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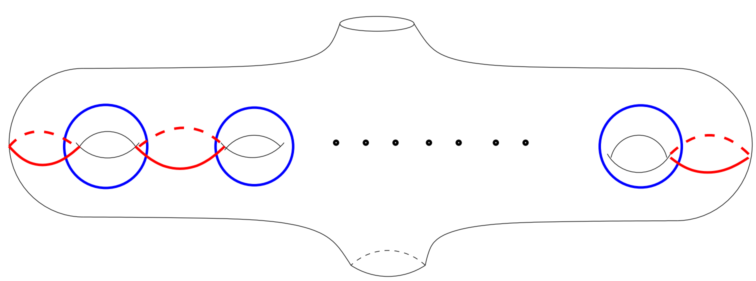
Riemann surfaces. Purely topologically the well-known classification theorem divides compact surfaces into those that are two-sided (orientable) and those that are one-sided (non-orientable), and up-to homeomorphisms any such surface is determined by its genus g and the number of boundary circles n it has. If we however follow Riemann and associate different complex structures to an oriented surface the situation becomes more multifaceted and discrete invariants do not suffice to distinguish between them. In his influential paper *Theorie der Abel'schen Funktionen* [R], he considers how the complex structure of a surface associated to a multi-valued complex function changes when one continuously varies the parameters of the function. He concludes that when the genus of the surface is $g \geq 2$ the isomorphism class *"hängt [...] von* $3g-3$ *stetig veränderlichen Grössen ab, welche die Moduln dieser Klasse genannt werden sollen."* Thus Riemann understands the complex dimension of his space to be $3g-3$ and introduces its name, moduli space, into the mathematical literature.

Since then, for more than 150 years, moduli spaces have been studied in nearly every branch of mathematics, be it complex analysis - its field of origin, algebraic geometry, topology and group theory, dynamics, mathematical physics, or arithmetic geometry and number theory.



A surface of genus g *with* 2 *boundary components and Dehn twists generating the image of* B_{2g+2} *in* $\Gamma_{g,2}$.

Case study: For a $(2g+2)$-tuple $\underline{a} = (a_1, a_2, \ldots, a_{2g+2})$ of distinct complex numbers inside the unit disk $D \subset \mathbb{C}$ consider the multi-valued function

$$f_{\underline{a}}(z) = \{(z-a_1)(z-a_2)\ldots(z-a_{2g+2})\}^{\frac{1}{2}}$$

where z is a complex variable. As every complex number with the exception of 0 has precisely two square roots, $f_{\underline{a}}$ is one-to-two everywhere apart for $z = a_1, a_2, \ldots, a_{2g+2}$. In order to have a well-defined function, one extends the domain to the double sheeted cover $\Sigma_{\underline{a}}$ of the disk D with branch points $a_1, a_2, \ldots a_{2g+2}$. A simple computation of the Euler characteristic gives

$$\chi(\Sigma_{\underline{a}}) = 2\chi(D) - (2g+2) = -2g$$

and thus identifies $\Sigma_{\underline{a}}$ topologically as a genus g surface with two boundary components as in the above figure. Its complex structure will vary with the position of the $a_i \in D$. The assignment $\underline{a} \mapsto \Sigma_{\underline{a}}$ defines a $2g+2$ dimensional sub-family, the hyperelliptic locus, inside the moduli space $\mathcal{M}_{g,2}$: each surface in this family admits an involution with $2g+2$ fixed-points, the maximal number possible.
The configuration space of $2g+2$ distinct points in the disk D is up to homotopy determined by its fundamental group, the braid group B_{2g+2} on $2g+2$ strands. Similarly, for $n > 0$, the moduli space $\mathcal{M}_{g,n}$ is determined up to homotopy by its fundamental group, the mapping class group $\Gamma_{g,n}$. The assignment $\underline{a} \mapsto \Sigma_{\underline{a}}$ corresponds to the embedding B_{2g+2} into $\Gamma_{g,2}$ that takes the standard generator of the braid group to the Dehn twists around the curves as indicated in the above figure. One can show that on stable cohomology this map carries no information [ST].

One way to construct the moduli spaces is by using marked Riemann surfaces and Teichmüller theory. For simplicity assume that there are no boundary components ($n = 0$) and $g > 1$. Fixing an oriented surface S_g of genus g, consider pairs (Σ, ϕ) of a Riemann surface and a diffeomorphism $\phi : \Sigma \to S_g$. Two such pairs (Σ, ϕ) and (Σ', ϕ') are equivalent if there exits a diffeomorphism $\varphi : \Sigma \to \Sigma'$ such that ϕ and $\varphi \circ \phi'$ are isotopic. The set of equivalence classes, $\mathcal{T}_g$, can be given a natural metric, the Teichmüller metric. [1] With this topology Teichmüller space $\mathcal{T}_g$ is homeomorphic to $\mathbb{C}^{3g-3}$ and hence contractible.

Let $\mathrm{Diff}(S_g)$ denote the group of orientation preserving diffeomorphisms of S_g. The mapping class group Γ_g is defined to be its group of connected components. This is a very rich infinite discrete group generated by Dehn twists around simple closed curves on the surfaces such as pictured in the above figure. It also carries all the homotopical information of the diffeomorphism group as the connected components of the latter are contractible.

A diffeomorphism $\alpha \in \mathrm{Diff}(S_g)$ acts on $\mathcal{T}_g$ via

$$\alpha.(\Sigma, \phi) = (\Sigma, \alpha\phi).$$

[1] Using the fact that any two Riemann surfaces can be mapped to each other via a conformal map that is isotopic to the identity, the distance between them is defined as the logarithm of K, where K is the infinum of the maximal dilations of such quasi-conformal maps.

This action descends to an action of the mapping class group. Moduli space is defined as the orbit space,

$$\mathcal{M}_g := \mathcal{T}_g/\Gamma_g.$$

The action of the mapping class group has at most finite stabilisers. Indeed, by Hurwitz's theorem a Riemann surface has at most $84(g-1)$ automorphisms. Thus the study of moduli spaces of Riemann surfaces and the study of the mapping class groups are closely related, and in particular there is an isomorphism of rational cohomology rings

$$H^*(\mathcal{M}_g;\mathbb{Q}) = H^*(\Gamma_g;\mathbb{Q}).$$

Riemann surfaces are also complex algebraic curves. From this point of view, the defining characteristic of the moduli space is the fact that it represents a universal family of algebraic curves: a map from a parameter space X to $\mathcal{M}_g$ defines a family of curves over X, and vice versa any such family can be defined via such a map which sends a point in X to the isomorphism class of the curve that it represents. This was only made rigorous through geometric invariant theory in the 1960s [Mum1]. It also suggested a natural compactification of $\mathcal{M}_g$ in which nodal curves are added to form the Deligne-Mumford compactification $\overline{\mathcal{M}}_g$, which itself is again a moduli space in the above sense.

In his landmark paper [Mum2] Mumford initiates the systematic study of the cohomology (and Chow ring) both for the moduli spaces and their compactifications. He introduces the most important classes and among them the κ_i that generate the so called tautological subring R_g^* of $H^*(\mathcal{M}_g;\mathbb{Q})$. In the light of explicit computations and Harer's cohomology stability theorem [H], Mumford conjectures that with increasing genus the rational cohomology ring of $\mathcal{M}_g$ is a polynomial ring on certain classes $\kappa_i, i = 1, 2, \ldots$. This conjecture remained open until it was reformulated and strengthened by Madsen and the reviewer [MT] in homotopic terms and then proved by Madsen and Weiss [MW]. The tautological ring R_g^* was described conjecturally by Faber in [F].

In a different direction, motivated by considerations in quantum field theory Witten conjectured a generating function for the intersection numbers of the κ_i classes for $\overline{\mathcal{M}}_g^n$ of genus g Riemann surfaces with n distinct marked points. This was first proved by Kontsevich [K] using a combinatorial description of the moduli space in terms of ribbon graphs. More recently, Mirzakhani used rather different techniques to give an alternative proof [Mir1, Mir2, Mir3]. She computed the Weil-Petersson volumes of moduli spaces of Riemann surfaces (with boundary) and expressed them as polynomials, the coefficients of which can be reinterpreted as intersection numbers. Both Kontsevich and Mirzakhani received a Fields medal for their work.

The book under review is a collection of lecture notes for nine research lecture courses delivered at the IAS/Park City Mathematics Institute during the two week programme "Moduli Spaces of Riemann Surfaces" in 2011. They vary from introductions addressed to the beginning graduate student to research expositions targeted at a specialised audience, from surveys of work over several decades to outlines of the proof of a recent theorem. Just as varied are the research methods represented with which the moduli spaces are studied, be it algebraic, topological, dynamical, geometric or arithmetic.

There is possibly one notable omission, that of mathematical physics, which has had a strong presence and keen interest in moduli spaces in the last thirty years. There is little mention of Gromov-Witten theory or conformal field theory. Still, reflecting the fields of interest of the organisers, the selection of topics represent a coherent choice centred around the 'pure' theory of moduli spaces.

We have mentioned above in particular cohomological aspects, biased by the reviewer's own interests and knowledge. But this is also at least partially justified by the fact that these topics are well reflected in the lecture notes and are the central theme for several of them. However, it does not do justice to the advances made on dynamical and geometric aspects of Teichmüller theory nor the leaps of progress made on arithmetic questions as touched on in the last two sets of lectures.

We will look at each of the sets of lecture notes individually.

The first set of lectures by **Minsky** gives a gentle introduction to the mapping class groups and includes a discussion three fundamental theorems. The first is the Nielsen-Thurston classification of elements in the mapping class group as elliptic, reducible or pseudo-Anasov. The second is Nielsen's realisation problem (proved by Kerckhoff) that every finite subgroup of the mapping class group has a realisation as a group of isometries for some Riemann surface. The third is the fact that the mapping class group satisfies the Tits alternative: either a subgroup is virtually abelian or it contains a free group of rank two. As part of this many fundamental topics are introduced such as the relation to hyperbolic geometry and measured laminations. A final section gives a survey of the literature.

A well written set of lecture notes by **Hamenstädt** discusses Teichmüller theory from the view of a differential geometer. Classical topics covered include the Fenchel-Nielsen coordinates, complex and hyperbolic structures, quadratic differentials and Jacobians. Using the curve complex she constructs the augmented Teichmüller space corresponding to $\overline{\mathcal{M}}_g$. She discusses both the Teichmüller and the Weil-Petersson metric, and finishes with a discussion of the dynamics of the Teichmüller flow thus foreshadowing some of the material covered by Wolpert.

Wahl and Galatius' lecture courses give an exposition of a proof of the Mumford conjecture. **Wahl** introduces the topic and concentrates on one ingredient of the proof, the homology stability theorem for mapping class groups orginally due to Harer. This is of interest in its own right and once again, the curve complex plays an important role. There are several expositions of this topic by the author, this one having the advantage of brevity. **Galatius'** lectures discuss the second part, the description of the homotopy type of the topological moduli space of surfaces and the computation of its rational homology following [GRW]. He draws on a very readable exposition of this material by Hatcher but adds precision in many places. Both Wahl and Galatius' lectures come with helpful exercises and together make an excellent basis for an autodidactic study or lecture course.

Putman's lectures in contrast concentrate on his recent computation of some low dimensional homology groups of certain finite index subgroups of the mapping class group, the mod p congruence subgroups [P]. The technical ingredients are similar to those used in Wahl's lectures, and the two papers could be read in conjunction.

Faber surveys what is known about the tautological subalgebra of the (unstable!) cohomology of moduli spaces and its compactification, starting with his conjectural description [F] of the tautological ring for $\mathcal{M}_g$. These are not so much lecture notes than a description of the state of the art. A long list of exercises is included, though these take often the form of research projects.

The subject of **Wolpert**'s carefully prepared lecture notes is the celebrated work of Mirzakhani [Mir1, Mir2, Mir3] giving a new proof of the Witten conjecture/Kontsevich theorem. These lectures are self-contained and include immediate background material on Teichmüller theory, moduli spaces and symplectic reduction. They are well-suited to form the basis for a graduate lecture course. The long list of exercises provided are a valuable addition.

Möller takes a look at Teichmüller curves from the point of view of an algebraic geometer. These are curves in moduli space that are geodesic with respect to the Teichmüller metric. He aims at relating the "quantities 'decomposition of the variation of Hodge structures', 'slopes ' and 'Lyapunov exponents' on Teichmüller curves." After some introductory sections, some of the author's own work [Möl], [CM] and several open problems are included.

Finally, **Matsumoto** takes an arithmetic view point. He explains how the absolute Galois group and its action on the profinite completion of the fundamental group of a surface are related to the mapping class group, and goes on to discuss various conjectures and results including the Deligne-Ihara conjecture recently proved by Brown [B].

There is a lot on offer in this collection of lecture notes. We hope that we have been able to convey that the whole of these proceedings is much more than the sum of its parts and, above all, that the moduli space of Riemann surfaces remains a fascinating object of study whatever tools are at one's disposal.

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