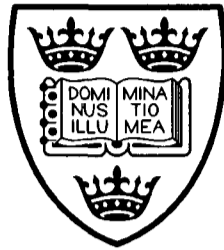


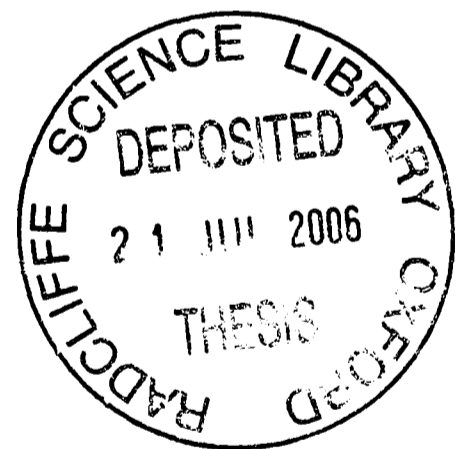
University of Oxford



Estimation of the Variation of Prices using
High-Frequency Financial Data

Carla Mariana Ysusi Mendoza

St Peter's College



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Department of Statistics, 1 South Parks Road, Oxford OX1 3TG, United Kingdom

Abstract

When high-frequency data is available, realised variance and realised absolute variation can be calculated from intra-day prices. In the context of a stochastic volatility model, realised variance and realised absolute variation can estimate the integrated variance and the integrated spot volatility respectively. A central limit theory enables us to do filtering and smoothing using model-based and model-free approaches in order to improve the precision of these estimators.

When the log-price process involves a finite activity jump process, realised variance estimates the quadratic variation of both continuous and jump components. Other consistent estimators of integrated variance can be constructed on the basis of realised multipower variation, i.e., realised bipower, tripower and quadpower variation. These objects are robust to jumps in the log-price process. Therefore, given adequate asymptotic assumptions, the difference between realised multipower variation and realised variance can provide a tool to test for jumps in the process.

Realised variance becomes biased in the presence of market microstructure effect, meanwhile realised bipower, tripower and quadpower variation are more robust in such a situation. Nevertheless there is always a trade-off between bias and variance; bias is due to market microstructure noise when sampling at high frequencies and variance is due to the asymptotic assumptions when sampling at low frequencies. By subsampling and averaging realised multipower variation this effect can be reduced, thereby allowing for calculations with higher frequencies.

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Chapter 1

Introduction

Analysing properties of financial structures is a major focus of financial theory. The issues at stake are the description of price dynamics and the development of concepts and theories to this effect. In the past two decades, markets have become increasingly volatile and new tools are required to quantify the magnitude of fluctuations.

Volatility is a measure of how rapidly the price of a financial asset fluctuates; it is the standard deviation of the returns per unit of time in a time series. Volatility measures the uncertainty of future price changes, and as such is a reflection of risk. Basic pricing models, such as in Black and Scholes (1973), assume constant volatility, but this hardly reflects reality. Stochastic volatility models deal with this issue, but their parameters are not easy to estimate. When high-frequency data is available, both realised variance and realised multipower variation can be used to estimate integrated variance.

Complete records of prices are available for many financial assets at a high-frequency and this enables us to calibrate continuous time models. More precisely, within a semimartingale process the sum of high-frequency squared returns can estimate quadratic variation. This is why realised variance can be used as an estimator of integrated variance in a stochastic volatility model. This result has been extensively studied in the recent literature, Andersen and Bollerslev (1998a), Barndorff-Nielsen and Shephard (2002a) and Comte and Renault (1998). However this is an asymptotic result and infinitesimal returns do not occur in real life. An asymptotic theory of estimation error was developed to distinguish between true underlying variability and measurement noise. This is how filtering and smoothing, carried out both with and without a model, can improve the estimations of integrated variance (see Barndorff-Nielsen, Nielsen, Shephard and Ysusi (2004)). In this thesis we will apply these models and methods to our INTEL dataset.

Now as far as integrated volatility is concerned, studies by Taylor (1986), Ding, Granger and Engle

(1993) and more recently by Forsberg and Ghysels (2004) showed that absolute returns were more persistent than squared returns. Realised absolute variation can be computed from high-frequency data as the sum of absolute intra-day returns in order to estimate the integrated volatility of a stochastic volatility model. Asymptotic considerations on the error distribution presented in Barndorff-Nielsen and Shephard (2003) will allow us to do filtering and smoothing to improve this estimation, with or without a model.

Although realised variance and realised absolute variation estimate different objects, a joint asymptotic distribution will enable us to estimate integrated variance and integrated spot volatility with both estimators. This combined estimation shows that absolute returns should be preferred to squared returns. These early results of the thesis have been recently reinforced by subsequent research (e.g. Forsberg and Ghysels (2004)). The problem with realised absolute variation is that it is unable to estimate integrated variance, however Barndorff-Nielsen and Shephard (2003) introduced a new estimator based on absolute returns that can estimate it, called realised multipower variation.

Even though realised variance consistently estimates integrated variance under a continuous log-price process, whenever log-prices incur jumps, realised variance will estimate the quadratic variations of both continuous and jump components. In Barndorff-Nielsen and Shephard (2004a) realised bipower variation (a special case of realised multipower variation) was introduced as a consistent estimator of integrated variance in the presence of finite activity jumps (see Barndorff-Nielsen, Shephard and Winkel (2005) for the infinite activity case). It only estimates the quadratic variation of the continuous component. In the case of a stochastic volatility process combined with jumps, the difference between realised variance and realised bipower variation gives an estimate of the quadratic variation of the jump component. These properties are at the basis of a test for jumps in the log-price sample path, c.f. Barndorff-Nielsen and Shephard (2006).

Still following the concept of realised multipower variation, realised tripower and quadpower variations can also estimate integrated variance. These estimators are robust to jumps in the log-price process, just as bipower variation, which is why they can also be used to test for the presence of jumps. Here such tests for jump will be established and studied.

Given that tripower and quadpower variation are built from more adjacent absolute returns, these estimators have the potential to be more robust to market microstructure noise than realised bipower variation. This property can also be secured with the skipped version of realised bipower variation.

True price processes are empirically known to be contaminated by market microstructure effects. The observed returns diverge from the true values owing to a variety of market frictions. Market microstructure effects cause mismatches between semimartingales and observations on a very fine time

scale. While it is theoretically necessary to use very small intervals in order to capture volatility, empirically there is an accumulation of noise. Asymptotic results based on the idea of increasing data frequency can be invalidated by microstructure noise. There exists a trade-off between bias and variance across the sampling frequencies.

Even though we show that realised tripower and quadpower variations are more robust to these microstructure effects than realised variance and realised bipower variation, alternative estimators are still needed. Zhang, Mykland and Aït-Sahalia (2005) proposed consistent subsampling and averaging estimators based on realised variance. Following this method, alternative estimators based on bipower, tripower and quadpower variations will be object of further scrutiny in this thesis.

The main contributions of this work are

- the combined estimation of integrated variance using realised variation and realised absolute variation and the analysis of its implications,
- testing for jumps in the log-price process using tripower, quadpower and the skipped version of bipower variation,
- the analysis of the effects of market frictions on multipower variation and the study of more robust estimators based on it.

The programming codes used in this thesis were written in S-PLUS and OX by the author. Selected parts of these codes are included in an Appendix.

1.1 Outline of the thesis

In a first stage, Chapter 2 sets the theoretical framework for this thesis, discusses the main characteristics of financial data and introduces the databases used later on. An overview of the evolution of financial price modelling up to stochastic volatility models is presented, along with the necessary probabilistic and statistical concepts.

Chapter 3 focuses on realised variance and realised absolute variation. Important results for these estimators are presented and the estimation of integrated variance and integrated volatility is carried out empirically based on asymptotic results. We then proceed to jointly estimate integrated variance and integrated volatility from realised variance and the square of realised absolute variation. The use of absolute returns is beneficial in some respects but also poses some problems. This is briefly analysed at the end of the chapter.

In Chapter 3 we have assumed that the price process was continuous. However in Chapter 4 we include a jump component and study realised tripower, quadpower and the skipped version of realised bipower variation as estimators of integrated variance. A hypothesis test for the presence of jumps is formulated based on these estimators as well as realised variance, and the asymptotic distribution theories necessary for these tests are developed in this chapter.

All the work carried out so far can be invalidated in the presence of market microstructure effect. In Chapter 5 this effect is described, along with its potential impact on our estimators. Realised bipower, tripower and quadpower variation are examined in presence of market microstructure effect and new, more robust estimators based on them are introduced.

We conclude in Chapter 6 and present further possible work and finally, some programming codes are included in the Appendix.

Chapter 2

Theoretical background

In this chapter we will describe our data (Section 2.1) and give a small introduction to financial price models (Section 2.2) and necessary probabilistic and statistical concepts (Section 2.3) to situate ourselves in the financial context.

2.1 Financial data

Theoretical and empirical finance is based mostly in our days in continuous time models. This is possible due to computer technology and the improvement of data collection and analysis. Today finance is not only based on yearly, monthly, weekly or daily observation, intra-day data is available and it gives an incessant flow of information. As high-frequency data is available, it allows an almost continuous recording of prices.

There is a crucial difference between daily, weekly,... data and high-frequency data. The first kind is received at equal, regular time intervals while intra-day data arrives at random instants with different time intervals between them. Therefore preliminary data processing is needed. Discretization is applied to fix the time intervals, abnormal observation need to be rejected and interpolation is needed to complete missing data (see for example Shiryaev (1999)).

Although almost continuous record of financial prices are available and can improve our statistical analysis, problems arise due to the market microstructure effects (e.g. discreteness of prices, bid/ask bounce, irregular trading, etc.). Market microstructure effects and possible robust estimators will be studied in chapter 5 of this thesis.

The analysis carried out in this thesis will be based on high-frequency data. We will use asset prices and exchange rates. We will focus on a single actively traded stock, INTEL stock transaction

prices and on the extensively employed United States Dollars/German Deutsche Mark series.

2.1.1 INTEL dataset

Intel stocks are traded on the NASDAQ exchange which predominantly focuses on high technology stocks. The dataset will be constructed from the Trade and Quotes (TAQ) Database. The TAQ Database is the collection of intra-day trades and quotes for all the securities listed in all the main United States of America equity markets. All this information is available on a collection of CD-ROMs, with each month having between 3 and 10 separate CDs. This is a very large database. The database includes information on:

- Process options: trades, quotes, daily statistics;
- Time periods: dates ranges, years, quarters, time ranges,...;
- Security: all securities are listed;
- Number format: no decimal, decimal;
- Time format: hour/min/sec, seconds,...;
- Date format: YY/MM/DD, DD/MM/YY,...;
- Quotes and Trades fields: exchange, time, bid, bid size,...;
- Exchanges: AMEX, BOSTON, CINCINNATI, MIDWEST, NYSE, PACIFIC, NASD, PHILADELPHIA, INSTINET, CBOE.

Further information can be found on the NYSE website (www.nyse.com).

We will work only with transaction prices although a similar analysis can be done based on quote data. The TAQ database includes each transaction price, but we will take the last recorded price every five minutes to have regularly spaced data (last tick method, e.g. Wasserfallen and Zimmermann(1985)). We take the prices every five minutes from 9:30 a.m. (when the market opens) until 4:00 in the afternoon (when the market closes) for every working day from the first of October of 1998 to the twenty-ninth of September of 2000.

Problems with missing data and split markets can be encountered in this dataset. It consists 39,816 observations (79 observations each day for 504 days), and there are 199 missing values. There are just seven days with missing values, and four of them have missing values because the market closed early

that day. To obtain a complete database, a linear interpolation or a Brownian bridge can be used. Figure 2.1a shows a day with missing values and Figure 2.1b illustrates both methods (observations 56-59 and 61-64 are missing). The problem with the linear interpolation is that the variance of the returns will be approximately zero for that interval, so we will use a Brownian bridge.

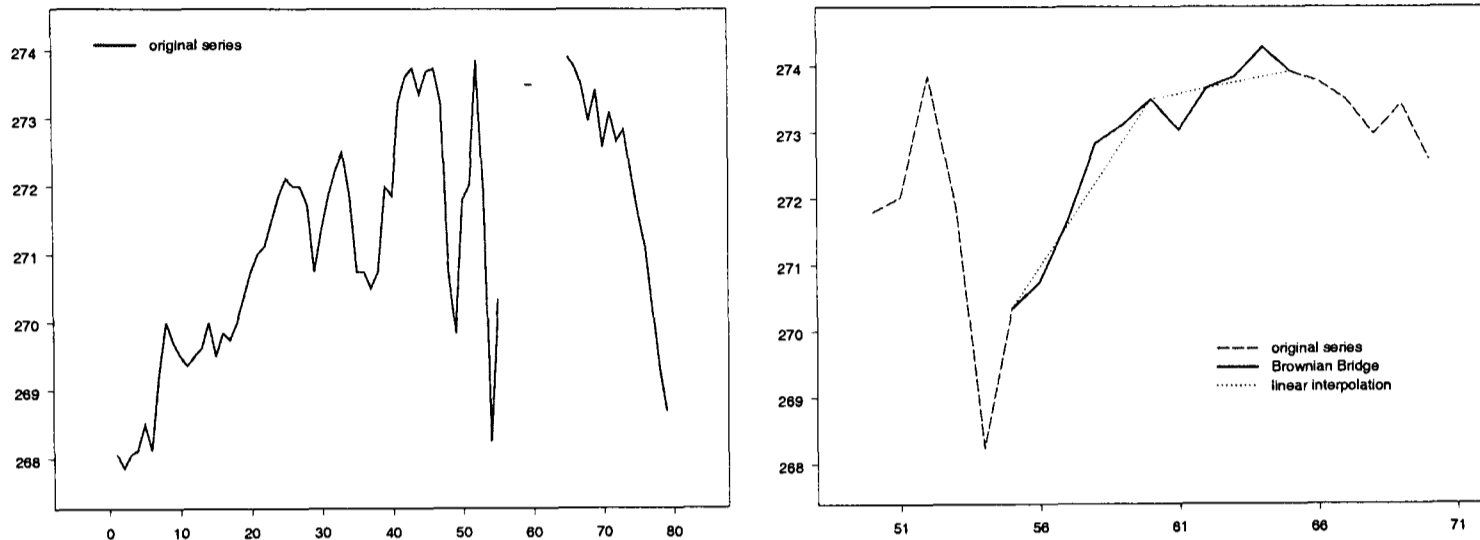


Figure 2.1: a) *One day with missing values and b) the Brownian Bridge and the linear interpolation methods to complete the observations.*

Equity prices used to be decreed by the New York stock exchange to have to be integer multiples of $1/8$ of a Dollar until June 24 of 1997. Afterwards and until January 29 of 2001, they were integer multiples of $1/16$ of a dollar. The commission charged by dealers was a fixed number of these ticks. Whenever the price was too high, the percentage of the commission was very small. The Stock Exchange tried to maintain the prices between ten and one hundred, so when the price was too high, the share was divided into two half-priced ones. Split markets are not a problem for our work because we can always multiply the series by two from the day the share was split on. Also we are working with intra-day prices and markets are split at the end of the day, so they do not affect our returns.

In Figure 2.2a the INTEL prices are plotted for the whole of our sample. We can observe how for the end of August of 2000 the prices had tripled, but afterwards, during the last month of our series, the prices fell abruptly. Figure 2.3a plots the intra-day prices every five minutes on a randomly selected day. With this data structure there are always 79 observations a day. In Figure 2.2b the five minutes returns of the dataset can be observed. The five-minutes intra-day returns, of the randomly selected day can be observed in Figure 2.3b.

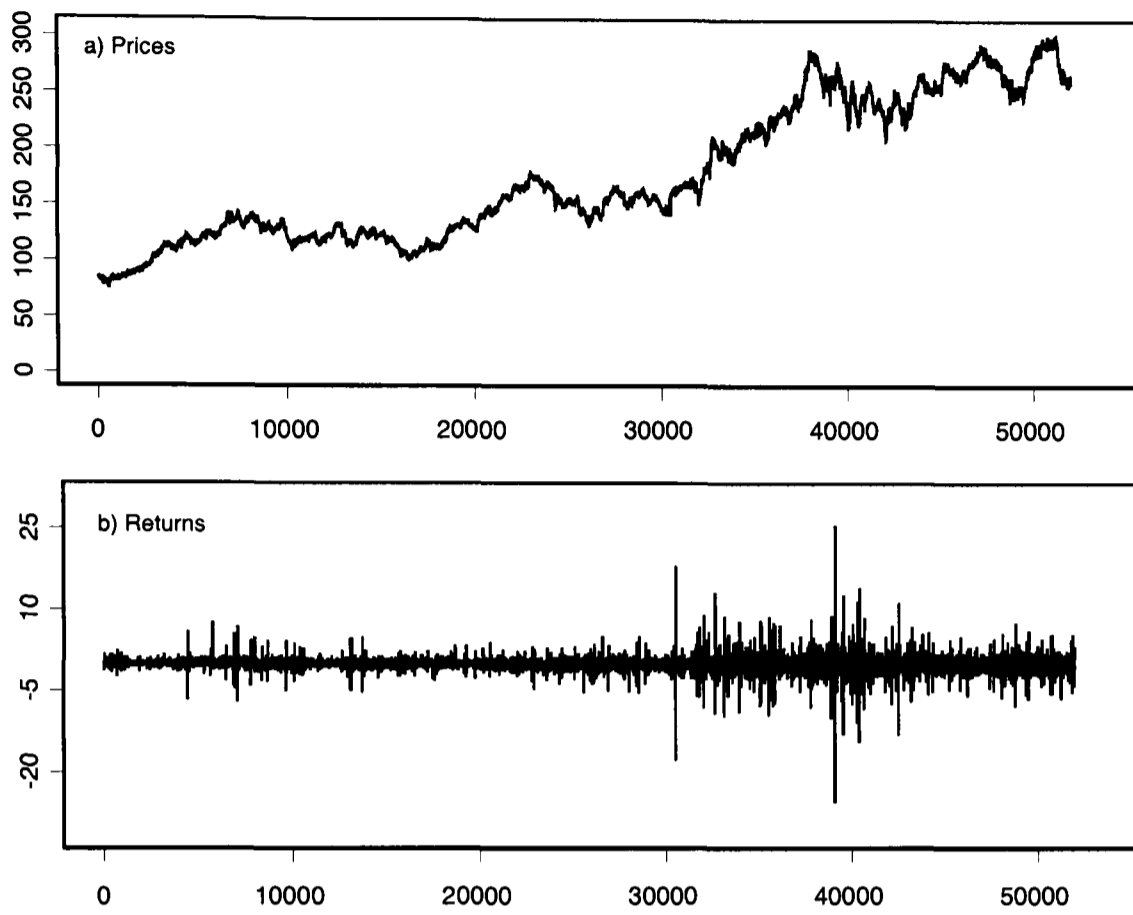


Figure 2.2: a) *INTEL prices without split market*, b) *Intel five minutes returns*.

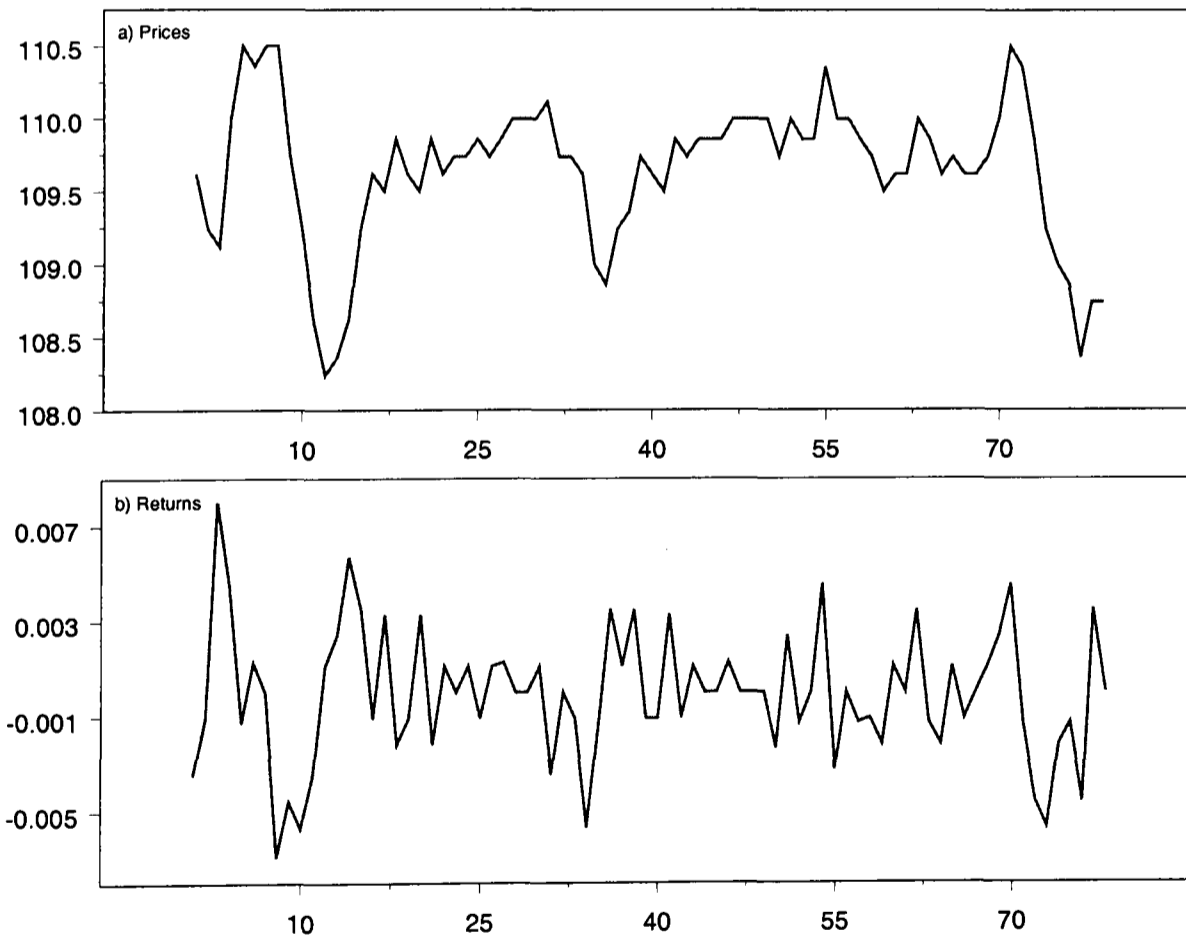


Figure 2.3: a) *Intra-day prices every five minutes* and b) *the corresponding intra-day returns of a randomly selected day of the INTEL series*.

2.1.2 Exchange rates dataset

In this thesis we will also work with the United States Dollars/German Deutsche Mark series employed extensively in previous papers (see for example Andersen, Bollerslev, Diebold and Labys(2001), Barndorff-Nielsen and Shephard(2002a)). It covers from the 1st of December 1986 until the 30th of November of 1996 and ignores weekend breaks. It reports every five minutes the prices computed by averaging the bid and ask log-quotes for the two closest ticks from the Reuters screen (see Dacorogna, Gencay, Müller, Olsen and Pictet (2001)). Missing data were interpolated and a Brownian bridge was added when there were sequences of very small price changes (see Barndorff-Nielsen and Shephard (2002a)). The dataset consists on 705,313 observations. Figure 2.4 shows the complete exchange rates series and the corresponding five minutes returns. The data was shifted for the exchange rate to start at zero at time zero. A random day was selected and the intra-day rates and intra-day five minutes returns are shown in Figure 2.5.

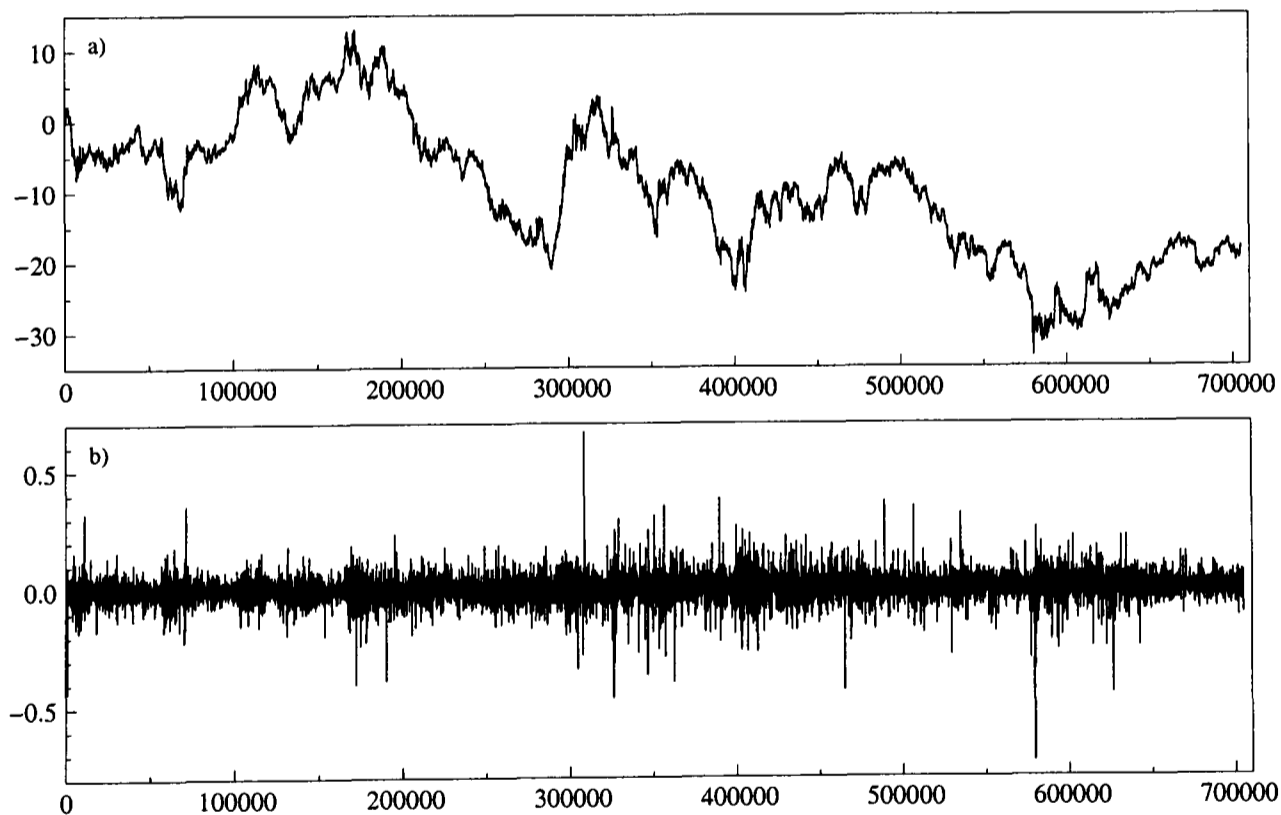


Figure 2.4: a) *United States Dollar/German Deutsche Mark series* and b) *the corresponding returns*.

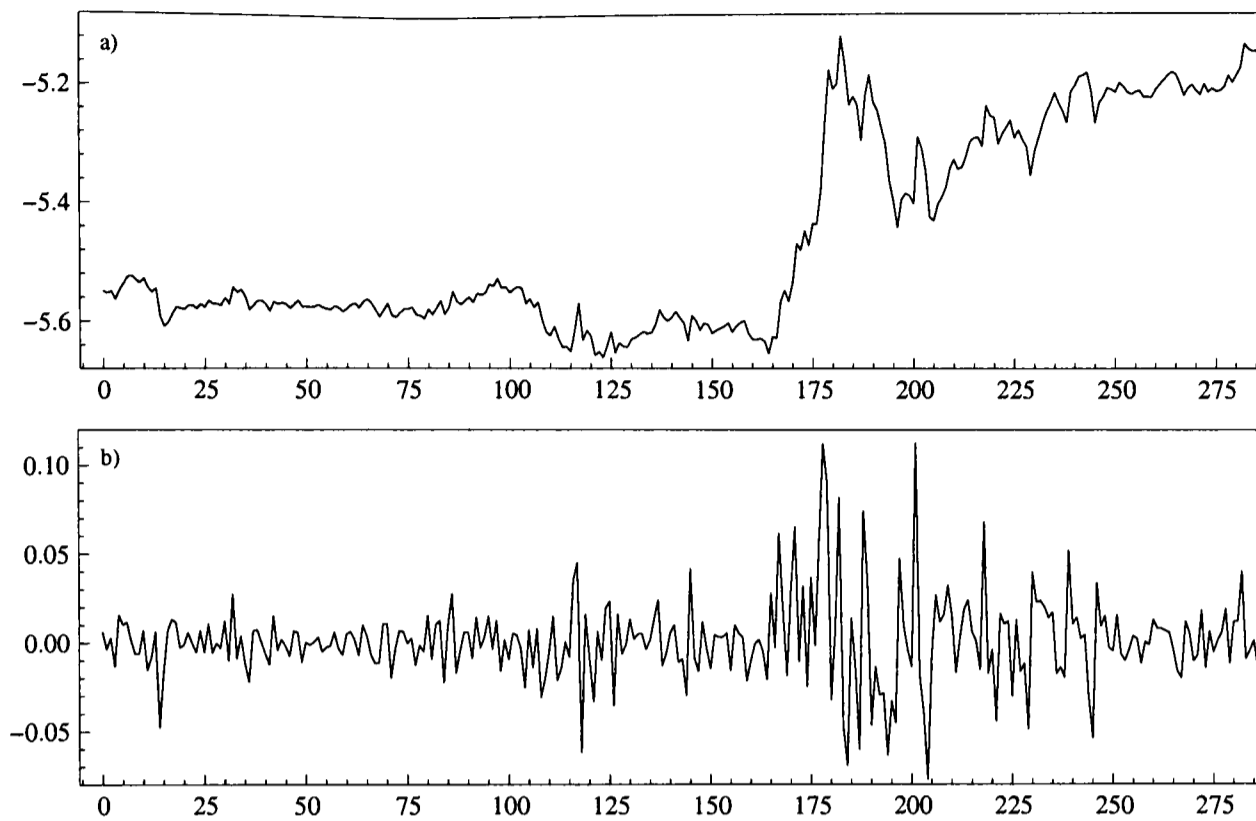


Figure 2.5: a) *Intra-day rates every five minutes and b) the corresponding intra-day returns of a randomly selected day of the exchange rate series.*

2.2 Modelling uncertainty of prices

Although we do not intend to give here a detailed account of the existing theories relating to modelling financial prices and returns, we shall briefly discuss those that will introduce our main topic of volatility. For a more complete discussion see for example Dothan (1990) or Duffie (1992).

Markowitz (1952) introduced the mean-variance analysis which reveals the role of covariance to determine the risk of portfolios and the importance of diversification. These ideas influenced the later theories of CAPM (Sharpe (1964)) and APT (Ross (1976)). These theories are related to the reduction of unsystematic risks, where unsystematic risk is the one that can be reduced by diversification, it is the risk that the investor can influence by his actions.

To understand risk we need to understand the stochastic behaviour of the prices. The first statisticians that addressed this problem were Cowles (1933), Working (1934) and Cowles and Jones (1937). Not much attention was paid to their work even though they arrived to interesting conclusions: independence of returns. Kendall (1953) also studied this problem. His empirical study on the speculative prices showed that they vary in a highly irregular way, so Kendall concluded that the series fluctuate as if "... *the Demon of Chance drew a random number ... and added it to the current price to*

determine the next ... price". The prices seem to behave as a random walk and the returns seem to be independent. Osborne (1959) concluded that the logarithms of the prices were the ones that behaved as a Brownian motion. Later Samuelson (1965) introduced the Geometric Brownian motion to describe the prices.

As it is well known, returns are approximately uncorrelated. This does not mean that they are independent as y_i^2 and y_{i+s}^2 or $|y_i|$ and $|y_{i+s}|$ are positively correlated. This positive correlation explains the clustering phenomenon. The essence of the clustering phenomenon is that large (small) values of y_i imply large (small) subsequent values (of uncertain sign).

Assume the returns can be described with the following model $y_i = \sigma \epsilon_i$ where the ϵ_i are independent standard normally distributed variables such that $\epsilon_i \sim N(0, 1)$ and the σ is a constant that is the standard deviation of the returns called volatility. However, this model is inconsistent with real data. The returns' empirical distribution densities are not normal; they have heavier tails and are more leptokurtotic. In Figure 2.6 a histogram of the INTEL daily returns shows that they are far from Gaussian.

More complicated models should be implemented to incorporate these properties. The assumption that the unconditional distribution of the returns is Gaussian can be analytical attractive but does not reflect the real behaviour of prices. So now we will assume that the conditional distribution of the returns is Gaussian, i.e.

$$Law(y_i | \mathcal{F}_{i-1}) = N(0, \sigma_i^2)$$

for some \mathcal{F}_{i-1} -measurable variable $\sigma_i^2 = \sigma_i^2(\omega)$ where \mathcal{F}_i represents the history of the y_i process. This will be formalised in the next section of this chapter. Then the unconditional distribution of the returns $Law(y_i)$ is a mixture of a conditionally normal distribution $Law(y_i | \mathcal{F}_{i-1})$ averaged over the distribution of σ_i^2 . This will extend the class of linear Gaussian models incorporating non-linear conditionally Gaussian models.

The first model of this type introduced into the literature was the ARCH model by Engle (1982). It incorporates the cluster phenomenon and defines

$$\sigma_i^2 = \alpha_0 + \alpha_1 y_{i-1}^2 + \dots + \alpha_q y_{i-q}^2.$$

Many other models were constructed based on the ARCH model trying to incorporate the other empirical distribution characteristics. The best known is the GARCH model by Bollerslev (1986) in which

$$\sigma_i^2 = \alpha_0 + \alpha_1 y_{i-1}^2 + \dots + \alpha_q y_{i-q}^2 + \beta_1 \sigma_{i-1}^2 + \dots + \beta_p \sigma_{i-p}^2.$$

There exist various generalizations of these models as EGARCH, AGARCH, NARCH, MARCH, HARCH, STARCH. For a extensive review about all these models see Bollerslev, Engle and Nelson (1994).

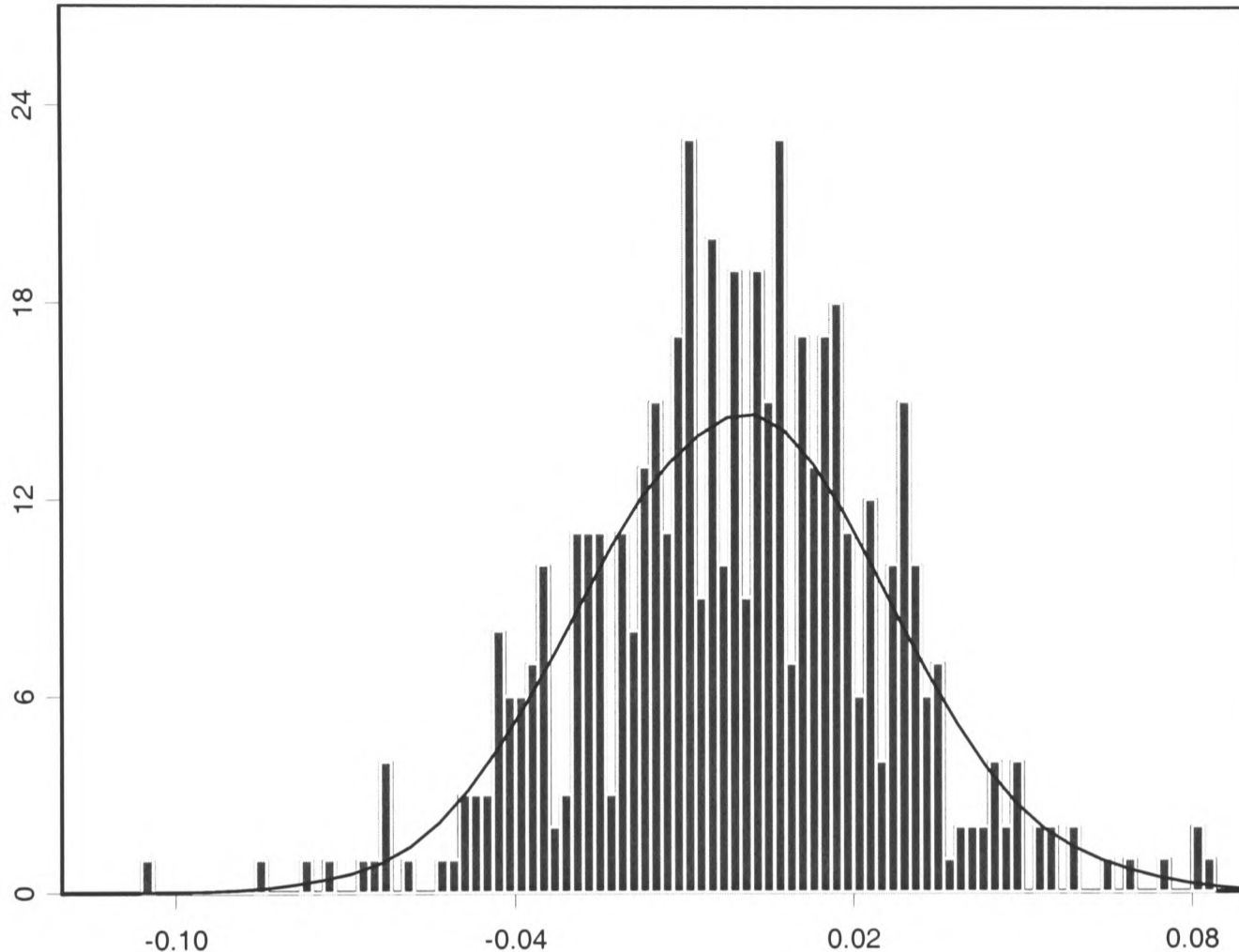


Figure 2.6: *INTEL daily returns histogram.*

In this ARCH-GARCH type of models there is a single source of randomness, the Gaussian sequence of independent ϵ variables. Stochastic volatility (SV) models involve two sources of randomness instead. In the stochastic volatility models σ_i^2 does not depend on past observations but on unobserved components or latent structure. The most popular of these stochastic volatility models from Taylor (1986) sets

$$y_i = \epsilon_i \exp(h_i/2), \quad h_{i+1} = \gamma_0 + \gamma_1 h_i + \eta_i$$

where $h_i = \log(\sigma_i^2)$ and ϵ_i and η_i are the two sources of randomness assumed to be independent standard Gaussian sequences. The main difficulty of SV models is that usually it lacks of an analytical one-step-ahead forecast density $y_i | Y_{i-1}$, so it is not clear how to evaluate the likelihood.

If high-frequency data is available, realised absolute variation, variance and power variation can

be constructed to estimate integrated variance and to estimate the SV models. We will define these concepts in the following chapters. More about stochastic volatility can be found in Taylor (1994), Ghysels, Harvey, and Renault (1996), Shephard (1996) and Shephard (2005).

2.3 Probabilistic and statistical concepts

Before proceeding we review some concepts and methods of probability theory following Mikosch (1998) which provide the framework for our study.

Taking Kolmogorov's probabilistic approach (Kolmogorov (1933)) we shall assume a probability space (Ω, \mathcal{F}, P) where Ω is the space of events ω ; \mathcal{F} is a σ -algebra of subsets of Ω ; and P is a probability on \mathcal{F} . In our context Ω can be interpreted as the market situation and \mathcal{F} as the observable market events.

We can define our probability space more specifically by assuming that we have a flow $(\mathcal{F}_t)_{t \geq 0}$ of σ -algebras such that

$$\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \dots \subseteq \mathcal{F}_t \subseteq \dots \subseteq \mathcal{F}.$$

This flow is called a filtration, the probability space is called a filtered probability space or stochastic basis and \mathcal{F}_t can be interpreted as the information that is available up to time t inclusive. Then if we say prices S_t are \mathcal{F}_t -measurable, we mean prices are formed on the basis of the developments observable on the market up to time t .

A stochastic process $X = (X_t)_{t \geq 0}$ on a stochastic basis is a collection X_t of random variables on a probability space such that for each t X_t is \mathcal{F}_t -measurable. The paths of a stochastic process (the random function $t \mapsto X_t = X(t, \omega)$) usually can be right-continuous with left limits (*càdlàg*) or left-continuous with right limits (*càglàd*). Sometimes the paths are jump-functions that change only at jump discontinuities.

A crucial difference between left-continuous functions and right-continuous ones is that with left-continuity, one can predict the value at t knowing the values before t . So a predictable σ -algebra \mathcal{P} is the σ -field generated by the adapted *càglàd* process.

A filtration is usually linked up with a stochastic process. The stochastic process X is said to be adapted to the filtration \mathcal{F}_t if

$$\sigma(X_t) \subset \mathcal{F}_t \text{ for all } t \geq 0.$$

The stochastic process X is always adapted to the natural filtration generated by X :

$$\mathcal{F}_t = \sigma(X_s, s \leq t).$$

Thus adaptedness of a stochastic process X means that the X_t s do not carry more information than \mathcal{F}_t .

Now consider a stochastic process X and some information \mathcal{F}_s at the present time, so let us ask how this information influences the knowledge of the future behaviour of X . If \mathcal{F}_s and X are dependent, X_t can be better predicted with the information \mathcal{F}_s than without it. The conditional expectation of X_t given \mathcal{F}_s

$$E(X_t | \mathcal{F}_s)$$

is the mathematical tool to describe this.

2.3.1 Martingales

The notion of martingale is crucial in stochastic finance. Martingales represent fair games; in price dynamics, the notion of martingale underlie the idea of markets without opportunities of arbitrage (no riskless profits are possible). We give now some basic definitions, but a more extensive description about martingales and stochastic calculus and the proofs of theorems and lemmas can be found in Mikosch (1998) and Shiryaev (1999) (with a financial approach) or Øksendal (1998) and Protter (1990) (for a more mathematical exposition).

Martingale. A stochastic process is a martingale with respect to the filtration \mathcal{F}_t if

- $E(|X_t|) < \infty$ for all $t \geq 0$
- X is adapted to \mathcal{F}_t
- $E(X_t | \mathcal{F}_s) = X_s$ for all $0 \leq s \leq t$.

Martingale Difference. We call a (discrete) stochastic process $x = (x_n)$ a martingale difference if

- $E(|x_n|) < \infty$ for all $n \geq 0$
- x is adapted to \mathcal{F}_n
- $E(x_n | \mathcal{F}_{n-1}) = 0$.

Clearly, if x is such a process then the corresponding process of sums X , where $X_n = X_0 + x_1 + \dots + x_n$, is a martingale.

A very important concept is that of a local martingale. For it we need the following definition of a stopping time. A random variable τ is a stopping time for the filtration \mathcal{F}_t if $\{\tau \leq t\} \in \mathcal{F}_t$ for all t . We can interpret τ as the time when an investor changes his holdings, he has to decide whether to sell or not at t with the information available until t .

Local Martingale. We call a stochastic process X a local martingale if there exists an increasing sequence of stopping times (τ_k) (called localizing sequence) and each stopped sequence

$$X^{\tau_k} = (X_{\tau_k \wedge t}, \mathcal{F}_t)$$

is a martingale.

A sufficiently general class of stochastic processes for financial economics is the semimartingale class. This class is not just fairly wide, but it is important as we can define stochastic integrals with respect to semimartingales.

Semimartingale. An adapted, *càdlàg* process is a semimartingale if it can be representable as

$$X_t = X_0 + A_t + M_t$$

where A is a process of locally bounded variation and M is a local martingale, both defined in some stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$. A function f is of bounded variation if the supremum of its variation $\sum_i |f(x_{i+1}) - f(x_i)|$ over partitions x_i is finite.

Quadratic Variation. The concept of quadratic variation will be essential in our later exposition, and it is one of the most important aspects of semimartingales. Quadratic variation is defined as

$$[X]_t = p - \lim_{r \rightarrow \infty} \sum \{X(t_{i+1}^r) - X(t_i^r)\}^2$$

for any partition $0 = t_0^r < t_1^r < \dots < t_{m_r}^r = t$ where $\sup_i \{t_{i+1}^r - t_i^r\} \rightarrow 0$ for $r \rightarrow \infty$. This series looks at the partial sums of squared increments over tiny intervals of time.

2.3.2 Lévy process and Brownian motion

Lévy process

The Lévy processes are certain stochastic processes with independent increments. They form a very important class of stochastic processes which include Brownian motions and Poisson processes. A Lévy process is a semimartingale.

Lévy Process. A càdlàg stochastic process $X = (X_t)_{t \geq 0}$ defined on the probability space (Ω, \mathcal{F}, P) is called a (d-dimensional) Lévy process if

- $X_0 = 0$ (P-a.s.);
- for each $n \geq 1$ and each collection t_0, t_1, \dots where $0 \leq t_0 < t_1 < \dots < t_n$, the variables $X_{t_0}, X_{t_1} - X_{t_0}, \dots, X_{t_n} - X_{t_{n-1}}$ are independent (independent increments);
- for all $s \geq 0$ and $t \geq 0$,

$$X_{t+s} - X_s \stackrel{d}{=} X_t - X_0$$

(the homogeneity property of increment).

A more formal discussion of these issues is given in Sato (1999).

Brownian motion

Brownian motion is named after the biologist Robert Brown whose research dates to the 1820's. Early in this century, Bachelier (1900), Einstein (1905) and Wiener (1923) started developing the mathematical theory of Brownian motion, but Wiener (1923) was the first to put Brownian motion on a firm mathematical basis.

Brownian Motion. We call a continuous Gaussian random process $B = (B_t)_{t \geq 0}$ a standard Brownian motion (BM) or a Wiener process if

- $B_0 = 0$;
- $E(B_t) = 0$;
- $Cov(B_s, B_t) = \min(s, t)$.

With this definition we can obtain that this is a process with homogeneous independent Gaussian increments. Hence, the Wiener process is a Lévy process with additional important characteristic of continuous trajectories.

Some properties of Brownian motion.

1. The sample paths of a Brownian Motion are non-differentiable. Brownian sample paths do not have bounded variation on any finite interval $[0, T]$.

2. **Brownian Scaling.** For any $c > 0$ and $t \geq 0$, write

$$B_t^{(c)} := c^{-1} B_{c^2 t}.$$

Then $B^{(c)}$ is Gaussian, with mean zero and variance t and covariance $\text{cov}(B_s^{(c)}, B_t^{(c)}) = \text{cov}(B_s, B_t)$.

3. **Time Inversion.** Write

$$X_t := t B_{1/t}.$$

Then X has mean zero and covariance $\text{cov}(X_s, X_t) = \min(s, t)$. So X is a Brownian motion obtained from B by time inversion.

4. **Quadratic Variation.** As defined before, take a partition of $[0, t]$ such that $0 = t_0^{(n)} < t_1^{(n)} < \dots < t_n^{(n)} = t$. The mesh of the partition $\sup_i (t_i^{(n)} - t_{i-1}^{(n)}) \rightarrow 0$ when $n \rightarrow \infty$, so the quadratic variation of the Brownian motion converge in mean square

$$\sum (B_{t_{i+1}} - B_{t_i})^2 \rightarrow t.$$

From here we can also obtain an essential result

$$(dB_t)^2 = dt.$$

Some processes derived from Brownian motion

Brownian Motion with Drift. If we form $\mu t + \sigma B_t$ (replace $N(0, t)$ by $N(\mu t, \sigma t)$ in the definition of Brownian increments), we obtain a Brownian motion with drift μ and diffusion coefficient σ , $BM(\mu, \sigma)$.

Brownian Bridge. This process is derived from a Brownian motion. Consider the process

$$X_t = B_t - t B_1$$

where B_t is a Brownian motion and $0 \leq t \leq 1$. Then we have that

$$X_0 = B_0 - 0 B_1 = 0 \quad \text{and} \quad X_1 = B_1 - 1 B_1 = 0,$$

for this reason it is called standard Brownian bridge or tied down Brownian motion.

Geometric Brownian Motion. A Brownian motion may assume negative values, which is not a property of financial prices. Paul Samuelson (1965) pointed out that the logarithms of prices, and not the prices themselves, varied in accordance with a Brownian motion giving the Geometric Brownian motion

$$S_t = S_0 \exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma B_t\right\}.$$

2.3.3 Itô Calculus

Here we will discuss some aspects of the theory of stochastic integrals, consider $\int_0^t H_s dM_s$ where the integrator M is a semimartingale and the integrand H is a previsible and bounded process. Generally M is a Brownian motion. This integral can be interpreted in an economic way. Think of the integrator M as a stock price. The increments over $[t, t + u]$ represent new information. Think of the integrand H as the amount of stock held. The investor has no advance warning of the price change over the immediate future but has to base his decisions in what he knows up to then. The value of the portfolio is the stochastic integral $\int_0^t H_s dM_s$.

Remember the sample paths of a Brownian motion are nowhere differentiable and have unbounded variation. Then the Brownian sample paths cannot be integrated with respect to themselves in a context of ordinary integrals. We must define in this section the Itô stochastic integral and the Itô lemma (it is the stochastic analogue of the ordinary chain rule of differentiation).

Itô stochastic integral

It is impossible to define the integral $\int_0^t B_s dB_s$ path by path as a Riemann or Riemann-Stieltjes integral. Usually we can define $\int_0^T f_t dB_t$ with $0 \leq t \leq T$ as the limit of $\sum_j f_{t_j^*} (B_{t_j} - B_{t_{j-1}})$ where $t_j^* \in [t_{j-1}, t_j]$ and when $n \rightarrow \infty$ in the partition $0 = t_0 < \dots < t_n = T$. Itô integral is defined when $t_j^* = t_{j-1}$. This fact is important in finance because the only available information is the past.

Itô (1944) extended the notion of stochastic integral. The Itô integral is defined as the mean square limit of suitable Riemann-Stieltjes sums. Let the process $C = (C_t, t \in [0, T])$ be the integrand of the stochastic integral where C is adapted to the Brownian motion (i.e. C_t is a function of $B_s, s \leq t$) and the integral $\int_0^T E(C_s^2) ds$ is finite. Then one can find a sequence $(C^{(n)})$ of simple processes (i.e. stochastic processes whose sample paths are step functions) such that

$$\int_0^T E(C_s - C_s^{(n)})^2 ds \rightarrow 0.$$

A process $I(C)$ exists on $[0, T]$ such that

$$E\left[\sup_{0 \leq t \leq T} (I_t(C) - I_t(C^{(n)}))^2\right] \rightarrow 0.$$

The mean square limit $I(C)$ is called the Itô stochastic integral of C . It is denoted by

$$I_t(C) = \int_0^t C_s dB_s \quad t \in [0, T].$$

For a given partition $0 = t_0 < t_1 < \dots < t_{n-1} < t_n = T$ and $t \in [t_{k-1}, t_k]$, $I_t(C)$ is close to the

Riemann-Stieltjes sum

$$\sum_{i=1}^{k-1} C_{t_{i-1}}(B_{t_i} - B_{t_{i-1}}) + C_{t_{k-1}}(B_t - B_{t_{k-1}}),$$

and this approximation is the closer (in mean square sense) to the value of $I_t(C)$ the more dense the partition is.

Much more could be explained about Itô stochastic integral, but that is not the aim of this work, so for further reference see for example Mikosch (1998), Shiryaev (1999), Øksendal (1998) or Protter (1990).

Itô lemma

Now that the Itô stochastic integral has been defined, we need a tool to calculate them and to do some operation. This tool is the Itô lemma.

The Itô lemma is the stochastic analogue of the classical chain rule of differentiation. Recall the deterministic chain rule: Let f and g be differentiable functions, then $[f(g_s)]' = f'(g_s)g'_s$. In integral form: $f(g_t) - f(g_0) = \int_0^t f'(g_s)dg_s$.

Now consider a stochastic process $X = (X_t)_{t \geq 0}$ defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$, it is an Itô process if there exist two nonanticipating processes $a = (a(t, \omega))_{t \geq 0}$ and $b = (b(t, \omega))_{t \geq 0}$ such that

$$P\left(\int_0^t |a(s, \omega)| ds < \infty\right) = 1, \quad t > 0,$$

$$P\left(\int_0^t b^2(s, \omega) ds < \infty\right) = 1, \quad t > 0,$$

and

$$X_t = X_0 + \int_0^t a(s, \omega) ds + \int_0^t b(s, \omega) dB_s,$$

where $B = (B_t, \mathcal{F}_t)_{t \geq 0}$ is a Brownian motion and X_0 is a \mathcal{F}_0 -measurable random variable.

For brevity we can consider the following stochastic differential notation:

$$dX_t = a(t, \omega)dt + b(t, \omega)dB_t.$$

Now let $F(t, x)$ be a function from the class $C^{1,2}$ and let X be a process with the previous stochastic differential. For each $t > 0$, we have the following Itô formula for $F(t, X_t)$,

$$F(t, X_t) = F(0, X_0) + \int_0^t \left[\frac{\delta F}{\delta s} + a(s, \omega) \frac{\delta F}{\delta x} + \frac{1}{2} b^2(s, \omega) \frac{\delta^2 F}{\delta x^2} \right] ds + \int_0^t \frac{\delta F}{\delta x} b(s, \omega) dB_s.$$

The proof of the lemma and further discussion can be found in many books; refer, for example, to Durrett (1984) or Rogers and Williams (1987).

Stochastic differential equations

A stochastic differential equation (SDE) is the equation

$$X_t = X_0 + \int_0^t a(s, X_s)ds + \int_0^t b(s, X_s)dB_s$$

where the first integral is a Riemann integral and the second one is an Itô stochastic integral. The Brownian motion B is the driving process of the SDE.

The SDE has two kinds of solutions, strong and weak solutions. A strong solution to the SDE $dX_t = a(t, X_t)dt + b(t, X_t)dB_t$ is a stochastic process $X = (X_t)_{t \geq 0}$ that satisfy the following conditions:

- X is adapted to Brownian motion, i.e. at time t it is a function of B_s , $s \leq t$,
- the Riemann and Itô integrals are well defined,
- X is a function of the underlying Brownian sample path and of the coefficient functions $a(t, x)$ and $b(t, x)$.

For weak solutions the path behaviour is not essential, we are only interested in the distribution of X . The initial condition X_0 and the coefficient functions $a(t, x)$ and $b(t, x)$ are given, so we have to find a Brownian motion such that the SDE holds. Weak solutions X are sufficient in order to determine the distributional characteristics of X ; we do not have to know the sample paths of X . A strong or weak solution X of the SDE is called a diffusion.

Ornstein-Uhlenbeck process

A Ornstein-Uhlenbeck (OU) process is a solution of the SDE

$$dX_t = -\lambda X_t dt + \sigma dB_t$$

with $\lambda > 0$. Using the Itô formula we can verify that the process $X = (X_t)_{t \geq 0}$ with

$$X_t = X_0 e^{-\lambda t} + \sigma \int_0^t e^{-\lambda(t-s)} dB_s$$

is its unique solution.

We can generalize this process by using a Lévy process instead of a Brownian motion obtaining the following SDE

$$dX_t = -\lambda X_t dt + dZ_t.$$

A key property of the OU process is mean reversion. OU processes are useful for modelling purposes as in stochastic volatility models.

Square root process

The Feller's square root process, also called the Cox-Ingersoll-Ross takes the form

$$dX_t = -\lambda\{X_t - \xi\}dt + \omega\sqrt{X_t}dB_t$$

with ξ , λ and $\omega > 0$ and the assumption that $\xi \geq \omega^2/2$. Notice that the process $(X_t)_{t \geq 0}$ cannot be negative.

The square root process is a special case of the constant elasticity of variance (CEV) process. See, for instance, Feller (1951) or Cox, Ingersoll and Ross (1985).

More about OU and CEV processes can be found in, for example, Rolski, Schmidli, Schmidt and Teugels (1999).

2.3.4 State space representation and Kalman filter

The state space representation is a useful tool when working with time series. It allows the application of the Kalman filter for predicting and smoothing. The state space models, as explained in Harvey (1993) specify the relationship between the state vector (mx1), α_t , and the vector (Nx1) of observed variables at time t, y_t , via a measurement equation

$$y_t = Z_t\alpha_t + d_t + \epsilon_t \quad t = 1, \dots, T$$

where Z_t is a matrix (Nxm), d_t is a vector (Nx1) and ϵ_t is a vector (Nx1) of serially uncorrelated disturbances with mean zero and covariance matrix H_t .

The elements of α_t are not observable but they are known to be generated by the transition equation of a Markov process

$$\alpha_t = T_t\alpha_{t-1} + c_t + \eta_t \quad t = 1, \dots, T$$

where T_t is a matrix (mxm), c_t is a vector (mx1) and η_t is a vector (mx1) of serially uncorrelated disturbances with mean zero and covariance matrix Q_t .

Further assumptions are needed to complete the specifications of the state space system: the initial state vector α_0 has a mean of a_0 and a covariance matrix P_0 ; the disturbances ϵ_t and η_t are uncorrelated with each other in all time periods and are uncorrelated with α_0 .

Representation of an ARMA(p,q) model

The ARMA(p,q) model

$$y_t = \phi_1 y_{t-1} + \cdots + \phi_p y_{t-p} + \epsilon_t + \theta_1 \epsilon_{t-1} + \cdots + \theta_q \epsilon_{t-q}$$

can be represented in the state space form as

$$\begin{aligned} y_t &= z_t' \alpha_t \\ z_t' &= (1 \quad 0_{m-1}') \\ \alpha_t &= \begin{pmatrix} \phi_1 \\ \phi_2 & I_{m-1} \\ \dots \\ \phi_m & 0' \end{pmatrix} \alpha_{t-1} + \begin{pmatrix} 1 \\ \theta_1 \\ \dots \\ \theta_{m-1} \end{pmatrix} \epsilon_t \end{aligned}$$

where $m = \max(p, q + 1)$. Unless $p = q + 1$, some of the AR or MA coefficients will equal zero.

Kalman filter

Once a model is in a state space form, the Kalman filter can be applied. It is a recursive procedure for calculating the optimal estimator of the state vector given all the information that is currently available. Once the end of the series is reached optimal predictions of future observations can be made. Smoothing, a backward recursion, enables optimal estimators of the state space to be calculated at all time using the complete sample.

Filtering

Let a_t be the optimal estimator of α_t based on all the observations up to y_t . Let P_t be the (mxm) covariance matrix of the estimation error, $P_t = E((\alpha_t - a_t)(\alpha_t - a_t)')$.

Assume that at time $t - 1$, a_{t-1} and P_{t-1} are given. The optimal estimator of α_t is given by the prediction equation

$$a_{t|t-1} = T_t a_{t-1} + c_t \quad \text{and} \quad P_{t|t-1} = T_t P_{t-1} T_t' + Q_t$$

for $t = 1, \dots, T$. The estimator of y_t is

$$\tilde{y}_{t|t-1} = Z_t a_{t|t-1} + d_t \quad t = 1, \dots, T.$$

The MSE of the prediction error vector

$$\nu_t = y_t - \tilde{y}_{t|t-1} \quad t = 1, \dots, T$$

is

$$F_t = Z_t P_{t|t-1} Z_t' + H_t.$$

Once the new observations become available, the estimator can be updated by

$$a_t = a_{t|t-1} + P_{t|t-1} Z_t' F_t^{-1} (y_t - Z_t a_{t|t-1} - d_t) \quad \text{and} \quad P_t = P_{t|t-1} - P_{t|t-1} Z_t' F_t^{-1} Z_t P_{t|t-1}$$

for $t = 1, \dots, T$. Given the initial conditions a_0 and P_0 , the Kalman filter gives the optimal estimators as each new observation is available. When the T observations have been processed, prediction and smoothing can be done.

Prediction

The optimal estimator to predict a_{T+l} based on the information available up to T is given by

$$a_{T+l|T} = T_{T+l} a_{T+l-1} + c_{T+l} \quad l = 1, 2, \dots$$

with $a_{T|T} = a_T$. The associated MSE matrix is obtained from

$$P_{T+l|T} = T_{T+l} P_{T+l-1|T} T_{T+l}' + Q_{T+l} \quad l = 1, 2, \dots$$

with $P_{T|T} = P_T$. The predictor is

$$\tilde{y}_{T+l|T} = Z_{T+l} a_{T+l|T} + d_{T+l} \quad l = 1, 2, \dots$$

with prediction MSE

$$MSE(\tilde{y}_{T+l|T}) = Z_{T+l} P_{T+l|T} Z_{T+l}' + H_{T+l}.$$

Smoothing

Once filtered, smoothing will allow us to take into account the information available after time t . The smoothed estimator is based on more information than the filtered one so generally its MSE is smaller. The algorithm consists of recursions which start with the final quantities given by the filter and go backwards. The equations are

$$a_{t|T} = a_t + P_t^* (a_{t+1|T} - T_{t+1} a_t - c_{t+1}) \quad \text{and} \quad P_{t|T} = P_t + P_t^* (P_{t+1|T} - P_{t+1|t}) P_t^{*'}.$$

where $P_t^* = P_t T_{t+1}' P_{t+1|t}^{-1}$ with $t = T-1, \dots, 1$.

See Harvey (1993) for further description of the state space representation and Kalman (1960), Kalman and Bucy (1961), Harvey (1993) and Hamilton (1994) for further explanation about the Kalman Filter.

Chapter 3

Realised Variance and Realised Absolute Variation

3.1 Introduction

Volatility is a measure of how rapidly the price of a financial asset fluctuates; it is the standard deviation of the returns per unit of time. Volatility measures the uncertainty of future price changes; it reflects the risk and is an essential aspect of pricing. Basic pricing models, such as the Black-Scholes model discussed in Black and Scholes (1973), assume constant volatility, but this assumption hardly reflects reality. Stochastic volatility models deal with this problem, but they are not easy to estimate. When high-frequency data is available, the sum of the squared increments of the intra-day log-prices (called realised variance) and the sum of the absolute values of these increments (called realised absolute variation) can be constructed to estimate integrated variance as explained in this chapter.

Here intra-day prices are used to calculate realised variance and realised absolute variation. Given a stochastic volatility model where the log-prices are a continuous semimartingale, realised variance and realised absolute variation are used to estimate the integrated variance and the integrated spot volatility respectively. As in Barndorff-Nielsen, Nielsen, Shephard and Ysusi (2004) a limit theory enables us to do filtering and smoothing using a model based and model free approach to improve the precision of the estimators. A combined estimation of integrated variance (using realised variance and realised absolute variation) gives guidance about which of these two estimators to use. Finally, an analysis about the benefits and faults of using realised absolute variation is done following the results in Forsberg and Ghysels (2004).

When an underlying continuous process is assumed for the log-prices, the use of high frequency

data to measure volatility can give misleading results because discrete observations are contaminated by market microstructure effects. Although the efficiency of realised variance and absolute variation is measured with asymptotic results, using data at the highest available frequency will not necessarily be the best approach. Nevertheless, in this chapter, we will assume that our observations are not affected by market microstructure noise in order to reach some conclusions; we will come back to this problem in Chapter 5, which will be entirely dedicated to market frictions.

The outline of this chapter is as follows. In Section 3.2 we present some definitions and results for realised variance and realised absolute variation given in the main literature. We apply the methods of estimation proposed in Barndorff-Nielsen, Nielsen, Shephard and Ysusi (2004) to our INTEL dataset in Section 3.3 and in Section 3.4 we extend these for the realised absolute variation. Finally in Section 3.5 we perform and analyse a combined estimation of integrated variance using both realised variance and realised absolute variation.

3.2 Framework and properties

A standard model in financial economics is a stochastic volatility (SV) model for log-prices Y_t which follows the equation

$$Y_t = \int_0^t a_u du + \int_0^t \sigma_s dW_s, \quad t \geq 0, \quad (3.1)$$

where we denote $A_t = \int_0^t a_u du$. The processes σ_t and A_t are assumed to be stochastically independent of the standard Brownian motion W . Here σ_t is called the instantaneous or spot volatility, σ_t^2 the corresponding spot variance and A_t the mean process. A simple example of this process is

$$A_t = \mu t + \beta \sigma_t^{2*}, \quad \text{where} \quad \sigma_t^{2*} = \int_0^t \sigma_s^2 ds.$$

The process σ_t^{2*} is called the integrated variance.

More generally A_t is assumed to have continuous locally bounded variation paths and it is set that $M_t = \int_0^t \sigma_s dW_s$, with the added condition that $\int_0^t \sigma_s^2 ds < \infty$ for all t . This is enough to guarantee that M_t is a local martingale. So the original equation (3.1) can be rewritten as

$$Y_t = A_t + M_t.$$

Under these assumptions Y_t is a continuous semimartingale.

It is essential to define a discretised version of Y_t based on intervals of time of length $\delta > 0$. Given the previous framework, let the δ -returns be

$$y_j = Y_{j\delta} - Y_{(j-1)\delta} \quad j = 1, 2, 3, \dots, \lfloor t/\delta \rfloor.$$

Then, independently of the model for the volatility, if A and σ are stochastically independent of W , this implies that

$$y_j | a_j, \nu_j^2 \sim N(a_j, \nu_j^2) \quad (3.2)$$

where $a_j = A_{j\delta} - A_{(j-1)\delta}$ and $\nu_j^2 = \sigma_{j\delta}^{2*} - \sigma_{(j-1)\delta}^{2*}$. Usually a_j is called the actual mean and ν_j^2 the actual variance.

One of the most important aspects of semimartingales is the quadratic variation (QV), defined as

$$[Y]_t = p \lim_{n \rightarrow \infty} \sum_{j=1}^n (Y_{t_j} - Y_{t_{j-1}})^2,$$

for any sequence of partitions $t_0^{(n)} = 0 < t_1^{(n)} < \dots < t_n^{(n)} = t$ with $\sup_j \{t_j^{(n)} - t_{j-1}^{(n)}\} \rightarrow 0$ as $n \rightarrow \infty$. As A_t is assumed to be continuous and of finite variation we obtain that

$$[Y]_t = [A]_t + 2[A, M]_t + [M]_t = [M]_t = \int_0^t \sigma_u^2 du$$

where

$$[X, Y]_t = p \lim_{n \rightarrow \infty} \sum_{j=1}^n (X_{t_j} - X_{t_{j-1}})(Y_{t_j} - Y_{t_{j-1}}).$$

This holds since the quadratic variation of any continuous, locally bounded variation process is zero (see Hull and White (1987)).

Now, based on the spot volatility, σ_t , we can define alternatively the integrated spot volatility

$$\sigma_t^{[1]*} = \int_0^t \sigma_s ds$$

and the actual spot volatility

$$\nu_j^{[1]} = \sigma_{j\delta}^{[1]*} - \sigma_{(j-1)\delta}^{[1]*} \quad j = 1, 2, 3, \dots, \lfloor t/\delta \rfloor.$$

It is important to notice that the actual spot volatility is different from the square root of the actual variance, so $(\nu_j^{[1]})^2 \neq \nu_j^2$.

More about SV models can be found in, for example, Shiryaev (1999) or Shephard (2005).

3.2.1 Realised variance

Realised variance has been used in financial econometrics for many years, examples include Rosenberg (1972), Merton (1980), Poterba and Summers (1986), Schwert (1989), Hsieh (1991), Zhou (1996), Taylor and Xu (1997), Christensen and Prabhala (1998), Andersen, Bollerslev, Diebold and Ebens (2001), Andersen, Bollerslev, Diebold and Labys (2001). Recent literature using quadratic variation for semimartingales has been the independently and concurrently development by Andersen and Bollerslev (1998a), Comte and Renault (1998) and Barndorff-Nielsen and Shephard (2001). Many other papers on realised variance exist which are discussed in Dacorogna et al. (2001) and in the reviews by Andersen, Bollerslev and Diebold (2005) and Barndorff-Nielsen and Shephard (2005). Here we will give an overview of the main definitions and results from these papers for applications later in this chapter.

The realised quadratic variation process is defined as

$$[Y_\delta]_t^{[2]} = \sum_{j=1}^{\lfloor t/\delta \rfloor} y_j^2,$$

so as $\delta \downarrow 0$

$$[Y_\delta]_t^{[2]} \xrightarrow{p} [Y]_t.$$

For daily series, suppose there are $\lfloor t/\delta \rfloor = M$ intra- h observations during each fixed h time period (here h denotes the period of a day) defined as

$$y_{j,i} = Y_{(i-1)h+j\delta} - Y_{(i-1)h+(j-1)\delta},$$

for the j -th intra- h return for the i -th period. Then the realised variance is defined by the summation of the squares of these M intra-period returns

$$[Y_M]_i^{[2]} = \sum_{j=1}^M y_{j,i}^2,$$

and the realised volatility by its square root

$$\sqrt{[Y_M]_i^{[2]}} = \sqrt{\sum_{j=1}^M y_{j,i}^2}.$$

In the case of the realised variance, under semimartingales, as $M \rightarrow \infty$ then

$$[Y_M]_i^{[2]} = \sum_{j=1}^M y_{j,i}^2 \xrightarrow{p} \nu_i^2 = [Y]_{ih} - [Y]_{(i-1)h},$$

so the realised variance can be used as an estimator of the actual variance.

The realised variance can be calculated for each day separately. Therefore a daily time series of length T can be constructed

$$\sum_{j=1}^M y_{j,1}^2, \sum_{j=1}^M y_{j,2}^2, \dots, \sum_{j=1}^M y_{j,T}^2$$

of realised variances.

Figure 3.1 illustrates a time series of realised variances. In the top part of this figure the realised variances and their ACF were computed using $M=78$, which corresponds to using five minutes returns data based on the INTEL dataset (described in Chapter 2). The realised variances computed using $M=6$, sixty minutes returns, and the correspondent ACF are illustrated in the bottom part of the figure. It can be observed that the realised variance series calculated using five minutes returns ($M=78$) is much less jagged than the series calculated using 60 minutes returns ($M=6$). All the correlograms have the usual slow decay behaviour and it starts at quite a low level for $M=6$.

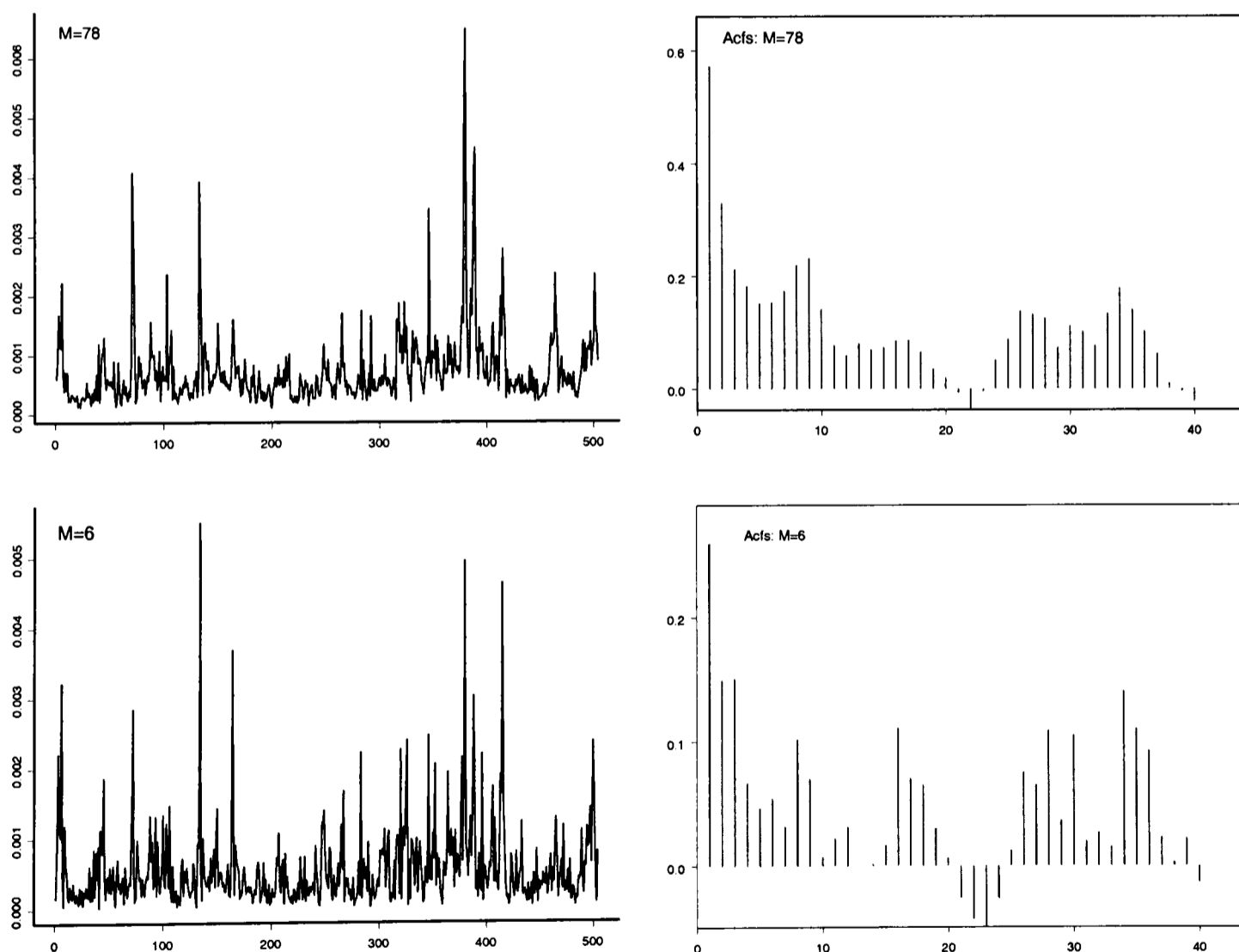


Figure 3.1: *Realised variance series computed using 5 and 60 minutes INTEL returns and its autocorrelation function.*

In Barndorff-Nielsen and Shephard (2002a), Barndorff-Nielsen and Shephard (2003) and Barndorff-Nielsen and Shephard (2004b) the previous theory has been extended to a Central Limit Theorem (CLT). In these papers the CLT is presented under somewhat restrictive assumptions. Recently Barndorff-Nielsen, Graversen, Jacod, Podolskij and Shephard (2005) and Barndorff-Nielsen, Graversen, Jacod and Shephard (2006) give weaker conditions on the log-price process which ensure that the CLT holds.

For the SV model (3.1), when $\delta \downarrow 0$

$$\frac{\delta^{-1/2}([Y_\delta]_t^{[2]} - [Y]_t)}{\sqrt{2 \int_0^t \sigma_s^4 ds}} \xrightarrow{L} N(0, 1),$$

under the assumptions that a is of locally bounded variation, $\int_0^t \sigma_u^2 du < \infty$ and that σ_t is càdlàg.

This limit theory enabled Barndorff-Nielsen and Shephard (2002a) to construct models in order to obtain better estimates of the actual variance using the realised variance.

3.2.2 Realised absolute variation

An alternative approach to the realised variance would be using the sum of the absolute value of the increments of the intra-day log-prices, $\sum_{j=1}^M |y_{j,i}|$, named realised absolute variation. This is empirically attractive for using absolute values is less sensitive to possible large movements in high-frequency data. Taylor (1986) and Ding, Granger and Engle (1993) recognized that empirically absolute returns are more persistent than squared returns. Andersen and Bollerslev (1997) and Andersen and Bollerslev (1998b) empirically studied the properties of realised absolute variation of speculative assets, nevertheless the approach was abandoned in their subsequent work due to the lack of appropriate theory. Ghysels, Santa-Clara and Valkanov (2003) and Forsberg and Ghysels (2004) retake the interest in absolute returns and provide empirical and theoretical explanations for the outperformance of realised absolute variation.

Barndorff-Nielsen and Shephard (2003) derived a limit theorem for the realised power variation process

$$[Y_\delta]_t^{[r]} = \sum_{j=1}^{\lfloor t/\delta \rfloor} |y_j|^r,$$

as $\delta \downarrow 0$. Given that realised absolute variation is a special case of realised power variation, $r = 1$, the asymptotic result can be used to study the properties of the integrated spot volatility.

Barndorff-Nielsen and Shephard (2003) gave the Central Limit Theorem for the realised power variation process with some restrictive conditions. Barndorff-Nielsen, Graversen, Jacod, Podolskij

and Shephard (2005) and Barndorff-Nielsen, Graversen, Jacod and Shephard (2006) provide some general limit results for realised power and bipower variation which are proved under much weaker assumptions.

For the SV model (3.1) where a is of locally bounded variation, $\int_0^t \sigma_u^2 du < \infty$ and σ_t is càdlàg, we have that for $\delta \downarrow 0$,

$$\mu_r^{-1} \delta^{1-r/2} [Y_\delta]_t^{[r]} \xrightarrow{p} \int_0^t \sigma_s^r ds$$

and

$$\frac{\mu_r^{-1} \delta^{1-r/2} [Y_\delta]_t^{[r]} - \int_0^t \sigma_s^r ds}{\mu_r^{-1} \delta^{1-r/2} \sqrt{\mu_{2r}^{-1} v_r [Y_\delta]_t^{[2r]}}} \xrightarrow{L} N(0, 1),$$

where $\mu_r = E(|X|^r)$ and $v_r = Var(|X|^r)$ with $X \sim N(0, 1)$.

In particular, this implies that

$$\frac{d[Y]_t^{[1]}}{dt} = \sigma_t.$$

From these, we can obtain the relevant result for our case ($r = 1$) where

$$\frac{\sqrt{\delta}}{\mu_1} [Y_\delta]_t^{[1]} \xrightarrow{p} [Y]_t^{[1]}$$

where $\mu_1 = \sqrt{2/\pi}$, and where

$$\frac{\frac{\sqrt{\delta}}{\mu_1} [Y_\delta]_t^{[1]} - [Y]_t^{[1]}}{\sqrt{\delta(\mu_1^{-2} - 1)[Y_\delta]_t^{[2]}}} \xrightarrow{L} N(0, 1).$$

If M is the number of intra-day observations during each day as before, then the realised absolute variation, i.e. a scaled sum of the absolute value of the intra-day changes of log-prices, will be defined here as

$$[Y_M]_i^{[1]} = \frac{1}{\sqrt{M} \mu_1} \sum_{j=1}^M |y_{j,i}|$$

and therefore, $[Y_M]_i^{[1]} \rightarrow \sigma_{ih}^{*[1]} - \sigma_{(i-1)h}^{*[1]}$ when $M \rightarrow \infty$. So, $\widehat{\nu}_i^{[1]} = [Y_M]_i^{[1]}$ converges to $\nu_i^{[1]}$.

Consequently we can use the realised absolute variation as an estimator of $\nu_i^{[1]}$, just as the realised variance is used as an estimator of ν_i^2 .

Using the intra-day INTEL prices, the realised absolute variation using five and sixty minutes returns were plotted, as well as their correspondent autocorrelation functions (Figure 3.2). As M gets smaller, the series become more jagged. Also, the autocorrelation is higher when the value of M is big, and it shows a slower decay. If we compare the ACF of the realised absolute variation with the one of the realised variance (Figure 3.1), we can notice it is marginally stronger.

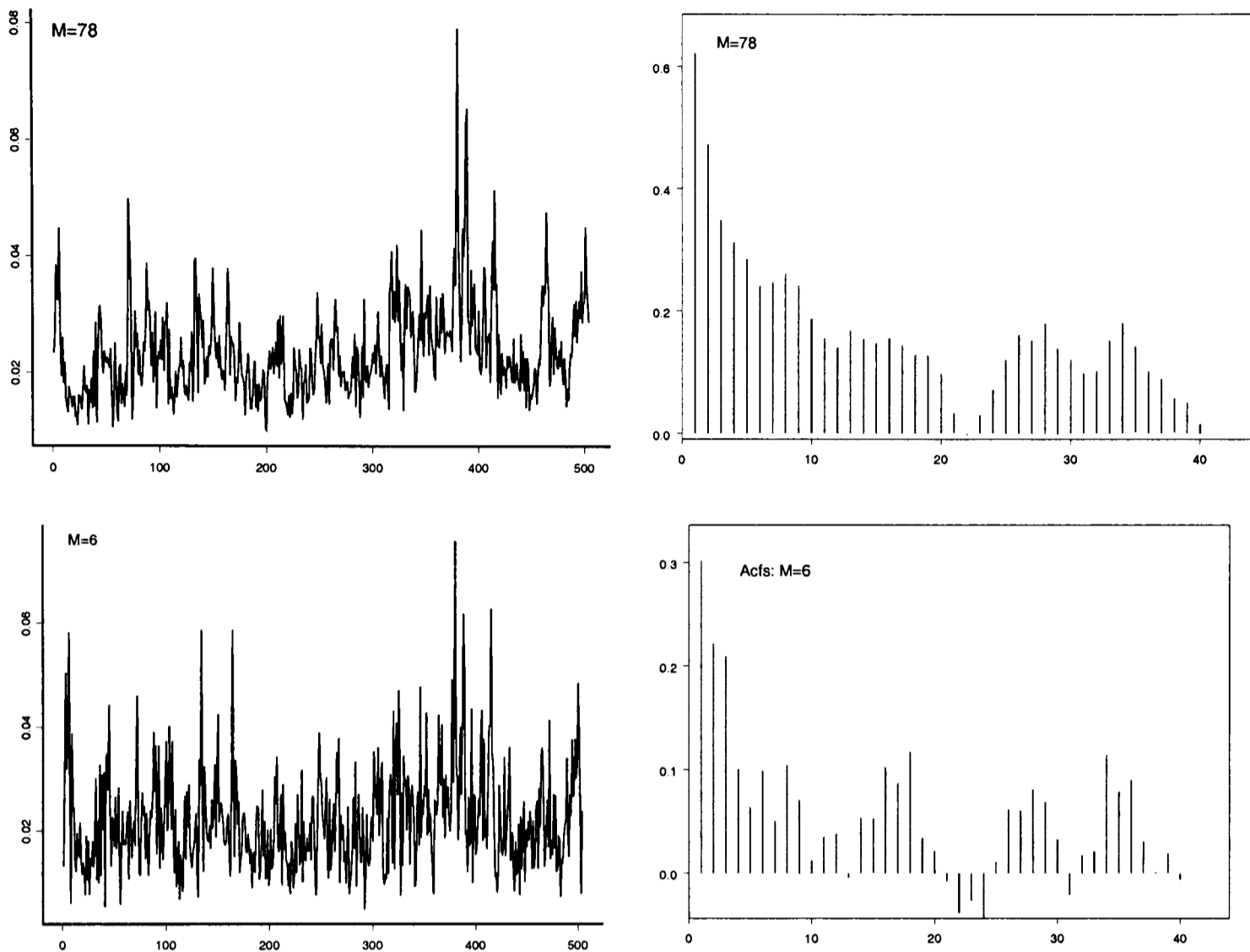


Figure 3.2: *Realised absolute variation computed using 5 and 60 minutes INTEL returns and their autocorrelation function.*

3.3 Estimations with realised variance

Although estimating ν_i^2 by using $[Y_M]_i^{[2]}$ has attractions, the variance of the error is quite high even when using large values of M .

From equation (3.2) we must expect that if the realised variance is a good estimator of the actual variance, then the daily returns divided by the square root of the realised variance must have a Gaussian distribution. In Figure 3.3 we can see from the qq-plots that there are some problems even when we use the five minutes returns to calculate the realised variance.

So while realised variance converges to the integrated variance when $M \rightarrow \infty$, the difference may not be negligible for a given frequency. Meddahi (2002) characterizes this difference for a given frequency using Itô Lemma. The process governing the volatility is more general than the one presented

here, including both leverage effect and drift. Given an Eigenfunction Stochastic Volatility model (Meddahi (2001)) he found the finite sample properties of the difference. Under leverage effect the correlation between the difference and the integrated variance was nonzero but so small that in practice it did not matter. Other authors as Goncalves and Meddahi (2005) and Nielsen and Frederiksen (2005) have studied the finite sample behaviour of realised variance.

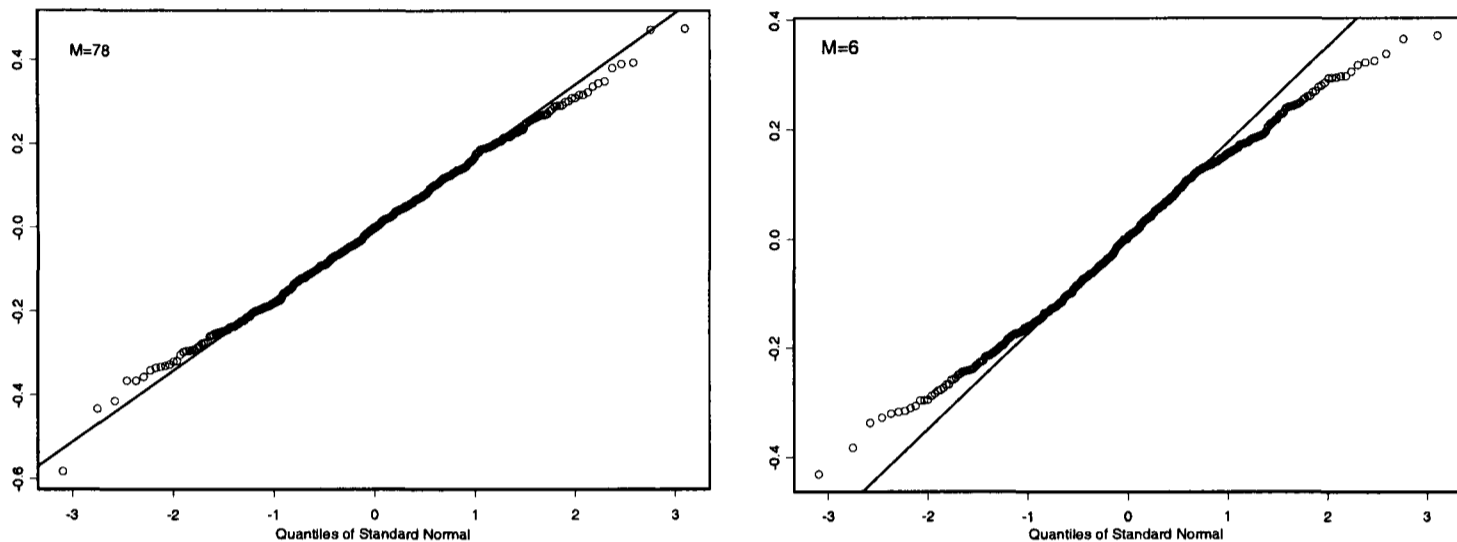


Figure 3.3: *QQ-plot of the daily returns divided by the square root of realised variance calculated using 5 and 60 minutes INTEL returns.*

It is convenient to try to use the complete time series structure to construct more efficient estimators of ν_i^2 and $\nu_i^{[1]}$ using the asymptotic results. More precise estimators can be obtained by pooling neighbouring time series observations, for realised variances and absolute variations tend to be highly correlated through time. So in this section we will apply the methods presented in Barndorff-Nielsen, Nielsen, Shephard and Ysusi (2004) on our INTEL dataset to estimate the actual variance with a linear model of realised variances.

Instead of estimating ν_i^2 using $[Y_M]_i^{[2]}$, the authors introduced a new class of linear estimators of a sequence of actual variances

$$\nu_{s:p}^2 = \left(\nu_s^2, \nu_{s+1}^2, \dots, \nu_p^2 \right)'$$

The estimator is of the form

$$\widehat{\nu}_{s:p}^2 = cE(\nu_{s:p}^2) + A[Y_M]_{s:p}^{[2]}$$

where

$$[Y_M]_{s:p}^{[2]} = \left(\sum_{j=1}^M y_{j,s}^2, \sum_{j=1}^M y_{j,s+1}^2, \dots, \sum_{j=1}^M y_{j,p}^2 \right)'$$

In these models the realised variance as well as the expectation of the actual variance are taken

into account, so if the realised variance does not explain it satisfactorily less weight will be given to the actual variance and more to the expectation. We shall expect from the asymptotic theory that when using a bigger value of M to calculate the realised variance, we will get a higher weight for it.

The asymptotic theory of realised variance implies that, as $M \rightarrow \infty$

$$\sqrt{\frac{M}{h}} \left([Y_M]_{s:p}^{[2]} - \nu_{s:p}^2 \right) | \nu_{s:p}^4 \xrightarrow{L} N \left(0, 2 \text{diag}(\nu_{s:p}^4) \right)$$

where the actual quarticity ν_i^4 is defined as $\int_{(i-1)h}^{ih} \sigma_s^4 ds$. If the A from the model is a matrix of non-stochastic weights, then

$$\sqrt{\frac{M}{h}} \left(A [Y_M]_{s:p}^{[2]} - A \nu_{s:p}^2 \right) | \nu_{s:p}^4 \xrightarrow{L} N \left(0, 2 A \text{diag}(\nu_{s:p}^4) A' \right). \quad (3.3)$$

To solve the model it needs to be assumed that the realised variances are a covariance stationary process, so stationarity is at the daily level. Then $E(\nu_{s:p}^2) = \iota E(\nu_i^2)$ where $\iota = (1, 1, \dots, 1)'$. We should note that it will be convenient if A is a matrix of non-negative weights.

The weighted least squares estimators (see Casella and Berger (1990)) require that

$$c = (I - A)\iota$$

and

$$\begin{aligned} A &= \text{Cov}(\nu_{s:p}^2, [Y_M]_{s:p}^{[2]}) \left[\text{Cov}([Y_M]_{s:p}^{[2]}) \right]^{-1} \\ &= \text{Cov}(\nu_{s:p}^2) \left[\text{Cov}([Y_M]_{s:p}^{[2]}) \right]^{-1}. \end{aligned}$$

This comes from the fact that

$$[Y_M]_i^{[2]} = \nu_i^2 + u_i$$

where u_i are the asymptotically uncorrelated errors.

The asymptotic results were applied by the authors to obtain the covariance of u_i , so

$$A = \text{Cov}(\nu_{s:p}^2) \left[\text{Cov}(\nu_{s:p}^2) + \frac{2h}{M} E(\nu_i^4) I \right]^{-1}.$$

Then as $M \rightarrow \infty$, $\hat{A} \rightarrow I$ and $\widehat{\nu_{s:p}^2} \xrightarrow{p} \nu_{s:p}^2$.

In practice, A should be estimated from the data. They carried it out in two different ways

1. by estimating A by using empirical averages from the data,
2. implying A from an estimated parametric model.

3.3.1 Model free estimator

When estimating A by using empirical averages from the realised variance time series, the assumptions that the realised variances are a covariance stationary process and that the daily process is ergodic are needed. By using the fact that

$$\frac{M}{h} \sum_{i=1}^M y_{j,i}^4 \xrightarrow{p} 3\nu_i^4$$

then

$$\widehat{E(\nu_i^4)} = \left(\frac{1}{T} \sum_{i=1}^T \frac{M}{3h} \sum_{j=1}^M y_{j,i}^4 \right) \xrightarrow{p} E(\nu_i^4),$$

as T and M go to infinity. Likewise $Cov([Y_M]_{s:p}^{[2]})$ can be estimated using averages of the time series.

The authors obtained A with

$$\hat{A} = \left(Cov(\widehat{[Y_M]_{s:p}^{[2]}}) - \frac{2h}{M} \widehat{E(\nu_i^4)} I \right) \left(Cov(\widehat{[Y_M]_{s:p}^{[2]}}) \right)^{-1}.$$

Then

$$\hat{c} = (I - \hat{A})\iota$$

and

$$\widehat{\nu_{s:p}^2} = \hat{c} \widehat{E(\nu_{s:p}^2)} + \hat{A} [Y_M]_{s:p}^{[2]}$$

where

$$\widehat{E(\nu_{s:p}^2)} = \left(\frac{1}{T} \sum_{i=1}^T \sum_{j=1}^M y_{j,i}^2 \right) \xrightarrow{p} E(\nu_i^2).$$

As illustration, a single actual variance is estimated using a single realised variance sequence, $s = p = i$. The weights \hat{c} and \hat{A} are given in Table 3.1 for the INTEL dataset for various values of M. For small values of M the regression estimator puts a small weight on the realised variance and a bigger one on the unconditional mean of the variance. As M increases, this fact reverses.

M	min	\hat{c}	\hat{A}
78	5	0.1313	0.8686
39	10	0.1843	0.8156
26	15	0.2556	0.7443
19	20	0.2829	0.7170
13	30	0.4442	0.5557
6	60	0.6007	0.3992

Table 3.1: Estimated weights for the estimation of ν_i^2 .

Models with lags and leads

In a dynamic case, lags and leads are used. When using one lag and one lead and the contemporaneous realised variance, \hat{A} will be a 3x3 matrix and \hat{c} a 3x1 vector. When using 5 minutes returns, which corresponds to setting $M=78$, we obtain that

$$\hat{A} = \begin{pmatrix} 0.80451 & 0.11113 & 0.00113 \\ 0.11113 & 0.74133 & 0.11113 \\ 0.00113 & 0.11113 & 0.80451 \end{pmatrix} \quad \hat{c} = \begin{pmatrix} 0.08323 \\ 0.03640 \\ 0.08323 \end{pmatrix},$$

so the second row gives the smoothed estimator

$$\hat{\nu}_i^2 = 0.036E(\nu_i^2) + 0.111[Y_{78}]_{i-1}^{[2]} + 0.741[Y_{78}]_i^{[2]} + 0.111[Y_{78}]_{i+1}^{[2]}. \quad (3.4)$$

The corresponding result for 60 minutes returns, $M=6$, is

$$\hat{\nu}_i^2 = 0.374E(\nu_i^2) + 0.153[Y_6]_{i-1}^{[2]} + 0.318[Y_6]_i^{[2]} + 0.153[Y_6]_{i+1}^{[2]}. \quad (3.5)$$

In both cases some weight is given to the neighbouring values of the realised variance, although the weight on the contemporaneous realised variance is not very much smaller than in the univariate case. The weight on the unconditional mean of the variance is the one reduced.

When using two lags and two leads, the following smoothed estimator for the sixty minutes returns is

$$\hat{\nu}_i^2 = 0.321E(\nu_i^2) + 0.043[Y_6]_{i-2}^{[2]} + 0.139[Y_6]_{i-1}^{[2]} + 0.312[Y_6]_i^{[2]} + 0.139[Y_6]_{i+1}^{[2]} + 0.043[Y_6]_{i+2}^{[2]}.$$

Here the usual decay in the weight, as we go further away in time, can be seen.

Figure 3.4 shows the fourth row of \hat{A} for the case of estimating ν_i^2 using three lags and three leads together with $[Y_M]_i^{[2]}$. It displays the weights drawn against lag length. It indicates how quickly the weights focus on $[Y_M]_i$ as M increases.

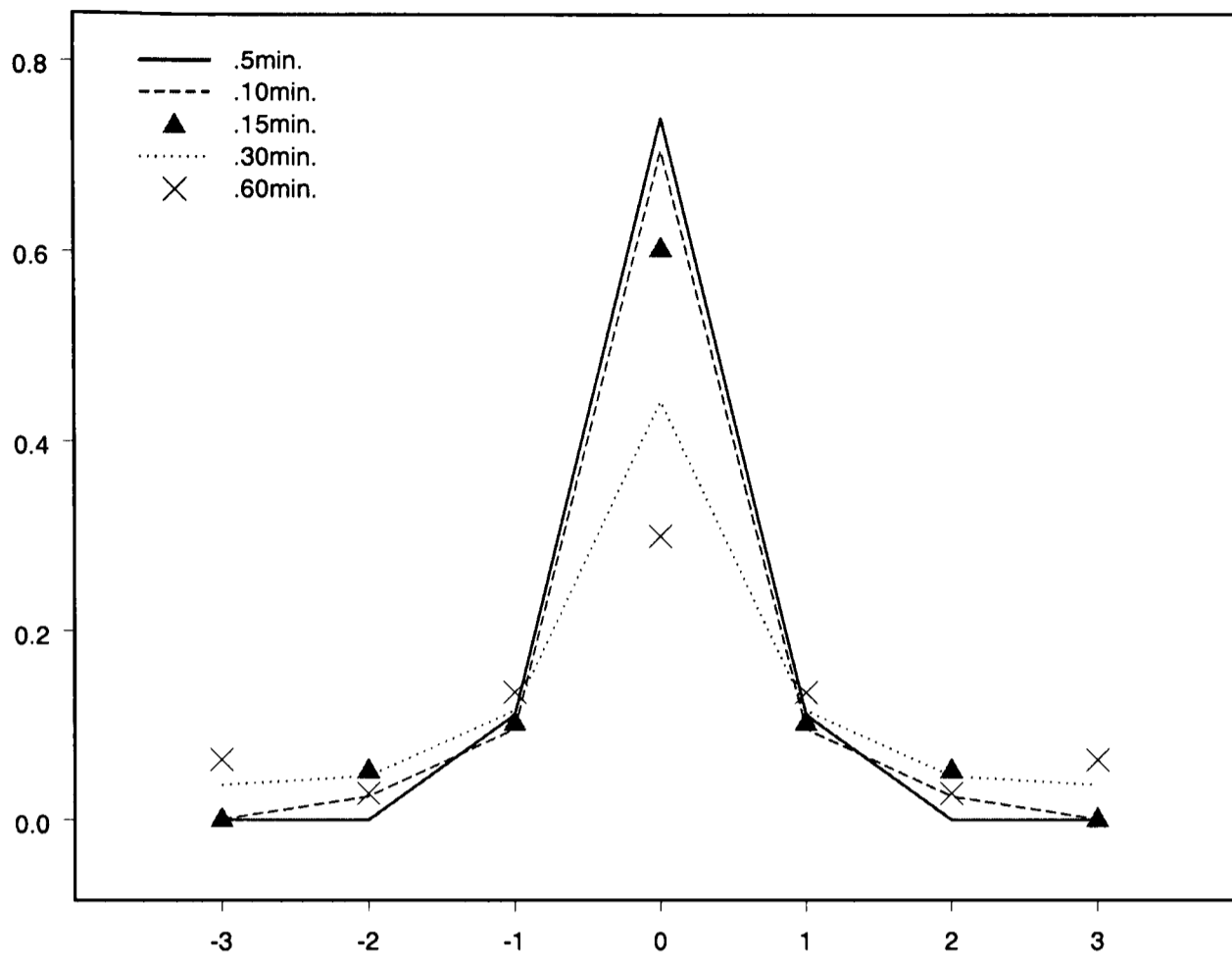


Figure 3.4: *Weights for estimating ν_i^2 using 3 lags and 3 leads for different values of M .*

Logarithms

The estimation can be improved by the use of logarithms. Logarithms can give us a better finite sample behaviour and the limit theorem may be more accurate on logarithmic scale. Theoretically the matrix A of non-stochastic weights should be non-negative, but in practice we can obtain negative values. If we use logarithms, there is no restriction on the signs of the elements of A .

By using log-realised variance

$$\log[Y_M]_{s:p}^{[2]} = \left(\log[Y_M]_s^{[2]}, \log[Y_M]_{s+1}^{[2]}, \dots, \log[Y_M]_p^{[2]} \right),$$

to estimate

$$\log \nu_{s:p}^2 = \left(\log \nu_s^2, \log \nu_{s+1}^2, \dots, \log \nu_p^2 \right)',$$

a similar model to the previous one was established in Barndorff-Nielsen, Nielsen, Shephard and Ysusi (2004)

$$\widehat{\log \nu_{s:p}^2} = cE(\log \nu_{s:p}^2) + A \log[Y_M]_{s:p}^{[2]}. \quad (3.6)$$

Using the asymptotic result of equation (3.3) and the Delta method, it is easy to find the new asymptotic distribution

$$\sqrt{\frac{M}{h}} \left(A \log([Y_M]_{s:p}^{[2]}) - A \log(\nu_{s:p}^2) \right) | \nu_{s:p}^4, \nu_{s:p}^2 \xrightarrow{L} N \left(0, 2AE \begin{pmatrix} \frac{\nu_s^4}{(\nu_s^2)^2} & 0 & \cdots & 0 \\ 0 & \ddots & & 0 \\ \vdots & & & \\ 0 & 0 & & \frac{\nu_p^4}{(\nu_p^2)^2} \end{pmatrix} A' \right).$$

They used the fact that

$$\left(\frac{1}{T} \sum_{i=1}^T \frac{\frac{M}{3h} \sum_{j=1}^M y_{j,i}^4}{\left(\sum_{j=1}^M y_{j,i}^2 \right)^2} \right) \xrightarrow{p} E \left(\frac{\nu_i^4}{(\nu_i^2)^2} \right)$$

and that

$$\frac{1}{T} \sum_{i=1}^T \log[Y_M]_i^{[2]} \xrightarrow{p} E(\log \nu_i^2)$$

to implement the estimator (3.6). Here the weighted least square statistic sets

$$\hat{c} = (I - \hat{A})\iota$$

and

$$\begin{aligned} \hat{A} &= \left(\text{Cov}(\log \nu_{s:p}^2, \log[Y_M]_{s:p}^{[2]}) \right) \left(\text{Cov}(\log[Y_M]_{s:p}^{[2]}) \right)^{-1} \\ &= \left(\text{Cov}(\log \nu_{s:p}^2) \right) \left(\text{Cov}(\log \nu_{s:p}^2) + \frac{2h}{M} E \left(\frac{\nu_i^4}{(\nu_i^2)^2} \right) I \right)^{-1}. \end{aligned}$$

So, as $M \rightarrow \infty$, $\widehat{\log \nu_{s:p}^2} \xrightarrow{p} \log \nu_{s:p}^2$.

In the case a single value of realised variance is used for the log-based estimation using the INTEL data, the weights are given in Table 3.2. Again for large values of M the regression estimator puts more weight on the log-realised variance and less on the unconditional mean. Although this situation reverses when M decreases, the realised variance is much more highly weighted when logarithms are used.

In the dynamic case, the smoothed estimator of $\log(\nu_i^2)$, in the case of M=78 using one lag, one lead and the contemporaneous log-realised variance, is

$$\widehat{\log(\nu_i^2)} = 0.040E(\log(\nu_i^2)) + 0.072\log([Y_{78}]_{i-1}^{[2]}) + 0.815\log([Y_{78}]_i^{[2]}) + 0.072\log([Y_{78}]_{i+1}^{[2]}).$$

and in the case M=6

$$\widehat{\log(\nu_i^2)} = 0.231E(\log(\nu_i^2)) + 0.075\log([Y_6]_{i-1}^{[2]}) + 0.617\log([Y_6]_i^{[2]}) + 0.075\log([Y_6]_{i+1}^{[2]}).$$

M	min	\hat{c}	\hat{A}
78	5	0.1027	0.8972
39	10	0.1594	0.8405
26	15	0.2074	0.7925
19	20	0.2569	0.7430
13	30	0.2870	0.7129
6	60	0.3458	0.6541

Table 3.2: Estimated weights for the estimation of $\log(\nu_i^2)$.

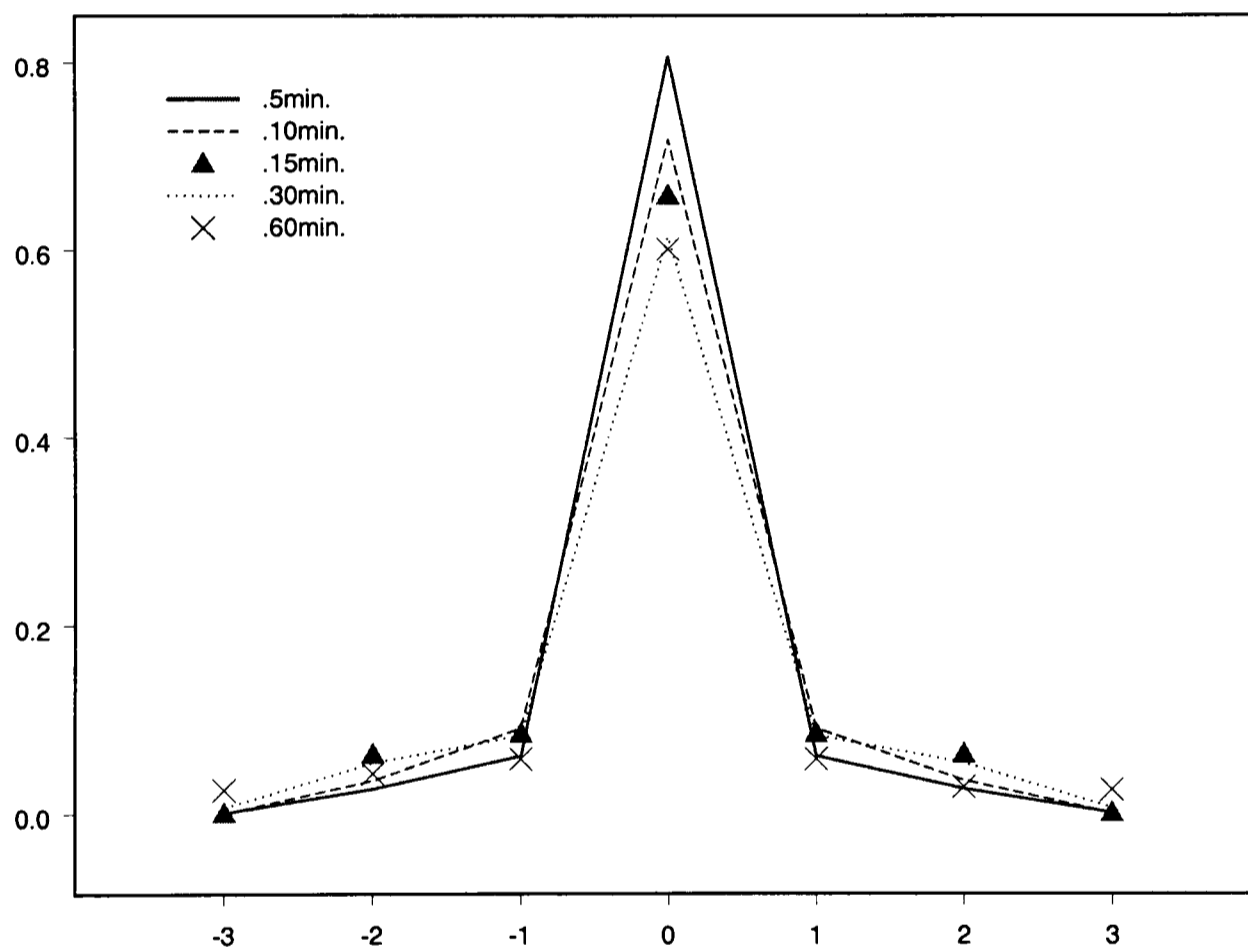


Figure 3.5: Weight vector for estimating $\log(\nu_i^2)$ using 3 lags, 3 leads and the contemporaneous observation for different values of M .

Even when M is small, few weight is given to the unconditional expectation of the log-variance and almost all of it is given to the log of the realised variance and its lag and lead. This fact can be observed as well from Figure 3.5. This graph shows the weights of the smoothed estimator of $\log(\nu_i^2)$ using 7 log-realised variances, 3 lags, 3 leads and the contemporaneous log-realised variance for different values of M . Compared to Figure 3.4, most of the weight is given to $\log([Y_M]_i^{[2]})$ independently of the value

of M . Also if we compare the previous equations (the smoothed estimators of $\log(\nu_i^2)$ using one lag and one lead) to equations (3.4) and (3.5) (the smoothed estimators of ν_i^2 using one lag and one lead), we observe that much more weight is given to the contemporaneous realised variance when logarithms are used.

3.3.2 Model based estimator

Properties of the error

To estimate A from a parametric model, Barndorff-Nielsen and Shephard (2002a) assume that ξ , ω^2 and r exist and are respectively the mean, variance and autocorrelation function of the continuous time stationary process σ_t^2 . As before let $u_i = [Y_M]_i^{[2]} - \nu_i^2$; then, they use the following properties of ν_i^2 to derive the second order properties of $[Y_M]_i^{[2]}$. They derived them under the assumptions that $A_t = 0$ and that σ is stochastically independent of W .

$$\begin{aligned} E(\nu_i^2) &= h\xi \\ \text{Var}(\nu_i^2) &= 2\omega^2 r_h^{**} \\ \text{Cov}\{\nu_i^2, \nu_{i+s}^2\} &= \omega^2 \diamond r_{hs}^{**} \end{aligned}$$

where

$$\begin{aligned} \diamond r_{hs}^{**} &= r_{s+h}^{**} - 2r_s^{**} + r_{s-h}^{**}, \\ r_t^* &= \int_0^t r_u du \quad \text{and} \quad r_t^{**} = \int_0^t r_u^* du. \end{aligned}$$

So with the second order properties of ν_i^2 , the second order properties of $[Y_M]_i^{[2]}$ can be fully established,

$$\begin{aligned} E([Y_M]_i^{[2]}) &= E(\nu_i^2) + E(u_i) = h\xi + o(1) \\ \text{Var}([Y_M]_i^{[2]}) &= \text{Var}(\nu_i^2) + \text{Var}(u_i) + o(1) \\ \text{Cov}([Y_M]_i^{[2]}, [Y_M]_{i+s}^{[2]}) &= \text{Cov}(\nu_i^2, \nu_{i+s}^2) + o(1). \end{aligned}$$

As $\text{Var}(\nu_i^2)$ and $\text{Cov}(\nu_i^2, \nu_{i+s}^2)$ have been fully established, it is only necessary to find $\text{Var}(u_i)$ to determine these second order properties. Using the asymptotic result where

$$\sqrt{M}u_i \xrightarrow{L} MN\left(0, 2h \int_{(i-1)h}^{ih} \sigma_u^4 du\right),$$

then

$$\text{Var}(\sqrt{M}u_i) \xrightarrow{P} 2h^2(\omega^2 + \xi^2)$$

as $M \rightarrow \infty$. So now it can be obtained that

$$\begin{aligned} \text{Var}([Y_M]_i^{[2]}) &= \frac{2h^2}{M}(\omega^2 + \xi^2) + 2\omega^2 r_h^{**} + o(1) \\ \text{Cov}([Y_M]_i^{[2]}, [Y_M]_{i+s}^{[2]}) &= \omega^2 \diamond r_{hs}^{**} + o(1). \end{aligned}$$

Model

Suppose that we wish to estimate $\nu_{s:p}$ using $[Y_M]_{s:p}^{[2]}$, then the best model based linear estimator is

$$\begin{aligned} \widehat{\nu_{s:p}^2} &= (I - A)\iota E(\nu_i^2) + A[Y_M]_{s:p}^{[2]} \\ &= h\xi\iota + \hat{A}([Y_M]_{s:p} - h\xi\iota), \end{aligned}$$

where

$$A = \left(\text{Cov}(\nu_{s:p}^2, [Y_M]_{s:p}^{[2]}) \right) \left(\text{Cov}([Y_M]_{s:p}^{[2]}) \right)^{-1}.$$

In order to implement this approach a model for σ_t^2 is needed to imply this A matrix. For modelling the stochastic volatility Barndorff-Nielsen and Shephard (2002a) use a process whose resulting autocorrelation function will be $r_t = \exp(-\lambda|t|)$. The non-Gaussian Ornstein-Uhlenbeck (OU) process, which is the solution to the SDE

$$d\sigma_t^2 = -\lambda\sigma_t^2 dt + dz_{\lambda t}$$

where z_t is a Levy process with non-negative increments, has the previous acf.

For this process

$$r_t^{**} = \lambda^{-2} \{e^{-\lambda t} - 1 + \lambda t\}$$

and

$$\diamond r_{hs}^{**} = \lambda^{-2} (1 - e^{-\lambda h})^2 e^{-\lambda h(s-1)}, \quad s > 0;$$

implying that the asymptotic moments are

$$\begin{aligned} E(\nu_i^2) &= \xi h \\ \text{Var}(\nu_i^2) &= \frac{2\omega^2}{\lambda^2} (e^{-\lambda h} - 1 + \lambda h) \\ \text{Cor}\{\nu_i^2, \nu_{i+s}^2\} &= \frac{(1 - e^{-\lambda h})^2 (e^{-\lambda h(s-1)})}{2(e^{-\lambda h} - 1 + \lambda h)}. \end{aligned}$$

The autocorrelation model for the ν_i^2 is that one for an ARMA(1,1).

In calculating $\nu_{1:T}^2$ Barndorff-Nielsen and Shephard (2002a) conveniently placed $[Y_M]_i^{[2]}$ into a linear state-space representation so the filtering, smoothing and forecasting can be carried out using the

Kalman filter. Set $\alpha_i = \nu_i^2 - h\xi$ and $u_i = \sigma_u v_{1i}$ then

$$\begin{aligned} [Y_M]_i^{[2]} &= h\xi + (1 \ 0) \alpha_i + \sigma_u v_{1i} \\ \alpha_{i+1} &= \begin{pmatrix} \phi & 1 \\ 0 & 0 \end{pmatrix} \alpha_i + \begin{pmatrix} \sigma_\sigma \\ \sigma_\sigma \theta \end{pmatrix} v_{2i}. \end{aligned}$$

Here v_i is a zero mean, white noise sequence uncorrelated with u_i with an identity covariance matrix. Also, ϕ , θ , and σ_σ are the autoregressive root, the moving average root and the variance of the innovation to the ARMA(1,1) representation of the ν_i^2 process, and finally $\sigma_u = \sqrt{2M^{-1}h^2(\omega^2 + \xi^2)}$ from the asymptotic result for the $Var(u_i)$.

Superposition of processes

Usually the OU volatility model is too simple to fit the long-range dependence financial time series have, so a superposition of processes to describe ν_i^2 should be used.

The authors used superposition of J OU independent process that are not necessarily identically distributed, so

$$\sigma_t^2 = \sum_{j=1}^J \sigma_t^{2(j)} \quad \text{and} \quad \sum_{j=1}^J w_j = 1$$

where w_j are the weights that must be larger or equal to zero and $\sigma_t^{2(j)}$ are the processes with memory λ_j . Assume that

$$E(\sigma_t^{2(j)}) = w_j \xi \quad \text{and} \quad Var(\sigma_t^{2(j)}) = w_j \omega^2$$

so

$$\begin{aligned} E(\sigma_t^2) &= \xi, & Var(\sigma_t^2) &= \omega^2, \\ Cov(\sigma_t^2, \sigma_{t+s}^2) &= \sum_{j=1}^J Cov(\sigma_t^{2(j)}, \sigma_{t+s}^{2(j)}) = \omega^2 \sum_{j=1}^J w_j exp(-\lambda_j |s|). \end{aligned}$$

The linearity of the superposition of the processes gives that

$$\nu_i^2 = \sum_{j=1}^J \nu_i^{2(j)}$$

where $\nu_i^{2(j)} = \sigma_{ih}^{2(j)*} - \sigma_{(i-1)h}^{2(j)*}$ and $\sigma_t^{2(j)*} = \int_0^t \sigma_u^{2(j)} du$.

An important fact is that each independent process of the superposition is itself an ARMA(1,1) so the autocovariance function of the superposition is just the addition of the autocovariance function of each OU process. Then as each process may have different parameters, the acf of the superposition will be the addition of slowly and quickly decaying components describing better the acf of ν_i^2 .

As an empirical illustration, a set of superposition based models have been fitted to the realised variance time series constructed using five minutes INTEL returns. The model allows different degrees of memory. Based on the prediction error decomposition obtained with the Kalman filter, the quasi-likelihood function (Harvey 1993) is used to estimate the parameters of the model.

A complication is that ϕ, θ, σ_u and σ_σ are needed to perform the Kalman filter but they are functions of the parameters we need to estimate (ξ, λ, ω) . To estimate these last parameters we need the output from the Kalman filter to construct the quasi-likelihood function. Therefore we need the Kalman filter output to estimate the parameters, but we need the parameters to do the Kalman filter. An optimising procedure is used giving initial values for ξ, λ, ω . The parameters obtained after this procedure are given in Table 3.3.

There is a dramatic shift when going from $J=1$ to $J=2$. The Box-Pierce statistic gives a measure of fit of the model which shows this large jump when going from $J=1$ to superposition model. The Box-Pierce statistic is used to test the null hypothesis that the first K autocorrelations of a time series are zero. It is based on $Q = n \sum_{k=1}^K R_k^2$ where R_k is the k th sample autocorrelation of the residual series. If the model is correct, asymptotically Q follows a Chi-square distribution with $K - p - q$ degrees of freedom (for an ARMA(p,q)).

J	ξ	ω^2	λ_1	λ_2	λ_3	λ_4	w_1	w_2	w_3	L_Q	BP_{20}
1	7.40	185.3	0.947	—	—	—	1.00	—	—	-34,860	34.13
2	7.52	185.3	0.067	1.322	—	—	0.193	—	—	-34,683	14.29
3	7.56	173.6	0.031	0.072	1.17	—	0.037	0.153	—	-34,676	13.71
4	7.51	173.9	0.043	0.116	0.97	6.72	0.138	0.033	0.729	-34,670	13.41

Table 3.3: *Fit of the superposition of J volatility processes for a SV model based on realised variance using $M=78$. Parameters, Quasi-likelihood (L_Q) and Box-Pierce (BP) statistic.*

In Figure 3.6 the autocorrelation function for the fitted series for different values of J together with the empirical correlogram are shown for $M=78$ and $M=6$. The model with $J=1$ is totally unable to fit the data. In both cases the superposition of three processes is the best, picking up the longer-range dependence in the data.

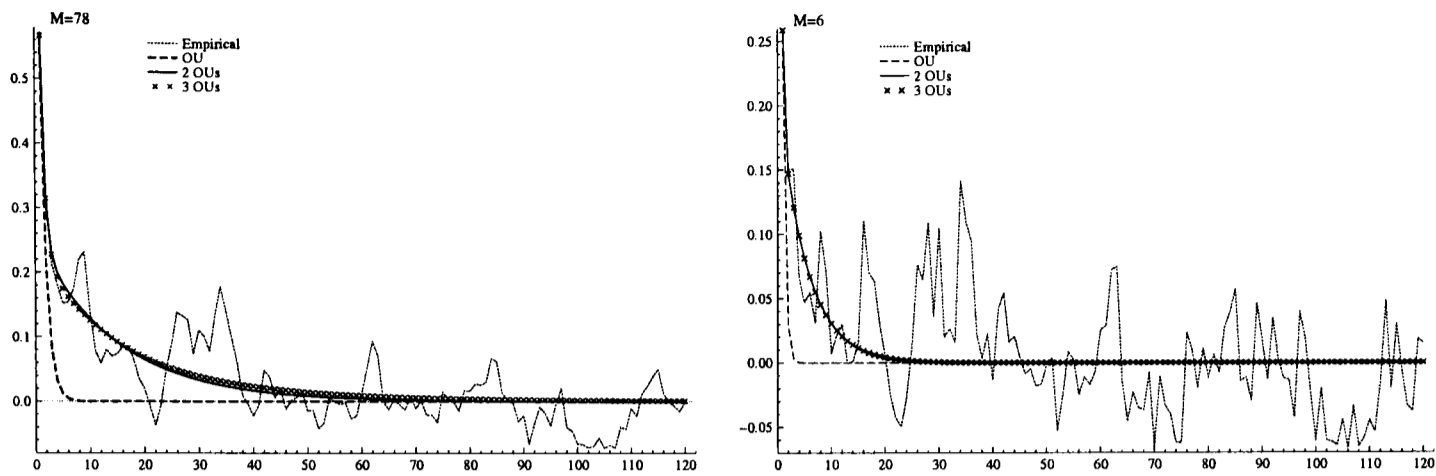


Figure 3.6: ACF of $[Y_M]_i^{[2]}$ and the fitted version for various values of J .

3.3.3 Comparison

Previously a model free and a model based estimator were used to obtain a better estimation of the actual variance. The model free estimator is quick and easy to compute while the model based estimation is computer intensive. So now it is important to make a comparison between the smoothed series obtained with each approach. In Table 3.4 the correlation between the model based series and the model free series (for different number of lags and leads) are displayed. As the number of lags and leads increases and as the value of M increases the correlation increases as well, so the connection between the estimators becomes stronger. In Figure 3.7 the first 105 values of the smoothed series are plotted. The model free series is calculated using 2 lags and 2 leads and the contemporaneous observation. These series are presented for $M=78$ and $M=6$. Although the model based series seems to be less sensitive to large values, there exists a strong connection between these two series.

	5min.	15min.	60min.
$s = p$	0.9854	0.9318	0.8028
1 lag 1 lead	0.9786	0.9599	0.8490
2 lags 2 leads	0.9783	0.9657	0.8628
logarithms	0.9860	0.9340	0.7920
logs 1 lag 1 lead	0.9841	0.9563	0.8231

Table 3.4: Correlation between the model free and model based smoothers.

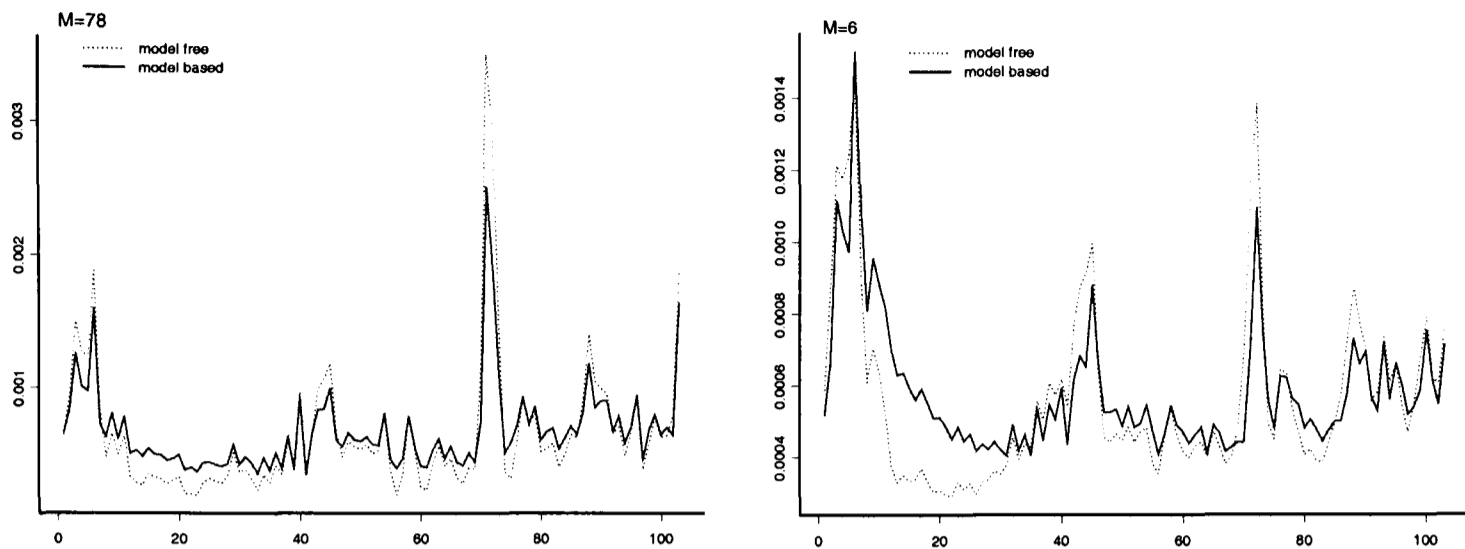


Figure 3.7: Model free using 2 lags and 2 leads and model based estimators of ν_i^2 for $M=78$ and $M=6$.

3.4 Estimations with Realised Absolute Variation

In this section we will repeat the modelling and estimation methods from Barndorff-Nielsen, Nielsen, Shephard and Ysusi (2004) applied in the previous section but now using the realised absolute variation.

When estimating $\nu_i^{[1]}$ with realised absolute variation, the variance of the error is quite high even when using large values of M , so a model should be established. As with the realised variance a linear model, now for $\nu_i^{[1]}$ can be defined as

$$\nu_{s:p}^{[1]} = cE(\nu_{s:p}^{[1]}) + A[Y_M]_{s:p}^{[1]}$$

where

$$[Y_M]_{s:p}^{[1]} = \frac{1}{\sqrt{M}\mu_1} \left(\sum_{j=1}^M |y_{j,s}|, \sum_{j=1}^M |y_{j,s+1}|, \dots, \sum_{j=1}^M |y_{j,p}| \right)',$$

and

$$\nu_{s:p}^{[1]} = \left(\nu_s^{[1]}, \nu_{s+1}^{[1]}, \dots, \nu_p^{[1]} \right)'$$

We know that

$$\sqrt{\frac{M}{h}} \left([Y_M]_{s:p}^{[1]} - \nu_{s:p}^{[1]} \right) | \nu_{s:p}^2 \xrightarrow{L} N \left(0, \left(\frac{\pi}{2} - 1 \right) A \text{diag}(\nu_{s:p}^2) A' \right).$$

Therefore, we can obtain for the realised absolute variation that

$$\sqrt{\frac{M}{h}} \left(A[Y_M]_{s:p}^{[1]} - A\nu_{s:p}^{[1]} \right) | \nu_{s:p}^2 \xrightarrow{L} N \left(0, \left(\frac{\pi}{2} - 1 \right) A \text{diag}(\nu_{s:p}^2) A' \right).$$

Assuming that the realised absolute variations are a covariance stationary process, the weighted least square estimator gives in this case that

$$\begin{aligned}\hat{c} &= (I - \hat{A})\iota \\ \hat{A} &= Cov(\nu_{s:p}^{[1]}) \left(Cov(\nu_{s:p}^{[1]}) + \left(\frac{\pi}{2} - 1\right) \frac{E(\nu_i^2)h}{M} I \right)^{-1}.\end{aligned}$$

Again we can obtain the estimators in a model free or a model based manner. We first look at the model free approach.

3.4.1 Sample based method

When using empirical averages, if we have a covariance stationary process of realised absolute variations and an ergodic daily process, then we know that

$$\frac{1}{T} \sum_{i=1}^T [Y_M]_i^{[1]} \xrightarrow{p} E(\nu_i^{[1]})$$

and

$$\left(\frac{1}{T} \sum_{i=1}^T \sum_{j=1}^M y_{j,i}^2 \right) \xrightarrow{p} E(\nu_i^2)$$

when M and T go to infinity. With these results we can obtain the estimates, as

$$\hat{A} = \left(Cov([Y_M]_{s:p}^{[1]}) - \left(\frac{\pi}{2} - 1\right) \frac{E(\nu_i^2)h}{M} I \right) \left(Cov([Y_M]_{s:p}^{[1]}) \right)^{-1}.$$

The $Cov([Y_M]_{s:p}^{[1]})$ is easily calculated from the data, and as we are working in daily bases we set $h = 1$.

Our first results are obtained using a single value of realised absolute variation in the model, $s = p = i$. From Table 3.5 we can appreciate that, as expected, as M gets larger the weight given to the realised absolute variation becomes more important. The difference in the weights when using 5 minute returns compared to when using 60 minutes returns is very illustrative in this sense.

If we go back to Table 3.1, we can notice that more weight is given to the realised absolute variation than to the realised variance, suggesting the realised absolute variation is more robust.

M	min	\hat{c}	\hat{A}
78	5	0.0802	0.9197
39	10	0.1413	0.8586
26	15	0.2050	0.7949
19	20	0.2513	0.7486
13	30	0.3652	0.6347
6	60	0.5536	0.4463

Table 3.5: *Estimated weights for the estimation of $\nu_i^{[1]}$.*

Models with lags and leads

Now in a dynamic approach, let us estimate $\nu_{s:p}^{[1]}$ using one lag, one lead and the contemporaneous realised absolute variation.

When working with five minutes returns we obtain the following matrices

$$\hat{A} = \begin{pmatrix} 0.86617 & 0.07172 & 0.01856 \\ 0.07172 & 0.83031 & 0.07172 \\ 0.01856 & 0.07172 & 0.86617 \end{pmatrix} \quad \hat{c} = \begin{pmatrix} 0.04354 \\ 0.02624 \\ 0.04354 \end{pmatrix},$$

so from the second row of the matrices the weights are obtained, giving the following estimator

$$\widehat{\nu}_i^{[1]} = 0.026E(\nu_i^{[1]}) + 0.071[Y_{78}]_{i-1}^{[1]} + 0.830[Y_{78}]_i^{[1]} + 0.071[Y_{78}]_{i+1}^{[1]}.$$

Observe that much more weight is given to the contemporaneous realised absolute variation than to the lag and lead. Nevertheless, if we compare this estimator to the previous one with no lags or leads, less weight is given to the expectation and the weight given as a total to the realised absolute variation (lag, contemporaneous and lead) has increased.

When the realised absolute variation is calculated with sixty minutes returns, the estimator is

$$\widehat{\nu}_i^{[1]} = 0.331E(\nu_i^{[1]}) + 0.159[Y_6]_{i-1}^{[1]} + 0.349[Y_6]_i^{[1]} + 0.159[Y_6]_{i+1}^{[1]}.$$

Again we can appreciate that as a smaller value of M is used, the less weight is given to the realised absolute variation. By using one lag and one lead, the weight in the expectation of $\nu_i^{[1]}$ decreases, and the weight given in total to the realised absolute variation increases compared to the estimation without lags or leads, independently of the value of M.

The number of lags and leads can be increased in the model. When using two lags and two leads for the estimation, we obtained a 5x5 matrix for \hat{A} and a 5x1 vector for \hat{c} . Using their third rows, the following models are obtained when using five and sixty minutes returns consecutively,

$$\widehat{\nu}_i^{[1]} = 0.015E(\nu_i^{[1]}) + 0.016[Y_{78}]_{i-2}^{[1]} + 0.062[Y_{78}]_{i-1}^{[1]} + 0.826[Y_{78}]_i^{[1]} + 0.062[Y_{78}]_{i+1}^{[1]} + 0.016[Y_{78}]_{i+2}^{[1]}$$

$$\widehat{\nu}_i^{[1]} = 0.253E(\nu_i^{[1]}) + 0.073[Y_6]_{i-2}^{[1]} + 0.133[Y_6]_{i-1}^{[1]} + 0.332[Y_6]_i^{[1]} + 0.133[Y_6]_{i+1}^{[1]} + 0.073[Y_6]_{i+2}^{[1]}$$

Here, again, as higher the value of M gets, the bigger the weight given to the contemporaneous realised absolute variation. Also, the closer the lags or the leads are to the contemporaneous realised absolute variation, the bigger the weight given to them. Now, the weights of the unconditional expectation are even smaller because the actual spot volatility can be better explained with the lags.

In Figure 3.8 the weights of a model using three lags, three leads and the contemporaneous realised absolute variation are plotted for different values of M. It shows how quickly the weights focus on the contemporaneous observation of the realised absolute variation as M increases (quicker than in the realised variance case in Figure 3.4).

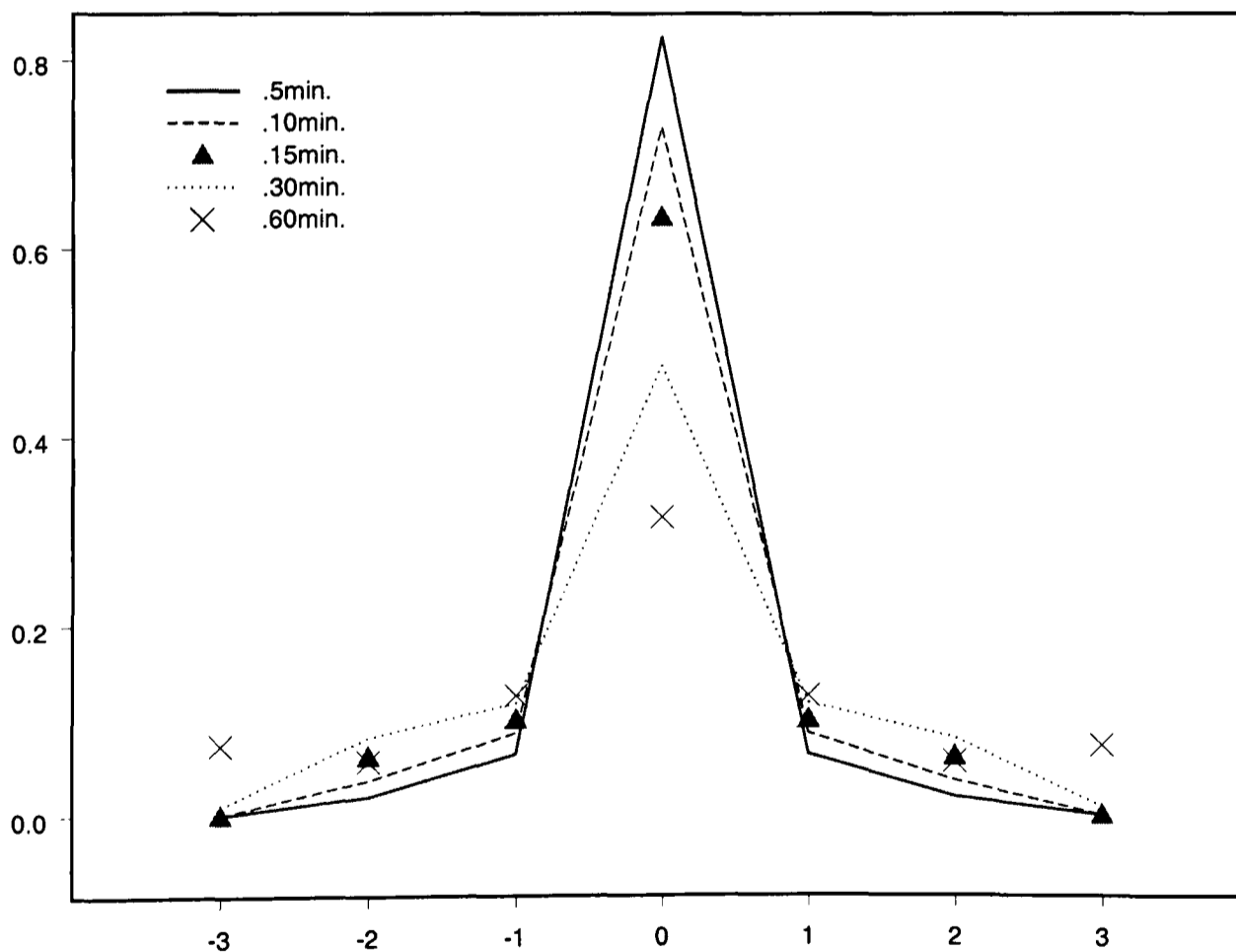


Figure 3.8: *Weights for estimating actual volatility using 3 lags and 3 leads for different values of M.*

Logarithms

A similar analysis can be established using log-realised absolute variations. The relevant asymptotic results obtained with the Delta method is the following

$$\sqrt{\frac{M}{h}} \left(A \log([Y_M]_{s:p}^{[1]}) - A \log(\nu_{s:p}^{[1]}) \right) | \nu_{s:p}^2, \nu_{s:p}^{[1]} \xrightarrow{L} N \left(0, \left(\frac{\pi}{2} - 1 \right) A E \begin{pmatrix} \frac{\nu_s^2}{(\nu_s^{[1]})^2} & 0 & \cdots & 0 \\ 0 & \ddots & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & \frac{\nu_p^2}{(\nu_p^{[1]})^2} \end{pmatrix} A' \right)$$

so now we can use the fact that

$$\left(\frac{1}{T} \sum_{i=1}^T \frac{\sum_{j=1}^M y_{j,i}^2}{\left(\frac{1}{\sqrt{M\mu}} \sum_{j=1}^M |y_{j,i}| \right)^2} \right) \xrightarrow{p} E \left(\frac{\nu_i^2}{(\nu_i^{[1]})^2} \right)$$

and that

$$\frac{1}{T} \sum_{i=1}^T \log[Y_M]_i^{[1]} \xrightarrow{p} E(\log \nu_i^{[1]})$$

to obtain the estimators.

This allows us to construct the estimate

$$\widehat{\log \nu_{s:p}^{[1]}} = \hat{c} E(\log \nu_{s:p}^{[1]}) + \hat{A} \log[Y_M]_{s:p}^{[1]},$$

and given that the realised absolute variations are a covariance stationary process, then the weighted least squares estimator set

$$\hat{c} = (I - \hat{A})\iota$$

and

$$\hat{A} = Cov(\log \nu_{s:p}^{[1]}) \left(Cov(\log \nu_{s:p}^{[1]}) + \left(\frac{\pi}{2} - 1 \right) \frac{h}{M} E \left(\frac{\nu_i^2}{(\nu_i^{[1]})^2} \right) I \right)^{-1}.$$

First, by setting $s = p = i$, i.e. using just the contemporaneous realised absolute variation, and using logarithms, the results shown in Table 3.6 are obtained. As before, for smaller values of M used, the weight given to the realised absolute variation gets smaller as well. Compared to Table 3.5, when using logarithms and M is small, higher values for the weight of the log-realised absolute variation are found. The weights given to the log-realised variances (Table 3.2) were larger to the weights given to the realised variances (Table 3.1) for every value of M. For the realised absolute variation we can see they increase just when M is small.

M	min	\hat{c}	\hat{A}
78	5	0.0862	0.9137
39	10	0.1465	0.8534
26	15	0.2045	0.7954
19	20	0.2633	0.7366
13	30	0.3337	0.6662
6	60	0.4718	0.5281

Table 3.6: Estimated weights for the estimation of $\log\nu_i^{[1]}$.

When using one lag and one lead in the logarithm model, the same conclusions can be obtained. For five minutes returns we get the model

$$\widehat{\nu}_i^{[1]} = 0.032E(\log\nu_i^{[1]}) + 0.067\log[Y_{78}]_{i-1}^{[1]} + 0.833\log[Y_{78}]_i^{[1]} + 0.067\log[Y_{78}]_{i+1}^{[1]},$$

and for sixty minutes

$$\widehat{\nu}_i^{[1]} = 0.316E(\log\nu_i^{[1]}) + 0.101\log[Y_6]_{i-1}^{[1]} + 0.479\log[Y_6]_i^{[1]} + 0.101\log[Y_6]_{i+1}^{[1]}.$$

In Figure 3.9 the weights of the log-model using three lags, three leads and the contemporaneous observation are shown for different values of M. When using logarithms the weights given for small values of M are much higher than those obtained for the estimator based on the raw realised absolute variation (compare to Figure 3.8).

3.4.2 Model based estimation

In order to construct a model based estimator of the actual spot volatility, we need to derive the first and second moments of $\nu_i^{[1]}$. If ξ is the mean of σ_t , ω is its variance, and r is its autocorrelation function, it is known that

$$E(\nu_i^{[1]}) = h\xi, \quad \text{Var}(\nu_i^{[1]}) = 2\omega^2 r_h^{**}, \quad \text{and} \quad \text{Cov}\{\nu_i^{[1]}, \nu_{i+s}^{[1]}\} = \omega^2 \diamond r_{hs}^{**}$$

where

$$\diamond r_h^{**} s = r_{s+h}^{**} - 2r_s^{**} + r_{s-h}^{**}, \quad r_t^* = \int_0^t r_u du,$$

$$\text{and} \quad r_t^{**} = \int_0^t r_u^* du.$$

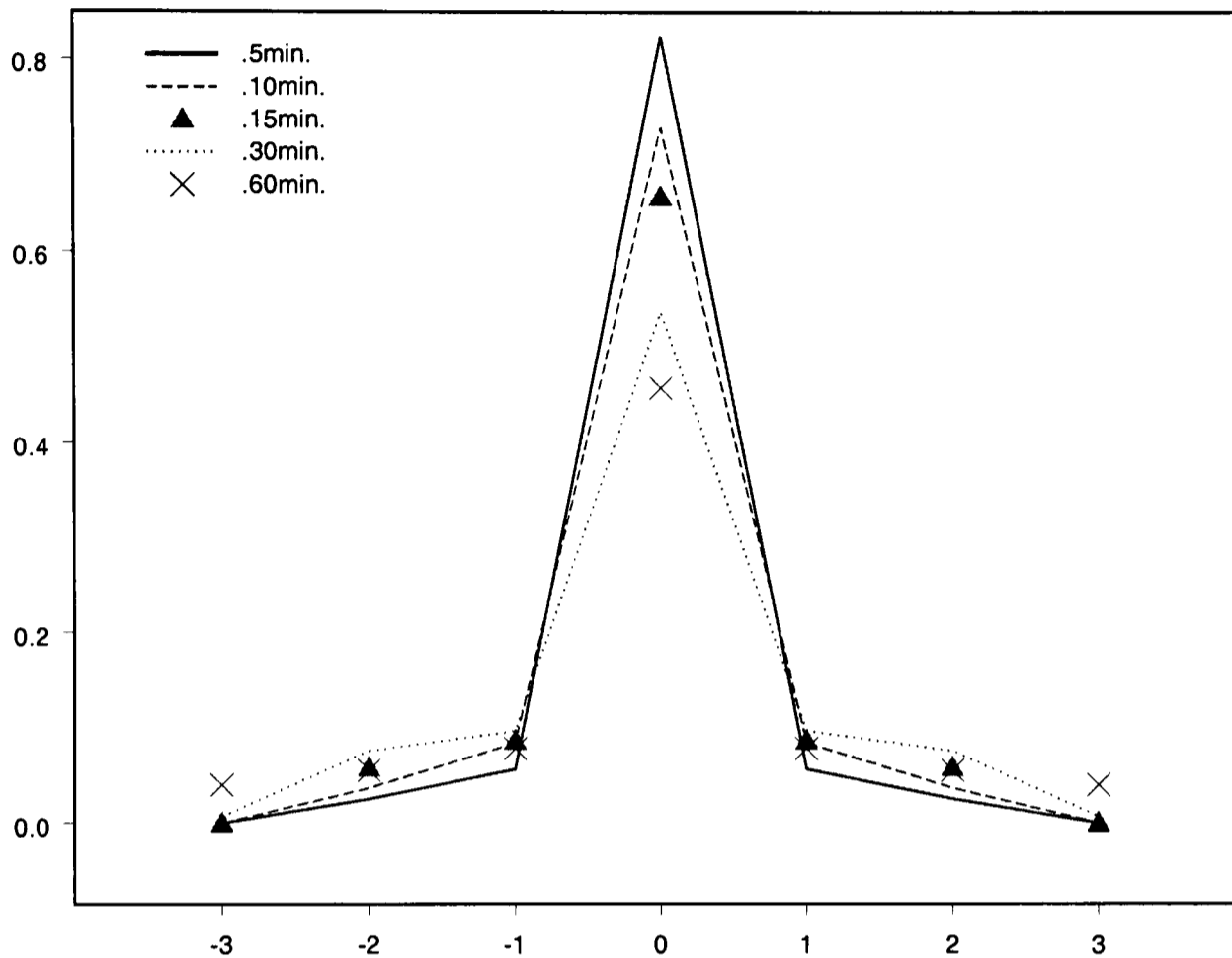


Figure 3.9: Weights for estimating $\log(\nu_i^{[1]})$ using 3 lags and 3 leads for different values of M .

So with the second order properties of σ_t , the second order properties of $\nu_i^{[1]}$ can be fully established.

In a stochastic volatility model context, realised absolute variation can be written as

$$[Y_M]_i^{[1]} = \nu_i^{[1]} + u_i,$$

so

$$u_i = [Y_M]_i^{[1]} - \nu_i^{[1]} = \frac{1}{\sqrt{M}\mu} \sum_{j=1}^M |\varepsilon_{j,i}| \sqrt{\nu_{j,i}^2} - \sum_{j=1}^M \nu_{j,i}^{[1]}$$

where

$$\begin{aligned} \varepsilon_{j,i} &\stackrel{iid}{\sim} N(0, 1) \\ \nu_{j,i}^2 &= \sigma_{(i-1)h + \frac{jh}{M}}^{2*} - \sigma_{(i-1)h + \frac{(j-1)h}{M}}^{2*}, \\ \nu_{j,i}^{[1]} &= \sigma_{(i-1)h + \frac{jh}{M}}^{[1]*} - \sigma_{(i-1)h + \frac{(j-1)h}{M}}^{[1]*}, \\ \sigma_t^{[1]*} &= \int_0^t \sigma_u du \quad \text{and} \quad \sigma_t^{2*} = \int_0^t \sigma_u^2 du. \end{aligned}$$

As stated before $\sqrt{\nu_{j,i}^2} \neq \nu_{j,i}^{[1]}$, so

$$E(\sqrt{M}u_i|\nu_i^{[1]}) \neq 0.$$

Nevertheless, it is known that $\sqrt{M}u_i \xrightarrow{L} MN(0, (\frac{\pi}{2} - 1)h \int_{(i-1)h}^{ih} \sigma_u^2 du)$ so

$$E(\sqrt{M}u_i|\nu_i^{[1]}) = o(1).$$

The second order properties of $[Y_M]_i^{[1]}$ can now be found. First set

$$\begin{aligned} E([Y_M]_i^{[1]}) &= E(\nu_i^{[1]}) + E(u_i) = h\xi + \frac{1}{\sqrt{M}}o(1), \\ \text{Var}([Y_M]_i^{[1]}) &= \text{Var}(\nu_i^{[1]}) + \text{Var}(u_i) + o(1), \\ \text{Cov}([Y_M]_i^{[1]}, [Y_M]_{i+s}^{[1]}) &= \text{Cov}(\nu_i^{[1]}, \nu_{i+s}^{[1]}) + o(1). \end{aligned}$$

Just $\text{Var}(u_i)$ is left to be found to determine these second order properties. Again from the fact that

$$\sqrt{M}u_i \xrightarrow{L} MN\left(0, \left(\frac{\pi}{2} - 1\right)h \int_{(i-1)h}^{ih} \sigma_u^2 du\right)$$

we have that

$$\begin{aligned} \text{Var}(\sqrt{M}u_i) &= \left(\frac{\pi}{2} - 1\right)hE\left(\int_{(i-1)h}^{ih} \sigma_u^2 du\right) + o(1) \\ &= \left(\frac{\pi}{2} - 1\right)h \int_{(i-1)h}^{ih} E(\sigma_u^2) du + o(1) \\ &= \left(\frac{\pi}{2} - 1\right)h^2 E(\sigma_u^2) + o(1) \\ &= \left(\frac{\pi}{2} - 1\right)h^2 (\text{Var}(\sigma_u) + E^2(\sigma_u)) + o(1) \end{aligned}$$

so

$$\text{Var}(\sqrt{M}u_i) = \left(\frac{\pi}{2} - 1\right)h^2(\omega^2 + \xi^2) + o(1).$$

With these results the variance of the realised variation and its covariance can be established,

$$\begin{aligned} \text{Var}([Y_M]_i^{[1]}) &= \left(\frac{\pi}{2} - 1\right)\frac{h^2}{M}(\omega^2 + \xi^2) + 2\omega^2 r_h^{**} + \frac{1}{M}o(1), \\ \text{Cov}([Y_M]_i^{[1]}, [Y_M]_{i+s}^{[1]}) &= \omega^2 \diamond r_{sh}^{**} + o(1). \end{aligned}$$

Model

Similar to the realised variance case, for modelling the stochastic volatility, a process with an autocorrelation function as $r_t = \exp(-\lambda|t|)$ will be used. The process, which is the solution to the stochastic differential equation

$$d\sigma_t = -\lambda\sigma_t dt + dz_{\lambda t} \tag{3.7}$$

where z_t is a Levy process with non-negative increments, has the previous acf.

For this process $r_t^{**} = \lambda^{-2}\{e^{-\lambda t} - 1 + \lambda t\}$ and $\diamond r_{sh}^{**} = \lambda^{-2}(1 - e^{-\lambda h})^2 e^{-\lambda(s-1)h}$, $s > 0$; implying that the asymptotic moments are

$$\begin{aligned} E(\nu_i^{[1]}) &= \xi h, \\ Var(\nu_i^{[1]}) &= \frac{2\omega^2}{\lambda^2}(e^{-\lambda h} - 1 + \lambda h) \\ Cor\{\nu_i^{[1]}, \nu_{i+s}^{[1]}\} &= \frac{(1 - e^{-\lambda h})^2(e^{-\lambda(s-1)h})}{2(e^{-\lambda h} - 1 + \lambda h)}. \end{aligned}$$

The autocorrelation model for the $\nu_i^{[1]}$ is that one for an ARMA(1,1), so the following linear state-space representation can be used:

$$\begin{aligned} [Y_M]_i^{[1]} &= \xi h + (1 \ 0) \alpha_i + \sigma_u v_{1i} \\ \alpha_{i+1} &= \begin{pmatrix} \phi & 1 \\ 0 & 0 \end{pmatrix} \alpha_i + \begin{pmatrix} \sigma_\sigma \\ \sigma_\sigma \theta \end{pmatrix} v_{2i} \end{aligned}$$

where

$$\alpha_i = \nu_i^{[1]} - \xi h \quad \text{and} \quad u_i = \sigma_u v_{1i}.$$

Here v_i is a zero mean, white noise sequence with an identity covariance matrix; ϕ , θ , and σ_σ are the autoregressive root, the moving average root and the variance of the innovation of the process. Finally σ_u^2 is found from the asymptotic result for the $Var(u_i)$. The Kalman filter can be used for predicting and smoothing $\nu_i^{[1]}$.

Again, just one of the previous volatility models is too simple to fit the long-range dependence financial time series have, so processes will be superposed to describe σ_u . In this case, we have

$$\sigma_t = \sum_{j=1}^J \sigma_t^{(j)} \quad \text{and} \quad \sum_{j=1}^J w_j = 1$$

where w_j are the weights that must be larger or equal to zero and $\sigma_t^{(j)}$ are the processes with memory λ_j .

Using the previous model $\nu_i^{[1]}$ can be estimated for the INTEL dataset. The model was fitted using five and sixty minutes returns, determining the number of processes needed in the superposition and estimating the parameters.

When using five minutes returns, the empirical ACF is best described with the superposition of three processes (Figure 3.10); adding an additional process to obtain a superposition of four processes does not contribute. Yet two processes were not enough to describe the acf, so this third component is essential. From Table 3.7 it can be seen how two of the processes have a lot of memory (low values

of λ) and the other one has very low memory (high value of λ). There is an important difference in the Box-Pierce statistic of the models based on one process and two processes. It can be noticed that the model-based estimator with only one process do not give a good fit. It is until the superposition of three processes is used when the long-range dependence of the data is picked up.

The model is fitted as well for smaller value of M , first for fifteen minutes returns. Again the superposition of three processes is the model that gives the best fit. Two of the components have small values of λ and just one a large value explaining the long-range dependence. The same happens when using sixty minutes returns; the best fit was found when using three processes. Also, two of the components have persistent variance (low value of λ), and the fit does not improve when changing to four the number of processes, but it does when changing from two to three.

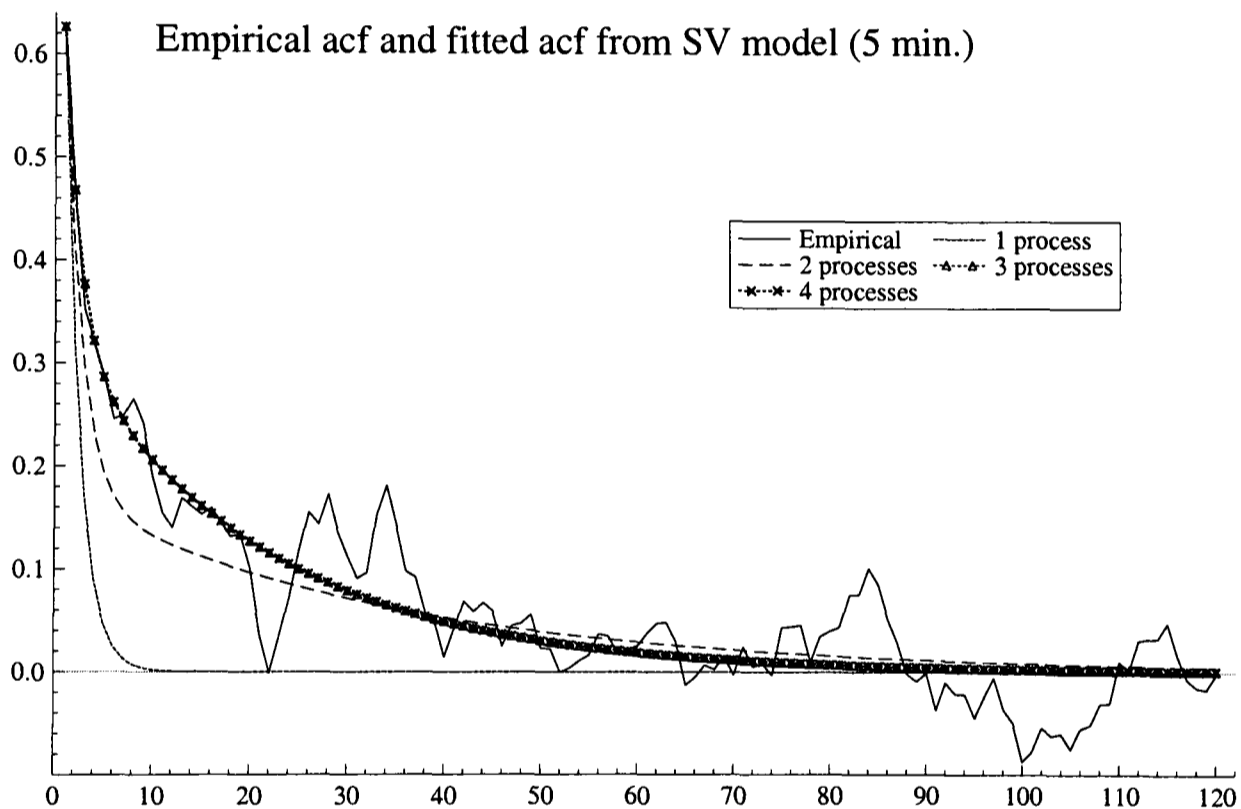


Figure 3.10: Using $M=78$, ACF of $[Y_M]_i^{[1]}$ and the fitted version for various values of J .

	ξ	ω^2	λ_1	λ_2	λ_3	λ_4	w_1	w_2	w_3	L_Q	BP_{20}
$J = 1$	3.41	5.39	0.64	-	-	-	1.00	-	-	-11,191	53.61
$J = 2$	3.48	5.08	0.03	0.57	-	-	0.18	-	-	-10,834	13.41
$J = 3$	3.48	7.02	0.04	0.61	291.9	-	0.15	0.26	-	-10,804	10.88
$J = 4$	3.49	7.05	0.04	0.61	57.3	291.9	0.19	0.32	0.42	-10,804	10.89

Table 3.7: Fit of the superposition of J volatility processes for a SV model based on realised absolute variation using $M=78$. Parameters, Quasi-Likelihood (L_Q) and Box-Pierce (BP) statistic.

Comparison

A comparison can be made of the smoothed series from the model free and the model based approach. Table 3.8 gives the correlation between the model based estimators and the model free estimators with different number of lags and leads and for different values of M . As the number of lags and leads increases they become more closely correlated, and also as M increases the connection between the estimators becomes stronger. In Figure 3.11 the first 200 estimated values of $\nu_i^{[1]}$ from the time series are shown. The time series correspond to the model free estimators using two lags and two leads and the smoothed model based estimator. It can be appreciated that there exists a close connection between these two estimators.

	5min.	15min.	60min.
$s = p$	0.9587	0.8941	0.7838
1 lag 1 lead	0.9805	0.9612	0.9508
2 lags 2 leads	0.9811	0.9692	0.9772
logarithms	0.9586	0.8925	0.7702
logs 1 lag 1 lead	0.9781	0.9507	0.8836

Table 3.8: Correlation between the model free and model based smoothers.

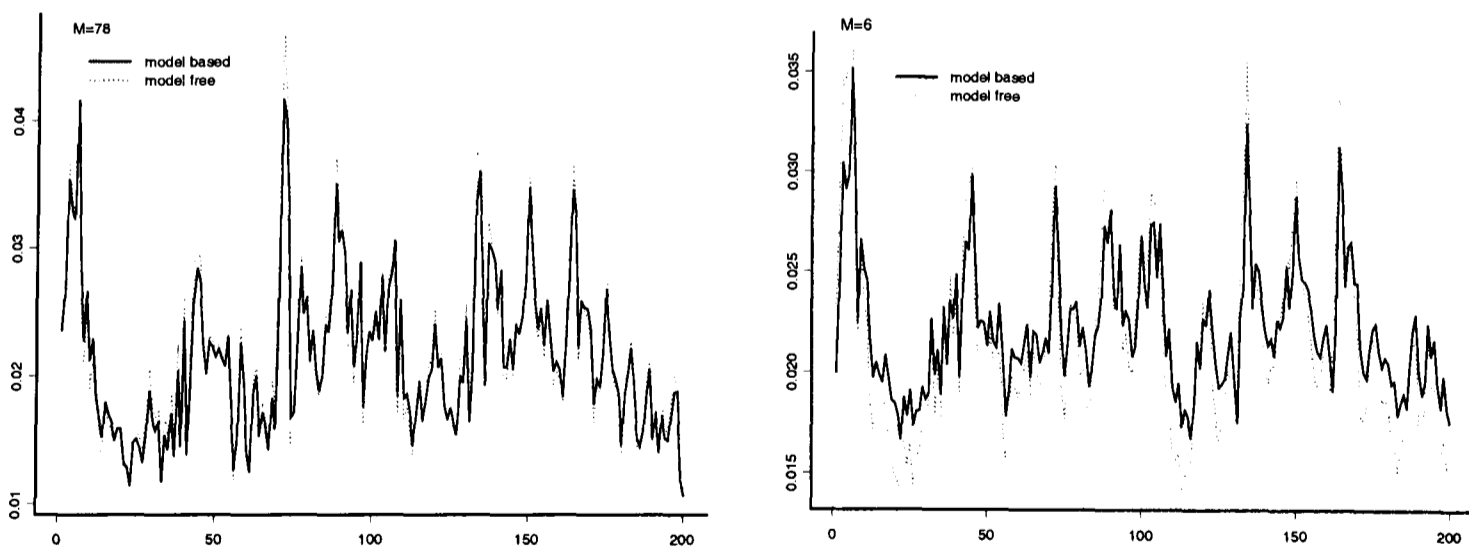


Figure 3.11: Model free using 2 lags and 2 leads and model based estimators of $\nu_i^{[1]}$ for $M=78$ and $M=6$.

3.5 Combined estimation

So far we have worked with separate and incompatible estimators: realised variance and realised absolute variation. A model to estimate ν_i^2 given $[Y_M]_i^{[2]}$ and a model to estimate $\nu_i^{[1]}$ given $[Y_M]_i^{[1]}$ have been studied. It is worth exploring the possibility of estimating ν_i^2 and $(\nu_i^{[1]})^2$ given both $[Y_M]_i^{[2]}$ and $([Y_M]_i^{[1]})^2$. In Figure 3.12 the squared realised absolute variation and the realised variance calculated from the INTEL returns are plotted. Although both variables are very similar for a given M, the realised variance is more jagged, and the values of the daily observations move in a wider range.

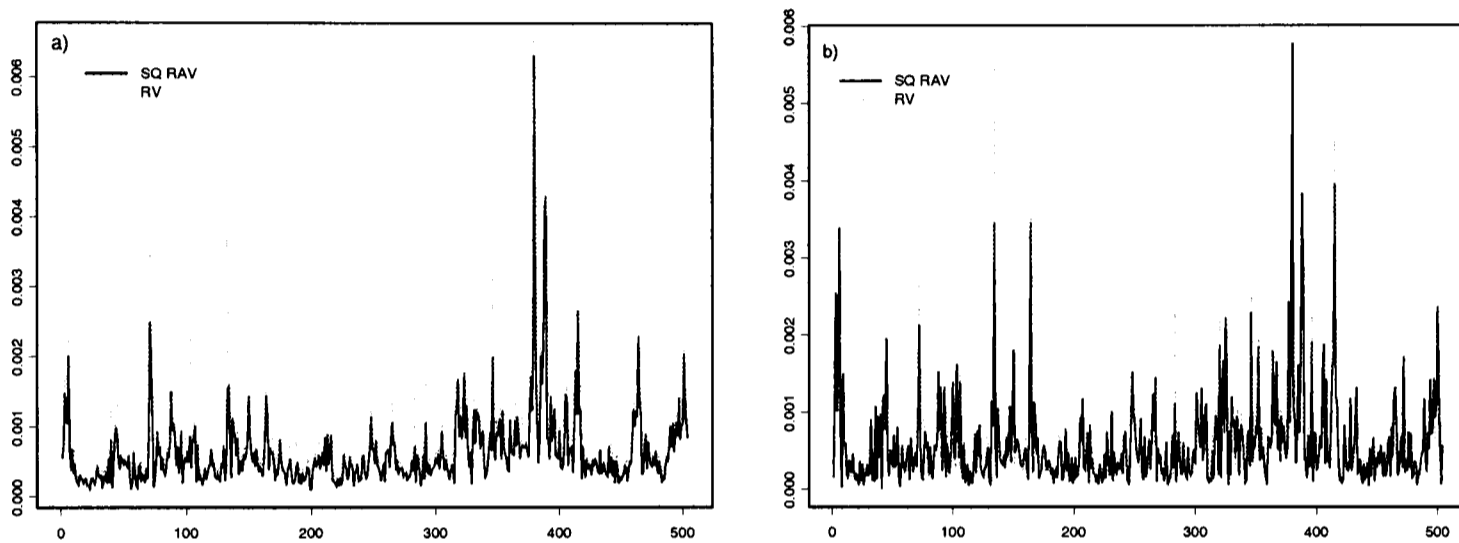


Figure 3.12: Realised variance and squared realised absolute variation for five (a) and sixty (b) minutes INTEL returns.

To get a combined estimation for

$$\theta_i = \begin{pmatrix} (\nu_i^{[1]})^2 \\ \nu_i^2 \end{pmatrix},$$

let us define

$$\Upsilon_i = \theta_i + u_i$$

$$\theta_i = \mu + \eta_i$$

where $u_i = \begin{pmatrix} u_i^{[1]} \\ u_i^{[2]} \end{pmatrix}$, $\eta_i = \begin{pmatrix} \eta_i^{[1]} \\ \eta_i^{[2]} \end{pmatrix}$ are the errors (where $u \perp \eta$), $\mu = E(\theta_i) = \begin{pmatrix} E((\nu_i^{[1]})^2) \\ E(\nu_i^2) \end{pmatrix}$ and

$$\Upsilon_i = \begin{pmatrix} ([Y_M]_i^{[1]})^2 \\ [Y_M]_i^{[2]} \end{pmatrix}.$$

To set a linear model, let us define

$$\hat{\theta}_i = \alpha_i + A\Upsilon_i$$

where

$$\alpha_i = \begin{pmatrix} E((\nu_i^{[1]})^2) \\ E(\nu_i^2) \end{pmatrix} - A \begin{pmatrix} E((\nu_i^{[1]})^2) \\ E(\nu_i^2) \end{pmatrix} = (I - A) \begin{pmatrix} E((\nu_i^{[1]})^2) \\ E(\nu_i^2) \end{pmatrix} = c\mu.$$

Then

$$\begin{aligned} A &= Cov(\theta_i, \Upsilon_i)Cov(\Upsilon_i)^{-1} \\ &= Cov(\theta_i, \theta_i + \epsilon_i)Cov(\Upsilon_i)^{-1} \\ &= \left(Cov(\theta_i) + Cov(\theta_i, \epsilon_i) \right) Cov(\Upsilon_i)^{-1} \\ &= Cov(\theta_i)Cov(\Upsilon_i)^{-1}. \end{aligned}$$

So for the estimator

$$\begin{pmatrix} \widehat{(\nu_i^{[1]})^2} \\ \widehat{\nu_i^2} \end{pmatrix} = c \begin{pmatrix} E((\nu_i^{[1]})^2) \\ E(\nu_i^2) \end{pmatrix} + A \begin{pmatrix} ([Y_M]_i^{[1]})^2 \\ [Y_M]_i^{[2]} \end{pmatrix},$$

the weighted least square estimation sets

$$\hat{c} = (I - \hat{A})\iota \quad \text{and} \quad \hat{A} = Cov(\theta_i)Cov(\Upsilon_i)^{-1}.$$

As

$$\hat{A} = Cov\left((\nu_i^{[1]})^2, \nu_i^2\right)Cov\left(\left([Y_M]_i^{[1]}\right)^2, [Y_M]_i^{[2]}\right)^{-1}$$

and

$$Cov\left((\nu_i^{[1]})^2, \nu_i^2\right) = Cov\left(\left([Y_M]_i^{[1]}\right)^2, [Y_M]_i^{[2]}\right) - Cov\left(u_i^{[1]}, u_i^{[2]}\right)$$

the following asymptotic result is needed to find the estimator. We know from Barndorff-Nielsen and Shephard (2003) that

$$\sqrt{\frac{M}{h}} \begin{pmatrix} [Y_M]_i^{[1]} - \nu_i^{[1]} \\ [Y_M]_i^{[2]} - \nu_i^2 \end{pmatrix} | \Omega \xrightarrow{L} N\left(0, \begin{pmatrix} \left(\frac{\pi}{2} - 1\right)\nu_i^2 & (\mu_3 - \mu_1)\mu_1^{-1}\nu_i^{[3]} \\ (\mu_3 - \mu_1)\mu_1^{-1}\nu_i^{[3]} & 2\nu_i^{[4]} \end{pmatrix}\right) = N(0, \Omega)$$

where $\mu_r = E|X|^r$ where $X \sim N(0, 1)$. By using the Delta method, the corresponding asymptotic result can be obtained

$$\sqrt{\frac{M}{h}} u_i | \Omega \xrightarrow{L} N\left(0, \begin{pmatrix} 4\left(\frac{\pi}{2} - 1\right)\nu_i^2(\nu_i^{[1]})^2 & 2(\mu_3 - \mu_1)\mu_1^{-1}\nu_i^{[3]}\nu_i^{[1]} \\ 2(\mu_3 - \mu_1)\mu_1^{-1}\nu_i^{[3]}\nu_i^{[1]} & 2\nu_i^{[4]} \end{pmatrix}\right) = N(0, \Omega)$$

from where

$$\hat{A} = \left(\begin{array}{cc} \text{Cov}\left(\left([Y_M]_i^{[1]}\right)^2, [Y_M]_i^{[2]}\right) - \left(\begin{array}{cc} 4\left(\frac{\pi}{2} - 1\right) \frac{h}{M} E(\nu_i^2 (\nu_i^{[1]})^2) & 2 \frac{(\mu_3 - \mu_1)}{(\mu_1)} \frac{h}{M} E(\nu_i^{[3]} \nu_i^{[1]}) \\ 2 \frac{(\mu_3 - \mu_1)}{(\mu_1)} \frac{h}{M} E(\nu_i^{[3]} \nu_i^{[1]}) & \frac{2h}{M} E(\nu_i^{[4]}) \end{array} \right) \\ \text{Cov}\left(\left([Y_M]_i^{[1]}\right)^2, [Y_M]_i^{[2]}\right)^{-1} \end{array} \right)$$

The $\text{Cov}\left(\left([Y_M]_i^{[1]}\right)^2, [Y_M]_i^{[2]}\right)$ can be estimated by averages of the time series. Also, as T and M go to infinity the following results from Barndorff-Nielsen and Shephard (2003) can be applied, with which we can fully calculate $\hat{\theta}$,

$$\begin{aligned} \frac{1}{T} \sum_{i=1}^T [Y_M]_i^{[1]} &\xrightarrow{p} E(\nu_i^{[1]}) & \text{where} & [Y_M]_i^{[1]} = \frac{1}{\sqrt{M\mu_1}} \sum_{j=1}^M |y_{j,i}|; \\ \frac{1}{T} \sum_{i=1}^T [Y_M]_i^{[2]} &\xrightarrow{p} E(\nu_i^2) & \text{where} & [Y_M]_i^{[2]} = \sum_{j=1}^M |y_{j,i}|^2; \\ \frac{1}{T} \sum_{i=1}^T [Y_M]_i^{[3]} &\xrightarrow{p} E(\nu_i^{[3]}) & \text{where} & [Y_M]_i^{[3]} = \frac{\sqrt{M}}{\mu_3} \sum_{j=1}^M |y_{j,i}|^3; \\ \frac{1}{T} \sum_{i=1}^T [Y_M]_i^{[4]} &\xrightarrow{p} E(\nu_i^{[4]}) & \text{where} & [Y_M]_i^{[4]} = \frac{M}{\mu_4} \sum_{j=1}^M |y_{j,i}|^4. \end{aligned}$$

In Table 3.9 the results are displayed. When estimating $(\nu_i^{[1]})^2$ and using large values of M, most of the weight is given to the realised absolute variation and almost no weight is given to the realised variance. When the M is small, more weight is given to the realised variance. When estimating ν_i^2 again the weight given to the squared realised absolute variation is higher than that given to the realised variance for large values of M, and smaller when M is small. These gives us a first evidence that working with absolute values can give better estimations than squaring.

	$\widehat{(\nu_i^{[1]})^2}$		$\widehat{\nu_i^2}$	
min	$\left([Y_M]_s^{[1]}\right)^2$	$[Y_M]_s^{[2]}$	$\left([Y_M]_s^{[1]}\right)^2$	$[Y_M]_s^{[2]}$
5	0.9858	-0.0593	0.5315	0.4482
10	0.8203	0.0206	0.4256	0.4494
15	0.6506	0.1066	0.5272	0.2641
20	0.6598	0.0744	0.2445	0.4918
30	0.2377	0.3035	0.2169	0.3607
60	-0.5233	0.8037	-0.5536	0.9122

Table 3.9: Weights for $\left([Y_M]_s^{[1]}\right)^2$ and $[Y_M]_s^{[2]}$ when estimating $(\nu_i^{[1]})^2$ and ν_i^2 .

Logarithms

A similar argument will be used for logarithms. In this case, $\theta_i = \begin{pmatrix} \log((\nu_i^{[1]})^2) \\ \log(\nu_i^2) \end{pmatrix}$,

and $\Upsilon_i = \begin{pmatrix} \log((Y_M)_i^{[1]})^2 \\ \log((Y_M)_i^{[2]}) \end{pmatrix}$. So now the linear model can be set as

$$\begin{pmatrix} \log((\widehat{\nu}_i^{[1]})^2) \\ \log(\widehat{\nu}_i^2) \end{pmatrix} = c \begin{pmatrix} E(\log((\nu_i^{[1]})^2)) \\ E(\log(\nu_i^2)) \end{pmatrix} + A \begin{pmatrix} \log((Y_M)_i^{[1]})^2 \\ \log((Y_M)_i^{[2]}) \end{pmatrix}.$$

Again, assuming that the different realised power variations are covariance stationary processes, the weighted least square statistic sets

$$\hat{c} = (I - \hat{A})$$

and

$$\hat{A} = Cov(\theta_i)Cov(\Upsilon_i)^{-1}.$$

Using the Delta method and the previous asymptotic results, it can be obtained that

$$\sqrt{\frac{M}{h}} \begin{pmatrix} \log((Y_M)_i^{[1]})^2 - \log((\nu_i^{[1]})^2) \\ \log((Y_M)_i^{[2]}) - \log(\nu_i^2) \end{pmatrix} | \Omega \xrightarrow{L} N \left(0, \begin{pmatrix} 4 \left(\frac{\pi}{2} - 1 \right) \frac{\nu_i^2}{(\nu_i^{[1]})^2} & 2 \frac{\mu_3 - \mu_1}{\mu_1} \frac{\nu_i^{[3]}}{\nu_i^{[1]} \nu_i^2} \\ 2 \frac{\mu_3 - \mu_1}{\mu_1} \frac{\nu_i^{[3]}}{\nu_i^{[1]} \nu_i^2} & 2 \frac{\nu_i^{[4]}}{(\nu_i^2)^2} \end{pmatrix} \right).$$

So by using the data, it will be possible to calculate

$$\hat{A} = \begin{pmatrix} Cov(\log((Y_M)_i^{[1]})^2, \log((Y_M)_i^{[2]})) \\ Cov(\log((Y_M)_i^{[1]})^2, \log((Y_M)_i^{[2]}))^{-1} \end{pmatrix} - \begin{pmatrix} \frac{4h}{M} \left(\frac{\pi}{2} - 1 \right) E \left(\frac{\nu_i^2}{(\nu_i^{[1]})^2} \right) & \frac{2h}{M} \frac{\mu_3 - \mu_1}{\mu_1} E \left(\frac{\nu_i^{[3]}}{\nu_i^{[1]} \nu_i^2} \right) \\ \frac{2h}{M} \frac{\mu_3 - \mu_1}{\mu_1} E \left(\frac{\nu_i^{[3]}}{\nu_i^{[1]} \nu_i^2} \right) & \frac{2h}{M} E \left(\frac{\nu_i^{[4]}}{(\nu_i^2)^2} \right) \end{pmatrix}$$

and obtain the estimators of $\begin{pmatrix} \log((\nu_i^{[1]})^2) \\ \log(\nu_i^2) \end{pmatrix}$.

In Table 3.10 the weights are displayed. Again the weights given to the square realised absolute variation are very high for both estimations. Nevertheless, when using logarithms it seems that the square realised absolute variation can give better estimations for bigger values of M, but for smaller values of M the realised variance can give better estimations.

	$\log(\widehat{(\nu_i^{[1]})^2})$		$\log(\widehat{\nu_i^2})$	
min	$\log(\widehat{([Y_M]_s^{[1]})^2})$	$\log(\widehat{[Y_M]_s^{[2]}})$	$\log(\widehat{([Y_M]_s^{[1]})^2})$	$\log(\widehat{[Y_M]_s^{[2]}})$
5	0.9199	-0.0067	0.3287	0.5757
10	0.8019	0.0534	0.3134	0.5266
15	0.6309	0.1718	0.2261	0.5659
20	0.5198	0.2249	0.0899	0.6533
30	0.1354	0.5417	-0.2797	0.9806
60	-0.6526	1.2140	-1.0464	1.6733

Table 3.10: Weights for $\log(\widehat{([Y_M]_s^{[1]})^2})$ and $\log(\widehat{[Y_M]_s^{[2]}})$ when estimating $\log(\widehat{(\nu_i^{[1]})^2})$ and $\log(\widehat{\nu_i^2})$.

In this section the combined estimation was only performed on a model free basis. This is due to the fact that model based estimation is technically challenging and its comparative advantages are not obvious. Specifying the driving process of the spot volatility (e.g. and OU process) will determine the properties of the spot variance. It is unclear which underlying processes are required for the squared integrated spot volatility and the integrated variance for us to apply the method.

3.5.1 Absolute returns vs squared returns

The combined estimations performed in this section indicate that absolute returns can give better results than squared returns for the estimation of integrated variance. This is consistent with Forsberg and Ghysels (2004) who provide a theoretical explanation for the fact that realised absolute variation outperforms realised variance when estimating integrated spot volatility and integrated variance respectively.

Using population moments, these authors prove that the use of absolute values yields a higher persistence. When using high-frequency data for the estimation, the bias due to sampling errors is much smaller with absolute returns (realised absolute variation) than with squared returns (realised variance). Although we do not account for jumps until Chapter 4, the authors did study the effect of discrete jumps on the log-price process and concluded that absolute returns were immune to their presence. They highlighted three reasons that made absolute returns more persistent than squared returns: 1) desirable population predictability features, 2) better sampling error behaviour and 3) immunity to jumps.

From this we can conclude that realised absolute variation is a better estimator of integrated spot volatility than realised variance is of integrated variance. Nevertheless, the main interest is to estimate

integrated variance. Let us compare the asymptotic error distributions for the realised variance process and the squared realised absolute variation process, as well as their logarithmic transformations. In the first case we compare

$$[Y_\delta]_t^{[2]} - \int_0^t \sigma_u^2 du \xrightarrow{L} MN\left(0, 2/\delta \int_0^t \sigma_u^4 du\right)$$

and

$$\left([Y_\delta]_t^{[1]}\right)^2 - \left(\int_0^t \sigma_u du\right)^2 \xrightarrow{L} MN\left(0, \frac{4}{\delta} \left(\frac{\pi}{2} - 1\right) \int_0^t \sigma_u^2 du \left(\int_0^t \sigma_u du\right)^2\right)$$

therefore the following inequality needs to hold for absolute returns to give better results than square returns,

$$\frac{2}{4\left(\frac{\pi}{2} - 1\right)} \int_0^t \sigma_u^4 du > \int_0^t \sigma_u^2 du \left(\int_0^t \sigma_u du\right)^2.$$

Whether the inequality will hold is unclear, so now we will use a logarithmic transformation where

$$\log([Y_\delta]_t^{[2]}) - \log\left(\int_0^t \sigma_u^2 du\right) \xrightarrow{L} MN\left(0, \frac{2}{\delta} \frac{\int_0^t \sigma_u^4 du}{\left(\int_0^t \sigma_u^2 du\right)^2}\right)$$

and

$$\log\left(\left([Y_\delta]_t^{[1]}\right)^2\right) - \log\left(\left(\int_0^t \sigma_u du\right)^2\right) \xrightarrow{L} MN\left(0, \frac{4\left(\frac{\pi}{2} - 1\right)}{\delta} \frac{\int_0^t \sigma_u^2 du}{\left(\int_0^t \sigma_u du\right)^2}\right)$$

to obtain that

$$\frac{2}{4\left(\frac{\pi}{2} - 1\right)} \int_0^t \sigma_u^4 du > \frac{\left(\int_0^t \sigma_u^2 du\right)^3}{\left(\int_0^t \sigma_u du\right)^2}.$$

Again it is unclear whether this is true. But even if it was, we could only say that squared realised absolute variation is a better estimator of the squared integrated spot volatility than realised variance is of the integrated variance. Nevertheless that is not exactly what we are looking for as $\left(\int_0^t \sigma_u du\right)^2$ will not always equalise our object of interest, $\int_0^t \sigma_u^2 du$. Therefore we would then need to estimate the integrated variance with $\left(\int_0^t \sigma_u du\right)^2$, adding up an extra error.

Forsberg and Ghysels (2004) also set some regression models to predict ν_{t+1}^2 with ν_t^2 or with $\nu_t^{[1]}$. They obtained a better fit after using $\nu_t^{[1]}$ as regressor. Ghysels, Santa-Clara and Valkanov (2003) obtained similar empirical results.

Although it is unclear whether realised absolute variation can give better estimations of the integrated variance than realised variance, studies point towards the use of absolute returns rather than squared returns. Forsberg and Ghysels (2004) and Andersen, Bollerslev and Diebold (2003) studied bipower variation, introduced by Barndorff-Nielsen and Shephard (2004a), as estimator and predictor of integrated variance and reported that it would improve upon realised variance.

3.6 Conclusions

In this chapter, given an SV model for the log-prices and the concepts of quadratic and power variation, the availability of high frequency data enabled us to use a time series of realised variances to estimate actual variances and a time series of realised absolute variation to estimate actual spot volatilities. When using the raw realised variances (or realised absolute variations) as estimators, the errors were large especially when M was small. Using the asymptotic distributions of these errors, we improved the estimations via a model based and a model free approach. Both approaches tended to give similar results when M was large and when several lags and leads were used in the model free estimator.

A combined estimation, using both realised variance and realised absolute variation to estimate the actual variance and the actual spot volatility gave interesting results. A higher weight is given to the realised absolute variation in most cases showing that better estimations can be achieved by taking absolute values instead of squaring. This gives us a first insight into the properties of realised multipower variation, that are explained in the next chapter.

Chapter 4

Multipower Variation and Test for Jumps

When the log-price process incorporates a jump component, realised variance will no longer estimate the integrated variance since its probability limit will be determined by the continuous and jump components. Instead realised bipower variation, tripower variation and quadpower variation are consistent estimators of integrated variance even in the presence of jumps. In this chapter we derive the limit distributions of realised tripower and quadpower variation, allowing us to compare these three estimators of integrated variance. Using the limit theories for the differences of the errors, a test for jumps is done with each estimator. As in the previous chapter, market microstructure noise will not be incorporated to our log-price process; its effect on our estimators will be studied in Chapter 5. The main contribution of this chapter is the use of tripower and quadpower variation to test for the presence of jumps in the log-price process.

4.1 Introduction

Recently, econometric methods have been developed to split the continuous and jump components of a log-price process using bipower variation (Barndorff-Nielsen and Shephard(2004a)). In this chapter we will consider alternative estimators of integrated variance in stochastic volatility models where the contribution to the quadratic variation of the log-prices process from the continuous and jump components can also be differentiated. Such estimators are the tripower and quadpower variation.

Firstly, we will derive the asymptotic distribution of these estimators and later, we will use these

asymptotic results to test for jumps in the log-price process. The power of our estimators can then be compared to those found in similar studies by Andersen, Bollerslev and Diebold (2003), Barndorff-Nielsen and Shephard (2006) and Huang and Tauchen (2005).

The outline of this chapter is as follows. Section 4.2 reviews some definitions and results for quadratic and bipower variation with corresponding results for tripower and quadpower variation given in Section 4.3. In Section 4.4 we present the asymptotic theory of these estimators and in Sections 4.5 and 4.6 we test for jumps on simulated and empirical data respectively. Section 4.7 concludes. The proofs of the asymptotic theories can be found in the Appendix.

4.2 Quadratic and bipower variation

4.2.1 Quadratic variation

A standard model in financial economics is a stochastic volatility (SV) model for log-prices Y_t which follows the equation

$$Y_t = \int_0^t a_s ds + \int_0^t \sigma_s dW_s, \quad t \geq 0, \quad (4.1)$$

where $A_t = \int_0^t a_s ds$. The processes σ and A are assumed to be stochastically independent of the standard Brownian motion W , σ is càdlàg and A is assumed to have locally bounded variation sample paths with the extra condition that $\int_0^t \sigma_s^2 ds < \infty$ for all t . As introduced in Chapter 3, σ is called the instantaneous or spot volatility, σ^2 the corresponding spot variance and A the mean process.

We will assume that Y is a semimartingale (SM). A SM is defined as the process that can be decomposed as $Y = A + M$ where A is a process with locally bounded variation paths and M is a local martingale (see Protter(1990)). If A is continuous and $M = \int_0^t \sigma_u dW_u$ then Y is a member of the continuous stochastic volatility semimartingale ($SVSM^c$) class. Here, a key role is played by the integrated variance

$$\sigma_t^{2*} = \int_0^t \sigma_s^2 ds,$$

and the quadratic variation

$$[Y]_t = p \lim_{n \rightarrow \infty} \sum_{j=1}^n (Y_{t_j} - Y_{t_{j-1}})^2$$

for any sequence of partitions $t_0^{(n)} = 0 < t_1^{(n)} < \dots < t_n^{(n)} = t$ with $\sup_j \{t_j^{(n)} - t_{j-1}^{(n)}\} \rightarrow 0$ for $n \rightarrow \infty$.

Assume that we have observations every $\delta > 0$ periods of time, so given the previous framework, let the j -th return be

$$y_j = Y_{j\delta} - Y_{(j-1)\delta} \quad j = 1, 2, 3, \dots, \lfloor t/\delta \rfloor.$$

If high-frequency financial data is available, the realised variance process

$$[Y_\delta]_t^{[2]} = \sum_{j=1}^{\lfloor t/\delta \rfloor} y_j^2,$$

can be defined.

The relationship between realised variance and quadratic variation was studied in the previous chapter and is well known to be

$$[Y_\delta]_t^{[2]} \xrightarrow{p} [Y]_t = \int_0^t \sigma_s^2 ds$$

if $Y \in SVSM^c$.

4.2.2 Bipower variation

Barndorff-Nielsen and Shephard (2004a) introduced the bipower variation process, defined as

$$\{Y\}_t^{[r,s]} = p \lim_{\delta \downarrow 0} \delta^{1-(r+s)/2} \sum_{j=1}^{\lfloor t/\delta \rfloor - 1} |y_j|^r |y_{j+1}|^s, \quad r, s \geq 0.$$

Here we will focus on the case where $r = s = 1$, so the realised bipower variation process

$$\{Y_\delta\}_t^{[1,1]} = \sum_{j=1}^{\lfloor t/\delta \rfloor - 1} |y_j| |y_{j+1}|$$

is an estimator of $\{Y\}_t^{[1,1]}$ and therefore of σ_t^{2*} as well. These authors show that in the $SVSM^c$ case under some regularity conditions

$$\mu_1^{-2} \{Y_\delta\}_t^{[1,1]} \xrightarrow{p} \int_0^t \sigma_s^2 ds.$$

Furthermore,

$$[Y_\delta]_t^{[2]} - \mu_1^{-2} \{Y_\delta\}_t^{[1,1]} \xrightarrow{p} 0,$$

giving them a tool to test for jumps.

4.2.3 Effect of jumps

We now consider the following process consisting of a continuous and a jump component

$$Y_t = Y_t^{(1)} + Y_t^{(2)}$$

where $Y^{(1)} \in SVSM^c$ and

$$Y_t^{(2)} = \sum_{i=1}^{N_t} c_i$$

with $N_t < \infty$ for all $t > 0$ a finite activity simple counting process and $\{c_i\}$ a collection of non-zero random variables.

Notice that in this case the quadratic variation is

$$\begin{aligned} [Y]_t &= \sigma_t^{2*} + \sum_{i=1}^{N_t} c_i^2 \\ &= [Y^{(1)}]_t + [Y^{(2)}]_t, \end{aligned}$$

so both components, the continuous and the jump one, contribute to the total quadratic variation.

In Barndorff-Nielsen and Shephard (2004a) it is shown that the realised variance incorporates the contribution to the quadratic variation of both the continuous and jump components

$$[Y_\delta]_t^{[2]} \xrightarrow{p} [Y]_t$$

but the realised bipower variation just incorporates the contribution of the continuous one

$$\mu_1^{-2} \{Y_\delta\}_t^{[1,1]} \xrightarrow{p} \int_0^t \sigma_s^2 ds.$$

These results imply that

$$[Y_\delta]_t^{[2]} - \mu_1^{-2} \{Y_\delta\}_t^{[1,1]} \xrightarrow{p} \sum_{i=1}^{N_t} c_i^2.$$

4.2.4 Testing for jumps using Realised bipower variation

By using realised variance and realised bipower variation, it is possible to identify the continuous component and the jump component of the log-price process, so to test for jumps in the log-price process we only need to know the convergence in distribution of the estimators. In Barndorff-Nielsen and Shephard (2004a) the joint distribution theory for the realised variance error and the realised bipower variation error was firstly given. Recently Barndorff-Nielsen, Graversen, Jacod, Podolskij

and Shephard (2005) and Barndorff-Nielsen, Graversen, Jacod and Shephard (2006) have unified the asymptotic treatment of some volatility measures, where the following result can be seen as a special case. The necessary limit theory for the difference of the realised variance and the realised bipower variation is

$$\frac{1}{\delta^{1/2} \sqrt{\int_0^t \sigma_s^4 ds}} \left([Y_\delta]_t^{[2]} - \mu_1^{-2} \{Y_\delta\}_t^{[1,1]} \right) \xrightarrow{L} N(0, \vartheta_{RV})$$

where $\vartheta_{RV} \simeq 0.6091$. This theory is later used by Andersen, Bollerslev and Diebold (2003), Barndorff-Nielsen and Shephard (2006) and Huang and Tauchen (2005) to carry out the testing.

In this thesis we assume that the jump component of the log-prices is a finite activity jump process, though Barndorff-Nielsen, Shephard and Winkel (2005) obtained the conditions for the convergence in probability and central limit theorem to hold under an infinite activity jump process.

4.3 Tripower and Quadpower variation

Barndorff-Nielsen and Shephard (2004a) generalised bipower variation by multiplying together a finite number of absolute returns raised to some non-negative power, calling this multipower variation. Tripower variation and quadpower variation are particular cases of this idea. The tripower variation process is defined as

$$\{Y\}_t^{[r,s,u]} = p \lim_{\delta \downarrow 0} \delta^{1-(r+s+u)/2} \sum_{j=1}^{\lfloor t/\delta \rfloor - 2} |y_j|^r |y_{j+1}|^s |y_{j+2}|^u, \quad r, s, u > 0,$$

and estimated with the realised tripower variation process

$$\{Y_\delta\}_t^{[r,s,u]} = \delta^{1-(r+s+u)/2} \sum_{j=1}^{\lfloor t/\delta \rfloor - 2} |y_j|^r |y_{j+1}|^s |y_{j+2}|^u, \quad r, s, u > 0.$$

We will focus on the case where $r = s = u = 2/3$. Analogously the quadpower variation process is defined as

$$\{Y\}_t^{[r,s,u,v]} = p \lim_{\delta \downarrow 0} \delta^{1-(r+s+u+v)/2} \sum_{j=1}^{\lfloor t/\delta \rfloor - 3} |y_j|^r |y_{j+1}|^s |y_{j+2}|^u |y_{j+3}|^v, \quad r, s, u, v > 0$$

and estimated with the realised quadpower variation process

$$\{Y_\delta\}_t^{[r,s,u,v]} = \delta^{1-(r+s+u+v)/2} \sum_{j=1}^{\lfloor t/\delta \rfloor - 3} |y_j|^r |y_{j+1}|^s |y_{j+2}|^u |y_{j+3}|^v, \quad r, s, u, v > 0.$$

We will consider the special case where $r = s = u = v = 1/2$.

Given that $Y \in SVSM^c$, we can get new estimators of the integrated variance based on the following results

$$\mu_{2/3}^{-3} \{Y_\delta\}_t^{[2/3, 2/3, 2/3]} \xrightarrow{p} \int_0^t \sigma_s^2 ds,$$

and

$$\mu_{1/2}^{-4} \{Y_\delta\}_t^{[1/2, 1/2, 1/2, 1/2]} \xrightarrow{p} \int_0^t \sigma_s^2 ds.$$

So we can also obtain that

$$[Y_\delta]_t^{[2]} - \mu_{2/3}^{-3} \{Y_\delta\}_t^{[2/3, 2/3, 2/3]} \xrightarrow{p} 0$$

and that

$$[Y_\delta]_t^{[2]} - \mu_{1/2}^{-4} \{Y_\delta\}_t^{[1/2, 1/2, 1/2, 1/2]} \xrightarrow{p} 0.$$

4.3.1 Effect of jumps

When a jump component is present in the log-price process, as in subSection 4.2.3, tripower variation and quadpower variation does not incorporate the contribution to quadratic variation of the jump component, so

$$\mu_{2/3}^{-3} \{Y_\delta\}_t^{[2/3, 2/3, 2/3]} \xrightarrow{p} \int_0^t \sigma_s^2 ds$$

and

$$\mu_{1/2}^{-4} \{Y_\delta\}_t^{[1/2, 1/2, 1/2, 1/2]} \xrightarrow{p} \int_0^t \sigma_s^2 ds$$

These results imply that

$$[Y_\delta]_t^{[2]} - \mu_{2/3}^{-3} \{Y_\delta\}_t^{[2/3, 2/3, 2/3]} \xrightarrow{p} \sum_{i=1}^{N_t} c_i^2$$

and that

$$[Y_\delta]_t^{[2]} - \mu_{1/2}^{-4} \{Y_\delta\}_t^{[1/2, 1/2, 1/2, 1/2]} \xrightarrow{p} \sum_{i=1}^{N_t} c_i^2.$$

As a consequence of this, tripower and quadpower variation provide an alternative to bipower variation in testing for jumps, we only need the asymptotic distributions which will be obtained in the next section.

Barndorff-Nielsen, Shephard and Winkel (2005) generalised these results, showing that multipower variation, $d_r \{Y_\delta\}_t^{\{r_1, \dots, r_s\}}$ with $\sum_{i=1}^s r_i = 2$, converges in probability to integrated variance, when a jump process of finite activity is added to the log-price process, if $\max\{r_1, \dots, r_s\} < 2$. Here d_r is a constant that only depends on r_1, \dots, r_s .

4.3.2 Daily time series

To produce daily time series, assume a fixed time interval $h > 0$ (here it denotes the period of a day) with $\lfloor t/\delta \rfloor = M$ intra- h returns, during each fixed h time period, defined as

$$y_{j,i} = Y_{(i-1)h+j\delta} - Y_{(i-1)h+(j-1)\delta},$$

for the $j - th$ intra- h return in the $i - th$ period.

The daily realised tripower variation (RTV), can then be defined as

$$\mu_{2/3}^{-3} \{Y_M\}_i^{[2/3,2/3,2/3]} = \mu_{2/3}^{-3} \sum_{j=1}^{M-2} |y_{j,i}|^{2/3} |y_{j+1,i}|^{2/3} |y_{j+2,i}|^{2/3}$$

and the daily realised quadpower variation (RQV) as

$$\mu_{1/2}^{-4} \{Y_M\}_i^{[1/2,1/2,1/2,1/2]} = \mu_{1/2}^{-4} \sum_{j=1}^{M-3} |y_{j,i}|^{1/2} |y_{j+1,i}|^{1/2} |y_{j+2,i}|^{1/2} |y_{j+3,i}|^{1/2}.$$

We will also study the skipped version of the daily realised bipower variation (RSBV)

$$\mu_1^{-2} \{Y_M\}_i^{[1,0,1]} = \sum_{j=1}^{M-2} |y_{j,i}| |y_{j+2,i}|$$

These three estimators converge in probability to the actual variance whether or not jumps are present in the log-price process. Furthermore, if such jumps are present their contribution to the quadratic variation, $\sum_{j=N_{i-1}+1}^{N_i} c_j^2$, can be estimated by the differences RV-RTV, RV-RQV and RV-RSBV.

4.4 Asymptotic theory

4.4.1 Asymptotic distributions for the difference between realised tripower, quadpower and skipped version of bipower variation and realised variance

To allow us to implement the test for jumps we need to extend from the convergence in probability to convergence in distribution.

As in Barndoff-Nielsen, Graversen, Jacod, Podolskij and Shephard (2005) and Barndoff-Nielsen, Graversen, Jacod and Shephard (2006) let us assume that given the SV model in equation (4.1) A is of locally bounded variation and σ is càdlàg. These and additional more technical conditions on the driving process of σ , also stated in the previous papers, enable us to obtain the asymptotic distributions below. Additionally, here we will set $A = 0$.

Theorem 1

If $Y \in SVSM^c$ then as $\delta \downarrow 0$

$$\frac{1}{\delta^{1/2} \sqrt{\int_0^t \sigma_s^4 ds}} \left(\mu_{2/3}^{-3} \{Y_\delta\}_t^{[2/3, 2/3, 2/3]} - [Y_\delta]_t^{[2]} \right) \xrightarrow{L} N(0, \vartheta_{TV})$$

where

$$\vartheta_{TV} = \mu_{4/3} \mu_{2/3}^{-2} (\mu_{4/3}^2 \mu_{2/3}^{-4} + 2\mu_{4/3} \mu_{2/3}^{-2} - 2) - 7 \simeq 1.0613$$

and $\mu_r = E(|x|^r) = 2^{r/2} \frac{\Gamma(\frac{1}{2}(r+1))}{\Gamma(\frac{1}{2})}$.

Proof. See appendix A4.1.

Theorem 2

If $Y \in SVSM^c$ then as $\delta \downarrow 0$

$$\frac{1}{\delta^{1/2} \sqrt{\int_0^t \sigma_s^4 ds}} \left(\mu_{1/2}^{-4} \{Y_\delta\}_t^{[1/2, 1/2, 1/2, 1/2]} - [Y_\delta]_t^{[2]} \right) \xrightarrow{L} N(0, \vartheta_{QV})$$

where

$$\vartheta_{QV} = \mu_1 \mu_{1/2}^{-2} (\mu_1^3 \mu_{1/2}^{-6} + 2\mu_1^2 \mu_{1/2}^{-4} + 2\mu_1 \mu_{1/2}^{-2} - 2) - 9 \simeq 1.37702$$

and $\mu_r = E(|x|^r)$.

Proof. See appendix A4.2.

Theorem 3

If $Y \in SVSM^c$ then as $\delta \downarrow 0$

$$\frac{1}{\delta^{1/2} \sqrt{\int_0^t \sigma_s^4 ds}} \left(\mu_1^{-2} \{Y_\delta\}_t^{[1, 0, 1]} - [Y_\delta]_t^{[2]} \right) \xrightarrow{L} N(0, \vartheta_{SBV})$$

where

$$\vartheta_{SBV} = \mu_1^{-4} + 2\mu_1^{-2} - 5 \simeq 0.60907$$

and $\mu_r = E(|x|^r)$.

Proof. See appendix A4.3.

A consequence of this last limit is that there is no difference in distribution between working with the realised bipower variation or working with the realised skipped version of bipower variation.

From the previous asymptotic distribution we can observe that

$$\vartheta_{BV} = \vartheta_{SBV} < \vartheta_{TV} < \vartheta_{QV};$$

i.e. realised bipower variation is more efficient than realised tripower and quadpower variation. So in ideal conditions, when $\delta \downarrow 0$ and there is no microstructure effect, we can expect that realised bipower variation will give us better results. Nevertheless, real financial series are finite and microstructure effect exists so it is uncertain which will give us the highest power in the test for jumps when working with empirical data.

Now that the three necessary asymptotic distributions have been obtained, the tests for jumps in the log-price process can be carried out.

4.4.2 Testing for jumps

Following Barndorff-Nielsen and Shephard (2006) two different tests for jumps can be established, a linear test and a ratio test. The linear tests are based on the previous limit theories while the ratio tests utilize the following convergence results. For the realised tripower variation process

$$\frac{\left(\frac{\mu_{2/3}^{-3} \{Y_\delta\}_t^{[2/3, 2/3, 2/3]}}{[Y_\delta]_t^{[2]}} - 1 \right)}{\delta^{1/2} \sqrt{\frac{\int_0^t \sigma_s^4 ds}{\left(\int_0^t \sigma_s^2 ds \right)^2}}} \xrightarrow{L} N(0, \vartheta_{TV})$$

where $\vartheta_{TV} \simeq 1.0613$; for the realised quadpower variation process

$$\frac{\left(\frac{\mu_{1/2}^{-4} \{Y_\delta\}_t^{[1/2, 1/2, 1/2, 1/2]}}{[Y_\delta]_t^{[2]}} - 1 \right)}{\delta^{1/2} \sqrt{\frac{\int_0^t \sigma_s^4 ds}{\left(\int_0^t \sigma_s^2 ds \right)^2}}} \xrightarrow{L} N(0, \vartheta_{QV})$$

where $\vartheta_{QV} \simeq 1.37702$; and for the realised skipped bipower variation process

$$\frac{\left(\frac{\mu_1^{-2} \{Y_\delta\}_i^{[1,0,1]}}{[Y_\delta]_i^{[2]}} - 1 \right)}{\delta^{1/2} \sqrt{\frac{\int_0^t \sigma_s^4 ds}{\left(\int_0^t \sigma_s^2 ds \right)^2}}} \xrightarrow{L} N(0, \vartheta_{SBV})$$

where $\vartheta_{SBV} \simeq 0.60907$.

When working with real data both the linear and ratio test are infeasible as $\int_0^t \sigma_s^4 ds$ cannot be observed even when high-frequency data is available. In such situations the following estimators are needed

1) Realised Quarticity (E1)

$$M \mu_4^{-1} \{Y_M\}_i^{[4]} = M \mu_4^{-1} \sum_{j=1}^M y_{j,i}^4$$

2) Realised Tripower Variation with $r = s = u = 4/3$ (E2)

$$M \mu_{4/3}^{-3} \{Y_M\}_i^{[4/3,4/3,4/3]} = M \mu_{4/3}^{-3} \sum_{j=1}^{M-2} |y_{j,i}|^{4/3} |y_{j+1,i}|^{4/3} |y_{j+2,i}|^{4/3}$$

3) Realised Quadpower Variation with $r = s = u = v = 1$ (E3)

$$M \mu_1^{-4} \{Y_M\}_i^{[1,1,1,1]} = M \mu_1^{-4} \sum_{j=1}^{M-3} |y_{j,i}| |y_{j+1,i}| |y_{j+2,i}| |y_{j+3,i}|.$$

As pointed out in Barndorff-Nielsen and Shephard (2006), the quadratic variation for the jump component cannot be negative, so the following estimators can be used

$$\begin{aligned} & \min(0, \mu_{2/3}^{-3} \{Y_M\}_i^{[2/3,2/3,2/3]} - [Y_M]_i^{[2]}), \\ & \min(0, \mu_{1/2}^{-4} \{Y_M\}_i^{[1/2,1/2,1/2,1/2]} - [Y_M]_i^{[2]}), \\ & \min(0, \mu_1^{-2} \{Y_M\}_i^{[1,0,1]} - [Y_M]_i^{[2]}). \end{aligned}$$

A bias is introduced to the test statistics because we use finite values of M, therefore our estimators will have less components in the summation compared to the realised variance. This problem can be dealt with the modified estimators

$$\begin{aligned} & \left(\frac{M}{M-2} \right) \mu_{2/3}^{-3} \{Y_M\}_i^{[2/3,2/3,2/3]}, \\ & \left(\frac{M}{M-3} \right) \mu_{1/2}^{-4} \{Y_M\}_i^{[1/2,1/2,1/2,1/2]}, \\ & \left(\frac{M}{M-2} \right) \mu_1^{-2} \{Y_M\}_i^{[1,0,1]}. \end{aligned}$$

In small samples in the feasible test, the result from Barndorff-Nielsen and Shepard(2003) given by Jensen's inequality,

$$\frac{\int_0^t \sigma_s^4}{(\int_0^t \sigma_s^2)^2} \geq 1$$

may not hold, so additionally we will use the adjusted ratio

$$\max\left(1, \frac{\widehat{\int_0^t \sigma_s^4 ds}}{(\int_0^t \sigma_s^2)^2}\right).$$

4.5 Simulations

The test for jumps will be applied to a simple simulated process. A constant elasticity of variance (CEV) process will be used. Specifically the Feller (1951) or Cox, Ingersoll and Ross (1985) square root process for σ^2

$$d\sigma_t^2 = -\lambda\{\sigma_t^2 - \xi\}dt + \omega\sigma_t dB_{\lambda t}, \quad \xi \geq \omega^2/2, \quad \lambda > 0,$$

where B is a standard Brownian motion process. The square root process has a marginal distribution

$$\sigma_t^2 \sim \Gamma(2\omega^{-2}\xi, 2\omega^{-2}) = \Gamma(\nu, a), \quad \nu \geq 1,$$

with a mean of $\xi = \nu/a$ and a variance of $\omega^2 = \nu/a^2$. We will take $A = 0$ and rule out the leverage effect by assuming $Cor\{B_{\lambda t}, W_t\} = 0$. We will take $h = 1$, $\lambda = 0.01$, $\nu = 4$ and $a = 8$. The jumps will be i.i.d. $N(0, \beta\nu/a)$, thus a jump has the same variance as that expected over a $(\beta \times 100)\%$ of a day when there are no jumps.

4.5.1 Finite sample behaviour of the estimators

Finite sample behaviour when there are no jumps

Firstly we will try to assess, given that there are no jumps, the accuracy of the convergence in probability for each of our estimators. To do so, we will simulate data corresponding to 1000 days from the previous process without jumps. The realised values for bipower, tripower, quadpower and the skipped version of bipower variation are calculated for different values of M ($M=12, 72$ and 288 , i.e. 120, 20 and 5 minutes returns). Figures 4.1, 4.2, 4.3 and 4.4 correspond to the first 250 observations of

realised bipower, tripower, quadpower and the skipped version of bipower variation respectively. The left hand graphs show the actual variance together with the realised values to assess the probability limit. As market microstructure noise is not added to our series, they become more accurate as the value of M increases. The magnitude of the difference seems to be large in times of high volatility. From these graphs, no particular estimator seems to outperform any of the others. The plots in the middle show the differences between the realised series and the actual variance, i.e. the errors, together with their confidence interval. The drastic fluctuations in the bands correspond to changes in the level of the variance process, with wider bands at levels of higher variance. The right hand graphs give the corresponding QQ-plot which should be a 45 degree line if the asymptotic results hold. There is an improvement in all the cases as M increases although the asymptotic theory does not provide an accurate guide to the finite sample behaviour. Even when using 5 minutes returns, there are problems due to heavy tails. Overall nothing can be concluded about which estimator shows better finite sample behaviour.

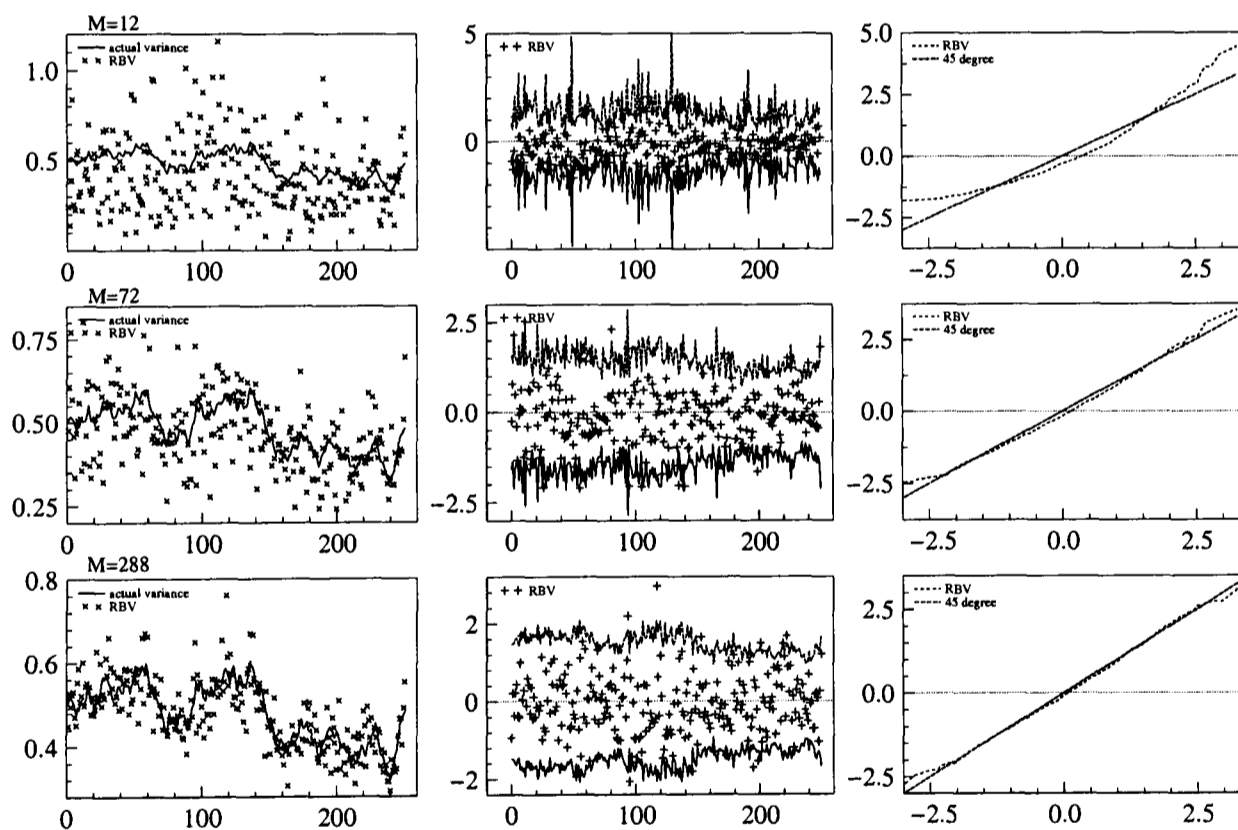


Figure 4.1: *Left graphs: Actual variance and realised bipower variation. Middle graphs: realised bipower variation minus actual variance and confidence interval. Right graphs: QQ-plot of the realised bipower variation error.*

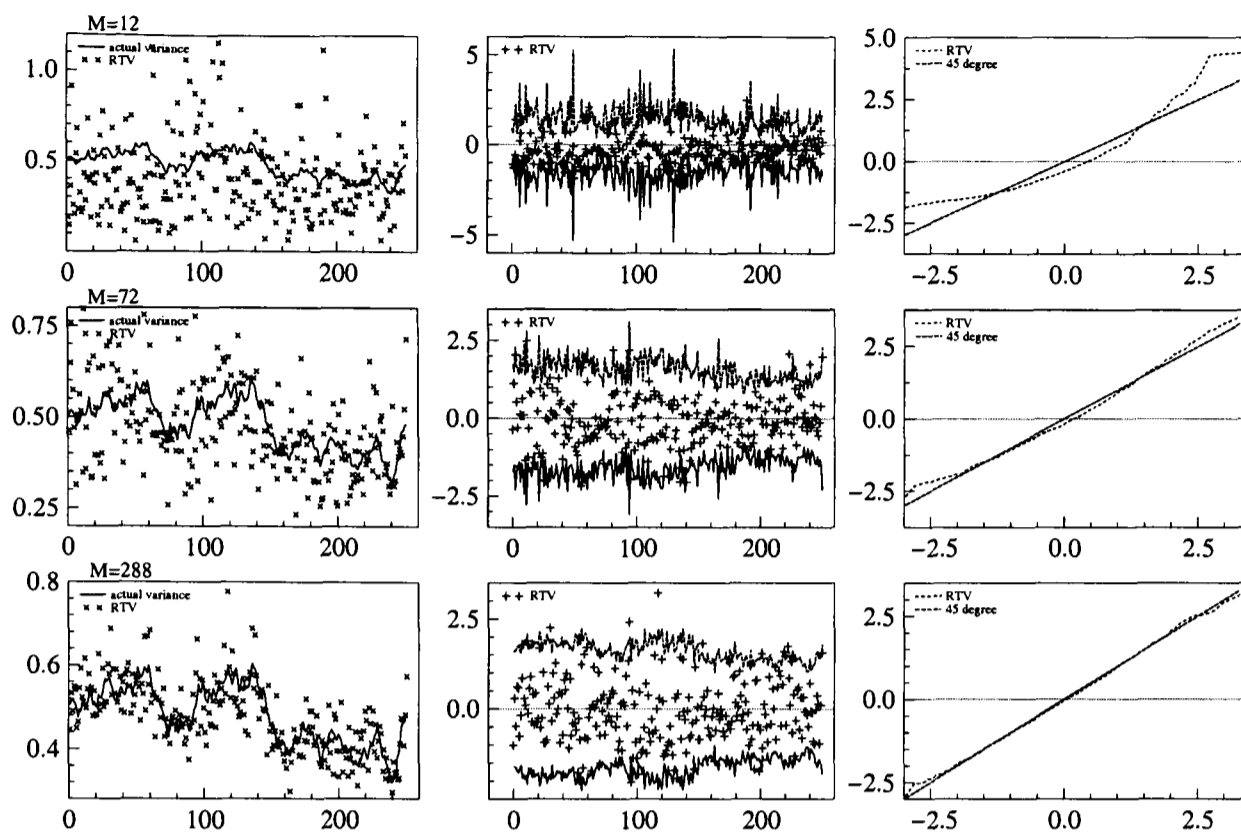


Figure 4.2: *Left graphs: Actual variance and realised tripower variation. Middle graphs: realised tripower variation minus actual variance and confidence interval. Right graphs: QQ-plot of the realised tripower variation error.*

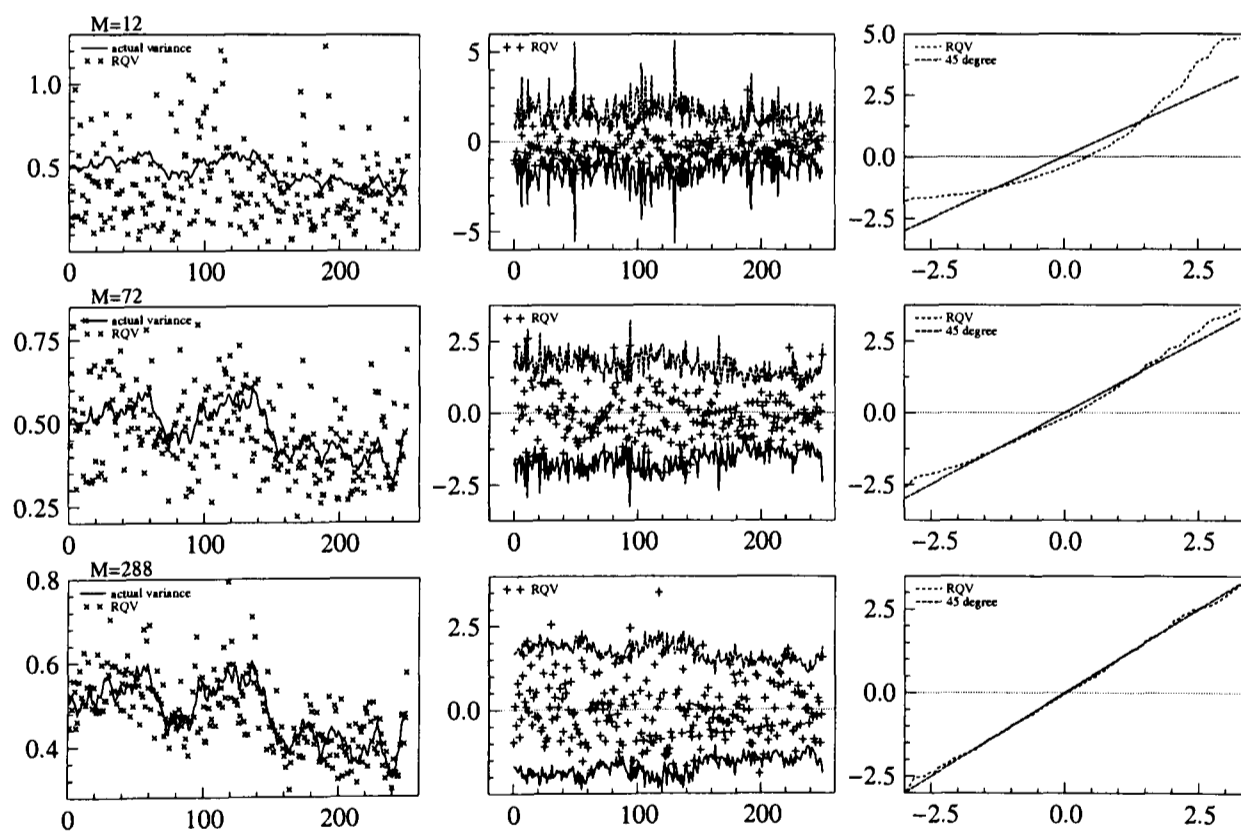


Figure 4.3: *Left graphs: Actual variance and realised quadpower variation. Middle graphs: realised quadpower variation minus actual variance and confidence interval. Right graphs: QQ-plot of the realised quadpower variation error.*

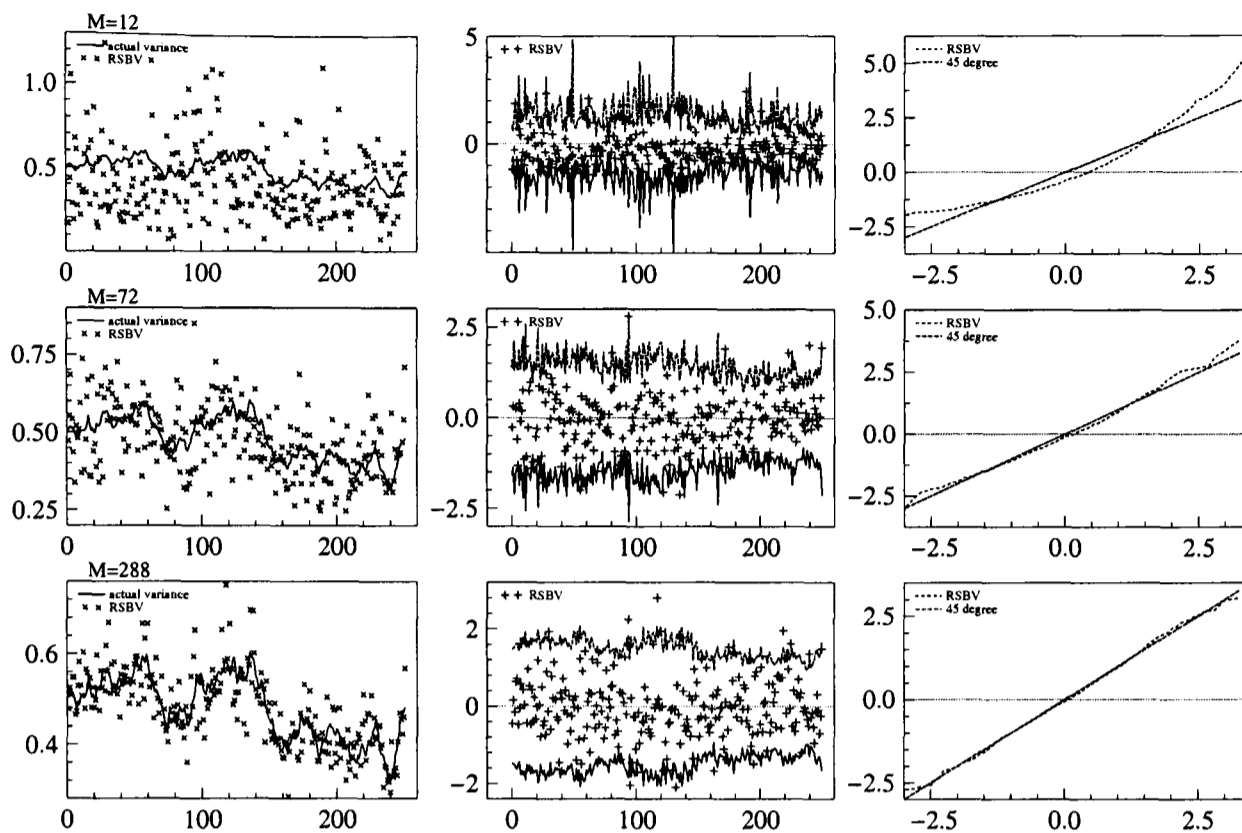


Figure 4.4: *Left graphs: Actual variance and skipped version of realised bipower variation. Middle graphs: skipped version of realised bipower variation minus actual variance and confidence interval. Right graphs: QQ-plot of the skipped version of realised bipower variation error.*

An alternative view to assess the finite sample behaviour is given in Tables 4.1 and 4.2. They record the bias from zero and standard deviation of the errors (the corresponding realised series minus the actual variance). The bias should be zero and the standard deviation should be around one if the limit theory describes properly the behaviour of our statistics. The coverage rate is also reported. Given the asymptotic distributions of each of our estimators (see appendix), this is the percent of the standardised statistics which are under, in absolute values, the 97.5% quantile (setting the confidence level at 95%) of a Normal distribution. Table 4.1 gives the infeasible results (given that we are using simulations) and Table 4.2 gives the feasible results. In this last table we use the realised quarticity (E1) to estimate the integrated quarticity. As we are not including market microstructure noise, the results improve for larger values of M. All of the estimators seem to have a similar and good finite sample behaviour.

M	BV			TV			QV			SBV		
	Bias	SD	Cov	Bias	SD	Cov	Bias	SD	Cov	Bias	SD	Cov
12	-.189	.950	96.8	-.203	.992	95.6	-.208	1.02	95.4	-.216	.989	96.2
24	-.148	.971	96.1	-.146	.969	96.0	-.143	.971	96.1	-.138	.997	96.0
72	-.108	.995	95.2	-.085	.988	95.2	-.076	.993	95.5	-.070	1.01	94.8
144	-.074	1.01	94.6	-.081	1.02	94.6	-.089	1.01	94.8	-.071	1.01	95.1
288	-.047	.986	94.6	-.028	.990	95.2	-.021	.989	95.3	-.020	1.00	94.3
576	-.016	.983	96.2	-.023	.987	95.5	-.033	.986	95.2	-.026	1.00	95.4

Table 4.1: *Bias, standard deviation and coverage of the infeasible standardised realised bipower, tripower, quadpower and skipped bipower variation error.*

M	BV			TV			QV			SBV		
	Bias	SD	Cov	Bias	SD	Cov	Bias	SD	Cov	Bias	SD	Cov
12	-.611	1.51	86.0	-.575	1.45	85.9	-.558	1.44	84.9	-.634	1.51	85.7
24	-.423	1.26	89.3	-.389	1.23	89.1	-.368	1.21	89.3	-.419	1.28	89.2
72	-.253	1.09	91.9	-.209	1.07	92.9	-.188	1.07	93.0	-.213	1.11	92.0
144	-.169	1.07	93.1	-.165	1.06	93.2	-.165	1.05	93.5	-.163	1.05	93.0
288	-.110	1.01	94.4	-.082	1.01	94.2	-.071	1.01	94.2	-.083	1.02	93.4
576	-.059	.994	95.9	-.061	.997	95.2	-.067	.996	95.0	-.071	1.01	95.3

Table 4.2: *Bias, standard deviation and coverage of the feasible standardised realised bipower, tripower, quadpower and skipped bipower variation error.*

Finite sample behaviour in the presence of jumps

As stated in this chapter, realised bipower variation and our three other estimators are robust to the presence of jumps. It will be convenient to study their finite sample behaviour in the presence of rare jumps. In Figure 4.5 we use RBV, RTV, RQV and RSBV to find the integrated variance and the quadratic variation of the jump component of the first 50 days of a series of simulated data. The data was simulated from the SV plus jump process explained previously. Realised values are calculated based on $M=288$. In the left hand side figures, the true integrated variance is shown together with the integrated variance estimated with the a)RBV, b)RTV, c)RQV and d)RSBV. For all the cases our estimators work rather well by not incorporating the variation due to the jumps. In the right hand side figures, the true quadratic variation of the jump component is shown together with the estimated

one using a)RV-RBV, b)RV-RTV, c)RV-RQV and d)RV-RSBV. All of them seem to give accurate estimations but we need to apply the asymptotic distributions to obtain stronger conclusions.

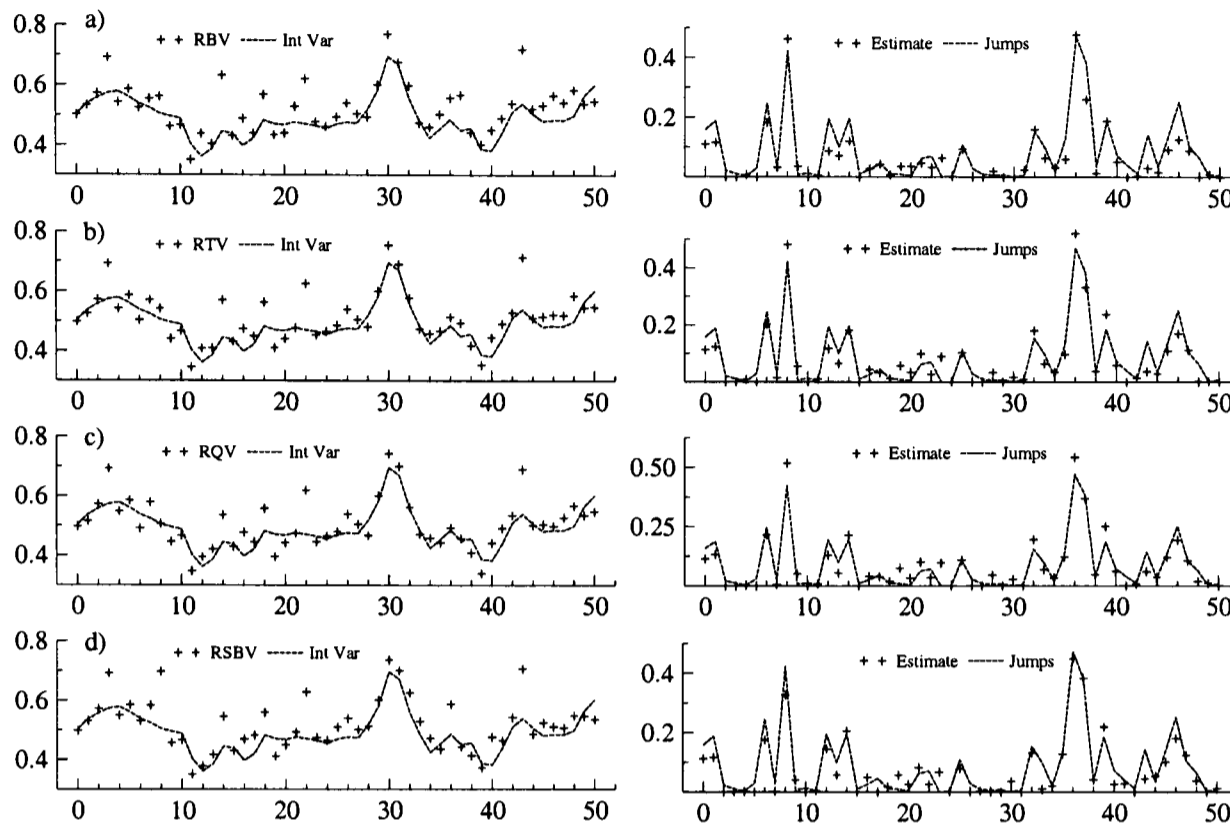


Figure 4.5: Simulation from a jump plus diffusion based SV model, estimating the integrated variance (left hand side figures) and the quadratic variation of the jump component (right hand side figures) with $M=288$. Rows correspond to RBV, RTV, RQV and RSBV respectively.

4.5.2 Null distribution

Infeasible test

Now instead of examining the finite sample behaviour of our estimators per se, we will investigate their accuracy in the tests for jumps. We will simulate a one thousand day process first without any jumps. We will do the infeasible linear and ratio tests for jumps using each one of the estimators (realised bipower variation, realised tripower variation, realised quadpower variation and realised skipped bipower variation) for different values of M , expecting not to reject the null hypothesis (the null hypothesis implies no jumps). The test statistics are plotted in Figure 4.6 for $M=288$ for a)RBV, b)RTV, c)RQV and d)RSBV. In the left hand side graph the linear test statistics are shown together with twice their standard error. In the middle graph this is repeated for the ratio test. The hypothesis of no jumps will be rejected every time a value falls below twice the standard error (setting a 95%

confidence level), therefore the ratio test seems to give better results than the linear one. The right hand side graph shows the normal QQ-plot of the two tests. If the asymptotic approximations were accurate they should lay on the 45 degree line. Both test have good QQ-plots, although the one of the ratio test is slightly better for all our estimators.

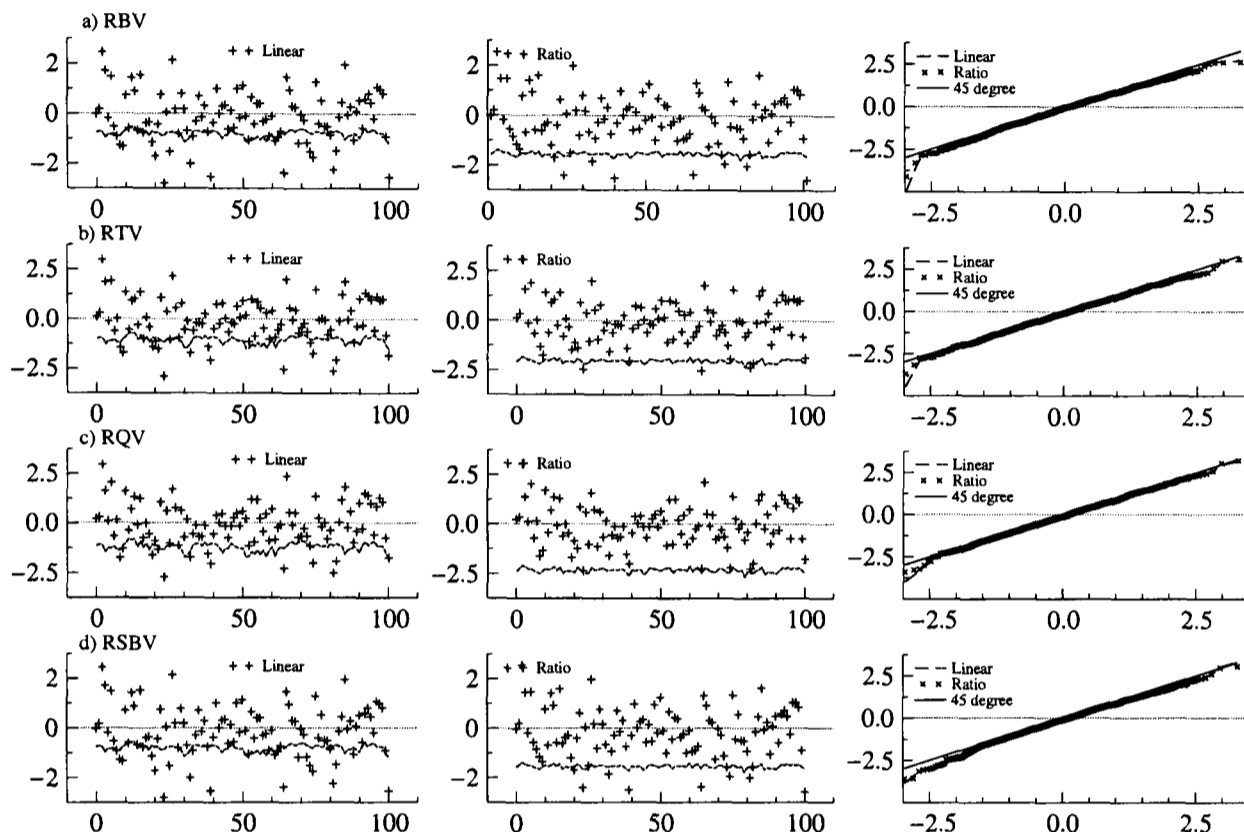


Figure 4.6: *T*-statistics plus twice their standard errors from simulations from the null distribution for the infeasible linear (left hand side) and ratio tests (middle) for $M=288$ and for a)RBV, b)RTV, c)RQV and d)RSBV. QQ-plots are given in the right hand side.

In Table 4.3 the bias from zero of the linear test statistics, their standard deviation and the coverage rates for one-sided linear tests are recorded for the series using the four estimators (RBV, RTV, RQV, RSBV). The bias should tend to zero and the standard deviation to one if the asymptotic theory is describing properly the behaviour of the t-test. The coverage rate tells the probability of not rejecting the null hypothesis when it is true, i.e., finding no jumps when the process is continuous. Here we are taking $\alpha = .05$ so the coverage rates should be $(1 - \alpha) * 100\%$ if the asymptotic distribution fit properly the finite sample test statistics. In this table the infeasible test is done, i.e. the actual quarticity is used to calculate the linear test statistic.

In Table 4.4 the bias, standard deviation and coverage rate are reported for the infeasible ratio test. In both the linear and ratio tests we obtain better results as the value of M gets larger. In this

case, the infeasible linear and ratio test give very similar results. Realised tripower and quadpower variation give slightly better results than the estimators based on bipower variation. Realised bipower variation and its skipped version overestimate the jumps.

M	12			72			288		
	Bias	SD	Cove	Bias	SD	Cove	Bias	SD	Cove
RBV	-.363	1.16	87.4	-.191	1.00	92.8	-.115	1.00	93.5
RTV	-.313	1.13	88.5	-.180	1.00	92.9	-.103	.994	94.1
RQV	-.313	1.14	89.0	-.181	1.00	93.3	-.104	.999	94.3
RSBV	-.424	1.22	85.9	-.199	1.03	92.2	-.129	1.01	93.6

Table 4.3: *Bias, standard deviation and coverage of the infeasible linear test.*

M	12			72			288		
	Bias	SD	Cove	Bias	SD	Cove	Bias	SD	Cove
RBV	-.379	1.08	85.2	-.196	.987	92.6	-.110	.993	93.3
RTV	-.328	1.07	89.5	-.185	.985	94.1	-.099	.985	93.7
RQV	-.329	1.08	89.7	-.183	.987	93.8	-.101	.992	94.4
RSBV	-.433	1.15	85.5	-.199	1.01	92.4	-.127	1.01	93.7

Table 4.4: *Bias, standard deviation and coverage of the infeasible ratio test.*

Feasible tests

As explained previously the use of the integrated quarticity is infeasible in practice, so estimators for it should be used. The feasible test statistics, using the quadpower variation (E3) as the estimator of the integrated quarticity, are plotted now in Figure 4.7 for M=288 for a)RBV, b)RTV, c)RQV and d)RSBV. The left hand side and middle graphs show the t-test statistics of the linear and ratio test respectively. The QQ-plots of both tests are shown in the right hand side graphs. Comparing this figure with Figure 4.6 (the feasible with the infeasible case), a deterioration is visible in the QQ-plots specially for the linear test.

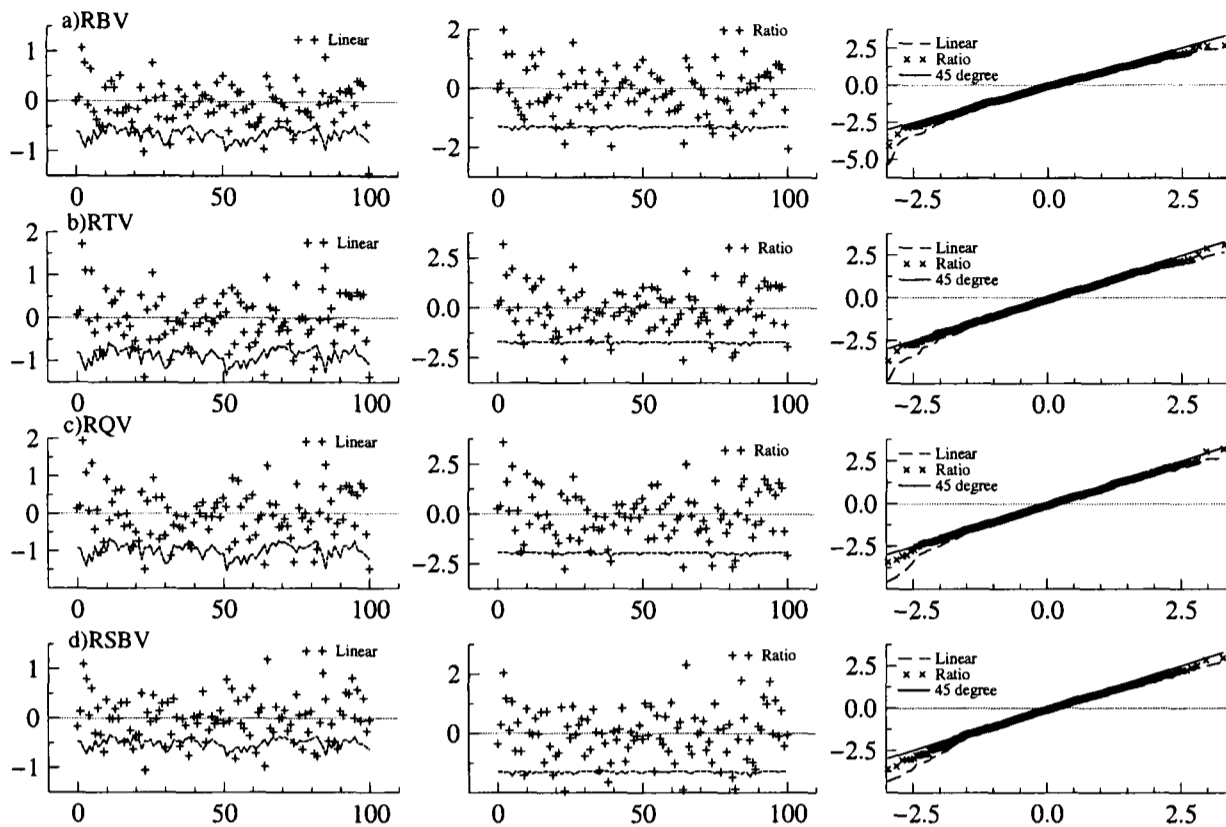


Figure 4.7: T -statistics plus twice their standard errors from simulations from the null distribution for the feasible linear (left hand side) and ratio tests (middle) for $M=288$ and for a)RBV, b)RTV, c)RQV and d)RSBV. QQ-plots are given in the right hand side.

In Table 4.5 the bias, standard deviation and coverage is reported for the linear test statistic using the realised quarticity (E1), the realised tripower variation with $r = s = u = 4/3$ (E2) and the realised quadpower variation with $r = s = u = v = 1$ (E3) given $M=288$. In Table 4.6 the results for the feasible ratio tests are shown for $M=288$ using E1, E2 and E3 to estimate the actual quarticity. In the feasible case, confirming the results from the previous QQ-plots, the ratio tests give much better results than the linear tests. Realised tripower and quadpower variation outperform the other estimators when testing for jumps if the process is continuous. Bipower variation and its skipped version overestimate the presence of jumps.

In the case where the integrated quarticity is infeasible and there are no jumps in the process, the realised quarticity seems to be its best estimator (nevertheless when a jump component is present it will be the worse one as explained later on). There is little to choose between its other two estimators, E2 and E3.

	E1			E2			E3		
	Bias	SD	Cove	Bias	SD	Cove	Bias	SD	Cove
RBV	-.086	.997	93.5	-.165	1.03	91.6	-.173	1.04	91.3
RTV	-.075	.992	94.3	-.154	1.02	92.1	-.167	1.03	91.6
RQV	-.076	.999	94.8	-.149	1.02	92.8	-.167	1.04	91.7
RSBV	-.103	1.01	94.4	-.161	1.04	92.9	-.175	1.05	92.2

Table 4.5: *Bias, standard deviation and coverage of the feasible linear test for $M=288$.*

	E1			E2			E3		
	Bias	SD	Cove	Bias	SD	Cove	Bias	SD	Cove
RBV	-.041	.986	94.9	-.110	.980	93.7	-.113	.980	93.6
RTV	-.015	.987	96.4	-.094	.971	94.1	-.099	.976	94.1
RQV	-.007	.998	97.1	-.087	.968	94.8	-.097	.980	94.4
RSBV	-.056	.994	95.0	-.117	.985	94.1	-.124	.989	94.0

Table 4.6: *Bias, standard deviation and coverage of the feasible ratio test with $M=288$.*

4.5.3 Alternative distribution

In this section we will add a jump component to the simulated process to assess the test for jumps when they are present in the log-price process. Each day the same number of jumps are added to the process (one thousand days will be simulated). The jumps will be i.i.d. $N(0, \beta\nu/a)$ and several simulations with a different β will be done to assess the performance of the test with different sizes of jumps. The one-sided hypothesis test is done every day and the level of significance is 5%. Table 4.7 reports the bias, standard deviation and the coverage for the linear infeasible test when one jump is added every day with $\beta = 50\%$ and $\beta = 20\%$. In this case the coverage reports the percent of times we do an error type II, i.e. we accept the null hypothesis when there are jumps. It can be seen that as the size of the jump increases, the easier it is to detect the jump. As M becomes larger, the results of the hypothesis test improve. Almost no jumps are detected when the jumps are very small, especially when the sample frequency is low.

M	12			72			288		
	Bias	SD	Cove	Bias	SD	Cove	Bias	SD	Cove
50%									
RBV	-.728	1.75	78.1	-1.55	3.21	69.8	-3.49	6.42	55.0
RTV	-.639	1.60	79.5	-1.36	2.70	70.5	-2.91	5.19	56.5
RQV	-.623	1.55	79.7	-1.27	2.47	71.1	-2.65	4.66	57.7
RSBV	-.796	1.82	76.9	-1.54	3.05	69.2	-3.51	6.48	54.9
20%									
RBV	-.469	1.25	85.2	-.190	1.17	89.9	-.389	1.49	87.3
RTV	-.407	1.23	86.2	-.143	1.13	91.4	-.327	1.35	88.5
RQV	-.384	1.22	86.8	-.139	1.14	90.7	-.291	1.28	90.1
RSBV	-.505	1.28	83.9	-.224	1.15	90.6	-.399	1.46	87.0

Table 4.7: Bias, standard deviation and coverage of the infeasible linear test when one jump is added every day.

M	12			72			288		
	Bias	SD	Cove	Bias	SD	Cove	Bias	SD	Cove
50%									
RBV	-.529	1.15	81.8	-.967	1.63	73.5	-2.18	2.99	57.0
RTV	-.467	1.11	84.3	-.862	1.44	75.2	-1.84	2.48	59.3
RQV	-.459	1.11	86.8	-.807	1.36	76.8	-1.68	2.26	60.5
RSBV	-.579	1.21	79.6	-.965	1.60	73.1	-2.19	3.03	57.8
20%									
RBV	-.421	1.06	86.7	-.172	1.08	90.8	-.331	1.24	88.2
RTV	-.361	1.04	88.4	-.129	1.06	92.4	-.278	1.15	90.0
RQV	-.347	1.04	89.9	-.123	1.07	92.0	-.246	1.12	91.0
RSBV	-.468	1.12	84.6	-.203	1.07	91.3	-.344	1.21	88.4

Table 4.8: Bias, standard deviation and coverage of the infeasible ratio test when one jump is added every day.

For the infeasible ratio test, the results for each one of the estimators (RV, RTV, RQV and RSBV) are reported in Table 4.8 when one jump is added each day . As under the null distribution, when we do the infeasible test there is not much difference between the linear and the ratio cases. Again if

the size of the jump is small, it is more difficult to detect. As expected, as the value of M increases, the rate of accepting the null falls. Notice that the tests based on realised bipower variation and its skipped version seem to have more power than the ones based on the other estimators. This result can be misleading because the real size of the tests may be different for each estimator (in the previous section we found that the null distribution of the tests based on realised tripower and quadpower variation fit better the test statistics).

Now we will do the feasible tests using the three estimators for the integrated quarticity ((E1), (E2) and (E3)). Table 4.9 shows the finite sample behaviour of the linear test when $M=288$. Immediately we can see how the realised quarticity is not giving adequate results when jumps are present in the process. Just as the realised variance does not give a good estimation of the integrated variance in the presence of a jump component, the realised quarticity does poorly in estimating the integrated quarticity. The realised quarticity incorporates the variation due to the jump component and not just the one due to the continuous component. Nevertheless, the other two estimators, E2 and E3 give good and similar results.

Table 4.10 shows the finite sample behaviour for the feasible ratio test when $M=288$. When using the estimators E2 and E3 for the integrated quarticity, the results are encouraging and very similar to the ones obtained with infeasible tests. In contrast, when using the realised quarticity as the estimator, there is a little number of rejections of the null hypothesis.

	Bias	E1 SD	Cove	Bias	E2 SD	Cove	Bias	E3 SD	Cove
50%									
RBV	-.851	1.05	72.9	-3.22	5.72	55.4	-3.38	6.06	54.5
RTV	-.725	.961	84.6	-2.69	4.61	56.8	-2.83	4.91	56.6
RQV	-.669	.927	90.4	-2.44	4.13	58.4	-2.58	4.40	57.7
RSBV	-.843	1.05	73.1	-3.19	5.71	55.9	-3.37	6.09	55.3
20%									
RBV	-.218	1.03	91.5	-.427	1.45	86.0	-.436	1.47	85.7
RTV	-.179	.998	93.8	-.365	1.31	87.9	-.379	1.33	87.3
RQV	-.156	.988	94.8	-.323	1.25	88.9	-.343	1.27	88.5
RSBV	-.235	.995	92.5	-.415	1.40	86.2	-.433	1.42	86.2

Table 4.9: *Bias, st. deviation and coverage of the feasible linear test when one jump is added every day for $M=288$.*

	E1			E2			E3		
	Bias	SD	Cove	Bias	SD	Cove	Bias	SD	Cove
50%									
RBV	-.667	.875	92.8	-2.10	2.88	58.2	-2.16	2.96	57.7
RTV	-.549	.808	96.6	-1.72	2.32	60.6	-1.79	2.42	59.9
RQV	-.498	.782	97.6	-1.54	2.07	62.4	-1.61	2.17	61.1
RSBV	-.656	.877	93.3	-2.07	2.87	59.2	-2.14	2.97	58.5
20%									
RBV	-.185	.956	94.2	-.328	1.22	88.4	-.331	1.23	88.3
RTV	-.143	.918	96.6	-.267	1.13	90.6	-.274	1.13	90.4
RQV	-.117	.906	97.3	-.228	1.08	91.9	-.239	1.02	91.3
RSBV	-.203	.924	95.8	-.327	1.16	88.7	-.337	1.18	88.5

Table 4.10: *Bias, standard deviation and coverage of the feasible ratio test when one jump is added every day for $M=288$.*

4.5.4 Real size tests

When doing the previous tests for jumps, even if we set a confidence level of 95% the real size of the tests will not be exactly 5%. This is due to the fact that the asymptotic distributions do not fit properly our test statistics when using finite samples. When using bipower variation and its skipped version this problem is accentuated. Therefore it is inadequate to compare the results obtained from the tests where we added jumps to the log-price process. Each test has a different size depending on the estimator and the sample size used.

The real critical values corresponding to the desired confidence level need to be found. Given these critical values, test for jumps with a real size of 5% can then be done and compared.

Using Monte Carlo simulations, the real critical values for the tests for jumps with size of 5% are found for each test statistic and different values of M . In Table 4.11 the critical values (CV) of the linear infeasible tests are displayed with the quantiles of the Normal distribution (NQ) that correspond to them. The better the fit of the asymptotic distribution, the closer the critical value will be to the value of the 95th quantile. Now, using the simulated series where one daily jump was added to the

log-price process the test for jumps can be repeated with a given fixed size of 5%. In Table 4.11 we give the power of the test (Pow) for different jump sizes ($\beta = 50\%$ and 20%), i.e. the percent of times we reject the null hypothesis given that there are jumps in the process. In Table 4.12 we display the results for the ratio infeasible tests.

From the tables we can confirm that the test statistics based in tripower and quadpower variation have better sample behaviour under the null distribution than the ones based on bipower variation and its skipped version. Also the ratio test statistics have better finite sample behaviour than the linear ones. Nevertheless it is not obvious how to conclude which of the tests have a better power. It seems that when the sample frequency is low, the tests based on tripower and quadpower variation have better power. Nevertheless it seems this is not true for higher frequencies, the tests based on bipower variation and its skipped version seem to have a bigger power. Also in the case when the smaller jumps are added, tripower and quadpower variation outperform the bipower variation and the opposite happens when big jumps are added to the process.

M	12				72				288			
	CV	NQ	Pow 50%	Pow 20%	CV	NQ	Pow 50%	Pow 20%	CV	NQ	Pow 50%	Pow 20%
RBV	2.46	99.3	0.092	0.063	1.93	97.3	0.199	0.057	96.8	1.85	0.343	0.085
RTV	2.27	98.8	0.095	0.064	1.84	96.7	0.195	0.061	96.1	1.75	0.346	0.080
RQV	2.09	98.2	0.106	0.067	1.80	96.4	0.184	0.069	95.7	1.71	0.337	0.077
RSBV	2.63	99.6	0.093	0.056	1.88	97.1	0.202	0.062	96.7	1.83	0.353	0.076

Table 4.11: Critical values, Normal quantiles and power of the infeasible linear tests when one jump is added every day.

M	12				72				288			
	CV	NQ	Pow 50%	Pow 20%	CV	NQ	Pow 50%	Pow 20%	CV	NQ	Pow 50%	Pow 20%
RBV	2.24	98.8	0.064	0.050	1.85	96.9	0.170	0.054	1.82	96.6	0.331	0.076
RTV	2.03	97.9	0.061	0.045	1.71	95.7	0.172	0.064	1.78	96.3	0.323	0.065
RQV	1.99	97.7	0.045	0.039	1.76	96.1	0.147	0.067	1.72	95.8	0.309	0.062
RSBV	2.36	98.1	0.068	0.042	1.84	96.8	0.174	0.061	1.79	96.4	0.336	0.073

Table 4.12: Critical values, Normal quantiles and power of the infeasible ratio tests when one jump is added every day.

In Tables 4.13 and 4.14 the real critical values, the corresponding Normal quantiles and the power of the feasible linear and ratio tests are given using each of the estimators of the integrated quarticity (E1, E2, E3) when $M=288$. It can be seen immediately that the lowest power is obtained when using E1 as the estimator. The other two estimators give better and very similar results as expected. Also there is not much to choose between the linear and ratio tests.

β	E1				E2				E3			
	CV	NQ	Pow 50%	Pow 20%	CV	NQ	Pow 50%	Pow 20%	CV	NQ	Pow 50%	Pow 20%
RBV	1.78	96.3	0.164	0.058	2.00	97.8	0.331	0.077	2.03	97.9	0.336	0.078
RTV	1.70	95.6	0.117	0.051	1.96	97.5	0.318	0.065	1.98	97.7	0.332	0.068
RQV	1.66	95.2	0.089	0.053	1.90	97.2	0.311	0.063	1.96	97.6	0.306	0.060
RSBV	1.77	96.2	0.167	0.048	1.96	97.5	0.333	0.068	1.98	97.7	0.338	0.070

Table 4.13: Critical values, Normal quantiles and power of the feasible linear tests for $M=288$ when one jump is added every day.

β	E1				E2				E3			
	CV	NQ	Pow 50%	Pow 20%	CV	NQ	Pow 50%	Pow 20%	CV	NQ	Pow 50%	Pow 20%
RBV	1.63	94.9	0.078	0.052	1.79	96.4	0.330	0.078	1.80	96.5	0.331	0.077
RTV	1.52	93.6	0.070	0.042	1.73	95.9	0.315	0.064	1.75	96.1	0.317	0.062
RQV	1.46	92.8	0.062	0.051	1.65	95.2	0.310	0.062	1.70	95.6	0.307	0.063
RSBV	1.63	94.9	0.090	0.046	1.78	96.3	0.327	0.067	1.77	96.2	0.337	0.068

Table 4.14: Critical values, Normal quantiles and power of the feasible ratio tests for $M=288$ when one jump is added every day.

From this section we can conclude that the realised tripower and quadpower variation give better results under the null hypothesis. Nevertheless once a jump component is added to the log-price process, the conclusion of which test gives the biggest power is unclear. Microstructure noise was not added to these simulations, so further research is needed to assess the power of test based on these estimators with contaminated series.

4.6 Empirical data

4.6.1 Testing for jumps

To illustrate empirically the test for jumps we will use the United States Dollar/ German Deutsche Mark series employed extensively in previous papers (see for example Andersen, Bollerslev, Diebold and Labys (2001), Barndorff-Nielsen and Shephard(2002a)). This series covers from 1st of December 1986 until 30th of November 1996 and reports every five minutes the most recent quote on the Reuters screen. This dataset was kindly supplied by Olsen and Associates in Zurich (see Dacorogna, Gencay, Müller, Olsen and Pictet (2001)).

In this section we will study the feasible ratio test for jumps using the realised quadpower variation with $r=s=u=v=1$ (E3) to estimate the actual quarticity. In Figure 4.8 the ratio statistic

$$\delta^{-1/2} \left(\frac{\mu_r^{-n} \{Y_\delta\}_t^{[r_1, \dots, r_n]}}{[Y_\delta]_t^{[2]}} - 1 \right),$$

where $r = r_1 = \dots = r_n$ and $\sum_{i=1}^n r_i = 2$, is plotted together with twice the standard error for the first 250 days for a)RBV, b)RTV, c)RQV and d)RSBV with $M=144$. In all of the cases, we can observe a considerable number of values below twice the standard error pointing out the existence of jumps in the series. Just by looking at the graphs we cannot conclude anything about the different estimators so further results are reported in Table 4.15.

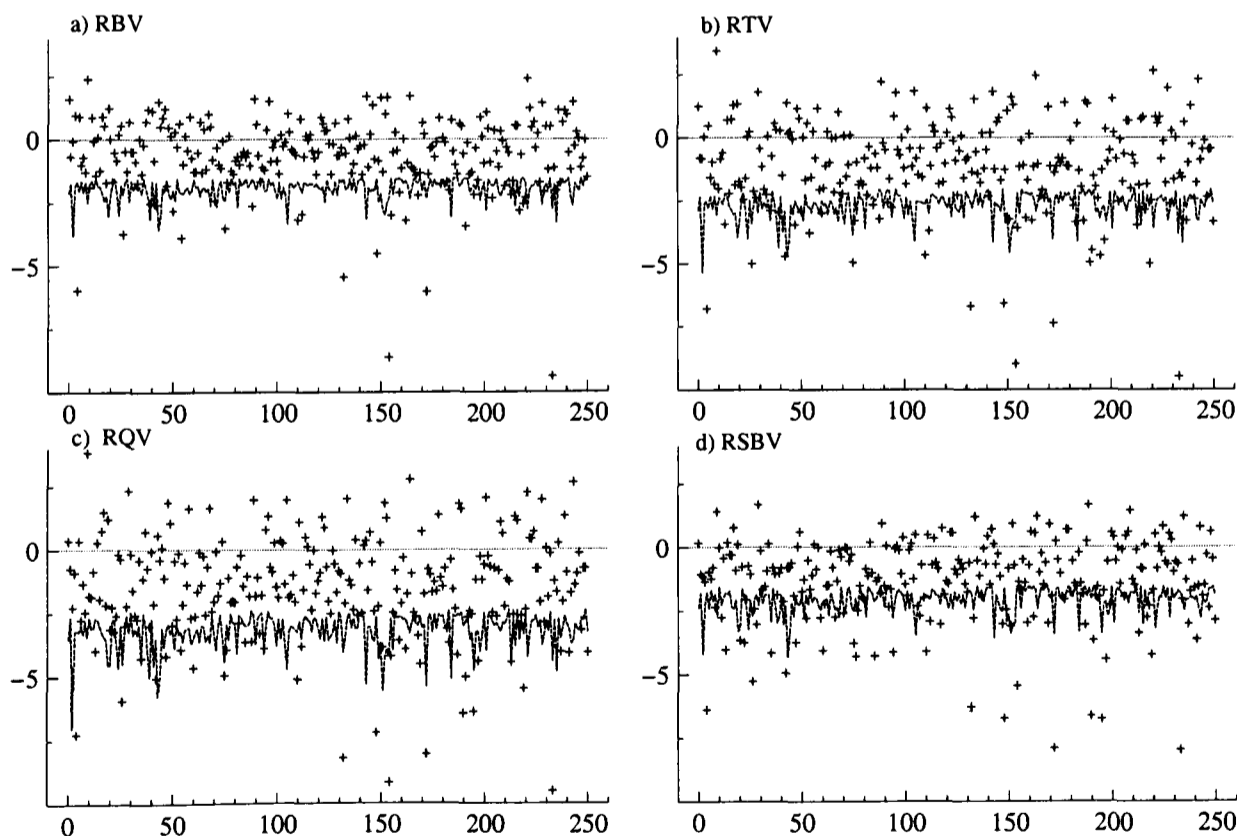


Figure 4.8: Ratio t -statistics plus twice their standard errors from the first 250 observations of the Dollar/DM series for a)RBV, b)RTV, c)RQV and d)RSBV with $M=144$.

Table 4.15 shows for different values of M the sum of the first five correlation coefficients of the Dollar/ DM series and the average values over the sample of the realised variance and of the bipower, tripower, quadpower and skipped version of bipower variation. It also shows for each of these estimators the proportions of rejections of the null hypothesis at the 5% level and 1% level.

M	corr	RV	BV	5%r	1%r	TV	5%r	1%r	QV	5%r	1%r	SB	5%r	1%r
12	.001	.459	.387	.183	.081	.357	.164	.043	.339	.140	.024	.357	.218	.098
24	.002	.463	.396	.207	.101	.372	.192	.078	.354	.177	.051	.373	.232	.118
72	-.001	.488	.442	.224	.120	.416	.233	.109	.402	.217	.089	.415	.283	.157
144	-.056	.511	.473	.228	.116	.448	.250	.126	.432	.255	.113	.442	.341	.186
288	-.092	.532	.502	.188	.095	.481	.227	.108	.467	.252	.107	.471	.368	.211

Table 4.15: *Sum of the first five correlation coefficients of the Dollar/ DM series (corr). Average value of realised variance (RV), realised bipower (BV), tripower (TV), quadpower (QV) and skipped bipower (SB) variation. Proportion of rejections of the null hypothesis at the 5% (5%r) and 1% (1%r) level.*

Firstly we can see how the sum of correlations increases when the value of M increases, reaching large values for M=144 (-.056) and M=288 (-.092) possibly explained by the microstructure effect. This fact surely is affecting the number of rejections of the null hypothesis as it decreases when M=288 for most of the estimators.

From the simulations in the previous section we know that the tests based on realised tripower and quadpower variation have bigger sizes (given a confidence level) compared to the tests based on realised bipower variation when using finite samples. This may explain the values in Table 4.15 for M=12, 24 and 72. This fact reverses when M=144 and 288 perhaps due to the microstructure effect, this gives us a first reason to think realised tripower and quadpower variation are more robust in the presence of such microstructure noise. The bias of the skipped version of realised bipower variation is almost the same as the one of realised tripower variation, nevertheless the proportion of rejections of the null hypothesis is higher. In fact, it is the highest compared to all the other estimators and it is the only estimator for which the number of rejections are not reduced when M=288.

Although these give us a preliminary insight into the behaviour of these estimators in the presence of microstructure noise, further research has to be done to obtain stronger conclusions. What we can conclude now however is that certainly the Dollar/DM series contains jumps.

4.6.2 Case study

As explained before a large value of the realised variance can be explained by the presence of jumps or by high volatility in the continuous component. As in Barndorff-Nielsen and Shephard (2006) we will contrast two days, one in which the extreme value of the quadratic variation is caused by the jump component and in another one where it is caused by the continuous component. We will compare the behaviour of each of our estimators of integrated variance on these days.

On one of the days, January 15th 1988, we have a huge increase in the Dollar/DM rate (Figure 4.9, upper graph). This increase is reflected in the realised variance and not in the other estimators of the integrated variance indicating the presence of a jump component. In contrast on August 19th 1991 (Figure 4.9, lower graph) there is a constant increase of the rate. Here the large estimated value of the quadratic variation is due to the integrated variance so we cannot expect it to be caused by a jump but by high volatility.

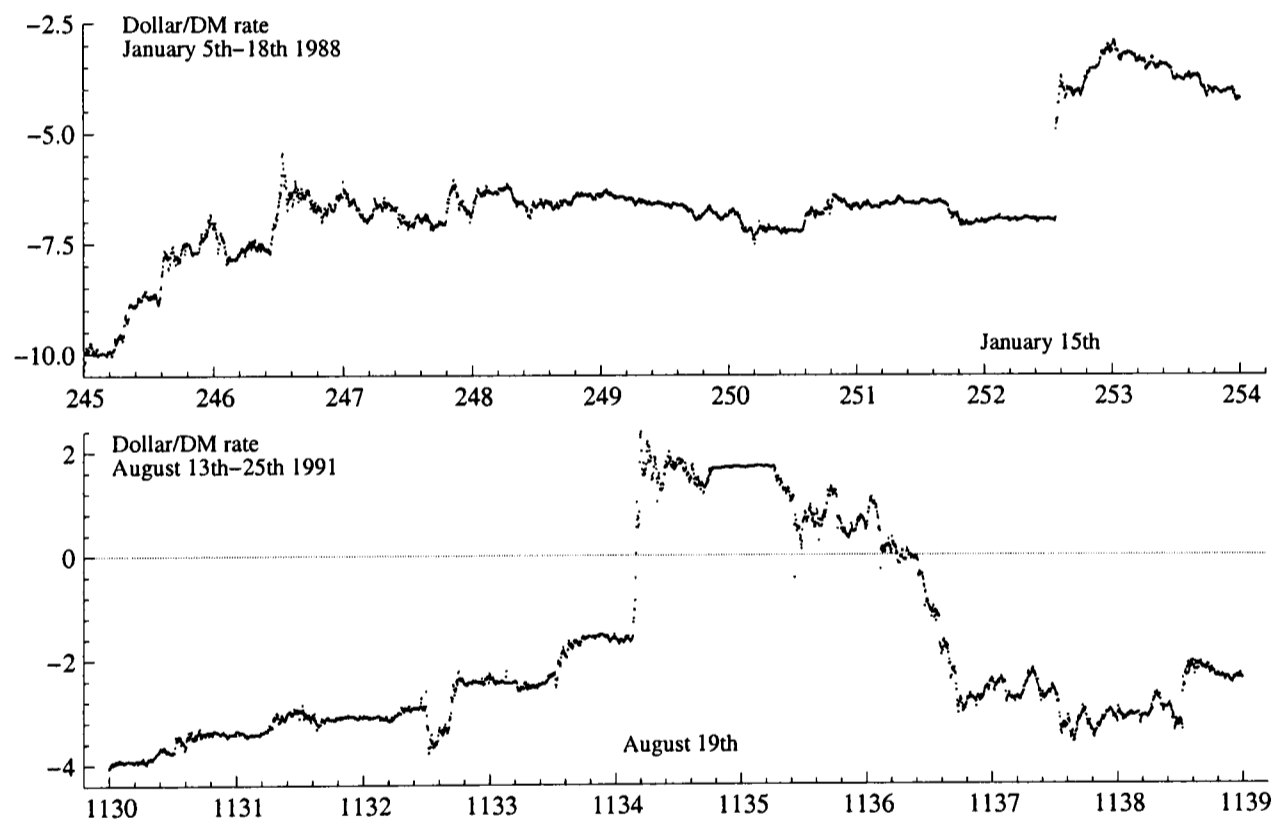


Figure 4.9: Dollar/DM rates from January 5th to January 18th 1988 (upper graph) and from August 13th to August 25th 1991 (lower graph). In the upper graph the large realised variance is due to a jump component and in the lower graph it is due to high volatility in the continuous component.

In Figure 4.10 we look carefully at the ratio t-statistics of the days around January 15th 1988. In all the cases (a)RBV, b)RTQ, c)RQV and d)RSBV) there is no doubt about the presence of a jump, the statistics are significantly below the 99% critical values. We reject the null hypothesis of no jumps

independently of the estimator used.

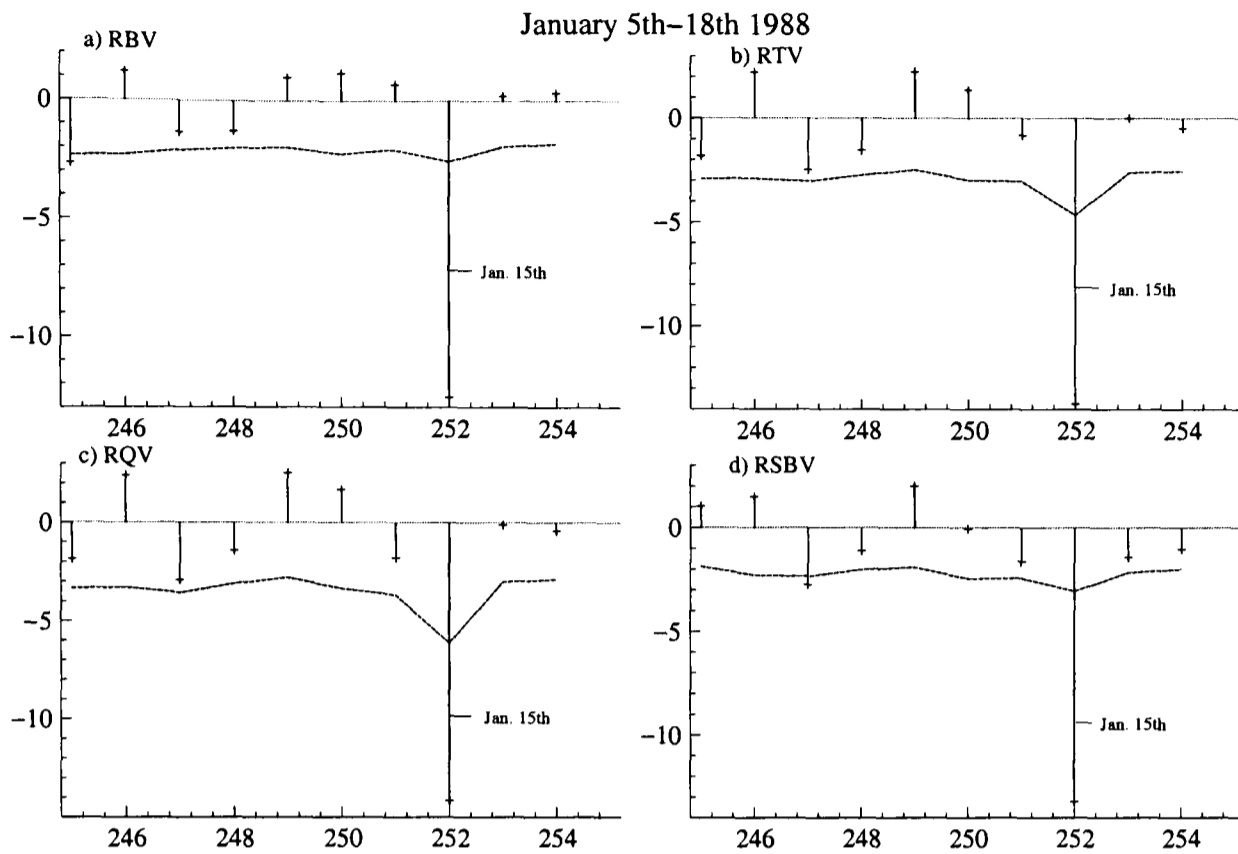


Figure 4.10: *January 5th to January 18th 1988. Ratio t-statistics and 99% critical values of the Dollar/DM series for a)RBV, b)RTV, c)RQV and d)RSBV with $M=288$. On January 15th a jump occurred.*

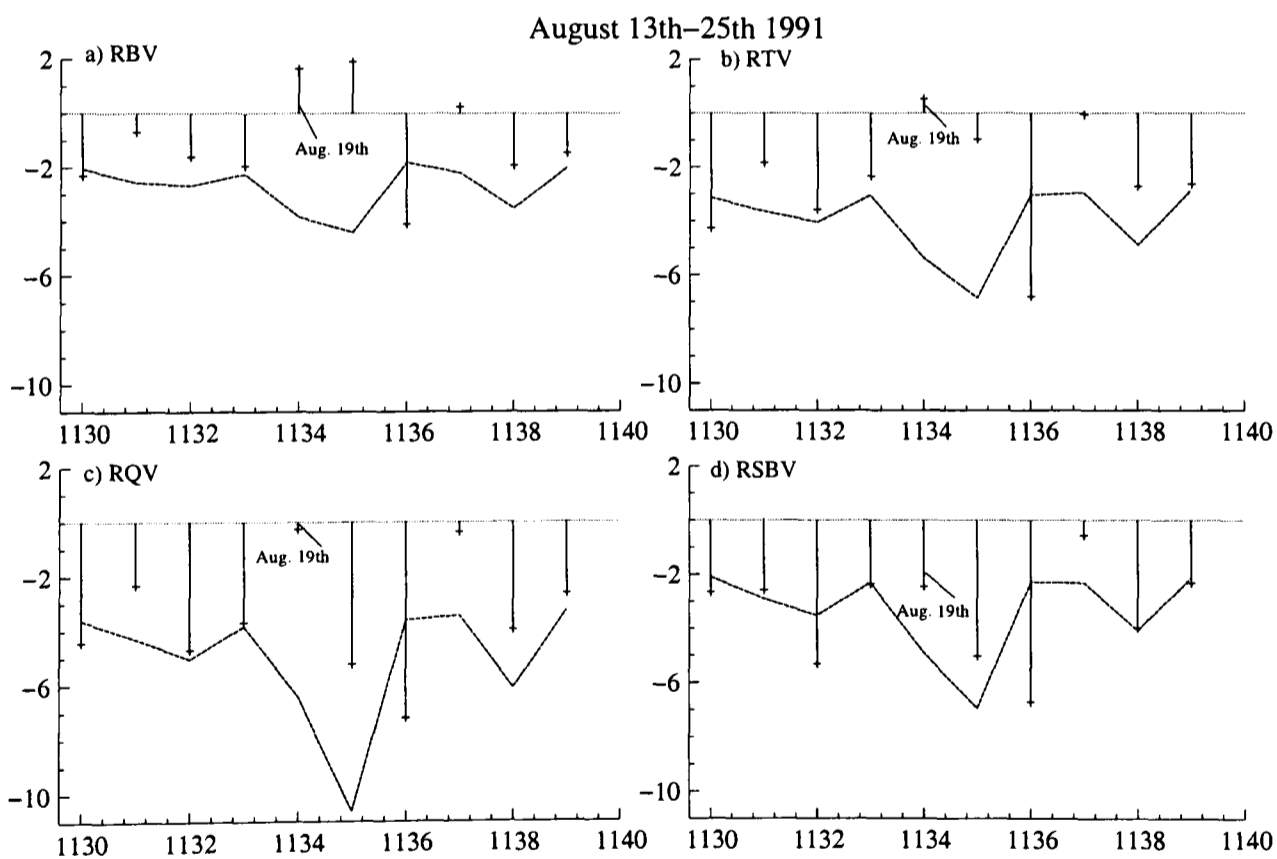


Figure 4.11: *August 13th to August 25th 1991. Ratio t-statistics and 99% critical values of the Dollar/DM series for a)RBV, b)RTV, c)RQV and d)RSBV with $M=288$. On August 19th the large value of the realised variance is caused by high volatility on the continuous component and not a jump.*

In Figure 4.11 we plot again the ratio t-statistic along with the 99% critical values using our four estimators but now for the days around August 19th 1991. In all the cases we reject the presence of jumps, instead the increase of the quadratic variation is due to a quick movement in the rates.

Note that all the estimators give the same result for August 19th but not for the surrounding days. When using the skipped version of the realised bipower variation three jumps are found during the period studied.

4.7 Conclusions

The aim of this chapter was to give alternative estimators from the realised bipower variation to test for the presence of jump in the log-price process. For that we provided the asymptotic distribution for the differences between realised variance and realised tripower variation, realised variance and realised quadpower variation and realised variance and realised skipped bipower variation. Using simulated data we found out that these estimators are reliable; giving even better results than the realised bipower variation under the null hypothesis. The test statistics based on realised tripower and quadpower variation have better finite sample behaviour, nevertheless it is unclear which of the tests is the most powerful. As expected, the performance of all the tests improves when using large values of M .

When using a feasible test, the integrated quarticity needs to be estimated. Under the presence of jumps, realised quarticity gives a great number of rejections of the null hypothesis as it estimates poorly the integrated quarticity. The other estimators give encouraging results.

The test for jumps was also done in empirical data, the Dollar/DM series. Results need to be interpreted carefully when working with real data, as it is well known that the price process is contaminated by market microstructure effects. Even though the previous tests for jumps were based on asymptotic results, its implementation using extremely large values of M can give misleading answers as this noise accumulates. The larger the value of M we use, the closer we get to the asymptotic results but also the more the microstructure effect will disturb the real process.

By using the skipped version of the realised bipower variation, i.e. realised skipped bipower variation and by giving less weight to adjacent observations, i.e. realised tripower and quadpower variation, we tried to solve this problem but further work is needed to assess the effectiveness of these estimators in the presence of microstructure noise.

4.8 Appendix: Proofs

For the following three proofs let us first define the generalised multipower variation as

$$Y^M(g)_t = \frac{1}{M} \sum_{i=1}^{\lfloor Mt \rfloor} \left(\prod_{i'=1}^{I \wedge (i+1)} g_{i'}(\sqrt{M}y_{i+i'-1}) \right).$$

As in Barndoff-Nielsen, Graversen, Jacod, Podolskij and Shephard (2005) and Barndoff-Nielsen, Graversen, Jacod and Shephard (2006) let us assume that all the g_i are continuous with at the most polynomial growth and that $Y \in SVSM^c$. So if $g_i(y) = |y|^{2/I}$, we get

$$Y(g)_t = \mu_{2/I}^I \int_0^t \sigma_s^2 ds,$$

Barndoff-Nielsen, Graversen, Jacod, Podolskij and Shephard (2005) show that

$$\sqrt{M}(Y^M(g)_t - Y(g)_t) \rightarrow \int_0^t \sqrt{\omega_I^2 \sigma_s^4} dB_s \quad (4.2)$$

where

$$\omega_I^2 = Var\left(\prod_{i=1}^I |u_i|^{2/I}\right) + 2 \sum_{j=1}^{I-1} Cov\left(\prod_{i=1}^I |u_i|^{2/I}, \prod_{i=1}^I |u_{i+j}|^{2/I}\right).$$

A4.1 Tripower Variation

We are mainly interested in the distribution of the following object

$$[Y_\delta]_t^{[2]} - \mu_{2/3}^{-3} \{Y_\delta\}_t^{[2/3, 2/3, 2/3]}.$$

To obtain it, we need to find the joint distribution of the realised variance error and the realised tripower variation error, i.e., the distribution of

$$\left(\begin{array}{c} \sum_{j=1}^{\lfloor t/\delta \rfloor} y_j^2 - \int_0^t \sigma_s^2 ds \\ \mu_{2/3}^{-3} \sum_{j=1}^{\lfloor t/\delta \rfloor - 2} |y_j|^{2/3} |y_{j+1}|^{2/3} |y_{j+2}|^{2/3} - \int_0^t \sigma_s^2 ds \end{array} \right).$$

By Barndorff-Nielsen and Shephard (2002a) we know that

$$\sqrt{\frac{t}{\delta \int_0^t \sigma_s^4 ds}} \left(\sum_{j=1}^{\lfloor t/\delta \rfloor} y_j^2 - \int_0^t \sigma_s^2 ds \right) \xrightarrow{L} N(0, 2).$$

A particular case of equation (4.2) gives us

$$\sqrt{\frac{t}{\delta \int_0^t \sigma_s^4 ds}} \left(\sum_{j=1}^{\lfloor t/\delta \rfloor - 2} |y_j|^{2/3} |y_{j+1}|^{2/3} |y_{j+2}|^{2/3} - \mu_{2/3}^3 \int_0^t \sigma_s^2 ds \right) \xrightarrow{L} N(0, \omega_3^2)$$

where

$$\begin{aligned} \omega_3^2 &= \text{Var}(|u|^{2/3} |u'|^{2/3} |u''|^{2/3}) \\ &+ 2\text{Cov}(|u|^{2/3} |u'|^{2/3} |u''|^{2/3}, |u'|^{2/3} |u''|^{2/3} |u'''|^{2/3}) \\ &+ 2\text{Cov}(|u|^{2/3} |u'|^{2/3} |u''|^{2/3}, |u''|^{2/3} |u'''|^{2/3} |u^{(IV)}|^{2/3}) \end{aligned}$$

where u, u', u'' and u''' are iid Normal(0,1).

Remember that $\nu_j^2 = \int_{\delta(j-1)}^{\delta j} \sigma_s^2 ds$, so if $A = 0$ we obtain that

$$\begin{aligned} \delta^{-1} \text{Cov} \left(\sum_{j=1}^{\lfloor t/\delta \rfloor - 2} \left(\begin{array}{c} \nu_j^2 (u_j^2 - 1) \\ \nu_j^{2/3} \nu_{j+1}^{2/3} \nu_{j+2}^{2/3} (|u_j|^{2/3} |u_{j+1}|^{2/3} |u_{j+2}|^{2/3} - \mu_{2/3}^3) \end{array} \right) \right) \\ \xrightarrow{p} \begin{pmatrix} k_1 & k_2 \\ k_3 & k_4 \end{pmatrix} \int_0^t \sigma_s^4 ds \end{aligned}$$

where

$$\begin{aligned} k_1 &= \text{Var}(u^2) \\ k_2 &= k_3 = 3\text{Cov}(u^2, |u|^{2/3} |u'|^{2/3} |u''|^{2/3}) \\ k_4 &= \omega_3^2 = \text{Var}(|u|^{2/3} |u'|^{2/3} |u''|^{2/3}) \\ &+ 2\text{Cov}(|u|^{2/3} |u'|^{2/3} |u''|^{2/3}, |u'|^{2/3} |u''|^{2/3} |u'''|^{2/3}) \\ &+ 2\text{Cov}(|u|^{2/3} |u'|^{2/3} |u''|^{2/3}, |u''|^{2/3} |u'''|^{2/3} |u^{(IV)}|^{2/3}). \end{aligned}$$

This implies that

$$\begin{aligned} \frac{1}{\delta^{1/2} \sqrt{\int_0^t \sigma_s^4 ds}} \left(\begin{array}{c} \sum_{j=1}^{\lfloor t/\delta \rfloor} y_j^2 - \int_0^t \sigma_s^2 ds \\ \mu_{2/3}^{-3} \sum_{j=1}^{\lfloor t/\delta \rfloor - 2} |y_j|^{2/3} |y_{j+1}|^{2/3} |y_{j+2}|^{2/3} - \int_0^t \sigma_s^2 ds \end{array} \right) \\ \xrightarrow{L} N \left(0, \begin{pmatrix} k_1 & \mu_{2/3}^{-3} k_2 \\ \mu_{2/3}^{-3} k_3 & \mu_{2/3}^{-6} k_4 \end{pmatrix} \right) \\ \simeq N \left(0, \begin{pmatrix} 2 & 2 \\ 2 & 3.0613 \end{pmatrix} \right) \end{aligned}$$

by using the results that

$$\begin{aligned} \text{Var}(|u|^{2/3} |u'|^{2/3} |u''|^{2/3}) &= \mu_{4/3}^3 - \mu_{2/3}^6 \\ \text{Cov}(u^2, |u|^{2/3} |u'|^{2/3} |u''|^{2/3}) &= \mu_{8/3} \mu_{2/3}^2 - \mu_{2/3}^3 \\ \text{Cov}(|u|^{2/3} |u'|^{2/3} |u''|^{2/3}, |u'|^{2/3} |u''|^{2/3} |u'''|^{2/3}) &= \mu_{4/3}^2 \mu_{2/3}^2 - \mu_{2/3}^6 \\ \text{Cov}(|u|^{2/3} |u'|^{2/3} |u''|^{2/3}, |u''|^{2/3} |u'''|^{2/3} |u^{(IV)}|^{2/3}) &= \mu_{4/3} \mu_{2/3}^4 - \mu_{2/3}^6 \end{aligned}$$

where $\mu_r = E(|x|^r)$.

With this joint distribution, the limit theory necessary for the test for jumps can be obtained

$$\frac{1}{\delta^{1/2} \sqrt{\int_0^t \sigma_s^4 ds}} \left([Y_\delta]_t^{[2]} - \mu_{2/3}^{-3} \{Y_\delta\}_t^{[2/3, 2/3, 2/3]} \right) \xrightarrow{L} N(0, \vartheta_{TV})$$

where

$$\vartheta_{TV} = \mu_{4/3} \mu_{2/3}^{-2} (\mu_{4/3}^2 \mu_{2/3}^{-4} + 2\mu_{4/3} \mu_{2/3}^{-2} - 2) - 7 \simeq 1.0613.$$

A4.2 Quadpower Variation

Now we are interested in the distribution of the following object

$$[Y_\delta]_t^{[2]} - \mu_{1/2}^{-4} \{Y_\delta\}_t^{[1/2, 1/2, 1/2, 1/2]}.$$

We need to find the following joint distribution

$$\left(\begin{array}{c} \sum_{j=1}^{\lfloor t/\delta \rfloor} y_j^2 - \int_0^t \sigma_s^2 ds \\ \sum_{j=1}^{\lfloor t/\delta \rfloor - 3} \sqrt{|y_j| |y_{j+1}| |y_{j+2}| |y_{j+3}|} - \mu_{1/2}^4 \int_0^t \sigma_s^2 ds \end{array} \right)$$

From equation (4.2) we can obtain that

$$\sqrt{\frac{t}{\delta \int_0^t \sigma_s^4 ds}} \left(\sum_{j=1}^{\lfloor t/\delta \rfloor - 3} |y_j|^{1/2} |y_{j+1}|^{1/2} |y_{j+2}|^{1/2} |y_{j+3}|^{1/2} - \mu_{1/2}^4 \int_0^t \sigma_s^2 ds \right) \xrightarrow{L} N(0, \omega_4^2)$$

where

$$\begin{aligned} \omega_4^2 &= \text{Var}(\sqrt{|u| |u'| |u''| |u'''|}) \\ &+ 2\text{Cov}(\sqrt{|u| |u'| |u''| |u'''|}, \sqrt{|u'| |u''| |u'''| |u^{(IV)}|}) \\ &+ 2\text{Cov}(\sqrt{|u| |u'| |u''| |u'''|}, \sqrt{|u''| |u'''| |u^{(IV)}| |u^{(V)}|}) \\ &+ 2\text{Cov}(\sqrt{|u| |u'| |u''| |u'''|}, \sqrt{|u'''| |u^{(IV)}| |u^{(V)}| |u^{(VI)}|}). \end{aligned}$$

If $\nu_j^2 = \int_{\delta(j-1)}^{\delta j} \sigma_s^2 ds$ and $A = 0$ then

$$\delta^{-1} \text{Cov} \left(\sum_{j=1}^{\lfloor t/\delta \rfloor - 3} \left(\frac{\nu_j^2(u_j^2 - 1)}{\sqrt{\nu_j \nu_{j+1} \nu_{j+2} \nu_{j+3}} (\sqrt{|u_j| |u_{j+1}| |u_{j+2}| |u_{j+3}|} - \mu_{1/2}^4)} \right) \right) \\ \xrightarrow{p} \begin{pmatrix} k_1 & k_2 \\ k_3 & k_4 \end{pmatrix} \int_0^t \sigma_s^4 ds$$

where

$$\begin{aligned} k_1 &= \text{Var}(u^2) \\ k_2 &= k_3 = 4 \text{Cov}(u^2, \sqrt{|u| |u'| |u''| |u'''|}) \\ k_4 &= \omega_4^2 = \text{Var}(\sqrt{|u| |u'| |u''| |u'''|}) \\ &\quad + 2 \text{Cov}(\sqrt{|u| |u'| |u''| |u'''|}, \sqrt{|u'| |u''| |u'''| |u^{(IV)}|}) \\ &\quad + 2 \text{Cov}(\sqrt{|u| |u'| |u''| |u'''|}, \sqrt{|u''| |u'''| |u^{(IV)}| |u^{(V)}|}) \\ &\quad + 2 \text{Cov}(\sqrt{|u| |u'| |u''| |u'''|}, \sqrt{|u'''| |u^{(IV)}| |u^{(V)}| |u^{(VI)}|}). \end{aligned}$$

Implying that

$$\frac{1}{\delta^{1/2} \sqrt{\int_0^t \sigma_s^4 ds}} \left(\begin{array}{c} \sum_{j=1}^{\lfloor t/\delta \rfloor} y_j^2 - \int_0^t \sigma_s^2 ds \\ \mu_{1/2}^{-4} \sum_{j=1}^{\lfloor t/\delta \rfloor - 3} \sqrt{|y_j| |y_{j-1}| |y_{j-2}| |y_{j-3}|} - \int_0^t \sigma_s^2 ds \end{array} \right) \\ \xrightarrow{L} N \left(0, \begin{pmatrix} k_1 & \mu_{1/2}^{-4} k_2 \\ \mu_{1/2}^{-4} k_3 & \mu_{1/2}^{-8} k_4 \end{pmatrix} \right) \\ \simeq N \left(0, \begin{pmatrix} 2 & 2 \\ 2 & 3.37702 \end{pmatrix} \right)$$

as

$$\begin{aligned} \text{Var}(|u|^{1/2} |u'|^{1/2} |u''|^{1/2} |u'''|^{1/2}) &= \mu_1^4 - \mu_{1/2}^8 \\ \text{Cov}(u^2, |u|^{1/2} |u'|^{1/2} |u''|^{1/2} |u'''|^{1/2}) &= \mu_{5/2} \mu_{1/2}^3 - \mu_{1/2}^4 \\ \text{Cov}(|u|^{1/2} |u'|^{1/2} |u''|^{1/2} |u'''|^{1/2}, |u'|^{1/2} |u''|^{1/2} |u'''|^{1/2} |u^{(IV)}|^{1/2}) &= \mu_1^3 \mu_{1/2}^2 - \mu_{1/2}^8 \\ \text{Cov}(|u|^{1/2} |u'|^{1/2} |u''|^{1/2} |u'''|^{1/2}, |u''|^{1/2} |u'''|^{1/2} |u^{(IV)}|^{1/2} |u^{(V)}|^{1/2}) &= \mu_1^2 \mu_{1/2}^4 - \mu_{1/2}^8 \\ \text{Cov}(|u|^{1/2} |u'|^{1/2} |u''|^{1/2} |u'''|^{1/2}, |u'''|^{1/2} |u^{(IV)}|^{1/2} |u^{(V)}|^{1/2} |u^{(VI)}|^{1/2}) &= \mu_1 \mu_{1/2}^6 - \mu_{1/2}^8. \end{aligned}$$

Finally we can obtain the limit distribution that we were originally looking for

$$\frac{1}{\delta^{1/2} \sqrt{\int_0^t \sigma_s^4 ds}} \left([Y_\delta]_t^{[2]} - \mu_{1/2}^{-4} \{Y_\delta\}_t^{[1/2, 1/2, 1/2, 1/2]} \right) \xrightarrow{L} N(0, \vartheta_{QV})$$

where

$$\vartheta_{QV} = \mu_1 \mu_{1/2}^{-2} (\mu_1^3 \mu_{1/2}^{-6} + 2\mu_1^2 \mu_{1/2}^{-4} + 2\mu_1 \mu_{1/2}^{-2} - 2) - 9 \simeq 1.37702$$

and $\mu_r = E(|x|^r)$.

A4.3 Skipped Bipower Variation

Lastly we will obtain the asymptotic distribution of the difference of the realised variance errors and the skipped version of the realised bipower variation errors,

$$[Y_\delta]_t^{[2]} - \mu_1^{-2} \{Y_\delta\}_t^{[1,0,1]}.$$

The joint distribution we need to find is

$$\left(\begin{array}{c} \sum_{j=1}^{\lfloor t/\delta \rfloor} y_j^2 - \int_0^t \sigma_s^2 ds \\ \sum_{j=1}^{\lfloor t/\delta \rfloor - 2} |y_j| |y_{j+2}| - \mu_1^2 \int_0^t \sigma_s^2 ds \end{array} \right).$$

From equation (4.2) we obtain that

$$\sqrt{\frac{t}{\delta \int_0^t \sigma_s^4 ds}} \left(\sum_{j=1}^{\lfloor t/\delta \rfloor - 2} |y_j| |y_{j+2}| - \mu_1^2 \int_0^t \sigma_s^2 ds \right) \xrightarrow{L} N(0, \omega_{2SV}^2)$$

where

$$\omega_{2SV}^2 = \text{Var}(|u| |u''|) + 2\text{Cov}(|u| |u''|, |u''| |u^{(IV)}|).$$

Note that the term $\text{Cov}(|u| |u''|, |u'| |u'''|)$ is not included as it equals zero.

Given that $\nu_j^2 = \int_{\delta(j-1)}^{\delta j} \sigma_s^2 ds$ and $A = 0$, now we can obtain that

$$\begin{aligned} \delta^{-1} \text{Cov} \left(\sum_{j=1}^{\lfloor t/\delta \rfloor - 2} \left(\frac{\nu_j^2 (u_j^2 - 1)}{\sqrt{\nu_j^2} \sqrt{\nu_{j+2}^2}} (|u_j| |u_{j+2}| - \mu_1^2) \right) \right) \\ \xrightarrow{p} \begin{pmatrix} k_1 & k_2 \\ k_3 & k_4 \end{pmatrix} \int_0^t \sigma_s^4 ds \end{aligned}$$

where

$$\begin{aligned} k_1 &= \text{Var}(u^2) \\ k_2 &= k_3 = 2\text{Cov}(u^2, |u| |u''|) \\ k_4 &= \omega_{2SV}^2 = \text{Var}(|u| |u''|) + 2\text{Cov}(|u| |u''|, |u''| |u^{(IV)}|). \end{aligned}$$

The joint distribution can now be expressed as

$$\frac{1}{\delta^{1/2} \sqrt{\int_0^t \sigma_s^4 ds}} \left(\begin{array}{c} \sum_{j=1}^{\lfloor t/\delta \rfloor} y_j^2 - \int_0^t \sigma_s^2 ds \\ \mu_1^{-2} \sum_{j=1}^{\lfloor t/\delta \rfloor - 2} |y_j| |y_{j+2}| - \int_0^t \sigma_s^2 ds \end{array} \right) \xrightarrow{L} N \left(0, \begin{pmatrix} k_1 & \mu_1^{-2} k_2 \\ \mu_1^{-2} k_3 & \mu_1^{-4} k_4 \end{pmatrix} \right) \\ \simeq N \left(0, \begin{pmatrix} 2 & 2 \\ 2 & 2.60907 \end{pmatrix} \right)$$

as

$$\begin{aligned} \text{Var}(u^2) &= 2 \\ \text{Var}(|u| |u''|) &= 1 - \mu_1^4 \\ \text{Cov}(|u| |u''|, |u''| |u^{(IV)}|) &= \mu_1^2(1 - \mu_1^2) \\ \text{Cov}(u^2, |u| |u''|) &= \mu_3 \mu_1 - \mu_1^2. \end{aligned}$$

Therefore the limit theory is

$$\frac{1}{\delta^{1/2} \sqrt{\int_0^t \sigma_s^4 ds}} \left([Y_\delta]_t^{[2]} - \mu_1^{-2} \{Y_\delta\}_t^{[1,0,1]} \right) \xrightarrow{L} N(0, \vartheta_{SBV})$$

where

$$\vartheta_{SBV} = \mu_1^{-4} + 2\mu_1^{-2} - 5 \simeq 0.60907$$

and $\mu_r = E(|x|^r)$.

Chapter 5

Market Microstructure Effect

5.1 Introduction to market microstructure effect

Realised variance and multipower variation may suffer from a bias problem due to autocorrelation in the intra-day returns. It has many sources referred to as market microstructure effects. These effects induce serial correlation in high-frequency returns, used to calculate realised variance or multipower variation; therefore they have an impact on the integrated variance estimation. To reduce this bias, small values of M (number of intra-day observations) can be chosen but by sampling at low frequencies we do not incorporate all the information in the data and our estimators of integrated variance will be inefficient/inconsistent. Realised variance gives a perfect estimation of integrated variance when prices are observed in continuous time, hence its calculation should be based on returns that are sampled at the highest possible frequency. So there is a trade-off between bias and variance; bias due to market frictions when sampling at high frequencies and variance due to the asymptotic assumptions that do not hold when sampling at low frequencies. The asymptotic results are based on the idea of samples of increasingly higher frequencies hence the presence of market microstructure effects can potentially invalidate them.

True prices could not equal the observed prices due to the interpolation method or market frictions. Equidistant price data must be interpolated from the observed prices and an error can arise from the econometric method used to construct this artificial price series (previous-tick or interpolation methods). Market microstructure noise can have many different origins. For stock indices the serial correlation can be caused by non-synchronous trading (Lo and MacKinlay (1990)). When individual securities are not traded simultaneously, they incorporate shocks non-synchronously to the common factor that is driving their price. This results in correlated price changes at the index level. For liquid

assets the bid/ask bounce (Roll (1984)) induces negative serial correlation. When there is no new information arriving at a given moment in time, the price bounces between bid and ask prices. This effect can be strong in high-frequency data. For less liquid assets inactive trading causes a positive serial correlation. Transaction costs, misrecorded prices and the discrete nature of data that implies rounding errors may also contribute to this effect. See Andersen, Bollerslev, Diebold and Labys (1999), Andersen, Bollerslev, Diebold and Labys (2000), Bai, Russell and Tiao (2001) and Oomen (2002) for a more complete description of the bias caused by market microstructure effects.

Because market microstructure effects are present in virtually all price series, we need to find a way for these effects to have a negligible impact on the estimation of actual variance when using high-frequency data.

A first approach is the selection of an optimal sampling frequency that minimizes the bias. Bandi and Russell (2003), Hansen and Lunde (2006) and Aït-Sahalia, Mykland and Zhang (2005) study different methods to obtain such a frequency. It needs to be high enough to produce a volatility estimate with negligible sampling variation, yet low enough to avoid market microstructure bias. In Andersen, Bollerslev, Diebold and Labys (1999) the volatility signature plot was introduced to provide some initial guidance. This is a plot of average realised variance against sampling frequency; the bias is expected to increase at high-frequency levels.

A second approach is the use of alternative estimators for the integrated variance which are unbiased in the presence of market microstructure effects. Zhang, Mykland and Aït-Sahalia (2005) developed a class of such estimators, based on realised variance and subsampling and averaging. They propose and compare different estimation methods: 1) calculating realised variance at the highest possible frequency and completely ignoring the noise, 2) sampling sparsely at a lower frequency, 3) using the optimal sampling frequency, 4) using the subsampling and averaging method, 5) bias-correcting the subsampling and averaging method. Andersen, Bollerslev, Diebold and Ebens (2001), Hansen and Lunde (2006), Oomen (2004) and Barndorff-Nielsen, Hansen, Lunde and Shephard (2004) give other approaches to correct the bias for the realised variance. Here, we will study alternative estimators based on realised multipower variation.

In this chapter, we shall first describe how the market microstructure effect will be incorporated into our original model. The contamination due to market microstructure effects will be treated as that of an observation error. We shall then be in a position to assess realised bipower, tripower and quadpower variation in the presence of these effects. We shall thereafter introduce and study new estimators based on multipower variation and the subsampling and averaging method. Finally a comparison between these two approaches will be made.

5.1.1 Modelling market microstructure noise

Empirically, it is well known that the price process is contaminated by market microstructure effects. This is why even though our previous tests for jumps were based on asymptotic results, their implementations using extremely large values of M can give misleading answers due to noise accumulation. The larger the value of M , the closer we get to the asymptotic results but also the more market microstructure effects disturb the real process.

We will try to combat the microstructure effect problem by using the skipped version of the realised bipower variation, and by taking more adjacent observations into account, i.e., realised tripower variation and realised quadpower variation (introduced in Chapter 4). Also new estimators based on subsampling and averaging will be defined. To assess the effectiveness of these estimators in the presence of microstructure noise we need to model a contaminated process.

The additive model is a popular one, which assumes that the noise is independent and identically distributed across time and also independent of the true price process. This model has been analysed by Bandi and Russell (2003), Corsi, Zumbach, Müller and Dacorogna (2001), Hansen and Lunde (2006) and Zhang, Mykland and Aït-Sahalia (2005). We will use it for simplicity although more general models have been already proposed by Aït-Sahalia, Mykland and Zhang (2005), Hansen and Lunde (2004) and Oomen (2002).

Let us define the contaminated log-price process as

$$\tilde{Y}_t = Y_t + \eta_t$$

where Y_t is the true log-price, η_t the microstructure noise and \tilde{Y}_t the observed log-prices. The returns are defined as

$$\begin{aligned} \tilde{Y}_{j\delta} - \tilde{Y}_{(j-1)\delta} &= \{Y_{j\delta} - Y_{(j-1)\delta}\} + \{\eta_{j\delta} - \eta_{(j-1)\delta}\} \\ \tilde{y}_j &= y_j + \varepsilon_j \end{aligned}$$

for $j = 1, 2, \dots, \lfloor t/\delta \rfloor$.

Given a fixed time period $h > 0$ (here h denotes the period of a day) with $\lfloor t/\delta \rfloor = M$ intra- h returns, let us now define

$$\tilde{y}_{j,i} = \tilde{Y}_{(i-1)h+j\delta} - \tilde{Y}_{(i-1)h+(j-1)\delta}$$

for $j = 1, 2, \dots, M$. Hence $\tilde{y}_{j,i}$ is the j -th intra-day observed return for the i -th day and can also be

seen as

$$\tilde{y}_{j,i} = y_{j,i} + \varepsilon_{j,i}.$$

For the previous model we will impose the following assumptions

- (1) The random shocks η_j are Gaussian i.i.d. with mean zero and variance σ_η^2 .
- (2) $y_{j,i} \perp \varepsilon_{j,i} \quad \forall i, j$.
- (3) The true log-price process Y_t will follow the same SV model as in the previous chapters of this thesis, i.e., Y_t follows the equation

$$Y_t = \int_0^t a_u du + \int_0^t \sigma_s dW_s, \quad t \geq 0,$$

where we denote $A_t = \int_0^t a_u du$. The processes σ_t and A_t are assumed to be stochastically independent of the standard Brownian motion W , σ_t is càdlàg, A_t is assumed to have continuous locally bounded variation paths and $\int_0^t \sigma_s^2 ds < \infty$ for all t .

Let us now simulate one thousand days of a contaminated log-price process. The true process will be based on the CEV process described in Chapter 4 of this thesis where $A = 0$ and the leverage effect is ruled out.

The true returns and the contaminated returns of the first five days of our simulated series are shown in Figure 5.1 and 5.2 with $\sigma_\eta^2 = 0.0001$ and $\sigma_\eta^2 = 0.001$ respectively. In both figures we plot the returns for $M=12$, $M=72$ and $M=288$. It can easily be seen that as the value of M gets larger, the series are substantially more affected by the market microstructure noise, i.e. higher frequencies are more affected than lower frequencies. Obviously the difference between the true and the observed returns also depends on the variance of the noise. In Figure 5.2, where the variance of the noise is quite big, the noise completely hides the true process when $M=288$.

The problems caused by the market microstructure noise will depend on σ_η^2 hence it is important to know some empirical values. For this we can use the empirical noise-to-signal ratio, the ratio between the noise variance and the estimated average integrated variance. In Bandi and Russell (2003) this ratio equals 0.0002829 for IBM stock prices; in Hansen and Lunde (2006) it equals 0.000177 for Alcoa Inc stock prices. To be coherent with these ratios in our simulations we need $\sigma_\eta^2 = 0.0001$. Therefore we will study the cases where $\sigma_\eta^2 = 0.00001$, $\sigma_\eta^2 = 0.0001$ and $\sigma_\eta^2 = 0.001$.

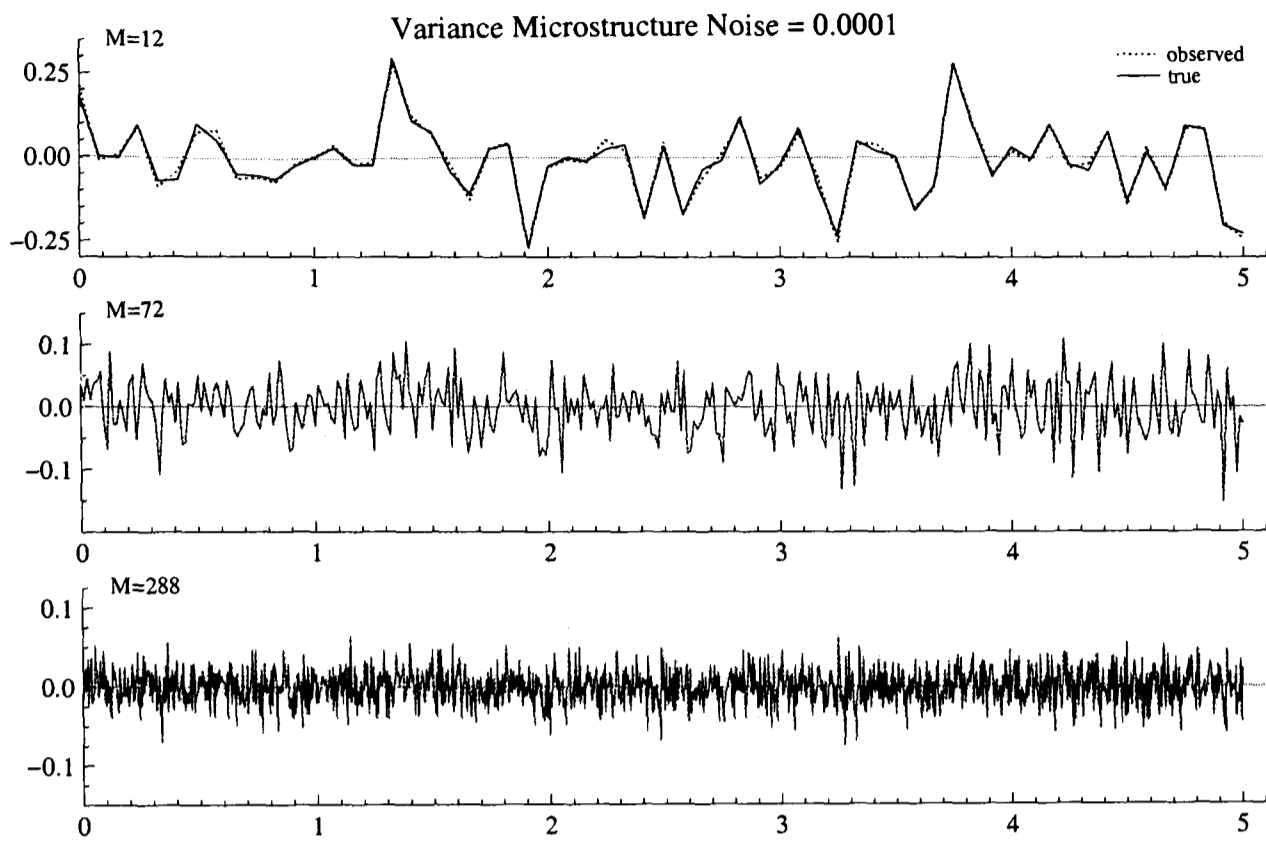


Figure 5.1. Plot of the true and observed return process with $\sigma_\eta^2 = 0.0001$.

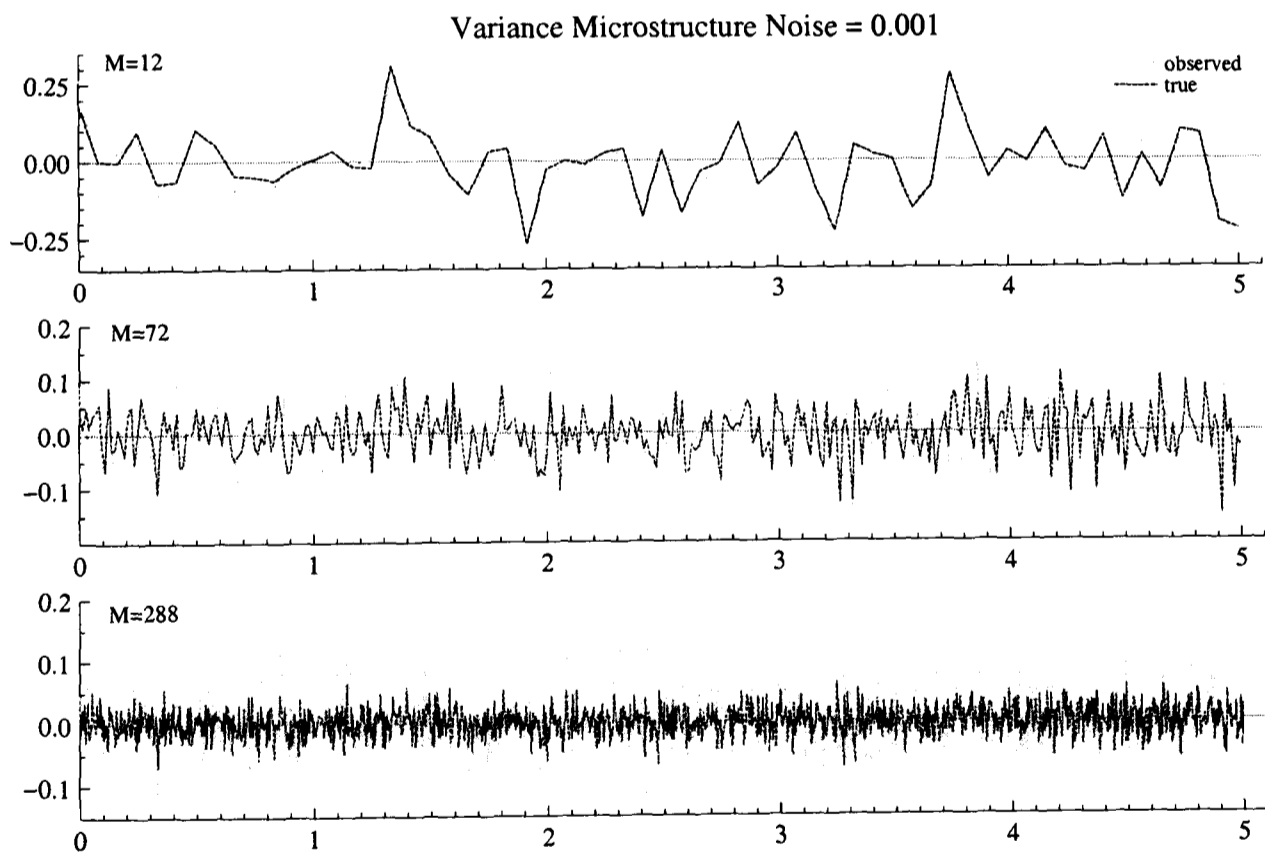


Figure 5.2. Plot of the true and observed return process with $\sigma_\eta^2 = 0.001$.

5.2 Bipower, tripower and quadpower variation in the presence of market microstructure noise

Realised bipower variation has recently been used to split the components of quadratic variation into one due to the continuous component and one due to the jump component of log-price processes. This allows us to test for the presence of jumps (see Barndorff-Nielsen and Shephard (2004a and 2006)). Realised tripower, quadpower and the skipped version of bipower variation were studied in the previous chapter of this thesis as alternative estimators which can be used to test for jumps in the price process.

An important issue to investigate is whether these estimators still give adequate results when market microstructure noise is present. The fact of skipping observations (skipped version of bipower variation) or using a number of adjacent observations to compute the estimators (tripower and quadpower variation) may help to reduce the bias caused by market microstructure effects. Using simulated contaminated data we shall compute realised bipower, tripower and quadpower variation and the skipped version of bipower variation and thereby attempt to assess the accuracy of these contaminated estimations.

5.2.1 Signature plots

As a first approach we will use signature plots where the average values of the estimators are displayed for different sampling frequencies. The sampling frequencies measured in minutes will be displayed on a logarithmic scale in our signature plots. In all this chapter we will use the modified estimators (see subSection 4.4.2 in this thesis) so each of them will be computed with the same number of observations. This therefore avoids the bias caused by finite values of M . If the estimators are affected by market microstructure effects, the bias will get bigger as the sampling frequency increases because these effects induce serial correlation in the high-frequency returns.

This problem is evident from Figure 5.3. The size of the bias will depend on the variance of the noise. When the variance is big (Figure 5.3 lower graph) all the estimators are biased for sampling frequencies above thirty minutes. There does not seem to be an evident difference between the estimators used. Only the skipped version of the realised bipower variation gives a slight improvement in the highest frequencies, but even for this estimator the bias is quite severe and it will not give reliable results.

Bias is still present in the highest frequencies (above ten minutes) when using a smaller variance for the noise (Figure 5.3 upper graph) although it seems to be quite smaller. Here there does not seem to be much difference between the estimators in any of the sampling frequencies.

It is impossible to say from the signature plots whether one estimator is more robust to market microstructure noise than the others, hence we need to study them further.

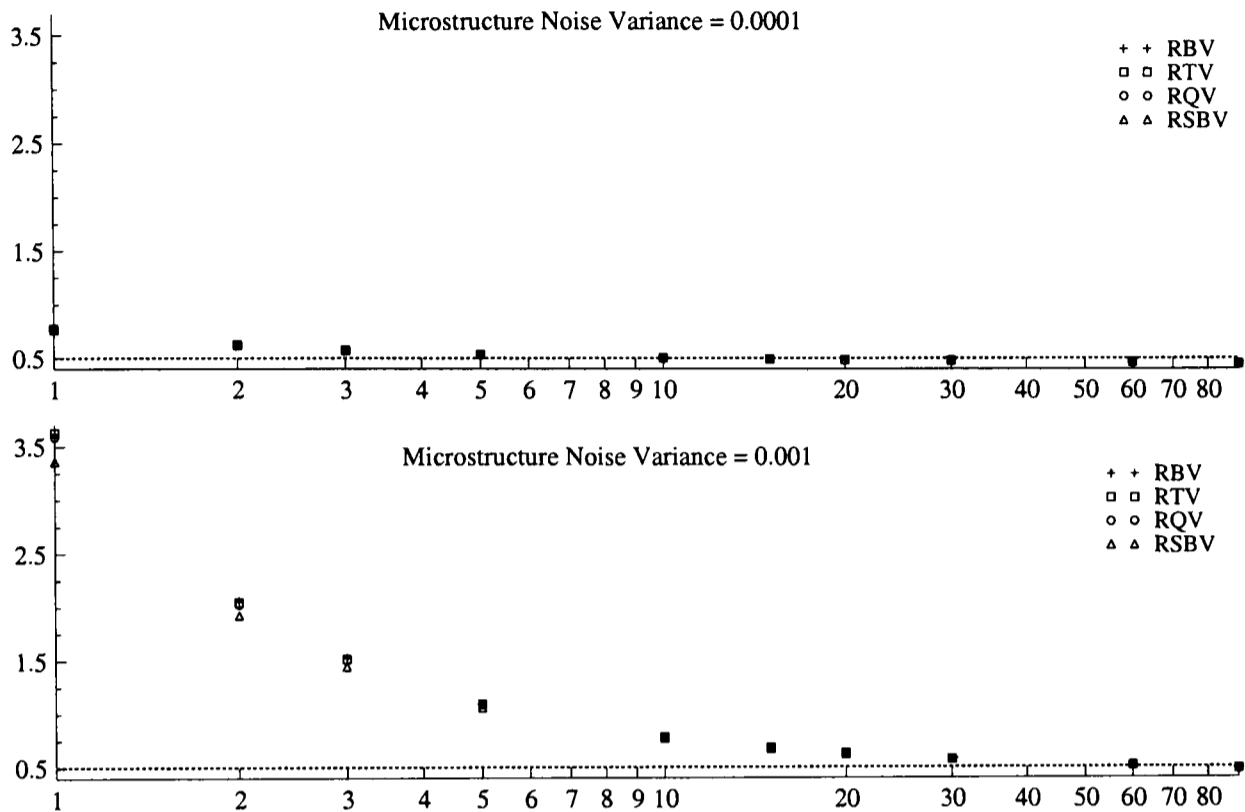


Figure 5.3. Signature plot when microstructure noise is present for simulated data.

5.2.2 Finite sample behaviour

As the signature plots just gave us a preliminary insight into the problem of autocorrelation in our estimators when market microstructure noise is present, we will now assess the accuracy of the mixed normal asymptotic approximation to their distribution in the case of contaminated observations.

We studied in Chapter 4 the finite sample behaviour of realised bipower, tripower and quadpower variation if computed with true data. Here we shall focus again on their finite sample behaviour but this time in the case when they are computed with contaminated data.

If market microstructure noise did not have any effect on our estimators, we would have the following limit distribution for the realised bipower variation error

$$\frac{1}{\delta^{1/2} \sqrt{\int_0^t \sigma_s^4 ds}} \left\{ \mu_1^{-2} \sum_{j=1}^{\lfloor t/\delta \rfloor - 1} |\tilde{y}_j| |\tilde{y}_{j+1}| - \int_0^t \sigma_s^2 ds \right\} \xrightarrow{L} N(0, \nu_{BV})$$

where $\nu_{BV} \simeq 2.60907$,

for the realised tripower variation error

$$\frac{1}{\delta^{1/2} \sqrt{\int_0^t \sigma_s^4 ds}} \left\{ \mu_{2/3}^{-3} \sum_{j=1}^{\lfloor t/\delta \rfloor - 2} |\tilde{y}_j|^{2/3} |\tilde{y}_{j+1}|^{2/3} |\tilde{y}_{j+2}|^{2/3} - \int_0^t \sigma_s^2 ds \right\} \xrightarrow{L} N(0, \nu_{TV})$$

where $\nu_{TV} \simeq 3.0613$,

for the realised quadpower variation error

$$\frac{1}{\delta^{1/2} \sqrt{\int_0^t \sigma_s^4 ds}} \left\{ \mu_{1/2}^{-4} \sum_{j=1}^{\lfloor t/\delta \rfloor - 3} \sqrt{|\tilde{y}_j| |\tilde{y}_{j+1}| |\tilde{y}_{j+2}| |\tilde{y}_{j+3}|} - \int_0^t \sigma_s^2 ds \right\} \xrightarrow{L} N(0, \nu_{QV})$$

where $\nu_{QV} \simeq 3.37702$

and for the skipped version of realised bipower variation error

$$\frac{1}{\delta^{1/2} \sqrt{\int_0^t \sigma_s^4 ds}} \left\{ \mu_1^{-2} \sum_{j=1}^{\lfloor t/\delta \rfloor - 2} |\tilde{y}_j| |\tilde{y}_{j+2}| - \int_0^t \sigma_s^2 ds \right\} \xrightarrow{L} N(0, \nu_{SBV})$$

where $\nu_{SBV} \simeq 2.60907$.

In Table 5.1 the bias, standard error and coverage rate of the previous infeasible errors are recorded for $\sigma_\eta^2 = 0.00001$, $\sigma_\eta^2 = 0.0001$ and $\sigma_\eta^2 = 0.001$. From this table we can see that realised tripower and quadpower variation seem to be the least affected by the market microstructure noise. Nevertheless, when the variance of the noise is too big the noise completely hides the true process and with high sampling frequencies none of the estimators give adequate results. When the noise is not that big, there are still some problems beyond 5 minutes returns. In all the cases low sampling frequencies do not seem to be heavily affected by the noise but our theory is based on asymptotic assumptions hence low frequencies lead to inefficient estimations. So far none of our estimators seem to solve the problem of microstructure noise if the noise's variance is sufficiently big.

5.2.3 Test for jumps in the presence of noise

The main interest of bipower variation is that a test for jumps in the price process can be established by subtracting realised variance from realised bipower variation. It can also be performed with realised tripower or quadpower variation. When market frictions are present the real power of this test is unknown as both the realised variance and the realised bipower (tripower or quadpower) variation

are affected. We shall study the effect of market microstructure noise on the test by applying it to simulated contaminated data.

M	BV			TV			QV			SBV		
	Bs	SD	Cov	Bs	SD	Cov	Bs	SD	Cov	Bs	SD	Cov
10^{-5}												
24	-.11	1.01	95.4	-.11	1.01	95.0	-.11	1.02	95.5	-.13	1.01	95.6
72	-.06	1.02	94.8	-.06	1.04	94.4	-.05	1.04	94.3	-.07	1.05	94.0
144	-.03	.97	96.3	-.03	.97	96.0	-.02	.98	95.5	-.01	1.01	95.0
288	.01	.98	94.6	.10	1.00	94.4	.09	1.02	95.0	.11	1.02	94.3
720	.51	1.02	91.0	.47	1.03	91.7	.45	1.03	92.4	.52	1.01	92.3
1440	1.50	1.11	67.0	1.37	1.10	70.7	1.29	1.09	72.4	1.49	1.12	67.3
10^{-4}												
24	-.08	1.00	95.1	-.09	1.01	95.2	-.09	1.01	95.7	-.09	1.01	95.3
72	.08	1.05	94.3	.08	1.05	93.7	.07	1.06	93.9	.08	1.06	93.6
144	.41	1.04	91.6	.39	1.03	91.4	.38	1.05	90.9	.43	1.06	91.0
288	1.32	1.13	72.5	1.23	1.14	74.3	1.17	1.15	77.0	1.31	1.17	71.5
720	5.39	1.89	1.7	4.95	1.81	2.8	4.69	1.77	4.6	5.22	1.80	1.5
1440	15.6	4.05	0	14.3	3.78	0	13.5	3.62	0	14.8	3.81	0
10^{-3}												
24	.19	1.08	93.1	.15	1.08	93.3	.12	1.09	93.5	.18	1.09	92.5
72	1.59	1.41	63.3	1.45	1.39	66.4	1.39	1.38	69.5	1.54	1.38	65.1
144	4.86	2.11	6.1	4.48	2.03	8.3	4.25	1.99	11.3	4.66	1.99	7.3
288	14.2	4.04	0	13.1	3.82	0	12.2	3.66	0	13.3	3.78	0
720	57.6	14.5	0	52.5	13.3	0	49.4	12.6	0	52.5	12.9	0
1440	164	38.9	0	150	35.4	0	141	33.3	0	148	34.5	0

Table 5.1: Bias, standard deviation and coverage rate of the infeasible standardised realised bipower, tripower, quadpower and skipped bipower variation error in the presence of microstructure noise.

If the test is not affected by market microstructure effects then the convergence results of Theorems 1, 2 and 3 of this thesis should hold. An infeasible ratio test for jumps can be based on these results using each one of our estimators.

Table 5.2 shows the bias and standard deviation of the test statistics as well as the coverage rate

of the test when there are no jumps in the real process for different values of the variance of market microstructure noise. Table 5.3 also shows the bias, standard deviation and coverage rate of the test but this time when a jump of given size is included. It appears from these tables that the presence of jumps is underestimated when prices are contaminated by market microstructure noise. Even when jumps are big, the noise can dominate the process and completely hide them.

M	24			144			1440		
	Bias	SD	Cove	Bias	SD	Cove	Bias	SD	Cove
10^{-5}									
RBV	-.308	1.06	89.6	-.099	.99	93.3	.982	1.15	97.8
RTV	-.258	1.02	91.1	-.070	0.98	93.9	.663	1.16	97.8
RQV	-.256	1.03	91.3	-.073	0.99	94.3	.499	1.14	97.5
RSBV	-.327	1.06	87.9	-.084	.99	93.8	-.004	0.96	95.6
10^{-4}									
RBV	-.283	1.05	89.2	.179	1.02	95.8	3.72	1.65	99.9
RTV	-.238	1.03	91.4	.101	1.01	96.3	2.45	1.32	99.9
RQV	-.230	1.04	92.7	.055	1.02	96.3	1.84	1.21	99.7
RSBV	-.317	1.08	88.4	-.112	1.01	94.2	.014	.969	95.8
10^{-3}									
RBV	-.101	1.04	92.4	1.03	1.02	99.1	5.71	0.89	100
RTV	-.098	1.05	93.3	.627	1.04	98.7	3.69	0.98	100
RQV	-.103	1.06	93.6	.445	1.04	98.1	2.77	1.01	100
RSBV	-.292	1.09	88.8	-.152	.996	93.5	.014	0.96	96.1

Table 5.2: Bias, standard deviation and coverage rate of the infeasible ratio test for jumps in the presence of microstructure noise.

When market microstructure noise is affecting the price process the test for jumps can be ineffective and give totally wrong answers. For low frequencies jumps are difficult to detect and for high frequencies they are hidden by the market microstructure noise. In the presence of jumps, the skipped version of realised bipower variation gives slightly better results, even though the improvement is negligible. If either the noise's variance is too big or the jumps are too small then none of our estimators will be reliable. We should therefore resort to an alternative estimator.

M		24			144			1440		
	Bias	SD	Cove		Bias	SD	Cove	Bias	SD	Cove
$\sigma_\eta^2 =$	10^{-4}									
50%										
RBV	-.426	1.14	85.0	-.327	1.36	85.5	3.18	1.74	99.7	
RTV	-.371	1.09	87.3	-.344	1.26	85.7	1.97	1.41	99.3	
RQV	-.350	1.08	87.6	-.342	1.21	86.8	1.40	1.28	98.9	
RSBV	-.479	1.16	84.2	-.578	1.38	83.6	-.501	1.25	83.4	
20%										
RBV	-.291	1.07	89.5	.096	1.06	94.4	3.65	1.64	100	
RTV	-.237	1.04	91.5	.023	1.02	94.4	2.36	1.31	100	
RQV	-.223	1.05	91.9	-.004	1.02	94.6	1.75	1.19	99.7	
RSBV	-.317	1.09	87.8	-.179	1.01	92.6	-.081	.998	94.3	
$\sigma_\eta^2 =$	10^{-3}									
50%										
RBV	-.211	1.08	90.5	.915	1.04	98.5	5.63	.909	100	
RTV	-.212	1.09	90.5	.573	1.05	98.1	3.61	1.01	100	
RQV	-.207	1.10	90.8	.413	1.06	98.2	2.69	1.03	100	
RSBV	-.384	1.14	85.6	-.174	1.01	93.3	-.055	.993	94.0	
20%										
RBV	-.161	1.07	91.5	.995	1.02	99.2	5.67	0.91	100	
RTV	-.162	1.07	91.5	.642	1.05	98.4	3.64	1.01	100	
RQV	-.160	1.09	91.6	.477	1.05	98.7	2.72	1.03	100	
RSBV	-.332	1.13	86.5	-.109	.984	94.4	-.029	0.99	94.6	

Table 5.3: Bias, standard deviation and coverage rate of the infeasible ratio test for jumps in the presence of microstructure noise when the price process includes a jump component.

5.3 Estimators based on subsampling and averaging

5.3.1 General estimator of actual variance

Realised variance computed from the highest possible frequency data ought to provide the best possible estimate for actual variance but market microstructure effects can really invalidate asymptotic results. Although realised bipower, tripower and quadpower variation seem to prove adequate when the variance of the market microstructure noise is small, they are unreliable whenever it is large. Zhang, Mykland and Aït-Sahalia (2005) have introduced a new estimator for the actual variance based on realised variance that gives better results. They point out that although sampling at low frequency merely reduces the impact of market microstructure effects rather than to correct them for the volatility estimations, subsampling and averaging seems to be the only way to deal with them. Hence in order to benefit from the low frequency data properties they propose to select a number of subgrids of the original grid of observation times. Then they average the estimators derived from the subgrids to obtain a new estimator which is less biased than realised variance in the presence of market microstructure effects.

The necessary assumptions are, as in the case of realised variance, that the log-prices Y_t follow the stochastic volatility model described previously in this thesis.

They suppose that the total grid of observation times $\mathcal{G} = \{t_0, \dots, t_n\}$ is partitioned in K non-overlapping subgrids $\mathcal{G}^{(k)}$ for $k = 1, \dots, K$. This can also be seen as $\mathcal{G} = \cup_{k=1}^K \mathcal{G}^{(k)}$ where $\mathcal{G}^{(k)} \cap \mathcal{G}^{(l)} = \emptyset$ when $k \neq l$.

To select the k^{th} subgrid $\mathcal{G}^{(k)}$ they start with t_{k-1} and then pick every K th sample point after that until the end of the series, T . In other words,

$$\mathcal{G}^{(k)} = (t_{k-1}, t_{k-1+K}, t_{k-1+2K}, \dots, t_{k-1+n_k K})$$

for $k = 1, \dots, K$ where n_k is the integer making $t_{k-1+n_k K}$ the last element in the corresponding subgrid.

The number of elements in the total grid is $n + 1$ whereas each subgrid has $n_k + 1$ elements. The n_k need not be the same for all k .

Afterwards they calculate the realised variance for each subgrid, i.e.

$$[Y_{K\delta}^{(k)}]_t = \sum_{t_j, t_{j+} \in \mathcal{G}^{(k)}} (Y_{t_{j+}} - Y_{t_j})^2$$

where if $t_j \in \mathcal{G}^{(k)}$ then t_{j+} denotes the following element in $\mathcal{G}^{(k)}$.

By taking the average over all the subgrids

$$[Y_\delta]_t^{(avg)} = \frac{1}{K} \sum_{k=1}^K [Y_{K\delta}^{(k)}]_t$$

they get a new estimator of the actual variance. Given a constant K , as $n \rightarrow \infty$ and so $\delta \downarrow 0$

$$[Y_\delta]_t^{(avg)} \xrightarrow{p} [Y]_t = \int_0^t \sigma_s^2 ds.$$

5.3.2 New estimators based on realised variance, bipower, tripower and quad-power variation

First, let us define

$$\begin{aligned} \sigma_{2\delta,j}^2 &= \int_{2\delta j}^{2\delta(j+1)} \sigma_u^2 du, \\ \lambda_{2\delta,j}^2 &= \int_{\delta(2j+1)}^{\delta(2j+3)} \sigma_u^2 du, \\ \sigma_{\delta,j}^2 &= \int_{\delta j}^{\delta(j+1)} \sigma_u^2 du. \end{aligned}$$

Following the idea of Zhang, Mykland and Aït-Sahalia (2005) and focusing on the case when $K=2$, we can define for realised variance

$$[Y_\delta]_t^{(avg)} = \frac{1}{2} \left([Y_{2\delta}^{(1)}]_t + [Y_{2\delta}^{(2)}]_t \right)$$

where

$$\begin{aligned} [Y_{2\delta}^{(1)}]_t &= \sum_{j=0}^{\lfloor t/2\delta \rfloor - 1} (Y_{(2j+2)\delta} - Y_{(2j)\delta})^2 = \sum_{j=0}^{\lfloor t/2\delta \rfloor - 1} (y_{2\delta,j}^{(1)})^2 \\ &\stackrel{law}{=} \sum_{j=0}^{\lfloor t/2\delta \rfloor - 1} (\sigma_{2\delta,j} \epsilon_{2\delta,j})^2 = \sum_{j=0}^{\lfloor t/2\delta \rfloor - 1} (\sigma_{\delta,2j} \epsilon_{\delta,2j} + \sigma_{\delta,2j+1} \epsilon_{\delta,2j+1})^2 \end{aligned}$$

given that $(y_{2\delta,j}^{(1)})^2 = (y_{2j} + y_{2j+1})^2$ and

$$[Y_{2\delta}^{(2)}]_t = \sum_{j=0}^{\lfloor t/2\delta \rfloor - 2} (Y_{(2j+3)\delta} - Y_{(2j+1)\delta})^2 = \sum_{j=0}^{\lfloor t/2\delta \rfloor - 2} (y_{2\delta,j}^{(2)})^2$$

$$\stackrel{law}{=} \sum_{j=0}^{\lfloor t/2\delta \rfloor - 2} (\lambda_{2\delta,j} \zeta_{2\delta,j})^2 = \sum_{j=0}^{\lfloor t/2\delta \rfloor - 2} (\sigma_{\delta,2j+1} \epsilon_{\delta,2j+1} + \sigma_{\delta,2j+2} \epsilon_{\delta,2j+2})^2$$

given that $(y_{2\delta,j}^{(2)})^2 = (y_{2j+1} + y_{2j+2})^2$.

Notice that

$$\sigma_{2\delta,j}^2 = \sigma_{\delta,2j}^2 + \sigma_{\delta,2j+1}^2.$$

As realised bipower, tripower and quadpower variation were constructed to obtain better estimations than realised variance when market microstructure effects were present, we may expect that new estimators using the subgrid and averaging technique on bipower, tripower and quadpower variation will also be less biased than the estimator based on realised variance. Here we define these alternative estimators.

Bipower Variation

Let us define the new estimator based on bipower variation,

$$[Y_{\delta}]_t^{[1,1](avg)} = \frac{1}{2} \left([Y_{2\delta}^{(1)}]_t^{[1,1]} + [Y_{2\delta}^{(2)}]_t^{[1,1]} \right)$$

where

$$\begin{aligned} [Y_{2\delta}^{(1)}]_t^{[1,1]} &= \sum_{j=0}^{\lfloor t/2\delta \rfloor - 2} | Y_{(2j+2)\delta} - Y_{(2j)\delta} || Y_{(2j+4)\delta} - Y_{(2j+2)\delta} | \\ &= \sum_{j=0}^{\lfloor t/2\delta \rfloor - 2} | y_{2\delta,j}^{(1)} || y_{2\delta,j+1}^{(1)} | \\ &\stackrel{law}{=} \sum_{j=0}^{\lfloor t/2\delta \rfloor - 2} | \sigma_{2\delta,j} \epsilon_{2\delta,j} || \sigma_{2\delta,j+1} \epsilon_{2\delta,j+1} | \\ &= \sum_{j=0}^{\lfloor t/2\delta \rfloor - 2} | \sigma_{\delta,2j} \epsilon_{\delta,2j} + \sigma_{\delta,2j+1} \epsilon_{\delta,2j+1} || \sigma_{\delta,2j+2} \epsilon_{\delta,2j+2} + \sigma_{\delta,2j+3} \epsilon_{\delta,2j+3} | \end{aligned}$$

and

$$[Y_{2\delta}^{(2)}]_t^{[1,1]} = \sum_{j=0}^{\lfloor t/2\delta \rfloor - 3} | Y_{(2j+3)\delta} - Y_{(2j+1)\delta} || Y_{(2j+5)\delta} - Y_{(2j+3)\delta} |$$

$$\begin{aligned}
&= \sum_{j=0}^{\lfloor t/2\delta \rfloor - 3} |y_{2\delta,j}^{(2)}| |y_{2\delta,j+1}^{(2)}| \\
&\stackrel{law}{=} \sum_{j=0}^{\lfloor t/2\delta \rfloor - 3} |\lambda_{2\delta,j}\zeta_{2\delta,j}| |\lambda_{2\delta,j+1}\zeta_{2\delta,j+1}| \\
&= \sum_{j=0}^{\lfloor t/2\delta \rfloor - 3} |\sigma_{\delta,2j+1}\epsilon_{\delta,2j+1} + \sigma_{\delta,2j+2}\epsilon_{\delta,2j+2}| |\sigma_{\delta,2j+3}\epsilon_{\delta,2j+3} + \sigma_{\delta,2j+4}\epsilon_{\delta,2j+4}|.
\end{aligned}$$

In this case notice that

$$\sigma_{2\delta,j}\sigma_{2\delta,j+1} = (\sigma_{\delta,2j}^2 + \sigma_{\delta,2j+1}^2)^{1/2}(\sigma_{\delta,2j+2}^2 + \sigma_{\delta,2j+3}^2)^{1/2}.$$

Tripower Variation

The tripower variation case can be defined as

$$[Y_{\delta}]_t^{[2/3,2/3,2/3](avg)} = \frac{1}{2} \left([Y_{2\delta}^{(1)}]_t^{[2/3,2/3,2/3]} + [Y_{2\delta}^{(2)}]_t^{[2/3,2/3,2/3]} \right)$$

where

$$\begin{aligned}
[Y_{2\delta}^{(1)}]_t^{[2/3,2/3,2/3]} &= \sum_{j=0}^{\lfloor t/2\delta \rfloor - 3} |Y_{(2j+2)\delta} - Y_{(2j)\delta}|^{2/3} |Y_{(2j+4)\delta} - Y_{(2j+2)\delta}|^{2/3} |Y_{(2j+6)\delta} - Y_{(2j+4)\delta}|^{2/3} \\
&= \sum_{j=0}^{\lfloor t/2\delta \rfloor - 3} |y_{2\delta,j}^{(1)}|^{2/3} |y_{2\delta,j+1}^{(1)}|^{2/3} |y_{2\delta,j+2}^{(1)}|^{2/3} \\
&\stackrel{law}{=} \sum_{j=0}^{\lfloor t/2\delta \rfloor - 3} |\sigma_{2\delta,j}\epsilon_{2\delta,j}|^{2/3} |\sigma_{2\delta,j+1}\epsilon_{2\delta,j+1}|^{2/3} |\sigma_{2\delta,j+2}\epsilon_{2\delta,j+2}|^{2/3} \\
&= \sum_{j=0}^{\lfloor t/2\delta \rfloor - 3} |\sigma_{\delta,2j}\epsilon_{\delta,2j} + \sigma_{\delta,2j+1}\epsilon_{\delta,2j+1}|^{2/3} |\sigma_{\delta,2j+2}\epsilon_{\delta,2j+2} + \sigma_{\delta,2j+3}\epsilon_{\delta,2j+3}|^{2/3} \\
&\quad |\sigma_{\delta,2j+4}\epsilon_{\delta,2j+4} + \sigma_{\delta,2j+5}\epsilon_{\delta,2j+5}|^{2/3}
\end{aligned}$$

and

$$\begin{aligned}
[Y_{2\delta}^{(2)}]_t^{[2/3,2/3,2/3]} &= \sum_{j=0}^{\lfloor t/2\delta \rfloor - 4} |Y_{(2j+3)\delta} - Y_{(2j+1)\delta}|^{2/3} |Y_{(2j+5)\delta} - Y_{(2j+3)\delta}|^{2/3} |Y_{(2j+7)\delta} - Y_{(2j+5)\delta}|^{2/3} \\
&= \sum_{j=0}^{\lfloor t/2\delta \rfloor - 4} |y_{2\delta,j}^{(2)}|^{2/3} |y_{2\delta,j+1}^{(2)}|^{2/3} |y_{2\delta,j+2}^{(2)}|^{2/3}
\end{aligned}$$

$$\begin{aligned}
&\stackrel{law}{=} \sum_{j=0}^{\lfloor t/2\delta \rfloor - 4} |\lambda_{2\delta,j} \zeta_{2\delta,j}|^{2/3} |\lambda_{2\delta,j+1} \zeta_{2\delta,j+1}|^{2/3} |\lambda_{2\delta,j+2} \zeta_{2\delta,j+2}|^{2/3} \\
&= \sum_{j=0}^{\lfloor t/2\delta \rfloor - 4} |\sigma_{\delta,2j+1} \epsilon_{\delta,2j+1} + \sigma_{\delta,2j+2} \epsilon_{\delta,2j+2}|^{2/3} |\sigma_{\delta,2j+3} \epsilon_{\delta,2j+3} + \sigma_{\delta,2j+4} \epsilon_{\delta,2j+4}|^{2/3} \\
&\quad |\sigma_{\delta,2j+5} \epsilon_{\delta,2j+5} + \sigma_{\delta,2j+6} \epsilon_{\delta,2j+6}|^{2/3}.
\end{aligned}$$

Here notice that

$$\sigma_{2\delta,j}^{2/3} \sigma_{2\delta,j+1}^{2/3} \sigma_{2\delta,j+2}^{2/3} = (\sigma_{\delta,2j}^2 + \sigma_{\delta,2j+1}^2)^{1/3} (\sigma_{\delta,2j+2}^2 + \sigma_{\delta,2j+3}^2)^{1/3} (\sigma_{\delta,2j+4}^2 + \sigma_{\delta,2j+5}^2)^{1/3}.$$

Quadpower Variation

Finally, let us define the following estimator based on quadpower variation,

$$[Y_{\delta}]_t^{[1/2,1/2,1/2,1/2](avg)} = \frac{1}{2} \left([Y_{2\delta}^{(1)}]_t^{[1/2,1/2,1/2,1/2]} + [Y_{2\delta}^{(2)}]_t^{[1/2,1/2,1/2,1/2]} \right)$$

where

$$\begin{aligned}
[Y_{2\delta}^{(1)}]_t^{[1/2,1/2,1/2,1/2]} &= \sum_{j=0}^{\lfloor t/2\delta \rfloor - 4} |Y_{(2j+2)\delta} - Y_{(2j)\delta}|^{1/2} |Y_{(2j+4)\delta} - Y_{(2j+2)\delta}|^{1/2} \\
&\quad |Y_{(2j+6)\delta} - Y_{(2j+4)\delta}|^{1/2} |Y_{(2j+8)\delta} - Y_{(2j+6)\delta}|^{1/2} \\
&= \sum_{j=0}^{\lfloor t/2\delta \rfloor - 4} |y_{2\delta,j}^{(1)}|^{1/2} |y_{2\delta,j+1}^{(1)}|^{1/2} |y_{2\delta,j+2}^{(1)}|^{1/2} |y_{2\delta,j+3}^{(1)}|^{1/2} \\
&\stackrel{law}{=} \sum_{j=0}^{\lfloor t/2\delta \rfloor - 4} |\sigma_{2\delta,j} \epsilon_{2\delta,j}|^{1/2} |\sigma_{2\delta,j+1} \epsilon_{2\delta,j+1}|^{1/2} |\sigma_{2\delta,j+2} \epsilon_{2\delta,j+2}|^{1/2} |\sigma_{2\delta,j+3} \epsilon_{2\delta,j+3}|^{1/2} \\
&= \sum_{j=0}^{\lfloor t/2\delta \rfloor - 4} |\sigma_{\delta,2j} \epsilon_{\delta,2j} + \sigma_{\delta,2j+1} \epsilon_{\delta,2j+1}|^{1/2} |\sigma_{\delta,2j+2} \epsilon_{\delta,2j+2} + \sigma_{\delta,2j+3} \epsilon_{\delta,2j+3}|^{1/2} \\
&\quad |\sigma_{\delta,2j+4} \epsilon_{\delta,2j+4} + \sigma_{\delta,2j+5} \epsilon_{\delta,2j+5}|^{1/2} |\sigma_{\delta,2j+6} \epsilon_{\delta,2j+6} + \sigma_{\delta,2j+7} \epsilon_{\delta,2j+7}|^{1/2}
\end{aligned}$$

and

$$\begin{aligned}
[Y_{2\delta}^{(2)}]_t^{[1/2,1/2,1/2,1/2]} &= \sum_{j=0}^{\lfloor t/2\delta \rfloor - 5} |Y_{(2j+3)\delta} - Y_{(2j+1)\delta}|^{1/2} |Y_{(2j+5)\delta} - Y_{(2j+3)\delta}|^{1/2} \\
&\quad |Y_{(2j+7)\delta} - Y_{(2j+5)\delta}|^{1/2} |Y_{(2j+9)\delta} - Y_{(2j+7)\delta}|^{1/2}
\end{aligned}$$

$$\begin{aligned}
&= \sum_{j=0}^{\lfloor t/2\delta \rfloor - 5} |y_{2\delta,j}^{(2)}|^{1/2} |y_{2\delta,j+1}^{(2)}|^{1/2} |y_{2\delta,j+2}^{(2)}|^{1/2} |y_{2\delta,j+3}^{(2)}|^{1/2} \\
&\stackrel{law}{=} \sum_{j=0}^{\lfloor t/2\delta \rfloor - 5} |\lambda_{2\delta,j}\zeta_{2\delta,j}|^{1/2} |\lambda_{2\delta,j+1}\zeta_{2\delta,j+1}|^{1/2} |\lambda_{2\delta,j+2}\zeta_{2\delta,j+2}|^{1/2} |\lambda_{2\delta,j+3}\zeta_{2\delta,j+3}|^{1/2} \\
&= \sum_{j=0}^{\lfloor t/2\delta \rfloor - 5} |\sigma_{\delta,2j+1}\epsilon_{\delta,2j+1} + \sigma_{\delta,2j+2}\epsilon_{\delta,2j+2}|^{1/2} |\sigma_{\delta,2j+3}\epsilon_{\delta,2j+3} + \sigma_{\delta,2j+4}\epsilon_{\delta,2j+4}|^{1/2} \\
&\quad |\sigma_{\delta,2j+5}\epsilon_{\delta,2j+5} + \sigma_{\delta,2j+6}\epsilon_{\delta,2j+6}|^{1/2} |\sigma_{\delta,2j+7}\epsilon_{\delta,2j+7} + \sigma_{\delta,2j+8}\epsilon_{\delta,2j+8}|^{1/2}.
\end{aligned}$$

Here we have that

$$\sigma_{2\delta,j}^{1/2}\sigma_{2\delta,j+1}^{1/2}\sigma_{2\delta,j+2}^{1/2}\sigma_{2\delta,j+3}^{1/2} = (\sigma_{\delta,2j}^2 + \sigma_{\delta,2j+1}^2)^{1/4} (\sigma_{\delta,2j+2}^2 + \sigma_{\delta,2j+3}^2)^{1/4} (\sigma_{\delta,2j+4}^2 + \sigma_{\delta,2j+5}^2)^{1/4} (\sigma_{\delta,2j+6}^2 + \sigma_{\delta,2j+7}^2)^{1/4}.$$

5.3.3 Daily time series

In order to produce daily time series, we need to consider a fixed time period h (corresponding to the period of one day) with $\lfloor t/\delta \rfloor = M$ intra-day observed (and contaminated) returns, during each day, defined as

$$\tilde{y}_{j,i} = \tilde{Y}_{(i-1)h+(j+1)\delta} - \tilde{Y}_{(i-1)h+j\delta},$$

for the j -th intra-day return in the i -th period.

Let us point out that a bias is introduced by using finite values of M because each estimator will have a different number of components in the summation. To avoid this the following modified estimators will be used throughout this chapter

$$[Y_M]_i^{[1,1](avg)} = \frac{1}{2} \left(\frac{M/2}{M/2-1} [Y_{M/2}^{(1)}]_i^{[1,1]} + \frac{M/2-1}{M/2-2} [Y_{M/2}^{(2)}]_i^{[1,1]} \right),$$

$$[Y_M]_i^{[2/3,2/3,2/3](avg)} = \frac{1}{2} \left(\frac{M/2}{M/2-2} [Y_{M/2}^{(1)}]_i^{[2/3,2/3,2/3]} + \frac{M/2-1}{M/2-3} [Y_{M/2}^{(2)}]_i^{[2/3,2/3,2/3]} \right),$$

$$[Y_M]_i^{[1/2,1/2,1/2,1/2](avg)} = \frac{1}{2} \left(\frac{M/2}{M/2-3} [Y_{M/2}^{(1)}]_i^{[1/2,1/2,1/2,1/2]} + \frac{M/2-1}{M/2-4} [Y_{M/2}^{(2)}]_i^{[1/2,1/2,1/2,1/2]} \right),$$

$$[Y_M]_i^{[1,0,1](avg)} = \frac{1}{2} \left(\frac{M/2}{M/2-2} [Y_{M/2}^{(1)}]_i^{[1,0,1]} + \frac{M/2-1}{M/2-3} [Y_{M/2}^{(2)}]_i^{[1,0,1]} \right).$$

5.3.4 Asymptotic distributions

To assess the behaviour of our new estimators under market microstructure noise we need them to converge in distribution.

Given the assumptions in Barndoff-Nielsen, Graversen, Jacod, Podolskij and Shephard (2005) and Barndoff-Nielsen, Graversen, Jacod and Shephard (2006), we can obtain the asymptotic distributions below by setting $A = 0$.

Result 1

If $Y \in SVSM^c$ then as $\delta \downarrow 0$

$$\sqrt{\frac{1}{2\delta}} \begin{pmatrix} [Y_{2\delta}^{(1)}]_t - \int_0^t \sigma_s^2 ds \\ [Y_{2\delta}^{(2)}]_t - \int_0^t \sigma_s^2 ds \end{pmatrix} \xrightarrow{L} MN \left(0, \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \int_0^t \sigma_s^4 ds \right).$$

Hence we get

$$\frac{1}{\delta^{1/2} \sqrt{\int_0^t \sigma_s^4 ds}} \left([Y_\delta]_t^{(avg)} - \int_0^t \sigma_s^2 ds \right) \xrightarrow{L} N(0, 3).$$

Result 2

If $Y \in SVSM^c$ then as $\delta \downarrow 0$

$$\begin{aligned} \sqrt{\frac{1}{2\delta}} \begin{pmatrix} [Y_{2\delta}^{(1)}]_t^{[1,1]} - \mu_1^2 \int_0^t \sigma_s^2 ds \\ [Y_{2\delta}^{(2)}]_t^{[1,1]} - \mu_1^2 \int_0^t \sigma_s^2 ds \end{pmatrix} &\xrightarrow{L} MN \left(0, \begin{pmatrix} k_{1,2} & k_{2,2} \\ k_{2,2} & k_{1,2} \end{pmatrix} \int_0^t \sigma_s^4 ds \right) \\ &\simeq MN \left(0, \begin{pmatrix} 2.592 & 0.988 \\ 0.988 & 2.592 \end{pmatrix} \int_0^t \sigma_s^4 ds \right). \end{aligned}$$

Hence we get

$$\frac{1}{\delta^{1/2} \sqrt{\int_0^t \sigma_s^4 ds}} \left([Y_\delta]_t^{[1,1](avg)} - \mu_1^2 \int_0^t \sigma_s^2 ds \right) \xrightarrow{L} N(0, \vartheta_{BVS})$$

where $\vartheta_{BVS} = (2k_{1,2} + 2k_{2,2})/2 \simeq 3.581$.

Result 3

If $Y \in SVSM^c$ then as $\delta \downarrow 0$

$$\begin{aligned} \sqrt{\frac{1}{2\delta}} \begin{pmatrix} [Y_{2\delta}^{(1)}]_t^{[2/3,2/3,2/3]} - \mu_{2/3}^3 \int_0^t \sigma_s^2 ds \\ [Y_{2\delta}^{(2)}]_t^{[2/3,2/3,2/3]} - \mu_{2/3}^3 \int_0^t \sigma_s^2 ds \end{pmatrix} &\xrightarrow{L} MN\left(0, \begin{pmatrix} k_{1,3} & k_{2,3} \\ k_{2,3} & k_{1,3} \end{pmatrix} \int_0^t \sigma_s^4 ds\right) \\ &\simeq MN\left(0, \begin{pmatrix} 3.049 & 1.018 \\ 1.019 & 3.049 \end{pmatrix} \int_0^t \sigma_s^4 ds\right). \end{aligned}$$

Hence we get

$$\frac{1}{\delta^{1/2} \sqrt{\int_0^t \sigma_s^4 ds}} \left([Y_\delta]_t^{[2/3,2/3,2/3](avg)} - \mu_{2/3}^3 \int_0^t \sigma_s^2 ds \right) \xrightarrow{L} N\left(0, \vartheta_{BTS}\right)$$

where $\vartheta_{BTS} = (2k_{1,3} + 2k_{2,3})/2 \simeq 4.067$.

Result 4

If $Y \in SVSM^c$ then as $\delta \downarrow 0$

$$\begin{aligned} \sqrt{\frac{1}{2\delta}} \begin{pmatrix} [Y_{2\delta}^{(1)}]_t^{[1/2,1/2,1/2,1/2]} - \mu_{1/2}^4 \int_0^t \sigma_s^2 ds \\ [Y_{2\delta}^{(2)}]_t^{[1/2,1/2,1/2,1/2]} - \mu_{1/2}^4 \int_0^t \sigma_s^2 ds \end{pmatrix} &\xrightarrow{L} MN\left(0, \begin{pmatrix} k_{1,4} & k_{2,4} \\ k_{2,4} & k_{1,4} \end{pmatrix} \int_0^t \sigma_s^4 ds\right) \\ &\simeq MN\left(0, \begin{pmatrix} 3.377 & 1.013 \\ 1.013 & 3.377 \end{pmatrix} \int_0^t \sigma_s^4 ds\right). \end{aligned}$$

Hence we get

$$\frac{1}{\delta^{1/2} \sqrt{\int_0^t \sigma_s^4 ds}} \left([Y_\delta]_t^{[1/2,1/2,1/2,1/2](avg)} - \mu_{1/2}^4 \int_0^t \sigma_s^2 ds \right) \xrightarrow{L} N\left(0, \vartheta_{BQS}\right)$$

where $\vartheta_{BQS} = (2k_{1,4} + 2k_{2,4})/2 \simeq 4.389$.

The derivations of these results are given in appendix A5.1

5.3.5 Signature plots

Now, in order to assess now the accuracy of our estimators we will use the simulated series described in Section 5.1.1. Firstly we shall check whether the use of subgridding and averaging in the calculations of the estimators reduces the bias caused by market microstructure effects.

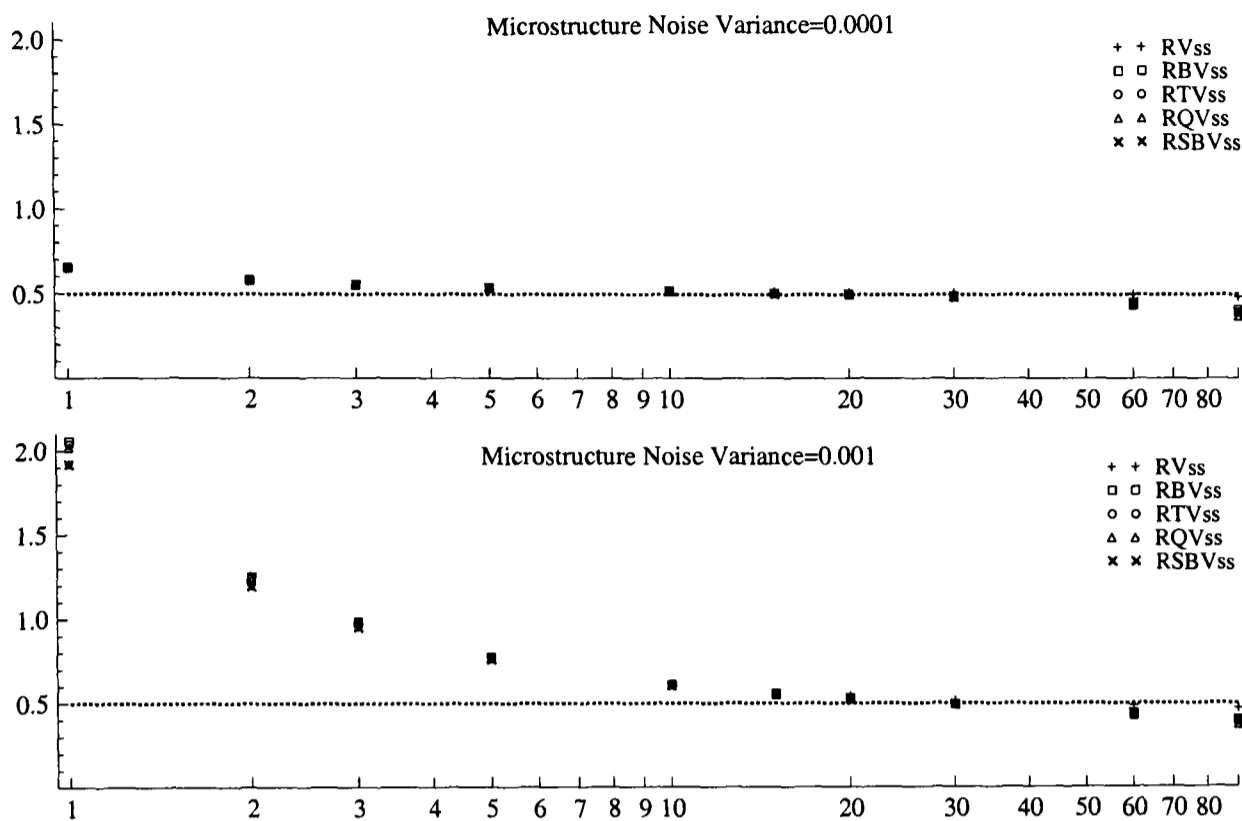


Figure 5.4. Signature plot for the subgridding and averaging estimators when microstructure noise is present.

Signature plots in logarithmic scale (Figure 5.4) show that the bias is still a problem when very high frequencies are used to calculate the estimators. Nevertheless, it appears that the use of subgrids and averaging reduces the bias. By comparing the upper graph in Figure 5.4 to the upper graph in Figure 5.3, we can see that now all the estimators seem robust to microstructure noise when using five minutes returns. There is still some bias for frequencies larger than two minutes but not as much as with the standard estimators. From the lower graph, where the variance of the microstructure noise is large, notice that the bias is still quite important for high frequencies but this time just for sampling frequencies above fifteen minutes. As in Figure 5.3, here again there is no obvious difference between the estimators. Although subgridding and averaging lowers the bias due to autocorrelation in high-frequency data, it increases the bias due to discretization in low frequency data. With ninety minutes returns a bias can be observed, certainly because of the subsampling.

5.3.6 Finite sample behaviour

To reinforce the results, Tables 5.4, 5.5 and 5.6 give an alternative and more complete view of the analysis. By using the asymptotic distributions of the estimators calculated using subgridding and averaging (results 1, 2, 3 and 4) we can compare these new estimators with the standard ones more

precisely. These tables report the bias, standard deviation and coverage rate of the infeasible standardised t-statistics.

M	RV			RV _{ss}			RBV			RBV _{ss}		
	Bias	SD	Cov	Bias	SD	Cov	Bias	SD	Cov	Bias	SD	Cov
0												
12	-.018	1.01	95.3	-.155	.995	96.4	-.172	.999	96.8	-.422	.953	97.2
72	-.047	1.03	94.5	-.082	.995	95.4	-.081	1.02	94.9	-.109	1.03	93.9
144	-.033	1.00	95.2	-.081	1.01	95.7	-.080	.962	96.4	-.086	1.02	94.5
288	-.013	.984	95.5	-.043	.999	96.2	-.027	.976	95.5	-.067	.996	95.8
720	-.004	.955	95.7	-.014	.982	95.7	-.015	.983	94.8	-.040	.974	95.5
1440	.041	.988	95.3	0.017	.960	96.1	.008	.978	95.3	.013	.981	95.6
10 ⁻⁴												
12	.040	1.04	95.1	-.147	.999	96.3	-.150	1.01	96.3	-.418	.952	97.3
72	.141	1.06	93.6	-.007	1.00	95.2	.082	1.05	94.3	-.046	1.03	93.6
144	.508	1.07	90.6	.145	1.03	94.6	.406	1.04	91.6	.123	1.05	94.2
288	1.53	1.15	63.5	.580	1.04	89.2	1.33	1.13	72.5	.512	1.05	90.1
720	5.99	1.91	0.2	2.42	1.21	35.8	5.39	1.88	1.7	2.23	1.21	42.3
1440	17.0	4.23	0	6.94	1.98	0	15.6	4.05	0	6.55	1.96	0.1
10 ⁻³												
12	.176	1.09	93.4	-.097	1.02	95.5	-.031	1.07	95.0	-.386	.966	97.1
72	1.85	1.41	57.1	.683	1.12	86.1	1.59	1.41	63.3	.571	1.14	88.5
144	5.36	2.11	2.8	2.13	1.34	47.1.4	4.85	2.10	6.1	1.98	1.36	52.4
288	15.3	4.11	0	6.16	2.05	0.6	14.2	4.04	0	5.87	2.11	1.5
720	60.1	14.6	0	24.4	6.05	0	57.5	14.5	0	24.1	6.22	0
1440	169	39.4	0	69.3	16.2	0	164	38.9	0	69.4	16.8	0

Table 5.4: Bias, standard deviation and coverage rate of the infeasible standardised realised variance error, the infeasible standardised realised bipower variation error and the subsampled ones in the absence and presence of microstructure noise.

M	RTV			RTV _{ss}			RQV			RQV _{ss}		
	Bs	SD	Cov	Bs	SD	Cov	Bs	SD	Cov	Bs	SD	Cov
0												
12	-.194	.998	96.4	-.518	.877	97.7	-.217	.997	96.9	-.707	.706	99.2
72	-.072	1.03	94.2	-.095	1.04	94.3	-.069	1.04	94.6	-.089	1.05	93.9
144	-.075	.963	96.2	-.073	1.05	94.5	-.070	.974	95.8	-.072	1.05	94.8
288	-.024	.990	95.1	-.057	1.00	95.3	-.026	1.01	94.6	-.052	1.02	94.7
720	-.003	.991	94.7	-.032	.964	95.9	.006	.993	94.9	-.023	.960	96.1
1440	.010	.982	95.5	.022	.980	95.8	.006	.980	95.5	.028	.985	95.3
10 ⁻⁴												
12	-.169	1.02	96.4	-.513	.875	97.7	-.193	1.02	96.4	-.699	.705	99.3
72	.078	1.06	93.7	-.037	1.04	94.2	.074	1.06	93.9	-.033	1.04	94.1
144	.391	1.04	91.4	.122	1.07	93.6	.384	1.05	90.9	.118	1.08	93.5
288	1.23	1.14	74.3	.497	1.05	90.8	1.17	1.15	77.0	.489	1.07	89.9
720	4.96	1.81	2.8	2.09	1.19	46.7	4.69	1.77	4.6	2.02	1.19	49.0
1440	14.3	3.78	0	6.13	1.89	0.1	13.55	3.62	0	5.87	1.83	0.4
10 ⁻³												
12	-.053	1.08	94.7	-.486	.889	97.6	-.080	1.09	95.1	-.682	.710	99.0
72	1.45	1.39	66.4	.524	1.15	89.1	1.39	1.38	69.5	.497	1.14	88.7
144	4.48	2.03	8.3	1.85	1.37	56.5	4.25	1.99	11.3	1.77	1.37	58.3
288	13.2	3.82	0	5.49	2.06	2.3	12.2	3.66	0	5.25	2.03	3.4
720	52.5	13.3	0	22.4	5.86	0	49.4	12.6	0	21.3	5.64	0
1440	150	35.5	0	64.3	15.6	0	141	33.3	0	61.3	14.9	0

Table 5.5: Bias, standard deviation and coverage rate of the infeasible standardised realised tripower variation error, the infeasible standardised realised quadpower variation error and the subsampled ones in the absence and presence of microstructure noise.

M	RSBV			RSBV _{ss}		
	Bs	SD	Cov	Bs	SD	Cov
0						
12	-.230	.968	97.1	-.551	.848	98.5
72	-.093	1.05	94.3	-.113	1.04	94.6
144	-.068	.994	94.9	-.095	1.07	94.3
288	-.023	1.01	94.6	-.057	1.03	93.5
720	.001	.970	95.2	-.010	.974	95.2
1440	.022	.991	95.2	.022	.975	95.2
10^{-4}						
12	-.208	.981	97.0	-.546	.848	98.3
72	.075	1.06	93.6	-.048	1.05	94.7
144	.429	1.06	91.0	.115	1.09	92.9
288	1.31	1.68	71.5	.526	1.07	89.7
720	5.22	1.80	1.5	2.22	1.21	43.0
1440	14.8	3.81	0	6.34	1.88	0.1
10^{-3}						
12	-.095	1.05	96.1	-.512	.863	98.1
72	1.54	1.38	65.1	.552	1.15	88.5
144	4.66	1.99	7.3	1.92	1.38	54.2
288	13.3	3.78	0	5.61	1.99	1.5
720	52.5	12.9	0	22.3	5.68	0
1440	148	34.5	0	63.3	15.0	0

Table 5.6: *Bias, standard deviation and coverage rate of the infeasible standardised skipped version of realised bipower variation error and the subsampled ones in the absence and presence of microstructure noise.*

As revealed by the asymptotic distribution, our new estimators are less efficient than the standard ones. The tables confirm this fact. When there is no market microstructure noise, the standard estimators give better results than the new estimators, i.e., they are less biased and their standard deviation is closer to one. When microstructure noise is added to the price series, the new estimators seem to be more robust for higher frequencies. When the variance of the noise equals 0.0001, the bias, standard deviation and coverage rate of the new estimators exhibit a significant improvement on frequencies above $M = 144$. Nevertheless for $M = 720$, even if the new estimators behave better

than the standard ones, the bias is too big. In the case when the variance of the noise is 0.001 the new estimators will be preferred for values of M higher than 24, but the noise will completely hide the process for frequencies higher than $M = 144$. Irrespectively of the size of the microstructure noise, the estimators based on realised tripower variation and realised quadpower variation give the best results although the differences between all the estimators are very subtle.

5.4 Comparison of the bias of the different estimators caused by market microstructure effects

When sampling at the highest frequencies, i.e. when δ becomes asymptotically small, market microstructure effects can invalidate the estimation of actual volatility. When working with very high frequencies the return process will be very heavily contaminated and hide the true variation of the log-prices. Although the efficiency of all the estimators studied in this thesis is based on asymptotic conditions ($\delta \downarrow 0$), the existence of microstructure noise introduces a bias due to noise accumulation. Here we will try to determine the bias of each of our estimators when market microstructure noise is present.

In order to obtain analytical results we need to use a simplified version of the model that incorporates microstructure noise into our price process. To this effect, let us define here the contaminated returns as

$$\tilde{y}_j = y_j + \varepsilon_j$$

for $j = 1, 2, \dots, \lfloor t/\delta \rfloor$ where y_j is the true return and ε_j is the market microstructure noise. Notice that the difference between this and the original model (Section 5.1.1) is that the noise is added to the returns and not to the log-prices.

Let us impose the following assumptions

- (1) The random shocks ε_j are Gaussian i.i.d. with mean zero and variance $w^2\Psi_j$ where w is a constant and $\Psi_j = \frac{1}{\delta} \int_{\delta(j-1)}^{\delta j} \sigma_u^2 du$.
- (2) The true returns are Gaussian i.i.d. with mean zero and variance $\delta\Psi_j$.
- (3) $y_j \perp \varepsilon_j$.

The bias of the realised variance, bipower, tripower and quadpower variation can be obtained from the following expressions. The derivation of these results can be consulted on the appendix A5.2. Let

us set $K = \delta^{-1/2}w$ then

$$E([Y_\delta]_t^{[2]}) = (1 + K^2)\delta \sum_{j=1}^{\lfloor t/\delta \rfloor} \Psi_j$$

$$\xrightarrow{p} \int_0^t \sigma_u^2 du + bias_{rv}$$

$$E([Y_\delta]_t^{[1,1]}) = \mu_1^2(1 + K^2)\delta \sum_{j=1}^{\lfloor t/\delta \rfloor - 1} \Psi_j^{1/2} \Psi_{j+1}^{1/2}$$

$$\xrightarrow{p} \int_0^t \sigma_u^2 du + bias_{bv}$$

$$E([Y_\delta]_t^{[2/3,2/3,2/3]}) = \mu_{2/3}^3(1 + K^2)\delta \sum_{j=1}^{\lfloor t/\delta \rfloor - 2} \Psi_j^{1/3} \Psi_{j+1}^{1/3} \Psi_{j+2}^{1/3}$$

$$\xrightarrow{p} \int_0^t \sigma_u^2 du + bias_{tv}$$

$$E([Y_\delta]_t^{[1/2,1/2,1/2,1/2]}) = \mu_{1/2}^4(1 + K^2)\delta \sum_{j=1}^{\lfloor t/\delta \rfloor - 3} \Psi_j^{1/4} \Psi_{j+1}^{1/4} \Psi_{j+2}^{1/4} \Psi_{j+3}^{1/4}$$

$$\xrightarrow{p} \int_0^t \sigma_u^2 du + bias_{qv}$$

The bias of the sampling and averaging version of our estimators are given below. In appendix A5.2 the derivation of these results are also given.

$$E([Y_\delta]_t^{[2]^{(avg)}}) = (1 + K^2)1/2 \left(\delta \sum_{j=1}^{\lfloor t/2\delta \rfloor - 1} (\Psi_{2j} + \Psi_{2j+1}) + \delta \sum_{j=1}^{\lfloor t/2\delta \rfloor - 2} (\Psi_{2j+1} + \Psi_{2j+2}) \right)$$

$$\xrightarrow{p} \int_0^t \sigma_u^2 du + bias_{rvss}$$

$$E([Y_\delta]_t^{[1,1]^{(avg)}}) = \mu_1^2(1 + K^2)2\delta \left(\sum_{j=1}^{\lfloor t/2\delta \rfloor - 2} \Psi_{2j}^{1/2} \Psi_{2j+2}^{1/2} + \sum_{j=1}^{\lfloor t/2\delta \rfloor - 3} \Psi_{2j+1}^{1/2} \Psi_{2j+3}^{1/2} \right)$$

$$\xrightarrow{p} \int_0^t \sigma_u^2 du + bias_{bvss}$$

$$E([Y_\delta]_t^{[2/3,2/3,2/3]^{(avg)}}) = \mu_{2/3}^3(1 + K^2)2\delta \left(\sum_{j=1}^{\lfloor t/2\delta \rfloor - 3} \Psi_{2j}^{1/3} \Psi_{2j+2}^{1/3} \Psi_{2j+4}^{1/3} + \sum_{j=1}^{\lfloor t/2\delta \rfloor - 4} \Psi_{2j+1}^{1/3} \Psi_{2j+3}^{1/3} \Psi_{2j+5}^{1/3} \right)$$

$$\xrightarrow{p} \int_0^t \sigma_u^2 du + bias_{tvss}$$

$$E([Y_\delta]_t^{[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}]}(avg)) = \mu_{\frac{1}{2}}^4 (1+K^2) 2\delta \left(\sum_{j=1}^{\lfloor t/2\delta \rfloor - 4} \Psi_{2j}^{1/4} \Psi_{2j+2}^{1/4} \Psi_{2j+4}^{1/4} \Psi_{2j+6}^{1/4} + \sum_{j=1}^{\lfloor t/2\delta \rfloor - 5} \Psi_{2j+1}^{1/4} \Psi_{2j+3}^{1/4} \Psi_{2j+5}^{1/4} \Psi_{2j+7}^{1/4} \right)$$

$$\xrightarrow{p} \int_0^t \sigma_u^2 du + bias_{qvss}$$

As previously described, we can observe how the bias grows by sampling more and more frequently. In all the cases we can observe how the bias explodes as the sampling frequency increases; as $\delta \downarrow 0$ the bias tends to infinity. Observe that the bias depends on the sampling frequency and on the variance of the noise added to the returns.

Here we obtained that

$$bias_{rv} \sim bias_{bv} \sim bias_{tv} \sim bias_{qv},$$

if we define that $a_\delta \sim b_\delta$ if $\frac{a_\delta}{b_\delta} \rightarrow 1$ when $\delta \rightarrow 0$. Similarly we obtained that

$$bias_{rvss} \sim bias_{bvss} \sim bias_{tvss} \sim bias_{qvss}.$$

A different relationship would have been expected. The use of adjacent observation is thought to reduce the bias. The problem is that we are using a simplified version of the original model that incorporates the market microstructure effects. By adding the noise to the returns and not to the prices, we are not including into our model the complete accumulation of noise. Nevertheless, these results give an preliminary insight of the problem.

5.5 Conclusions

The aim of this chapter was to introduce market microstructure effects and determine how they affect the estimation of the actual variance when using high-frequency data. It is well known that estimators of actual variance become biased in the presence of market frictions when increasing the sampling frequency, nevertheless their asymptotic properties oblige us to sample at the highest possible frequency. Alternative approaches are needed to overcome the bias-variance trade-off.

When high-frequency data is available, the usual financial practice is to sample sparsely to reduce the bias caused by the market microstructure effect in the realised variance. In this chapter we claimed that using adjacent observations, i.e. realised bipower, tripower and quadpower variation reduces this bias without throwing away too much information. Although these estimators improved the results given by realised variance, the market microstructure noise can completely hide the true process if its variance is big enough. Therefore we constructed alternative estimators based on the sampling

and averaging technique introduced by Zhang, Mykland and Aït-Sahalia (2005). These estimators are considerably more robust to the noise although a bias still exists when using very high frequencies.

Recently a lot of research is being focused on market microstructure noise, its effects, its quantification and correction. Here we just gave some alternative estimators that improved the use of realised variance, although they did not turn out to be completely robust to the noise. Further research is needed in this area.

5.6 Appendix: Derivations

A5.1 Derivation of the asymptotic distributions for the subsampling and averaged based estimators

For the derivation of the asymptotic distributions of this chapter we will set the assumptions in Barndoff-Nielsen, Graversen, Jacod, Podolskij and Shephard (2005) and Barndoff-Nielsen, Graversen, Jacod and Shephard (2006) to hold. Also we set $A = 0$.

Derivation for Realised Variance

We need to find the covariance matrix of

$$\sqrt{\frac{1}{2\delta}} \begin{pmatrix} [Y_{2\delta}^{(1)}]_t^{[2]} - \int_0^t \sigma_s^2 ds \\ [Y_{2\delta}^{(2)}]_t^{[2]} - \int_0^t \sigma_s^2 ds \end{pmatrix}.$$

Notice the following results when $\delta \downarrow 0$,

$$\begin{aligned} & \sqrt{\frac{1}{2\delta}} \left([Y_{2\delta}^{(1)}]_t^{[2]} - \int_0^t \sigma_s^2 ds \right) \\ & \stackrel{law}{=} \sqrt{\frac{1}{2\delta}} \left(\sum_{j=0}^{\lfloor t/2\delta \rfloor - 2} (\sigma_{\delta,2j} \epsilon_{\delta,2j} + \sigma_{\delta,2j+1} \epsilon_{\delta,2j+1})^2 - (\sigma_{\delta,2j}^2 + \sigma_{\delta,2j+1}^2) \right) \\ & = \sqrt{\frac{1}{2\delta}} \left(\sum_{j=0}^{\lfloor t/2\delta \rfloor - 2} ((\sigma_{\delta,2j}^2 (\epsilon_{\delta,2j}^2 - 1) + \sigma_{\delta,2j+1}^2 (\epsilon_{\delta,2j+1}^2 - 1) + 2\sigma_{\delta,2j} \sigma_{\delta,2j+1} \epsilon_{\delta,2j} \epsilon_{\delta,2j+1})) \right) \end{aligned}$$

$$\frac{1}{2\delta} \sum_{j=0}^{\lfloor t/2\delta \rfloor - 2} \sigma_{2\delta,j}^4 = \frac{1}{2\delta} \sum_{j=0}^{\lfloor t/2\delta \rfloor - 2} (\sigma_{\delta,2j}^2 + \sigma_{\delta,2j+1}^2)^2$$

$$\begin{aligned}
&= \frac{1}{2\delta} \sum_{j=0}^{\lfloor t/2\delta \rfloor - 2} (\sigma_{\delta,2j}^4 + \sigma_{\delta,2j+1}^4 + 2\sigma_{\delta,2j}^2\sigma_{\delta,2j+1}^2) \\
&\xrightarrow{p} \int_0^t \sigma_s^4 ds.
\end{aligned}$$

This comes from the fact that

$$\begin{aligned}
\frac{1}{2\delta} \sum_{j=0}^{\lfloor t/2\delta \rfloor - 2} (\sigma_{\delta,2j}^4 + \sigma_{\delta,2j+1}^4 + 2\sigma_{\delta,2j}^2\sigma_{\delta,2j+1}^2) &= \frac{1}{2} \left(\frac{1}{\delta} \sum_{j=0}^{\lfloor t/2\delta \rfloor - 2} (\sigma_{\delta,2j}^4 + \sigma_{\delta,2j+1}^4) \right) + \frac{1}{2} \left(\frac{1}{\delta} \sum_{j=0}^{\lfloor t/2\delta \rfloor - 2} 2\sigma_{\delta,2j}^2\sigma_{\delta,2j+1}^2 \right) \\
&\xrightarrow{p} 1/2 \int_0^t \sigma_s^4 ds + 1/2 \int_0^t \sigma_s^4 ds
\end{aligned}$$

as

$$\begin{aligned}
\frac{1}{\delta} \sum_{j=0}^{\lfloor t/\delta \rfloor - 1} \sigma_{\delta,j}^4 &= \frac{1}{\delta} \sum_{j=0}^{\lfloor t/2\delta \rfloor - 2} (\sigma_{\delta,2j}^4 + \sigma_{\delta,2j+1}^4) \\
&\xrightarrow{p} \int_0^t \sigma_s^4 ds
\end{aligned}$$

and

$$\begin{aligned}
\frac{1}{\delta} \sum_{j=0}^{\lfloor t/\delta \rfloor - 2} \sigma_{\delta,j}^2\sigma_{\delta,j+1}^2 &= \frac{1}{\delta} \left(\sum_{j=0}^{\lfloor t/2\delta \rfloor - 2} \sigma_{\delta,2j}^2\sigma_{\delta,2j+1}^2 + \sum_{j=0}^{\lfloor t/2\delta \rfloor - 2} \sigma_{\delta,2j+1}^2\sigma_{\delta,2j+2}^2 \right) \\
&\xrightarrow{p} \int_0^t \sigma_s^4 ds.
\end{aligned}$$

With the previous results it is easy to obtain the variance,

$$\begin{aligned}
&\text{Var} \left(\sqrt{\frac{1}{2\delta}} \left([Y_{2\delta}^{(1)}]_t^{[2]} - \int_0^t \sigma_s^2 ds \right) \right) \\
&\stackrel{law}{=} \frac{1}{2\delta} \sum_{j=0}^{\lfloor t/2\delta \rfloor - 2} \sigma_{\delta,2j}^4 \text{Var}(\epsilon_{\delta,2j}^2 - 1) + \sigma_{\delta,2j+1}^4 \text{Var}(\epsilon_{\delta,2j+1}^2 - 1) + 4\sigma_{\delta,2j}^2\sigma_{\delta,2j+1}^2 \text{Var}(\epsilon_{\delta,2j}\epsilon_{\delta,2j+1}) \\
&= 2 \left(\frac{1}{2\delta} \sum_{j=0}^{\lfloor t/2\delta \rfloor - 2} \sigma_{\delta,2j}^4 + \sigma_{\delta,2j+1}^4 + 2\sigma_{\delta,2j}^2\sigma_{\delta,2j+1}^2 \right) \\
&\xrightarrow{p} 2 \int_0^t \sigma_s^4 ds.
\end{aligned}$$

Analogously, we can obtain that

$$\text{Var} \left(\sqrt{\frac{1}{2\delta}} \left([Y_{2\delta}^{(2)}]_t^{[2]} - \int_0^t \sigma_s^2 ds \right) \right) \xrightarrow{p} 2 \int_0^t \sigma_s^4 ds.$$

For the covariance, we have that

$$\begin{aligned} \text{Cov} & \left(\sqrt{\frac{1}{2\delta}} \left([Y_{2\delta}^{(1)}]_t^{[2]} - \int_0^t \sigma_s^2 ds \right), \sqrt{\frac{1}{2\delta}} \left([Y_{2\delta}^{(2)}]_t^{[2]} - \int_0^t \sigma_s^2 ds \right) \right) \\ &= E \left(\sqrt{\frac{1}{2\delta}} \left([Y_{2\delta}^{(1)}]_t^{[2]} - \int_0^t \sigma_s^2 ds \right) \cdot \sqrt{\frac{1}{2\delta}} \left([Y_{2\delta}^{(2)}]_t^{[2]} - \int_0^t \sigma_s^2 ds \right) \right) - \\ & E \left(\sqrt{\frac{1}{2\delta}} \left([Y_{2\delta}^{(1)}]_t^{[2]} - \int_0^t \sigma_s^2 ds \right) \right) E \left(\sqrt{\frac{1}{2\delta}} \left([Y_{2\delta}^{(2)}]_t^{[2]} - \int_0^t \sigma_s^2 ds \right) \right) \\ &\stackrel{\text{law}}{=} \frac{1}{2\delta} E \left(\left\{ \left(\sigma_{\delta,0}^2 (\epsilon_{\delta,0}^2 - 1) + \sigma_{\delta,1}^2 (\epsilon_{\delta,1}^2 - 1) + 2\sigma_{\delta,0}\sigma_{\delta,1}\epsilon_{\delta,0}\epsilon_{\delta,1} \right) + \left(\sigma_{\delta,2}^2 (\epsilon_{\delta,2}^2 - 1) + \sigma_{\delta,3}^2 (\epsilon_{\delta,3}^2 - 1) \right. \right. \right. \\ & \left. \left. \left. + 2\sigma_{\delta,2}\sigma_{\delta,3}\epsilon_{\delta,2}\epsilon_{\delta,3} \right) + \dots \right\} \cdot \left\{ \left(\sigma_{\delta,1}^2 (\epsilon_{\delta,1}^2 - 1) + \sigma_{\delta,2}^2 (\epsilon_{\delta,2}^2 - 1) + 2\sigma_{\delta,1}\sigma_{\delta,2}\epsilon_{\delta,1}\epsilon_{\delta,2} \right) \right. \right. \\ & \left. \left. \left. + \left(\sigma_{\delta,3}^2 (\epsilon_{\delta,3}^2 - 1) + \sigma_{\delta,4}^2 (\epsilon_{\delta,4}^2 - 1) + 2\sigma_{\delta,3}\sigma_{\delta,4}\epsilon_{\delta,3}\epsilon_{\delta,4} \right) + \dots \right\} \right) \\ &= \frac{1}{2\delta} \sum_{j=0}^{\lfloor t/2\delta \rfloor - 3} \sigma_{\delta,2j}^4 E(\epsilon_{\delta,2j}^2 - 1)^2 + \sigma_{\delta,2j+1}^4 E(\epsilon_{\delta,2j+1}^2 - 1)^2 \\ &= 2 \frac{1}{2\delta} \left(\sum_{j=0}^{\lfloor t/2\delta \rfloor - 3} \sigma_{\delta,2j}^4 + \sigma_{\delta,2j+1}^4 \right) \\ &\xrightarrow{p} \int_0^t \sigma_s^4 ds \end{aligned}$$

Then

$$\text{Cov} \left(\sqrt{\frac{1}{2\delta}} \left([Y_{2\delta}^{(1)}]_t^{[2]} - \int_0^t \sigma_s^2 ds \right), \sqrt{\frac{1}{2\delta}} \left([Y_{2\delta}^{(2)}]_t^{[2]} - \int_0^t \sigma_s^2 ds \right) \right) \xrightarrow{p} \int_0^t \sigma_s^4 ds$$

Derivation for Realised Bipower Variance

We need to find the covariance matrix of

$$\sqrt{\frac{1}{2\delta}} \begin{pmatrix} \mu_1^{-2} [Y_{2\delta}^{(1)}]_t^{[1,1]} - \int_0^t \sigma_s^2 ds \\ \mu_1^{-2} [Y_{2\delta}^{(2)}]_t^{[1,1]} - \int_0^t \sigma_s^2 ds \end{pmatrix}$$

If $\delta \downarrow 0$, we have that

$$\text{Var} \left(\sqrt{\frac{1}{2\delta}} \begin{pmatrix} \mu_1^{-2} [Y_{2\delta}^{(1)}]_t^{[1,1]} - \int_0^t \sigma_s^2 ds \\ \mu_1^{-2} [Y_{2\delta}^{(2)}]_t^{[1,1]} - \int_0^t \sigma_s^2 ds \end{pmatrix} \right)$$

$$\begin{aligned}
&\stackrel{law}{=} \frac{1}{2\delta} \mu_1^{-4} Var \left(\sum_{j=0}^{\lfloor t/2\delta \rfloor - 4} |y_{2\delta,j}^{(1)}| |y_{2\delta,j+1}^{(1)}| - \mu_1^2 \sum_{j=0}^{\lfloor t/2\delta \rfloor - 4} \sigma_{2\delta,j} \sigma_{2\delta,j+1} \right) \\
&\stackrel{law}{=} \frac{1}{2\delta} \mu_1^{-4} Var \left(\sum_{j=0}^{\lfloor t/2\delta \rfloor - 4} |\sigma_{2\delta,j} \epsilon_{2\delta,j}| |\sigma_{2\delta,j+1} \epsilon_{2\delta,j+1}| - \mu_1^2 \sum_{j=0}^{\lfloor t/2\delta \rfloor - 4} \sigma_{2\delta,j} \sigma_{2\delta,j+1} \right) \\
&= \frac{1}{2\delta} \mu_1^{-4} Var \left(\sum_{j=0}^{\lfloor t/2\delta \rfloor - 4} |\sigma_{\delta,2j} \epsilon_{\delta,2j} + \sigma_{\delta,2j+1} \epsilon_{\delta,2j+1}| |(\sigma_{\delta,2j+2} \epsilon_{\delta,2j+2} + \sigma_{\delta,2j+3} \epsilon_{\delta,2j+2})| \right. \\
&\quad \left. - \mu_1^2 (\sigma_{\delta,2j}^2 + \sigma_{\delta,2j+1}^2)^{1/2} (\sigma_{\delta,2j+2}^2 + \sigma_{\delta,2j+3}^2)^{1/2} \right)
\end{aligned}$$

Assume that as $\delta \downarrow 0$ then $\sigma_{\delta,2j} = \sigma_{\delta,2j+1}$, therefore

$$\begin{aligned}
&\frac{1}{2\delta} \mu_1^{-4} Var \left(\sum_{j=0}^{\lfloor t/2\delta \rfloor - 4} |\sigma_{\delta,2j} \epsilon_{\delta,2j} + \sigma_{\delta,2j+1} \epsilon_{\delta,2j+1}| |\sigma_{\delta,2j+2} \epsilon_{\delta,2j+2} + \sigma_{\delta,2j+3} \epsilon_{\delta,2j+2}| \right. \\
&\quad \left. - \mu_1^2 (\sigma_{\delta,2j}^2 + \sigma_{\delta,2j+1}^2)^{1/2} (\sigma_{\delta,2j+2}^2 + \sigma_{\delta,2j+3}^2)^{1/2} \right) \\
&\stackrel{law}{=} \frac{1}{2\delta} \mu_1^{-4} Var \left(\sum_{j=0}^{\lfloor t/2\delta \rfloor - 4} \sigma_{\delta,2j} \sigma_{\delta,2j+2} |\epsilon_{\delta,j} + \epsilon_{\delta,2j+1}| |\epsilon_{\delta,2j+2} \epsilon_{\delta,2j+3}| - 2\mu_1^2 \sigma_{\delta,2j} \sigma_{\delta,2j+2} \right)
\end{aligned}$$

and

$$\begin{aligned}
\frac{1}{2\delta} \sum_{j=0}^{\lfloor t/2\delta \rfloor - 4} \sigma_{2\delta,j}^2 \sigma_{2\delta,j+1}^2 &= \frac{1}{2\delta} \sum_{j=0}^{\lfloor t/2\delta \rfloor - 4} (\sigma_{\delta,2j}^2 + \sigma_{\delta,2j+1}^2)^{2/2} (\sigma_{\delta,2j+2}^2 + \sigma_{\delta,2j+3}^2)^{2/2} \\
&\stackrel{law}{=} \frac{1}{2\delta} \sum_{j=0}^{\lfloor t/2\delta \rfloor - 4} 4\sigma_{\delta,2j}^2 \sigma_{\delta,2j+2}^2 \\
&\xrightarrow{p} \int_0^t \sigma_s^2 ds.
\end{aligned}$$

The variance can now be obtained

$$\begin{aligned}
&\frac{1}{2\delta} \mu_1^{-4} Var \left(\sum_{j=0}^{\lfloor t/2\delta \rfloor - 4} \sigma_{\delta,2j} \sigma_{\delta,2j+2} |\epsilon_{\delta,j} + \epsilon_{\delta,2j+1}| |\epsilon_{\delta,2j+2} \epsilon_{\delta,2j+3}| - 2\mu_1^2 \sigma_{\delta,2j} \sigma_{\delta,2j+2} \right) \\
&= \frac{1}{2\delta} \mu_1^{-4} \sum_{j=0}^{\lfloor t/2\delta \rfloor - 4} \sigma_{\delta,2j}^2 \sigma_{\delta,2j+2}^2 \left(Var(|\epsilon_{\delta,j} + \epsilon_{\delta,2j+1}| |\epsilon_{\delta,2j+2} \epsilon_{\delta,2j+3}| - 2\mu_1^2) \right. \\
&\quad \left. + 2Cov(|\epsilon_{\delta,j} + \epsilon_{\delta,2j+1}| |\epsilon_{\delta,2j+2} \epsilon_{\delta,2j+3}| - 2\mu_1^2, |\epsilon_{\delta,j+2} + \epsilon_{\delta,2j+3}| |\epsilon_{\delta,2j+4} \epsilon_{\delta,2j+5}| - 2\mu_1^2) \right) \\
&= \frac{1}{2\delta} \mu_1^{-4} (C_1 + 2C_2) \sum_{j=0}^{\lfloor t/2\delta \rfloor - 4} \sigma_{\delta,2j}^2 \sigma_{\delta,2j+2}^2
\end{aligned}$$

$$\begin{aligned} &\xrightarrow{p} 1/4\mu_1^{-4}(C_1 + 2C_2) \int_0^t \sigma_s^4 ds \\ &\simeq 2.592 \int_0^t \sigma_s^4 ds, \end{aligned}$$

where

$$\begin{aligned} C_1 &= \text{Var}(| u' + u'' || u''' + u^{(IV)} |) \\ C_2 &= \text{Cov}(| u' + u'' || u''' + u^{(IV)} |, | u''' + u^{(IV)} || u^{(V)} + u^{(VI)} |). \end{aligned}$$

The same procedure holds to obtain

$$\begin{aligned} \text{Var}\left(\sqrt{\frac{1}{2\delta}}\left(\mu_1^{-2}[Y_{2\delta}^{(2)}]_t^{[1,1]} - \int_0^t \sigma_s^2 ds\right)\right) &\xrightarrow{p} 1/4\mu_1^{-4}(C_1 + 2C_2) \int_0^t \sigma_s^4 ds \\ &\simeq 2.592 \int_0^t \sigma_s^4 ds. \end{aligned}$$

Now we should obtain the covariance. We need to assume that when $\delta \downarrow 0$ then $\sigma_{\delta,2j} = \sigma_{\delta,2j+1}$, so

$$\begin{aligned} &\text{Cov}\left(\sqrt{\frac{1}{2\delta}}\left(\mu_1^{-2}[Y_{2\delta}^{(1)}]_t^{[1,1]} - \int_0^t \sigma_s^2 ds\right), \sqrt{\frac{1}{2\delta}}\left(\mu_1^{-2}[Y_{2\delta}^{(2)}]_t^{[1,1]} - \int_0^t \sigma_s^2 ds\right)\right) \\ &\stackrel{\text{law}}{=} \frac{1}{2\delta}\mu_1^{-4}\text{Cov}\left(\sum_{j=0}^{\lfloor t/2\delta \rfloor - 4} \sigma_{\delta,2j}\sigma_{\delta,2j+2}(|\epsilon_{\delta,2j} + \epsilon_{\delta,2j+1} || \epsilon_{\delta,2j+2}\epsilon_{\delta,2j+3} | - 2\mu_1^2), \right. \\ &\quad \left. \sum_{j=0}^{\lfloor t/2\delta \rfloor - 5} \sigma_{\delta,2j+1}\sigma_{\delta,2j+3}(|\epsilon_{\delta,2j+1} + \epsilon_{\delta,2j+2} || \epsilon_{\delta,2j+3}\epsilon_{\delta,2j+4} | - 2\mu_1^2)\right) \\ &\stackrel{\text{law}}{=} \frac{1}{2\delta}\mu_1^{-4}\sum_{j=0}^{\lfloor t/2\delta \rfloor - 5} \sigma_{\delta,2j}^2\sigma_{\delta,2j+2}^2\left(2\text{Cov}(|\epsilon_{\delta,2j} + \epsilon_{\delta,2j+1} || \epsilon_{\delta,2j+2}\epsilon_{\delta,2j+3} | - 2\mu_1^2, \right. \\ &\quad |\epsilon_{\delta,2j+1} + \epsilon_{\delta,2j+2} || \epsilon_{\delta,2j+3}\epsilon_{\delta,2j+4} | - 2\mu_1^2) + 2\text{Cov}(|\epsilon_{\delta,2j} + \epsilon_{\delta,2j+1} || \epsilon_{\delta,2j+2}\epsilon_{\delta,2j+3} | - 2\mu_1^2, \\ &\quad \left. |\epsilon_{\delta,2j+3} + \epsilon_{\delta,2j+4} || \epsilon_{\delta,2j+5}\epsilon_{\delta,2j+6} | - 2\mu_1^2)\right) \\ &= \frac{1}{2\delta}\mu_1^{-4}\sum_{j=0}^{\lfloor t/2\delta \rfloor - 5} \sigma_{\delta,2j}\sigma_{\delta,2j+2}(2C_3 + 2C_4) \\ &\xrightarrow{p} 1/4\mu_1^{-4}(2C_3 + 2C_4) \int_0^t \sigma_s^4 ds \\ &\simeq 0.988 \int_0^t \sigma_s^4 ds \end{aligned}$$

where

$$\begin{aligned} C_3 &= \text{Cov}(| u' + u'' || u''' + u^{(IV)} |, | u'' + u''' || u^{(IV)} + u^{(V)} |) \\ C_4 &= \text{Cov}(| u' + u'' || u''' + u^{(IV)} |, | u^{(IV)} + u^{(V)} || u^{(VI)} + u^{(VII)} |). \end{aligned}$$

Derivation for Realised Tripower Variance

We need to find the covariance matrix of

$$\sqrt{\frac{1}{2\delta}} \begin{pmatrix} \mu_{2/3}^{-3} [Y_{2\delta}^{(1)}]_t^{[2/3,2/3,2/3]} - \int_0^t \sigma_s^2 ds \\ \mu_{2/3}^{-3} [Y_{2\delta}^{(2)}]_t^{[2/3,2/3,2/3]} - \int_0^t \sigma_s^2 ds \end{pmatrix}$$

If $\delta \downarrow 0$, we have that

$$\begin{aligned} & \text{Var} \left(\sqrt{\frac{1}{2\delta}} \left(\mu_{2/3}^{-3} [Y_{2\delta}^{(1)}]_t^{[2/3,2/3,2/3]} - \int_0^t \sigma_s^2 ds \right) \right) \\ & \stackrel{\text{law}}{=} \frac{1}{2\delta} \mu_{2/3}^{-6} \text{Var} \left(\sum_{j=0}^{\lfloor t/2\delta \rfloor - 6} |y_{2\delta,j}^{(1)}|^{2/3} |y_{2\delta,j+1}^{(1)}|^{2/3} |y_{2\delta,j+2}^{(1)}|^{2/3} - \mu_{2/3}^3 \sum_{j=0}^{\lfloor t/2\delta \rfloor - 6} \sigma_{2\delta,j}^{2/3} \sigma_{2\delta,j+1}^{2/3} \sigma_{2\delta,j+2}^{2/3} \right) \\ & \stackrel{\text{law}}{=} \frac{1}{2\delta} \mu_{2/3}^{-6} \text{Var} \left(\sum_{j=0}^{\lfloor t/2\delta \rfloor - 6} |\sigma_{2\delta,j} \epsilon_{2\delta,j}|^{2/3} |\sigma_{2\delta,j+1} \epsilon_{2\delta,j+1}|^{2/3} |\sigma_{2\delta,j+2} \epsilon_{2\delta,j+2}|^{2/3} - \mu_{2/3}^3 \sigma_{2\delta,j}^{2/3} \sigma_{2\delta,j+1}^{2/3} \sigma_{2\delta,j+2}^{2/3} \right) \\ & = \frac{1}{2\delta} \mu_{2/3}^{-6} \text{Var} \left(\sum_{j=0}^{\lfloor t/2\delta \rfloor - 6} |\sigma_{\delta,2j} \epsilon_{\delta,2j} + \sigma_{\delta,2j+1} \epsilon_{\delta,2j+1}|^{2/3} (\sigma_{\delta,2j+2} \epsilon_{\delta,2j+2} + \sigma_{\delta,2j+3} \epsilon_{\delta,2j+3})^{2/3} \right. \\ & \quad \left. |(\sigma_{\delta,2j+4} \epsilon_{\delta,2j+4} + \sigma_{\delta,2j+5} \epsilon_{\delta,2j+5})|^{2/3} - \mu_{2/3}^3 (\sigma_{\delta,2j}^2 + \sigma_{\delta,2j+1}^2)^{2/6} (\sigma_{\delta,2j+2}^2 + \sigma_{\delta,2j+3}^2)^{2/6} \right. \\ & \quad \left. (\sigma_{\delta,2j+4}^2 + \sigma_{\delta,2j+5}^2)^{2/6} \right) \end{aligned}$$

Assume that as $\delta \downarrow 0$ then $\sigma_{\delta,2j} = \sigma_{\delta,2j+1}$, so we obtain that

$$\begin{aligned} & \frac{1}{2\delta} \mu_{2/3}^{-6} \text{Var} \left(\sum_{j=0}^{\lfloor t/2\delta \rfloor - 6} |\sigma_{\delta,2j} \epsilon_{\delta,2j} + \sigma_{\delta,2j+1} \epsilon_{\delta,2j+1}|^{2/3} |\sigma_{\delta,2j+2} \epsilon_{\delta,2j+2} + \sigma_{\delta,2j+3} \epsilon_{\delta,2j+3}|^{2/3} \right. \\ & \quad \left. |(\sigma_{\delta,2j+4} \epsilon_{\delta,2j+4} + \sigma_{\delta,2j+5} \epsilon_{\delta,2j+5})|^{2/3} - \mu_{2/3}^3 (\sigma_{\delta,2j}^2 + \sigma_{\delta,2j+1}^2)^{2/6} (\sigma_{\delta,2j+2}^2 + \sigma_{\delta,2j+3}^2)^{2/6} (\sigma_{\delta,2j+4}^2 + \sigma_{\delta,2j+5}^2)^{2/6} \right) \\ & \stackrel{\text{law}}{=} \frac{1}{2\delta} \mu_{2/3}^{-6} \text{Var} \left(\sum_{j=0}^{\lfloor t/2\delta \rfloor - 6} \sigma_{\delta,2j}^{2/3} \sigma_{\delta,2j+2}^{2/3} \sigma_{\delta,2j+4}^{2/3} (|\epsilon_{\delta,j} + \epsilon_{\delta,2j+1}|^{2/3} |\epsilon_{\delta,2j+2} \epsilon_{\delta,2j+3}|^{2/3} |\epsilon_{\delta,j+4} + \epsilon_{\delta,2j+5}|^{2/3} \right. \\ & \quad \left. - 2\mu_{2/3}^3) \right) \end{aligned}$$

and

$$\begin{aligned} \frac{1}{2\delta} \sum_{j=0}^{\lfloor t/2\delta \rfloor - 6} \sigma_{2\delta,j}^{4/3} \sigma_{2\delta,j+1}^{4/3} \sigma_{2\delta,j+2}^{4/3} & = \frac{1}{2\delta} \sum_{j=0}^{\lfloor t/2\delta \rfloor - 6} (\sigma_{\delta,2j}^2 + \sigma_{\delta,2j+1}^2)^{2/3} (\sigma_{\delta,2j+2}^2 + \sigma_{\delta,2j+3}^2)^{2/3} (\sigma_{\delta,2j+4}^2 + \sigma_{\delta,2j+5}^2)^{2/3} \\ & \stackrel{\text{law}}{=} \frac{1}{2\delta} \sum_{j=0}^{\lfloor t/2\delta \rfloor - 6} 4\sigma_{\delta,2j}^{4/3} \sigma_{\delta,2j+2}^{4/3} \sigma_{\delta,2j+4}^{4/3} \\ & \xrightarrow{p} \int_0^t \sigma_s^4 ds. \end{aligned}$$

So then we find that

$$\begin{aligned}
& \frac{1}{2\delta} \mu_{2/3}^{-6} \text{Var} \left(\sum_{j=0}^{\lfloor t/2\delta \rfloor - 6} \sigma_{\delta,2j}^{2/3} \sigma_{\delta,2j+2}^{2/3} \sigma_{\delta,2j+4}^{2/3} (|\epsilon_{\delta,j} + \epsilon_{\delta,2j+1}|^{2/3} |\epsilon_{\delta,2j+2} \epsilon_{\delta,2j+3}|^{2/3} |\epsilon_{\delta,2j+3} \epsilon_{\delta,2j+4}|^{2/3} \right. \\
& \quad \left. - 2\mu_{2/3}^3) \right) \\
&= \frac{1}{2\delta} \mu_{2/3}^{-6} \sum_{j=0}^{\lfloor t/2\delta \rfloor - 6} \sigma_{\delta,2j}^{4/3} \sigma_{\delta,2j+2}^{4/3} \sigma_{\delta,2j+4}^{4/3} \\
& \quad \times \left(\text{Var}(|\epsilon_{\delta,j} + \epsilon_{\delta,2j+1}|^{2/3} |\epsilon_{\delta,2j+2} \epsilon_{\delta,2j+3}|^{2/3} |\epsilon_{\delta,2j+4} \epsilon_{\delta,2j+5}|^{2/3} - 2\mu_{2/3}^3) \right. \\
& \quad + 2\text{Cov}(|\epsilon_{\delta,j} + \epsilon_{\delta,2j+1}|^{2/3} |\epsilon_{\delta,2j+2} \epsilon_{\delta,2j+3}|^{2/3} |\epsilon_{\delta,2j+4} \epsilon_{\delta,2j+5}|^{2/3} - 2\mu_{2/3}^3, \\
& \quad \quad |\epsilon_{\delta,2j+2} + \epsilon_{\delta,2j+3}|^{2/3} |\epsilon_{\delta,2j+4} \epsilon_{\delta,2j+5}|^{2/3} |\epsilon_{\delta,2j+6} \epsilon_{\delta,2j+7}|^{2/3} - 2\mu_{2/3}^3) \\
& \quad + 2\text{Cov}(|\epsilon_{\delta,j} + \epsilon_{\delta,2j+1}|^{2/3} |\epsilon_{\delta,2j+2} \epsilon_{\delta,2j+3}|^{2/3} |\epsilon_{\delta,2j+4} \epsilon_{\delta,2j+5}|^{2/3} - 2\mu_{2/3}^3, \\
& \quad \quad |\epsilon_{\delta,2j+4} + \epsilon_{\delta,2j+5}|^{2/3} |\epsilon_{\delta,2j+6} \epsilon_{\delta,2j+7}|^{2/3} |\epsilon_{\delta,2j+8} \epsilon_{\delta,2j+9}|^{2/3} - 2\mu_{2/3}^3) \left. \right) \\
&= \frac{1}{2\delta} \mu_{2/3}^{-6} (C_5 + 2C_6 + 2C_7) \sum_{j=0}^{\lfloor t/2\delta \rfloor - 6} \sigma_{\delta,2j}^{4/3} \sigma_{\delta,2j+2}^{4/3} \sigma_{\delta,2j+4}^{4/3} \\
&\stackrel{p}{\rightarrow} 1/4 \mu_{2/3}^{-6} (C_5 + 2C_6 + 2C_6) \int_0^t \sigma_s^4 ds \\
&\simeq 3.049 \int_0^t \sigma_s^4 ds,
\end{aligned}$$

where

$$\begin{aligned}
C_5 &= \text{Var}(|u' + u''|^{2/3} |u''' + u^{(IV)}|^{2/3} |u^{(V)} + u^{(VI)}|^{2/3}) \\
C_6 &= \text{Cov}(|u' + u''|^{2/3} |u''' + u^{(IV)}|^{2/3} |u^{(V)} + u^{(VI)}|^{2/3}, \\
& \quad |u''' + u^{(IV)}|^{2/3} |u^{(V)} + u^{(VI)}|^{2/3} |u^{(VII)} + u^{(VIII)}|^{2/3}). \\
C_7 &= \text{Cov}(|u' + u''|^{2/3} |u''' + u^{(IV)}|^{2/3} |u^{(V)} + u^{(VI)}|^{2/3}, \\
& \quad |u^{(V)} + u^{(VI)}|^{2/3} |u^{(VII)} + u^{(VIII)}|^{2/3} |u^{(IX)} + u^{(X)}|^{2/3}).
\end{aligned}$$

The same procedure holds to obtain

$$\begin{aligned}
\text{Var} \left(\sqrt{\frac{1}{2\delta}} \left(\mu_{2/3}^{-6} [Y_{2\delta}^{(2)}]_t^{[2/3,2/3,2/3]} - \int_0^t \sigma_s^2 ds \right) \right) &\stackrel{p}{\rightarrow} 1/4 \mu_{2/3}^{-6} (C_5 + 2C_6 + 2C_7) \int_0^t \sigma_s^4 ds \\
&\simeq 3.049 \int_0^t \sigma_s^4 ds.
\end{aligned}$$

Now to obtain the covariance, we need to assume that when $\delta \downarrow 0$ then $\sigma_{\delta,2j} = \sigma_{\delta,2j+1}$, so

$$\text{Cov} \left(\sqrt{\frac{1}{2\delta}} \left(\mu_{2/3}^{-3} [Y_{2\delta}^{(1)}]_t^{[2/3,2/3,2/3]} - \int_0^t \sigma_s^2 ds \right), \sqrt{\frac{1}{2\delta}} \left(\mu_{2/3}^{-3} [Y_{2\delta}^{(2)}]_t^{[2/3,2/3,2/3]} - \int_0^t \sigma_s^2 ds \right) \right)$$

$$\begin{aligned}
&\stackrel{law}{=} \frac{1}{2\delta} \mu_{2/3}^{-6} Cov \left(\sum_{j=0}^{\lfloor t/2\delta \rfloor - 6} \sigma_{\delta,2j}^{2/3} \sigma_{\delta,2j+2}^{2/3} \sigma_{\delta,2j+4}^{2/3} (|\epsilon_{\delta,2j} + \epsilon_{\delta,2j+1}|^{2/3} |\epsilon_{\delta,2j+2} + \epsilon_{\delta,2j+3}|^{2/3} |\epsilon_{\delta,2j+4} + \epsilon_{\delta,2j+5}|^{2/3} - 2\mu_{2/3}^3), \right. \\
&\quad \left. \sum_{j=0}^{\lfloor t/2\delta \rfloor - 7} \sigma_{\delta,2j+1}^{2/3} \sigma_{\delta,2j+3}^{2/3} \sigma_{\delta,2j+5}^{2/3} (|\epsilon_{\delta,2j+1} + \epsilon_{\delta,2j+2}|^{2/3} |\epsilon_{\delta,2j+3} \epsilon_{\delta,2j+4}|^{2/3} |\epsilon_{\delta,2j+5} \epsilon_{\delta,2j+6}|^{2/3} - 2\mu_{2/3}^3) \right) \\
&\stackrel{law}{=} \frac{1}{2\delta} \mu_{2/3}^{-6} \sum_{j=0}^{\lfloor t/2\delta \rfloor - 7} \sigma_{\delta,2j}^{4/3} \sigma_{\delta,2j+2}^{4/3} \sigma_{\delta,2j+4}^{4/3} \left(2Cov(|\epsilon_{\delta,2j} + \epsilon_{\delta,2j+1}|^{2/3} |\epsilon_{\delta,2j+2} \epsilon_{\delta,2j+3}|^{2/3} |\epsilon_{\delta,2j+4} \epsilon_{\delta,2j+5}|^{2/3} - 2\mu_{2/3}^3, \right. \\
&\quad |\epsilon_{\delta,2j+1} + \epsilon_{\delta,2j+2}|^{2/3} |\epsilon_{\delta,2j+3} \epsilon_{\delta,2j+4}|^{2/3} |\epsilon_{\delta,2j+5} \epsilon_{\delta,2j+6}|^{2/3} - 2\mu_{2/3}^3) \\
&\quad + 2Cov(|\epsilon_{\delta,2j} + \epsilon_{\delta,2j+1}|^{2/3} |\epsilon_{\delta,2j+2} \epsilon_{\delta,2j+3}|^{2/3} |\epsilon_{\delta,2j+4} \epsilon_{\delta,2j+5}|^{2/3} - 2\mu_{2/3}^3, \\
&\quad |\epsilon_{\delta,2j+3} \epsilon_{\delta,2j+4}|^{2/3} |\epsilon_{\delta,2j+5} \epsilon_{\delta,2j+6}|^{2/3} |\epsilon_{\delta,2j+7} + \epsilon_{\delta,2j+8}|^{2/3} - 2\mu_{2/3}^3) \\
&\quad \left. + 2Cov(|\epsilon_{\delta,2j} + \epsilon_{\delta,2j+1}|^{2/3} |\epsilon_{\delta,2j+2} \epsilon_{\delta,2j+3}|^{2/3} |\epsilon_{\delta,2j+4} \epsilon_{\delta,2j+5}|^{2/3} - 2\mu_{2/3}^3, \right. \\
&\quad \left. |\epsilon_{\delta,2j+5} \epsilon_{\delta,2j+6}|^{2/3} |\epsilon_{\delta,2j+7} + \epsilon_{\delta,2j+8}|^{2/3} |\epsilon_{\delta,2j+9} \epsilon_{\delta,2j+10}|^{2/3} - 2\mu_{2/3}^3) \right) \\
&= \frac{1}{2\delta} \mu_{2/3}^{-6} \sum_{j=0}^{\lfloor t/2\delta \rfloor - 5} \sigma_{\delta,2j}^{4/3} \sigma_{\delta,2j+2}^{4/3} \sigma_{\delta,2j+4}^{4/3} (2C_8 + 2C_9 + 2C_{10}) \\
&\stackrel{p}{\rightarrow} 1/4 \mu_{2/3}^{-6} (2C_8 + 2C_9 + 2C_{10}) \int_0^t \sigma_s^4 ds \\
&\simeq 1.028 \int_0^t \sigma_s^4 ds
\end{aligned}$$

where

$$\begin{aligned}
C_8 &= Cov(|u' + u''|^{2/3} |u''' + u^{(IV)}|^{2/3} |u^{(V)} + u^{(VI)}|^{2/3}, \\
&\quad |u'' + u'''|^{2/3} |u^{(IV)} + u^{(V)}|^{2/3} |u^{(VI)} + u^{(VII)}|^{2/3}) \\
C_9 &= Cov(|u' + u''|^{2/3} |u''' + u^{(IV)}|^{2/3} |u^{(V)} + u^{(VI)}|^{2/3}, \\
&\quad |u^{(IV)} + u^{(V)}|^{2/3} |u^{(VI)} + u^{(VII)}|^{2/3} |u^{(VII)} + u^{(IX)}|^{2/3}) \\
C_{10} &= Cov(|u' + u''|^{2/3} |u''' + u^{(IV)}|^{2/3} |u^{(V)} + u^{(VI)}|^{2/3}, \\
&\quad |u^{(VI)} + u^{(VII)}|^{2/3} |u^{(VIII)} + u^{(IX)}|^{2/3} |u^{(X)} + u^{(XI)}|^{2/3}).
\end{aligned}$$

Derivation for Realised Quadpower Variance

We need to find the covariance matrix of

$$\sqrt{\frac{1}{2\delta}} \begin{pmatrix} \mu_{1/2}^{-4} [Y_{2\delta}^{(1)}]_t^{[1/2, 1/2, 1/2, 1/2]} - \int_0^t \sigma_s^2 ds \\ \mu_{1/2}^{-4} [Y_{2\delta}^{(2)}]_t^{[1/2, 1/2, 1/2, 1/2]} - \int_0^t \sigma_s^2 ds \end{pmatrix}.$$

If $\delta \downarrow 0$, we have that

$$\begin{aligned}
& \text{Var} \left(\sqrt{\frac{1}{2\delta}} \left(\mu_{1/2}^{-4} [Y_{2\delta}^{(1)}]_t^{[1/2, 1/2, 1/2, 1/2]} - \int_0^t \sigma_s^2 ds \right) \right) \\
& \stackrel{\text{law}}{=} \frac{1}{2\delta} \mu_{1/2}^{-8} \text{Var} \left(\sum_{j=0}^{\lfloor t/2\delta \rfloor - 8} \sqrt{|y_{2\delta, j}^{(1)}| |y_{2\delta, j+1}^{(1)}| |y_{2\delta, j+2}^{(1)}| |y_{2\delta, j+3}^{(1)}|} - \mu_{1/2}^4 \sum_{j=0}^{\lfloor t/2\delta \rfloor - 8} \sqrt{\sigma_{2\delta, j} \sigma_{2\delta, j+1} \sigma_{2\delta, j+2} \sigma_{2\delta, j+3}} \right) \\
& \stackrel{\text{law}}{=} \frac{1}{2\delta} \mu_{1/2}^{-8} \text{Var} \left(\sum_{j=0}^{\lfloor t/2\delta \rfloor - 8} |\sigma_{2\delta, j} \epsilon_{2\delta, j}|^{1/2} |\sigma_{2\delta, j+1} \epsilon_{2\delta, j+1}|^{1/2} |\sigma_{2\delta, j+2} \epsilon_{2\delta, j+2}|^{1/2} |\sigma_{2\delta, j+3} \epsilon_{2\delta, j+3}|^{1/2} \right. \\
& \quad \left. - \mu_{1/2}^4 \sigma_{2\delta, j}^{1/2} \sigma_{2\delta, j+1}^{1/2} \sigma_{2\delta, j+2}^{1/2} \sigma_{2\delta, j+3}^{1/2} \right) \\
& = \frac{1}{2\delta} \mu_{1/2}^{-8} \text{Var} \left(\sum_{j=0}^{\lfloor t/2\delta \rfloor - 8} |\sigma_{\delta, 2j} \epsilon_{\delta, 2j} + \sigma_{\delta, 2j+1} \epsilon_{\delta, 2j+1}|^{1/2} (\sigma_{\delta, 2j+2} \epsilon_{\delta, 2j+2} + \sigma_{\delta, 2j+3} \epsilon_{\delta, 2j+3})^{1/2} \right. \\
& \quad |(\sigma_{\delta, 2j+4} \epsilon_{\delta, 2j+4} + \sigma_{\delta, 2j+5} \epsilon_{\delta, 2j+5})^{1/2} (\sigma_{\delta, 2j+6} \epsilon_{\delta, 2j+6} + \sigma_{\delta, 2j+7} \epsilon_{\delta, 2j+7})^{1/2}| \\
& \quad \left. - \mu_{1/2}^4 (\sigma_{\delta, 2j}^2 + \sigma_{\delta, 2j+1}^2)^{1/4} (\sigma_{\delta, 2j+2}^2 + \sigma_{\delta, 2j+3}^2)^{1/4} (\sigma_{\delta, 2j+4}^2 + \sigma_{\delta, 2j+5}^2)^{1/4} (\sigma_{\delta, 2j+6}^2 + \sigma_{\delta, 2j+7}^2)^{1/4} \right)
\end{aligned}$$

Assume that as $\delta \downarrow 0$ then $\sigma_{\delta, 2j} = \sigma_{\delta, 2j+1}$, so

$$\begin{aligned}
& \frac{1}{2\delta} \mu_{1/2}^{-8} \text{Var} \left(\sum_{j=0}^{\lfloor t/2\delta \rfloor - 8} |\sigma_{\delta, 2j} \epsilon_{\delta, 2j} + \sigma_{\delta, 2j+1} \epsilon_{\delta, 2j+1}|^{1/2} |\sigma_{\delta, 2j+2} \epsilon_{\delta, 2j+2} + \sigma_{\delta, 2j+3} \epsilon_{\delta, 2j+3}|^{1/2} \right. \\
& \quad |(\sigma_{\delta, 2j+4} \epsilon_{\delta, 2j+4} + \sigma_{\delta, 2j+5} \epsilon_{\delta, 2j+5})^{1/2} (\sigma_{\delta, 2j+6} \epsilon_{\delta, 2j+6} + \sigma_{\delta, 2j+7} \epsilon_{\delta, 2j+7})^{1/2}| \\
& \quad \left. - \mu_{1/2}^4 (\sigma_{\delta, 2j}^2 + \sigma_{\delta, 2j+1}^2)^{1/4} (\sigma_{\delta, 2j+2}^2 + \sigma_{\delta, 2j+3}^2)^{1/4} (\sigma_{\delta, 2j+4}^2 + \sigma_{\delta, 2j+5}^2)^{1/4} (\sigma_{\delta, 2j+6}^2 + \sigma_{\delta, 2j+7}^2)^{1/4} \right) \\
& \stackrel{\text{law}}{=} \frac{1}{2\delta} \mu_{1/2}^{-8} \text{Var} \left(\sum_{i=0}^{\lfloor t/2\delta \rfloor - 6} \sigma_{\delta, 2j}^{1/2} \sigma_{\delta, 2j+2}^{1/2} \sigma_{\delta, 2j+4}^{1/2} \sigma_{\delta, 2j+6}^{1/2} (|\epsilon_{\delta, j} + \epsilon_{\delta, 2j+1}|^{1/2} |\epsilon_{\delta, 2j+2} \epsilon_{\delta, 2j+3}|^{1/2} \right. \\
& \quad \left. |\epsilon_{\delta, j+4} + \epsilon_{\delta, 2j+5}|^{1/2} |\epsilon_{\delta, j+5} + \epsilon_{\delta, 2j+7}|^{1/2} - 2\mu_{1/2}^4) \right)
\end{aligned}$$

therefore

$$\begin{aligned}
& \frac{1}{2\delta} \sum_{j=0}^{\lfloor t/2\delta \rfloor - 8} \sigma_{2\delta, j} \sigma_{2\delta, j+1} \sigma_{2\delta, j+2} \sigma_{2\delta, j+3} \\
& = \frac{1}{2\delta} \sum_{j=0}^{\lfloor t/2\delta \rfloor - 8} \sqrt{(\sigma_{\delta, 2j}^2 + \sigma_{\delta, 2j+1}^2)(\sigma_{\delta, 2j+2}^2 + \sigma_{\delta, 2j+3}^2)(\sigma_{\delta, 2j+4}^2 + \sigma_{\delta, 2j+5}^2)(\sigma_{\delta, 2j+6}^2 + \sigma_{\delta, 2j+7}^2)} \\
& \stackrel{\text{law}}{=} \frac{1}{2\delta} \sum_{j=0}^{\lfloor t/2\delta \rfloor - 8} 4\sigma_{\delta, 2j} \sigma_{\delta, 2j+2} \sigma_{\delta, 2j+4} \sigma_{\delta, 2j+6} \\
& \xrightarrow{p} \int_0^t \sigma_s^4 ds.
\end{aligned}$$

So now the variance can be obtained

$$\begin{aligned}
\frac{1}{2\delta}\mu_{1/2}^{-8}Var & \left(\sum_{j=0}^{\lfloor t/2\delta \rfloor - 8} \sigma_{\delta,2j}^{1/2}\sigma_{\delta,2j+2}^{1/2}\sigma_{\delta,2j+4}^{1/2}\sigma_{\delta,2j+6}^{1/2} (|\epsilon_{\delta,j} + \epsilon_{\delta,2j+1}|^{1/2} |\epsilon_{\delta,2j+2}\epsilon_{\delta,2j+3}|^{1/2} \right. \\
& \left. |\epsilon_{\delta,2j+3}\epsilon_{\delta,2j+4}|^{1/2} |\epsilon_{\delta,2j+5}\epsilon_{\delta,2j+6}|^{1/2} - 2\mu_{1/2}^4) \right) \\
= \frac{1}{2\delta}\mu_{1/2}^{-8} & \sum_{j=0}^{\lfloor t/2\delta \rfloor - 8} \sigma_{\delta,2j}\sigma_{\delta,2j+2}\sigma_{\delta,2j+4}\sigma_{\delta,2j+6} \left(\right. \\
& Var(\sqrt{|\epsilon_{\delta,j} + \epsilon_{\delta,2j+1}| |\epsilon_{\delta,2j+2}\epsilon_{\delta,2j+3}| |\epsilon_{\delta,2j+4}\epsilon_{\delta,2j+5}| |\epsilon_{\delta,2j+6}\epsilon_{\delta,2j+7}| - 2\mu_{1/2}^4}) \\
& + 2Cov(\sqrt{|\epsilon_{\delta,j} + \epsilon_{\delta,2j+1}| |\epsilon_{\delta,2j+2}\epsilon_{\delta,2j+3}| |\epsilon_{\delta,2j+4}\epsilon_{\delta,2j+5}| |\epsilon_{\delta,2j+6}\epsilon_{\delta,2j+7}| - 2\mu_{1/2}^4}, \\
& \quad \sqrt{|\epsilon_{\delta,2j+2} + \epsilon_{\delta,2j+3}| |\epsilon_{\delta,2j+4}\epsilon_{\delta,2j+5}| |\epsilon_{\delta,2j+6}\epsilon_{\delta,2j+7}| |\epsilon_{\delta,2j+8}\epsilon_{\delta,2j+9}| - 2\mu_{1/2}^4}) \\
& + 2Cov(\sqrt{|\epsilon_{\delta,j} + \epsilon_{\delta,2j+1}| |\epsilon_{\delta,2j+2}\epsilon_{\delta,2j+3}| |\epsilon_{\delta,2j+4}\epsilon_{\delta,2j+5}| |\epsilon_{\delta,2j+6}\epsilon_{\delta,2j+7}| - 2\mu_{1/2}^4}, \\
& \quad \sqrt{|\epsilon_{\delta,2j+4} + \epsilon_{\delta,2j+5}| |\epsilon_{\delta,2j+6}\epsilon_{\delta,2j+7}| |\epsilon_{\delta,2j+8}\epsilon_{\delta,2j+9}| |\epsilon_{\delta,2j+10}\epsilon_{\delta,2j+11}| - 2\mu_{1/2}^4}) \\
& + 2Cov(\sqrt{|\epsilon_{\delta,j} + \epsilon_{\delta,2j+1}| |\epsilon_{\delta,2j+2}\epsilon_{\delta,2j+3}| |\epsilon_{\delta,2j+4}\epsilon_{\delta,2j+5}| |\epsilon_{\delta,2j+6}\epsilon_{\delta,2j+7}| - 2\mu_{1/2}^4}, \\
& \quad \left. \sqrt{|\epsilon_{\delta,2j+6}\epsilon_{\delta,2j+7}| |\epsilon_{\delta,2j+8}\epsilon_{\delta,2j+9}| |\epsilon_{\delta,2j+10}\epsilon_{\delta,2j+11}| |\epsilon_{\delta,2j+12} + \epsilon_{\delta,2j+13}| - 2\mu_{1/2}^4}) \right) \\
= \frac{1}{2\delta}\mu_{1/2}^{-8} & (C_{11} + 2C_{12} + 2C_{13} + 2C_{14}) \sum_{j=0}^{\lfloor t/2\delta \rfloor - 8} \sigma_{\delta,2j}\sigma_{\delta,2j+2}\sigma_{\delta,2j+4}\sigma_{\delta,2j+6} \\
\stackrel{p}{\rightarrow} & 1/4\mu_{1/2}^{-8} (C_{11} + 2C_{12} + 2C_{13} + 2C_{14}) \int_0^t \sigma_s^4 ds \\
\approx & 3.378 \int_0^t \sigma_s^4 ds,
\end{aligned}$$

where

$$\begin{aligned}
C_{11} & = Var(\sqrt{|u' + u''| |u''' + u^{(IV)}| |u^{(V)} + u^{(VI)}| |u^{(VII)} + u^{(VIII)}|}) \\
C_{12} & = Cov(\sqrt{|u' + u''| |u''' + u^{(IV)}| |u^{(V)} + u^{(VI)}| |u^{(VII)} + u^{(VIII)}|}, \\
& \quad \sqrt{|u''' + u^{(IV)}| |u^{(V)} + u^{(VI)}| |u^{(VII)} + u^{(VIII)}| |u^{(IX)} + u^{(X)}|}). \\
C_{13} & = Cov(\sqrt{|u' + u''| |u''' + u^{(IV)}| |u^{(V)} + u^{(VI)}| |u^{(VII)} + u^{(VIII)}|}, \\
& \quad \sqrt{|u^{(V)} + u^{(VI)}| |u^{(VII)} + u^{(VIII)}| |u^{(IX)} + u^{(X)}| |u^{(XI)} + u^{(XII)}|}). \\
C_{14} & = Cov(\sqrt{|u' + u''| |u''' + u^{(IV)}| |u^{(V)} + u^{(VI)}| |u^{(VII)} + u^{(VIII)}|}, \\
& \quad \sqrt{|u^{(VII)} + u^{(VIII)}| |u^{(IX)} + u^{(X)}| |u^{(XI)} + u^{(XII)}| |u^{(XIII)} + u^{(XIV)}|}).
\end{aligned}$$

The same procedure holds to obtain

$$\begin{aligned}
Var\left(\sqrt{\frac{1}{2\delta}}\left(\mu_{1/2}^{-8}[Y_{2\delta}^{(2)}]_t^{[1/2,1/2,1/2,1/2]} - \int_0^t \sigma_s^2 ds\right)\right) & \stackrel{p}{\rightarrow} 1/4\mu_{1/2}^{-8}(C_{11} + 2C_{12} + 2C_{13} + 2C_{14}) \int_0^t \sigma_s^4 ds \\
& \approx 3.378 \int_0^t \sigma_s^4 ds.
\end{aligned}$$

Now to obtain the covariance, we need to assume that when $\delta \downarrow 0$ then $\sigma_{\delta,2j} = \sigma_{\delta,2j+1}$, so

$$\begin{aligned}
& Cov\left(\sqrt{\frac{1}{2\delta}}\left(\mu_{1/2}^{-4}[Y_{2\delta}^{(1)}]_{[1/2,1/2,1/2,1/2]}^t - \int_0^t \sigma_s^2 ds\right), \sqrt{\frac{1}{2\delta}}\left(\mu_{1/2}^{-4}[Y_{2\delta}^{(2)}]_{[1/2,1/2,1/2,1/2]}^t - \int_0^t \sigma_s^2 ds\right)\right) \\
& \stackrel{law}{=} \frac{1}{2\delta}\mu_{1/2}^{-4}Cov\left(\sum_{j=0}^{\lfloor t/2\delta \rfloor - 7} \sqrt{\sigma_{\delta,2j}\sigma_{\delta,2j+2}\sigma_{\delta,2j+4}\sigma_{\delta,2j+6}}\left(\right. \right. \\
& \quad \left. \left. \sqrt{|\epsilon_{\delta,2j} + \epsilon_{\delta,2j+1} \parallel \epsilon_{\delta,2j+2} + \epsilon_{\delta,2j+3} \parallel \epsilon_{\delta,2j+4} + \epsilon_{\delta,2j+5} \parallel \epsilon_{\delta,2j+6} + \epsilon_{\delta,2j+7} \parallel - 2\mu_{1/2}^4}, \right. \right. \\
& \quad \left. \left. \sum_{j=0}^{\lfloor t/2\delta \rfloor - 8} \sqrt{\sigma_{\delta,2j+1}\sigma_{\delta,2j+3}\sigma_{\delta,2j+5}\sigma_{\delta,2j+7}}\left(\right. \right. \right. \\
& \quad \left. \left. \left. \sqrt{|\epsilon_{\delta,2j+1} + \epsilon_{\delta,2j+2} \parallel \epsilon_{\delta,2j+3}\epsilon_{\delta,2j+4} \parallel \epsilon_{\delta,2j+5}\epsilon_{\delta,2j+6} \parallel \epsilon_{\delta,2j+7}\epsilon_{\delta,2j+8} \parallel - 2\mu_{1/2}^4}\right)\right)\right) \\
& \stackrel{law}{=} \frac{1}{2\delta}\mu_{1/2}^{-8} \sum_{j=0}^{\lfloor t/2\delta \rfloor - 8} \sigma_{\delta,2j}\sigma_{\delta,2j+2}\sigma_{\delta,2j+4}\sigma_{\delta,2j+6} \left(\right. \\
& \quad 2Cov\left(\sqrt{|\epsilon_{\delta,2j} + \epsilon_{\delta,2j+1} \parallel \epsilon_{\delta,2j+2}\epsilon_{\delta,2j+3} \parallel \epsilon_{\delta,2j+4}\epsilon_{\delta,2j+5} \parallel \epsilon_{\delta,2j+6}\epsilon_{\delta,2j+7} \parallel - 2\mu_{1/2}^4}, \right. \\
& \quad \left. \sqrt{|\epsilon_{\delta,2j+1} + \epsilon_{\delta,2j+2} \parallel \epsilon_{\delta,2j+3}\epsilon_{\delta,2j+4} \parallel \epsilon_{\delta,2j+5}\epsilon_{\delta,2j+6} \parallel \epsilon_{\delta,2j+7}\epsilon_{\delta,2j+8} \parallel - 2\mu_{1/2}^4}\right) \\
& \quad + 2Cov\left(\sqrt{|\epsilon_{\delta,2j} + \epsilon_{\delta,2j+1} \parallel \epsilon_{\delta,2j+2}\epsilon_{\delta,2j+3} \parallel \epsilon_{\delta,2j+4}\epsilon_{\delta,2j+5} \parallel \epsilon_{\delta,2j+6}\epsilon_{\delta,2j+7} \parallel - 2\mu_{1/2}^4}, \right. \\
& \quad \left. \sqrt{|\epsilon_{\delta,2j+3}\epsilon_{\delta,2j+4} \parallel \epsilon_{\delta,2j+5}\epsilon_{\delta,2j+6} \parallel \epsilon_{\delta,2j+7}\epsilon_{\delta,2j+8} \parallel \epsilon_{\delta,2j+9} + \epsilon_{\delta,2j+10} \parallel - 2\mu_{1/2}^4}\right) \\
& \quad + 2Cov\left(\sqrt{|\epsilon_{\delta,2j} + \epsilon_{\delta,2j+1} \parallel \epsilon_{\delta,2j+2}\epsilon_{\delta,2j+3} \parallel \epsilon_{\delta,2j+4}\epsilon_{\delta,2j+5} \parallel \epsilon_{\delta,2j+6}\epsilon_{\delta,2j+7} \parallel - 2\mu_{1/2}^4}, \right. \\
& \quad \left. \sqrt{|\epsilon_{\delta,2j+5}\epsilon_{\delta,2j+6} \parallel \epsilon_{\delta,2j+7}\epsilon_{\delta,2j+8} \parallel \epsilon_{\delta,2j+9} + \epsilon_{\delta,2j+10} \parallel \epsilon_{\delta,2j+11}\epsilon_{\delta,2j+12} \parallel - 2\mu_{1/2}^4}\right) \\
& \quad + 2Cov\left(\sqrt{|\epsilon_{\delta,2j} + \epsilon_{\delta,2j+1} \parallel \epsilon_{\delta,2j+2}\epsilon_{\delta,2j+3} \parallel \epsilon_{\delta,2j+4}\epsilon_{\delta,2j+5} \parallel \epsilon_{\delta,2j+6}\epsilon_{\delta,2j+7} \parallel - 2\mu_{1/2}^4}, \right. \\
& \quad \left. \sqrt{|\epsilon_{\delta,2j+7}\epsilon_{\delta,2j+8} \parallel \epsilon_{\delta,2j+9} + \epsilon_{\delta,2j+10} \parallel \epsilon_{\delta,2j+11}\epsilon_{\delta,2j+12} \parallel \epsilon_{\delta,2j+12}\epsilon_{\delta,2j+13} \parallel - 2\mu_{1/2}^4}\right) \\
& \left. = \frac{1}{2\delta}\mu_{1/2}^{-8} \sum_{j=0}^{\lfloor t/2\delta \rfloor - 7} \sigma_{\delta,2j}\sigma_{\delta,2j+2}\sigma_{\delta,2j+4}\sigma_{\delta,2j+6}(2C_{14} + 2C_{15} + 2C_{16} + 2C_{17}) \right) \\
& \stackrel{p}{\rightarrow} 1/4\mu_{1/2}^{-8}(2C_{14} + 2C_{15} + 2C_{16} + 2C_{17}) \int_0^t \sigma_s^4 ds \\
& \simeq 1.013 \int_0^t \sigma_s^4 ds
\end{aligned}$$

where

$$\begin{aligned}
C_{14} &= Cov\left(\sqrt{|\ u' + u'' \parallel u''' + u^{(IV)} \parallel u^{(V)} + u^{(VI)} \parallel u^{(VII)} + u^{(VIII)} \parallel \right. \\
& \quad \left. \sqrt{|\ u'' + u''' \parallel u^{(IV)} + u^{(V)} \parallel u^{(VI)} + u^{(VII)} \parallel u^{(VIII)} + u^{(IX)} \parallel \right)} \\
C_{15} &= Cov\left(\sqrt{|\ u' + u'' \parallel u''' + u^{(IV)} \parallel u^{(V)} + u^{(VI)} \parallel u^{(VII)} + u^{(VIII)} \parallel \right. \\
& \quad \left. \sqrt{|\ u^{(IV)} + u^{(V)} \parallel u^{(VI)} + u^{(VII)} \parallel u^{(VIII)} + u^{(IX)} \parallel u^{(X)} + u^{(XI)} \parallel \right)} \\
C_{16} &= Cov\left(\sqrt{|\ u' + u'' \parallel u''' + u^{(IV)} \parallel u^{(V)} + u^{(VI)} \parallel u^{(VII)} + u^{(VIII)} \parallel \right. \\
& \quad \left. \sqrt{|\ u^{(VI)} + u^{(VII)} \parallel u^{(VIII)} + u^{(IX)} \parallel u^{(X)} + u^{(XI)} \parallel u^{(XII)} + u^{(XIII)} \parallel \right)} \\
C_{17} &= Cov\left(\sqrt{|\ u' + u'' \parallel u''' + u^{(IV)} \parallel u^{(V)} + u^{(VI)} \parallel u^{(VII)} + u^{(VIII)} \parallel \right)}
\end{aligned}$$

$$\sqrt{|u^{(VIII)} + u^{(IX)} || u^{(X)} + u^{(XI)} || u^{(XII)} + u^{(XIII)} || u^{(XIV)} + u^{(XV)} |)}.$$

A5.2 Bias in the presence of market microstructure effects

Here we will obtain the bias of all our estimators given market microstructure noise. For that, let us define here the contaminated returns as $\tilde{y}_j = y_j + \varepsilon_j$ for $j = 1, 2, \dots, \lfloor t/\delta \rfloor$ where y_j is the true return and ε_j is the market microstructure noise. Notice that the difference between this and the original model (Section 5.1.1) is that the noise is added to the returns and not to the log-prices.

Let us impose the following assumptions

- (1) The random shocks ε_j are Gaussian i.i.d. with mean zero and variance $w^2\Psi_j$ where w is a constant and $\Psi_j = \frac{1}{\delta} \int_{\delta(j-1)}^{\delta j} \sigma_u^2 du$.
- (2) The true returns are Gaussian i.i.d. with mean zero and variance $\delta\Psi_j$.
- (3) $y_j \perp \varepsilon_j$.

For all the cases assume u_t and v_t are uncorrelated, i.i.d standard Normal random variables also let us set $k = w\delta^{-1/2}$.

Realised Variance

$$\begin{aligned} E\left([\tilde{Y}_\delta]_t^{[2]}\right) &= E\left(\sum_{j=1}^{\lfloor t/\delta \rfloor} \tilde{y}_j^2\right) \\ &= E\left(\sum_{j=1}^{\lfloor t/\delta \rfloor} (y_j^2 + 2y_j\varepsilon_j + \varepsilon_j^2)\right) \\ &= E\left(\sum_{j=1}^{\lfloor t/\delta \rfloor} y_j^2\right) + E\left(\sum_{j=1}^{\lfloor t/\delta \rfloor} 2y_j\varepsilon_j\right) + E\left(\sum_{j=1}^{\lfloor t/\delta \rfloor} \varepsilon_j^2\right) \\ &\xrightarrow{p} (1 + w^2\delta^{-1}) \int_0^t \sigma_u^2 du. \end{aligned}$$

Given that

$$\begin{aligned} E\left(\sum_{j=1}^{\lfloor t/\delta \rfloor} y_j^2\right) &= \sum_{j=1}^{\lfloor t/\delta \rfloor} \delta\Psi_j \\ E\left(\sum_{j=1}^{\lfloor t/\delta \rfloor} 2y_j\varepsilon_j\right) &= 0 \end{aligned}$$

$$E\left(\sum_{j=1}^{\lfloor t/\delta \rfloor} \varepsilon_j^2\right) = w^2 \delta^{-1} \sum_{j=1}^{\lfloor t/\delta \rfloor} \delta \Psi_j.$$

Realised bipower variation

$$\begin{aligned} E\left([\tilde{Y}_\delta]_t^{[1,1]}\right) &= E\left(\sum_{j=1}^{\lfloor t/\delta \rfloor - 1} |\tilde{y}_j| |\tilde{y}_{j+1}|\right) \\ &= E\left(\sum_{j=1}^{\lfloor t/\delta \rfloor - 1} |y_j + \varepsilon_j| |y_{j+1} + \varepsilon_{j+1}|\right) \\ &\stackrel{law}{=} E\left(\sum_{j=1}^{\lfloor t/\delta \rfloor - 1} |\delta^{1/2} \Psi_j^{1/2} u_j + w \Psi_j^{1/2} v_j| |\delta^{1/2} \Psi_{j+1}^{1/2} u_{j+1} + w \Psi_{j+1}^{1/2} v_{j+1}|\right) \\ &= \sum_{j=1}^{\lfloor t/\delta \rfloor - 1} \delta \Psi_j^{1/2} \Psi_{j+1}^{1/2} E\left(|u_j + kv_j| |u_{j+1} + kv_{j+1}|\right) \\ &= \sum_{j=1}^{\lfloor t/\delta \rfloor - 1} \delta \Psi_j^{1/2} \Psi_{j+1}^{1/2} \mu_1 \sqrt{1+k^2} \mu_1 \sqrt{1+k^2} \\ &\xrightarrow{p} \mu_1^2 (1+k^2) \int_0^t \sigma_u^2 du. \end{aligned}$$

Realised tripower variation

$$\begin{aligned} E\left([\tilde{Y}_\delta]_t^{[2/3,2/3,2/3]}\right) &= E\left(\sum_{j=1}^{\lfloor t/\delta \rfloor - 2} |\tilde{y}_j|^{2/3} |\tilde{y}_{j+1}|^{2/3} |\tilde{y}_{j+2}|^{2/3}\right) \\ &= E\left(\sum_{j=1}^{\lfloor t/\delta \rfloor - 2} |y_j + \varepsilon_j|^{2/3} |y_{j+1} + \varepsilon_{j+1}|^{2/3} |y_{j+2} + \varepsilon_{j+2}|^{2/3}\right) \\ &\stackrel{law}{=} E\left(\sum_{j=1}^{\lfloor t/\delta \rfloor - 2} |\delta^{1/2} \Psi_j^{1/2} u_j + w \Psi_j^{1/2} v_j|^{2/3} |\delta^{1/2} \Psi_{j+1}^{1/2} u_{j+1} + w \Psi_{j+1}^{1/2} v_{j+1}|^{2/3} \right. \\ &\quad \left. |\delta^{1/2} \Psi_{j+2}^{1/2} u_{j+2} + w \Psi_{j+2}^{1/2} v_{j+2}|^{2/3}\right) \\ &= \sum_{j=1}^{\lfloor t/\delta \rfloor - 2} \delta \Psi_j^{1/3} \Psi_{j+1}^{1/3} \Psi_{j+2}^{1/3} E\left(|u_j + kv_j|^{2/3} |u_{j+1} + kv_{j+1}|^{2/3} |u_{j+2} + kv_{j+2}|^{2/3}\right) \\ &= \sum_{j=1}^{\lfloor t/\delta \rfloor - 2} \delta \Psi_j^{1/3} \Psi_{j+1}^{1/3} \Psi_{j+2}^{1/3} \mu_{2/3}^3 \left((\sqrt{1+k^2})^{2/3}\right)^3 \\ &\xrightarrow{p} \mu_{2/3}^3 (1+k^2) \int_0^t \sigma_u^2 du. \end{aligned}$$

Realised quadpower variation

$$\begin{aligned}
E \left([\tilde{Y}_\delta]_t^{[1/2,1/2,1/2,1/2]} \right) &= E \left(\sum_{j=1}^{\lfloor t/\delta \rfloor - 3} \sqrt{|\tilde{y}_j| |\tilde{y}_{j+1}| |\tilde{y}_{j+2}| |\tilde{y}_{j+3}|} \right) \\
&= E \left(\sum_{j=1}^{\lfloor t/\delta \rfloor - 3} \sqrt{|y_j + \varepsilon_j| |y_{j+1} + \varepsilon_{j+1}| |y_{j+2} + \varepsilon_{j+2}| |y_{j+3} + \varepsilon_{j+3}|} \right) \\
&\stackrel{law}{=} E \left(\sum_{j=1}^{\lfloor t/\delta \rfloor - 3} \left| \delta^{1/2} \Psi_j^{1/2} u_j + w \Psi_j^{1/2} v_j \right|^{1/2} \left| \delta^{1/2} \Psi_{j+1}^{1/2} u_{j+1} + w \Psi_{j+1}^{1/2} v_{j+1} \right|^{1/2} \right. \\
&\quad \left. \left| \delta^{1/2} \Psi_{j+2}^{1/2} u_{j+2} + w \Psi_{j+2}^{1/2} v_{j+2} \right|^{1/2} \left| \delta^{1/2} \Psi_{j+3}^{1/2} u_{j+3} + w \Psi_{j+3}^{1/2} v_{j+3} \right|^{1/2} \right) \\
&= \sum_{j=1}^{\lfloor t/\delta \rfloor - 3} \delta \Psi_j^{1/4} \Psi_{j+1}^{1/4} \Psi_{j+2}^{1/4} \Psi_{j+3}^{1/4} E \left(\sqrt{|u_j + kv_j| |u_{j+1} + kv_{j+1}| |u_{j+2} + kv_{j+2}| |u_{j+3} + kv_{j+3}|} \right) \\
&= \sum_{j=1}^{\lfloor t/\delta \rfloor - 3} \delta \Psi_j^{1/4} \Psi_{j+1}^{1/4} \Psi_{j+2}^{1/4} \Psi_{j+3}^{1/4} \mu_{1/2}^4 \left((\sqrt{1+k^2})^{1/2} \right)^4 \\
&\xrightarrow{p} \mu_{1/2}^4 (1+k^2) \int \sigma_u^2 du.
\end{aligned}$$

Estimators based on subsampling and averaging

Subsampled and averaged realised variance

$$\begin{aligned}
E \left([\tilde{Y}_\delta]_t^{[2](avg)} \right) &= \frac{1}{2} E \left([\tilde{Y}_{2\delta}^{(1)}]_t^{[2]} + [\tilde{Y}_{2\delta}^{(2)}]_t^{[2]} \right) \\
&\stackrel{law}{=} \frac{1}{2} (1 + w^2 \delta^{-1}) \left(\sum_{j=1}^{\lfloor t/2\delta \rfloor - 1} \delta (\Psi_{2j} + \Psi_{2j+1}) + \sum_{j=1}^{\lfloor t/2\delta \rfloor - 2} \delta (\Psi_{2j+1} + \Psi_{2j+2}) \right) \\
&\xrightarrow{p} (1 + w^2 \delta^{-1}) \int_0^t \sigma_u^2 du.
\end{aligned}$$

Given that

$$\begin{aligned}
E \left([\tilde{Y}_{2\delta}^{(1)}]_t^{[2]} \right) &= E \left(\sum_{j=1}^{\lfloor t/2\delta \rfloor - 1} (\tilde{y}_{2\delta,j}^{(1)})^2 \right) \\
&= E \left(\sum_{j=1}^{\lfloor t/2\delta \rfloor - 1} (\tilde{y}_{2j} + \tilde{y}_{2j+1})^2 \right) \\
&= E \left(\sum_{j=1}^{\lfloor t/2\delta \rfloor - 1} (\tilde{y}_{2j}^2 + 2\tilde{y}_{2j}\tilde{y}_{2j+1} + \tilde{y}_{2j+1}^2) \right) \\
&= E \left(\sum_{j=1}^{\lfloor t/2\delta \rfloor - 1} (y_{2j} + \varepsilon_{2j})^2 \right) + 2E \left(\sum_{j=1}^{\lfloor t/2\delta \rfloor - 1} (y_{2j} + \varepsilon_{2j})(y_{2j+1} + \varepsilon_{2j+1}) \right) + E \left(\sum_{j=1}^{\lfloor t/2\delta \rfloor - 1} (y_{2j+1} + \varepsilon_{2j+1})^2 \right) \\
&\stackrel{law}{=} (1 + w^2 \delta^{-1}) \sum_{j=1}^{\lfloor t/2\delta \rfloor - 1} \delta (\Psi_{2j} + \Psi_{2j+1}).
\end{aligned}$$

Subsampled and averaged realised bipower variation

$$\begin{aligned}
E\left([\tilde{Y}_\delta]_t^{[1,1](avg)}\right) &= \frac{1}{2}E\left([\tilde{Y}_{2\delta}^{(1)}]_t^{[1,1]} + [\tilde{Y}_{2\delta}^{(2)}]_t^{[1,1]}\right) \\
&\stackrel{law}{=} \frac{1}{2}\mu_1^2(1+w^2\delta^{-1})4\delta\left(\sum_{j=1}^{\lfloor t/2\delta\rfloor-2}\Psi_{2j}^{1/2}\Psi_{2j+2}^{1/2} + \sum_{j=1}^{\lfloor t/2\delta\rfloor-3}\Psi_{2j+1}^{1/2}\Psi_{2j+3}^{1/2}\right) \\
&\xrightarrow{p} \mu_1^2(1+w^2\delta^{-1})\int_0^t\sigma_u^2du.
\end{aligned}$$

Given that

$$\begin{aligned}
E\left([\tilde{Y}_{2\delta}^{(1)}]_t^{[1,1]}\right) &= E\left(\sum_{j=1}^{\lfloor t/2\delta\rfloor-2}|\tilde{y}_{2\delta,j}^{(1)}||\tilde{y}_{2\delta,j+1}^{(1)}|\right) \\
&= E\left(\sum_{j=1}^{\lfloor t/2\delta\rfloor-2}|\tilde{y}_{2j} + \tilde{y}_{2j+1}||\tilde{y}_{2j+2} + \tilde{y}_{2j+3}|\right) \\
&= E\left(\sum_{j=1}^{\lfloor t/2\delta\rfloor-2}|y_{2j} + \varepsilon_{2j} + y_{2j+1} + \varepsilon_{2j+1}||y_{2j+2} + \varepsilon_{2j+2} + y_{2j+3} + \varepsilon_{2j+3}|\right) \\
&\stackrel{law}{=} E\left(\sum_{j=1}^{\lfloor t/2\delta\rfloor-2}|\delta^{1/2}\Psi_{2j}^{1/2}u_{2j} + w\Psi_{2j}^{1/2}v_{2j} + \delta^{1/2}\Psi_{2j+1}^{1/2}u_{2j+1} + w\Psi_{2j+1}^{1/2}v_{2j+1}| \right. \\
&\quad \left. |\delta^{1/2}\Psi_{2j+2}^{1/2}u_{2j+2} + w\Psi_{2j+2}^{1/2}v_{2j+2} + \delta^{1/2}\Psi_{2j+3}^{1/2}u_{2j+3} + w\Psi_{2j+3}^{1/2}v_{2j+3}|\right) \\
&\stackrel{law}{=} \sum_{j=1}^{\lfloor t/2\delta\rfloor-2}\delta\Psi_{2j}^{1/2}\Psi_{2j+2}^{1/2}E\left(|u_{2j} + u_{2j+1} + kv_{2j} + kv_{2j+1}||u_{2j+2} + u_{2j+3} + kv_{2j+2} + kv_{2j+3}|\right) \\
&\stackrel{law}{=} \sum_{j=1}^{\lfloor t/2\delta\rfloor-2}\delta\Psi_{2j}^{1/2}\Psi_{2j+2}^{1/2}2\mu_1\sqrt{1+k^2}2\mu_1\sqrt{1+k^2} \\
&= \mu_1^2(1+k^2)4\delta\sum_{j=1}^{\lfloor t/2\delta\rfloor-2}\Psi_{2j}^{1/2}\Psi_{2j+2}^{1/2}.
\end{aligned}$$

Subsampled and averaged realised tripower variation

$$\begin{aligned}
E\left([\tilde{Y}_\delta]_t^{[2/3,2/3,2/3](avg)}\right) &= \frac{1}{2}E\left([\tilde{Y}_{2\delta}^{(1)}]_t^{[2/3,2/3,2/3]} + [\tilde{Y}_{2\delta}^{(2)}]_t^{[2/3,2/3,2/3]}\right) \\
&\stackrel{law}{=} \frac{1}{2}\mu_{2/3}^3(1+w^2\delta^{-1})4\delta\left(\sum_{j=1}^{\lfloor t/2\delta\rfloor-3}\Psi_{2j}^{1/3}\Psi_{2j+2}^{1/3}\Psi_{2j+4}^{1/3} + \sum_{j=1}^{\lfloor t/2\delta\rfloor-4}\Psi_{2j+1}^{1/3}\Psi_{2j+3}^{1/3}\Psi_{2j+5}^{1/3}\right) \\
&\xrightarrow{p} \mu_{2/3}^3(1+w^2\delta^{-1})\int_0^t\sigma_u^2du.
\end{aligned}$$

Given that

$$E\left([\tilde{Y}_{2\delta}^{(1)}]_t^{[2/3,2/3,2/3]}\right) = E\left(\sum_{j=1}^{\lfloor t/2\delta\rfloor-3}|\tilde{y}_{2\delta,j}^{(1)}|^{2/3}|\tilde{y}_{2\delta,j+1}^{(1)}|^{2/3}|\tilde{y}_{2\delta,j+2}^{(1)}|^{2/3}\right)$$

$$\begin{aligned}
&= E \left(\sum_{j=1}^{\lfloor t/2\delta \rfloor - 3} |\tilde{y}_{2j} + \tilde{y}_{2j+1}|^{2/3} |\tilde{y}_{2j+2} + \tilde{y}_{2j+3}|^{2/3} |\tilde{y}_{2j+4} + \tilde{y}_{2j+5}|^{2/3} \right) \\
&= E \left(\sum_{j=1}^{\lfloor t/2\delta \rfloor - 3} |y_{2j} + \varepsilon_{2j} + y_{2j+1} + \varepsilon_{2j+1}|^{2/3} |y_{2j+2} + \varepsilon_{2j+2} + y_{2j+3} + \varepsilon_{2j+3}|^{2/3} \right. \\
&\quad \left. |y_{2j+4} + \varepsilon_{2j+4} + y_{2j+5} + \varepsilon_{2j+5}|^{2/3} \right) \\
&\stackrel{law}{=} E \left(\sum_{j=1}^{\lfloor t/2\delta \rfloor - 3} |\delta^{1/2} \Psi_{2j}^{1/2} u_{2j} + w \Psi_{2j}^{1/2} v_{2j} + \delta^{1/2} \Psi_{2j+1}^{1/2} u_{2j+1} + w \Psi_{2j+1}^{1/2} v_{2j+1}|^{2/3} \right. \\
&\quad |\delta^{1/2} \Psi_{2j+2}^{1/2} u_{2j+2} + w \Psi_{2j+2}^{1/2} v_{2j+2} + \delta^{1/2} \Psi_{2j+3}^{1/2} u_{2j+3} + w \Psi_{2j+3}^{1/2} v_{2j+3}|^{2/3} \\
&\quad \left. |\delta^{1/2} \Psi_{2j+4}^{1/2} u_{2j+4} + w \Psi_{2j+4}^{1/2} v_{2j+4} + \delta^{1/2} \Psi_{2j+5}^{1/2} u_{2j+5} + w \Psi_{2j+5}^{1/2} v_{2j+5}|^{2/3} \right) \\
&\stackrel{law}{=} \sum_{j=1}^{\lfloor t/2\delta \rfloor - 3} \delta \Psi_{2j}^{1/3} \Psi_{2j+2}^{1/3} \Psi_{2j+4}^{1/3} E \left(|u_{2j} + u_{2j+1} + kv_{2j} + kv_{2j+1}|^{2/3} \right. \\
&\quad \left. |u_{2j+2} + u_{2j+3} + kv_{2j+2} + kv_{2j+3}|^{2/3} |u_{2j+4} + u_{2j+5} + kv_{2j+4} + kv_{2j+5}|^{2/3} \right) \\
&\stackrel{law}{=} \sum_{j=1}^{\lfloor t/2\delta \rfloor - 3} \delta \Psi_{2j}^{1/3} \Psi_{2j+2}^{1/3} \Psi_{2j+4}^{1/3} 4\mu_{2/3}^3 \left((\sqrt{1+k^2})^{2/3} \right)^3 \\
&= \mu_{2/3}^3 (1+k^2) 4\delta \sum_{j=1}^{\lfloor t/2\delta \rfloor - 3} \Psi_{2j}^{1/3} \Psi_{2j+2}^{1/3} \Psi_{2j+4}^{1/3}.
\end{aligned}$$

Subsampled and averaged realised quadpower variation

$$\begin{aligned}
E \left([\tilde{Y}_\delta]_t^{[1/2,1/2,1/2,1/2](avg)} \right) &= \frac{1}{2} E \left([\tilde{Y}_{2\delta}^{(1)}]_t^{[1/2,1/2,1/2,1/2]} + [\tilde{Y}_{2\delta}^{(2)}]_t^{[1/2,1/2,1/2,1/2]} \right) \\
&\stackrel{law}{=} \frac{1}{2} \mu_{1/2}^4 (1+w^2\delta^{-1}) 4\delta \left(\sum_{j=1}^{\lfloor t/2\delta \rfloor - 4} \Psi_{2j}^{1/4} \Psi_{2j+2}^{1/4} \Psi_{2j+4}^{1/4} \Psi_{2j+6}^{1/4} + \sum_{j=1}^{\lfloor t/2\delta \rfloor - 5} \Psi_{2j+1}^{1/4} \Psi_{2j+3}^{1/4} \Psi_{2j+5}^{1/4} \Psi_{2j+7}^{1/4} \right) \\
&\xrightarrow{p} \mu_{1/2}^4 (1+w^2\delta^{-1}) \int_0^t \sigma_u^2 du.
\end{aligned}$$

Given that

$$\begin{aligned}
E \left([\tilde{Y}_{2\delta}^{(1)}]_t^{[1/2,1/2,1/2,1/2]} \right) &= E \left(\sum_{j=1}^{\lfloor t/2\delta \rfloor - 4} \sqrt{|\tilde{y}_{2\delta,j}^{(1)}| |\tilde{y}_{2\delta,j+1}^{(1)}| |\tilde{y}_{2\delta,j+2}^{(1)}| |\tilde{y}_{2\delta,j+3}^{(1)}|} \right) \\
&= E \left(\sum_{j=1}^{\lfloor t/2\delta \rfloor - 4} \sqrt{|\tilde{y}_{2j} + \tilde{y}_{2j+1}| |\tilde{y}_{2j+2} + \tilde{y}_{2j+3}| |\tilde{y}_{2j+4} + \tilde{y}_{2j+5}| |\tilde{y}_{2j+6} + \tilde{y}_{2j+7}|} \right) \\
&= E \left(\sum_{j=1}^{\lfloor t/2\delta \rfloor - 4} |y_{2j} + \varepsilon_{2j} + y_{2j+1} + \varepsilon_{2j+1}|^{1/2} |y_{2j+2} + \varepsilon_{2j+2} + y_{2j+3} + \varepsilon_{2j+3}|^{1/2} \right. \\
&\quad \left. |y_{2j+4} + \varepsilon_{2j+4} + y_{2j+5} + \varepsilon_{2j+5}|^{1/2} |y_{2j+6} + \varepsilon_{2j+6} + y_{2j+7} + \varepsilon_{2j+7}|^{1/2} \right)
\end{aligned}$$

$$\begin{aligned}
&\stackrel{law}{=} E \left(\sum_{j=1}^{\lfloor t/2\delta \rfloor - 4} \left| \delta^{1/2} \Psi_{2j}^{1/2} u_{2j} + w \Psi_{2j}^{1/2} v_{2j} + \delta^{1/2} \Psi_{2j+1}^{1/2} u_{2j+1} + w \Psi_{2j+1}^{1/2} v_{2j+1} \right|^{1/2} \right. \\
&\quad \left| \delta^{1/2} \Psi_{2j+2}^{1/2} u_{2j+2} + w \Psi_{2j+2}^{1/2} v_{2j+2} + \delta^{1/2} \Psi_{2j+3}^{1/2} u_{2j+3} + w \Psi_{2j+3}^{1/2} v_{2j+3} \right|^{1/2} \\
&\quad \left| \delta^{1/2} \Psi_{2j+4}^{1/2} u_{2j+4} + w \Psi_{2j+4}^{1/2} v_{2j+4} + \delta^{1/2} \Psi_{2j+5}^{1/2} u_{2j+5} + w \Psi_{2j+5}^{1/2} v_{2j+5} \right|^{1/2} \\
&\quad \left. \left| \delta^{1/2} \Psi_{2j+6}^{1/2} u_{2j+6} + w \Psi_{2j+6}^{1/2} v_{2j+6} + \delta^{1/2} \Psi_{2j+7}^{1/2} u_{2j+7} + w \Psi_{2j+7}^{1/2} v_{2j+7} \right|^{1/2} \right) \\
&\stackrel{law}{=} \sum_{j=1}^{\lfloor t/2\delta \rfloor - 4} \delta \Psi_{2j}^{1/4} \Psi_{2j+2}^{1/4} \Psi_{2j+4}^{1/4} \Psi_{2j+6}^{1/4} E \left(\left| u_{2j} + u_{2j+1} + kv_{2j} + kv_{2j+1} \right|^{1/2} \right. \\
&\quad \left. \left| u_{2j+2} + u_{2j+3} + kv_{2j+2} + kv_{2j+3} \right|^{1/2} \left| u_{2j+4} + u_{2j+5} + kv_{2j+4} + kv_{2j+5} \right|^{1/2} \right. \\
&\quad \left. \left| u_{2j+6} + u_{2j+7} + kv_{2j+6} + kv_{2j+7} \right|^{1/2} \right) \\
&\stackrel{law}{=} \sum_{j=1}^{\lfloor t/2\delta \rfloor - 4} \delta \Psi_{2j}^{1/4} \Psi_{2j+2}^{1/4} \Psi_{2j+4}^{1/4} \Psi_{2j+6}^{1/4} 4\mu_{1/2}^4 \left((\sqrt{1+k^2})^{1/2} \right)^4 \\
&= \mu_{1/2}^4 (1+k^2) 4\delta \sum_{j=1}^{\lfloor t/2\delta \rfloor - 4} \Psi_{2j}^{1/4} \Psi_{2j+2}^{1/4} \Psi_{2j+4}^{1/4} \Psi_{2j+6}^{1/4}.
\end{aligned}$$

Chapter 6

Conclusion

The main objective of this thesis is the measurement of the variability of log-prices based on the innovative use of high-frequency financial data and its inherent challenges. There is a need to exploit the availability of these data to estimate the volatility process efficiently. The pricing of options and other derivatives, asset allocation weights and many investment decisions rely on its accurate estimation. Different estimators were studied throughout this thesis with a view to overcome various obstacles; we need efficient and consistent estimators, we need to be able to separate the continuous component from the jump component of the log-price process and we need estimators robust to the market microstructure effect given the very fine time intervals we are working with.

A commonly used and well studied estimator of integrated variance is the realised variance, the sum of squared high-frequency returns. Its probability limit and distribution theory, based on the theory of quadratic variation, is well known and implemented (e.g. Andersen and Bollerslev (1998a), Comte and Renault (1998), Barndorff-Nielsen and Shephard (2001)). Nevertheless, the estimation error can be very large and is likely to be so if the volatility in the market is high. This is why we need its asymptotic analysis and modelling. A linear state space representation in conjunction with a model for the dynamics of volatility has previously been used to provide a more accurate estimation of the actual variance. Here, we extended the theory of realised variance to provide similar results using the sum of absolute returns, realised absolute variation. We studied its statistical properties as an estimator of integrated spot volatility in the context of stochastic volatility models. High-frequency data allowed us to use a model based and a model free approach to the problem of measuring integrated spot volatility; the estimation error was thereby considerably reduced. Both approaches tended to be quite similar when M was large and we employed lags and leads. When using a logarithmic transformation, the finite sample performance of the asymptotic approximation of the distribution improved even for

moderate values of M . It is important to point out that leverage effect, jumps in the process and market microstructure effect were not considered in the modelling. Recent literature (e.g. Meddahi (2002)) established that leverage effect would not affect the estimation, nevertheless ignoring the jumps in the process or the market microstructure effect can completely invalidate the results. Therefore these issues were studied carefully in this thesis.

Realised absolute variation was considered as an estimator of integrated spot volatility given that absolute returns are known to be more robust to extreme observations and jumps than squared returns. After a joint estimation of integrated variance and squared integrated spot volatility using realised variance and the squared realised absolute variation, we could confirm that more weight was given to the latter. After some more analysis we could conclude that the squared realised absolute variation estimated the squared integrated spot volatility better than realised variance did integrated variance. Nevertheless with absolute returns we can only estimate integrated spot volatility and our real aim is to estimate integrated variance. The main contribution of Chapter 3 was the combined estimation of integrated variance using realised variance and realised absolute variation and the analysis of its implications. Nevertheless further research is needed to determine how realised absolute variation can be used to estimate realised variance and how efficient this estimation could be.

The fact that absolute returns should be preferred when estimating the variability of returns forces us to look at the generalisation of quadratic variation called multipower variation. This measure, specifically bipower variation, was originally proved to be robust to jumps in Barndorff-Nielsen and Shephard (2004a). In the case of a stochastic volatility plus infrequent jump process, the quadratic variation of the jump component can be estimated by the difference between the realised variance and realised bipower variation. This fact provides a tool to test for the presence of jumps in the log-price process (as in Andersen, Bollerslev and Diebold (2003), Barndorff-Nielsen and Shephard (2005) and Huang and Tauchen (2005)). Alternative estimators based on multipower variation are also robust to jumps, i.e. tripower and quadpower variation. We found their asymptotic distributions and developed a linear and ratio test for jumps both for these estimators and for the skipped version of bipower variation. To our knowledge this is the first time test for jumps have been implemented using tripower, quadpower and the skipped version of bipower variation. We could determine realised tripower and quadpower variation gave better results under the null hypothesis. Once jumps were added to the process the power of the tests will depend on the sample frequency and the size of the jumps, so it is not possible to give a conclusion about which of the estimators gave a better power. The addition of finite or infinite activity jump processes and the magnitude of the jumps will obviously determine the power of our test and the robustness of our asymptotic theories. This issue needs further

attention and has recently been examined in Barndorff-Nielsen, Shephard and Winkel (2005).

Once jumps have been accounted for, a perfect estimation of the integrated variance will be possible in the hypothetical situation where prices are observed in continuous time. This fact suggests that returns should be sampled at the highest possible frequency, but this leads to a bias problem due to market microstructure effects that can completely invalidate our previous estimations and tests. These effects cause intra-day returns to be autocorrelated. There is therefore a trade-off between bias and variance when choosing the sample frequency. There is an urgent need for estimators of the integrated variance which are robust to these effects. We studied realised bipower, tripower and quadpower variation under this perspective. These estimators were found to be more robust than realised variance, so by using absolute returns and adjacent observations the effect of the market microstructure noise can be reduced. Nevertheless with large noise variance or at very high frequencies, none of the estimators provided reliable results.

Based on the idea of Zhang, Mykland and Aït-Sahalia (2005), alternative estimators based on multipower variation were defined using the subsampling and averaging method. Their asymptotic distributions were derived, enabling us to look at their finite sample behaviour. This approach improved the results significantly and showed that the subsampled and averaged versions of realised tripower and quadpower variation outperformed all the other estimators. Notice the important improvements achieved when using two subgrids to define the estimators. It would be interesting to determine how much can be gained by increasing the number of subgrids.

By using a simplified model for the market microstructure noise, the bias for each estimator was found analytically. As expected, their bias tended to infinity as the sampling frequency increased. Nevertheless, because of the model used, we could not compare the robustness of the estimators under market microstructure noise.

We should point out that we assumed the noise to be independent and identically distributed across time and independent of the true price process. It has been proven that empirically this was not always true. In order to either confirm or invalidate our findings, a similar research should be carried out under a more general specification where the noise may be autocorrelated and need not be independent of the latent price process.

At the moment many efforts are concentrated on the correct estimation of integrated variance using high-frequency data. So far all the various difficulties encountered in this estimation have been addressed on the basis of realised variance. It has been the main focus of ongoing research. Our objective in this thesis was to prove how other estimators based on multipower variation could be as effective as realised variance, but with extra advantages (robustness to jumps and market microstruc-

ture effects). More work needs to be done but we can conclude estimators based on absolute returns seem to outperform those based on squared returns.

Chapter 7

Appendix: SPLUS and OX codes

In this appendix selected parts of the codes used in this thesis is included.

SPLUS code

I. Brownian bridge

SPLUS code to put the data in matrix form adding a Brownian Bridge wherever there are missing values.

```
matpricintbb<-matrix(0,79,504)
i<-1
s<-0
for (d in 14153:14882)
{m<-114
if (date[i]==d)
  s<-s+1
while (date[i]==d)
  {if ((min[i]<193)&&(min[i]>113))
    {if (min[i]==m)
      {matpricintbb[m-113,s]<-priceintelbb[i]
      i<-i+1
      m<-m+1}
    else
      { j<-0
      if (m==114)
        pr1<-matpricintbb[192,s-1]
```

```

else
  pr1<-matpricintbb[m-114,s]
while(min[i]!=m)
  {j<-j+1
  m<-m+1}
pri<-matrix(0,j+2,1)
pri[1]<-pr1; pri[j+2]<-priceintelbb[i]
n<-j+2
value<-matrix(0,n,1)
value[1]<-0
for (i in 2:n){
  ad<-rnorm(1,0,3/(n-1))
  value[i]<-value[i-1]+ad}
subs<-value[n]/(n-1)
for (i in 2:n){
  value[i]<-value[i]-((i-1)*subs)}
adpr<-(pri[n]-pri[1])/(n-1)
for (i in 2:n){
  value[i]<-value[i]+((i-1)*adpr)
pri[i]<-pri[1]+value[i]}
for (w in 1:j+1)
  matpricintbb[m-j+w-2-113,s]<-pri[w] } }
else
{if (i>1)
  y<-min[i-1]+1
if (i==1)
  y<-0
if (((min[i]>192)&&(y<193)))
  {j<-0
  pr1<-matpricintbb[y-114,s]
  while((y<193))
    {j<-j+1
    y<-y+1
    m<-m+1}
  pri<-matrix(0,j+2,1)
  pri[1]<-pr1; pri[j+2]<-priceintelbb[i]
  n<-j+2

```

```

value<-matrix(0,n,1)
value[1]<-0
for (i in 2:n){
  ad<-rnorm(1,0,5/(n-1))
  value[i]<-value[i-1]+ad}
subs<-value[n]/(n-1)
for (i in 2:n){
  value[i]<-value[i]-((i-1)*subs)}
adpr<-(pri[n]-pri[1])/(n-1)
for (i in 2:n){
  value[i]<-value[i]+((i-1)*adpr)
  pri[i]<-pri[1]+value[i]}
for (w in 1:j+1)
  {matpricintbb[m-j+w-2-113,s]<-pri[w]} }
if (((min[i]<114)&&(y<193)&&(y>114)))
  {j<-0
  pr1<-matpricintbb[y-114,s-1]
  while((y<193))
    {j<-j+1
    y<-y+1}
  pri<-matrix(0,j+2,1)
  pri[1]<-pr1; pri[j+2]<-priceintelbb[i]
  n<-j+2
  value<-matrix(0,n,1)
  value[1]<-0
  for (i in 2:n){
    ad<-rnorm(1,0,5/(n-1))
    value[i]<-value[i-1]+ad}
  subs<-value[n]/(n-1)
  for (i in 2:n){
    value[i]<-value[i]-((i-1)*subs)}
  adpr<-(pri[n]-pri[1])/(n-1)
  for (i in 2:n){
    value[i]<-value[i]+((i-1)*adpr)
    pri[i]<-pri[1]+value[i]}
  for (w in 1:j+1)
    {matpricintbb[y-j+w-2-113,s-1]<-pri[w]} }

```

```
i<-i+1 } } }
```

II. Combined estimation

SPLUS code for the model free combined estimation. It obtains the values for the matrix A and the vector c.

```
tshatsqcombsqreabvr<-matrix(0,504,60)
tshatsqcombrevl<-matrix(0,504,60)

m<-5
mu3<-sqrt(8/pi)
covsqravrv<-matrix(0,2,2)
covsqravrv[1,1]<-var(sqreabvr[,m])
covsqravrv[1,2]<-var(sqreabvr[,m],revl[,m])
covsqravrv[2,1]<-var(sqreabvr[,m],revl[,m])
covsqravrv[2,2]<-var(revl[,m])
invcovsqravrv<-solve(covsqravrv)
covsqerror<-matrix(0,2,2)
covsqerror[1,1]<-4*(((pi/2)-1)*erevlsqreabvr[m])/div[m]
covsqerror[1,2]<-2*(((mu3-sqrt(2/pi))/sqrt(2/pi))*ereabvarthrrev[m])/div[m]
covsqerror[2,1]<-2*(((mu3-sqrt(2/pi))/sqrt(2/pi))*ereabvarthrrev[m])/div[m]
covsqerror[2,2]<-(2*esqrevl[m])/div[m]
hatsqA<-(covsqravrv-covsqerror)%*%(invcovsqravrv)
hatsqc<-diag(c(1,1))-hatsqA
```

OX

III. Common functions

These functions are called in several OX programs. For convenience they will just be presented here.

```
dailysum(const multifact, const M, const leads, const n)
{ decl i,j;
decl dailyest=zeros(1,n);
for(i=0; i<n; i++)
  { for (j=M*i; j<((M*(i+1))-1-leads); j++)
```

```

    { dailyest[0][i]+=multifact[0][j]; } }
return dailyest; }

RVSG2F(const multifact, const M, const n)
{ decl i,j;
decl dailyest=zeros(1,n);
for(i=0; i<n; i++)
    { for (j=(M*i)+1; j<((M*(i+1))-2); j++)
        { dailyest[0][i]+=(multifact[0][j]+multifact[0][j+1])^2;
          j++; } }
return dailyest; }

CIR( const n, const v1, const a1, const lambda1, const delta, const xstart, xstop)
decl mY = zeros(2,n);
decl i,u1,u2;
decl phil = exp(-lambda1*delta);
decl omegal = sqrt( v1 /sqr(a1));
decl psil = v1/a1;
decl x = xstart;
for (i=0; i<n; i++)
    { u1 = (rann(2,1))';
      mY[0][i] = x*delta;
      x += (-lambda1*(x-psil)*delta) + (omegal*sqrt(x*lambda1*delta)*u1[0][0]);
      mY[1][i] = sqrt(mY[0][i])*u1[0][1];
    }
xstop[0] = x;
return mY; }

mr_r(const r)
{ return (2.0^(0.5*r))*exp(loggamma(0.5+(0.5*r))-loggamma(0.5)); }

meansecov(const tstat, const por)
{ decl biass4=meanr(tstat);
decl SEs4=sqrt(varr(tstat));
decl matres = (fabs(tstat).<quann(por)); }

maxi(const comp1, const comp2)
{ return comp1.*(comp1.>comp2) + comp2.*(comp1.<comp2); }

```

IV. Test for jumps

OX code to simulate a process with a jump component and to perform a test for jumps on it.

```
#include <oxstd.h>
#include <oxfloat.h>
main()
{decl Delta = 1.0, psil, omega1, delta, gamma, xstop;
decl i, M,n = 1000, v1 = 4.0, a1 = 8.0, mStar=1440;
decl lambda1 = -log(0.99)*ones(1,1);
decl fabsylag,RV,RAPV,ymax; decl j,z,z1,z2,z3,x,st,yt,yp;

decl noise = CIR(n*mStar,v1,a1,lambda1,1.0/mStar,(v1/a1),&xstop);
decl sseed = noise[0][];
decl yseed = noise[1][];
decl s0 = sseed;
decl y0=yseed;
decl q, nj, ij;
for (q=0; q<8; q++)
    if (q==0) nj=0, ij=0;
    if (q==1) nj=1, ij=0.5;
    if (q==2) nj=1, ij=0.2;
    if (q==3) nj=1, ij=0.1;
    if (q==4) nj=2, ij=0.5;
    if (q==5) nj=2, ij=0.2;
    if (q==6) nj=2, ij=0.1;
    if (q==7) nj=5, ij=0.1;

decl y1=y0; decl s1=s0;
decl jump_proc = zeros(1,n*mStar);
for (i=0; i<n; i++)
    { decl inp=mStar*i;
    decl finp= (mStar*(i+1))-1;
    decl ijump = ranmultinomial(nj,ones(1,1*mStar)/(1*mStar));
    jump_proc [inp:finp] = ij*((v1/a1)+(v2/a2)).*rann(1,1*mStar).*ijump; }
y1 += jump_proc;

decl RVerror,IVerror,CPPEerror, Aerror,Rjump;
```

```

decl qqx,logqqx,qqtstat,logqqtstat;
decl rtest1=new matrix[5][1],logrtest1=new matrix[5][1];
decl xresults1=new matrix[5][3];

for (i=1; i<6; i++)
  { if (i==1) M = 12;
    ...
    if (i==5) M = 576;

    jump2i = jump_proc.*jump_proc;
    jump2 = aggregator(jump2i,mStar/M);
    y = aggregator(y1,mStar/M);
    s=aggregator(s1,mStar/M);

    decl fabsylag1 = fabs(lag0(y',-1)'); decl fabsylag2 = fabs(lag0(y',-2)');
    decl fabsylag3 = fabs(lag0(y',-3)');
    decl RV = aggregator(y.*y,M);
    decl RAPV1 = (M/(M-1)).*dailysum((fabs(y).*fabsylag1),M,1,n)./sqr(mr_r(1.0));
    decl RTriV = (M/(M-2)).*(dailysum((fabs(y).^((2/3)).*(fabsylag1.^((2/3))
      .*fabsylag2.^((2/3))),M,2,n))./(mr_r(2.0/3.0)*sqr(mr_r(2.0/3.0))));
    decl RQuadV =(M/(M-3)).*(dailysum(sqrt(fabs(y).*fabsylag1
      .*fabsylag2.*fabsylag3),M,3,n))./(sqr(sqr(mr_r(1.0/2.0))));
    decl RModBV =(M/(M-2)).*dailysum((fabs(y).*fabsylag2),M,2,n)./sqr(mr_r(1.0));
    decl Rjump = RAPV1-RV;
    decl RjumpTV = RTriV-RV;
    decl RjumpQV = RQuadV-RV;
    decl RjumpMBV = RModBV-RV;
    decl porl=0.95; decl por=1-porl;

    decl s4=M.*M.*s.*s;
    decl s2=aggregator(s,M);
    decl s4a=(1.0/M)*aggregator(s4,M);
    decl E1s4 = (M*aggregator(y.*y.*y.*y,M))./(mr_r(4.0));
    decl E2s4 = ((M/(M-2))*M*dailysum((fabs(y).^((4/3)).*(fabsylag1.^((4/3))
      .*fabsylag2.^((4/3))),M,2,n))./(mr_r(4.0/3.0)*sqr(mr_r(4.0/3.0))));
    decl E3s4 = ((M/(M-3))*M*dailysum(fabs(y).*fabsylag1
      .*fabsylag2.*fabsylag3,M,3,n))./(sqr(sqr(mr_r(1.0))));

```

```

decl tstat1=((sqrt(M))./(sqrt((0.6091)*E1s4))).*Rjump;
decl rattstat1=((sqrt(M))./(sqrt((0.6091)*(maxi(1,E1s4./(RAPV1.*RAPV1)))))).*((RAPV1./RV)-1);
decl tstat1TV=((sqrt(M))./(sqrt((1.06131)*E1s4))).*RjumpTV;
decl rattstat1TV=((sqrt(M))./(sqrt((1.06131)*(maxi(1,E1s4./(RTriV.*RTriV)))))).*((RTriV./RV)-1);
decl tstat1QV=((sqrt(M))./(sqrt((1.37702)*E1s4))).*RjumpQV;
decl rattstat1QV=((sqrt(M))./(sqrt((1.37702)
      *(maxi(1,E1s4./(RQuadV.*RQuadV)))))).*((RQuadV./RV)-1);
decl tstat1MBV=((sqrt(M))./(sqrt((0.6091)*E1s4))).*RjumpMBV;
decl rattstat1MBV=((sqrt(M))./(sqrt((0.6091)
      *(maxi(1,E1s4./(RModBV.*RModBV)))))).*((RModBV./RV)-1);

meansecov(tstat1, por); meansecov(tstat1TV, por);
meansecov(tstat1QV, por); meansecov(tstat1MBV, por);
meansecov(rattstat1,por); meansecov(rattstat1TV,por);
meansecov(rattstat1QV,por); meansecov(rattstat1MBV,por);

RVErr = (aggregater(s,mStar)+aggregater(jump2,M))-RV;
RVErr = (aggregater(s,mStar))-RV;
IVErr = aggregater(s,mStar)-RAPV1;
CPPErr = aggregater(jump2,M)-(Rjump);
}}}
```

V. Real quantiles

We need to add the following OX code to the simulation to obtain the Normal quantile correspondent to the real size test (see Chapter 4).

```

decl matres= (rattstat3MBV.<quann(por));
decl meanw=meanr(matres);
while (meanw>0.05)
  { por-=0.0005;
  matres= (rattstat3MBV.<quann(por));
  meanw= meanr(matres); }
```

VI. Microstructure noise and subsampling

After simulating our price process, this is the part of the OX program that incorporates market microstructure noise and calculates the subsampling and averaging estimators.

```
...
decl varnoise=0.001;
decl p=cumulate(y)';
decl micrnoise=sqrt(varnoise).*rann(rows(p),columns(p));
decl pn=p+micrnoise;
decl yn=(diff0(pn',1))';
yn[][0]=yo[][0];
decl y1=yn;

for (i=1; i<7; i++)
    { if (i==1) M = 12;
      ...
      if (i==6) M = 1440;

      y = aggregator(y1,mStar/M);
      decl ysg=aggregator(y,2);
      decl y1s=y, y2s, y3s,j;
      for(j=0; j<n; j++)
          y2s=dropc(y1s,(j*M)-(2*j));
          y3s=dropc(y2s,(j*M)-(2*j)+(M-2));
          y1s=y3s;
      decl ysg2=aggregator(y1s,2);

      decl fabsy1lag1 = fabs(lag0(ysg',-1)'); decl fabsy1lag2 = fabs(lag0(ysg',-2)');
      decl fabsy1lag3 = fabs(lag0(ysg',-3)');
      decl fabsy2lag1 = fabs(lag0(ysg2',-1)'); decl fabsy2lag2 = fabs(lag0(ysg2',-2)');
      decl fabsy2lag3 = fabs(lag0(ysg2',-3)');

      decl RVSG1=aggregator((ysg.*ysg),M/2);
      decl RVSG2=aggregator((ysg2.*ysg2),(M-2)/2);
      decl RVSG=(RVSG1+RVSG2)/2;
      decl RBVSG1 = ((M/2)/((M/2)-1)).*dailysum((fabs(ysg).*fabsy1lag1),M/2,1,n)./sqr(mr_r(1.0));
      decl RBVSG2 = ((M/2)/((M/2)-2)).*dailysum((fabs(ysg2).*fabsy2lag1),(M/2-1),1,n)./sqr(mr_r(1.0));
```

```

decl RBVSG=(RBVSG1+RBVSG2)/2;
decl RTriVSG1 = ((M/2)/((M/2)-2)).*(dailysum((fabs(ysg).^(2/3)).*(fabsy1lag1.^(2/3))
      *(fabsy1lag2.^(2/3)),M/2,2,n))./(mr_r(2.0/3.0)*sqr(mr_r(2.0/3.0)));
decl RTriVSG2 = ((M/2)/((M/2)-3)).*(dailysum((fabs(ysg2).^(2/3)).*(fabsy2lag1.^(2/3))
      *(fabsy2lag2.^(2/3)),(M/2-1),2,n))./(mr_r(2.0/3.0)*sqr(mr_r(2.0/3.0)));
decl RTriVSG=(RTriVSG1+RTriVSG2)/2;
decl RQuadVSG1 =((M/2)/((M/2)-3)).*(dailysum(sqrt(fabs(ysg).*fabsy1lag1
      *fabsy1lag2.*fabsy1lag3),M/2,3,n))./(sqr(sqr(mr_r(1.0/2.0))));
decl RQuadVSG2 =((M/2)/((M/2)-4)).*(dailysum(sqrt(fabs(ysg2).*fabsy2lag1
      *fabsy2lag2.*fabsy2lag3),(M/2-1),3,n))./(sqr(sqr(mr_r(1.0/2.0))));
decl RQuadVSG=(RQuadVSG1+RQuadVSG2)/2;
decl RModBVSG1 =((M/2)/((M/2)-2)).*dailysum((fabs(ysg).*fabsy1lag2),M/2,2,n)./sqr(mr_r(1.0));
decl RModBVSG2 =((M/2)/((M/2)-3)).*dailysum((fabs(ysg2).*fabsy2lag2),(M/2-1),2,n)./sqr(mr_r(1.0));
decl RModBVSG=(RModBVSG1+RModBVSG2)/2;

decl stSGerror= (1.0./(sqrt((3/2)*s4a))).*sqrt(M/2).*(RVSG-s2);
decl stSGerrorB= (1.0./(sqrt((3.581/2)*s4a))).*sqrt(M/2).*(RBVSG-s2);
decl stSGerrorT= (1.0./(sqrt((4.067/2)*s4a))).*sqrt(M/2).*(RTriVSG-s2);
decl stSGerrorQ= (1.0./(sqrt((4.389/2)*s4a))).*sqrt(M/2).*(RQuadVSG-s2);
decl stSGerrorMB= (1.0./(sqrt((3.581/2)*s4a))).*sqrt(M/2).*(RModBVSG-s2);

meansecov(stSGerror, por); meansecov(stSGerrorB, por);
meansecov(stSGerrorT, por); meansecov(stSGerrorQ, por);
meansecov(stSGerrorMB, por); }

```

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