V\Gamma_1$ corresponding to the block Λ_1 .

Figure 5.1: The trees T_1 and T_2

The following argument is illustrated in Figure 5.1. Fix $\alpha \in V\Lambda_1$, and let C be the connected component of $T_2 \setminus \{\alpha\}$ containing ϵ . Since ϵ is an end of Λ_1 and ϵ is contained in C , there must be infinitely many vertices in $V\Lambda_1$ that lie in C . Choose $\alpha' \in V\Lambda_1 \setminus \{\alpha\}$ such that $\alpha' \in C$, and fix any vertex $\delta \in [\alpha, \epsilon)_{T_2} \cap [\alpha', \epsilon)_{T_2}$. Let β be the vertex adjacent to x in $[x, \delta]_{T_1}$. Since β is adjacent to x in T_1 , we have $\beta \in V\Lambda_1$. If $\beta = \alpha$ then redefine $\alpha := \alpha'$. In this way, we can be sure $\beta \neq \alpha$.

Since $\beta \in V\Lambda_1$ and $G_{\{\Lambda_1\}}$ is primitive but not regular on $V\Lambda_1$, there is an element $g \in G_{\alpha, \{\Lambda_1\}}$ such that $\beta^g \neq \beta$. Now $\beta \in [\alpha, \delta]_{T_1}$, so $[\alpha, \delta]_{T_1} \neq [\alpha, \delta]_{T_1}^g$. However, $g \in G_\alpha$, so $[\alpha, \delta]_{T_1}^g = [\alpha, \delta^g]_{T_1}$ and therefore $\delta \neq \delta^g$. There is only one half-line in T_1 with initial vertex α contained in the end ϵ , so if $g \in G_{\alpha, \epsilon}$ then g must fix the half-line $[\alpha, \epsilon)_{T_1}$ pointwise. The vertex δ lies on this half-line, so $G_{\alpha, \epsilon} \leq G_\delta$. Since $\delta \neq \delta^g$, the automorphism g cannot lie in G_ϵ . This, however, is absurd, as ϵ is the end of Λ_1 , and $g \in G_{\alpha, \{\Lambda_1\}} \leq G_{\alpha, \epsilon}$. \square

If Γ is a locally finite connected graph, an automorphism $g \in \text{Aut } \Gamma$ is called *hyperbolic* if g fixes precisely two thin ends of Γ and leaves no non-empty finite subset of $V\Gamma$ invariant.

Theorem 5.10. ([31, Proposition 3.4]) *If T is an undirected tree and G is a group of automorphism of T containing no hyperbolic elements, then either G fixes some vertex or leaves some edge invariant, or G fixes some end of the tree T .* \square

Lemma 5.11. *Suppose G is a primitive group of permutations of an infinite set and Γ_1 and Γ_2 are canonical orbital graphs of G . Let T_2 be the block-cut-vertex tree of Γ_2 .*

If Λ_1 is a block of Γ_1 then there exists $x \in VT$ such that $G_{\{\Lambda_1\}} \leq G_x$, and for all $\alpha_1, \alpha_2 \in V\Lambda_1$ we have $d_{T_2}(\alpha_1, x) = d_{T_2}(\alpha_2, x)$.

Proof. If Λ_1 is infinite, then it contains exactly one block-end ϵ of Γ_1 , which by Lemma 5.9 is a block-end of Γ_2 . Let Λ_2 be the block of Γ_2 whose end is ϵ and let x be the vertex of T_2 corresponding to the block Λ_2 . Then $G_{\{\Lambda_1\}} = G_\epsilon = G_{\{\Lambda_2\}} = G_x$, so we have $G_{\{\Lambda_1\}}$ fixes some vertex in T_2 .

Now assume $V\Lambda_1$ is finite. Since $G_{\{\Lambda_1\}}$ cannot contain a hyperbolic element, we may use Theorem 5.10 to deduce that $G_{\{\Lambda_1\}}$ must fix a vertex of T_2 , or leave some edge of T_2 invariant, or fix an end of T_2 . We show that, in the latter two cases, the group must still fix some vertex of T_2 . Indeed, since no element of G may interchange two adjacent vertices of T_2 , if the group $G_{\{\Lambda_1\}}$ leaves some edge $\{\alpha, x\}$ of T_2 invariant, then it must fix both α and x pointwise. Furthermore, if ϵ is some end of T_2 fixed by the group $G_{\{\Lambda_1\}}$, take x to be any vertex in T_2 such that the connected component of $T_2 \setminus \{x\}$ containing the end ϵ contains no element in $V\Lambda_1$. Since $V\Lambda_1$ is finite, there are infinitely many choices for such a vertex. Then any half-line in ϵ whose root lies in $V\Lambda_1$ must contain the vertex x . Thus, if $G_{\{\Lambda_1\}}$ fixes the end ϵ , it must also fix the vertex x .

Finally we observe that, since $G_{\{\Lambda_1\}}$ is transitive on $V\Lambda_1$, if $\alpha_1, \alpha_2 \in V\Lambda_1$ then $d_{T_2}(\alpha_1, x) = d_{T_2}(\alpha_2, x)$. \square

Henceforth, if Λ_1 is a block of Γ_1 , the vertex x in the above lemma will be referred to as the *centroid* of Λ_1 in T_2 . Since the above arguments are symmetric in Γ_1 and Γ_2 , we can also make reference to the *centroid* of Λ_2 in T_1 , where Λ_2 is any block of Γ_2 , and T_1 is the block-cut-vertex tree of Γ_1 .

Let Ω be some fixed infinite set and let G be a primitive group of permutations of Ω . Recall that, given any locally finite primitive connectivity-one orbital graph Γ_1 of G , if each vertex lies in m distinct blocks and Λ is a block of Γ_1 , we write the graph Γ_1 as $\Gamma(m, \Lambda)$. All locally finite primitive connectivity-one orbital graphs of G can be written in this way. Since G acts transitively on the set of blocks of $\Gamma(m, \Lambda)$, there is a natural equivalence relation on the set of such graphs: the orbital graph

Γ_2 is equivalent to Γ_1 if $\Gamma_2 = \Gamma(m, \Lambda')$ for some block Λ' of Γ_2 satisfying $V\Lambda = V\Lambda'$. It is under this equivalence relation that a canonical orbital graph of a primitive permutation group can be considered essentially unique.

Theorem 5.12. *If G is a locally finite primitive group of permutations of an infinite set Ω then the canonical orbital graphs of G are equivalent.*

Proof. Suppose the group G has two canonical orbital graphs $\Gamma_1 = \Gamma(m_1, \Lambda_1)$ and $\Gamma_2 = \Gamma(m_2, \Lambda_2)$. By Theorem 4.9, both m_1 and m_2 are at least two, and the graphs Λ_1 and Λ_2 are primitive but not automorphism-regular, and have at least three vertices. Let T_1 and T_2 be the block-cut-vertex trees of Γ_1 and Γ_2 respectively. We wish to show there is a block Λ'_2 of Γ_2 such that $V\Lambda_1 = V\Lambda'_2$. From this we will deduce that the graphs Γ_1 and Γ_2 are equivalent. Set $d := \sup\{d_{T_2}(\alpha, \beta) \mid \alpha, \beta \in V\Lambda_1\}$. This is finite, by Lemma 5.11, so we may fix $\alpha, \beta \in V\Lambda_1$ such that $d_{T_2}(\alpha, \beta) = d$. Let x_2 be the vertex of T_2 that is adjacent to α in the line $[\alpha, \beta]_{T_2}$. Since x_2 is adjacent to α in T_2 , it must lie in $VT_2 \setminus V\Gamma_2$, and therefore corresponds to some block Λ'_2 of Γ_2 . Let c_1 be the centroid of Λ_1 in T_2 .

We begin by showing $G_{\alpha, \{\Lambda_1\}} = G_{\alpha, \{\Lambda'_2\}}$. Now c_1 and β must lie in the same connected component of $T_2 \setminus \{\alpha\}$, and β and x_2 lie in the same component of $T_2 \setminus \{\alpha\}$, so c_1 and x_2 are in the same component of $T_2 \setminus \{\alpha\}$. Since x_2 is adjacent to α in $[\alpha, \beta]_{T_2}$, it follows that $x_2 \in [\alpha, c_1]_{T_2}$, and therefore $G_{\alpha, c_1} \leq G_{x_2}$. Thus

$$G_{\alpha, \{\Lambda_1\}} \leq G_{\alpha, c_1} \leq G_{\alpha, x_2} = G_{\alpha, \{\Lambda'_2\}}.$$

Now choose $\gamma \in \Lambda'_2 \setminus \{\alpha\}$ and repeat this argument, replacing the vertex β with γ , the tree T_2 with T_1 and the block Λ_1 with Λ'_2 . Whence, there exists a block Λ'_1 of Γ_1 containing α such that $G_{\alpha, \{\Lambda'_2\}} \leq G_{\alpha, \{\Lambda'_1\}}$, and therefore

$$G_{\alpha, \{\Lambda_1\}} \leq G_{\alpha, \{\Lambda'_2\}} \leq G_{\alpha, \{\Lambda'_1\}}.$$

Since G_α is transitive on the blocks of Γ_1 containing α , there exists $g \in G_\alpha$ such that $\Lambda_1^g = \Lambda'_1$. As there are only finitely many such blocks, $\Lambda_1^{g^n} = \Lambda_1$ for some natural number n . Now $G_{\alpha, \{\Lambda_1\}} \leq G_{\alpha, \{\Lambda_1\}^g}$, and therefore $G_{\alpha, \{\Lambda_1\}^g} \leq G_{\alpha, \{\Lambda_1\}^{g^2}}$. If we continue

in this way we eventually obtain

$$G_{\alpha, \{\Lambda_1\}} \leq G_{\alpha, \{\Lambda_1\}^g} \leq \cdots \leq G_{\alpha, \{\Lambda_1\}^{g^n}};$$

however, $\Lambda_1^{g^n} = \Lambda_1$, so it follows that $G_{\alpha, \{\Lambda_1\}} = G_{\alpha, \{\Lambda_1\}^g}$. Thus

$$G_{\alpha, \{\Lambda_1\}} = G_{\alpha, \{\Lambda'_2\}}.$$

We now show $G_{\{\Lambda_1\}} = G_{\{\Lambda'_2\}}$. Let γ be the vertex in $[x_2, \beta]_{T_2}$ adjacent to x_2 , and observe $\gamma \in V\Lambda'_2$. Now $G_{\{\Lambda'_2\}}$ does not act regularly on $V\Lambda'_2$, so there is an element $g \in G_{\alpha, \{\Lambda'_2\}}$ such that $\gamma^g \neq \gamma$. Therefore x_2 is the only vertex in $[x_2, \beta]_{T_2} \cap [x_2, \beta^g]_{T_2}$. Thus $d_{T_2}(\beta, \beta^g) = d_{T_2}(\beta, x_2) + d_{T_2}(\beta^g, x_2) = (d-1) + (d-1) = 2d-2$. Now $g \in G_{\alpha, \{\Lambda'_2\}} = G_{\alpha, \{\Lambda_1\}}$, and $\beta \in V\Lambda_1$, so $\beta^g \in V\Lambda_1$. Furthermore, $d = d_{T_2}(\alpha, \beta)$ must be even, so if $d > 2$, then $d \geq 4$, and therefore $d_{T_2}(\beta, \beta^g) = 2d-2 > d$. This is not possible, as d was chosen to be maximal. It must therefore be the case that $d = 2$. Since no two vertices of Λ_1 are at distance greater than d in T_2 , all vertices of Λ_1 are adjacent to x_2 in T_2 , and therefore lie in Λ'_2 . Now Λ_1 contains at least three vertices, each of which is adjacent to x_2 in T_2 , and so, since $G_{\{\Lambda_1\}}$ is transitive on $V\Lambda_1$, it must be the case that $G_{\{\Lambda_1\}}$ fixes the vertex x_2 , and therefore fixes $V\Lambda'_2$ setwise. Thus $G_{\{\Lambda_1\}} \leq G_{\{\Lambda'_2\}}$. If $G_{\{\Lambda_1\}} \neq G_{\{\Lambda'_2\}}$ then $G_{\alpha, \{\Lambda'_2\}} = G_{\alpha, \{\Lambda_1\}} < G_{\{\Lambda_1\}} < G_{\{\Lambda'_2\}}$, which cannot happen since $G_{\{\Lambda'_2\}}$ is primitive on $V\Lambda_2$. Hence $G_{\{\Lambda_1\}} = G_{\{\Lambda'_2\}}$.

Finally, since $G_{\{\Lambda_1\}}$ and $G_{\{\Lambda'_2\}}$ are transitive on $V\Lambda_1$ and $V\Lambda'_2$ respectively, and $\alpha \in V\Lambda_1 \cap V\Lambda'_2$, we have $V\Lambda'_2 = \alpha^{G_{\{\Lambda'_2\}}} = \alpha^{G_{\{\Lambda_1\}}} = V\Lambda_1$. Furthermore, since G_α is transitive on the set of blocks of Γ_1 that contain α , and on the set of blocks of Γ_2 containing α , we have $m_1 = |G_\alpha : G_{\alpha, \{\Lambda_1\}}| = |G_\alpha : G_{\alpha, \{\Lambda'_2\}}| = m_2$. Hence $\Gamma_2 = \Gamma(m_1, \Lambda'_2)$ and $V\Lambda'_2 = V\Lambda_1$, so the graphs Γ_1 and Γ_2 are equivalent. \square

It is now possible to describe every locally finite connectivity-one orbital graph of a primitive group G .

Theorem 5.13. *If G is a locally finite primitive group of permutations of an infinite set Ω with more than one permutation-end then G has a canonical orbital graph $\Gamma(m, \Lambda)$, and the canonical orbital graphs of G are precisely the graphs $\Gamma(m, \Lambda')$, where Λ' is an orbital graph of $G_{\{\Lambda\}}$ acting on the set $V\Lambda$.*

Proof. Suppose G has a locally finite orbital graph with more than one end. Then by Theorem 5.8, G has a canonical orbital graph $\Gamma(m, \Lambda)$. Set $\Gamma := \Gamma(m, \Lambda)$, and fix vertices $\alpha, \beta \in V\Lambda$ such that $\Gamma = (\Omega, (\alpha, \beta)^G)$. Suppose Γ' is also a canonical orbital graph of G . We show there exists an orbital graph Λ' of $G_{\{\Lambda\}}$ acting on $V\Lambda$ such that Λ' is a block of Γ' and $\Gamma' = \Gamma(m, \Lambda')$. By Theorem 5.12, the graph Γ' must be equivalent to Γ , so there exists a block Λ' of Γ' such that $\Gamma' = \Gamma(m, \Lambda')$ and $V\Lambda' = V\Lambda$. Let (α, β') be an edge in Λ' . Since Γ' is edge-transitive, $\Gamma' = (\Omega, (\alpha, \beta')^G)$. If (γ, δ) is any edge in Λ' , then there exists an element $g \in G$ such that $(\gamma, \delta) = (\alpha, \beta')^g$, but such an automorphism must fix the block Λ' , and therefore lies in $G_{\{\Lambda'\}} = G_{\{\Lambda\}}$. Hence Λ' is an orbital graph of $G_{\{\Lambda\}}$ acting on the set $V\Lambda$.

Conversely, suppose Λ' is an orbital graph of $G_{\{\Lambda\}}$ acting on $V\Lambda$. We show the connectivity-one graph $\Gamma(m, \Lambda')$ is a canonical orbital graph of G .

It is simple to check that $G \leq \text{Aut } \Gamma(m, \Lambda')$, so we begin by showing G is edge-transitive on $\Gamma(m, \Lambda')$. Suppose Λ'_1 and Λ'_2 are blocks of $\Gamma(m, \Lambda')$. Then there exist blocks Λ_1 and Λ_2 of $\Gamma(m, \Lambda)$ such that $V\Lambda_1 = V\Lambda'_1$ and $V\Lambda_2 = V\Lambda'_2$. Since G acts edge-transitively on $\Gamma(m, \Lambda)$, it acts transitively on its blocks. Therefore, there exists an automorphism $g \in G$ such that $\Lambda_1^g = \Lambda_2$. Since $G \leq \text{Aut } \Gamma(m, \Lambda')$, we must also have $\Lambda'_1{}^g = \Lambda'_2$, so G permutes the blocks of $\Gamma(m, \Lambda')$ transitively. Thus G is transitive on the blocks of $\Gamma(m, \Lambda)$ and acts edge-transitively on each block, so it must act edge-transitively on the whole graph $\Gamma(m, \Lambda')$.

It remains to show the blocks of $\Gamma(m, \Lambda')$ have at most one end. This follows immediately from Lemma 2.6, since both Λ and Λ' are orbital graphs of $G_{\{\Lambda\}}$ acting on $V\Lambda$, they must have the same ends.

Hence, $\Gamma(m, \Lambda)$ is a connectivity-one orbital graph of G in which each block has at most one end, and is therefore a canonical orbital graph of the group G . \square

Thus, for any primitive group G possessing a canonical orbital graph Γ , when speaking of the permutation-ends of G as being the ends of Γ , it makes sense to refer to the *block-ends* and *tree-ends* of G as being, respectively, the block-ends and tree-ends of its canonical orbital graph. This can be extended to include any group G with just one permutation-end ϵ , by defining the *block-end* of G to be the end ϵ .

Chapter 6

Paired suborbits of infinite primitive permutation groups

It was shown in Theorem 3.1 that if G is a primitive group of permutations of an infinite set Ω , possessing a finite suborbit whose pair is also finite, then Ω is countably infinite and every suborbit of G is finite. For many years it was an open question whether the pair of every finite suborbit of an infinite primitive group G was also finite. This question was answered when, in [7], for each infinite cardinal κ , Evans constructs a primitive permutation group with a finite suborbit whose pair has size κ .

While Evans' impressive result sheds light on the size of paired suborbits, it is still not known which infinite primitive permutation groups possessing a finite suborbit have the property that every suborbit is self-paired. Such groups are called *generously transitive*. Finite generously transitive groups have been studied extensively by Neumann in [25], in which he showed that generous transitivity implies a strong condition on the centraliser ring and the permutation character of the group. We consider the infinite case, and, by examining the possible orbital graphs of G , obtain a complete characterisation of the locally finite generously transitive groups with more than one permutation-end.

Lemma 6.1. *If G is a locally finite infinite distance-transitive group then every sub-*

orbit of G is self-paired.

Proof. If Δ is an orbital of G then there exist vertices $\alpha, \beta \in V\Gamma$ such that $\Delta = (\alpha, \beta)^G$. Since G is distance-transitive, there exists $g \in G$ such that $(\alpha, \beta)^g = (\beta, \alpha)$, so $\Delta^* = (\beta, \alpha)^G = (\alpha, \beta)^{gG} = (\alpha, \beta)^G = \Delta$. Hence, every orbital of G is self-paired. If Υ is a suborbit of G , then there is a vertex $\alpha \in V\Gamma$, and an orbital Δ of G , such that $\Upsilon = \Delta(\alpha)$. Whence $\Upsilon^* = \Delta^*(\alpha) = \Delta(\alpha) = \Upsilon$. \square

Lemma 6.2. *Suppose Γ is a canonical orbital graph of a primitive group of permutations G of an infinite set Ω , and that G has more than one permutation-end. If all the suborbits of G are all self-paired then Γ is distance-transitive.*

Proof. Since Γ is a canonical orbital graph of G , any block Λ is primitive but not automorphism-regular, with at least three vertices and at most one end, and $\Gamma = \Gamma(m, \Lambda)$ for some $m \geq 2$. Suppose Γ is not distance transitive. By Theorem 2.5, Λ is not complete, so its diameter must be at least two. We may therefore choose a geodesic $\alpha\alpha_1\alpha_2\alpha_3$ in Γ , such that α, α_1 and α_2 all lie in the block Λ , but α_3 does not. We will show the orbital $(\alpha, \alpha_3)^G$ is not self-paired.

Suppose the orbital $(\alpha, \alpha_3)^G$ is self-paired. Then there exists $g \in G$ such that $(\alpha, \alpha_3)^g = (\alpha_3, \alpha)$. Let T be the block-cut-vertex tree of Γ . Then there is a vertex $x_1 \in VT$ corresponding to the block Λ , and a vertex $x_2 \in VT$ corresponding to the block of Γ containing α_2 and α_3 but not α and α_1 . The T -geodesic from α to α_3 is the path $\alpha x_1 \alpha_2 x_2 \alpha_3$. Since $G \leq \text{Aut } T$ we have $g \in \text{Aut } T$. Now the T -geodesic from α to α_3 is unique, so $(\alpha, \alpha_3)^g = (\alpha_3, \alpha)$ implies $(\alpha, x_1, \alpha_2, x_2, \alpha_3)^g = (\alpha_3, x_2, \alpha_2, x_1, \alpha)$; that is, $g \in G_{\alpha_2}$. However, $d_\Gamma(\alpha_2, \alpha) \neq d_\Gamma(\alpha_2, \alpha_3)$, so no such $g \in G_{\alpha_2}$ exists. \square

Theorem 6.3. *If G is a locally finite infinite primitive permutation group with more than one permutation-end, then every suborbit of G is self-paired if and only if G is distance-transitive.*

Proof. Suppose G is an infinite primitive group with a locally finite orbital graph with more than one permutation-end. If G acts distance-transitively on some distance-transitive orbital graph $\Gamma(m, K_t)$ then all suborbits of G are self-paired by Lemma 6.1.

Now consider the converse. By Theorem 5.8, there is a canonical orbital graph Γ of G with connectivity one. If every suborbit of G is self-paired then, by Lemma 6.2, Γ must be distance-transitive, so for some $m, t \geq 2$ we have $\Gamma = \Gamma(m, K_t)$.

We claim that if every suborbit of G is self-paired, then G must act distance-transitively on Γ . Fix $\alpha \in V\Gamma$ and suppose G does not act distance-transitively on Γ . Then there exists $r \geq 1$ such that $S_r(\alpha, \Gamma)$ contains distinct vertices β, γ with $\beta \notin \gamma^{G_\alpha}$. If G_α acts transitively on the blocks containing α , then β and γ may be chosen in distinct components of $\Gamma \setminus \{\alpha\}$. If G_α does not act transitively on the blocks containing α then we may choose β and γ in $S_1(\alpha, \Gamma)$ lying in distinct components of $\Gamma \setminus \{\alpha\}$, satisfying $\beta \notin \gamma^{G_\alpha}$. In both cases, α lies on every Γ -geodesic between β and γ and $d_\Gamma(\alpha, \beta) = d_\Gamma(\alpha, \gamma)$. Thus $(\beta, \gamma)^G = (\gamma, \beta)^G$ implies that there exists $g \in G_\alpha$ such that $\beta^g = \gamma$, which is not possible. We therefore conclude the orbital $(\beta, \gamma)^G$ is not self-paired. Hence, if $\Delta := (\beta, \gamma)^G$, then the suborbit $\Delta(\alpha)$ is not self-paired. \square

If one removes the stipulation that G have an orbital graph with more than one end from the above theorem then the hypothesis is no longer true. To see this, let Γ be the distance-transitive graph $\Gamma(m, K_{t+1})$ and let G be the group $\text{Aut } \Gamma \text{ Wr Sym}(2)$. This is a primitive group of permutations of $V\Gamma \times V\Gamma$. By Theorem 3.6, every orbital graph of G is locally finite and has one end. Since all locally finite distance-transitive graphs have infinitely many ends, G does not have a distance-transitive orbital graph; however, every suborbit of G is self-paired. Indeed, fix an orbital $((\gamma_1, \gamma_2), (\delta_1, \delta_2))^G$ of G in $V\Gamma \times V\Gamma$. Since $\text{Aut } \Gamma$ is distance-transitive, all its suborbits are self-paired, so there exist $g_1, g_2 \in \text{Aut } \Gamma$ such that $(\gamma_1, \delta_1)^{g_1} = (\delta_1, \gamma_1)$ and $(\gamma_2, \delta_2)^{g_2} = (\delta_2, \gamma_2)$. Taking $g \in \text{Aut } \Gamma \text{ Wr Sym}(2)$ to be the permutation (g_1, g_2) we have $((\gamma_1, \gamma_2), (\delta_1, \delta_2))^g = ((\delta_1, \delta_2), (\gamma_1, \gamma_2))$. Hence, every orbital and therefore every suborbit of G is self-paired.

Chapter 7

The structure of infinite primitive groups

While the classification of the finite simple groups and the O’Nan–Scott Theorem describe the structure of finite primitive permutation groups, their infinite cousins remain at best abstruse. Infinitary versions of the O’Nan–Scott Theorem, [18] for example, lack generality and are further weakened by our incomplete understanding of the infinite simple groups. When considering primitive groups whose suborbits are all finite, rather than requiring the existence of a minimal normal subgroup, one may instead use the theory developed thus far to determine their structure.

Suppose we are given a non-empty set of groups $\{G_\psi \mid \psi \in \Psi\}$. The *free product* of the G_ψ is a group G which, together with a collection of homomorphisms $i_\psi : G_\psi \rightarrow G$, satisfies the following. Given a set of homomorphisms $\varphi_\psi : G_\psi \rightarrow H$ into some group H there is a unique homomorphism $\varphi : G \rightarrow H$ such that $i_\psi \varphi = \varphi_\psi$. The group G will be denoted by $\text{Fr}_{\psi \in \Psi} G_\psi$.

If X is a non-empty subset of the group G , the *normal closure* of X in G is the intersection of all normal subgroups of G that contain X , and is itself a normal subgroup of G . Let H be a group and $\{G_\psi \mid \psi \in \Psi\}$ be a non-empty set of groups such that, for all $\psi \in \Psi$, the group H is isomorphic to a subgroup H_ψ of G_ψ via a

monomorphism

$$\varphi_\psi : H \rightarrow G_\psi.$$

We define the *free product of the G_ψ with amalgamated subgroup H* to be the group

$$G := F/N,$$

where F is the free product $\text{Fr}_{\psi \in \Psi} G_\psi$, and N is the normal closure in F of the set $\{(h^\psi)^{-1}h^\mu \mid \psi, \mu \in \Psi, h \in H\}$. Intuitively, one may think of the free product of the G_ψ with amalgamated subgroup H as being the largest group generated by the G_ψ in which the subgroups H_ψ are considered to be equal.

The most common example of an amalgamated free product is when there are two groups G_1 and G_2 , with subgroups H_1 and H_2 that are isomorphic via $\varphi : H_1 \rightarrow H_2$. In this case the free product of G_1 and G_2 with amalgamated subgroup H_1 is denoted by $G_1 *_{H_1} G_2$. A detailed introduction to free groups and free products can be found in [16], or, more recently, in [29].

Let G be a group acting on a graph Γ . An inversion is a pair consisting of an element $g \in G$ and a directed edge $(x, y) \in E\Gamma$ such that $(x, y)^g = (y, x)$; if there is no such pair the group G is said to act *without inversion* on Γ . If G acts on Γ without inversion, we define the *quotient graph* G/Γ to be the graph whose vertex and edge set are the quotients of $V\Gamma$ and $E\Gamma$ respectively under the action of G ; a *fundamental domain* of $\Gamma \bmod G$ is a subgraph of Γ that is isomorphic to G/Γ . A *segment* of Γ is a subgraph of Γ consisting two adjacent vertices and a directed edge of Γ between them.

Theorem 7.1. ([30, Theorem 6]) *Let G be a group acting on a graph Γ and let $P = \{\{x, y\}, (x, y)\}$ be a segment of Γ . If P is a fundamental domain of $\Gamma \bmod G$ then Γ is a tree if and only if $G \cong G_x *_{G_{x,y}} G_y$. \square*

Using this powerful theorem, we may apply the results of Chapter 5 to determine the structure of locally finite infinite primitive permutation groups with more than one permutation-end.

Theorem 7.2. *If G is a locally finite infinite primitive group of permutations of an infinite set Ω with more than one permutation-end, then G has a canonical orbital graph $\Gamma(m, \Lambda)$ and*

$$G \cong G_\alpha *_{G_{\alpha, \{\Lambda\}}} G_{\{\Lambda\}},$$

where $\alpha \in V\Lambda$, and $G_{\alpha, \{\Lambda\}}$ is a non-trivial maximal proper subgroup of $G_{\{\Lambda\}}$ that fixes no element in $V\Lambda \setminus \{\alpha\}$.

Proof. Let G be an infinite primitive group possessing a finite suborbit whose pair is also finite, and suppose G has an orbital graph with more than one end. By Theorem 5.8, G has a canonical orbital graph Γ of the form $\Gamma(m, \Lambda)$, where $m \geq 2$ and Λ is a primitive but not automorphism-regular graph, with at least three vertices and at most one end. Let T be the block-cut-vertex tree of Γ . Then $G \leq \text{Aut } T$. There is a natural bipartition of T , one part containing the vertices of Γ , and the other containing vertices corresponding to the blocks of Γ . Since G preserves this bipartition, the group G acts on T without inversion, and has two orbits on the vertices of T .

Let $x \in VT$ correspond with the block Λ of Γ , and fix $\alpha \in V\Lambda$. Then (α, x) is an edge in T . If we define $T' := (VT, (\alpha, x)^G)$, then T' is an edge-transitive tree upon which G acts without inversion, with fundamental domain the segment $\{\{\alpha, x\}, (\alpha, x)\}$. Hence, by Theorem 7.1,

$$G \cong G_\alpha *_{G_{\alpha, x}} G_x.$$

Since Λ is the block of Γ corresponding to the vertex x of T , the stabiliser G_x is equal to $G_{\{\Lambda\}}$. As $G_{\{\Lambda\}}$ acts primitively on $V\Lambda$, the group $G_{\alpha, \{\Lambda\}}$ is maximal in $G_{\{\Lambda\}}$. By Theorem 4.6, $G_{\alpha, \{\Lambda\}}$ fixes no vertex in $V\Lambda \setminus \{\alpha\}$. \square

Chapter 8

Primitive groups and ends of graphs

We now consider groups acting edge-transitively on a locally finite primitive graph and investigate their action on the end space.

Let G be a primitive and edge-transitive group of automorphisms of a locally finite primitive directed graph Γ with more than one end. By Theorem 5.8, G has a canonical orbital graph Γ' . Furthermore, Γ and Γ' have the same ends by Lemma 2.6, so without loss of generality, one may assume Γ is a canonical orbital graph of G . We assume G is closed. Recall G and its closure \overline{G} in $\text{Sym}(\Omega)$ have the same orbits on $V\Gamma$. Let T be the block-cut-vertex tree of Γ .

In this chapter we will show every orbit of the closed group G on the set of tree-ends of Γ has size 2^{\aleph_0} . The following result will be fundamental to our argument.

Theorem 8.1. ([6, Theorem 2.8]) *If G and H are closed subgroups of $\text{Sym}(\Omega)$, where Ω is a countable set and $H \leq G$, then either $|G : H| \leq \aleph_0$ or $|G : H| = 2^{\aleph_0}$. The former holds if and only if H contains the pointwise stabiliser in G of some finite subset of Ω . \square*

Lemma 8.2. *For any tree-end ϵ of Γ , if $\underline{\alpha}$ is an n -tuple of vertices with $G_{\underline{\alpha}} \leq \overline{G}_{\epsilon}$, then $G_{\underline{\alpha}} \leq G_{\epsilon}$.*

Proof. Enumerate the set $V\Gamma$ as $\{\gamma_1, \gamma_2, \gamma_3, \dots\}$, and recall that an element $g \in G$ is

the limit of a sequence of permutations (g_n) of $V\Gamma$ if and only if, given $k \geq 1$, there exists an integer N_k such that, for all $n \geq N_k$ we have $\gamma_i^{g_n} = \gamma_i^g$ and $\gamma_i^{g_n^{-1}} = \gamma_i^{g^{-1}}$ whenever $1 \leq i \leq k$.

Suppose $G_{\underline{\alpha}} \leq \overline{G_\epsilon}$ but $G_{\underline{\alpha}} \not\leq G_\epsilon$, and fix

$$g \in G_{\underline{\alpha}} \setminus G_\epsilon.$$

Since $g \in \overline{G_\epsilon}$, there exists a sequence (g_n) of permutations in G_ϵ such that

$$g_n \rightarrow g.$$

If $\underline{\alpha} = (\alpha_1, \dots, \alpha_n)$ then choose $k \geq 1$ such that $\{\alpha_1, \dots, \alpha_n\} \subseteq \{\gamma_1, \dots, \gamma_k\}$. Since $g_n \rightarrow g$, there exists an integer N_k such that, for all $n \geq N_k$ we have $\gamma_i^{g_n} = \gamma_i^g$ whenever $1 \leq i \leq k$. Hence, for all $n \geq N_k$,

$$\underline{\alpha}^{g_n} = \underline{\alpha}.$$

Let T be the block-cut-vertex tree of Γ , and let $[\alpha_1, \epsilon)_T$ be the unique half-line of T rooted at α_1 lying in the end ϵ . For all $n \geq N_k$ we have $g_n \in G_{\alpha_1, \epsilon}$, therefore g_n fixes every vertex on $[\alpha_1, \epsilon)_T$. Since g lies in G_{α_1} but not G_ϵ , there exists $x \in [\alpha_1, \epsilon)_T$ such that $x^g \neq x$; furthermore, if $y \in [x, \epsilon)_T$ then $y^g \neq y$. Therefore, we may choose $\beta \in [x, \epsilon)_T \cap V\Gamma$ such that $\beta^g \neq \beta$. Since $\beta \in V\Gamma$, there is an integer k' such that $\beta = \gamma_{k'}$; because $g_n \rightarrow g$, there exists $N_{k'} \geq 1$ such that for all $n \geq N_{k'}$ we have $\gamma_{k'}^{g_n} = \gamma_{k'}^g$. Fix $N := \max(N_k, N_{k'})$. Then, for all $n \geq N$ we have $g_n \in G_{\underline{\alpha}}$ but $\beta^{g_n} = \beta^g \neq \beta$, so $g_n \notin G_\epsilon$, a contradiction. \square

Lemma 8.3. *If there is an end ϵ of Γ and an n -tuple $\underline{\alpha} = (\alpha_1, \dots, \alpha_n)$ of vertices of Γ such that $G_{\underline{\alpha}} \leq G_\epsilon$, then, for $1 \leq i \leq n$, the orbit $\epsilon^{G_{\alpha_i}}$ is finite.*

Proof. Suppose $G_{\underline{\alpha}} \leq G_\epsilon$. Then, for all i satisfying $1 \leq i \leq n$ we have $G_{\underline{\alpha}} \leq G_{\alpha_i, \epsilon}$. Therefore

$$\begin{aligned} |G_{\alpha_i} : G_{\underline{\alpha}}| &= |G_{\alpha_i} : G_{\alpha_i, \epsilon}| |G_{\alpha_i, \epsilon} : G_{\underline{\alpha}}| \\ &= |\epsilon^{G_{\alpha_i}}| |G_{\alpha_i, \epsilon} : G_{\underline{\alpha}}|. \end{aligned}$$

Observe

$$\begin{aligned} |G_{\alpha_1} : G_{\underline{\alpha}}| &= \prod_{m=1}^{n-1} |G_{\alpha_1 \dots \alpha_m} : G_{\alpha_1 \dots \alpha_{m+1}}| \\ &= \prod_{m=1}^{n-1} |\alpha_{m+1}^{G_{\alpha_1 \dots \alpha_m}}|. \end{aligned}$$

Since $G \leq \text{Aut } \Gamma$ and Γ is locally finite, each orbit $|\alpha_{m+1}^{G_{\alpha_1 \dots \alpha_m}}|$ is finite, so $|G_{\alpha_1} : G_{\underline{\alpha}}|$ is finite. A similar argument shows $|G_{\alpha_i} : G_{\underline{\alpha}}|$ is finite for each i satisfying $1 \leq i \leq n$.

We have already seen

$$|G_{\alpha_i} : G_{\underline{\alpha}}| = |\epsilon^{G_{\alpha_i}}| |G_{\alpha_i, \epsilon} : G_{\underline{\alpha}}|,$$

so one may deduce $|\epsilon^{G_{\alpha_i}}|$ is finite. \square

Lemma 8.4. *If ϵ is a tree-end of Γ with $|\epsilon^G| < 2^{\aleph_0}$ then, for all $\alpha \in V\Gamma$, the orbit ϵ^{G_α} is finite.*

Proof. Let ϵ be a tree-end of Γ , and let $H := \overline{G_\epsilon}$. Now

$$\begin{aligned} |\epsilon^G| &= |G : G_\epsilon| \\ &= |G : H| |H : G_\epsilon|. \end{aligned}$$

If $|\epsilon^G| < 2^{\aleph_0}$ then $|G : H| < 2^{\aleph_0}$, and, since G and H are both closed, we may apply Theorem 8.1 to deduce that there exists an n -tuple of vertices $\underline{\alpha}$ such that $G_{\underline{\alpha}} \leq H$, and $|G : H| \leq \aleph_0$.

Suppose $|\epsilon^G| < 2^{\aleph_0}$ and $\alpha = (\alpha_1, \dots, \alpha_n)$ is an n -tuple of vertices of Γ such that $G_{\underline{\alpha}} \leq \overline{G_\epsilon}$. By Lemma 8.2 we have

$$G_{\underline{\alpha}} \leq G_\epsilon.$$

Fix $\alpha \in V\Gamma$ and let $\underline{\alpha}' := (\alpha, \alpha_1, \dots, \alpha_n)$; then $G_{\underline{\alpha}'} \leq G_{\underline{\alpha}} \leq G_\epsilon$, and so, from Lemma 8.3, the orbit ϵ^{G_α} is finite. \square

Theorem 8.5. ([28]) *Let Γ be an infinite connected vertex- and edge-transitive directed graph with finite but unequal in-valency and out-valency. Then there is an epimorphism φ from the vertex set of Γ to the set of integers \mathbb{Z} such that (α, β) is an edge of Γ only if $\varphi(\beta) = \varphi(\alpha) + 1$.* \square

Corollary 8.6. *Suppose G is a primitive group of permutations of an infinite set Ω , and every suborbit of G is finite. If $\alpha \in \Omega$ and $\Delta(\alpha)$ and $\Delta^*(\alpha)$ are paired α -suborbits then $|\Delta(\alpha)| = |\Delta^*(\alpha)|$.*

Proof. Suppose G is a primitive group of permutations of an infinite set Ω , and every suborbit of G is finite. Fix an element $\alpha \in \Omega$ and let $\Delta(\alpha)$ and $\Delta^*(\alpha)$ be paired α -suborbits. Suppose $|\Delta(\alpha)| \neq |\Delta^*(\alpha)|$. We will show this implies G cannot be primitive, contradicting our original assumption.

Choose $\beta \in \Delta(\alpha)$ and let Γ be the orbital graph $(\Omega, (\alpha, \beta)^G)$. As G is primitive, this graph is connected, and because $\Delta(\alpha)$ and $\Delta^*(\alpha)$ are finite, Γ is locally finite, with in-valency $|\Delta(\alpha)|$ and out-valency $|\Delta^*(\alpha)|$. As these are not equal, we may apply Theorem 8.5 to deduce that there is an epimorphism φ from the vertex set of Γ to the set of integers \mathbb{Z} such that (γ, δ) is an edge of Γ only if $\varphi(\delta) = \varphi(\gamma) + 1$.

Without loss of generality, we may assume $\varphi(\alpha) = 0$. If $\alpha_0\alpha_1 \cdots \alpha_n$ is any cycle in Γ with $\alpha_0 = \alpha$, we must have $\sum_{i=0}^n \varphi(\alpha_i) = 0$. Thus, there can be no odd cycles in Γ containing α . Since Γ is vertex-transitive, it contains no odd cycles. Hence the G -congruence given by $\gamma \cong \delta$ if and only if $d_\Gamma(\gamma, \delta)$ is even, is non-trivial and non-universal. Whence, G is not primitive. \square

Recall that in light of Lemma 2.6, the permutation-ends of G are defined to be the ends of any orbital graph of G .

Theorem 8.7. *Let G be a locally finite primitive group of permutations of an infinite set Ω that is closed in $\text{Sym}(\Omega)$. If ϵ is a permutation-end of G then precisely one of the following holds.*

- (i) $|\epsilon^G| = 1$ and G has just one permutation-end;
- (ii) $|\epsilon^G| = \aleph_0$ and ϵ is a block-end of G ;
- (iii) $|\epsilon^G| = 2^{\aleph_0}$ and ϵ is a tree-end of G .

Proof. If G has precisely one permutation-end, then it is clear (i) holds, so suppose G has more than one permutation-end.

By Theorem 5.8, G has a canonical orbital graph Γ . If ϵ is a block-end of G then every block of Γ contains a block-end. Since G permutes the blocks of Γ transitively, it acts transitively on its set of block-ends. There are countably many blocks in Γ , so $|\epsilon^G| = \aleph_0$ and (ii) holds.

It remains to show that if ϵ is a tree-end of G then $|\epsilon^G| = 2^{\aleph_0}$. Suppose ϵ is a tree-end of G and $|\epsilon^G| < 2^{\aleph_0}$.

Let T be the block-cut-vertex tree of Γ . Since ϵ is a tree-end of G , it is an end of T . By Lemma 8.4, if $|\epsilon^G| < 2^{\aleph_0}$ then for all $\alpha \in V\Gamma$, the orbit ϵ^{G_α} is finite. Fix $\alpha \in V\Gamma$. Since ϵ^{G_α} is finite, there exists a vertex $\alpha' \in V\Gamma$ that lies in $[\alpha, \epsilon)_T$ such that $G_{\alpha, \alpha'} \leq G_\epsilon$. Put $m := |\alpha'^{G_\alpha}|$, and note $m = |\epsilon^{G_\alpha}|$. By Corollary 8.6, $|\alpha^{G_{\alpha'}}| = m$.

Now choose $\beta \in (\alpha', \epsilon)_T$. A similar argument shows there exists $\beta' \in (\beta, \epsilon)_T$ such that $G_{\beta, \beta'} \leq G_\epsilon$.

In choosing β and β' in this way we ensure $G_{\alpha, \beta'} \leq G_{\beta, \beta'}$ and $|\beta'^{G_\alpha}| = |\epsilon^{G_\alpha}| = m$. We again apply Corollary 8.6 to deduce $|\alpha^{G_{\beta'}}| = m$. Hence

$$\begin{aligned} m &= |\alpha^{G_{\beta'}}| \\ &= |G_{\beta'} : G_{\alpha, \beta'}| \\ &= |G_{\beta'} : G_{\beta, \beta'}| |G_{\beta, \beta'} : G_{\beta' \alpha}|. \end{aligned}$$

If $|G_{\beta, \beta'} : G_{\beta' \alpha}| = 1$ then $G_{\beta, \beta'} \leq G_\alpha$ and so $G_{\alpha, \alpha'} \leq G_{\beta, \beta'}$ and $G_{\beta, \beta'} \leq G_{\alpha, \alpha'}$, with $\alpha', \beta \in (\alpha, \beta')_T$ and $[\alpha, \alpha']_T \cap (\beta, \beta')_T = \emptyset$. Applying Theorem 4.5 we see G is not primitive, which contradicts our original assumption; thus, we must have $|G_{\beta, \beta'} : G_{\beta' \alpha}| > 1$. This implies $|G_{\beta'} : G_{\beta, \beta'}| < m$; that is, $|\beta^{G_{\beta'}}| < m$. However, $|\beta^{G_{\beta'}}| = |\beta'^{G_\beta}|$ by Corollary 8.6; furthermore, β and β' we chosen so that $G_{\beta, \beta'} \leq G_\epsilon$. Hence $|\epsilon^{G_\beta}| < m$.

Since α was chosen arbitrarily, we may now set $\alpha := \beta$ and repeat the above argument. Eventually, we will find $|\epsilon^{G_\alpha}| = 1$. However, this is also a contradiction. Indeed, let x be the vertex of T that is adjacent to α in $[\alpha, \epsilon)_T$. This vertex corresponds to some block Λ of Γ that contains α . By Theorem 4.6, the group $G_{\alpha, \{\Lambda\}}$ fixes no vertex in $V\Lambda \setminus \{\alpha\}$, and therefore cannot fix the half-line $[\alpha, \epsilon)_T$. Whence, we must have $|\epsilon^G| = 2^{\aleph_0}$. \square

Chapter 9

Subdegree growth rates

In this chapter we give bounds on the growth of the lower and upper subdegree sequences of infinite locally finite primitive groups, and show that, when the growth of the lower subdegree sequences is fast enough, the rate of growth uniquely determines the group. Following this, we show that in many cases the number of permutation ends of a group is determined by its multiset of subdegrees.

9.1 Subdegree growth and group structure

Let G be a primitive group of permutations of an infinite set Ω , with every subdegree of G finite, and suppose Γ is an orbital graph of G . Fix $\alpha \in \Omega$. An upper bound can be found for the growth of the lower subdegree sequences by bounding the growth of $s_r := |S_r(\alpha, \Gamma)|$, since G_α fixes $S_r(\alpha, \Gamma)$ setwise. The following simple observation gives a useful upper bound.

Lemma 9.1. *If Γ is an infinite locally finite primitive graph then, for all $r \geq 1$,*

$$s_r \leq s_1(s_1 - 1)^{r-1}.$$

Proof. Since Γ is vertex-primitive, it is connected and every vertex has valency s_1 . Thus for all $r > 1$, any vertex in the sphere $S_r(\alpha, \Gamma)$ is connected to at least one vertex in $S_{r-1}(\alpha, \Gamma)$, and is therefore adjacent to at most $s_1 - 1$ vertices in S_{r+1} . Whence $s_r \leq s_{r-1}(s_1 - 1)$. □

A corresponding bound is easily obtained for the growth of the upper subdegree sequence.

Lemma 9.2. *Let G be a locally finite primitive group of permutations of an infinite set Ω . If (M_r) is the upper subdegree sequence of G , then for all $r \geq 1$,*

$$M_r \leq 2M_1(2M_1 - 1)^{r-1}.$$

Proof. Fix $\alpha \in \Omega$ and choose $\beta \in \Omega$ such that the suborbit β^{G_α} is of size M_1 . Let Γ be the orbital graph $(\Omega, (\alpha, \beta)^G)$, and let $\Delta(\alpha)$ denote the suborbit β^{G_α} .

If $\Delta^*(\alpha)$ is the suborbit paired with $\Delta(\alpha)$ then $|\Delta^*(\alpha)|$ is also equal to M_1 by Corollary 8.6. Since $S_1 = \Delta(\alpha) \cup \Delta^*(\alpha)$, the valency of Γ is at most $2M_1$.

If the upper subdegree sequence contains just one subdegree then there is nothing to prove, so suppose this is not the case. Choose $r \geq 1$ such that M_r and M_{r+1} are elements of the upper subdegree sequence of G ; since all members of this sequence are distinct, there exists an integer $t \geq 1$ such that M_r is the largest subdegree in the sphere S_t but not in S_{t+1} . Thus, there exists a vertex $\gamma \in S_t$ such that γ is adjacent to a vertex $\delta \in S_{t+1}$ with $|\delta^{G_\alpha}| > M_r$.

Since $|\delta^{G_\alpha}| > M_r$, we have $|\delta^{G_\alpha}| \geq M_{r+1}$, and therefore

$$M_{r+1} \leq |\delta^{G_\alpha}| \leq (s_1 - 1)|\gamma^{G_\alpha}| \leq M_r(s_1 - 1). \quad \square$$

If Γ is a locally finite primitive graph with connectivity one, and T is the block-cut-vertex-tree of Γ , we define the *block-distance* between two vertices α and β in Γ to be

$$bd(\alpha, \beta) := \frac{d_T(\alpha, \beta)}{2}.$$

Informally, one may think of the block-distance between α and β as being the number of blocks of Γ through which any geodesic between α and β must pass.

If Γ is an infinite locally finite primitive graph and $\alpha \in V\Gamma$, we define $n_r(\Gamma)$ to be the number of α -suborbits of $\text{Aut } \Gamma$ in $S_r(\alpha, \Gamma)$. Since G acts transitively on Γ , this definition is independent of our choice of α . When there can be no ambiguity as to the identity of the graph in question, $n_r(\Gamma)$ will be written as n_r .

Let $\{f_r\}_{r \geq 0}$ be the sequence of *Fibonacci numbers* defined by the relation $f_{r+1} = f_r + f_{r-1}$ for $r \geq 1$ and $f_0 = f_1 = 1$.

Theorem 9.3. *If Γ is a locally finite primitive graph with connectivity one that is not distance-transitive then $n_r \geq f_r$ for all $r \geq 1$.*

Proof. Fix $\alpha \in V\Gamma$. To each vertex β in $V\Gamma$ choose a geodesic in Γ from α to β and assign a label $\ell(\beta) := (l_1, \dots, l_k)$, where $k = bd(\alpha, \beta)$, and for all r satisfying $1 \leq r \leq k$ the integer l_r is the number of vertices at block-distance r from α in the geodesic. It is simple to check that, since Γ has connectivity one, this is independent of the geodesic chosen. Let L_r be the set of labels for vertices in $S_r(\alpha, \Gamma)$ and put $k_r := |L_r|$. It is very easy to see two vertices $\beta, \beta' \in S_r(\alpha, \Gamma)$ lie in the same α -suborbit only if $\ell(\beta) = \ell(\beta')$. Thus $n_r(\Gamma) \geq k_r$.

By Theorem 2.5, if Γ is not distance-transitive then the blocks of Γ have diameter at least 2. We claim that, in this case,

$$k_{r+1} \geq k_r + k_{r-1}. \quad (9.1)$$

Indeed, for all $r \geq 1$ there is an injective correspondence from L_r into L_{r+1} via the map sending $(l_1, \dots, l_k) \in L_r$ to $(l_1, \dots, l_k, 1) \in L_{r+1}$. Fix $r \geq 1$ and note there are k_{r-1} labels in L_r whose last entry is 1. If $(l_1, \dots, l_{k-1}, 1)$ is such a label, then $(l_1, \dots, l_{k-1}, 2)$ is a label in L_{r+1} . Therefore, there are at least k_{r-1} labels in L_{r+1} whose final entry is 2; we have already seen there are k_r labels in L_{r+1} whose final entry is 1, so we must have $k_{r+1} \geq k_r + k_{r-1}$, as claimed. Since $k_0 = k_1 = 1$ we have $n_r \geq k_r \geq f_r$ for all $r \geq 0$. \square

It is well known that

$$\lim_{r \rightarrow \infty} f_r^{1/r} = \frac{1 + \sqrt{5}}{2};$$

consequently it is possible to find a lower bound for the growth of the sequence $(n_r^{1/r})$.

Corollary 9.4. *If Γ is a locally finite primitive graph with connectivity one that is not distance-transitive then*

$$\liminf_{r \rightarrow \infty} n_r^{1/r} \geq \frac{1 + \sqrt{5}}{2}. \quad \square$$

The bound given in the above corollary is sharp; that is, it cannot be improved upon. Indeed, consider the Peterson graph P_5 . This is a finite primitive distance-transitive graph with diameter 2. Let $\Gamma := \Gamma(2, P_5)$ and $G := \text{Aut } \Gamma$, and fix $\alpha \in V\Gamma$. It is hopefully clear that two vertices $\beta, \beta' \in S_r(\alpha, \Gamma)$ lie in the same orbit of G_α if and only if $\ell(\beta) = \ell(\beta')$. Thus $n_r = k_r$. We claim, for all $r \geq 1$,

$$k_{r+1} = k_r + k_{r-1}. \quad (9.2)$$

Indeed, let L'_r be the elements of L_r whose last entry is 1, let L''_r be those elements whose last entry is 2 and put $k'_r := |L'_r|$ and $k''_r := |L''_r|$. Since P_5 has diameter 2 we have $k_r = k'_r + k''_r$. We may assign to each label $\ell \in L''_r$ the unique label in L'_{r-1} obtained by changing the last entry in ℓ from 2 to 1. Hence $k''_r \geq k'_{r-1}$. We may also assign to each label $\ell \in L'_{r-1}$ a unique label in L''_r obtained by changing the last entry in ℓ to 2, so $k'_{r-1} \geq k''_r$. Whence $k''_r = k'_{r-1}$. Similarly, we assign to each label $(l_1, \dots, l_k) \in L_{r-1}$ the label $(l_1, \dots, l_k, 1) \in L'_r$, and to each label $(l_1, \dots, l_k, 1) \in L'_r$ the label $(l_1, \dots, l_k) \in L_{r-1}$. Such a correspondence is bijective, so $k_{r-1} = k'_r$. Thus $k_{r+1} = k'_{r+1} + k''_{r+1} = k'_{r+1} + k'_r = k_r + k_{r-1}$, as claimed.

Now $k_0 = k_1 = 1$ and so by (9.2), for all $r \geq 0$ we have $n_r = k_r = f_r$. Thus

$$\lim_{r \rightarrow \infty} n_r^{1/r} = \frac{1 + \sqrt{5}}{2}.$$

Let Γ be any infinite locally finite connected vertex- and edge-transitive graph, and fix $\alpha \in V\Gamma$. For $\gamma \in S_r(\alpha)$ we define

$$\begin{aligned} a(\gamma) &:= |S_1(\alpha) \cap S_r(\gamma)|; \\ b(\gamma) &:= |S_1(\alpha) \cap S_{r+1}(\gamma)|; \\ c(\gamma) &:= |S_1(\alpha) \cap S_{r-1}(\gamma)|. \end{aligned}$$

The following lemma is an extension of an observation by Macpherson in [17]. The argument presented here is based on that given by Dicks and Dunwoody in [4].

Lemma 9.5. *If there exists a natural number R_0 such that, for all $\beta, \gamma \in V\Gamma$ with $d(\alpha, \beta) > R_0$ and $d(\alpha, \gamma) > R_0$, we have $c(\beta) = c(\gamma)$ and $b(\beta) = b(\gamma)$, then Γ has more than one end.*

Proof. Suppose, for all $\beta, \gamma \in V\Gamma$ with $d(\alpha, \beta) > R_0$ and $d(\alpha, \gamma) > R_0$, we have $c(\beta) = c(\gamma)$ and $b(\beta) = b(\gamma)$. We will describe an infinite set of vertices s , such that the complement $s^* := V\Gamma \setminus s$ is infinite, and the set of edges δs from s to s^* is finite, thus showing Γ must have more than one end.

Fix an edge $(\alpha_0, \alpha_1) \in V\Gamma$ and define $s := \{\gamma \in V\Gamma \mid R_0 + 1 \leq d(\alpha_0, \gamma) = d(\alpha_1, \gamma) + 1\}$. This set and its complement are both infinite. Indeed, given a positive integer n one may choose a vertex $\gamma \in V\Gamma$ with $d(\alpha_0, \gamma) = 2n + 1$. There exists a geodesic $\alpha_0\beta_1\beta_2 \dots \beta_{2n}\gamma$ of length $2n + 1$ between α_0 and γ . If e is the edge between β_n and β_{n+1} then, since Γ is edge-transitive, there exists an automorphism of Γ mapping e to the edge (α_0, α_1) . Therefore, both s and s^* contain vertices in $S_n(\alpha_0, \Gamma) \cup S_{n+1}(\alpha_0, \Gamma)$. Whence s and s^* are infinite.

We now show δs is finite. Suppose $e \in E$ is an edge between $\beta \in s$ and $\gamma \in s^*$. Write $i = d(\alpha_0, \beta)$.

We claim $d(\alpha_0, \gamma) = 1 + d(\alpha_1, \gamma)$. Observe $d(\alpha_0, \gamma) \leq d(\alpha_0, \beta) + d(\beta, \gamma) = i + 1$, and $d(\alpha_0, \beta) \leq d(\alpha_0, \gamma) + d(\beta, \gamma)$, so $d(\alpha_0, \gamma) \geq d(\alpha_0, \beta) - d(\beta, \gamma) = i - 1$. Hence,

$$i - 1 \leq d(\alpha_0, \gamma) \leq i + 1.$$

We now consider three cases: when $d(\alpha_0, \gamma) = i - 1$, when $d(\alpha_1, \gamma) = i$ and finally when $d(\alpha_0, \gamma) \geq i$ and $d(\alpha_1, \gamma) \leq i - 1$, and in each case show the claim is true.

Suppose $d(\alpha_0, \gamma) = i - 1$. Then $d(\alpha_0, \beta) = d(\alpha_0, \gamma) + d(\gamma, \beta)$, so there is a geodesic from α_0 to β that contains γ . Hence

$$S_1(\alpha_0) \cap S_{i-2}(\gamma) \subseteq S_1(\alpha_0) \cap S_{i-1}(\beta);$$

however, since $i - 1 \geq R_0$ we have $|S_1(\alpha_0) \cap S_{i-2}(\gamma)| = c(\gamma) = c(\beta) = |S_1(\alpha_0) \cap S_{i-1}(\beta)|$, so the two sets must be equal. Now $\alpha_1 \in S_1(\alpha_0) \cap S_{i-1}(\beta)$, so $\alpha_1 \in S_1(\alpha_0) \cap S_{i-2}(\gamma)$; that is, $d(\alpha_1, \gamma) = i - 2 = d(\alpha_0, \gamma) - 1$ as claimed.

Next, suppose $d(\alpha_1, \gamma) = i$. Then $d(\alpha_1, \gamma) = d(\alpha_1, \beta) + d(\beta, \gamma)$, so there is a geodesic from α_1 to γ that contains β . Hence

$$S_1(\alpha_1) \cap S_{i+1}(\gamma) \subseteq S_1(\alpha_1) \cap S_i(\beta);$$

however, since $i - 1 \geq R_0$ we have $|S_1(\alpha_1) \cap S_{i+1}(\gamma)| = b(\gamma) = b(\beta) = |S_1(\alpha_1) \cap S_i(\beta)|$, so the two sets must be equal. Now $\alpha_0 \in S_1(\alpha_1) \cap S_i(\beta)$, so $\alpha_0 \in S_1(\alpha_1) \cap S_{i+1}(\gamma)$; that is, $d(\alpha_0, \gamma) = i + 1 = d(\alpha_1, \gamma) + 1$ as claimed.

Finally, if $d(\alpha_0, \gamma) \geq i$ and $d(\alpha_1, \gamma) \leq i - 1$, then $d(\alpha_1, \gamma) + 1 \leq i \leq d(\alpha_0, \gamma)$. In fact, since $d(\alpha_0, \gamma) \leq d(\alpha_1, \gamma) + d(\alpha_1, \alpha_0) = d(\alpha_1, \gamma) + 1$, we have $d(\alpha_0, \gamma) \geq 1 + d(\alpha_1, \gamma)$, so $d(\alpha_0, \gamma) = 1 + d(\alpha_1, \gamma)$ as claimed.

Now $\gamma \in s^*$, so either $d(\alpha_0, \gamma) < R_0 + 1$ or $d(\alpha_0, \gamma) \neq d(\alpha_1, \beta) + 1$. The latter is not true by the above argument, so we must have $d(\alpha_0, \gamma) < R_0 + 1$. Hence, there are only finitely many $\gamma \in s^*$ that are adjacent to a vertex in s , so δs is finite, and Γ has more than one end. \square

Theorem 9.6. *If Γ is an infinite vertex- and edge-transitive locally finite graph with one end then $n_r(\Gamma) \geq 2$ for all large enough r .*

Proof. Let Γ be an infinite locally finite graph with one end, and fix $\alpha \in V\Gamma$. We claim there is an integer R_0 such that for all $r \geq R_0$ there exist vertices $\gamma_1, \gamma_2 \in S_r(\alpha)$ with $c(\gamma_1) \neq c(\gamma_2)$ or $b(\gamma_1) \neq b(\gamma_2)$.

Suppose no such R_0 exists. Then there exists an infinite sequence (r_i) such that for all $\gamma_1, \gamma_2 \in S_{r_i}(\alpha)$ we have $c(\gamma_1) = c(\gamma_2)$ and $b(\gamma_1) = b(\gamma_2)$. For each $i \geq 1$ choose $\gamma \in S_{r_i}(\alpha)$ and set $c_{r_i} := c(\gamma)$ and $b_{r_i} := b(\gamma)$. It is easy to see $c_{r_i} \geq c_{r_{i-1}}$ and $b_{r_i} \leq b_{r_{i-1}}$. Since $1 \leq c_{r_i} \leq |S_1(\alpha)|$ and $1 \leq b_{r_i} \leq |S_1(\alpha)|$ for all $i \geq 0$, there exists constants k, c and b such that, for all $i \geq k$ we have $c_{r_i} = c$ and $b_{r_i} = b$. Put $R_0 := r_k$. Suppose there exists $r \geq R_0$ such that $S_r(\alpha)$ contains two vertices γ_1 and γ_2 with $c(\gamma_1) \neq c(\gamma_2)$. Without loss of generality, one may suppose $c(\gamma_1) > c(\gamma_2)$. However, $r > R_0$ so $c(\gamma_2) \geq c$ and $c(\gamma_1) > c$. If i is chosen so $r_i > r$ then $c_{r_i} \geq c(\gamma_1) > c$ which is a contradiction.

It must therefore be the case that, for all $r \geq R_0$, we have $c(\gamma) = c$ for all $\gamma \in S_r(\alpha)$. A similar argument shows that for all $r \geq R_0$, we have $b(\gamma) = b$ for all $\gamma \in S_r(\alpha)$. Hence, by Lemma 9.5, the graph Γ must have more than one end. Since this is not the case, our claim must be true.

Let $G := \text{Aut } \Gamma$. It is clear that two vertices $\gamma_1, \gamma_2 \in S_r(\alpha)$ lie in the same

orbit of G_α only if $c(\gamma_1) = c(\gamma_2)$ and $b(\gamma_1) = b(\gamma_2)$. Hence, for all $r > R_0$ we have $n_r(\Gamma) \geq 2$. \square

Theorem 9.7. *Let G be a primitive group of permutations of an infinite set Ω . If G is locally finite with more than one permutation-end then the lower subdegree sequence of G grows exponentially if and only if G is distance-transitive. In this case, G has height ω , and its subdegree sequence (m_r) satisfies*

$$\lim_{r \rightarrow \infty} m_r^{1/r} = \left(\frac{m-1}{m} \right) m_1,$$

where G acts distance-transitively on the infinite locally finite distance-transitive graph $\Gamma(m, K_{t+1})$. Furthermore, if the growth of the lower subdegree sequence of G is not exponential, then it is polynomial.

Proof. Suppose G is a primitive group of permutations of an infinite set Ω , possessing an orbital graph with more than one end, and G has a finite suborbit whose pair is also finite. By Theorem 3.1, every suborbit of G is finite. Let (m_r) be the lower subdegree sequence of G . Note that, if G acts distance-transitively on the locally finite infinite distance-transitive graph $\Gamma(m, K_{t+1})$, then every suborbit is self-paired, G has height ω , so (m_r) is equal to the subdegree sequence of G , and the subdegree growth of G is exponential with

$$\begin{aligned} \lim_{r \rightarrow \infty} m_r^{1/r} &= \left(\frac{m-1}{m} \right) m t_1 \\ &= \left(\frac{m-1}{m} \right) m_1. \end{aligned}$$

Now consider the converse. Suppose G does not act distance-transitively on any locally finite infinite distance-transitive graph. We will show that the lower subdegree growth of G is bounded above by some polynomial.

By Corollary 9.4, $\liminf n_r^{1/r} \geq (1 + \sqrt{5})/2 > 3/2$. Hence, there exists an integer R such that, for all $r > R$, we have $n_r > (3/2)^r$, and thus $N_r > (3/2)^r$.

Fix $r > R$ and observe that $m_{N_r} \leq s_1(s_1 - 1)^{r-1}$ by Lemma 9.1. We may choose an integer N such that, for all $n \geq N$, we have $(3/2)^n \geq s_1(s_1 - 1)$. Thus, for all $n \geq N$,

$$m_{N_r} < N_r^n.$$

Furthermore, given any integer s with $N_{r-1} \leq s \leq N_r$, the subdegree m_s satisfies $m_s \leq s_1(s_1 - 1)^{r-1}$. Since $s \geq N_{r-1}$ we also have $s^n \geq N_{r-1}^n > (3/2)^{n(r-1)} > s_1(s_1 - 1)^{r-1} \geq m_s$, so the growth of the lower subdegree sequence of G is polynomial. \square

If one removes the condition that G have more than one permutation-end then the following is obtained.

Theorem 9.8. *Suppose G is an infinite locally finite group of permutations of an infinite set Ω , and G is not distance-transitive. If (m_r) is the lower subdegree sequence of G then*

$$\liminf_{r \rightarrow \infty} m_r^{1/r} \leq \sqrt{2m_1 - 1}.$$

Proof. Suppose G is a locally finite primitive group of permutations of an infinite set Ω and does not act distance-transitively on any graph. If G has an orbital graph with more than one end then its lower subdegree growth is subexponential by Theorem 9.7. So, suppose G has an orbital graph with precisely one end; every orbital graph of G therefore has one permutation-end. Let $\Delta(\alpha)$ be a suborbit of size m_1 , and let Γ be the orbital graph (Ω, Δ) . Then $S_1(\alpha, \Gamma) = \Delta(\alpha) \cup \Delta^*(\alpha)$. Let $s_r := |S_r(\alpha, \Gamma)|$ and $n_r := n_r(\Gamma)$, and let N_r be the sum $\sum_{i=1}^r n_i$. Since all suborbits of G are finite, one may deduce from Corollary 8.6 that $\Delta^*(\alpha)$, the suborbit paired with $\Delta(\alpha)$, also has cardinality m_1 . Thus, $s_1 \leq 2m_1$. Our aim is to show

$$\liminf_{r \rightarrow \infty} m_r^{1/r} \leq \sqrt{2m_1 - 1}. \quad (9.3)$$

From the proof of Theorem 9.7, $m_{N_r} \leq s_1(s_1 - 1)^{r-1}$. We again note it is sufficient to show (9.3) holds when $G = \text{Aut } \Gamma$.

By Theorem 9.6, there exists an integer R_0 such that $n_r \geq 2$ for all $r \geq R_0$. Hence, for sufficiently large r ,

$$N_r \geq R_0 + 2(r - R_0).$$

Thus, if $a > 1$,

$$\begin{aligned} \limsup_{r \rightarrow \infty} m_{N_r}^{1/N_r} &\leq \limsup_{r \rightarrow \infty} (s_1 - 1)^{r/N_r} \\ &\leq \limsup_{r \rightarrow \infty} (s_1 - 1)^{r/(2r - R_0)} \\ &= \sqrt{2m_1 - 1}. \end{aligned}$$

Hence $\liminf_{r \rightarrow \infty} m_r^{1/r} \leq \sqrt{2m_1 - 1}$ as required. \square

Corollary 9.9. *If G is a group acting primitively on an infinite set Ω with a finite suborbit whose pair is also finite, then the subdegrees of G are all finite. If (m_r) is the lower subdegree sequence of G then*

$$\liminf_{r \rightarrow \infty} m_r^{1/r} > \sqrt{2m_1 - 1}$$

if and only if G acts distance-transitively on some locally finite infinite distance-transitive graph $\Gamma(m, K_{t+1})$ with $m > 2$ and $t \geq 2$, or $m = 2$ and $t \geq 4$.

Proof. Suppose G is a locally finite primitive group of permutations of an infinite set Ω . If G does not act distance-transitively on any locally finite orbital graph and (m_r) is the lower subdegree sequence of G then

$$\liminf_{r \rightarrow \infty} m_r^{1/r} \leq \sqrt{2m_1 - 1}$$

by Theorem 9.8.

Now suppose G acts distance-transitively on a locally finite distance-transitive graph Γ . By Theorem 2.5, we may write $\Gamma = \Gamma(m, K_{t+1})$ for some $m \geq 2$ and $t \geq 2$. Observe that $\liminf_{r \rightarrow \infty} m_r^{1/r} = (m - 1)t$, so $\liminf_{r \rightarrow \infty} m_r^{1/r} > \sqrt{2m_1 - 1}$ precisely when $m > 2$ and $t \geq 2$, or $m = 2$ and $t \geq 4$. \square

All known examples of primitive groups with locally finite one-ended orbital graphs exhibit subexponential lower subdegree growth; furthermore, it seems highly unlikely that examples exhibiting exponential growth exist.

Conjecture 9.10. *If G is a group acting primitively on an infinite set Ω with a finite suborbit whose pair is also finite, then the subdegrees of G are all finite and G has exponential lower subdegree growth if and only if G is distance-transitive.*

9.2 Subdegree growth and ends of orbital graphs

We begin by observing that the relationship between the subdegree growth of a primitive group G and its permutation-end structure is more subtle than one might expect.

Theorem 9.11. *If (m_r) is the lower subdegree sequence of a locally finite infinite primitive group $G \leq \text{Sym}(\Omega)$ then there exist infinite primitive groups G' and G'' whose suborbits are all finite, with lower subdegree sequences (m'_r) and (m''_r) respectively, such that G' has one permutation-end, and G'' has infinitely many permutation-ends, with*

$$m'_r \leq 2m_r$$

and

$$m''_r \leq 2m_r$$

for all $r \geq 1$.

Proof. Fix $\alpha \in \Omega$, and write $\underline{\alpha} := (\alpha, \alpha) \in \Omega^2$. Take $G' := G \text{ Wr } \text{Sym}(2)$ and consider its action on Ω^2 . By Theorem 3.6, all suborbits of G' are finite and every orbital graph has one end. Furthermore, for every suborbit β^{G_α} of G , the set $(\beta, \alpha)^{G'_\alpha}$ is a suborbit of G' ; since $|(\beta, \alpha)^{G'_\alpha}| = 2|\beta^{G_\alpha}|$, we have $m'_r \leq 2m_r$.

Let G'' be the group $G(2, G)$ constructed in Chapter 3.2.2, and let Λ be an orbital graph of G acting on Ω with connectivity greater than one; by Theorem 3.3, G'' acts primitively on the vertex set of the graph $\Gamma(2, \Lambda)$. Indeed, this graph is an orbital graph of G'' . Since $\Gamma(2, \Lambda)$ has infinitely many ends, every orbital graph of G'' has infinitely many permutation-ends. Furthermore, the action of $G''_{\{\Lambda\}}$ on $V\Lambda$ is isomorphic to the action of G on $V\Lambda = \Omega$; thus, for each subdegree m_r of G , the group $G''_{\underline{\alpha}, \{\Lambda\}}$ has an orbit on $V\Lambda$ of size m_r . Hence, for each subdegree m_r of G , the group G'' has a suborbit of size $2m_r$. Whence, $m''_r \leq 2m_r$. \square

It should be noted that both the difference between m_r and m'_r , and the difference between m_r and m''_r , may grow arbitrarily large, leaving gaps in the range of possible rates of growth.

Some growth rates are only exhibited by groups with precisely one end. The following is immediate from Theorem 9.7.

Theorem 9.12. *If G is a locally finite primitive group of permutations of an infinite set Ω , and the lower subdegree growth of G is subexponential but not polynomial, then G has precisely one permutation-end.* \square

Example 3.17 illustrates that groups with subexponential, non-polynomial lower subdegree growth exist.

All known examples of locally finite primitive groups with exponential lower subdegree growth have infinitely many permutation-ends. Indeed, it seems highly likely that such growth rates cannot be achieved by groups with just one permutation-end. If true then the above theorem and the following conjecture would allow one to naturally partition the non-polynomial rates of growth of infinite primitive groups according to the number of permutation-ends possessed by each group.

Conjecture 9.13. *If G is an infinite primitive permutation group whose subdegrees are all finite, and the lower subdegree sequence of G grows exponentially, then G has 2^{\aleph_0} permutation-ends.*

In Section 3.3.1 it was shown that there exist examples of infinite primitive groups with precisely one permutation-end, and infinite primitive groups with infinitely many permutation-ends, both possessing bounded lower subdegree sequences. If, instead of examining just the lower subdegree sequence, one considers the whole subdegree sequence, it is again possible to determine the permutation-end structure of those infinite primitive groups exhibiting specific rates of subdegree growth.

Theorem 9.14. *If G is an infinite primitive permutation group whose subdegrees are all finite and bounded above, then G has precisely one permutation-end.*

Proof. Suppose G is an infinite primitive permutation group with more than one permutation-end whose subdegrees are all finite. By Theorem 5.8, G has an orbital graph of the form $\Gamma(m, \Lambda)$ for some integer $m \geq 2$, and for some primitive graph Λ . Let Γ denote the graph $\Gamma(m, \Lambda)$ and let T be the block-cut-vertex tree of Γ .

Observe that if α and β are vertices in Γ and $x \in VT$ lies on the T -geodesic $[\alpha, \beta]_T$ between α and β , then $G_{\alpha, \beta} \leq G_x$ and therefore

$$|G_\alpha : G_{\alpha, \beta}| = |G_\alpha : G_{\alpha, x}| |G_{\alpha, x} : G_{\alpha, \beta}|.$$

Hence, the cardinality of the suborbit β^{G_α} is equal to the product $|x^{G_\alpha}| |\beta^{G_{\alpha, x}}|$.

Thus, if all subdegrees of G are bounded above, then there exists a finite number k such that any automorphism in G fixing the sphere $S_k(\alpha, \Gamma)$ pointwise must also fix every vertex in Γ , and therefore every vertex in T . However, in light of Lemma 4.5, such a group cannot be primitive. \square

Using a similar argument, it is sometimes possible to determine the permutation-end structure of a primitive group by knowing just one subdegree.

Theorem 9.15. *If G is an infinite primitive permutation group whose subdegrees are all finite, and at least one subdegree is prime, then every orbital graph of G has precisely one end.*

Proof. Again suppose that G is an infinite primitive permutation group with more than one permutation-end whose subdegrees are all finite. Let Γ be a connectivity-one orbital graph of G of the form $\Gamma(m, \Lambda)$, the existence of which is assured by Theorem 5.8, and let T be the block-cut-vertex tree of this graph.

Recall that, given $\alpha, \beta \in V\Gamma$ and a vertex $x \in VT$ lying on the T -geodesic $[\alpha, \beta]_T$ between α and β , the cardinality of the suborbit x^{G_α} is equal to $|x^{G_\alpha}| |\beta^{G_{\alpha, x}}|$.

Since G acts edge-transitively on Γ , it permutes the blocks of Γ . By Theorem 4.6, the setwise stabiliser in G of each block acts primitively on the vertices of the block; whence, for each vertex $\alpha \in V\Gamma$, the stabiliser G_α transitively permutes the blocks of Γ that contain α . Thus G_α acts transitively on the sphere $S_1(\alpha, T)$, which has cardinality $m \geq 2$.

If Λ is a block of Γ , then by Theorem 5.8, Λ has at least three vertices, and $G_{\{\Lambda\}}$ acts primitively but not regularly on $V\Lambda$. Thus if $\alpha \in V\Lambda$, then $G_{\alpha, \{\Lambda\}}$ fixes no vertex in $\Lambda \setminus \alpha$. The block Λ corresponds to some vertex $x \in S_1(\alpha, T)$, so $G_{\alpha, x}$ fixes no vertex

in $S_1(x, T) \setminus \{\alpha\}$. Therefore, for each vertex γ in $S_1(x, T) \setminus \{\alpha\}$, there exists a prime number p dividing the cardinality of the orbit $\gamma^{G_{\alpha, x}}$. Hence mp divides $\gamma^{G_{\alpha}}$.

Since G_{α} acts transitively on the sphere $S_1(\alpha, T)$, for all vertices γ in $S_2(\alpha, T)$ there exist primes p and q such that pq divides the subdegree $|\gamma^{G_{\alpha}}|$; therefore, the same is true of all vertices in T lying at distance greater than two from α . Since this includes all vertices in $\Gamma \setminus \{\alpha\}$, no subdegree of G is prime. \square

In this section, our discussion of the relationship between the permutation-end structure of a primitive group and its subdegree growth rate contained an obvious omission: groups exhibiting polynomial growth. Groups with just one permutation end, and groups with infinitely many permutation-ends, are well represented in the class of groups with polynomial growth. It may be that it is possible to determine the permutation-end structure of such groups from the order of their growth; however, at present too little is known to offer anything more than spurious conjecture.

Chapter 10

Conclusions

By constructing examples of primitive groups whose suborbits are all finite, with at least one occurring with infinite multiplicity, we have shown the conjecture of Adeleke and Neumann was incorrect in its stated form. The methods used to create these examples can be utilised to manufacture bespoke examples of infinite primitive groups; these construction methods will form a useful tool in any further investigation of the subdegree growth rates of such groups.

By extending the concept of subdegree growth to include primitive groups of height greater than ω , we were able to show that primitive groups exhibiting an exponential subdegree growth rate above a prescribed bound were totally determined by the rate of growth. In addition, it was shown that in many cases it is possible to classify the possible subdegree growth rates of infinite primitive groups according to the end structure of their orbital graphs.

In order to significantly improve upon these results, it will be necessary to address the recondite nature of one-ended primitive graphs. It may be possible to obtain results analogous to Theorems 4.9 and Theorem 5.8, characterising a class of one-ended graphs and showing that every infinite primitive group possessing a locally finite one-ended orbital graph has an orbital graph in this class. Little has been done in this area. In [15], Jung defines several different equivalence relations on the set of half-lines of a locally finite graph, and shows the equivalence classes of these relations are similar to ends in many respects; this approach may prove fruitful, as one-ended

primitive graphs possess infinitely many of these end-like structures.

In Chapter 6 we classified the generously transitive primitive groups possessing a locally finite orbital graph with more than one end. An improved understanding of locally finite one-ended primitive graphs could lead to a classification of the generously transitive primitive groups possessing a locally finite orbital graph with one permutation-end, and thus a complete classification of the locally finite generously transitive primitive groups.

In Chapter 8 the permutation-end structure of a locally finite closed primitive group was fully characterised by orbit length.

Much of the theory underpinning this thesis was presented in Chapters 4 and 5 in which we classified the infinite locally finite primitive directed graphs with connectivity one, and showed that any locally finite primitive group with more than one permutation-end has a connectivity one orbital graph whose blocks have at most one end, and that this graph is essentially unique. This thesis is not an exhaustive account of the potential applications of these results; indeed, through their careful use it may be possible to extend the contents of Chapter 7 and determine necessary and sufficient conditions for the primitivity of a group acting vertex- and edge-transitively on a locally finite graph with more than one end. Such a result would be extremely interesting, and would lead naturally to a result analogous to Theorem 3.5, giving necessary and sufficient conditions for an amalgamated free product $G := A *_B C$ to act primitively on the set of right cosets of A in G . Some progress has already been made in this area by the author, which he intends to continue in his post-doctoral work.

The author intends to publish this work over four papers; the first classifying the infinite locally finite primitive directed graphs, the second describing the canonical orbital graph of an infinite locally finite group with more than one permutation-end, proving its existence and uniqueness. The latter will also contain both the classification of the infinite locally finite generously transitive groups with more than one permutation-end, and the observation that infinite locally finite primitive groups with more than one permutation-end can be written as amalgamated free products,

as illustrations of possible applications. In the third paper the construction of groups with height strictly greater than ω , and the revised definition of subdegree growth will be published, along with the contents of Chapter 9. A fourth and final paper is planned, containing the complete characterisation of the permutation-end structure of infinite locally finite primitive groups.

Bibliography

- [1] S. A. Adeleke and P. M. Neumann, ‘Relations related to betweenness: their structure and automorphisms’, *Mem. Amer. Math. Soc.* (623) 131 (1998).
- [2] M. Bhattacharjee, D. Macpherson, R. G. Möller and P. M. Neumann, *Notes on infinite permutation groups*, Lecture Notes in Mathematics 1698 (Springer-Verlag, Berlin, 1998).
- [3] P. J. Cameron, *Oligomorphic permutation groups*, London Mathematical Society Lecture Notes Series 152 (Cambridge University Press, Cambridge, 1990).
- [4] W. Dicks and M. J. Dunwoody, *Groups acting on graphs*, Cambridge Studies in Advanced Mathematics 17 (Cambridge University Press, Cambridge, 1989).
- [5] J. Dixon and B. Mortimer, *Permutation groups*, Graduate Texts in Mathematics 163 (Springer-Verlag, New York, 1996).
- [6] D. M. Evans, ‘A note on automorphism groups of countably infinite structures’, *Arch. Math. (Basel)* (6) 49 (1987) 479–483.
- [7] D. M. Evans, ‘Suborbits in infinite primitive permutation groups’, *Bull. London Math. Soc.* (5) 33 (2001) 583–590.
- [8] C. Godsil and G. Royle, *Algebraic graph theory*, Graduate Texts in Mathematics 207 (Springer-Verlag, New York, 2001).
- [9] R. Halin, ‘Automorphisms and endomorphisms of infinite locally finite graphs’, *Abh. Math. Sem. Univ. Hamburg* 39 (1973) 251–283.

- [10] P. J. Higgins, *An introduction to topological groups*, London Mathematical Society Lecture Notes 15 (Cambridge University Press, Cambridge, 1974).
- [11] D. G. Higman, 'Intersection matrices for finite permutation groups', *J. Algebra* 6 (1967) 22–42.
- [12] W. Imrich and N. Seifter, 'A note on the growth of transitive graphs', *Discrete Math.* (1-2) 73 (1988/89) 111–117.
- [13] A. A. Ivanov, 'Bounding the diameter of a distance-regular graph', *Dokl. Akad. Nauk SSSR* (4) 28 (1983) 789–792.
- [14] H. A. Jung and M. E. Watkins, 'On the structure of infinite vertex-transitive graphs', *Discrete Math.* (1) 18 (1977) 45–53.
- [15] H. A. Jung, 'Notes on rays and automorphisms of locally finite graphs', *Contemp. Math.* 147 (1993) 477–484.
- [16] R. C. Lyndon and P. E. Schupp, *Combinatorial group theory* (Springer-Verlag, Berlin, 1977).
- [17] H. D. Macpherson, 'Infinite distance transitive graphs of finite valency', *Combinatorica* (1) 2 (1982) 63–69.
- [18] D. Macpherson and C. E. Praeger, 'Infinitary versions of the O'Nan–Scott Theorem', *Proc. London Math. Soc.* (3) 68 (1994) 518–540.
- [19] R. G. Möller, 'The automorphism groups of regular trees', *J. London Math. Soc.* (2) 43 (1991) 236–252.
- [20] R. G. Möller, 'Ends of graphs', *Math. Proc. of the Cambridge Philos. Soc.* (3) 111 (1992) 255–266.
- [21] R. G. Möller, 'Ends of graphs II', *Math. Proc. of the Cambridge Philos. Soc.* (3) 111 (1992) 455–460.

- [22] R. G. Möller, 'Groups acting on locally finite graphs — a survey of the infinitely ended case', *Groups '93, Galway / St. Andrews, Vol. 2*, London Mathematical Society Lecture Note Series (Cambridge Univ. Press, Cambridge, 1993), pp. 426–456.
- [23] R. G. Möller, 'Distance-transitivity in infinite graphs', *J. Combin. Theory Ser. B* (1) 60 (1994) 36–39.
- [24] R. G. Möller, 'Primitivity and ends of graphs', *Combinatorica* (4) 14 (1994) 477–484.
- [25] P. M. Neumann, 'Generosity and characters of multiply transitive permutation groups' *Proc. London Math. Soc.* (3) 31 (1975) 457–481.
- [26] P. M. Neumann, 'Some questions of Edjvet and Pride about infinite groups', *Illinois J. Math.* (2) 30 (1986) 301–316.
- [27] A. Yu. Ol'Shanskiĭ, *Geometry of defining relations in groups*, Mathematics and its Applications (Soviet Series) 70 (Kluwer Acad. Publ., Dordrecht, 1991).
- [28] C. E. Praeger, *On homomorphic images of edge-transitive directed graphs*, Australas. J. Combin. 3 (1991) 207–210.
- [29] D. J. S. Robinson, *A course in the theory of groups*, Graduate Texts in Mathematics 80 2nd edn. (Springer-Verlag, New York, 1996).
- [30] J-P. Serre, *Trees* (Springer-Verlag, Berlin, 2003).
- [31] J. Tits, 'Sur le groupe des automorphismes d'un arbre', *Essays on topology and related topics*, Mémoires dédiés à Georges de Rham (eds A. Haefliger and R. Narasimhan, Springer-Verlag, New York, 1970), pp. 188–211.
- [32] V. I. Trofimov, 'Graphs with polynomial growth', *Math. USSR Sbornik* (2) 51 (1985) 405–417.

- [33] M. E. Watkins, ‘Ends and automorphisms of infinite graphs’, *Graph symmetry* (eds G. Hahn and G. Sabidussi), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci. 497 (Kluwer Acad. Publ., Dordrecht, 1997), pp. 379–414.

