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Corresponding Author: Prof. Gregory Seregin,

Corresponding Author's Institution:

First Author: Gregory Seregin

Order of Authors: Gregory Seregin; Vladimir Sverak

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# On global weak solutions to the Cauchy problem for the Navier-Stokes equations with large $L_3$ -initial data

G. Seregin<sup>\*</sup>, V. Sverak<sup>†</sup>

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## Abstract

The aim of the note is to discuss different definitions of solutions to the Cauchy problem for the Navier-Stokes equations with the initial data belonging to the Lebesgue space  $L_3(\mathbb{R}^3)$

*Dedicated to Professor Nicola Fusco on the occasion of his 60th birthday.*

## 1 Introduction

We consider the classical Cauchy problem for the Navier-Stokes system, describing the flow of a viscous incompressible fluid:

$$\left. \begin{aligned} \partial_t v + v \cdot \nabla v - \Delta v &= -\nabla q \\ \operatorname{div} v &= 0 \end{aligned} \right\} \quad \text{in } Q_\infty = \mathbb{R}^3 \times ]0, \infty[ , \quad (1.1)$$

with

$$|v(x, t)| \rightarrow 0, \quad t > 0, |x| \rightarrow \infty \quad (1.2)$$

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and

$$v(\cdot, 0) = v_0(\cdot) \in L_3(\mathbb{R}^3) \quad (1.3)$$

where  $\operatorname{div} v_0 = 0$ .

There are essentially two methods for constructing the solutions: the perturbation theory and the energy method. In the first approach, we treat the non-linear term as a perturbation and try to find the best spaces in which such treatment is possible. The scaling symmetry of the equation

$$\begin{aligned} v(x, t) & \rightarrow v_\lambda(x, t) = \lambda v(\lambda x, \lambda^2 t) \\ q(x, t) & \rightarrow q_\lambda(x, t) = \lambda^2 q(\lambda x, \lambda^2 t) \end{aligned} \quad (1.4)$$

plays an important rôle in the choice of the function spaces, with the scale-invariant spaces being at the borderline of various families of spaces for which the method works. The most general result in this direction is due to Koch and Tataru [13]. The choice of  $L_3(\mathbb{R}^3)$  in (1.3) represents a well-known simple example of such a border-line space. The perturbation method cannot work for  $L_{3-\delta}(\mathbb{R}^3)$  for any  $\delta > 0$ . The perturbation approach goes back to the papers of Oseen and Leray [18, 17], but in the context of the scale-invariant spaces it was pioneered by Kato [7].

The energy method is based on the natural *a-priori* energy estimate

$$\int_{\mathbb{R}^3} |v(x, t)|^2 dx + \int_0^t \int_{\mathbb{R}^3} 2|\nabla v(x, t')|^2 dx dt' \leq \int_{\mathbb{R}^3} |v_0(x)|^2 dx, \quad (1.5)$$

and was pioneered by Leray in [17]. The natural condition on the initial data in the context of the energy method is  $v_0 \in L_2(\mathbb{R}^3)$ . The energy method gives global weak solutions for any initial data in  $L_2$ , but the regularity and, more importantly, uniqueness of the solutions is unknown, and possibly does not hold, see [8].

In many cases it is desirable to have a good theory of the weak solution for initial data  $v_0 \in L_3(\mathbb{R}^3)$ , but the original theory of the weak solutions, which needs  $v_0 \in L_2(\mathbb{R}^3)$ , does not cover that case. Various approaches have been developed to adapt the theory of the weak solutions so that it would allow  $v_0 \in L_3(\mathbb{R}^3)$ . For example, in the paper of Calderon [2] the author decomposes an  $L_3$  initial data  $v_0$  as

$$v_0 = v_0^1 + v_0^2 \quad (1.6)$$

so that  $v_0^1$  is small in  $L_3$  and  $v_0^2$  belongs to  $L_2 \cap L_3$ . Due to the smallness, the initial data  $v_0^1$  generates a global smooth solution  $v^1$

by perturbation theory, and we can write down the equation for  $v^2 = v - v^1$  and solve it via the energy method.

A more general approach, to be discussed in some detail below, was developed by Lemarie-Rieusset, see [16].

Here we consider another method for constructing global weak solutions for  $v_0 \in L_3(\mathbb{R}^3)$ . The method is very simple and, moreover, is easily extendable to problems in unbounded domains with boundaries. The method of Calderon probably also allows such extensions quite easily, whereas the extension of the (more general) concepts from [16] does not appear to be straightforward.

The main idea is as follows. Let  $v^1$  be the solution of the linear version of our problem (obtained from the original system simply by omitting the non-linear term). We now seek the solution  $v$  of the original non-linear problem as

$$v = v^1 + v^2. \quad (1.7)$$

It is easy to see that the ‘‘correction’’  $v^2$  should be in the energy class. The ‘‘first correction’’  $v^{21}$  is given by

$$v_t^{21} - \Delta v^{21} + \nabla q^{21} = -\operatorname{div} v^1 \otimes v^1. \quad (1.8)$$

We have

$$v^1 \otimes v^1 \in L_\infty([0, \infty[ , L_{\frac{3}{2}}(\mathbb{R}^3)) \cap L_{\frac{5}{2}}([0, \infty[ , L_{\frac{5}{2}}(\mathbb{R}^3)) \subset L_4([0, \infty[ , L_2(\mathbb{R}^3)), \quad (1.9)$$

Hence

$$v^1 \otimes v^1 \in L_2(\mathbb{R}^3 \times (0, T)) \quad (1.10)$$

for every  $T > 0$ , which is enough to have  $v^{21}$  in the energy class on every bounded time interval. From this it is heuristically clear that we should have  $v = v^1 + v^2$ , where  $v^2$  is in the energy class on every bounded time interval. The general idea that the correction  $v^2$  might be easier to deal with than the full solution  $v$  is standard, and has been already suggested by considerations in Leray’s classical paper [17], and also been often used in the works on other PDEs.

We now discuss more technical details. We start with the definition of the *mild solutions* solutions, which is usually considered in connection with the perturbation method for the problem (1.1) – (1.3).

**Definition 1.1.** A function  $u \in C([0, T]; L_3(\mathbb{R}^3)) \cap L_5(Q_T)$ , satisfying the identity

$$v(x, t) = \int_{\mathbb{R}^3} \Gamma(x-y, t) v_0(x) dx + \int_0^t \int_{\mathbb{R}^3} K(x, y, t-s) : v(y, s) \otimes v(y, s) dy ds \quad (1.11)$$

for all  $z = (x, t) \in Q_T := \mathbb{R}^3 \times ]0, T[$ , is called a mild solution to problem (1.1)–(1.3) in  $Q_T$ .

Here,  $\Gamma$  is the known heat kernel and a kernel  $K$  is derived with the help of  $\Gamma$  as follows:

$$\Delta_y \Phi(x, y, t) = \Gamma(x - y, t),$$

$$K_{mjs}(x, y, t) = \delta_{mj} \frac{\partial^3 \Phi}{\partial y_i \partial y_i \partial y_s}(x, y, t) - \frac{\partial^3 \Phi}{\partial y_m \partial y_j \partial y_s}(x, y, t).$$

Although mild solutions are known to be unique, their global existence is an open problem. So, they exist locally in time and, moreover, in proofs that are available to the authors, the time interval of the existence of mild solutions depends not only on the value  $\|v_0\|_{3, \mathbb{R}^3}$  but on the integral modulus of continuity of  $v_0$  in  $L_3(\mathbb{R}^3)$  as well.

The classical existence results about the weak solutions rely upon relatively simple considerations based on the energy estimate (1.5). The main issue in this approach is the uniqueness, which at the moment can only be proved via regularity. For our set up, such a notion of weak solutions is already known due to Lemarie-Rieusset, see [16]. In this paper, we shall call them local energy Leray-Hopf solutions or just local energy solutions or even just Lemarier-Rieusset solutions. The important feature of those solutions is that the very rich  $\varepsilon$ -regularity theory developed by Caffarelli-Kohn-Nirenberg is applicable to them. Here, it is a definition, which is essentially given by Lemarie-Rieusset, see also [9].

**Definition 1.2.** We call a pair of functions  $v$  and  $q$  defined in the space-time cylinder  $Q_T = \mathbb{R}^3 \times ]0, T[$  a local energy weak Leray-Hopf solution to the Cauchy problem (1.1)–(1.3) if they satisfy the following conditions:

$$v \in L_\infty(0, T; L_{2, \text{unif}}), \nabla v \in L_{2, \text{unif}}(0, T), q \in L_{\frac{3}{2}}(0, T; L_{\frac{3}{2}, \text{loc}}(\mathbb{R}^3)) \quad (1.12)$$

$$v \text{ and } q \text{ meet (1.1) in the sense of distributions;} \quad (1.13)$$

the function  $t \mapsto \int_{\mathbb{R}^3} v(x, t) \cdot w(x) dx$  is continuous on  $[0, T]$  (1.14)

for any compactly supported function  $w \in L_2(\mathbb{R}^3)$ ;  
for any compact  $K$ ,

$$\|v(\cdot, t) - v_0(\cdot)\|_{L_2(K)} \rightarrow 0 \quad \text{as } t \rightarrow +0; \quad (1.15)$$

$$\int_{\mathbb{R}^3} \varphi |v(x, t)|^2 dx + 2 \int_0^t \int_{\mathbb{R}^3} \varphi |\nabla v|^2 dx dt \leq \int_0^t \int_{\mathbb{R}^3} \left( |v|^2 (\partial_t \varphi + \Delta \varphi) + v \cdot \nabla \varphi (|v|^2 + 2q) \right) dx dt \quad (1.16)$$

for a.a.  $t \in ]0, T[$  and for all nonnegative smooth functions  $\varphi$  vanishing in a neighborhood of the parabolic boundary of the space-time cylinder  $\mathbb{R}^3 \times ]0, T[$ ;

for any  $x_0 \in \mathbb{R}^3$ , there exists a function  $c_{x_0} \in L_{\frac{3}{2}}(0, T)$  such that

$$q_{x_0}(x, t) := q(x, t) - c_{x_0}(t) = q_{x_0}^1(x, t) + q_{x_0}^2(x, t), \quad (1.17)$$

for  $(x, t) \in B(x_0, 3/2) \times ]0, T[$ , where

$$q_{x_0}^1(x, t) = -\frac{1}{3} |v(x, t)|^2 + \frac{1}{4\pi} \int_{B(x_0, 2)} K(x - y) : v(y, t) \otimes v(y, t) dy,$$

$$q_{x_0}^2(x, t) = \frac{1}{4\pi} \int_{\mathbb{R}^3 \setminus B(x_0, 2)} (K(x - y) - K(x_0 - y)) : v(y, t) \otimes v(y, t) dy$$

and  $K(x) = \nabla^2(1/|x|)$ .

Here, marginal Morrey spaces

$$L_{m,unif} := \{u \in L_{m,loc}(\mathbb{R}^3) : \|u\|_{2,unif} = \sup_{x_0 \in \mathbb{R}^3} \|u\|_{m,B(x_0,1)} < \infty\}$$

and

$$L_{m,unif}(0, T) := \{u \in L_m(0, T; L_{m,loc}(\mathbb{R}^3)) : \sup_{x_0 \in B(x_0, 1)} \int_0^T \int_{B(x_0, 1)} |u(x, t)|^m dx dt < \infty\}$$

$< \infty\}$

have been used.

Lemarie-Rieusset proved local in time existence of a local energy solution for  $v_0 \in L_{2,unif}$ . But, what seems to be more important, he showed that if  $v_0 \in E_2$ , where  $E_m$  is the completion of  $C_{0,0}^\infty(\mathbb{R}^3) := \{v \in C_0^\infty(\mathbb{R}^3) : \operatorname{div} v = 0\}$  in  $L_m(\mathbb{R}^3)$ , then the above solution exists globally, i.e., for any  $T > 0$ . The corresponding uniqueness theorem is also true saying that if one has two local energy solutions to (1.1)–(1.3) with the same initial data and one of them belongs to  $C([0, T]; E_3)$ , then they coincide on the interval  $]0, T[$ . It should be noticed that the space  $L_3(\mathbb{R}^3)$  is continuously imbedded into  $E_3$  and of course into  $E_2$ . So, in this sense, local energy solutions can be regarded as a possible tool to study the case of initial data belonging to  $L_3(\mathbb{R}^3)$ . That has been exploited in the paper [25] on the behaviour of  $L_3$ -norm of a solution as time tends to a possible blow up. Moreover, as it has been shown there, the limit of a sequence of solutions with weakly converging  $L_3$ -initial data is a local energy solution as well. By the way, the same has been proven for initial data from  $H^{\frac{1}{2}}$  in papers, see [19] and [23]. However, the aforesaid scheme does not work in the case of unbounded domains say as a half space  $\mathbb{R}_+^3$ . The reason is simple: it is unknown how to construct local energy solutions in unbounded domains that are different from  $\mathbb{R}^3$ .

The aim of the presented note is to give a definition of global weak solutions to the Cauchy problem for the Navier-Stokes system with  $L_3$ -initial data that it is not based on the conception of local energy solutions. This approach seems to be interesting itself and certainly simplifies the above mentioned proofs in papers [23] and [25]. Moreover, it works well for other unbounded domains.

The new definition relies on two simple facts. Consider a Stokes problem:

$$\partial_t v^1 - \Delta v^1 = -\nabla q^1, \quad \operatorname{div} v^1 = 0 \quad (1.18)$$

in  $\Omega \times ]0, \infty[$ ,

$$v^1(x, t) = 0 \quad (1.19)$$

for all  $(x, t) \in \partial\Omega \times [0, \infty[$ , and

$$v^1(\cdot, 0) = v_0(\cdot) \in L_3(\Omega). \quad (1.20)$$

Assume that  $\Omega \subset \mathbb{R}^3$  is so good that  $v^1$  obeys the following estimates:

$$\|v^1\|_{3,\infty,\Omega \times ]0,\infty[} + \|v^1\|_{5,\Omega \times ]0,\infty[} < c\|v_0\|_{3,\Omega} \quad (1.21)$$

and

$$\|\nabla v^1(\cdot, t)\|_{3,\Omega} \leq \frac{c}{\sqrt{t}} \|v_0\|_{3,\Omega} \quad (1.22)$$

for all  $t > 0$ . It is well known that (1.21) and (1.22) are satisfied if  $\Omega = \mathbb{R}^3$  or if  $\Omega = \mathbb{R}_+^3$  (for other cases, see [6]).

In what follows, it is assumed that  $\Omega = \mathbb{R}^3$  and thus we may let  $q^1 = 0$ . The general case will be discussed elsewhere.

**Definition 1.3.** *Let  $v_0 \in L_3(\mathbb{R}^3)$  and let  $v^1$  be a solution to problem (1.18)–(1.20). A function  $v$ , defined in  $Q_\infty = \mathbb{R}^3 \times ]0, \infty[$ , is called a weak  $L_3$ -solution to problem (1.1)–(1.3) if  $v = v^1 + v^2$ , where  $v^2 \in L_{2,\infty}(Q_T) \cap W_2^{1,0}(Q_T)$  with  $Q_T = \mathbb{R}^3 \times ]0, T[$  for any  $T > 0$  and satisfies the following conditions:*

$$\partial_t v^2 + v^2 \cdot \nabla v^2 - \Delta v^2 + \nabla q^2 = -v^1 \cdot \nabla v^2 - v^2 \cdot \nabla v^1 - v^1 \cdot \nabla v^1, \quad \operatorname{div} v^2 = 0 \quad (1.23)$$

in  $Q_\infty$  in the sense of distributions with  $q^2 \in L_{\frac{3}{2}}(0, T; L_{\frac{3}{2},loc}(\mathbb{R}^3))$  for  $T > 0$ ;

for all  $w \in L_2(\mathbb{R}^3)$ , the function

$$t \mapsto \int_{\mathbb{R}^3} v^2(x, t) \cdot w(x) dx \quad (1.24)$$

is continuous at any  $t \in [0, \infty[$ ;

$$\|v^2(\cdot, t)\|_{2,\mathbb{R}^3} \rightarrow 0 \quad (1.25)$$

as  $t \downarrow 0$ ;

$$\frac{1}{2} \int_{\mathbb{R}^3} |v^2(x, t)|^2 dx + \int_0^t \int_{\mathbb{R}^3} |\nabla v^2|^2 dx ds \leq \int_0^t \int_{\mathbb{R}^3} v^1 \otimes v : \nabla v^2 dx ds \quad (1.26)$$

for all  $t > 0$ ;

for a.a.  $t > 0$

$$\begin{aligned} & \int_{\mathbb{R}^3} \varphi(x, t) |v^2(x, t)|^2 dx + 2 \int_0^t \int_{\mathbb{R}^3} \varphi |\nabla v^2|^2 dx ds \leq \\ & \leq \int_0^t \int_{\mathbb{R}^3} \left( 2v^1 \otimes v : \nabla v^2 \varphi + |v^2|^2 (\Delta \varphi + \partial_t \varphi) \right) dx ds \end{aligned} \quad (1.27)$$

$$+v \cdot \nabla \varphi(|v^2|^2 + 2q^2 + 2v^1 \cdot v^2) dx ds$$

for any non-negative function  $\varphi \in C_0^\infty(Q_\infty)$ .

**Remark 1.4.** If  $\|v_0\|_{3,\mathbb{R}^3} \leq M$ , then

$$\|v^2\|_{2,Q_T}^2 := \|v^2\|_{2,\infty,Q_T}^2 + \|\nabla v^2\|_{2,Q_T}^2 \leq c(M)\sqrt{T} \quad (1.28)$$

**Remark 1.5.** Indeed, any weak  $L_3$ -solution is a local energy solution.

**Theorem 1.6.** Problem (1.1)–(1.3) has at least one weak  $L_3$ -solution.

The following important property of weak  $L_3$ -solutions, in fact, can be regarded as another strong motivation for introducing them. To explain it, let us consider a sequence  $v_0^{(m)} \in L_3(\mathbb{R}^3)$  such that

$$v_0^{(m)} \rightharpoonup v_0$$

in  $L_3(\mathbb{R}^3)$ . We denote by  $u^{(m)}$  a weak solution to the Cauchy problem (1.1)–(1.3) with initial data  $v_0^{(m)}$ .

**Theorem 1.7.** There exists a subsequence of  $u^{(m)}$  (still denoted by  $u^{(m)}$ ) such that:

$$u^{(m)} \rightharpoonup u, \quad \nabla u^{(m)} \rightharpoonup \nabla u$$

in  $L_2(Q_T)$  for any  $T > 0$  and

$$u^{(m)} \rightarrow u$$

in  $L_{3,\text{loc}}(Q_\infty)$ , where  $u$  is a weak  $L_3$ -solution to the Cauchy problem (1.1)–(1.3) with initial data  $v_0$ .

Now, let us discuss some uniqueness issues related to weak  $L_3$ -solutions to the Cauchy problem (1.1)–(1.3). We start with the following auxiliary statement:

**Proposition 1.8.** Let  $v$  and  $\tilde{v}$  be two weak  $L_3$ -solutions to the the Cauchy problem for the Navier-Stokes equations corresponding to the initial data  $v_0 \in L_3(\mathbb{R}^3)$ . Suppose that  $v \in L_{3,\infty,Q_T}$ . There exists an absolute constant  $\mu > 0$  such that if, for some number  $0 < T_1 \leq T$ ,

$$\|v - v_0\|_{3,\infty,Q_{T_1}} \leq \mu, \quad (1.29)$$

then  $v = \tilde{v}$  in  $Q_T$ .

**Remark 1.9.** *Condition (1.29) holds if*

$$\lim_{t \downarrow 0} \|v(\cdot, t) - v_0(\cdot)\|_{3, \mathbb{R}^3} = 0. \quad (1.30)$$

An elementary modification of the final part of the proof of Proposition 1.8 gives the following statement.

**Theorem 1.10.** *Let  $v$  and  $\tilde{v}$  be two weak  $L_3$ -solutions to the the Cauchy problem for the Navier-Stokes equations corresponding to the initial data  $v_0 \in L_3(\mathbb{R}^3)$ . Suppose that  $v^2 \in L_5(Q_T)$ . Then  $v = \tilde{v}$  in  $Q_T$ .*

**Theorem 1.11.** *Let  $v$  be a weak  $L_3$ -solutions to the Cauchy problem for the Navier-Stokes equations corresponding to the initial data  $v_0 \in L_3(\mathbb{R}^3)$ . Then there exists  $T_0 = T(v_0) > 0$  such that  $v \in L_5(Q_{T_0})$ .*

**Remark 1.12.** *From the proof of Theorem 1.11, it follows also that  $v^2 \in L_{3, \infty}(Q_{T_0})$ .*

Our final result is (see also [12] for a different set up):

**Theorem 1.13.** *Let  $v$  and  $\tilde{v}$  be weak  $L_3$ -solutions to the Cauchy problem (1.1)–(1.3) with the same initial data  $v_0$  from  $L_3(\mathbb{R}^3)$ . Let  $v \in L_{3, \infty}(Q_T)$ . Then  $v = \tilde{v}$  in  $Q_T$ .*

Our paper is rather expository and some statements in it have been already known. We prove them in order to demonstrate how our new conception of weak  $L_3$ -solutions works and that it is in a good accordance with the previous definitions of solutions to the Cauchy problem (1.1)–(1.3). We recommend papers [3], [5], [6], [7] and monographs [14] and [16] for more details and references.

The paper is organized as follows. In the second section, the existence of weak  $L_3$ -solutions is proven. Sequences of weak  $L_3$ -solutions are studied in the third section. The uniqueness of weak  $L_3$ -solutions and related questions are discussed in fourth section. To make the paper more or less self-contained, we give a simple proof of the existence of mild solutions with the initial data from  $L_3(\mathbb{R}^3)$  in the Appendix.

## 2 Existence

In this section, we are going to prove Theorem 1.6.

The first step of our proof is to solve the problem in bounded domains  $\Omega = B(R)$ . We do this in a standard way by considering several simple linear problems and applying Leray-Schauder principle.

Assume that

$$a \in \mathring{J}(\Omega), \quad (2.1)$$

where the space  $\mathring{J}(\Omega)$  is the completion of  $C_{0,0}^\infty(\Omega)$  in  $L_2(\Omega)$ .

**Proposition 2.1.** *Let  $Q_T = \Omega \times ]0, T[$  and*

$$w^1, w \in L_\infty(Q_T), \quad \operatorname{div} w = 0 \quad \text{in } Q_T. \quad (2.2)$$

*There exists a unique solution  $u$  to the initial boundary value problem*

$$\begin{aligned} \partial_t u - \Delta u + \operatorname{div} u \otimes w + \nabla p = \\ = -\operatorname{div}(w^1 \otimes u + u \otimes w^1 + w^1 \otimes w^1), \quad \operatorname{div} u = 0 \quad \text{in } Q_T, \\ u|_{\partial\Omega \times [0, T]} = 0, \quad (2.3) \\ u|_{t=0} = a \end{aligned}$$

*in the following sense:*

$$u \in C([0, T]; L_2(\Omega)) \cap L_2(0, T; \mathring{J}_2^1(\Omega)), \quad \partial_t u \in L_2(0, T; (\mathring{J}_2^1(\Omega))');$$

*for a.a.  $t \in [0, T]$*

$$\begin{aligned} & \int_{\Omega} (\partial_t u(x, t) \cdot \tilde{v}(x) + \nabla u(x, t) : \nabla \tilde{v}(x)) dx \\ &= \int_{\Omega} (u(x, t) \otimes w(x, t) + w^1(x, t) \otimes u(x, t) + \\ &+ u(x, t) \otimes w^1(x, t) + w^1(x, t) \otimes w^1(x, t)) : \nabla \tilde{v}(x) dx \end{aligned} \quad (2.4)$$

*for all  $\tilde{v} \in \mathring{J}_2^1(\Omega)$ ;*

$$\|u(\cdot, t) - a(\cdot)\|_{2, \Omega} \rightarrow 0 \quad (2.5)$$

*as  $t \rightarrow +0$ .*

Here, we have used the notation  $\mathring{J}_r^1(\Omega)$  for the completion of  $C_{0,0}^\infty(\Omega)$  in the Sobolev space  $W_r^1(\Omega)$ .

*Proof.* We are going to apply the Leray-Schauder principle. To this end, let

$$X = L_2(0, T; \mathring{J}(\Omega)).$$

Given  $u \in X$ , define  $v = A(u)$  as a solution to the following problem:

$$v \in C([0, T]; L_2(\Omega)) \cap L_2(0, T; \mathring{J}_2^1(\Omega)), \quad \partial_t v \in L_2(0, T; (\mathring{J}_2^1(\Omega))'); \quad (2.6)$$

for a.a.  $t \in [0, T]$

$$\begin{aligned} \int_{\Omega} (\partial_t v(x, t) \cdot \tilde{v}(x) + \nabla v(x, t) : \nabla \tilde{v}(x)) dx \\ = \int_{\Omega} \tilde{f}(x, t) \cdot \tilde{v}(x) dx \end{aligned} \quad (2.7)$$

for all  $\tilde{v} \in \mathring{J}_2^1(\Omega)$ ;

$$\|v(\cdot, t) - a(\cdot)\|_{2, \Omega} \rightarrow 0 \quad (2.8)$$

as  $t \rightarrow +0$ . Here,  $\tilde{f} = -\operatorname{div}(u \otimes w + w^1 \otimes u + u \otimes w^1 + w^1 \otimes w^1)$ .

Such a function  $v$  exists and is unique (for given  $u$ ) since

$$\tilde{f} \in L_2(0, T; (\mathring{J}_2^1(\Omega))').$$

So, the operator  $A$  is well defined. Let us check that it satisfies all the requirements of the Leray-Schauder principle.

Continuity: Let  $v^1 = A(u^1)$  and  $v^2 = A(u^2)$ . Then

$$\begin{aligned} \int_{\Omega} (\partial_t(v^1 - v^2) \cdot \tilde{v} + \nabla(v^1 - v^2) : \nabla \tilde{v}) dx = \int_{\Omega} \left( (u^1 - u^2) \otimes w + w^1 \otimes (u^1 - u^2) + \right. \\ \left. + (u^1 - u^2) \otimes w^1 \right) : \nabla \tilde{v} dx \end{aligned}$$

and letting  $\tilde{v} = v^1 - v^2$ , we find

$$\frac{1}{2} \partial_t \|v^1 - v^2\|_{2, \Omega}^2 + \|\nabla v^1 - \nabla v^2\|_{2, \Omega}^2 \leq c(w, w^1) \|u^1 - u^2\|_{2, \Omega} \|\nabla v^1 - \nabla v^2\|_{2, \Omega}$$

and thus

$$\sup_{0 < t < T} \|v^1 - v^2\|_{2, \Omega} \leq c(w, w^1) \|u^1 - u^2\|_{2, Q_T}.$$

The latter implies continuity.

Compactness: As in the previous case, we use the energy estimate

$$\sup_{0 < t < T} \|v\|_{2,\Omega}^2 + \|\nabla v\|_{2,\Omega}^2 \leq c(w, w^1)(\|u\|_{2,\Omega}^2 + 1).$$

The second estimate comes from (2.7) and has the form

$$\|\partial_t v\|_{L_2(0,T;(\mathring{J}_2^1(\Omega))')}^2 \leq \|\nabla v\|_{Q_T}^2 + c(w, w^1)(\|u\|_{2,\Omega}^2 + 1).$$

Combining the above bounds, we observe that sets, which are bounded in  $X$ , remain to be bounded in

$$W = \{w \in L_2(0, T; \mathring{J}_2^1(\Omega)), \quad \partial_t w \in L_2(0, T; (\mathring{J}_2^1(\Omega))')\}.$$

Now, for  $v = \lambda A(v)$  with  $\lambda \in [0, 1]$ , after integration by parts, we find that, for a.a.  $t \in [0, T]$ , the identity

$$\int_{\Omega} (\partial_t v \cdot \tilde{v} + \nabla v : \nabla \tilde{v}) dx = \lambda \int_{\Omega} (w^1 \otimes v + v \otimes w^1 + w^1 \otimes w^1) : \nabla \tilde{v} - (w \cdot \nabla v) \cdot \tilde{v} dx$$

holds for any  $\tilde{v} \in \mathring{J}_2^1(\Omega)$ . If we insert  $\tilde{v}(\cdot) = v(\cdot, t)$  into the latter relation, then another identity

$$\int_{\Omega} (w \cdot \nabla v) \cdot v dx = 0$$

ensures the following estimate:

$$\frac{1}{2} \partial_t \int_{\Omega} |v|^2 dx + \int_{\Omega} |\nabla v|^2 dx \leq c(w^1)(\|v\|_{2,\Omega} + 1) \|\nabla v\|_{2,\Omega}.$$

Hence,

$$\partial_t \int_{\Omega} |v|^2 dx \leq c(w^1)(\|v\|_{2,\Omega}^2 + 1)$$

and thus

$$\|v\|_{Q_T}^2 \leq c(\|w^1\|_{\infty,\Omega}, T, \|a\|_{2,\Omega}^2) = R^2.$$

Now, all the statements of Proposition 2.1 follow from the Leray-Schauder principle. Proposition 2.1 is proved.  $\square$

Let  $\omega_\varrho$  be a standard mollifier and let

$$(u)_\varrho(x, t) = \int_{\Omega} \omega_\varrho(x-x')u(x', t)dx', \quad (v^1)_\varrho(x, t) = \int_{\mathbb{R}^3} \omega_\varrho(x-x')v^1(x', t)dx'.$$

It is easy to check that  $\operatorname{div}(u)_\varrho(\cdot, t) = 0$  if  $t \mapsto u(\cdot, t) \in \mathring{J}(\Omega)$ .

Now, we wish to show that, given  $\varrho > 0$ , there exists at least one function  $u^\varrho$  such that:

$$u^\varrho \in C([0, T]; L_2(\Omega)) \cap L_2(0, T; \mathring{J}_2^1(\Omega)), \quad \partial_t u^\varrho \in L_2(0, T; (\mathring{J}_2^1(\Omega))'); \quad (2.9)$$

for a.a.  $t \in [0, T]$

$$\begin{aligned} & \int_{\Omega} (\partial_t u^\varrho(x, t) \cdot \tilde{v}(x) + \nabla u^\varrho(x, t) : \nabla \tilde{v}(x)) dx \\ &= \int_{\Omega} (u^\varrho(x, t) \otimes (u^\varrho)_\varrho(x, t) + (v^1)_\varrho(x, t) \otimes u^\varrho(x, t) + \\ & u^\varrho(x, t) \otimes (v^1)_\varrho(x, t) + (v^1)_\varrho(x, t) \otimes (v^1)_\varrho(x, t)) : \nabla \tilde{v}(x) dx \end{aligned} \quad (2.10)$$

for all  $\tilde{v} \in \mathring{J}_2^1(\Omega)$ ;

$$\|u^\varrho(\cdot, t) - a(\cdot)\|_{2, \Omega} \rightarrow 0 \quad (2.11)$$

as  $t \rightarrow +0$ .

We notice that (2.9)-(2.11) can be regarded as a weak form of the following initial boundary value problem

$$\begin{aligned} \partial_t u^\varrho - \Delta u^\varrho + (u^\varrho)_\varrho \cdot \nabla u^\varrho + \nabla p^\varrho &= f, \quad \operatorname{div} u^\varrho = 0 \quad \text{in } Q_T, \\ u^\varrho|_{\partial\Omega \times [0, T]} &= 0, \quad (2.12) \\ u^\varrho|_{t=0} &= a \end{aligned}$$

with  $f = -\operatorname{div}(u^\varrho \otimes (v^1)_\varrho + (v^1)_\varrho \otimes u^\varrho + (v^1)_\varrho \otimes (v^1)_\varrho)$ .

**Proposition 2.2.** *There exists a unique function  $u^\varrho$  defined on  $Q_\infty$  such that it satisfies (2.9)-(2.11) for any  $T > 0$ .*

*Proof.* Let us fix an arbitrary  $T > 0$ .

To simplify our notation, let us drop upper index  $\varrho$  for a moment. The idea is the same as in Proposition 2.1: to use the Leray-Schauder principle. The space  $X$  is the same as in the proof of Proposition 2.1.

But the operator  $A$  will be defined in a different way: given  $u \in X$ , we are looking for  $w = A(u)$  so that

$$w \in C([0, T]; L_2(\Omega)) \cap L_2(0, T; \mathring{J}_2^1(\Omega)), \quad \partial_t w \in L_2(0, T; (\mathring{J}_2^1(\Omega))'); \quad (2.13)$$

for a.a.  $t \in [0, T]$

$$\begin{aligned} & \int_{\Omega} (\partial_t w(x, t) \cdot \tilde{v}(x) + \nabla w(x, t) : \nabla \tilde{v}(x)) dx \\ &= \int_{\Omega} (w(x, t) \otimes (u)_{\varrho}(x, t) + (v^1)_{\varrho}(x, t) \otimes w(x, t) + \\ &+ w(x, t) \otimes (v^1)_{\varrho}(x, t) + (v^1)_{\varrho}(x, t) \otimes (v^1)_{\varrho}(x, t)) : \nabla \tilde{v}(x) dx \end{aligned} \quad (2.14)$$

for all  $\tilde{v} \in \mathring{J}_2^1(\Omega)$ ;

$$\|w(\cdot, t) - a(\cdot)\|_{2, \Omega} \rightarrow 0 \quad (2.15)$$

as  $t \rightarrow +0$ . By Proposition 2.1, such a function exists and is unique.

Continuity: Do the same as in Proposition 2.1:

$$\begin{aligned} I &:= \frac{1}{2} \partial_t \|w^2 - w^1\|_{2, \Omega}^2 + \|\nabla(w^2 - w^1)\|_{2, \Omega}^2 \\ &= \int_{\Omega} \left( w^2 \otimes (u^2)_{\varrho} - w^1 \otimes (u^1)_{\varrho} \right) : \nabla(w^2 - w^1) dx \\ &+ \int_{\Omega} \left( (v^1)_{\varrho} \otimes (w^2 - w^1) + (w^2 - w^1) \otimes (v^1)_{\varrho} \right) : \nabla(w^2 - w^1) dx \\ &= \int_{\Omega} (w^2 - w^1) \otimes (u^2)_{\varrho} : \nabla(w^2 - w^1) dx \\ &+ \int_{\Omega} w^1 \otimes (u^2 - u^1)_{\varrho} : \nabla(w^2 - w^1) dx \\ &+ \int_{\Omega} \left( (v^1)_{\varrho} \otimes (w^2 - w^1) + (w^2 - w^1) \otimes (v^1)_{\varrho} \right) : \nabla(w^2 - w^1) dx. \end{aligned}$$

The first integral in the right hand side of the above identity vanishes.

Hence,

$$\begin{aligned} I &\leq \|(u^2 - u^1)_{\varrho}\|_{\infty, \Omega} \|w^1\|_{2, \Omega} \|\nabla(w^2 - w^1)\|_{2, \Omega} \\ &+ c \|(v^1)_{\varrho}\|_{\infty, \Omega} \|(w^2 - w^1)\|_{2, \Omega} \|\nabla(w^2 - w^1)\|_{2, \Omega}. \end{aligned}$$

It follows from the Hölder inequality that

$$\begin{aligned} & \partial_t \|w^2 - w^1\|_{2,\Omega}^2 + \|\nabla(w^2 - w^1)\|_{2,\Omega}^2 \leq \\ & \leq c(\varrho) \|w^1\|_{2,\Omega}^2 \|u^2 - u^1\|_{2,\Omega}^2 + c(\varrho) \|v_0\|_{3,\mathbb{R}^3}^2 \|(w^2 - w^1)\|_{2,\Omega}^2 \end{aligned}$$

and thus

$$\|w^2 - w^1\|_{2,Q_T}^2 \leq c(\varrho, T, \|w^1\|_{2,\infty,Q_T}, \|v_0\|_{3,\mathbb{R}^3}) \|u^2 - u^1\|_{2,Q_T}^2.$$

The latter gives us continuity.

Compactness: In our case, the usual energy estimate implies the following:

$$\begin{aligned} & \frac{1}{2} \partial_t \|w\|_{2,\Omega}^2 + \|\nabla w\|_{2,\Omega}^2 = - \int_{\Omega} \left( w \otimes (u)_\varrho + (v^1)_\varrho \otimes w + \right. \\ & \quad \left. + w \otimes (v^1)_\varrho + (v^1)_\varrho \otimes (v^1)_\varrho \right) : \nabla w dx = \\ & = - \int_{\Omega} \left( (v^1)_\varrho \otimes w + w \otimes (v^1)_\varrho + (v^1)_\varrho \otimes (v^1)_\varrho \right) : \nabla w dx \\ & \leq c(\|(v^1)_\varrho\|_{5,\Omega} \|w\|_{\frac{10}{3},\Omega} + \|(v^1)_\varrho\|_{4,\Omega}^2) \|\nabla w\|_{2,\Omega} \leq \quad (2.16) \\ & \leq c(\|v^1\|_{5,\Omega} \|w\|_{\frac{10}{3},\Omega} + \|v^1\|_{4,\Omega}^2) \|\nabla w\|_{2,\Omega} \leq \\ & \leq c\|v^1\|_{5,\Omega} \|w\|_{2,\Omega}^{\frac{2}{5}} \|\nabla w\|_{2,\Omega}^{\frac{8}{5}} + c\|v^1\|_{4,\Omega}^2 \|\nabla w\|_{2,\Omega} \leq \\ & \leq \frac{1}{2} \|\nabla w\|_{2,\Omega}^2 + c\|v^1\|_{5,\Omega}^5 \|w\|_{2,\Omega}^2 + c\|v^1\|_{4,\Omega}^4. \end{aligned}$$

Since

$$\|v^1\|_{4,\Omega} \leq \|v^1\|_{3,\Omega}^{\frac{3}{8}} \|v^1\|_{5,\Omega}^{\frac{5}{8}},$$

we have

$$\partial_t \|w\|_{2,\Omega}^2 + \|\nabla w\|_{2,\Omega}^2 \leq c\|v^1\|_{5,\Omega}^5 \|w\|_{2,\Omega}^2 + c\|v^1\|_{3,\Omega}^{\frac{3}{2}} \|v^1\|_{5,\Omega}^{\frac{5}{2}}.$$

Then

$$\begin{aligned} & \partial_t \left( \|w\|_{2,\Omega}^2 \exp \left( -c \int_0^t \|v^1\|_{5,\Omega}^5 ds \right) \right) \leq \\ & \leq c\|v^1\|_{3,\Omega}^{\frac{3}{2}} \|v^1\|_{5,\Omega}^{\frac{5}{2}} \exp \left( -c \int_0^t \|v^1\|_{5,\Omega}^5 ds \right) \end{aligned}$$

and thus

$$\begin{aligned}
\|w\|_{2,\Omega}^2 &\leq \|a\|_{2,\Omega}^2 \exp\left(c \int_0^t \|v^1\|_{5,\Omega}^5 ds\right) + \\
&+ c \int_0^t \|v^1(\tau)\|_{3,\Omega}^{\frac{3}{2}} \|v^1(\tau)\|_{5,\Omega}^{\frac{5}{2}} \exp\left(c \int_\tau^t \|v^1(s)\|_{5,\Omega}^5 ds\right) d\tau \leq \\
&\leq \|a\|_{2,\Omega}^2 \exp\left(c \int_0^t \|v^1\|_{5,\Omega}^5 ds\right) + \\
&+ c \|v^1\|_{3,\infty,Q_T}^{\frac{3}{2}} \int_0^t \|v^1(\tau)\|_{5,\Omega}^{\frac{5}{2}} \exp\left(c \int_\tau^t \|v^1(s)\|_{5,\Omega}^5 ds\right) d\tau.
\end{aligned}$$

Now, a rough estimate looks like:

$$\begin{aligned}
\|w\|_{2,\Omega}^2 &\leq \exp\left(c \int_0^t \|v^1\|_{5,\Omega}^5 ds\right) \left(\|a\|_{2,\Omega}^2 + c \|v^1\|_{3,\infty,Q_T}^{\frac{3}{2}} \int_0^t \|v^1(\tau)\|_{5,\Omega}^{\frac{5}{2}} d\tau\right) \leq \\
&\leq \exp\left(c \int_0^t \|v^1\|_{5,\Omega}^5 ds\right) \left(\|a\|_{2,\Omega}^2 + c \|v^1\|_{3,\infty,Q_T}^{\frac{3}{2}} \sqrt{t} \|v^1\|_{5,Q_T}^{\frac{5}{2}}\right).
\end{aligned}$$

So, we have

$$\|w\|_{2,\infty,Q_T}^2 \leq \exp\left(c \|v^1\|_{5,Q_T}^5\right) \left(\|a\|_{2,\Omega}^2 + c \|v^1\|_{3,\infty,Q_T}^{\frac{3}{2}} \sqrt{T} \|v^1\|_{5,Q_T}^{\frac{5}{2}}\right)$$

and

$$\|\nabla w\|_{2,Q_T}^2 \leq \|a\|_{2,\Omega}^2 + c \|w\|_{2,\infty,Q}^2 \|v^1\|_{5,Q_T}^5 + c \|v^1\|_{3,\infty,Q_T}^{\frac{3}{2}} \sqrt{T} \|v^1\|_{5,Q_T}^{\frac{5}{2}}.$$

Hence, the required energy estimate takes the form

$$|w|_{2,Q_T}^2 \leq c(\|v^1\|_{5,Q_T}, \|v^1\|_{3,\infty,Q_T}, T, \|a\|_{2,\Omega}), \quad (2.17)$$

where a constant  $c$  is independent of  $\varrho$ .

**Remark 2.3.** *If  $a = 0$ , then we can see how constants depends on  $T$*

$$|w|_{2,Q_T}^2 \leq c(\|v_0\|_{3,\infty}) \sqrt{T}. \quad (2.18)$$

Now, we need to evaluate the first derivative in time. Indeed,

$$\begin{aligned}
\|\partial_t w\|_{(J_2^1(\Omega))'}^2 &\leq c\|\nabla w\|_{2,\Omega}^2 + c \int_{\Omega} (|w|^2|(v^1)_\varrho|^2 + |(v^1)_\varrho|^2|w|^2 + |(v^1)_\varrho|^4) dx \\
&\leq c\|\nabla w\|_{2,\Omega}^2 + c(\varrho)\|w\|_{2,\infty,Q_T}^2 \|u\|_{2,\Omega}^2 + c \int_{\Omega} (|(v^1)_\varrho|^2|w|^2 + |(v^1)_\varrho|^4) dx \leq \\
&\leq c\|\nabla w\|_{2,\Omega}^2 + c(\varrho)\|w\|_{2,\infty,Q_T}^2 \|u\|_{2,\Omega}^2 + c(\|(v^1)_\varrho\|_{5,\Omega}^2 \|w\|_{\frac{10}{3},\Omega}^2 + \|(v^1)_\varrho\|_{4,\Omega}^4) \leq \\
&\leq c\|\nabla w\|_{2,\Omega}^2 + c(\varrho)\|w\|_{2,\infty,Q_T}^2 \|u\|_{2,\Omega}^2 + c(\|v\|_{5,\Omega}^2 \|w\|_{\frac{10}{3},\Omega}^2 + \|v^1\|_{4,\Omega}^4)
\end{aligned}$$

Then

$$\begin{aligned}
\|\partial_t w\|_{L_2(0,T;(J_2^1(\Omega))')}^2 &\leq c\|\nabla w\|_{2,Q_T}^2 + c(\varrho)\|w\|_{2,\infty,Q_T}^2 \|u\|_{2,Q_T}^2 + \\
&\quad + c\|v\|_{5,Q_T}^2 \|w\|_{\frac{10}{3},Q_T}^2 + c\|v^1\|_{3,\infty,Q_T}^{\frac{3}{2}} \sqrt{T} \|v^1\|_{5,Q_T}^{\frac{5}{2}}
\end{aligned}$$

and thus

$$\begin{aligned}
&\|\partial_t w\|_{L_2(0,T;(J_2^1(\Omega))')}^2 \leq \\
&\leq c(\varrho, \|v^1\|_{5,Q_T}, \|v^1\|_{3,\infty,Q_T}, T, \|a\|_{2,\Omega}) \left(1 + \|u\|_{2,Q_T}^2\right). \quad (2.19)
\end{aligned}$$

Making use of similar arguments as in the proof of Proposition 2.1, we conclude that for each fixed  $\varrho > 0$  the operator  $A$  is compact.

Now, for  $w = \lambda A(w)$  with  $\lambda \in [0, 1]$ , after integration by parts, we find that, for a.a.  $t \in [0, T]$ ,

$$\begin{aligned}
\int_{\Omega} (\partial_t w \cdot \tilde{v} + \nabla w : \nabla \tilde{v}) dx &= \lambda \int_{\Omega} ((v^1)_\varrho \otimes w + w \otimes (v^1)_\varrho + \\
&\quad + (v^1)_\varrho \otimes (v^1)_\varrho) : \nabla \tilde{v} - ((w)_\varrho \cdot \nabla w) \cdot \tilde{v} dx
\end{aligned}$$

for any  $\tilde{v} \in \overset{\circ}{J}_2^1(\Omega)$ . If we insert  $\tilde{v}(\cdot) = w(\cdot, t)$  into the latter relation, then the identity

$$\int_{\Omega} ((w)_\varrho \cdot \nabla w) \cdot w dx = 0$$

and previous arguments ensure the estimate (2.17). Now, let us prove the uniqueness for fixed  $\varrho$  and  $T$ . Coming back to the proof of continuity of the operator  $A$ , we find

$$\partial_t \|u^2 - u^1\|_{2,\Omega}^2 + \|\nabla(u^2 - u^1)\|_{2,\Omega}^2 \leq$$

$$\leq c(\varrho)\|u^1\|_{2,\Omega}^2\|u^2 - u^1\|_{2,\Omega}^2 + c(\varrho)\|v_0\|_{3,\mathbb{R}^3}^2\|(u^2 - u^1)\|_{2,\Omega}^2.$$

From this, it follows that  $u^1 = u^2$  on the interval  $]0, T[$ . Selecting a sequence of  $T_k \rightarrow \infty$  we can construct a unique function  $u$  satisfying all statements of the proposition. Proposition 2.2 is proved.  $\square$

Now, we wish to extend statements of Proposition 2.2 to  $\Omega = \mathbb{R}^3$ . From now on, let us assume that  $a = 0$ .

**Proposition 2.4.** *Given  $\varrho > 0$ , there exists a unique function  $u$ , defined on  $Q_\infty = \mathbb{R}^3 \times ]0, \infty[$ , such that, for any  $T > 0$ ,  $u \in W_{2,\infty}^{1,0}(Q_T)$  and  $\partial_t u \in L_2(Q_T)$ , and  $u$  satisfies the identity*

$$\begin{aligned} & \int_{Q_\infty} (-u \cdot \partial_t w + \nabla u : \nabla w) dx dt = \\ & = \int_{Q_\infty} (u \otimes (u)_\varrho + (v^1)_\varrho \otimes u + u \otimes (v^1)_\varrho + (v^1)_\varrho \otimes (v^1)_\varrho) : \nabla w dx dt \end{aligned} \quad (2.20)$$

for any  $w \in C_{0,0}^\infty(Q_\infty)$  and the initial condition  $u(\cdot, 0) = 0$ .

*Proof.* Let  $\varrho$  be fixed and  $u^{(k)}$  is a sequence of solutions from Proposition 2.2 for  $\Omega_k = B(R_k)$  with  $R_k \rightarrow \infty$ . According to (2.18), the energy estimate

$$\|u^{(k)}\|_{2,Q_T^k}^2 \leq c(\|v_0\|_{3,\mathbb{R}^3})\sqrt{T}. \quad (2.21)$$

holds for any  $T > 0$ . Here,  $Q_T^k = \Omega_k \times ]0, T[$ .

Now, let us derive some additional estimates. First, we have

$$\begin{aligned} & \int_{\Omega_k} (|\partial_t u^{(k)}|^2 + \frac{1}{2}\partial_t |\nabla u^{(k)}|^2) dx = - \int_{\Omega_k} \left( (u^{(k)})_\varrho \cdot \nabla u^{(k)} + (v^1)_\varrho \cdot \nabla u^{(k)} + \right. \\ & \left. + u^{(k)} \cdot \nabla (v^1)_\varrho + (v^1)_\varrho \cdot \nabla (v^1)_\varrho \right) \cdot \partial_t u^{(k)} dx. \end{aligned}$$

Moreover, it is easy to check

$$\begin{aligned} |(u^{(k)})_\varrho| & \leq c(\varrho)\|u^{(k)}\|_{2,\infty,Q_T^k}, \\ |(v^1)_\varrho| & \leq c(\varrho)\|v^1\|_{3,\infty,Q_T^k} \leq c(\varrho)\|v_0\|_{3,\mathbb{R}^3}, \\ |\nabla(v^1)_\varrho| & \leq c(\varrho)\|v^1\|_{3,\infty,Q_T^k} \leq c(\varrho)\|v_0\|_{3,\mathbb{R}^3}. \end{aligned}$$

Then, we find

$$\begin{aligned}
& \int_{\Omega_k} (|\partial_t u^{(k)}|^2 + \partial_t |\nabla u^{(k)}|^2) dx \leq c(\varrho) \|u^{(k)}\|_{2,\infty,Q_T^k}^2 \|\nabla u^{(k)}\|_{2,\Omega_k}^2 + \\
& + c(\varrho) \|v_0\|_{3,\mathbb{R}^3}^2 \|\nabla u^{(k)}\|_{2,\Omega_k}^2 + c(\varrho) \|v_0\|_{3,\mathbb{R}^3}^2 \|u^{(k)}\|_{2,\infty,Q_T^k}^2 + \\
& + c(\varrho) \|v_0\|_{3,\mathbb{R}^3} \int_{\Omega_k} |(v^1)_\varrho|^2 |\nabla(v^1)_\varrho| dx \leq \\
& \leq \dots + c(\varrho) \|v_0\|_{3,\mathbb{R}^3} \|(v^1)_\varrho\|_{3,\Omega_k}^2 \|(\nabla v^1)_\varrho\|_{3,\Omega_k} \leq \\
& \leq \dots + c(\varrho) \|v_0\|_{3,\mathbb{R}^3} \|v^1\|_{3,\Omega_k}^2 \|\nabla v^1\|_{3,\Omega_k} \leq \\
& \leq \dots + c(\varrho) \frac{1}{\sqrt{t}} \|v_0\|_{3,\mathbb{R}^3}^4.
\end{aligned}$$

Integration in  $t$  gives

$$\|\partial_t u^{(k)}\|_{2,Q_T^k}^2 + \|\nabla u^{(k)}\|_{2,\infty,Q_T^k} \leq c(\varrho, T, \|v_0\|_{3,\mathbb{R}^3}). \quad (2.22)$$

As usual, let us assume that functions  $u^{(k)}$  are extended by zero to the whole space  $\mathbb{R}^3$ . Then, according to (2.21) and (2.22), one can select a subsequence (still denoted by  $u^{(k)}$ ) such that

$$u^{(k)} \rightharpoonup u, \quad \nabla u^{(k)} \rightharpoonup \nabla u, \quad \partial_t u^{(k)} \rightharpoonup \partial_t u$$

in  $L_2(Q_T)$  with  $Q_T = \mathbb{R}^3 \times ]0, T[$  and

$$u^{(k)} \rightarrow u$$

for any set of the form  $K \times ]0, T[$ , where  $K$  is a compact in  $\mathbb{R}^3$ . Moreover, the limit function  $u$  satisfies estimates

$$|u|_{2,Q_T}^2 \leq c(\|v_0\|_{3,\mathbb{R}^3}) \sqrt{T} \quad (2.23)$$

and

$$\|\partial_t u\|_{2,Q_T}^2 + \|\nabla u\|_{2,\infty,Q_T} \leq c(\varrho, T, \|v_0\|_{3,\mathbb{R}^3})$$

for any  $T > 0$  and the identity (2.20).

It remains to prove the uniqueness. We have the same inequality as in the proof of the previous proposition

$$\begin{aligned}
& \partial_t \|u^2 - u^1\|_{2,\mathbb{R}^3}^2 + \|\nabla(u^2 - u^1)\|_{2,\mathbb{R}^3}^2 \leq \\
& \leq c(\varrho) \|u^1\|_{2,\mathbb{R}^3}^2 \|u^2 - u^1\|_{2,\mathbb{R}^3}^2 + c(\varrho) \|v_0\|_{3,\mathbb{R}^3}^2 \|(u^2 - u^1)\|_{2,\mathbb{R}^3}^2,
\end{aligned}$$

which implies the required property. Proposition 2.4 is proven.  $\square$

Now, we are going to prove the main theorem by passing to the limit as  $\varrho \rightarrow 0$ . To this end, we shall split  $u$  into four parts in the following way:

$$u = u^{2,1} + u^{2,2} + u^{2,3} + u^{2,4}$$

so that, for  $i = 1, 2, 3, 4$ ,

$$\partial_t u^{2,i} - \Delta u^{2,i} + \nabla p^{2,i} = f^i, \quad \operatorname{div} u^{2,i} = 0$$

in  $Q_\infty$ ,

$$u^{2,i}(x, 0) = 0$$

for  $x \in \mathbb{R}^3$ , where

$$f^1 := -(u)_\varrho \cdot \nabla u, \quad f^2 := -u \cdot \nabla(v^1)_\varrho,$$

$$f^3 := -(v^1)_\varrho \cdot \nabla u, \quad f^4 := -(v^1)_\varrho \cdot \nabla(v^1)_\varrho.$$

We now can introduce the pressure  $p = p^\varrho$  so that

$$p = p^{2,1} + p^{2,2} + p^{2,3} + p^{2,4}.$$

Let us start with evaluation of  $u^{2,1}$ . Here, our main tool is the Solonnikov coercive estimates of the linear theory. One can use the standard consequences of the energy bound, the multiplicative inequalities, and Hölder inequality and find

$$\begin{aligned} & \|\partial_t u^{2,1}\|_{s,l,Q_T} + \|\nabla^2 u^{2,1}\|_{s,l,Q_T} + \|\nabla p^{2,1}\|_{s,l,Q_T} \leq \\ & \leq c(s) \|f^1\|_{s,l,Q_T} \leq C(s, \|v_0\|_{3,\mathbb{R}^3}) \sqrt{T} \end{aligned} \quad (2.24)$$

provided

$$\frac{3}{s} + \frac{2}{l} = 4.$$

For  $i = 2$ , we may apply the known estimate of the heat potential

$$\|\nabla v^1(\cdot, t)\|_{s,\mathbb{R}^3} \leq \frac{c}{t^{\frac{1}{r} + \frac{1}{2}}} \|v_0(\cdot)\|_{3,\mathbb{R}^3} \quad (2.25)$$

with

$$\frac{1}{r} = \frac{3}{2} \left( \frac{1}{3} - \frac{1}{s} \right).$$

We take  $s = 4$  and try to estimate  $\|f^2\|_{\frac{4}{3}, \frac{3}{2}, Q_T}$ . Indeed, we have

$$\|f^2(\cdot, t)\|_{\frac{4}{3}, \mathbb{R}^3} \leq \|u(\cdot, t) \cdot \nabla(v^1)_\varrho(\cdot, t)\|_{\frac{4}{3}} \leq$$

$$\begin{aligned}
&\leq \|u(\cdot, t)\|_{2, \mathbb{R}^3} \|\nabla(v^1)_\rho(\cdot, t)\|_{4, \mathbb{R}^3} \leq \\
&\leq \|u(\cdot, t)\|_{2, \mathbb{R}^3} \|\nabla v^1(\cdot, t)\|_{4, \mathbb{R}^3} \leq \\
&\leq \|u\|_{2, \infty, Q_T} c \frac{1}{t^{\frac{5}{8}}} \|v_0(\cdot)\|_{3, \mathbb{R}^3}.
\end{aligned}$$

Hence,

$$\|f^2\|_{\frac{4}{3}, \frac{3}{2}, \mathbb{R}^3} \leq \|u\|_{2, \infty, Q_T} c T^{\frac{1}{24}} \|v_0(\cdot)\|_{3, \mathbb{R}^3} \leq c(\|v_0\|_{3, \mathbb{R}^3}) T^{\frac{7}{24}},$$

which implies

$$\begin{aligned}
&\|\partial_t u^{2,2}\|_{4/3, 3/2, Q_T} + \|\nabla^2 u^{2,2}\|_{4/3, 3/2, Q_T} + \|\nabla p^{2,2}\|_{4/3, 3/2, Q_T} \leq \\
&\leq c \|f^2\|_{4/3, 3/2, Q_T} \leq C(\|v_0\|_{3, \mathbb{R}^3}) T^{\frac{7}{24}}.
\end{aligned}$$

Next, let  $i = 3$ . Then, by (2.23),

$$\begin{aligned}
\|f^3(\cdot, t)\|_{6/5, \mathbb{R}^3} &= \|(v^1)_\rho(\cdot, t) \cdot \nabla u(\cdot, t)\|_{6/5, \mathbb{R}^3} \leq \|v^1(\cdot, t)\|_{3, \mathbb{R}^3} \|\nabla u(\cdot, t)\|_{2, \mathbb{R}^3} \leq \\
&\leq c(\|v_0\|_{3, \mathbb{R}^3}) \frac{1}{t^{\frac{1}{4}}}
\end{aligned}$$

and thus

$$\|f^3\|_{6/5, 3/2, \mathbb{R}^3} \leq T^{\frac{5}{12}} c(\|v_0\|_{3, \mathbb{R}^3}).$$

Applying a Solonnikov estimate one more time, we find

$$\begin{aligned}
&\|\partial_t u^{2,3}\|_{6/5, 3/2, Q_T} + \|\nabla^2 u^{2,3}\|_{6/5, 3/2, Q_T} + \|\nabla p^{2,3}\|_{6/5, 3/2, Q_T} \leq \\
&\leq c \|f^3\|_{6/5, 3/2, Q_T} \leq C(\|v_0\|_{3, \mathbb{R}^3}) T^{\frac{5}{12}}.
\end{aligned}$$

Finally, for the last term, we have

$$\begin{aligned}
\|f^4(\cdot, t)\|_{3/2, \mathbb{R}^3} &:= \|(v^1)_\rho(\cdot, t) \cdot \nabla(v^1)_\rho(\cdot, t)\|_{3/2, \mathbb{R}^3} \leq \\
&\leq \|v^1(\cdot, t)\|_{3, \mathbb{R}^3} \|\nabla v^1\|_{3, \mathbb{R}^3} \leq \\
&\leq \frac{c}{t^{\frac{1}{2}}} \|v_0\|_{3, \mathbb{R}^3}^2
\end{aligned}$$

and thus

$$\|f^4\|_{3/2, Q_T} \leq c T^{\frac{1}{6}} \|v_0\|_{3, \mathbb{R}^3}^2.$$

Coercive Solonnikov estimates give

$$\|\partial_t u^{2,4}\|_{3/2, Q_T} + \|\nabla^2 u^{2,4}\|_{3/2, Q_T} + \|\nabla p^{2,4}\|_{3/2, Q_T} \leq$$

$$\leq c\|f^4\|_{3/2, Q_T} \leq C(\|v_0\|_{3, \mathbb{R}^3})T^{\frac{1}{6}}. \quad (2.26)$$

In what follows, we are going to use the following Poincare type inequalities:

$$\int_0^T \int_{B(x_0, R)} |p^{2,1} - [p^{2,1}]_{B(x_0, R)}|^{\frac{3}{2}} dx dt \leq cR^{\frac{1}{2}} \int_0^T \left( \int_{B(x_0, R)} |\nabla p^{2,1}|^{\frac{9}{8}} dx \right)^{\frac{4}{3}} dt; \quad (2.27)$$

$$\int_0^T \int_{B(x_0, R)} |p^{2,2} - [p^{2,2}]_{B(x_0, R)}|^{\frac{3}{2}} dx dt \leq cR^{\frac{9}{8}} \int_0^T \left( \int_{B(x_0, R)} |\nabla p^{2,2}|^{\frac{4}{3}} dx \right)^{\frac{9}{8}} dt; \quad (2.28)$$

$$\int_0^T \int_{B(x_0, R)} |p^{2,3} - [p^{2,3}]_{B(x_0, R)}|^{\frac{3}{2}} dx dt \leq cR^{\frac{3}{4}} \int_0^T \left( \int_{B(x_0, R)} |\nabla p^{2,3}|^{\frac{6}{5}} dx \right)^{\frac{5}{4}} dt; \quad (2.29)$$

$$\int_0^T \int_{B(x_0, R)} |p^{2,4} - [p^{2,4}]_{B(x_0, R)}|^{\frac{3}{2}} dx dt \leq cR^{\frac{3}{2}} \int_0^T \int_{B(x_0, R)} |\nabla p^{2,4}|^2 dx dt. \quad (2.30)$$

Now, let us see what happens if  $\varrho \rightarrow 0$ . We can select a subsequence (still denoted as the whole sequence) with the following properties: for any  $T > 0$ ,

$$u^\varrho \rightharpoonup u, \quad \nabla u^\varrho \rightharpoonup \nabla u$$

in  $L_2(Q_T)$ ,

$$u^\varrho \rightarrow u$$

in  $L_3(0, T; L_{3, \text{loc}}(\mathbb{R}^3))$ ,

$$p^\varrho \rightharpoonup p$$

in  $L_{\frac{3}{2}}(0, T; L_{\frac{3}{2}, \text{loc}}(\mathbb{R}^3))$ . Moreover, limit functions  $u$  and  $p$  satisfy the energy estimate

$$|u|_{2, Q_T} \leq c(\|v_0\|_{3, \mathbb{R}^3})T^{\frac{1}{4}} \quad (2.31)$$

and the Navier-Stokes equations in  $Q_\infty$  in the sense of distributions. From the estimates above, it follows that the function

$$t \rightarrow \int_{\mathbb{R}^3} u(x, t) \cdot w(x) dx$$

is continuous on  $[0, \infty[$  for all  $w \in L_2(\mathbb{R}^3)$ .

Let us show that  $u$  and  $p$  satisfy the local energy inequality. Indeed, we have

$$\begin{aligned}
& \int_{\mathbb{R}^3} \varphi^2(x, t) |u^\varrho(x, t)|^2 dx + 2 \int_0^t \int_{\mathbb{R}^3} \varphi^2 |\nabla u^\varrho|^2 dx ds \leq \\
& \leq \int_0^t \int_{\mathbb{R}^3} \left( |u^\varrho|^2 (\Delta \varphi^2 + \partial_t \varphi^2) + (u^\varrho)_\varrho \cdot \nabla \varphi^2 (|u^\varrho|^2 + 2p^\varrho) + \right. \\
& \quad + (v^1)_\varrho \otimes u^\varrho : \nabla u^\varrho \varphi^2 + (v^1)_\varrho \otimes u^\varrho : u^\varrho \otimes \nabla \varphi^2 + \\
& \quad + u^\varrho \otimes (v^1)_\varrho : \nabla u^\varrho \varphi^2 + u^\varrho \otimes (v^1)_\varrho : u^\varrho \otimes \nabla \varphi^2 + \\
& \quad \left. + (v^1)_\varrho \otimes (v^1)_\varrho : \nabla u^\varrho \varphi^2 + (v^1)_\varrho \otimes (v^1)_\varrho : u^\varrho \otimes \nabla \varphi^2 \right) dx ds
\end{aligned}$$

for all  $\varphi \in C_0^\infty(\mathbb{R}^4)$ .

Further, we observe that

$$\begin{aligned}
\| (u^\varrho)_\varrho - u \|_{3, B(R) \times ]0, T[} &= \| (u^\varrho - u)_\varrho - (u)_\varrho - u \|_{3, B(R) \times ]0, T[} \leq \\
&\leq \| (u^\varrho - u)_\varrho \|_{3, B(R) \times ]0, T[} + \| (u)_\varrho - u \|_{3, B(R) \times ]0, T[} \leq \\
&\leq \| (u^\varrho - u) \|_{3, B(2R) \times ]0, T[} + \| (u)_\varrho - u \|_{3, B(R) \times ]0, T[} \rightarrow 0
\end{aligned}$$

as  $\varrho \rightarrow 0$ . This, of course, implies

$$\begin{aligned}
& \int_0^t \int_{\mathbb{R}^3} \left( |u^\varrho|^2 (\Delta \varphi^2 + \partial_t \varphi^2) + (u^\varrho)_\varrho \cdot \nabla \varphi^2 (|u^\varrho|^2 + 2p^\varrho) \right) dx ds \rightarrow \\
& \int_0^t \int_{\mathbb{R}^3} \left( |u|^2 (\Delta \varphi^2 + \partial_t \varphi^2) + u \cdot \nabla \varphi^2 (|u|^2 + 2p) \right) dx ds.
\end{aligned}$$

Next, first, we notice that

$$\int_0^t \int_{\mathbb{R}^3} \left( (v^1)_\varrho - v^1 \right) \otimes u^\varrho : \nabla u^\varrho \varphi^2 + \left( (v^1)_\varrho - v^1 \right) \otimes u^\varrho : u^\varrho \otimes \nabla \varphi^2 dx ds \rightarrow 0$$

and

$$\int_0^t \int_{\mathbb{R}^3} \left( v^1 \otimes u^\varrho : \nabla u^\varrho \varphi^2 + v^1 \otimes u^\varrho : u^\varrho \otimes \nabla \varphi^2 \right) dx ds =$$

$$= - \int_0^t \int_{\mathbb{R}^3} (u^\varrho \cdot \nabla v^1) \cdot u^\varrho \varphi^2 dx ds.$$

Now, let us consider the case

$$\varphi \in C_0^\infty(\mathbb{R}^3 \times ]0, \infty[).$$

Here, of course, we have

$$\begin{aligned} - \int_0^t \int_{\mathbb{R}^3} (u^\varrho \cdot \nabla v^1) \cdot u^\varrho \varphi^2 dx ds &\rightarrow - \int_0^t \int_{\mathbb{R}^3} (u \cdot \nabla v^1) \cdot u \varphi^2 dx ds = \\ &= \int_0^t \int_{\mathbb{R}^3} \left( v^1 \otimes u : \nabla u \varphi^2 + v^1 \otimes u : u \otimes \nabla \varphi^2 \right) dx ds. \end{aligned}$$

The same arguments give

$$\begin{aligned} \int_0^t \int_{\mathbb{R}^3} \left( u^\varrho \otimes (v^1)_\varrho : \nabla u^\varrho \varphi^2 + u^\varrho \otimes (v^1)_\varrho : u^\varrho \otimes \nabla \varphi^2 \right) dx ds &\rightarrow \\ \int_0^t \int_{\mathbb{R}^3} \left( u \otimes v^1 : \nabla u \varphi^2 + u \otimes v^1 : u \otimes \nabla \varphi^2 \right) dx ds & \end{aligned}$$

for any  $\varphi \in C_0^\infty(\mathbb{R}^3 \times ]0, \infty[)$ . The last term can be treated in the same manner and thus the following estimate comes out:

$$\begin{aligned} \int_{\mathbb{R}^3} \varphi^2(x, t) |u(x, t)|^2 dx + 2 \int_0^t \int_{\mathbb{R}^3} \varphi^2 |\nabla u|^2 dx ds &\leq \\ \leq \int_0^t \int_{\mathbb{R}^3} \left( |u|^2 (\Delta \varphi^2 + \partial_t \varphi^2) + u \cdot \nabla \varphi^2 (|u|^2 + 2p) + \right. & \\ \quad + v^1 \otimes u : \nabla u \varphi^2 + v^1 \otimes u : u \otimes \nabla \varphi^2 + & \\ \quad + u \otimes v^1 : \nabla u \varphi^2 + u \otimes v^1 : u \otimes \nabla \varphi^2 + & \\ \quad \left. + v^1 \otimes v^1 : \nabla u \varphi^2 + v^1 \otimes v^1 : u \otimes \nabla \varphi^2 \right) dx ds & \end{aligned}$$

for any  $\varphi \in C_0^\infty(\mathbb{R}^3 \times ]0, \infty[)$ .

In order to extend our local energy inequality to all function  $\varphi \in C_0^\infty(\mathbb{R}^4)$ , we take a function  $\chi(t)$  such that  $\chi(t) = 1$  if  $t > \varepsilon > 0$  and  $\chi(t) = 0$  if  $t < \varepsilon/2$  with  $0 \leq \chi'(t) \leq c/\varepsilon$ . Consider  $\psi = \chi\varphi$  with  $\varphi \in C_0^\infty(\mathbb{R}^4)$  as a cut-off function in the local energy inequality and see what happens if  $\varepsilon \rightarrow 0$ . The only term we should care of is the term containing the derivative in time:

$$\int_0^t \int_{\mathbb{R}^3} |u|^2 \partial_t \psi^2 dx ds = \int_\varepsilon^t \int_{\mathbb{R}^3} \chi^2 |u|^2 \partial_t \varphi^2 dx ds + \int_0^\varepsilon \int_{\mathbb{R}^3} \varphi^2 |u|^2 \partial_t \chi^2 dx ds = I_1 + I_2.$$

Obviously,

$$I_1 \rightarrow \int_0^t \int_{\mathbb{R}^3} |u|^2 \partial_t \varphi^2 dx ds.$$

To estimate the second term, we can use the energy estimate:

$$|I_2| \leq c |u|_{2,\infty,Q_\varepsilon}^2 \leq C(\|v_0\|_{3,\mathbb{R}^3}) \sqrt{\varepsilon} \rightarrow 0.$$

So, the local energy inequality is proven.

From the last inequality, we can deduce that

$$\begin{aligned} & \int_{\mathbb{R}^3} \varphi^2(x) |u(x,t)|^2 dx + 2 \int_0^t \int_{\mathbb{R}^3} \varphi^2 |\nabla u|^2 dx ds \leq \\ & \leq \int_0^t \int_{\mathbb{R}^3} \left( |u|^2 \Delta \varphi^2 + u \cdot \nabla \varphi^2 (|u|^2 + 2p) + \right. \\ & \quad + v^1 \otimes u : \nabla u \varphi^2 + v^1 \otimes u : u \otimes \nabla \varphi^2 + \\ & \quad + u \otimes v^1 : \nabla u \varphi^2 + u \otimes v^1 : u \otimes \nabla \varphi^2 + \\ & \quad \left. + v^1 \otimes v^1 : \nabla u \varphi^2 + v^1 \otimes v^1 : u \otimes \nabla \varphi^2 \right) dx ds \end{aligned}$$

for any  $\varphi \in C_0^\infty(\mathbb{R}^3)$ .

Now, we wish to get a global energy inequality. We can take a function  $\varphi$  satisfying  $0 \leq \varphi \leq 1$  such that  $\varphi(x) = 1$  if  $|x| < R$  and  $\varphi(x) = 0$  if  $|x| > 2R$  and  $|\nabla \varphi(x)| < c/R$ .

The only term to be treated carefully is

$$I = \int_0^t \int_{\mathbb{R}^3} u \cdot \nabla \varphi^2 p dx ds.$$

Indeed, we have

$$I = I_1 + I_2 + I_3 + I_4,$$

where

$$I_i = \int_0^t \int_{\mathbb{R}^3} u \cdot \nabla \varphi^2 (p^{2,i} - [p^{2,i}]_{A(R)}) dx ds,$$

where  $A(R) := B(2R) \setminus \overline{B}(R)$ . Then, by (2.27), we have

$$\begin{aligned} |I_1| &\leq \frac{c}{R} \|u\|_{3, A(R) \times ]0, t]} \left( \int_0^t \int_{A(R)} |p^{2,1} - [p^{2,1}]_{A(R)}|^{\frac{3}{2}} dx ds \right)^{\frac{2}{3}} \leq \\ &\leq \frac{c}{R} \|u\|_{3, A(R) \times ]0, t]} R^{\frac{1}{3}} \left( \int_0^t \left( \int_{A(R)} |\nabla p^{2,1}|^{\frac{9}{8}} dx \right)^{\frac{4}{3}} dt \right)^{\frac{2}{3}} \rightarrow 0 \end{aligned}$$

as  $R \rightarrow \infty$ .

As to the second term, we use (2.28) and show

$$\begin{aligned} |I_2| &\leq \frac{c}{R} \|u\|_{3, A(R) \times ]0, t]} \left( \int_0^t \int_{A(R)} |p^{2,2} - [p^{2,2}]_{A(R)}|^{\frac{3}{2}} dx ds \right)^{\frac{2}{3}} \leq \\ &\leq \frac{c}{R} \|u\|_{3, A(R) \times ]0, t]} R^{\frac{3}{4}} \left( \int_0^t \left( \int_{A(R)} |\nabla p^{2,2}|^{\frac{4}{3}} dx \right)^{\frac{9}{8}} dt \right)^{\frac{2}{3}} \rightarrow 0 \end{aligned}$$

as  $R \rightarrow \infty$ .

From (2.29) it follows that

$$\begin{aligned} |I_3| &\leq \frac{c}{R} \|u\|_{3, A(R) \times ]0, t]} \left( \int_0^t \int_{A(R)} |p^{2,3} - [p^{2,3}]_{A(R)}|^{\frac{3}{2}} dx ds \right)^{\frac{2}{3}} \leq \\ &\leq \frac{c}{R} \|u\|_{3, A(R) \times ]0, t]} R^{\frac{1}{2}} \left( \int_0^t \left( \int_{A(R)} |\nabla p^{2,3}|^{\frac{6}{5}} dx \right)^{\frac{5}{4}} dt \right)^{\frac{2}{3}} \rightarrow 0 \end{aligned}$$

as  $R \rightarrow \infty$ .

Finally, for the fourth term, we derive from (2.30)

$$\begin{aligned} |I_4| &\leq \frac{c}{R} \|u\|_{3, A(R) \times ]0, t[} \left( \int_0^t \int_{A(R)} |p^{2,4} - [p^{2,4}]_{A(R)}|^{\frac{3}{2}} dx ds \right)^{\frac{2}{3}} \leq \\ &\leq \frac{c}{R} \|u\|_{3, A(R) \times ]0, t[} R \left( \int_0^t \int_{A(R)} |\nabla p^{2,4}|^{\frac{3}{2}} dx dt \right)^{\frac{2}{3}} \rightarrow 0 \end{aligned}$$

as  $R \rightarrow \infty$ .

The global energy inequality has been proven. That's all.

### 3 Weak Convergence of Initial Data

Here, we are going to prove Theorem 1.7.

We know that  $v^{2(m)}$  satisfies the energy estimate

$$|v^{2(m)}|_{2, Q_T}^2 \leq c(M) \sqrt{T}$$

for any  $T > 0$ , where

$$M := \sup_m \|v_0^{(m)}\|_{3, \mathbb{R}^2} < \infty.$$

Then, one may proceed as in the proof of Theorem 1.6, splitting  $v^{2(m)}$  and  $q^{2(m)}$  so that:

$$v^{2(m)} = u^{2,1} + u^{2,2} + u^{2,3} + u^{2,4}$$

and

$$q^{2(m)} = p^{2,1} + p^{2,2} + p^{2,3} + p^{2,4},$$

and, for  $i = 1, 2, 3, 4$ ,

$$\partial_t u^{2,i} - \Delta u^{2,i} + \nabla p^{2,i} = f^i, \quad \operatorname{div} u^{2,i} = 0$$

in  $Q_\infty$ ,

$$u^{2,i}(x, 0) = 0$$

for  $x \in \mathbb{R}^3$ , where

$$f^1 := -v^{2(m)} \cdot \nabla v^{2(m)}, \quad f^2 := -v^{2(m)} \cdot \nabla v^{1(m)},$$

$$f^3 := -v^{1(m)} \cdot \nabla v^{2(m)}, \quad f^4 := -v^{1(m)} \cdot \nabla v^{1(m)}.$$

To evaluate  $u^{2,1}$ , Solonnikov's coercive estimates and the energy bound are used. As a result, we find

$$\begin{aligned} \|\partial_t u^{2,1}\|_{s,l,Q_T} + \|\nabla^2 u^{2,1}\|_{s,l,Q_T} + \|\nabla p^{2,1}\|_{s,l,Q_T} &\leq \\ &\leq c(s) \|f^1\|_{s,l,Q_T} \leq C(s, M) \sqrt{T} \end{aligned} \quad (3.1)$$

provided

$$\frac{3}{s} + \frac{2}{l} = 4.$$

If  $i = 2$ , one can exploit estimates (2.25) for  $v^1$  with  $s = 4$  and show

$$\begin{aligned} \|f^2(\cdot, t)\|_{\frac{4}{3}, \mathbb{R}^3} &\leq \|v^{2(m)}(\cdot, t) \cdot \nabla v^{1(m)}(\cdot, t)\|_{\frac{4}{3}} \leq \\ &\leq \|v^{2(m)}(\cdot, t)\|_{2, \mathbb{R}^3} \|\nabla v^{1(m)}(\cdot, t)\|_{4, \mathbb{R}^3} \leq \\ &\leq \|v^{2(m)}\|_{2, \infty, Q_T} c \frac{1}{t^{\frac{5}{8}}} \|v_0^{(m)}(\cdot)\|_{3, \mathbb{R}^3}. \end{aligned}$$

Hence,

$$\|f^2\|_{\frac{4}{3}, \frac{3}{2}, \mathbb{R}^3} \leq \|v^{2(m)}\|_{2, \infty, Q_T} c T^{\frac{1}{24}} \|v_0^{(m)}(\cdot)\|_{3, \mathbb{R}^3} \leq c(M) T^{\frac{7}{24}},$$

which implies

$$\begin{aligned} \|\partial_t u^{2,2}\|_{4/3, 3/2, Q_T} + \|\nabla^2 u^{2,2}\|_{4/3, 3/2, Q_T} + \|\nabla p^{2,2}\|_{4/3, 3/2, Q_T} &\leq \\ &\leq c \|f^2\|_{4/3, 3/2, Q_T} \leq C(M) T^{\frac{7}{24}}. \end{aligned}$$

If  $i = 3$ , then similar arguments lead to the bound

$$\begin{aligned} \|\partial_t u^{2,3}\|_{6/5, 3/2, Q_T} + \|\nabla^2 u^{2,3}\|_{6/5, 3/2, Q_T} + \|\nabla p^{2,3}\|_{6/5, 3/2, Q_T} &\leq \\ &\leq c \|f^3\|_{6/5, 3/2, Q_T} \leq C(M) T^{\frac{5}{12}}. \end{aligned}$$

Finally, for the last term, we have

$$\begin{aligned} \|\partial_t u^{2,4}\|_{3/2, Q_T} + \|\nabla^2 u^{2,4}\|_{3/2, Q_T} + \|\nabla p^{2,4}\|_{3/2, Q_T} &\leq \\ &\leq c \|f^4\|_{3/2, Q_T} \leq C(M) T^{\frac{1}{6}}. \end{aligned} \quad (3.2)$$

Now, let  $m \rightarrow \infty$ . We can select a subsequence (still denoted as the whole sequence) such that, for any  $T > 0$ ,

$$v^{2(m)} \rightharpoonup v^2, \quad \nabla v^{2(m)} \rightharpoonup \nabla v^2$$

in  $L_2(Q_T)$ ,

$$v^{2(m)} \rightharpoonup v^2$$

in  $L_3(0, T; L_{3,\text{loc}}(\mathbb{R}^3))$ ,

$$q^{2(m)} \rightharpoonup q^2$$

in  $L_{\frac{3}{2}}(0, T; L_{\frac{3}{2},\text{loc}}(\mathbb{R}^3))$ . Moreover, limit functions  $v = v^1 + v^2$  and  $q = q^2$  satisfy the estimate

$$|v^2|_{2, Q_T} \leq c(M)T^{\frac{1}{4}}$$

and the Navier-Stokes equations in  $Q_\infty$  in the sense of distributions. It is easy to see that the function

$$t \rightarrow \int_{\mathbb{R}^3} v^2(x, t) \cdot w(x) dx$$

is continuous on  $[0, \infty[$  for all  $w \in L_2(\mathbb{R}^3)$ .

Let us show that  $v^2$  and  $q^2$  satisfy the local energy inequality. Indeed, we have

$$\begin{aligned} & \int_{\mathbb{R}^3} \varphi^2(x, t) |v^{2(m)}(x, t)|^2 dx + 2 \int_0^t \int_{\mathbb{R}^3} \varphi^2 |\nabla v^{2(m)}|^2 dx ds \\ & \leq \int_0^t \int_{\mathbb{R}^3} \left( |v^{2(m)}|^2 (\varphi^2 + \partial_t \varphi^2) + v^{2(m)} \cdot \nabla \varphi^2 (|v^{2(m)}|^2 + 2q^{2(m)}) + \right. \\ & \quad + v^{1(m)} \otimes v^{2(m)} : \nabla v^{2(m)} \varphi^2 + v^{1(m)} \otimes : v^{2(m)} \otimes \nabla \varphi^2 + \\ & \quad + v^{2(m)} \otimes v^{1(m)} : \nabla v^{2(m)} \varphi^2 + v^{2(m)} \otimes v^{1(m)} : v^{2(m)} \otimes \nabla \varphi^2 + \\ & \quad \left. + v^{1(m)} \otimes v^{1(m)} : \nabla v^{2(m)} \varphi^2 + v^{1(m)} \otimes v^{1(m)} : v^{2(m)} \otimes \nabla \varphi^2 \right) dx ds \end{aligned}$$

for all  $\varphi \in C_0^\infty(\mathbb{R}^4)$ .

The first thing to notice is:

$$\begin{aligned} & \int_0^t \int_{\mathbb{R}^3} \left( |v^{2(m)}|^2 (\Delta \varphi^2 + \partial_t \varphi^2) + v^{2(m)} \cdot \nabla \varphi^2 (|v^{2(m)}|^2 + 2q^{2(m)}) \right) dx ds \rightarrow \\ & \int_0^t \int_{\mathbb{R}^3} \left( |v^2|^2 (\Delta \varphi^2 + \partial_t \varphi^2) + u \cdot \nabla \varphi^2 (|v^2|^2 + 2q^2) \right) dx ds. \end{aligned}$$

As in the previous section, let us consider two cases. In the first one, it is assumed that

$$\varphi \in C_0^\infty(\mathbb{R}^3 \times ]0, \infty[).$$

Then

$$\begin{aligned} & \int_0^t \int_{\mathbb{R}^3} \left( (v^{1(m)} - v^1) \otimes v^{2(m)} : \nabla v^{2(m)} \varphi^2 + \right. \\ & \left. + (v^{1(m)} - v^1) \otimes v^{2(m)} : v^{2(m)} \otimes \nabla \varphi^2 \right) dx ds \rightarrow 0 \end{aligned}$$

and

$$\begin{aligned} & \int_0^t \int_{\mathbb{R}^3} \left( v^1 \otimes v^{2(m)} : \nabla v^{2(m)} \varphi^2 + v^1 \otimes v^{2(m)} : v^{2(m)} \otimes \nabla \varphi^2 \right) dx ds = \\ & = - \int_0^t \int_{\mathbb{R}^3} (v^{2(m)} \cdot \nabla v^1) \cdot v^{2(m)} \varphi^2 dx ds. \end{aligned}$$

Next, one can observe that

$$\begin{aligned} & - \int_0^t \int_{\mathbb{R}^3} (v^{2(m)} \cdot \nabla v^1) \cdot v^{2(m)} \varphi^2 dx ds \rightarrow - \int_0^t \int_{\mathbb{R}^3} (v^2 \cdot \nabla v^1) \cdot v^2 \varphi^2 dx ds = \\ & = \int_0^t \int_{\mathbb{R}^3} \left( v^1 \otimes v^2 : \nabla v^2 \varphi^2 + v^1 \otimes v^2 : v^2 \otimes \nabla \varphi^2 \right) dx ds. \end{aligned}$$

The same arguments imply

$$\begin{aligned} & \int_0^t \int_{\mathbb{R}^3} \left( v^{2(m)} \otimes v^{1(m)} : \nabla v^{2(m)} \varphi^2 + v^{2(m)} \otimes v^{1(m)} : v^{2(m)} \otimes \nabla \varphi^2 \right) dx ds \rightarrow \\ & \int_0^t \int_{\mathbb{R}^3} \left( v^2 \otimes v^1 : \nabla v^2 \varphi^2 + v^2 \otimes v^1 : v^2 \otimes \nabla \varphi^2 \right) dx ds \end{aligned}$$

for any  $\varphi \in C_0^\infty(\mathbb{R}^3 \times ]0, \infty[)$ . The last term is treated in the same manner. Hence,

$$\int_{\mathbb{R}^3} \varphi^2(x, t) |v^2(x, t)|^2 dx + 2 \int_0^t \int_{\mathbb{R}^3} \varphi^2 |\nabla v^2|^2 dx ds \leq$$

$$\begin{aligned}
&\leq \int_0^t \int_{\mathbb{R}^3} \left( |v^2|^2 (\Delta \varphi^2 + \partial_t \varphi^2) + v^2 \cdot \nabla \varphi^2 (|v^2|^2 + 2q^2) + \right. \\
&\quad + v^1 \otimes v^2 : \nabla v^2 \varphi^2 + v^1 \otimes v^2 : v^2 \otimes \nabla \varphi^2 + \\
&\quad + v^2 \otimes v^1 : \nabla v^2 \varphi^2 + v^2 \otimes v^1 : v^2 \otimes \nabla \varphi^2 + \\
&\quad \left. + v^1 \otimes v^1 : \nabla v^2 \varphi^2 + v^1 \otimes v^1 : v^2 \otimes \nabla \varphi^2 \right) dx ds
\end{aligned}$$

for any  $\varphi \in C_0^\infty(\mathbb{R}^3 \times ]0, \infty[)$ . In order to extend our local energy inequality to all function  $\varphi \in C_0^\infty(\mathbb{R}^4)$ , we can exploit the same cut-off function  $\chi(t)$  as in the proof of the existence theorem. Letting  $\psi = \chi\varphi$  with  $\varphi \in C_0^\infty(\mathbb{R}^4)$ , we observe that the only term that should be treated carefully is the term containing the derivative in time. Indeed, for example, consider the term

$$\int_0^t \chi^2 g(s) ds,$$

where

$$g(t) = \int_{\mathbb{R}^3} v^1 \otimes v^2 : \nabla v^2 \varphi^2 dx.$$

We need to show that

$$A := \int_0^t |g(s)| ds < \infty.$$

Indeed,

$$\begin{aligned}
A &\leq \int_0^t \int_{\mathbb{R}^3} |v^1| |v^2| |\nabla v^2| dx ds \leq \int_0^t \|v^1\|_{5, \mathbb{R}^3} \|v^2\|_{\frac{10}{3}, \mathbb{R}^3} \|\nabla v^2\|_{2, \mathbb{R}^3} ds \leq \\
&\leq \int_0^t \|v^1\|_{5, \mathbb{R}^3} \|v^2\|_{2, \mathbb{R}^3}^{\frac{2}{5}} \|\nabla v^2\|_{2, \mathbb{R}^3}^{\frac{8}{5}} ds \leq \|v^2\|_{2, \infty, \mathbb{R}^3}^{\frac{2}{5}} \|v^1\|_{5, Q_t} \|\nabla v^2\|_{2, Q_t}^{\frac{8}{5}}.
\end{aligned}$$

Now, let us make the evaluation of the most important term

$$\int_0^t \int_{\mathbb{R}^3} |v^2|^2 \partial_t \psi^2 dx ds = \int_\varepsilon^t \int_{\mathbb{R}^3} \chi^2 |v^2|^2 \partial_t \varphi^2 dx ds + \int_0^\varepsilon \int_{\mathbb{R}^3} \varphi^2 |v^2|^2 \partial_t \chi^2 dx ds =$$

$$= I_1 + I_2.$$

Obviously,

$$I_1 \rightarrow \int_0^t \int_{\mathbb{R}^3} |v^2|^2 \partial_t \varphi^2 dx ds.$$

To estimate the second term, the energy estimate is used and, therefore,

$$|I_2| \leq c |v^2|_{2,\infty,Q_\varepsilon}^2 \leq C(M) \sqrt{\varepsilon} \rightarrow 0.$$

So, the local energy inequality has been proven and takes the form:

$$\begin{aligned} & \int_{\mathbb{R}^3} \varphi^2(x) |v^2(x,t)|^2 dx + 2 \int_0^t \int_{\mathbb{R}^3} \varphi^2 |\nabla v^2|^2 dx ds \leq \\ & \leq \int_0^t \int_{\mathbb{R}^3} \left( |v^2|^2 \Delta \varphi^2 + v^2 \cdot \nabla \varphi^2 (|v^2|^2 + 2q^2) + \right. \\ & \quad + v^1 \otimes v^2 : \nabla v^2 \varphi^2 + v^1 \otimes v^2 : v^2 \otimes \nabla \varphi^2 + \\ & \quad + v^2 \otimes v^1 : \nabla v^2 \varphi^2 + v^2 \otimes v^1 : v^2 \otimes \nabla \varphi^2 + \\ & \quad \left. + v^1 \otimes v^1 : \nabla v^2 \varphi^2 + v^1 \otimes v^1 : v^2 \otimes \nabla \varphi^2 \right) dx ds \end{aligned}$$

for any  $\varphi \in C_0^\infty(\mathbb{R}^3)$ .

From the latter relation we can deduce the global energy inequality, using the same arguments as in the proof of the existence theorem.

Hence, the limit function  $v = v^1 + v^2$  is a weak  $L_3$ -solution starting with initial data  $v_0$ .

## 4 Uniqueness

Let us start with a proof of Proposition 1.8.

*Proof.* Our first remark is that, given  $\varepsilon > 0$  and  $R > 0$ , there exists a number  $R_*(T, R, \varepsilon) > 0$  such that if  $B(x_0, R) \subset \mathbb{R}^3 \setminus B(R_*)$  and  $t_0 - R^2 > 0$  then

$$\frac{1}{R^2} \int_{Q(z_0, R)} (|v|^3 + |q - [q]_{B(x_0, R)}|^{\frac{3}{2}}) dx dt \leq \varepsilon.$$

For  $v$  and  $q^1 = 0$ , it is certainly true. For  $q^2$ , we can use arguments similar to those used in the previous section. Indeed, if  $q^2 = p^{2,1} + p^{2,2} + p^{2,3} + p^{2,4}$ , then, for example, we have

$$\begin{aligned} & \frac{1}{R^2} \int_{Q(z_0, R)} |p^{2,1} - [p^{2,1}]_{B(x_0, R)}|^{\frac{3}{2}} dx ds \leq \\ & \leq \frac{1}{R^2} \int_0^T \int_{B(x_0, R)} |p^{2,1} - [p^{2,1}]_{B(x_0, R)}|^{\frac{3}{2}} dx ds \leq \frac{1}{R^{\frac{3}{2}}} \int_0^T \left( \int_{B(x_0, R)} |\nabla p^{2,1}|^{\frac{9}{8}} dx \right)^{\frac{4}{3}} dt \leq \\ & \leq \frac{1}{R^{\frac{3}{2}}} \int_0^T \left( \int_{\mathbb{R}^3 \setminus B(R_*)} |\nabla p^{2,1}|^{\frac{9}{8}} dx \right)^{\frac{4}{3}} dt \rightarrow 0 \end{aligned}$$

as  $R_* \rightarrow \infty$  for any fixed  $R > 0$ . Since  $v \in L_{3, \infty}(Q_T)$ , it is not so difficult to show that the pair  $v$  and  $q^2$  satisfies the local energy inequality (in fact, the local energy identity) and thus, by  $\varepsilon$ -regularity theory developed in [1], we can claim that

$$|v(z_0)| \leq \frac{c}{R}$$

as long as  $z_0$  and  $R$  satisfy the conditions above. According to [4],  $v$  is locally bounded as it belongs to  $L_{3, \infty}(Q_T)$ . Therefore, we can ensure that  $v \in L_{\infty}(Q_{\delta, T})$  for any  $\delta > 0$ . Here,  $Q_{\delta, T} = \mathbb{R}^3 \times ]\delta, T[$ . Then, we can easily show that, for any  $\delta > 0$ ,  $v^2 \in W_2^{2,1}(Q_{\delta, T})$ ,  $\nabla v^2 \in L_{2, \infty}(Q_{\delta, T})$ , and  $\nabla q^2 \in L_2(Q_{\delta, T})$ . The latter allows us to state that the energy identity

$$\frac{1}{2} \int_{\mathbb{R}^3} |v^2(x, t)|^2 dx + \int_0^t \int_{\mathbb{R}^3} |\nabla v^2|^2 dx ds = \int_0^t \int_{\mathbb{R}^3} v^1 \otimes v : \nabla v^2 dx ds$$

holds for any  $t > 0$  and, moreover,

$$\int_{\mathbb{R}^3} \left( \partial_t v^2(x, t) \cdot w(x) + (v(x, t) \cdot \nabla v(x, t)) \cdot w(x) + \nabla v^2(x, t) : \nabla w(x) \right) dx = 0$$

for any  $w \in C_{0,0}^{\infty}(\mathbb{R}^3)$  and for a.a.  $t \in ]0, t[$ .

As  $\tilde{v}^2$ , we have

$$\int_{Q_T} \left( -\tilde{v}^2 \cdot \partial_t w - \tilde{v} \otimes \tilde{v} : \nabla w + \nabla \tilde{v}^2 : \nabla w \right) dx dt = 0$$

for any  $w \in C_{0,0}^\infty(Q_T)$ . Using known approximative arguments, see for example [26], we deduce from the last identity that

$$\begin{aligned} & \int_{\delta}^{t_0} \left( -\tilde{v}^2 \partial_t v^2 - \tilde{v} \otimes \tilde{v} : \nabla v^2 + \nabla \tilde{v}^2 : \nabla v^2 \right) dx dt + \\ & + \int_{\mathbb{R}^3} \tilde{v}^2(x, t_0) \cdot v^2(x, t_0) dx - \int_{\mathbb{R}^3} \tilde{v}^2(x, \delta) \cdot v^2(x, \delta) dx = 0 \end{aligned}$$

for any  $t_0 \in [\delta, T]$ .

Let  $w = \tilde{v}^2 - v^2$ . Then

$$\begin{aligned} & \int_{\delta}^{t_0} \int_{\mathbb{R}^3} \partial_t v^2 \cdot \tilde{v}^2 dx dt - \frac{1}{2} \int_{\mathbb{R}^3} |v^2(x, t_0)|^2 dx + \frac{1}{2} \int_{\mathbb{R}^3} |v^2(x, \delta)|^2 dx + \\ & + \int_{\delta}^{t_0} \int_{\mathbb{R}^3} \left( -v \otimes v : \nabla w + \nabla v^2 : \nabla w \right) dx dt = 0. \end{aligned}$$

Adding the last two identities, we find

$$\int_{\mathbb{R}^3} \left[ \tilde{v}^2(x, t_0) \cdot v^2(x, t_0) - \tilde{v}^2(x, \delta) \cdot v^2(x, \delta) - \frac{1}{2} |v^2(x, t_0)|^2 + \frac{1}{2} |v^2(x, \delta)|^2 \right] dx +$$

$$+ \int_{\delta}^{t_0} \int_{\mathbb{R}^3} \left( -v \otimes v : \nabla w - \tilde{v} \otimes \tilde{v} : \nabla v^2 + \nabla v^2 : \nabla w + \nabla \tilde{v}^2 : \nabla v^2 \right) dx dt = 0$$

or

$$\begin{aligned} & \int_{\mathbb{R}^3} \left[ \tilde{v}^2(x, t_0) \cdot v^2(x, t_0) - \frac{1}{2} |v^2(x, t_0)|^2 \right] dx + \\ & + \int_0^{t_0} \int_{\mathbb{R}^3} \left( -v \otimes v : \nabla w - \tilde{v} \otimes \tilde{v} : \nabla v^2 + \nabla v^2 : \nabla w + \nabla \tilde{v}^2 : \nabla v^2 \right) dx dt = \alpha(\delta), \end{aligned}$$

where  $\alpha(\delta) \rightarrow 0$  as  $\delta \rightarrow 0$ .

We also know that  $\tilde{v}^2$  satisfies the energy inequality

$$\frac{1}{2} \int_{\mathbb{R}^3} |\tilde{v}^2(x, t_0)|^2 dx + \int_0^{t_0} \int_{\mathbb{R}^3} |\nabla \tilde{v}^2|^2 dx dt \leq$$

$$\leq \int_0^{t_0} \int_{\mathbb{R}^3} \tilde{v} \otimes \tilde{v} : \nabla \tilde{v}^2 dx dt.$$

Subtracting the previous identity from the last inequality, we show

$$\begin{aligned} & \frac{1}{2} \int_{\mathbb{R}^3} |w(x, t_0)|^2 dx + \int_0^{t_0} \int_{\mathbb{R}^3} |\nabla \tilde{v}^2|^2 dx dt \leq \\ & \leq \int_0^{t_0} \int_{\mathbb{R}^3} \tilde{v} \otimes \tilde{v} : \nabla \tilde{v}^2 dx dt + \\ & + \int_0^{t_0} \int_{\mathbb{R}^3} \left( -v \otimes v : \nabla w - \tilde{v} \otimes \tilde{v} : \nabla v^2 + \nabla v^2 : \nabla w + \nabla \tilde{v}^2 : \nabla v^2 \right) dx dt - \alpha(\delta) \end{aligned}$$

and then, passing to the limit as  $\delta \rightarrow 0$ , we find

$$\begin{aligned} & \frac{1}{2} \int_{\mathbb{R}^3} |w(x, t_0)|^2 dx + \int_0^{t_0} \int_{\mathbb{R}^3} |\nabla w|^2 dx dt \leq \\ & \leq \int_0^{t_0} \int_{\mathbb{R}^3} \left( \tilde{v} \otimes \tilde{v} : \nabla w - v \otimes v : \nabla w \right) dx dt = \int_0^{t_0} \int_{\mathbb{R}^3} (w \otimes v + v \otimes w) : \nabla w dx dt. \end{aligned}$$

So, finally,

$$\begin{aligned} & \int_{\mathbb{R}^3} |w(x, t_0)|^2 dx + \int_0^{t_0} \int_{\mathbb{R}^3} |\nabla w|^2 dx dt \leq \\ & \leq c \int_0^{t_0} \int_{\mathbb{R}^3} |v^1|^2 |w|^2 dx dt + c \int_0^{t_0} \int_{\mathbb{R}^3} |v^2|^2 |w|^2 dx dt = cI_1 + cI_2. \end{aligned}$$

Estimate for  $I_1$  has been already derived:

$$\begin{aligned} I_1 & \leq c \left( \int_0^{t_0} \int_{\mathbb{R}^3} |v^1(y, t)|^5 dy \int_{\mathbb{R}^3} |w(x, t)|^2 dx dt \right)^{\frac{2}{5}} \left( \int_0^{t_0} \int_{\mathbb{R}^3} |\nabla w|^2 dx dt \right)^{\frac{3}{5}} \leq \\ & \leq c \left( \int_0^{t_0} g^1(t) \int_{\mathbb{R}^3} |w(x, t)|^2 dx dt \right)^{\frac{2}{5}} \left( \int_0^{t_0} \int_{\mathbb{R}^3} |\nabla w|^2 dx dt \right)^{\frac{3}{5}}, \end{aligned}$$

where

$$g^1(t) := \int_{\mathbb{R}^3} |v^1(y, t)|^5 dy.$$

To estimate the second term, we are going to exploit condition (1.29) that implies the existence of  $\delta \in ]0, T_1]$  with the following property:

$$\|v^2\|_{3, \infty, Q_\delta} \leq 2\mu.$$

Then for  $t_0 \leq \delta$ , we have

$$\begin{aligned} I_2 &\leq \int_0^{t_0} \|v^2\|_{3, \mathbb{R}^3}^2 \|w\|_{6, \mathbb{R}^3}^2 dt \leq \\ &\leq c \|v^2\|_{3, \infty, Q_{t_0}}^2 \int_0^{t_0} \int_{\mathbb{R}^3} |\nabla w|^2 dx dt \leq 4\mu^2 c \int_0^{t_0} \int_{\mathbb{R}^3} |\nabla w|^2 dx dt. \end{aligned}$$

Letting

$$8\mu^2 c = 1,$$

we find

$$\int_{\mathbb{R}^3} |w(x, t_0)|^2 dx + \int_0^{t_0} \int_{\mathbb{R}^3} |\nabla w|^2 dx dt \leq c \int_0^{t_0} g^1(t) \int_{\mathbb{R}^3} |w(x, t)|^2 dx dt$$

for  $0 < t_0 \leq \delta$ . Hence,  $w = 0$  on the interval  $]0, \delta[$ . On the other hand, we know that  $v \in L_5(Q_{\delta, T})$  and thus  $v^2 \in L_5(Q_{\delta, T})$  and the same arguments as above give  $w = 0$  on the whole interval  $]0, T[$ .  $\square$

Now, we wish to prove Theorem 1.11.

*Proof.* It is enough to show that there exists a weak  $L_3$ -solution  $u$  that belongs to  $L_5(Q_{T_0})$  for some  $T_0 > 0$  with the same initial data. Indeed, by Theorem 1.10,  $v = u$  in  $Q_{T_0}$ . To this end, let us go back to our approximating solution  $v^\ell$ . For this smooth solution, we have the known estimate

$$\|v^{2, \ell}\|_{5, Q_{T_0}} \leq c \|((v^1)_\ell + (v^{2, \ell})_\ell) \otimes ((v^1)_\ell + v^{2, \ell})\|_{5, Q_{T_0}}$$

with an absolute constant  $c$ . So, we have

$$\|v^{2, \ell}\|_{5, Q_{T_0}} \leq c (\|v^{2, \ell}\|_{5, Q_{T_0}}^2 + \|v^{2, \ell}\|_{5, Q_{T_0}} \|v^1\|_{5, Q_{T_0}} + \|v^1\|_{5, Q_{T_0}}^2)$$

If we assume that

$$\|v^1\|_{5, Q_{T_0}} \leq \frac{1}{5c}, \quad (4.1)$$

then it is not difficult to show that

$$\|v^{2,e}\|_{5, Q_{T_0}} \leq \frac{1}{2} \|v^1\|_{5, Q_{T_0}}.$$

And the same bound is true for the limit function. So,  $\|v^2\|_{5, Q_{T_0}} < \infty$ . Condition (4.1) gives an estimate on  $T_0$ .  $\square$

The statement of Theorem 1.13 follows immediately from Proposition 1.8 and Corollary 5.1 reading that any weak  $L_3$ -solution is a mild solution on a short time interval.

## 5 Appendix

The aim of this section is to give an elementary proof of the existence of a mild solution to the Cauchy problem (1.1)–(1.3). To this end, let us consider first the following Stokes problem

$$\partial_t w - \Delta w + \nabla r = -\operatorname{div} F, \quad \operatorname{div} w = 0$$

in  $Q_\infty$  and

$$w(\cdot, 0) = 0$$

in  $\mathbb{R}^3$ .

Given  $F \in C_0^\infty(Q_T; \mathbb{R}^{3 \times 3})$ , there exists a unique function  $w$  such that  $w \in C([0, T]; L_3(\mathbb{R}^3)) \cap L_5(Q_T)$  and  $\sqrt{|w|} \nabla w \in L_2(Q_T)$  with the estimate

$$\|w\|_{3, \infty, Q_T} + \|w\|_{5, Q_T} \leq c \|F\|_{\frac{5}{2}, Q_T},$$

where  $c > 0$  is an absolute constant. Moreover,  $w$  can be expressed in the following way

$$w(x, t) = \int_0^t \int_{\mathbb{R}^3} K(x - y, t - \tau) F(y, \tau) dy d\tau.$$

So, we have the linear integral operator  $\mathcal{G} : C_0^\infty(Q_T) \subset L_{\frac{5}{2}}(Q_T) \rightarrow L_{\frac{5}{2}}(Q_T) \cap C([0, T]; L_3(\mathbb{R}^3))$ . We denote by the same symbol the extension of this operator to the whole space  $L_{\frac{5}{2}}(Q_T)$ . We wish to show

that

$$\mathcal{G}F(x, t) = \mathcal{G}_0F(x, t) := \int_0^t \int_{\mathbb{R}^3} K(x - y, t - \tau) F(y, \tau) dy d\tau$$

for  $F \in L_{\frac{5}{2}}(\mathbb{R}^3)$ . To this end, we first shall show that  $\mathcal{G}_0 : L_{\frac{5}{2}}(\mathbb{R}^3) \rightarrow L_4(\mathbb{R}^3)$  is bounded. We know, see for example [10], that

$$|K(x, t)| \leq K_0(x, t) := \frac{c}{(|x|^2 + t)^2}.$$

Let

$$g(x, t) := \int_0^t \int_{\mathbb{R}^3} K_0(x - y, t - \tau) |F(y, \tau)| dy d\tau$$

and

$$s = \frac{20}{17}.$$

Then we have

$$\begin{aligned} |K_0(x - y, t - \tau) F(y, \tau)| &= |K_0(x - y, t - \tau)|^{\frac{5}{17}} |F(y, \tau)|^{\frac{5}{8}} \times |F(y, \tau)|^{\frac{3}{8}} \times \\ &\quad \times |K_0(x - y, t - \tau)|^{\frac{12}{17}}. \end{aligned}$$

By Hölder inequality, we find

$$\begin{aligned} g(x, t) &\leq c \left( \int_0^t \int_{\mathbb{R}^3} \frac{1}{(|x - y|^2 + t - \tau)^{\frac{40}{17}}} |F(y, \tau)|^{\frac{5}{2}} dy d\tau \right)^{\frac{1}{4}} \times \\ &\quad \times \left( \int_0^t \int_{\mathbb{R}^3} |F(y, \tau)|^{\frac{5}{2}} dy d\tau \right)^{\frac{2}{5} - \frac{1}{4}} \times \left( \int_0^t \int_{\mathbb{R}^3} \frac{1}{(|x - y|^2 + t - \tau)^{\frac{40}{17}}} dy d\tau \right)^{\frac{3}{5}}. \end{aligned}$$

The last factor can be evaluated as follows:

$$\int_0^t \int_{\mathbb{R}^3} \frac{1}{(|x - y|^2 + t - \tau)^{\frac{40}{17}}} dy d\tau = c \int_0^t (t - \tau)^{-\frac{29}{34}} d\tau \int_0^\infty \frac{r^2 dr}{(r^2 + 1)^{\frac{40}{17}}} \leq C(T).$$

So, we have

$$\|g(\cdot, t)\|_{4, \mathbb{R}^3}^4 \leq C(T) \|F\|_{\frac{3}{2}, Q_T}^{\frac{3}{2}} \int_{\mathbb{R}^3} dx \int_0^t \int_{\mathbb{R}^3} \frac{1}{(|x - y|^2 + t - \tau)^{\frac{40}{17}}} |F(y, \tau)|^{\frac{5}{2}} dy d\tau =$$

$$\begin{aligned}
&= C(T) \|F\|_{\frac{3}{2}, Q_T}^{\frac{3}{2}} \int_0^t (t-\tau)^{-\frac{29}{34}} \int_{\mathbb{R}^3} |F(y, \tau)|^{\frac{5}{2}} dy d\tau \leq \\
&\leq C(T) \|F\|_{\frac{3}{2}, Q_T}^{\frac{3}{2}} \int_0^T |t-\tau|^{-\frac{29}{34}} \int_{\mathbb{R}^3} |F(y, \tau)|^{\frac{5}{2}} dy d\tau.
\end{aligned}$$

Hence,

$$\begin{aligned}
\|g(\cdot, t)\|_{4, Q_T}^4 &\leq C(T) \|F\|_{\frac{3}{2}, Q_T}^{\frac{3}{2}} \int_0^T \int_{\mathbb{R}^3} |F(y, \tau)|^{\frac{5}{2}} dy d\tau \int_0^T |t-\tau|^{-\frac{29}{34}} dt \leq \\
&\leq C(T) \|F\|_{\frac{3}{2}, Q_T}^4.
\end{aligned}$$

Next, our arguments are as follows. One can find a sequence  $F^{(m)} \in C_0^\infty(Q_T)$  such that  $F^{(m)} \rightarrow F$  in  $L_{\frac{5}{2}}(Q_T)$  as  $m \rightarrow \infty$ . Let  $w^{(m)}$  be a solution to the above Stokes system. It is easy to check that

$$w^{(m)}(x, t) = \int_0^t \int_{\mathbb{R}^3} K(x-y, t-\tau) F^{(m)}(y, \tau) dy d\tau.$$

As we know, the following estimate is valid:

$$\|w^{(m)} - w^{(k)}\|_{3, \infty, Q_T} + \|w^{(m)} - w^{(k)}\|_{5, Q_T} \leq c \|F^{(m)} - F^{(k)}\|_{\frac{5}{2}, Q_T}.$$

Let  $w \in C([0, T]; L_3(\mathbb{R}^3)) \cap L_5(Q_T)$  be a limit function. It is a unique solution to the Stokes problem with the limit function  $F$ . We need to show that  $w = \mathcal{G}_0 F$ . Indeed,

$$\|\mathcal{G}_0 F - \mathcal{G}_0 F^{(m)}\|_{4, Q_T} = \|\mathcal{G}_0 F - w^{(m)}\|_{4, Q_T} \leq C(T) \|F - F^{(m)}\|_{\frac{5}{2}, Q_T}.$$

Simply by interpolation, we can state that  $w \in L_4(Q_T)$  and  $\|w - w^{(m)}\|_{4, Q_T} \rightarrow 0$  and, hence,

$$w = \mathcal{G}(F).$$

Now, our further arguments are quite standard. We let

$$V(\cdot, t) = \Gamma(\cdot, t) \star v_0(\cdot), \quad \kappa(T) = \|V\|_{5, Q_T}$$

and let

$$v^{(k+1)} = V + \mathcal{G}(v^{(k)} \otimes v^{(k)})$$

with  $v^{(0)} = 0$ . Using previous estimates, we have

$$\|v^{(k+1)} - V\|_{3,\infty,Q_T} + \|v^{(k+1)} - V\|_{5,Q_T} \leq c\|v^{(k)}\|_{5,Q_T}^2.$$

Thus

$$\|v^{(k+1)}\|_{5,Q_T} \leq \|V\|_{5,Q_T} + c\|v^{(k)}\|_{5,Q_T}^2.$$

The objective, now, is to show

$$\|v^{(k+1)}\|_{5,Q_T} \leq 2\kappa(T).$$

Arguing by induction, we arrive at the estimate

$$\|v^{(k+1)}\|_{5,Q_T} \leq c4\kappa^2(T) + \kappa(T) = \kappa(T)(4c\kappa(T) + 1).$$

Let us impose the additional assumption

$$\kappa(T) \leq \frac{1}{16c}.$$

Later on, we shall show that it is possible. Now, assume that the above condition holds. We have

$$v^{(k+1)} - v^{(k)} = \mathcal{G}(v^{(k)} \otimes v^{(k)} - v^{(k-1)} \otimes v^{(k-1)})$$

and thus

$$\begin{aligned} & \|v^{(k+1)} - v^{(k)}\|_{3,\infty,Q_T} + \|v^{(k+1)} - v^{(k)}\|_{5,Q_T} \leq \\ & \leq 2c\|v^{(k)} - v^{(k-1)}\|_{5,Q_T} (\|v^{(k)}\|_{5,Q_T} + \|v^{(k-1)}\|_{5,Q_T}) \leq \\ & \leq 8c\kappa(T)\|v^{(k)} - v^{(k-1)}\|_{5,Q_T} \leq \frac{1}{2}\|v^{(k)} - v^{(k-1)}\|_{5,Q_T} \leq \frac{1}{4}\|v^{(k-1)} - v^{(k-2)}\|_{5,Q_T} \leq \\ & \leq \frac{1}{2^{k-1}}\|v^{(1)}\|_{5,Q_T} = \frac{1}{2^{k-1}}\kappa(T). \end{aligned}$$

Therefore,

$$\begin{aligned} \|v^{(k)} - v^{(m)}\|_{5,Q_T} & \leq \sum_{i=m}^{k-1} \|v^{(i+1)} - v^{(i)}\|_{5,Q_T} \leq \\ & \leq \sum_{i=m}^{k-1} \frac{1}{2^i} \kappa(T). \end{aligned}$$

So,  $v^{(m)} \rightarrow v$  in  $L_5(Q_T)$ . Then

$$\|v^{(k)} - v^{(m)}\|_{3,\infty,Q_T} \leq 2c\|v^{(k)} - v^{(m)}\|_{5,Q_T} (\|v^{(k)}\|_{5,Q_T} + \|v^{(m)}\|_{5,Q_T}) \leq$$

$$\leq 8c\|v^{(k)} - v^{(m)}\|_{5, Q_T} \kappa(T).$$

This means that  $v^{(m)} \rightarrow v$  in  $C([0, T]; L_3(\mathbb{R}^3))$ . So, the existence of mild solution has been proven. Uniqueness follows easily from the same arguments as above. See the additional assumption when proving strong convergence of the whole sequence.

Now, going back to the assumption on the smallness of  $\kappa(T)$ , we have

$$\kappa(T) = I_\varrho^1 + I_\varrho^2,$$

where

$$I_\varrho^1 = \|(V)_\varrho\|_{5, Q_T}, \quad I_\varrho^2 = \|V - V_\varrho\|_{5, Q_T}, \quad V_\varrho(\cdot, t) = \Gamma(\cdot, t) \star (v_0)_\varrho(\cdot),$$

and  $(v_0)_\varrho$  is a standard mollification of  $v_0$ . With  $I_\varrho^2$ , we proceed as follows

$$I_\varrho^2 \leq c\|v_0 - (v_0)_\varrho\|_{3, \mathbb{R}^3} \leq \frac{1}{32c}$$

for some fixed  $\varrho > 0$ . Next,

$$\|(V)_\varrho(\cdot, t)\|_{5, \mathbb{R}^3} \leq ct^{-\frac{1}{r}} \|(v_0)_\varrho\|_{4, \mathbb{R}^3},$$

where

$$\frac{1}{r} = \frac{3}{2} \left( \frac{1}{4} - \frac{1}{5} \right) = \frac{3}{40}.$$

Hence,

$$I_\varrho^1 = \|(V)_\varrho\|_{5, Q_T} \leq cT^{\frac{1}{8}} \|(v_0)_\varrho\|_{4, \mathbb{R}^3}.$$

The right hand side of the latter inequality can be made small for a given  $\varrho$  at the expense of  $T$ .

Now, I wish to show that the constructed above mild solution is in fact a weak  $L_3$ -solution in  $Q_T$ . To this end we need to show that  $w := v - v^1 \in L_{2, \infty}(Q_T) \cap W_2^{1, 0}(Q_T)$  and satisfy the energy inequality. We start with local energy inequality

$$\begin{aligned} I &:= \frac{1}{2} \int_{\mathbb{R}^3} \varphi^2(x) |w(x, t)|^2 dx + \int_0^t \int_{\mathbb{R}^3} \varphi^2 |\nabla w|^2 dx ds \leq \\ &\leq \int_0^t \int_{\mathbb{R}^3} \left( \frac{1}{2} |w|^2 \Delta \varphi^2 + \frac{1}{2} w \cdot \nabla \varphi^2 (|w|^2 + 2r) + \right. \\ &\quad \left. + v^1 \otimes w : \nabla w \varphi^2 + v^1 \otimes w : w \otimes \nabla \varphi^2 + \right. \end{aligned}$$

$$\begin{aligned}
& +w \otimes v^1 : \nabla w \varphi^2 + w \otimes v^1 : w \otimes \nabla \varphi^2 + \\
& +v^1 \otimes v^1 : \nabla w \varphi^2 + v^1 \otimes v^1 : w \otimes \nabla \varphi^2) dx ds
\end{aligned}$$

for any  $\varphi \in C_0^\infty(\mathbb{R}^3)$ . Here, the pressure  $r$  is defined by the equation

$$\partial_t w - \Delta w + \nabla r = -\operatorname{div} v \otimes v$$

and we know

$$\|r\|_{\frac{5}{2}, Q_T} + \|r\|_{\frac{3}{2}, \infty, Q_T} < \infty.$$

Assuming that  $0 \leq \varphi \leq 1$ ,  $\varphi(x) = 1$ , if  $|x| < R$ ,  $\varphi(x) = 0$  if  $|x| > 2R$ , and  $|\nabla \varphi| \leq c/R$ , we find

$$\begin{aligned}
I & \leq c \frac{1}{R^2} \|w\|_{3, Q_T}^2 R + c \frac{1}{R} \|w\|_{3, Q_T}^3 + c \frac{1}{R} \|w\|_{3, Q_T} \|r\|_{\frac{3}{2}, Q_T} + \\
& + c(\|v^1\|_{4, Q_T} \|w\|_{4, Q_T} + \|v^1\|_{4, Q_T}^2) I^{\frac{1}{2}} + \\
& + c \frac{1}{R} (\|v^1\|_{3, Q_T} \|w\|_{3, Q_T}^2 + \|v^1\|_{3, Q_T}^2 \|w\|_{3, Q_T})
\end{aligned}$$

From the latter bound, we can easily get all the statements.

**Corollary 5.1.** *Let be  $v$  be a weak  $L_3$ -solution. There exists  $T > 0$  such that  $v$  is a mild solution in  $Q_T$ .*

Indeed, we know that there exist a mild solution  $u$  in  $Q_T$  for some  $T > 0$  depending on  $v_0$ . By the previous observation, it is a weak  $L_3$ -solution in  $Q_T$ . By the uniqueness theorem,  $v = u$ .

## References

- [1] Caffarelli, L., Kohn, R.-V., Nirenberg, L., *Partial regularity of suitable weak solutions of the Navier-Stokes equations*, Comm. Pure Appl. Math., Vol. XXXV (1982), pp. 771–831.
- [2] Caldern, C. P., Existence of weak solutions for the Navier-Stokes equations with initial data in  $L_p$ . Trans. Amer. Math. Soc. 318 (1990), no. 1, 179–200.
- [3] Cannone, M., Harmonic analysis tools for solving the incompressible Navier-Stokes equations. Handbook of mathematical fluid dynamics. Vol. III, 161–244, North-Holland, Amsterdam, 2004.

- [4] Escauriaza, L.; Seregin, G.; Šverák, V.  $L_{3,\infty}$ -solutions of Navier-Stokes equations and backward uniqueness. (Russian) *Uspekhi Mat. Nauk* 58 (2003), no. 2(350), 3–44; translation in *Russian Math. Surveys* 58 (2003), no. 2, 211–250.
- [5] Galdi, Giovanni P., An introduction to the Navier-Stokes initial-boundary value problem. *Fundamental directions in mathematical fluid mechanics*, 1–70, *Adv. Math. Fluid Mech.*, Birkhuser, Basel, 2000.
- [6] Giga, Y., Solutions for semilinear parabolic equations in  $L_p$  and regularity of weak solutions of the Navier Stokes system, *J. Differential Equations*, 62 (1986), pp. 186–212.
- [7] Kato, T., Strong  $L^p$ -solutions of the Navier-Stokes equation in  $R^m$ , with applications to weak solutions. *Math. Z.* 187 (1984), no. 4, 471–480.
- [8] Jia, H., Sverak, V., Are the incompressible 3d Navier-Stokes equations locally ill-posed in the natural energy space? *J. Funct. Anal.* 268 (2015), no. 12, 3734–3766.
- [9] Kikuchi, N., Seregin, G., Weak solutions to the Cauchy problem for the Navier-Stokes equations satisfying the local energy inequality, *AMS translations, Series 2, Volume 220*, pp. 141–164.
- [10] Koch, G., Nadirashvili, N., Seregin, A., Šverák, V., Liouville theorems for the Navier-Stokes equations and applications. *Acta Math.* 203 (2009), no. 1, 83–105.
- [11] Koch, H., Solonnikov, V. A.,  $L_p$ -estimates for a solution to the nonstationary Stokes equations. *Function theory and phase transitions. J. Math. Sci. (New York)* 106 (2001), no. 3, 3042–3072.
- [12] Kozono, H.; Sohr, H., Remark on uniqueness of weak solutions to the Navier-Stokes equations. *Analysis* 16 (1996), no. 3, 255–271.
- [13] Koch, H., Tataru, D., *Well-posedness for the Navier-Stokes equations*, *Adv. Math.* 157 (2001), no. 1, 22–35.
- [14] Ladyzhenskaya, O. A. The mathematical theory of viscous incompressible flow. Second English edition, revised and enlarged. Translated from the Russian by Richard A. Silverman and John Chu. *Mathematics and its Applications, Vol. 2* Gordon and Breach, Science Publishers, New York-London-Paris 1969 xviii+224 pp.

- [15] Ladyzhenskaya, O. A., Solonnikov, V. A., Uralt'seva, N. N., Linear and quasi-linear equations of parabolic type, Moscow, 1967; English translation, American Math. Soc., Providence 1968.
- [16] Lemarie-Rieusset, P. G., Recent developemnets in the Navier-Stokes problem, Chapman&Hall/CRC reseacrh notes in mathematics series, 431.
- [17] J. Leray, *Sur le mouvement d'un liquide visqueux emplissant l'espace*, Acta Math. **63** (1934), 193–248.
- [18] Oseen, C. W., Sur let formules de Green généralisées qui se présentent dans l'hydrodynamique et sur quelques-unes de leus application, Acta Math. 34 (1911), no. 1, 205–284.
- [19] Rusin, W., Šverák, V. Minimal initial data for potential Navier–Stokes singularities. J. Funct. Anal. 260 (2011), no. 3, 879891.
- [20] Seregin, G.A., *Local regularity of suitable weak solutions to the Navier-Stokes equations near the boundary*, J. math. fluid mech., 4(2002), no.1,1–29.
- [21] Seregin, G. A., *On smoothness of  $L_{3,\infty}$ -solutions to the Navier-Stokes equations up to boundary*, Mathematische Annalen, 332(2005), pp. 219-238.
- [22] Seregin, G., *A note on local boundary regularity for the Stokes system*, Zapiski Nauchn. Seminar., POMI, 370 (2009), pp. 151-159.
- [23] Seregin, G., *A note on necessary conditions for blow-up of energy solutions to the Navier-Stokes equations*, Progress in Nonlinear Differential Equations and Their Applications, 2011 Springer Basel AG, Vol. 60, 631–645.
- [24] Seregin, G., *Necessary conditions of potential blow up for the Navier-Stokes equations*, Zapiski Nauchn.Seminar. POMI, 385(2010), 187-199.
- [25] Seregin, G., *A certain necessary condition of potential blow up for Navier-Stokes equations*, Comm. Math. Phys. 312 (2012), no. 3, 833–845.
- [26] Seregin, G., *Lecture notes on regularity theory for the Navier-Stokes equations*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2015. x+258 pp. ISBN: 978-981-4623-40-7

- [27] Solonnikov, V. A., *Estimates of solutions to the non-stationary Navier-Stokes system*, Zapiski Nauchn. Seminar. LOMI 28(1973), 153–231.
- [28] Solonnikov, V. A. Estimates for solutions of the nonstationary Stokes problem in anisotropic Sobolev spaces and estimates for the resolvent of the Stokes operator. (Russian) Uspekhi Mat. Nauk 58 (2003), no. 2(350), 123–156; translation in Russian Math. Surveys 58 (2003), no. 2, 331-365

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Dear Editors,

Please, will you consider the paper "On global weak solutions to the Cauchy problem for the Navier-Stokes equations with large  $L_3$ -initial data" by G. Seregin and V. Sverak for publication in the journal Nonlinear Analysis.

Sincerely,

G. Seregin and V. Sverak.