

On local constraints and regularity of
PDE in electromagnetics.
Applications to hybrid imaging
inverse problems



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This thesis is dedicated to
my wife Gessica
for her great love and fundamental support

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Abstract

The first contribution of this thesis is a new regularity theorem for time harmonic Maxwell's equations with less than Lipschitz complex anisotropic coefficients. By using the L^p theory for elliptic equations, it is possible to prove H^1 and Hölder regularity results, provided that the coefficients are $W^{1,p}$ for some $p > 3$. This improves previous regularity results, where the assumption $W^{1,\infty}$ for the coefficients was believed to be optimal. The method can be easily extended to the case of bi-anisotropic materials, for which a separate approach turns out to be unnecessary.

The second focus of this work is the boundary control of the Helmholtz and Maxwell equations to enforce local constraints inside the domain. More precisely, we look for suitable boundary conditions such that the corresponding solutions and their derivatives satisfy certain local non-zero constraints. Complex geometric optics solutions can be used to construct such illuminations, but are impractical for several reasons. We propose a constructive approach to this problem based on the use of multiple frequencies. The suitable boundary conditions are explicitly constructed and give the desired constraints, provided that a finite number of frequencies, given a priori, are chosen in a fixed range. This method is based on the holomorphicity of the solutions with respect to the frequency and on the regularity theory for the PDE under consideration.

This theory finds applications to several hybrid imaging inverse problems, where the unknown coefficients have to be imaged from internal measurements. In order to perform the reconstruction, we often need to find suitable boundary conditions such that the corresponding solutions satisfy certain non-zero constraints, depending on the particular problem under consideration. The multiple frequency approach introduced in this thesis represents a valid alternative to the use of complex geometric optics solutions to construct such boundary conditions. Several examples are discussed.

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Chapter 1

Introduction

Two of the main questions in the analysis of partial differential equations are the regularity and the boundary control of solutions. Regularity theory allows to infer smoothness of solutions from suitable smoothness assumptions on the coefficients. Broadly speaking, boundary control of PDE is the search for suitable boundary conditions such that the corresponding solutions satisfy certain properties. This thesis deals with these two problems for stationary PDE in electromagnetics, namely the Helmholtz and Maxwell equations. Moreover, several applications to hybrid imaging inverse problems are discussed.

As far as regularity is concerned, the theory is very developed for elliptic PDEs, whence for the Helmholtz equation. On the other hand, for time-harmonic Maxwell's equations the situation is more complicated. For example, the assumption of Lipschitz continuity on the coefficients was believed to be optimal to have Hölder continuous solutions. However, by using the L^p theory for elliptic PDE and a formulation of Maxwell's equations in terms of a coupled elliptic system, we show that it is possible to significantly lower the regularity assumption on the coefficients. Moreover, the method can be easily applied to the case of complex bi-anisotropic material.

In this thesis, by boundary control we mean the search for suitable boundary conditions such that the solutions to the Helmholtz and Maxwell equations and their derivatives satisfy certain non-zero constraints inside the domain. The problem is not trivial since we are interested in studying the non-constant coefficient case. Moreover, the construction of these boundary conditions should be independent of the coefficients for the applications to inverse problems we consider. Classically, these boundary conditions are constructed by means of the complex geometric optics (CGO) solutions. However, this construction depends on the coefficients and is numerically impracticable for several reasons. In this work, we show how a multiple frequency approach can solve these issues. This theory relies upon the regularity properties for Maxwell's equations we have derived.

Finally, we show how this multi-frequency method can be applied to several hybrid imaging inverse problems. As detailed below, these techniques have been established in the last years in order to tackle the issues of the current imaging modalities, in particular to obtain images with high resolution and high contrast. The problem is usually modelled

by the Helmholtz or Maxwell equations, and it is fundamental to be able to control the behaviour of the solutions inside the domain. Since the coefficients are unknown, the multi-frequency approach discussed in this thesis represents a valid alternative to the use of CGO solutions.

In the following three sections, these three topics are illustrated in more detail, and full details are given in Chapters 2, 3 and 4, respectively. Finally, in Chapter 5 some open problems are discussed and Appendix A contains some of the codes used for the numerical simulations.

Most of the content of this thesis is also contained in these journal articles:

- [1] G. S. Alberti. On multiple frequency power density measurements. *Inverse Problems*, 29(11):115007, 25, 2013.
- [2] G. S. Alberti. On multiple frequency power density measurements II. The full Maxwell's equations. *arXiv preprint*, arXiv:1311.7603, 2013.
- [3] G. S. Alberti. Enforcing local non-zero constraints in PDEs and applications to hybrid imaging problems. *arXiv preprint*, arXiv:1406.3248, 2014.
- [4] G. S. Alberti and Y. Capdeboscq. Elliptic Regularity Theory Applied to Time Harmonic Anisotropic Maxwell's Equations with Less than Lipschitz Complex Coefficients. *SIAM J. Math. Anal.*, 46(1):998–1016, 2014.

1.1 Regularity theorems for Maxwell's equations

Let $\Omega \subseteq \mathbb{R}^3$ be a bounded domain (connected open set) in $\mathbb{R}^3$, with $C^{1,1}$ boundary. Let $\varepsilon, \sigma \in L^\infty(\Omega; \mathbb{R}^{3 \times 3})$ and $\mu \in L^\infty(\Omega; \mathbb{C}^{3 \times 3})$ be matrix-valued functions such that

$$(1.1) \quad \begin{aligned} \Lambda^{-1} |\xi|^2 &\leq \xi \cdot (\Re \mu) \xi, \quad \Lambda^{-1} |\xi|^2 \leq \xi \cdot \sigma \xi, \quad \xi \in \mathbb{R}^3, \\ \varepsilon^T &= \varepsilon, \quad \Im \mu^T = \Im \mu, \quad \|(\mu, \sigma, \varepsilon)\|_{L^\infty(\Omega)^3} \leq \Lambda \end{aligned}$$

for some $\Lambda > 0$. The 3×3 matrix ε represents the electric permittivity, σ is the electric conductivity and μ stands for the complex magnetic permeability.

Consider a given frequency $\omega \in \mathbb{C}$ and current sources

$$(1.2) \quad \varphi \in H(\text{curl}, \Omega), \quad J_e, J_m \in L^2(\Omega; \mathbb{C}^3), \quad \text{div} J_m = 0 \text{ in } \Omega, \quad (\text{curl} \varphi - J_m) \cdot \nu = 0 \text{ on } \partial\Omega,$$

where ν denotes the outward unit normal to $\partial\Omega$. We are interested in the regularity of the time-harmonic electromagnetic fields E and H , that is, the weak solutions $E, H \in H(\text{curl}, \Omega)$ of the time-harmonic Maxwell's equations

$$(1.3) \quad \begin{cases} \text{curl} E = \mathbf{i} \omega \mu H + J_m & \text{in } \Omega, \\ \text{curl} H = -\mathbf{i}(\omega \varepsilon + \mathbf{i} \sigma) E + J_e & \text{in } \Omega, \\ E \times \nu = \varphi \times \nu & \text{on } \partial\Omega. \end{cases}$$

For simplicity, we set $q_\omega = \omega\varepsilon + \mathbf{i}\sigma$. Our focus is the dependence of the regularity of E and H on the coefficients μ , ε and σ , the current sources J_e and J_m , and the boundary condition φ . The precise dependence on the regularity of the boundary of Ω is beyond the scope of this thesis. We refer the reader to [23, 59, 48] where domains with rougher boundaries are considered. For $\kappa \in \mathbb{N}^*$ and $p > 1$ we denote by $W^{\kappa,p}(\text{curl}, \Omega)$ and $W^{\kappa,p}(\text{div}, \Omega)$ the Banach spaces

$$\begin{aligned} W^{\kappa,p}(\text{curl}, \Omega) &= \{v \in W^{\kappa-1,p}(\Omega; \mathbb{C}^3) : \text{curl} v \in W^{\kappa-1,p}(\Omega; \mathbb{C}^3)\}, \\ W^{\kappa,p}(\text{div}, \Omega) &= \{v \in W^{\kappa-1,p}(\Omega; \mathbb{C}^3) : \text{div} v \in W^{\kappa-1,p}(\Omega; \mathbb{C})\}, \end{aligned}$$

equipped with canonical norms. The space $W^{1,2}(\text{curl}, \Omega)$ is the space $H(\text{curl}, \Omega)$ mentioned above, whereas $W^{1,2}(\text{div}, \Omega)$ is commonly denoted by $H(\text{div}, \Omega)$ and $H^\mu(\text{curl}, \Omega)$ is defined in (2.21).

It is very well known that when the domain is a cylinder $\Omega' \times (0, L)$, the electric field E has only one component, $E = (0, 0, u)^T$, the physical parameters are real, scalar and do not depend on the third variable, then u satisfies a second order elliptic equation in the first two variables

$$\text{div}(\mu^{-1} \nabla u) + \omega^2 \varepsilon u = 0 \text{ in } \Omega'.$$

In such a case, the regularity of u follows from classical elliptic regularity theory (see Proposition 2.5). In particular, u is Hölder continuous due to the De Giorgi–Nash Theorem (at least in the interior). The regularity of E and H is less clear in the general situation, when the material parameters are anisotropic and/or complex valued. Assuming that the coefficients are real, anisotropic, suitably smooth matrices, Leis [83] established well-posedness in $H^1(\Omega)$. The regularity of the coefficients was reduced to globally Lipschitz in Weber [109], for a C^2 smooth boundary, and C^1 for a $C^{1,1}$ domain in Costabel [58].

Neither the H^1 nor the Hölder regularity of the electric and magnetic fields for complex anisotropic less than Lipschitz media have been addressed so far. Anisotropic dielectric parameters have received a renewed attention in the last decades. They appear for example in the mathematical theory of liquid crystals, in optically chiral media, and in meta-materials. In this thesis we show that the theory of elliptic boundary value problems can be used to study the general case of complex anisotropic coefficients.

The main result reads as follows.

Theorem 1.1. *Assume that (1.1) and (1.2) hold, and that*

$$\begin{aligned} \varepsilon, \sigma &\in W^{1,3+\delta}(\Omega; \mathbb{R}^{3 \times 3}), \quad \mu \in W^{1,3+\delta}(\Omega; \mathbb{C}^{3 \times 3}) \text{ for some } \delta > 0, \\ J_m &\in L^p(\Omega; \mathbb{C}^3), \quad J_e \in W^{1,p}(\text{div}, \Omega) \text{ and } \varphi \in W^{1,p}(\Omega; \mathbb{C}^3) \text{ for some } p \geq 2. \end{aligned}$$

If $(E, H) \in H(\text{curl}, \Omega) \times H^\mu(\text{curl}, \Omega)$ is a weak solution of (1.3) with $|\omega| \leq M$ for some $M > 0$, then $E, H \in W^{1,q}(\Omega; \mathbb{C}^3)$ with $q = \min(p, 3 + \delta)$ and

$$\|(E, H)\|_{W^{1,q}(\Omega; \mathbb{C}^3)^2} \leq C(\|(E, H)\|_{L^2(\Omega; \mathbb{C}^3)^2} + \|\varphi\|_{W^{1,p}(\Omega; \mathbb{C}^3)} + \|J_e\|_{W^{1,p}(\text{div}, \Omega)} + \|J_m\|_{L^p(\Omega; \mathbb{C}^3)})$$

for some constant C depending on Ω , Λ , δ , M , q and $\|(q_\omega, \mu)\|_{W^{1,3+\delta}(\Omega; \mathbb{C}^{3 \times 3})^2}$ only. In particular, if $p > 3$, then $E, H \in C^{0,\alpha}(\bar{\Omega}; \mathbb{C}^3)$ with $\alpha = \min(1 - \frac{3}{p}, \frac{\delta}{3+\delta})$.

Our approach is classical and fundamentally scalar. It is oblivious of the fact that Maxwell's equations is posed on vectors, as we consider the problem component per component, just like it is done by Leis in [84]. Namely, Maxwell's equations are written as a coupled elliptic system, to which we apply the L^p theory for elliptic equations. A general L^p theory for vector potentials has been developed very recently by Amrouche & Seloula [24, 25]. Applying their results would lead to similar regularity results for scalar coefficients.

We then study the case when only one of the two coefficients is complex-valued. We consider the case when $\varepsilon, \sigma \in W^{1,3+\delta}(\Omega; \mathbb{R}^{3 \times 3})$ and $\mu \in L^\infty(\Omega; \mathbb{R}^{3 \times 3})$. In that situation, a Helmholtz decomposition of the magnetic field into $H = T + \nabla h$, where $T \in H^1(\Omega)$ is divergence free, provides additional insight on the regularity of H . Indeed, the potential h then satisfies a real scalar second order elliptic equation, and therefore enjoys additional regularity properties.

Theorem 1.2. *Assume that (1.1) and (1.2) hold, that Ω is simply connected, $\Im \mu = 0$, $M > 0$ and that*

$$\begin{aligned} \varepsilon, \sigma &\in W^{1,3+\delta}(\Omega; \mathbb{R}^{3 \times 3}) \text{ for some } \delta > 0, \\ J_m &\in L^p(\Omega; \mathbb{C}^3), J_e \in W^{1,p}(\text{div}, \Omega) \text{ and } \varphi \in W^{1,p}(\Omega; \mathbb{C}^3) \text{ for some } p > 3. \end{aligned}$$

If $(E, H) \in H(\text{curl}, \Omega) \times H^\mu(\text{curl}, \Omega)$ is a weak solution of (1.3) with $|\omega| \leq M$, then there exists $0 < \alpha \leq \min(1 - \frac{3}{p}, \frac{\delta}{3+\delta})$ depending only on Ω and Λ given in (1.1) such that $E \in C^{0,\alpha}(\bar{\Omega}; \mathbb{C}^3)$ with

$$\|E\|_{C^{0,\alpha}(\bar{\Omega}; \mathbb{C}^3)} \leq C(\|E\|_{L^2(\Omega)} + \|\varphi\|_{W^{1,p}(\Omega; \mathbb{C}^3)} + \|J_e\|_{W^{1,p}(\text{div}, \Omega)} + \|J_m\|_{L^p(\Omega; \mathbb{C}^3)})$$

for some constant C depending on Ω , Λ , δ , M and $\|q_\omega\|_{W^{1,3+\delta}(\Omega; \mathbb{C}^{3 \times 3})}$ only.

This is a generalisation of the result proved by Yin [112] who assumed instead $\varepsilon, \sigma \in W^{1,\infty}(\Omega; \mathbb{R})$ and $\mu \in L^\infty(\Omega; \mathbb{R})$ and claimed optimality: this is not the minimal regularity requirement to prove Hölder continuity of the electric field.

Finally, we show that, as far as interior regularity is concerned, the analogue of Theorem 1.1 holds for more general constitutive relations, for which Maxwell's equations read

$$(1.4) \quad \begin{cases} \text{curl} E = \mathbf{i}\omega(\zeta E + \mu H) + J_m & \text{in } \Omega, \\ \text{curl} H = -\mathbf{i}\omega(\varepsilon E + \xi H) + J_e & \text{in } \Omega, \end{cases}$$

provided that $\zeta, \xi \in L^\infty(\Omega; \mathbb{C}^{3 \times 3})$ are small enough to preserve the underlying elliptic structure of the corresponding coupled elliptic system. These constitutive relations are commonly used to model the so called bi-anisotropic materials [64].

We do not claim that requiring that (one of) the parameters is in $W^{1,3+\delta}$ for some $\delta > 0$ is optimal. We are confident that it is sufficient to assume $W^{1,3}$ regularity. However, this does not seem to work with this proof: the bootstrap argument used stalls in this case.

1.2 Using multiple frequencies to enforce local constraints in PDE

1.2.1 The Helmholtz equation

Let $d = 2$ or $d = 3$ be the dimension of the ambient space, $\Omega \subseteq \mathbb{R}^d$ be a smooth bounded domain and consider the Helmholtz equation

$$(1.5) \quad \begin{cases} -\operatorname{div}(a \nabla u_\omega^\varphi) - (\omega^2 \varepsilon + \mathbf{i} \omega \sigma) u_\omega^\varphi = 0 & \text{in } \Omega, \\ u_\omega^\varphi = \varphi & \text{on } \partial\Omega, \end{cases}$$

where $a \in L^\infty(\Omega; \mathbb{R}^{d \times d})$ is a real and uniformly elliptic symmetric tensor with ellipticity constant $\Lambda > 0$ and $\varepsilon, \sigma \in L^\infty(\Omega; \mathbb{R})$ satisfy $\Lambda^{-1} \leq \varepsilon \leq \Lambda$ and either $\sigma = 0$ or $\Lambda^{-1} \leq \sigma \leq \Lambda$. Let $\mathcal{A} = [K_{\min}, K_{\max}] \subseteq \mathbb{R}_+$ represent the set of admissible frequencies ω for some $0 < K_{\min} < K_{\max}$.

We want to find suitable illuminations φ_i such that the corresponding solutions to (1.5) satisfy certain non-zero constraints in Ω . For example, we may look for $d + 1$ illuminations $\varphi_1, \dots, \varphi_{d+1}$ such that for some $C > 0$

$$(1.6) \quad |u_\omega^{\varphi_1}(x)| \geq C, \quad \det [\nabla u_\omega^{\varphi_2} \quad \dots \quad \nabla u_\omega^{\varphi_{d+1}}](x) \geq C, \quad \det \begin{bmatrix} u_\omega^{\varphi_1} & \dots & u_\omega^{\varphi_{d+1}} \\ \nabla u_\omega^{\varphi_1} & \dots & \nabla u_\omega^{\varphi_{d+1}} \end{bmatrix}(x) \geq C,$$

or, more generally, for b illuminations $\varphi_1, \dots, \varphi_b$ such that the corresponding solutions verify r conditions given by

$$(1.7) \quad |\zeta^j(u_\omega^{\varphi_1}, \dots, u_\omega^{\varphi_b})(x)| \geq C, \quad j = 1, \dots, r,$$

where the maps ζ^j depend on $u_\omega^{\varphi_i}$ and their derivatives. As discussed in Section 1.3, these conditions are motivated by the reconstruction algorithms of several hybrid imaging techniques. More precisely, we are interested in the following class of sets of measurements.

Definition 1.3. Take $\Omega' \subseteq \Omega$. Let $b, r \in \mathbb{N}^*$ and $C > 0$. Given a finite subset $K \subseteq \mathcal{A}$, a set of measurements $K \times \{\varphi_1, \dots, \varphi_b\}$ is (ζ, C) -complete in Ω' if for every $x \in \overline{\Omega'}$ there exists $\omega_x \in K$ such that

$$(1.8) \quad |\zeta^j(u_{\omega_x}^{\varphi_1}, \dots, u_{\omega_x}^{\varphi_b})(x)| \geq C, \quad j = 1, \dots, r.$$

In particular, the constraints given in (1.6) characterise $(\zeta_{\det}, C)$ -complete sets, where $\zeta_{\det}$ is defined by

$$\begin{aligned} \zeta_{\det}^1(u^1, \dots, u^{d+1}) &= u^1, \\ \zeta_{\det}^2(u^1, \dots, u^{d+1}) &= \det [\nabla u^2 \quad \dots \quad \nabla u^{d+1}], \\ \zeta_{\det}^3(u^1, \dots, u^{d+1}) &= \det \begin{bmatrix} u^1 & \dots & u^{d+1} \\ \nabla u^1 & \dots & \nabla u^{d+1} \end{bmatrix}. \end{aligned}$$

The problem of constructing (ζ, C) -complete sets is usually set for a fixed frequency $\omega \in \mathcal{A}$. The classical way to tackle this problem is by means of the so called complex

geometric optics solutions. Introduced by Calderón [51] and developed by Sylvester and Uhlmann [102], CGO solutions are particular highly oscillatory solutions of the Helmholtz equation in $\mathbb{R}^d$ such that for $t \gg 1$ ($a = 1$, $d = 2$)

$$u^{(t)}(x) \approx e^{tx_1} (\cos(tx_2) + \mathbf{i} \sin(tx_2)) \quad \text{in } C^1(\overline{\Omega}; \mathbb{C}),$$

and can be used to determine suitable illuminations by using the estimates proved by Bal and Uhlmann [36] (see also [31, 30, 19]). For example, setting $\varphi_1 \approx u|_{\partial\Omega}^{(t)}$, $\varphi_2 \approx \Re u|_{\partial\Omega}^{(t)}$ and $\varphi_3 \approx \Im u|_{\partial\Omega}^{(t)}$ gives an open set of illuminations whose solutions satisfy the first two constraints of (1.6). Thus, CGO solutions represent a very important theoretical tool. However, this approach presents several drawbacks. First, the suitable boundary conditions can only be constructed when the parameters are very smooth. Second, since $t \gg 1$, the exponential decay in the first variable gives very small lower bounds C and the high oscillations make this approach hardly implementable. Furthermore, the construction depends on the coefficients a , ε and σ , that are usually unknown in inverse problems.

In this thesis we propose an alternative constructive strategy to this issue based on the use of multiple frequencies in the fixed range $\mathcal{A} = [K_{\min}, K_{\max}]$. Namely, given the maps ζ^j , we shall give conditions on the illuminations such that the corresponding solutions satisfy the required properties, provided that a finite number of frequencies are used in the range $\mathcal{A}$. These conditions may depend on a , and never depend on ε and σ . More precisely, we state the main results regarding the constraints in (1.6). Let $K^{(n)}$ be the uniform partition of $\mathcal{A}$ into $n - 1$ intervals, so that $\#K^{(n)} = n$. We start with the two-dimensional case.

Theorem 1.4. *Assume $a \in C^{0,1}(\overline{\Omega}; \mathbb{R}^{2 \times 2})$ and that $\Omega \subseteq \mathbb{R}^2$ is convex. If $\Omega' \Subset \Omega$ then there exist $C > 0$ and $n \in \mathbb{N}$ depending Ω , Ω' , Λ , $\mathcal{A}$ and $\|a\|_{C^{0,1}(\overline{\Omega}; \mathbb{R}^{2 \times 2})}$ such that*

$$K^{(n)} \times \{1, x_1, x_2\}$$

is $(\zeta_{\det}, C)$ -complete in Ω' .

In three dimensions the situation is more complicated and we shall assume that a is close to a constant matrix. For simplicity, here we take $a = 1$.

Theorem 1.5. *Assume $a = 1$ and that $\Omega \subseteq \mathbb{R}^3$. There exist $C > 0$ and $n \in \mathbb{N}$ depending on Ω , Λ and $\mathcal{A}$ such that*

$$K^{(n)} \times \{1, x_1, x_2, x_3\}$$

is $(\zeta_{\det}, C)$ -complete in Ω .

The main idea behind this method is simple: if the illuminations are suitably chosen then the zero level sets of functionals depending on u_ω^φ move when the frequency changes. The main steps of the proof are as follows.

1. The map $\omega \mapsto u_\omega^\varphi \in C^1(\overline{\Omega}; \mathbb{C})$ is holomorphic: this is a consequence of classical elliptic regularity theory and of the structure of the equation.

2. The constraints (1.6) are satisfied in $\omega = 0$ for some $C_0 > 0$: in the two-dimensional case, this is true for some $C_0 > 0$ depending on Ω and Λ only by a result of Alessandrini [6] (see also [10, 41]), and in three dimensions this is trivial if a is constant, with $C_0 = 1$.
3. Momm's lemma [85]: Let $0 < r < R$ and g be holomorphic in $\{|\omega| < R\}$. If $|g(0)| \geq C_0$ then there exists $\omega \in (r, R)$ such that $|g(\omega)| \geq C(r, R, C_0, \sup |g|) > 0$.
For every $x \in \Omega$, we apply this lemma to $\omega \mapsto \zeta_{\det}^j(u_{\omega}^{\varphi^1}, \dots, u_{\omega}^{\varphi^{d+1}})(x)$, and obtain that there exists $\omega_x \in \mathcal{A}$ such that $|\zeta_{\det}^j(u_{\omega_x}^{\varphi^1}, \dots, u_{\omega_x}^{\varphi^{d+1}})(x)| \geq C > 0$.
4. Finally, a bound on $\|\partial_{\omega} u_{\omega}^{\varphi}\|_{C^1(\bar{\Omega}; \mathbb{C})}$ shows that the above condition is satisfied for ω close to ω_x , and this gives the result if n is chosen big enough.

Step 2 clarifies the difference between the two and the three dimensional cases. In three dimensions, results regarding critical points similar to the ones discussed in [6] are false [47], and the assumption $a \approx 1$ is necessary for this proof to work (see also the recent work by Bal and Courdurier [32] for a different approach to the case $\omega = 0$ in 3D).

With this method, the drawbacks of CGO solutions are mostly solved:

- The smoothness of the coefficients required for the multi-frequency method is lower than the smoothness required to construct CGO solutions. Indeed, consider for simplicity the constraints introduced in (1.6) and suppose $a = 1$ and $\sigma = 0$. The CGO approach requires $\varepsilon \in C^1(\bar{\Omega})$ [36], while with this method we only need to assume $\varepsilon \in L^{\infty}(\Omega; \mathbb{R})$. In general, the regularity assumption for this theory is in some sense minimal: we only need the constraints given by (1.7) to be meaningful in every $x \in \Omega$, namely $\zeta^j(u_{\omega}^{\varphi^1}, \dots, u_{\omega}^{\varphi^b}) \in C(\Omega)$. In other words, if the maps ζ^j depend on the derivatives of $u_{\omega}^{\varphi^i}$ up to the κ -th order, then we shall assume that the coefficients a , ε and σ are smooth enough so that $u_{\omega}^{\varphi^i} \in C^{\kappa}(\Omega; \mathbb{C})$.
- The lower bound C given by the CGO solutions may be very small and depend on the unknowns a , ε and σ . This method gives a lower bound C that depends on the coefficients only through the ellipticity constant and their smoothness, which can be considered as a priori data.
- The construction of suitable CGO illuminations strongly depends on the coefficients. With this approach, the construction of good illuminations is always independent of ε and σ , since ε and σ disappear when $\omega = 0$, and may depend on a . In particular, when dealing with the constraints in (1.6), we have seen that in two dimensions the construction is also independent of a .

As it is clear from the sketch of the proof, this machinery can be used in different other situations. In particular, there is no need to consider the particular constraints given in (1.6). General constraints as in (1.7) can be considered, as long as they are satisfied in $\omega = 0$ and the maps ζ^j depend holomorphically on $u_{\omega}^{\varphi^i}$ and satisfy a certain growth condition, that is verified by all relevant examples.

It is worth mentioning that this approach has been recently successfully adapted to the conductivity equation with complex coefficients by Ammari et al. in [20].

1.2.2 Maxwell's equations

It is natural to generalise this approach to the full Maxwell's equations, as the Helmholtz model often is merely an approximation of this system for hybrid imaging. Maxwell's system of equations reads

$$(1.9) \quad \begin{cases} \operatorname{curl} E_\omega^\varphi = \mathbf{i}\omega\mu H_\omega^\varphi & \text{in } \Omega, \\ \operatorname{curl} H_\omega^\varphi = -\mathbf{i}(\omega\varepsilon + \mathbf{i}\sigma)E_\omega^\varphi & \text{in } \Omega, \\ E_\omega^\varphi \times \nu = \varphi \times \nu & \text{on } \partial\Omega. \end{cases}$$

As before, we look for illuminations φ_i and frequencies ω such that the corresponding solutions verify r conditions given by

$$(1.10) \quad |\zeta^j((E_\omega^{\varphi_1}, H_\omega^{\varphi_1}), \dots, (E_\omega^{\varphi_b}, H_\omega^{\varphi_b}))(x)| \geq C > 0, \quad j = 1, \dots, r,$$

where the maps ζ^j depend on $(E_\omega^{\varphi_i}, H_\omega^{\varphi_i})$ and their derivatives. An example of such conditions with $r = 1$ and $b = 3$ is given by

$$(1.11) \quad |\det [E_\omega^{\varphi_1} \quad E_\omega^{\varphi_2} \quad E_\omega^{\varphi_3}](x)| \geq C > 0.$$

More precisely, we are interested in the following class of sets of measurements.

Definition 1.6. Let $b, r \in \mathbb{N}^*$, $C > 0$ and K be a finite subset of $\mathcal{A}$. A set of measurements $K \times \{\varphi_1, \dots, \varphi_b\}$ is (ζ, C) -complete if for every $x \in \overline{\Omega}$ there exists $\omega_x \in K$ such that

$$(1.12) \quad |\zeta^j((E_{\omega_x}^{\varphi_1}, H_{\omega_x}^{\varphi_1}), \dots, (E_{\omega_x}^{\varphi_b}, H_{\omega_x}^{\varphi_b}))(x)| \geq C, \quad j = 1, \dots, r.$$

In particular, the constraints given in (1.11) define $(\zeta_{\det}^M, C)$ -complete sets, where $\zeta_{\det}^M$ is defined by

$$(1.13) \quad \zeta_{\det}^M((u^1, v^1), (u^2, v^2), (u^3, v^3)) = \det [u^1 \quad u^2 \quad u^3].$$

Complex geometric optics solutions for Maxwell's equations have been studied by Colton and Päiväranta [57]. As before, they can be used to obtain suitable solutions [55], but have the drawbacks discussed before.

By using the regularity results discussed in the previous section, it is possible to extend the multi-frequency approach to this case with some changes. In particular, the case $\omega = 0$ has to be carefully studied since $\omega = 0$ is an eigenvalue with infinite multiplicity of the above problem.

As in the previous case, step 2 of the proof requires that the constraints given by (1.10) are satisfied in $\omega = 0$. In this case, ε disappears from system (1.9) and so the construction will always be independent of ε but may depend on μ and σ . However, if the constraints are independent of H then the construction will also be independent of μ .

In the next section, we illustrate how this theory can be applied to several hybrid imaging inverse problems.

1.3 Applications to hybrid imaging inverse problems

Nowadays, many medical imaging modalities are available in our hospitals and laboratories. Among the most known, we can mention the X-ray Computerised Tomography (CT), Magnetic Resonance Imaging (MRI) and Ultrasound Imaging (UI). Other techniques, such as Optical Tomography (OT), Electrical Impedance Tomography (EIT) and Microwave Imaging, are less widespread but well developed. The reader is referred to [13, 95] for a survey on the general topic of imaging techniques. Although this variety of modalities, over the last years much research has been done on the physical and mathematical aspects of medical imaging techniques to overcome their drawbacks. One of the main problems relies on the need for images with high resolution and high contrast. Unfortunately, modalities such as CT, MRI and UI provide high resolution images but fail to exhibit the contrast between different types of tissues. On the other hand, other modalities like OT, EIT and Microwave Imaging display good contrast but low resolution, since the related inverse problems are severely ill-posed. For instance, the mathematical formulation of the EIT leads to the well-known Calderón problem [75, 76, 102, 46, 27, 8, 51, 9, 43].

In order to tackle this problem, *hybrid* (or *coupled physics*) imaging techniques have been developed. By combining measurements coming from two different modalities it is possible to obtain high-resolution and high-contrast images. The reader is referred to the works by Ammari [22], Kuchment [77], Arridge and Scherzer [26] and Bal [30] for a review of the state of the art in hybrid techniques. Many possible couplings have been studied over the last decade, such as optical with ultrasonic waves [36, 35, 19], electric currents with ultrasonic waves [14, 53, 79, 18, 31, 111], EIT with MRI [98, 99, 97] and microwaves with ultrasounds [105, 16, 15].

Generally, a hybrid problem involves two steps. First, internal functionals are measured inside the domain and, second, from their knowledge the unknown coefficients of the PDE have to be reconstructed. In other words, the Calderón problem is characterised by boundary measurements, whereas hybrid problems rely upon internal measurements.

Many hybrid problems are governed by the Helmholtz equation (1.5), e.g. microwave imaging by ultrasound deformation [105, 16, 5], quantitative thermo-acoustic [35, 19], transient elastography and magnetic resonance elastography [17, 82, 37] (for which the Helmholtz equation is used as a one-dimensional approximation). The internal measurements are always linear or quadratic functionals of u_ω^φ and of ∇u_ω^φ . For example, in microwave imaging by ultrasound deformation, that is modelled by (1.5) with a scalar-valued a and $\sigma = 0$, the internal measurements have the form

$$a(x) |\nabla u_\omega^\varphi|^2(x), \quad \varepsilon(x) |u_\omega^\varphi|^2(x), \quad x \in \Omega,$$

in thermo-acoustic, modelled by (1.5) with $a = \varepsilon = 1$ and $\sigma > 0$, we measure

$$\sigma(x) |u_\omega^\varphi|^2(x), \quad x \in \Omega,$$

and with transient elastography the internal measurements are

$$u_{\omega}^{\varphi}(x), \quad x \in \Omega.$$

In order for these measurements to be meaningful at every $x \in \Omega$, they need to be non-zero: otherwise, we would measure only noise. Moreover, we shall see that conditions like (1.6) or, more generally, (1.7) for some map ζ , are necessary to reconstruct the unknown parameters a , ε and/or σ or to obtain good stability estimates [105, 80, 37]. Thus, being able to determine suitable illuminations independently of the unknown parameters is fundamental, and these can be given by the multi-frequency approach developed in this thesis and discussed in the previous section. It should be mentioned that stability of Hölder type has been proved by Alessandrini in the context of microwave imaging by ultrasound deformation with $a = 1$ without requiring any non-zero constraint [7].

It is worth observing that the nature of this approach, for which the frequency ω depends on the points in the domain, is better suited in cases where the internal data are collected locally, as for instance in microwave imaging by ultrasound deformation. On the other hand, it is more restrictive when the internal data are measured in the whole domain, as in thermo-acoustic imaging. Namely, we will have redundant measurements in some parts of the domain, where (1.7) is satisfied for two or more frequencies.

Similarly, several problems are modelled by the Maxwell's equations (1.9) [97, 33, 38, 55], and the second step usually requires the availability of solutions satisfying certain non-zero constraints inside the domain, such as (1.11) or, more generally, (1.10), for some maps ζ^j depending on the particular problem under consideration. As above, the multi-frequency approach developed in this work can be applied to all these situations.

It is worth mentioning that the underlying physical principle was employed by Renzhiglova et al. in an experimental study on magneto-acousto-electrical tomography, where dual-frequency ultrasounds were used to obtain non-zero internal data [92].

In the following subsections, some examples of applications are discussed. (Full details are given in Chapter 4, where further techniques are considered.) In all these cases, CGO solutions can be used to obtain suitable illuminations. Considering their several drawbacks, we believe that the multi-frequency approach described in this thesis represents a valid, or possibly better, alternative.

1.3.1 Microwave imaging by ultrasound deformation

We consider the hybrid problem arising from the combination of microwaves and ultrasounds that was introduced in [16]. In addition to the previous assumptions, we suppose that a is scalar-valued. In microwave imaging, a is the inverse of the magnetic permeability, ε is the electric permittivity and $\mathcal{A} = [K_{\min}, K_{\max}]$ represent the admissible frequencies in the microwave regime.

Given a set of measurements $K \times \{\varphi_i\}$ we consider internal measurements of the form

$$(1.14) \quad e_{\omega}^{ij} = \varepsilon u_{\omega}^{\varphi_i} u_{\omega}^{\varphi_j}, \quad E_{\omega}^{ij} = a \nabla u_{\omega}^{\varphi_i} \cdot \nabla u_{\omega}^{\varphi_j},$$

where $u_\omega^{\varphi_i}$ are given by (1.5). For simplicity, we denote $e_\omega = (e_\omega^{ij})_{ij}$ and similarly for E . These internal energies have to be considered as known functions in $\Omega' \Subset \Omega$.

We need to choose a suitable set of measurements $K \times \{\varphi_i\}$ and find a and ε in Ω' from the knowledge of e_ω^{ij} and E_ω^{ij} in Ω' . This can be achieved via two reconstruction formulae for a/ε and ε , respectively, which we shall now describe. Their applicability is guaranteed if $K \times \{\varphi_i\}$ is a $(\zeta_{\det}, C)$ -complete set of measurements in Ω' . Therefore, the multi-frequency method discussed in this work can be easily applied.

Theorem 1.7. *Let $K \times \{\varphi_i\}$ be a $(\zeta_{\det}, C)$ -complete set in Ω' . Take $x \in \Omega'$ and $\omega_x \in K$ as in Definition 1.3. Then there exists $c > 0$ depending on a priori data only such that*

$$|\nabla(e_{\omega_x}/\text{tr}(e_{\omega_x}))(x)|_2^2 \geq cC^6,$$

and a/ε is given in terms of the data by

$$|\nabla(e_{\omega_x}/\text{tr}(e_{\omega_x}))(x)|_2^2 \frac{a}{\varepsilon}(x) = 2 \frac{\text{tr}(e_{\omega_x}) \text{tr}(E_{\omega_x}) - \text{tr}(e_{\omega_x} E_{\omega_x})}{\text{tr}(e_{\omega_x})^2}(x).$$

Moreover, if $\varepsilon \in H^1(\Omega; \mathbb{R})$ then $\log \varepsilon$ is the unique solution to the problem

$$\begin{cases} -\text{div} \left(\frac{a}{\varepsilon} \sum_\omega e_\omega^{11} \nabla u \right) = -\text{div} \left(\frac{a}{\varepsilon} \nabla \left(\sum_\omega e_\omega^{11} \right) \right) + 2 \sum_\omega (E_\omega^{11} - \omega e_\omega^{11}) & \text{in } \Omega', \\ u = \log \varepsilon|_{\partial\Omega'} & \text{on } \partial\Omega'. \end{cases}$$

The formula for a/ε was first derived in [16] for the two-dimensional case, and we have extended it to any dimension. The formula to reconstruct ε is new.

1.3.2 Magnetic resonance electrical impedance tomography (MREIT)

In this example, we model the problem with the Maxwell's equations (1.9). Combining electric currents with an MRI scanner, we can measure the internal magnetic fields $H_\omega^{\varphi_i}$ [98, 99, 97]. Assuming $\mu = 1$, the electromagnetic parameters to image are ε and σ , and both are assumed isotropic.

Let $K \times \{\varphi_1, \varphi_2, \varphi_3\}$ be a $(\zeta_{\det}^M, C)$ -complete set of measurements. We shall show that $q_\omega = \omega\varepsilon + \mathbf{i}\sigma$ satisfies a first order partial differential equation in Ω . This equation is of the form

$$\nabla q_\omega M_\omega = F(\omega, q_\omega, H_\omega^{\varphi_i}, \Delta H_\omega^{\varphi_i}) \quad \text{in } \Omega,$$

where M_ω is the 3×6 matrix-valued function given by

$$M_\omega = \begin{bmatrix} \text{curl} H_\omega^{\varphi_1} \times \mathbf{e}_1 & \text{curl} H_\omega^{\varphi_1} \times \mathbf{e}_2 & \cdots & \text{curl} H_\omega^{\varphi_3} \times \mathbf{e}_1 & \text{curl} H_\omega^{\varphi_3} \times \mathbf{e}_2 \end{bmatrix},$$

and F is a given vector-valued function. If

$$(1.15) \quad |\det [E_\omega^{\varphi_1} \ E_\omega^{\varphi_2} \ E_\omega^{\varphi_3}](x)| > 0,$$

then $M_\omega(x)$ admits a right inverse $M_\omega^{-1}(x)$. The equation for q_ω becomes

$$(1.16) \quad \nabla q_\omega(x) = F(\omega, q_\omega, H_\omega^{\varphi_i}, \Delta H_\omega^{\varphi_i}) M_\omega^{-1}(x).$$

Since $K \times \{\varphi_1, \varphi_2, \varphi_3\}$ is $(\zeta_{\text{det}}^M, C)$ -complete, (1.15) is satisfied everywhere in Ω for some $\omega = \omega_x$. We shall see that in this case it is possible to integrate (1.16) and reconstruct q_ω uniquely, provided that q_ω is known at one point of $\overline{\Omega}$.

Therefore, we have seen that $(\zeta_{\text{det}}^M, C)$ -complete sets are sufficient to be able to image the electromagnetic parameters, and can be constructed with the multiple frequency approach discussed in this thesis.

Chapter 2

Existence and regularity results for the Helmholtz and Maxwell equations

In this chapter we study well-posedness and regularity for the Helmholtz and Maxwell equations. The results regarding the Helmholtz equation (Section 2.1) and the well-posedness for Maxwell's equation (Section 2.2) are classical, whereas the regularity theory for Maxwell's equations discussed in Section 2.3 is new. One of the main aims of this careful analysis is to show that the maps $\omega \mapsto u_\omega^\varphi \in C^\kappa$ and $\omega \mapsto (E_\omega^\varphi, H_\omega^\varphi) \in C^\kappa$ are holomorphic.

2.1 The Helmholtz equation

Let $\Omega \subseteq \mathbb{R}^d$ be a smooth bounded domain for some $d \geq 2$. In this section we study the Dirichlet boundary value problem for the Helmholtz equation

$$(2.1) \quad \begin{cases} -\operatorname{div}(a \nabla u_\omega) - (\omega^2 \varepsilon + \mathbf{i} \omega \sigma) u_\omega = 0 & \text{in } \Omega, \\ u_\omega = \varphi & \text{on } \partial\Omega, \end{cases}$$

where $a \in L^\infty(\Omega; \mathbb{R}^{d \times d})$ and $\varepsilon \in L^\infty(\Omega; \mathbb{R})$ and satisfy

$$(2.2a) \quad a = a^T, \quad \Lambda^{-1} |\xi|^2 \leq \xi \cdot a \xi \leq \Lambda |\xi|^2, \quad \xi \in \mathbb{R}^d,$$

$$(2.2b) \quad \Lambda^{-1} \leq \varepsilon \leq \Lambda \text{ almost everywhere,}$$

$\sigma \in L^\infty(\Omega; \mathbb{R})$ and satisfies

$$(2.3) \quad 0 \leq \sigma \leq \Lambda \text{ almost everywhere}$$

for some $\Lambda > 0$. In some situations, we shall assume either

$$(2.4) \quad \sigma = 0, \text{ or}$$

$$(2.5) \quad \sigma \geq \Lambda^{-1} \text{ almost everywhere.}$$

2.1.1 Well-posedness

2.1.1.1 The real case – $\sigma = 0$

We study here the case of real coefficients, namely we assume that $\sigma = 0$.

First of all, we study existence, uniqueness and stability with homogeneous Dirichlet boundary conditions. The space $H^{-1}(\Omega; \mathbb{C})$ denotes the continuous antidual of $H_0^1(\Omega; \mathbb{C})$ and given $\Sigma \subseteq \mathbb{R}_+$ we denote $\sqrt{\Sigma} = \{\omega \in \mathbb{C} : \omega^2 \in \Sigma\}$.

Proposition 2.1. *Assume that (2.2) holds. There exists $\Sigma = \{\lambda_l : l \in \mathbb{N}^*\} \subseteq \mathbb{R}_+$ with $\lambda_l \rightarrow +\infty$ such that for $\omega \in \mathbb{C} \setminus \sqrt{\Sigma}$ and $f \in H^{-1}(\Omega; \mathbb{C})$ the equation*

$$(2.6) \quad -\operatorname{div}(a \nabla u) - \omega^2 \varepsilon u = f$$

has a unique solution $u \in H_0^1(\Omega; \mathbb{C})$ satisfying

$$(2.7) \quad \|u\|_{H_0^1(\Omega; \mathbb{C})} \leq C(\Omega, \Lambda) \left[1 + \sup_{l \in \mathbb{N}^*} \frac{|\omega^2|}{|\lambda_l - \omega^2|} \right] \|f\|_{H^{-1}(\Omega; \mathbb{C})}.$$

Moreover, for fixed $f \in H^{-1}(\Omega; \mathbb{C})$, the map $\omega \in \mathbb{C} \setminus \Sigma \mapsto u \in H_0^1(\Omega; \mathbb{C})$ is holomorphic.

Proof. Consider the sesquilinear form A on $H_0^1(\Omega; \mathbb{C})$ defined by $A(u, v) = \int_{\Omega} a \nabla u \cdot \nabla \bar{v} \, dx$ for any $u, v \in H_0^1(\Omega; \mathbb{C})$, and the associated operator $L = -\operatorname{div}(a \nabla \cdot) : H_0^1(\Omega; \mathbb{C}) \rightarrow H^{-1}(\Omega; \mathbb{C})$ defined by $\langle Lu, v \rangle = A(u, v)$, where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $H_0^1(\Omega; \mathbb{C})$ and $H^{-1}(\Omega; \mathbb{C})$. By (2.2a), the form A is continuous and satisfies the coercivity condition

$$A(u, u) \geq \Lambda^{-1} \int_{\Omega} |\nabla u|^2 \, dx \geq C(\Omega) \Lambda^{-1} \|u\|_{H_0^1(\Omega; \mathbb{C})}^2, \quad u \in H_0^1(\Omega; \mathbb{C}),$$

by Poincaré inequality. Therefore, in view of the Lax-Milgram Theorem [104, Chapter VI, Theorem 1.4] the operator L is invertible with

$$(2.8) \quad \|L^{-1}\| \leq C(\Omega, \Lambda).$$

We introduce the operator $M_{\varepsilon} : L^2(\Omega; \mathbb{C}) \rightarrow L^2(\Omega; \mathbb{C})$ defined by $f \mapsto \varepsilon f$. By the chain

$$H_0^1(\Omega; \mathbb{C}) \hookrightarrow L^2(\Omega; \mathbb{C}) \xrightarrow{M_{\varepsilon}} L^2(\Omega; \mathbb{C}) \hookrightarrow H^{-1}(\Omega; \mathbb{C}) \xrightarrow{L^{-1}} H_0^1(\Omega; \mathbb{C})$$

and using Kondrachov Compactness Theorem [68, Theorem 7.22] we obtain that $S := L^{-1}M_{\varepsilon} : H_0^1(\Omega; \mathbb{C}) \rightarrow H_0^1(\Omega; \mathbb{C})$ is compact. For $u, v \in H_0^1(\Omega; \mathbb{C})$ we have

$$A(Su, v) = \langle M_{\varepsilon}u, v \rangle = \int_{\Omega} \varepsilon u \bar{v} \, dx = \langle u, M_{\varepsilon}v \rangle = A(u, Sv),$$

and $A(Su, u) = \int_{\Omega} \varepsilon |u|^2 \, dx$. We have thus proven that S is compact, self-adjoint and positive. By the spectral theory for compact and self-adjoint operators [104, Chapter VI, Theorem 4.2] S has a countable set of eigenvalues $\{\eta_l > 0 : l \in \mathbb{N}^*\}$, with $\eta_l \rightarrow 0$. Define

$\Sigma = \{\lambda_l = 1/\eta_l : l \in \mathbb{N}^*\}$ and take $\omega \in \mathbb{C} \setminus \sqrt{\Sigma}$. If $\omega = 0$ the claim follows from (2.8). If $\omega \neq 0$, (2.6) is equivalent to $(\omega^{-2} - S)u = \omega^{-2}L^{-1}f$, whence

$$(2.9) \quad u = \omega^{-2} (\omega^{-2} - S)^{-1} L^{-1}f.$$

Moreover, [104, (4-5), Section VI.4] gives

$$(2.10) \quad \|(\omega^{-2} - S)^{-1}\| \leq |\omega^2| \left[1 + \sup_{l \in \mathbb{N}^*} \left| \frac{1/\lambda_l}{1/\omega^2 - 1/\lambda_l} \right| \right] = |\omega^2| \left[1 + \sup_{l \in \mathbb{N}^*} \left| \frac{\omega^2}{\lambda_l - \omega^2} \right| \right].$$

Combining (2.8), (2.9) and (2.10) we obtain (2.7).

Finally, the holomorphicity of $\omega \mapsto u$ is a consequence of the so called Analytic Fredholm Theorem [91, Theorem VI.14]. $\square$

As a consequence, we immediately obtain the following result regarding the Dirichlet boundary value problem (2.1).

Corollary 2.2. *Assume that (2.2) holds and take $\omega \in \mathbb{C} \setminus \sqrt{\Sigma}$ with $|\omega| \leq M$. Then for every $f \in H^{-1}(\Omega; \mathbb{C})$ and $\varphi \in H^1(\Omega; \mathbb{C})$ the problem*

$$(2.11) \quad \begin{cases} -\operatorname{div}(a \nabla u) - \omega^2 \varepsilon u = f & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \end{cases}$$

has a unique solution $u \in H^1(\Omega; \mathbb{C})$ with

$$(2.12) \quad \|u\|_{H^1(\Omega; \mathbb{C})} \leq C(\Omega, \Lambda, M) \left[1 + \sup_{l \in \mathbb{N}^*} \frac{1}{|\lambda_l - \omega^2|} \right] (\|\varphi\|_{H^1(\Omega; \mathbb{C})} + \|f\|_{H^{-1}(\Omega; \mathbb{C})}).$$

Moreover, for fixed $f \in H^{-1}(\Omega; \mathbb{C})$ and $\varphi \in H^1(\Omega; \mathbb{C})$, the map $\omega \in \mathbb{C} \setminus \Sigma \mapsto u \in H^1(\Omega; \mathbb{C})$ is holomorphic.

Proof. Write $u = v + \varphi$ with $v \in H_0^1(\Omega; \mathbb{C})$. Then (2.11) is equivalent to

$$-\operatorname{div}(a \nabla v) - \omega^2 \varepsilon v = f + \operatorname{div}(a \nabla \varphi) + \omega^2 \varepsilon \varphi.$$

Since $\|\operatorname{div}(a \nabla \varphi) + \omega^2 \varepsilon \varphi\|_{H^{-1}(\Omega; \mathbb{C})} \leq C(\Omega, \Lambda) [1 + |\omega^2|] \|\varphi\|_{H^1(\Omega; \mathbb{C})}$, the result is an immediate consequence of Proposition 2.1. $\square$

2.1.1.2 The complex case – $\sigma > 0$

We study here the case of complex coefficients, namely we assume (2.3) and (2.5).

Proposition 2.3. *Assume that (2.2), (2.3) and (2.5) hold and take $M > 0$. There exists $\eta > 0$ depending on Ω and Λ only such that for $\omega \in \mathbb{C}$ with $|\omega| \leq M$ and $\Im \omega \geq -\eta$ and $f \in H^{-1}(\Omega; \mathbb{C})$ the equation*

$$(2.13) \quad -\operatorname{div}(a \nabla u) - (\omega^2 \varepsilon + \mathbf{i} \omega \sigma) u = f$$

has a unique solution $u \in H_0^1(\Omega; \mathbb{C})$ satisfying

$$\|u\|_{H_0^1(\Omega; \mathbb{C})} \leq C \|f\|_{H^{-1}(\Omega; \mathbb{C})}$$

for some $C > 0$ depending on Ω , Λ and M only.

Proof. Let $T_\omega: H_0^1(\Omega; \mathbb{C}) \rightarrow H^{-1}(\Omega; \mathbb{C})$ be the continuous operator defined by $T_\omega u = -\operatorname{div}(a \nabla u) - (\omega^2 \varepsilon + \mathbf{i} \omega \sigma) u$ and consider the associated continuous bilinear form

$$A_\omega(u, v) = \int_\Omega a \nabla u \cdot \nabla \bar{v} \, dx - \int_\Omega (\omega^2 \varepsilon + \mathbf{i} \omega \sigma) u \bar{v} \, dx, \quad u, v \in H_0^1(\Omega; \mathbb{C}).$$

We claim that A_ω is coercive if $\Im \omega \geq -\eta$ for some $\eta > 0$ small enough. A straightforward calculation shows that

$$\begin{aligned} \Re A_\omega(u, u) &= \int_\Omega a \nabla u \cdot \nabla \bar{u} \, dx + ((\Im \omega)^2 - (\Re \omega)^2) \int_\Omega \varepsilon |u|^2 \, dx + \Im \omega \int_\Omega \sigma |u|^2 \, dx, \\ \Im A_\omega(u, u) &= -\Re \omega \left(2\Im \omega \int_\Omega \varepsilon |u|^2 \, dx + \int_\Omega \sigma |u|^2 \, dx \right). \end{aligned}$$

Suppose now that $|\omega| \leq M$ and $\Im \omega \geq -\eta$ for some $\eta > 0$. There holds

$$\begin{aligned} \Re A_\omega(u, u) &\geq \int_\Omega a \nabla u \cdot \nabla \bar{u} \, dx - (\Re \omega)^2 \int_\Omega \varepsilon |u|^2 \, dx + \min(\Im \omega, 0) \int_\Omega \sigma |u|^2 \, dx \\ &\geq \Lambda^{-1} \|\nabla u\|_{L^2(\Omega; \mathbb{C}^3)}^2 - (\Re \omega)^2 \Lambda \|u\|_{L^2(\Omega; \mathbb{C})}^2 - \eta \Lambda \|u\|_{L^2(\Omega; \mathbb{C})}^2 \\ &\geq (\Lambda^{-1} - \eta \Lambda c(\Omega)) \|\nabla u\|_{L^2(\Omega; \mathbb{C}^3)}^2 - (\Re \omega)^2 \Lambda \|u\|_{L^2(\Omega; \mathbb{C})}^2 \\ &\geq c_1(\Omega, \Lambda) \|\nabla u\|_{L^2(\Omega; \mathbb{C}^3)}^2 - (\Re \omega)^2 \Lambda \|u\|_{L^2(\Omega; \mathbb{C})}^2, \end{aligned}$$

provided that $\eta \leq \eta_1$ for some $\eta_1 > 0$ depending on Ω and Λ only. Similarly we have

$$\begin{aligned} |\Im A_\omega(u, u)| &= |\Re \omega| \left| 2\Im \omega \int_\Omega \varepsilon |u|^2 \, dx + \int_\Omega \sigma |u|^2 \, dx \right| \\ &\geq |\Re \omega| \left(\int_\Omega \sigma |u|^2 \, dx - 2\eta \int_\Omega \varepsilon |u|^2 \, dx \right) \\ &\geq |\Re \omega| (\Lambda^{-1} - 2\eta \Lambda) \|u\|_{L^2(\Omega; \mathbb{C})}^2 \\ &\geq |\Re \omega| c_2(\Lambda) \|u\|_{L^2(\Omega; \mathbb{C})}^2, \end{aligned}$$

provided that $\eta \leq \eta_2$ for some $\eta_2 > 0$ depending on Λ only.

Define $\eta = \min(\eta_1, \eta_2)$. If $|\Re \omega| \leq c_2 \Lambda^{-1}$, the previous inequalities yield

$$\begin{aligned} |\Re A_\omega(u, u)| + |\Im A_\omega(u, u)| &\geq c_1 \|\nabla u\|_{L^2(\Omega; \mathbb{C}^3)}^2 - (\Re \omega)^2 \Lambda \|u\|_{L^2(\Omega; \mathbb{C})}^2 + |\Re \omega| c_2 \|u\|_{L^2(\Omega; \mathbb{C})}^2 \\ &\geq c_1 \|\nabla u\|_{L^2(\Omega; \mathbb{C}^3)}^2. \end{aligned}$$

Similarly, if $|\Re \omega| \geq c_2 \Lambda^{-1}$ there holds

$$\begin{aligned} |\Re A_\omega(u, u)| + |\Im A_\omega(u, u)| &= \frac{c_2 \Lambda^{-1}}{|\Re \omega|} \left(\frac{|\Re \omega|}{c_2 \Lambda^{-1}} |\Re A_\omega(u, u)| + \frac{|\Re \omega|}{c_2 \Lambda^{-1}} |\Im A_\omega(u, u)| \right) \\ &\geq \frac{c_2 \Lambda^{-1}}{M} \left(|\Re A_\omega(u, u)| + \frac{|\Re \omega|}{c_2} \Lambda |\Im A_\omega(u, u)| \right) \\ &\geq \frac{c_2 \Lambda^{-1}}{M} \left(c_1 \|\nabla u\|_{L^2(\Omega; \mathbb{C}^3)}^2 - (\Re \omega)^2 \Lambda \|u\|_{L^2(\Omega; \mathbb{C})}^2 + \Lambda |\Re \omega|^2 \|u\|_{L^2(\Omega; \mathbb{C})}^2 \right) \\ &= \frac{c_2 c_1 \Lambda^{-1}}{M} \|\nabla u\|_{L^2(\Omega; \mathbb{C}^3)}^2. \end{aligned}$$

Combining the last two inequalities we obtain

$$|A_\omega(u, u)| \geq C(\Omega, \Lambda, M) \|u\|_{H_0^1(\Omega; \mathbb{C})}^2, \quad \Im \omega \geq -\eta.$$

Finally, Lax-Milgram theorem [104, Chapter VI, Theorem 1.4] gives the result. $\square$

As a consequence, we immediately obtain the following result regarding the Dirichlet boundary value problem (2.1).

Corollary 2.4. *Assume that (2.2), (2.3) and (2.5) hold and take $M > 0$. There exists $\eta > 0$ depending on Ω and Λ only such that the following is true. For any $\omega \in \mathbb{C}$ with $\Im \omega \geq -\eta$ and $|\omega| \leq M$, $f \in H^{-1}(\Omega; \mathbb{C})$ and $\varphi \in H^1(\Omega; \mathbb{C})$ the problem*

$$(2.14) \quad \begin{cases} -\operatorname{div}(a \nabla u) - (\omega^2 \varepsilon + \mathbf{i} \omega \sigma) u = f & \text{in } \Omega, \\ u = \varphi & \text{on } \partial \Omega, \end{cases}$$

has a unique solution $u \in H^1(\Omega; \mathbb{C})$ with

$$(2.15) \quad \|u\|_{H^1(\Omega; \mathbb{C})} \leq C \left[\|\varphi\|_{H^1(\Omega; \mathbb{C})} + \|f\|_{H^{-1}(\Omega; \mathbb{C})} \right]$$

for some $C > 0$ depending on Ω , Λ and M only.

Proof. Write $u = v + \varphi$ with $v \in H_0^1(\Omega; \mathbb{C})$. Then (2.14) is equivalent to

$$-\operatorname{div}(a \nabla v) - (\omega^2 \varepsilon + \mathbf{i} \omega \sigma) v = f + \operatorname{div}(a \nabla \varphi) + (\omega^2 \varepsilon + \mathbf{i} \omega \sigma) \varphi.$$

Since $\|\operatorname{div}(a \nabla \varphi) + (\omega^2 \varepsilon + \mathbf{i} \omega \sigma) \varphi\|_{H^{-1}(\Omega; \mathbb{C})} \leq C(\Omega, \Lambda, M) \|\varphi\|_{H^1(\Omega; \mathbb{C})}$, the result is an immediate consequence of Proposition 2.3. $\square$

2.1.2 Regularity

Standard elliptic regularity theory allows us to study the regularity of the solution $u \in H^1(\Omega; \mathbb{C})$ to (2.11) and (2.14). We shall assume

$$(2.16) \quad a \in C^{\kappa-1, \alpha}(\overline{\Omega}; \mathbb{R}^{d \times d}), \quad \varepsilon, \sigma \in W^{\kappa-1, \infty}(\Omega; \mathbb{R})$$

for some $\kappa \in \mathbb{N}$ and $\alpha \in (0, 1)$. (For simplicity of notation, $C^{-m, \alpha}$ denotes L^∞ for $m \in \mathbb{N}^*$, and $W^{-1, \infty}$ denotes L^∞ , with corresponding norms.)

Proposition 2.5. *Take $\kappa \in \mathbb{N}$, $\alpha \in (0, 1)$ and $M > 0$. Assume that (2.2), (2.3) and (2.16) hold. Take $\omega \in \mathbb{C}$ with $|\omega| \leq M$, $f \in C^{\kappa-2, \alpha}(\overline{\Omega}; \mathbb{C})$, $F \in C^{\kappa-1, \alpha}(\overline{\Omega}; \mathbb{C}^3)$ and $\varphi \in C^{\kappa, \alpha}(\overline{\Omega}; \mathbb{C})$. Let $u \in H^1(\Omega; \mathbb{C})$ be a solution to*

$$\begin{cases} -\operatorname{div}(a \nabla u) - (\omega^2 \varepsilon + \mathbf{i} \omega \sigma) u = \operatorname{div} F + f & \text{in } \Omega, \\ u = \varphi & \text{on } \partial \Omega. \end{cases}$$

Then $u \in C^{\kappa, \alpha}(\overline{\Omega}; \mathbb{C})$ and

$$\|u\|_{C^\kappa(\overline{\Omega}; \mathbb{C})} \leq C \left(\|u\|_{H^1(\Omega; \mathbb{C})} + \|\varphi\|_{C^{\kappa, \alpha}(\overline{\Omega}; \mathbb{C})} + \|f\|_{C^{\kappa-2, \alpha}(\overline{\Omega}; \mathbb{C})} + \|F\|_{C^{\kappa-1, \alpha}(\overline{\Omega}; \mathbb{C}^3)} \right)$$

for some $C > 0$ depending on Ω , Λ , κ , α , M , $\|a\|_{C^{\kappa-1, \alpha}(\overline{\Omega}; \mathbb{R}^{d \times d})}$ and $\|(\varepsilon, \sigma)\|_{W^{\kappa-1, \infty}(\Omega; \mathbb{R})^2}$ only.

Proof. It follows from [67, Theorem 5.21]. $\square$

2.1.3 Holomorphicity in ω

In this subsection we study the dependence of the solutions u_ω on ω . We come back to the original problem

$$(2.1) \quad \begin{cases} -\operatorname{div}(a \nabla u_\omega) - (\omega^2 \varepsilon + \mathbf{i} \omega \sigma) u_\omega = 0 & \text{in } \Omega, \\ u_\omega = \varphi & \text{on } \partial\Omega, \end{cases}$$

for a fixed $\varphi \in C^{\kappa, \alpha}(\overline{\Omega}; \mathbb{C})$. We now want to show that the map $\omega \mapsto u_\omega \in C^\kappa(\overline{\Omega}; \mathbb{C})$ is holomorphic.

We first give a more general result, that will be used also for Maxwell's equations. Given two Banach spaces Y_1 and Y_2 , we denote the Banach space of bounded linear operators from Y_2 to Y_1 by $\mathcal{B}(Y_2, Y_1)$.

Lemma 2.6. *Let Y be a Banach space. Take an operator $T: \mathcal{D}(T) \subseteq Y \rightarrow Y$, let $D \subseteq \mathbb{C}$ be an open set and $E: D \rightarrow \mathcal{B}(Y_2, Y_1)$ be holomorphic. Take two subsets $Y_1 \subseteq Y$ and $Y_2 \subseteq \mathcal{D}(T) \cap Y_1$ equipped with norms $\|\cdot\|_{Y_1}$ and $\|\cdot\|_{Y_2}$, respectively. Assume that for all $\omega \in D$ the operator $(T - E_\omega)^{-1}: Y_1 \rightarrow Y_2$ is well-defined and bounded. Take $N \in Y_2$ and let $g: D \rightarrow Y_1$ be such that $g(\omega) - g(\omega_0) = (E_\omega - E_{\omega_0})N$ for all $\omega, \omega_0 \in D$. Then the map*

$$\omega \in D \mapsto (T - E_\omega)^{-1}g(\omega) \in Y_2$$

is holomorphic.

Proof. Denote the map $\omega \in D \mapsto (T - E_\omega)^{-1}g(\omega) \in Y_2$ by f . Take $\omega_0 \in D$: we shall prove that f is holomorphic in ω_0 . In view of the holomorphicity of the map E we have

$$(2.17) \quad \lim_{\omega \rightarrow \omega_0} \frac{E_\omega - E_{\omega_0}}{\omega - \omega_0} = F_{\omega_0}$$

for some $F_{\omega_0} \in \mathcal{B}(Y_2, Y_1)$. A straightforward calculation shows that for any $\omega \in D$ we have $(T - E_{\omega_0})(f(\omega) - f(\omega_0)) = (E_\omega - E_{\omega_0})(N + f(\omega))$, where the equality makes sense in Y_1 . Therefore

$$\frac{f(\omega) - f(\omega_0)}{\omega - \omega_0} = (T - E_{\omega_0})^{-1} \frac{E_\omega - E_{\omega_0}}{\omega - \omega_0} (N + f(\omega)),$$

whence by (2.17) the limit

$$\lim_{\omega \rightarrow \omega_0} \frac{f(\omega) - f(\omega_0)}{\omega - \omega_0} = (T - E_{\omega_0})^{-1} F_{\omega_0} (N + f(\omega_0))$$

exists and is finite in Y_2 . Namely, the map f is holomorphic in ω_0 . $\square$

Proposition 2.7. *Take $\kappa \in \mathbb{N}$ and $\alpha \in (0, 1)$. Assume that (2.2), (2.3) and (2.16) hold and take $\varphi \in C^{\kappa, \alpha}(\overline{\Omega}; \mathbb{C})$.*

If (2.4) holds then the map

$$\mathbb{C} \setminus \sqrt{\Sigma} \longrightarrow C^\kappa(\overline{\Omega}; \mathbb{C}), \quad \omega \longmapsto u_\omega$$

is holomorphic, where u_ω denotes the unique solution to (2.1) given by Corollary 2.2.

If (2.5) holds then the map

$$\{\omega \in \mathbb{C} : |\Im \omega| < \eta\} \longrightarrow C^\kappa(\overline{\Omega}; \mathbb{C}), \quad \omega \longmapsto u_\omega$$

is holomorphic, where η and u_ω , the unique solution to (2.1), are given by Corollary 2.4.

Proof. Write $u_\omega = v_\omega + \varphi$, and without loss of generality suppose that $\operatorname{div}(a \nabla \varphi) = 0$. Thus $-\operatorname{div}(a \nabla v_\omega) - (\omega^2 \varepsilon + \mathbf{i} \omega \sigma) v_\omega = (\omega^2 \varepsilon + \mathbf{i} \omega \sigma) \varphi$. Setting $Lv = -\operatorname{div}(a \nabla v)$ and $M_\omega v = (\omega^2 \varepsilon + \mathbf{i} \omega \sigma) v$, this equation can be recast as

$$(L - M_\omega) v_\omega = M_\omega \varphi.$$

We apply Lemma 2.6 with $Y = L^2(\Omega; \mathbb{C})$, $D = \mathbb{C} \setminus \sqrt{\Sigma}$ if (2.4) holds and $D = \{\omega \in \mathbb{C} : |\Im \omega| < \eta\}$ if (2.5) holds, $\mathcal{D}(T) = H_0^1(\Omega; \mathbb{C})$, $T = L$, $Y_1 = C^{\kappa-2, \alpha}(\overline{\Omega}; \mathbb{C})$, $Y_2 = C^{\kappa, \alpha}(\overline{\Omega}; \mathbb{C}) \cap H_0^1(\Omega; \mathbb{C})$, $E_\omega = M_\omega$, $N = \varphi$ and $g(\omega) = M_\omega \varphi$.

If (2.4) holds, the assumptions are verified by Propositions 2.1 and 2.5. If (2.5) holds, the assumptions are verified by Propositions 2.3 and 2.5. This concludes the proof. $\square$

2.2 Maxwell's equations

Let $\Omega \subseteq \mathbb{R}^3$ be a smooth bounded domain with a simply connected boundary $\partial\Omega$. In this section we study the Dirichlet boundary value problem for the Maxwell's system

$$\begin{aligned} (2.18a) \quad & \begin{cases} \operatorname{curl} E_\omega = \mathbf{i} \omega \mu H_\omega & \text{in } \Omega, \\ \operatorname{curl} H_\omega = -\mathbf{i}(\omega \varepsilon + \mathbf{i} \sigma) E_\omega & \text{in } \Omega, \\ E_\omega \times \nu = \varphi \times \nu & \text{on } \partial\Omega, \end{cases} \\ (2.18b) \quad & \\ (2.18c) \quad & \end{aligned}$$

where $\mu, \varepsilon, \sigma \in L^\infty(\Omega; \mathbb{R}^{3 \times 3})$ are symmetric real tensors satisfying the conditions

$$(2.19) \quad \begin{aligned} \Lambda^{-1} |\xi|^2 &\leq \xi \cdot \mu \xi, \quad \Lambda^{-1} |\xi|^2 \leq \xi \cdot \varepsilon \xi, \quad \Lambda^{-1} |\xi|^2 \leq \xi \cdot \sigma \xi, \quad \xi \in \mathbb{R}^3, \\ \|(\sigma, \varepsilon, \mu)\|_{L^\infty(\Omega; \mathbb{R}^{3 \times 3})^3} &\leq \Lambda, \quad \mu = \mu^T, \quad \varepsilon = \varepsilon^T, \quad \sigma = \sigma^T \end{aligned}$$

for some $\Lambda > 0$. The natural functional space associated to (2.18) is

$$H(\operatorname{curl}, \Omega) = \{u \in L^2(\Omega; \mathbb{C}^3) : \operatorname{curl} u \in L^2(\Omega; \mathbb{C}^3)\}.$$

We shall study problem (2.18) with an illumination satisfying

$$(2.20) \quad \varphi \in H(\operatorname{curl}, \Omega), \quad \operatorname{curl} \varphi \cdot \nu = 0 \text{ on } \partial\Omega.$$

The second of these conditions is required to make the problem well-posed if $\omega = 0$ (see Subsection 2.2.1). For the same reason, we need to add some constraints on H_ω . Namely, we look for solutions $(E_\omega, H_\omega) \in H(\operatorname{curl}, \Omega) \times H^\mu(\operatorname{curl}, \Omega)$, where

$$(2.21) \quad H^\mu(\operatorname{curl}, \Omega) = \{v \in H(\operatorname{curl}, \Omega) : \operatorname{div}(\mu v) = 0 \text{ in } \Omega, \mu v \cdot \nu = 0 \text{ on } \partial\Omega\}.$$

Note that $\operatorname{div}(\mu v) = 0$ is a formulation of Gauss's law and is implicit in (2.18) when $\omega \neq 0$.

2.2.1 Well-posedness

We first study well-posedness for the problem at hand. The case $\omega \neq 0$ was studied in [101], and the problem is well-posed except for a discrete set of complex resonances. Well-posedness in the case $\omega = 0$ will follow from a standard argument involving the Helmholtz decomposition.

We now justify the introduction of the additional constraints $\operatorname{curl} \varphi \cdot \nu = \mu H \cdot \nu = 0$ on $\partial\Omega$ in order to make (2.18) well-posed in the case $\omega = 0$. In view of [86, (3.52)] we have

$$(2.22) \quad \operatorname{curl} w \cdot \nu = \operatorname{div}_{\partial\Omega}(w \times \nu), \quad w \in H(\operatorname{curl}, \Omega),$$

whence from (2.18a) we obtain

$$\mathbf{i}\omega\mu H_\omega \cdot \nu = \operatorname{curl} E_\omega \cdot \nu = \operatorname{div}_{\partial\Omega}(E_\omega \times \nu) = \operatorname{div}_{\partial\Omega}(\varphi \times \nu) = \operatorname{curl} \varphi \cdot \nu \quad \text{on } \partial\Omega.$$

This suggests to assume (2.20) and impose $\mu H_\omega \cdot \nu = 0$ on $\partial\Omega$.

The main well-posedness result reads as follows.

Proposition 2.8. *Assume that (2.19) and (2.20) hold and take $M > 0$. There exist $\eta, C > 0$ depending on Ω , Λ and M such that for all $\omega \in \mathbb{C}$ with $|\Im \omega| \leq \eta$ and $|\omega| \leq M$ the problem*

$$(2.23) \quad \begin{cases} \operatorname{curl} E_\omega = \mathbf{i}\omega\mu H_\omega & \text{in } \Omega, \\ \operatorname{curl} H_\omega = -\mathbf{i}(\omega\varepsilon + \mathbf{i}\sigma)E_\omega & \text{in } \Omega, \\ \operatorname{div}(\mu H_\omega) = 0 & \text{in } \Omega, \\ E_\omega \times \nu = \varphi \times \nu & \text{on } \partial\Omega, \\ \mu H_\omega \cdot \nu = 0 & \text{on } \partial\Omega. \end{cases}$$

admits a unique solution $(E_\omega, H_\omega) \in H(\operatorname{curl}, \Omega) \times H^\mu(\operatorname{curl}, \Omega)$ satisfying

$$\|(E_\omega, H_\omega)\|_{H(\operatorname{curl}, \Omega)^2} \leq C \|\varphi\|_{H(\operatorname{curl}, \Omega)}.$$

The rest of this subsection is devoted to the proof of Proposition 2.8. As far as the case $\omega \neq 0$ is concerned, we follow [101]. The case $\omega = 0$ is standard, but requires additional care.

To simplify our study of (2.23), we first do a lifting of the boundary condition φ . Namely, write

$$(2.24) \quad E_\omega = \tilde{E}_\omega + \varphi,$$

where $\tilde{E}_\omega \in H_0(\operatorname{curl}, \Omega) = \{u \in H(\operatorname{curl}, \Omega) : u \times \nu = 0 \text{ on } \partial\Omega\}$, and obtain

$$(2.25) \quad \begin{cases} \operatorname{curl} \tilde{E}_\omega = \mathbf{i}\omega\mu H_\omega - \operatorname{curl} \varphi & \text{in } \Omega, \\ \operatorname{curl} H_\omega = -\mathbf{i}(\omega\varepsilon + \mathbf{i}\sigma)\tilde{E}_\omega - \mathbf{i}(\omega\varepsilon + \mathbf{i}\sigma)\varphi & \text{in } \Omega. \end{cases}$$

Introduce the space

$$X = L^2(\Omega; \mathbb{C}^3) \times \{v \in L^2(\Omega; \mathbb{C}^3) : \operatorname{div}(\mu v) = 0 \text{ in } \Omega, \mu v \cdot \nu = 0 \text{ on } \partial\Omega\},$$

equipped with the norm $\|(u, v)\|_X^2 = \|u\|_{L^2(\Omega; \mathbb{C}^3)}^2 + \|v\|_{L^2(\Omega; \mathbb{C}^3)}^2$. Consider its subspace $\mathcal{D}(T) = H_0(\text{curl}, \Omega) \times H^\mu(\text{curl}, \Omega)$ and the operator

$$T: \mathcal{D}(T) \longrightarrow X, \quad T(u, v) = \mathbf{i} \left(\varepsilon^{-1}(\text{curl} v - \sigma u), -\mu^{-1} \text{curl} u \right).$$

In view of (2.22) we have $\text{curl} u \cdot \nu = 0$ for all $u \in H_0(\text{curl}, \Omega)$. Therefore $T(u, v) \in X$ for all $(u, v) \in \mathcal{D}(T)$, and so T is well-defined.

The following lemma states that (2.25) can be recast as a Fredholm-type equation involving the operator T .

Lemma 2.9. *Assume that (2.19) and (2.20) hold and take $\omega \in \mathbb{C}$ and $(\tilde{E}_\omega, H_\omega) \in \mathcal{D}(T)$. Then $(\tilde{E}_\omega, H_\omega)$ is a solution to (2.25) if and only if*

$$(2.26) \quad (T - \omega)(\tilde{E}_\omega, H_\omega) = ((\omega + \mathbf{i}\varepsilon^{-1}\sigma)\varphi, \mathbf{i}\mu^{-1}\text{curl}\varphi).$$

Proof. Showing this equivalence is just a matter of writing down the relevant identities, and the details are left to the reader. $\square$

The previous lemma states the equivalence between (2.25) and the Fredholm-type equation (2.26). The first natural step towards the study of the latter is the characterisation of the spectrum of T , which we will denote by $\sigma(T)$.

The spectrum of an extension $\tilde{T}$ of T was studied in [101]. Consider the space $\tilde{X} = L^2(\Omega; \mathbb{C}^3) \times L^2(\Omega; \mathbb{C}^3)$ equipped with the norm $\|(u, v)\|_{\tilde{X}}^2 = \|u\|_{L^2(\Omega; \mathbb{C}^3)}^2 + \|v\|_{L^2(\Omega; \mathbb{C}^3)}^2$, its subspace $\mathcal{D}(\tilde{T}) = H_0(\text{curl}, \Omega) \times H(\text{curl}, \Omega)$ and the operator

$$\tilde{T}: \mathcal{D}(\tilde{T}) \longrightarrow \tilde{X}, \quad \tilde{T}(u, v) = \mathbf{i} \left(\varepsilon^{-1}(\text{curl} v - \sigma u), -\mu^{-1} \text{curl} u \right).$$

Lemma 2.10. *Assume that (2.19) holds. The spectrum of $\tilde{T}$ is discrete and is a pure point spectrum.*

Proof. Proposition 3.1 in [101] states that the spectrum of $\tilde{T}$ is discrete. Moreover, a careful look at the proof, that is based on the Analytic Fredholm Theorem [91, Theorem VI.14], shows that every element of the spectrum is an eigenvalue. $\square$

As it has already been already pointed out, $\tilde{T}(0, \nabla p) = 0$ for every $p \in H^1(\Omega; \mathbb{C})$, namely $\tilde{T}$ is not injective. Therefore $0 \in \sigma(\tilde{T})$. The restriction we set in this work to the domain and the codomain of the operator T are motivated by the need of studying (2.25), whence (2.26), also in the case $\omega = 0$. Thus, we shall now prove that $0 \notin \sigma(T)$.

Lemma 2.11. *Assume that (2.19) holds true. The operator T is invertible and $T^{-1}: X \rightarrow \mathcal{D}(T)$ is continuous, namely*

$$\|(u, v)\|_X \leq C \|T(u, v)\|_X, \quad (u, v) \in X$$

for some $C > 0$ depending on Ω and Λ only.

Proof. Let $(F, G) \in X$. We need to show that there exists a unique $(u, v) \in \mathcal{D}(T)$ such that $T(u, v) = (F, G)$ and that $\|(u, v)\|_X \leq C \|(F, G)\|_X$, for some $C > 0$ depending on Ω and Λ only. In the following we shall denote different such constants with the same letter c .

Let us rewrite $T(u, v) = (F, G)$ as

$$(2.27) \quad \begin{cases} \sigma u - \operatorname{curl} v = \mathbf{i}\varepsilon F & \text{in } \Omega, \\ \operatorname{curl} u = \mathbf{i}\mu G & \text{in } \Omega. \end{cases}$$

In view of the Helmholtz decomposition [69, Chapter I, Corollary 3.4], we can write $u = \nabla p + \operatorname{curl} \Phi$ for some $p \in H^1(\Omega; \mathbb{C})$ and $\Phi \in H^1(\Omega; \mathbb{C}^3)$ such that $\operatorname{div} \Phi = 0$ in Ω and $\Phi \times \nu = 0$ on $\partial\Omega$. Since $\operatorname{curl}(\operatorname{curl} \Phi) = \nabla(\operatorname{div} \Phi) - \Delta \Phi = -\Delta \Phi$, the second equation of (2.27) yields

$$\begin{cases} -\Delta \Phi = \mathbf{i}\mu G & \text{in } \Omega, \\ \operatorname{div} \Phi = 0 & \text{in } \Omega, \\ \Phi \times \nu = 0 & \text{on } \partial\Omega. \end{cases}$$

Thus Φ is uniquely determined by G and in view of [69, Chapter I, Theorem 3.8] there holds

$$(2.28) \quad \|\operatorname{curl} \Phi\|_{H^1(\Omega; \mathbb{C}^3)} \leq c(\|\operatorname{curl} \operatorname{curl} \Phi\|_{L^2(\Omega; \mathbb{C}^3)} + \|\operatorname{div} \operatorname{curl} \Phi\|_{L^2(\Omega; \mathbb{C})}) \leq c\|G\|_{L^2(\Omega; \mathbb{C}^3)}.$$

We now want to find suitable boundary conditions satisfied by p . We denote the surface gradient by $\nabla_{\partial\Omega}$, the surface divergence by $\operatorname{div}_{\partial\Omega}$ and the surface scalar curl by $\operatorname{curl}_{\partial\Omega}$ [86, Section 3.4]. By (2.22) and [86, (3.15)] we have

$$0 = \mathbf{i}\mu G \cdot \nu = \operatorname{curl} \operatorname{curl} \Phi \cdot \nu = \operatorname{div}_{\partial\Omega}(\operatorname{curl} \Phi \times \nu) = \operatorname{curl}_{\partial\Omega} \operatorname{curl} \Phi \quad \text{on } \partial\Omega.$$

(Note that $\operatorname{curl} \Phi \cdot \nu = \operatorname{div}_{\partial\Omega}(\Phi \times \nu) = 0$ on $\partial\Omega$, so that $\operatorname{curl} \Phi$ is a tangential vector field, and so we can apply to it the surface scalar curl.) As a result, since $\partial\Omega$ is simply connected, there exists a unique $r \in H^1(\partial\Omega; \mathbb{C})$ such that $\operatorname{curl} \Phi = -\nabla_{\partial\Omega} r$ on $\partial\Omega$ and $\int_{\partial\Omega} r \, ds = 0$. Poincaré inequality gives

$$(2.29) \quad \|r\|_{H^1(\partial\Omega; \mathbb{C})} \leq c\|\nabla_{\partial\Omega} r\|_{L_t^2(\partial\Omega; \mathbb{C}^3)} = c\|\operatorname{curl} \Phi\|_{L_t^2(\partial\Omega; \mathbb{C}^3)},$$

where $L_t^2(\partial\Omega, \mathbb{C}^3)$ denotes the space of tangential vector fields in $L^2(\partial\Omega, \mathbb{C}^3)$. As $u \times \nu = 0$ on $\partial\Omega$ we have $\nabla_{\partial\Omega} r \times \nu = -\operatorname{curl} \Phi \times \nu = \nabla p \times \nu = \nabla_{\partial\Omega} p \times \nu$ on $\partial\Omega$, whence $\nabla_{\partial\Omega} p = \nabla_{\partial\Omega} r$ on $\partial\Omega$. Since p is defined up to a constant, we can set $p = r$ on $\partial\Omega$. Thus, in view of the first equation of (2.27), we must look for a solution to

$$\begin{cases} -\operatorname{div}(\sigma \nabla p) = \operatorname{div}(\sigma \operatorname{curl} \Phi - \mathbf{i}\varepsilon F) & \text{in } \Omega, \\ p = r & \text{on } \partial\Omega. \end{cases}$$

Therefore, p is uniquely determined by Φ , F and r and the following estimate holds

$$(2.30) \quad \begin{aligned} \|p\|_{H^1(\Omega; \mathbb{C})} &\leq c \left(\|\operatorname{curl} \Phi\|_{L^2(\Omega; \mathbb{C}^3)} + \|F\|_{L^2(\Omega; \mathbb{C}^3)} + \|r\|_{H^1(\partial\Omega; \mathbb{C})} \right) \\ &\leq c \left(\|\operatorname{curl} \Phi\|_{H^1(\Omega; \mathbb{C}^3)} + \|F\|_{L^2(\Omega; \mathbb{C}^3)} \right) \\ &\leq c \left(\|G\|_{L^2(\Omega; \mathbb{C}^3)} + \|F\|_{L^2(\Omega; \mathbb{C}^3)} \right), \end{aligned}$$

where the second inequality is a consequence of (2.29) and the third one follows from (2.28). We have proven that u is uniquely determined by F and G and combining (2.28) and (2.30) gives the estimate

$$\|u\|_{L^2(\Omega; \mathbb{C}^3)} \leq c \|(F, G)\|_X.$$

The well-posedness for v follows from [69, Chapter I, Theorem 3.5] applied to the first equation of (2.27). Indeed, $\operatorname{div}(\sigma u - \mathbf{i}\varepsilon F) = 0$ in Ω and so there exists a unique $v \in H(\operatorname{curl}, \Omega)$ such that

$$\begin{cases} \operatorname{curl} v = \sigma u - \mathbf{i}\varepsilon F & \text{in } \Omega, \\ \operatorname{div}(\mu v) = 0 & \text{in } \Omega, \\ \mu v \cdot \nu = 0 & \text{on } \partial\Omega, \end{cases}$$

and the norm estimate $\|v\|_{L^2(\Omega; \mathbb{C}^3)} \leq c(\|u\|_{L^2(\Omega; \mathbb{C}^3)} + \|F\|_{L^2(\Omega; \mathbb{C}^3)})$ holds.

Finally, we have proven that there exists a unique $(u, v) \in \mathcal{D}(T)$ such that (2.27) holds true. Moreover, thanks to the estimates on the norms of u and v , T^{-1} is continuous. $\square$

We now show that the spectrum of T is purely imaginary.

Lemma 2.12. *Assume that (2.19) holds. There exists $\eta > 0$ depending on Ω and Λ such that $\sigma(T) \subseteq \{\omega \in \mathbb{C} : |\Im \omega| > \eta\}$.*

Proof. In the proof, we shall denote several constants depending on Ω and Λ by C .

Take $\omega \in \sigma(T)$ and assume that $|\Im \omega| \leq \eta$ for some $\eta > 0$. Since $T \subseteq \tilde{T}$ there holds $\sigma(T) \subseteq \sigma(\tilde{T})$. Hence $\omega \in \sigma(\tilde{T})$. By Lemma 2.10, ω is an eigenvalue of $\tilde{T}$. Therefore there exists $(u, v) \in \mathcal{D}(\tilde{T}) \setminus \{0\}$ such that $\tilde{T}(u, v) = \omega(u, v)$, namely

$$(2.31) \quad \begin{cases} \operatorname{curl} v - \sigma u = -\omega \mathbf{i}\varepsilon u & \text{in } \Omega, \\ \operatorname{curl} u = \mathbf{i}\omega \mu v & \text{in } \Omega. \end{cases}$$

Since $u \times \nu = 0$ on $\partial\Omega$ and $\omega \neq 0$ by Lemma 2.11, in view of (2.22) we obtain

$$\begin{cases} \operatorname{curl} v = \sigma u - \omega \mathbf{i}\varepsilon u & \text{in } \Omega, \\ \operatorname{div}(\mu v) = 0 & \text{in } \Omega, \\ \mu v \cdot \nu = 0 & \text{on } \partial\Omega. \end{cases}$$

Therefore [69, Chapter I, Theorem 3.5] yields

$$(2.32) \quad \|v\|_{L^2(\Omega; \mathbb{C}^3)} \leq C(1 + |\omega|) \|u\|_{L^2(\Omega; \mathbb{C}^3)}.$$

From the second equation of (2.31) and [86, Corollary 3.51] we obtain

$$\|u\|_{L^2(\Omega; \mathbb{C}^3)} \leq C \|\operatorname{curl} u\|_{L^2(\Omega; \mathbb{C}^3)} \leq C |\omega| \|v\|_{L^2(\Omega; \mathbb{C}^3)}.$$

Hence, in view of (2.32) there holds $\|u\|_{L^2(\Omega; \mathbb{C}^3)} \leq C |\omega| (1 + |\omega|) \|u\|_{L^2(\Omega; \mathbb{C}^3)}$, whence

$$(2.33) \quad |\omega| \geq C,$$

since $u \neq 0$. Thus, we can take η small enough so that $\Re \omega \neq 0$.

We now integrate the first equation of (2.31) against $\bar{u}$ and the second equation against $\bar{v}$. An integration by parts gives

$$\int_{\Omega} \sigma u \cdot \bar{u} \, dx = \omega \mathbf{i} \int_{\Omega} \varepsilon u \cdot \bar{u} \, dx + \overline{\omega \mathbf{i}} \int_{\Omega} \mu v \cdot \bar{v} \, dx.$$

Taking the imaginary part of this equality, since $\Re \omega \neq 0$, we infer that $\int_{\Omega} \varepsilon u \cdot \bar{u} \, dx = \int_{\Omega} \mu v \cdot \bar{v} \, dx$. Therefore, taking the real part we obtain

$$-2\Im \omega = \int_{\Omega} \sigma u \cdot \bar{u} \, dx \left(\int_{\Omega} \varepsilon u \cdot \bar{u} \, dx \right)^{-1} \geq C \|u\|_{L^2(\Omega; \mathbb{C}^3)}^2 \|u\|_{L^2(\Omega; \mathbb{C}^3)}^{-2} = C,$$

whence $\eta \geq |\Im \omega| \geq C$. Choosing $\eta < C$ we obtain a contradiction. $\square$

The following result gives an estimate on $\|(T - \omega)^{-1}\|$ for $|\Im \omega| \leq \eta$.

Lemma 2.13. *Assume that (2.19) holds and take $M > 0$. There exist $C, \eta > 0$ depending on Ω, Λ and M such that $\sigma(T) \subseteq \{\omega \in \mathbb{C} : |\Im \omega| > \eta\}$ and*

$$\|(T - \omega)^{-1}\| \leq C, \quad |\Im \omega| \leq \eta, \quad |\omega| \leq M.$$

Proof. In the proof, we shall denote several constants depending on Ω, Λ and M by C .

In view of Lemma 2.12 there exists $\eta' > 0$ depending on Ω and Λ such that $\{\omega \in \mathbb{C} : |\Im \omega| \leq \eta'\} \subseteq \rho(T)$. Take now $\omega \in \mathbb{C}$ such that $|\Im \omega| \leq \eta$ for some $\eta \leq \eta'$.

Take $(u, v) \in \mathcal{D}(T)$ and $(F, G) \in X$ such that $(T - \omega)(u, v) = (F, G)$. Then

$$(2.34) \quad \begin{cases} -\operatorname{curl} v + \sigma u - \mathbf{i} \varepsilon \omega u = \mathbf{i} \varepsilon F & \text{in } \Omega, \\ \operatorname{curl} u - \mathbf{i} \omega \mu v = \mathbf{i} \mu G & \text{in } \Omega. \end{cases}$$

Arguing as in the proof of Lemma 2.12 we obtain

$$(2.35) \quad \|v\|_{L^2(\Omega; \mathbb{C}^3)} \leq C(\|u\|_{L^2(\Omega; \mathbb{C}^3)} + \|F\|_{L^2(\Omega; \mathbb{C}^3)}).$$

Testing the first equation of (2.34) against $\bar{u}$ and the second equation against $\bar{v}$, integrating by parts gives and taking the real part give

$$(2.36) \quad \int_{\Omega} \sigma u \cdot \bar{u} \, dx + \Im \omega \left(\int_{\Omega} \varepsilon u \cdot \bar{u} \, dx + \int_{\Omega} \mu v \cdot \bar{v} \, dx \right) = -\Im \left(\int_{\Omega} \varepsilon F \cdot \bar{u} - \mu \bar{G} \cdot v \, dx \right).$$

Estimate (2.35) yields

$$\begin{aligned} \left| \Im \left(\int_{\Omega} \varepsilon F \cdot \bar{u} - \mu \bar{G} \cdot v \, dx \right) \right| &\leq C(\|(F, G)\|_{L^2(\Omega; \mathbb{C}^3)^2} \|u\|_{L^2(\Omega; \mathbb{C}^3)} + \|F\|_{L^2(\Omega; \mathbb{C}^3)} \|G\|_{L^2(\Omega; \mathbb{C}^3)}) \\ &\leq C(\|(F, G)\|_{L^2(\Omega; \mathbb{C}^3)^2} \|u\|_{L^2(\Omega; \mathbb{C}^3)} + \|(F, G)\|_{L^2(\Omega; \mathbb{C}^3)^2}^2), \end{aligned}$$

and

$$\begin{aligned} \left| \int_{\Omega} \sigma u \cdot \bar{u} \, dx + \Im \omega \left(\int_{\Omega} \varepsilon u \cdot \bar{u} \, dx + \int_{\Omega} \mu v \cdot \bar{v} \, dx \right) \right| &\geq \int_{\Omega} \sigma u \cdot \bar{u} \, dx - |\Im \omega| \left(\int_{\Omega} \varepsilon u \cdot \bar{u} + \mu v \cdot \bar{v} \, dx \right) \\ &\geq (\Lambda^{-1} - \eta \Lambda) \|u\|_{L^2(\Omega; \mathbb{C}^3)}^2 - \eta \Lambda \|v\|_{L^2(\Omega; \mathbb{C}^3)}^2 \\ &\geq (\Lambda^{-1} - C \eta \Lambda) \|u\|_{L^2(\Omega; \mathbb{C}^3)}^2 - \eta C \|F\|_{L^2(\Omega; \mathbb{C}^3)}^2. \end{aligned}$$

Combining these two inequalities with (2.36) we obtain

$$(\Lambda^{-1} - C\eta\Lambda) \|u\|_{L^2(\Omega; \mathbb{C}^3)}^2 \leq C(\|(F, G)\|_{L^2(\Omega; \mathbb{C}^3)^2} \|u\|_{L^2(\Omega; \mathbb{C}^3)} + \|(F, G)\|_{L^2(\Omega; \mathbb{C}^3)}^2).$$

As a consequence, choosing η sufficiently small such that $\Lambda^{-1} - C\eta\Lambda > 0$, there holds $\|u\|_{L^2(\Omega; \mathbb{C}^3)}^2 \leq C(\|(F, G)\|_{L^2(\Omega; \mathbb{C}^3)^2} \|u\|_{L^2(\Omega; \mathbb{C}^3)} + \|(F, G)\|_{L^2(\Omega; \mathbb{C}^3)}^2)$, which with simple algebra yields $\|u\|_{L^2(\Omega; \mathbb{C}^3)} \leq C \|(F, G)\|_{L^2(\Omega; \mathbb{C}^3)^2}$. Combining this bound with (2.35) gives the result. $\square$

We are now in a position to prove Proposition 2.8.

Proof of Proposition 2.8. It follows immediately from (2.24) and Lemmata 2.9 and 2.13. $\square$

2.2.2 Regularity

As far as the regularity is concerned, we assume that

$$(2.37) \quad \mu, \varepsilon, \sigma \in W^{\kappa+1,p}(\Omega; \mathbb{R}^{3 \times 3}), \quad \varphi \in W^{\kappa+1,p}(\Omega; \mathbb{C}^3)$$

for some $p > 3$ and $\kappa \in \mathbb{N}$. The main regularity result for problem (2.23) is given in Proposition 2.15. We need the following preliminary result, that follows from the general regularity theory for Maxwell's equations discussed in Section 2.3

Proposition 2.14. *Assume that (2.19) and (2.37) hold true with $p > 3$ and $\kappa \in \mathbb{N}$. Let $\eta > 0$ be as in Lemma 2.13 and take $\omega \in \mathbb{C}$ with $|\Im \omega| \leq \eta$ and $|\omega| \leq M$ and $(F, G) \in X$ such that $F \in W^{\kappa+1,p}(\Omega; \mathbb{C}^3)$ and $G \in W^{\kappa,p}(\Omega; \mathbb{C}^3)$. Let $(u, v) \in \mathcal{D}(T)$ be a solution to*

$$(2.38) \quad (T - \omega)(u, v) = (F, G).$$

Then $u, v \in W^{\kappa+1,p}(\Omega; \mathbb{C}^3)$ and

$$(2.39) \quad \|(u, v)\|_{W^{\kappa+1,p}(\Omega; \mathbb{C}^3)^2} \leq C \left(\|(u, v)\|_{L^2(\Omega; \mathbb{C}^3)^2} + \|F\|_{W^{\kappa+1,p}(\Omega; \mathbb{C}^3)} + \|G\|_{W^{\kappa,p}(\Omega; \mathbb{C}^3)} \right)$$

for some $C > 0$ depending on Ω , M , Λ , κ , p and $\|(\mu, \varepsilon, \sigma)\|_{W^{\kappa+1,p}(\Omega; \mathbb{R}^{3 \times 3})^3}$ only.

Proof. Since (2.38) can be rewritten as

$$\begin{cases} \operatorname{curl} u = \mathbf{i}\omega\mu v + \mathbf{i}\mu G & \text{in } \Omega, \\ \operatorname{curl} v = -\mathbf{i}(\omega\varepsilon + \mathbf{i}\sigma)u - \mathbf{i}\varepsilon F & \text{in } \Omega, \end{cases}$$

the result immediately follows from Theorem 1.1 if $\kappa = 0$ and from its higher regularity counterpart (which will be given in Section 2.3) if $\kappa \geq 1$. $\square$

Proposition 2.15. *Assume that (2.19), (2.20) and (2.37) hold for some $p > 3$ and $\kappa \in \mathbb{N}$. Take $\eta, M > 0$ as in Proposition 2.8. For $\omega \in \mathbb{C}$ with $|\Im \omega| \leq \eta$ and $|\omega| \leq M$ let $(E_\omega, H_\omega) \in H(\operatorname{curl}, \Omega)^2$ be the unique solution to (2.23). Then $(E_\omega, H_\omega) \in C^\kappa(\bar{\Omega}; \mathbb{C}^6)$ and*

$$\|(E_\omega, H_\omega)\|_{C^\kappa(\bar{\Omega}; \mathbb{C}^6)} \leq C \|\varphi\|_{W^{\kappa+1,p}(\Omega; \mathbb{C}^3)}$$

for some $C > 0$ depending on Ω , Λ , M , κ , p and $\|(\mu, \varepsilon, \sigma)\|_{W^{\kappa+1,p}(\Omega; \mathbb{R}^{3 \times 3})^3}$ only.

Proof. We have already seen that setting $E_\omega = \tilde{E}_\omega + \varphi$, $(\tilde{E}_\omega, H_\omega)$ is a solution to (2.38) with $F = \varepsilon^{-1}q_\omega\varphi$ and $G = \mathbf{i}\mu^{-1}\text{curl}\varphi$. Thus, in view of Proposition 2.14, $\tilde{E}_\omega, H_\omega \in W^{\kappa+1,p}(\Omega; \mathbb{C}^3)$ and

$$\begin{aligned} \|(\tilde{E}_\omega, H_\omega)\|_{W^{\kappa+1,p}(\Omega; \mathbb{C}^3)^2} &\leq C \left(\|(\tilde{E}_\omega, H_\omega)\|_{L^2(\Omega; \mathbb{C}^3)^2} + \|\varepsilon^{-1}q_\omega\varphi\|_{W^{\kappa+1,p}(\Omega; \mathbb{C}^3)} \right. \\ &\quad \left. + \|\mu^{-1}\text{curl}\varphi\|_{W^{\kappa,p}(\Omega; \mathbb{C}^3)} \right) \end{aligned}$$

for some $C > 0$ depending on Ω , M , Λ , κ , p and $\|(\mu, \varepsilon, \sigma)\|_{W^{\kappa+1,p}(\Omega; \mathbb{R}^{3 \times 3})^3}$ only. Combining this inequality with the estimate for $\|(\tilde{E}_\omega, H_\omega)\|_{H(\text{curl}, \Omega)^2}$ given in Lemma 2.13 we obtain the result, as $W^{\kappa+1,p}(\Omega; \mathbb{C}^3)^2$ is continuously embedded into $C^\kappa(\bar{\Omega}; \mathbb{C}^6)$. $\square$

2.2.3 Holomorphicity in ω

We finally show that $\omega \mapsto (E_\omega, H_\omega)$ is holomorphic, as a consequence of Lemma 2.6.

Proposition 2.16. *Under the hypotheses of Proposition 2.15 the map*

$$\{\omega \in \mathbb{C} : |\Im \omega| < \eta, |\omega| < M\} \longrightarrow C^\kappa(\bar{\Omega}; \mathbb{C}^6), \quad \omega \longmapsto (E_\omega, H_\omega)$$

is holomorphic, where (E_ω, H_ω) is the unique solution to (2.23).

Proof. Since $E_\omega = \tilde{E}_\omega + \varphi$, it is enough to show the holomorphicity of the map $\omega \mapsto (\tilde{E}_\omega, H_\omega)$. By Lemma 2.9 there holds

$$(\tilde{E}_\omega, H_\omega) = (T - \omega)^{-1}((\omega + \mathbf{i}\varepsilon^{-1}\sigma)\varphi, \mathbf{i}\mu^{-1}\text{curl}\varphi).$$

We want to apply Lemma 2.6 with $Y = X$, $D = \{\omega \in \mathbb{C} : |\Im \omega| < \eta, |\omega| < M\}$, $Y_1 = \{(F, G) \in X : F \in W^{\kappa+1,p}(\Omega; \mathbb{C}^3), G \in W^{\kappa,p}(\Omega; \mathbb{C}^3)\}$ with the norm $\| \cdot \|_{W^{\kappa+1,p}(\Omega; \mathbb{C}^3) \times W^{\kappa,p}(\Omega; \mathbb{C}^3)}$, $Y_2 = \mathcal{D}(T) \cap W^{\kappa+1,p}(\Omega; \mathbb{C}^3)^2$ equipped with the norm $\| \cdot \|_{W^{\kappa+1,p}(\Omega; \mathbb{C}^3)^2}$, $E_\omega = \omega i$ where $i: Y_2 \rightarrow Y_1$ is the inclusion, $N = (0, \varphi)$ and $g(\omega) = (\varepsilon^{-1}q_\omega\varphi, \mathbf{i}\mu^{-1}\text{curl}\varphi)$. Let us now check that the assumptions of the lemma are verified. The continuity of the inclusion $Y_2 \subseteq Y_1$ is trivial. The continuity of $(T - \omega)^{-1}: Y_1 \rightarrow Y_2$ follows from Lemma 2.13 and Proposition 2.14. Finally, a direct computation shows that $g(\omega) - g(\omega_0) = (\omega - \omega_0)N$. Therefore, the result follows by Lemma 2.6, as $W^{\kappa+1,p}(\Omega; \mathbb{C}^3)^2$ is continuously embedded into $C^\kappa(\bar{\Omega}; \mathbb{C}^6)$. $\square$

2.3 Regularity theorems for Maxwell's equations

Let us recall the main hypotheses. Let $\Omega \subseteq \mathbb{R}^3$ be a bounded domain in $\mathbb{R}^3$, with $C^{1,1}$ boundary. Let $\varepsilon, \sigma \in L^\infty(\Omega; \mathbb{R}^{3 \times 3})$ and $\mu \in L^\infty(\Omega; \mathbb{C}^{3 \times 3})$ be matrix-valued functions such that

$$(2.40) \quad \begin{aligned} \Lambda^{-1}|\xi|^2 &\leq \xi \cdot (\Re \mu) \xi, \quad \Lambda^{-1}|\xi|^2 \leq \xi \cdot \sigma \xi, \quad \xi \in \mathbb{R}^3, \\ \varepsilon^T &= \varepsilon, \quad \Im \mu^T = \Im \mu, \quad \|(\mu, \sigma, \varepsilon)\|_{L^\infty(\Omega; \mathbb{R}^{3 \times 3})^3} \leq \Lambda \end{aligned}$$

for some $\Lambda > 0$. Consider a given frequency $\omega \in \mathbb{C}$ and current sources

$$(2.41) \quad \varphi \in H(\operatorname{curl}, \Omega), \quad J_e, J_m \in L^2(\Omega; \mathbb{C}^3), \quad \operatorname{div} J_m = 0 \text{ in } \Omega, \quad (\operatorname{curl} \varphi - J_m) \cdot \nu = 0 \text{ on } \partial\Omega.$$

We are interested in the regularity of the weak solution $(E, H) \in H(\operatorname{curl}, \Omega) \times H^\mu(\operatorname{curl}, \Omega)$ of

$$(2.42) \quad \begin{cases} \operatorname{curl} E = \mathbf{i}\omega\mu H + J_m & \text{in } \Omega, \\ \operatorname{curl} H = -\mathbf{i}(\omega\varepsilon + \mathbf{i}\sigma)E + J_e & \text{in } \Omega, \\ E \times \nu = \varphi \times \nu & \text{on } \partial\Omega. \end{cases}$$

In Section 1.1 we stated some of the main results. We anticipate here the other two main results of this section. The first one deals with the $H^1(\Omega)$ regularity for E . Note that no regularity assumption is made on μ , apart from (2.40).

Theorem 2.17. *Assume that (2.40) holds, and that ε and σ also satisfy*

$$(2.43) \quad \varepsilon, \sigma \in W^{1,3+\delta}(\Omega; \mathbb{R}^{3 \times 3}) \text{ for some } \delta > 0.$$

Suppose that the source terms J_m, J_e and φ satisfy (2.41), $J_e \in H(\operatorname{div}, \Omega)$ and $\varphi \in H^1(\Omega; \mathbb{C}^3)$. Take $|\omega| \leq M$ for some $M > 0$. If $(E, H) \in H(\operatorname{curl}, \Omega) \times H^\mu(\operatorname{curl}, \Omega)$ is a weak solution of (2.42), then $E \in H^1(\Omega; \mathbb{C}^3)$ and

$$(2.44) \quad \|E\|_{H^1(\Omega; \mathbb{C}^3)} \leq C(\|(E, H)\|_{L^2(\Omega; \mathbb{C}^3)^2} + \|\varphi\|_{H^1(\Omega; \mathbb{C}^3)} + \|J_m\|_{L^2(\Omega; \mathbb{C}^3)} + \|J_e\|_{H(\operatorname{div}, \Omega)})$$

for some constant C depending on $\Omega, \Lambda, \delta, M$ and $\|(\sigma, \varepsilon)\|_{W^{1,3+\delta}(\Omega; \mathbb{R}^{3 \times 3})^2}$ only.

Our second result is devoted to the $H^1(\Omega)$ regularity of H . Naturally, interior regularity for H follows from the interior regularity of E , due to the (almost) symmetrical role of the pairs $(E, \omega\varepsilon + \mathbf{i}\sigma)$ and $(H, \omega\mu)$ in Maxwell's equations. The difference between Theorem 2.17 and Theorem 2.18 comes from the fact that (2.42) involves a boundary condition on E , not on H .

Theorem 2.18. *Assume that (2.40) holds, and that μ also satisfies*

$$(2.45) \quad \mu \in W^{1,3+\delta}(\Omega; \mathbb{C}^{3 \times 3}) \text{ for some } \delta > 0.$$

Suppose that the source terms J_e, J_m and φ satisfy (2.41) and $\varphi \in H^1(\Omega; \mathbb{C}^3)$. Take $|\omega| \leq M$ for some $M > 0$. If $(E, H) \in H(\operatorname{curl}, \Omega) \times H^\mu(\operatorname{curl}, \Omega)$ is a weak solution of (2.42), then $H \in H^1(\Omega; \mathbb{C}^3)$ and

$$\|H\|_{H^1(\Omega; \mathbb{C}^3)} \leq C(\|(E, H)\|_{L^2(\Omega; \mathbb{C}^3)^2} + \|\varphi\|_{H^1(\Omega; \mathbb{C}^3)} + \|(J_e, J_m)\|_{L^2(\Omega; \mathbb{C}^3)^2})$$

for some constant $C > 0$ depending on $\Omega, \Lambda, \delta, M$ and $\|\mu\|_{W^{1,3+\delta}(\Omega; \mathbb{C}^{3 \times 3})}$ only.

This section is structured as follows. Subsection 2.3.1 is devoted to the proof of Theorems 1.1, 2.17, 2.18 and of Theorem 2.23, the $W^{\kappa,p}$ counterpart of Theorem 1.1 with appropriately smooth coefficients in a domain with $C^{\kappa,1}$ boundary. Then, § 2.3.2 focuses on the particular case when μ is real-valued and is devoted to the proof of Theorem 1.2. Finally, § 2.3.3 is devoted to the result for the generalised bi-anisotropic Maxwell's equations.

2.3.1 $W^{1,p}$ regularity for E and H

Our strategy is to consider a coupled elliptic system satisfied by each component of the electric and magnetic field, where in each equation, only one component appears in the leading order term. In a first step, we show that the electric and magnetic fields are very weak solutions of such a system. This system was already introduced, in its strong form, in Leis [84], and was used recently in Nguyen & Wang [89].

Proposition 2.19. *Assume that (2.40) and (2.41) hold true. Let (E, H) in $H(\text{curl}, \Omega) \times H^\mu(\text{curl}, \Omega)$ be a weak solution of (2.42).*

- If (2.43) holds and $J_e \in H(\text{div}, \Omega)$, for each $k = 1, 2, 3$, E_k is a very weak solution of

$$(2.46) \quad -\text{div}(q_\omega \nabla E_k) = \text{div}((\partial_k q_\omega) E - q_\omega (\mathbf{e}_k \times (J_m + \mathbf{i}\omega\mu H)) + \mathbf{i}\mathbf{e}_k \text{div} J_e) \text{ in } \Omega,$$

where $\mathbf{e}_k$ is the unit vector in the k -th direction. More precisely, E_k satisfies for any $\Phi \in W^{2,2}(\Omega; \mathbb{C})$

$$(2.47) \quad \int_{\Omega} E_k \text{div}(q_\omega^T \nabla \bar{\Phi}) \, dx = \int_{\partial\Omega} (\partial_k \bar{\Phi}) q_\omega E \cdot \nu \, d\sigma - \int_{\partial\Omega} (\mathbf{e}_k \times (E \times \nu)) \cdot (q_\omega^T \nabla \bar{\Phi}) \, d\sigma \\ + \int_{\Omega} ((\partial_k q_\omega) E - q_\omega (\mathbf{e}_k \times (J_m + \mathbf{i}\omega\mu H)) + \mathbf{i}\mathbf{e}_k \text{div} J_e) \cdot \nabla \bar{\Phi} \, dx.$$

- If (2.45) holds, for each $k = 1, 2, 3$, H_k is a very weak solution of

$$(2.48) \quad -\text{div}(\mu \nabla H_k) = \text{div}((\partial_k \mu) H - \mu (\mathbf{e}_k \times (J_e - \mathbf{i}q_\omega E))) \text{ in } \Omega.$$

More precisely, H_k satisfies for any $\Phi \in W^{2,2}(\Omega; \mathbb{C})$

$$(2.49) \quad \int_{\Omega} H_k \text{div}(\mu^T \nabla \bar{\Phi}) \, dx = - \int_{\partial\Omega} (\mathbf{e}_k \times (H \times \nu)) \cdot (\mu^T \nabla \bar{\Phi}) \, d\sigma \\ + \int_{\Omega} ((\partial_k \mu) H - \mu (\mathbf{e}_k \times (J_e - \mathbf{i}q_\omega E))) \cdot \nabla \bar{\Phi} \, dx.$$

Proof. We detail the derivation of (2.47) for the sake of completeness. The derivation of (2.49) is similar, thanks to the intrinsic symmetry of Maxwell's equations (2.42).

We multiply the identity $\text{curl} E = \mathbf{i}\omega\mu H + J_m$ by $\bar{\Phi} = \bar{g}\mathbf{e}_l$ for some $g \in W^{1,2}(\Omega; \mathbb{C})$, integrate by parts and multiply the result by $\mathbf{e}_l$. We obtain

$$\mathbf{e}_l \int_{\Omega} \bar{g} (\mathbf{i}\omega\mu H + J_m) \cdot \mathbf{e}_l \, dx = \mathbf{e}_l \int_{\Omega} E \cdot (\nabla \times \bar{\Phi}) \, dx - \mathbf{e}_l \int_{\partial\Omega} (E \times \nu) \cdot \bar{\Phi} \, d\sigma,$$

which can be written also as

$$\int_{\Omega} \bar{g} (\mathbf{i}\omega\mu H + J_m) \, dx + \int_{\partial\Omega} \bar{g} (E \times \nu) \, d\sigma = \int_{\Omega} E \times \nabla \bar{g} \, dx.$$

Note that since $E \in H(\text{curl}, \Omega)$ by assumption, $E \times \nu$ is well defined in $H^{-\frac{1}{2}}(\partial\Omega; \mathbb{C}^3)$ and this formulation is valid. Next, we cross product this identity with $\mathbf{e}_k$, and take the scalar

product with $\mathbf{e}_i$. Using the vector identity $a \times (b \times c) = (a \cdot c)b - (a \cdot b)c$ on the right-hand side, we obtain

$$(2.50) \quad \mathbf{e}_i \cdot \int_{\Omega} \bar{g} \mathbf{e}_k \times (\mathbf{i}\omega\mu H + J_m) dx + \mathbf{e}_i \cdot \int_{\partial\Omega} \bar{g} \mathbf{e}_k \times (E \times \nu) d\sigma \\ = \int_{\Omega} E_i \partial_k \bar{g} - E_k \partial_i \bar{g} dx,$$

for any i and k in $\{1, 2, 3\}$ and $g \in W^{1,2}(\Omega; \mathbb{C})$. In view of (2.43), we have that $\bar{q}_\omega^T \nabla \Phi \in H^1(\Omega; \mathbb{C}^3)$ for any $\Phi \in W^{2,2}(\Omega; \mathbb{C})$. Thus, applying (2.50) with $g = (\bar{q}_\omega^T \nabla \Phi)_i$ for any $i = 1, 2, 3$ and $\Phi \in W^{2,2}(\Omega; \mathbb{C})$ we find that

$$\int_{\Omega} E_i \partial_k (q_\omega^T \nabla \bar{\Phi})_i dx = \int_{\Omega} E_k \partial_i (q_\omega^T \nabla \bar{\Phi})_i dx + \mathbf{e}_i \cdot \int_{\partial\Omega} (q_\omega^T \nabla \bar{\Phi})_i \mathbf{e}_k \times (E \times \nu) d\sigma \\ + \mathbf{e}_i \cdot \int_{\Omega} (q_\omega^T \nabla \bar{\Phi})_i \mathbf{e}_k \times (\mathbf{i}\omega\mu H + J_m) dx.$$

Summing over i , this yields

$$(2.51) \quad \int_{\Omega} E \cdot \partial_k (q_\omega^T \nabla \bar{\Phi}) dx = \int_{\Omega} E_k \operatorname{div} (q_\omega^T \nabla \bar{\Phi}) dx \\ + \int_{\partial\Omega} (\mathbf{e}_k \times (E \times \nu)) \cdot (q_\omega^T \nabla \bar{\Phi}) d\sigma + \int_{\Omega} q_\omega (\mathbf{e}_k \times (\mathbf{i}\omega\mu H + J_m)) \cdot \nabla \bar{\Phi} dx.$$

We then use the second part of Maxwell's equations. We test $\operatorname{curl} H - J_e = -\mathbf{i}q_\omega E$ against $\nabla (\partial_k \bar{\Phi}) \mathbf{i}$ for $\Phi \in W^{2,2}(\Omega; \mathbb{C})$ and obtain

$$\int_{\Omega} q_\omega E \cdot \partial_k (\nabla \bar{\Phi}) dx = \mathbf{i} \int_{\Omega} \operatorname{curl}(H) \cdot \nabla (\partial_k \bar{\Phi}) dx - \mathbf{i} \int_{\Omega} J_e \cdot \nabla (\partial_k \bar{\Phi}) dx \\ = \mathbf{i} \left(\int_{\partial\Omega} (\partial_k \bar{\Phi}) \operatorname{curl} H \cdot \nu d\sigma - \int_{\partial\Omega} (\partial_k \bar{\Phi}) J_e \cdot \nu d\sigma + \int_{\Omega} \operatorname{div} J_e \partial_k \bar{\Phi} dx \right) \\ = \mathbf{i} \left(-\mathbf{i} \int_{\partial\Omega} (\partial_k \bar{\Phi}) q_\omega E \cdot \nu d\sigma + \int_{\Omega} \operatorname{div} J_e \partial_k \bar{\Phi} dx \right).$$

Since $J_e \in H(\operatorname{div}, \Omega)$, the boundary term is well defined. Writing the left-hand side of the above identity in the form

$$\int_{\Omega} q_\omega E \cdot \partial_k (\nabla \bar{\Phi}) dx = \int_{\Omega} E \cdot \partial_k (q_\omega^T \nabla \bar{\Phi}) dx - \int_{\Omega} (\partial_k q_\omega) E \cdot \nabla \bar{\Phi} dx,$$

we obtain

$$- \int_{\Omega} (\partial_k q_\omega) E \cdot \nabla \bar{\Phi} dx + \int_{\Omega} E \cdot \partial_k (q_\omega^T \nabla \bar{\Phi}) dx = \int_{\partial\Omega} (\partial_k \bar{\Phi}) q_\omega E \cdot \nu d\sigma + \mathbf{i} \int_{\Omega} \operatorname{div} J_e \partial_k \bar{\Phi} dx$$

Inserting this identity in (2.51) we obtain (2.47). $\square$

To transform the very weak identities given by Proposition 2.19 into regular weak formulations, we shall use the following lemma. Given $r \in (1, \infty)$, we write r' the solution of $\frac{1}{r} + \frac{1}{r'} = 1$.

Lemma 2.20. *Assume that (2.40) and (2.45) hold with $\delta = 0$. Given $r \geq 6/5$, $u \in L^2(\Omega; \mathbb{C}) \cap L^r(\Omega; \mathbb{C})$, $F \in (W^{1,r'}(\Omega; \mathbb{C}))'$, let B be the trace operator given either by $B\Phi = \Phi$ on $\partial\Omega$ or by $B\Phi = \mu^T \nabla \bar{\Phi} \cdot \nu$ on $\partial\Omega$ for $\Phi \in W^{2,2}(\Omega; \mathbb{C})$.*

If for all $\Phi \in W^{2,2}(\Omega; \mathbb{C})$ such that $B\Phi = 0$ there holds

$$(2.52) \quad \int_{\Omega} u \operatorname{div}(\mu^T \nabla \bar{\Phi}) dx = \langle F, \Phi \rangle,$$

then $u \in W^{1,r}(\Omega; \mathbb{C})$ and

$$(2.53) \quad \|\nabla u\|_{L^r(\Omega; \mathbb{C}^3)} \leq C \|F\|_{(W^{1,r'}(\Omega; \mathbb{C}))'}$$

for some constant C depending on Ω , Λ , $\|\mu\|_{W^{1,3}(\Omega; \mathbb{C}^{3 \times 3})}$ and r only.

Proof. We first observe that, since $r \geq 6/5$, then both terms of identity (2.52) are well defined as $W^{2,2}(\Omega; \mathbb{C}) \subset W^{1,6}(\Omega; \mathbb{C})$ and $\frac{1}{6} + \frac{1}{6/5} = 1$. Let $\psi \in \mathcal{D}(\Omega)$ be a test function and fix $i = 1, 2$ or 3 . Let $\Phi^* \in W^{1,2}(\Omega; \mathbb{C})$ be the unique solution to the problem

$$\begin{cases} \operatorname{div}(\mu^T \nabla \bar{\Phi}^*) = \partial_i \psi & \text{in } \Omega, \\ B\Phi^* = 0 & \text{on } \partial\Omega. \end{cases}$$

In the case of Neumann boundary conditions, we add the normalisation condition $\int_{\Omega} \Phi^* dx = 0$. Since $\mu \in W^{1,3}(\Omega; \mathbb{C}^{3 \times 3})$, it is known [28, Theorem 1] that for any $q \in (1, \infty)$ there holds

$$(2.54) \quad \|\Phi^*\|_{W^{1,q}(\Omega; \mathbb{C})} \leq C \|\psi\|_{L^q(\Omega; \mathbb{C})}$$

for some $C = C(q, \Omega, \Lambda, \|\mu\|_{W^{1,3}(\Omega; \mathbb{C}^{3 \times 3})}) > 0$. In particular, $\Phi^* \in W^{1,q}(\Omega; \mathbb{C})$ for all $q < \infty$. The usual difference quotient argument (see e.g. [70, 67]) shows in turn that $\Phi^* \in W^{2,2}(\Omega; \mathbb{C})$, as ψ is regular. Thus, by assumption we have

$$\left| \int_{\Omega} u \partial_i \psi dx \right| = \left| \int_{\Omega} u \operatorname{div}(\mu^T \nabla \bar{\Phi}^*) dx \right| = |\langle F, \Phi^* \rangle| \leq \|F\|_{(W^{1,r'}(\Omega; \mathbb{C}))'} \|\Phi^*\|_{W^{1,r'}(\Omega; \mathbb{C})},$$

which in view of (2.54) gives

$$\left| \int_{\Omega} u \partial_i \psi dx \right| \leq C \|F\|_{(W^{1,r'}(\Omega; \mathbb{C}))'} \|\psi\|_{L^{r'}(\Omega; \mathbb{C})},$$

as required. $\square$

We are now equipped to write the main regularity proposition for E , which will lead to the proof of Theorem 2.17 by a bootstrap argument.

Proposition 2.21. *Assume that (2.40), (2.41) and (2.43) hold and*

$$(2.55) \quad J_m \in L^p(\Omega; \mathbb{C}^3), J_e \in W^{1,p}(\operatorname{div}, \Omega) \text{ and } \varphi \in W^{1,p}(\Omega; \mathbb{C}^3)$$

for some $p \geq 2$. Assume that $(E, H) \in H(\operatorname{curl}, \Omega) \times H^\mu(\operatorname{curl}, \Omega)$ is a weak solution of (2.42) with $\varphi = 0$ and $|\omega| \leq M$ for some $M > 0$.

Suppose that $E \in L^q(\Omega; \mathbb{C}^3)$ and $H \in L^s(\Omega; \mathbb{C})$, with $2 \leq q, s < \infty$ and write $r = \min((3q + q\delta)(q + 3 + \delta)^{-1}, p, s)$. Then $E \in W^{1,r}(\Omega; \mathbb{C}^3)$ and

$$(2.56) \quad \|E\|_{W^{1,r}(\Omega; \mathbb{C}^3)} \leq C(\|E\|_{L^q(\Omega; \mathbb{C}^3)} + \|H\|_{L^s(\Omega; \mathbb{C})} + \|J_e\|_{L^2(\Omega)} + \|J_m\|_{L^p(\Omega; \mathbb{C}^3)} + \|\operatorname{div} J_e\|_{L^p(\Omega; \mathbb{C})})$$

for some constant C depending on Ω , Λ , M , $\|(\varepsilon, \sigma)\|_{W^{1,3+\delta}(\Omega; \mathbb{R}^{3 \times 3})^2}$ and r only.

The corresponding proposition regarding H is as follows.

Proposition 2.22. Assume that (2.40), (2.41) and (2.45) hold and

$$J_e \in L^p(\Omega; \mathbb{C}^3) \quad \text{and} \quad \varphi \in W^{1,p}(\Omega; \mathbb{C}^3)$$

for some $p \geq 2$. Assume that $(E, H) \in H(\operatorname{curl}, \Omega) \times H^\mu(\operatorname{curl}, \Omega)$ is a weak solution of (2.42) with $\varphi = 0$ and $|\omega| \leq M$ for some $M > 0$.

Suppose that $E \in L^s(\Omega; \mathbb{C}^3)$ and $H \in L^q(\Omega; \mathbb{C})$, with $2 \leq q, s < \infty$ and write $r = \min((3q + q\delta)(q + 3 + \delta)^{-1}, p, s)$. Then $H \in W^{1,r}(\Omega; \mathbb{C}^3)$ and

$$(2.57) \quad \|H\|_{W^{1,r}(\Omega; \mathbb{C}^3)} \leq C(\|H\|_{L^q(\Omega; \mathbb{C}^3)} + \|E\|_{L^s(\Omega; \mathbb{C}^3)} + \|J_m\|_{L^2(\Omega)} + \|J_e\|_{L^p(\Omega; \mathbb{C}^3)})$$

for some constant C depending on Ω , Λ , M , $\|\mu\|_{W^{1,3+\delta}(\Omega; \mathbb{C}^{3 \times 3})}$ and r only.

We prove both propositions below. We first prove Theorems 2.17 and 2.18.

Proof of Theorems 2.17 and 2.18. Let us prove Theorem 2.17 first. Considering the system satisfied by $E - \varphi$ and H , we may assume $\varphi = 0$. Since $H \in L^2(\Omega; \mathbb{C}^3)$, we may apply Proposition 2.21 with $p = s = 2$ a finite number of times with increasing values of q . For $q_n \geq 2$ we obtain $E \in W^{1,r_n}(\Omega; \mathbb{C}^3)$, with $r_n = \min(q_n(3 + \delta)(q_n + 3 + \delta)^{-1}, 2)$. If $r_n = 2$, the result is proved. If $r_n < 2$, Sobolev embeddings show that $E \in L^{q_{n+1}}(\Omega; \mathbb{C}^3)$ with

$$q_{n+1} = q_n + \frac{\delta q_n^2}{9 + \delta(3 - q_n)} \geq q_n + \frac{4\delta}{9 + \delta},$$

using the bounds $q_n \geq 2$ and $9 + \delta(3 - q_n) > 0$, which follows from $r_n < 2$. Thus the sequence r_n converges to 2 in a finite number of steps.

The proof of Theorem 2.18 is similar, using Proposition 2.22 in lieu of Proposition 2.21 to bootstrap. The details are left to the reader. $\square$

We now prove Theorem 1.1.

Proof of Theorem 1.1. Suppose first $p \leq 3$ and $\delta < 3$. From Theorem 2.17 (resp. Theorem 2.18) and Sobolev Embeddings, we have $E \in L^6(\Omega; \mathbb{C}^3)$ (resp. $H \in L^6(\Omega; \mathbb{C}^3)$). We apply Propositions 2.21 and 2.22 a finite number of times, with $q = s$. Starting with $q_n \geq 6 = q_0$ we obtain E (and H) $\in W^{1,r_n}(\Omega; \mathbb{C}^3)$, with $r_n = \min(q_n(3 + \delta)(3 + \delta + q_n)^{-1}, p)$.

If $r_n = p$, the result is proved. If $r_n < p$, Sobolev embeddings imply that E and H belong to $L^{q_{n+1}}(\Omega; \mathbb{C}^3)$, with

$$q_{n+1} = q_n + \frac{\delta q_n^2}{9 + \delta(3 - q_n)} \geq q_n + \frac{q_0^2 \delta}{9 + \delta(3 - q_n)} \geq q_n + \frac{12\delta}{3 - \delta},$$

since $q_n \geq 6$, $\delta < 3$ and $9 + \delta(3 - q_n) > 0$ (as $r_n < 3$). Thus the sequence r_n converges to p in a finite number of steps.

Suppose now $p > 3$ and $\delta \in (0, \infty)$. The previous argument shows that E and H are in $W^{1,3}(\Omega; \mathbb{C}^3)$. One more iteration of the argument concludes the proof if $p < 3 + \delta$, and shows otherwise that E and H are in $L^\infty(\Omega; \mathbb{C}^3)$, and the result is obtained by a final application of Propositions 2.21 and 2.22. $\square$

We now prove Proposition 2.21.

Proof of Proposition 2.21. We subdivide the proof into four steps.

Step 1. Variational formulation. Since $E \times \nu = 0$ on $\partial\Omega$, multiplying identity (2.47) by $-\mathbf{i}$ shows that for every $\Phi \in W^{2,2}(\Omega; \mathbb{C})$ and $k = 1, 2, 3$ there holds

$$(2.58) \quad \int_{\Omega} E_k \operatorname{div}((\sigma - \mathbf{i}\omega\varepsilon)^T \nabla \bar{\Phi}) dx = \int_{\Omega} F_k \cdot \nabla \bar{\Phi} dx - \mathbf{i} \int_{\partial\Omega} (\partial_k \bar{\Phi}) q_\omega E \cdot \nu d\sigma,$$

where we set

$$F_k = -\mathbf{i}(\partial_k q_\omega)E + \mathbf{i}q_\omega(\mathbf{e}_k \times (J_m + \mathbf{i}\omega\mu H)) + \mathbf{e}_k \operatorname{div} J_e.$$

Since $(\partial_k q_\omega)E \in L^{q(3+\delta)(q+3+\delta)^{-1}}(\Omega; \mathbb{C}^3)$, we have that $F_k \in L^r(\Omega; \mathbb{C}^3)$.

Step 2. Interior regularity. Given a smooth open subdomain Ω_0 such that $\overline{\Omega_0} \subset \Omega$, we consider a cut-off function $\chi \in C_0^\infty(\Omega; \mathbb{R})$ such that $\chi = 1$ in Ω_0 . A computation gives for $\Phi \in W^{2,2}(\Omega; \mathbb{C})$

$$\int_{\Omega} \chi E_k \operatorname{div}((\sigma - \mathbf{i}\omega\varepsilon)^T \nabla \bar{\Phi}) dx = \int_{\Omega} E_k \operatorname{div}((\sigma - \mathbf{i}\omega\varepsilon)^T \nabla (\chi \bar{\Phi})) dx + T_k(\Phi),$$

where $T_k(\Phi) = -\int_{\Omega} E_k (\operatorname{div}((\sigma - \mathbf{i}\omega\varepsilon)^T \bar{\Phi} \nabla \chi) + (\sigma - \mathbf{i}\omega\varepsilon) \nabla \chi \cdot \nabla \bar{\Phi}) dx$. Thus, by (2.58) we obtain

$$\int_{\Omega} \chi E_k \operatorname{div}((\sigma - \mathbf{i}\omega\varepsilon)^T \nabla \bar{\Phi}) dx = \int_{\Omega} F_k \cdot \nabla (\chi \bar{\Phi}) dx + T_k(\Phi),$$

since χ is compactly supported. Using Sobolev embeddings and the fact that F_k is in $L^r(\Omega; \mathbb{C}^3)$, we verify that $\Phi \mapsto \int_{\Omega} F_k \cdot \nabla (\chi \bar{\Phi}) dx + T_k(\Phi)$ is in $(W^{1,r'}(\Omega; \mathbb{C}))'$. Thanks to Lemma 2.20 we conclude that $\chi E_k \in W^{1,r}(\Omega; \mathbb{C})$, namely $E \in W^{1,r}(\Omega_0; \mathbb{C}^3)$.

Step 3. Boundary regularity. Take now $x_0 \in \partial\Omega$. Since $\partial\Omega$ is of class $C^{1,1}$ there exists a ball B centred in x_0 and an orthogonal change of coordinates $\Psi \in C^{1,1}(B; \mathbb{R}^3)$ such that in the new system $u_i = \Psi_i(x)$ we have $\Psi(B \cap \Omega) = \{u_3 < 0\} \cap B(0, R)$, where $B(x, R) = \{y : |y - x| < R\}$. We can now express the relevant quantities with respect to the coordinates u_1, u_2, u_3 . Let the components of vectors be marked by tildes if they are expressed in the u_i coordinate system. Denoting $L = \operatorname{curl} E$ we have (see [109, Lemma 3.1])

$$\tilde{E} = (\nabla \Psi)E, \quad \tilde{L} = (\nabla \Psi)L = (\partial_{u_1}, \partial_{u_2}, \partial_{u_3}) \times \tilde{E},$$

and the corresponding identities for H , as $\nabla\Psi$ is an orthogonal matrix chosen so that $\det \nabla\Psi = 1$. Therefore, using the notation $\tilde{\nabla} = (\partial_{u_1}, \partial_{u_2}, \partial_{u_3})$, (2.42) implies

$$\begin{cases} \tilde{\nabla} \times \tilde{E} = \mathbf{i}\omega\tilde{\mu}\tilde{H} + \tilde{J}_m, \\ \tilde{\nabla} \times \tilde{H} = -\mathbf{i}(\omega\tilde{\varepsilon} + \mathbf{i}\tilde{\sigma})\tilde{E} + \tilde{J}_e, \\ \tilde{E}_1 = \tilde{E}_2 = 0 \quad \text{on } u_3 = 0. \end{cases}$$

where $\tilde{\varepsilon} = (\nabla\Psi)\varepsilon(\nabla\Psi)^T$, $\tilde{\sigma} = (\nabla\Psi)\sigma(\nabla\Psi)^T$, $\tilde{J}_e = (\nabla\Psi)J_e$, $\tilde{\mu} = (\nabla\Psi)\mu(\nabla\Psi)^T$ and $\tilde{J}_m = (\nabla\Psi)J_m$. Namely, Maxwell's equations (2.42) in the new coordinates u_i can be written in the same form. Note that $\tilde{\varepsilon}$, $\tilde{\sigma}$ and $\tilde{\mu}$ satisfy condition (2.40) for some $\tilde{\Lambda} > 0$. Moreover, since $\nabla\Psi \in W^{1,\infty}(B; \mathbb{R}^{3 \times 3})$, the regularity assumptions (2.43) and (2.55) hold for $\tilde{\varepsilon}$, $\tilde{\sigma}$ and for the sources $\tilde{J}_e$, $\tilde{J}_m$. Furthermore, $E \in W^{1,r}(B \cap \Omega; \mathbb{C}^3)$ if $\tilde{E} \in W^{1,r}(\{u_3 < 0\} \cap B(0, R); \mathbb{C}^3)$.

We have shown that without loss of generality we can assume that around x_0 the boundary is flat. More precisely, suppose that $B \cap \Omega = \{x \cdot \mathbf{e}_3 < 0\} \cap B(0, R)$ and take $\chi \in \mathcal{D}(B; \mathbb{R})$ such that $\chi = 1$ in a neighbourhood $\tilde{B}$ of x_0 .

Let us first consider the two tangential components of E , that is, E_j with $j = 1, 2$. Proceeding as in step 2 we obtain for every $\Phi \in W^{2,2}(\Omega; \mathbb{C}) \cap W_0^{1,2}(\Omega; \mathbb{C})$

$$\int_{\Omega} \chi E_j \operatorname{div}((\sigma - \mathbf{i}\omega\varepsilon)^T \nabla \bar{\Phi}) dx = \int_{\Omega} E_j \operatorname{div}((\sigma - \mathbf{i}\omega\varepsilon)^T \nabla(\chi \bar{\Phi})) dx + T_j(\Phi),$$

where $T_j(\Phi) = -\int_{\Omega} E_j (\operatorname{div}((\sigma - \mathbf{i}\omega\varepsilon)^T \bar{\Phi} \nabla \chi) + (\sigma - \mathbf{i}\omega\varepsilon) \nabla \chi \cdot \nabla \bar{\Phi}) dx$. In view of identity (2.58), since $\chi \bar{\Phi} = 0$ on $\partial\Omega$, we have

$$\int_{\Omega} \chi E_j \operatorname{div}((\sigma - \mathbf{i}\omega\varepsilon)^T \nabla \bar{\Phi}) dx = \int_{\Omega} F_j \cdot \nabla(\chi \bar{\Phi}) dx + T_j(\Phi).$$

As in step 2, thanks to Lemma 2.20 we conclude that $\chi E_j \in W^{1,r}(\Omega; \mathbb{C})$ for $j = 1, 2$.

Let us now turn to the normal component E_3 . Consider the first part of Maxwell's equations (2.42), $\operatorname{curl} E = \mathbf{i}\omega\mu H + J_m$ in the quotient space where every element of $L^r(\tilde{B}; \mathbb{C}^3)$ is identified with nought, that is, $W^{-1,2}(\tilde{B}; \mathbb{C}^3)/L^r(\tilde{B}; \mathbb{C}^3)$. We find $0 = -\operatorname{curl} E = -\operatorname{curl}(E_3 \mathbf{e}_3) = \mathbf{e}_3 \times \nabla E_3$, since $\mathbf{i}\omega\mu H + J_m \in L^r(\Omega; \mathbb{C}^3)$ and $E_1, E_2 \in W^{1,r}(\tilde{B}; \mathbb{C})$. In other words,

$$(2.59) \quad \nabla E_3 = \mathbf{e}_3 (\mathbf{e}_3 \cdot \nabla E_3) \text{ in } W^{-1,2}(\tilde{B}; \mathbb{C}^3)/L^r(\tilde{B}; \mathbb{C}^3).$$

Therefore, taking now the divergence of the second identity in Maxwell's equations (2.42), and using the fact that $\operatorname{div} J_e \in L^r(\Omega; \mathbb{C})$ and $E \in L^r(\Omega; \mathbb{C}^3)$ we obtain, in the quotient space $W^{-1,2}(\tilde{B}; \mathbb{C})/L^r(\tilde{B}; \mathbb{C})$,

$$0 = \operatorname{div}((\sigma - \mathbf{i}\omega\varepsilon)E) = \operatorname{div}(E_3(\sigma - \mathbf{i}\omega\varepsilon)\mathbf{e}_3) = \nabla E_3 \cdot ((\sigma - \mathbf{i}\omega\varepsilon)\mathbf{e}_3) = (\mathbf{e}_3 \cdot \nabla E_3) \mathbf{e}_3 \cdot ((\sigma - \mathbf{i}\omega\varepsilon)\mathbf{e}_3).$$

Hence, in view of (2.40) and (2.59) we obtain $\nabla E_3 \in L^r(\tilde{B}; \mathbb{C}^3)$, whence $E \in W^{1,r}(\tilde{B}; \mathbb{C}^3)$.

Step 4. Global regularity. Combining the interior and the boundary regularities, a standard ball covering argument shows $E \in W^{1,r}(\Omega; \mathbb{C}^3)$. The estimate given in (2.56) follows from Lemma 2.20. $\square$

We now turn to the proof of Proposition 2.22. Naturally, the interior estimates can be obtained in the exact same way, substituting the very weak formulations for the components of E by the corresponding identities for the components of H . The boundary estimates require different arguments, and we detail this step below.

Proof of Proposition 2.22. Boundary regularity. By (2.49), for every $\Phi \in W^{2,2}(\Omega; \mathbb{C})$ and $k = 1, 2, 3$ there holds

$$(2.60) \quad \int_{\Omega} H_k \operatorname{div}(\mu^T \nabla \bar{\Phi}) \, dx = \int_{\Omega} G_k \cdot \nabla \bar{\Phi} \, dx - \int_{\partial\Omega} (\mathbf{e}_k \times (H \times \nu)) \cdot (\mu^T \nabla \bar{\Phi}) \, d\sigma,$$

where

$$G_k = (\partial_k \mu) H - \mu (\mathbf{e}_k \times (J_e - \mathbf{i} q_{\omega} E)) \in L^r(\Omega; \mathbb{C}^3),$$

since $(\partial_k \mu) H \in L^{q(3+\delta)(q+3+\delta)^{-1}}(\Omega; \mathbb{C}^3)$.

Take $x_0 \in \partial\Omega$. As in the proof of Proposition 2.21, we can assume that $\partial\Omega$ is the plane $x \cdot \mathbf{e}_3 = 0$ in a neighbourhood B of x_0 . Again let us focus on the tangential components first. Take $\chi \in \mathcal{D}(B; \mathbb{R})$ such that $\chi = 1$ in a neighbourhood $\tilde{B}$ of x_0 and $j = 1, 2$.

We choose a test function satisfying a Neumann type boundary condition, that is $\Phi \in W^{2,2}(\Omega; \mathbb{C})$ such that $\mu^T \nabla \bar{\Phi} \cdot \nu = 0$ on $\partial\Omega$. We have

$$\int_{\Omega} \chi H_j \operatorname{div}(\mu^T \nabla \bar{\Phi}) \, dx = \int_{\Omega} H_j \operatorname{div}(\mu^T \nabla (\chi \bar{\Phi})) \, dx + R(\Phi),$$

where $R(\Phi) = - \int_{\Omega} H_j (\operatorname{div}(\mu^T \bar{\Phi} \nabla \chi) + \mu \nabla \chi \cdot \nabla \bar{\Phi}) \, dx$. From identity (2.60) we obtain

$$(2.61) \quad \int_{\Omega} \chi H_j \operatorname{div}(\mu^T \nabla \bar{\Phi}) \, dx = \int_{\Omega} G_j \cdot \nabla (\chi \bar{\Phi}) \, dx + R(\Phi) + S(\Phi),$$

where

$$S(\Phi) = - \int_{\partial\Omega} (\mathbf{e}_j \times (H \times \mathbf{e}_3)) \cdot (\mu^T \nabla (\chi \bar{\Phi})) \, d\sigma.$$

As before, the functional $\Phi \mapsto \int_{\Omega} G_j \cdot \nabla (\chi \bar{\Phi}) \, dx + R(\Phi)$ is in $(W^{1,r'}(\Omega; \mathbb{C}))'$. We shall now prove that $S \in (W^{1,r'}(\Omega; \mathbb{C}))'$. Since $\mu^T \nabla \bar{\Phi} \cdot \nu = 0$ on $\partial\Omega$ and $\nu = \mathbf{e}_3$ on B , we have $\chi (\mathbf{e}_j \times (H \times \mathbf{e}_3)) \cdot (\mu^T \nabla \bar{\Phi}) = 0$, thus

$$S(\Phi) = - \int_{\partial\Omega} (\mathbf{e}_j \times (H \times \mathbf{e}_3)) \cdot (\mu^T \nabla \chi) \bar{\Phi} \, d\sigma.$$

By hypothesis we have $H \in W^{1,r}(\operatorname{curl}, \Omega)$, whence $H \times \nu \in W^{-1/r,r}(\partial\Omega; \mathbb{C}^3)$. It follows that $(\mathbf{e}_j \times (H \times \mathbf{e}_3)) \cdot (\mu^T \nabla \chi) \in W^{-1/r,r}(\partial\Omega; \mathbb{C})$. As a result (see [70, Theorem 1.5.1.2]),

$$|S(\Phi)| \leq C \|\Phi\|_{W^{1-\frac{1}{r'},r'}(\partial\Omega;\mathbb{C})} \leq C \|\Phi\|_{W^{1,r'}(\Omega;\mathbb{C})}$$

for some $C > 0$ independent of Φ ; in other words $S \in (W^{1,r'}(\Omega; \mathbb{C}))'$. We can now apply Lemma 2.20 to (2.61) and obtain $\chi H_j \in W^{1,r}(\Omega; \mathbb{C})$. The rest of the proof follows faithfully that of Proposition 2.21. $\square$

To conclude this section, we point out that higher regularity results follow naturally under appropriate assumptions.

Theorem 2.23. *Suppose that (2.40) and (2.41) hold and take $\kappa \in \mathbb{N}^*$, $M > 0$ and $|\omega| \leq M$. Assume additionally that $\partial\Omega$ is of class $C^{\kappa,1}$ and that*

$$(\omega\varepsilon + \mathbf{i}\sigma), \mu \in W^{\kappa,p}(\Omega; \mathbb{C}^{3 \times 3}), \quad J_e, J_m \in W^{\kappa,p}(\text{div}, \Omega), \quad \varphi \in W^{\kappa,p}(\Omega; \mathbb{C}^3)$$

for some $p > 3$. If $(E, H) \in H(\text{curl}, \Omega) \times H^\mu(\text{curl}, \Omega)$ is a weak solution of (2.42), then $E, H \in W^{\kappa,p}(\Omega; \mathbb{C}^3)$ and there holds

$$\|(E, H)\|_{W^{\kappa,p}(\Omega; \mathbb{C}^3)^2} \leq C \left(\|(E, H)\|_{L^2(\Omega; \mathbb{C}^3)^2} + \|\varphi\|_{W^{\kappa,p}(\Omega; \mathbb{C}^3)} + \|(J_e, J_m)\|_{W^{\kappa,p}(\text{div}, \Omega)^2} \right)$$

for some constant C depending on Ω , Λ , κ , p , M , $\|(\varepsilon, \sigma)\|_{W^{\kappa,p}(\Omega; \mathbb{R}^{3 \times 3})^2}$ and $\|\mu\|_{W^{\kappa,p}(\Omega; \mathbb{C}^{3 \times 3})}$.

Proof. The proof is done by induction. Theorem 1.1 corresponds to $\kappa = 1$. Assume that for some $\kappa \geq 2$, Theorem 2.23 holds for $\kappa - 1$.

For simplicity, we shall consider (2.42) in its strong form, but every step can be made rigorous by passing to the suitable weak formulation.

By using a change of coordinates as in the proof of Proposition 2.21, we can assume without loss of generality that $\Omega \cap B(0, R) = \{x \cdot \mathbf{e}_3 < 0\} \cap B(0, R)$. Indeed, the assumption $\partial\Omega \in C^{\kappa,1}$ implies that the regularity assumptions on the coefficients and on the source terms and the conditions $E, H \in W^{\kappa,p}$ are insensitive to a $C^{\kappa,1}$ change of coordinates.

For $i = 1, 2$ we have

$$\begin{cases} \text{curl } \partial_i E = \mathbf{i}\omega\mu\partial_i H + J'_m & \text{in } \Omega \cap B(0, R), \\ \text{curl } \partial_i H = -\mathbf{i}q_\omega\partial_i E + J'_e & \text{in } \Omega \cap B(0, R), \\ \partial_i E \times \mathbf{e}_3 = \partial_i \varphi \times \mathbf{e}_3 & \text{on } \partial\Omega \cap \{x \cdot \mathbf{e}_3 = 0\} \cap \bar{B}(0, R), \end{cases}$$

where $J'_e = \partial_i J_e + \mathbf{i}(\partial_i q_\omega)E$ and $J'_m = \partial_i J_m + \mathbf{i}\omega(\partial_i \mu)H$. By assumption, we have $E, H \in W^{\kappa-1,p}(\Omega; \mathbb{C}^3)$, therefore $J'_e, J'_m \in W^{\kappa-1,p}(\text{div}, \Omega)$ and $J'_m \cdot \nu \in W^{\kappa-1-\frac{1}{p},p}(\partial\Omega; \mathbb{C})$. Applying Theorem 2.23 with $\kappa - 1$ in lieu of κ to the above system shows that $\partial_i E, \partial_i H \in W^{\kappa-1,p}(\Omega; \mathbb{C}^3)$.

An argument similar to the one given in the third step of the proof of Proposition 2.21 allows us to infer that $\partial_3 E, \partial_3 H \in W^{\kappa-1,p}(\Omega; \mathbb{C}^3)$, whence $E, H \in W^{\kappa,p}(\Omega; \mathbb{C}^3)$. The corresponding norm estimate follows by Theorem 1.1 and the argument given above. $\square$

2.3.2 Proof of Theorem 1.2 using Campanato estimates

The purpose of this section is to prove Theorem 1.2. We shall apply classical Campanato estimates for elliptic equations to (2.46), namely the elliptic equations satisfied by E . We first state the properties of Campanato spaces that we shall use, and then proceed to the proof of Theorem 1.2.

For $\lambda \geq 0$ and $p \geq 1$ we denote the Campanato space by $L^{p,\lambda}(\Omega; \mathbb{C})$ [52], namely the Banach space of functions $u \in L^p(\Omega; \mathbb{C})$ such that

$$[u]_{p,\lambda;\Omega}^p := \sup_{x \in \Omega, 0 < \rho < \text{diam} \Omega} \rho^{-\lambda} \int_{\Omega(x,\rho)} \left| u(y) - \frac{1}{|\Omega(x,\rho)|} \int_{\Omega(x,\rho)} u(z) dz \right|^p dy < \infty,$$

where $\Omega(x, \rho) = \Omega \cap \{y \in \mathbb{R}^3 : |y - x| < \rho\}$, equipped with the norm

$$\|u\|_{L^{p,\lambda}(\Omega; \mathbb{C})} = \|u\|_{L^p(\Omega; \mathbb{C})} + [u]_{p,\lambda;\Omega}.$$

Lemma 2.24. *Take $\lambda \geq 0$.*

1. *If $\lambda > 3$ then $L^{2,\lambda}(\Omega; \mathbb{C})$ is isomorphic to $C^{0, \frac{\lambda-3}{2}}(\overline{\Omega}; \mathbb{C})$.*
2. *Suppose $\lambda < 3$. If $u \in L^2(\Omega; \mathbb{C})$ and $\nabla u \in L^{2,\lambda}(\Omega; \mathbb{C}^3)$ then $u \in L^{2,2+\lambda}(\Omega; \mathbb{C})$, and the embedding is continuous.*
3. *Suppose $\delta > 0$ and $\lambda \neq 1$. If $f \in L^{3+\delta}(\Omega; \mathbb{C})$ and $u \in L^2(\Omega; \mathbb{C})$ with $\nabla u \in L^{2,\lambda}(\Omega; \mathbb{C}^3)$ then $fu \in L^{2,\lambda'}(\Omega; \mathbb{C})$ with $\lambda' = \min(\lambda + 2\delta(3+\delta)^{-1}, 3(1+\delta)(3+\delta)^{-1})$, and the embedding is continuous.*

Proof. Statements (1) and (2) are classical, see e.g. [106, Chapter 1]. For (3), note that Hölder's inequality implies that $f \in L^{2,3(1+\delta)(3+\delta)^{-1}}(\Omega; \mathbb{C})$. When $\lambda < 1$, the result follows from [60, Lemma 4.1]. When $\lambda > 1$, (3) follows from (1) and (2). $\square$

We now state the regularity result regarding Campanato estimates we will use. It can be found in [106] (Theorems 2.19 and 3.16).

Proposition 2.25. *Assume (2.40) and $\Im \mu = 0$. There exists $\lambda_\mu \in (1, 2]$ depending only on Ω and on Λ , such that if $F \in L^{2,\lambda}(\Omega; \mathbb{C}^3)$ for some $\lambda \in [0, \lambda_\mu)$, and $u \in W^{1,2}(\Omega; \mathbb{C})$ satisfies*

$$\begin{cases} -\text{div}(\mu \nabla u) = \text{div} F & \text{in } \Omega, \\ \mu \nabla u \cdot \nu = F \cdot \nu & \text{on } \partial\Omega, \end{cases}$$

then $\nabla u \in L^{2,\lambda}(\Omega; \mathbb{C}^3)$ and

$$(2.62) \quad \|\nabla u\|_{L^{2,\lambda}(\Omega; \mathbb{C}^3)} \leq C \|F\|_{L^{2,\lambda}(\Omega; \mathbb{C}^3)},$$

where the constant C depends only on Λ and Ω .

Alternatively, assume (2.40) and (2.45). For all $\lambda \in [0, 2]$, if $F \in L^{2,\lambda}(\Omega; \mathbb{C}^3)$, $f \in L^2(\Omega; \mathbb{C})$, and $u \in W^{1,2}(\Omega; \mathbb{C})$ satisfies

$$\begin{cases} -\text{div}(\mu \nabla u) = \text{div}(F) + f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

then $\nabla u \in L^{2,\lambda}(\Omega; \mathbb{C}^3)$ and

$$(2.63) \quad \|\nabla u\|_{L^{2,\lambda}(\Omega; \mathbb{C}^3)} \leq C \left(\|F\|_{L^{2,\lambda}(\Omega; \mathbb{C}^3)} + \|f\|_{L^2(\Omega; \mathbb{C})} \right),$$

where the constant C depends on Λ , Ω , and $\|\mu\|_{W^{1,3+\delta}(\Omega; \mathbb{C}^{3 \times 3})}$ only.

We first study the regularity of H following a variant of an argument given in [112].

Proposition 2.26. *Assume that Ω is simply connected, that (2.40) and (2.41) hold with $\Im\mu = 0$ and $J_m \in L^{2,\lambda}(\Omega)$ with $1 < \lambda < \lambda_\mu$, where λ_μ is given by Proposition 2.25. Let $(E, H) \in H(\text{curl}, \Omega) \times H^\mu(\text{curl}, \Omega)$ be a weak solution of (2.42) with $\varphi = 0$ and $|\omega| \leq M$. Then $H \in L^{2,\lambda}(\Omega; \mathbb{C}^3)$ and*

$$(2.64) \quad \|H\|_{L^{2,\lambda}(\Omega; \mathbb{C}^3)} \leq C(\|E\|_{L^2(\Omega; \mathbb{C}^3)} + \|J_e\|_{L^2(\Omega; \mathbb{C}^3)}),$$

where the constant C depends only on Λ , λ , M and Ω .

Proof. Since $-\mathbf{i}q_\omega E + J_e$ is divergence free in Ω , and Ω is $C^{1,1}$ and simply connected, it is well known that there exists $T \in H^1(\Omega; \mathbb{C}^3)$ such that $-\mathbf{i}q_\omega E + J_e = \text{curl}T$, satisfying

$$(2.65) \quad \|T\|_{H^1(\Omega; \mathbb{C}^3)} \leq C(\|J_e\|_{L^2(\Omega; \mathbb{C}^3)} + \|E\|_{L^2(\Omega; \mathbb{C}^3)})$$

where C depends on Ω , Λ , and M only, see e.g. [69, Chapter I, Theorem 3.5]. Thanks to Lemma 2.24, this implies $\mu T \in L^{2,2}(\Omega; \mathbb{C}^3) \subset L^{2,\lambda}(\Omega; \mathbb{C}^3)$.

As $H - T$ is curl free in Ω , in view of [69, Chapter I, Theorem 2.9] there exists $h \in H^1(\Omega; \mathbb{C})$ such that $H - T = \nabla h$. The potential h is defined up to a constant by

$$\begin{cases} \text{div}(\mu \nabla h) = -\text{div}(\mu T) & \text{in } \Omega, \\ \mu \nabla h \cdot \nu = -\mu T \cdot \nu & \text{on } \partial\Omega. \end{cases}$$

Note that the boundary condition follows from $H \in H^\mu(\text{curl}, \Omega)$. Thanks to estimate (2.62) in Proposition 2.25, we have

$$(2.66) \quad \|\nabla h\|_{L^{2,\lambda}(\Omega; \mathbb{C}^3)} \leq C \|\mu T\|_{L^{2,\lambda}(\Omega; \mathbb{C}^3)} \leq \tilde{C} \|T\|_{H^1(\Omega; \mathbb{C}^3)}.$$

The conclusion follows from the identity $H = T + \nabla h$ and the estimates (2.65) and (2.66). $\square$

We now adapt Proposition 2.21 to be able to use Campanato estimates in the bootstrap argument.

Proposition 2.27. *Assume that Ω is simply connected, that (2.40) and (2.41) hold with $\Im\mu = 0$ and that (2.43) holds. Suppose $J_m \in L^{2,\tilde{\lambda}}(\Omega; \mathbb{C}^3)$ and $\text{div}J_e \in L^{2,\tilde{\lambda}}(\Omega; \mathbb{C})$ for some $1 < \tilde{\lambda} < \lambda_\mu$, where λ_μ is given by Proposition 2.25. Let $(E, H) \in H(\text{curl}, \Omega) \times H^\mu(\text{curl}, \Omega)$ be a weak solution of (2.42) with $\varphi = 0$ and $|\omega| \leq M$.*

If $\nabla E \in L^{2,\lambda_0}(\Omega; \mathbb{C}^{3 \times 3})$ for some $\lambda_0 \in [0, \infty) \setminus \{1\}$ then $\nabla E \in L^{2,\lambda_1}(\Omega; \mathbb{C})^9$, with $\lambda_1 = \min(\tilde{\lambda}, \lambda_0 + 2\delta(3 + \delta)^{-1}, 3(1 + \delta)(3 + \delta)^{-1})$. Moreover there holds

$$(2.67) \quad \|\nabla E\|_{L^{2,\lambda_1}(\Omega; \mathbb{C})^9} \leq C \left(\|E\|_{L^2(\Omega; \mathbb{C}^3)} + \|\nabla E\|_{L^{2,\lambda_0}(\Omega; \mathbb{C})^9} + \|J_e\|_{L^2(\Omega; \mathbb{C}^3)} \right. \\ \left. + \|J_m\|_{L^{2,\tilde{\lambda}}(\Omega; \mathbb{C}^3)} + \|\text{div}J_e\|_{L^{2,\tilde{\lambda}}(\Omega; \mathbb{C})} \right),$$

where the constant C depends only on Ω , Λ , λ_1 , M and $\|(\varepsilon, \sigma)\|_{W^{1,3+\delta}(\Omega; \mathbb{R}^{3 \times 3})^2}$.

Proof. In view of Theorem 2.17 and Proposition 2.19, for each $k = 1, 2, 3$, $E_k \in H^1(\Omega; \mathbb{C})$ is a weak solution of

$$(2.68) \quad -\operatorname{div}((\sigma - \mathbf{i}\omega\varepsilon)\nabla E_k) = \operatorname{div}(-\mathbf{i}\partial_k q_\omega E + S_k) \text{ in } \Omega,$$

with

$$S_k = \mathbf{i}q_\omega (\mathbf{e}_k \times (J_m + \mathbf{i}\omega\mu H)) + \mathbf{e}_k \operatorname{div} J_e.$$

Thanks to Proposition 2.26 we have that $S_k \in L^{2,\tilde{\lambda}}(\Omega; \mathbb{C})$. Furthermore there holds $\partial_k q_\omega E \in L^{2,\tilde{\lambda}_0}(\Omega; \mathbb{C}^3)$ with $\tilde{\lambda}_0 = \min(\lambda_0 + 2\delta(3+\delta)^{-1}, 3(1+\delta)(3+\delta)^{-1})$ in view of Lemma 2.24. Thus

$$(2.69) \quad -\mathbf{i}\partial_k q_\omega \varepsilon E + S_k \in L^{2,\lambda_1}(\Omega; \mathbb{C}^3).$$

Interior regularity. Given a smooth open subdomain Ω_0 such that $\overline{\Omega_0} \subset \Omega$, introduce a cut-off function $\chi \in \mathcal{D}(\Omega)$ such that $\chi = 1$ in Ω_0 . From (2.68) we deduce

$$(2.70) \quad -\operatorname{div}((\sigma - \mathbf{i}\omega\varepsilon)\nabla(\chi E_k)) = \operatorname{div}(\chi(-\mathbf{i}\partial_k q_\omega E + S_k)) + f_k \text{ in } \Omega,$$

where

$$f_k = -\nabla\chi \cdot (-\mathbf{i}\partial_k q_\omega E + S_k) - (\sigma - \mathbf{i}\omega\varepsilon)\nabla E_k \cdot \nabla\chi - \operatorname{div}((\sigma - \mathbf{i}\omega\varepsilon)E_k \nabla\chi) \in L^2(\Omega; \mathbb{C}).$$

As $\lambda_1 < 2$ and $(\sigma - \mathbf{i}\omega\varepsilon)$ satisfies (2.43), we may apply Proposition 2.25 (with $(\sigma - \mathbf{i}\omega\varepsilon)$ in lieu of μ) to show that $\nabla(\chi E_k)$ is in $L^{2,\lambda_1}(\Omega; \mathbb{C}^3)$, which implies $\nabla E \in L^{2,\lambda_1}(\Omega_0; \mathbb{C}^{3 \times 3})$.

Boundary regularity. By using a change of coordinates as in the proof of Proposition 2.21, we can assume without loss of generality that $\Omega \cap B(0, R) = \{x \cdot \mathbf{e}_3 < 0\} \cap B(0, R)$. Indeed, the assumption $\partial\Omega \in C^{1,1}$ implies that the regularity assumptions on ε and on the source terms and the condition $\nabla E \in L^{2,\lambda_1}$ are insensitive to a $C^{1,1}$ change of coordinates, as L^∞ is a multiplier space for L^{2,λ_1} .

Let us focus on the tangential components first. Take $\chi \in \mathcal{D}(B(0, R); \mathbb{R})$ such that $\chi = 1$ in a neighbourhood $\tilde{B}$ of 0 and $j \in \{1, 2\}$. Identity (2.68) yields, for $j = 1, 2$

$$-\operatorname{div}((\sigma - \mathbf{i}\omega\varepsilon)\nabla(\chi E_j)) = \operatorname{div}(\chi(-\mathbf{i}\partial_j q_\omega E + S_j)) + f_j \text{ in } \Omega,$$

where $f_j = -\nabla\chi \cdot (-\mathbf{i}\partial_j q_\omega E + S_j) - (\sigma - \mathbf{i}\omega\varepsilon)\nabla E_j \cdot \nabla\chi - \operatorname{div}((\sigma - \mathbf{i}\omega\varepsilon)E_j \nabla\chi) \in L^2(\Omega; \mathbb{C})$. Note that $E \times \nu = 0$ on $\partial\Omega$ implies $\chi E_1 = \chi E_2 = 0$ on $\partial\Omega$. Proposition 2.25 together with (2.69) then shows that $\nabla(\chi E_j)$ belongs to $L^{2,\lambda_1}(\Omega; \mathbb{C}^3)$ for $j = 1, 2$. Arguing as in the proof of Proposition 2.21, we also derive that $\nabla(\chi E_3) \in L^{2,\lambda_1}(\Omega; \mathbb{C}^3)$. Therefore $\nabla(\chi E) \in L^{2,\lambda_1}(\tilde{B}; \mathbb{C}^{3 \times 3})$, and in turn $\nabla E \in L^{2,\lambda_1}(\tilde{B}; \mathbb{C}^{3 \times 3})$.

Global regularity. Combining the interior and the boundary estimates we obtain that ∇E is in $L^{2,\lambda_1}(\Omega; \mathbb{C}^{3 \times 3})$, together with (2.67). $\square$

We are now ready to prove the global Hölder regularity result.

Proof of Theorem 1.2. Considering the system satisfied by $E - \varphi$ and H , we may assume $\varphi = 0$. Choose any $\tilde{\lambda} > 1$ such that $\tilde{\lambda} < \lambda_\mu$ and $\tilde{\lambda} \leq 3^{\frac{p-2}{p}}$. Hölder's inequality shows that $J_m \in L^{2,\tilde{\lambda}}(\Omega; \mathbb{C}^3)$ and $\operatorname{div} J_e \in L^{2,\tilde{\lambda}}(\Omega; \mathbb{C})$. We apply Proposition 2.27 a finite number of times, starting with $\nabla E \in L^{2,\lambda_n}(\Omega; \mathbb{C}^{3 \times 3})$ for some $\lambda_n < 1$ (in the initial step we take $\lambda_0 = 0$, in view of Theorem 2.17), and obtain that $\nabla E \in L^{2,\lambda_{n+1}}(\Omega; \mathbb{C}^{3 \times 3})$, with $\lambda_{n+1} = \min(\tilde{\lambda}, (n+1)2\delta(\delta+3)^{-1})$. If $\lambda_{n+1} = 1$, Proposition 2.27 could not be applied another time (as $\lambda_0 = 1$ is excluded). An easy workaround of course is to reduce δ to a nearby irrational (just for this step), and proceed. We stop the iterative procedure as soon as $\lambda_{n+1} > 1$ and we infer that $\nabla E \in L^{2,\lambda}(\Omega; \mathbb{C}^{3 \times 3})$ for some $1 < \lambda \leq \tilde{\lambda}$. A final application of Proposition 2.27 gives $\nabla E \in L^{2,\min(\tilde{\lambda}, 3(1+\delta)(3+\delta)^{-1})}(\Omega; \mathbb{C}^{3 \times 3})$; the result then follows from Lemma 2.24. $\square$

2.3.3 Bi-anisotropic materials

In this subsection, we investigate the interior regularity of the solutions of the time harmonic Maxwell's equations for bi-anisotropic materials

$$(2.71) \quad \begin{cases} \operatorname{curl} E = \mathbf{i}\omega(\zeta E + \mu H) + J_m & \text{in } \Omega, \\ \operatorname{curl} H = -\mathbf{i}\omega(\varepsilon E + \xi H) + J_e & \text{in } \Omega, \end{cases}$$

for $\zeta, \mu, \varepsilon, \xi \in L^\infty(\Omega; \mathbb{C}^{3 \times 3})$ and $\omega \in \mathbb{C} \setminus \{0\}$. Note that in this subsection we use a different notation for the material parameters.

As far as the author is aware, this question was previously studied only recently in [63], where the parameters are assumed to be at least Lipschitz continuous. In this more general context, hypothesis (2.40) is not sufficient to ensure ellipticity. As we will see in Proposition 2.31, the leading order parameter for the coupled elliptic system is the tensor

$$(2.72) \quad A = A_{ij}^{\alpha\beta} = \begin{bmatrix} \Re \varepsilon & -\Im \varepsilon & \Re \xi & -\Im \xi \\ \Im \varepsilon & \Re \varepsilon & \Im \xi & \Re \xi \\ \Re \zeta & -\Im \zeta & \Re \mu & -\Im \mu \\ \Im \zeta & \Re \zeta & \Im \mu & \Re \mu \end{bmatrix},$$

where the Latin indices $i, j = 1, \dots, 4$ identify the different 3×3 block sub-matrices, whereas the Greek letters $\alpha, \beta = 1, 2, 3$ span each of these 3×3 block sub-matrices. We assume that A is in $L^\infty(\Omega; \mathbb{R})^{12 \times 12}$ and satisfies a strong Legendre condition (as in [56, 67]), that is, there exists $\Lambda > 0$ such that

$$(2.73) \quad A_{ij}^{\alpha\beta} \eta_\alpha^i \eta_\beta^j \geq \Lambda^{-1} |\eta|^2, \quad \eta \in \mathbb{R}^{12} \quad \text{and} \quad \|A\|_{L^\infty(\Omega; \mathbb{R})^{12 \times 12}} \leq \Lambda.$$

The following result gives a sufficient condition for (2.73) to hold true.

Lemma 2.28. *Assume that $\varepsilon_0, \mu_0, \kappa, \chi$ are real constants, with $\varepsilon_0 > 0$ and $\mu_0 > 0$. Let*

$$(2.74) \quad \varepsilon = \varepsilon_0 I_3, \quad \mu = \mu_0 I_3, \quad \xi = (\chi - \mathbf{i}\kappa) I_3, \quad \zeta = (\chi + \mathbf{i}\kappa) I_3,$$

where I_3 is the 3×3 identity matrix, and construct the matrix A as in (2.72). If

$$(2.75) \quad \chi^2 + \kappa^2 < \varepsilon_0 \mu_0,$$

then A satisfies (2.73).

Remark 2.29. This result shows that a wide class of materials satisfy the strong Legendre condition (2.73). Considering for simplicity the case of constant and isotropic parameters, the constitutive relations given in (2.74) describe the so-called chiral materials. It turns out that (2.75) is satisfied for natural materials [108].

Proof. A direct calculation shows that the smallest eigenvalue of A is $(\varepsilon_0 + \mu_0 - (\varepsilon_0^2 - 2\varepsilon_0\mu_0 + \mu_0^2 + 4\chi^2 + 4\kappa^2)^{1/2})/2$, which is strictly positive since $\chi^2 + \kappa^2 < \varepsilon_0\mu_0$. $\square$

We now give the regularity assumptions on the parameters. In contrast to the previous situation, here the mixing coefficients ξ and ζ fully couple electric and magnetic properties. We are thus led to assume that

$$(2.76) \quad \varepsilon, \xi, \mu, \zeta \in W^{1,3+\delta}(\Omega; \mathbb{C}^{3 \times 3}) \text{ for some } \delta > 0.$$

The theorem below shows that at least as far as interior regularity is concerned, Theorem 1.1 also applies in this more general setting.

Theorem 2.30. *Assume that (2.73) and (2.76) hold. Suppose that the current sources J_e and J_m are in $W^{1,p}(\text{div}, \Omega)$ for some $p \geq 2$.*

If E and H in $H(\text{curl}, \Omega)$ are weak solutions of (2.71), then $E, H \in W_{loc}^{1,q}(\Omega; \mathbb{C}^3)$ with $q = \min(p, 3 + \delta)$. Furthermore, for any open set Ω_0 such that $\overline{\Omega_0} \subset \Omega$ there holds

$$\|(E, H)\|_{W^{1,q}(\Omega_0; \mathbb{C}^3)^2} \leq C(\|(E, H)\|_{L^2(\Omega; \mathbb{C}^3)^2} + \|(J_e, J_m)\|_{W^{1,p}(\text{div}, \Omega)^2}),$$

where C is a constant depending on Ω , Ω_0 , q , Λ , ω and $\|(\varepsilon, \mu, \xi, \zeta)\|_{W^{1,3+\delta}(\Omega; \mathbb{C}^{3 \times 3})^4}$ only. In particular, if $p > 3$ then $E, H \in C_{loc}^{0,1-\frac{3}{q}}(\Omega; \mathbb{C}^3)$.

The proof of this result is a variant of the proof of Theorem 1.1. In this case, the system is written in $\mathbb{R}^{12}$ (instead of a weakly coupled system of 6 complex unknowns). The first step is to derive an appropriate very weak formulation.

Proposition 2.31. *Under the hypotheses of Theorem 2.30, let $E, H \in H(\text{curl}, \Omega)$ be a weak solution of (2.71).*

Then for each $k = 1, 2, 3$, (E_k, H_k) is a very weak solution of the elliptic system

$$\begin{cases} -\text{div}(\varepsilon \nabla E_k + \xi \nabla H_k) = \text{div}((\partial_k \varepsilon)E + (\partial_k \xi)H - \varepsilon(\mathbf{e}_k \times (\mathbf{i}\omega \zeta E + \mathbf{i}\omega \mu H + J_m))) \\ \quad + \text{div}(\xi(\mathbf{e}_k \times (\mathbf{i}\omega \varepsilon E + \mathbf{i}\omega \xi H - J_e)) + \mathbf{i}\omega^{-1} \mathbf{e}_k \text{div} J_e) \text{ in } \Omega. \\ -\text{div}(\zeta \nabla E_k + \mu \nabla H_k) = \text{div}((\partial_k \zeta)E + (\partial_k \mu)H + \mu(\mathbf{e}_k \times (\mathbf{i}\omega \varepsilon E + \mathbf{i}\omega \xi H - J_e))) \\ \quad + \text{div}(\zeta(\mathbf{e}_k \times (-\mathbf{i}\omega \zeta E - \mathbf{i}\omega \mu H - J_m)) - \mathbf{i}\omega^{-1} \mathbf{e}_k \text{div} J_m) \text{ in } \Omega. \end{cases}$$

More precisely, for any $\Phi \in W^{2,2}(\Omega; \mathbb{C})$ there holds

$$\begin{aligned}
 (2.77) \quad & \int_{\Omega} E_k \operatorname{div} (\varepsilon^T \nabla \bar{\Phi}) \, dx + \int_{\Omega} H_k \operatorname{div} (\xi^T \nabla \bar{\Phi}) \, dx = \int_{\Omega} ((\partial_k \varepsilon) E + (\partial_k \xi) H) \cdot \nabla \bar{\Phi} \, dx \\
 & - \int_{\Omega} (\varepsilon (\mathbf{e}_k \times (\mathbf{i}\omega \zeta E + \mathbf{i}\omega \mu H + J_m)) + \xi (\mathbf{e}_k \times (-\mathbf{i}\omega \varepsilon E - \mathbf{i}\omega \xi H + J_e))) \cdot \nabla \bar{\Phi} \, dx \\
 & + \int_{\Omega} (\mathbf{i}\omega^{-1} \operatorname{div} J_e \mathbf{e}_k) \cdot \nabla \bar{\Phi} \, dx + \int_{\partial\Omega} (\partial_k \bar{\Phi}) (\varepsilon E + \xi H) \cdot \nu \, d\sigma \\
 & - \int_{\partial\Omega} (\mathbf{e}_k \times (H \times \nu)) \cdot (\xi^T \nabla \bar{\Phi}) \, d\sigma - \int_{\partial\Omega} (\mathbf{e}_k \times (E \times \nu)) \cdot (\varepsilon^T \nabla \bar{\Phi}) \, d\sigma,
 \end{aligned}$$

and

$$\begin{aligned}
 (2.78) \quad & \int_{\Omega} E_k \operatorname{div} (\zeta^T \nabla \bar{\Phi}) \, dx + \int_{\Omega} H_k \operatorname{div} (\mu^T \nabla \bar{\Phi}) \, dx = \int_{\Omega} ((\partial_k \zeta) E + (\partial_k \mu) H) \cdot \nabla \bar{\Phi} \, dx \\
 & + \int_{\Omega} (\mu (\mathbf{e}_k \times (\mathbf{i}\omega \varepsilon E + \mathbf{i}\omega \xi H - J_e)) + \zeta (\mathbf{e}_k \times (\mathbf{i}\omega \zeta E + \mathbf{i}\omega \mu H + J_m))) \cdot \nabla \bar{\Phi} \, dx \\
 & - \int_{\Omega} (\mathbf{i}\omega^{-1} \operatorname{div} J_m \mathbf{e}_k) \cdot \nabla \bar{\Phi} \, dx + \int_{\partial\Omega} (\partial_k \bar{\Phi}) (\zeta E + \mu H) \cdot \nu \, d\sigma \\
 & - \int_{\partial\Omega} (\mathbf{e}_k \times (E \times \nu)) \cdot (\zeta^T \nabla \bar{\Phi}) \, d\sigma - \int_{\partial\Omega} (\mathbf{e}_k \times (H \times \nu)) \cdot (\mu^T \nabla \bar{\Phi}) \, d\sigma.
 \end{aligned}$$

Proof. We detail the derivation of (2.77) for the sake of completeness. Then, (2.78) is a consequence of the intrinsic symmetry of (2.71).

We test $\operatorname{curl} E = \mathbf{i}\omega \zeta E + \mathbf{i}\omega \mu H + J_m$ against $\bar{\Phi} = \bar{g} \mathbf{e}_l$ for some $g \in W^{1,2}(\Omega; \mathbb{C})$, integrate by parts and multiply the result by $\mathbf{e}_l$. We obtain

$$\mathbf{e}_l \int_{\Omega} \bar{g} (\mathbf{i}\omega \zeta E + \mathbf{i}\omega \mu H + J_m) \cdot \mathbf{e}_l \, dx = \mathbf{e}_l \int_{\Omega} E \cdot (\nabla \times \bar{\Phi}) \, dx - \int_{\partial\Omega} (E \times \bar{\Phi}) \cdot \nu \, d\sigma$$

Which can be written also

$$\int_{\Omega} \bar{g} (\mathbf{i}\omega \zeta E + \mathbf{i}\omega \mu H + J_m) \, dx + \int_{\partial\Omega} \bar{g} (E \times \nu) \, d\sigma = \int_{\Omega} E \times \nabla \bar{g} \, dx$$

Note that since $E \in W^{1,2}(\operatorname{curl}, \Omega)$ by assumption, $E \times \nu$ is well defined in $W^{-\frac{1}{2},2}(\partial\Omega; \mathbb{C}^3)$ and this formulation is valid. Next, we cross product this identity with $\mathbf{e}_k$, and take the scalar product with $\mathbf{e}_i$. Using the vector identity $a \times (b \times c) = (a \cdot c)b - (a \cdot b)c$ on the right-hand side, we obtain

$$(2.79) \quad \mathbf{e}_i \cdot \int_{\Omega} \bar{g} \mathbf{e}_k \times (\mathbf{i}\omega \zeta E + \mathbf{i}\omega \mu H + J_m) \, dx + \mathbf{e}_i \cdot \int_{\partial\Omega} \bar{g} \mathbf{e}_k \times (E \times \nu) \, d\sigma = \int_{\Omega} E_i \partial_k \bar{g} - E_k \partial_i \bar{g} \, dx,$$

for any i and k in $\{1, 2, 3\}$. Proceeding in a similar way with $\operatorname{curl} H = -\mathbf{i}\omega (\varepsilon E + \xi H) + J_e$ we obtain

$$(2.80) \quad \mathbf{e}_i \cdot \int_{\Omega} \bar{g} \mathbf{e}_k \times (-\mathbf{i}\omega \varepsilon E - \mathbf{i}\omega \xi H + J_e) \, dx + \mathbf{e}_i \cdot \int_{\partial\Omega} \bar{g} \mathbf{e}_k \times (H \times \nu) \, d\sigma = \int_{\Omega} H_i \partial_k \bar{g} - H_k \partial_i \bar{g} \, dx,$$

Next, we use the second part of Maxwell's equations. We test $\operatorname{curl}(H) - J_e = -\mathbf{i}\omega \varepsilon E - \mathbf{i}\omega \xi H$ against $\nabla(\partial_k \bar{\Phi}) \frac{1}{\mathbf{i}\omega}$ for $\Phi \in H^2(\Omega; \mathbb{C})$ and obtain

$$\begin{aligned}
 & - \int_{\Omega} (\varepsilon E + \xi H) \cdot \partial_k (\nabla \bar{\Phi}) \, dx = -\mathbf{i}\omega^{-1} \int_{\Omega} \operatorname{curl} H \cdot \nabla (\partial_k \bar{\Phi}) \, dx + \mathbf{i}\omega^{-1} \int_{\Omega} J_e \cdot \nabla (\partial_k \bar{\Phi}) \, dx \\
 & = -\mathbf{i}\omega^{-1} \left(\int_{\partial\Omega} (\partial_k \bar{\Phi}) \operatorname{curl} H \cdot \nu \, ds - \int_{\partial\Omega} (\partial_k \bar{\Phi}) J_e \cdot \nu \, ds + \int_{\Omega} \operatorname{div} J_e \partial_k \bar{\Phi} \, dx \right) \\
 & = \mathbf{i}\omega^{-1} \left(\mathbf{i}\omega \int_{\partial\Omega} (\partial_k \bar{\Phi}) (\varepsilon E + \xi H) \cdot \nu \, ds - \int_{\Omega} \operatorname{div} J_e \partial_k \bar{\Phi} \, dx \right).
 \end{aligned}$$

Since $J_e \in W^{1,2}(\operatorname{div}, \Omega)$ the boundary term is well defined. Therefore, since

$$\begin{aligned} \int_{\Omega} (\varepsilon E + \xi H) \cdot \partial_k (\nabla \bar{\Phi}) \, dx &= \int_{\Omega} E \cdot \partial_k (\varepsilon^T \nabla \bar{\Phi}) \, dx - \int_{\Omega} (\partial_k \varepsilon) E \cdot \nabla \bar{\Phi} \, dx \\ &\quad + \int_{\Omega} H \cdot \partial_k (\xi^T \nabla \bar{\Phi}) \, dx - \int_{\Omega} (\partial_k \xi) H \cdot \nabla \bar{\Phi} \, dx, \end{aligned}$$

we have

$$\begin{aligned} (2.81) \quad \int_{\Omega} E \cdot \partial_k (\varepsilon^T \nabla \bar{\Phi}) \, dx &- \int_{\Omega} ((\partial_k \varepsilon) E + (\partial_k \xi) H) \cdot \nabla \bar{\Phi} \, dx + \int_{\Omega} H \cdot \partial_k (\xi^T \nabla \bar{\Phi}) \, dx \\ &= \int_{\partial\Omega} (\partial_k \bar{\Phi}) (\varepsilon E + \xi H) \cdot \nu \, ds + \mathbf{i}\omega^{-1} \int_{\Omega} \operatorname{div} J_e \partial_k \bar{\Phi} \, dx \end{aligned}$$

Applying (2.79) with $g = (\varepsilon^T \nabla \Phi)_i$ for some $i = 1, 2, 3$ we find

$$\begin{aligned} - \int_{\Omega} E_i \cdot \partial_k (\varepsilon^T \nabla \bar{\Phi})_i \, dx &= - \int_{\Omega} E_k \cdot \partial_i (\varepsilon^T \nabla \bar{\Phi})_i \, dx - \mathbf{e}_i \cdot \int_{\partial\Omega} (\varepsilon^T \nabla \bar{\Phi})_i \mathbf{e}_k \times (E \times \nu) \, d\sigma \\ &\quad - \mathbf{e}_i \cdot \int_{\Omega} (\varepsilon^T \nabla \bar{\Phi})_i \mathbf{e}_k \times (\mathbf{i}\omega \zeta E + \mathbf{i}\omega \mu H + J_m) \, dx. \end{aligned}$$

Summing over i , this yields

$$\begin{aligned} (2.82) \quad - \int_{\Omega} E \cdot \partial_k (\varepsilon^T \nabla \bar{\Phi}) \, dx &= - \int_{\Omega} \varepsilon (\mathbf{e}_k \times (\mathbf{i}\omega \zeta E + \mathbf{i}\omega \mu H + J_m)) \cdot \nabla \bar{\Phi} \, dx \\ &\quad - \int_{\Omega} E_k \cdot \operatorname{div} (\varepsilon^T \nabla \bar{\Phi}) \, dx - \int_{\partial\Omega} (\mathbf{e}_k \times (E \times \nu)) \cdot (\varepsilon^T \nabla \bar{\Phi}) \, d\sigma. \end{aligned}$$

Similarly, applying (2.80) with $g = (\xi^T \nabla \Phi)_i$ and summing over i we find

$$\begin{aligned} (2.83) \quad - \int_{\Omega} H \cdot \partial_k (\xi^T \nabla \bar{\Phi}) \, dx &= - \int_{\Omega} \xi (\mathbf{e}_k \times (-\mathbf{i}\omega \varepsilon E - \mathbf{i}\omega \xi H + J_e)) \cdot \nabla \bar{\Phi} \, dx \\ &\quad - \int_{\Omega} H_k \cdot \operatorname{div} (\xi^T \nabla \bar{\Phi}) \, dx - \int_{\partial\Omega} (\mathbf{e}_k \times (H \times \nu)) \cdot (\xi^T \nabla \bar{\Phi}) \, d\sigma. \end{aligned}$$

Inserting (2.82) and (2.83) in (2.81) we obtain (2.77). $\square$

We only study interior regularity for the problem at hand. The boundary regularity does not follow easily from the method used in § 2.3.1. Indeed, mixed boundary terms appear in (2.77) and (2.78), and the technique used in Proposition 2.21 and in Proposition 2.22, with test functions satisfying either Dirichlet or Neumann boundary conditions, does not apply, as both conditions would be required simultaneously.

The “very weak to weak” Lemma 2.20 adapted to this mixed system is given below.

Lemma 2.32. *Assume (2.73) and (2.76) hold, and let A be given by (2.72).*

Given $r \geq \frac{6}{5}$, $u \in L^2(\Omega; \mathbb{R}^4) \cap L^r(\Omega; \mathbb{R}^4)$ and $F \in W^{1,r'}(\Omega; \mathbb{R}^4)'$, if

$$(2.84) \quad \int_{\Omega} u^j \partial_{\alpha} (A_{ij}^{\alpha\beta} \partial_{\beta} \Phi^i) \, dx = \langle F_i, \Phi^i \rangle, \quad \Phi \in W^{2,2}(\Omega; \mathbb{R}^4) \cap W_0^{1,2}(\Omega; \mathbb{R}^4),$$

then $u \in W^{1,r}(\Omega; \mathbb{R}^4)$ and

$$(2.85) \quad \|\nabla u\|_{L^r(\Omega; \mathbb{R}^{4 \times 3})} \leq C \|F\|_{W^{1,r'}(\Omega; \mathbb{R}^4)'}.$$

for some constant $C = C(r, \Omega, \Lambda, \|(\varepsilon, \xi, \mu, \zeta)\|_{W^{1,3}(\Omega; \mathbb{C}^{3 \times 3})^4})$.

Proof. Let $\psi \in \mathcal{D}(\Omega; \mathbb{R})$ be a test function and take $\alpha^* \in \{1, 2, 3\}$ and $j^* \in \{1, \dots, 4\}$. Since A satisfies the strong Legendre condition (2.73), the system

$$(2.86) \quad \begin{cases} \partial_\alpha(A_{ij}^{\alpha\beta} \partial_\beta \Phi_*^i) = \delta_{jj^*} \partial_{\alpha^*} \psi & \text{in } \Omega, \\ \Phi_* = 0 & \text{on } \partial\Omega, \end{cases}$$

has a unique solution $\Phi_* \in H_0^1(\Omega; \mathbb{R}^4)$ (see e.g. [56, 67]). Further, since $A_{ij}^{\alpha\beta} \in W^{1,3}(\Omega; \mathbb{R})$, by [50, Theorem 1.7, Remark 1.8] for any $q \in (1, \infty)$

$$(2.87) \quad \|\Phi_*\|_{W^{1,q}(\Omega; \mathbb{R}^4)} \leq c \|\psi\|_{L^q(\Omega; \mathbb{R}^4)}$$

for some $c = c(q, \Omega, \Lambda, \|(\varepsilon, \xi, \mu, \zeta)\|_{W^{1,3}(\Omega; \mathbb{C}^{3 \times 3})^4}) > 0$. Hence, the usual difference quotient argument given in [67] shows that $\Phi_* \in W^{2,2}(\Omega; \mathbb{R}^4)$. Therefore, by assumption we have

$$\left| \int_\Omega u^{j^*} \partial_{\alpha^*} \psi \, dx \right| = \left| \int_\Omega u^j \partial_\alpha (A_{ij}^{\alpha\beta} \partial_\beta \Phi_*^i) \, dx \right| = |\langle F_i, \Phi_*^i \rangle| \leq \|F\|_{W^{1,r'}(\Omega; \mathbb{R}^4)'} \|\Phi_*\|_{W^{1,r'}(\Omega; \mathbb{R}^4)},$$

which in view of (2.87) gives

$$\left| \int_\Omega u^{j^*} \partial_{\alpha^*} \psi \, dx \right| \leq c \|F\|_{W^{1,r'}(\Omega; \mathbb{R}^4)'} \|\psi\|_{L^{r'}(\Omega; \mathbb{R}^4)},$$

whence the result. $\square$

The following proposition mirrors Propositions 2.21 and 2.22. Theorem 2.30 then follows by the bootstrap argument used in the proof of Theorem 1.1.

Proposition 2.33. *Under the hypotheses of Theorem 2.30 and given $q \in [2, \infty)$, set $r = \min((3q + q\delta)(q + 3 + \delta)^{-1}, p)$. Let E and H in $H(\text{curl}, \Omega)$ be weak solutions of (2.71).*

Suppose $E, H \in L^q(\Omega; \mathbb{C}^3)$. Then $E, H \in W_{loc}^{1,r}(\Omega; \mathbb{C}^3)$ and for any open subdomain Ω_0 such that $\overline{\Omega_0} \subset \Omega$ there holds

$$(2.88) \quad \|(E, H)\|_{W^{1,r}(\Omega_0; \mathbb{C}^3)} \leq C(\|(E, H)\|_{L^q(\Omega; \mathbb{C})^6} + \|(J_e, J_m)\|_{W^{1,p}(\text{div}, \Omega)^2})$$

for some constant $C = C(r, \Omega, \Omega_0, \Lambda, \omega, \|(\varepsilon, \xi, \mu, \zeta)\|_{W^{1,\delta+3}(\Omega; \mathbb{C}^{3 \times 3})^4})$.

Proof. From (2.77) we see that for every compactly supported $\Phi^1, \Phi^2 \in W^{2,2}(\Omega; \mathbb{C})$ and $k = 1, 2, 3$ there holds

$$(2.89) \quad \begin{cases} \int_\Omega E_k \text{div} \left(\varepsilon^T \nabla \overline{\Phi^1} \right) + H_k \text{div} \left(\xi^T \nabla \overline{\Phi^1} \right) \, dx = \int_\Omega F_k \cdot \nabla \overline{\Phi^1} \, dx, \\ \int_\Omega E_k \text{div} \left(\zeta^T \nabla \overline{\Phi^2} \right) + H_k \text{div} \left(\mu^T \nabla \overline{\Phi^2} \right) \, dx = \int_\Omega G_k \cdot \nabla \overline{\Phi^2} \, dx, \end{cases}$$

with

$$F_k = (\partial_k \varepsilon) E + (\partial_k \xi) H - \varepsilon (\mathbf{e}_k \times (\mathbf{i}\omega \zeta E + \mathbf{i}\omega \mu H + J_m)) \\ + \xi (\mathbf{e}_k \times (\mathbf{i}\omega \varepsilon E + \mathbf{i}\omega \xi H - J_e)) + \mathbf{i}\omega^{-1} \operatorname{div} J_e \mathbf{e}_k,$$

and

$$G_k = (\partial_k \zeta) E + (\partial_k \mu) H + \mu (\mathbf{e}_k \times (\mathbf{i}\omega \varepsilon E + \mathbf{i}\omega \xi H - J_e)) \\ - \zeta (\mathbf{e}_k \times (\mathbf{i}\omega \zeta E + \mathbf{i}\omega \mu H + J_m)) - \mathbf{i}\omega^{-1} \operatorname{div} J_m \mathbf{e}_k.$$

By construction, $F_k, G_k \in L^r(\Omega; \mathbb{C}^3)$.

Given a smooth subdomain Ω_0 , we consider a cut-off function $\chi \in \mathcal{D}(\Omega)$ such that $\chi = 1$ in Ω_0 . A straightforward computation shows

$$\begin{cases} \int_{\Omega} \chi E_k \operatorname{div}(\varepsilon^T \nabla \overline{\Phi^1}) + \chi H_k \operatorname{div}(\xi^T \nabla \overline{\Phi^1}) dx = \int_{\Omega} F_k \cdot \nabla(\chi \overline{\Phi^1}) dx + T_k(\Phi^1), \\ \int_{\Omega} \chi E_k \operatorname{div}(\zeta^T \nabla \overline{\Phi^2}) + \chi H_k \operatorname{div}(\mu^T \nabla \overline{\Phi^2}) dx = \int_{\Omega} G_k \cdot \nabla(\chi \overline{\Phi^2}) dx + R_k(\Phi^2), \end{cases}$$

where

$$T_k(\Phi^1) = - \int_{\Omega} E_k (\operatorname{div}(\varepsilon^T \overline{\Phi^1} \nabla \chi) + \varepsilon \nabla \chi \cdot \nabla \overline{\Phi^1}) + H_k (\operatorname{div}(\xi^T \overline{\Phi^1} \nabla \chi) + \xi \nabla \chi \cdot \nabla \overline{\Phi^1}) dx,$$

and

$$R_k(\Phi^2) = - \int_{\Omega} E_k (\operatorname{div}(\zeta^T \overline{\Phi^2} \nabla \chi) + \zeta \nabla \chi \cdot \nabla \overline{\Phi^2}) + H_k (\operatorname{div}(\mu^T \overline{\Phi^2} \nabla \chi) + \mu \nabla \chi \cdot \nabla \overline{\Phi^2}) dx.$$

This last system can be reformulated in the form (2.84), with A given by (2.72). We then apply Lemma 2.32 and obtain $\chi E_k, \chi H_k \in W^{1,r}(\Omega; \mathbb{C})$, namely $E, H \in W^{1,r}(\Omega_0; \mathbb{C}^3)$. Finally, (2.88) follows from (2.85). $\square$

Chapter 3

Using multiple frequencies to enforce local constraints in PDE

The focus of this chapter is the multiple frequency approach to the problem of finding suitable boundary conditions such that the corresponding solutions to the Helmholtz and Maxwell equations satisfy certain non-zero constraints. In Section 3.1 we describe the standard approach, based on complex geometric optics solutions. In Section 3.2, some preliminary results on holomorphic functions and elliptic equations are discussed. Then, Sections 3.3 and 3.4 contain the main results for the Helmholtz equation and Maxwell's equations, respectively. Finally, in Section 3.5 some additional results are discussed.

3.1 Complex geometric optics solutions

In this section we discuss the classical approach to construct illuminations such that the corresponding solutions to the Helmholtz equation satisfy certain non-zero local constraints. The illuminations are traces on the boundary of particular highly oscillatory solutions to the Helmholtz equation, called complex geometric optics (CGO) solutions. This technique can be extended to Maxwell's equations: CGO solutions for Maxwell's equations have been introduced by Colton and Päiväranta [57], and can be used to satisfy non-zero constraints. For examples of applications, see [55, 33].

The rest of this section is devoted to the study of CGO solutions for the Helmholtz equation. In particular, we shall show how these can be used to satisfy local constraints by analysing a particular example. For simplicity, we study the two-dimensional case, but everything can be extended to three dimensions. We consider the constraints

$$(3.1a) \quad |u_\omega^1| \geq C,$$

$$(3.1b) \quad |\nabla u_\omega^2 \times \nabla u_\omega^3| \geq C.$$

These characterise $(\zeta_\times, C)$ -complete sets of measurements (Definition 1.3), where

$$\zeta_\times(u^1, u^2, u^3) = (u^1, \nabla u^2 \times \nabla u^3).$$

These constraints are weaker than the ones considered for $(\zeta_{\det}, C)$ -complete sets (§ 1.2.1) and naturally arise in the context of microwave imaging by ultrasound deformation (§ 4.1.1).

Introduced by Calderón [51] and developed by Sylvester and Uhlmann [102], CGO solutions have been extensively used over the last years in inverse problems to exhibit solutions with good properties to the equation

$$\Delta u + q u = 0 \quad \text{in } \Omega.$$

The reader is referred to [30] for a review on this topic. The CGO solutions we are interested in are complex solutions to

$$(3.2) \quad -\operatorname{div}(a \nabla u) - \omega_0^2 \varepsilon u = 0 \quad \text{in } \mathbb{R}^2$$

of the form

$$(3.3) \quad u_t(x) = \frac{1}{\sqrt{a}} e^{\rho_t \cdot x} (1 + \psi_t)$$

where $\rho_t = t(e_1 + \mathbf{i}e_2)$, $t > 0$ and $\psi_t \in C^1(\mathbb{R}^2)$. Before going into the details, let us explain the rationale behind this construction. The main idea is that for sufficiently smooth a and ε , if t is big enough then ψ_t is negligible and so

$$(3.4) \quad u_t(x) \approx \frac{1}{\sqrt{a}} e^{\rho_t \cdot x} = \frac{1}{\sqrt{a}} e^{t x_1} (\cos(t x_2) + \mathbf{i} \sin(t x_2)),$$

whose real and imaginary parts satisfy the required properties (3.1), as we shall see later on. As a result, the traces of the real and imaginary parts constitute a $(\zeta_\times, C)$ -complete set of measurements.

More formally, let $a, \varepsilon \in L^\infty(\mathbb{R}^2; \mathbb{R})$ satisfy (2.2) and take $\omega_0 \in \mathbb{R} \setminus \sqrt{\Sigma}$. In this section the frequency ω_0 will be kept fixed. We need to introduce the spaces H_1^s for $s \geq 0$ as the completion of $\mathcal{D}(\mathbb{R}^2)$ with respect to the norm

$$\|u\|_{H_1^s} = \left(\int_{\mathbb{R}^2} (1 + |x|^2) |(I - \Delta)^{\frac{s}{2}} u|^2 dx \right)^{\frac{1}{2}},$$

where $(I - \Delta)^{\frac{s}{2}} u = \mathcal{F}^{-1} \left((1 + |\xi|^2)^{\frac{s}{2}} \mathcal{F} u \right)$ and $\mathcal{F}$ denotes the Fourier transform. The following result, due to Bal and Uhlmann [36], shows that ψ_t and its derivatives are actually small in Ω for big values of t , provided that a and ε satisfy the regularity assumption

$$(3.5) \quad -\frac{\Delta \sqrt{a}}{\sqrt{a}} + \omega_0^2 \frac{\varepsilon}{a} \in H_1^{\frac{n}{2} + 1 + \delta}$$

for some $\delta > 0$.

Proposition 3.1 ([36, Proposition 3.3]). *Let $\delta > 0$, $a, \varepsilon \in L^\infty(\mathbb{R}^2; \mathbb{R})$ satisfy (2.2) and (3.5) and take $\omega_0 \in \mathbb{R} \setminus \sqrt{\Sigma}$. Then there exist $c, t_0 > 0$ satisfying the following property: for every $t \geq t_0$ there exists $\psi_t \in C^1(\mathbb{R}^2; \mathbb{C})$ such that u_t given by (3.3) is a solution to (3.2) and*

$$(3.6) \quad \|\psi_t\|_{C^1(\overline{\Omega}; \mathbb{C})} \leq \frac{c}{t}.$$

For $t \geq t_0$ and u_t as in Proposition 3.1, denote the real and imaginary parts of u_t by $u_t^r = \Re u_t$ and $u_t^i = \Im u_t$, respectively. Let us now prove that the traces on $\partial\Omega$ of these solutions form a $(\zeta_\times, C)$ -complete set of measurements with fixed wavenumber ω_0 , provided that t is big enough.

Proposition 3.2. *Under the assumptions of Proposition 3.1, there exist $\bar{t} \geq t_0$ and $C > 0$ such that*

$$\{\omega_0\} \times \{u_t|_{\partial\Omega}, u_t^r|_{\partial\Omega}, u_t^i|_{\partial\Omega}\}$$

is a $(\zeta_\times, C)$ -complete set of measurements in Ω .

Proof. In the following, the O symbol will involve constants independent of $x \in \bar{\Omega}$. In view of (3.3) and the bound (3.6), we have for $x \in \bar{\Omega}$ as $t \rightarrow \infty$

$$|u_t(x)|^2 = \frac{1}{a} e^{2tx_1} (1 + |\psi_t(x)|^2 + 2|\psi_t(x)|) = \frac{1}{a} e^{2tx_1} (1 + O(1/t)).$$

Therefore there exist $t_1 \geq t_0$ and $C > 0$ such that $|u_t(x)| \geq C$ for any $t \geq t_1$ and $x \in \bar{\Omega}$, and so (3.1a) is satisfied.

Similarly, for the gradient we have

$$\nabla u_t(x) = \frac{e^{\rho_t \cdot x}}{\sqrt{a}} \left(-\frac{\nabla a}{2a} (1 + \psi_t(x)) + \rho_t (1 + \psi_t(x)) + \nabla \psi_t(x) \right) = \frac{e^{\rho_t \cdot x}}{\sqrt{a}} (t(e_1 + \mathbf{i}e_2) + O(1)),$$

whence

$$\begin{aligned} \nabla u_t^r(x) &= t \frac{e^{tx_1}}{\sqrt{a}} (\cos(tx_2)e_1 - \sin(tx_2)e_2 + O(1/t)), \\ \nabla u_t^i(x) &= t \frac{e^{tx_1}}{\sqrt{a}} (\sin(tx_2)e_1 + \cos(tx_2)e_2 + O(1/t)). \end{aligned}$$

As a consequence

$$\nabla u_t^r \times \nabla u_t^i(x) = t^2 \frac{e^{2tx_1}}{a} (\cos(tx_2)^2 + \sin(tx_2)^2 + O(1/t)) = t^2 \frac{e^{2tx_1}}{a} (1 + O(1/t)).$$

Therefore there exist $t_3 \geq t_2$ and $C > 0$ such that $|\nabla u_t^r \times \nabla u_t^i(x)| \geq C$ for any $t \geq t_3$ and $x \in \bar{\Omega}$, and so (3.1b) is satisfied.

Finally, in order to obtain the result we choose $\bar{t} = \max\{t_1, t_3\}$. $\square$

The previous result can be slightly strengthened with the following argument. Since by Proposition 2.5 $u_{\omega_0}^\varphi$ depends continuously on the boundary condition, there exists $\delta > 0$ such that $\{\omega_0\} \times \{\varphi_1, \varphi_2, \varphi_3\}$ is a $(\zeta_\times, C)$ -complete set for some $C > 0$, provided that $\|\varphi_1 - u_t|_{\partial\Omega}\|_{C^{1,\alpha}(\bar{\Omega};\mathbb{C})} < \delta$, $\|\varphi_2 - u_t^r|_{\partial\Omega}\|_{C^{1,\alpha}(\bar{\Omega};\mathbb{C})} < \delta$ and $\|\varphi_3 - u_t^i|_{\partial\Omega}\|_{C^{1,\alpha}(\bar{\Omega};\mathbb{C})} < \delta$ for a fixed $t \geq \bar{t}$.

In summary, in this subsection we have proved that CGO solutions are a very powerful theoretical tool to construct complete sets of measurements for sufficiently smooth a and ε and for any frequency ω_0 . However, this construction has the drawbacks discussed in Section 1.2.

3.2 Preliminaries

3.2.1 Holomorphic functions

Holomorphic functions in a Banach space setting were studied in [103]. Let E and E' be complex Banach spaces, $D \subseteq E$ be an open set and take $f: D \rightarrow E'$. We say that f admits a Gateaux differential in $x_0 \in D$ with respect to the direction $y \in E$ if the limit

$$\lim_{\tau \rightarrow 0} \frac{f(x_0 + \tau y) - f(x_0)}{\tau}$$

exists in E' . We say that f is holomorphic in x_0 if it is continuous in x_0 and admits a Gateaux differential in x_0 with respect to every direction $y \in E$. We say that f is holomorphic in D (or simply holomorphic) if it is holomorphic in every point of D . With this definition, it is clear that this notion extends the classical notion of holomorphicity for functions of complex variable.

The following lemma summarises some of the basic properties of holomorphic functions that are of interest to us.

Lemma 3.3. *Let $E_1, \dots, E_r$, E and E' be complex Banach spaces. Let $D \subseteq E$ be an open set.*

1. *If $f: E_1 \times \dots \times E_r \rightarrow E'$ is multilinear and bounded then f is holomorphic.*
2. *If $f: D \rightarrow E_1$ and $g: E_1 \rightarrow E'$ are holomorphic then $g \circ f: D \rightarrow E'$ is holomorphic.*
3. *Take $f = (f^1, \dots, f^r): D \rightarrow E_1 \times \dots \times E_r$. Then f is holomorphic if and only if f^j is holomorphic for every $j = 1, \dots, r$.*

Proof. Parts 1 and 3 trivially follow from the definition. Part 2 is shown in [110]. □

The following result is a quantitative version of the unique continuation property for holomorphic functions of one complex variable.

Lemma 3.4. *Take $C_0, D > 0$, $\theta \in (0, 1)$ and $r \in (0, \theta]$. Let g be a holomorphic function in $B(0, 1)$ such that $|g(0)| \geq C_0$ and $\sup_{B(0, 1)} |g| \leq D$. There exists $\omega \in [r, 1)$ such that*

$$|g(\omega)| \geq C$$

for some constant $C > 0$ depending on θ , C_0 and D only.

Proof. Since $[\theta, (1 + \theta)/2] \subseteq [r, 1)$, it is sufficient to show that there exists $C > 0$ depending on θ , C_0 and D only such that

$$\max_{[\theta, (1+\theta)/2]} |g| \geq C.$$

By contradiction, suppose that there exists a sequence $(g_n)_n$ of holomorphic functions in $B(0, 1)$ such that $\sup_{B(0, 1)} |g_n| \leq D$, $|g_n(0)| \geq C_0$ and $\max_{[\theta, (1+\theta)/2]} |g_n| \rightarrow 0$. Since $\sup_{B(0, 1)} |g_n| \leq D$, by standard complex analysis, up to a subsequence $g_n \rightarrow g_\infty$ for some g_∞ holomorphic in $B(0, 1)$. As $\max_{[\theta, (1+\theta)/2]} |g_n| \rightarrow 0$, we obtain $g_\infty = 0$ on $[\theta, (1 + \theta)/2]$, whence $g_\infty = 0$, which contradicts $|g_\infty(0)| \geq C_0$. □

Remark 3.5. Although elementary, the proof of Lemma 3.4 does not give the dependence of the constant C on the parameters θ , C_0 and D . Even though we shall not need the precise estimate, we present a possible derivation for the sake of completeness.

In view of [85] there is a Jordan curve Γ in $r < |\omega| < 1$ around the origin such that

$$\log |g(\omega)/g(0)| \geq -\frac{\tilde{C}}{1-r} \left(\int_0^1 \left(\frac{\log \sup_{B(0,t)} |g/g(0)|}{1-t} \right)^{1/2} dt \right)^2, \quad \omega \in \Gamma,$$

for an absolute constant $\tilde{C} > 0$. By the Jordan curve theorem there exists $\omega \in (r, 1)$ such that

$$\begin{aligned} \log |g(\omega)/g(0)| &\geq -\frac{\tilde{C}}{1-r} \left(\int_0^1 \left(\frac{\log \sup_{B(0,t)} |g/g(0)|}{1-t} \right)^{1/2} dt \right)^2, \\ &\geq -\frac{\tilde{C} \log \sup_{B(0,1)} |g/g(0)|}{1-r} \left(\int_0^1 \left(\frac{1}{1-t} \right)^{1/2} dt \right)^2 \\ &\geq -\frac{\tilde{C} \log(DC_0^{-1})}{1-r}. \end{aligned}$$

Therefore

$$|g(\omega)| \geq |g(0)| (DC_0^{-1})^{-\frac{\tilde{C}}{1-r}} \geq C_0 (DC_0^{-1})^{-\frac{\tilde{C}}{1-r}} \geq C_0 (DC_0^{-1})^{-\frac{\tilde{C}}{1-\theta}},$$

whence the constant given in Lemma 3.4 is $C = C_0 (DC_0^{-1})^{-\frac{\tilde{C}}{1-\theta}}$.

It is possible to generalise the previous result to holomorphic functions defined in an ellipse. The proof is elementary, but needed to show the precise dependence of C on $R_1 - r$.

Lemma 3.6. *Take $0 < r < R_1 \leq M$ and $0 < \eta \leq R_2$. Let g be a holomorphic function in the ellipse*

$$E = \left\{ \omega \in \mathbb{C} : \frac{(\Re \omega)^2}{R_1^2} + \frac{(\Im \omega)^2}{R_2^2} < 1 \right\}$$

such that $|g(0)| \geq C_0 > 0$ and $\sup_E |g| \leq D$. There exists $\omega \in (r, R_1)$ such that

$$|g(\omega)| \geq C$$

for some constant $C > 0$ depending on M , $R_1 - r$, η , C_0 and D only.

Proof. Several positive constants depending on M , $R_1 - r$, η , C_0 and D will be denoted by c . Without loss of generality, we can always suppose $R_2 \leq R_1$.

Set $\beta := \sqrt{R_1^2 + R_2^2} \leq \sqrt{2}M$, $r_i = R_i/\beta$ and $\tilde{E} := \left\{ \omega \in \mathbb{C} : \frac{(\Re \omega)^2}{r_1^2} + \frac{(\Im \omega)^2}{r_2^2} < 1 \right\}$. The map $\psi_1: \tilde{E} \rightarrow E$, $\omega \mapsto \beta \omega$ is bi-holomorphic and the segment $(r, R_1) \subseteq E$ is transformed via ψ_1^{-1} into $(r/\beta, R_1/\beta) \subseteq \tilde{E}$. Consider now a bi-holomorphic transformation $\psi_2: B(0, 1) \rightarrow \tilde{E}$. The existence of this map is a consequence of the Riemann mapping theorem, and an explicit

formula is given in [88, page 296]. In particular, Ψ_2 can be chosen so that $\Psi_2(0) = 0$ and $\Psi_2^{-1}((r/\beta, R_1/\beta)) = (r', 1)$ for some $r' \in (0, 1)$. Since $(R_1 - r)/\beta \geq c$ and $1 \leq r_1/r_2 = R_1/R_2 \leq M/\eta$ we have $1 - r' \geq c$, as the ratio r_1/r_2 determines the deformation carried out by ψ_2 . Hence $r' \leq \theta$ with $\theta = 1 - c$.

Consider now the map $g': B(0, 1) \rightarrow \mathbb{C}$ defined by $g' = g \circ \psi_1 \circ \psi_2$. We have that g' is holomorphic in $B(0, 1)$, $|g'(0)| = |g(0)| \geq C_0$ and $\sup_{B(0, 1)} |g'| = \sup_E |g| \leq D$. By Lemma 3.4 applied to g' and r' we obtain the result. $\square$

3.2.2 Elliptic equations

3.2.2.1 Asymptotic distribution of the eigenvalues of elliptic operators

Let $\Omega \subseteq \mathbb{R}^d$ be a bounded smooth domain. As in Proposition 2.1, we consider the eigenvalues λ_l for the problem

$$-\operatorname{div}(a \nabla z_l) = \lambda_l \varepsilon z_l, \quad z_l \in H_0^1(\Omega; \mathbb{R}),$$

where a and ε satisfy (2.2). We suppose that $0 < \lambda_1 \leq \lambda_2 \leq \dots$ and count the eigenvalues with multiplicity. We have the following result, regarding the asymptotic distribution of the eigenvalues. The result is classical, and in the case $a = \varepsilon = 1$ it is known as Weyl's lemma.

Lemma 3.7. *Assume that (2.2) holds true. There exist $C_1, C_2 > 0$ depending on Ω and Λ such that*

$$C_1 l^{\frac{2}{d}} \leq \lambda_l \leq C_2 l^{\frac{2}{d}}, \quad l \in \mathbb{N}^*.$$

Proof. Let $\mathfrak{F}_l$ denote the set of all l -dimensional subspaces of $H_0^1(\Omega; \mathbb{R})$. In view of the Courant–Fischer–Weyl min-max principle [96, Exercise 12.4.2] we have

$$\lambda_l = \min_{D \in \mathfrak{F}_l} \max_{u \in D \setminus \{0\}} \frac{\int_{\Omega} a \nabla u \cdot \nabla u \, dx}{\int \varepsilon u^2 \, dx}, \quad l \in \mathbb{N}^*.$$

Therefore we have

$$(3.7) \quad \Lambda^{-2} \mu_l \leq \lambda_l \leq \Lambda^2 \mu_l, \quad l \in \mathbb{N}^*,$$

where

$$\mu_l = \min_{D \in \mathfrak{F}_l} \max_{u \in D \setminus \{0\}} \frac{\int_{\Omega} \nabla u \cdot \nabla u \, dx}{\int u^2 \, dx}.$$

By the min-max principle, μ_l are the eigenvalues of the Laplace operator on Ω , and so they satisfy

$$c_1 l^{\frac{2}{d}} \leq \mu_l \leq c_2 l^{\frac{2}{d}}, \quad l \in \mathbb{N}^*$$

for some $c_1, c_2 > 0$ depending on Ω (see [96, Theorem 12.14] or [74, Chapter 5, Lemma 3.1]). Combining this inequality with (3.7) yields the result. $\square$

3.2.2.2 Critical points of solutions to elliptic equations in two dimensions

We state here a result due to Alessandrini regarding the absence of critical points of solutions to elliptic PDE in two dimensions.

Lemma 3.8. *Let $\Omega \subseteq \mathbb{R}^2$ be a bounded smooth domain and take $\Omega' \Subset \Omega$. Let $a \in C^{0,1}(\overline{\Omega}; \mathbb{R}^{2 \times 2})$ satisfy (2.2a) and $\varphi \in C^2(\overline{\Omega}; \mathbb{R})$ such that $\varphi_{\partial\Omega}$ has one minimum and one maximum. Let $u \in H^1(\Omega; \mathbb{R})$ be the solution to*

$$\begin{cases} -\operatorname{div}(a \nabla u) = 0 & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega. \end{cases}$$

Then there exists $C > 0$ depending on Ω , Ω' , Λ , $\operatorname{osc}_{\partial\Omega}\varphi$ and $\|\varphi\|_{C^2(\overline{\Omega}; \mathbb{R})}$ such that

$$|\nabla u(x)| \geq C, \quad x \in \overline{\Omega'}.$$

Proof. We adopt Einstein summation convention. Set $c = \sqrt{\det a} \in C^{0,1}(\overline{\Omega}; \mathbb{R})$, $\tilde{a} = c^{-1}a$ and $\tilde{b}_j = c^{-1}\partial_i a_{ij}$. Hence $\det \tilde{a} = 1$. Since $a \in C^{0,1}(\overline{\Omega}; \mathbb{R}^{2 \times 2})$ and $u \in H^2(\Omega; \mathbb{R})$ by standard elliptic regularity theory, a trivial calculation gives

$$\tilde{a}_{ij}\partial_i\partial_j u + \tilde{b}_j\partial_j u = 0 \quad \text{in } \Omega.$$

The result readily follows from [6, Theorems 1.1 and 2.1]. $\square$

3.3 The Helmholtz equation

3.3.1 The main result

Given a smooth bounded domain $\Omega \subseteq \mathbb{R}^d$, $d = 2, 3$, in this section we consider problem (2.1)

$$(3.8) \quad \begin{cases} -\operatorname{div}(a \nabla u_\omega^\varphi) - (\omega^2 \varepsilon + i\omega \sigma) u_\omega^\varphi = 0 & \text{in } \Omega, \\ u_\omega^\varphi = \varphi & \text{on } \partial\Omega. \end{cases}$$

Let us recall the assumptions on the coefficients. We assume that $a \in L^\infty(\Omega; \mathbb{R}^{d \times d})$ and $\varepsilon \in L^\infty(\Omega; \mathbb{R})$ and satisfy

$$(3.9a) \quad a = a^T, \quad \Lambda^{-1} |\xi|^2 \leq \xi \cdot a \xi \leq \Lambda |\xi|^2, \quad \xi \in \mathbb{R}^d,$$

$$(3.9b) \quad \Lambda^{-1} \leq \varepsilon \leq \Lambda \quad \text{almost everywhere}$$

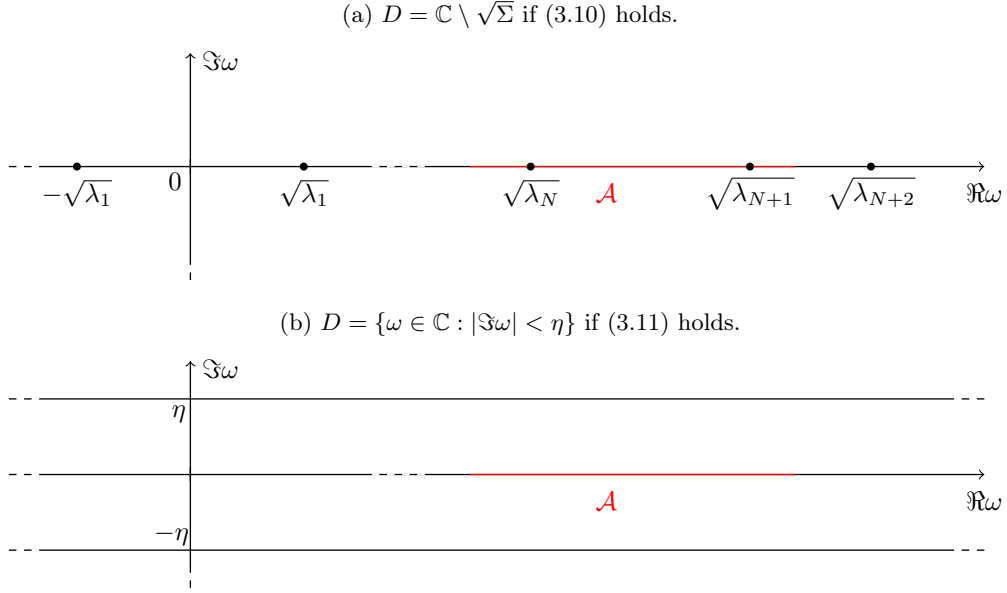
for some $\Lambda > 0$ and that $\sigma \in L^\infty(\Omega; \mathbb{R})$ and satisfies either

$$(3.10) \quad \sigma = 0, \quad \text{or}$$

$$(3.11) \quad \Lambda^{-1} \leq \sigma \leq \Lambda \quad \text{almost everywhere.}$$

Moreover, we suppose

$$(3.12) \quad a \in C^{\kappa-1, \alpha}(\overline{\Omega}; \mathbb{R}^{d \times d}), \quad \varepsilon, \sigma \in W^{\kappa-1, \infty}(\Omega; \mathbb{R})$$

Figure 3.1: The domain D and the admissible set $\mathcal{A}$.

for some $\kappa \in \mathbb{N}$ and $\alpha \in (0, 1)$.

Let $\mathcal{A} = [K_{min}, K_{max}] \subseteq B(0, M)$ represent the frequencies we have access to, for some $0 < K_{min} < K_{max} \leq M$. In view of Corollaries 2.2 and 2.4, problem (3.8) is well-posed for every $\omega \in D$, where

$$(3.13) \quad D = \begin{cases} \mathbb{C} \setminus \sqrt{\Sigma} & \text{if (3.10) holds,} \\ \{\omega \in \mathbb{C} : |\Im \omega| < \eta\} & \text{if (3.11) holds.} \end{cases}$$

Here we have used the notation $\sqrt{\Sigma} = \{\omega \in \mathbb{C} : \omega^2 \in \Sigma\}$. Figure 3.1 represents the domain D and the admissible set of frequencies $\mathcal{A}$. Note that $u_\omega^i \in C^\kappa(\overline{\Omega}; \mathbb{C})$ by Proposition 2.5.

Definition 3.9. Given a finite set $K \subseteq \mathcal{A}$ and $\varphi_1, \dots, \varphi_b \in C^{\kappa, \alpha}(\overline{\Omega}; \mathbb{C})$, we say that $K \times \{\varphi_1, \dots, \varphi_b\}$ is a *set of measurements*.

We shall study a particular class of sets of measurements, namely those whose corresponding solutions u_ω^i to

$$(3.14) \quad \begin{cases} -\operatorname{div}(a \nabla u_\omega^i) - (\omega^2 \varepsilon + i\omega \sigma) u_\omega^i = 0 & \text{in } \Omega, \\ u_\omega^i = \varphi_i & \text{on } \partial\Omega, \end{cases}$$

and their derivatives up to the κ -th order satisfy some non-zero constraints inside the domain. Let $b \in \mathbb{N}^*$ be the number of illuminations and $r \in \mathbb{N}^*$ be the number of constraints. These are described by a map ζ , which we now introduce. For $b, r \in \mathbb{N}^*$ let

$$(3.15a) \quad \zeta = (\zeta^1, \dots, \zeta^r) : C^\kappa(\overline{\Omega}; \mathbb{C})^b \longrightarrow C(\overline{\Omega}; \mathbb{C})^r \text{ be holomorphic, such that}$$

$$(3.15b) \quad \|\zeta(u^1, \dots, u^b)\|_{C(\overline{\Omega}; \mathbb{C})^r} \leq c_\zeta (1 + \|(u^1, \dots, u^b)\|_{C^\kappa(\overline{\Omega}; \mathbb{C})^b}^s) \text{ and}$$

$$(3.15c) \quad \|D\zeta(u^1, \dots, u^b)\|_{B(C^\kappa(\overline{\Omega}; \mathbb{C})^b, C(\overline{\Omega}; \mathbb{C})^r)} \leq c_\zeta (1 + \|(u^1, \dots, u^b)\|_{C^\kappa(\overline{\Omega}; \mathbb{C})^b}^s)$$

for some $c_\zeta > 0$ and $s \in \mathbb{N}^*$. For simplicity, we shall use the notation $C_\zeta = (c_\zeta, s, r, \kappa, \alpha)$.

We now consider some examples of maps ζ 's.

Example 3.10. The following example has already been introduced in Section 1.2, and it turns out to be the most relevant one. Take $b = d + 1$, $r = 3$ and $\kappa = 1$ and let $\zeta_{\det}: C^1(\overline{\Omega}; \mathbb{C})^{d+1} \rightarrow C(\overline{\Omega}; \mathbb{C})^3$ be defined by

$$\begin{aligned}\zeta_{\det}^1(u^1, \dots, u^{d+1}) &= u^1, \\ \zeta_{\det}^2(u^1, \dots, u^{d+1}) &= \det \begin{bmatrix} \nabla u^2 & \dots & \nabla u^{d+1} \end{bmatrix}, \\ \zeta_{\det}^3(u^1, \dots, u^{d+1}) &= \det \begin{bmatrix} u^1 & \dots & u^{d+1} \\ \nabla u^1 & \dots & \nabla u^{d+1} \end{bmatrix}.\end{aligned}$$

In view of Lemma 3.3, the map $\zeta_{\det}$ is holomorphic. Simple calculations show that (3.15b) holds true with $s_b = d + 1$ and (3.15c) with $s_c = d$, and so we can set $s = d + 1$.

In § 3.3.2 we shall study the construction of $(\zeta_{\det}, C)$ -complete sets of measurements. Then, as simple consequences, it will be possible to construct complete sets associated to weaker constraints, which are introduced in the following two examples and are useful in the applications.

Example 3.11. We remove here the third constraint given by $\zeta_{\det}^3$ and we weaken $\zeta_{\det}^2$ to require only two linearly independent gradients instead of d . Namely, take $b = 3$, $r = 2$ and $\kappa = 1$ and let $\zeta_\times: C^1(\overline{\Omega}; \mathbb{C})^3 \rightarrow C(\overline{\Omega}; \mathbb{C})^2$ be defined by

$$\zeta_\times(u^1, u^2, u^3) = \begin{cases} (u^1, \nabla u^2 \times \nabla u^3) & \text{if } d = 2, \\ (u^1, (\nabla u^2 \times \nabla u^3)_3) & \text{if } d = 3. \end{cases}$$

Note that $\zeta_\times^2 = \zeta_{\det}^2$ in two dimensions. In view of Lemma 3.3, the map $\zeta_\times$ is holomorphic. Similarly, simple calculations show that (3.15b) holds true with $s_b = 2$ and (3.15c) with $s_c = 1$, and so we can set $s = 2$.

The map $\zeta_\times$ has already been introduced in Section 3.1. The constraints given by this map will be used in § 4.1.1.

Example 3.12. Here we simply remove the second constraint given by $\zeta_{\det}^2$. Namely, take $b = d + 1$, $r = 2$ and $\kappa = 1$ and let $\zeta'_{\det}: C^1(\overline{\Omega}; \mathbb{C})^{d+1} \rightarrow C(\overline{\Omega}; \mathbb{C})^2$ be defined by

$$\zeta'_{\det}(u^1, \dots, u^{d+1}) = \left(u^1, \det \begin{bmatrix} u^1 & \dots & u^{d+1} \\ \nabla u^1 & \dots & \nabla u^{d+1} \end{bmatrix} \right).$$

Obviously, this map still satisfies assumption (3.15).

The constraints characterised by this map will be used in § 4.1.2.

We now give the precise definition of (ζ, C) -complete sets of measurements, already introduced in Section 1.2.

Definition 3.13. Take $\Omega' \subseteq \Omega$. Let $b, r \in \mathbb{N}^*$ be two positive integers, $C > 0$ and let ζ be as in (3.15). A set of measurements $K \times \{\varphi_1, \dots, \varphi_b\}$ is (ζ, C) -complete in Ω' if for every $x \in \overline{\Omega'}$ there exists $\omega_x \in K \cap D$ such that

$$(3.16) \quad \begin{array}{l} 1. \quad |\zeta^1(u_{\omega_x}^1, \dots, u_{\omega_x}^b)(x)| \geq C, \\ \vdots \\ r. \quad |\zeta^r(u_{\omega_x}^1, \dots, u_{\omega_x}^b)(x)| \geq C. \end{array}$$

An equivalent characterisation of (ζ, C) -complete sets is given in the following remark.

Remark 3.14. Let $K \times \{\varphi_1, \dots, \varphi_b\}$ be (ζ, C) -complete in Ω' and define

$$\Omega_\omega = \{x \in \overline{\Omega'} : \min_{j=1, \dots, r} |\zeta^j(u_\omega^1, \dots, u_\omega^b)(x)| > C/2\}, \quad \omega \in K.$$

Since $\zeta^j(u_\omega^1, \dots, u_\omega^b)$ is continuous, Ω_ω are open sets in $\overline{\Omega'}$. Moreover, (3.16) yields

$$\overline{\Omega'} = \bigcup_{\omega \in K} \Omega_\omega.$$

In other words, a (ζ, C) -complete set of measurements naturally gives a cover of the domain Ω' into $\#K$ subdomains, such that the constraints (3.16) are satisfied in each subdomain for different frequencies.

We now describe how to choose the frequencies in the admissible set $\mathcal{A}$. Let $K^{(n)}$ be the uniform partition of $\mathcal{A}$ into $n - 1$ intervals so that $\#K^{(n)} = n$, namely

$$(3.17) \quad K^{(n)} = \{\omega_1^{(n)}, \dots, \omega_n^{(n)}\}, \quad \omega_i^{(n)} = K_{min} + \frac{(i-1)}{(n-1)}(K_{max} - K_{min}).$$

The main result of this section reads as follows.

Theorem 3.15. Assume that (3.9), (3.12) and either (3.10) or (3.11) hold. Let ζ be as in (3.15) and $\varphi_1, \dots, \varphi_b \in C^{\kappa, \alpha}(\overline{\Omega}; \mathbb{C})$. If

$$(3.18) \quad |\zeta^j(u_0^1, \dots, u_0^b)(x)| \geq C_0, \quad j = 1, \dots, r, \quad x \in \overline{\Omega'}$$

for some $C_0 > 0$ then there exist $C > 0$ and $n \in \mathbb{N}$ depending on Ω , Λ , $|\mathcal{A}|$, M , C_ζ , $\|a\|_{C^{\kappa-1, \alpha}(\overline{\Omega}; \mathbb{R}^{d \times d})}$, $\|(\varepsilon, \sigma)\|_{W^{\kappa-1, \infty}(\Omega; \mathbb{R})^2}$, $\|\varphi_i\|_{C^{\kappa, \alpha}(\overline{\Omega}; \mathbb{C})}$ and C_0 such that

$$K^{(n)} \times \{\varphi_1, \dots, \varphi_b\}$$

is (ζ, C) -complete in Ω' .

In the following remark we discuss assumption (3.18) and the dependence of the construction of the illuminations on the electromagnetic parameters.

Remark 3.16. This result allows an a priori construction of (ζ, C) -complete sets, since C and n depend only on a priori data, provided that the illuminations $\varphi_1, \dots, \varphi_b$ are chosen in such a way that (3.18) holds true. It is in general much easier to satisfy (3.18) than (3.16), as $\omega = 0$ makes problem (3.14) simpler. More precisely, there exist many results regarding the conductivity equation [10, 53, 111, 37, 32, 31, 34] (see also § 3.2.2.2 and § 3.3.2).

Note that (3.14) with $\omega = 0$ does not depend on ε and σ , so that the construction of the illuminations $\varphi_1, \dots, \varphi_b$ is always independent of ε and σ . For the same reason, their construction may depend on a .

We shall see in § 3.3.2 (Examples 3.30 and 3.31) that there exist *occulting* illuminations, that is, boundary conditions for which a finite number of frequencies is not sufficient, and so assumption (3.18) cannot be completely removed. Yet, it is very likely that this assumption can be considerably weakened (see § 3.5.1)

We now comment on the dependence of C on $|\mathcal{A}|$ and M .

Remark 3.17. The proof of this result is based on Lemma 3.6. Thus, by Remark 3.5, the constant C goes to zero as $|\mathcal{A}| \rightarrow 0$ or $M \rightarrow \infty$. In particular, this approach gives good estimates for frequencies in a moderate regime (e.g. with microwaves), but these estimates get worse for very high frequencies.

Finally, we comment on the regularity assumption on the coefficients.

Remark 3.18. The regularity of the coefficients required for this approach is much lower than the regularity required if CGO solutions are used (see Section 1.2).

Before proving this result, we first weaken assumption (3.18).

Corollary 3.19. *Assume that (3.9), (3.12) and either (3.10) or (3.11) hold. Let $\hat{a} \in L^\infty(\Omega; \mathbb{R}^{d \times d})$ satisfy (3.9a). Let ζ be as in (3.15) and $\varphi_1, \dots, \varphi_b \in C^{\kappa, \alpha}(\overline{\Omega}; \mathbb{C})$. Suppose that for some $C_0 > 0$*

$$(3.19) \quad |\zeta^j(\hat{u}_0^1, \dots, \hat{u}_0^b)(x)| \geq C_0, \quad j = 1, \dots, r, \quad x \in \overline{\Omega'},$$

where $\hat{u}_0^i \in H^1(\Omega; \mathbb{C})$ is the solution to (3.14) with $\hat{a}$ in lieu of a and $\omega = 0$, namely

$$\begin{cases} -\operatorname{div}(\hat{a} \nabla \hat{u}_0^i) = 0 & \text{in } \Omega, \\ \hat{u}_0^i = \varphi_i & \text{on } \partial\Omega. \end{cases}$$

There exist $\delta, C > 0$ and $n \in \mathbb{N}$ depending on Ω , Λ , $|\mathcal{A}|$, M , C_ζ , $\|a\|_{C^{\kappa-1, \alpha}(\overline{\Omega}; \mathbb{R}^{d \times d})}$, $\|(\varepsilon, \sigma)\|_{W^{\kappa-1, \infty}(\Omega; \mathbb{R}^2)}$, $\|\varphi_i\|_{C^{\kappa, \alpha}(\overline{\Omega}; \mathbb{C})}$ and C_0 such that if $\|a - \hat{a}\|_{C^{\kappa-1, \alpha}(\overline{\Omega}; \mathbb{R}^{d \times d})} \leq \delta$ then

$$K^{(n)} \times \{\varphi_1, \dots, \varphi_b\}$$

is (ζ, C) -complete in Ω' .

Remark 3.20. A typical application of the corollary is in the case where a is a small perturbation of a known constant tensor $\hat{a}$. Then, the construction of the illuminations $\varphi_1, \dots, \varphi_b$ is independent of a .

Proof. Several positive constants depending on Ω , Λ , $|\mathcal{A}|$, M , C_ζ , C_0 , $\|a\|_{C^{\kappa-1,\alpha}(\overline{\Omega};\mathbb{R}^{d\times d})}$, $\|(\varepsilon, \sigma)\|_{W^{\kappa-1,\infty}(\Omega;\mathbb{R})^2}$ and $\|\varphi_i\|_{C^{\kappa,\alpha}(\overline{\Omega};\mathbb{C})}$ will be denoted by c .

An immediate calculation shows that

$$\begin{cases} -\operatorname{div}(\hat{a} \nabla(\hat{u}_0^i - u_0^i)) = \operatorname{div}((\hat{a} - a) \nabla u_0^i) & \text{in } \Omega, \\ \hat{u}_0^i - u_0^i = 0 & \text{on } \partial\Omega. \end{cases}$$

In view of Propositions 2.5 and 2.1 or 2.3 we have

$$\|\hat{u}_0^i - u_0^i\|_{C^\kappa(\overline{\Omega};\mathbb{C})} \leq c \|(\hat{a} - a) \nabla u_0^i\|_{C^{\kappa-1,\alpha}(\overline{\Omega};\mathbb{C}^3)} \leq c\delta,$$

provided that $\|a - \hat{a}\|_{C^{\kappa-1,\alpha}(\overline{\Omega};\mathbb{R}^{d\times d})} \leq \delta$. Therefore, since the map ζ is holomorphic we obtain

$$\|\zeta^j(\hat{u}_0^1, \dots, \hat{u}_0^b) - \zeta^j(u_0^1, \dots, u_0^b)\|_{C(\overline{\Omega};\mathbb{C})} \leq c\delta, \quad j = 1, \dots, r.$$

By choosing δ so that $c\delta \leq C_0/2$, by (3.19) we obtain that (3.18) is satisfied with $C_0/2$ in lieu of C_0 . As a consequence, the result immediately follows from Theorem 3.15. $\square$

The rest of this subsection is devoted to the proof of Theorem 3.15. For simplicity, we shall say that a positive constant depends on a priori data if it depends on Ω , Λ , $|\mathcal{A}|$, M , C_ζ , $\|a\|_{C^{\kappa-1,\alpha}(\overline{\Omega};\mathbb{R}^{d\times d})}$, $\|(\varepsilon, \sigma)\|_{W^{\kappa-1,\infty}(\Omega;\mathbb{R})^2}$, $\|\varphi_i\|_{C^{\kappa,\alpha}(\overline{\Omega};\mathbb{C})}$ and C_0 only.

Lemma 3.21. *Under the hypotheses of Theorem 3.15, set*

$$\theta^j : D \rightarrow C(\overline{\Omega};\mathbb{C}), \quad \omega \mapsto \zeta^j(u_\omega^1, \dots, u_\omega^b).$$

The map $\theta^j : D \rightarrow C(\overline{\Omega};\mathbb{C})$ is holomorphic for all j .

Proof. It immediately follows from Proposition 2.7, (3.15a) and Lemma 3.3, parts 2 and 3. $\square$

We next study some a priori bounds on θ^j and $\partial_\omega \theta^j$.

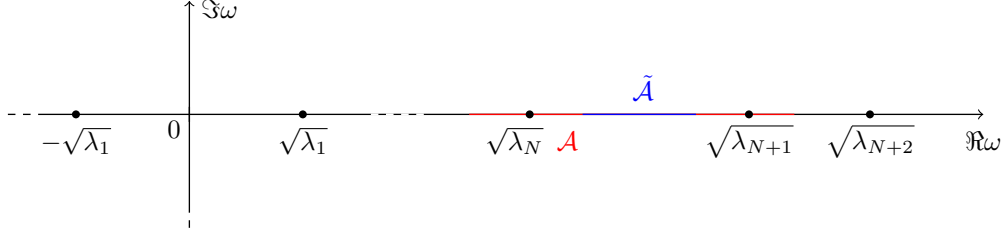
Lemma 3.22. *Assume that the hypotheses of Theorem 3.15 hold true and take $j = 1, \dots, r$ and $\omega \in B(0, M) \cap D$.*

1. *If (3.10) holds true then there exists $C > 0$ depending on a priori data such that*

$$\begin{aligned} (a) \quad & \|\theta_\omega^j\|_{C(\overline{\Omega};\mathbb{C})} \leq C \left[1 + \sup_{l \in \mathbb{N}^*} \frac{1}{|\lambda_l - \omega^2|} \right]^s; \\ (b) \quad & \|\partial_\omega \theta_\omega^j\|_{C(\overline{\Omega};\mathbb{C})} \leq C \left[1 + \sup_{l \in \mathbb{N}^*} \frac{1}{|\lambda_l - \omega^2|} \right]^{s+2}. \end{aligned}$$

2. *If (3.11) holds true then there exists $C > 0$ depending on a priori data such that*

$$\begin{aligned} (a) \quad & \|\theta_\omega^j\|_{C(\overline{\Omega};\mathbb{C})} \leq C; \\ (b) \quad & \|\partial_\omega \theta_\omega^j\|_{C(\overline{\Omega};\mathbb{C})} \leq C. \end{aligned}$$

Figure 3.2: The admissible sets $\mathcal{A}$ and $\tilde{\mathcal{A}}$.

Proof. In view of Corollary 2.2 and Proposition 2.5 we have

$$(3.20) \quad \|u_\omega^i\|_{C^\kappa(\bar{\Omega}; \mathbb{C})} \leq C \left[1 + \sup_{l \in \mathbb{N}^*} \frac{1}{|\lambda_l - \omega^2|} \right], \quad \omega \in B(0, M) \cap D,$$

whence we obtain part 1a from (3.15b).

It can be easily seen that $\partial_\omega u_\omega^i$ is the solution to

$$\begin{cases} -\operatorname{div}(a \nabla(\partial_\omega u_\omega^i)) - \omega^2 \varepsilon \partial_\omega u_\omega^i = 2\omega \varepsilon u_\omega^i & \text{in } \Omega, \\ \partial_\omega u_\omega^i = 0 & \text{on } \partial\Omega. \end{cases}$$

Arguing as before, from Corollary 2.2 and Proposition 2.5 we obtain

$$(3.21) \quad \|\partial_\omega u_\omega^i\|_{C^\kappa(\bar{\Omega}; \mathbb{C})} \leq C \left[1 + \sup_{l \in \mathbb{N}^*} \frac{1}{|\lambda_l - \omega^2|} \right]^2.$$

Since $\partial_\omega \theta_\omega^j = D\zeta_{(u_\omega^1, \dots, u_\omega^b)}^j(\partial_\omega u_\omega^1, \dots, \partial_\omega u_\omega^b)$ we have

$$\begin{aligned} \|\partial_\omega \theta_\omega^j\|_{C(\bar{\Omega}; \mathbb{C})} &= \|D\zeta_{(u_\omega^1, \dots, u_\omega^b)}^j(\partial_\omega u_\omega^1, \dots, \partial_\omega u_\omega^b)\|_{C(\bar{\Omega}; \mathbb{C})} \\ &\leq \|D\zeta_{(u_\omega^1, \dots, u_\omega^b)}^j\|_{\mathcal{B}(C^\kappa(\bar{\Omega}; \mathbb{C})^b, C(\bar{\Omega}; \mathbb{C}))} \|(\partial_\omega u_\omega^1, \dots, \partial_\omega u_\omega^b)\|_{C^\kappa(\bar{\Omega}; \mathbb{C})^b} \\ &\leq C \left[1 + \sup_{l \in \mathbb{N}^*} \frac{1}{|\lambda_l - \omega^2|} \right]^{s+2}, \end{aligned}$$

where the last inequality follows from (3.15c), (3.20) and (3.21). Part 1b is now proved.

Part 2 can be proved analogously, by using Corollary 2.4 in lieu of Corollary 2.2. The details are left to the reader. $\square$

In the following two lemmata we study the case where (3.10) holds true, and how to deal with the presence of the eigenvalues (see Figure 3.2).

Lemma 3.23. *Under the hypotheses of Theorem 3.15, assume that (3.10) holds true. Then there exist $N \in \mathbb{N}^*$, $\delta > 0$ and $\beta > 0$ depending on Ω , Λ , $|\mathcal{A}|$ and M only and a closed interval $\tilde{\mathcal{A}} = [\tilde{K}_{\min}, \tilde{K}_{\max}] \subseteq \mathcal{A}$ such that*

$$d(\tilde{\mathcal{A}}^2, \Sigma) \geq \delta, \quad \tilde{\mathcal{A}}^2 \subseteq (\lambda_l, \lambda_{l+1}), \quad |\tilde{\mathcal{A}}| \geq \beta$$

for some $l \leq N$.

Proof. In view of Lemma 3.7 there exists $N \in \mathbb{N}^*$ depending on Ω , Λ and M only such that $[0, K_{max}^2] \cap \Sigma \subseteq \{\lambda_1, \dots, \lambda_N\}$. In particular, $\#(\mathcal{A}^2 \cap \Sigma) \leq N$. Therefore there exists $l \leq N$ such that $|\mathcal{A}^2 \cap (\lambda_l, \lambda_{l+1})| \geq |\mathcal{A}^2| (N+1)^{-1}$. Write $\mathcal{A}^2 \cap (\lambda_l, \lambda_{l+1}) = [p, q]$ and define $\tilde{\mathcal{A}}$ by $\tilde{\mathcal{A}}^2 = [p + \frac{|\mathcal{A}^2|}{3(N+1)}, q - \frac{|\mathcal{A}^2|}{3(N+1)}]$. This concludes the proof, since $|\mathcal{A}^2|$ depends on $|\mathcal{A}|$ and N only. $\square$

Thanks to Lemma 3.23, by taking a subinterval of the original admissible set $\mathcal{A}$, without loss of generality we can assume that

$$(3.22) \quad d(\mathcal{A}^2, \Sigma) \geq \delta, \quad \mathcal{A}^2 \subseteq (\lambda_l, \lambda_{l+1}), \quad l \leq N$$

for some $\delta > 0$ and $N \in \mathbb{N}^*$ depending on Ω , Λ , $|\mathcal{A}|$ and M only. Moreover, the new size of $\mathcal{A}$ is comparable with the size of the original $\mathcal{A}$ by means of constants depending on Ω , Λ , $|\mathcal{A}|$ and M only.

The main idea is to apply Lemma 3.6 to the maps $\omega \mapsto \theta_\omega^j(x)$ and use the fact that in $\omega = 0$ they are non-zero. However, in the case where (3.10) holds true we first need to remove the singularities in the poles $\pm\sqrt{\lambda_1}, \dots, \pm\sqrt{\lambda_N}$. This is done with the following result.

Lemma 3.24. *Under the hypotheses of Theorem 3.15, if (3.10) and (3.22) hold true then for any $x \in \bar{\Omega}$ the function*

$$(3.23) \quad \omega \in B(0, K_{max}) \mapsto g_x^j(\omega) := \theta_\omega^j(x) \prod_{l=1}^N \frac{(\lambda_l - \omega^2)^s}{\lambda_l^s},$$

is holomorphic in $B(0, K_{max})$ and

$$\sup_{B(0, K_{max})} |g_x^j| \leq C$$

for some $C > 0$ depending on a priori data.

Proof. Different positive constants depending on a priori data will be denoted by C . In view of Lemma 3.21, the map $\omega \in \mathbb{C} \setminus \sqrt{\Sigma} \mapsto \theta_\omega^j(x) \in \mathbb{C}$ is holomorphic and by Lemma 3.22, part 1a, it is meromorphic in $B(0, K_{max})$. For $\omega \in B(0, K_{max}) \cap D$ we have

$$\begin{aligned} |g_x^j(\omega)| &\leq |\theta_\omega^j(x)| \prod_{l=1}^N \frac{|\lambda_l - \omega^2|^s}{\lambda_l^s} \\ &\leq C \lambda_1^{-Ns} \prod_{l=1}^N |\lambda_l - \omega^2|^s \left[1 + \sup_{l \in \mathbb{N}^*} \frac{1}{|\lambda_l - \omega^2|^s} \right] \\ &\leq C \prod_{l=1}^N |\lambda_l - \omega^2|^s \left[1 + \sup_{l \leq N} \frac{1}{|\lambda_l - \omega^2|^s} + \sup_{l > N} \frac{1}{|\lambda_l - \omega^2|^s} \right], \end{aligned}$$

where the second inequality follows from Lemma 3.22, part 1a. As a consequence

$$\begin{aligned} |g_x^j(\omega)| &\leq C \prod_{l=1}^N |\lambda_l - \omega^2|^s \left[1 + \sup_{l \leq N} \frac{1}{|\lambda_l - \omega^2|^s} \right] \\ &\leq C \left[\prod_{l=1}^N |\lambda_l - \omega^2|^s + \frac{\prod_{l=1}^N |\lambda_l - \omega^2|^{d+1}}{\inf_{l \leq N} |\lambda_l - \omega^2|^{d+1}} \right] \\ &\leq C, \end{aligned}$$

where the first inequality follows from

$$|\lambda_l - \omega^2| \geq \delta, \quad l > N,$$

and the third inequality from

$$|\lambda_l - \omega^2| \leq 2M^2, \quad l \leq N.$$

Therefore the map g_x^j is holomorphic in $B(0, K_{max})$ and the bound on $\sup_{B(0, K_{max})} |g_x^j|$ holds. $\square$

The next lemma is the last step needed for the proof of Theorem 3.15.

Lemma 3.25. *Under the hypotheses of Theorem 3.15, assume that if (3.10) holds then (3.22) holds. Then for every $x \in \overline{\Omega'}$ there exists $\omega_x \in \mathcal{A}$ such that*

$$|\theta_{\omega_x}^j(x)| \geq C, \quad j = 1, \dots, r$$

for some $C > 0$ depending on a priori data.

Proof. Different positive constants depending on a priori data will be denoted by C .

First case – Assumption (3.10). Take $x \in \overline{\Omega'}$ and define g_x^j as in (3.23), where N is given by (3.22). Set

$$g_x = \prod_{j=1}^r g_x^j.$$

In view of Lemma 3.24 the map g_x is holomorphic in $B(0, K_{max})$ and $\max_{B(0, K_{max})} |g_x| \leq C$. Moreover, $|g_x(0)| \geq C_0^r$ by (3.18). Therefore, by Lemma 3.6 with $r = K_{min}$ and $R_1 = R_2 = K_{max}$ there exists $\omega_x \in [r, R] = \mathcal{A}$ such that $|g_x(\omega_x)| \geq C$. As a consequence, in view of (3.23) we obtain

$$\left| \prod_{j=1}^r \theta_{\omega_x}^j(x) \right| = |g_x(\omega_x)| \prod_{l=1}^N \frac{\lambda_l^{rs}}{|\lambda_l - \omega_x^2|^{rs}} \geq C,$$

since $\lambda_l \geq \lambda_1 \geq C(\Omega, \Lambda)$ and $|\lambda_l - \omega_x^2| \leq 2M^2$. The result now follows from Lemma 3.22, part 1a.

Second case – Assumption (3.11). Take $x \in \overline{\Omega'}$ and define

$$g_x(\omega) = \prod_{j=1}^r \theta_{\omega}^j(x), \quad \omega \in D.$$

In view of Lemma 3.21, the map g_x is holomorphic in D and by Lemma 3.22, part 2a, $\max_{B(0,M) \cap D} |g_x| \leq C$. Moreover, $|g_x(0)| \geq C_0^r$ by (3.18). Therefore, by Lemma 3.6 with $r = K_{\min}$, $R_1 = K_{\max}$, and $R_2 = \eta$ there exists $\omega_x \in \mathcal{A}$ such that $|g_x(\omega_x)| \geq C$. The result now follows from Lemma 3.22, part 2a. $\square$

We are now ready to prove Theorem 3.15.

Proof of Theorem 3.15. Different positive constants depending on a priori data will be denoted by C or Z .

If (3.10) holds true, by Lemma 3.23 we can assume (3.22). Thus, in view of Lemma 3.25, for every $x \in \overline{\Omega'}$ there exists $\omega_x \in \mathcal{A}$ such that

$$|\theta_{\omega_x}^j(x)| \geq C, \quad j = 1, \dots, r.$$

Thus, by Lemma 3.22, parts 1b and 2b, there exists $Z > 0$ such that

$$(3.24) \quad |\theta_{\omega}^j(x)| \geq C, \quad \omega \in [\omega_x - Z, \omega_x + Z] \cap \mathcal{A}, \quad j = 1, \dots, r.$$

Recall that $\mathcal{A} = [K_{\min}, K_{\max}]$ and that $\omega_i^{(n)} = K_{\min} + \frac{(i-1)}{(n-1)}(K_{\max} - K_{\min})$. It is trivial to see that there exists $P = P(Z, |\mathcal{A}|) \in \mathbb{N}$ such that

$$(3.25) \quad \mathcal{A} \subseteq \bigcup_{p=1}^P I_p, \quad I_p = [K_{\min} + (p-1)Z, K_{\min} + pZ].$$

Choose now $n \in \mathbb{N}$ big enough so that for every $p = 1, \dots, P$ there exists $i_p = 1, \dots, n$ such that $\omega(p) := \omega_{i_p}^{(n)} \in I_p$. Note that n depends on Z and $|\mathcal{A}|$ only.

Take now $x \in \overline{\Omega'}$. Since $|\omega_x - Z, \omega_x + Z| = 2Z$ and $|I_p| = Z$, in view of (3.25) there exists $p = 1, \dots, P$ such that $I_p \subseteq [\omega_x - Z, \omega_x + Z]$. Therefore $\omega(p) \in [\omega_x - Z, \omega_x + Z] \cap \mathcal{A}$, whence by (3.24) there holds $|\theta_{\omega(p)}^j(x)| \geq C$ for all $j = 1, \dots, r$. Recalling the definition of θ^j this implies

$$|\zeta^j(u_{\omega(p)}^1, \dots, u_{\omega(p)}^b)(x)| \geq C, \quad j = 1, \dots, r.$$

As a consequence, $K^{(n)} \times \{\varphi_1, \dots, \varphi_b\}$ is (ζ, C) -complete in Ω' since $\omega(p) \in K^{(n)}$. The theorem is proved. $\square$

3.3.2 Construction of $(\zeta_{\det}, C)$ -complete sets of measurements

In this subsection we apply Theorem 3.15 and Corollary 3.19 to construct $(\zeta_{\det}, C)$ -complete sets of measurements (Example 3.10). Let us recall the definition of the map $\zeta_{\det}$. Take $b = d + 1$, $r = 3$ and $\kappa = 1$ and let $\zeta_{\det} = (\zeta_{\det}^1, \zeta_{\det}^2, \zeta_{\det}^3): C^1(\overline{\Omega}; \mathbb{C})^{d+1} \longrightarrow C(\overline{\Omega}; \mathbb{C})^3$ be defined by

$$\begin{aligned} \zeta_{\det}^1(u^1, \dots, u^{d+1}) &= u^1, \\ \zeta_{\det}^2(u^1, \dots, u^{d+1}) &= \det \begin{bmatrix} \nabla u^2 & \dots & \nabla u^{d+1} \end{bmatrix}, \\ \zeta_{\det}^3(u^1, \dots, u^{d+1}) &= \det \begin{bmatrix} u^1 & \dots & u^{d+1} \\ \nabla u^1 & \dots & \nabla u^{d+1} \end{bmatrix}, \end{aligned}$$

for $u^i \in C^1(\overline{\Omega}; \mathbb{C})$.

The construction of $(\zeta_{\det}, C)$ -complete sets of measurements depends on the dimension. Indeed, the validity of (3.18) for $\zeta_{\det}^2$ and $\zeta_{\det}^3$ depends on the dimension, as we shall see later.

As far as the two-dimensional case is concerned, the precise statement of Theorem 1.4 reads as follows.

Theorem 3.26. *Assume that (3.9) holds with $d = 2$, that $a \in C^{0,1}(\overline{\Omega}; \mathbb{R}^{2 \times 2})$ and that Ω is convex. Assume that either (3.10) or (3.11) holds. If $\Omega' \Subset \Omega$ then there exist $C > 0$ and $n \in \mathbb{N}$ depending on Ω , Ω' , Λ , $|\mathcal{A}|$, M and $\|a\|_{C^{0,1}(\overline{\Omega}; \mathbb{R}^{2 \times 2})}$ such that*

$$K^{(n)} \times \{1, x_1, x_2\}$$

is $(\zeta_{\det}, C)$ -complete in Ω' .

Proof. The main point of the proof of this theorem is satisfying (3.18). Then, the result will follow immediately from Theorem 3.15. Thus, it remains to prove that

$$|\zeta_{\det}^j(u_0^1, u_0^2, u_0^3)(x)| \geq C_0, \quad j = 1, \dots, 3, \quad x \in \overline{\Omega'}$$

for some $C_0 > 0$ depending on Ω , Ω' , Λ and $\|a\|_{C^{0,1}(\overline{\Omega}; \mathbb{R}^{2 \times 2})}$.

Several positive constants depending on Ω , Ω' , Λ and $\|a\|_{C^{0,1}(\overline{\Omega}; \mathbb{R}^{2 \times 2})}$ will be denoted by C . Recall that, setting “ $x_0 = 1$ ”, we have

$$\begin{cases} -\operatorname{div}(a \nabla u_0^i) = 0 & \text{in } \partial\Omega, \\ u_0^i = x_{i-1} & \text{on } \partial\Omega. \end{cases}$$

Since $u_0^1 = 1$, the thesis is equivalent to show that

$$(3.26) \quad |\gamma(x)| := |\det [\nabla u_0^2 \quad \nabla u_0^3](x)| \geq C, \quad x \in \Omega'.$$

Fix now $x \in \Omega'$. Since Ω is convex, in view of Lemma 3.8 we have $\beta := |\nabla u_0^2(x)| \geq C$. Set $\nabla^\perp u_0^2 = (-\partial_2 u_0^2, \partial_1 u_0^2)$. Therefore $\{\beta^{-1} \nabla u_0^2(x), \beta^{-1} \nabla^\perp u_0^2(x)\}$ is an orthonormal basis of $\mathbb{R}^2$. As a consequence there holds

$$\nabla u_0^3(x) = (\nabla u_0^3(x) \cdot \beta^{-2} \nabla u_0^2(x)) \nabla u_0^2(x) + (\nabla u_0^3(x) \cdot \beta^{-2} \nabla^\perp u_0^2(x)) \nabla^\perp u_0^2(x).$$

Setting $\alpha = \nabla u_0^3(x) \cdot \beta^{-2} \nabla u_0^2(x)$ and $v = u_0^3 - \alpha u_0^2$, since $\gamma(x) = \nabla u_0^3(x) \cdot \nabla^\perp u_0^2(x)$ we have $\beta^{-2} \gamma(x) \nabla^\perp u_0^2(x) = \nabla v(x)$, whence

$$(3.27) \quad |\gamma(x)| = \beta |\nabla v(x)|.$$

Since Ω is convex and v is the solution to

$$\begin{cases} -\operatorname{div}(a \nabla v) = 0 & \text{in } \partial\Omega, \\ v = x_2 - \alpha x_1 & \text{on } \partial\Omega, \end{cases}$$

we can apply again Lemma 3.8 and obtain $|\nabla v(x)| \geq C$ (note that $|\alpha| \leq C$ by standard elliptic regularity theory – see Proposition 2.5). As a consequence, in view of (3.27) we obtain (3.26). $\square$

Remark 3.27. It is possible to remove the assumption on the convexity on the domain, provided that the boundary conditions are chosen in such a way that the assumptions of Lemma 3.8 are satisfied. For further details, the reader is referred to [41, 11, 1].

Remark 3.28. In order to satisfy the constraints corresponding to $\zeta_{\det}^1$, by the strong maximum principle it is enough to choose $\varphi_1 \geq C_0 > 0$. Indeed, in this way assumption (3.18) for $j = 1, 2$ is easily satisfied (see Proposition 4.1). However, we decided to avoid this generalisation here since if $\varphi_1 \neq 1$ then it is slightly more complicated to satisfy (3.18) for $\zeta_{\det}^3$. It can be seen that in the general case it is sufficient to set $\varphi_2 = x_1\varphi_1$ and $\varphi_3 = x_2\varphi_1$ [1].

As far as the three-dimensional case is concerned, results similar to Lemma 3.8 are not true [47, 29, 31]. As a consequence, in order to satisfy (3.18) we assume that a is close to a constant matrix. In § 3.5.1, we shall investigate if it is possible to remove this assumption (see also [32]).

The precise statement of Theorem 1.5 reads as follows.

Theorem 3.29. *Assume that (3.9) and (3.12) hold with $d = 3$, $\kappa = 1$ and $\alpha \in (0, 1)$. Assume that either (3.10) or (3.11) holds. Let $\hat{a} \in \mathbb{R}^{3 \times 3}$ satisfy (3.9a). There exist $\delta, C > 0$ and $n \in \mathbb{N}$ depending on Ω , Λ , α , $|\mathcal{A}|$, M and $\|a\|_{C^{0,\alpha}(\bar{\Omega}; \mathbb{R}^{3 \times 3})}$ such that if $\|a - \hat{a}\|_{C^{0,\alpha}(\bar{\Omega}; \mathbb{R}^{3 \times 3})} \leq \delta$ then*

$$K^{(n)} \times \{1, x_1, x_2, x_3\}$$

is $(\zeta_{\det}, C)$ -complete in Ω .

Proof. This result is an immediate consequence of Corollary 3.19, since in the case $\omega = 0$ we have $\hat{u}^1 = 1$ and $\hat{u}^i = x_{i-1}$ for $i = 2, \dots, d+1$ as $\hat{a}$ is constant, and so

$$\begin{aligned} \zeta_{\det}^1(\hat{u}_0^1, \dots, \hat{u}_0^{d+1}) &= 1, \\ \zeta_{\det}^2(\hat{u}_0^1, \dots, \hat{u}_0^{d+1}) &= \det \begin{bmatrix} e_1 & \cdots & e_d \end{bmatrix} = 1, \\ \zeta_{\det}^3(\hat{u}_0^1, \dots, \hat{u}_0^{d+1}) &= \det \begin{bmatrix} 1 & x_1 & \cdots & x_d \\ 0 & e_1 & \cdots & e_d \end{bmatrix} = 1, \end{aligned}$$

whence (3.19) is satisfied with $C_0 = 1$. □

We shall now comment on assumption (3.18) and give some examples to show that it cannot be completely removed. In other words, the boundary conditions φ_i have to be properly chosen since there exist *occluding* illuminations for which a finite number of frequencies are not sufficient to satisfy the desired constraint. For simplicity, we study the two-dimensional case.

As a model case, we consider the first two constraints relative to $(\zeta_{\det}, C)$ -complete sets of measurements. First, we provide a counterexample for condition $|u_\omega^1| \geq C$ and show that it is not enough to assume $\varphi_1 \neq 0$.

Example 3.30. Suppose $d = 2$, $\Omega = B(0, 1)$, $a = \varepsilon = 1$ and $\sigma = 0$. We choose the boundary condition $\varphi_1(x_1, x_2) = x_1$, so that $u_0^1 = x_1$ does not satisfy (3.18), as $|u_0^1| \not\geq C_0 > 0$. In this case the Helmholtz equation (3.8) can be written in polar coordinates (r, θ) and reads

$$u_{rr} + \frac{u_r}{r} + \frac{u_{\theta\theta}}{r^2} + ku = 0.$$

It is straightforward to see that

$$u_\omega^1(r, \theta) = \frac{J_1(\omega r)}{J_1(\omega)} \cos \theta$$

satisfies the above equation, where J_1 is the Bessel function of the first kind of order 1 and $\omega^2 > 0$ is not an eigenvalue of the problem. Thus, $\{u_\omega^1\}$ represents a family of solution to the Helmholtz equation with fixed boundary condition and varying frequency. However, condition $|u_\omega^1| \geq C$ cannot hold since $u_\omega^1(0, x_2) = 0$ for every ω and for every $x_2 \in (-1, 1)$.

Next, we study condition $|\det [\nabla u_\omega^2 \quad \nabla u_\omega^3]| \geq C$. Since it expresses the linear independence of the gradient of the solutions inside the domain, we shall see that it is not possible to require φ_2 and φ_3 to be just linearly independent.

Example 3.31. We consider the situation of Example 3.30. Suppose $\Omega = B(0, 1)$, $a = \varepsilon = 1$ and $\sigma = 0$. We choose the boundary conditions $\varphi_2(x_1, x_2) = x_1$ and $\varphi_3 = 1$, so that (3.18) is not verified, since $\det [\nabla u_0^2 \quad \nabla u_0^3] = \det [e_1 \quad 0] = 0$. It is straightforward to see that the corresponding solutions to (3.14) are

$$u_\omega^2(r, \theta) = \frac{J_1(\omega r)}{J_1(\omega)} \cos \theta, \quad u_\omega^3(r, \theta) = \frac{J_0(\omega r)}{J_0(\omega)},$$

where J_n is the Bessel function of the first kind of order n and $\omega^2 > 0$ is not an eigenvalue of the problem.

Take a matrix-valued function $A: \Omega \rightarrow GL(2)$, where $GL(2)$ denotes the set of 2×2 invertible matrices. By viewing $A(x)$ as a change of coordinates in $T_x\Omega$, the tangent space in x to Ω , we get

$$\det [A \nabla u_\omega^2 A^{-1} \quad A \nabla u_\omega^3 A^{-1}] = \det (A [\nabla u_\omega^2 \quad \nabla u_\omega^3] A^{-1}) = \det [\nabla u_\omega^2 \quad \nabla u_\omega^3].$$

Therefore, as far as $\det [\nabla u_\omega^2 \quad \nabla u_\omega^3]$ is concerned, we can express the gradient in any system of coordinates.

In this case, writing ∇u_ω^i with respect to e_θ and e_r we have $\nabla u_\omega^i = \frac{1}{r} \frac{\partial u_\omega^i}{\partial \theta} e_\theta + \frac{\partial u_\omega^i}{\partial r} e_r$. Hence

$$\det [\nabla u_\omega^2 \quad \nabla u_\omega^3] = \det \begin{bmatrix} \frac{1}{r} \frac{\partial u_\omega^2}{\partial \theta} & \frac{1}{r} \frac{\partial u_\omega^3}{\partial \theta} \\ \cdot & \cdot \end{bmatrix} = \det \begin{bmatrix} -\frac{1}{r} \frac{J_1(\omega r)}{J_1(\omega)} \sin \theta & 0 \\ \cdot & \cdot \end{bmatrix}.$$

Thus, we have $\det [\nabla u_\omega^2 \quad \nabla u_\omega^3](x_1, 0) = 0$ for every ω and for every $x_1 \in (-1, 1)$ and so $|\det [\nabla u_\omega^2 \quad \nabla u_\omega^3]| \geq C$ cannot be satisfied by using many measurements with these fixed illuminations and varying wavenumbers.

3.4 Maxwell's equations

Given a smooth bounded domain $\Omega \subseteq \mathbb{R}^3$ with a simply connected boundary $\partial\Omega$, in this section we consider Maxwell's equations with $\mu, \varepsilon, \sigma \in L^\infty(\Omega; \mathbb{R}^{3 \times 3})$ and $\varphi \in H(\text{curl}, \Omega)$ satisfying

$$(3.28a) \quad \Lambda^{-1} |\xi|^2 \leq \xi \cdot \mu \xi, \quad \Lambda^{-1} |\xi|^2 \leq \xi \cdot \varepsilon \xi, \quad \Lambda^{-1} |\xi|^2 \leq \xi \cdot \sigma \xi, \quad \xi \in \mathbb{R}^3,$$

$$(3.28b) \quad \|(\sigma, \varepsilon, \mu)\|_{L^\infty(\Omega; \mathbb{R}^{3 \times 3})^3} \leq \Lambda, \quad \mu = \mu^T, \quad \varepsilon = \varepsilon^T, \quad \sigma = \sigma^T, \quad \mu, \varepsilon, \sigma \in W^{\kappa+1,p}(\Omega; \mathbb{R}^{3 \times 3}),$$

$$(3.28c) \quad \text{curl} \varphi \cdot \nu = 0 \text{ on } \partial\Omega \text{ and } \varphi \in W^{\kappa+1,p}(\Omega; \mathbb{C}^3)$$

for some $\Lambda > 0$, $\kappa \in \mathbb{N}$ and $p > 3$.

As before, let $\mathcal{A} = [K_{\min}, K_{\max}] \subseteq B(0, M)$ be the frequencies we have access to.

Definition 3.32. Given a finite set $K \subseteq \mathcal{A}$ and $\varphi_1, \dots, \varphi_b \in W^{\kappa+1,p}(\Omega; \mathbb{C}^3)$ satisfying (3.28c), we say that $K \times \{\varphi_1, \dots, \varphi_b\}$ is a *set of measurements*.

We shall study a particular class of sets of measurements, namely those whose corresponding solutions $(E_\omega^i, H_\omega^i) \in H(\text{curl}, \Omega) \times H^\mu(\text{curl}, \Omega)$ to

$$(3.29) \quad \begin{cases} \text{curl} E_\omega^i = i\omega \mu H_\omega^i & \text{in } \Omega, \\ \text{curl} H_\omega^i = -i(\omega \varepsilon + i\sigma) E_\omega^i & \text{in } \Omega, \\ E_\omega^i \times \nu = \varphi_i \times \nu & \text{on } \partial\Omega, \end{cases}$$

and their derivatives up to the κ -th order satisfy some non-zero constraints inside the domain. Let $b \in \mathbb{N}^*$ be the number of illuminations and $r \in \mathbb{N}^*$ be the number of constraints. As before, these are described by a map ζ , which we now introduce. For $b, r \in \mathbb{N}^*$ let

$$(3.30a) \quad \zeta = (\zeta^1, \dots, \zeta^r): C^\kappa(\overline{\Omega}; \mathbb{C}^6)^b \longrightarrow C(\overline{\Omega}; \mathbb{C})^r \text{ be holomorphic, such that}$$

$$(3.30b) \quad \|\zeta((u^1, v^1), \dots, (u^b, v^b))\|_{C(\overline{\Omega}; \mathbb{C})^r} \leq c_\zeta (1 + \|((u^1, v^1), \dots, (u^b, v^b))\|_{C^\kappa(\overline{\Omega}; \mathbb{C}^6)^b})^s,$$

$$(3.30c) \quad \|D\zeta_{((u^1, v^1), \dots, (u^b, v^b))}\|_{\mathcal{B}(C^\kappa(\overline{\Omega}; \mathbb{C}^6)^b, C(\overline{\Omega}; \mathbb{C})^r)} \leq c_\zeta (1 + \|((u^1, v^1), \dots, (u^b, v^b))\|_{C^\kappa(\overline{\Omega}; \mathbb{C}^6)^b})^s$$

for some $c_\zeta > 0$ and $s \in \mathbb{N}^*$. For simplicity, we shall use the notation $C_\zeta = (c_\zeta, s, r, \kappa, p)$.

We now consider some examples of maps ζ 's. In Section 4.2, we shall see that these examples are motivated by some hybrid inverse problems.

Example 3.33. This example has already been introduced in Section 1.2. Take $b = 3$, $r = 1$, $\kappa = 0$ and let $\zeta_{\det}^M$ be defined by

$$\zeta_{\det}^M((u_1, v_1), (u_2, v_2), (u_3, v_3)) = \det \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix}, \quad (u_i, v_i) \in C(\overline{\Omega}; \mathbb{C}^6).$$

The map $\zeta_{\det}^M$ is multilinear and bounded, whence holomorphic. Assumptions (3.30b) and (3.30c) are obviously verified. In this case, the condition characterising $(\zeta_{\det}^M, C)$ -complete sets of measurements is

$$(3.31) \quad |\det [E_\omega^1 \quad E_\omega^2 \quad E_\omega^3](x)| \geq C.$$

In other words, (3.31) signals the availability, in every point, of three independent electric fields and, in particular, of one non-vanishing electric field.

Example 3.34. Take $b = 6$, $r = 1$ and $\kappa = 1$. First consider the function $\eta: C^1(\overline{\Omega}; \mathbb{C}^3)^2 \rightarrow C(\overline{\Omega}; \mathbb{C}^3)$ given by

$$\eta(u_1, u_2) = (\nabla u_1)u_2 - (\nabla u_2)u_1 + \operatorname{div} u_1 u_2 - \operatorname{div} u_2 u_1 - 2^t(\nabla u_1)u_2 + 2^t(\nabla u_2)u_1.$$

Define now $\zeta^{(2)}: C^1(\overline{\Omega}; \mathbb{C}^6)^6 \rightarrow C(\overline{\Omega}; \mathbb{C})$ by

$$\zeta^{(2)}((u_1, v_1), \dots, (u_6, v_6)) = \det \begin{bmatrix} \eta(u_1, u_2) & \eta(u_3, u_4) & \eta(u_5, u_6) \end{bmatrix}.$$

As before, the map $\zeta^{(2)}$ satisfies (3.30). The interpretation of the corresponding constraint is not immediate, and the reader is referred to § 4.2.1 for an application.

Example 3.35. Take $b = 6$, $r = 2$ and $\kappa = 1$. The map $\zeta^{(3)}$ considered here involves two conditions. The first one is given by $\zeta^{(2)}$, and the second one refers to the availability of one non-vanishing electric field. More precisely, we define $\zeta^{(3)}: C^1(\overline{\Omega}; \mathbb{C}^6)^6 \rightarrow C(\overline{\Omega}; \mathbb{C})^2$ by

$$\zeta^{(3)}((u_1, v_1), \dots, (u_6, v_6)) = (\zeta^{(2)}((u_1, v_1), \dots, (u_6, v_6)), (u_1)_2).$$

As before, the map $\zeta^{(3)}$ satisfies (3.30). The interpretation of the corresponding constraint is not immediate, and the reader is referred to § 4.2.2 for an application.

We now give the precise definition of (ζ, C) -complete sets of measurements for Maxwell's equations. The only difference with the Helmholtz equation (Definition 3.13) is that here, for simplicity, we require the constraints to hold in the whole domain Ω .

Definition 3.36. Let $b, r \in \mathbb{N}^*$ be two positive integers, $C > 0$ and let ζ be as in (3.30). A set of measurements $K \times \{\varphi_1, \dots, \varphi_b\}$ is (ζ, C) -complete if for every $x \in \overline{\Omega}$ there exists $\omega_x \in K$ such that

$$(3.32) \quad |\zeta^j((E_{\omega_x}^1, H_{\omega_x}^1), \dots, (E_{\omega_x}^b, H_{\omega_x}^b))(x)| \geq C, \quad x \in \overline{\Omega}, j = 1, \dots, r.$$

Let $K^{(n)}$ be as in (3.17). The main result of this section reads as follows.

Theorem 3.37. Assume that (3.28) holds. Let $\hat{\sigma} \in W^{\kappa, p}(\Omega; \mathbb{R}^{3 \times 3})$ be an arbitrary matrix valued function satisfying (3.28a). Let ζ be as in (3.30) and $\varphi_1, \dots, \varphi_b \in W^{\kappa+1, p}(\Omega; \mathbb{C}^3)$ satisfy (3.28c). Suppose that

$$(3.33) \quad |\zeta^j((\hat{E}_0^1, \hat{H}_0^1), \dots, (\hat{E}_0^b, \hat{H}_0^b))(x)| \geq C_0, \quad x \in \overline{\Omega}, j = 1, \dots, r$$

for some $C_0 > 0$, where $(\hat{E}_0^i, \hat{H}_0^i) \in H(\operatorname{curl}, \Omega) \times H^\mu(\operatorname{curl}, \Omega)$ is the solution to (3.29) with $\hat{\sigma}$ in lieu of σ and $\omega = 0$, namely

$$(3.34) \quad \begin{cases} \operatorname{curl} \hat{E}_0^i = 0 & \text{in } \Omega, \\ \operatorname{div}(\hat{\sigma} \hat{E}_0^i) = 0 & \text{in } \Omega, \\ \hat{E}_0^i \times \nu = \varphi_i \times \nu & \text{on } \partial\Omega, \end{cases} \quad \begin{cases} \operatorname{curl} \hat{H}_0^i = \hat{\sigma} \hat{E}_0^i & \text{in } \Omega, \\ \operatorname{div}(\mu \hat{H}_0^i) = 0 & \text{in } \Omega, \\ \mu \hat{H}_0^i \cdot \nu = 0 & \text{on } \partial\Omega. \end{cases}$$

There exist $\delta, C > 0$ and $n \in \mathbb{N}$ depending on Ω , Λ , $|\mathcal{A}|$, M , C_ζ , $\|(\varepsilon, \sigma, \mu)\|_{W^{\kappa+1, p}(\overline{\Omega}; \mathbb{R}^{3 \times 3})}$, $\|\varphi_i\|_{W^{\kappa+1, p}(\overline{\Omega}; \mathbb{C}^3)}$ and C_0 such that if $\|\sigma - \hat{\sigma}\|_{W^{\kappa+1, p}(\overline{\Omega}; \mathbb{R}^{3 \times 3})} \leq \delta$ then

$$K^{(n)} \times \{\varphi_1, \dots, \varphi_b\}$$

is (ζ, C) -complete.

Proof. The proof of this theorem is very similar to the proof of Corollary 3.19 in the case where (3.11) holds true. The results of Section 2.2 have to be used in place of the corresponding results of Section 2.1. The details are left to the reader. $\square$

In the following remark we discuss assumption (3.33) and the dependence of the construction of the illuminations on the electromagnetic parameters (cfr. Remark 3.16)

Remark 3.38. Suppose that we are in the simpler case $\hat{\sigma} = \sigma$. This result states that we can construct a ζ -complete set of measurements, if the illuminations $\varphi_1, \dots, \varphi_b$ are chosen in such a way that (3.33) holds true. It is in general much easier to satisfy (3.33) than (3.32), as $\omega = 0$ makes problem (3.29) simpler. Note that (3.34) does not depend on ε , so that the construction of the illuminations $\varphi_1, \dots, \varphi_b$ is always independent of ε . For the same reason, their construction may depend on σ and μ . However, in the cases where the maps ζ^j involve only the electric field E , it depends on σ , and not on ε and μ (see Corollary 3.40).

A typical application of the theorem is in the case where σ is a small perturbation of a known constant tensor $\hat{\sigma}$. Then, the construction of the illuminations $\varphi_1, \dots, \varphi_b$ is independent of σ (see Corollary 3.40). A similar argument would work if μ is a small perturbation of a constant tensor $\hat{\mu}$. We have decided to omit it for simplicity, since in the applications we have in mind the maps ζ^j do not depend on the magnetic field H , thereby making assumption (3.33) independent of μ .

Finally, we comment on the regularity assumption on the coefficients (cfr. Remark 3.18).

Remark 3.39. The regularity of the coefficients required for this approach is much lower than the regularity required if CGO solutions are used. Indeed, if the conditions depend on the derivatives up to the $(\kappa - 1)$ -th order, with this approach we require the parameters to be in $W^{\kappa,p}$ for some $p > 3$, while with CGO we need at least $W^{\kappa+2,p}$ [55].

In the case where the conditions given by the map ζ are independent of the magnetic field H , Theorem 3.37 can be rewritten in the following form.

Corollary 3.40. *Assume that (3.28) holds. Let $\hat{\sigma} \in W^{\kappa,p}(\Omega; \mathbb{R}^{3 \times 3})$ be an arbitrary matrix valued function satisfying (3.28a) and ζ be as in (3.30) and independent of H . Take $\psi_1, \dots, \psi_b \in W^{\kappa+2,p}(\Omega; \mathbb{C})$. Suppose that*

$$(3.35) \quad |\zeta^j(\nabla w^1, \dots, \nabla w^b)(x)| \geq C_0, \quad x \in \bar{\Omega}, j = 1, \dots, r$$

for some $C_0 > 0$, where $w^i \in H^1(\Omega; \mathbb{C})$ is the solution to

$$\begin{cases} \operatorname{div}(\hat{\sigma} \nabla w^i) = 0 & \text{in } \Omega, \\ w^i = \psi_i & \text{on } \partial\Omega. \end{cases}$$

There exist $\delta, C > 0$ and $n \in \mathbb{N}$ depending on Ω , Λ , $|\mathcal{A}|$, M , C_ζ , $\|(\varepsilon, \sigma, \mu)\|_{W^{\kappa+1,p}(\bar{\Omega}; \mathbb{R}^{3 \times 3})}$, $\|\psi_i\|_{W^{\kappa+2,p}(\bar{\Omega}; \mathbb{C}^3)}$ and C_0 such that if $\|\sigma - \hat{\sigma}\|_{W^{\kappa+1,p}(\bar{\Omega}; \mathbb{R}^{3 \times 3})} \leq \delta$ then

$$K^{(n)} \times \{\nabla \psi_1, \dots, \nabla \psi_b\}$$

is (ζ, C) -complete.

In other words, if the required constraints do not depend on H , then the problem of finding ζ -complete sets is completely reduced to satisfying the same conditions for the gradients of solutions to the conductivity equation, as in the Helmholtz case. This last problem has received a considerable attention in the literature (see Remark 3.16).

3.5 Additional results

3.5.1 A genericity property

In § 3.3.2 we showed that the construction of $(\zeta_{\det}, C)$ -complete sets of measurements in three dimensions requires the assumption $a \approx a_0$ for some constant matrix a_0 . As we saw before, the presence of critical points in the case $\omega = 0$ cannot be excluded in the three-dimensional case. As a result, it may be hard to satisfy assumption (3.18) if a is arbitrary [31, 32]. However, we shall show in this section that assumption (3.18) is not necessary, and that for generic boundary conditions the zero set still moves when the frequency varies. In other words, the occulting illuminations considered in Examples 3.30 and 3.31 are chosen in a residual set.

Let us consider problem (3.8) with $\sigma = 0$. We shall substitute $\omega^2 \rightarrow \omega$ in order to make some calculations simpler:

$$(3.36) \quad \begin{cases} -\operatorname{div}(a \nabla u_\omega^\varphi) - \omega \varepsilon u_\omega^\varphi = 0 & \text{in } \Omega, \\ u_\omega^\varphi = \varphi & \text{on } \partial\Omega. \end{cases}$$

The current technique does not allow to consider constraints such as the determinant. We restrict ourselves to consider the constraint

$$|\partial_{x_1} u_\omega^\varphi(x)| \geq C.$$

Namely, we study the construction of (ζ_{∂_1}, C) -complete sets of measurements, where $\zeta_{\partial_1}(u) = \partial_1 u$. This constraint is weaker than the one given by $\zeta_{\det}^2$ but stronger than requiring the absence of critical points.

For a C^2 bounded domain $\Omega \subseteq \mathbb{R}^3$, let $\operatorname{Diff}^2(\Omega)$ denote the open subset of $C^2(\Omega; \mathbb{R}^3)$ consisting of maps F which are diffeomorphisms to their images $\Omega^F := F(\Omega)$.

It is convenient to change the way frequencies are chosen in the admissible set $\mathcal{A}$. For $n \in \mathbb{N}^*$ let $\tilde{K}^{(n)} := K^{(2^n+1)}$ denote the uniform partition of $\mathcal{A}$ such that $\#\tilde{K}^{(n)} = 2^n + 1$. With this definition, there holds $\tilde{K}^{(n)} \subseteq \tilde{K}^{(n+1)}$.

The main result of this subsection reads as follows.

Theorem 3.41. *Let $a \in C^2(\mathbb{R}^3; \mathbb{R}^{3 \times 3})$ and $\varepsilon \in C^2(\mathbb{R}^3)$ be such that $a = a^T$ and (3.9) holds true and $\Omega \subseteq \mathbb{R}^3$ be a C^2 bounded domain. Take a generic $F \in \operatorname{Diff}^2(\Omega)$ and $\Omega' \Subset \Omega^F$. For a generic $\varphi \in C^2(\overline{\Omega^F}; \mathbb{R})$ there exist $n \in \mathbb{N}$ and $C > 0$ such that*

$$\tilde{K}^{(n)} \times \{\varphi\}$$

is a (ζ_{∂_1}, C) -complete set in Ω' for problem (3.36) in Ω^F .

Remark 3.42. In this case, we have no quantitative a priori estimates on n and C , in contrast to the results given in Sections 3.3 and 3.4. This is natural and unavoidable with generic boundary conditions. Indeed, we have shown in Examples 3.30 and 3.31 that there are occulting illuminations for which the result would not be true. As a consequence, as the chosen generic illumination approaches an occulting illumination, $n \rightarrow \infty$ and $C \rightarrow 0$.

The rest of this subsection is devoted to the proof of this theorem.

It can be easily checked that Proposition 2.7 still holds for problem (3.36), so that the map $\omega \in \mathbb{C} \setminus \Sigma \mapsto u_\omega^\varphi \in C^1(\overline{\Omega}; \mathbb{C})$ is holomorphic (see [1]).

As in Proposition 2.1, we consider the system of eigenvalues $\Sigma = \{\lambda_l\}_l$ and eigenfunctions $\{z_l\}_l$ given by

$$(3.37) \quad -\operatorname{div}(a \nabla z_l) = \lambda_l \varepsilon z_l, \quad z_l \in H_0^1(\Omega; \mathbb{R}),$$

subject to the condition $\|z_l\|_{L^2(\Omega; \mathbb{R})} := \int_\Omega \varepsilon z_l^2 dx = 1$. Classical elliptic theory [62] gives that $\{z_l\}$ is an orthonormal basis of $L^2(\Omega; \mathbb{R})$ and an orthogonal basis of $H_0^1(\Omega; \mathbb{C})$. Moreover, Proposition 2.5 yields $z_l \in C^1(\overline{\Omega}; \mathbb{R})$ and

$$(3.38) \quad \|z_l\|_{C^1(\overline{\Omega})} \leq c \lambda_l^P$$

for some $P, c > 0$ depending on Ω and Λ only.

We now introduce a good set of boundary conditions:

$$B(\Omega) = \{\varphi \in C^2(\overline{\Omega}; \mathbb{R}) : \text{for all } x \in \Omega \text{ there exists } \omega \in \mathbb{C} \setminus \Sigma \text{ such that } \partial_{x_1} u_\omega^\varphi(x) \neq 0\}.$$

In the following result we show that if φ is chosen in $B(\Omega)$ then a finite number of frequencies are sufficient to obtain measurements with a non-vanishing first partial derivative.

Proposition 3.43. *Let $\Omega \subseteq \mathbb{R}^3$ be a C^2 bounded domain and $\Omega' \Subset \Omega$. Take $\varphi \in B(\Omega)$. Then there exists $n \in \mathbb{N}$ and $C > 0$ such that $\tilde{K}^{(n)} \times \{\varphi\}$ is a (ζ_{∂_1}, C) -complete set of measurements in Ω' .*

Proof. By construction of $\tilde{K}^{(n)}$, it is possible to choose $\omega_n \in \tilde{K}^{(n)} \setminus \Sigma$ and $\omega \in \mathcal{A} \setminus \Sigma$ such that $\omega_n \rightarrow \omega$ and $\omega_n \neq \omega$ for all $n \in \mathbb{N}$. Indeed, it is enough to choose $\omega \in \mathcal{A} \setminus \cup_n \tilde{K}^{(n)}$ and use the fact that $\cup_n \tilde{K}^{(n)} = \mathcal{A}$. For $x \in \overline{\Omega'}$ the map $\omega \in \mathbb{C} \setminus \Sigma \mapsto \partial_{x_1} u_\omega^\varphi(x) \in \mathbb{C}$ is holomorphic and non-zero, as $\varphi \in B(\Omega)$. Therefore by the analytic continuation theorem there exists $n_x \in \mathbb{N}$ such that $\partial_{x_1} u_{\omega_{n_x}}^\varphi(x) \neq 0$. Since $\partial_{x_1} u_{\omega_{n_x}}^\varphi$ is continuous, there exists $r_x > 0$ such that

$$(3.39) \quad \partial_{x_1} u_{\omega_{n_x}}^\varphi(y) \neq 0, \quad y \in B(x, r_x).$$

Since $\overline{\Omega'} \subseteq \cup_{x \in \overline{\Omega'}} B(x, r_x)$ and $\overline{\Omega'}$ is compact, there exist $x_1, \dots, x_N \in \overline{\Omega'}$ such that $\overline{\Omega'} \subseteq \cup_{i=1}^N B(x_i, r_{x_i})$. Therefore, choosing $n = \max\{n_{x_1}, \dots, n_{x_N}\}$, in view of $\tilde{K}^{(n)} \subseteq \tilde{K}^{(n+1)}$ and (3.39) we obtain

$$\sum_{\omega \in \tilde{K}^{(n)} \setminus \Sigma} |\partial_{x_1} u_\omega^\varphi(x)| > 0, \quad x \in \overline{\Omega'}.$$

By using the continuity of $\partial_{x_1} u_\omega^\varphi$ we obtain that $\sum_{\omega \in \tilde{K}^{(n)} \setminus \Sigma} |\partial_{x_1} u_\omega^\varphi(x)| \geq C > 0$ for all $x \in \overline{\Omega'}$ for some $C > 0$, which gives the result. $\square$

We now need a preliminary result that follows from the pointwise convergence of the Fourier series associated to the eigenfunctions z_l of a compactly supported smooth function.

Lemma 3.44. *Let $\Omega \subseteq \mathbb{R}^3$ be a C^2 bounded domain and take $x_0 \in \Omega$. Then there exists $l^* \in \mathbb{N}$ such that $\partial_{x_1} z_{l^*}(x_0) \neq 0$.*

Proof. It is enough to show that if $g \in C_0^\infty(\Omega; \mathbb{R})$ then $g = \sum_l (g, z_l)_{L^2(\Omega; \mathbb{R})} z_l$ converges in $C^1(\overline{\Omega}; \mathbb{R})$. Indeed, then take $g \in C_0^\infty(\Omega; \mathbb{R})$ such that $\partial_{x_1} g(x_0) \neq 0$.

Define $g_n = \sum_{l \geq n} (g, z_l)_{L^2(\Omega; \mathbb{R})} z_l \in H_0^1(\Omega; \mathbb{R})$. We shall show that $g_n \rightarrow 0$ in $C^1(\overline{\Omega}; \mathbb{R})$. Let $f \in H_0^1(\Omega; \mathbb{R})$ such that $-\operatorname{div}(a \nabla g) = \varepsilon f$ in Ω . Since $\lambda_l(g, z_l)_{L^2(\Omega; \mathbb{R})} = (f, z_l)_{L^2(\Omega; \mathbb{R})}$, we have

$$-\operatorname{div}(a \nabla g_n) = \varepsilon f_n \quad \text{in } \Omega,$$

where $f_n = \sum_{l \geq n} (f, z_l)_{L^2(\Omega; \mathbb{R})} z_l$. Since $f \in H_0^1(\Omega; \mathbb{R})$, then $f_n \rightarrow 0$ in $H_0^1(\Omega; \mathbb{R})$. By the regularity assumption on a and ε , this implies $g_n \rightarrow 0$ in $H^3(\Omega)$. Finally, the result follows by the Sobolev embedding theorem. $\square$

Let us introduce the set of boundary conditions

$$O_l(\Omega) = \{v \in C^2(\overline{\Omega}; \mathbb{R}) : (v, \psi_l)_{L^2(\partial\Omega)} \neq 0\},$$

for any $l \in \mathbb{N}^*$, where $\psi_l = (a \nabla z_l) \cdot \nu \in C^0(\partial\Omega; \mathbb{R})$.

Lemma 3.45. *Let $\Omega \subseteq \mathbb{R}^3$ be a C^2 bounded domain and take $l \in \mathbb{N}^*$.*

1. *The set $O_l(\Omega)$ is open and dense in $C^2(\overline{\Omega}; \mathbb{R})$ for every $l \in \mathbb{N}^*$.*
2. *There exists $C > 0$ depending on Ω and Λ such that $\|\psi_l\|_{H^{-1/2}(\partial\Omega; \mathbb{R})} \leq C \lambda_l$.*

Proof. 1. Since the map $v \mapsto (v, \psi_l)_{L^2(\partial\Omega)}$ is continuous, the set $O_l(\Omega)$ is open. We now show that it is dense. Take $v \in C^2(\overline{\Omega}; \mathbb{R}) \setminus O_l(\Omega)$. By unique continuation for the Cauchy problem for elliptic equations [12, Theorem 1.7], we have that $\psi_l \neq 0$. Therefore there exists $\psi \in C^2(\overline{\Omega}; \mathbb{R})$ such that $(\psi, \psi_l)_{L^2(\partial\Omega)} \neq 0$. As a consequence, $v_n = v + \psi/n \in O_l(\Omega)$ and $v_n \rightarrow v$ in $C^2(\overline{\Omega}; \mathbb{R})$.

2. Let $\varphi \in H^{1/2}(\partial\Omega; \mathbb{R})$ and take $v \in H^1(\Omega; \mathbb{R})$ such that $\Delta v = 0$ and $v = \varphi$ on $\partial\Omega$. Simple integrations by parts yield $-\langle \psi_l, \varphi \rangle_{H^{-1/2}(\partial\Omega; \mathbb{R}) \times H^{1/2}(\partial\Omega; \mathbb{R})} = \lambda_l (z_l, v)_{L^2(\Omega; \mathbb{R})}$, whence

$$|\langle \psi_l, \varphi \rangle_{H^{-1/2}(\partial\Omega; \mathbb{R}) \times H^{1/2}(\partial\Omega; \mathbb{R})}| \leq \lambda_l \|z_l\|_{L^2(\Omega; \mathbb{R})} \|\varphi\|_{H^{1/2}(\partial\Omega; \mathbb{R})} \leq c(\Omega, \Lambda) \lambda_l \|\varphi\|_{H^{1/2}(\partial\Omega; \mathbb{R})},$$

as desired. $\square$

Since the map $\omega \mapsto u_\omega^\varphi$ is holomorphic, by differentiating the equation satisfied by u_ω^φ it can be easily proven by induction that for all $m \in \mathbb{N}^*$

$$(3.40) \quad \begin{cases} -\operatorname{div}(a \nabla (\partial_\omega^m u_\omega^\varphi)) - \omega \varepsilon \partial_\omega^m u_\omega^\varphi = m \varepsilon \partial_\omega^{m-1} u_\omega^\varphi & \text{in } \Omega, \\ \partial_\omega^m u_\omega^\varphi = 0 & \text{on } \partial\Omega. \end{cases}$$

We shall now show that if the eigenvalues λ_l are all simple then the set $B(\Omega)$ is generic.

Proposition 3.46. *Let $\Omega \subseteq \mathbb{R}^3$ be a C^2 bounded domain. The set $G(\Omega) := \cap_l O_l(\Omega)$ is generic in $C^2(\overline{\Omega}; \mathbb{R})$. Moreover, if the eigenvalues λ_l are all simple then $G(\Omega) \subseteq B(\Omega)$.*

Proof. The first part immediately follows from Lemma 3.45, part 1.

Let us now prove the second part. Take $\varphi \in G(\Omega)$. Hence

$$(3.41) \quad (\varphi, \psi_l)_{L^2(\partial\Omega)} \neq 0, \quad l \in \mathbb{N}^*.$$

By contradiction, assume that $\varphi \notin B(\Omega)$, i.e. there exists $x_0 \in \Omega$ such that $\partial_{x_1} u_\omega^\varphi(x_0) = 0$ for every $\omega \in \mathbb{C} \setminus \Sigma$. Therefore for any $m \in \mathbb{N}$ there holds

$$(3.42) \quad \partial_{x_1}(\partial_\omega^m u_\omega^\varphi)(x_0) = 0, \quad \omega \in \mathbb{C} \setminus \Sigma.$$

By Lemma 3.44, there exists $l^* \in \mathbb{N}^*$ such that $\partial_{x_1} z_{l^*}(x_0) \neq 0$. Moreover, from (3.36) and (3.40) we obtain

$$(3.43) \quad (\partial_\omega^m u_\omega^\varphi, z_l)_{L^2(\Omega; \mathbb{R})} = \frac{m!}{(\lambda_l - \omega)^m} (u_\omega^\varphi, z_l)_{L^2(\Omega; \mathbb{R})} = (-1)^m \frac{m!}{(\lambda_l - \omega)^{m+1}} (\varphi, \psi_l)_{L^2(\partial\Omega; \mathbb{R})}.$$

Therefore, for any $\omega \in B(0, \lambda_{l^*}) \setminus \Sigma$ and $l > l^*$ there holds

$$\begin{aligned} \|(\partial_\omega^m u_\omega^\varphi, z_l)_{L^2(\Omega; \mathbb{R})} z_l\|_{C^1(\overline{\Omega}; \mathbb{R})} &\leq c(\Omega, \Lambda) |(\partial_\omega^m u_\omega^\varphi, z_l)_{L^2(\Omega; \mathbb{R})}| \lambda_l^P \\ &\leq c(\Omega, \Lambda, m) \frac{|\langle \psi_l, \varphi \rangle_{H^{-1/2}(\partial\Omega; \mathbb{R}) \times H^{1/2}(\partial\Omega; \mathbb{R})}| \lambda_l^P}{|\lambda_l - \omega|^{m+1}} \\ &\leq c(\Omega, \Lambda, m, \varphi) \frac{\lambda_l^{P+1}}{|\lambda_l - \omega|^{m+1}} \\ &\leq c(\Omega, \Lambda, m, \varphi) \frac{\lambda_l^{P+1}}{(\lambda_l - \lambda_{l^*})^{m+1}} \\ &\leq c(\Omega, \Lambda, m, \varphi) \frac{l^{2(P+1)/3}}{(l^{2/3} - \lambda_{l^*})^{m+1}}, \end{aligned}$$

where the first inequality follows from (3.38), the third inequality from Lemma 3.45, part 2, and the fifth from Lemma 3.7. As a consequence, we can choose m large enough such that the Fourier series $\partial_\omega^m u_\omega^\varphi = \sum_l (\partial_\omega^m u_\omega^\varphi, z_l)_{L^2(\Omega; \mathbb{R})} z_l$ converges in $C^1(\overline{\Omega})$. As a result, combining (3.42) and (3.43) we have $0 = \sum_l \frac{1}{(\lambda_l - \omega)^{m+1}} (\varphi, \psi_l)_{L^2(\partial\Omega)} \partial_{x_1} z_l(x_0)$, whence

$$-\frac{(\varphi, \psi_{l^*})_{L^2(\partial\Omega)}}{(\lambda_{l^*} - \omega)^{m+1}} \partial_{x_1} z_{l^*}(x_0) = \sum_{l \neq l^*} \frac{1}{(\lambda_l - \omega)^{m+1}} (\varphi, \psi_l)_{L^2(\partial\Omega)} \partial_{x_1} z_l(x_0),$$

for all $\omega \in B(0, \lambda_{l^*}) \setminus \Sigma$. By assumption, $\lambda_l \neq \lambda_{l^*}$ for every $l \neq l^*$. As a consequence, since $\partial_{x_1} z_{l^*}(x_0) \neq 0$, letting ω go to λ_{l^*} we obtain $(\varphi, \psi_{l^*})_{L^2(\partial\Omega)} = 0$, which contradicts (3.41). $\square$

Remark 3.47. The assumption on the simplicity of the eigenvalues seems essential, or at least not easily removable. In the general case, the set $G(\Omega)$ would be

$$G(\Omega) = \bigcap_{x_0 \in \Omega} \{ \varphi \in C^2(\overline{\Omega}; \mathbb{R}) : \text{there exists } l^* \text{ such that } \sum_{\alpha} (\varphi, \psi_{l^*}^\alpha)_{L^2(\partial\Omega)} \partial_{x_1} z_{l^*}^\alpha(x_0) \neq 0 \},$$

where α identifies all the eigenfunctions corresponding to the same eigenvalue. Therefore, the set $G(\Omega)$ does not seem to be generic at a first glance.

For general domains Ω , the eigenvalues λ_l may have multiplicity bigger than one. However, this is a consequence of special symmetries in Ω and in the parameters. Indeed, for a generic domain all the eigenvalues are simple.

Proposition 3.48 ([73, Example 6.3]). *Let $\Omega \subseteq \mathbb{R}^3$ be a C^2 bounded domain. For a generic $F \in \text{Diff}^2(\Omega)$, all the eigenvalues of (3.37) in Ω^F are simple.*

We are now ready to prove Theorem 3.41.

Proof of Theorem 3.41. The result immediately follows from Propositions 3.43, 3.46 and 3.48. $\square$

3.5.2 On the number of needed frequencies with real analytic coefficients

In this section we show that, in the case of real analytic coefficients, we can give a precise estimate on the number n of needed frequencies in Theorems 3.15 and 3.37.

3.5.2.1 The Helmholtz equation

We first study the Helmholtz equation

$$(3.44) \quad \begin{cases} -\text{div}(a \nabla u_\omega^i) - (\omega^2 \varepsilon + i\omega \sigma) u_\omega^i = 0 & \text{in } \Omega, \\ u_\omega^i = \varphi_i & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subseteq \mathbb{R}^d$ is a smooth bounded domain and a, ε and σ satisfy (3.9), (3.12) and either (3.10) or (3.11) and are real analytic in Ω , namely

$$(3.45) \quad a \in C^\omega(\Omega; \mathbb{R}^{d \times d}), \quad \varepsilon, \sigma \in C^\omega(\Omega; \mathbb{R}).$$

This assumption guarantees that $u_\omega^i \in C^\omega(\Omega; \mathbb{C})$. Define D as in (3.13), so that (3.44) is well-defined for $\omega \in D$.

Lemma 3.49 ([87]). *Assume that (3.9), (3.45) and either (3.10) or (3.11) hold true. If $\omega \in D$ and $\varphi_i \in H^1(\Omega; \mathbb{R})$ then $u_\omega^i \in C^\omega(\Omega; \mathbb{C})$.*

We shall consider (ζ, C) -complete sets of measurements, where for $\kappa \in \mathbb{N}$ and $b, r \in \mathbb{N}^*$ we let

$$(3.46a) \quad \zeta = (\zeta^1, \dots, \zeta^r): C^\kappa(\overline{\Omega}; \mathbb{C})^b \longrightarrow C(\overline{\Omega}; \mathbb{C})^r \text{ be holomorphic, such that}$$

$$(3.46b) \quad \zeta(C^\omega(\Omega; \mathbb{C})^b) \subseteq C^\omega(\Omega; \mathbb{C})^r.$$

Note that these assumptions are slightly different from the ones considered before and are satisfied for all the maps ζ we have in mind (see Section 3.3). The main result of this subsection concerning the Helmholtz equation reads as follows.

Theorem 3.50. *Assume that (3.9), (3.12), (3.45) and either (3.10) or (3.11) hold true. Suppose that $\mathcal{A} = [K_{\min}, K_{\max}] \subseteq D$ and take ζ as in (3.46). Let $\varphi_1, \dots, \varphi_b \in H^1(\Omega; \mathbb{R})$ be such that*

$$(3.47a) \quad \zeta^j(u_\omega^1, \dots, u_\omega^b) \neq 0, \quad j = 1, \dots, r, \omega \in D,$$

$$(3.47b) \quad \zeta^j(u_0^1, \dots, u_0^b)(x) \neq 0, \quad j = 1, \dots, r, x \in \Omega.$$

If $\Omega' \Subset \Omega$ then

$$\left\{ (\omega_1, \dots, \omega_{d+1}) \in \mathcal{A}^{d+1} : \{\omega_p\}_p \times \{\varphi_i\}_i \text{ is } (\zeta, C)\text{-complete in } \Omega' \text{ for some } C > 0 \right\}$$

is open and dense in $\mathcal{A}^{d+1}$.

Note that assumption (3.47b) corresponds to (3.18), and (3.47a) simply represents the non-triviality of ζ^j .

Remark 3.51. This result shows that almost any $d+1$ frequencies in $\mathcal{A}$ give a (ζ, C) -complete set for some $C > 0$. An a priori estimate on C cannot be obtained by choosing frequencies in an open and dense set in $\mathcal{A}^{d+1}$. Indeed, $C \rightarrow 0$ when the frequencies are chosen near the residual set.

Remark 3.52. The assumption $\Omega' \Subset \Omega$ is made only for simplicity, and the result is still true if $\Omega' = \Omega$. However, the proof would be slightly more involved, since we would need to study the analyticity of u_ω^i up to the boundary. Such a level of sophistication was unjustified in this work, since assumption (3.45) is far too strong for the applications, and it is expected that it can be considerably weakened (see Chapter 5). However, the real analyticity assumption is crucial for this proof, that is based on the structure of analytic varieties.

We now discuss the corresponding result for Maxwell's equations. The proof of these two results will be given at the end of this subsection and will follow from a general result that does not involve partial differential equations.

3.5.2.2 The Maxwell's equations

Let $\Omega \subseteq \mathbb{R}^3$ be a smooth bounded domain and consider problem

$$(3.48) \quad \begin{cases} \operatorname{curl} E_\omega^i = \mathbf{i}\omega\mu H_\omega^i & \text{in } \Omega, \\ \operatorname{curl} H_\omega^i = -\mathbf{i}(\omega\varepsilon + \mathbf{i}\sigma)E_\omega^i & \text{in } \Omega, \\ E_\omega^i \times \nu = \varphi_i \times \nu & \text{on } \partial\Omega, \end{cases}$$

where μ, ε, σ and φ_i satisfy (3.28) and

$$(3.49) \quad \mu, \varepsilon, \sigma \in C^\omega(\Omega; \mathbb{R}^{3 \times 3}).$$

This assumption guarantees that $(E_\omega^i, H_\omega^i) \in C^\omega(\Omega; \mathbb{C}^6)$. Define $D = \{\omega \in \mathbb{C} : |\Im\omega| < \eta, |\omega| < M\}$ as in Proposition 2.16, so that (3.48) is well-defined for $\omega \in D$.

Lemma 3.53. *Assume that (3.28) and (3.49) hold true. If $\omega \in D$ then $(E_\omega^i, H_\omega^i) \in C^\omega(\Omega; \mathbb{C}^6)$.*

Proof. In view of Proposition 2.19, (E_ω^i, H_ω^i) are solutions to the following elliptic system

$$\begin{cases} -\operatorname{div}(q_\omega \nabla (E_\omega^i)_k) = \operatorname{div}((\partial_k q_\omega) E_\omega^i - \mathbf{i} \omega q_\omega (\mathbf{e}_k \times \mu H_\omega^i)) & \text{in } \Omega, \\ -\operatorname{div}(\mu \nabla (H_\omega^i)_k) = \operatorname{div}((\partial_k \mu) H_\omega^i - \mathbf{i} \mu (\mathbf{e}_k \times q_\omega E_\omega^i)) & \text{in } \Omega. \end{cases}$$

whose coefficients are real analytic functions. Therefore the result follows from [87]. An alternative proof is given in [63]. $\square$

We shall consider (ζ, C) -complete sets of measurements, where for $\kappa \in \mathbb{N}$ and $b, r \in \mathbb{N}^*$

$$(3.50) \quad \zeta : C^\kappa(\overline{\Omega}; \mathbb{C}^6)^b \longrightarrow C(\overline{\Omega}; \mathbb{C})^r \text{ is holomorphic and } \zeta(C^\omega(\Omega; \mathbb{C}^6)^b) \subseteq C^\omega(\Omega; \mathbb{C})^r.$$

Note that these assumptions are satisfied for all the maps ζ we have in mind (see Section 3.4). The main result of this subsection concerning the Maxwell's equations reads as follows.

Theorem 3.54. *Assume that (3.28) and (3.49) hold true and take ζ as in (3.50). Let $\varphi_1, \dots, \varphi_b$ be as in (3.28c) and such that*

$$\begin{aligned} \zeta^j((E_\omega^1, H_\omega^1), \dots, (E_\omega^b, H_\omega^b)) &\neq 0, & j = 1, \dots, r, \omega \in D, \\ \zeta^j((E_0^1, H_0^1), \dots, (E_0^b, H_0^b))(x) &\neq 0, & j = 1, \dots, r, x \in \Omega. \end{aligned}$$

If $\Omega' \Subset \Omega$ then

$$\left\{ (\omega_1, \dots, \omega_{d+1}) \in \mathcal{A}^{d+1} : \{\omega_p\}_p \times \{\varphi_i\}_i \text{ is } (\zeta, C)\text{-complete in } \Omega' \text{ for some } C > 0 \right\}$$

is open and dense in $\mathcal{A}^{d+1}$.

Analogous comments to those given in Remarks 3.51 and 3.52 are applicable also in this context.

3.5.2.3 Proof of Theorems 3.50 and 3.54

Theorems 3.50 and 3.54 will be a consequence of the following result.

Proposition 3.55. *Let $\Omega, \Omega' \subseteq \mathbb{R}^d$ be smooth domains such that $\Omega' \Subset \Omega$. Let $D \subseteq \mathbb{C}$ be an open set such that $0 \in D$ and $\mathcal{A} \subseteq D$. Consider a map*

$$\theta : D \rightarrow C^\omega(\Omega; \mathbb{C}) \setminus \{0\}, \quad \omega \mapsto \theta_\omega$$

such that for all $x \in \Omega$, $\omega \in D \mapsto \theta_\omega(x) \in \mathbb{C}$ is holomorphic and $\theta_0(x) \neq 0$. The set

$$\left\{ (\omega_1, \dots, \omega_{d+1}) \in \mathcal{A}^{d+1} : \min_{\Omega'} (|\theta_{\omega_1}| + \dots + |\theta_{\omega_{d+1}}|) > 0 \right\}$$

is open and dense in $\mathcal{A}^{d+1}$.

We first show how to prove Theorems 3.50 and 3.54 from this result.

Proof of Theorems 3.50 and 3.54 . We prove only Theorem 3.50, the proof of Theorem 3.54 is very similar, and the details are left to the reader.

Define θ by

$$\theta_\omega = \prod_{j=1}^r \zeta^j(u_\omega^1, \dots, u_\omega^b).$$

In view of Lemma 3.49 and (3.46b) the map $\theta: D \rightarrow C^\omega(\Omega; \mathbb{C})$ is well-defined and by Proposition 2.7 and (3.46a), we obtain that $\omega \in D \mapsto \theta_\omega(x) \in \mathbb{C}$ is holomorphic for all $x \in \Omega$. Suppose now that $\theta_\omega = 0$ for some $\omega \in D$. Since $\zeta^j(u_\omega^1, \dots, u_\omega^b)$ is real analytic in Ω , this implies that $\zeta^j(u_\omega^1, \dots, u_\omega^b) = 0$ for some j , which contradicts (3.47a). As a consequence, we have that $\theta_\omega \neq 0$ for all $\omega \in D$. Moreover, $\theta_0(x) \neq 0$ for all $x \in \Omega$ by (3.47b). We can apply Proposition 3.55 and obtain that

$$\left\{ (\omega_1, \dots, \omega_{d+1}) \in \mathcal{A}^{d+1} : \min_{\overline{\Omega'}} \sum_{p=1}^{d+1} \prod_{j=1}^r |\zeta^j(u_{\omega_p}^1, \dots, u_{\omega_p}^b)| > 0 \right\}$$

is open and dense in $\mathcal{A}^{d+1}$. Note that the condition $\min_{\overline{\Omega'}} \sum_{p=1}^{d+1} \prod_{j=1}^r |\zeta^j(u_{\omega_p}^1, \dots, u_{\omega_p}^b)| > 0$ is equivalent to

$$\text{for all } x \in \overline{\Omega'} \text{ there exists } p \text{ such that } \prod_{j=1}^r |\zeta^j(u_{\omega_p}^1, \dots, u_{\omega_p}^b)| > 0,$$

namely

$$\text{for all } x \in \overline{\Omega'} \text{ there exists } p \text{ such that for all } j = 1, \dots, r, |\zeta^j(u_{\omega_p}^1, \dots, u_{\omega_p}^b)| > 0.$$

This last condition is equivalent to the fact that $\{\omega_p\}_p \times \{\varphi_i\}_i$ is a (ζ, C) -complete set of measurements in Ω' for some $C > 0$, as $\zeta^j(u_{\omega_p}^1, \dots, u_{\omega_p}^b)$ is continuous. This concludes the proof. $\square$

The rest of this subsection is devoted to the proof of Proposition 3.55, which based on the structure of analytic varieties.

An *analytic variety in Ω* is the set of common zeros of a finite collection of real analytic functions in Ω , namely $\{x \in \Omega : f_1(x) = \dots = f_N(x) = 0\}$, for some $f_1, \dots, f_N \in C^\omega(\Omega; \mathbb{C})$. For $\omega_1, \dots, \omega_N \in \mathcal{A}$ we shall consider the analytic variety

$$Z(\omega_1, \dots, \omega_N) = \{x \in \Omega : \theta_{\omega_1}(x) = \dots = \theta_{\omega_N}(x) = 0\} = \bigcap_{i=1}^N Z(\omega_i).$$

Analytic varieties can be stratified into submanifolds of different dimensions.

Lemma 3.56 ([45, Corollary 2.11]). *Let X be an analytic variety in Ω . There exists a locally finite collection $\{A_l\}_l$ of pairwise disjoint connected analytic submanifolds of Ω such that*

$$X = \bigcup_l A_l.$$

The decomposition $X = \cup_l A_l$ is called *stratification* of X . With this in mind, we can define the dimension of an analytic variety $X = \cup_l A_l$ by

$$(3.52) \quad \dim X := \max_l \dim A_l.$$

The main result leading to the proof of Proposition 3.55 is the following

Lemma 3.57. *Under the hypotheses of Proposition 3.55, let Ω'' be a smooth domain such that $\Omega' \Subset \Omega'' \Subset \Omega$ and X be an analytic variety in Ω such that $X \cap \Omega'' \neq \emptyset$. Then the set*

$$\{\omega \in \mathcal{A} : \dim(Z(\omega) \cap X \cap \Omega'') = \dim(X \cap \Omega'')\}$$

is finite.

Proof. By contradiction, suppose that the set is infinite. Since $\mathcal{A}$ is compact, there exist $\omega_n, \omega \in \mathcal{A}$, $\omega_n \rightarrow \omega$ such that $\dim(Z(\omega_n) \cap X \cap \Omega'') = \dim(X \cap \Omega'')$ and $\omega_n \neq \omega$ for all $n \in \mathbb{N}$. Therefore, in view of (3.52), there exist connected analytic submanifolds S_n such that

$$(3.53) \quad S_n \subseteq Z(\omega_n) \cap X \cap \Omega''$$

and

$$(3.54) \quad \dim S_n = \dim(X \cap \Omega'').$$

Since $X \cap \Omega'' \neq \emptyset$, we can take arbitrary $x_n \in S_n$ for all $n \in \mathbb{N}$. Up to a subsequence, we have $x_n \rightarrow x$, for some $x \in \overline{\Omega''}$. By Lemma 3.56 applied to X , there exists an open neighbourhood U of x and a finite collection $\{A_l\}_l$ of analytic submanifolds of Ω such that

$$X \cap U = \cup_l A_l.$$

Therefore, as $S_n \subseteq X$, up to a subsequence we have

$$(3.55) \quad S_n \cap U \subseteq A_{\bar{l}} \cap \Omega'', \quad n \in \mathbb{N},$$

for some $\bar{l}$. Moreover, since $x_n \in S_n$ and $x_n \rightarrow x$, up to a subsequence we have $S_n \cap U \neq \emptyset$ for all $n \in \mathbb{N}$. Thus, by (3.54), we obtain $\dim(S_n \cap U) = \dim(X \cap \Omega'')$ for all $n \in \mathbb{N}$, whence by (3.55)

$$(3.56) \quad \dim(S_n \cap U) = \dim A_{\bar{l}}, \quad n \in \mathbb{N}.$$

In view of (3.53) we have $\theta_{\omega_n}(y) = 0$ for all $y \in S_n \cap U$. Therefore, by (3.55), (3.56), [61, Theorem 1.2] and $\theta_\omega \in C^\omega(\Omega; \mathbb{C})$ we obtain

$$A_{\bar{l}} \subseteq Z(\omega_n), \quad n \in \mathbb{N}.$$

Moreover, since $x_n \in A_{\bar{l}}$ for all n , we have $x \in \overline{A_{\bar{l}}}$. As a consequence, as $Z(\omega_n)$ is closed, we infer that $x \in Z(\omega_n)$ for all $n \in \mathbb{N}$, namely $\theta_{\omega_n}(x) = 0$ for all $n \in \mathbb{N}$. Since $\omega \mapsto \theta_\omega(x)$ is holomorphic, this implies $\theta_0(x) = 0$, which contradicts the assumptions. $\square$

We are now in a position to prove Proposition 3.55.

Proof of Proposition 3.55. Let $G = \{(\omega_1, \dots, \omega_{d+1}) \in \mathcal{A}^{d+1} : \min_{\overline{\Omega'}}(|\theta_{\omega_1}| + \dots + |\theta_{\omega_{d+1}}|) > 0\}$. Since the map $(\omega_1, \dots, \omega_{d+1}) \mapsto \min_{\overline{\Omega'}}(|\theta_{\omega_1}| + \dots + |\theta_{\omega_{d+1}}|)$ is continuous, G is open. It remains to show that G is dense in $\mathcal{A}^{d+1}$.

Take $(\tilde{\omega}_1, \dots, \tilde{\omega}_{d+1}) \in \mathcal{A}^{d+1}$ and $\varepsilon > 0$ and let Ω'' be as in Lemma 3.57. We equip $\mathcal{A}^{d+1}$ with the norm

$$\|(\omega_1, \dots, \omega_{d+1})\| = \max_p |\omega_p|.$$

We now want to construct an element $(\omega_1, \dots, \omega_{d+1}) \in G$ such that

$$(3.57) \quad \|(\omega_1, \dots, \omega_{d+1}) - (\tilde{\omega}_1, \dots, \tilde{\omega}_{d+1})\| < \varepsilon.$$

Set $\omega_1 = \tilde{\omega}_1$; obviously we have $|\omega_1 - \tilde{\omega}_1| < \varepsilon$. Suppose now that we have constructed $\omega_1, \dots, \omega_p$ such that $|\omega_j - \tilde{\omega}_j| < \varepsilon$ for all $j = 1, \dots, p$. Let us describe how to construct ω_{p+1} . If $Z(\omega_1, \dots, \omega_p) \cap \Omega'' = \emptyset$, then it is enough to choose $\omega_{p+1} = \tilde{\omega}_{p+1}$. Otherwise, applying Lemma 3.57 with $X = Z(\omega_1, \dots, \omega_p)$, we obtain that the set

$$\{\omega \in \mathcal{A} : \dim(Z(\omega) \cap Z(\omega_1, \dots, \omega_p) \cap \Omega'') = \dim(Z(\omega_1, \dots, \omega_p) \cap \Omega'')\}$$

is finite. Therefore, we can choose $\omega_{p+1} \in \mathcal{A}$ such that

$$\dim(Z(\omega_1, \dots, \omega_{p+1}) \cap \Omega'') < \dim(Z(\omega_1, \dots, \omega_p) \cap \Omega'')$$

and $|\omega_{p+1} - \tilde{\omega}_{p+1}| < \varepsilon$. Therefore, as by $\theta_{\omega_1} \in C^\omega(\Omega; \mathbb{C}) \setminus \{0\}$ we have $\dim Z(\omega_1) \leq d-1$, we obtain $\dim(Z(\omega_1, \dots, \omega_{d+1}) \cap \Omega'') < 0$, namely $Z(\omega_1, \dots, \omega_{d+1}) \cap \Omega'' = \emptyset$. In other words, $(\omega_1, \dots, \omega_{d+1}) \in G$. By construction, (3.57) is satisfied. This concludes the proof. $\square$

3.5.3 Numerical experiments on the number of needed frequencies

In this subsection we will describe some numerical experiments to investigate the number of needed frequencies with non-analytic coefficients. FreeFem++ has been used [72, 71] (see Listing A.2 in Appendix A for the codes).

For simplicity, we consider the Helmholtz equation in 2D with $\sigma = 0$ and the map $\zeta_\times$ given by $b = 3$, $r = 2$ and

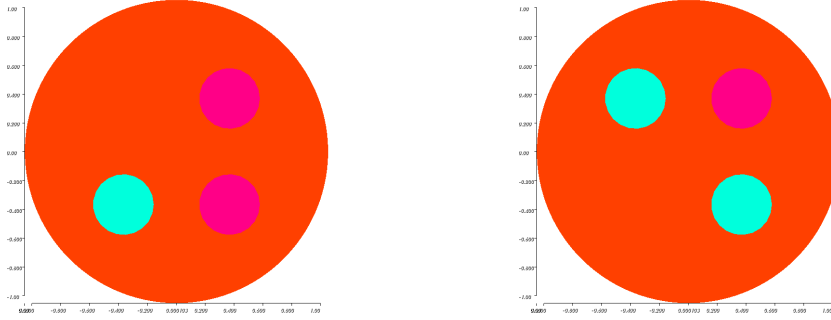
$$\zeta_\times^1(u^1, u^2, u^3) = u^1, \quad \zeta_\times^2(u^1, u^2, u^3) = \det [\nabla u^2 \quad \nabla u^3],$$

(see Example 3.11). We have performed a numerical test in $3^8 = 6561$ different cases, namely for different choices of the parameter distributions in the material.

Take $\Omega = B(0, 1)$. We now describe the coefficients we have used. Set $c = 0.35$, $r = 0.2$ and $P_1 = (-c, -c)$, $P_2 = (-c, c)$, $P_3 = (c, -c)$ and $P_4 = (c, c)$. Let χ_i be the characteristic function of $B(P_i, r)$ (or, more precisely, a smooth approximation of it). Then we set

$$a = 1 + \sum_{i=1}^4 \alpha_i \chi_i, \quad \varepsilon = 1 + \sum_{i=1}^4 \beta_i \chi_i,$$

Figure 3.3: A particular combination of the coefficients.



where $\alpha_i, \beta_i \in \{0, 1, 2\}$ (see Figure 3.3). This construction gives 3^8 different combinations.

In view of Theorem 3.26, we choose the illuminations $\varphi_1 = 1$, $\varphi_2 = x_1$ and $\varphi_3 = x_2$. The sequence $(\omega_p)_p$ of Theorem 3.50 is given by

$$(3.58) \quad \omega_p^2 = \lambda_1 + \alpha + \frac{\beta}{p},$$

for some $\alpha, \beta > 0$ such that $\lambda_1 + \alpha + \beta < \lambda_2$. (As in Proposition 2.1, the sequence $(\lambda_l)_l$ denotes the Dirichlet eigenvalues of the problem.) These conditions ensure that the chosen frequencies lie between the first and the second Dirichlet eigenvalue, which simplifies the numerical computation. For any combination of the coefficients, we compute the minimum number $\#K$ of needed frequencies such that $\{\omega_p : p = 1, \dots, \#K\} \times \{\varphi_i\}_i$ is a $(\zeta_\times, C)$ -complete set of measurements in Ω for some $C > 0$. The results are summarised in Table 3.1. These figures suggest that in practical applications the number of needed frequencies could be quite small. In particular, these figures correspond to the theoretical result given in Theorem 3.50, as $d + 1 = 3$ in two dimensions.

Table 3.1: Number of combinations of coefficients per number of needed frequencies to obtain a $(\zeta_\times, C)$ -complete set of measurements in Ω .

Needed frequencies ($\#K$)	2	3	≥ 4
Combinations of coefficients	1609	4952	0

Chapter 4

Applications to hybrid imaging inverse problems

In this chapter we show how to apply the multi-frequency method developed in the previous chapter to several hybrid imaging inverse problems. In Section 4.1, we consider two problems modelled by the Helmholtz equation. More precisely, in § 4.1.1 we consider the case $\sigma = 0$, whereas in § 4.1.2 we consider the case $\sigma > 0$. In Section 4.2, we consider two problems modelled by Maxwell's equations.

4.1 Problems modelled by the Helmholtz equation

4.1.1 Microwave imaging by ultrasound deformation

This section is devoted to the discussion of the hybrid problem introduced in § 1.3.1 and to the proofs of the results therein presented.

4.1.1.1 Formulation of the Problem

The object under examination is a smooth bounded domain $\Omega \subseteq \mathbb{R}^d$, for $d = 2$ or $d = 3$. In the microwave regime, the electric field u_ω^φ in Ω is assumed to satisfy the Helmholtz equation

$$(4.1) \quad \begin{cases} -\operatorname{div}(a \nabla u_\omega^\varphi) - \omega^2 \varepsilon u_\omega^\varphi = 0 & \text{in } \Omega, \\ u_\omega^\varphi = \varphi & \text{on } \partial\Omega, \end{cases}$$

where $a \in \mathcal{C}^{0,\alpha}(\overline{\Omega}; \mathbb{R})$ is the inverse of the magnetic permeability, $\varepsilon \in L^\infty(\Omega; \mathbb{R})$ is the electric permittivity and ω is the frequency. We assume (3.9). The use of the Helmholtz equation is a very common scalar approximation of the Maxwell equations [49, 107, 16, 30]. As we have seen in Section 2.1, problem (4.1) is well-posed provided that ω is not a resonant frequency.

Practitioners can choose a frequency $\omega \in \mathcal{A} \subseteq \mathbb{R}_+$ in a fixed range (not a resonant frequency), a real illumination φ on the boundary and measure the generated current on the boundary $a \frac{\partial u_\omega^\varphi}{\partial \nu}$. As described in [16], these measurements are combined with localised ultrasonic waves focusing in small regions $B = B(z, r)$ inside Ω , for $z \in \Omega$ and $r \ll |\Omega|$. Focusing ultrasounds is possible by using time reversal techniques [66, 40, 39]. We assume

that the electromagnetic parameters are affected linearly with respect to the amplitude of the ultrasonic perturbation, that is supposed to be small. Moreover, this modification is localised only in the region B . Under these assumptions, denoting the modified coefficients by $\bar{a}$ and $\bar{\varepsilon}$ we have

$$\begin{cases} \bar{a} = a (1 + c_a \alpha \chi_B), \\ \bar{\varepsilon} = \varepsilon (1 + c_\varepsilon \alpha \chi_B), \end{cases}$$

where α is the amplitude of the ultrasonic signal and c_a and c_ε are known functions. The corresponding electric field $\bar{u}_\omega^\varphi$ is the solution to

$$\begin{cases} -\operatorname{div}(\bar{a} \nabla \bar{u}_\omega^\varphi) - \omega \bar{\varepsilon} \bar{u}_\omega^\varphi = 0 & \text{in } \Omega, \\ \bar{u}_\omega^\varphi = \varphi & \text{on } \partial\Omega. \end{cases}$$

The density current $a \frac{\partial \bar{u}_\omega^\varphi}{\partial \nu}$ on the boundary of the domain is a measurable datum.

We now see how the internal energies can be determined by studying the change between $a \frac{\partial u_\omega^\varphi}{\partial \nu}$ and $a \frac{\partial \bar{u}_\omega^\varphi}{\partial \nu}$. We consider a wavenumber ω and two fixed illuminations φ and ψ . By asymptotic analysis techniques [21, 16], there holds

$$(4.2) \quad \int_{\partial\Omega} a \left(\frac{\partial \bar{u}_\omega^\varphi}{\partial \nu} - \frac{\partial u_\omega^\varphi}{\partial \nu} \right) \psi d\sigma \\ = |B| \frac{d c_a(z) \alpha}{c_a(z) \alpha + d} a(z) \nabla u_\omega^\varphi(z) \cdot \nabla u_\omega^\psi(z) + |B| \omega^2 c_\varepsilon(z) \alpha \varepsilon(z) u_\omega^\varphi(z) u_\omega^\psi(z) + o(|B|),$$

where $d\sigma$ denotes the integration with respect to the surface area. Since the left hand side is known, by choosing different values for α the internal power density data

$$E_\omega^{\varphi\psi}(z) = a(z) \nabla u_\omega^\varphi(z) \cdot \nabla u_\omega^\psi(z), \quad e_\omega^{\varphi\psi}(z) = \varepsilon(z) u_\omega^\varphi(z) u_\omega^\psi(z)$$

are recovered for every $z \in \Omega'$, where $\Omega' \Subset \Omega$ is the set of all possible centres where the ultrasonic beams are focused. Note that the ‘‘polarised’’ data with $\varphi \neq \psi$ is available for a fixed ω , but this argument does not allow the reconstruction of the cross-frequency data

$$E_{\omega_1 \omega_2}^{\varphi\psi}(z) = a(z) \nabla u_{\omega_1}^\varphi(z) \cdot \nabla u_{\omega_2}^\psi(z), \quad e_{\omega_1 \omega_2}^{\varphi\psi}(z) = \varepsilon(z) u_{\omega_1}^\varphi(z) u_{\omega_2}^\psi(z).$$

We thus chose not to use cross-frequency data for the reconstruction, in contrast to [16].

Let us now precisely describe the inverse problem under consideration. Given a set of measurement $K \times \{\varphi_i\}$ we consider the unique solution $u_\omega^i \in H^1(\Omega; \mathbb{R})$ to the problem

$$(4.3) \quad \begin{cases} -\operatorname{div}(a \nabla u_\omega^i) - \omega^2 \varepsilon u_\omega^i = 0 & \text{in } \Omega, \\ u_\omega^i = \varphi_i & \text{on } \partial\Omega. \end{cases}$$

We define the internal data by

$$(4.4) \quad e_{\omega_1 \omega_2}^{ij} = \varepsilon u_{\omega_1}^i u_{\omega_2}^j, \quad E_{\omega_1 \omega_2}^{ij} = a \nabla u_{\omega_1}^i \cdot \nabla u_{\omega_2}^j.$$

For simplicity, we denote $e_{\omega_1 \omega_2} = (e_{\omega_1 \omega_2}^{ij})_{ij}$ and $e_\omega := e_{\omega \omega}$ and similarly for E . The matrices e_ω and E_ω are to be considered as known matrix-valued functions. We study the inverse problem of determining the parameters a and ε in Ω' from the knowledge in Ω' of e_ω and

E_ω with a properly chosen set of measurements. Note that this reconstruction problem is slightly different to the one studied in [16], where the full matrices

$$(4.5) \quad e = \begin{bmatrix} e_{\omega_1\omega_1} & \cdots & e_{\omega_1\omega_n} \\ \vdots & \ddots & \vdots \\ e_{\omega_n\omega_1} & \cdots & e_{\omega_n\omega_n} \end{bmatrix}, \quad E = \begin{bmatrix} E_{\omega_1\omega_1} & \cdots & E_{\omega_1\omega_n} \\ \vdots & \ddots & \vdots \\ E_{\omega_n\omega_1} & \cdots & E_{\omega_n\omega_n} \end{bmatrix},$$

are supposed to be known, with $K = \{\omega_1, \dots, \omega_n\}$. In our case, we suppose that only the diagonal blocks are measurable.

4.1.1.2 Reconstruction Algorithm

We now discuss the exact formulae given in Subsection 1.3.1. In the statement of Theorem 1.7, the class of $(\zeta_{\det}, C)$ -complete sets of measurements was required. However, as we shall see in the proofs, weaker constraints are sufficient. In particular, the relevant map is the map $\zeta_\times$ introduced in Example 3.11. Recall that $\zeta_\times$ is given by $\kappa = 1$, $b = 3$, $r = 2$ and

$$(4.6) \quad \zeta_\times(u^1, u^2, u^3) = \begin{cases} (u^1, \nabla u^2 \times \nabla u^3) & \text{if } d = 2, \\ (u^1, (\nabla u^2 \times \nabla u^3)_3) & \text{if } d = 3. \end{cases}$$

Let us recall the constraints given by the map $\zeta_{\det}$ studied in § 3.3.2:

$$\begin{aligned} \zeta_{\det}^1(u^1, \dots, u^{d+1}) &= u^1, \\ \zeta_{\det}^2(u^1, \dots, u^{d+1}) &= \det \begin{bmatrix} \nabla u^2 & \cdots & \nabla u^{d+1} \end{bmatrix}, \\ \zeta_{\det}^3(u^1, \dots, u^{d+1}) &= \det \begin{bmatrix} u^1 & \cdots & u^{d+1} \\ \nabla u^1 & \cdots & \nabla u^{d+1} \end{bmatrix}. \end{aligned}$$

We have $\zeta_\times^1 = \zeta_{\det}^1$ and $\zeta_{\det}^3$ has simply been removed. As far as $\zeta_\times^2$ is concerned, we have that $\zeta_\times^2 = \zeta_{\det}^2$ if $d = 2$, and if $d = 3$ the constraint $|(\nabla u^2 \times \nabla u^3)_3| \geq C$ is weaker than $|\det \begin{bmatrix} \nabla u^2 & \nabla u^3 & \nabla u^4 \end{bmatrix}| \geq C$ since only two linearly independent gradients are required. As a consequence, the construction of $(\zeta_\times, C)$ -complete sets can be done by adapting the results of § 3.3.2. In particular we have the following

Proposition 4.1. *Assume that (3.9) and (3.10) hold. Take $s \in \mathbb{R}$ and $\varphi_1 \in C^{1,\alpha}(\overline{\Omega}; \mathbb{R})$ such that $\min_{\partial\Omega} \varphi_1 \geq c > 0$.*

- *Suppose $d = 2$, $a \in C^{0,1}(\overline{\Omega}; \mathbb{R}^{2 \times 2})$ and that Ω is convex. If $\Omega' \Subset \Omega$ then there exist $C > 0$ and $n \in \mathbb{N}$ depending on Ω , Ω' , Λ , $|\mathcal{A}|$, M , c , s , $\|\varphi_1\|_{C^{1,\alpha}(\overline{\Omega}; \mathbb{R})}$ and $\|a\|_{C^{0,1}(\overline{\Omega}; \mathbb{R}^{2 \times 2})}$ such that*

$$K^{(n)} \times \{\varphi_1, x_1 + s, x_2 + s\}$$

is $(\zeta_\times, C)$ -complete in Ω' .

- *Suppose that $d = 3$ and that (3.12) hold with $\kappa = 1$ and $\alpha \in (0, 1)$. Let $\hat{a} \in \mathbb{R}^{3 \times 3}$ satisfy (3.9a). There exist $\delta, C > 0$ and $n \in \mathbb{N}$ depending on Ω , Λ , $|\mathcal{A}|$, α , M , c , s , $\|\varphi_1\|_{C^{1,\alpha}(\overline{\Omega}; \mathbb{R})}$ and $\|a\|_{C^{0,\alpha}(\overline{\Omega}; \mathbb{R}^{3 \times 3})}$ such that if $\|a - \hat{a}\|_{C^{\kappa-1,\alpha}(\overline{\Omega}; \mathbb{R}^{d \times d})} \leq \delta$ then*

$$K^{(n)} \times \{\varphi_1, x_1 + s, x_2 + s\}$$

is $(\zeta_\times, C)$ -complete in Ω .

Proof. The proof is very similar to the proofs of Theorems 3.26 and 3.29.

We start with the two-dimensional case. In view of Theorem 3.15, it is enough to show that

$$|\zeta_\times^j(u_0^1, u_0^2, u_0^3)| \geq C_0, \quad j = 1, 2, \quad x \in \overline{\Omega'}$$

for some $C_0 > 0$ depending on Ω , Ω' , Λ , c and $\|a\|_{C^{0,1}(\overline{\Omega}; \mathbb{R}^{2 \times 2})}$ only. For $j = 1$, this follows trivially by $\min_{\partial\Omega} \varphi_1 \geq c$ and the strong maximum principle [68, Theorem 8.19]. For $j = 2$, this has already been proved in the proof of Theorem 3.26, as $\nabla u_0^2 \times \nabla u_0^3 = \det [\nabla u_0^2 \quad \nabla u_0^3]$. Note that this condition is independent of s , since $\nabla s = 0$.

The three-dimensional case is a simple consequence of Corollary 3.19. $\square$

We restate here the first part of Theorem 1.7. The new contribution of this work is the proof of the bounds (4.8), since (4.9) was already obtained in [16].

Theorem 4.2. *Take $C > 0$ and let $K \times \{\varphi_1, \varphi_2, \varphi_3\}$ be a $(\zeta_\times, C)$ -complete set in Ω' . Suppose*

$$(4.7) \quad |E_\omega^{ii}(x)|, |e_\omega^{ii}(x)| \leq b, \quad x \in \Omega', \quad \omega \in K, \quad i = 1, 2, 3$$

for some $b > 0$. Take $x \in \Omega'$ and ω_x as in Definition 3.13. Then there exists $c > 0$ depending on Λ and b such that

$$(4.8) \quad \frac{\text{tr}(e_{\omega_x}) \text{tr}(E_{\omega_x}) - \text{tr}(e_{\omega_x} E_{\omega_x})}{\text{tr}(e_{\omega_x})^2}(x) \geq cC^6,$$

and a/ε is given in terms of the data by

$$(4.9) \quad |\nabla(e_{\omega_x}/\text{tr}(e_{\omega_x}))(x)|_2^2 \frac{a}{\varepsilon}(x) = 2 \frac{\text{tr}(e_{\omega_x}) \text{tr}(E_{\omega_x}) - \text{tr}(e_{\omega_x} E_{\omega_x})}{\text{tr}(e_{\omega_x})^2}(x).$$

Remark 4.3. It is worth noting that in presence of noise the dependence of ω on x does not constitute a source of instability during the reconstruction performed in (4.9). As in Remark 3.14, we define for each $\omega \in K$ the set $G_\omega = \{x \in \Omega' : |u_\omega^1(x)| > C/2 \text{ and } |\nabla u_\omega^2(x) \times \nabla u_\omega^3(x)| > C/2\}$. In view of Proposition 2.5, G_ω is open. Since $K \times \{\varphi_i\}$ is a $(\zeta_\times, C)$ -complete set, Ω' can be written as $\Omega' = \cup_{\omega \in K} G_\omega$, and so we can consider a smooth partition of unity $\{\psi_\omega\}_{\omega \in K}$ subject to the cover $\{G_\omega\}_{\omega \in K}$. Denote by r_ω the value for a/ε obtained via (4.9) with ω . In view of Theorem 4.2, r_ω is a well-defined function in G_ω . We can now reconstruct a/ω in Ω' using

$$\frac{a}{\varepsilon}(x) = \sum_{\omega \in K} r_\omega(x) \psi_\omega(x), \quad x \in \Omega'.$$

Thus, with this formulation we have removed the dependence of ω on x and so no additional instability occurs.

Proof. For simplicity we write simply $\omega = \omega_x$. In view of Definition 3.13 and (4.6) we have

$$(4.10) \quad |u_\omega^1(x)| \geq C, \quad |\nabla u_\omega^2(x)| |\nabla u_\omega^3(x)| |\sin \theta_{\nabla u_\omega^2, \nabla u_\omega^3}(x)| = |\nabla u_\omega^2 \times \nabla u_\omega^3(x)| \geq C.$$

In particular, $\text{tr}(e_\omega)(x) > 0$, and so we may divide by $\text{tr}(e_\omega)$. Following the argument given in the proof of Proposition 3.3 in [16] we obtain

$$|\nabla(e_\omega/\text{tr}(e_\omega))|_2^2 \frac{a}{\varepsilon} = 2 \frac{\text{tr}(e_\omega) \text{tr}(E_\omega) - \text{tr}(e_\omega E_\omega)}{\text{tr}(e_\omega)^2}.$$

We now prove (4.8). For cleanliness of notation we shall denote several positive constants depending on Λ and b simply by $c = c(\Lambda, b) > 0$. Until the end of the proof, all the functions will be considered evaluated in x . We equip the space of real symmetric matrices with the Hilbert-Schmidt scalar product defined by $\langle A, B \rangle = \text{tr}(AB)$. We claim that

$$(4.11) \quad \left| (E_\omega^{2,3})^2 - E_\omega^{2,2} E_\omega^{3,3} \right| \leq c \|E_\omega\| \left(1 + \frac{\text{tr}(E_\omega)}{\text{tr}(e_\omega)} \right) \sqrt{1 - \cos \theta_{e_\omega, E_\omega}}.$$

As a consequence of this inequality, which we shall prove later, we get

$$(4.12) \quad \sqrt{1 - \cos \theta_{e_\omega, E_\omega}} \|E_\omega\| \geq c \frac{C^2}{1 + \frac{\text{tr}(E_\omega)}{\text{tr}(e_\omega)}},$$

since by (4.10) there holds

$$\left| (E_\omega^{2,3})^2 - E_\omega^{2,2} E_\omega^{3,3} \right| = a^2 \left| (\nabla u_\omega^2 \cdot \nabla u_\omega^3)^2 - |\nabla u_\omega^2|^2 |\nabla u_\omega^3|^2 \right| \geq c C^2.$$

Note that, from the definition of e_ω and E_ω , we have $(e_\omega^{ij})^2 = e_\omega^{ii} e_\omega^{jj}$ and $(E_\omega^{ij})^2 \leq E_\omega^{ii} E_\omega^{jj}$. In particular,

$$(4.13) \quad \|e_\omega\| = \sqrt{\sum_{i,j} (e_\omega^{ij})^2} = \text{tr}(e_\omega), \quad \|E_\omega\| = \sqrt{\sum_{i,j} (E_\omega^{ij})^2} \leq \text{tr}(E_\omega).$$

Combining (4.12) and (4.13) we obtain

$$\begin{aligned} \frac{\text{tr}(e_\omega) \text{tr}(E_\omega) - \text{tr}(e_\omega E_\omega)}{\text{tr}(e_\omega)^2} &\geq \frac{\text{tr}(e_\omega)(1 - \cos \theta_{e_\omega, E_\omega}) \|E_\omega\|}{\text{tr}(e_\omega)^2} \\ &\geq \frac{c C^4 \text{tr}(e_\omega)}{\text{tr}(e_\omega)^2 \text{tr}(E_\omega) (1 + \frac{\text{tr}(E_\omega)}{\text{tr}(e_\omega)})^2}, \end{aligned}$$

which gives the desired result since the denominator is bounded by a constant depending on b , and $\text{tr}(e_\omega) \geq c C^2$.

Let us now turn to the proof of (4.11). We use the notation $g = E_\omega / \|E_\omega\| - e_\omega / \|e_\omega\|$. By (4.7) and (4.13) we have

$$\begin{aligned} \left| (E_\omega^{2,3})^2 - E_\omega^{2,2} E_\omega^{3,3} \right| &\leq \left| (E_\omega^{2,3})^2 - \frac{\|E_\omega\|^2}{\|e_\omega\|^2} (e_\omega^{2,3})^2 \right| + \left| \frac{\|E_\omega\|^2}{\|e_\omega\|^2} e_\omega^{2,2} e_\omega^{3,3} - \frac{\|E_\omega\|}{\|e_\omega\|} e_\omega^{2,2} E_\omega^{3,3} \right| \\ &\quad + \left| \frac{\|E_\omega\|}{\|e_\omega\|} e_\omega^{2,2} E_\omega^{3,3} - E_\omega^{2,2} E_\omega^{3,3} \right| \\ &= \|E_\omega\| \left| E_\omega^{2,3} + \frac{\|E_\omega\|}{\|e_\omega\|} e_\omega^{2,3} \right| |g_{2,3}| + \frac{\|E_\omega\|^2}{\|e_\omega\|} e_\omega^{2,2} |g_{3,3}| + \|E_\omega\| E_\omega^{3,3} |g_{2,2}| \\ &\leq c \|E_\omega\| \left(1 + \frac{\text{tr}(E_\omega)}{\text{tr}(e_\omega)} \right) \sqrt{1 - \cos \theta_{e_\omega, E_\omega}}, \end{aligned}$$

since $|g_{2,3}| + |g_{3,3}| + |g_{2,2}| \leq c \|g\| \leq c \sqrt{1 - \cos \theta_{e_\omega, E_\omega}}$. $\square$

We now derive the reconstruction formula for ε given in Theorem 1.7. It is based on the knowledge of the ratio $G := a/\varepsilon$, that can be computed via the formula (4.9). Since this reconstruction involves the derivative of the data, we need a stable way to obtain ε from G .

The following lemma reviews the derivatives and the trace of products of functions in Sobolev Spaces.

Lemma 4.4. *Take $u, v \in H^1(\Omega; \mathbb{C}) \cap L^\infty(\Omega; \mathbb{C})$. Then $uv \in H^1(\Omega; \mathbb{C})$, $\nabla(uv) = u \nabla v + v \nabla u$ and $(uv)|_{\partial\Omega} = u|_{\partial\Omega} v|_{\partial\Omega}$. If in addition $v \geq c > 0$ almost everywhere then $u/v \in H^1(\Omega; \mathbb{C})$, $\nabla(u/v) = \nabla u/v - u \nabla v/v^2$ and $(u/v)|_{\partial\Omega} = u|_{\partial\Omega}/v|_{\partial\Omega}$.*

Proof. It easily follows from [62, Section 5.5, Theorem 1] and [68, Lemma 7.5]. $\square$

Recall that if we consider a $(\zeta_\times, C)$ -complete set of measurements $K \times \{\varphi_i\}$ we have that

$$\sum_{\omega} e_\omega^{11} \geq C > 0 \quad \text{in } \Omega'.$$

Theorem 4.5. *Let $K \times \{\varphi_1, \varphi_2, \varphi_3\}$ be a $(\zeta_\times, C)$ -complete set. Suppose $\varepsilon \in H^1(\Omega; \mathbb{R})$. Then $\log \varepsilon$ is the unique solution to the problem*

$$\begin{cases} -\operatorname{div} (G \sum_{\omega} e_\omega^{11} \nabla u) = -\operatorname{div} (G \nabla (\sum_{\omega} e_\omega^{11})) + 2 \sum_{\omega} (E_\omega^{11} - \omega^2 e_\omega^{11}) & \text{in } \Omega', \\ u = \log \varepsilon|_{\partial\Omega'} & \text{on } \partial\Omega'. \end{cases}$$

Proof. By Proposition 2.5 we infer that $u_\omega^i, u_\omega^j \in H^1(\Omega; \mathbb{R}) \cap L^\infty(\Omega; \mathbb{R})$. Therefore, by Lemma 4.4 we get that $u_\omega^i u_\omega^j \in H^1(\Omega; \mathbb{R}) \cap L^\infty(\Omega; \mathbb{R})$. Since $\varepsilon \in H^1(\Omega; \mathbb{R}) \cap L^\infty(\Omega; \mathbb{R})$ we obtain that $e_\omega^{ij} \in H^1(\Omega; \mathbb{R})$. Hence by Lemma 7.5 in [68] we have

$$\nabla(u_\omega^1 u_\omega^1) = \nabla(e_\omega^{11}/\varepsilon) = (\nabla e_\omega^{11} - e_\omega^{11} \nabla(\log \varepsilon))/\varepsilon,$$

with $\log \varepsilon \in H^1(\Omega; \mathbb{R})$. Moreover, by Lemma 4.4 and Proposition 2.5 an easy calculation yields

$$-\operatorname{div} (a \nabla(u_\omega^1 u_\omega^1)) = 2\omega^2 e_\omega^{11} - 2E_\omega^{11} \quad \text{in } \Omega,$$

Combining the last two identities it is immediate to show that

$$2\omega^2 e_\omega^{11} - 2E_\omega^{11} = -\operatorname{div}(G \nabla e_\omega^{11}) + \operatorname{div}(G e_\omega^{11} \nabla(\log \varepsilon)),$$

whence the result follows by summing all these equations with $\omega \in K$. $\square$

4.1.1.3 Numerical Experiments

Here we describe some numerical simulations of the reconstruction algorithm discussed in the last section. FreeFem++ has been used [72, 71] (see Listing A.1 in Appendix A for the codes). The exact formulae given in Theorem 4.2 and Theorem 4.5 will be used to image the electromagnetic parameters a and ε .

In both cases, the construction of $(\zeta_\times, C)$ -complete sets of measurements by means of the multi-frequency approach turns out to be effective. Further, in two dimensions, the reconstruction procedure gives better results than the one described in [16]: with about one seventh of the available data, the reconstruction errors are about half of the previous ones.

Two-Dimensional Example Since the two-dimensional case has been tested thoroughly in [16], we have decided to study the same example in order to be able to make a comparison. There are two main differences. First, the formula for the reconstruction of ε is not the same. Second, in our case the available data is smaller since we do not use the cross-frequency data.

Let $\Omega = B(0, 1)$ be the unit disk. We use a uniform mesh of the disk with about 3000 triangles and 1600 vertices. The coefficients are given by

$$a = \begin{cases} 2 & \text{in } B, \\ 1.2 & \text{in } C, \\ 2.5 & \text{in } E, \\ 1 & \text{otherwise,} \end{cases} \quad \varepsilon = \begin{cases} 2 & \text{in } B, \\ 1.8 & \text{in } C, \\ 1.2 & \text{in } E, \\ 1 & \text{otherwise.} \end{cases}$$

The set B is the rectangle with diagonal $(0, 0.4) - (0.3, 0.5)$. The set C is the area delimited by the curve $t \mapsto (0.3 + \rho(t) \cos(t), -0.2 + \rho(t) \sin(t))$, where $\rho(t) = (20 + 3 \sin(5t) - 2 \sin(15t) + \sin(25t))/100$. The set E is the ellipse of centre $(-0.3, 0.1)$, with vertical axis of length 0.3 and horizontal axis of length 0.2. These parameters represent three inclusions of different contrast in a homogeneous background medium. This is the typical practical situation, since cancerous tissues have typically higher values in the parameters [111]. The coefficients a and ε are shown in Figure 4.1.

Let $\Omega' = B(0, 0.8)$ be the subdomain where the internal energies are constructed (see § 4.1.1.1). We also suppose that a and ε are known in $\Omega \setminus \Omega'$. We choose the illuminations $\varphi_1 = \varphi_2 = x_1 + 2$ and $\varphi_3 = x_2 + 2$ and $K = \{1, \sqrt{3}, \sqrt{7}\}$. The illuminations satisfy the assumptions of Proposition 4.1. The set K is chosen in a different way, in order to be able to compare these results with [16]. In [16], the boundary conditions 1, x_1, x_2 and the same K were chosen: these illuminations satisfy the hypotheses of Proposition 4.1.

Let a^* and ε^* denote the approximated coefficients. We first reconstruct $G = a^*/\varepsilon^*$ in $x \in \Omega'$ by means of the formula (4.9)

$$\frac{a^*}{\varepsilon^*}(x) = 2 \frac{\text{tr}(e_\omega) \text{tr}(E_\omega) - \text{tr}(e_\omega E_\omega)}{|\nabla(e_\omega/\text{tr}(e_\omega))|_2^2 \text{tr}(e_\omega)^2}(x),$$

averaging over all the $\omega \in K$ such that $|\nabla(e_\omega/\text{tr}(e_\omega))|_2^2 > 10^{-2}$. Since for every $x \in \Omega'$ the set of such ω is not empty, the chosen set of measurements turns out to be $(\zeta_\times, C)$ -complete in Ω' .

Then, we use Theorem 4.5 to image $\log \varepsilon^* \in H_0^1(\Omega')$:

$$-\text{div}\left(G \sum_{\omega} e_{\omega}^{11} \nabla(\log \varepsilon^*)\right) = -\text{div}\left(G \nabla\left(\sum_{\omega} e_{\omega}^{11}\right)\right) + 2 \sum_{\omega \in K} (E_{\omega}^{11} - \omega^2 e_{\omega}^{11}) \quad \text{in } \Omega'.$$

Finally, a^* is given by $a^* = G\varepsilon^*$, which, in absence of noise, gives a good approximation. The reconstructed coefficients are shown in Figure 4.2.

In Table 4.1 we compare these findings with the numerical experiments performed in [16]. Even if the non-availability of the cross-frequency data makes the number of measurable internal energies much smaller (about one seventh), the L^2 norms of the errors $a - a^*$ and $\varepsilon - \varepsilon^*$ in this work are about half the corresponding norms obtained in [16]. This is due to the better reconstruction formula used for ε .

Figure 4.1: The reference parameters.

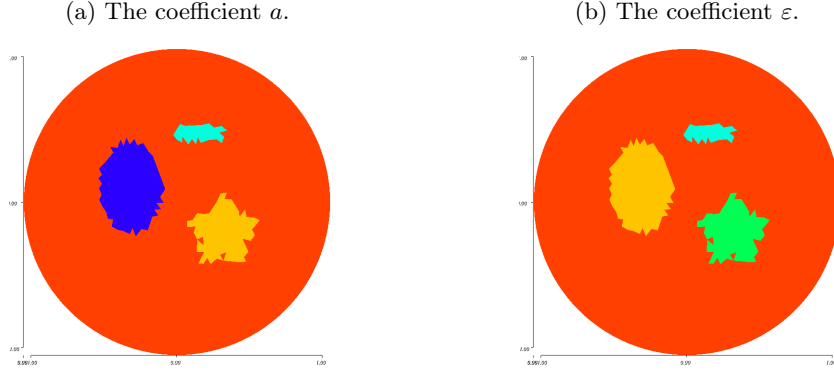


Figure 4.2: The reconstructed parameters.

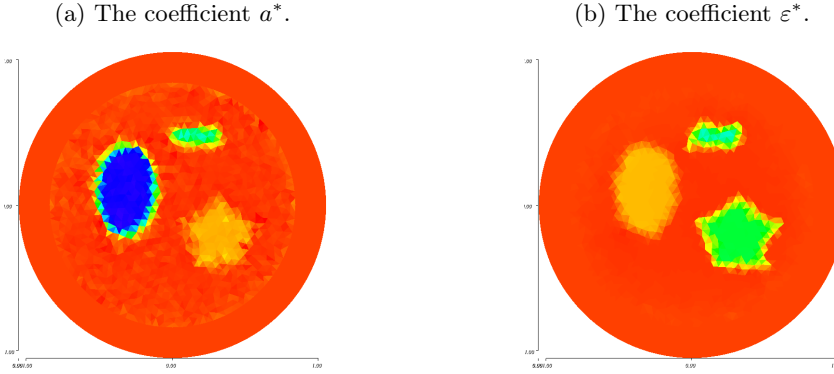
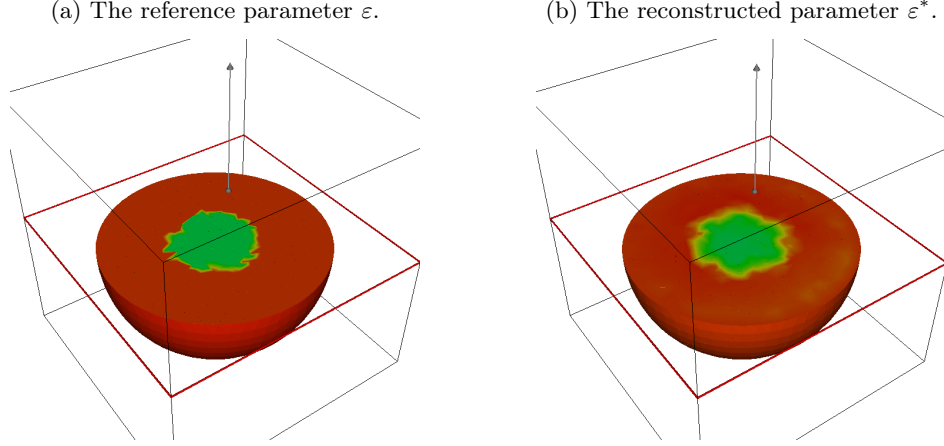


Table 4.1: A comparison with the numerical experiments carried out in [16].

	Numerical experiments in [16]	Numerical experiments in this work
Illuminations	$1, x_1, x_2$	$x_1 + 2, x_2 + 2$
Frequencies K	$1, \sqrt{3}, \sqrt{7}$	$1, \sqrt{3}, \sqrt{7}$
Number of energies	81	12
$\ a - a^*\ _2$	$3.5 \cdot 10^{-1}$	$1.6 \cdot 10^{-1}$
$\ \varepsilon - \varepsilon^*\ _2$	$1.5 \cdot 10^{-1}$	$0.8 \cdot 10^{-1}$

Figure 4.3: A section of the electric permittivity ε in Ω' .

Three-Dimensional Example Take $\Omega = B(0, 1)$ and $\Omega' = B(0, 0.8)$. We use a mesh with about 30000 tetrahedra and 6000 vertices. For simplicity, we choose $a = 1$ and $\varepsilon = 1 + 0.8 \chi_{B(0, 0.3)}$ and are interested in imaging the electric permittivity ε . As before, we choose the set of measurements $\{1, \sqrt{3}, \sqrt{7}\} \times \{x_1 + 2, x_2 + 2\}$, which a posteriori turns out to be (ζ_x, C) -complete in Ω' . The same reconstruction procedure described in the two-dimensional example is used. The reconstruction error is $\|\varepsilon - \varepsilon^*\|_2 \approx 1.3 \cdot 10^{-1}$. A comparison between the reference and the reconstructed parameters is shown in Figure 4.3.

4.1.2 Quantitative thermo-acoustic tomography (QTAT)

In thermo-acoustic tomography [90], the combination of acoustic waves and microwaves is carried out in a different way, if compared to the hybrid problem studied in the previous subsection. The absorption of the microwaves inside the object under investigation results in local heating, and so in a local expansion of the medium. This creates acoustic waves that propagate outside the domain, where they can be measured. In a first step [65, 78, 35], it is possible to measure the amount of absorbed radiation, which is given by

$$e_\omega^\varphi(x) = \sigma(x) |u_\omega^\varphi(x)|^2, \quad x \in \Omega,$$

where $\Omega \subseteq \mathbb{R}^d$ is a smooth bounded domain, for $d = 2, 3$, u_ω^φ is the unique solution to

$$(4.14) \quad \begin{cases} -\Delta u_\omega^\varphi - (\omega^2 + i\omega\sigma)u_\omega^\varphi = 0 & \text{in } \Omega, \\ u_\omega^\varphi = \varphi & \text{on } \partial\Omega, \end{cases}$$

and $\sigma \in L^\infty(\Omega; \mathbb{R})$ satisfies (3.11). The problem of QTAT is to reconstruct σ from the knowledge of e_ω^φ .

Let $\mathcal{A} \subseteq \mathbb{R}_+ \cap B(0, M)$ denote the range of the admissible frequencies and let $K \times \{\varphi_1, \dots, \varphi_{d+1}\}$ be a set of measurements. By using the polarisation formula, we can measure the quantities

$$e_\omega^{ij}(x) = \sigma(x) u_\omega^i(x) \overline{u_\omega^j(x)}, \quad x \in \Omega.$$

We shall see that it is possible to reconstruct σ if $K \times \{\varphi_1, \dots, \varphi_{d+1}\}$ is a $(\zeta'_{\det}, C)$ -complete, where $\zeta'_{\det}: C^1(\bar{\Omega}; \mathbb{C})^{d+1} \rightarrow C(\bar{\Omega}; \mathbb{C})^2$ was introduced in Example 3.12 and is given by

$$\zeta'_{\det}(u^1, \dots, u^{d+1}) = \left(u^1, \det \begin{bmatrix} u^1 & \dots & u^{d+1} \\ \nabla u^1 & \dots & \nabla u^{d+1} \end{bmatrix} \right).$$

Since $a = 1$, the construction of $(\zeta'_{\det}, C)$ -complete sets of measurements can be easily achieved with the multi-frequency approach in any dimensions.

Proposition 4.6. *Assume that $a = \varepsilon = 1$ and that $\sigma \in L^\infty(\Omega; \mathbb{R})$ satisfies (3.11). Then there exist $C > 0$ and $n \in \mathbb{N}$ depending on Ω , Λ , M and $|\mathcal{A}|$ only such that*

$$K^{(n)} \times \{1, x_1, \dots, x_d\}$$

is a $(\zeta'_{\det}, C)$ -complete set of measurements in Ω .

Proof. It follows immediately from Theorem 3.15, since the assumption $a = 1$ yields (3.18) with $C_0 = 1$. $\square$

If $K \times \{\varphi_1, \dots, \varphi_{d+1}\}$ is a $(\zeta'_{\det}, C)$ -complete set then for any $x \in \Omega$ there exists $\omega_x \in K$ such that

$$|u_{\omega_x}^1(x)| \geq C, \quad \left| \det \begin{bmatrix} u_{\omega_x}^1 & \dots & u_{\omega_x}^{d+1} \\ \nabla u_{\omega_x}^1 & \dots & \nabla u_{\omega_x}^{d+1} \end{bmatrix} (x) \right| \geq C.$$

With this assumption, it is possible to apply the following reconstruction formula. We use the notation $\alpha_\omega^i = e_\omega^{i1}/e_\omega^{11}$ wherever $u_\omega^1 \neq 0$ and $A_\omega = [\nabla \alpha_\omega^2 \quad \dots \quad \nabla \alpha_\omega^{d+1}]$.

Proposition 4.7 ([19, Theorem 3.3]). *Assume that $a = \varepsilon = 1$ and let $\sigma \in L^\infty(\Omega; \mathbb{R})$ satisfy (3.11). Let $K \times \{\varphi_1, \dots, \varphi_{d+1}\}$ be a $(\zeta'_{\det}, C)$ -complete set in Ω . There exists $c > 0$ depending on Ω , Λ and M such that*

$$|\det A_{\omega_x}(x)| \geq cC, \quad x \in \Omega.$$

Moreover, σ can be reconstructed via

$$\sigma(x) = \frac{-\Re v_{\omega_x}(x) \cdot \Im v_{\omega_x}(x) + \operatorname{div} \Im v_{\omega_x}(x)}{2\omega}, \quad x \in \Omega,$$

where $v_\omega = A_\omega^{-1} \operatorname{div}(A_\omega)^T$ (the divergence acts on each column).

In [19], in order to find the suitable illuminations, complex geometric optics solutions are used; these have several drawbacks, as it was discussed in Section 1.2. The multi-frequency approach allows to choose a priori simple illuminations and a finite number of frequencies to satisfy the desired constraints.

4.2 Problems modelled by Maxwell's equations

4.2.1 Magnetic resonance electrical impedance tomography (MREIT)

Let us recall the problem discussed in § 1.3.2. In MREIT, the combination of electric currents with an MRI scanner allows to measure the generated magnetic fields. The reader is referred to [98, 99, 97] and to the references therein for the details of this hybrid technique. From the knowledge of H , the electromagnetic parameters have to be imaged. Consider problem (3.29)

$$(4.15) \quad \begin{cases} \operatorname{curl} E_\omega^i = \mathbf{i}\omega H_\omega^i & \text{in } \Omega, \\ \operatorname{curl} H_\omega^i = -\mathbf{i}q_\omega E_\omega^i & \text{in } \Omega, \\ E_\omega^i \times \nu = \varphi_i \times \nu & \text{on } \partial\Omega. \end{cases}$$

Recall that $q_\omega = \omega\varepsilon + \mathbf{i}\sigma$. We assume $\mu = 1$ and $\varepsilon, \sigma > 0$, namely we study the isotropic case. Given a set of measurements $K \times \{\varphi_1, \dots, \varphi_b\}$, we measure H_ω^i in Ω and want to reconstruct ε and σ .

Two interesting issues of practical importance are not considered in this work. First, the case where only one or two components of the magnetic fields are measured. In such a case, the rotation of the MRI scanner is avoided. The reader is referred to [100], where a low frequency approximation is considered. Second, it is possible to consider anisotropic coefficients, which in some cases are a better model for human tissues [33].

4.2.1.1 First method

We now describe the first method to reconstruct σ and ε . Let $K \times \{\varphi_1, \varphi_2, \varphi_3\}$ be a $(\zeta_{\det}^M, C)$ -complete set of measurement in Ω , where $\zeta_{\det}^M$ is given by

$$\zeta_{\det}^M((u^1, v^1), (u^2, v^2), (u^3, v^3)) = \det \begin{bmatrix} u^1 & u^2 & u^3 \end{bmatrix},$$

as in Example 3.33.

By using the multi-frequency approach developed in this thesis, $(\zeta_{\det}^M, C)$ -complete sets can be explicitly constructed.

Proposition 4.8. *Assume that (3.28) holds with $\kappa = 0$ and let $\hat{\sigma} \in \mathbb{R}^{3 \times 3}$ satisfy (3.28a). There exist $\delta > 0$ and $C > 0$ depending on Ω , Λ , $|\mathcal{A}|$, M and $\|(\varepsilon, \sigma)\|_{W^{1,p}(\overline{\Omega}; \mathbb{R}^{3 \times 3})}$ such that if $\|\sigma - \hat{\sigma}\|_{W^{1,p}(\overline{\Omega}; \mathbb{R}^{3 \times 3})} \leq \delta$ then*

$$K^{(n)} \times \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$$

is $(\zeta_{\det}^M, C)$ -complete in Ω .

Proof. We want to apply Corollary 3.40 with $\zeta = \zeta_{\det}^M$ and $\psi_i = x_i$ for $i = 1, 2, 3$. We only need to show that (3.35) holds. Since $w^i = x_i$, for every $x \in \overline{\Omega}$ there holds

$$\zeta(\nabla w^1, \nabla w^2, \nabla w^3)(x) = \det \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \end{bmatrix} = 1,$$

as desired. □

We then show why these sets of measurements are useful to reconstruct the parameters.

Since $K \times \{\varphi_1, \varphi_2, \varphi_3\}$ is a $(\zeta_{\det}^M, C)$ -complete set of measurement, for every $x \in \overline{\Omega}$ there exists $\omega_x \in K$ such that

$$|\det [\begin{smallmatrix} E_{\omega_x}^1 & E_{\omega_x}^2 & E_{\omega_x}^3 \end{smallmatrix}](x)| \geq C.$$

Thus, (4.15) implies

$$(4.16) \quad |\det [\begin{smallmatrix} \operatorname{curl} H_{\omega_x}^1 & \operatorname{curl} H_{\omega_x}^2 & \operatorname{curl} H_{\omega_x}^3 \end{smallmatrix}](x)| = |q_{\omega_x}^3 \det [\begin{smallmatrix} E_{\omega_x}^1 & E_{\omega_x}^2 & E_{\omega_x}^3 \end{smallmatrix}](x)| \geq C',$$

with $C' = C\Lambda^{-3}$. This inequality will be necessary in the following.

We now proceed to eliminate the unknown electric field from system (4.15), in order to obtain an equation with only ε and σ as unknowns and the magnetic field as a known datum. An immediate calculation shows that for any $\omega \in K$ and $i = 1, 2, 3$ there holds $\operatorname{curl}(\mathbf{i}q_{\omega}^{-1}\operatorname{curl}H_{\omega}^i) = \mathbf{i}\omega H_{\omega}^i$ in Ω , whence

$$\nabla q_{\omega} \times \operatorname{curl} H_{\omega}^i = q_{\omega} \operatorname{curl} \operatorname{curl} H_{\omega}^i - q_{\omega}^2 \omega H_{\omega}^i = -q_{\omega} \Delta H_{\omega}^i - q_{\omega}^2 \omega H_{\omega}^i \quad \text{in } \Omega,$$

where the last identity is a consequence of the fact that H_{ω}^i is divergence free, since $\mu = 1$. Taking now scalar product with $\mathbf{e}_j$ for $j = 1, 2$ we have

$$\nabla q_{\omega} \cdot (\operatorname{curl} H_{\omega}^i \times \mathbf{e}_j) = -q_{\omega} \Delta (H_{\omega}^i)_j - q_{\omega}^2 \omega (H_{\omega}^i)_j \quad \text{in } \Omega.$$

We can now write these 6 equations in a more compact form. By introducing the 3×6 matrix

$$M_{\omega}^{(1)} = [\operatorname{curl} H_{\omega}^1 \times \mathbf{e}_1 \quad \operatorname{curl} H_{\omega}^1 \times \mathbf{e}_2 \quad \cdots \quad \operatorname{curl} H_{\omega}^3 \times \mathbf{e}_1 \quad \operatorname{curl} H_{\omega}^3 \times \mathbf{e}_2]$$

and the six-dimensional horizontal vector

$$v_{\omega} = ((H_{\omega}^1)_1, (H_{\omega}^1)_2, \dots, (H_{\omega}^3)_1, (H_{\omega}^3)_2)$$

we obtain

$$(4.17) \quad \nabla q_{\omega} M_{\omega}^{(1)} = -q_{\omega} \Delta v_{\omega} - q_{\omega}^2 \omega v_{\omega} \quad \text{in } \Omega.$$

We now want to right invert the matrix $M_{\omega}^{(1)}$ to obtain a well-posed first order PDE for q_{ω} . The following lemma gives a sufficient condition for the matrix $M_{\omega}^{(1)}$ to admit a right inverse.

Lemma 4.9. *Let $G_1, G_2, G_3 \in \mathbb{C}^3$ be linearly independent. Then the 3×6 matrix*

$$[\begin{smallmatrix} G_1 \times \mathbf{e}_1 & G_1 \times \mathbf{e}_2 & \cdots & G_3 \times \mathbf{e}_1 & G_3 \times \mathbf{e}_2 \end{smallmatrix}]$$

has rank three.

Proof. Take $u \in \mathbb{C}^3$ such that $G_i \times \mathbf{e}_j \cdot u = 0$ for every $i = 1, 2, 3$ and $j = 1, 2$. We need to prove that $u = 0$. Since $\mathbf{e}_j \times u \cdot G_i = 0$ for all i and j and $\{G_i : i = 1, 2, 3\}$ is a basis of $\mathbb{C}^3$ by assumption, we obtain $\mathbf{e}_j \times u = 0$ for $j = 1, 2$, namely $u = 0$. $\square$

As in Remark 3.14, define now for any $\omega \in K$ the set

$$\Omega_\omega = \{x \in \bar{\Omega} : |\det [\operatorname{curl} H_\omega^1 \quad \operatorname{curl} H_\omega^2 \quad \operatorname{curl} H_\omega^3](x)| > \frac{C'}{2}\}.$$

Since $\frac{C'}{2} < C'$, in view of (4.16) we obtain the cover

$$\bar{\Omega} = \bigcup_{\omega \in K} \Omega_\omega.$$

As the sets Ω_ω are relatively open in $\bar{\Omega}$, they must overlap, and this will be exploited below in the reconstruction. For any $\omega \in K$ and $x \in \Omega_\omega$, in view of Lemma 4.9 the matrix $M_\omega^{(1)}(x)$ admits a right inverse, which with an abuse of notation we denote by $(M_\omega^{(1)})^{-1}(x)$. Therefore, problem (4.17) becomes

$$(4.18) \quad \nabla q_\omega = -q_\omega \Delta v_\omega (M_\omega^{(1)})^{-1} - q_\omega^2 \omega v_\omega (M_\omega^{(1)})^{-1} \quad \text{in } \Omega_\omega.$$

It is now possible to integrate this PDE and reconstruct ε and σ in every $x \in \Omega$ if these are known for one value $x_0 \in \bar{\Omega}$. It is the nature of the multi-frequency approach that the relevant conditions are satisfied only locally for a fixed value of the frequency $\omega \in K$. In other words, (4.18) is not satisfied everywhere but only in Ω_ω . As a consequence, it is not possible to reconstruct ε and σ in x after one simple integration of (4.18). The process is more involved, and is similar to the algorithm described in [31].

Suppose now that $q_\omega(x_0)$ is known for some $x_0 \in \bar{\Omega}$ and take $x \in \bar{\Omega}$. Let Ω_ω^j for $j \in J_\omega$ be the connected components of Ω_ω , for a suitable set J_ω . Since $\bar{\Omega}$ is compact, from the cover $\bar{\Omega} = \bigcup_{\omega \in K} \bigcup_{j \in J_\omega} \Omega_\omega^j$, we can extract a finite subcover

$$\bar{\Omega} = \bigcup_{\omega \in K} \bigcup_{j \in J'_\omega} \Omega_\omega^j,$$

where $J'_\omega \subseteq J_\omega$ is finite. Hence, as $\bar{\Omega}$ is connected and Ω_ω^j are relatively open in $\bar{\Omega}$ and connected, we can find a smooth path $\gamma: [0, 1] \rightarrow \bar{\Omega}$ such that $\gamma(0) = x_0$, $\gamma(1) = x$ and

$$\gamma([0, 1]) = \bigcup_{m=0}^{M-1} \gamma([t_m, t_{m+1}]),$$

for some $M \in \mathbb{N}^*$, where $t_0 = 0$, $t_M = 1$ and $\gamma([t_m, t_{m+1}]) \subseteq \Omega_{\omega_m}^{j_m}$ for some $\omega_m \in K$ and $j_m \in J'_{\omega_m}$. Starting with $m = 0$, we integrate (4.18) with $\omega = \omega_m$ along $\gamma([t_m, t_{m+1}])$ and obtain q_{ω_m} in $\gamma(t_{m+1})$. Thus, we can reconstruct σ and ε in $\gamma(t_{m+1})$ and so $q_{\omega_{m+1}}(\gamma(t_{m+1}))$. Repeating this process $M - 1$ times we obtain $\varepsilon(x)$ and $\sigma(x)$, as desired.

4.2.1.2 Second method

We discuss here another method to reconstruct ε and σ from the knowledge of H .

The relevant constraints are given by $b = 6$, $r = 1$, $\kappa = 2$ and

$$\zeta^{(2)}((u_1, v_1), \dots, (u_6, v_6)) = \det \begin{bmatrix} \eta(u_1, u_2) & \eta(u_3, u_4) & \eta(u_5, u_6) \end{bmatrix},$$

where $\eta: C^1(\overline{\Omega}; \mathbb{C}^3)^2 \longrightarrow C(\overline{\Omega}; \mathbb{C}^3)$ is given by

$$(4.19) \quad \eta(u_1, u_2) = (\nabla u_1)u_2 - (\nabla u_2)u_1 + \operatorname{div} u_1 u_2 - \operatorname{div} u_2 u_1 - 2^t(\nabla u_1)u_2 + 2^t(\nabla u_2)u_1,$$

as in Example 3.34.

As in the previous case, $\zeta^{(2)}$ -complete sets can be explicitly constructed using Corollary 3.40. Let I denote the 3×3 identity matrix.

Proposition 4.10. *Assume that (3.28) holds with $\kappa = 1$ and take $\hat{\sigma} \geq \Lambda^{-1}$. There exist $\delta, C > 0$ depending on $\Omega, \Lambda, |\mathcal{A}|, M$ and $\|(\varepsilon, \sigma)\|_{W^{2,p}(\overline{\Omega}; \mathbb{R}^{3 \times 3})}$ such that if $\|\sigma - \hat{\sigma}I\|_{W^{2,p}(\overline{\Omega}; \mathbb{R}^{3 \times 3})} \leq \delta$ then*

$$K^{(n)} \times \{\mathbf{e}_2, \nabla(x_1 x_2), \mathbf{e}_3, \nabla(x_2 x_3), \mathbf{e}_1, \nabla(x_1 x_3)\}$$

is $(\zeta^{(2)}, C)$ -complete in Ω .

Proof. We want to apply Corollary 3.40 with $\zeta = \zeta^{(2)}$ and $\psi_1 = x_2, \psi_2 = x_1 x_2, \psi_3 = x_3, \psi_4 = x_2 x_3, \psi_5 = x_1$ and $\psi_6 = x_1 x_3$. We only need to show that (3.35) holds. Since $w^i = \psi_i$, a trivial calculation shows that for every $x \in \overline{\Omega}$

$$\zeta^{(2)}(\nabla w^1, \dots, \nabla w^6)(x) = \det \begin{bmatrix} \eta(\mathbf{e}_2, \nabla(x_1 x_2)) & \eta(\mathbf{e}_3, \nabla(x_2 x_3)) & \eta(\mathbf{e}_1, \nabla(x_1 x_3)) \end{bmatrix} = 1,$$

as desired. $\square$

Remark 4.11. In [33], complex geometric optics solutions are used to construct suitable boundary conditions. With this approach, the six illuminations are given explicitly and do not depend on the coefficients. However, it has to be noted that the assumption $\|\sigma - \hat{\sigma}I\|_{W^{2,p}(\Omega; \mathbb{R}^{3 \times 3})} \leq \delta$ can be restrictive (see Remark 3.38).

We now show how the coefficients can be reconstructed by using $(\zeta^{(2)}, C)$ -complete sets.

Let $K \times \{\varphi_1, \dots, \varphi_6\}$ be a $(\zeta^{(2)}, C)$ -complete set of measurements. Namely, for every $x \in \overline{\Omega}$ there exists $\omega_x \in K$ such that

$$(4.20) \quad \left| \det \begin{bmatrix} \eta(E_{\omega_x}^1, E_{\omega_x}^2) & \eta(E_{\omega_x}^3, E_{\omega_x}^4) & \eta(E_{\omega_x}^5, E_{\omega_x}^6) \end{bmatrix} (x) \right| \geq C$$

for some $C > 0$ independent of x , where η is given by (4.19).

In view of (4.15), for $\omega \in K$ there holds

$$\operatorname{curl} \operatorname{curl} E_\omega^1 \cdot E_\omega^2 - \operatorname{curl} \operatorname{curl} E_\omega^2 \cdot E_\omega^1 = 0 \quad \text{in } \Omega.$$

An easy calculation shows that substituting $E_\omega^i = \mathbf{i} q_\omega^{-1} \operatorname{curl} H_\omega^i$ in this identity we obtain

$$(4.21) \quad \nabla q_\omega \cdot \eta(\operatorname{curl} H_\omega^1, \operatorname{curl} H_\omega^2) = q_\omega \gamma(\operatorname{curl} H_\omega^1, \operatorname{curl} H_\omega^2) \quad \text{in } \Omega,$$

where $\gamma: C^1(\overline{\Omega}; \mathbb{C}^3)^2 \rightarrow C(\overline{\Omega}; \mathbb{C})$ is defined by

$$\gamma(u_1, u_2) = (\nabla \operatorname{div} u_1) \cdot u_2 - (\nabla \operatorname{div} u_2) \cdot u_1 - \Delta u_1 \cdot u_2 + \Delta u_2 \cdot u_1.$$

Repeating the same argument with the other illuminations and combining the resulting equations we have

$$(4.22) \quad \nabla q_\omega M_\omega^{(2)} = q_\omega (\gamma(\operatorname{curl} H_\omega^1, \operatorname{curl} H_\omega^2), \gamma(\operatorname{curl} H_\omega^3, \operatorname{curl} H_\omega^4), \gamma(\operatorname{curl} H_\omega^5, \operatorname{curl} H_\omega^6)) \quad \text{in } \Omega,$$

where $M_\omega^{(2)}$ is the 3×3 matrix-valued function given by

$$M_\omega^{(2)} = \begin{bmatrix} \eta(\operatorname{curl} H_\omega^1, \operatorname{curl} H_\omega^2) & \eta(\operatorname{curl} H_\omega^3, \operatorname{curl} H_\omega^4) & \eta(\operatorname{curl} H_\omega^5, \operatorname{curl} H_\omega^6) \end{bmatrix}$$

By definition of η and since $\operatorname{curl} H_\omega^i = -\mathbf{i}q_\omega E_\omega^i$ we have $\eta(\operatorname{curl} H_\omega^i, \operatorname{curl} H_\omega^j) = -q_\omega^2 \eta(E_\omega^i, E_\omega^j)$, whence

$$M_\omega^{(2)} = -q_\omega^2 \begin{bmatrix} \eta(E_\omega^1, E_\omega^2) & \eta(E_\omega^3, E_\omega^4) & \eta(E_\omega^5, E_\omega^6) \end{bmatrix}.$$

Therefore, inverting $M_\omega^{(2)}$ in (4.22) we obtain

$$\nabla q_\omega = q_\omega (\gamma(\operatorname{curl} H_\omega^1, \operatorname{curl} H_\omega^2), \dots, \gamma(\operatorname{curl} H_\omega^5, \operatorname{curl} H_\omega^6)) (M_\omega^{(2)})^{-1} \quad \text{in } \Omega_\omega,$$

where

$$\Omega_\omega = \{x \in \overline{\Omega} : |\det \begin{bmatrix} \eta(E_\omega^1, E_\omega^2) & \eta(E_\omega^3, E_\omega^4) & \eta(E_\omega^5, E_\omega^6) \end{bmatrix} (x)| > \frac{C}{2}\}.$$

In view of (4.20) we have $\overline{\Omega} = \bigcup_{\omega \in K} \Omega_\omega$, and so q_ω can be reconstructed everywhere in Ω following the algorithm discussed previously, provided that q_ω is known at one point in $\overline{\Omega}$.

4.2.2 Inverse problem of electro-seismic conversion

Electro-seismic conversion is the generation of a seismic wave in a fluid-saturated porous material when an electric field is applied [113]. The problem is modelled by the coupling of Maxwell's equations and Biot's equations. We consider the hybrid inverse problem introduced in [55]. In the first step, by inverting Biot's equation, the quantity $D_\omega^i = L E_\omega^i$ is recovered in Ω , where $L > 0$ is a possibly varying coefficient representing the coupling between electromagnetic and mechanic effects. In a second step, the electromagnetic parameters ε and σ have to be imaged from the knowledge of D_ω^i . As in the previous cases, we assume that $\mu = 1$ and that ε and σ are isotropic.

Let $K \times \{\varphi_1, \dots, \varphi_6\}$ be a set of measurements. Since $\operatorname{curl} H_\omega^i = -\mathbf{i}q_\omega E_\omega^i$, this problem is very similar to the one discussed in § 4.2.1.2, and L plays the role of $-\mathbf{i}q_\omega$. Therefore, if $K \times \{\varphi_1, \dots, \varphi_6\}$ is $\zeta^{(2)}$ -complete, it is possible to reconstruct the coefficient L . Once L is known, the electric field can be easily obtained as $E_\omega^i = L^{-1} D_\omega^i$. Finally, ε and σ can be reconstructed via

$$\omega q_\omega E_\omega^1(x) = \operatorname{curl} \operatorname{curl} E_\omega^1(x),$$

provided that E_ω^1 is non vanishing. In particular, this is true if $|(E_\omega^1)_2|(x) > 0$. This suggests to consider $(\zeta^{(3)}, C)$ -complete sets, where $b = 6$, $r = 2$, $\kappa = 1$ and $\zeta^{(3)}: C^1(\overline{\Omega}; \mathbb{C}^6)^6 \rightarrow C(\overline{\Omega}; \mathbb{C})^2$ is given by

$$\zeta^{(3)}((u_1, v_1), \dots, (u_6, v_6)) = (\zeta^{(2)}((u_1, v_1), \dots, (u_6, v_6)), (u_1)_2),$$

as in Example 3.35. If $K \times \{\varphi_1, \dots, \varphi_6\}$ is $\zeta^{(3)}$ -complete then it is possible to uniquely reconstruct σ and ε if these are known at one point in $\overline{\Omega}$.

The construction of $\zeta^{(3)}$ -complete sets of measurements is analogous to the construction of $\zeta^{(2)}$ -complete sets, and the proof of the following result is left to the reader.

Proposition 4.12. *Assume that (3.28) holds with $\kappa = 1$ and take $\hat{\sigma} \geq \Lambda^{-1}$. There exist $\delta, C > 0$ depending on $\Omega, \Lambda, |\mathcal{A}|, M$ and $\|(\varepsilon, \sigma)\|_{W^{2,p}(\overline{\Omega}; \mathbb{R}^{3 \times 3})}$ such that if $\|\sigma - \hat{\sigma}I\|_{W^{2,p}(\overline{\Omega}; \mathbb{R}^{3 \times 3})} \leq \delta$ then*

$$K^{(n)} \times \{\mathbf{e}_2, \nabla(x_1x_2), \mathbf{e}_3, \nabla(x_2x_3), \mathbf{e}_1, \nabla(x_1x_3)\}$$

is $(\zeta^{(3)}, C)$ -complete in Ω .

The same comment given in Remark 4.11 is relevant here: in [55], complex geometric optics solutions are used to make this problem solvable.

Chapter 5

Conclusions

In this section the main contributions of this thesis are summarised and some open problems are discussed.

5.1 Regularity theory for Maxwell's equations

Previous results concerning the regularity of the time harmonic Maxwell's equations require coefficients in $W^{1,\infty}$ to have solutions in H^1 and/or in $C^{0,\alpha}$, and the Lipschitz continuity assumption was believed to be optimal. In this thesis we have shown that the result is still true with coefficients in $W^{1,3+\delta}$ for some $\delta > 0$. Boundary regularity and higher regularity have been proven as well. The proof is based on the L^p theory for elliptic systems and on the reduction of Maxwell's equations into a coupled elliptic systems. The same method applies to the case of bi-anisotropic materials, for which a different approach is not needed.

The case $\delta = 0$ cannot be handled with the current method, as the bootstrap argument used fails in this case. A natural question arises: what is the minimal regularity assumption on the coefficients to have H^1 or Hölder continuous solutions? By using the Helmholtz decomposition, this problem is strictly related to optimal regularity for solutions to elliptic equations. With this approach, $W^{1,3}$ seems to be sufficient, but possibly not optimal.

5.2 Local constraints in PDE

Motivated by several hybrid imaging inverse problems, we studied the boundary control of solutions of the Helmholtz and Maxwell equations to enforce local non-zero constraints inside the domain. Classically, suitable boundary conditions are determined as traces of complex geometric optics solutions. However, this approach has several drawbacks: the coefficients need to be very smooth, the construction depends on the (unknown) coefficients and is numerically unstable.

In this thesis we have proposed a new multiple frequency approach to this problem and have shown its effectiveness in several contexts. More precisely, if the required constraints are verified in the case $\omega = 0$, which corresponds to the conductivity equation, then they are verified also in the case $\omega > 0$, provided that a finite number of frequencies, given a priori,

are chosen in a fixed range. The proof is based on the holomorphicity of the solutions with respect to ω , and does not depend on the particular structure of the Helmholtz equation or of Maxwell's equations.

In § 3.5.2 we showed that, under the assumption of real analytic coefficients, almost any $d + 1$ frequencies in a fixed range give the required constraints, where d is the dimension of the ambient space. The proof is based on the structure of analytic varieties, and so the hypothesis of real analytic coefficients is crucial. However, this assumption is far too strong for the applications. Thus, a very natural question to ask is whether it is possible to lower the assumption of real analytic coefficients. The strategy to tackle this problem may be to use local expansions of solutions to elliptic PDE in harmonic polynomials [44, 93, 94].

Satisfying the constraints in the case $\omega = 0$ is usually straightforward in two dimensions, but may present difficulties in 3D if a (or σ in the case of Maxwell's equations) is not constant. We have seen in Section 3.5.1 that the method may work even if the constraint is not verified in the case $\omega = 0$: when dealing with the constraints $|\nabla u_\omega^\varphi| \geq C$, a generic choice of the boundary condition φ is sufficient. However, choosing a generic boundary condition may give a very low constant C and a very high number of frequencies, if the boundary condition is chosen near the residual set. Therefore, an open problem is to find an alternative to the study of the constraints in $\omega = 0$. In particular, as far as the Helmholtz equation is concerned, an asymptotic of u_ω^φ for high frequencies ω [42, 54] may give the required non-zero constraints, and by holomorphicity this would still give the desired result.

Appendix A

Codes

This appendix contains some of the FreeFem++ [72] codes that have been written by the author for the numerical simulations of this thesis.

Listing A.1 contains the code used in § 4.1.1.3 to reconstruct a and ε in microwave imaging by ultrasound deformation in two dimensions. The corresponding code for the three-dimensional case is very similar and thus has been omitted. The main relevant difference is in the construction of the three-dimensional mesh of the sphere, which is achieved as in [81].

Listing A.1: Microwave imaging by ultrasound deformation – code used in §4.1.1.3

```
1 // 1. Meshes
2 verbosity=0;
3
4 border BDEXT(t= 0, 2*pi){x= 1.0*cos(t); y= 1.0*sin(t);};
5 border BDEIT(t= 0, 2*pi){x= 0.9*cos(t); y= 0.9*sin(t);};
6 border BDINT(t= 0, 2*pi){x= 0.8*cos(t); y= 0.8*sin(t);};
7
8 real XEL=-0.3, YEL= 0.1,
9       XCA= 0.3, YCA=-0.2,
10      RXE= 0.2, RYE= 0.3,
11      XB1= 0.0, YB1= 0.4,
12      LXB= 0.3, LYB= 0.1;
13
14 func real rca(real t)
15 {return 0.2 +0.03*sin(5*t)-0.02*sin(15*t)+ 0.01*sin(25*t);}
16
17 border el(t=0,2*pi){x= XEL+ RXE*cos(t); y=YEL+ RYE*sin(t);};
18 border can(t=0,2*pi){x= XCA+rca(t)*cos(t);y=YCA+rca(t)*sin(t);};
19 border bo1(t=0, 1){x= XB1+LXB*t ; y=YB1 ; };
20 border bo2(t=0, 1){x= XB1+LXB ; y=YB1 +LYB*t ; };
21 border bo3(t=0, 1){x= XB1+LXB-LXB*t; y=YB1 +LYB ; };
22 border bo4(t=0, 1){x= XB1 ; y=YB1 +LYB -LYB*t; };
23
24 // NaturalMesh is adapted to the coefficients and is used to collect
   the internal data.
25 int NP =80; int NP2=10;
26 mesh NaturalMesh=buildmesh(BDEXT(NP) + BDEIT(NP) + BDINT(NP)
27   +el(4*NP2) + can(NP) + bo1(NP2) + bo2(-NP) + bo3(NP2)+ bo4(-NP));
```

```

28
29 mesh NaturalMeshInt=buildmesh(BDINT(NP) + e1(4*NP2)
30     +can(NP) + bo1(NP2) + bo2(-NP) + bo3(NP2) + bo4(-NP));
31
32 // MeasureMesh is uniform and is used for the reconstruction.
33 NP=56;
34 mesh MeasureMesh=buildmesh(BDEXT(2*NP)+BDEIT(2*NP)+BDINT(2*NP));
35 mesh MeasureMeshInt = buildmesh(BDINT(2*NP));
36
37 plot(NaturalMesh); plot(MeasureMesh);
38
39 // 2. The Finite Element Spaces
40 // P1 is used for the solutions, and P0 is used for the coefficients
41 // and the derivatives of the elements in P1.
42
43 fespace CNath(NaturalMesh,P0);
44 fespace CMeash(MeasureMesh,P0);
45 fespace CMeashInt(MeasureMeshInt,P0);
46 fespace PNath(NaturalMesh,P1);
47 fespace PMeash(MeasureMesh,P1);
48 fespace PMeashInt(MeasureMeshInt,P1);
49
50 // 3. Construction of the coefficients  $a$  and  $\varepsilon$ 
51
52 func ellipse = ((x-XEL)*(x-XEL)/(RXE*RXE) + (y-YEL)*(y-YEL)/(RYE*RYE)
53     ) <= 1);
54 func canada = (sqrt((x-XCA)*(x-XCA)+(y-YCA)*(y-YCA)) - rca(atan(y-
55     YCA,x-XCA)) <= 0 );
56 func recta = (((x-XB1-LXB)*(x-XB1) <= 0)*((y-YB1-LYB)*(y-YB1) <=0));
57
58 CMeash reg = region;
59 CMeash Ring = 1.0*(region==reg(0.95,0.0));
60 CMeash Ring2 = Ring + 1.0*(region==reg(0.85,0.0));
61 CMeash Interior = 1.0 - Ring;
62 CMeash Interior2 = 1.0 - Ring2;
63
64 real a0=1.0, a1=2.50, a2=2.00, a3=1.2,
65     ep0=1.0, ep1=1.20, ep2=2.00, ep3= 1.8;
66
67 CNath aRef,epsiRef; // The reference parameters
68 CNath FE=ellipse, CA=canada, B0=recta;
69
70 aRef = a0 + (a1 -a0)*FE + (a3 -a0)*CA + (a2 -a0)*B0;
71 epsiRef = ep0 + (ep1 -ep0)*FE + (ep3 -ep0)*CA + (ep2 -ep0)*B0;
72
73 CMeash aplot=aRef, epsiplot=epsiRef;
74
75 // 4. Variational problems
76
77 PNath us,vs,phis;
78 CMeash test;
79
80 test[]=1; int le=test[].sum; // length of the vectors in CMeash
81 int bc=2; // number of boundary conditions

```

```

79 real omega; // parameter for the frequency
80 func phi1=x+2; func phi2=y+2; // boundary conditions
81
82 // Variational formulation of the Helmholtz equation
83 problem Helx(us,vs) = int2d(NaturalMesh)(aRef*(dx(us)*dx(vs)+dy(us)*
      dy(vs)))-int2d(NaturalMesh)(omega^2*epsiRef*us*vs)+on(1,us=phi1);
84 problem Hely(us,vs) = int2d(NaturalMesh)(aRef*(dx(us)*dx(vs)+dy(us)*
      dy(vs)))-int2d(NaturalMesh)(omega^2*epsiRef*us*vs)+on(1,us=phi2);
85
86 // 5. The frequencies K
87
88 int dim=3; real[int] KInit=[1,sqrt(3),sqrt(7)]; // chosen frequencies
89 cout<<"The chosen set of frequencies is: "<<KInit<<endl;
90
91 //uu contains all the solutions to the equation
92 int dim2=dim*bc; PNath[int] uu(dim2);
93
94 // With the following code, the frequencies that are too close to an
      eigenvalue of the problem are removed from "KInit". The set of
      the remaining valid frequencies is "K".
95 int c=0; int c2=0; int now=dim;
96 for (int i=0;i<now;i++) {
97     omega=KInit(i); Helx;
98     if (int2d(NaturalMesh)(us^2) <800) {
99         uu[c]=us; c++; Hely; uu[c]=us; c++;KInit(c2)=KInit(i);c2++;
100     }
101     if (int2d(NaturalMesh)(us^2) >= 800) dim=dim-1;
102 }
103
104 real[int] K(dim);
105 for (int i=0;i<dim;i++) K(i)=KInit(i);
106 cout<<"The new set of frequencies is: "<<K<<endl;
107
108 // 6. Construction of the internal measurements
109
110 macro createeE(u,v,e,E) {
111     PMeash udeux = u*v;
112     CMeash dudeux = dx(u)*dx(v)+dy(u)*dy(v);
113     E = aRef*dudeux*Interior + Ring;
114     e = epsiRef*udeux*Interior + Ring;
115 }//
116
117 // We now construct the vectors e and E containing the full matrices
      e and E (see (4.4) and (4.5)). However, only the non-diagonal blocks
       $e_\omega$  and  $E_\omega$  will be used for the reconstruction.
118 dim2=dim*bc; c=0;
119 PMeash[int] e(dim2^2); CMeash[int] E(dim2^2);
120 for (int i=0;i<dim2;i++) {
121     for (int j=0;j<dim2; j++) {createeE(uu[i],uu[j],e[c],E[c]);c++;}
122 }
123
124 // 7. Reconstruction of  $G = a/\varepsilon$ 
125
126 PMeash[int] eb(bc^2); CMeash[int] Eb(bc^2); // "eb" and "Eb"

```

represent a single diagonal block of the matrices e and E , namely
 $eb = e_\omega$ and $Eb = E_\omega$.

```

127
128 // The macro "blocks" creates the left hand side "LHSb" and the
    right hand side "RHSb" in formula (4.9) for a fixed frequency  $\omega$ .
    More precisely, given as inputs  $eb = e_\omega$  and  $Eb = E_\omega$ , we obtain as
    outputs  $LHSb = |\nabla(e_\omega / \text{tr}(e_\omega))|_2^2$  and  $RHSb = 2 \frac{\text{tr}(e_\omega) \text{tr}(E_\omega) - \text{tr}(e_\omega E_\omega)}{\text{tr}(e_\omega)^2}$ .
129 macro blocks(eb,Eb,LHSb,RHSb) {
130     PMeash treb; CMeash trEb,trPEb; PMeash[int] Pb(bc^2);
131     CMeash[int] PbC(bc^2); CMeash trebC;
132     treb=eb[0]+eb[3];trebC=treb; trEb=Eb[0]+Eb[3];
133     for (int i=0;i<bc^2;i++) {Pb[i]=eb[i]/treb; PbC[i]=Pb[i];}
134     trPEb=PbC[0]*Eb[0]+PbC[1]*Eb[2]+PbC[2]*Eb[1]+PbC[3]*Eb[3];
135     RHSb=2*(trEb-trPEb)/trebC;
136     LHSb=dx(Pb[0])*dx(Pb[0])+dy(Pb[0])*dy(Pb[0])+dx(Pb[1])*dx(Pb
        [1])+dy(Pb[1])*dy(Pb[1])+dx(Pb[2])*dx(Pb[2])+dy(Pb[2])*dy(
        Pb[2])+dx(Pb[3])*dx(Pb[3])+dy(Pb[3])*dy(Pb[3]);
137 }//
138
139 // The macro "createblock" creates the numb-th diagonal block "out"
    of the full matrix "in".
140 macro createblock(in,numb,out) {
141     int c=0;
142     for (int i=0;i<bc;i++) {
143         for (int j=0;j<bc;j++) {
144             int ind=(dim*bc+1)*2*numb+j+i*dim*bc;out[c]=in[ind];c++;
145         };
146     };
147 }//
148
149 // With the following code the left and right hand sides of (4.9) are
    computed for every  $\omega$  and saved in the vectors "LHS" and "RHS".
150 CMeash[int] LHS(dim), RHS(dim);
151 for (int l=0;l<dim;l++) {
152     createblock(e,l,eb); createblock(E,l,Eb);
153     blocks(eb,Eb,LHS[l],RHS[l]);
154 }
155
156 // Now the function  $G = a/\varepsilon$  is reconstructed. We apply formula (4.9)
    and average over the set of frequencies  $\omega$  that give a left hand
    side bigger than 0.01. If such a set is empty, we set  $G = 1$ .
157 CMeash G,Gref=aRef/epsiRef;
158 int bad=0; real[int] lhs(1e);real[int] rhs(1e); real[int] Gvect(1e);
159 for (int i=0;i<1e;i++) {
160     c=0;
161     for (int j=0;j<dim;j++) {
162         lhs=LHS[j][];rhs=RHS[j][];
163         if (lhs[i]>0.01) { Gvect[i]=Gvect[i]+rhs[i]/lhs[i];c++; }
164     }
165     if (c>0) Gvect[i]=Gvect[i]/c;
166     if (c=0) { Gvect[i]=1;bad++; }
167 }
168 cout<<"Bad points: "<<bad<<endl;
169 G[]=Gvect; G=G*Interior2 + Ring2*Gref;

```

```

170
171 // The macro "smooth" smoothens the function "in" of the finite
    element space of type P0 over the mesh "Mesh"
172 macro smooth(Mesh,in,1) {
173     fespace TEMP(Mesh,P0);TEMP out; int nbt=Mesh.nt;
174     int[int] tr(3);int ee=0;
175     for (int i=0;i<nbt;i++) {
176         tr(0)=Mesh[i].adj(ee);ee=1; tr(1)=Mesh[i].adj(ee); ee=2;
177         tr(2)=Mesh[i].adj(ee); out[](i)=(1*in[][i]+in[][tr(0)]+in
            [][tr(1)]+in[][tr(2)])/(1+3);
178     }
179     in=out*Interior2 + Ring2*in;
180 }//
181
182 smooth(MeasureMesh,G,3);
183
184 // 8. Reconstruction of  $\varepsilon$ 
185
186 // We now compute  $w := \sum_{\omega} \omega^2 e_{\omega}^{11}$ 
187 PNath w=K(0)^2*e[0]; c=0;
188 for (int i=1;i<dim2;i++){w=w+K(c)^2*e[i*(dim2+1)];if (i % 2==1)c++;}
189
190 // Theorem 4.5 is applied. Notation:  $tre = \sum_{\omega} e_{\omega}^{11}$ ,  $trE = \sum_{\omega} E_{\omega}^{11}$ .
191 PMeash tre=e[0]; CMeash trE=E[0];
192 for (int i=1;i<dim2;i++) {
193     tre=tre+e[i*(dim2+1)]; trE=trE+E[i*(dim2+1)];
194 }
195 CMeash epsi,a; // the reconstructed parameters
196 PMeasInt u,v,epsInt,epsIntRef=epsInt,aRefInt=aRef;
197 problem logepsi(u,v)=
198     int2d(MeasureMeshInt)(G*tre*(dx(u)*dx(v)+dy(u)*dy(v)))
199     -int2d(MeasureMeshInt)(G*(dx(tre)*dx(v)+dy(tre)*dy(v)))
200     -int2d(MeasureMeshInt)((2*trE-2*w)*v)+on(3,u=0);
201 logepsi; epsInt=exp(u);
202
203 // a is reconstructed via  $a = G\varepsilon$ 
204 a=G*epsInt; smooth(MeasureMesh,a,4); epsi=epsInt;
205
206 // The reference and reconstructed parameters are plotted
207 real[int] colors(.9:.01:3.3);
208 plot(aplot,fill=1,nbiso=500, viso=colors,ps="aRef.ps");
209 plot(epsiplot,fill=1,nbiso=500,viso=colors,ps="epsiRef.ps");
210 plot(a,fill=1,nbiso=500,viso=colors,ps="a.ps");
211 plot(epsi,fill=1,nbiso=500,viso=colors,ps="epsi.ps");
212
213 // The reconstruction errors are computed
214 cout<<endl<<
215 "Error in epsi: "<<sqrt(int2d(MeasureMeshInt)((epsIntRef-epsInt)
    ^2))<<
216 ". Error in a: "<<sqrt(int2d(MeasureMeshInt)((aRefInt-a)^2))<<endl;

```

We now describe the code used in § 3.5.3 to compute the number of needed frequencies to obtain a $(\zeta_{\times}, C)$ -complete set of measurements. For simplicity, Listing A.2 contains only

the computation for a fixed combination of coefficients a and ε . The complete code can then be obtained with a simple iteration.

Listing A.2: Numerical simulations on the number of needed frequencies – code used in 3.5.3

```

1 // 1. Macros
2
3 // The macro "refinemesh" refines the mesh "Mesh" near the zeros.
4 macro refinemesh(Mesh,lalpha,beta,af,epsif) {
5     real omega,t1,t2;
6     int nbt=Mesh.nt; int[int] Z1(nbt);
7     fespace M0m(Mesh,P0); fespace M1m(Mesh,P1);
8     M0m a=af,epsi=epsif,zeros; M1m u,v,u1,u2,u3,det,phi;
9     problem Hel(u,v)= int2d(Mesh)(a*(dx(u)*dx(v)+dy(u)*dy(v)))
10                     -int2d(Mesh)(omega*epsi*u*v)+on(1,u=phi);
11
12     Z1=1; omega=lalpha+beta; phi=1; Hel; u1=u; phi=x; Hel; u2=u;
13     phi=y; Hel; u3=u; det=dx(u2)*dy(u3)-dx(u3)*dy(u2);
14     createZ(Mesh,u1,det,Z1);
15     for (int j=0;j<nbt;j++) zeros[j]=Z1(j);
16     Mesh=splitmesh(Mesh,1+1*zeros);
17
18     u=0;v=0; zeros=0;
19     a=af;epsi=epsif; nbt=Mesh.nt; int[int] Z2(nbt);Z2=1;
20     phi=1; Hel; u1=u; phi=x; Hel; u2=u;phi=y; Hel;
21     u3=u; det=dx(u2)*dy(u3)-dx(u3)*dy(u2);
22     createZ(Mesh,u1,det,Z2);
23     for (int j=0;j<nbt;j++) zeros[j]=Z2(j);
24     omega=lalpha+beta/2;Hel; phi=1; Hel; u1=u; phi=x; Hel;
25     u2=u;phi=y; Hel; u3=u; det=dx(u2)*dy(u3)-dx(u3)*dy(u2);
26     createZ(Mesh,u1,det,Z2);
27     for (int j=0;j<nbt;j++) zeros[j]=Z2(j);
28     Mesh=splitmesh(Mesh,1+2*zeros);
29
30     u=0;v=0; zeros=0;
31     a=af;epsi=epsif; nbt=Mesh.nt; int[int] Z3(nbt);Z3=1;
32     omega=lalpha+beta; phi=1; Hel; u1=u; phi=x; Hel; u2=u;
33     phi=y; Hel; u3=u; det=dx(u2)*dy(u3)-dx(u3)*dy(u2);
34     createZ(Mesh,u1,det,Z3);
35     for (int j=0;j<nbt;j++) zeros[j]=Z3(j);
36     omega=lalpha+beta/2;Hel; phi=1; Hel; u1=u; phi=x; Hel;
37     u2=u;phi=y; Hel; u3=u; det=dx(u2)*dy(u3)-dx(u3)*dy(u2);
38     createZ(Mesh,u1,det,Z3);
39     for (int j=0;j<nbt;j++) zeros[j]=Z3(j);
40     omega=lalpha+beta/3;phi=1; Hel; u1=u; phi=x; Hel; u2=u;
41     phi=y; Hel; u3=u; det=dx(u2)*dy(u3)-dx(u3)*dy(u2);
42     createZ(Mesh,u1,det,Z3);
43     for (int j=0;j<nbt;j++) zeros[j]=Z3(j);
44     Mesh=splitmesh(Mesh,1+3*zeros);
45
46     plot(Mesh);
47 }//
48
49 // The macro "createZ" finds the zeros of the functions "u" and "v".

```

More precisely, the inputs are the mesh "Mesh" with "nbt" triangles, two functions "u" and "v" in the finite element space of type P1 associated to "Mesh" and a vector $Z \in \{0,1\}^{nbt}$. The vector Z is modified according to the following rule: if "u" and "v" have constant sign values on the vertices of the j-th triangle of "Mesh" then we set $Z(j) := 0$.

```

50 macro createZ(Mesh,u,v,Z){
51     int nbt=Mesh.nt; real t1,t2,t3,t4;
52     for (int j=0;j<nbt;j++) {
53         t1=u[][Mesh[j][0]]*u[][Mesh[j][1]];
54         t2=u[][Mesh[j][0]]*u[][Mesh[j][2]];
55         t3=v[][Mesh[j][0]]*v[][Mesh[j][1]];
56         t4=v[][Mesh[j][0]]*v[][Mesh[j][2]];
57         if (min(min(t1,t2),min(t3,t4))>0) Z(j)=0;
58     }
59 }//
60
61 // The macro "findl" computes the number of needed frequencies "K"
    with the mesh "Mesh". If compared with the notation in (3.58), we
    have  $\text{lalpha} = \lambda_1 + \alpha$  and  $\text{beta} = \beta$  so that  $\omega_i = \lambda_1 + \alpha + \beta/(i+1)$ . Denote
    the triangles of "Mesh" by  $t_j$  for  $j = 0, \dots, nbt - 1$ . For  $i = 0, \dots, 6$ 
    define  $\chi(i) = \#\{j : \sum_{s=0}^i |u_{\omega_s}^1| \text{ or } \sum_{s=0}^i |\det(u_{\omega_s}^2, u_{\omega_s}^3)| \text{ has a zero in } t_j\}$  so that
    the desired output is  $K = \min\{i : \chi(i) = 0\} + 1$ .
62 macro findl(Mesh,lalpha,beta,af,epsif,K) {
63     int[int] chi(7); real omega,ii;K=0;
64     int nbt=Mesh.nt; int[int] Z(nbt); Z=1;
65     fespace M0m(Mesh,P0); fespace M1m(Mesh,P1);
66     M0m a=af,epsif,zeros; M1m u,u1,u2,u3,v,phi,det;
67     problem Hel(u,v)= int2d(Mesh)(a*(dx(u)*dx(v)+dy(u)*dy(v)))
68         -int2d(Mesh)(omega*epsif*u*v)+on(1,u=phi);
69     for (int i=0;i<7;i++) {
70         ii=i;omega=lalpha+beta/(ii+1); phi=1; Hel; u1=u; phi=x;
71         Hel;u2=u;phi=y; Hel; u3=u;det=dx(u2)*dy(u3)-dx(u3)*dy(u2);
72         createZ(Mesh,u1,det,Z);
73         for (int j=0;j<nbt;j++) zeros[(j)]=Z(j);
74         chi(i)=Z.sum;
75         if (chi(i)==0) {K=i+1;break;}
76     }
77     cout<<"chi: " <<chi<<endl;
78 }//
79
80 // With the macro "eigenvalues" the first two eigenvalues of the
    problem are computed and stored in the vector "lambda".
81 macro eigenvalues(Mesh,af,epsif,lambda) {
82     fespace Vh1(Mesh,P1); fespace Vh(Mesh,P1);
83     Vh1 a=af,epsif; Vh u1,u2;
84     real sigma = 10;
85     varf op(u1,u2)= int2d(Mesh)(a*(dx(u1)*dx(u2) + dy(u1)*dy(u2))
86         - sigma*epsif* u1*u2 ) + on(1,u1=0) ;
87     varf b([u1],[u2]) = int2d(Mesh)(epsif* u1*u2 );
88     matrix OP= op(Vh,Vh,solver=Crout,factorize=1);
89     matrix B= b(Vh,Vh,solver=CG,eps=1e-20);
90     lambda=1; int nev=lambda.sum; lambda=0; Vh[int] eV(nev);
91     int kk=EigenValue(OP,B,sym=true,sigma=sigma,value=lambda,

```

```

vector=eV,tol=1e-10,maxit=0,ncv=0);
92 }//
93
94 // 2. Meshes
95
96 verbosity=0;
97 border BORD(t= 0, 2*pi){x= 1.0*cos(t); y= 1.0*sin(t); };
98 int NP = 155; real r=.2; real p=.35;
99 real[int] xel(4); xel=[-p,-p,p,p];
100 real[int] yel(4); yel=[-p,p,-p,p];
101
102 border el0(t= 0, 2*pi){x= xel(0)+ r*cos(t);y=yel(0)+r*sin(t);};
103 border el1(t= 0, 2*pi){x= xel(1)+ r*cos(t);y=yel(1)+r*sin(t);};
104 border el2(t= 0, 2*pi){x= xel(2)+ r*cos(t);y=yel(2)+r*sin(t);};
105 border el3(t= 0, 2*pi){x= xel(3)+ r*cos(t);y=yel(3)+r*sin(t);};
106
107 func elli0 = ((x-xel(0))^2/r^2 + (y-yel(0))^2/r^2 <= 1 );
108 func elli1 = ((x-xel(1))^2/r^2 + (y-yel(1))^2/r^2 <= 1 );
109 func elli2 = ((x-xel(2))^2/r^2 + (y-yel(2))^2/r^2 <= 1 );
110 func elli3 = ((x-xel(3))^2/r^2 + (y-yel(3))^2/r^2 <= 1 );
111
112 mesh Mesh = buildmesh (BORD(NP)); plot(Mesh);
113 fespace M0(Mesh,P0);
114 fespace M1(Mesh,P1);
115
116 // 3. Coefficients  $a$  and  $\varepsilon$ 
117
118 // The following is simply an arbitrary choice for the coefficients.
119 int c0=2; int b0=2; int c1=1; int b1=3;
120 int c2=3; int b2=3; int c3=1; int b3=2;
121 func af = 1+b0*elli0+b1*elli1+b2*elli2+b3*elli3;
122 func epsif = 1+c0*elli0+c1*elli1+c2*elli2+c3*elli3;
123
124 // 4. Computation of the number of needed frequencies
125
126 int K; real M=15; real[int] lambda(2);
127 eigenvalues(Mesh,af,epsif,lambda);
128 real lalpha=(lambda(1)-lambda(0))/M+lambda(0);
129 real beta=(lambda(1)-lambda(0))*(1-2/M);
130
131 // With the following iteration, the final number of needed
// frequencies "K" is computed. If at a first attempt  $K > 3$  then the
// mesh is refined and the computation repeated, up to three times.
132 for (int i=1;i<4;i++) {
133     cout<<"Refinement iteration: "<<i<<endl;
134     int nbt=Mesh.nt; cout<<"Number of triangles: "<<nbt<<endl;
135     refinemesh(Mesh,lalpha,beta,af,epsif);
136     findl(Mesh,lalpha,beta,af,epsif,K);
137     if (K>0) if (K<4) break;
138 }
139
140 if (K>0) cout<<"The number of frequencies is "<<K<<endl;
141 if (K==0) cout<<"The number of frequencies is bigger than 8."<<endl;

```


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