Generalised Young Measures and lower semi-continuity of integral functionals



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Abstract

The purpose of this thesis is to study representations of integral functionals of the form

$$u \mapsto \int_\Omega f(u(x))d\mu(x) =: F(u), \tag{0.0.1}$$

when

- $u \in L^p(\Omega, \mu, \mathbb{R}^d)$ varies in different topologies,
- μ is a non-negative finite Borel measure,
- $f\colon \mathbb{R}^d \to \mathbb{R}$ is continuous and has p-growth

$$|f(z)| \leq C(1 + |z|^p).$$

If a sequence $(u_j)_j$ converges strongly in L^p to some $u \in L^p(\Omega, \mu, \mathbb{R}^m)$, one can always pass to the limit under the integral sign in (0.0.1) and obtain that $F(u_j) \to F(u)$. However, a sequence $(u_j)_j$ bounded in $L^p(\Omega, \mu, \mathbb{R}^m)$ is not always relatively compact in the strong topology, and this happens exactly when $(u_j)_j$ oscillates or when $(|u_j|^p)_j$ concentrates. In this case, $F(u_j)$, in general, does not converge to $F(u)$. After constructing a formula representing accumulation points of $F(u_j)$ for sequences $(u_j)_j$ that are only bounded in $L^p(\Omega, \mu, \mathbb{R}^m)$, and so allowed to oscillate and concentrate, we use it to solve classical problems in the calculus of variations that involve gradients.

We mainly focus on classes of functions whose derivatives are integrable, for example, when

$$u = \nabla v \quad \text{or} \quad u = (v, \nabla v),$$

with $v \in W^{1,p}(\Omega, \mathbb{R}^m)$. This problem has been largely studied for $1 < p \leq +\infty$, and lower semi-continuity of (0.0.1) with respect to weak convergence

has been fully characterised via a condition on f called *quasi-convexity*, which is

$$f(z) \leq \frac{1}{\mathcal{L}^n(\Omega)} \int_\Omega f(z + \nabla\phi(x))dx$$

for all $z \in \mathbb{R}^{m\times n}$ and all $\phi\colon \Omega \to \mathbb{R}^m$ that are smooth and compactly supported. The aim of this thesis is to answer some of the unsolved questions, mostly in the less-known case $p = 1$.

When a sequence $(u_j)_j$ is bounded in $L^p(\Omega, \mu, \mathbb{R}^m)$, it can happen that $(|u_j|^p)_j$, which is a sequence bounded in $L^1(\Omega, \mu)$, converges to a measure singular with respect to μ. There is extensive literature on representations of (0.0.1) when $f(z)$ has a homogeneous-like behaviour for large values of z, but the general case has not been studied in depth.

Here we prove a new representation of limits of bounded sequence $(u_j)_j$ in $L^p(\Omega, \mu, \mathbb{R}^d)$ of (0.0.1), which are called *Young Measures*. Such a representation is an improvement of the well-known construction by DiPerna and Majda in [36] and enables us to manipulate such limits similarly to Alibert and Bouchitté [6], but for any continuous f that has p-growth. This is necessary as quasi-convexity does not entail a homogeneous-like behaviour of $f(z)$ for large values of $z \in \mathbb{R}^{m\times n}$, when $m, n > 1$, so the classical theory of Young Measures cannot always be used.

Via our representation, we are able to prove a characterisation of Gradient Young Measures, which are generalised limits of sequence $(\nabla u_j)_j$ with $(u_j)_j$ bounded in $W^{1,1}(\Omega, \mathbb{R}^m)$, in greater generality than previously achieved. We are also able to improve the lower semi-continuity result for measure derivatives in [65], to the case $v = \left(u, \frac{Du}{\mu}\right)$ in (0.0.1), and we also show that minimising sequences of the Dirichlet problem of (0.0.1) in $W^{1,p}$, under a sharp p-coercivity condition, are equi-integrable when $p > 1$.

The importance and originality of our framework are that we can show that many properties of different classes of generalised Young Measures are essentially the same as in the classical case developed in [6], where only a small class of integrands was considered. Our framework also enables us to perform arithmetic-like operations on objects that were previously too abstract to be manipulated.

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Chapter 1

Intro

1.1 Introduction

Young Measures were first introduced by Young in [100] to study the minima of integral energies of the form

$$u \mapsto \int_0^1 f(u(t), u'(t))dt, \tag{1.1.1}$$

where $u \in C^1([0,1])$ with $u(0) = a$, $u(1) = b$ and $\|u'\|_\infty \leq C, C > 0$ fixed and $f\colon \mathbb{R}^2 \to \mathbb{R}$ is continuous, and for related problems in optimal control theory. The author wanted to understand what conditions on f would guarantee the existence of a minimising curve $u(t)$. Young had the intuition that, for a general class of functions f, minimising sequences $(u_j)_j$ in $C^1([0,1])$ always admit a subsequence (non-relabelled) and $u \in C([0,1])$ with $u' \in L^\infty([0,1])$ so that $u_j \to u$ uniformly and u_j' converges to a *generalised curve*

$$\nu = \nu_t\colon [0,1] \to \mathcal{M}_1^+(\mathbb{R}).$$

In particular, the derivatives distribute their values and become a probability measure on $\mathbb{R}$. This translates to the following equality

$$\inf_{u(0)=a, u(1)=b} \int_0^1 f(u(t), u'(t))dt = \lim_j \int_0^1 f(u_j(t), u_j'(t))dt = \int_0^1 \int_{\mathbb{R}} f(u(t), z)d\nu_t(z)dt,$$

for every continuous $f\colon \mathbb{R}^2 \to \mathbb{R}$. The question of the existence of a minimiser can be reformulated as to whether (1.1.1) is *lower semi-continuous* along $(u_j)_j$, meaning

$$\liminf_j \int_0^1 f(u_j(t), u_j'(t))dx \geq \int_0^1 f(u(x), u'(t))dt.$$

But because of the existence of a generalised curve ν, the above lower semi-continuity is equivalent to

$$\int_0^1 \int_{\mathbb{R}} f(u(t), z) d\nu_t(z) dt \geq \int_0^1 f(u(x), u'(t)) dt.$$

Therefore, the question of the existence of a minimiser can be reformulated in two different ways.

1. Are minimising sequences so well behaved that every generalised curve ν is, actually, just the gradient of a function u? In mathematical terms, we have to understand if $\nu_t = \delta_{u'(t)}$ at $\mathcal{L}^1$ almost every $t \in (0,1)$, where $\mathcal{L}^1$ is the Lebesgue measure.

2. Is f well behaved so that

$$\int_{\mathbb{R}} f(u(t), z) d\nu_t(z) \geq f(u(t), u'(t))$$

at $\mathcal{L}^1$ almost every $t \in (0,1)$?

Let $\mathcal{M}^+(\Omega)$ be the set of non-negative finite Borel measures, and for $\mu \in \mathcal{M}^+(\Omega)$ let

$$L^p(\Omega, \mu, \mathbb{R}^d) = \left\{ u \colon \Omega \to \mathbb{R}^d : u \text{ Borel measurable and } \|u\|^p_{\Omega,\mu,p} < +\infty, \right\},$$

where

$$\|u\|^p_{\Omega,\mu,p} := \int_\Omega |u(x)|^p d\mu(x) \quad \text{if } 1 \leq p < \infty \text{ and}$$
$$\|u\|_{\Omega,\mu,\infty} := \inf\{t > 0 : \mu(\{x \in \Omega : |u(x)| > t\}) = 0\},$$

where $|\cdot|$ is the 2-norm on $\mathbb{R}^d$. The natural reformulation of (1.1.1) in the setting of $L^p(\Omega, \mu, \mathbb{R}^d)$ functions is to require that f is continuous and has *p-growth*, i.e. there is a constant $C > 0$ so that

$$|f(z)| \leq C(1 + |z|^p) \quad \forall z \in \mathbb{R}^d,$$

and the integral

$$u \mapsto \int_\Omega f(u(x)) d\mu(x)$$

is minimised over a subset of $L^p(\Omega, \mu, \mathbb{R}^d)$. For a continuous $f\colon \mathbb{R}^d \to \mathbb{R}$ as above, the first question translates as to whether or not a minimising sequence $(u_j)_j$ bounded in $L^p(\Omega, \mu, \mathbb{R}^d)$ converges in μ-measure to some $u \in L^p(\Omega, \mu, \mathbb{R}^d)$, meaning

$$\mu(\{x \in \Omega : |u_j(x) - u(x)| > \varepsilon) \xrightarrow{j\to\infty} 0 \quad \forall \varepsilon > 0,$$

and when this does not occur we say that $(u_j)_j$ *oscillates*. In general, convergence in μ measure does not imply strong convergence in $L^p(\Omega, \mu, \mathbb{R}^m)$ (unless μ is a sum of Dirac masses), where, by strong convergence, we mean convergence in the norm $\|\cdot\|_{\Omega,\mu,p}$. One needs to check that $(u_j)_j$ is also p-equi-integrable, meaning

$$\lim_{k\to+\infty} \left(\sup_{j\in\mathbb{N}} \int_{\{x\in\Omega : |u_j(x)|>k\}} |u_j(x)|^p d\mu(x) \right) = 0,$$

and if this is not the case we say that $(u_j)_j$ *concentrates*. By the Vitali Theorem A.1.1, and since μ is a finite measure, these are the only two obstructions to strong convergence in $L^p(\Omega, \mu, \mathbb{R}^m)$ for every $1 \leq p < +\infty$. To visualise, heuristically, what oscillation and concentration mean, we look at the graph of $(u_j)_j$ and study what happens when converge fails in a classical sense.

We illustrate oscillation and concentration in the toy case $n = m = 1$, $\Omega = \left[-\frac{1}{2}, \frac{1}{2}\right]$, $\mu = \mathcal{L}^1$ and $p = 1$ with the two following examples.

- *Oscillation* - $u_j(x)$ assumes frequently multiple values around points $x \in \Omega$ and, in the limit, generates probability distributions

  $$x \mapsto \nu_x \in \mathcal{M}_1^+(\mathbb{R}^d)$$

  Example - take

  $$u(x) = -\chi_{\left[-\frac{1}{2},0\right]} + \chi_{\left[0,\frac{1}{2}\right]} \quad x \in \left[-\frac{1}{2}, \frac{1}{2}\right],$$

  extended by 1-periodicity to the whole of $\mathbb{R}$ and define $u_j(x) = u\left(\frac{x}{j}\right)$. Then, if $f\colon \mathbb{R} \to \mathbb{R}$ is continuous, we immediately obtain that

  $$\int_{-\frac{1}{2}}^{\frac{1}{2}} f(u_j(x))dx = \frac{f(-1)}{2} + \frac{f(1)}{2} = \int_{\mathbb{R}} f(z) d\left(\frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1\right).$$

- *Concentration* - $|u_j|^p$ concentrates to some measure $\lambda \in \mathcal{M}^+(\overline{\Omega})$, meaning

  $$\lim_k \lim_j \int_{\{x\in U : |u_j|>k\}} \phi(x)|u_j(x)|^p d\mu(x) = \int_{\overline{\Omega}} \phi(x) d\lambda,$$

for all smooth $\phi\colon \mathbb{R}^n \to \mathbb{R}$. The values of $u_j(x)$, on the other side, escape to infinity on a set where λ is concentrated, and so their limit can be seen by embedding $u_j \in \mathbb{R}^d$ into a compact set K containing $\mathbb{R}^d$.

Example - think of u_j as a step function approximating a Dirac delta,

$$u_j(x) = j\chi_{\left[-\frac{1}{2j},\frac{1}{2j}\right]}.$$

The integral becomes

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} f(u_j(x))dx = \frac{f(j)}{j}.$$

Because $p = 1$, we assume that f has linear growth and the above quantity is always bounded, so we can find a subsequence $(u_{j_i})_i$ - in fact, many different subsequences - such that $\lim_i \frac{f(j_i)}{j_i} = f^\infty(\{j_i\}_i)$, where f^∞ depends on the particular subsequence we considered. But then we can write

$$\lim_i \int_{-\frac{1}{2}}^{\frac{1}{2}} f(u_{j_i}(x))dx = \lim_i \frac{f(j_i)}{j_i} = \int_{\left[-\frac{1}{2},\frac{1}{2}\right]} f^\infty(\{j_i\}_i)d\delta_0.$$

So, the corresponding representation of a generalised curve in $L^p(\Omega, \mu, \mathbb{R}^d)$, called *generalised Young Measure*, is an object that acts on suitable fs by assigning the value

$$\int_\Omega \int_{\mathbb{R}^d} f(z)d\nu_x(z)d\mu(x) + \int_{\overline{\Omega}} \int_{\partial K} f^\infty(w)d\nu_x^\infty(w)d\lambda(x),$$

where $\partial K = K \setminus \mathbb{R}^d$ is the "boundary" of the compact set K containing $\mathbb{R}^d$, and we are able to formulate the second question as a question about the validity of a Jensen-type inequality on some abstract space.

This thesis is devoted to studying different representations of such generalised Young Measures in different classes of integrable functions and relating them to the problem of lower semi-continuity and coercivity of integral functionals under different conditions.

1.1.1 Generalised Young Measures

Young's original work focused on the case $n = 1$ and was carried out via functional analytic methods. This approach was later extended by many authors, including [79, 80, 26, 18, 15, 29, 16] to higher dimensions. We call these generalised functions *Young Measures*. The method developed by Young is not powerful enough to tackle

problems arising in modern mathematics, as it essentially only describes oscillations in sequences $(v_j)_{j\in\mathbb{N}}$ that are bounded in L^∞. In [99], Tartar pioneered the use of Young Measures as a tool to study non-linear partial differential equations, which moved the development of classical Young Measures to the study of Young Measures in Lebesgue spaces, as they are the natural spaces for many questions related to partial differential equations. The first systematic attempt to represent generalised limits of weakly converging sequences of integrable functions that also takes concentration effects into account is due to DiPerna and Majda, [36]. Functions $v_j \in L^p(\Omega, \mu, \mathbb{R}^d)$ are identified with the measures on $\Omega \times \mathbb{R}^d$ that result by integrating over their graph, namely

$$\int_\Omega \Phi(x, v_j(x))d\mu(x) = \int_{\Omega\times\mathbb{R}^d} \Phi(x, y)d\left(\left(\cdot, v_j(\cdot)\right)_\# \mu\right) =: \langle \xi_{v_j}, \Phi\rangle,$$

where "#" is called the *pushforward operator* and $\left(\cdot, v_j(\cdot)\right)_\# \mu \in \mathcal{M}(\Omega \times \mathbb{R}^d)$. The product space $\Omega \times \mathbb{R}^d$ is subsequently embedded into a compact set $K \supset \Omega \times \mathbb{R}^d$, and assuming that the sequence $(v_j)_j$ is bounded in $L^p(\Omega, \mu, \mathbb{R}^d)$, an accumulation point, in a weak* sense, is then found

$$\xi_{v_{j_i}} \rightharpoonup^* \xi \in \mathcal{M}(K). \tag{1.1.2}$$

By weak* convergence $\mu_j \to \mu$ in $\mathcal{M}(U, \mathbb{R}^d)$ (or equivalently $C_0(U, \mathbb{R}^d)^*$), where U is an arbitrary locally compact metric space, we mean

$$\int_U \phi(x)d\mu_j(x) \to \int_\Omega \phi(x)d\mu(x) \quad \forall \phi \in C_0(U, \mathbb{R}^d),$$

where

$$C_0(U, \mathbb{R}^d) = \{u \in C(U, \mathbb{R}^d) : \forall \varepsilon > 0 \text{ there is } K \subset U \text{ compact such that } \|u\|_{\infty,K} \leq \varepsilon\}.$$

Such an accumulation point is a measure defined on K, where K is an abstract compact set containing $\Omega \times \mathbb{R}^d$, called *compactification*, and as such, it is not clear how to conveniently represent ξ. In [6], an explicit and natural formula for such accumulation points was obtained for a class of integrands that behave like homogeneous functions at infinity, whereas in other instances such as [95, 73], ξ as in (1.1.2) has been used in its bare form to prove results in the calculus of variations.

In Chapter 2, we give a general representation formula for describing Young Measures for a large class of integrands in the spirit of [6]. The construction of Young Measures follows mainly the work by DiPerna and Majda [36], Alibert and Bouchitté

[6], and a procedural approach developed in [68], see also [95, 88, 92]. This generalisation is based on the canonical way of constructing Hausdorff compactifications starting from continuous functions, see [30]. A small reduction lemma gives a clearer and more geometrical interpretation of such limits. This formula captures concentrations at infinity, which are now allowed to occur without restrictions. We also prove a few structure theorems that relate different compactifications and their Young Measure representations to each other. Formally, we show that given a bounded sequence $(v_j)_j$ in $L^p(\Omega, \mu, \mathbb{R}^d)$ and a family of integrands of p-growth $\{f_i, i \in \mathbb{N}\}$, there is a subsequence (non-relabelled) and a triple

$$(\nu_x, \lambda, \nu_x^\infty)$$

with

$$\nu_x \colon \Omega \to \mathcal{M}_1^+(\mathbb{R}^d),\ \lambda \in \mathcal{M}^+(\overline{\Omega}) \quad \text{and} \quad \nu_x^\infty \colon \overline{\Omega} \to \mathcal{M}_1^+(\partial e_{f_i, i\in\mathbb{N}}) \tag{1.1.3}$$

such that

$$\lim_j \int_\Omega f(v_j(x))d\mu(x) = \int_\Omega \int_{\mathbb{R}^d} f(z)d\nu_x(z)d\mu(x) + \int_{\overline{\Omega}} \int_{\partial e_{f_i, i\in\mathbb{N}}} f^\infty(w)d\nu_x^\infty(w)d\lambda(x).$$

Here $e_{f_i, i\in\mathbb{N}}$ is a compactification of $\mathbb{R}^d$ naturally derived from $\{f_i, i \in \mathbb{N}\}$ and f^∞ is the extension of f to such compactification. This compactification has the topology of a product space, in many cases of interest finite-dimensional, and so it is easier to work with compared to previous representations of generalised Young Measures.

It is important to mention that such generalised Young Measures, defined as the limit (1.1.2), were known to exist since DiPerna and Majda, and have been studied and used in applications ever since, see for example in [73, 95, 56]. However, the lack of clear representation and geometric interpretation always hindered the use of generalised Young Measures to solve problems from the calculus of variations that involve gradients, some of which we solve, in this thesis, thanks to our new representation.

1.1.2 Characterisation of Gradient Young Measures

This generalisation of Young Measures is then applied to study extensions and variations of energies that depend on gradients, the prototype of which is

$$u \mapsto \int_\Omega f(\nabla u(x))dx, \tag{1.1.4}$$

where u is initially taken in a small class of functions, such as $C^\infty(\Omega,\mathbb{R}^m)\cap W^{1,p}(\Omega,\mathbb{R}^m)$, where

$$C^\infty(\Omega,\mathbb{R}^m)=\{u\colon \Omega\to\ R^m : u \text{ is infinitely differentiable }\},$$

and for $1\le p\le\infty$

$$W^{1,p}(\Omega,\mathbb{R}^m)=\{u\in L^p(\Omega,\mathcal{L}^n,\mathbb{R}^m) : \nabla u\in L^p(\Omega,\mathcal{L}^n,\mathbb{R}^{m\times n}\},$$

$\Omega\subset\mathbb{R}^n$ is a bounded domain with Lipschitz boundary, and $f\in C(\mathbb{R}^{m\times n})$ has p-growth (or just continuous when $p=\infty$). Here, for a function $u\in L^1(\Omega)$, by $Du=[\partial_j u]_j$ we mean its derivative in the sense of distribution, namely for all

$$\phi\in\mathcal{D}(\Omega)=\{\phi\in C^\infty(\Omega,\mathbb{R}^m) : \phi \text{ has compactly support }\}$$

we have

$$\langle\partial_j u,\phi\rangle=-\int_\Omega u\partial_j\phi dx.$$

Reconnecting to the problem of finding minimisers, or similarly studying lower semi-continuity of (1.1.4), we notice that the space $C^\infty(\Omega,\mathbb{R}^m)\cap W^{1,1}(\Omega,\mathbb{R}^m)$ is dense in many spaces of interest in applications, like $W^{1,p}(\Omega,\mathbb{R}^m)$ with the weak topology, where $u_j\rightharpoonup u$ weakly in $W^{1,p}(\Omega,\mathbb{R}^m)$ if $u_j\to u$ strongly in $L^p(\Omega,\mathbb{R}^m)$ and $Du_j\rightharpoonup Du$ weakly in $L^p(\Omega,\mathbb{R}^m)$ (weakly* if $p=\infty$), or

$$BV(\Omega,\mathbb{R}^m)=\{u\in L^1(\Omega,\mathbb{R}^m) : Du\in\mathcal{M}(\Omega,\mathbb{R}^{m\times n})\},$$

with the weak* topology, where we say that $u_j\to u$ weak* in $BV(\Omega,\mathbb{R}^m)$ provided $u_j\to u$ strongly in $L^1(\Omega,\mathbb{R}^m)$ and $Du_j\to Du$ weak* in $C_0(\Omega,\mathbb{R}^{m\times n})^*$.

This latter topology can be especially hard to work with, but it is the more interesting one in this class of problems. For example, it can be shown that for any non-linear $f\in C(\mathbb{R}^{m\times n})$ of linear growth, (1.1.4) is not sequentially weak* continuous in $BV(\Omega,\mathbb{R}^{m\times n})$, see [17, 19]. We can however find an extension which is lower semi-continuous for certain fs, under suitable growth conditions.

In [83], Morrey established the equivalence of lower semi-continuity of (1.1.4) to a condition that he called *quasi-convexity*, which can be written as a Jensen-type inequality:

$$f(z)\le\frac{1}{\mathcal{L}^n(\Omega)}\int_\Omega f(z+\nabla\phi(x))dx\quad\forall\phi\in\mathcal{D}(\Omega,\mathbb{R}^m) \text{ and } z\in\mathbb{R}^{m\times n}. \quad (1.1.5)$$

The original result by Morrey applies to the setting of weak* convergence in $W^{1,\infty}(\Omega,\mathbb{R}^m)$ and weak convergence in $W^{1,p}(\Omega,\mathbb{R}^m)$ with $1 \leq p < +\infty$, when f has p-growth and it is bounded from below. This was later extended to functions that depend on more than one variable in [81], and to Carathéodory integrands in [78, 1, 77]. The aforementioned results hold when f is positive and has appropriate growth. In the case of signed integrands, one must work within the Dirichlet class

$$W^{1,p}_g(\Omega,\mathbb{R}^m) = \{u \in W^{1,p}(\Omega,\mathbb{R}^m) : u\chi_\Omega + g\chi_{\Omega^c} \in W^{1,p}(\mathbb{R}^n,\mathbb{R}^m)\}$$

for some $g \in W^{1,p}(\mathbb{R}^n,\mathbb{R}^m)$, where χ_U (or equivalently $x \mapsto \chi_{x\in U}$) is the indicator function of $U \subset \mathbb{R}^n$. In this case, [22] is one of the first examples where Young Measures are employed for proving lower semi-continuity for functionals of the type (1.1.4). To be more specific, (1.1.5) can be rephrased as a Jensen-type inequality for measures of the form

$$\left\{\nu : \nu = \frac{1}{\mathcal{L}^n(\Omega)}(\nabla\phi)_{\#}d\mathcal{L}^n \llcorner \Omega, \quad \phi \in \mathcal{D}(\Omega,\mathbb{R}^m)\right\}, \tag{1.1.6}$$

where the measure ν acts on f in the following way:

$$\frac{1}{\mathcal{L}^n(\Omega)}\int_\Omega f(z+\nabla\phi(x))dx = \frac{1}{\mathcal{L}^n(\Omega)}\int_{\mathbb{R}^{m\times n}} f(z+w)d\nu(w) \equiv \langle \nu, f(z+\cdot)\rangle.$$

It turns out that gradients of a weakly convergent sequence $(u_j)_j$ in $W^{1,p}(\Omega,\mathbb{R}^m)$ with $u_j \rightharpoonup u$ in $W^{1,p}$ if $1 \leq p < +\infty$ or $u_j \overset{*}{\rightharpoonup} u$ in $W^{1,\infty}(\Omega,\mathbb{R}^m)$, subsequentially generate an oscillation Young Measure

$$\nu_x \colon \Omega \to \mathcal{M}^+_1(\mathbb{R}^{m\times n}),$$

where ν_x acts on functions $f \in C_0(\mathbb{R}^{m\times n})$ and $\phi \in C_0(\Omega)$ via the pairing

$$\lim_j \int_\Omega \phi(x) f(\nabla u_j(x))dx = \int_\Omega \phi(x)\int_{\mathbb{R}^m} f(z)d\nu_x(z)dx =: \int_\Omega \phi(x)\langle \nu_x, f\rangle dx,$$

and satisfies the moment condition

$$\int_\Omega \langle \nu_x, |\cdot|^p\rangle dx < +\infty$$

if $1 \leq p < +\infty$, or $\operatorname{supp}(\nu_x) \subset \mathbb{B}_R(0)$ for some $R > 0$ and $\mathcal{L}^n$-a.e. $x \in \Omega$ if $p = +\infty$. The lower semi-continuity of (1.1.4) reduces to a Jensen-type inequality of the form

$$\langle \nu_x, f\rangle \geq f(\nabla u(x)) \quad \text{and} \quad \nabla u(x) = \int_{\mathbb{R}^{m\times n}} z d\nu_x \quad \mathcal{L}^n\text{-a.e.},$$

when f has p-growth and is quasi-convex. In this case, we call $(\nu_x)_{x\in\Omega}$ a *Gradient Young Measure*, which can be seen as an inhomogeneous version of the Young Measure (1.1.6). This class can be seen as the closure of the set (1.1.6) in a suitable weak* topology of measures over the graph of f. The opposite is also true and was proven in [58, 60] when $1 \leq p \leq +\infty$ (see also [56, 72, 71] for the case with concentration), i.e. every Young Measure $(\nu_x)_{x\in\Omega}$, $\nu_x \colon \Omega \to \mathcal{M}_1^+(\mathbb{R}^{m\times n})$ that satisfies the moment condition

$$\int_\Omega \langle \nu_x, |\cdot|^p \rangle dx < +\infty$$

if $1 \leq p < +\infty$ or $\operatorname{supp}(\nu_x) \subset \mathbb{B}_R(0)$ for some $R > 0$ and $\mathcal{L}^n$-a.e. $x \in \Omega$, and there is $u \in W^{1,p}(\Omega, \mathbb{R}^m)$ such that

$$\overline{\nu_x} = \nabla u(x) \quad \mathcal{L}^n\text{-a.e.}$$

for which a Jensen-type inequality holds against quasi-convex functions of suitable growth, is the limit of a sequence of gradients of $(u_j)_j$ in $W^{1,p}(\Omega, \mathbb{R}^m)$ so that $u_j \rightharpoonup u$ weakly in $W^{1,p}$ (weakly* in $W^{1,\infty}$), where we insist that the convergence is weak when $p = 1$.

Most of the above theory is not specific to gradients only. It is well known that for a matrix field $U \in L^1(\mathbb{B}^n, \mathbb{R}^{m\times n})$, then $\operatorname{curl}(U) = 0$ row-wise if and only if there is $u \in W^{1,1}(\mathbb{B}^n, \mathbb{R}^m)$ so that $\nabla u = U$ $\mathcal{L}^n$-almost everywhere. Inspired by this well-known result, the above theory has been largely extended to integral functionals

$$U \mapsto \int_\Omega f(U(x))dx$$

where $U \in L^1(\Omega, \mathbb{R}^d)$ solves some system of linear constant-coefficient homogeneous differential equations

$$P(\partial)(U) = 0,$$

see for example [85, 47, 70, 12].

The aforementioned results hold in the setting of weak convergence in $W^{1,p}, 1 \leq p < +\infty$ and weak* convergence in $W^{1,\infty}$. This is a natural condition to assume when $p > 1$, but not when $p = 1$, as the Lebesgue space $L^1(\Omega, \mathcal{L}^n)$ is not the dual space of any Banach space. In particular, a bounded sequence in $L^1(\Omega, \mathcal{L}^n)$ can concentrate and converge to measures that are singular with respect to the Lebesgue measure. In terms of gradients, the closure of $W^{1,1}(\Omega, \mathbb{R}^m)$ so that its unit ball is weak*

compact is the set of functions of bounded variations $BV(\Omega, \mathbb{R}^m)$. This is the set of functions whose distributional derivatives are measures, meaning $u \in BV(\Omega, \mathbb{R}^m)$ if $u \in L^1(\Omega, \mathbb{R}^m)$ and $Du \in \mathcal{M}(\Omega, \mathbb{R}^{m\times n})$. It can be shown (see Theorem A.2.1) that if $(u_j)_j$ is bounded in $W^{1,1}(\Omega, \mathbb{R}^m)$, then there exists a subsequence $(u_{j_i})_i$ such that $u_{j_i} \rightharpoonup^* u$ in BV, which means that

$$u_{j_i} \to u \quad \text{in } L^1(\Omega, \mathbb{R}^m) \quad \text{and} \quad \nabla u_{j_i}(x)dx \rightharpoonup^* Du \quad \text{in } C_0(\Omega, \mathbb{R}^{m\times n})^*.$$

Du needs not be absolutely continuous with respect to the Lebesgue measure, as $\nabla u_{j_i}(x)dx$ could concentrate. Because this concentration phenomenon is exclusive of the case $p = 1$, it concerns integrands that have linear growth at infinity. It turns out that when f is non-negative, has linear growth and is quasi-convex, then if $(u_j)_j \subset W^{1,1}(\Omega, \mathbb{R}^m)$ is so that $u_j \rightharpoonup^* u$ in BV for some $u \in BV(\Omega, \mathbb{R}^m)$, then

$$\liminf_j \int_\Omega f(\nabla u_j(x))dx \geq \int_\Omega f(\nabla u(x))dx, \tag{1.1.7}$$

see [40, 45]. However, one can prove that the above is in general a strict inequality because for a sequence $u_j \rightharpoonup^* u$ in BV with $Du = \nabla u(x)dx + D^s u, D^s u \perp \mathcal{L}^n$ and $D^s u \neq 0$, formula (1.1.7) is unable to capture the singular part $D^s u$. Consider a function $f\colon \mathbb{R}^d \to \mathbb{R}$ having *regular recession* at infinity, meaning that the limit

$$f^\infty(z) = \lim_{t\to\infty, z'\to z} \frac{f(tz')}{t} \tag{1.1.8}$$

exists locally uniformly in $z \in \mathbb{R}^d$, the lower semi-continuous envelope of (1.1.4) in the space $BV(\Omega, \mathbb{R}^m)$, with respect to sequential weak* convergence as in Theorem A.2.1, is

$$u \mapsto \int_\Omega f(\nabla u(x))dx + f^\infty\left(\frac{D^s u}{|D^s u|}(x)\right) d|D^s u|(x). \tag{1.1.9}$$

References to this result with different conditions on f can be found in [46, 9, 69, 91, 24]. In this work, by writing $\frac{\eta}{\mu}$ where $\mu \in \mathcal{M}^+(\Omega)$ and $\eta \in \mathcal{M}(\Omega, \mathbb{R}^d)$ we mean the Radon-Nikodym derivative of η with respect to μ. In particular, for any vector-valued measure $\eta \in \mathcal{M}(\Omega, \mathbb{R}^d)$ we can consider its *polar decomposition* $\eta = \frac{\eta}{|\eta|}d|\eta|$.

In the above, the Assumption (1.1.8) is, despite first appearances, not necessary. More precisely, two important theorems show that (1.1.9) makes sense without the assumption that f has regular recession, in other words without assuming that f^∞ exists everywhere:

- the rank-1 Theorem [4] states that every $u \in BV(\Omega, \mathbb{R}^m)$ is so that

$$\frac{D^s u}{|D^s u|}(x) \in \mathbb{R}^{m \times n} \quad \text{has rank 1 for} \quad |D^s u|\text{-a.e. } x \in \Omega;$$

- if f has linear growth and it is quasi-convex it is also rank-1 convex (convex along matrices of rank 1), and so $f^\infty(z)$ exists, in a classical sense, for all $\text{rank}(z) = 1$.

These two facts show that (1.1.9) makes sense for all quasi-convex functions of linear growth and it turns out that (1.1.9) is still lower semi-continuous in this general setting. It is worth mentioning that some old results concerning the lower semi-continuity of (1.1.9) made use of the fact that f^∞ exists everywhere and had to be deeply changed to cover the general case of f that is just quasi-convex and of linear growth (in contrast, see [91] for a proof of lower semi-continuity that uses (1.1.8) but not the rank-1 property of BV functions). However, a Young Measure formulation of the Jensen-type inequality (1.1.5) has to take into account not only the singular part of Du - which has rank 1 - but also the concentration along any possible direction. This can be made into a proper statement using the characterisation Theorem [68] together with the rank-1 auto convexity result [62]. In the spirit of the previous results, one is tempted to prove a duality-type characterisation of Young Measures with concentrations and quasi-convex functions. As just mentioned, one must be careful about the assumptions on f^∞ (1.1.8). As shown in [84], quasi-convex functions can oscillate at infinity, meaning that $f^\infty(z)$ defined as in (1.1.8) may not exist for some $z \in \mathbb{R}^m$. This suggests that to obtain a Jensen-type inequality and characterisation result for Gradient Young Measures in the case $p = 1$, it is necessary to specify a compactification at infinity.

The characterisation for Gradient Young Measures when $p = 1$ has been already obtained on the so-called *sphere compactification* - functions for which $f^\infty(z)$ exists in the sense of (1.1.8) for all z - see [68] and [14] for a refinement concerning concentration on the boundary $\partial\Omega$ of Ω. See also [48, 56] for the case with concentration when $p > 1$. In Chapter 3, after showing that the class of quasi-convex functions of linear growth is too big to be included within any separable compactification, we prove the characterisation result for Gradient Young Measures on separable compactifications of subclasses of quasi-convex functions. Roughly, we show that given a compactification $e_{f_i, i \in \mathbb{N}}$ generated by a countable family of quasi-convex functions of linear growth, any triple $(\nu_x, \lambda, \nu_x^\infty)$ that satisfies (1.1.3),

$$\overline{\nu_x} dx + \overline{\nu_x^\infty} d\lambda = Du$$

for some $u \in BV(\Omega, \mathbb{R}^m)$, the moment condition

$$\int_\Omega \langle \nu_x, |\cdot| \rangle dx < +\infty,$$

and the Jensen-type inequality

$$\begin{cases} \langle \nu_x, f \rangle dx + \langle \nu_x^\infty, f^\infty \rangle \frac{\lambda}{\mathcal{L}^n}(x) dx \geq f(\nabla u(x)) dx, \\ \langle \nu_x^\infty, f^\infty \rangle d\lambda \geq f^\infty \left(\frac{D^s u}{|D^s u|} \right) |D^s u| \end{cases}$$

in the sense of measures, for all quasi-convex functions f (suitably adapted to the compactification $e_{f_i, i \in \mathbb{N}}$), is generated by a sequence $(\nabla u_j)_j$ with $(u_j)_j$ bounded in $W^{1,1}(\Omega, \mathbb{R}^m)$. Here f^∞ is not defined as in (1.1.8), but it is the suitably defined extension of f to the remainder of the compactification $e_{f_i, i \in \mathbb{N}}$. Moreover, $e_{f_i, i \in \mathbb{N}}$ cannot contain all quasi-convex functions from the above family, so we are restricted to working with subfamilies of quasi-convex functions at once. However, it is inevitable because the compactification containing all quasi-convex functions of linear growth is so large that its topology fails to be metrisable and separable, and, in that case, the Young Measure theory at its core would require us to work with nets instead of sequences.

In the above characterisation result, we have confined our attention to first-order gradients. However, the argument can be extended to also cover the case of higher-order gradients.

1.1.3 Oscillations generated by derivatives

The characterisation theorem states that generalised Young Measures $(\nu_x, \lambda, \nu_x^\infty)$ are generated by a sequence of gradients $(\nabla u_j)_j$, $u_j \in W^{1,1}(\Omega, \mathbb{R}^m)$, if a specific set of inequalities is satisfied. However, the result is non-constructive and does not show what type of oscillation Young Measures $(\nu_x)_{x \in \Omega}$ can be the oscillation of a Gradient Young Measure $(\nu_x, \lambda, \nu_x^\infty)$.

After presenting the characterisation result, we proceed with studying oscillations generated by first and second-order derivatives. Because oscillations are always generated by equi-integrable functions, one can, roughly speaking, focus on how to generate integrable functions instead. The problem of approximating arbitrary vector fields via gradients was initiated by Alberti in [2]. It is well known that given a simply connected domain $\Omega \subset \mathbb{R}^n$ and $f \in L^1(\Omega, \mathbb{R}^n)$ so that $\operatorname{curl}(f) = 0$, there exists a potential $u \in W^{1,1}(\Omega)$ so that $\nabla u = f$.

To cover the general case, we introduce a subclass of $BV(\Omega, \mathbb{R}^m)$. Let $u \in BV(\Omega, \mathbb{R}^m)$ and write $Du = \nabla u(x) dx + D^s u, D^s u \perp \mathcal{L}^n$, and further decompose

$D^s u = D^c u + D^j u$, where $D^j u$ (*jump* part) is concentrated on a set of Hausdorff dimension $n-1$ and $D^c u$ (*Cantor* part) is so that $|D^c u|(U) = 0$ when $\mathcal{H}^{n-1}(U) < +\infty$. Here, given a set $E \subset \Omega$, its Hausdorff measure is

$$\mathcal{H}^\alpha(E) = \sup_{\delta>0} \left(\inf \left\{ \sum_{i=1}^{\infty} \operatorname{diam}(U_i)^\alpha : \cup_i U_i \supset E \text{ and } \operatorname{diam}(U_i) \leq \delta \right\} \right),$$

the infimum taken over all countable collections $(U_i)_{i\in\mathbb{N}}$, and its Hausdorff dimension is

$$\inf\{\alpha \geq 0 : \mathcal{H}^\alpha(E) = 0\}.$$

We say that $u \in SBV(\Omega, \mathbb{R}^m)$ - function of *special bounded variation* - when $D^c u = 0$. In case Ω is not simply connected but the relation $\operatorname{curl}(f) = 0$ still holds, it is not hard to show that there is $u \in SBV(\Omega)$ so that $\nabla u = f$ and $|D^j u|(\Omega) \leq C\|f\|_1$, for a constant $C > 0$ that depends only on Ω. The groundbreaking result presented in [2] shows that such $u \in SBV(\Omega)$ exists for all vector fields $f \in L^1(\Omega, \mathbb{R}^n)$ (see also [76] for an alternative proof of this result).

In the case of higher-order derivatives, given an integrable function $(f^\alpha)_{|\alpha|=m}$, $\alpha \in \mathbb{N}_0^n$, $|\alpha| = \alpha_1 + \ldots + \alpha_n$, if one tries to approximate it with higher-order gradients $\nabla^m u$ in a similar way to [2], the singular part of the gradients diverges to infinity and only a partial result is possible, as was shown by the same author in [3]. Using a different method, in [44] an approximation result in the case $m = 2$ was proven. For $m \geq 3$ the question remains open. As shown in [49], a subtle modification of the argument in [2] allows one to find a function $u \in C^{m-1}$ such that $\nabla^{m-1} u$ is differentiable almost everywhere, and

$$\nabla^m u(x) = f(x) \quad \mathcal{L}^n\text{-a.e.}$$

However, the full derivative $D^m u$ needs not exist in any classical sense. See also [82] for a similar result in this direction.

In Chapter 4, we present a more concise proof of the full approximation result in the case $m = 2$, namely that any vector field $f \in L^1(\mathbb{R}^{n\times n}_{sym})$ is the absolutely continuous part of the distributional Hessian of a function $u \in SBH(\Omega)$, where

$$SBH(\Omega) = \{u \in W^{1,1}(\Omega) : \nabla u \in SBV(\Omega, \mathbb{R}^n)\},$$

that is

$$D^2 u = f \, dx + D^j \nabla u.$$

Our proof improves the result in [44] as it also provides infinitesimal bounds for lower derivatives of u, i.e. for a fixed $\varepsilon > 0$, we can also arrange that

$$\|u\|_{W^{1,1}} \leq \varepsilon \|f\|_1$$

The strategy we adopted is to approximate an integrable field $f \in L^1(\Omega, \mathbb{R}^{n\times n}_{sym})$ via a sequence $(Du_j)_j$ with $(u_j)_j$ in $SBV(\Omega, \mathbb{R}^n)$ in a way so that the singular part does not diffuse and $Du_j = Du_j^T$. The latter condition is equivalent to $\operatorname{curl}(u_j) = 0$, which in turn implies that u_j are themselves gradients. In light of this result, a classical lower semi-continuity result in BH, see for example [43, 96], and an application of the characterisation result for Young Measure generated by $\mathcal{A}$-free vector fields (vector fields satisfying certain system of constant coefficient partial differential equations), see [12, 67], we are able to show that for any oscillation Young Measure $(\nu_x)_{x\in\Omega}$ on $\mathbb{R}^{n\times n}_{sym}$ there are $\lambda \in \mathcal{M}^+(\Omega)$ and $(\nu_x^\infty)_{x\in\Omega}$, supported on a suitable compactification $e_{f_i, i\in\mathbb{N}}$, such that

$$D^2 u_j \to (\nu_x, \lambda, \nu_x^\infty),$$

for a not-so-arbitrary sequence $(u_j)_j$ in $SBH(\Omega)$. The result is then refined to remove exceeding mass from λ.

1.1.4 Lower semi-continuity of measure derivatives

Next, we study the semi-continuity of functionals of the type

$$u \mapsto \int_\Omega f\left(u(x), \frac{Du}{\mu}(x)\right) d\mu(x) \tag{1.1.10}$$

where $f = f(y, z) \in C(\mathbb{R}^m \times \mathbb{R}^{m\times n})$ is non-negative, $\mu \in \mathcal{M}^+(\Omega)$ is a positive measure that does not charge sets of Hausdorff dimension at most $n-1$, and u belongs to the class

$$W^{1,1}_\mu(\Omega, \mathbb{R}^m) = \{u \in BV(\Omega, \mathbb{R}^m) : |Du| \ll \mu\},$$

where given two finite measures $\mu_1, \mu_2 \in \mathcal{M}^+(\Omega)$ we say that $\mu_1 \ll \mu_2$ if $\mu_2(U) = 0$ implies $\mu_1(U) = 0$ for all Borel sets $U \subset \Omega$.

By considering the Radon-Nikodym derivative of Du_j with respect to μ, $\left(\frac{Du_j}{\mu}\right)_j$, we are imposing the gradients Du_j on a set where the singular part μ^s is concentrated. Also, because μ does not charge sets of Hausdorff dimension at most $n-1$, a function $u \in W^{1,1}_\mu$ does not have jump discontinuities, meaning $D^j u = 0$.

To understand the expected behaviour of (1.1.10) one needs to break it down into all its components.

The case $\mu = \mathcal{L}^n$, which corresponds to $W^{1,1}_\mu = W^{1,1}$, is classical. The lower semi-continuity of (1.1.10) with respect to sequences $(u_j)_j$ that are weakly convergent in $W^{1,p}$ for $1 \leq p < +\infty$, or weak* convergent in $W^{1,\infty}$, when f is non-negative, quasi-convex and has, accordingly, p-growth in z, was studied in [1, 77, 22]. However, when $p = 1$, weak convergence is too strong and many problems require working with the weak* topology of BV instead. A sequence $(u_j)_j$ in $W^{1,1}(\Omega, \mathbb{R}^m)$ with $u_j \rightharpoonup^* u$ in BV to some function $u \in BV(\Omega, \mathbb{R}^m)$, could be ill-behaved and have $(\nabla u_j)_j$ concentrate, which causes severe difficulties around discontinuity points of u. The jump discontinuity of u gives birth to a new measure called *surface area* $K_f[u]$. To understand this measure, one needs to look at the pointwise behaviour of a function $u \in BV(\Omega, \mathbb{R}^m)$. It is well known (see for example [10]), that there is a set J_u of Hausdorff dimension $n-1$ and $\eta(x)\colon J_u \to \partial\mathbb{B}^n$ (outer normal) so that, for all $x \in J_u$,

$$\fint_{B_r(x)\cap\{\langle y-x,\eta(x)\rangle\geq 0\}} u(y)dy \to u^+(x),$$

u^- similarly defined on the other half of $\mathbb{B}^n$. This means that we have "upper" (u^+) and "lower" (u^-) limits of u around the curve J_u. We set

$$K_f[u](x) = \inf_{\phi|_{\partial\mathbb{B}^n}=u^\pm} \int_{\mathbb{B}^n} f^\infty(\phi(x), \nabla\phi(x))dx, \tag{1.1.11}$$

where by $\phi|_{\partial\mathbb{B}^n} = u^\pm$ we mean $\phi \in C^\infty(\mathbb{B}^n, \mathbb{R}^m) \cap W^{1,1}(\mathbb{B}^n, \mathbb{R}^m)$ and, in the sense of trace,

$$\phi(y) = u^+, \quad y \in \partial\mathbb{B}^n \text{ with } \langle y, \eta(x)\rangle \geq 0,$$

similarly on the other half $\partial\mathbb{B}^n$ and u^-. Formula (1.1.11) somehow expresses the value of the minimal displacement for f of u on the set J_u where it jumps from two different values (u^+ and u^-). A better representation of $K_f[u]$ is still the object of active research. The first results in the direction of studying lower semi-continuity in BV were established in [33, 46]. Recent improvements are contained in [93, 94], where, using Young Measures, the authors achieved lower semi-continuity when integrands are partially unbounded from below and have critical growth in y, meaning

$$-h(y,z) \leq f(y,z) \leq C(1 + |y|^{\frac{n}{n-1}} + |z|)$$

for some constant $C > 0$, where $h(y,z)$ is a non-negative function that has sufficiently slow growth in y and z. In order to introduce the correct generalised Young Measures

suited for representing limits as in (1.1.10), one needs to analyse the behaviour of certain measures $(\cdot, u(\cdot))_{\#}(\nabla u(x)dx)$ under small variations in the weak* topology of BV, where, by $(\cdot, u(\cdot))_{\#}(\nabla u(x)dx)$, $u \in C^{\infty}(\Omega, \mathbb{R}^m) \cap W^{1,1}(\Omega, \mathbb{R}^m)$, we mean

$$\int_{\times\times\mathbb{R}^m} f(x,y)d((\cdot, u(\cdot))_{\#}\nabla u) = \int_{\Omega} f(x, u(x))d(\nabla u(x)dx), \quad f \in C_0(\Omega \times \mathbb{R}^m)$$

This preliminary step captures how weak* oscillations of ∇u affect u point-wisely. Generalised limits of $(\cdot, u(\cdot))_{\#}(\nabla u(x)dx)$ are called *Liftings* and are measures $\gamma \in \mathcal{M}(\Omega \times \mathbb{R}^m, \mathbb{R}^{m\times n})$, whose definition and representation were established in [55], and systematically developed [93, 94]. By embedding such Liftings into the set of Young Measures, one obtains Lifting Young Measures, which were introduced in [94].

On the other side, the case of $\mu \neq \mathcal{L}^n$ and $f = f(y)$ has also been explored in different instances. In [8], the authors proved lower semi-continuity of (1.1.10) when f has p-growth and $\frac{\mu}{\mathcal{L}^n} \in L^{\infty}(\Omega, \mathcal{L}^n)$, see also [64] where the general case was treated. In the case of f having linear growth, the study of lower semi-continuity properties was considered in [52] and solved in [65] when f is non-negative and has a regular recession.

In Chapter 5, we prove the weak* sequential lower semi-continuous relaxation formula for (1.1.10) when $f = f(y,z)$ is non-negative, has a strong regular recession in the sense that

$$\lim_j \frac{f(y_j, t_j z_j)}{t_j} = \sigma f^{\infty}(y,z) \in C(\sigma\mathbb{R}^m \times \mathbb{R}^{m\times n}) \tag{1.1.12}$$

where $\sigma\mathbb{R}^m$ is the sphere compactification of $\mathbb{R}^m$, and the limit is uniform along sequences $y_j \to y \in \sigma\mathbb{R}^m$, $z_j \to z \in \mathbb{R}^{m\times n}$ and $t_j \uparrow \infty$ (see Assumption 1 in Chapter 5), $f(y, \cdot)$ is quasi-convex, and μ is so that $\mathcal{L}^n \ll \mu$, meaning

$$\mu = a(x)dx + \mu^s, \quad \mu^s \perp \mathcal{L}^n \text{ and } a(x) > 0 \ \mathcal{L}^n\text{-a.e.},$$

and there is a constant $C > 0$ so that

$$\mu(\mathbb{B}_r(x) \cap \Omega) \leq Cr^{p(n-1)} \quad \forall x \in \Omega, r > 0$$

and $1 < p \leq \frac{n}{n-1}$. Under these assumptions, we show that

$$\begin{aligned} F_{**}(u) &:= \inf\left\{\liminf_j F(u_j) : u_j \in W^{1,1}_{\mu}(\Omega, \mathbb{R}^m), u_j \rightharpoonup^* u \text{ in } BV\right\} \\ &= \int_{\Omega} f\left(u, \frac{Du}{\mu}\right) d\mu + K_f[u]d\mathcal{H}^{n-1} + f^{\infty}\left(u, \frac{D^{c,\mu}u}{|D^{c,\mu}u|}\right) d|D^{c,\mu}u|, \end{aligned}$$

where $D^c u = \frac{D^c u}{\mu} d\mu + D^{c,\mu} u$ and $D^{c,\mu} u \perp \mu$. Our proof is based on the methodology developed in [65] and a generalisation of the Lifting Young Measures constructed in [94] to functions $f(y, z)$ such that

$$|f(y, z)| \leq C(1 + |z|)$$

and so that $f(y, z)$ has a regular recession in z, uniformly in y, and satisfies (1.1.12). We make the standing assumption that $\mu \in \mathcal{M}^+(\Omega)$ satisfies the Morrey condition

$$\mu(\mathbb{B}_r(x) \cap \Omega) \leq C r^{p(n-1)}$$

for some $1 < p \leq \frac{n}{n-1}$ because, for such values of p, we can show that if a sequence $(u_j)_j$ and u in $BV(\Omega, \mathbb{R}^m)$ are so that $u_j \rightharpoonup^* u$ in BV, then

$$u_j \to u \quad \text{in } \mu \text{ measure.}$$

1.1.5 Regularity of quasi-convex envelopes and equi-integrability of minimising sequences when $p > 1$

The last chapter of this thesis is devoted to the study of the regularity of quasi-convex envelopes and equi-integrability of minimising sequences. It is an illustration of the possible applications of Young Measures and the viewpoints exposed in the previous chapters.

Given a C^1 function $f\colon \mathbb{R}^{m\times n} \to \mathbb{R}$ of p-growth with $1 < p < +\infty$, we seek to understand the necessary conditions for which the quasi-convex envelope of f

$$f^{qc}(z) = \sup\{g(z) \leq f(z) : g\colon \mathbb{R}^{m\times n} \to \mathbb{R} \text{ is quasi-convex and } \ g \leq f\}$$

is also C^1. Here, we use the convention that $\sup \emptyset = -\infty$. The quasi-convex envelope is conveniently represented via *Dacorogna's formula*

$$f^{qc}(z) = \inf \left\{ \int_Q f(z + \nabla\phi(x))dx, \ \phi \in W_0^{1,\infty}(Q, \mathbb{R}^m) \right\},$$

where $Q = \left[-\frac{1}{2}, \frac{1}{2}\right]^n$ is the unit cube in $\mathbb{R}^n$.

In order to understand the necessary conditions that give the regularity of the C^1 envelope, we look at the case of the convex envelope first. It was proven in [61] that if $f\colon \mathbb{R}^d \to \mathbb{R}$ is continuous, C^1 and coercive

$$f(x) \to +\infty \text{ as } x \to \infty,$$

then the convex envelope of f is C^1. More results on the smoothness of convex envelopes and the properties of their subdifferentials are contained in [51, 25, 86, 87, 34]

In the case of quasiconvex integrands, the regularity of the envelope was first proven in [20]. Here, the authors showed, for instance, that if $f\colon \mathbb{R}^{m\times n} \to \mathbb{R}$ is of class C^1 and there is $1 < p < +\infty$ so that f has p-growth and

$$\frac{f(z)}{1+|z|^{p-1}} \to \infty \text{ as } |z| \to \infty,$$

then the quasi-convex envelope $f^{qc}\colon \mathbb{R}^{m\times n} \to \mathbb{R}$ is also of class C^1. The above growth conditions on f are necessary even in the case $n = N = 2$; for a counterexample see [20].

In Chapter 6, we show that if $f\colon \mathbb{R}^{m\times n} \to \mathbb{R}$ is C^1 and there is $1 < p < +\infty$ so that f is $p-1$ *mean-coercive*, in the sense that

$$(f - C|\cdot|^{p-1})^{qc} > -\infty \tag{1.1.13}$$

for some constant $C > 0$, and

$$\limsup_{|z|\to\infty} \frac{|f(z)|}{1+|z|^p} = 0,$$

then f^{qc} is also C^1.

After proving the regularity of the quasi-convex envelope, we present a concise proof that when $1 < p < +\infty$ and $g \in W^{1,p}(\mathbb{R}^n, \mathbb{R}^m)$, minimising sequences for

$$u \mapsto \int_\Omega f(\nabla u(x))dx$$

in the Dirichlet class

$$W^{1,p}_g(\Omega, \mathbb{R}^m) = \{u \in W^{1,p}(\Omega, \mathbb{R}^m) : u\chi_\Omega + g\chi_{\Omega^c} \in W^{1,p}(\mathbb{R}^n, \mathbb{R}^m)\}$$

have p-equi-integrable derivatives when f satisfies $f - c|\cdot|^p$ is quasi-convex at z_0 for some $z_0 \in \mathbb{R}^{m\times n}$ and $c > 0$, which is a slightly stronger version of the coercivity condition (1.1.13). This result completes part of the work carried out in [31], which shows that the reverse of our implication is also true.

We use generalised Young Measures to represent limits of minimising sequences, as we do not assume any particular asymptotic on f at infinity other than p-growth. In particular, we combine the Jensen-type inequality of generalised p-Gradient Young Measures obtained in [56] with our representation of Young Measures developed in

Chapter 2. Here, by generalised p-Gradient Young Measures we mean generalised Young Measures on separable compactifications (as in Chapter 2) that are generated by gradients of sequence $(u_j)_j$ in $W^{1,p}(\Omega, \mathbb{R}^m)$.

In our notation, we take a countable family of continuous functions $\{f_i, i \in \mathbb{N}\}$ of p-growth, and $e_{f_i, i\in\mathbb{N}}$ its associated Hausdorff compactification. We can show that if a sequence $(u_j)_j$ bounded in $W^{1,p}(\Omega, \mathbb{R}^m)$ is so that (see Chapter 2 for the precise meaning of the notation)

$$\nabla u_j \xrightarrow{Y^p(e_{f_i, i\in\mathbb{N}})} (\nu_x, \lambda, \nu_x^\infty),$$

then for all quasi-convex functions $f \in e_{f_i, i\in\mathbb{N}}$, we have

$$\begin{cases} \langle \nu_x, f\rangle \geq f(\nabla u(x)) & \mathcal{L}^n \llcorner \Omega\text{-a.e.} \\ \langle \nu_x^\infty, f^\infty\rangle \geq 0 & \lambda \llcorner \Omega\text{-a.e.} \end{cases}$$

which is a crucial piece of information for proving that $\langle \nu_x^\infty, |\cdot|\rangle d\lambda = 0$, equivalently $(\nabla u_j)_j$ is p-equi-integrable

1.2 Outline of the thesis

This thesis can be summarised as follows.

- In Chapter 1, we introduce the core concepts and notations of the thesis.
- In Chapter 2, we construct generalised Young Measures on separable compactifications and discuss their properties in connection to classical Young Measures.
- In Chapter 3, we prove a characterisation result for Gradient Young Measures on separable compactifications.
- In Chapter 4, we show that a large class of oscillation Young Measures can be generated by first and second-order derivatives.
- In Chapter 5, we prove a lower semi-continuity result for measure derivatives for integrands that depend on functions and their derivatives.
- In Chapter 6, we prove the regularity of quasi-convex envelopes and equi-integrability of minimising sequences.
- In Chapter 7, we draw the conclusion on the thesis and present an open problem.

Chapters 2 and 3 are part of a paper which is submitted for publication, the preprint can be found in [98]. Chapter 6 is part of a paper, in the process of being finalised, in collaboration with Prof. Jan Kristensen and Prof. Bernd Kirchheim.

1.3 Preliminaries and notation

We list here the notation and terminology that will be used in this thesis.

- For a vector $v \in \mathbb{R}^m$, we write $|v| = \sqrt{\sum_{i=1}^m v_i^2}$ otherwise specified.
- Given a function $f\colon X \to Z$ and W an arbitrary set, we call

$$\begin{aligned}\text{graph}_X(f) &= \{(x, f(x)) \in X \times Z \text{ such that } x \in X\},\\ \text{graph}_{X\times W}(f) &= \{(x, w, f(x)) \in X \times W \times Z \text{ such that } x \in X, w \in W\}.\end{aligned}$$

  When X is clear from the context we just write $\text{graph}(f)$.
- We say that a function $f\colon \mathbb{R}^d \to \mathbb{R}$ has p-growth if there is a constant $C > 0$ such that

$$|f(z)| \leq C(1 + |z|^p) \quad \forall z \in \mathbb{R}^d.$$

- The identity matrix is indicated by $\mathbb{1}$, or $\mathbb{1}_d \in \mathbb{R}^{d\times d}$ if we need to specify the dimension.
- The Lebesgue measure is indicated by dx, or $\mathcal{L}^n$ depending on the context. The set of finite Borel d-vector measures is indicated by $\mathcal{M}(\Omega, \mathbb{R}^d)$. For $E \subset \mathcal{M}(\Omega)$, E^+ is the set of positive Borel measures that belong to E.
- The $n-1$-dimensional Hausdorff measure is indicated by $\mathcal{H}^{n-1}$.
- For a measure $\mu \in \mathcal{M}(\Omega)^+$, we write $L^p(\Omega, \mu, \mathbb{R}^d)$ to mean the space of μ-measurable functions $f\colon \Omega \to \mathbb{R}^d$ such that

$$\int_\Omega |f(x)|^p d\mu(x) < +\infty.$$

- For $\mu \in \mathcal{M}(\Omega, \mathbb{R}^d)$, we write its restriction to a μ-measurable set $E \subset \Omega$ as

$$\mu \llcorner E\colon A \text{ Borel set} \mapsto \mu(E \cap A).$$

- If $\mu \in \mathcal{M}^+(\Omega)$ and $f \in L^1(\Omega, \mu, \mathbb{R}^d)$, we call $f d\mu$ the measure in $\mathcal{M}(\Omega, \mathbb{R}^d)$ defined by

$$U \mapsto \int_U f d\mu,$$

  where U runs through all μ-measurable sets.

- If X is any space, the Dirac delta is indicated, for $x \in X$, by

$$\int_X f(y) d\delta_x(y) = f(x),$$

where $f \colon X \to \mathbb{R}^d$ is an arbitrary function.

- Let X be a metric space, $\mu \in \mathcal{M}^+(\Omega)$ and $(f_j)_{j\in\mathbb{N}}$ in $L^p(\Omega, \mu, \mathbb{R}^d)$. We say that the sequence $(f_j)_j$ is p-equi-integrable (or simply equi-integrable if p is clear from the context) if it is norm bounded and

$$\lim_{k\uparrow\infty} \sup_{j\in\mathbb{N}} \int_{|f_j|^p > k} |f_j|^p d\mu = 0.$$

- The set of $\mathbb{R}^d$-values test functions is

$$\mathcal{D}(\Omega, \mathbb{R}^d) = C_c^\infty(\Omega, \mathbb{R}^d),$$

that is, infinitely differentiable functions with compact support in Ω. We do not insist on the topology this space is endowed with. We often abbreviate $\mathcal{D}(\Omega, \mathbb{R}^d) = \mathcal{D}(\Omega)$ when the target space is clear from the context.

- By the derivative of a function $u \in L^1(\Omega, \mathbb{R}^m)$ we mean the $m \times n$ matrix-valued distribution $Du = [\partial_j u_i]_{i,j} \in \mathcal{D}(\Omega, \mathbb{R}^{m\times n})^*$ such that

$$\int_\Omega u_i \partial_j \phi dx = -\langle \partial_j u_i, \phi \rangle \quad \forall \phi \in \mathcal{D}(\Omega).$$

- The Sobolev space of functions with integrable derivatives is

$$W^{1,1}(\Omega, \mathbb{R}^m) = \{u \in L^1(\Omega, \mathbb{R}^m) : Du \in L^1(\Omega, \mathbb{R}^{m\times n})\}.$$

In this case, we write $Du = \nabla u$ in analogy with classical derivatives.

- The set of functions of bounded variation is

$$BV(\Omega, \mathbb{R}^m) = \{u \in L^1(\Omega, \mathbb{R}^m) : Du \in \mathcal{M}(\Omega, \mathbb{R}^{m\times n})\},$$

and the set of functions of bounded variation with zero average is

$$BV_\#(\Omega, \mathbb{R}^m) = \left\{ u \in BV(\Omega, \mathbb{R}^m) : \fint_\Omega u(x) dx = 0 \right\}.$$

- For a function $u \in BV(\Omega, \mathbb{R}^m)$ we can write

$$Du = \nabla u d\mathcal{L}^n \llcorner \Omega + D^s u, \quad D^s u = D^j u \llcorner J_u + D^c u$$

where $\nabla u d\mathcal{L}^n$ is the absolutely continuous part, $D^j u$ is the jump part concentrated on a $n-1$ rectifiable set J_u, and D^s is the Cantor part, which is absolutely continuous with respect to $\mathcal{H}^{n-1}$, see [10], Chapter 3.

- For a function $U \in BV(\Omega, \mathbb{R}^m)$, we write

$$BV_U(\Omega, \mathbb{R}^m) = \left\{ \begin{array}{c} u \in BV(\Omega, \mathbb{R}^m) : \text{there is a sequence } (u_j)_j \text{ in } \mathcal{D}(\Omega, \mathbb{R}^m) \\ \text{such that } u_j \xrightarrow{\text{weak* in BV}} u - U \end{array} \right\}.$$

In this thesis, we will work with domains $\Omega \subset \mathbb{R}^n$ bounded and with a Lipschitz boundary, so the above definition is consistent with trace theorems and other literature on the subject, see for example [97] and [10], Chapter 3.

- The set of special functions of bounded variation is

$$SBV(\Omega, \mathbb{R}^m) = \{u \in BV(\Omega, \mathbb{R}^m) : D^s u = D^j u, \text{ or equivalently } D^c u = 0\},$$

see [10], Chapter 4.

- The set of functions whose derivative has bounded variation is

$$BH(\Omega, \mathbb{R}^m) = \{u \in W^{1,1}(\Omega, \mathbb{R}^m) : \nabla u \in BV(\Omega, \mathbb{R}^{m \times n})\}.$$

- The set of functions whose derivative is of special bounded variation is

$$SBH(\Omega, \mathbb{R}^m) = \{u \in BH(\Omega, \mathbb{R}^m) : \nabla u \in SBV(\Omega, \mathbb{R}^{m \times n})\}.$$

Chapter 2

Generalised Young Measures on separable compactifications

In this chapter, we construct generalised Young Measures and provide a new geometric representation. Even though generalised Young Measures have been known to exist for a long time, our new representation captures how concentration "oscillates with different amplitudes at infinity" and gives a geometric insight into what these measures look like. To achieve such a result, we embed spaces of functions into bigger compact sets and use the theory of Hausdorff compactifications to study their topological properties.

This chapter is the first part of a paper submitted for publication, the preprint can be found in [98].

2.1 Generalised limits and parametrised measures

Generalised Young Measures are objects that were known to exist since Majda and Di Perna [36], and have been used in a few instances, see for example [41, 73, 95, 56]. However, their existence per se does not give enough clarity on their properties, and so makes it hard to work with such objects.

We give a new interpretation and geometric representation that better captures oscillation and concentration effects that occur in limits of the form

$$\lim_j \int_\Omega f(v_j(x))d\mu(x),$$

where $(v_j)_j$ is a norm-bounded sequence in $L^p(\Omega, \mu, \mathbb{R}^d)$ and $f \in C(\mathbb{R}^d)$ has p-growth. Under these assumptions, it is easy to see that, up to a subsequence, $f \circ v_j$ converges

to a measure ν; formally, this means that

$$\int_\Omega f(v_j(x))\phi(x)d\mu(x) \to \int_\Omega \phi(x)d\nu(x)$$

for all $\phi \in C_0(\Omega)$. It is clear that $\nu = \nu(f)$ is a linear function of f. What is not clear is how such dependence can be represented in terms of μ and f. Without a clear representation, it is not possible to set up a system of calculus. This section is dedicated to working out a geometric interpretation of the relation between ν and f, which is called Young Measure. We will mainly concentrate on the more interesting and harder case of $p = 1$ and $v_j = \nabla u_j$ gradients, where concentration effects create rather complicated structures, and cannot in general be separated from oscillation.

Definition 2.1.1. *A function $f\colon \mathbb{R}^d \to \mathbb{R}$ is said to have p-growth if there is a constant $C > 0$ such that*

$$|f(z)| \leq C(1 + |z|^p) \quad \forall z \in \mathbb{R}^m.$$

When $p = 1$, we say that such functions have linear growth.

We emphasise that when $v_j \to v$ strongly in $L^1(\mu)$, the limit Young Measure is trivial, meaning that

$$\int_\Omega f(v_j(x))\phi(x)d\mu(x) \to \int_\Omega f(v(x))\phi(x)d\mu(x)$$

for all $f \in C(\mathbb{R}^d)$ of linear growth and $\phi \in C_0(\Omega)$. This is a simple consequence of the Vitali convergence Theorem A.1.1 (or the generalised dominated convergence theorem).

In order to construct Young Measures, we regard functions as maps from a domain Ω into the set of probability measures over a target space $\mathbb{R}^d$. Ordinary functions $f\colon \Omega \to \mathbb{R}^d, x \mapsto f(x)$ are embedded into maps $\Omega \to \mathcal{M}_1^+(\mathbb{R}^d), x \mapsto \delta_{f(x)}$. Preliminary to the construction, we introduce two basic concepts that are at the core of this theory.

Definition 2.1.2. *Let X and Z be locally compact, separable metric spaces and $\lambda \in \mathcal{M}^+(X)$. A map $\nu\colon X \to \mathcal{M}^+(Z)$ is said to be λ-measurable if for each $\phi \in C_0(Z)$ the function $x \mapsto \langle \nu(x), \phi \rangle$ is λ-measurable.*

We shall often write the measure-valued map ν as a *parametrized measure* $(\nu_x)_{x\in X}$, where $\nu_x := \nu(x)$. Given a measure ν on a product space $X \times Z$, it can always be decomposed as a product of its projection onto X and its cross section on Z.

Theorem 2.1.3 (Disintegration of measures)**.** *Let X and Z be compact metric spaces and denote by $\pi\colon X \times Y \to Z$ the projection mapping onto the first coordinate $\pi(x,y) = x$. For $u \in \mathcal{M}^+(X \times Z)$ and $\lambda = \pi_\# \nu \in \mathcal{M}^+(X)$ (the pushforward of ν via π) there exists a unique λ-measurable parametrized measure $(\eta_x)_{x\in X}, \eta_x \in \mathcal{M}_1^+(Z)$ such that for all $\phi \in C(X), \psi \in C(Z)$ we have*

$$\langle \nu, \phi \otimes \psi \rangle = \int_X \langle \eta_x, \psi \rangle \phi(x) d\lambda(x) = \int_X \int_Z \psi(z) d\eta_x(z) \phi(x) d\lambda(x).$$

For a proof of this result see [10]. In this case, we write

$$\nu = \eta_x d\lambda.$$

2.2 Generalised Young Measures

In what follows, we show how to obtain a good representation of Young Measures on separable compactifications. The procedure is adapted from [68], where it was developed to construct Young Measures on a very specific compactification, called *sphere compactification* (see 2.4.1 for the connection to our general setting). Some of the results can also be found in [92], Chapter 12, where they were also proven on the sphere compactification.

Throughout this section, $\Omega \subset \mathbb{R}^n$ is open and bounded, $\mu \in \mathcal{M}^+(\Omega)$ and $(v_j)_j$ is a bounded sequence in $L^p(\Omega, \mu, \mathbb{R}^d)$, with $1 \leq p < +\infty$. Assume

$$v_j \rightharpoonup v \text{ in } L^p \quad \text{when} \quad 1 < p < +\infty$$

and

$$v_j \rightharpoonup^* v \text{ in } C_0(\Omega, \mathbb{R}^d)^* \quad \text{when} \quad p = 1.$$

Given a continuous integrand $\Phi\colon \Omega \times \mathbb{R}^d \to \mathbb{R}$ satisfying the p-growth condition

$$|\Phi(x,z)| \leq C(1 + |z|)^p \quad \forall (x,z) \in \Omega \times \mathbb{R}^d,$$

we seek to represent limits of $\left(\int_\Omega \Phi(x, v_j(x)) dx\right)_j$ as $j \to \infty$, possibly passing through suitable subsequences.

Remark 2.2.1. *For each j, the map Φ acts on the graph of v_j, i.e. $\Phi(x, v_j(x)) = \Phi \circ (x, v_j(x))$. Therefore, we look for the limiting distribution of $(x, v_j(x))$ as $j \to \infty$, and more precisely the Φ-moment of this limiting distribution.*

Morally speaking, since $(\Phi(\cdot, v_j))_j$ is bounded in $L^1(\Omega, \mu)$, $(\Phi(\cdot, v_j)d\mu)_j$ is bounded in $\mathcal{M}(\overline{\Omega}) \approx C(\overline{\Omega})^*$, so by the abstract compactness principle A.1.4, there exists a limit measure which depends on the integrand Φ. To better understand the way the limit measure depends on the integrand Φ, we need to change the setup of the problem.

Let $z \in \mathbb{R}^d \mapsto \hat{z} = \frac{z}{1+|z|} \in \mathbb{B}^d$ be a homeomorphism $\mathbb{R}^d \mapsto \mathbb{B}^d$. Define the class of functions of p-growth in the z variable to be

$$\mathbb{G}_p = \mathbb{G}_p(\Omega, \mathbb{R}^d) := \left\{ \Phi \in C(\Omega \times \mathbb{R}^d) : \sup_{(x,z)} \frac{|\Phi(x,z)|}{(1+|z|)^p} < +\infty \right\},$$

and for $\Phi \in \mathbb{G}_p$ put

$$(T\Phi)(x, \hat{z}) := (1 - |\hat{z}|)^p \Phi\left(x, \frac{\hat{z}}{1-|\hat{z}|}\right).$$

Then $T \colon \mathbb{G}_p \to BC(\Omega \times \mathbb{B}^d)$ is an isometric isomorphism provided $\mathbb{G}_p$ is normed by $\|T\Phi\|_\infty$ and $BC(\Omega \times \mathbb{B}^d)$ by $\|\cdot\|_\infty$. The inverse operator is

$$(T^{-1}\Psi)(x, z) = (1 + |z|)^p \Psi\left(x, \frac{z}{1+|z|}\right),$$

where $\Psi \in BC(\Omega \times \mathbb{B}^d)$.

The dual operator $T^* \colon BC(\Omega \times \mathbb{B}^d)^* \to \mathbb{G}_p^*$ is again an isometric isomorphism. We are interested in the limits of $\int_\Omega \Phi(x, v_j(x))dx$ for $\Phi \in \mathbb{G}_p$ and may define $\xi_{v_j} \in \mathbb{G}_p^*$ by

$$\xi_{v_j}(\Phi) := \int_\Omega \Phi(x, v_j(x))dx, \Phi \in \mathbb{G}_p.$$

Note

$$\|\xi_{v_j}\| = \sup_{\|\Phi\|_{\mathbb{G}_p}} \xi_{v_j}(\Phi) = \int_\Omega (1 + |v_j|)^p dx$$

so (ξ_{v_j}) is a bounded sequence in $\mathbb{G}_p^*$. But $\mathbb{G}_p^* \approx BC(\Omega \times \mathbb{B}^d)^*$, and because BC (hence $\mathbb{G}_p$) is not separable we do not necessarily have sequential compactness. We must restrict the integrands Φ to a separable subspace of $\mathbb{G}_p$.

2.3 Hausdorff compactification

In this section, we present how to construct a compactification of the space $X = \Omega \times \mathbb{R}^d$ from a family of bounded and continuous functions $F \subset BC(X)$. Roughly speaking, such compactification is a compact set $e_F X$ that contains X as a dense subset and on which all $f \in F$ admit a continuous extension. The idea behind such construction

is to look at the graph of each $f \in F$. Because of the boundedness assumption, each function has its image contained in a closed bounded interval of $\mathbb{R}$. Therefore, the graph is embedded into a closed subset of an (infinite-dimensional) hypercube, which is compact in the product topology by the Tychonoff Theorem, A.1.3.

Most of the results will be stated without proof, which can be found in [30], Chapters 1 and 2, and [39], Chapter 4.

We first define three classes of functions that are rich enough to determine the topological structure of their domains:

Definition 2.3.1. *Consider a family of functions $F \subset BC(X)$, we say that F separates points from closed sets if for each $C \subset X$ closed and $x \in X \setminus C$ there exists $f \in C(X)$ such that $f(x) \notin \overline{f(C)}$.*

Next, we define what a compactification of a topological space is.

Definition 2.3.2. *A compactification of X is a compact Hausdorff space αX and an embedding $\alpha \colon X \to \alpha X$ (continuous and so that $\alpha^{-1} \colon \alpha X \to X$ exists and is continuous) such that $\alpha(X)$ is dense in αX.*

It is useful to remark that because α is continuous, any function $f \in C(\alpha X)$ can be restricted to a continuous function on X. Indeed, $f \circ \alpha$ is the composition of a bounded continuous function with a continuous function, and thus it belongs to $BC(X)$. On the other hand, because $\alpha(X)$ is dense in αX, each $f \in C(\alpha X)$ is uniquely recovered from $f \circ \alpha \in C(X)$.

Given a family $F \subset BC(X)$ that separates points from closed sets, there is a canonical way of generating a compactification αX on which every $f \in F$ has a continuous extension.

Theorem 2.3.3. *To each family $F \subset BC(X)$ that separates points from closed sets, we associate a canonical embedding*

$$e_F \colon X \to \Pi_{f \in F} \big[\inf f, \sup f \big], \ x \mapsto \{f(x)\}_{f \in F}.$$

$e_F X := \overline{e_F(X)}$ is a compactification of X

The above theorem is a direct implication of Tychonoff's Theorem. When $F \subset BC(X)$ is a family that separates points from closed sets, then

$$e_F \colon X \mapsto \Pi_{f \in F} \big[\inf f, \sup f \big]$$

is open and continuous, and so it is an embedding.

However, the map e_F makes sense even if F does not separate points from closed sets, and $e_F X$ is always a compact subset of $\Pi_{f \in F} \big[\inf f, \sup f \big]$.

Lemma 2.3.4. *Let $F \subset BC(X)$ be a family that separates points from closed sets and $e_F X$ its induced compactification. Each $f \in F$ embeds into $C(e_F(X))$ in an obvious way and admits a unique extension $\overline{f} \in C(e_F X)$.*

Every $y \in e_F X$ is an accumulation point of $e_F(X)$, so we can find a net $y_\lambda = \Pi_{f \in F} f(x_\lambda)$ such that $y_\lambda \to y$. A way of extending $f \in F$ is by setting

$$\overline{f} \colon e_F X \to \mathbb{R}, y \left(= \lim_\lambda \Pi_{f \in F} f(x_\lambda) \right) \mapsto \overline{f}(y) = \lim_\lambda f(x_\lambda),$$

which does not depend on the choice of x_γ as far as $\lim_\gamma f(x_\gamma) = \lim_\lambda f(x_\lambda)$ for each $f \in F$.

Suppose that we have given a family F and its associated compactification $e_F X$, and we consider the compactification of $F \cup \{f\}$, where $f \in BC(X)$. We expect the latter compactification to be bigger than the former, i.e. to be a space where all the previous extensions can be further extended to continuous functions.

Definition 2.3.5. *Given two compactifications αX and γX of X, we say that $\alpha X \geq \gamma X$ if there exists a continuous function $f \colon \alpha X \to \gamma X$ such that $f \circ \alpha = \gamma$. Moreover, we write $\alpha X \approx \gamma X$ if $\alpha X \geq \gamma X$ and $\gamma X \geq \alpha X$, or equivalently if $f \colon \alpha X \to \gamma X$ is a homeomorphism.*

In the following work, we will sometimes refer to a generic compactification K without specifying the underlying family generating it. The reason why is contained in the following result.

Theorem 2.3.6. *Given a compactification αX of X, there exists a family $F \subset BC(X)$ that separates points from closed sets such that $e_F X \approx \alpha X$.*

2.3.1 Representation of compactifications

Consider a family $F \subset BC(X)$ that separates points from closed sets. According to the Hausdorff compactification theory, its induced compactification can be written as a subset of the hypercube $\Pi_{f \in F} \left[\inf f, \sup f \right]$, where the sides of this cube are as many as the functions $f \in F$. Because F can be uncountable, its compactification could be hard to deal with from an analytical point of view. We seek a better representation of such space.

The idea behind the following result is that if we know the limits of functions $f, g \in BC(\Omega, \mathbb{R})$, we also know the limits of $f^n + g^m$, $n, m \in \mathbb{N}$.

Definition 2.3.7. *Let $\mathcal{F}$ be a family of functions $f\colon X \to \mathbb{R}$. We call $\mathbb{A}(\mathcal{F})$ the algebra generated by $\mathcal{F}$, i.e.*

$$\mathbb{A}(\mathcal{F}) = \{f^n + g^m : f, g \in \mathcal{F},\ n, m \in \mathbb{N}\}.$$

Theorem 2.3.8 (Representation Theorem)**.** *Let $F \subset BC(X)$ be a closed sub-algebra that separates points from closed sets and let $F' \subset F$ be such that $\overline{\mathbb{A}(F')} = F$. Then $e_{F'}X$ is a compactification of X and $e_{F'}X \approx e_F X$.*

Proof. Let $e_{F'}X$ be the (formal) compactification of F'. Clearly $e_F X \geq e_{F'}X$. To prove the opposite inclusion, we must find a continuous function $T\colon e_{F'}X \to e_F X$ such that $T \circ e_{F'} = e_F$. Fix $y = \lim_\lambda \Pi_{f\in F'} f(x_\lambda) \in e_F X$. If $g \in \mathbb{A}(F')$, $\lim_\lambda g(x_\lambda)$ exists and coincides on all nets x_γ such that $y = \lim_\gamma \Pi_{f\in F'} f(x_\gamma)$. Next, let $g \in \overline{\mathbb{A}(F')}$ and find a sequence $\{f_n\}_{n\in\mathbb{N}} \subset \mathbb{A}(F')$ such that $f_n \to g$ uniformly. Because $\|f_n\|_\infty$ is bounded, so is $\{\lim_\lambda f_n(x_\lambda)\}_{n\in\mathbb{N}}$, and so we can extract a subsequence $\{f_{n_k}\}_{k\in\mathbb{N}}$ such that $\lim_k \lim_\lambda f_{n_k}(x_\lambda) = L \in \mathbb{R}$. Fix $\varepsilon > 0$ and $N \in \mathbb{N}$ such that $\|f_{n_N} - g\|_\infty < \frac{\varepsilon}{3}$ and find $\tilde{\lambda} \in \Lambda$ such that $|f_{n_N}(x_\lambda) - \lim_\lambda f_{n_N}(x_\lambda)| < \frac{\varepsilon}{3}$ for all $\lambda \geq \tilde{\lambda}$. Finally

$$|g(x_\lambda) - L| \leq |g(x_\lambda) - f_{n_N}(x_\lambda)| + |f_{n_N}(x_\lambda) - \lim_\lambda f_{n_N}(x_\lambda)| + |\lim_\lambda f_{n_N}(x_\lambda) - L| < \varepsilon$$

for all $\lambda \geq \tilde{\lambda}$, i.e. the net $g(x_\lambda)$ converges to $L \in \mathbb{R}$. In particular, by the uniqueness of $\lim_\lambda g(x_\lambda)$, we conclude that the original sequence $\{\lim_\lambda f_n(x_\lambda)\}_{n\in\mathbb{N}}$ converges to L, which also proves that the limit does not depend on the particular net x_γ as far as $\lim_\lambda f(x_\lambda) = \lim_\gamma f(x_\gamma)$ for all $f \in F'$. This shows that the map

$$T\colon e_{F'}X \to e_F X, y = \lim_\lambda \Pi_{f\in F'} f(x_\lambda) \mapsto Ty = \lim_\lambda \Pi_{f\in F} f(x_\lambda).$$

is a well-defined isomorphism. Because its inverse is the projection map $\pi_{F'}|_{T(e_{F'}X)}$, which is continuous and open, then

$$T \circ e_{F'}\colon x \mapsto \Pi_{f\in F'} f(x) \mapsto \Pi_{f\in F} f(x) = e_F(x)$$

is a homeomorphism. To prove that $e_{F'}X$ is a compactification of X, we notice that F separates points, as so does F'. If $U \subset X$ is open, so is

$$e_{F'}(U) = T^{-1}(e_F(U)),$$

and $e_{F'}$ is an injective continuous open map, thus it is an embedding onto its image. □

In this work, the set of functions we work with must be separable (has a countable and dense subset). Let $F \subset BC(\Omega \times \mathbb{B}^d)$ be a closed separable algebra that separates points from closed sets and let $e_F\Omega \times \mathbb{B}^d$ be its compactification. Because $e_F\Omega \times \mathbb{B}^d$ is a compact Hausdorff space we have the following isometric isomorphism of its dual

$$C(e_F\Omega \times \mathbb{B}^d)^* \approx \mathcal{M}(e_F\Omega \times \mathbb{B}^d).$$

Lemma 2.3.9. *If $F \subset BC(X)$ is separable, so is $C(e_F X)$.*

Proof. Let $\{f_n\}_{n\in\mathbb{N}}$ be dense in F. By the Stone-Weierstrass Theorem, the algebra generated by $\{1\} \cup \{\overline{f}_n\}_{n\in\mathbb{N}}$ is dense in $C(e_F X)$, and so $C(e_F X)$ is separable. □

Because $C(e_F\Omega \times \mathbb{B}^d)$ is separable we also have the abstract sequential compactness principle A.1.4 on its dual $\mathcal{M}(e_F\Omega \times \mathbb{B}^d)$. Let $T^{-1}F \subset \mathbb{G}_p$ the corresponding algebra, with respect to

$$(f \times^p g)(z) = \frac{f(z)g(z)}{1 + |z|^p}$$

on the set of continuous functions with p-growth. There is an isometric isomorphism

$$T^{-1}F \overset{\tilde{T}}{\approx} C(e_F\Omega \times \mathbb{B}^d).$$

By the Riesz representation Theorem, we can write its adjoint as

$$\tilde{T}^* \colon \mathcal{M}(e_F\Omega \times \mathbb{B}^d) \to (T^{-1}F)^*, \nu \mapsto \left(\Phi \mapsto (\tilde{T}^*\nu, \Phi) = (\nu, \tilde{T}\Phi) = \int_{e_F\Omega\times\mathbb{B}^d} \tilde{T}\Phi d\nu.\right)$$

Lemma 2.3.10. *Let X, Z be completely regular Hausdorff spaces and let $F \subset BC(X)$ and $G \subset BC(Z)$ be closed sub-algebras that separate points from closed sets and contain the constant function. The following spaces are all isometrically isomorphic to each other*

$$C(e_{F\cup G}X \times Z) \approx C(e_F X \times e_G Z) \approx \overline{\mathbb{A}\Big(C(e_F X) \times C(e_G Z)\Big)} \approx \overline{\mathbb{A}(F \cup G)},$$

where each $f \in F$ and $g \in G$ is extended to a function on the product space by keeping constant the other variable. Moreover, $F \cup G \subset BC(X \times Z)$ separates points from closed sets.

The proof of the above lemma is a straightforward application of the Stone-Weierstrass Theorem.

Remark 2.3.11. *It is important to take families F, G defined exclusively on each respective space, and the theorem is false if we instead add $f = f(x, y)$ that is not of the form above. Specifically, in this case, we will work with compactifications $e_f X \times Y$ which are still separable and metrisable.*

Morally speaking, what the previous lemma says is that on product spaces it is enough to work with the compactifications in each coordinate separately. Moreover, their dual elements (generalised measures) can be tested against tensor products of functions that depend on each variable independently.

2.4 Representation of Young Measures

Let $\Omega \subset \mathbb{R}^n$ be open and bounded and $G' \subset BC(\mathbb{B}^d)$ and $F' \subset BC(\Omega)$ be closed separable sub-algebras that separate points from closed sets. Let $T^{-1}F = A \subset \mathbb{G}_p$ the corresponding algebra, with respect to the $\times^p$ product, in the set of functions having p-growth. F is isometrically isomorphic to $C(e_F\Omega \times \mathbb{B}^d)$.

With abuse of notation, we are going to call T the isomorphism between A and $C(e_F\Omega \times \mathbb{B}^d)$. To each function $u \in L^p(\Omega, \mathbb{R}^d)$ we associate an *elementary Young Measure* $\xi_u \in A^*$ by setting

$$\xi_u \colon A \to \mathbb{R}, \Phi \mapsto \int_\Omega \Phi(x, u(x)) d\mu(x).$$

Next, consider a sequence $\{u_n\}_{n\in\mathbb{N}}$ in $L^p(\Omega, \mathbb{R}^d)$ with $\sup_n \|u_n\|_p \leq C$. As $\Phi \in \mathbb{G}_p$, the sequence of elementary Young Measures is also bounded

$$\|\xi_{u_n}\| = \sup_{\|\Phi\|\leq 1} \left| \int_\Omega \Phi(x, u_n(x)) d\mu(x) \right| = \int_\Omega (1 + |u_n|^p) d\mu \leq \mu(\Omega) + C^p.$$

Because of the isomorphism $A^* \eqsim C(e_F\Omega \times \mathbb{B}^d)^* \eqsim \mathcal{M}(e_F\Omega \times \mathbb{B}^d)$, there exists a subsequence, relabelled in the same way, and $\nu \in A^*$ such that $\xi_{u_n} \rightharpoonup^* \nu$ in A^*. Set

$$L := (T^*)^{-1}\nu \in \mathcal{M}(e_F\Omega \times \mathbb{B}^d). \tag{2.4.1}$$

We now study the measure L to find a better representation for $\nu \in A^*$. Let $\psi \in C(e_F\Omega \times \mathbb{B}^d)$, we immediately notice that $L \in \mathcal{M}^+(e_F\Omega \times \mathbb{B}^d)$ as a consequence of the following equality

$$\ll \nu, T^{-1}\psi \gg = \lim_n \int_\Omega (T^{-1}\psi)(x, u_n(x)) d\mu(x).$$

Because the constant functions belong to G', we can plug $T^{-1}\psi = \phi(x)(1+|z|)^p, \phi \in C(e_{F'}\Omega)$ into the previous equation and obtain the identity

$$\int_{e_F\Omega\times\mathbb{B}^d} \phi(x)dL(x,\overline{z}) = \lim_n \int_\Omega \phi(x)(1+|u_n(x)|)^p d\mu(x) = \int_{e_{F'}} \phi(x)d\lambda(x).$$

By Lemma 2.3.10, the projection

$$\pi\colon e_F\Omega \times \mathbb{B}^d \to e_{F'}\Omega$$

is well-defined, and we can write $\tilde{\lambda} = \pi_\# L$. Note that hereby $\tilde{\lambda} \in \mathcal{M}^+(e_{F'}(\Omega))$.

Find the unique $\tilde{\lambda}$-measurable parametrized family $\{\tilde{\nu}_x\}_{x\in e_{F'}\Omega}$ such that $\nu_x \in \mathcal{M}_1^+(e_{G'}\mathbb{B}^d)$ $\tilde{\lambda}$-almost every x and

$$\langle L, \Phi\rangle = \int_{e_{F'}\Omega} \langle\tilde{\nu}_x, \Phi(x,\cdot)\rangle d\tilde{\lambda}(x) \quad \forall\ \Phi \in C(e_F\Omega\times\mathbb{B}^d).$$

For any $\phi \in C_0(\Omega)$ take $\Phi = \phi(1-|\cdot|)^p$. We compute

$$\begin{aligned}\int_{e_{F'}\Omega} \phi(x)\langle\tilde{\nu}_x, (1-|\cdot|)^p\rangle d\tilde{\lambda}(x) &= \int_{e_F\Omega\times\mathbb{B}^d} \phi(x)(1-|z|)^p dL(x,z)\\ &= \lim_n \int_\Omega (\phi \mathbb{1}_{\mathbb{R}^d})(x,u_n(x))d\mu(x) = \int_\Omega \phi(x)d\mu(x).\end{aligned}$$

Because $\mu \in C_0(\Omega)^*$ we immediately conclude that

$$\mu = \langle\tilde{\nu}_x, (1-|\cdot|)^p\rangle\tilde{\lambda}\llcorner\Omega,$$

where μ is extended on $e_{F'}\Omega$ by $\mu(E) \equiv \mu(e_{F'}^{-1}(E)), E \subset e_{F'}\Omega$ Borel. Apply the Radon-Nikodym Theorem and write

$$\tilde{\lambda} = \frac{\tilde{\lambda}}{\mu}d\mu + \tilde{\lambda}^s.$$

From the previous identification, we get

$$\begin{cases}\langle\tilde{\nu}_x, (1-|\cdot|)^p\rangle\frac{\tilde{\lambda}}{\mu} = 1 & \mu - \text{a.e.}\\ \langle\tilde{\nu}_x, (1-|\cdot|)^p\rangle = 0 & \tilde{\lambda}^s - \text{a.e.},\end{cases}$$

where the second condition implies that $\tilde{\nu}_x(e_{G'}(\mathbb{B}^d)) = 0$ $\tilde{\lambda}^s$-a.e., i.e. the measures are concentrated on the boundary $\partial e_{G'}(\mathbb{B}^d)$. On $e_{G'}(\mathbb{B}^d)$ we have $0 < (1-|z|)^p \leq 1$, whereas $|z| = 1$ on $\partial e_{G'}(\mathbb{B}^d)$. In particular

$$\begin{cases}\frac{\tilde{\lambda}}{\mu} = \frac{1}{\langle\tilde{\nu}_x, (1-|\cdot|)^p\rangle} \geq 1 & \mu - \text{a.e.}\\ \tilde{\nu}_x(\partial e_{G'}(\mathbb{B}^d)) = 1 & \tilde{\lambda}^s - \text{a.e.}\end{cases}$$

Now let $\phi \in BC(\mathbb{R}^d)$ and define

$$\begin{aligned}\langle \nu_x, \phi \rangle &= \frac{\tilde{\lambda}}{\mu}(x) \int_{e_{G'}(\mathbb{B}^d)} (1-|z|)^p \phi\left(\frac{z}{1-|z|}\right) d\tilde{\nu}_x(z) \\ &= \frac{\tilde{\lambda}}{\mu}(x) \int_{\mathbb{B}^d} (1-|z|)^p \phi\left(\frac{z}{1-|z|}\right) d(e_{G'})_{\#}\tilde{\nu}_x(z)\end{aligned}$$

In particular $\nu_x \in \mathcal{M}_1^+(\mathbb{R}^d)$ and $\{\nu_x\}_{x\in\Omega}$ is μ-measurable. Let $\lambda = \tilde{\nu}_x(\partial e_{G'}(\mathbb{B}^d))\tilde{\lambda}$. Then $\lambda \in \mathcal{M}^+(e_{F'}\Omega)$ and it decomposes into

$$\begin{aligned}\lambda &= \tilde{\nu}_x(\partial e_{G'}(\mathbb{B}^d))\frac{\tilde{\lambda}}{\mu}\mu + \tilde{\nu}_x(\partial e_{G'}(\mathbb{B}^d))\tilde{\lambda}^s \\ &= \tilde{\nu}_x(\partial e_{G'}(\mathbb{B}^d))\frac{\tilde{\lambda}}{\mu}\mu + \tilde{\lambda}^s.\end{aligned}$$

For λ-almost every $x \in e_{F'}\Omega$ and for $\psi \in C(\partial e_{G'}\mathbb{B}^d)$ set

$$\langle \nu_x^\infty, \psi \rangle = \frac{1}{\tilde{\nu}_x(\partial e_{G'}(\mathbb{B}^d))} \int_{\partial e_{G'}(\mathbb{B}^d)} \psi(z) d\tilde{\nu}_x(z),$$

hereby

$$\nu_x^\infty \in \mathcal{M}_1^+(\partial e_{G'}(\mathbb{B}^d)).$$

For each $\Phi \in A$, its recession function is defined to be the restriction

$$\phi^\infty = \phi|_{e_{F'}\Omega \times \partial e_{G'}(\mathbb{B}^d)}.$$

Finally, we obtain the formula

$$\begin{aligned}\langle L, T\Phi \rangle &= \int_{e_{F'}(\Omega)} \langle \tilde{\nu}_x, T\Phi(x,\cdot)\rangle d\tilde{\lambda} \\ &= \int_{e_{F'}\Omega} \int_{e_{G'}(\mathbb{B}^d)} T\Phi d\tilde{\nu}_x \left(\frac{\tilde{\lambda}}{\mu} d\mu + \overbrace{d\tilde{\lambda}^s}^{=0} \right) + \int_{e_{F'}\Omega} \fint_{\partial e_{G'}(\mathbb{B}^d)} T\Phi d\tilde{\nu}_x \ (\tilde{\nu}_x(\partial e_{G'}(\mathbb{B}^d)) d\tilde{\lambda}) \\ &= \int_\Omega \langle \nu_x, \Phi(x,\cdot)\rangle d\mu + \int_{e_{F'}\Omega} \langle \nu_x^\infty, \Phi^\infty(x,\cdot)\rangle d\lambda\end{aligned}$$

and the representation of the Young Measure as the triple

$$\nu = \big(\{\nu_x\}_{x\in\Omega}, \lambda, \{\nu_x^\infty\}_{x\in e_{F'}\Omega}\big).$$

where

oscillation:	$\nu_x \in \mathcal{M}_1^+(\mathbb{R}^d)$ μ-almost every $x \in \Omega$,
concentration:	$\lambda \in \mathcal{M}^+(e_{F'}\Omega)$,
angle-concentration:	$\nu_x^\infty \in \mathcal{M}_1^+(\partial e_{G'}(\mathbb{B}^d))$ λ-almost every $x \in \overline{\Omega}$.

We say that u_n converges in the sense of Young Measures to ν, and write

$$u_n \xrightarrow{Y^p(\mu,e_F)} \nu,$$

where μ is the measure that "regulates and weights" oscillation and concentration of u_n, and e_F is the compactification at infinity, generated by the family F.

From this point onwards the family F' in Ω will always be the set of functions $C(\overline{\Omega})$.

Remark 2.4.1. *It is interesting to compare our representation with the classical formula by DiPerna and Majda [36] and later developments such as those in [73, 95, 56]. In these publications, the authors developed and used Young Measures by taking the object $L \in \mathcal{M}(\overline{\Omega} \times e_{f_i,i\in\mathbb{N}}\mathbb{B}^d)$ as in* (2.4.1) *and testing it against functions $f \in e_{f_i,i\in\mathbb{N}}$ and $\phi \in C_0(\Omega)$. Even though such measure L is, by all means, the same object as what we construct, our deeper analysis reveals a triple-like structure $(\nu_x, \lambda, \nu_x^\infty) \in Y(\mu, e_{f_i,i\in\mathbb{N}})$, where ν_x^∞ is the only part that reflects the use of the compactification $e_{f_i,i\in\mathbb{N}}$. Thanks to this representation, we are able to prove structural theorems about generalised Young Measures and systematically visualise and compare them when changing the measure $\mu \in \mathcal{M}^+(\Omega)$ and the family $\{f_i, i \in \mathbb{N}\}$ (and so the associated Hausdorff compactification $e_{f_i,i\in\mathbb{N}}$).*

2.4.1 Properties of generalised Young Measures

Here we show how the above construction generalises the more classical setting of Young Measures on the sphere, see [90] for the original idea behind their representation, and [6] for their modern implementation in the calculus of variations. We then study how the new representation for generalised Young Measures behaves geometrically, and its properties.

As a reminder, we state here the definition of integrands with a regular recession at infinity.

Definition 2.4.2. *The set of integrands admitting a regular recession at infinity is*

$$\mathbb{E}_p(\Omega, \mathbb{R}^d) = \left\{ \Phi \in C(\overline{\Omega} \times \mathbb{R}^d) : \lim_{t\to\infty} \frac{\Phi(x,tz)}{t^p} \in \mathbb{R} \text{ loc. uniformly in } (x,z) \in \overline{\Omega} \times \mathbb{R}^d \right\}.$$

Because we intend to generalise the theory of Young Measures on functions with a regular recession, we need to extend the above class and at the same time preserve good topological properties of such a larger class. To do so, consider countably

many functions $(g_i)_i$ in $BC(\mathbb{B}^d)$ and their representations as integrands of p-growth $g_i(\frac{z}{1+|z|})(1+|z|)^p$. We are interested in understanding how to represent, in a simple way, Young Measures relative to the compactification generated by $\mathbb{E}_p \cup \{g_i(\frac{z}{1+|z|})(1+|z|)^p\}$. In the language of Hausdorff compactifications, set $G' = C(\overline{\mathbb{B}^d}) \cup \{g_i, i \in \mathbb{N}\}$ and $F' = C(\overline{\Omega})$. Because $C(\overline{\mathbb{B}^d}) \subset G'$, the closure of the algebra generated by either family is separable, separates points from closed sets and contains the constants. Call $F = G' \cup F'$, and without loss of generality, we can assume that $\|g_i\| \leq 1$ for all i.

Lemma 2.4.3. *$C(e_F\Omega \times \mathbb{B}^d)$ is isometrically isomorphic to $C(\overline{\Omega} \times \overline{\mathrm{graph}(g_i)})$, where $(g_i)_i \colon \mathbb{B}^d \to [-1,1]^{\mathbb{N}}, z \mapsto (g_i(z))_{i\in\mathbb{N}}$ and the topology on the target space is the product topology. It is metrised by*

$$d(z,w) = |z-w| + \sum_i 2^{-i}|g_i(z) - g_i(w)|.$$

Proof. By Lemma 2.3.10 it is enough to prove that $C(e_{g_i,i\in\mathbb{N}}\mathbb{B}^d) \backsimeq C(\overline{\mathrm{graph}(g_i)})$. Theorem 2.3.8 provides the isomorphism

$$\overline{\mathbb{A}\Big(C(\overline{\mathbb{B}^d}) \cup \{g_i, i \in \mathbb{N}\}\Big)} = \overline{\mathbb{A}\Big(1, z^1, \dots, z^d, g_i, i \in \mathbb{N}\Big)},$$

and we conclude by noticing that $\overline{(1, z^1, \dots, z^d, g_i, i \in \mathbb{N})(\mathbb{B}^d)}$ is homeomorphic to $\overline{\mathrm{graph}(g_i)}$. The topological equivalence between such metrics and the product topology is standard. □

When $g_i \in C(\overline{\mathbb{B}^d})$ for all $i \in \mathbb{N}$, then $C(e_F\Omega \times \mathbb{B}^d) \backsimeq C(\overline{\Omega} \times \overline{\mathbb{B}^d})$, and therefore we recover the usual sphere representation for the compactification induced by $\mathbb{E}_p$. This means the obvious, that we can add functions that have a regular recession and we still obtain the same space (up to homeomorphisms).

By definition, the compactification of $\mathbb{B}^d$ can be represented by the space Γ of sequences $\{z_n\}_{n\in\mathbb{N}}$ in $\mathbb{B}^d$ such that $z_n \to z \in \overline{\mathbb{B}^d}$ and $g_i(z_n)$ converges for all $i \in \mathbb{N}$, and two such sequences $\{z_n\}_{n\in\mathbb{N}}$ and $\{w_n\}_{n\in\mathbb{N}}$ are identified provided

$$\lim_n |z_n - w_n| + \sum_i 2^{-i}|g_i(z_n) - g_i(w_n)| = 0.$$

Definition 2.4.4. *We call $e_{g_i,i\in\mathbb{N}}$ the compactification, and*

$$\partial e_{g_i,i\in\mathbb{N}} = e_{g_i,i\in\mathbb{N}} \setminus \mathrm{graph}(g_i, i \in \mathbb{N}),$$

we can write the triple Young Measure as

$$\nu = \big(\{\nu_x\}_{x\in\Omega}, \lambda, \{\nu_x^\infty\}_{x\in\overline{\Omega}}\big),$$

where $\nu_x \in \mathcal{M}_1^+(\mathbb{R}^d)$ for μ-almost every $x \in \Omega$, $\lambda \in \mathcal{M}^+(\overline{\Omega})$, and $\nu_x^\infty \in \mathcal{M}_1^+(\partial e_{g_i,i\in\mathbb{N}})$ for λ-almost every $x \in \overline{\Omega}$.

Notice that $\partial e_{g_i, i\in\mathbb{N}}$ is an abuse of notation and refers to the boundary of the embedded space within the compactification.

So far we have constructed compactifications by glueing g_is on top of the functions $z^1, \dots, z^d$; that is to say on top of the unit ball. It is sometimes useful to iterate this argument, to stack another countable family $\{f_i, i \in \mathbb{N}\}$ on top of the compactification $e_{g_i, i\in\mathbb{N}}$. This process gives the same compactification as if we were considering the two families at once, as the following lemma shows.

Lemma 2.4.5. *Let $e_{g_i, i\in\mathbb{N}}$ be a compactification of $\mathbb{B}^d$ and $f_i \in BC(\mathbb{B}^d)$ for all $i \in \mathbb{N}$. Then*

$$e_{g_i, f_i, i\in\mathbb{N}} \approx \overline{\text{graph}_{\text{graph}(g_i)} f_i}.$$

Proof. This is a trivial consequence of the fact that, after extending f_i to a constant in the variable $w = (w^1, w^2, \dots)$,

$$\begin{aligned}
&\{(z^1, \dots, z^d, g_i(z), f_i(z)), z \in \mathbb{B}^d\} \\
&\quad = \{(z^1, \dots, z^d, w, z) : w = g_i(z), y = f_i(z), z \in \mathbb{B}^d\} \\
&\quad = \{(z^1, \dots, z^d, w, z) : y = f_i(z, w), w = g_i(z), z \in \mathbb{B}^d\} = \text{graph}_{\text{graph}(g_i)}(f_i), z \in \mathbb{B}^d.
\end{aligned}$$

□

A standard application of the disintegration lemma yields the following.

Corollary 2.4.6. *Consider a compactification $e_{g_i, f_i, i\in\mathbb{N}}$ and $\nu^\infty \in \mathcal{M}(\partial e_{g_i, f_i, i\in\mathbb{N}})$, then*

$$\nu^\infty = P_{(z_n)_{n\in\mathbb{N}}} d\tilde{\nu}^\infty$$

where $\tilde{\nu}^\infty \in \mathcal{M}(\partial e_{g_i})$, $(z_n)_n \in \partial e_{g_i}$, and $P_{(z_n)_n}$ is a probability measure defined on the space of subsequences $(z_{n_i})_i$ of $(z_n)_n$ so that $f_i(z_{n_i})$ converges for all $i \in \mathbb{N}$ (with sequences being equivalents if all the limits are).

For the case of oscillating functions $\{f_i, i \in \mathbb{N}\}$, we also write the compactification as $e_{f_i, i\in\mathbb{N}} X \equiv e_{f_i, i\in\mathbb{N}}$ and the convergence as

$$v_j \xrightarrow{Y^p(\mu, e_{f_i, i\in\mathbb{N}})} \nu.$$

When working with the sphere compactification, we will simply write

$$v_j \xrightarrow{Y^p(\mu, \mathbb{B}^d)} \nu, \quad \text{or just } v_j \xrightarrow{Y^p(\mu)} \nu.$$

Also, when working with $p = 1$, we omit the superscript p in Y^p and write

$$v_j \xrightarrow{Y(\mu, e_{f_i}, i\in\mathbb{N})} \nu.$$

We now study the relation of Young Measures with respect to different compactifications and different underlying measures $\mu \in \mathcal{M}^+(\Omega)$. Using Chacon's Lemma A.1.5, we can prove the following structure theorems.

Lemma 2.4.7. *Let $v_j \xrightarrow{Y(\mu, e_{f_i}, i\in\mathbb{N})} (\nu_x, \lambda, \nu_x^\infty)$. Then for all $\psi \in C_0(\mathbb{R}^d)$, we have*

$$\psi(v_j) \rightharpoonup \langle \nu_x, \psi \rangle \quad \textit{weakly in } L^1(\mu).$$

Proof. Because

$$\|\psi(v_j)\|_\infty \leq \|\psi\|_\infty,$$

then the sequence is equi-integrable and there is a subsequence that converges weakly in L^1 to v. Because $\psi^\infty = 0$, testing against $\phi \in \mathcal{D}(\Omega)$ we get

$$\int_\Omega \phi\psi(u_j)d\mu \to \int_\Omega \phi v d\mu = \int_\Omega \phi\langle \nu_x, \psi\rangle d\mu.$$

□

We can improve the above weak convergence result to show the following.

Lemma 2.4.8. *Let $u_j \xrightarrow{Y(\mu)} \big(\nu_x, 0, N/A\big)$. For every $a \in L^1(\Omega, \mu)$ such that $a > 0$ μ-a.e. and for all $\psi \in C_0(\mathbb{R}^d)$ we have*

$$\psi\left(\frac{u_j}{a}\right) a \rightharpoonup \left\langle \nu_x, \psi\left(\frac{\cdot}{a(x)}\right)\right\rangle a(x) \quad \textit{in } L^1(\Omega, \mu).$$

As expected, this implies that oscillations do not depend on the particular compactification chosen.

Before proving the above results we show the following uniform approximation result:

Lemma 2.4.9. *Let $\mu \in \mathcal{M}^+(\Omega)$ and $a \in L^1(\Omega, \mu), a > 0$ μ-almost everywhere. There exists $a_n \in L^1(\Omega), 0 < a_n < a$ so that $a_n(x) \in \mathbb{Q}$ for all $x \in \Omega$ and*

$$\|a_n - a\|_\infty + \left\|\frac{a}{a_n} - 1\right\|_\infty \xrightarrow{n\to\infty} 0.$$

Proof. Let

$$a_n = \sum_{k\in\mathbb{N},k\geq 1} \chi_{a^{-1}\left([\frac{k}{n},\frac{k+1}{n})\right)} \frac{k}{n}.$$

Because $a_n \leq a$ then $a_n \in L^1$ for all $n \in \mathbb{N}$ and it also assumes countably many values at a time. Also $|a_n(x) - a(x)| \leq \frac{1}{n}$ so it converges uniformly to a. Furthermore

$$1 = \frac{\frac{k}{n}}{\frac{k}{n}} \leq \frac{a(x)}{a_n(x)} \leq \frac{\frac{k+1}{n}}{\frac{k}{n}} = \frac{k+1}{k}$$

and so $\frac{a}{a_n}$ converges uniformly to 1. □

Now we can prove Lemma 2.4.8

Proof. (of Lemma 2.4.8).

Using the previous approximation, we write, for 1-Lipschitz $\psi\colon \mathbb{R}^d \to \mathbb{R}$,

$$\int_\Omega \psi\left(\frac{u_j}{a}\right) a = \int_\Omega \overbrace{\psi\left(\frac{u_j}{a}\right) a - \psi\left(\frac{u_j}{a_n}\right) a}^{=I} + \overbrace{\psi\left(\frac{u_j}{a_n}\right) a - \psi\left(\frac{u_j}{a_n}\right) a_n}^{=II} + \psi\left(\frac{u_j}{a_n}\right) a_n.$$

The first two terms are bounded by

$$|I| \leq \int_\Omega a\left|\frac{u_j}{a} - \frac{u_j}{a_n}\right| = \int_\Omega |u_j|\left|1 - \frac{a}{a_n}\right| \leq \sup_j \|u_j\| \left\|1 - \frac{a}{a_n}\right\|_\infty$$
$$|II| \leq \int_\Omega \left(1 + \frac{|u_j|}{a_n}\right)|a - a_n| \leq (1 + \sup_j |u_j|)\left\|1 - \frac{a}{a_n}\right\|_\infty,$$

which goes to 0 as $n \to \infty$ uniformly in j. As for the third term, calling $E_k = a^{-1}\left([\frac{k}{n}, \frac{k+1}{n})\right)$, we can use dominated convergence theorem to pass to the limit

$$\lim_j \int_\Omega \psi\left(\frac{u_j}{a_n}\right) a_n = \lim_j \sum_k \int_{E_k} \psi\left(\frac{u_j}{k/n}\right) \frac{k}{n}$$
$$= \sum_k \int_{E_k} \left\langle \nu_x, \psi\left(\frac{\cdot}{k/n}\right)\right\rangle \frac{k}{n} = \int_\Omega \left\langle \nu_x, \psi\left(\frac{\cdot}{a_n}\right)\right\rangle a_n.$$

Another application of the dominated convergence theorem will let us conclude the result. □

We can finally conclude with a structure theorem regarding concentration.

Proposition 2.4.10. *Consider two separable algebras (that separate points from closed sets) A and B of $\mathbb{G}_1$ and let $a \in L^1(\Omega,\mu), a > 0$ μ-a.e. Let $(v_j)_j$ in $L^1(\Omega,\mu)$ be so that*

$$v_j \xrightarrow{Y(\mu,e_A)} \big(\nu_x, \lambda_\nu, \nu_x^\infty\big) \quad \text{and} \quad \frac{v_j}{a} \xrightarrow{Y(a\,d\mu,e_B)} \big(\eta_x, \lambda_\eta, \eta_x^\infty\big).$$

Then $\nu_x = \left(\frac{\cdot}{a(x)}\right)_{\#} \eta_x$, $\lambda_\nu = \lambda_\eta = \lambda$. Moreover, decomposing

$$\nu_x^\infty = P^\nu_{(z_n)_n} d\tilde{\nu}_x^\infty \quad \text{and} \quad \eta_x^\infty = P^\eta_{(z_n)_n} d\tilde{\eta}_x^\infty,$$

where $\tilde{\nu}_x^\infty$ and $\tilde{\eta}_x^\infty$ are the projections on the sphere according to the decomposition 2.4.6, then $\tilde{\nu}_x^\infty = \tilde{\eta}_x^\infty$ λ-a.e. with

$$v_j \xrightarrow{Y(\mu,\mathbb{B}^d)} \big(\nu_x, \lambda, \tilde{\nu}_x^\infty\big).$$

Proof. For all $\psi \in C_0(\mathbb{R}^d)$ we have that $\psi(v_j)$ is equi-integrable so that

$$\psi(v_j) \rightharpoonup \langle \eta_x, \psi\rangle \quad \text{in } L^1(\mu)$$

and

$$\psi\left(\frac{v_j}{a}\right) \rightharpoonup \langle \nu_x, \psi\rangle \quad \text{in } L^1(a d\mu).$$

Next, identify v_j with its subsequence and find E_k so that $v_j \rightharpoonup v$ in $L^1(E_k,\mu)$ for all k. By inner approximation, we can assume that E_k is compact for all k. Consider now the sequence $v_j\chi_{E_k}$. Then

$$v_j\chi_{E_k} \xrightarrow{Y(\mu,e_A)} \big(\eta_x\chi_{E_k} + \delta_0\chi_{E_k^c}, 0, N/A\big).$$

By Lemma 2.4.8 we then have, for $\phi \in C_0(\Omega)$ and $\psi \in C_0(\mathbb{R}^d)$, because $\Omega \setminus E_k$ is open,

$$\begin{aligned}
\int_\Omega \phi\langle \nu_x, \psi\rangle a(x) d\mu &= \lim_j \int_\Omega \phi\psi\left(\frac{v_j}{a}\right) a d\mu \\
&= \lim_j \int_{\Omega\setminus E_k} \phi\psi\left(\frac{v_j}{a}\right) a d\mu + \int_{E_k} \phi\psi\left(\frac{v_j}{a}\right) a d\mu \\
&= \int_{\Omega\setminus E_k} \phi\langle \nu_x, \psi\rangle a d\mu + \int_{E_k} \phi \left\langle \eta_x, \psi\left(\frac{\cdot}{a}\right)\right\rangle a d\mu.
\end{aligned}$$

Next, let $\phi \in C(\overline{\Omega})$, then

$$\begin{aligned}
\lim_j \int_\Omega \phi|v_j| d\mu &= \int_\Omega \langle \nu_x, |\cdot|\rangle \phi d\mu + \int_{\overline{\Omega}} \phi d\lambda_\nu \\
&= \lim_j \int_\Omega \phi \left|\frac{v_j}{a}\right| a d\mu = \int_\Omega \left\langle \eta_x, \frac{|\cdot|}{a(x)}\right\rangle \phi a(x) d\mu + \int_{\overline{\Omega}} \phi d\lambda_\eta.
\end{aligned}$$

Using the previous part we conclude that $\lambda_\nu = \lambda_\eta = \lambda$.

Finally, let $f \in C(\partial\mathbb{B}^d)$ and extending by 1-homogeneity we obtain that

$$\begin{aligned}
&\lim_j \int_\Omega \phi f(v_j) d\mu \\
&= \int_\Omega \phi \langle \nu_x, f\rangle d\mu + \int_{\overline{\Omega}} \phi \langle \nu_x^\infty, f^\infty\rangle d\lambda = \int_\Omega \phi \langle \nu_x, f\rangle d\mu + \int_{\overline{\Omega}} \phi \int\int f^\infty dP^\nu_{(z_n)} d\tilde{\nu}_x^\infty d\lambda_\nu \\
&= \int_\Omega \phi \langle \nu_x, f\rangle d\mu + \int_{\overline{\Omega}} \phi \int f^\infty d\tilde{\nu}_x^\infty d\lambda_\nu \\
&= \int_\Omega \phi \left\langle \nu_x, f\left(\frac{\cdot}{a(x)}\right)\right\rangle a(x) d\mu + \int_{\overline{\Omega}} \phi \langle \eta_x^\infty, f^\infty\rangle d\lambda \\
&= \int_\Omega \phi \langle \nu_x, f\rangle d\mu + \int_{\overline{\Omega}} \phi \int f^\infty dP^\eta_{(z_n)} d\tilde{\eta}_x^\infty d\lambda_\eta = \int_\Omega \phi \langle \nu_x, f\rangle d\mu + \int_{\overline{\Omega}} \phi \int f^\infty d\tilde{\eta}_x^\infty d\lambda_\eta.
\end{aligned}$$

□

Notice that the previous identification with the concentration angle measure fails if we only consider $\Gamma = A \cap B$ which does not necessarily generate the sphere compactification. This is so because sequences $(u_j)_j$ can concentrate around values of a that are measure-discontinuous. However, equality holds if $a = 1$.

Lemma 2.4.11. *Under the assumptions of Proposition 2.4.10 and $a = 1$, we let $\Gamma = A \cap B$. Writing*

$$\nu_x^\infty = P^\nu_{(z_n)_n} d(\gamma^\nu)^\infty_x \quad \text{and} \quad \eta_x^\infty = P^\eta_{(z_n)_n} d(\gamma^\eta)^\infty_x,$$

where γ^η and γ^ν are the projections onto the compactification generated by Γ, then

$$(\gamma^\nu)^\infty_x = (\gamma^\eta)^\infty_x \quad \lambda\text{-a.e.}$$

Proof. This is proven similarly at the end of Proposition 2.4.10 and testing against functions belonging in $\overline{\mathbb{A}(\Gamma)}$ and using the decomposition of angle Young Measures. □

Next, we show that the lack of concentration is equivalent to the equi-integrability of the generating sequence.

Theorem 2.4.12. *Let $(v_j)_j$ in $L^1(\Omega, \mu, \mathbb{R}^d)$ be so that*

$$v_j \xrightarrow{Y(\mu, e_{f_i}, i\in\mathbb{N})} \left(\nu_x, \lambda, \nu_x^\infty\right).$$

Then the sequence $(v_j)_j$ is equi-integrable if and only if $\lambda = 0$.

Moreover $v_j \to v$ strongly in $L^1(\Omega, \mu, \mathbb{R}^d)$ if and only if $\lambda = 0$ and $\nu_x = \delta_{v(x)}$ for μ-a.e. $x \in \Omega$.

Proof. Because λ does not depend on the compactification (see Proposition 2.4.10), we can apply the same theorem from the sphere compactification, [92], Lemma 12.14 and [92], Corollary 12.15, to conclude. □

Compare the above theorem to the Vitali convergence Theorem A.1.1 in the appendix.

Before stating the next two structure results, we prove that T^{-1} is a bounded operator from $\mathrm{Lip}(e_{f_i,i\in\mathbb{N}})$ to $\mathrm{Lip}(\mathbb{R}^d)$, provided the compactification is generated by Lipschitz functions. In this case, by $\mathrm{Lip}(\mathbb{R}^d)$ we mean the weighted norm

$$\|f\|_{\mathrm{Lip}(\mathbb{R}^d)} := \|Tf\|_\infty + \sup_{x\neq y}\frac{|f(x)-f(y)|}{|x-y|} = \left\|\frac{f}{1+|\cdot|}\right\|_\infty + \sup_{x\neq y}\frac{|f(x)-f(y)|}{|x-y|}.$$

As for the compactification, the metric is always intended as in Lemma 2.4.3.

Lemma 2.4.13. *Let $e_{f_i,i\in\mathbb{N}}$ be a separable compactification metrised by the usual metric, where, for every $i\in\mathbb{N}$, $f_i\in\mathrm{Lip}(\mathbb{R}^d)$ are normalised so that $\|f\|_{\mathrm{Lip}(\mathbb{R}^d)}\leq 1$. Then*

$$\sup_{x\neq y}\frac{|g(x)-g(y)|}{|x-y|}\leq 5\mathrm{Lip}(Tg, e_{f_i,i\in\mathbb{N}})$$

for all maps $g\colon\mathbb{R}^d\to\mathbb{R}$.

Proof. Without loss of generality assume that $\|Tg\|_{\mathrm{Lip}}\leq 1$, i.e. for all $|x|,|y|<1$,

$$\left|g\left(\frac{x}{1-|x|}\right)(1-|x|)-g\left(\frac{y}{1-|y|}\right)(1-|y|)\right|\leq |x-y|+\sum_i 2^{-i}|Tf_i(x)-Tf_i(y)|.$$

Then

$$\begin{aligned}
&|g(x)-g(y)|\\
&=\left|\frac{g(x)}{1+|x|}(1+|x|)-\frac{g(x)}{1+|x|}(1+|y|)+\frac{g(x)}{1+|x|}(1+|y|)-\frac{g(y)}{1+|y|}(1+|y|)\right|\\
&=\frac{|g(x)|}{1+|x|}\Big|1+|x|-1-|y|\Big|+(1+|y|)\left|\frac{g(x)}{1+|x|}-\frac{g(y)}{1+|y|}\right|\\
&\leq |x-y|+(1+|y|)\left(\left|\frac{x}{1+|x|}-\frac{y}{1+|y|}\right|+\sum_i 2^{-i}\left|Tf_i(\frac{x}{1+|x|})-Tf_i(\frac{y}{1+|y|})\right|\right).
\end{aligned}$$

For all i we have that

$$\left|Tf_i\left(\frac{x}{1+|x|}\right)-Tf_i\left(\frac{y}{1+|y|}\right)\right|=\left|\frac{f_i(x)}{1+|x|}-\frac{f_i(y)}{1+|y|}\right|,$$

and therefore after multiplying by $1+|y|$ we obtain

$$\left|\frac{f_i(x)}{1+|x|}\Big(1+|x|+(|y|-|x|)\Big)-f_i(y)\right| = \left|f_i(x)-f_i(y)+\frac{f_i(x)}{1+|x|}(|y|-|x|)\right|$$
$$\leq |f_i(x)-f_i(y)|+\frac{|f_i(x)|}{1+|x|}|x-y| \leq 2|x-y|.$$

□

We now show that the above lemma allows us to test Young Measures on Lipschitz compactifications against Lipschitz functions of $\mathbb{R}^d$.

Definition 2.4.14 (Kantorovich semi-norm)**.** *Let X be a metric space and $\mu \in \mathcal{M}(X)$, then the (formal) Kantorovich norm of μ is*

$$\|\mu\|_K = \sup_{\|\phi\|_{\mathrm{Lip}}\leq 1}\int_X \phi d\mu.$$

The above formula induces a pseudo-distance between measures by setting

$$d(\mu,\eta)_K = \|\mu-\eta\|_K.$$

It turns out that this is indeed a metric on the positive cone of non-negative Measures, see Lemma A.1.6. In particular, by taking $\Psi \in \mathrm{Lip}(e_{f_i,i\in\mathbb{N}})$ and the pull-back T^{-1} we deduce the following.

Lemma 2.4.15. *Let $e_{f_i,i\in\mathbb{N}}$ be a separable compactification. Then, every $\nu \in Y(\mu, e_{f_i,i\in\mathbb{N}})$ is defined by testing it against Lipschitz functions of the form*

$$\phi\otimes\psi, \quad \text{where } \|\phi\|_{\mathrm{Lip}(\Omega)}\leq 1,\ \|\psi\|_{\mathrm{Lip}(\mathbb{R}^d)}\leq 1.$$

For this reason, we remind once again of the norm we will be using on the space $e_{f_i,i\in\mathbb{N}}$ throughout this thesis.

Definition 2.4.16. *We say that $e_{f_i,i\in\mathbb{N}}$ is a Lipschitz compactification if each f_i is Lipschitz continuous, and renormalised so that $\mathrm{Lip}(Tf)\leq 1$. The norm on $\mathrm{Lip}(e_{f_i,i\in\mathbb{N}})$ will always be*

$$\|g\|_{\mathrm{Lip}(e_{f_i,i\in\mathbb{N}})} := \sup_{x\in e_{f_i,i\in\mathbb{N}}}|g(x)| + \sup_{x\neq y}\frac{|g(x)-g(y)|}{d_{e_{f_i,i\in\mathbb{N}}}(x,y)}$$

where

$$d_{e_{f_i,i\in\mathbb{N}}}(x,y) = |x-y| + \sum_i 2^{-i}|Tf_i(x)-Tf_i(y)|.$$

To conclude this section, we state decomposition results for Young Measures regarding oscillation and concentration. Originally proven in the context of the sphere compactification, [67], we here extend them to separable compactifications.

Lemma 2.4.17. *Let $(v_j)_j$ in $L^1(\Omega,\mu)$ so that*

$$v_j \xrightarrow{Y(\mu,e_{f_i,i\in\mathbb{N}})} \big(\nu_x,\lambda,\nu_x^\infty\big).$$

We can write $v_j = o_j + c_j$, where for all $j\in\mathbb{N}$, $o_j\in L^1(\Omega,\mu)$ is equi-integrable,

$$o_j \xrightarrow{Y(\mu,e_{f_i,i\in\mathbb{N}})} \big(\nu_x,0,N/A\big)$$

and $c_j\in L^1(\Omega,\mu)$ so that

$$c_j \xrightarrow{Y(\mu,e_{f_i,i\in\mathbb{N}})} \big(\delta_0,\lambda,\nu_x^\infty\big).$$

The converse is also true, for each such sequence o_j, c_j as above, their sum converges to the former Young Measure.

Proof. A standard diagonal argument gives us $k_j\uparrow\infty$ so that $o_j = v_j\chi_{|v_j|}\leq k_j$ is equi-integrable and generates $o_j \xrightarrow{Y(\mu,e_{f_i,i\in\mathbb{N}})} \big(\nu_x,0,N/A\big)$. Then, letting $c_j = v_j - o_j$, for $\eta\in C(\overline{\Omega})$, $T\psi\in C(e_{f_i,i\in\mathbb{N}})$ we have

$$\begin{aligned}\int_\Omega \eta(\psi(c_j)-\psi(v_j)) &= \int_{|v_j|\leq k_j}\eta(\psi(0)-\psi(o_j)) + \int_{|v_j|>k_j}\eta(\psi(v_j)-\psi(v_j))\\ &= \int_\Omega \eta(\psi(0)-\psi(o_j)) \to \int_\Omega \eta(\psi(0)-\langle\nu_x,\psi\rangle).\end{aligned}$$

Writing $\psi(c_j) = \Big(\psi(c_j)-\psi(v_j)\Big)+\psi(v_j)$ and letting $j\to\infty$ we conclude. □

We remark here that when considering certain subsets of E of Y (for example Young Measures generated by gradients, see next section) o_j and c_j might generate different Young Measures that do not belong to E.

The next lemma is an extension of the previous result.

Lemma 2.4.18. *Let $(v_j)_j$ and $(w_j)_j$ in $L^1(\mu)$ generate*

$$v_j \xrightarrow{Y(\mu,e_{f_i,i\in\mathbb{N}})} \big(\delta_{v(x)},\lambda_\eta,\eta_x^\infty\big) \quad \textit{and} \quad w_j \xrightarrow{Y(\mu,e_{f_i,i\in\mathbb{N}})} \big(\nu_x,\lambda_\nu,\nu_x^\infty\big)$$

with $\lambda_\eta\perp\lambda_\nu$, for some $v\in L^1(\mu)$. Then the sum of the sequence generates

$$v_j + w_j \xrightarrow{Y(\mu,e_{f_i,i\in\mathbb{N}})} \big(\delta_{v(x)}*\nu_x,\lambda_\nu+\lambda_\eta,k_x^\infty\big),$$

where

$$k_x^\infty = \begin{cases}\nu_x^\infty & \lambda_\nu\text{-a.e.}\\ \eta_x^\infty & \lambda_\eta\text{-a.e.}\end{cases}$$

Proof. Write $w_j = o_j + c_j$ as in the previous lemma and put $b_j = v_j - v$. We claim that

$$b_j + c_j \xrightarrow{Y(\mu, e_{f_i, i\in\mathbb{N}})} \big(\delta_0, \lambda_\nu + \lambda_\eta, k_x^\infty\big).$$

No oscillation is a consequence of the fact that $b_j + c_j \to 0$ in μ-measure. Next, let $\phi \in C(\overline{\Omega})$, $\|\phi\|_{\text{Lip}} \leq 1$ and $\Psi \in e_{f_i, i\in\mathbb{N}}$, $\|T\Phi\|_{\text{Lip}} \leq 1$ with $\Psi(0) = 0$. Let E_ν and E_η be sets where λ_ν and λ_η are concentrated, respectively. For $\varepsilon > 0$ find $C_\nu \subset E_\nu, C_\eta \subset E_\eta$ compact sets and $O_\nu \supset C_\nu, O_\eta \supset C_\eta$ open sets such that

$$\lambda_\eta(\overline{\Omega} \setminus C_\eta) + \lambda_\nu(O_\eta) + \lambda_\nu(\overline{\Omega} \setminus C_\nu) + \lambda_\eta(O_\nu) < \varepsilon.$$

Consider a function $\rho \in C(\mathbb{R}^n)$ with $\chi_{C_\eta} \leq \rho \leq \chi_{O_\eta}$. We write

$$\int_\Omega \phi\big(\Psi(b_j + c_j) - \Psi(b_j) - \Psi(c_j)\big)(\overbrace{\rho}^{=I} + \overbrace{1-\rho}^{II})d\mu.$$

We estimate the first term by

$$\limsup_j |I| \leq \limsup_j 2\int_\Omega \rho|b_j| = 2\int_{\overline{\Omega}} \rho d\lambda_\nu \leq 2\lambda_\nu(\overline{\Omega} \cap O_\eta) \leq 2\varepsilon.$$

Similarly for the second term

$$\limsup_j |II| \leq \limsup_j 2\int_\Omega (1-\rho)|c_j| \leq 2\lambda_\eta(\overline{\Omega} \setminus C_\eta) \leq 2\varepsilon.$$

But the first term converges to 0 and therefore we conclude for the representation of Young Measures. □

2.4.2 Characterisation of generalised Young Measures

We dedicate this part to clarifying which Young Measure can be generated by integrable functions. We start by giving a clear definition of what generalised Young Measures are.

Definition 2.4.19. *Given a family $\{f_i, i \in \mathbb{N}\}$, with $f_i \in \mathbb{G}_p$ for all $i \in \mathbb{N}$, that separates points from closed sets, a p-Young Measure is a triple*

$$\nu = \big((\nu_x)_{x\in\Omega}, \lambda, (\nu_x^\infty)_{x\in\overline{\Omega}}\big)$$

where

1. *$(\nu_x)_{x\in\Omega}$ is μ-measurable and $\nu_x \in \mathcal{M}_1^+(\mathbb{R}^d)$ μ-a.e. $x \in \Omega$.*

   *We call it **oscillation**;*

2. $\lambda \in \mathcal{M}^+(\overline{\Omega})$.

   We call it ***concentration****;*

3. $(\nu_x^\infty)_{x\in\overline{\Omega}}$ *is* λ*-measurable and* $\nu_x^\infty \in \mathcal{M}_1^+(\partial e_{f_i,i\in\mathbb{N}})$ λ*-a.e.* $x \in \overline{\Omega}$.

   We call it ***angle-concentration****;*

4. *the moment condition*

$$\int_\Omega \int_{\mathbb{R}^d} |z|^p d\nu_x(z) < +\infty$$

   must hold.

The collection of all such triples is denoted by $Y^p = Y^p(\Omega, \mu, e_{f_i,i\in\mathbb{N}})$.

When obvious from the context, we will not specify the domain Ω, the measure μ, the family F or the target space $\mathbb{R}^d$. Also, when $\lambda = 0$, i.e. when there is no concentration, there is no point in specifying the compactification we are working with.

From every triple $\nu = \left(\nu_x, \lambda, \nu_x^\infty\right)$ one can construct the measure $L \in C(e_F\Omega \times \mathbb{R}^d)$ and vice-versa.

Lemma 2.4.20. *The set* Y^p *equals*

$$T^*\left\{L \in \mathcal{M}^+(e_F\Omega \times \mathbb{B}^d) : \int_{e_F\Omega\times\mathbb{B}^d} \phi(x)(1-|\hat{z}|)^p dL = \int_\Omega \phi(x) d\mu(x) \ \forall \phi \in C(e_G\Omega)\right\}.$$

In particular Y^p *is a weak* closed and convex subset of* F^*.

Proof. Let $L \in \mathcal{M}^+(e_F\Omega \times \mathbb{R}^d)^+$ as above. It was already shown that $T^*L \in Y^p$. On the other side, if $\nu = \left(\nu_x, \lambda, \nu_x^\infty\right)$, we let

$$L = \nu_x d\mu + \nu_x^\infty d\lambda.$$

Testing against a test function $\phi = \phi(x)$ that only depends on x,

$$\int_{e_F\Omega\times\mathbb{B}^d} T\phi dL = \int_{e_F\Omega\times\mathbb{B}^d} \phi(x)(1-|\hat{z}|)^p dL = \int_\Omega \phi d\mu.$$

Conclude by noticing that the above characterisation amounts to

$$Y^p = T^*\left(\bigcup_{\phi\in C(e_F\Omega)} \left\{L \in \mathcal{M}^+(e_F\Omega \times \mathbb{B}^d)^+ : \int_{e_F\Omega\times\mathbb{B}^d} \phi(x)(1-\hat{z})^p dL = \int_\Omega \phi d\mu\right\}\right).$$

□

We now prove that every Young Measure $\nu = (\nu_x, \lambda, \nu_x^\infty)$ can be generated by a sequence of integrable functions. We will show this by constructing two sequences, one that generates oscillation and one that generates concentration only.

Lemma 2.4.21. *Let $\mu \in \mathcal{M}^+(\Omega)$ non-atomic and $\nu_x \colon \Omega \to \mathcal{M}_1^+(\mathbb{R}^d)$ weak* μ-measurable and so that*

$$\int_\Omega \langle \nu_x, |\cdot| \rangle d\mu < +\infty.$$

There exists a sequence $(v_j)_j$ in $L^1(\Omega, \mu, \mathbb{R}^d)$ such that

$$v_j \xrightarrow{Y} (\nu_x, 0, N/A).$$

and $v_j(x) \in \cup_y \text{supp}(\nu_y)$ for μ-a.e. $x \in \Omega$.

Proof. Notice that $\mathcal{M}_1^+(\mathbb{R}^d)$ is a metric space and denote by d_k its Kantorovich metric A.1.6, and define

$$d(\mu_1, \mu_2) = d_K(\mu_1, \mu_2) + \left| \int_{\mathbb{R}^d} |\cdot| d(\mu_1 - \mu_2) \right|$$

Fix an arbitrary point $\tilde{z} \in \cup_y \text{supp}(\nu_y)$. Find $K_j \subset \Omega$ compact sets such that $\mu(\Omega \setminus K_j) < j^{-1}$ and

$$x \mapsto \nu_x \in C(K_j, (\mathcal{M}_1^+, d)).$$

For each $j \in \mathbb{N}$, by compactness, we can find finitely many sets $(E_s^j)_s, \mu(E_s^j) > 0$ so that $\cup_s E_s^j = K_j$ and $diam(E_s^j)$ is so small that, fixing an arbitrary point $x_s^j \in E_s^j$,

$$d(\nu_{x_j^s}, \nu_x) < j^{-1} \ \forall x \in E_s^j.$$

By the density of Dirac measures, we can find finitely many values $(z_{s,t}^j, \lambda_{s,t}^j)_t \subset \mathbb{R}^d \times \mathbb{R}^+$ so that $\sum_t \lambda_{s,t}^j = 1$ and

$$d\left(\sum_t \lambda_{s,t}^j \delta_{z_{s,t}^j}, \nu_{x_s^j} \right) < j^{-1}.$$

In the above, we can assume that each $z_{s,t}^j \in \text{supp}(\nu_{x_s^j})$. Finally, on each E_s^j, because $\mu(E_s^j) > 0$ and μ is non-atomic, we can divide the set into t many subsets of measure $\mu(E_{s,t}^j) = \lambda_{s,t}^j \mu(E_s^j)$ and put v_j so that

$$v_j = \sum_s \sum_t \chi_{E_{s,t}^j} z_{s,t}^j + \tilde{z} \chi_{K_j^c}.$$

Because the above sets are disjoint, then $v_j(x) \in \cup_y \text{supp}(\nu_y)$ for μ-a.e. $x \in \Omega$. To prove convergence, let $\Psi \in C_0(\mathbb{R}^d)$ with $\text{Lip}(\Psi) \leq 1$, or $\Psi(z) = |z|$, and compute

$$\left| \int_{E_j} \Psi(v_j(x)) d\mu(x) - \langle \nu_x, \Psi \rangle d\mu(x) \right| = \left| \sum_s \sum_t \int_{E^j_{s,t}} \Psi(z^j_{s,t}) d\mu(x) - \langle \nu_x, \Psi \rangle d\mu(x) \right|$$
$$= \left| \sum_s \int_{E^j_s} \int_{\mathbb{R}^d} \Psi(z) \left(\left(d \sum_t \lambda^j_{s,t} \delta_{z^j_{s,t}} \right) - d\nu_{x^k_s} + d\nu_{x^k_s} - d\nu_x \right) d\mu(x) \right| \leq 2j^{-1}\mu(E),$$

where we remind that $d_K \leq d$. Outside of E_j we have

$$\int_{\Omega \setminus E_j} \Psi(v_j(x)) d\mu(x) - \langle \nu_x, \Psi \rangle d\mu(x) = \Psi(\tilde{z})\mu(\Omega \setminus E_j) + \int_{\Omega \setminus E_j} \langle \nu_x, \Psi \rangle d\mu(x) \to 0.$$

By density, this holds for all $\Psi \in C_0(\mathbb{R}^d)$. To show that v_j is equi-integrable, we consider an approximation of the unity $\Psi \in C_c(\mathbb{B}_k), \mathbb{B}_k \subset \mathbb{R}^d$ so that $0 \leq \Psi \leq k-1$ and $\Psi(z) = |z|$ if $|z| \leq k-1$.

$$\limsup_j \int_{|v_j|>k} |v_j| d\mu \leq \limsup_j \int_\Omega |v_j| d\mu(x) - \int_\Omega \Psi_k(v_j) d\mu = \int_\Omega \langle \nu_x, |\cdot| - \Psi_k(\cdot) \rangle d\mu$$

which converges to 0 as $k \to \infty$ by the dominated convergence theorem. □

We now show that any angle concentration can also be generated by integrable functions. Notice that, in this case, we need to approximate λ and $(\nu^\infty_x)_{x \in \overline{\Omega}}$ at the same time.

Theorem 2.4.22. *Let $\mu \in \mathcal{M}^+(\Omega)$ be non-atomic, $\lambda \in \mathcal{M}^+(\overline{\Omega})$ so that $\text{supp}(\lambda) \subset \text{supp}(\mu)$ and $\nu^\infty_x \colon \Omega \to \mathcal{M}^+_1(\partial e_{f_i, i \in \mathbb{N}})$ weak* λ-measurable, where, for all $i \in \mathbb{N}$, f_i are continuous and have linear growth. Then, there is a sequence $(v_j)_j$ in $L^1(\Omega, \mu, \mathbb{R}^d)$ so that*

$$v_j \xrightarrow{Y(\mu, e_{f_i, i \in \mathbb{N}})} (\delta_0, \lambda, \nu^\infty_x).$$

Moreover, $v_j(x)$ satisfies

$$d\left(\frac{v_j(x)}{1 + v_j(x)}, \pi_{\partial \mathbb{B}^d}(\cup_y \text{supp}(\nu^\infty_y)) \right) \xrightarrow{j \to \infty} 0 \quad \text{or} \quad v_j(x) = 0.$$

We first prove the result when λ is just a Dirac delta. In the following d is the Kantorovich metric (see Lemma A.1.6) on $\mathcal{M}^+_1(\partial e_{f_i, i \in \mathbb{N}})$.

Lemma 2.4.23. *Under the assumptions of the main theorem, let $\lambda = a\delta_z, a > 0, z \in \text{supp}(\mu)$, then there is $\phi_j \in L^1(\mathbb{B}_{j^{-1}}, \mu, \mathbb{R}^d)$ so that*

$$\int_\Omega f(\phi_j) d\mu \to f(0) d\mu(\Omega) + a \langle \nu^\infty_0, f \rangle$$

uniformly in $Tf \in \text{Lip}(e_{f_i, i \in \mathbb{N}})$ with $\text{Lip}(Tf) \leq 1$.

Proof. Assume $z = 0$. Find $(w_h^j)_h = (w_h)_h \in e_{f_i, i\in\mathbb{N}}$ and $\theta_h^j = \theta_h$ with $\sum_h \theta_h = 1$ so that

$$d\left(\sum_h \theta_h \delta_{w_h}, \nu_0^\infty\right) < j^{-1}.$$

Notice that $w_h = (T(\omega_n^h))_n$ for some $\omega_n \to \infty$ in $\mathbb{R}^d$, so can find $n = n_j$ so that $\tilde{d}(T(\omega_{n_j}^h), w_h) < j^{-1}$ for all h, where $\tilde{d}$ is the distance on the compact metric space $e_{f_i, i\in\mathbb{N}}$. By assumption, $\mu(\mathbb{B}_r) > 0$ for all $r > 0$. Therefore, because μ is non-atomic, we can find h many μ-measurable sets $E_h^j \subset \mathbb{B}_{j^{-1}}$ so that $\mu(E_h^j) > 0$ and

$$\mu(E_h^j) = \theta_h \frac{a}{|\omega_{n_j}^h|},$$

where, if necessary, we increased n_j. Finally, set

$$\phi_j = \sum_h \chi_{E_h^j} \omega_{n_j}^h.$$

Taking f so that $\mathrm{Lip}_{e_{f_i, i\in\mathbb{N}}}(Tf) \leq 1$ we compute

$$\int_\Omega f(\phi_j) d\mu = \int_{\Omega \setminus \cup_h E_h^j} f(0) d\mu + a \sum_h \theta_h \frac{f(\omega_{n_j}^h)}{|\omega_{n_j}^h|}.$$

The left term converges to $f(0)\mu(\Omega)$. Because Tf is one Lipschitz, we have that

$$\left| \frac{f(\omega_{n_j}^h)}{|\omega_{n_j}^h|} - Tf(w^h) \right| \leq \tilde{d}(T(\omega_{n_j}^h), w^h) \leq j^{-1}.$$

Therefore we conclude that

$$\left| a \sum_h \theta_h \frac{f(\omega_{n_j}^h)}{|\omega_{n_j}^h|} - a \sum_h \theta_h f^\infty(w^h) \right| \leq j^{-1}$$

and the right-hand side converges to the desired quantity. Moreover,

$$\lim_h w_h \in \pi_{\partial \mathbb{B}^d} \cup_y \mathrm{supp}(\nu_y^\infty).$$

□

We now further approximate λ by Dirac deltas to prove the full statement.

Proof. (of Theorem 2.4.22).

Upong metrising $\mathcal{M}_1^+(\partial e_{f_i, i\in\mathbb{N}})$ with the Kantorivich metric, which we just call d, we can find $K_j \subset \mathrm{supp}(\lambda) \subset \mathrm{supp}(\mu)$ compact so that $\lambda(\Omega \setminus K_j) \to 0$ and

$$x \mapsto \nu_x^\infty \in C(K_j, (\mathcal{M}_1^+(e_{f_i, i\in\mathbb{N}}), d)).$$

Then find finitely many disjoint closed balls $(\mathbb{B}_h^j)_h$ of radius bounded by $r > 0$ to be decided later, so that

$$\lambda(K^j \setminus \cup_h \mathbb{B}_h^j) < j^{-1}.$$

Call $K^j \cap \mathbb{B}_h^j = E_h^j$. By uniform continuity, letting $x_h^j \in E_h^j$, if $r > 0$ is small enough we have that

$$d(\nu_x^\infty, \nu_{x_h^j}) < j^{-1} \ \forall x \in E_h^j.$$

Then the following inequality holds uniformly for $\mathrm{Lip}(Tf) \leq 1$:

$$\begin{aligned}&\left| \int_{K^j} \langle \nu_x^\infty, f^\infty \rangle d\lambda - \sum_h \lambda(E_h^j) \langle \nu_{x_h^j}, f^\infty \rangle \right| \\ &\leq \left| \int_{\cup_h E_h^j} \langle \nu_x^\infty, f^\infty \rangle d\lambda - \sum_h \lambda(E_h^j) \langle \nu_{x_h^j}, f^\infty \rangle \right| + \lambda(K^j \setminus \cup_h E_h^j) \leq 2j^{-1}.\end{aligned}$$

Finally, because $\sum_h \lambda(E_h^j) \langle \nu_{x_h^j}, \cdot \rangle$, as a Young Measure, can be written as

$$\left(\delta_0, \sum_h \lambda(E_h^j) \delta_{x_h^j}, \nu_x^\infty \right),$$

we can use the lemma to obtain a uniform approximation on each x_h^j via $\sum_h v_h^j$ over $\mathrm{Lip}(Tf) \leq 1$, where the terms in the summation have disjoint support. □

Corollary 2.4.24. *Let μ be non-atomic and consider the Young Measure*

$$(\nu_x, \lambda, \nu_x^\infty) \in Y(\mu, e_{f_i, i \in \mathbb{N}}).$$

There exist two sequences $(v_j)_j$ and $(u_j)_j$ in $L^1(\Omega, \mu, \mathbb{R}^d)$ so that

$$\xi_{v_j} \xrightarrow{Y(\mu)} (\nu_x, 0, N/A), \quad \xi_{u_j} \xrightarrow{Y(\mu, e_{f_i, i \in \mathbb{N}})} (\delta_0, \lambda, \nu_x^\infty)$$

and

$$\xi_{v_j + u_j} \xrightarrow{Y(\mu, e_{f_i, i \in \mathbb{N}})} \nu.$$

Moreover,

$$\begin{cases} v_j(x) \in \cup_y \mathrm{supp}(\nu_y) \quad \mu\text{-a.e. } x \in \Omega, \\ d\left(\frac{u_j(x)}{1 + u_j(x)}, \pi_{\partial \mathbb{B}^d}(\cup_y \mathrm{supp}(\nu_y^\infty)) \right) \xrightarrow{j \to \infty} 0 \quad \text{or} \quad u_j(x) = 0. \end{cases}$$

Proof. Combine Lemma 2.4.21 and Theorem 2.4.22 via Lemma 2.4.18. □

Remark 2.4.25. *If $\mu = \mathcal{L}^n$, we can mollify the approximating sequence so that it belongs to $\mathcal{D}(\Omega, \mathbb{R}^d)$. In this case $v_j(x) \in \text{cohull}(\cup_y \text{supp}(\nu_y))$ for $\mathcal{L}^n$-a.e.*

We already saw how to embed functions $v \in L^1(\mu)$ into generalised Young Measures on arbitrary compactifications. We write down such definitions and show how to embed measures $\mathcal{M}(\Omega, \mathbb{R}^d)$ as Young Measures on the sphere compactification.

Definition 2.4.26. *Let $\mu \in \mathcal{M}^+(\Omega)$ and $v \in L^p(\Omega, \mu, \mathbb{R}^d), 1 \leq p \leq \infty$. The corresponding elementary p-Young Measure is*

$$\xi_v := \left((\delta_{v(x)})_{x\in\Omega}, 0, N/A\right) \in Y(\mu).$$

When $p = 1$, we extend the definition to $l \in \mathcal{M}(\Omega, \mathbb{R}^d)$ by setting, for $l = \frac{l}{\mu} d\mu + l^{s,\mu}$,

$$\xi_l := \left(\left(\delta_{\frac{l(x)}{\mu(x)}}\right)_{x\in\Omega}, |l^{s,\mu}|, \left(\delta_{\frac{l^{s,\mu}}{|l^{s,\mu}|}}\right)_{x\in\Omega}\right) \in Y(\mu, \mathbb{B}^d)$$

Note that for $\Phi \in \mathbb{E}_p$ we have

$$\ll \xi_v, \Phi \gg = \int_\Omega \Phi(x, v(x)) d\nu(x)$$

and

$$\ll \xi_l, \Phi \gg = \int_\Omega \Phi\left(x, \frac{l}{\mu}(x)\right) d\mu(x) + \int_\Omega \Phi^\infty\left(x, \frac{l^{s,\mu}}{|l^{s,\mu}|}(x)\right) d|l^{s,\mu}|(x).$$

In general, there is no clear way of defining elementary Young Measures on compactifications that are larger than the sphere. We will see later, however, that this can be done in very specific cases when we have more information on both the family $\{f_i, i \in \mathbb{N}\}$ and the measure $l \in \mathcal{M}(\Omega, \mathbb{R}^d)$ we are trying to embed.

Definition 2.4.27 (Barycentre of a p-Young Measure)**.** *Let*

$$\nu = \left((\nu_x)_{x\in\Omega}, \lambda, (\nu_x^\infty)_{x\in\overline{\Omega}}\right) \in Y^p(\mu, e_{f_i, i\in\mathbb{N}}),$$

where $\{f_i, i \in \mathbb{N}\}$ is so that $e_{f_i,i\in\mathbb{N}} \geq \overline{\mathbb{B}^d}$ (in the sense of compactifications, see 2.3.5). We call its barycentre

$$\overline{\nu} = \begin{cases} \overline{\nu_x} & 1 < p < +\infty \\ \overline{\nu_x}\mu + \overline{\nu_x^\infty}\lambda & p = 1, \end{cases}$$

which is the following quantity

$$\overline{\nu_x} = \int_{\mathbb{R}^d} z d\nu_x(z)$$
$$\overline{\nu_x^\infty} = \int_{\partial e_{f_i,i\in\mathbb{N}}} z d\nu_x^\infty.$$

In the above definition, "$\geq$" is the ordering over the set of Hausdorff compactifications of a topological space, see Definition 2.3.5. Moreover, the barycentre does not depend on the compactification, as far as $e_{f_i, i\in\mathbb{N}} \geq \overline{\mathbb{B}^d}$. Indeed $z \mapsto z = (z^j)_{j=1,\dots,d}$ extended to $e_{f_i,i\in\mathbb{N}}$ coordinate-wise, and so

$$\int_{\partial e_{f_i,i\in\mathbb{N}}} z d\nu_x^\infty = \int_{\partial\mathbb{B}^d} \int_{\{(w_n)_n\}} z dP_z((w_n)_n) d\pi_{\partial\mathbb{B}^d}\nu_x^\infty = \int_{\partial\mathbb{B}^d} z d\pi_{\partial\mathbb{B}^d}\nu_x^\infty,$$

as the coordinate map $z \mapsto z^j$ is constant on sequences $(w_n)_n$ that converge to the same value $w \in \partial\mathbb{B}^d$.

Notice that $x \mapsto \overline{\nu_x}$ is μ-measurable and $x \mapsto \overline{\nu_x^\infty}$ is λ-measurable. In particular,

$$\overline{\nu} \in L^p(\Omega, \mu, \mathbb{R}^d) \quad \text{for } 1 < p < +\infty$$

and

$$\overline{\nu} \in \mathcal{M}(\Omega, \mathbb{R}^d) \quad \text{for } p = 1.$$

It is easy to see that when $1 < p < +\infty$, $v_j \rightharpoonup v \in L^p(\Omega, \mu, \mathbb{R}^d)$ we have $v = \overline{\nu_v}$, and when $p = 1$, $\rho_j \rightharpoonup^* \rho \in C_0(\Omega, \mathbb{R}^d)^*$, then $\rho = \overline{\xi_\rho}$.

2.5 Stronger notions of convergence

To conclude the discussion about generalised Young Measures, we mention some stronger notions of convergence such as strict convergence and μ-strict convergence, where μ is any positive finite Borel measure. These modes of convergence explain why the above embedding of measures in the set of Young Measures is the canonical one. Moreover, we give a simple example of why such canonical embedding has no meaning when the compactification is larger than the sphere one.

Definition 2.5.1. *Let $(\eta_j)_j$ and η in $\mathcal{M}(\Omega, \mathbb{R}^d)$, we say that $\eta_j \xrightarrow{s} \eta$ (η_j converges strictly to η) if $\eta_j \to \eta$ weakly* in $C_0(\Omega, \mathbb{R}^d)^*$ and $|\eta_j|(\Omega) \to |\eta|(\Omega)$.*

It is easy to see that if $\eta_j \to \eta$ strictly, then $|\eta_j| \rightharpoonup^* |\eta|$ in $C_0(\Omega, \mathbb{R}^d)^*$. Strict convergence prevents cancellations of sign and concentration at the boundary of Ω. However, it does not prevent oscillation. To prevent oscillation, we must choose a weight $\mu \in \mathcal{M}^+(\Omega)$ and ask for convergence of η_j on the graph (μ, η_j). We thus obtain a notion of μ-strict convergence.

Definition 2.5.2. *We say that $\eta_j \xrightarrow{\mu-s} \eta$ (η_j converges μ-strictly to η) if $\eta_j \to \eta$ weakly* in $C_0(\Omega, \mathbb{R}^d)^*$ and $(\mu, \eta_j) \xrightarrow{s} (\mu, \eta)$ in $C_0(\Omega, \mathbb{R} \times \mathbb{R}^d)^*$.*

Similarly to what was observed in the case of strict convergence, such a notion implies that $|(\mu, \eta_j)| \rightharpoonup^* |(\mu, \eta_j)|$ in $C_0(\Omega)^*$. Moreover, μ-strict convergence of η_j to η simply amounts to weak* convergence and additional convergence of the following quantity: writing $\eta_j = \frac{\eta_j}{\mu} d\mu + \eta_j^{s,\mu}, \eta_j^{s,\mu} \perp \mu$,

$$|(\eta_j, \mu)|(\Omega) = \left| \left(\frac{\eta_j}{\mu} d\mu, \mu \right) + (\eta_j^{s,\mu}, 0) \right| (\Omega) = \int_\Omega \sqrt{1 + \left| \frac{\eta_j}{\mu} \right|^2} d\mu + |\eta_j^{s,\mu}|(\Omega)$$
$$\to \int_\Omega \sqrt{1 + \left| \frac{\eta}{\mu} \right|^2} d\mu + |\eta^{s,\mu}|(\Omega) = |(\eta, \mu)|(\Omega).$$

When $\mu = \mathcal{L}^n$, we refer to such convergence as *area-strict convergence*, in analogy with the area formula for smooth functions.

Reshetnyak continuity Theorem shows that strict convergence is equivalent to convergence through 1-homogeneous functions.

Theorem 2.5.3 ([90])**.** *Let $f(x, z) \in C(\Omega \times \mathbb{R}^d)$ be 1-homogeneous in z. If $\eta_j \to \eta$ strictly in the sense of measures, then*

$$\int_\Omega f\left(x, \frac{\eta_j}{|\eta_j|}\right) d|\eta_j|(x) \to \int_\Omega f\left(x, \frac{\eta}{|\eta|}\right) d|\eta|(x)$$

In case f is not 1-homogeneous but has an extension on the sphere compactification, we can obtain the following continuity result by requiring μ-strict convergence instead.

Corollary 2.5.4. *Let $f \in \mathbb{E}(\Omega \times \mathbb{R}^d)$. If $\eta_j \xrightarrow{\mu-s} \eta$ in $C_0(\Omega, \mathbb{R}^d)^*$ then*

$$\int_\Omega f\left(x, \frac{\eta_j}{\mu}\right) d\mu + f^\infty\left(x, \frac{\eta_j^{s,\mu}}{|\eta_j^{s,\mu}|}\right) d|\eta_j^{s,\mu}| \to \int_\Omega f\left(x, \frac{\eta}{\mu}\right) d\mu + f^\infty\left(x, \frac{\eta^{s,\mu}}{|\eta^{s,\mu}|}\right) d|\eta^{s,\mu}|.$$

These are well-known results, but we write down a proof of the latter because it gives an insight into how to move from one type of convergence to the other.

Proof. Consider the so-called *perspective functional*

$$\tilde{f}(x, z, t) = \begin{cases} f(x, \frac{z}{t})|t| & t \neq 0 \\ f^\infty(x, z) & t = 0 \end{cases}$$

which is positively 1-homogeneous in the last variable. By the Reshetnyak continuity Theorem, we obtain that

$$\int_\Omega \tilde{f}(x, (\eta, \mu)) = \int_\Omega f\left(x, \frac{\eta}{d\mu}\right) d\mu + f^\infty\left(x, \frac{\eta^{s,\mu}}{|\eta^{s,\mu}|}\right) d|\eta^{s,\mu}|$$

is sequentially continuous in the μ-strict topology in $\eta \in \mathcal{M}(\Omega, \mathbb{R}^d)$. □

Upon taking $f(x,z) = |z|$ we get that μ-strict convergence implies strict convergence, for every $\mu \in \mathcal{M}^+(\Omega)$.

In light of these results, for a fixed measure $\mu \in \mathcal{M}^+(\Omega)$, the canonical embedding of measures $\eta \in \mathcal{M}(\Omega, \mathbb{R}^d)$ into the set of Young Measures on the sphere

$$\eta \in \mathcal{M}(\Omega, \mathbb{R}^d) \mapsto \xi_\eta = \left(\delta_{\frac{\eta}{\mu}}, |\eta^{s,\mu}|, \delta_{\frac{\eta^{s,\mu}}{|\eta^{s,\mu}|}}\right)$$

is sequentially μ-strictly continuous. This is a solid justification for this choice of embedding. For the same reason, we can show why on larger compactifications we don't have, in general, a canonical embedding of measures.

Theorem 2.5.5. *Let $\mu \in \mathcal{M}^+(\Omega)$ with the property that there is $x \in \operatorname{supp}(\mu \llcorner \overline{\Omega})$ with $\delta_x \perp \mu$, and let $f \in C(\mathbb{R}^d)$ of linear growth, $f \notin \mathbb{E}(\mathbb{R}^d)$ (it doesn't have a regular recession). There is $(u_j)_j$ in $\mathcal{D}(\Omega, \mathbb{R}^d)$, $u_j \xrightarrow{\mu-strictly} z\delta_x$ in $\mathcal{M}(\overline{\Omega}, \mathbb{R}^d)$ for some $z \in \partial\mathbb{B}^d$, but*

$$\int_\Omega f(u_j(x))d\mu(x)$$

does not converge.

The above theorem implies that for all $e_{f_i, i\in\mathbb{N}} \geq e_f$ (in the sense of compactifications), ξ_{u_j} does not converge in $Y(\mu, e_{f_i,i\in\mathbb{N}})$.

Proof. Because of the assumptions on μ, we can find $x \in \operatorname{supp}(\mu \llcorner \overline{\Omega})$ so that $\delta_x \perp \mu$. Notice that the result is unchanged if we instead consider $f(x) + C_1|z| + C_2$, so taking $C_1, C_2 > 0$ big enough we can assume that $f \geq 0$ everywhere. Moreover, after multiplying by another function $g(z) \in C^\infty(\mathbb{R}^d)$ so that $g(\mathbb{B}_2^c) = 1$ and $g(\mathbb{B}_1) = 0$ we can also assume that $f(0) = 0$. After all such operations, it is still true that $f \notin \mathbb{E}(\mathbb{R}^d)$. Find $z \in \partial\mathbb{B}^d$ and $z_j, w_j \to z$ so that

$$\lim_n Tf(z_j) = M > m = \lim_j Tf(w_j)$$

Next, because $x \in \operatorname{supp}(\mu)$ then $\mu(\mathbb{B}_r(x)) > 0$ for all r. There are two possibilities. Either $x \in \Omega$, in which case we consider only those balls $\mathbb{B}_r(x)$ so that $\mathbb{B}_{2r}(x) \subset \Omega$, or $x \in \partial\Omega$, and we take a function $\delta(r)\colon (0,\infty) \to (0,\infty)$ so that $\delta(r) \downarrow 0$ as $r \downarrow 0$ and $\mu(\mathbb{B}_r(x) \cap \Omega_{-\delta(r)}) > 0$, where we remind that

$$\Omega_{-\delta(r)} = \{x \in \Omega : d(x, \partial\Omega) > \delta(r)\}.$$

Either way, we call $\mathbb{B}_r(x)$ or $\mathbb{B}_r(x) \cap \Omega_{-\delta(r)}$ simply $\mathbb{B}_r$. Furthermore, we can find an increasing and positive function $\varepsilon = \varepsilon(r)$ such that $\varepsilon(r) \to 0$ as $r \to 0$ and, setting

$$\mathbb{B}_r^{\varepsilon(r)} = (\mathbb{B}_r)_{\varepsilon(r)} = \{x \in \mathbb{R}^n : d(x, \mathbb{B}_r) < \varepsilon(r)\} \subset \Omega,$$

then

$$\lim_{r \downarrow 0} \frac{\mu(\mathbb{B}_r^{\varepsilon(r)})}{\mu(\mathbb{B}_r)} = 1.$$

For each such r find $\phi_r \in \mathcal{D}(\mathbb{B}_r^{\varepsilon(r)}), 0 \leq \phi_r \leq 1$ so that $\phi_r(y) = 1$ for all $y \in \mathbb{B}_r$. Put $u_r = \frac{\phi_r}{\int \phi_r d\mu}$. Clearly $u_r \rightharpoonup^* \delta_x$ in $C(\overline{\Omega})^*$ as $r \to 0$. Next, refine the sequences $(z_j)_j$ and $(w_j)_j$ so that there is a sequence $r_j \downarrow 0$ so that

$$\int_{\mathbb{B}_{r_{2j}}} \phi_{2j} d\mu = 1 - |z_j| \quad \text{and} \quad \int_{\mathbb{B}_{r_{2j+1}}^{\varepsilon(r_{2j+1})}} \phi_{2j+1} d\mu = 1 - |w_j|,$$

and put

$$u_j = \begin{cases} u_{r_j} z_{\frac{j}{2}} & \text{if j is even,} \\ u_{r_j} w_{\frac{j-1}{2}} & \text{if j is odd.} \end{cases}$$

First we show that $u_j \xrightarrow{\mu-strictly} z\delta_x$ in $C(\overline{\Omega}, \mathbb{R}^d)^*$ as $j \to \infty$. Because $z_j, w_j \to z$, it is enough to show that $u_r \xrightarrow{\mu-strictly} \delta_x$ in $C(\overline{\Omega})^*$ as $r \downarrow 0$. Because $u_r \rightharpoonup^* \delta_x$ in $C(\overline{\Omega})^*$ then

$$\liminf_{r \to 0} |(u_r d\mu, \mu)|(\overline{\Omega}) \geq |(\delta_x, \mu)|(\overline{\Omega}).$$

To achieve the opposite inequality, we calculate

$$\begin{aligned}
|(u_r d\mu, \mu)|(\Omega) &= |(0, \mu)|(\Omega \setminus \mathbb{B}_r^{\varepsilon(r)}) + |(u_r, 1) d\mu|(\Omega \cap \mathbb{B}_r^{\varepsilon(r)}) \\
&= \mu(\Omega \setminus \mathbb{B}_r^{\varepsilon(r)}) + \int_{\mathbb{B}_r^{\varepsilon(r)}} \sqrt{1 + u_r^2} d\mu \\
&= \mu(\Omega \setminus \mathbb{B}_r^{\varepsilon(r)}) + \frac{\int_{\mathbb{B}_r^{\varepsilon(r)}} \sqrt{\left(\int \phi_r d\mu\right)^2 + \phi_r^2} d\mu}{\int \phi_r d\mu} \\
&\leq \mu(\Omega \setminus \mathbb{B}_r^{\varepsilon(r)}) + \frac{\int_{\mathbb{B}_r^{\varepsilon(r)}} \sqrt{\mu(\mathbb{B}_r^{\varepsilon(r)})^2 + 1} d\mu}{\mu(\mathbb{B}_r)} \\
&= \mu(\Omega \setminus \mathbb{B}_r^{\varepsilon(r)}) + \frac{\mu(B_r^{\varepsilon}) \sqrt{\mu(\mathbb{B}_r^{\varepsilon(r)})^2 + 1}}{\mu(\mathbb{B}_r)} \xrightarrow{r \to 0} \mu(\Omega) + 1 = |(\delta_x, \mu)|(\overline{\Omega}).
\end{aligned}$$

Next, we study how the integral behaves on alternating integers of the sequence $(u_j)_j$. If $j = 2i$ then

$$\liminf_i \int_\Omega f(u_{2i}(x))d\mu(x) \geq \liminf_i \int_{\mathbb{B}_{r_{2i}}} f\left(\frac{z_i}{1-|z_i|}\right) d\mu = \lim_i Tf(z_i) = M.$$

On the other side, if $j = 2i+1$ we get the upper bound

$$\limsup_i \int_\Omega f(u_{2i+1}(x))d\mu(x) \leq \limsup_i \int_{\mathbb{B}_{r_{2i+1}}^{\varepsilon(r_{2i+1})}} f\left(\frac{w_i}{1-|w_i|}\right) d\mu = \lim_i Tf(w_i) = m.$$

□

In the following remark, we show that the assumption of μ is sharp.

Remark 2.5.6. *If for all $x \in \operatorname{supp}(\mu \llcorner \overline{\Omega})$ we have $\delta_x \not\perp \mu$, then all such xs belong to Ω. We can then consider the atomic decomposition of μ:*

$$\mu = \mu^a + \mu^{n-a} = \sum_n \mu(x_n)\delta_{x_n} + \mu^{n-a},$$

see the atomic decomposition A.1.8. If $\mu(\Omega \setminus \{x_n, n \in \mathbb{N}\}) > 0$ then we could find $x \in \Omega \setminus \{x_n, n \in \mathbb{N}\}$ and $\delta_x \perp \mu$, so we immediately conclude that $\mu^{n-a} = 0$. Then $\mu = \sum_n \mu(x_n)\delta_{x_n}$, and the set $\{x_n, n \in \mathbb{N}\}$ contains its accumulation points. In particular $\{x_n, n \in \mathbb{N}\} = \operatorname{supp}(\mu)$ is a compact subset of Ω. It is easy to see that, in this setting, if $(\phi_j)_{j\in\mathbb{N}}$ in $L^1(\mu)$ is bounded in norm and

$$\phi_j \xrightarrow{Y(\mu, e_{f_i, i\in\mathbb{N}})} \left(\nu_x, \lambda, \nu_x^\infty\right),$$

then $\lambda \ll \mu$, i.e. $\lambda = \sum_n \lambda(x_n)\delta_{x_n}$ (because the space is countable and compact). Also

$$\phi_j \rightharpoonup^* \phi = \left(\overline{\nu_x} + \overline{\nu_x^\infty}\frac{\lambda}{\mu}\right) d\mu$$

in $C(X)^*$, $X = \{x_n, n \in \mathbb{N}\}$. Assume also that $\phi_j \to \phi$ μ-strictly. This amounts to the following

$$\int_X f(\phi_j)d\mu \to \int_X \langle \nu_x, f\rangle + \langle \nu_x^\infty, f^\infty\rangle \frac{\lambda}{\mu} d\mu = \int_X f(\phi)d\mu, \tag{2.5.1}$$

where $f(z) = \sqrt{1+|z|^2}$. f is a strictly convex function, therefore the inequality

$$f(x+y) = f(x) + \left.\frac{f(x+ty)-f(x)}{t}\right|_{t=1} \leq f(x) + f^\infty(y)$$

is strict unless $y = 0$. *We have*

$$\langle \nu_x, f \rangle + \langle \nu_x^\infty, f^\infty \rangle \frac{\lambda}{\mu} \geq f(\overline{\nu_x}) + f^\infty \left(\overline{\nu_x^\infty} \frac{\lambda}{\mu} \right) > f \left(\overline{\nu_x} + \overline{\nu_x^\infty} \frac{\lambda}{\mu} \right) = f(\phi)$$

unless $\overline{\nu_x^\infty} = 0$ λ*-a.e. So* $\phi = \overline{\nu_x}$ μ*-a.e., and using convexity once again in* (2.5.1), *and the fact that* $f^\infty = 1$, *we get*

$$f(\phi) = \langle \nu_x, f \rangle + \langle \nu_x^\infty, f^\infty \rangle \frac{\lambda}{\mu} \geq f(\overline{\nu_x}) + \frac{\lambda}{\mu} = f(\phi) + \frac{\lambda}{\mu}.$$

So $\lambda = 0$, *which implies that the sequence* ϕ_j *does not concentrate. Moreover,* $\phi(x) = \overline{\nu_x}$, *which means that* $\phi_j \to \phi$ *in measure. Then* $\phi_j \to \phi$ *strongly in* $L^1(\mu)$, *and Theorem 2.5.5 cannot be true.*

Chapter 3

Characterisation of Gradient Young Measure on separable compactifications

In this section, we show that generalised Gradient Young Measures $\nu = (\nu_x, \lambda, \nu_x^\infty)$ which are integrable,

$$\int_\Omega \langle \nu_x, |\cdot| \rangle dx + \int_{\overline{\Omega}} \langle \nu_x^\infty, |\cdot| \rangle d\lambda < +\infty$$

are characterised by a set of integral inequalities that include quasi-convex functions $f\colon \mathbb{R}^{m\times n} \to \mathbb{R}$ and their extension f^∞ on certain compactifications. A characterisation result was previously obtained in [68] the context of the sphere compactification and it is here extended to separable compactifications. See also [12, 67] for a similar characterisation in the setting of $\mathcal{A}$-free vector fields.

This chapter is the second part of a paper submitted for publication, the preprint can be found in [98].

3.1 Non-separability of the space of quasi-convex functions

We start this chapter by showing that the set of quasi-convex functions having linear growth is not separable. The lack of separability prevents sequential compactness and other essential properties that were used to develop the theory of generalised Young Measures. Therefore, we are forced to consider only smaller countable collections of quasi-convex functions at the time, and cannot work with the entire class. To show that the class is not separable, we modify the example given by Muller in [84]

and generate quasi-convex functions that oscillate at different amplitudes in the same direction.

Theorem 3.1.1. *The space of quasi-convex functions $f\colon \mathbb{R}^{2\times 2} \to \mathbb{R}$ having linear growth is not separable with respect to $\|T\cdot\|_{\infty,\mathbb{B}^{2\times 2}}$.*

We remind that

$$T\colon \mathbb{G}_1(\mathbb{R}^{m\times n}) \to C(\mathbb{B}^{m\times n}), f \mapsto Tf = f\left(\frac{\cdot}{1-|\cdot|}\right)(1-|\cdot|).$$

The core of the proof is to construct an uncountable family $\{f_\Lambda\}_\Lambda$ so that $\|Tf_\Lambda - Tf_\Gamma\|_\infty \geq c$ for some universal constant $c > 0$ and all $\Lambda \neq \Gamma$. We will split the proof of the above result into two different parts. First, we show that we have quasi-convex functions that oscillate along every possible sequence of natural numbers.

Proposition 3.1.2. *There is $c > 0$ such that for every $\Lambda \subset \mathbb{N}$ infinite with Λ^c also infinite there exists $f_\Lambda\colon \mathbb{R}^{2\times 2} \to \mathbb{R}$ quasi-convex and having linear growth such that*

1. *$f_\Lambda(3^j\mathbb{1}) = 0$ for all $j \in \Lambda$ and*
2. *$\frac{f_\Lambda(3^j\mathbb{1})}{3^j} \geq c$ for all $j \in \Lambda^c$ sufficiently large.*

To prove this theorem we first need the following preliminary lemma.

Lemma 3.1.3 ([84], Lemma 4)**.** *For $k > 0$ we let*

$$g_k\colon \mathbb{R}^{2\times 2} \to \mathbb{R}, F \mapsto |F_{1,1} - F_{2,2}| + |F_{1,2} + F_{2,1}| + (2k - |F_{1,1} + F_{2,2}|)^+.$$

There exists $c_1 > 0$ such that

$$Qg_k(0) \geq c_1 k$$

for all sufficiently large ks.

We can then prove Proposition 3.1.2.

Proof. Let $\Lambda \subset \mathbb{N}$ infinite and set $f_\Lambda = Qg_\Lambda$, where

$$g_\Lambda(F) = |F_{1,1} - F_{2,2}| + |F_{1,2} + F_{2,1}| + \inf\{|F_{1,1} + F_{2,2} - 2\cdot 3^i|, i \in \Lambda\}.$$

We have that $g_\Lambda(3^j\mathbb{1}) = 0$ for all $j \in \Lambda$ and so $f_\Lambda(3^j\mathbb{1}) = 0$ as well. We first derive the following lower bound: compute

$$g_\Lambda(3^j\mathbb{1} + G) = |G_{1,1} - G_{2,2}| + |G_{1,2} + G_{2,1}| + \inf\{|G_{1,1} + G_{2,2} + 2(3^j - 3^i)| : i \in \Lambda\};$$

because 3^j is increasing we estimate, for arbitrary $\beta \in \mathbb{R}$,

$$\inf_{i\in\Lambda} |2(3^j - 3^i) + \beta| \geq \left(\inf_{i\neq j} |2(3^j - 3^i)| - |\beta|\right)^+ = \left(2(3^j - 3^{j-1}) - |\beta|\right)^+,$$

and so $g_\Lambda(3^j\mathbb{1} + G) \geq g_k(G)$ for all matrices G and $k = 3^j - 3^{j-1}$, and g_k given by

$$g_k(F) = |F_{1,1} - F_{2,2}| + |F_{1,2} + F_{2,1}| + (2k - |F_{1,1} + F_{2,2}|)^+$$

does not depend on Λ. Apply Lemma 3.1.3 to find $c > 0$ (independent of Λ) such that

$$f_\Lambda(3^j\mathbb{1}) = Qg_\Lambda(3^j\mathbb{1}) \geq Qg_k(0) \geq c_1(3^j - 3^{j-1})$$

for j big enough. Dividing everything by 3^j, we obtain the bound

$$\frac{f_\Lambda(3^j\mathbb{1})}{3^j} \geq \frac{2}{3}c_1 \equiv c$$

and conclude the proof. □

It is not a priori clear if these functions are "far away from each other at infinity", given that different subsets of natural numbers could intersect infinitely many times. To show that there is a wide variety of sequences that differ at infinity, we start by defining the following relation.

Definition 3.1.4. *Given two sequences Γ, Λ (not necessarily subsets of $\mathbb{N}$), we say that $\Gamma \leq \Lambda$ provided Γ is eventually a subset of Λ, i.e.*

$$\Lambda = (a_i)_{,i\in\mathbb{N}}, \Gamma = (b_i)_{i\in\mathbb{N}}, \quad \Lambda \leq \Gamma$$

if and only if there exists $k > 0$ such that $(a_i)_{i\geq k}$ is a subsequence of $(b_i)_{i\geq 0}$.

Let $[\Lambda]$ be the equivalence class of Λ with respect to $\leq$, i.e.

$$\Gamma \in [\Lambda] \quad \text{if} \quad \Gamma \leq \Lambda \leq \Gamma.$$

If $\Lambda' \in [\Lambda]$ and $\Gamma' \in [\Gamma]$ then $\Lambda \leq \Gamma$ if and only if $\Lambda' \leq \Gamma'$. We use $\left(\mathcal{F}, \leq\right)$ to indicate the set of equivalence classes with the inherited order.

Our strategy is to use Zorn's Lemma to find a maximal set, and then show that such a set is uncountable. We remark that one could also reason that the Stone-Cech compactification of natural numbers is not metrisable and prove by contradiction using a suitably adapted version of the Young Measure representation 2.3.8.

Lemma 3.1.5. *There exists an uncountable family $G \subset \mathcal{F}$ such that for every different pair $\Lambda, \Gamma \in G$, Λ is not comparable to either Γ nor Γ^c with respect to $\leq$.*

The above means that we can find a set G such that given any two sequences of natural numbers in $\Lambda, \Gamma \in G$, one is frequently in the other sequence and its complement, i.e. $\Lambda \cap \Gamma$ and $\Lambda \cap \Gamma^c$ are both infinite.

Lemma 3.1.6. *Given any countable or finite collection of infinite natural numbers $\{c_i^j, i \in \mathbb{N}\}_{j\in\mathbb{N}}$ in $\mathbb{N}$, there is $\{c_i, i \in \mathbb{N}\} \subset \mathbb{N}$ such that*

$$\{c_i^j, i \in \mathbb{N}\} \cap \{c_i, i \in \mathbb{N}\} \quad \textit{and} \quad \{c_i^j, i \in \mathbb{N}\} \setminus \{c_i, i \in \mathbb{N}\}$$

are both infinite subsets of $\mathbb{N}$.

The statement is equivalent to finding $\{c_i, \in \mathbb{N}\}$ such that

$$\begin{aligned} &\{c_i^j, i \in \mathbb{N}\} \setminus \{c_i, i \in \mathbb{N}\} < \{c_i^j, i \in \mathbb{N}\} \quad \text{and} \\ &\{c_i^j, i \in \mathbb{N}\} \cap \{c_i, i \in \mathbb{N}\} < \{c_i^j, i \in \mathbb{N}\} \end{aligned}$$

for all $j \in \mathbb{N}$.

Proof. Consider the isomorphism

$$N \colon \mathcal{F} \to \{0,1\}^{\mathbb{N}}, \{c_k, k \in \mathbb{N}\} \mapsto \left(Nc_i = \begin{cases} 1 & \text{if } i \in \{c_k, k \in \mathbb{N}\} \\ 0 & \text{otherwise} \end{cases} \right)_{i\in\mathbb{N}}$$

We are replacing subsets of natural numbers to sequences that take values 1 when the i-th number is in the subset, 0 otherwise, i.e. $Nc_i = \chi_{i\in\{c_k,k\in\mathbb{N}\}}$ is the indicator function.

Set initially $Nc_i = 0$ for all i. At i_1 so that $Nc_{i_1}^1 = 1$ for the first time put $Nc_{i_1} = 1$. We then iterate diagonally in the following way. At the n-th iteration find i_{n+1} so that $Nc_{k_j}^j = 1$ for all $1 \leq j \leq n$ and some $i_n < k_j$ with k_j increasing and $Nc_{t_j}^j = 1$ for all $1 \leq j \leq n+1$ and $k_n < t_j$ with t_j increasing and $t_{n+1} = i_{n+1}$. Set $Nc_{t_j} = 1$ for all $1 \leq j \leq n+1$. Iterate countably many times.

This way we guarantee that $Nc_{k_j} = 0$ for $i_n < k_j$ and $Nc_{k_j}^j = 1$, which means that Nc_i skips infinitely many 1s from each sequence $(Nc_i^j)_{i\in\mathbb{N}}$, for all $j \in \mathbb{N}$. On the other side $Nc_{t_j} = 1 = Nc_{t_j}^j$, $t_j \leq i_{n+1}$, so Nc_i is also frequently in every sequence $(Nc_i^j)_{i\in\mathbb{N}}$. □

Now we can prove the main lemma.

Proof. (of Lemma 3.1.5).

Consider the set of subsets

$$\mathcal{G} = \{G \subset \mathcal{F} : \text{no pair within } G \text{ is comparable according to } \leq\},$$

ordered by inclusion $\subset$. The above set is non-empty, which can be seen by taking $\Lambda = 2\mathbb{N}$ and $\Gamma = 4\mathbb{N} \cup (4\mathbb{N}+1)$. Every chain in $\mathcal{G}$ has an upper limit given by its union. By Zorn's Lemma, there exists a maximal element $\overline{G} \subset \mathcal{G}$ which we assume is countable or finite. Writing $\overline{G} = \left\{\{a_i^j, i \in \mathbb{N}\}, j \in \mathbb{N}\right\}$, we can apply the previous lemma to find $\{c_i, i \in \mathbb{N}\} \in \mathcal{F}$ generated by the countable family

$$\{c_i^j, i \in \mathbb{N}\}_{j\in\mathbb{N}} = \left\{\{a_i^j, i \in \mathbb{N}\}\ ,\ \mathbb{N} \setminus \{a_i^j, i \in \mathbb{N}\}\right\}_{j\in\mathbb{N}}$$

Because $\{c_i, i \in \mathbb{N}\}$ is frequently and properly in $\{a_i^j, i \in \mathbb{N}\}$ and its complement $\mathbb{N} \setminus \{a_i^j, i \in \mathbb{N}\}$ for all j, then $\{c_i, i \in \mathbb{N}\}$ is not comparable to any member of the family and this contradicts the maximality of our set. □

We are now ready to prove Lemma 3.1.2.

Proof. (of Lemma 3.1.2).

By the lemma, we can find an uncountable set of uncomparable subsequences of $\mathbb{N}$, call it $\overline{G}$. I claim that if $\Lambda, \Gamma \in \overline{G}, \Lambda \neq \Gamma$ then f_Λ and f_Γ have different recessions at infinity. Find $\{x_n, n \in \mathbb{N}\} \in e_{f_\Lambda}$ with $\{x_n, n \in \mathbb{N}\} \geq \{3^j\mathbb{1}, j \in \Lambda\}$, where the inequality could be strict given that there might be more zero points at infinity. Given our construction, we immediately have that $\{x_n, n \in \mathbb{N}\} \not\geq \{3^j\mathbb{1}, j \in \mathbb{N} \setminus \Lambda\}$ as $f_\Lambda(3^j\mathbb{1}) \geq c3^j$ for all $j \in \mathbb{N} \setminus \Lambda$. Also, Γ intersects $\mathbb{N} \setminus \Lambda$ and Λ infinitely many times, and vice versa, so that

$$\limsup_n \frac{f_\Gamma(x_n)}{|x_n|} \geq \limsup_{j\in\Lambda\cap\Gamma} \frac{f_\Gamma(3^j\mathbb{1})}{3^j} \geq c, \quad \text{and} \quad \liminf_n \frac{f_\Gamma(x_n)}{|x_n|} \leq \liminf_{j\in\Lambda\cap\Gamma} \frac{f_\Gamma(3^j\mathbb{1})}{3^j} = 0,$$

which shows that $\{x_n, n \in \mathbb{N}\} \notin e_{f_\Gamma}$. In terms of the non-separability, by the definition of x_n we have $\lim_n \frac{f_\Lambda(x_n)}{|x_n|} = 0$, thus

$$\|Tf_\Lambda - Tf_\Gamma\|_\infty \geq \limsup_n \left|\frac{f_\Gamma(x_n)}{|x_n|} - \frac{f_\Lambda(x_n)}{|x_n|}\right| = \limsup_n \left|\frac{f_\Gamma(x_n)}{|x_n|}\right| \geq c,$$

where c is independent of Λ or Γ. □

Given that the space is metric, it can not be separable. We can end this section with the following corollary that covers the higher dimensional case.

Corollary 3.1.7. *The set of quasi-convex functions having linear growth $f\colon \mathbb{R}^{m\times n} \to \mathbb{R}$ is separable in the topology induced by $\|T(\cdot)\|_\infty$ if and only if $min(m,n) = 1$.*

Proof. If m or n is 1, quasi-convex functions are convex and therefore the space is separable. This is because convex functions admit a limit at infinity in every direction; in this case, we actually recover the sphere compactification.

On the other side, for a function $g\colon \mathbb{R}^{2\times 2} \to \mathbb{R}$ we let

$$gP \equiv g \circ P\colon \mathbb{R}^{n\times m} \to \mathbb{R}, \quad P\colon \mathbb{R}^{m\times n} \to \mathbb{R}^{2\times 2}, M \mapsto \begin{bmatrix} M_{1,1}, M_{1,2} \\ M_{2,1}, M_{2,2} \end{bmatrix}.$$

If g is quasi-convex and locally bounded we let $\phi \in \mathcal{D}(Q, \mathbb{R}^m)$ and compute

$$\begin{aligned}\int_Q gP(\nabla\phi + z) &= \int_{Q\subset\mathbb{R}^{n-2}} d\mathcal{L}^{n-2} \int_{[0,1]^2} dx_1 x_2 g(P\nabla\phi + Pz) \\ &\geq \int_{Q\subset\mathbb{R}^{n-2}} d\mathcal{L}^{n-2} gP(z) = gP(z),\end{aligned}$$

which means that gP is quasi-convex. Then the set $f_\Lambda P$ is uncountable and

$$\|T(f_\Lambda P - f_\Gamma P)\|_{\infty,\mathbb{B}^{m\times n}} = \|T(f_\Lambda - f_\Gamma)\|_{\infty,\mathbb{B}^{2\times 2}} \geq c$$

if $\Gamma \neq \Lambda$, so the space cannot be separable. □

3.2 Characterisation of Gradient Young Measures

In this section, we characterise Gradient Young Measures on separable compactification via certain Jensen-like integral inequalities. It is worth mentioning that the result for the sphere compactification, achieved in [68], can be easily improved in consideration of the fact that $\limsup_{t\to\infty} f(tz)t^{-1}$ is 1-homogeneous rank one convex and so convex at points $\text{rank}(z) = 1$ (see Theorem A.3.3, taken from [62]). In what follows, we cannot use this type of auto-convexity. In our context, f^∞ lives on a separable compactification and convexity at points of rank one, in the form of a Jensen-type inequality, is not necessarily true anymore.

To prove our result, we will adopt the same strategy as in [67]. Preliminarily to stating the theorem, we define the upper recession of a function relative to a separable compactification.

Definition 3.2.1. *Let $e_{f_i, i\in\mathbb{N}}$ be a separable compactification. For any f having linear growth, we define*

$$f^{\sharp, e_{f_i,i\in\mathbb{N}}}((z_n)) = \sup_{(w_n)_n \in [(z_n)_n]} \limsup_n Tf(w_n),$$

where $(w_n)_n$ are sequences belonging to the equivalence class of $(z_n)_n$ within $e_{f_i,i\in\mathbb{N}}$.

The reason why we introduce this notion is that the strategy for proving the characterisation theorem makes use of the trivial fact that $f \geq f^{qc}$, where f^{qc} is the quasi-convex envelope of f. Dacorogna's formula [32] gives the representation

$$f^{qc}(z) = \inf_{\phi \in \mathcal{D}(\Omega)} \frac{1}{|\Omega|} \int_{\Omega} f(z + \nabla\phi(x))dx$$

under suitable assumptions on f. However, f^{qc} needs not to live in the same class of separable quasi-convex functions, so

$$\lim_n \frac{f(z_n)}{|z_n|} \text{ exists} \quad \not\Rightarrow \quad \lim_n \frac{f^{qc}(z_n)}{|z_n|} \quad \text{in general.}$$

Fix any compactification $e_{f_i, i\in\mathbb{N}}$ and a function $Tg \in e_{f_i,i\in\mathbb{N}}$. If $g \geq f$ then

$$g^{\infty}((z_n)_n) = \lim_n Tg(z_n) \geq \sup_{(z_n)_n \in [(z_n)_n]} \limsup_n Tf(z_n) = f^{\sharp, e_{f_i,i\in\mathbb{N}}}((z_n)_n).$$

$f^{\sharp, e_{f_i,i\in\mathbb{N}}}$ does not need to be continuous on $e_{f_i,i\in\mathbb{N}}$ with respect to its product topology. However, we can show that it is still upper semi-continuous.

Lemma 3.2.2. *Let $e_{f_i,i\in\mathbb{N}}$ be a separable compactification and $f\colon \mathbb{R}^d \to \mathbb{R}$ a continuous function having linear growth, then $f^{\sharp, e_{f_i,i\in\mathbb{N}}}$ is upper semi-continuous on $e_{f_i,i\in\mathbb{N}}$.*

Proof. Notice that $e_{f_i,i\in\mathbb{N}}$ is metrisable, so it is enough to check sequential upper semi-continuity. Moreover, we only have to prove continuity on $\partial e_{f_i,i\in\mathbb{N}}$. Let $(z_n^j)_n \in e_{f_i,i\in\mathbb{N}}$ so that $(z_n^j)_n \xrightarrow{j} (z_n)_n \in \partial e_{f_i,i\in\mathbb{N}}$. By the very definition of $g^{\sharp, e_{f_i,i\in\mathbb{N}}}((z_n^j)_n)$, for fixed $\varepsilon > 0$ we can find $n_j \geq k_j$, where k_j is a natural number to be selected later, so that $g^{\sharp, e_{f_i,i\in\mathbb{N}}}((z_n^j)_n) \leq \varepsilon + Tg((z_{n_j}^j))$, where $(z_{n_j}^j)$ is a constant sequence and belongs to $e_{f_i,i\in\mathbb{N}}$. We want to show that we can select k_j so that $(z_{n_j}^j)_j$ belongs in the equivalence class of $(z_n)_n$. By applying the dominated convergence theorem we get

$$\lim_j \sum_i \lim_n 2^{-i}|f_i(z_n^j) - f_i(z_n)| = 0 = \lim_j \lim_n \sum_i 2^{-i}|f_i(z_n^j) - f_i(z_n)|,$$

and so can find k_j so that

$$\sum_i 2^{-i}|f_i(z_n^j) - f_i(z_n)| \leq \varepsilon_j \quad \forall n \geq k_j,$$

where $0 \leq \varepsilon_j \downarrow 0$. This shows that the above sequence $(z_{n_j}^j)_j \in [(z_n)_n]$ (the equivalence class), thus

$$\limsup_j g^{\sharp, e_{f_i,i\in\mathbb{N}}}((z_n^j)_n) \leq \varepsilon + \limsup_j Tg((z_{n_j}^j)) \leq \varepsilon + g^{\sharp, e_{f_i,i\in\mathbb{N}}}((z_n)_n).$$

By the arbitrariness of $\varepsilon > 0$ we conclude upper semi-continuity of $g^{\sharp, e_{f_i,i\in\mathbb{N}}}$ □

In particular, $g^{\sharp,e_{f_i,i\in\mathbb{N}}}$ is Borel measurable on $e_{f_i,i\in\mathbb{N}}$. The above statement can be generalised to extensions of functions over more general compact metric spaces, but this version suffices for our scopes.

We are interested in studying those Young Measures that are generated by gradients. So we define the following class.

Definition 3.2.3. *We say that $\nu \in Y(e_{f_i,i\in\mathbb{N}})$ is a generalised Gradient Young Measure if there exists a sequence $(u_j)_j$ in $BV(\Omega,\mathbb{R}^m)$ such that*

$$Du_j \xrightarrow{Y(e_{f_i,i\in\mathbb{N}})} \left(\nu_x, \lambda, \nu_x^\infty\right).$$

We write $GY(e_{f_i,i\in\mathbb{N}})$ to refer to this subset of generalised Young Measures.

The convergence of measure derivatives has not been fully comprehended yet and it is still the subject of active research. This means that it is not so clear how rich the above class is, how gradients oscillate and concentrate - at least with respect to the weak* convergence of measures. We will partially answer the first question in the next chapter.

Remark 3.2.4. *We can use the characterisation lemma for Young Measure on the sphere to show that the class of generalised Gradient Young Measures is numerous. Fix any $z = a \otimes b \in \mathbb{R}^{m\times n}$ and $\nu^\infty \in \mathcal{M}_1^+(\partial\mathbb{B}^{m\times n})$ with $\overline{\nu^\infty} = z$. The following Gradient Young Measure on the sphere*

$$\left(\delta_0, \mathcal{H}^{n-1} \llcorner (\mathbb{B}^n \cap b^\perp), \nu^\infty\right)$$

satisfies the characterisation theorem from [68], with $u = a\chi_{x\cdot b\geq 0}$ and so it is generated by a sequence of gradients of $(u_j)_j$ in $BV(\mathbb{B}_1(0))$. Because Du_j is bounded in BV, we can find a subsequence $(u_{j_k})_{k\in\mathbb{N}}$ such that

$$Du_{j_k} \xrightarrow{Y(e_{f_i,i\in\mathbb{N}})} \left(\delta_0, \mathcal{H}^{n-1} \llcorner (\mathbb{B}^n \cap b^\perp), \eta^\infty\right),$$

where $\pi_{\partial\mathbb{B}^{m\times n}}\eta^\infty = \nu^\infty$.

Using Corollary 2.4.6 to decompose η^∞ as $\eta^\infty = P_z d\nu^\infty, z \in \partial\mathbb{B}^d$, it remains an open question to understand how many probabilities P_z over subsequences $z_n \to z$ can be generated by gradients.

We now state the main theorem of this section.

Theorem 3.2.5. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and $e_{f_i, i\in\mathbb{N}}$ be a separable compactification of quasi-convex functions and consider a generalised Young Measure $\nu \in Y(e_{f_i,i\in\mathbb{N}})$ that satisfies $\lambda(\partial\Omega) = 0$.*

Then $\nu \in Y(e_{f_i,i\in\mathbb{N}})$ is a Young Measure generated by a sequence

$$(\phi_j \star (Du \llcorner \Omega) + \nabla u_j) \xrightarrow{Y(e_{f_i,i\in\mathbb{N}})} \left(\nu_x, \lambda, \nu_x^\infty\right),$$

where $u \in BV(\Omega, \mathbb{R}^m)$, $(u_j)_j$ is in $\mathcal{D}(\Omega, \mathbb{R}^m)$ and $\|u_j\|_1 \to 0$, and ϕ_j is any sequence of mollifiers with $\phi_j \rightharpoonup^* \delta_0$, if and only if there is $u \in BV(\Omega, \mathbb{R}^m)$ such that

1. *$\int_\Omega \langle \nu_x, |\cdot| \rangle dx < +\infty$, and for all f quasi-convex and having linear growth,*
2. *$f(\nabla u(x))dx \leq \langle \nu_x, f\rangle dx + \langle \nu_x^\infty, f^{\sharp, e_{f_i,i\in\mathbb{N}}}\rangle \frac{\lambda}{\mathcal{L}^n} dx$ and*
3. *$f^\infty\left(\frac{D^s u}{|D^s u|}\right) d|D^s u| \leq \langle \nu_x^\infty, f^{\sharp, e_{f_i,i\in\mathbb{N}}}\rangle d\lambda^s$,*

where the inequalities are in the sense of measures.

We divide the proof into different parts and put them together at the end of the chapter.

We can adjust the above theorem to fix the boundary of the converging sequence so that its trace equals u.

Lemma 3.2.6. *If $\nu \in Y(e_{f_i,i\in\mathbb{N}})$ is generated by a sequence*

$$\phi_j \star (Du \llcorner \Omega) + \nabla u_j$$

as above, there exists another sequence $(v_j)_j$ in $C^\infty(\Omega, \mathbb{R}^m) \cap W^{1,1}_u(\Omega, \mathbb{R}^m)$ such that

$$\nabla(v_j + u_j) \xrightarrow{Y(e_{f_i,i\in\mathbb{N}})} \nu.$$

In particular, $\nu \in GY(e_{f_i,i\in\mathbb{N}})$.

Proof. We can find $u_j \to u$ strictly in $BV(\Omega, \mathbb{R}^m)$ with $(u_j)_j$ in $C^\infty(\Omega, \mathbb{R}^m) \cap W^{1,1}_u(\Omega, \mathbb{R}^m)$, see for example [68], Lemma 1 for a proof of this fact. In the construction of the u_js just mentioned, it is possible to select $\phi_j \star (Du \llcorner \Omega)$ on $\Omega_{-\varepsilon}$ for j big enough, ϕ_j as in Theorem 3.2.5. Also, without loss of generality, we can assume that $|Du|(\partial\Omega_{-\varepsilon}) = |Du_j|(\partial\Omega_{-\varepsilon}) = 0$. Because the f_is are all Lipschitz, it is enough to test against $f \in \mathrm{Lip}(\mathbb{R}^{m\times n})$ with $\mathrm{Lip}(f) \leq 1$. We then compute

$$\begin{aligned}\int_\Omega |f(\phi_j \star (Du \llcorner \Omega) + \nabla v_j) - f(\nabla u_j + \nabla v_j)| dx &\leq \int_\Omega |\phi_j \star (Du \llcorner \Omega) - \nabla u_j| \\ &\leq \int_{\Omega\setminus\Omega_{-\varepsilon}} |\phi_j \star (Du \llcorner \Omega)| + |\nabla u_j|.\end{aligned}$$

By strict convergence of both integrands, we have that

$$\limsup_j \int_\Omega |f(\phi_j \star (Du \llcorner \Omega) + \nabla v_j) - f(\nabla u_j + \nabla v_j)|dx \leq 2|Du|(\Omega \setminus \Omega_{-\varepsilon}),$$

and so by a diagonal argument conclude the existence and equality of the limit Young Measure. □

It is implicit in Theorem 3.2.5 that

$$Du = \overline{\nu} = \langle \nu_x, \cdot \rangle dx + \langle \nu_x^\infty, \cdot \rangle d\lambda.$$

Also, the above inequalities can be written as one inequality in the sense of distribution, in the form

$$\begin{aligned} &\int_\Omega \phi(x) \langle \nu_x, f \rangle dx + \int_\Omega \phi(x) \langle \nu_x^\infty, f^{\sharp, e_{f_i}, i \in \mathbb{N}} \rangle d\lambda \\ &\qquad \geq \int_\Omega \phi(x) f(\nabla u(x)) dx + \phi(x) f^\infty \left(\frac{D^s u}{|D^s u|} \right) d|D^s u| \end{aligned}$$

for all $\phi \in \mathcal{D}(\Omega), \phi \geq 0$.

To prove the characterisation result we will follow the same strategy as in [67]. We initially prove the result for homogeneous Gradient Young Measures and then extend the theorem to the inhomogeneous case.

3.2.1 Homogeneous Gradient Young Measures

Notice that Young Measures that act on functions $f = f(z)$ that only depend on z can be represented by

$$\int_\Omega \langle \nu_x, f \rangle dx + \int_{\overline{\Omega}} \langle \nu_x^\infty, f^\infty \rangle d\lambda = \int_{\mathbb{R}^{m \times n}} f d\nu^0 + \int_{\partial e_{f_i}, i \in \mathbb{N}} f^\infty d\nu^\infty,$$

where

$$\nu^0 = \nu_x d\mathcal{L}^n \in \mathcal{M}^+(\mathbb{R}^{m \times n}) \quad \text{and} \quad \nu^\infty = \nu_x^\infty d\lambda \in \mathcal{M}^+(\partial e_{f_i, i \in \mathbb{N}}).$$

The (push-forward) Kantorovich metric is then

$$\|(\nu^0, \nu^\infty)\|_K = \sup_{\Phi \in H, \|T\Phi\|_{\mathrm{Lip}(e_{f_i, i \in \mathbb{N}})} \leq 1} \left| \int_{\mathbb{R}^{m \times n}} \Phi d\nu^0 + \int_{e_{f_i}, i \in \mathbb{N}} \Phi^\infty d\nu^\infty \right|.$$

For $z \in \mathbb{R}^{m \times n}$ we let $\mathcal{Y} = \mathcal{Y}(z)$ be the set of pairs $\left(\nu^0, \nu^\infty\right) \in \mathcal{M}_1^+(\mathbb{R}^{m \times n}) \times \mathcal{M}^+(e_{f_i, i \in \mathbb{N}})$ such that there is a sequence $(u_j)_j$ in $\mathcal{D}(Q, \mathbb{R}^m)$, where Q is the unit cube, so that $z + \nabla u_j \xrightarrow{Y(e_{f_i, i \in \mathbb{N}})} \left(\nu^0, \nu^\infty\right)$ and $\|u_j\|_1 \to 0$. The following proposition follows from simple variations of the proofs contained in [67], Lemmas 5.2, 5.3, and 5.4, which we combine in one lemma.

Lemma 3.2.7. *The family* $\{\varepsilon_{z+\nabla u} : u \in \mathcal{D}(Q,\mathbb{R}^m)\}$ *is weakly* dense in* $\mathcal{Y}$*, and* $\mathcal{Y}$ *is a weak* closed and convex subset of homogeneous Young Measures.*

Proof. **Step 1: weak* density**. It is enough to prove that for all $u \in \mathcal{D}(Q,\mathbb{R}^m)$, $\varepsilon_{z+\nabla u}$ belongs to $\mathcal{Y}$. To show this, extend u to 0 outside of Q and divide each side of Q into $j \in \mathbb{N}$ disjoint congruent open intervals, and consider the resulting mesh of j^n disjoint congruent cubes. Fix an arrangement of these cubes

$$Q_i = x_i + j^{-1}Q, \quad i = 1, \dots, j^n,$$

and define

$$\phi_j(x) = \sum_{i=1}^{j^n} \frac{u(j(x - x_i))}{j} \in \mathcal{D}(Q,\mathbb{R}^m).$$

For any $\Phi \in e_{f_i, i\in\mathbb{N}}$ we have

$$\int_Q \Phi(z + \nabla\phi_j(x))dx = \int_Q \Phi(z + \nabla u(x))dx,$$

and also $\|\phi_j\|_1 \le j^{-1}\|u\|_1 \to 0$.

Step 2: weak* closedness. Take a pair $(\nu^0, \nu^\infty) \in \overline{Y}^{w*}$, which is the weak* closure of Y within the set of measures $(e_{f_i,i\in\mathbb{N}})^*$. Take $(\Phi_j)_j$ to be a countable dense subset of $e_{f_i,i\in\mathbb{N}}$, where $|\cdot| = \Phi_j$ for some j. For each $i \in \mathbb{N}$ we can find $v_i \in \mathcal{D}(Q,\mathbb{R}^m)$ so that

$$\|v_i\|_1 + \max_{0\le i\le j} \left| \int_Q \Phi_j(z + \nabla v_i(x))dx - \int_{\mathbb{R}^m} \Phi d\nu^0 - \int_{\partial e_{f_i,i\in\mathbb{N}}} \Phi^\infty d\nu^\infty \right| \le \frac{1}{i}.$$

Because, by construction, $(z + \nabla v_i)_i$ is bounded in L^1, for every subsequence there is a further subsequence that converges to a functional on $(e_{f_i,i\in\mathbb{N}})^*$. By the density of $(\Phi_j)_j$ in $e_{f_i,i\in\mathbb{N}}$ we know that such functional is (ν^0, ν^∞), and so the full sequence converges to (ν^0, ν^∞), which then belongs to $\mathcal{Y}$.

Step 3: convexity. Instead of proving convexity in $\mathcal{Y}$, by the previous 2 steps it is enough to show that for $u_0, u_1 \in \mathcal{D}(Q,\mathbb{R}^m)$, setting

$$\varepsilon_0 = \varepsilon_{z+\nabla u_0} \quad \text{and} \quad \varepsilon_1 = \varepsilon_{z+\nabla u_1},$$

then $(1-t)\varepsilon_0 + t\varepsilon_1 \in \mathcal{Y}$ for $t = \left(\frac{p}{q}\right)^n$, where p and q are coprime and $p < q$. Similarly to step 1, we divide the side of Q by q many intervals and write

$$Q_i = x_i + q^{-1}Q, \quad i = 1, \dots, q^n,$$

and set, after extending u to 0 outside of Q,

$$\phi(x) = \sum_{i=1}^{p^n} \frac{u_1(q(x-x_i))}{q} + \sum_{i=1+p^n}^{q^n} \frac{u_0(q(x-x_i))}{q}.$$

Then $\phi \in \mathcal{D}(Q, \mathbb{R}^m)$ and it is readily seen that, for all $\Phi \in e_{f_i, i\in\mathbb{N}}$, then $\varepsilon_{z+\nabla\phi} = (1-t)\varepsilon_0 + t\varepsilon_1$. Conclude using step 1. □

We can now prove the main theorem in the case of homogeneous Gradient Young Measures.

Proposition 3.2.8. *Let $\nu = \left(\nu^0, \nu^\infty\right) \in M_1^+(\Omega) \times \mathcal{M}^+(\partial e_{f_i, i\in\mathbb{N}})$. Then $\nu \in \mathcal{Y}$ if and only if there is $z \in \mathbb{R}^{m\times n}$ such that*

$$\int_{\mathbb{R}^{m\times n}} f d\nu^0 + \int_{\partial e_{f_i, i\in\mathbb{N}}} f^{\sharp, e_{f_i, i\in\mathbb{N}}} d\nu^\infty \geq f(z)$$

for all $f\colon \mathbb{R}^{m\times n} \to \mathbb{R}$ quasi-convex and having linear growth.

Proof. Suppose that $\nu \in \mathcal{Y}$ and let $(z + \nabla u_j)_j$ with $(u_j)_j$ in $\mathcal{D}(Q, \mathbb{R}^m)$ be the generating sequence, i.e. for all $\Phi\colon \mathbb{R}^{m\times n} \to \mathbb{R}$ of linear growth and belonging to the compactification $e_{f_i, i\in\mathbb{N}}$ (in short, $\Phi \in T^{-1} e_{f_i, i\in\mathbb{N}}$),

$$\int_Q \Phi(z + \nabla u_j) dx \to \langle \nu, \Phi \rangle = \int_{\mathbb{R}^{m\times n}} \Phi d\nu^0 + \int_{\partial e_{f_i, i\in\mathbb{N}}} \Phi d\nu^\infty.$$

Fix an arbitrary f having linear growth and quasi-convex and let $e_{f, f_i, i\in\mathbb{N}}$ the bigger compactification containing both $\{f_i, i \in \mathbb{N}\}$ and f. Upon extracting a subsequence we have that

$$z + \nabla u_j \xrightarrow{Y(e_{f, f_i, i\in\mathbb{N}})} \left(\nu^0, \tilde{\nu}^\infty\right),$$

where we identify the Gradient Young Measure with its products as explained before. By quasi-convexity, we have

$$\int_{\mathbb{R}^{m\times n}} f d\nu^0 + \int_{\partial e_{f, f_i, i\in\mathbb{N}}} f^\infty d\tilde{\nu}^\infty = \limsup_j \int_Q f(z + \nabla u_j) dx \geq f(z).$$

On the other side, using the decomposition of angle concentration Young Measure decomposition 2.4.6 we also obtain that

$$\int_{\partial e_{f, f_i, i\in\mathbb{N}}} f^\infty d\tilde{\nu}^\infty = \int_{\partial e_{f_i, i\in\mathbb{N}}} \int f^\infty dP_{(z_n)_n} d\nu^\infty \leq \int_{\partial e_{f_i, i\in\mathbb{N}}} f^{\sharp, e_{f_i, i\in\mathbb{N}}} d\nu^\infty.$$

For the other implication, because $\mathcal{Y}$ is weakly* closed and convex, we can write $\mathcal{Y} = \cap H$ where H are half-spaces containing $\mathcal{Y}$. By the Hahn-Banach Theorem, each H can be written as

$$H = \{l \in H^* : l(\Phi) \geq t\} \quad \text{for some } \Phi \in T^{-1}e_{f_i, i\in\mathbb{N}}.$$

In particular, we can test the above inequality against $\varepsilon_{z+\nabla u}$ and get

$$t \leq \varepsilon_{z+\nabla u}(\Phi) \leq \int_Q \Phi(z + \nabla u)dx$$

for all $u \in \mathcal{D}(Q, \mathbb{R}^m)$. Passing to the infimum over all such u we deduce $t \leq \Phi^{qc}(z)$ and so

$$\langle \nu, \Phi \rangle = \int_{\mathbb{R}^{m\times n}} \Phi d\nu^0 + \int_{\partial e_{f_i, i\in\mathbb{N}}} \Phi^\infty d\nu^\infty$$

$$\geq \int_{\mathbb{R}^{m\times n}} \Phi^{qc} d\nu^0 + \int_{\partial e_{f_i, i\in\mathbb{N}}} (\Phi^{qc})^{\sharp, e_{f_i, i\in\mathbb{N}}} d\nu^\infty \geq \Phi^{qc}(z) \geq t$$

which shows that $\nu \in H$. □

We remind the reader that, even if Φ belongs to the compactification $e_{f_i, i\in\mathbb{N}}$, it is not guaranteed that its convex envelope belongs to the same compactification, and so, in general,

$$\Phi \geq \Phi^{qc} \quad \Rightarrow \quad \Phi^\infty \geq (\Phi^{qc})^{\sharp, e_{f_i, i\in\mathbb{N}}},$$

where $(\Phi^{qc})^{\sharp, e_{f_i, i\in\mathbb{N}}}$ is defined as in 3.2.1.

3.2.2 Inhomogenization

In what follows, we will prove a semi-approximation result for the absolutely continuous and singular parts separately and then put them together via Lemma 2.4.18. In each case, we will use a covering argument to break it down so that we can use the result for the homogeneous case, which was solved in the above section. This part is adapted from [67], where it was worked out on the sphere compactification.

Consider a family standard mollifier $\phi_t(x) = t^{n-1}\phi(\frac{x}{t}), t > 0$, where $\phi \in \mathcal{D}(Q)$ and let $M = \|\nabla\phi\|_\infty$. Also, unless otherwise specified, the norm on $\mathbb{R}^n$ is the maximum norm $\|x\| = \max_i |x_i|$.

Lemma 3.2.9. *Given $\varepsilon > 0$ there is $t_\varepsilon > 0$ such that for all $t \in (0, t_\varepsilon]$ there is $\varphi_t \in \mathcal{D}(\Omega, \mathbb{R}^m)$ with $\|\varphi_t\|_1 \leq \varepsilon$ so that*

$$\left| \int_\Omega \eta\Phi(0) + \eta\langle \Phi^\infty, \nu_x^\infty \rangle d\lambda^s - \int_\Omega \eta\Phi\big(\phi_t \star (\overline{\nu^\infty} d\lambda^s) + \nabla\varphi_t\big) dx \right| < \varepsilon$$

uniformly for all $\eta \colon \Omega \to \mathbb{R}$ and $\Phi \colon \mathbb{R}^{m\times n} \to \mathbb{R}$ satisfying

$$\|\eta\|_{\mathrm{Lip}} \leq 1 \quad \text{and} \quad \|T\Phi\|_{\mathrm{Lip},\mathrm{graph}(f)} \leq 1.$$

The idea behind this approximation is the following. The singular part of the centre of mass $\overline{\nu} = Du \in \mathcal{M}(\Omega, \mathbb{R}^{m\times n})$ is approximated by mollification. Such a procedure generates area-strictly convergent smooth approximations. At the same time, we generate angle concentration and oscillation via compactly supported functions. Because the first type of convergence is very strong, and the latter doesn't concentrate, the two modes of convergence don't interfere with each other. Notice that this strategy would not be possible using only weak* convergence because of the lack of quantifiability, whereas the Kantorovich metric gives us an exact quantity to approximate.

Before proving the above statement we remind that according to Lemma 2.4.13, T pulls back bounded sets of Lipschitz functions on $e_{f_i, i\in\mathbb{N}}$ to bounded sets of Lipschitz functions on $\mathbb{R}^{m\times n}$ (provided $f_i \colon \mathbb{R}^{m\times n} \to \mathbb{R}$ are all Lipschitz continuous). Therefore, all the functions in the following theorem can be taken to be, after renormalisation, of Lipschitz constant less than or equal to 1 in both spaces.

From now on, after fixing a compactification, we will always identify

$$\|T\Phi\|_{\mathrm{Lip}} = \|T\Phi\|_{\mathrm{Lip}(e_{f_i,i\in\mathbb{N}})} = \|T\Phi\|_\infty + \sup_{x\neq y} \frac{|T\Phi(x) - T\Phi(y)|}{|x-y| + \sum_i 2^{-i}|Tf_i(x) - Tf_i(y)|}.$$

Proof. Fix $\varepsilon > 0$ and apply Lusin's Theorem to the λ^s-measurable map

$$x \in \overline{\Omega} \to (\delta_0, \nu_x^\infty) \in \mathcal{M}_1^+(\mathbb{R}^{m\times n}) \times \mathcal{M}^+(\partial e_{f_i,i\in\mathbb{N}}) \hookrightarrow \big((T^{-1}e_{f_i,i\in\mathbb{N}})^*\big)^+$$

to find a compact set $C = C_\varepsilon \subset \overline{\Omega}$ with $\lambda^s(\overline{\Omega} \setminus C) < \lambda^s(\overline{\Omega})\varepsilon$ restricted to which the above map is uniformly continuous, and has modulus of continuity $\omega = \omega_\varepsilon$. Without loss of generality, assume that $\mathcal{L}^n(C^s) = 0$ and because $\lambda^s(\partial\Omega) = 0$ then

$$\Delta = \Delta_\varepsilon = d(C^s, \partial\Omega) > 0.$$

For the moment, fix two integers $a, b \in \mathbb{N}$ and put $t = 2^{-a}$, so that $\phi_t > 0$ if and only if $\|x\| < t$. Let a be so large that

$$2t \leq \Delta \quad \text{and} \quad a \geq \log_2\left(\frac{2}{\Delta}\right).$$

Denote by $\mathcal{F}$ the collection of $a+b$-th generation dyadic cubes $\mathbb{Q}$ in $\mathbb{R}^n$ so that $d(\mathbb{Q},\partial\Omega)>2^{-a}$, and for each such $\mathbb{Q}\in\mathcal{F}$ we define

$$r_{\mathbb{Q}}=\fint_{\mathbb{Q}}\phi\star(\lambda^s\llcorner C^s)dx.$$

Notice that $r_{\mathbb{Q}}>0$ means that $dist(\mathbb{Q},C^s)<t$, and so for each such $\mathbb{Q}$ we can find $x_{\mathbb{Q}}\in C^s$ so that $d(x_{\mathbb{Q}},\mathbb{Q})<t$. Denote by $\mathcal{F}^s$ the set of those $\mathbb{Q}\in\mathcal{F}$ for which $r_{\mathbb{Q}}>0$. In particular, if $\mathbb{Q}\in\mathcal{F}^s$ we can find $x_{\mathbb{Q}}\in C^s$ so that $\sup_{\mathbb{Q}}\|x-x_{\mathbb{Q}}\|<2t$.

For every quasi-convex function having linear growth we have

$$f(z+w)\leq f(z)+f^{\infty}(w)$$

for all $z\in\mathbb{R}^{m\times n}$ and $\operatorname{rank}(w)=1$, see Theorem A.3.4. So we have the upper bound

$$f(r_{\mathbb{Q}}\overline{\nu^{\infty}_{x_{\mathbb{Q}}}})\leq f(0)+r_{\mathbb{Q}}f^{\infty}(\overline{\nu^{\infty}_{x_{\mathbb{Q}}}})\leq f(0)+r_{\mathbb{Q}}\int_{\partial e_{f_i},i\in\mathbb{N}}f^{\sharp,e_{f_i},i\in\mathbb{N}}d\nu^{\infty}_{x_{\mathbb{Q}}}$$

for all f quasi-convex and having linear growth.

Going back to the homogeneous result 3.2.8, we can select $\varphi^{\mathbb{Q}}\in\mathcal{D}(\mathbb{Q},\mathbb{R}^m)$ with $\|\varphi^{\mathbb{Q}}\|_1<\varepsilon\lambda^s(\mathbb{Q})$ such that

$$\|(\delta_0,\nu^{\infty}_{x_{\mathbb{Q}}}r_{\mathbb{Q}})-\varepsilon_{r_{\mathbb{Q}}\overline{\nu^{\infty}_{x_{\mathbb{Q}}}}+\nabla\varphi^{\mathbb{Q}}}\|_K<\varepsilon.$$

Define $\varphi=\sum_{\mathbb{Q}\in\mathcal{F}^d}\varphi^{\mathbb{Q}}\in\mathcal{D}(\Omega,\mathbb{R}^m)$ and $\|\varphi\|_1\leq\varepsilon\lambda^s(\Omega)$. We claim that the sought-after map is

$$\xi^s=\phi\star(\overline{\nu^{\infty}_x}d\lambda^s+\nabla\varphi)\in\mathcal{D}(\mathbb{R}^n,\mathbb{R}^{m\times n}).$$

To prove that this function is the desired one, we fix $\|\eta\|_{\mathrm{Lip}}\leq 1,\|\Psi\|_{\mathrm{Lip}(e_{f_i},i\in\mathbb{N})}\leq 1$ as in the statement. We have

$$\int_{\Omega}\eta\langle\Phi^{\infty},\phi\star(\overline{\nu^{\infty}}d\lambda^s)\rangle dx=\int_{\Omega}\eta\langle\Phi^{\infty},\phi\star(\overline{\nu^{\infty}}d\lambda^s\llcorner C^s)\rangle dx+\overbrace{\int_{\Omega}\eta\langle\Phi^{\infty},\phi\star(\overline{\nu^{\infty}}d\lambda^s\llcorner\Omega\setminus C^s)\rangle dx}^{=\mathcal{E}_1},$$

and $|\mathcal{E}_1|\leq\varepsilon\lambda^s(\Omega)$. Notice that here

$$\int_{\Omega}\eta\langle\Phi^{\infty},\phi\star(\nu^{\infty}d\lambda^s\llcorner U)\rangle dx=\int_{\Omega}\eta(x)\int_U\phi(x-y)\int_{\partial e_{f_i},i\in\mathbb{N}}\Phi^{\infty}d\nu^{\infty}_y d\lambda^s(y)dx$$

where $U = \Omega$ or C^s. Since for each $\mathbb{Q} \in \mathcal{F}$ with $r_{\mathbb{Q}} = 0$

$$\int_{\mathbb{Q}} \eta \langle \Phi^\infty, \phi \star (\nu^\infty d\lambda^s \llcorner C^s) \rangle dx = 0$$

and $dist(\cup \mathcal{F}, \partial\Omega) > 2t$, we get

$$\begin{aligned}\int_\Omega \langle \Phi^\infty, \phi \star (\nu^\infty d\lambda^s \llcorner C^s) \rangle dx &= \sum_{\mathbb{Q} \in \mathcal{F}^s} \int_{\mathbb{Q}} \eta \langle \Phi^\infty, \phi \star (\nu^\infty d\lambda^s \llcorner C^s) \rangle dx + \mathcal{E}_2 \\ &= \sum_{\mathbb{Q} \in \mathcal{F}^s} \left(\int_{\mathbb{Q}} \eta dx \langle \Phi^\infty, \nu^\infty_{x_{\mathbb{Q}}} \rangle r_{\mathbb{Q}} + \mathcal{E}_3^{\mathbb{Q}} \right) + \mathcal{E}_2,\end{aligned}$$

where $|\mathcal{E}_2| \leq \lambda^s(C^s \cap (\partial\Omega)_{2t})$. The third error is estimated in the following way:

$$\begin{aligned}|\mathcal{E}_3^Q| &\leq \left| \int_{\mathbb{Q}} \left(\eta - \int_{\mathbb{Q}} \eta \right) \langle \Phi^\infty, \phi \star (\nu^\infty d\lambda^s \llcorner C^s) \rangle dx \right| \\ &\quad + \left| \int_{\mathbb{Q}} \eta \left(\int_{\mathbb{Q}} \langle \Phi^\infty, \phi \star (\nu^\infty d\lambda^s \llcorner C^s) \rangle dx - \langle \Phi^\infty, \nu^\infty_{x_{\mathbb{Q}}} \rangle r_{\mathbb{Q}} \right) \right| \\ &\leq \|\eta\|_{\mathrm{Lip}} \, \mathcal{L}^n(\mathbb{Q})^{\frac{1}{n}} \, \|\Phi^\infty\| \int_{\mathbb{Q}} \phi \star \lambda^s dx \\ &\quad + \|\eta\|_{\mathrm{Lip}} \int_{\mathbb{Q}} \int_{C^s} \phi(x-y) \langle \Phi^\infty, \nu^\infty_y - \nu^\infty_{x_{\mathbb{Q}}} \rangle d\lambda^s(y) dx.\end{aligned}$$

In particular, we obtain

$$\begin{aligned}|\mathcal{E}_3^Q| &\leq t \int_{\mathbb{Q}} \phi \star \lambda^s dx + \int_{\mathbb{Q}} \int_{C^s} \phi(x-y) \omega(\|y - x_{\mathbb{Q}}\|) d\lambda^s(y) dx \\ &\leq (t + \omega(3t)) \int_{\mathbb{Q}} \phi \star \lambda^s dx.\end{aligned}$$

From each $\mathbb{Q} \in \mathcal{F}^s$ we get

$$\Phi(0) + \langle \Phi^\infty, \nu^\infty_{x_{\mathbb{Q}}} \rangle r_{\mathbb{Q}} = \int_{\mathbb{Q}} \Phi(r_{\mathbb{Q}} \overline{\nu^\infty_{x_{\mathbb{Q}}}} + \nabla \phi^{\mathbb{Q}}) dx + \overbrace{\mathcal{E}_4^{\mathbb{Q}}}^{|\cdot| \leq \varepsilon}.$$

Further computations show that

$$\begin{aligned}\int_{\mathbb{Q}} |\Phi(r_{\mathbb{Q}} \overline{\nu^\infty_{x_{\mathbb{Q}}}} + \nabla \phi^{\mathbb{Q}}) - \Phi(0)| dx &\leq \|\varepsilon_{r_{\mathbb{Q}} \overline{\nu^\infty_{x_{\mathbb{Q}}}} + \nabla \phi^{\mathbb{Q}}}\|_K \\ &\leq \|(\delta_0, \nu^\infty_{x_{\mathbb{Q}}} r_{\mathbb{Q}})\|_K + \varepsilon \leq 1 + r_{\mathbb{Q}} + \varepsilon\end{aligned}$$

and consequently

$$\int_{\mathbb{Q}} \eta \int_{\mathbb{Q}} \Phi(r_{\mathbb{Q}} \overline{\nu^\infty_{x_{\mathbb{Q}}}} + \nabla \phi^{\mathbb{Q}}) dx = \int_{\mathbb{Q}} \eta \Phi(r_{\mathbb{Q}} \overline{\nu^\infty_{x_{\mathbb{Q}}}} + \nabla \phi^{\mathbb{Q}}) dx + \mathcal{E}_5^{\mathbb{Q}},$$

where the error term is upper bounded by

$$|\mathcal{E}_5^{\mathbb{Q}}| \leq \sup_{\mathbb{Q}} \left|\eta - \int_{\mathbb{Q}} \eta\right| \mathcal{L}^n(\mathbb{Q})(1 + r_{\mathbb{Q}} + \varepsilon) \leq \mathcal{L}^n(\mathbb{Q})^{\frac{1}{n}} \int_{\mathbb{Q}} (2 + \phi \star \lambda^s) dx.$$

Finally, we estimate the last term with

$$\int_{\mathbb{Q}} \eta \Phi(r_{\mathbb{Q}} \overline{\nu^\infty_{x_{\mathbb{Q}}}} + \nabla \phi^{\mathbb{Q}}) dx = \int_{\mathbb{Q}} \eta \Phi(\phi \star \overline{\nu^\infty} d\lambda^s) + \nabla \phi^{\mathbb{Q}}) dx + \mathcal{E}_6^{\mathbb{Q}}.$$

To bound the 6th error term we introduce an extra quantity

$$\begin{aligned}
&\left| \int_{\mathbb{Q}} \eta \Big(\Phi(\phi \star \overline{\nu^\infty} d\lambda^s) + \nabla \phi^{\mathbb{Q}}) dx - \Phi(\phi \star \overline{\nu^\infty} d\lambda^s \llcorner C^s) + \nabla \phi^{\mathbb{Q}}) \Big) dx \right| \\
&\qquad \leq \int_{\mathbb{Q}} \phi \star (\lambda^s \llcorner \Omega \setminus C^s) dx,
\end{aligned}$$

and

$$\begin{aligned}
&\left| \int_{\mathbb{Q}} \eta \Big(\Phi(r_{\mathbb{Q}} \overline{\nu^\infty_{x_{\mathbb{Q}}}} + \nabla \phi^{\mathbb{Q}}) dx - \Phi(\phi \star \overline{\nu^\infty} d\lambda^s \llcorner C^s) + \nabla \phi^{\mathbb{Q}}) \Big) dx \right| \\
&\quad \leq \int_{\mathbb{Q}} |r_{\mathbb{Q}} \overline{\nu^\infty_{x_{\mathbb{Q}}}} - \phi \star (\overline{\nu^\infty} d\lambda^s \llcorner C^s)| dx \\
&\quad \leq \left| \int_{\mathbb{Q}} \phi \star \big((\overline{\nu^\infty_{x_{\mathbb{Q}}}} - \overline{\nu^\infty}) d\lambda^s \llcorner C^s) dx \right| \\
&\qquad + \int_{\mathbb{Q}} \left| \fint_{\mathbb{Q}} \phi \star (\overline{\nu^\infty} d\lambda^s \llcorner C^s) dx' - \phi \star (\overline{\nu^\infty} d\lambda^s \llcorner C^s) \right| dx \\
&\quad \leq \int_{\mathbb{Q}} \int_{C^s} \phi(x - y) |\overline{\nu^\infty_{x_{\mathbb{Q}}}} - \overline{\nu^\infty_y}| d\lambda^s dx \\
&\qquad + \int_{\mathbb{Q}} \left| \fint_{\mathbb{Q}} \phi \star (\overline{\nu^\infty} d\lambda^s \llcorner C^s) dx' - \phi \star (\overline{\nu^\infty} d\lambda^s \llcorner C^s) \right| dx \\
&\quad \leq \omega(3t) \int_{\mathbb{Q}} \phi \star \lambda^s dx + \mathcal{E}_7^{\mathbb{Q}}.
\end{aligned}$$

The 7th error term is estimated by

$$\begin{aligned}
\mathcal{E}_7^{\mathbb{Q}} &\leq \int_{\mathbb{Q}} \fint_{\mathbb{Q}} \int_{C^s} \big| \phi(x' - y) - \phi(x - y) \big| |\overline{\nu^\infty_y}| d\lambda^s(y) dx' dx \\
&\leq \sqrt{n} \int_{\mathbb{Q}} \fint_{\mathbb{Q}} \int_{\mathbb{Q} + tQ} \|x - x'\| \int_0^1 \big| \nabla \phi(x + (x' - x)\tau)(x' - x) \big| d\tau d\lambda^s(y) dx' dx.
\end{aligned}$$

Given the scaling of $\phi = \phi_t$ with respect to t, we have $|\nabla \phi| \leq t^{-1} M(\chi_{\mathbb{Q}})_t$, so the above can be bounded by

$$\mathcal{E}_7^{\mathbb{Q}} \leq \sqrt{n} M \frac{\mathcal{L}^n(\mathbb{Q})^{\frac{1}{n}}}{t} \int_{\mathbb{Q}} (\chi_{2Q})_t \star \lambda^s dx.$$

For our choice of a and b, we have that $\mathcal{L}^n(Q) = 2^{-n(a+b)}$. With ξ^s defined above we obtain that

$$\int_\Omega \eta\langle \Phi^\infty, \phi \star (\nu^\infty d\lambda^s)\rangle dx = \int_{\cup \mathcal{F}^s} \eta\Phi(\xi^s) + \mathcal{E},$$

where

$$\begin{aligned}|\mathcal{E}| &\leq \varepsilon\lambda^s(\Omega) + (t + \omega(3t)) \int_{\cup\mathcal{F}^s} \Big(\phi \star \lambda^s + 2^{-a-b}(2 + \phi \star \lambda^s) \\ &\quad + \phi \star (\lambda^s \llcorner \Omega \setminus C^s) + \omega(3t)\ \phi \star \lambda^s dx + \sqrt{n} M 2^{-b} (\chi_{2Q})_t \star \lambda^s\Big) dx \\ &\leq \big(2\varepsilon + 2^{-a} + 2\omega(32^{-a}) + 2^{-a-b} + c_n M 2^{-b}\big)\lambda^s(\Omega) + 2^{1-a-b}\mathcal{L}^n(\Omega).\end{aligned}$$

To conclude, we add

$$\int_{\Omega\setminus\cup\mathcal{F}^s} \eta\Phi(\xi^s)dx = \int_{\Omega\setminus\cup\mathcal{F}^s} \eta dx\Phi(0)$$

to both sides and obtain

$$\int_\Omega \eta\big(\Phi(0) + \langle \Phi^\infty, \phi \star (\nu^\infty d\lambda^s \llcorner \Omega)\rangle\big)dx = \int_\Omega \Phi(\xi^s)dx + \mathcal{E} + \int_{\cup\mathcal{F}^s} \eta dx\Phi(0).$$

Because $\cup\mathcal{F}^s \subset (C^s)_{2t}$ and since $\mathcal{L}^n(C^s) = 0$ we can find $a_\varepsilon \geq a, b_\varepsilon \geq b$ such that

$$|\mathcal{E}| + \left|\int_{\cup\mathcal{F}^s} \eta dx\Phi(0)\right| \leq 3\varepsilon(\mathcal{L}^n + \lambda^s)(\Omega).$$

The left-hand side tends to

$$\int_\Omega \eta dx\Phi(0) + \int_\Omega \eta\langle \Phi^\infty, \nu_x^\infty\rangle d\lambda^s(x)$$

as $a \to \infty$, uniformly in η and Φ, and this concludes the proof. □

We now move on to the absolutely continuous part, which is proven similarly and is slightly easier to construct.

Lemma 3.2.10. *Let $\varepsilon > 0$, there is $t_\varepsilon > 0$ such that for all $t \in (0, t_\varepsilon]$ there is $\psi_t \in \mathcal{D}(\Omega, \mathbb{R}^m)$ with $\|\psi_t\|_1 \leq \varepsilon$ so that*

$$\left|\int_\Omega \eta(\langle \Phi, \nu_x\rangle + \lambda^a(x)\langle \Phi^\infty, \nu_x^\infty\rangle)dx - \int_\Omega \eta\Phi\big(\phi_t \star (\overline{\nu} + \overline{\nu^\infty}\lambda^a(x))dx \llcorner \Omega + \nabla\psi_t\big)dx\right| < \varepsilon$$

uniformly for all $\eta\colon \Omega \to \mathbb{R}$ and $\Phi\colon \mathbb{R}^{m\times n} \to \mathbb{R}$ satisfying

$$\|\eta\|_{\mathrm{Lip}} \leq 1 \quad \text{and} \quad \|T\Phi\|_{\mathrm{Lip}(e_{f_i}, i\in\mathbb{N})} \leq 1.$$

Proof. Fix $\varepsilon \in (0,1)$ and apply Lusin's Theorem to the $\mathcal{L}^n$-measurable map

$$\Omega \ni x \mapsto (\nu_x, \lambda^a(x)\nu_x^\infty) \in \mathcal{M}_1^+(\mathbb{R}^{m\times n}) \times \mathcal{M}^+(\partial e_{f_i, i\in\mathbb{N}}) \hookrightarrow \big((T^{-1}e_{f_i,i\in\mathbb{N}})^*\big)^+$$

to find a compact set $C^a \subset \Omega$ such that

$$\int_{\Omega\setminus C^a} \overbrace{\langle \nu_x, |\cdot|\rangle + \lambda^a(x)}^{=M(x)} dx < \varepsilon$$

and such map is continuous on C^a, and let ω be the modulus of continuity

$$\|(\nu_x, \lambda^a(x)\nu_x^\infty) - (\nu_y, \lambda^a(y)\nu_y^\infty)\|_K \le \omega(\|x-y\|) \quad \forall\ x, y \in C^a.$$

Fix $d \in \mathbb{N}$ and $s \in (0,1)$ and let $\mathcal{F}^a$ be the family of dyadic cubes in $\mathbb{R}^n$ of side length $t = 2^{-d}$

$$\mathcal{F}^a = \{\mathbb{Q} \in \mathcal{D}_d : d(\mathbb{Q}, \partial\Omega) > t,\ \mathcal{L}^n(\mathbb{Q}\cap C^a) > s\mathcal{L}^n(\mathbb{Q})\},$$

where the distance is induced by $\|\cdot\|_\infty$ over vectors in $\mathbb{R}^n$, and d and s will be selected later in the proof. For every $\mathbb{Q} \in \mathcal{F}^a$ select $x_\mathbb{Q}$. By Proposition 3.2.8 we have $\psi^Q \in \mathcal{D}(Q, \mathbb{R}^{n\times m})$ with $\|\psi^\mathbb{Q}\|_1 < \varepsilon\frac{\mathcal{L}^n(\mathbb{Q})}{\mathcal{L}^n(\Omega)}$ and

$$\|(\nu_{x_\mathbb{Q}}, \lambda^a(x_\mathbb{Q})\nu_{x_\mathbb{Q}}^\infty) - \varepsilon_{\overline{\nu_{x_\mathbb{Q}}} + \lambda^a(x_\mathbb{Q})\overline{\nu_{x_\mathbb{Q}}^\infty} + \nabla\psi^\infty}\|_K < \varepsilon.$$

Let $\psi = \sum_\mathbb{Q} \psi^\mathbb{Q} \in \mathcal{D}(\Omega, \mathbb{R}^{m\times n})$ and $\|\psi\|_1 \le \varepsilon$. Then

$$\int_\Omega \eta(\langle\Psi, \nu_x\rangle + \lambda^a(x)\langle\Psi^\infty, \nu_x^\infty\rangle)dx = \sum_{\mathbb{Q}\in\mathcal{F}^a}\int_\mathbb{Q} \eta(\langle\Psi, \nu_x\rangle + \lambda^a(x)\langle\Psi^\infty, \nu_x^\infty\rangle)dx + \mathcal{E}_1$$

with $|\mathcal{E}_1| \le \int_{\Omega\setminus\cup\mathcal{F}^a} M(x)dx \le \varepsilon$ for large enough d. Next

$$\sum_{\mathbb{Q}\in\mathcal{F}^a}\int_\mathbb{Q} \eta(\langle\Psi, \nu_x\rangle + \lambda^a(x)\langle\Psi^\infty, \nu_x^\infty\rangle)dx = \sum_{\mathbb{Q}\in\mathcal{F}^a} \fint_\mathbb{Q} \eta \int_\mathbb{Q} (\langle\Psi, \nu_x\rangle + \lambda^a(x)\langle\Psi^\infty, \nu_x^\infty\rangle)dx + \mathcal{E}_2,$$

where

$$|\mathcal{E}_2| \le t\sum_{\mathbb{Q}\in\mathcal{F}^a}\int_\mathbb{Q} |\langle\Phi, \nu_x\rangle + \lambda^a(x)\langle\Phi^\infty, \nu_x^\infty\rangle|dx \le t\int_\Omega M(x)dx.$$

We further estimate, on every set $\mathbb{Q} \in \mathcal{F}^a$,

$$\begin{aligned}
&\left|\int_\mathbb{Q} \langle\Phi, \nu_x\rangle + \lambda^a(x)\langle\Phi^\infty, \nu_x^\infty\rangle - \mathcal{L}^n(\mathbb{Q})\Big(\langle\Phi, \nu_{x_\mathbb{Q}}\rangle + \lambda^a(x_\mathbb{Q})\langle\Phi^\infty, \nu_{x_\mathbb{Q}}^\infty\rangle\Big)\right| \\
&\quad\le \frac{\omega(t)}{s}\mathcal{L}^n(\mathbb{Q}) + \frac{1-s}{s}\int_\mathbb{Q} |\langle\Phi, \nu_x\rangle + \lambda^a(x)\langle\Phi^\infty, \nu_x^\infty\rangle|dx \\
&\qquad + \int_{\mathbb{Q}\setminus C^a} |\langle\Phi, \nu_x\rangle + \lambda^a(x)\langle\Phi^\infty, \nu_x^\infty\rangle|dx.
\end{aligned}$$

By the linear growth of f, we get that

$$\sum_{\mathbb{Q}\in\mathcal{F}^a} \fint_{\mathbb{Q}} \eta \int_{\mathbb{Q}} \langle \Phi, \nu_x\rangle + \lambda^a(x)\langle \Phi^\infty, \nu_x^\infty\rangle dx = \sum_{\mathbb{Q}\in\mathcal{F}^a} \Big(\langle \Phi, \nu_{x_{\mathbb{Q}}}\rangle + \lambda^a(x_{\mathbb{Q}})\langle \Phi^\infty, \nu_{x_{\mathbb{Q}}}^\infty\rangle\Big) \int_{\mathbb{Q}} \eta + \mathcal{E}_3$$

where

$$|\mathcal{E}_3| \leq \frac{\omega(t)}{s}\mathcal{L}^n(\mathbb{Q}) + \frac{1-s}{s}\int_{\mathbb{Q}} M(x)dx + \varepsilon.$$

For every $\mathbb{Q}\in\mathcal{F}^a$ we have that

$$f(x_{\mathbb{Q}}) = \fint_{\mathbb{Q}} \Phi(\overline{\nu_{x_{\mathbb{Q}}}} + \lambda^a(x_{\mathbb{Q}})\overline{\nu_{x_{\mathbb{Q}}}^\infty} + \nabla\phi^{\mathbb{Q}}) + \mathcal{E}_4^{\mathbb{Q}}, \text{ with } |\mathcal{E}_4^{\mathbb{Q}}| \leq \varepsilon.$$

Set $v^a(x) = \overline{\nu_x} + \lambda^a(x)\overline{\nu_x^\infty}$. Letting $\Phi = z\cdot e^i$, where $(e^i)_{i=1}^{m\times n}$ is the canonical basis of $\mathbb{R}^{m\times n}$, we obtain, from continuity over C^a, $|v^a - v^a(x_{\mathbb{Q}})| \leq \omega(t)$ on $\mathbb{Q}\cap C^a$ for each $\mathbb{Q}\in\mathcal{F}^a$. Consequently,

$$\int_{\mathbb{Q}} |v^a - v^a(x_{\mathbb{Q}})|dx \leq \frac{\omega(t)}{s}\mathcal{L}^n(\mathbb{Q}) + \int_{\mathbb{Q}\setminus C^a} |v^a|dx + \frac{1-s}{s}\int_{\mathbb{Q}} |v^a|dx$$

for all $\mathbb{Q}\in\mathcal{F}^a$. Because $|v^a|\leq M(x)$ and $\mathrm{Lip}(\Phi)\leq 1$, then

$$\sum_{\mathbb{Q}\in\mathcal{F}^a} \fint_{\mathbb{Q}} \eta dx \int_{\mathbb{Q}} \Phi(v^a(x_{\mathbb{Q}}) + \nabla\psi^{\mathbb{Q}})dx = \sum_{\mathbb{Q}\in\mathcal{F}^a} \fint_{\mathbb{Q}} \eta dx \int_{\mathbb{Q}} \Phi(v^a + \nabla\psi^{Q})dx + \mathcal{E}_5$$

with

$$|\mathcal{E}_5| \leq \frac{\omega(t)}{s}\mathcal{L}^n(Q) + \varepsilon + \frac{1-s}{s}\int_\Omega M(x)dx.$$

Combining some of the previous estimates, we get

$$\sum_{\mathbb{Q}\in\mathcal{F}^a} \fint_{\mathbb{Q}} \eta dx \int_{\mathbb{Q}} \Phi(v^a + \nabla\psi^{\mathbb{Q}})dx = \sum_{\mathbb{Q}\in\mathcal{F}^a} \int_{\mathbb{Q}} \eta\Phi(v^a + \nabla\psi^{\mathbb{Q}})dx + \mathcal{E}_6,$$

and

$$\begin{aligned}
|\mathcal{E}_6| &\leq t\sum_{\mathbb{Q}\in\mathcal{F}^a}\int_{\mathbb{Q}} |\Phi(v^a + \nabla\psi^{\mathbb{Q}})|dx \leq t|\mathcal{E}_5| + t\sum_{\mathbb{Q}\in\mathcal{F}^a}\int_{\mathbb{Q}} |\Phi(v^a(x_{\mathbb{Q}}) + \nabla\psi^{\mathbb{Q}})|dx \\
&\leq t|\mathcal{E}_5| + t\varepsilon\mathcal{L}^n(\Omega) + t\sum_{\mathbb{Q}\in\mathcal{F}^a} \mathcal{L}^n(\mathbb{Q})(\langle|\Phi|, \nu_{x_{\mathbb{Q}}}\rangle + \lambda^a(x_{\mathbb{Q}})\langle|\Phi|^\infty, \nu_{x_{\mathbb{Q}}}^\infty\rangle) \\
&\leq t|\mathcal{E}_5| + t\varepsilon\mathcal{L}^n(\Omega) + t\frac{\omega(t)}{s}\mathcal{L}^n(\Omega) + \left(\int_{Q\setminus C^a} + \frac{1-s}{s}\int_\Omega\right) M(x)dx.
\end{aligned}$$

Finally, if ϕ_t is a standard mollifier, then $\phi_t \star v^a \llcorner \Omega \xrightarrow{L^1(\Omega)} v^a$ as $t \to 0$, and so

$$\int_\Omega \eta\Phi(v^a + \nabla\psi)dx = \int_\Omega \eta\Phi(\phi_t \star v^a \llcorner \Omega + \nabla\psi)dx + \mathcal{E}_7,$$

where using again that Φ is Lipschitz over $\mathbb{R}^{m\times n}$,

$$|\mathcal{E}_7| \leq \mathrm{Lip}(\Phi) \int_\Omega |\phi_t \star v^a \llcorner \Omega - v^a|dx \leq \int_\Omega |\phi_t \star v^a \llcorner \Omega - v^a|dx.$$

This concludes the proof. □

We are now ready to prove Theorem 3.2.5.

Proof. (Theorem 3.2.5) The proof of $\Rightarrow$ is similar to that of 3.2.8, so we only show the reverse implication. Let $(\nu_x, \lambda, \nu_x^\infty)$ satisfy Assumptions 1,2 and 3 of the theorem and fix $j \in \mathbb{N}$. By Lemma 3.2.9 and Lemma 3.2.10, we can find $\phi_j \in \mathcal{D}(\Omega)$ and $u_j^{ac}, u_j^s \in \mathcal{D}(\Omega, \mathbb{R}^m)$ so that

$$\left| \int_\Omega \eta\Phi(0) + \eta\langle\Phi^\infty, \nu_x^\infty\rangle d\lambda^s - \int_\Omega \eta\Phi\big(\phi_j \star (\overline{\nu^\infty} d\lambda^s) + \nabla u_j^{ac}\big)dx \right| < \frac{1}{j},$$

and

$$\left| \int_\Omega \eta(\langle\Phi, \nu_x\rangle + \lambda^a(x)\langle\Phi^\infty, \nu_x^\infty\rangle)dx - \int_\Omega \eta\Phi\big(\phi_j \star (\overline{\nu} + \overline{\nu^\infty}\lambda^a(x))dx \llcorner \Omega + \nabla u_j^s\big)dx \right| < \frac{1}{j},$$

where both inequalities hold uniformly for all η and Φ satisfying $\|\eta\|_{\mathrm{Lip}} \leq 1$ and $\|T\Phi\|_{\mathrm{Lip,graph}(f)} \leq 1$. Let

$$o_j = \phi_j \star (\overline{\nu^\infty} d\lambda^s) + \nabla u_j^{ac} \quad \text{and} \quad c_j = \phi_j \star (\overline{\nu} + \overline{\nu^\infty}\lambda^a(x))dx \llcorner \Omega + \nabla u_j^s.$$

Then $(o_j)_j$ and $(c_j)_j$ satisfy the assumption of Lemma 2.4.18, and because

$$o_j + c_j = \phi_j \star \overline{\nu} + \nabla(\overbrace{u_j^{ac} + u_j^s}^{=:u_j}) = \phi_j \star (Du \llcorner \Omega) + \nabla u_j,$$

we conclude that

$$\phi_j \star (Du \llcorner \Omega) + \nabla u_j \xrightarrow{Y(e_{f_i}, i\in\mathbb{N})} (\nu_x, \lambda^a(x)dx + \lambda^s, \nu_x^\infty) = (\nu_x, \lambda, \nu_x^\infty),$$

where we also used Lemma 2.4.15 to deduce full convergence to the Young Measure from uniform convergence against the Lipschitz functions η and Φ. □

Chapter 4

Oscillations generated by derivatives

In this chapter, we discuss how to approximate vector-valued integrable fields with higher-order gradients. More specifically, we study when and how a function $v \in L^1(\Omega, \mathbb{R}^d)$, $\Omega \subset \mathbb{R}^n$ and $d = m \times n$ or $d = m \times n \times n$, can be approximated by, respectively, first and second-order derivatives. As a consequence, in the spirit of results like [2, 3, 49], we show that symmetric oscillations Young Measures $(\nu_x)_{x\in\Omega}$ can be generated by first and second-order derivatives of functions. At the end of this chapter, we partially extend this theory to functions U satisfying a constant coefficient homogeneous differential equation $\mathcal{A}U = 0$.

4.1 Approximation via first-order derivatives

The following result is attributed to Alberti and shows that any integrable vector field $f \in L^1(\Omega, \mathbb{R}^{m\times n})$ is the absolutely continuous part of a gradient of a function in BV.

Theorem 4.1.1 ([2], Theorem 3)**.** *Let $\Omega \subset \mathbb{R}^n$ be open and $f \in L^1(\Omega, \mathbb{R}^n)$. There exists $u \in SBV(\mathbb{R}^n)$ such that*

$$Du = f dx + D^j u,$$

with

$$|D^j u|(\Omega) \leq C\|f\|_1,$$

where $C > 0$ is a universal constant that depends only on the dimension n.

The idea behind the above theorem is that every vector $v \in \mathbb{R}^n$ is the derivative of a function $f_v(x) = x \cdot v$. Approximating first f by continuous functions via Lusin's Theorem, it is then possible to approximate the latter by piecewise linear functions that have jump discontinuities. A careful limiting argument that does not create any diffuse concentration in the singular part is finally employed.

The above result immediately implies that every oscillation Young Measure $(\nu_x)_{x\in\Omega}$ with $\nu_x \colon \Omega \to \mathcal{M}_1^+(\mathbb{R}^n)$ can be generated by gradients. Moreover, convergence can be extended to integrands that do not have a regular recession at infinity but admit recession on rank-1 directions, such as quasi-convex functions.

Lemma 4.1.2. *Let $(\nu_x)_{x\in\Omega}$ be so that $\nu_x \colon \Omega \to \mathcal{M}_1^+(\mathbb{R}^{m\times n})$ is weak* measurable and $\int_\Omega \langle \nu_x, |\cdot|\rangle dx < +\infty$. There are $g \in L^1(\Omega), g \geq 0$, $(u_j)_j$ in $C^\infty(\overline{\Omega}, \mathbb{R}^m)$ and $u \in SBV(\Omega, \mathbb{R}^m)$, $u_j \rightharpoonup^* u$ in BV so that*

$$Du_j \xrightarrow{Y(e_f)} \nu = (\nu_x, g dx + |D^j u|, \nu_x^\infty), \quad \nu_x^\infty = \delta_{\frac{D^j u}{|D^j u|}} \quad |D^j u|\text{-a.e.},$$

and

$$\operatorname{supp}(\nu_x^\infty) \subset \{a \otimes b \in \partial\mathbb{R}^{m\times n} : a \in \mathbb{R}^m,\ b \in \mathbb{R}^n\} \quad g dx + |D^j u|\text{-a.e.}$$

The above convergence holds for every $f \colon \mathbb{R}^{m\times n} \to \mathbb{R}$ of linear growth such that $f^\infty(z)$ exists for all $\operatorname{rank}(z) = 1$.

Moreover, if $m = 1$ we can take $g = 0$.

Proof. We only prove the case $m = 1$ because for $m \geq 2$ the proof is almost identical to that of Theorem 4.2.9. See also the remark following the proof of Theorem 4.2.9, which gives a good representation of the generating sequence.

Use Alberti Theorem 4.1.1 to find $u \in SBV(\Omega)$ such that

$$Du = \nabla u\, dx + D^j u = \overline{\nu_x}\, dx + D^j u.$$

Because $m = 1$, quasi-convex functions are convex, and so for any convex f of linear growth we have

$$\begin{cases} \langle \nu_x, f\rangle \geq f(\overline{\nu_x}) = f(\nabla u(x)) & \mathcal{L}^n\text{-a.e.} x \in \Omega \\ f\left(\frac{D^j u}{|D^j u|}(x)\right) \equiv f\left(\frac{D^j u}{|D^j u|}(x)\right) & |D^j u|\text{-a.e.} x \in \Omega. \end{cases}$$

and the characterisation theorem yields the sought sequence. □

Remark 4.1.3. *The previous result is sharp. This is because when $m \geq 2$ quasi-convex functions are not necessarily convex and therefore the function "g" forces convexity of the quantity*

$$\langle \nu_x, f \rangle + g(x)\langle \nu_x^\infty, f^\infty \rangle \geq f(\overline{\nu_x} + \overline{\nu_x^\infty} g(x))$$

at $\mathcal{L}^n$-a.e. $x \in \Omega$. Moreover, it is interesting to compare the approximation result 4.1.2 to [66], Theorem 1, which roughly speaking says that given any Gradient Young Measure $(\nu_x, \lambda, \nu_x^\infty)$ on the sphere compactification, the Young Measure obtained by suppressing the oscillation, i.e. $(\overline{\nu_x}, \lambda, \nu_x^\infty)$, is still a Gradient Young Measure. In our case, this does not add anything, as we already know that

$$\left(\overline{\nu_x}, |D^j u|, \delta_{\frac{D^j u}{|D^j u|}}\right) = \left(\delta_{f(x)}, |D^j u|, \delta_{\frac{D^j u}{|D^j u|}}\right)$$

is a Gradient Young Measure by construction since $\overline{\nu_x} = \nabla u(x)$. Therefore, our result can be seen as complementary to [66].

4.2 Approximation via second-order derivatives

The exact approximation for second-order derivatives was achieved for the first time in [44]. The original proof is rather complicated and makes use of a geometric argument to extract a nice mesh on the domain. Our contribution was to develop a more concise proof that uses certain symmetries, which in addition gives an infinitesimal estimate on lower derivatives. More specifically, a symmetric valued vector field $f \in L^1(\Omega, \mathbb{R}^{m\times n})$ is approximated by functions u_j that satisfy $Du_j = Du_j^T$. Such condition implies $\operatorname{curl}(u) = 0$, and therefore it provides the existence of a potential for u_j.

Before stating and proving the theorem, we define the two main classes of functions we are using in this section.

Definition 4.2.1. *The set of functions of bounded Hessian will be called*

$$BH(\Omega, \mathbb{R}^m) = \{u \in W^{1,1}(\Omega, \mathbb{R}^m) : \nabla u \in BV(\Omega, \mathbb{R}^{m\times n})\},$$

and its subset consisting of those functions whose Hessian has no Cantor part will be called

$$SBH(\Omega, \mathbb{R}^m) = \{u \in W^{1,1}(\Omega, \mathbb{R}^m) : \nabla u \in SBV(\Omega, \mathbb{R}^{m\times n})\}.$$

Each of these spaces is normed by

$$\|u\|_{BH} = \int_\Omega |u(x)|dx + |\nabla u(x)|dx + |D^2 u|(\Omega).$$

The unit ball in BH is weak* closed as the following lemma shows.

Lemma 4.2.2. *Let $(u_j)_j$ in $BH(\Omega, \mathbb{R}^m)$ be so that $\sup_j \|u_j\|_{BH} < +\infty$. There exists a subsequence $(u_{j_i})_i$ and $u \in BH(\Omega, \mathbb{R}^m)$ so that $u_{j_i} \rightharpoonup^* u$ in BH, meaning that*

$$\|u_{j_i} - u\|_1 + \|\nabla u_{j_i} - \nabla u\|_1 \to 0$$

and

$$D^2 u_{j_i} \rightharpoonup^* D^2 u \quad in \quad C_0(\Omega, \mathbb{R}^{m\times n\times n})^*.$$

In light of these definitions, let's state the main result of this section.

Theorem 4.2.3. *Let $n \geq 2$, $\Omega \subset \mathbb{R}^n$ open and $f \in L^1(\Omega, \mathbb{R}^{n\times n}_{sym})$. For every $\varepsilon > 0$ there is $u \in SBH(\Omega)$ such that*

$$D^2 u = f dx + \nabla u \otimes \nu_{\nabla u} d\mathcal{H}^{n-1} \llcorner_{J_{\nabla u}},$$

and the following bounds hold

$$\|u\|_{W^{1,1}} \leq \varepsilon \|f\|_1 \quad and \quad |D^j \nabla u|(\Omega) \leq C\|f\|_1,$$

for a universal constant $C > 0$ which depends only on Ω (and so n).

Remark 4.2.4. *After applying the above result coordinate-wise, we can prove the same statement for when $f \in L^1(\Omega, \mathbb{R}^m \otimes \mathbb{R}^{n\times n}_{sym})$, and obtain $u \in SBH(\Omega, \mathbb{R}^m)$ as above.*

The first step to proving the main result of this section is to find an L^1 approximation when f is continuous and compactly supported.

Proposition 4.2.5. *Let $f \in C_c(\Omega, \mathbb{R}^{n\times n}_{sym})$ and set $F = f(x) + 2\mathbb{1}|f(x)|_2$. There exists $C > 0$ such that for each $\varepsilon > 0$ there is $u \in SBH(\Omega)$, u compactly supported, such that*

$$\|u\|_{W^{1,1}} \leq \varepsilon \|f\|_1, \quad |D^j \nabla u|(\mathbb{R}^n) \leq C\|f\|_1 \quad and \quad \|\nabla^2 u - F\|_1 \leq \varepsilon.$$

Here, C depend neither on f nor on ε.

Proof. Let ω be the modulus of continuity of f, so that 3ω is the modulus of continuity of F. Fix $0 < \delta < 1$ and an arbitrary point $x \in \Omega$. If $F(x) = 0$ then $f(x)$ is also zero, and we set

$$E_{x,r} = \mathbb{B}_r(x)$$

so that $\mathbb{B}_r(x) \subset \Omega$ and $0 < r < \delta$. Otherwise $f(x) \neq 0$, and so $F(x)$ is strictly positive definite and we can diagonalise

$$F(x) = S(x)^{-1} D(x) S(x).$$

Without loss of generality, we assume $x = 0$ and omit the dependence on x. Also, we write $M \cdot (S)$ or MS to mean the matrix multiplication between $M \in \mathbb{R}^{a \times b}$ and $S \in \mathbb{R}^{b \times c}$ with $a, b, c \in \mathbb{N}$, and $M(S)$ to mean that M is a function of S. We want to find a convex domain E, centred at 0 and with outer normal $\nu(y), y \in \partial E$, so that $Fy \parallel \nu(y)$. Changing coordinates $y = S^{-1}z$ we get

$$DSy \parallel S\nu(y) \iff Dz \parallel S\nu(S^{-1}z) \iff Dz = \gamma(z)\eta(z)$$

where $\gamma(z)$ is a scalar function and $\eta(z) = S\nu(S^{-1}z)$ is the outer normal in the S coordinates. We guess that η is normal to an ellipse and $\gamma = 1$. Consider

$$G(z) = \frac{z_1^2}{a_1} + \ldots + \frac{z_n^2}{a_n} = 1,$$

its gradient is normal to the surface, i.e.

$$\nabla G(z) = 2\mathrm{diag}\left(\frac{z_1}{a_1}, \ldots, \frac{z_n}{a_n}\right),$$

and so $Dz = \eta(z)$ is equivalent to

$$\lambda_i z_i = 2\frac{z_i}{a_i} \quad \iff \quad a_i = \frac{2}{\lambda_i}.$$

Keeping track of the x dependence once again, we have found a convex domain E, centred at x and with unit normal $\nu(y), y \in \partial E$, such that $F(x) \cdot (y - x) \parallel \nu(y)$, which implies that $\big(F(x) \cdot (y - x)\big) \otimes \nu(y)$ is symmetric. Set $\lambda_{\min} = \min_i \lambda_i$ and $\lambda_{\max} = \max_i \lambda_i$. Because S is orthonormal, the diameter of E is $\frac{2}{\lambda_{\min}}$ and its closest point to the origin has magnitude $\frac{2}{\lambda_{\max}}$. Let

$$E_{x,r} = (tE - x)r + x,$$

where $t > 0$ is to be decided. By the construction of the function F we have

$$|f(x)|_2 \leq \lambda_{\min}(x) \leq \lambda_{\max}(x) \leq 3|f(x)|_2,$$

so that $\lambda_{\max} \leq 3\lambda_{\min}$. In particular,

$$\mathbb{B}_{\lambda_{\max}/3}(x) \subset E \subset \mathbb{B}_{\lambda_{\max}}(x),$$

and we can change t so that $\mathbb{B}_{1/3}(x) \subset E \subset \mathbb{B}(x)$, and so $\mathbb{B}_{r/3} \subset E_{x,r} \subset \mathbb{B}_{x,r}$.

Letting

$$E = \{\overline{E_{x,r}} : x \in \Omega \text{ and } r < \delta_x\},$$

we can apply Lemma A.1.10 with $\eta = \frac{1}{3}$ and find finitely many disjoint sets $\overline{E_{x_j,r_j}}, j = 1, \dots, m$ so that, letting $E^c = \Omega \setminus \cup_j E_{x_j,r_j}$,

$$\int_{E^c} |f(x)|dx < \delta.$$

We can also choose $0 < \delta_x < \delta$ so that $|f(x)| \leq 2|f(y)|$ for all $y \in E_{x,\delta_x}$. Let

$$v(y) = \sum_{j=1}^{m} F(x_j) \cdot (y - x_j)\chi_{E_{x_j,r_j}}(y).$$

We have that

$$\begin{aligned}\|v\|_1 &\leq \sum_{j=1}^{m} |F(x_j)| \int_{E_{x_j,r_j}} |y - x_j|dy \\ &\leq \delta \int_{\cup_j E_{x_j,r_j}} |F(y)| + 3\omega(\delta)dy \leq \delta\Big(3\omega(\delta)|\Omega| + \|f\|_1 + \delta\Big).\end{aligned}$$

We decrease $\delta > 0$ so that the right-hand side can be bounded by $\varepsilon\|f\|_1$. On the other side, the derivative of v is

$$Dv = \sum_{j=1}^{m} F(x_j)\chi_{E_{x_j,r_j}} + \sum_{j=1}^{m} \big(F(x_j) \cdot (y - x_j)\big) \otimes \nu_j(y)d\mathcal{H}^{n-1} \llcorner \partial E_{x_j,r_j}.$$

The absolutely continuous part can be estimated by

$$\|\nabla v - F\|_1 = \sum_{j=1}^{m} \int_{E_{x_j,r_j}} |F(x_j) - F(y)|dy + \int_{E^c} |F(y)|dy \leq 3\omega(\delta)|\Omega| + 3\delta \leq \varepsilon$$

upon further restricting δ. For the singular part, in consideration that $2\frac{r_j}{\lambda_{\min}(x_j)}$ is the diameter of the ellipses, we compute

$$\begin{aligned}\int_{\partial E_{x_j,r_j}} |F(x_j) \cdot (y - x_j)|d\mathcal{H}^{n-1} &\leq |F(x_j)| \int_{\partial E_{x_j,r_j}} |y - x_j|d\mathcal{H}^{n-1} \\ &\leq 6|f(x_j)|\frac{r_j}{\lambda_{\min}(x_j)}\mathcal{H}^{n-1}(\partial E_{x_j,r_j}).\end{aligned}$$

By construction $E_{x_j,r_j} \subset \mathbb{B}_{r_j}(x_j)$ and so, because E_{x_j,r_j} is convex, we can upper-bound its perimeter by

$$\mathcal{H}^{n-1}(\partial E_{x_j,r_j}) \leq C\frac{r_j^{n-1}}{\lambda_{\min}^{n-1}},$$

where $C > 0$ only depends on the dimension n. We can then complete the above upper bound to reach the inequality

$$6|f(x_j)|\frac{r_j}{\lambda_{\min}(x_j)}\mathcal{H}^{n-1}(\partial E_{x_j,r_j}) \leq C6|f(x_j)|\frac{r_j^n}{\lambda_{\min}^n}$$
$$\leq C|f(x_j)||E_{x_j,r_j}| \leq C\int_{E_{x_j,r_j}} |f(y)|dy,$$

where the last $C > 0$ is a new universal constant. Summing over j we finally obtain the bound

$$|D^s v|(\mathbb{R}^n) \leq C\|f\|_1,$$

where we used that v is also compactly supported.

At last, we show that Dv is symmetric: the symmetry of the absolutely continuous part is inherited by F. On the other side, the singular part is the sum of functions of the form

$$\big(F(x)\cdot(y-x)\big)\otimes\nu(y)d\mathcal{H}^{n-1}.$$

Notice that by construction $F(x)\cdot(y-x) \parallel \nu(y)$ and therefore the singular part is also symmetric. In particular $Dv = Dv^T$ and so $\operatorname{curl}(v) = 0$ in $\mathcal{D}'$. Apply Lemma A.2.9 (as the domain might not be simply connected) to find $u \in W^{1,1}(\Omega)$ such that $\nabla u = v$ and $\|u\|_1 \leq C\|v\|_1$, where C depends only on Ω, and u is compactly supported. □

Upon applying the previous result to $x \mapsto \frac{1}{3}\mathbb{1}|f(x)|$, we immediately deduce the following.

Corollary 4.2.6. *Let f as before and set $F = 2\mathbb{1}|f(x)|_2$. For $\varepsilon > 0$ there is $u \in SBH(\Omega)$, u compactly supported, such that*

$$\|u\|_{W^{1,1}} \leq \varepsilon\|f\|_1, \quad |D^j\nabla u|(\mathbb{R}^n) \leq C\|f\|_1 \quad \textit{and} \quad \|\nabla^2 u - F\|_1 \leq \varepsilon.$$

We now combine the two previous approximation results.

Proposition 4.2.7. *Let $f \in L^1(\Omega, \mathbb{R}^{n\times n}_{sym})$. For each $\varepsilon > 0$, there is $u \in SBH(\Omega)$, u compactly supported, and a constant C that depends only on Ω such that*

$$\|u\|_{W^{1,1}} \leq \varepsilon\|f\|_1, \quad |D^j\nabla u|(R^n) \leq C\|f\|_1 \quad \textit{and} \quad \|\nabla^2 u - f\|_1 \leq \varepsilon.$$

Proof. First, assume that $f(x) \in C_c(\Omega, \mathbb{R}^{n\times n})$ is symmetric. Apply the previous results to $f(x) + 2\mathbb{1}|f(x)|$ and $2\mathbb{1}|f(x)|$ to obtain u_1 and u_2 respectively and put $u = u_1 - u_2$. When $f \in L^1(\Omega, \mathbb{R}^{n\times n}_{sym})$, it is enough to find $g \in C_c$ such that $\|f-g\|_1 \leq \varepsilon$ and $\|g\|_1 \leq \|f\|_1$, for example

$$g_n = (\chi_{\Omega_{-2\delta}} f) \star \rho_\delta$$

for $\delta > 0$ small enough, where $\Omega_{-a} = \{x \in \Omega : d(x, \partial\Omega) > a\}$ and $\{\rho_\eta\}_{\eta>0}$ is a family of standard mollifiers. □

To prove the main theorem we follow an iteration argument as in [2].

Proof. (of Theorem 4.2.3).

Let $u_0 = 0$ and $f_0 = f$. Apply Proposition 4.2.7 iteratively to find $(u_n)_n$ in $SBH(\Omega)$ so that

$$\|\nabla^2 u_n - f_{n-1}\|_1 \leq 2^{-n} \min(1, \|f\|_1), \quad \|u_n\|_{W^{1,1}} \leq 2^{-n}\|f\|_1$$

and

$$|D^j \nabla u_n|(\Omega) \leq C\|f_{n-1}\|_1,$$

and let $f_n = \nabla^2 u_n - f_{n-1}$. Because $\|f_n\| \leq 2^{-n}$, both summations $\sum_n \nabla^2 u_n$ and $\sum_n D^j \nabla u_n$ converge strongly in the sense of measures. Then, the sum $u = \sum_n u_n$ converges strongly in $BV(\Omega, \mathbb{R}^m)$. Its derivative is $\sum_n \nabla u_n$ and the second derivative is

$$D^2 u = \sum_n \nabla^2 u_n + \sum_n D^j \nabla u_n.$$

Because $f = f_m + \sum_{n=1}^m \nabla^2 u_m$ then

$$\|f - \nabla^2 u\|_1 \leq \|f_m\|_1 + \sum_{n\geq m+1} \|\nabla^2 u_m\|_1 \leq 2^{-m} + \sum_{n\geq m+1} \|\nabla^2 u_m\|_1 \xrightarrow{m\to\infty} 0.$$

On the other side, the singular part is so that

$$|D^s \nabla u|(\Omega) \leq \sum_{n\geq 1} |D^s \nabla u_n|(\Omega) \leq \sum_{n\geq 1} C\|f_n\|_1 \leq \|f\|_1 + \sum_{n\geq 1} C 2^{-n}\|f\|_1 = (C+1)\|f\|_1.$$

□

Similarly to the case of first-order gradients, we can show that any symmetric oscillation Young Measure on $\mathbb{R}^{n\times n}$ is the limit of second-order derivates.

Proposition 4.2.8. *Let $(\nu_x)_{x\in\Omega}$ be so that $\nu_x\colon \Omega \to \mathcal{M}_1^+(\mathbb{R}^m \otimes \mathbb{R}^{n\times n}_{sym})$ is a weak* measurable map such that*

$$\int_\Omega \langle \nu_x, |\cdot|\rangle dx < +\infty.$$

Then there exist $\lambda \in \mathcal{M}^+(\Omega), \nu_x^\infty\colon \Omega \to \mathcal{M}_1^+(\partial e_f)$ weak λ-measurable and $(u_j)_j$ in $SBH(\Omega,\mathbb{R}^m), u \in BH(\Omega,\mathbb{R}^m)$ such that $u_j \overset{*}{\rightharpoonup} u$ in BH and*

$$D^2 u_j \xrightarrow{Y(e_f)} (\nu_x, \lambda, \nu_x^\infty)$$

for all f with a regular recession on the set $\{w\otimes v\otimes v \in \partial\mathbb{B}^{m\times n\times n} : w\in\mathbb{R}^m, v\in\mathbb{R}^n\}$. Moreover,

$$\operatorname{supp}(\nu_x^\infty) \subset \{w\otimes v\otimes v \in \partial\mathbb{B}^{m\times n\times n} : w\in\mathbb{R}^m, v\in\mathbb{R}^n\} \quad \lambda\text{-a.e.}$$

The above theorem implies that

$$\overline{\nu} = \overline{\nu_x}dx + \overline{\nu_x^\infty}d\lambda = D^2 u,$$

however $|D^s\nabla u| \le \lambda$ in $C(\overline{\Omega})^*$, and the inequality could a priori be strict. We will sharpen this inequality in the next theorem.

Proof. Find $(v_j)_j$ in $L^1(\Omega,\mathbb{R}^m\otimes\mathbb{R}^{n\times n}_{sym})$ so that $v_j \xrightarrow{Y} (\nu_x, 0, N/A)$ and $\sup_j \|v_j\|_1 \le 2\int_\Omega\langle\nu_x,|\cdot|\rangle dx$, where the symmetry of every row of v_j is guaranteed by Corollary 2.4.24. Apply Proposition 4.2.7 row-wise to find a sequence $(u_j)_j$ in $SBH(\Omega,\mathbb{R}^m)$ so that

$$\|\nabla^2 u_j - v_j\|_1 \le \frac{1}{j}, \quad \|u_j\|_{W^{1,1}} \le \|f\|_1$$

and

$$|D^j\nabla u_j|(\mathbb{R}^n) \le C\|v_j\|_1.$$

Also, u_j is bounded in BH since

$$\sup_j \|u_j\|_{BH} \le C\sup_j\|v_j\|_1 \le C\int_\Omega\langle\nu_x,|\cdot|\rangle dx < +\infty,$$

and so, upon extracting a subsequence, $u_j \overset{*}{\rightharpoonup} u$ in BH. Here, $\nabla^2 u_j \rightharpoonup f$ weakly in L^1 but we still have diffuse concentration if its singular part as $\frac{\lambda}{\mathcal{L}^n} \neq 0$ in general. Conclude that

$$D^2 u_j \xrightarrow{Y(\mathbb{B}^{m\times n\times n})} (\nu_x, \lambda, \nu_x^\infty).$$

As for the support of ν_x^∞ and convergence on fs having regular recession on the same set just notice that $\frac{D^j\nabla u}{|D^j\nabla u|}$ belongs to

$$\{w\otimes v\otimes v\in\partial\mathbb{B}^{m\times n\times n}: w\in\mathbb{R}^m, v\in\mathbb{R}^n\}$$

for $|D^j\nabla u_j|$-a.e. $x\in\Omega$ and such a set is compact. □

With the aid of Theorem A.3.5 we can improve the above result.

Theorem 4.2.9. *Let $(\nu_x)_{x\in\Omega}$ be so that $\nu_x\colon\Omega\to\mathcal{M}_1^+(\mathbb{R}^m\otimes\mathbb{R}^{n\times n}_{sym})$ is a weak* measurable map with*

$$\int_\Omega\langle\nu_x,|\cdot|\rangle dx<+\infty.$$

Then there exist $g\in L^1(\Omega), g\geq 0$, $(u_j)_j$ in $C^\infty(\overline{\Omega},\mathbb{R}^m)$ and $u\in SBH(\Omega,\mathbb{R}^{m\times n})$ such that $u_j\overset{*}{\rightharpoonup}u$ in BH and

$$D^2u_j\xrightarrow{Y(e_f)}(\nu_x,gdx+|D^j\nabla u|,\nu_x^\infty),\quad \nu_x^\infty=\delta_{\frac{D^j\nabla u}{D^j\nabla u}}\quad |D^j\nabla u|\text{-a.e.},$$

for all f with regular recession on the set $\{w\otimes v\otimes v\in\partial\mathbb{B}^{m\times n\times n}: w\in\mathbb{R}^m, v\in\mathbb{R}^n\}$ and

$$\operatorname{supp}(\nu_x^\infty)\subset\{w\otimes v\otimes v\in\partial\mathbb{B}^{m\times n\times n}: w\in\mathbb{R}^m, v\in\mathbb{R}^n\}\quad g(x)dx\text{-a.e.}$$

Remark 4.2.10. *The sequence u_j can be taken as in [67], Theorem 1.2.*

Proof. Apply first Proposition 4.2.8 to find $v_j\overset{*}{\rightharpoonup}v$ in $BH(\Omega,\mathbb{R}^m)$ with

$$D^2v_j\xrightarrow{Y}(\nu_x,\lambda,\eta_x^\infty).$$

By Theorem A.3.6 we have that, for $\mathcal{L}^n$-a.e. $x\in\Omega$,

$$\langle\nu_x,f\rangle+\langle\eta_x^\infty,f^\infty\rangle\frac{\lambda}{\mathcal{L}^n}(x)\geq f(\nabla^2v(x))$$

for all curl-free $f\colon\mathbb{R}^m\otimes\mathbb{R}^{n\times n}_{sym}\to\mathbb{R}$ of linear growth. Such inequality can also be proven via convexity of $\mathcal{A}$-free functions on $\Lambda_{\mathcal{A}}$, given that $\operatorname{supp}(\eta_x^\infty)\subset\Lambda_{\mathcal{A}}$, as a consequence of which we also have $f^\infty=f^{\sharp\partial\mathbb{B}^{m\times n\times n}}$ on $\operatorname{supp}(\eta_x^\infty)$, see [62].

Next, by Theorem 4.2.3 we find $u\in SBH(\Omega,\mathbb{R}^m)$ such that

$$D^2u=\left(\overline{\nu_x}+\frac{\lambda}{\mathcal{L}^n}\overline{\eta_x^\infty}\right)dx+D^j\nabla u.$$

The Young Measure

$$\nu = \left(\nu_x, g(x)dx + |D^j \nabla u|, \nu_x^\infty\right)$$

with

$$g(x) = \frac{\lambda}{\mathcal{L}^n}(x) \quad \text{and} \quad \nu_x^\infty = \begin{cases} \eta_x^\infty & g(x)dx\text{-a.e.} \\ \delta_{\frac{D^j \nabla u}{|D^j \nabla u|}} & |D^j \nabla u|\text{-a.e.} \end{cases}$$

satisfies $\overline{\nu} = D^2 u$ and

$$\begin{aligned} &\langle \nu_x, f \rangle + \langle \nu_x^\infty, f^\infty \rangle g(x) \geq f(\nabla^2 u(x)) && g(x)dx\text{-a.e.} \\ &\langle \nu_x^\infty, f^\infty \rangle = f^\infty \left(\frac{D^j \nabla u}{|D^j \nabla u|} \right) && |D^j \nabla u|\text{-a.e.} \end{aligned}$$

We can apply Theorem A.3.5 to obtain a sequence $(u_j)_j$ in $C^\infty(\overline{\Omega}, \mathbb{R}^m)$ that generates ν, and so ν is our sought Gradient Young Measure. A priori, this sequence generates ν on the sphere, but after noticing that ν_x^∞ has support contained in $\Lambda_{\mathcal{A}}$ we conclude that convergence holds for all fs having regular recession on the set

$$\{w \otimes v \otimes v \in \partial \mathbb{B}^{m \times n \times n} : w \in \mathbb{R}^m, v \in \mathbb{R}^n\}.$$

□

4.3 Approximation via $\mathcal{A}$-free fields

When trying to generalise Alberti's Theorem 4.1.1 to arbitrary high-order derivatives, one comes across an integrability problem. Specifically, the procedure adopted to prove this result does not prevent diffuse concentration of derivatives of order greater than 1. A scheme for approximating vector fields with higher-order derivatives can be found in the appendix of [3]. It was subsequently refined in [49] (see also [82]) in the form of the following result.

Theorem 4.3.1 ([49], Theorem 1.1)**.** *Let Ω be an open set and $(f^\alpha)_{|\alpha|=m}$ a Lebesgue measurable vector-valued function on Ω. There exists $u \in C^{m-1}(\mathbb{R}^n)$ such that $\nabla^{m-1} u$ is differentiable $\mathcal{L}^n$ almost everywhere, and*

$$\nabla^m u = f \quad \mathcal{L}^n\text{-a.e.}$$

The above makes use of the same approximation procedure as in [2] but doesn't preserve the integrability of higher-order derivatives even when f is itself integrable. This means that $D^m u$, where the derivative is interpreted in the sense of distribution, might not be a measure.

In the following, we show how to generalise Theorem 4.3.1 to the case of constant coefficient partial differential operators.

Definition 4.3.2. *Consider a linear, k-homogeneous, partial differential operator*

$$\mathcal{A} = \sum_{|\alpha|=k} \partial^\alpha A_\alpha, \ A_\alpha \in Lin(W, X),$$

from a linear space W to a linear space X, which maps $w \in C^\infty(\mathbb{R}^n, W)$ to $\mathcal{A}w \in C^\infty(\mathbb{R}^n, X)$.

We let

$$\mathcal{A}(\xi) = \sum_{|\alpha|=k} \xi^\alpha A_\alpha \in Lin(W, X),$$

where $\xi^\alpha := \xi_1^{\alpha_1} \times \ldots \times \xi_n^{\alpha_n}$ are the symbols of $\mathcal{A}$, and the convention $0^0 = 1$ is held.

Also, denote by

$$\Lambda_{\mathcal{A}} := \cup_{\xi \in \mathbb{S}^{n-1}} \ker(\mathcal{A}(\xi)) \subset W$$

the wave cone of $\mathcal{A}$.

If $\Lambda_{\mathcal{A}}$ is the wave cone of $\mathcal{A}$, then

$$\operatorname{span}(\Lambda_{\mathcal{A}}) = \left\{ \sum_j \lambda_j v_j, \quad v_j \in \Lambda_{\mathcal{A}}, \lambda_j \in \mathbb{R} \right\}$$

is the spanning cone of $\mathcal{A}$.

Remark 4.3.3.

- *For first-order derivative, $W = \mathbb{R}^{m \times n}$ and $\mathcal{A} = \operatorname{curl}$ and the wave cone is*

$$\{v \otimes w : v \in \mathbb{R}^m, w \in \mathbb{R}^n\}.$$

- *First second-order derivative, $W = \mathbb{R}^m \otimes \mathbb{R}^{n \times n}_{sym}$ and $\mathcal{A} = \operatorname{curl}$ (row-wise) and the wave cone is*

$$\{v \otimes w \otimes w : v \in \mathbb{R}^m, w \in \mathbb{R}^n\}.$$

We refer to appendix A.3 for further results on $\mathcal{A}$-free vector fields.

Definition 4.3.4. *We say that $\mathcal{A}$ is* constant rank *if*

$$\operatorname{rank}(\mathcal{A}(\xi)) = r \quad \text{is constant for all} \quad \xi \in \mathbb{R}^n \setminus \{0\}.$$

Under the assumption of constant rank, we obtain the existence of a potential, as the following result shows.

Lemma 4.3.5 ([89], Theorem 1)**.** *Let $\mathcal{A}$ be linear, constant coefficient, k-homogeneous differential operator with constant rank, there exists $\mathcal{B}$ linear, constant coefficient, homogeneous differential operator such that*

$$\ker(\mathcal{A}(\xi)) = \operatorname{img}(\mathcal{B}(\xi))$$

for all $\xi \in \mathbb{R}^n \setminus \{0\}$.

The order of $\mathcal{B}$ is not necessarily k.

Remark 4.3.6. *Let $\mathcal{A}, \mathcal{B}$ as before, then there are finitely many $\xi_1, \dots, \xi_b$ so that*

$$\operatorname{span}(\Lambda_{\mathcal{A}}) = \operatorname{span}(\mathcal{B}(\xi_1), \dots, \mathcal{B}(\xi_b)).$$

where $\mathcal{B}(\xi_1), \dots, \mathcal{B}(\xi_b)$ can be taken to be a basis for $\operatorname{span}(\Lambda_{\mathcal{A}})$. In particular, given any $v \in \operatorname{span}(\Lambda_{\mathcal{A}})$ written as

$$v = \sum_i \mathcal{B}(\xi_i) v_i,$$

we have that $|v_i| \leq C\|v\|$ for some constant C that depends only on $\mathcal{B}$.

Here, we show that for $\mathcal{A}$ with constant rank, we can find $U \in \ker(\mathcal{A})$ that approximates any integrable vector field in measure.

Lemma 4.3.7. *Let $\Omega \subset \mathbb{R}^n$ open and bounded, $\mathcal{A}$ a linear, constant coefficient, k-homogeneous differential operator with constant rank and $f \in C_c(\Omega, \operatorname{span}(\Lambda_{\mathcal{A}}))$. Given $\eta, \varepsilon > 0$, we can find $K \subset \Omega$ compact and $U \in C_c(\Omega, W)$ with $\mathcal{A}U = 0$, so that*

$$\begin{aligned} &\mathcal{L}^n(\Omega \setminus K) < \mathcal{L}^n(\Omega)\varepsilon, \\ &\|f - U\|_{\infty, K} < \eta, \\ &\|U\|_1 \leq C\varepsilon^{1-l}\|f\|_1, \end{aligned}$$

where l is the degree of $\mathcal{B}$, the constant C depends only on the dimension n and l, and $\mathcal{B}$ is a potential for $\mathcal{A}$ according to Lemma 4.3.5. In this case, $U = \mathcal{B}u$ for some $u \in \mathcal{D}(\Omega, W)$.

The following proof is inspired by [3], Theorem 5.10.

Proof. Apply 4.3.5 to find $\mathcal{B}$ so that

$$\ker(\mathcal{A}(\xi)) = \operatorname{img}(\mathcal{B}(\xi))$$

where

$$\mathcal{B} = \sum_{|\alpha|=l} \partial^\alpha B_\alpha.$$

Fix an arbitrary $v \in \operatorname{span}(\Lambda_\mathcal{A})$. We can then write

$$v = \sum_i \mathcal{B}(\xi_i) v_i = \sum_i \sum_\alpha \xi_i^\alpha B_\alpha v_i,$$

with $\xi \in \mathbb{R}^n$ and v_i scalar. Also, for $v \in span(\Lambda_\mathcal{A})$ define the following polynomial

$$P_v(x) = \sum_i \sum_{|\gamma|=l} \xi_i^\gamma \frac{x^\gamma}{\gamma!} v_i,$$

where we remind that

$$x^\gamma = x_1^{\gamma_1} \cdot \ldots \cdot x_n^{\gamma_n} \text{ and } \gamma! = \gamma_1! \cdot \ldots \cdot \gamma_n!.$$

Notice that ξ depends on v_i. A simple computation shows that

$$\begin{aligned}\mathcal{B}P_v &= \sum_i \sum_\alpha \sum_\gamma B_\alpha \xi_i^\gamma \frac{\partial^\alpha x^\gamma}{\gamma!} v_i \\ &= \sum_i \sum_\alpha \sum_\gamma B_\alpha \xi_i^\gamma \frac{\alpha! \chi_{\alpha=\gamma}}{\gamma!} v_i = \sum_i \sum_\alpha B_\alpha \xi_i^\alpha v_i = v.\end{aligned}$$

For all $x \in \Omega$ let $0 < \delta_x < \delta$ (to be selected later) be so small that $|f(y) - f(x)| \leq \eta$ and $|f(y)| \leq 2|f(x)|$ for all $y \in \mathbb{B}_{\delta'}(x), 0 < \delta' < \delta_x$ and $\overline{\mathbb{B}_{\delta_x}(x)} \subset \Omega$. Because it is a fine cover, we can find finitely balls $\mathbb{B}_i$ so that $\mathcal{L}^n(\cup_i \mathbb{B}_i) > \mathcal{L}^n(\Omega)(1-\delta)$. Furthermore, on a generic ball $\mathbb{B}_r$ we let $\phi_r \in \mathcal{D}(\mathbb{B}_r)$ so that $\phi_r(\mathbb{B}_{r(1-\delta)}) = 1$ and

$$\|\nabla^h \phi_r\|_\infty \leq C(r\delta)^{-|h|} \quad |h| \leq l,$$

where C depends on the dimension n and l. Call ϕ_r relative to $\mathbb{B}_i$ simply ϕ_i and $P_{f(x_i)} = P_i$, and letting x_i be the centre of $\mathbb{B}_i$, we define the smooth and compactly supported function

$$u = \sum_i \phi_i P_i.$$

Then the set where $\phi_i = 1$ is $\cup_i \mathbb{B}_i^\delta$ where $\mathbb{B}_i^\delta$ is the $1-\delta$ dilation of $\mathbb{B}_i$. We can immediately estimate its measure as these balls are disjoint: first notice that

$$(1-\delta)^{n+1} \geq 1 - (n+1)\delta,$$

therefore

$$\mathcal{L}^n(\cup_i \mathbb{B}_i^\delta) = (1-\delta)^n \mathcal{L}^n(\cup_i \mathbb{B}_i) \geq (1-\delta)^{n+1} \mathcal{L}^n(\Omega) \geq \mathcal{L}^n(\Omega)(1-\varepsilon)$$

if $\delta = \varepsilon/(n+1)$. Set $K = \cup_i \mathbb{B}_i$. For all $x \in K$ then $x \in \mathbb{B}_i$ for some i and $\phi_i(x) = 1$, therefore

$$|\mathcal{B}u(x) - f(x)| = |\mathcal{B}P_i - f(x)| = |f_{x_i} - f(x)| \leq \eta.$$

To compute the integral bound we first notice that

$$\begin{aligned}\mathcal{B}u = \sum_i \mathcal{B}(P_i\phi_i) &\leq \sum_i \sum_{|\alpha|=l} B_\alpha \sum_{s+t=\alpha} \partial_s P_i \partial_t \phi_i \\ &\leq \sum_i \sum_{t \leq \alpha} \partial_t \phi_i \sum_{|\alpha|=l} B_\alpha \partial_{\alpha-t} P_i\end{aligned}$$

where the sum is to be considered on indices. After renaming $P = P_i$, expand the above summation into

$$\begin{aligned}\left|\sum_{|\alpha|=l} B_\alpha \partial_{\alpha-t} P\right| = \left|\sum_{|\alpha|=l} B_\alpha \partial_{\alpha-t} \sum_i \sum_{|\gamma|=l} \xi_i^\gamma \frac{x^\gamma}{\gamma!} v_i\right| &= \left|\sum_i \sum_{|\alpha|=l} B_\alpha v_i \sum_{|\gamma|=l} \xi_i^\gamma \partial_{\alpha-t} \frac{x^\gamma}{\gamma!}\right| \\ \leq \sum_i \sum_{|\alpha|=l} \|B_\alpha\| |v_i| \sum_{|\gamma|=l} |\xi^\gamma| \frac{1}{\gamma!} |\partial_{\alpha-t} x^\gamma|.\end{aligned}$$

Using that $|\xi| \leq 1$, $\gamma! \geq 1$, $|\partial_{\alpha-t} x^\gamma| \leq |x|^{|t|}$, the above can be further upper estimated by

$$\ldots \leq \sup_\alpha \|B_\alpha\| \sum_i |v_i| |x|^{|t|} \leq C\|v\| |x|^{|t|},$$

where the constant C depends only on $\mathcal{B}$. To compute the L^1 norm, first, we write

$$|U| = \left|\sum_i \sum_{|\alpha|=l} B_\alpha \sum_{s+t=\alpha} \partial_s P_i \partial_t \phi_i\right| \leq \sum_i \sum_t |\partial_t \phi_i| \left|\sum_{|\alpha|=l} B_\alpha \partial_{\alpha-t} P_i\right|,$$

and notice that the above is only supported on $\cup_i \mathbb{B}_i \setminus \mathbb{B}_i^\delta$, with $\delta = \varepsilon/(1+n)$. Then the L^1 norm of the above is

$$\|U\|_1 = \|U\|_{1, \cup_i \mathbb{B}_i^\delta} + \|U\|_{1, \cup_i \mathbb{B}_i \setminus \mathbb{B}_i^\delta} \leq \|f\|_{1, \cup_i \mathbb{B}_i^\delta} + \|U\|_{1, \cup_i \mathbb{B}_i \setminus \mathbb{B}_i^\delta}.$$

So the second term just becomes, on each $\mathbb{B}_i$ separately,

$$\sum_t C(r_i\varepsilon)^{-|t|}|f(x_i)|\int_{\mathbb{B}_i\setminus\mathbb{B}_i^\delta}|x|^{|t|}dx \leq C\varepsilon^{-l}|f(x_i)|\ \varepsilon|\mathbb{B}_i| \leq C\frac{1}{\varepsilon^{l-1}}\|f\|_{1,\mathbb{B}_i},$$

where we used that

$$\mathcal{L}^n(\mathbb{B}_i \setminus \mathbb{B}_i^\delta) \leq C\varepsilon\mathcal{L}^n(\mathbb{B}_i)$$

for a universal constant C, and $\delta = \delta(\varepsilon)$ as before. Conclude by summing over i. $\square$

Chapter 5

Lower semi-continuity of measure derivatives

In this chapter, we study the sequential lower semi-continuity of functionals of the form

$$u \mapsto \int_\Omega f\left(u, \frac{Du}{\mu}\right) d\mu, \tag{5.0.1}$$

where $\Omega \subset \mathbb{R}^n$ is open and bounded, $n > 1$ and u belongs to the Sobolev class

$$W^{1,1}_\mu(\Omega, \mathbb{R}^m) = \{u \in BV(\Omega, \mathbb{R}^m) : |Du| \ll \mu\}$$

with $m \geq 1$. Conditions on $f\colon \mathbb{R}^m \times \mathbb{R}^{m\times n} \to \mathbb{R}$ and $\mu \in \mathcal{M}^+(\Omega)$ will be discussed later.

This type of problem originated from [7], where the author studied the lower semi-continuity of integral functionals defined on $SBV(\Omega, \mathbb{R}^m)$ (see Chapter 4)

$$u \in SBV(\Omega, \mathbb{R}^m) \mapsto \int_\Omega f(u(x), \nabla u(x))dx + \int_{J_u} \phi(u^+, u^-, \eta)d\mathcal{H}^{n-1},$$

where u^+ and u^- are the upper and lower measure-theoretic limits of u on its jump set J_u, η is the outer normal to J_u and f and ϕ satisfy suitable growth assumptions. Here, the "crack site" (cit. [8]) J_u is penalised via the function ϕ, but it is not imposed a priori anywhere within Ω, which is different from the formulation (5.0.1) where the crack site is imposed, via μ, on a set where the singular part μ^s concentrates. For example, when $\Omega = \mathbb{B} \subset \mathbb{R}^2$ and $\mu = \mathcal{L}^2 + \mathcal{H}^1 \llcorner \{y = 0\}$, then $u \in W^{1,1}_\mu(\mathbb{B}, \mathbb{R})$ has the property that it belongs to $W^{1,1} = W^{1,1}_{\mathcal{L}^n}$ on the two halves $\mathbb{B} \cap \{y < 0\}$ and $\mathbb{B} \cap \{y > 0\}$ and its gradient is not a Lebesgue integrable function only on the "crack" $\{y = 0\}$. In this chapter, we will consider measures μ that can be fractional Hausdorff

measures, and so we no longer talk about "crack site" as the Cantor part of Du, for $u \in BV(\Omega, \mathbb{R}^m)$, could also be imposed a priori in the integral functional (5.0.1).

The main result of this chapter is based on the work carried out in [65], where the authors studied (5.0.1) when $f = f(z)$ does not depend on y, see also [8, 64]. To represent accumulation points of (5.0.1) when $(u_j)_j$ is a sequence in $W^{1,1}_\mu(\Omega, \mathbb{R}^m)$ that converges weakly* in $BV(\Omega, \mathbb{R}^m)$, we use a generalisation of Young Measures introduced in [94] called *Lifting Young Measures*, see also [55, 93].

Before stating the main theorem of this chapter, we define the sequential envelope of (5.0.1) to be

$$F_{**}(u) = \inf\left\{\liminf_j \int_\Omega f\left(u_j, \frac{Du_j}{\mu}\right) d\mu : u_j \in W^{1,1}_\mu, u_j \rightharpoonup^* u \text{ in } BV\right\}, \quad (5.0.2)$$

and we say that $\mu \in \mathcal{M}^+(\Omega)$ has α growth if there is $C > 0$ so that for all $r > 0$

$$\mu(\Omega \cap \mathbb{B}_r(x)) \leq Cr^\alpha \quad \forall x \in \Omega, r > 0.$$

Given the tools we rely upon, we need f to have a strong type of recession at infinity in the variable $y \in \mathbb{R}^m$.

Assumption 1. $f\colon \mathbb{R}^m \times \mathbb{R}^{m\times n} \to \mathbb{R}$ *satisfies*

- $f \in C(\mathbb{R}^m \times \mathbb{R}^{m\times n})$,
- *there is a constant* $C > 0$ *such that*

$$0 \leq f(z, y) \leq C(1 + |z|) \quad \forall y \in \mathbb{R}^m, z \in \mathbb{R}^{m\times n},$$

- *the limit*

$$\lim_{y'\to y, z'\to z, t\uparrow\infty} \frac{f(y', tz')}{t} = \sigma f^\infty(y, z) \in C(\sigma\mathbb{R}^m \times \mathbb{R}^{m\times n})$$

exists uniformly for all sequences $(y', t, z') \to (y, \infty, z) \in \sigma\mathbb{R}^m \times \{\infty\} \times \partial\mathbb{B}^{m\times n}$.

Here, $\sigma\mathbb{R}^m$ is the sphere compactification of $\mathbb{R}^m$ with its interior.

Given the strong assumptions on the recession of f, we can show the following.

Theorem 5.0.1 (Main)**.** *Let* $\mu \in \mathcal{M}^+(\Omega)$ *have* $p(n-1)$ *growth for some* $1 < p \leq \frac{n}{n-1}$ *and* $\mathcal{L}^n \ll \mu$. *Let* $f\colon \mathbb{R}^m \times \mathbb{R}^{m\times n} \to \mathbb{R}$ *be so that*

- $f(y, \cdot)$ *is quasi-convex for all* $y \in \mathbb{R}^m$,
- f *satisfies Assumption 1.*

Then

$$F_{**}(u) = \int_\Omega f\left(u, \frac{Du}{\mu}\right) d\mu + K_f[u] d\mathcal{H}^{n-1} \llcorner J_u + f^\infty\left(u, \frac{D^{c,\mu}u}{|D^{c,\mu}u|}\right) d|D^{c,\mu}u|. \quad (5.0.3)$$

This theorem is an extension of results such as [46, 94] to measure derivatives of BV functions, with the further restriction of f being non-negative and having no growth in y.

Regarding the formulation of (5.0.3), we underline that $D^{c,\mu}u$ is the singular part to μ of the Cantor part of u, meaning

$$D^c u = \frac{D^c u}{\mu} d\mu + D^{c,\mu}u,$$

with $D^{c,\mu}u \perp \mu$. On the other side, when we write $D^{s,\mu}u$ we mean the singular part to μ of the singular part of Du with respect to the Lebesgue measure, i.e.

$$Du = \nabla u(x)dx + D^s u, \quad D^s u = D^j u + D^c u,$$

and

$$D^s u = \frac{D^s u}{\mu} d\mu + D^{s,\mu}u,$$

with $D^{s,\mu}u \perp \mu$. The term $K_f(u)$ is called the *surface energy* of u and is defined by the formula

$$K_f[u] = \inf_{\phi|_{\partial\mathbb{B}^n} = u^\pm} \int_{\mathbb{B}^n} f^\infty(\phi(x), \nabla\phi(x))dx$$

where, letting u^+ and $u^- \in \mathbb{R}^m$ are upper and lower measure theoretic limits of u and $\eta(x) : J_u \to \mathbb{R}^n$ the outer normal to the jump set J_u of u (discussed in deeper detail in Section 5.1.3), ϕ belongs to the class $C^\infty(\mathbb{B}^n, \mathbb{R}^m) \cap W^{1,1}(\mathbb{B}^n, \mathbb{R}^m)$ and satifies, in the sense of trace,

$$\phi(y) = \begin{cases} u^+ & y \in \partial\mathbb{B}^n \text{ with } \langle y, \eta(x)\rangle \geq 0 \\ u^- & y \in \partial\mathbb{B}^n \text{ with } \langle y, \eta(x)\rangle < 0. \end{cases}$$

The assumption $\mathcal{L}^n \ll \mu$ is equivalent to $\mu = adx + \mu^s$ with $a > 0$ $\mathcal{L}^n$ almost everywhere and $\mu^s \perp \mathcal{L}^n$. This assumption guarantees that (5.0.1) behaves like $\mu = \mathcal{L}^n$ (which is classical) almost everywhere in Ω, and cannot be removed as the following example shows.

Example 5.0.1. *If $\Omega = \mathbb{B} \subset \mathbb{R}^2$ and $\mu = \mathcal{L}^n \llcorner (\mathbb{B} \cap \{y < 0\})$, then any $u \in W^{1,1}_\mu(\Omega)$ satisfies $Du \llcorner \mathbb{B} \cap \{y > 0\} = 0$, and because this set is connected, u is constant on $\mathbb{B} \cap \{y > 0\}$. Then $W^{1,1}_\mu(\Omega)$ is not dense in $BV(\Omega)$ and the result cannot hold true.*

5.0.1 μ-integrability of BV functions

In order to understand the behaviour of

$$u \mapsto \int_\Omega f\left(u, \frac{Du}{\mu}\right) d\mu$$

under small variations of u, it is first necessary to analyse the simpler case where $f = f(y)$ depends only on u. Thus, we first consider the integral functional

$$\int_\Omega f(u(x)) d\mu(x)$$

for $u \in BV(\Omega, \mathbb{R}^m)$. Since u is integrable with respect to the Lebesgue measure, the above integral, a priori, makes no sense, unless $\mu = adx$ for some $a \in L^\infty(\Omega, \mathcal{L}^n)$. However, with the help of a Sobolev-type embedding Theorem, we can assign values of u on a larger set than just $\mathcal{L}^n$-a.e.

Definition 5.0.2. *Let $\mu \in \mathcal{M}^+(\Omega)$. We say that μ has α growth if there is a constant $C > 0$ such that for all $x \in \Omega, r > 0$ we have*

$$\mu(\mathbb{B}_r(x) \cap \Omega) \leq Cr^\alpha. \tag{5.0.4}$$

It is easy to see that $\mu \in \mathcal{M}^+(\Omega)$ has α growth if and only if (5.0.4) is satisfied for all $x \in \mathbb{R}^n$, after possibly increasing C.

When μ has $p(n-1)$ growth with $1 < p \leq \frac{n}{n-1}$ we recover a Sobolev-type inequality as in the following theorem.

Theorem 5.0.3 ([101], Lemma 5.10.4, Theorem 5.13.1, Theorem 5.13.2)**.** *Let*

$$\mu \in \mathcal{M}^+(\Omega)$$

have $p(n-1)$ growth for some $1 < p \leq \frac{n}{n-1}$.

There are constants $C_1, C_2, C_3 > 0$ such that:

- *for all $u \in BV(\mathbb{R}^n, \mathbb{R}^m)$*

$$\|u\|_{L^p(\mu)} \leq C_1 |Du|(\mathbb{R}^m),$$

- *if $\Omega \subset \mathbb{R}^n$ is a connected domain, for all $u \in BV(\Omega, \mathbb{R}^m)$*

$$\left\| u - \fint_\Omega u d\mu \right\|_{L^p(\mu)} \leq C_2 |Du|(\Omega),$$

- *if $\Omega \subset \mathbb{R}^n$ is Lipschitz, for all $u \in BV(\Omega, \mathbb{R}^m)$*

$$\|u\|_{L^p(\mu)} \leq C_3 \|u\|_{BV(\Omega,\mathbb{R}^m)}.$$

Remark 5.0.4. *When $p = 1$ the above is false. However, it is still true that $\mu \in \mathcal{M}^+(\Omega)$ having $n-1$ growth belongs to the dual of $BV(\Omega)$, see [101], Theorem 5.12.4. The problem is that the precise representative of u fails to be continuous on $n-1$ hyper-surfaces, which might have non-zero μ measure.*

It makes sense to consider only those $\mu \in \mathcal{M}^+(\Omega)$ for which Theorem 5.0.3 holds, and for convenience we call such measures "p-admissible".

Definition 5.0.5. *When $\mu \in \mathcal{M}^+(\Omega)$ has $p(n-1)$ growth with $1 < p \leq \frac{n}{n-1}$, we say that μ is* p-admissible.

Example 5.0.2. *The example presented at the beginning of this chapter, $\mu = \mathcal{L}^2 + \mathcal{H}^1 \llcorner \{y = 0\}$, is not p-admissible because*

$$\mu(\mathbb{B}_r(0)) = \mathcal{L}^2(\mathbb{B}_1(0))r^2 + 2r \geq 2r.$$

On the other side, we can take a Borel set $E \subset \Omega$ with $0 < \mathcal{H}^\alpha(E) < +\infty$, $1 < \alpha < 2$, and so that $\mathcal{H}^\alpha(\mathbb{B}_r(x) \cap E) \leq Cr^\alpha$ for some $C > 0$ and all $x \in \Omega$ and $r > 0$. Then, set $\mu = \mathcal{L}^2 + \mathcal{H}^\alpha \llcorner E$ which satisfies

$$\mu(\mathbb{B}_r(x)) \leq \tilde{C} r^\alpha$$

for some $\tilde{C} > 0$, and so is p admissible with $p = \alpha$.

We now show that if μ is p-admissible, then weak* convergence in BV implies convergence in μ-measure of the precise representative. The proof of this statement is inspired by [37], Theorem 9.

Lemma 5.0.6. *Let $\mu \in \mathcal{M}^+(\Omega)$ be p-admissible. If $u_j \rightharpoonup^* u$ in BV, then $u_j \to u$ in μ-measure.*

Proof. Fix $\varepsilon > 0$ and find $K \subset \Omega$ compact and $\phi \in \mathcal{D}(\Omega), 0 \leq \phi \leq 1$ so that $\phi(K) = 1$ and $\mu(\Omega \setminus K) < \varepsilon$. Let $v_j = \phi(u_j - u)$ so that $v_j \rightharpoonup^* 0$ in $BV(\mathbb{R}^n)$. By Theorem 5.0.3 there is $C > 0$ such that

$$\int |v_j|^p d\mu \leq C^p |Dv_j|(\mathbb{R}^n)^p.$$

Because $|v_j|^p d\mu$ and $|Dv_j|$ are bounded sequences of measures, we can find a subsequence, identified with the original sequence, such that $|v_j|^p d\mu \rightharpoonup^* \eta$ and $|Dv_j| \rightharpoonup^* \gamma$ in $C_0(\mathbb{R}^n)^*$. Take $\phi \in \mathcal{D}(\mathbb{R}^n)^+$, then

$$\int \phi |v_j|^p d\mu \leq C^p \left(\int |\nabla \phi| |v_j| dx + \int \phi d|Dv_j| \right)^p,$$

and letting $j \to \infty$ we obtain

$$\int \phi d\eta \leq C^p \left(\int \phi d\gamma \right)^p.$$

In particular $\eta \ll \gamma$. For any $x \in \mathbb{R}^n$, we can approximate the unit ball $\overline{\mathbb{B}_r}$ by $\phi \in \mathcal{D}(\Omega)$ and obtain

$$\frac{\eta(\overline{\mathbb{B}_r})}{\gamma(\overline{\mathbb{B}_r})} \leq C^p \gamma(\overline{\mathbb{B}_r})^{p-1}.$$

Letting now $r \downarrow 0$ and using the Besicovich differentiation Theorem we get

$$\eta = \sum_i \eta_i \delta_{x_i}$$

where

$$\gamma \geq \sum_i \left(\frac{\eta_i}{C^p} \right)^{\frac{1}{p-1}} \delta_{x_i}.$$

In particular, $|v_j|^p d\mu \rightharpoonup^* \sum_i \eta_i \delta_{x_i}$ which is singular with respect to μ. I claim that such orthogonality implies convergence to 0 in μ-measure. Indeed, consider the Young Measure generated by $|v_j|^p d\mu$

$$|v_j|^p \xrightarrow{Y(\mu)} (\nu_x, \lambda, \nu_x^\infty).$$

Letting $\phi \in \mathcal{D}(\Omega)$ with $\phi \geq 0$ and $j \to \infty$ in $\int \phi |v_j|^p d\mu$ we infer that

$$\int \phi \, d \left(\sum_i \eta_i \delta_{x_i} \right) = \int \phi \langle \nu_x, id \rangle d\mu + \phi d\lambda.$$

So $\langle \nu_x, id \rangle = 0$ μ-a.e., and because $|v_j|^p \geq 0$ we have that $\nu_x = \delta_0 = \overline{\nu}_x$, i.e. $|v_j|^p \to 0$ in μ-measure. Now fix $\delta > 0$, we have that

$$\mu(|u_j - u| > \delta) \leq \varepsilon + \mu(v_j > \delta) \to \varepsilon.$$

Letting $\varepsilon \downarrow 0$ and using that convergence in μ measure is topological, we conclude that $u_j \to u$ in μ-measure. $\square$

See also [23], where convergence is proven to be pointwise outside of a set of Hausdorff-dimension $n-1$.

Remark 5.0.7. *In light of the above proof, we see that proving strong convergence of $(u_j)_j$ in $L^p(\Omega,\mu,\mathbb{R}^m)$, under weak* convergence in BV, is equivalent to asking that $(|u_j|^p)_j$ is μ equi-integrable.*

As remarked in 5.0.7, the fact alone that $u_j \rightharpoonup^* u$ in BV is not enough to imply that $u_j \to u$ strongly $L^p(\mu)$. In the following, we show that strict convergence in BV is what we need. We remind the reader that strict convergence of $(u_j)_j$ to u in $BV(\Omega,\mathbb{R}^m)$ amounts to

$$u_j \to u \text{ in } L^1(\Omega,\mathbb{R}^m) \quad \text{and} \quad Du_j \rightharpoonup^* Du \text{ in } C_0(\Omega,\mathbb{R}^{m\times n})^*.$$

The following result is not necessary for proving the lower semi-continuity Theorem 5.0.1 and it is only included for completeness.

Lemma 5.0.8. *Let $u_j \to u$ strictly in $BV(\Omega,\mathbb{R}^m)$. If $\mu \in \mathcal{M}^+(\Omega)$ is p-admissible, then $u_j \to u$ in $L^p(\Omega,\mu,\mathbb{R}^m)$.*

Proof. By Lemma 5.0.6 we have that $u_j \to u$ in μ-measure. Therefore, we only have to show that u_j is μ equi-integrable, that is

$$\lim_{k\to\infty}\left(\limsup_{j\to\infty}\int_{|u_j|\geq k}|u_j|^p d\mu = 0\right).$$

Notice that, because the trace is continuous with respect to strict convergence (see Lemma A.2.7), by extending u_j and u to elements of $BV(\mathbb{R}^{m\times n})$ by setting them to be 0 outside of Ω we can assume that $(u_j)_j$ and u belong to $BV(\mathbb{R}^n,\mathbb{R}^m)$.

The previous expression, as a function of k, is decreasing. So is enough to prove that it converges to zero for a subsequence of ks. If $v \geq 0$, by the co-area formula A.2.3

$$|Dv|(\Omega) = \int_0^{+\infty}\mathcal{H}^{n-1}(\partial\{v>t\})dt.$$

Therefore, we can find a set $\tilde{K}\subset\mathbb{R}$ so that $\mathcal{L}^1(\mathbb{R}^+\setminus\tilde{K})=0$ and $\mathcal{H}^{n-1}(\partial\{v>t\})<+\infty$ for all $t\in\tilde{K}$ and all $v\in\{|u|,|u_j|,j\in\mathbb{N}\}$.

Let v as above and mollify to obtain v_ε. By Sard's Theorem, there is $0<\delta_\varepsilon<\varepsilon$ so that $\{x : v_\varepsilon+\delta_\varepsilon > k\}$ has a smooth boundary and $\mu(\{x : v_\varepsilon+\delta_\varepsilon = k\})=0$. Call $w_\varepsilon = v_\varepsilon+\delta_\varepsilon$. Then

$$\int_{v_\varepsilon\geq k}|v_\varepsilon|^p d\mu \leq \int |w_\varepsilon|^p \chi_{w_\varepsilon\geq k}d\mu.$$

Next, notice that $w_\varepsilon \chi_{w_\varepsilon \geq k}$ is the precise representative of its class μ-a.e., so that by Theorem 5.0.3 we obtain the bound

$$\left(\int |w_\varepsilon|^p \chi_{w_\varepsilon \geq k} d\mu \right)^{\frac{1}{p}} \leq C \|w_\varepsilon \chi_{w_\varepsilon \geq k}\|_{BV} = C \left(\int_{w_\varepsilon \geq k} w_\varepsilon dx + |D(w_\varepsilon \chi_{w_\varepsilon \geq k})|(\mathbb{R}^n) \right).$$

About the first term, we have

$$\int_{w_\varepsilon \geq k} w_\varepsilon dx - \int_{v \geq k} v dx \leq \|w_\varepsilon - v\|_1 + \int v(\chi_{w_\varepsilon \geq k} - \chi_{v \geq k}).$$

The first term converges to 0 by strict convergence in BV. As for the second term, by the dominated convergence theorem we have

$$\limsup_\varepsilon \int v(\chi_{w_\varepsilon \geq k} - \chi_{v \geq k}) \leq 0.$$

In particular, the first term can be immediately estimated by

$$\limsup_\varepsilon \int_{w_\varepsilon \geq k} w_\varepsilon dx \leq \int_{v \geq k} v dx.$$

Using that $\{x : w_\varepsilon > k\}$ has a smooth boundary and the trace theorem,

$$\begin{aligned} |D(w_\varepsilon \chi_{w_\varepsilon > k})|(\mathbb{R}^n) &\leq |Dw_\varepsilon|(\{x : w_\varepsilon \geq k\}) + \int_{\partial\{w_\varepsilon > k\}} w_\varepsilon d\mathcal{H}^{n-1} \\ &= |Dw_\varepsilon|(\{x : w_\varepsilon \geq k\}) + k\mathcal{H}^{n-1}(\partial\{w_\varepsilon > k\}). \end{aligned}$$

In order to estimate the first term, use Theorem 5.1.4 (see also p. 973 of [93]) and strict convergence of w_ε to u_j in BV to conclude

$$\limsup_\varepsilon |Dw_\varepsilon|(\{x : w_\varepsilon \geq k\}) \leq |Du_j|(\{x : |u_j| \geq k\}).$$

As for the second term, first consider the following inequality:

$$\begin{aligned} |Du_j|(\mathbb{R}^n) = \lim_\varepsilon |Dw_\varepsilon|(\mathbb{R}^n) &= \lim_\varepsilon \int_0^\infty \mathcal{H}^{n-1}(\partial\{w_\varepsilon > t\}) dt \\ &\overset{\text{Fatou}}{\geq} \int_0^\infty \liminf_\varepsilon \mathcal{H}^{n-1}(\partial\{w_\varepsilon > t\}) dt \\ &\overset{\text{lsc}}{\geq} \int_0^\infty \mathcal{H}^{n-1}(\partial\{|u_j| > t\}) dt = |Du_j|(\mathbb{R}^n). \end{aligned}$$

Therefore, we can find a subset of $\tilde{K}$ that still has full measure and so that, for all $k \in \tilde{K}$,

$$\lim_\varepsilon \mathcal{H}^{n-1}(\partial\{w_\varepsilon > t\}) = \mathcal{H}^{n-1}(\partial\{|u_j| > t\})$$

for all $j \in \mathbb{N} \cup \{0\}$. Finally, for all $k \in \tilde{K}$, collecting the above inequalities together we obtain

$$\limsup_j \int_{|u_j|>k} |u_j| d\mu \leq \limsup_j |Du_j|(\{x : |u_j|(x) \geq k\}) + k\mathcal{H}^{n-1}(\partial\{|u_j| > k\}).$$

Once again, apply the same argument as before to reduce $\tilde{K}$ to a set of full measure for which

$$\mathcal{H}^{n-1}(\partial\{|u_j| > k\}) \to \mathcal{H}^{n-1}(\partial\{|u| > k\}),$$

because, by assumption, $u_j \to u$ strictly in BV. So we arrive at the formula

$$\limsup_j \int_{|u_j|>k} |u_j| d\mu \leq |Du|(\{x : |u|(x) \geq k\}) + k\mathcal{H}^{n-1}(\partial\{|u| > k\}).$$

Because the function

$$k \mapsto \mathcal{H}^{n-1}(\partial\{|u| > k\})$$

is integrable and $\mathcal{L}^1(\mathbb{R}^+ \setminus \tilde{K}) = 0$, we can find a sequence of $(k_s)_s$ in $\tilde{K}$ with $k_s \uparrow \infty$ and so that

$$k_s \mathcal{H}^{n-1}(\partial\{|u| > k_s\}) \to 0.$$

Then the above tends to 0 as $k \to \infty$.

To conclude, because $u_j \to u$ pointwise μ-a.e. and it is equi-integrable, we conclude that $u_j \to u$ strongly in $L^p(\Omega, \mu, \mathbb{R}^m)$ and the following embedding is continuous

$$(BV(\Omega, \mathbb{R}^m), strict) \hookrightarrow L^p(\Omega, \mu, \mathbb{R}^m).$$

□

We refer to [74] for recent developments on the pointwise convergence of $(u_j)_j$ when $u_j \rightharpoonup^* u$ in BV.

5.1 μ-Lifting Young Measures

The theory of Liftings was formulated in [55], and subsequently improved in [93] and [94], where it was applied to the study of generalised Gradient Young Measures.

Liftings are generalised measures that capture the interaction, in the limit, between a sequence of functions $(u_j)_j$ and their gradients $(\nabla u_j)_j$, where, say, $(u_j)_j$ in

$C^\infty(\Omega, \mathbb{R}^m)$ is weakly* pre-compact in BV. To be more precise, such Liftings allow one to understand small variations of functionals of the type

$$u \mapsto \int_\Omega f(x, u(x))\nabla u(x)dx \tag{5.1.1}$$

where $f \in C_0(\Omega \times \mathbb{R}^m)$, via the natural embedding

$$\begin{aligned}\int_\Omega f(x, u(x))\nabla u(x)dx &= \int_\Omega f(x, u(x))d(\nabla u(x)dx) \\ &= \int_{\Omega\times\mathbb{R}^m} f(x, y)d(\cdot, u(\cdot))_\#(\nabla u(x)dx).\end{aligned}$$

The expression $(\cdot, u(\cdot))_\#(\nabla u(x)dx) \in \mathcal{M}(\Omega \times \mathbb{R}^m, \mathbb{R}^{m\times n})$ stands for the pushforward of the measure $\nabla u(x)dx \in \mathcal{M}(\Omega, \mathbb{R}^{m\times n})$ by the map $x \mapsto (x, u(x))$.

Lifting Young Measures are generalised measures obtained by embedding Liftings into the space of Young Measures on $\mathbb{R}^{m\times n}$. These new generalised Young Measures were introduced in [94], and let one represent limits of

$$u \mapsto \int_\Omega f(x, u(x), \nabla u(x))dx,$$

which is a non-linear version of (5.1.1).

In this thesis, we do not work with classical Liftings as in [55, 93] and classical Liftings Young Measures as in [94], but a slight variation of them that fits our problem. Moreover, we only use these tools to obtain a representation of limits but do not use their intrinsic properties.

5.1.1 Preliminaries on Liftings

In what follows, we briefly introduce the theory of Liftings as in [55, 93]. We remind the reader that by writing $u \in BV_\#(\Omega, \mathbb{R}^m)$ we mean $u \in BV(\Omega, \mathbb{R}^m)$ with

$$\int_\Omega u(x)dx = 0.$$

Definition 5.1.1 ([55])**.** *A measure $\gamma \in \mathcal{M}(\Omega \times \mathbb{R}^m, \mathbb{R}^{m\times n})$ is said to be a Lifting if there exists $u \in BV_\#(\Omega, \mathbb{R}^m)$ such that*

$$\int_\Omega D_x\phi(x, u(x))dx + \int_{\Omega\times\mathbb{R}^m} D_y\phi(x, y)d\gamma(x, y) = 0 \quad \forall\phi \in \mathcal{D}(\Omega).$$

We will write $L(\Omega \times \mathbb{R}^m, \mathbb{R}^{m\times n})$ to indicate the set of Liftings.

Clearly, to every function $u \in C^\infty(\Omega, \mathbb{R}^m) \cap BV_\#(\Omega, \mathbb{R}^m)$ corresponds the Lifting

$$\gamma[u] := (\cdot, u(\cdot))_\# \nabla u(x) dx.$$

In the following, π is the projection from $\Omega \times \mathbb{R}^{m\times n} \to \Omega$, $\pi(x, y) = x$.

Lemma 5.1.2 ([55], Lemma 2.3)**.** *Let $\gamma \in L(\Omega \times \mathbb{R}^m, \mathbb{R}^{m\times n})$ be a Lifting. Then*

$$\pi_\# \gamma = Du \quad \textit{and} \quad \pi_\# |\gamma| \geq |Du|.$$

The function $u \in BV_\#(\Omega, \mathbb{R}^m)$ is uniquely associated with a Lifting γ, and we call u the *barycentre* of γ, and write $[\gamma] = u$. Clearly, for a fixed barycenter u there might be more Liftings γ with $[\gamma] = u$.

Lemma 5.1.3. *The set of Liftings $L(\Omega \times \mathbb{R}^m, \mathbb{R}^{m\times n})$ is sequentially weak* compact in the space of measures $\mathcal{M}(\Omega \times \mathbb{R}^m, \mathbb{R}^{m\times n})$. Furthermore, the map $\gamma \mapsto [\gamma]$ is sequentially weak* continuous in $BV_\#(\Omega, \mathbb{R}^m)$.*

We now discuss an important class of Liftings on functions $u \in BV_\#(\Omega, \mathbb{R}^m)$ that are not necessarily smooth. According to Theorem A.2.2, there is a Borel set $J_u \subset \Omega$, called the *jump set* of u, which is $n-1$ rectifiable and such that for all $x \in J_u$ there are $\eta(x) \in \partial\mathbb{B}^n$, $u^+(x) \in \mathbb{R}^m$ and $u^-(x) \in \mathbb{R}^m$ so that

$$⨍_{\mathbb{B}_r(x) \cap \{\langle x-y, \eta(x)\rangle \geq 0\}} u(y) dy \to u^+(x)$$

and

$$⨍_{\mathbb{B}_r(x) \cap \{\langle x-y, \eta(x)\rangle \leq 0\}} u(y) dy \to u^-(x).$$

There is also a Borel set $N \subset \Omega \setminus J_u, \mathcal{H}^{n-1}(N) = 0$ so that

$$⨍_{\mathbb{B}_r(x)} u(y) dy \to \frac{u^+(x) + u^-(x)}{2} =: \tilde{u}(x) \quad \forall x \in J_u$$
$$⨍_{\mathbb{B}_r(x)} u(y) dy \to \tilde{u}(x) \quad x \in \Omega \setminus (J_u \cup N)$$

and $\tilde{u} = u$ $\mathcal{L}^n$-a.e. This shows that u can be written as a linear interpolation of u^+ and u^-

$$\frac{u^+ + u^-}{2} = \tilde{u} \tag{5.1.2}$$

which is valid $\mathcal{H}^{n-1}$ almost everywhere. So, given a function $f \in C_0(\mathbb{R}^m)$, it is only natural to consider the Lifting

$$\int_{\Omega\times\mathbb{R}^m} f(x,y)d\gamma(x,y) = \int_\Omega \int_0^1 f(x,u^\theta(x))dDu(x), \tag{5.1.3}$$

where

$$u^\theta(x) = \theta u^-(x) + (1-\theta)u^+(x) \quad \theta \in (0,1)$$

$\mathcal{H}^{n-1}$-a.e., and so $|Du|$-a.e. as well.

Because u is evaluated through a non-linear function f, equation (5.1.3) represents a non-linear version of the averaging principle (5.1.2). We call these elementary representations *elementary Liftings*, and we indicate them by $\gamma[u]$.

When a sequence $(u_j)_j$ weak* converges to a function u in $BV(\Omega,\mathbb{R}^m)$, the weak* limit of the elementary Liftings $(\gamma[u_j])_j$ might not be elementary. To guarantee that the limit is still elementary, we need to ask that $u_j \to u$ strictly in BV.

Theorem 5.1.4 ([93], Lemma 3.2)**.** *Let $(u_j)_j$ and u in $BV_\#(\Omega,\mathbb{R}^m)$. If $u_j \to u$ strictly in BV, then $\gamma[u_j] \to \gamma[u]$ strictly in $\mathcal{M}(\Omega\times\mathbb{R}^m,\mathbb{R}^{m\times n})$.*

Remark 5.1.5. *If $u \in BV(\Omega,\mathbb{R}^m)$, we only talked about the* elementary Lifting *(or just Lifting) associated to u when*

$$\int_\Omega u(x)dx = 0.$$

However, it still makes sense to consider

$$\gamma[u] = (\cdot, u(\cdot))_\# \nabla u(x)dx,$$

and, with abuse of terminology, we still call $\gamma[u]$ the Lifting associated with u. Clearly, some of the classical theorems for Liftings might not apply to such measures. In the next section, we will always consider Liftings of arbitrary $u \in BV(\Omega,\mathbb{R}^m)$, as classical results about Liftings and Lifting Young Measures as in [93, 94] are never used directly.

5.1.2 μ-Lifting Young Measures

This section presents an extension of the work done in [94], with suited modifications to make it work in our context of measure derivatives. Specifically, we show how to define Lifting Young Measures whose oscillation is controlled by a given p-admissible measure $\mu \in \mathcal{M}^+(\Omega)$. Remember that $\pi(x,y) = x$ is the projection of $\Omega\times\mathbb{R}^m$ to Ω.

The following definition is a more general version of [94], Definition 4.1 which suffices for our scope. We decided to keep the name for the sake of consistency.

Definition 5.1.6. *Let $\mu \in \mathcal{M}^+(\Omega)$ be p-admissible. A $\nu \in LY(\Omega, \mu, \mathbb{R}^m)$ is a quadruple $(\iota, \nu, \lambda, \nu^\infty)$ such that*

- *$\iota \in \mathcal{M}^+(\Omega \times \mathbb{R}^m)$ satisfies $\pi_\# \iota = \mu$ and $\lambda \in \mathcal{M}^+(\overline{\Omega} \times \sigma\mathbb{R}^m)$,*
- *$\nu \colon \Omega \times \mathbb{R}^m \to \mathcal{M}_1^+(\mathbb{R}^{m\times n})$ and $\nu^\infty \colon \overline{\Omega} \times \sigma\mathbb{R}^m \to \mathcal{M}_1^+(\partial\mathbb{B}^{m\times n})$ are parametrised measure which are weak* measurable with respect to ι and λ respectively,*
- *the map $(x, y) \mapsto \langle \nu_{x,y}, |\cdot| \rangle$ belongs to $L^1(\Omega \times \mathbb{R}^m, \iota)$.*

By $\sigma\mathbb{R}^m$ we mean the compactification of $\mathbb{R}^m$ on the sphere and its interior,

$$\sigma\mathbb{R}^m \backsimeq \mathbb{R}^m \bar{\cup} \partial\mathbb{B}^m,$$

where $\bar{\cup}$ stands for the disjoint union.

The above class of Young Measures are nothing other than a way of representing limits of Liftings embedded in the set of Young Measures, which we define in the following way.

Definition 5.1.7. *Let $\gamma \in L(\Omega \times \mathbb{R}^m, \mathbb{R}^{m\times n})$. The* elementary μ-Lifting Young Measure *(or simply μ-Lifting Young Measure) associated to γ, with respect to a p-admissible measure $\mu \in \mathcal{M}^+(\Omega)$,*

$$\delta = \delta[\gamma] = (\iota_\delta, \delta, \lambda_\delta, \delta^\infty)$$

is given by the relations

$$\iota_\delta = \mathrm{gr}^{[\gamma]}_\#(\mu \llcorner \Omega), \quad \delta_{x,y} = \delta_{\frac{D[\gamma]}{\mu}}, \quad \lambda_\delta = |\gamma^{s,\mu}| \quad \text{and} \quad \delta^\infty_{x,y} = \delta_{\frac{\gamma^{s,\mu}}{|\gamma^{x,\mu}|}},$$

where

$$\mathrm{gr}^f = (\cdot, f(\cdot)), \quad \mathrm{gr}^f_\# \mu = (\cdot, f(\cdot))_\# \mu,$$

and

$$\gamma^{s,\mu} = \gamma - \mathrm{gr}^{[\gamma]}_\# \left(\frac{D[\gamma]}{\mu} d\mu \right).$$

The definition of elementary μ-Lifting Young Measure is independent of the compactification $\sigma\mathbb{R}^m$ as it is only defined in its interior $\mathbb{R}^m$.

To motivate where such a definition comes from, we introduce the following class of functions that can be used as test functions for our μ-Lifting Young Measures.

Assumption 2. *$f \colon \Omega \times \mathbb{R}^m \times \mathbb{R}^{m\times n} \to \mathbb{R}$ satisfies*

- $f \in C(\Omega \times \mathbb{R}^m \times \mathbb{R}^{m\times n})$,
- *there is a constant* $C > 0$ *such that*

$$0 \leq f(x, z, y) \leq C(1 + |z|) \quad \forall x \in \Omega, y \in \mathbb{R}^m, z \in \mathbb{R}^{m\times n},$$

- *the limit*

$$\lim_{x'\to x, y'\to y, z'\to z, t\uparrow\infty} \frac{f(x', y', tz')}{t} = \sigma f^\infty(x, y, z) \in C(\overline{\Omega} \times \sigma\mathbb{R}^m \times \mathbb{R}^{m\times n})$$

exists uniformly for all sequences

$$(x', y', t, z') \to (x, y, \infty, z) \in \overline{\Omega} \times \sigma\mathbb{R}^m \times \{\infty\} \times \partial\mathbb{B}^{m\times n}.$$

When $f = f(y, z)$, then Assumption 2 is the same as Assumption 1.

Compared to Definition 2.11 of [94], one sees that 2 is much stronger, and so fewer functions belong to this class. We do so because we want to use tools from the theory of generalised Young Measures introduced in Chapter 2.

To show how μ-Lifting Young Measures act on functions belonging to this class, we define the perspective integrand of f.

Definition 5.1.8. *Let f satisfy Assumption 2. We define the perspective integrand to be*

$$Pf(x, y, z, t) = \begin{cases} f\left(x, y, \frac{z}{|t|}\right)|t| & t \neq 0 \\ f^\infty(x, y, z) & t = 0. \end{cases}$$

Pf can be extended to the compactification of $\Omega \times \mathbb{R}^m \times \mathbb{R}^{m\times n}$ and we write

$$\sigma f^\infty(x, y, z) \in \overline{\Omega} \times \sigma\mathbb{R}^m \times \mathbb{R}^{m\times n}.$$

On the other side, we can elevate a Lifting to a μ-Lifting Young Measure via the action of P.

Definition 5.1.9. *Let $\mu \in \mathcal{M}^+(\Omega)$ be p-admissible and $\gamma \in L(\Omega \times \mathbb{R}^m, \mathbb{R}^{m\times n})$. We define the μ-perspective Lifting to be*

$$P_\mu\gamma = (\gamma, \mathrm{gr}^{[\gamma]}_{\#}\mu).$$

The Radon-Nykodym decomposition of γ with respect to $\mathrm{gr}^{[\gamma]}_{\#}\mu$ is

$$\gamma = \frac{D[\gamma]}{\mu}\mathrm{gr}^{[\gamma]}_{\#}\mu + \gamma^{s,\mu},$$

whereas, for the μ perspective measure, we obtain

$$P_\mu\gamma = \left(\frac{D[\gamma]}{\mu}, 1\right)\mathrm{gr}^{[\gamma]}_{\#}\mu + (\gamma^{s,\mu}, 0).$$

Lifting Young Measures act on integrands f satisfying Assumption 2 via the pairing

$$\ll \nu, f \gg = \int_{\Omega\times\mathbb{R}^m} \langle \nu_{x,y}, f(x,y,\cdot)\rangle d\iota(x,y) + \int_{\overline{\Omega}\times\sigma\mathbb{R}^m} \langle \nu^\infty_{x,y}, \sigma f^\infty(x,y,\cdot)\rangle d\lambda(x,y),$$

and when ν is elementary, i.e. for some $u \in BV(\Omega,\mathbb{R}^m)$ we have

$$\nu = \delta(\gamma[u]),$$

then

$$\langle Pf, P\gamma\rangle = \langle f, \delta(\gamma[u])\rangle.$$

This is analogous to the extension of measures to Young Measures performed in Chapter 2 and should be compared to Corollary 2.5.4.

We now state the sequential compactness theorem for $LY(\Omega,\mu,\mathbb{R}^m)$.

Theorem 5.1.10. *If $(u_j)_j$ is a sequence in $W^{1,1}_\mu(\Omega,\mathbb{R}^m)$ bounded in $BV(\Omega,\mathbb{R}^m)$, there is a subsequence (non-relabelled) and $\nu \in LY(\Omega,\mu,\mathbb{R}^m)$ so that*

$$\frac{\gamma[u_j]}{\mu} \xrightarrow{LY(\mu)} \nu \tag{5.1.4}$$

Proof. The proof is essentially the same as our construction of generalised Young Measures in Chapter 2, see also Propositions 4.10, 4.11, and 4.13 in [94]. □

As mentioned before, by (5.1.4) we mean

$$\delta(\gamma[u_j]) \xrightarrow{LY(\mu)} \nu,$$

where $\delta(\gamma[u_j])$ is as in 5.1.7.

We make here an important remark regarding our notation and definitions, which could result unfamiliar if compared to [94].

Remark 5.1.11. *When γ is an elementary Lifting, $\gamma = \gamma[u]$ for some $u \in BV(\Omega, \mathbb{R}^m)$, we write*

$$\frac{\gamma[u]}{\mu}$$

to indicate the elementary μ-Lifting Young Measure with respect to the measure μ. This is an abuse of notation, as $\gamma[u]$ is a measure on the product space $\Omega \times \mathbb{R}^m$ and cannot be differentiated with respect to μ, so we always interpret $\frac{\gamma[u]}{\mu}$ as in Definition 5.1.7. Notice that this abuse of notation is consistent with the notation from Chapter 2. Moreover, referring back to Remark 5.1.5 we insist that

$$(\cdot, u(\cdot))_{\#} \nabla u(x) dx$$

is not a Lifting in a classical sense because $u \in BV(\Omega, \mathbb{R}^m)$ might have non-zero average. Since we only use our definition of μ-Lifting Young Measures to obtain good representations of limits later in this chapter and never use intrinsic properties of these measures, we will not stress this abuse of notation any more.

We will only work with μ-Lifting Young Measures that are generated by a sequence of elementary μ-Lifting Young Measures. In this case, we write

$$\frac{\gamma[u_j]}{\mu} \xrightarrow{LY(\mu)} (\iota, \nu, \lambda, \nu^\infty),$$

or simply say that a sequence $(u_j)_j$ in $BV(\Omega, \mathbb{R}^m)$ generates $\nu \in LY(\Omega, \mu, \mathbb{R}^m)$.

Because on elementary Liftings we have the following equality $\iota = \mathrm{gr}_{\#}^{[\gamma]}\mu$, and because $u_j \rightharpoonup^* u$ in BV implies $u_j \to u$ in μ-measure, we can immediately recover the oscillation of μ-Lifting Young Measures when generated by BV functions.

Corollary 5.1.12. *Let $(u_j)_j$ in $BV(\Omega, \mathbb{R}^m)$ generate $\nu \in LY(\Omega, \mu, \mathbb{R}^m)$ and $u_j \rightharpoonup^* u$ in BV for some $u \in BV(\Omega, \mathbb{R}^m)$, then*

$$\iota = \mathrm{gr}_{\#}^{u} d\mu.$$

Proof. For $\psi \in \mathcal{D}(\Omega \times \mathbb{R}^m)$ we have

$$\langle \psi, \mathrm{gr}_{\#}^{[\gamma_j]}\mu \rangle = \int_{\Omega \times \mathbb{R}^m} \psi(x, u_j(x)) d\mu \to \int_{\Omega \times \mathbb{R}^m} \psi(x, u(x)) d\mu = \langle \psi, \mathrm{gr}_{\#}^{[\gamma]}\mu \rangle$$

provided $u_j \rightharpoonup^* u$ in BV. □

5.1.3 Surface energy and jump set of $u \in W^{1,1}_{\mu}(\Omega)$

Measure derivatives are always taken with respect to a measure $\mu \in \mathcal{M}^+(\Omega)$ which is p-admissible, i.e.

$$\mu(\mathbb{B}_r(x) \cap \Omega) \leq Cr^{p(n-1)} \quad \forall x \in \Omega, r > 0,$$

where $C > 0$ is a constant and $1 < p \leq \frac{n}{n-1}$. Because $p > 1$, it is easy to see that $\mu(E) = 0$ for every C^1 surface $E \subset \Omega$ of Hausdorff dimension $n-1$. Thus, if $u \in BV(\Omega, \mathbb{R}^m)$ and J_u is its jump set, we always have that $\mu(J_u) = 0$, and if

$$u \in W^{1,1}_{\mu}(\Omega, \mathbb{R}^m) = \{u \in BV(\Omega, \mathbb{R}^m) : |Du| \ll \mu\}$$

then $|D^j u| = 0$. So the elementary Lifting $\gamma[u] \in L(\Omega \times \mathbb{R}^m, \mathbb{R}^{m \times n})$ of $u \in W^{1,1}_{\mu}(\Omega, \mathbb{R}^m)$ is given by the formula

$$\int_{\Omega \times \mathbb{R}^m} f(x,y) d\gamma[u](x,y) = \int_{\Omega} f(x, u(x)) dDu(x) \quad \forall f \in C_0(\Omega \times \mathbb{R}^m),$$

and no jump part appears in the equation.

This means that, in our setting of measure derivatives, the behaviour of a sequence of μ-Lifting Young Measures $\left(\frac{\gamma[u_j]}{\mu}\right)_j$ in $LY(\mu)$ that converges to $\nu \in LY(\mu)$, with $u_j \rightharpoonup^* u$ in BV, is essentially unchanged from the classical case $\mu = \mathcal{L}^n$ around the jump set J_u of u.

Definition 5.1.13. *Let f satisfy Assumption 1, $u \in BV(\Omega, \mathbb{R}^m)$ and J_u its jump set. We define*

$$K_f[u](x) = \inf_{\phi|_{\partial \mathbb{B}^n} = u^{\pm}} \int_{\mathbb{B}^n} f^{\infty}(\phi(y), \nabla \phi(y)) dy,$$

where by $\phi|_{\partial \mathbb{B}^n} = u^{\pm}$ we mean $\phi \in C^{\infty}(\mathbb{B}^n, \mathbb{R}^m) \cap W^{1,1}(\mathbb{B}^n, \mathbb{R}^m)$ so that, in the sense of trace,

$$\phi(y) = \begin{cases} u^+ & y \in \partial \mathbb{B}^n \text{ with } \langle y, \eta(x) \rangle \geq 0 \quad \text{and} \\ u^- & y \in \partial \mathbb{B}^n \text{ with } \langle y, \eta(x) \rangle < 0 \end{cases}$$

This definition is clearly independent of μ.

5.2 Lower semi-continuous envelope

We divide this section into proving first the lower bound and then the upper bound for F_{**} as defined in (5.0.2). We mainly follow the strategy adopted in [65], which is to approximate a quasi-convex function by special quasi-convex functions (see Definition 5.2.1) and then use the partial homogeneity of functions belonging to this last class.

5.2.1 Special quasi-convex functions

Our approach, as in [65], is based on an approximation via quasi-convex functions that behave like homogeneous functions for large values of z.

Definition 5.2.1. *A function $f \in C(\mathbb{R}^m \times \mathbb{R}^{m\times n})$ is said to be special quasi-convex if*

1. *$0 \leq f(y,z) \leq C(1+|z|)$ for some $C > 0$ and all $z \in \mathbb{R}^{m\times n}$,*
2. *$f(y,\cdot)$ is quasi-convex for all $y \in \mathbb{R}^m$,*
3. *there are $R, i > 0$ so that for all $|z| \geq R$ and $y \in \mathbb{R}^m$ we have*
$$f_i(y,z) = f_i^\infty(y,z) - i.$$

The set of special quasi-convex functions is indicated by $SQ(\mathbb{R}^m \times \mathbb{R}^{m\times n})$.

Proving the main theorem on this subclass of function turns out to be much easier. In order to extend it to a function $f \in C(\mathbb{R}^m \times \mathbb{R}^{m\times n})$ satisfying Assumption 1, we will employ the following approximation result.

Lemma 5.2.2. *Let $f \in C(\mathbb{R}^m \times \mathbb{R}^{m\times n})$ satisfy Assumption 1. There exists a sequence $(f_i)_i$ in $SQ(\mathbb{R}^m \times \mathbb{R}^{m\times n})$ such that $f_i(y,z) \downarrow f(y,z)$ and $\sigma f_i(y,z)^\infty \downarrow \sigma f(y,z)^\infty$.*

Proof. Define
$$f_i(y,z) = \max\left\{ f(y,z), f^\infty(y,z) + \frac{|z|}{i} - i \right\}.$$
Condition (1) and (2) are trivially satisfied and
$$f_i(y,z) \downarrow f(y,z) \quad \text{as } i \uparrow \infty.$$
To prove that condition number (3), first notice that f_i extends to a continuous function on the set $\sigma\mathbb{R}^m \times \mathbb{R}^{m\times n}$. So assume that $f(y_j,z_j) > f^\infty(y_j,z_j) + \frac{|z|}{i} - i$ for a sequence $|z_j| \geq j$ and an arbitrary sequence $y_j \in \mathbb{R}^m$. Upon passing to a convergent subsequence, we let $y_j \to y \in \sigma\mathbb{R}^m$ and $\frac{z_j}{|z_j|} \to z \in \partial\mathbb{B}^{m\times n}$. Then
$$\frac{f(y_j,z_j)}{|z_j|} > \frac{f^\infty(y_j,z_j)}{|z_j|} + \frac{1}{i} - \frac{i}{|z_j|},$$
and letting $j \to \infty$ we reach a contradiction. □

Given f satisfying
$$f(y,z) \leq C(1+|z|) \quad \forall y \in \mathbb{R}^m, z \in R^{m\times n}$$
for some $C > 0$, it is clear that
$$f_i(y,z) \leq 2C(1+|z|) \quad \forall y \in \mathbb{R}^m, z \in R^{m\times n}$$
for all $i \in \mathbb{N}$.

5.2.2 Bound from below

The proof of the lower bound is divided into three parts. We first show how to obtain a lower bound on the absolutely continuous part if F_{**}, then prove a lower bound on the singular part and, finally, glue both together via a covering argument. We insist that, in this section, $\mu \in \mathcal{M}^+(\Omega)$ is always p-admissible. Moreover, we decompose μ into its absolutely continuous part and its singular part with respect to the Lebesgue measure,

$$\mu = a dx + \mu^s, \quad a \in L^1(\Omega, \mathcal{L}^n) \quad \text{and} \quad \mu^s \perp \mathcal{L}^n,$$

and so the condition $\mathcal{L}^n \ll \mu$ becomes $a > 0$ $\mathcal{L}^n$-a.e. Also, when we say that $f \in SQ(\mathbb{R}^m \times \mathbb{R}^{m\times n})$ we mean that f satisfies 5.2.1, and in particular it is non-negative.

5.2.2.1 Absolutely continuous part

Here, we show how to obtain the lower bound of the μ-absolutely continuous part of F_{**}. The remainder coming from the following lemma will be removed using a covering argument.

Lemma 5.2.3. *Let $U \subset \Omega$ be open, bounded, $\mu \in \mathcal{M}^+(\Omega)$ p-admissible with $\mathcal{L}^n \ll \mu$ and $f \in SQ(\mathbb{R}^m \times \mathbb{R}^{m\times n})$. If $(u_j)_j$ in $W^{1,1}_\mu(U, \mathbb{R}^m)$ and $u \in BV(\Omega, \mathbb{R}^m)$ are so that $u_j \rightharpoonup^* u$ in BV, then*

$$\begin{aligned}&\liminf_j \int_U f\left(u_j, \frac{Du_j}{\mu}\right) d\mu \\ &\geq \int_U f\left(u, \frac{\nabla u}{a}\right) a dx + K_f[u] d\mathcal{H}^{n-1} \llcorner J_u + f^\infty\left(u, \frac{D^c u}{|D^c u|}\right) d|D^c u| - (CR + i)\mu^s(U),\end{aligned}$$

where R and i are as in 5.2.1.

Proof. Write

$$f\left(u_j, \frac{Du_j}{\mu}\right) d\mu = f\left(u_j, \frac{\nabla u_j}{a}\right) a dx + f\left(u_j, \frac{D^s u_j}{\mu^s}\right) d\mu^s.$$

We start by looking at the first term. For $M > 0$ we let

$$f_M(x, y, z) = f\left(y, \frac{z}{a(x) \wedge M}\right) \; a(x) \wedge M,$$

where $s \wedge t := \min(s,t)$. This function is Carathéodory, quasi-convex and admits a continuous extension on the sphere compactification of $\mathbb{R}^{m\times n}$ as defined in Assumption 4, that is

$$(x,y,z) \mapsto f^\infty(y,z).$$

We underline that this condition is weaker than the assumption in our theorem (Assumption 1), but is enough so that we can use the classical result for $\mu = \mathcal{L}^n$, which is contained in the appendix A.4.1.

Next, we evaluate the difference with the absolutely continuous part of the original integral:

$$\begin{aligned}
&\int_U f\left(u_j, \frac{\nabla u_j}{a}\right) a - f_M(x,u_j,\nabla u_j)dx \\
&= \int_U f\left(u_j, \frac{\nabla u_j}{a}\right) a - f\left(u_j, \frac{\nabla u_j}{a\wedge M}\right) a\wedge M dx \\
&= \left(\overbrace{\int_{U_M\cap|\nabla u_j|<aR}}^{:=I_j} + \overbrace{\int_{U_M\cap|\nabla u_j|\geq aR}}^{:=II_j}\right) f\left(u_j, \frac{\nabla u_j}{a}\right) a - f\left(u_j, \frac{\nabla u_j}{M}\right) M dx,
\end{aligned}$$

where $U_M = \{x : a(x) > M\}$. The first term is estimated by

$$\begin{aligned}
|I_j| &\leq \int_{U_M\cap|\nabla u_j|<aR} f\left(u_j, \frac{\nabla u_j}{a}\right) a + f\left(u_j, \frac{\nabla u_j}{M}\right) M dx \\
&\leq C\int_{U_M\cap|\nabla u_j|<aR} (a+|\nabla u_j|) + (M+|\nabla u_j|)dx \leq 2C(1+R)\int_{U_M} a dx,
\end{aligned}$$

which converges to zero as $M\to\infty$ uniformly in j. On the other side, the second part can be estimated by

$$\begin{aligned}
|II_j| &= \left|\int_{U_M\cap|\nabla u_j|\geq aR} f^\infty\left(u_j, \frac{\nabla u_j}{a}\right) a - ia - f^\infty\left(u_j, \frac{\nabla u_j}{M}\right) M + iM dx\right| \\
&= i\int_{U_M\cap|\nabla u_j|\geq aR} (a-M)dx
\end{aligned}$$

which is also infinitesimal as $M\to\infty$, uniformly in j. Call the difference we just estimated ε_M.

Next, we evaluate the component on the singular part of μ. We compute

$$\begin{aligned}
&\left(\int_{U\cap\left|\frac{Du_j}{\mu^s}\right|<R}+\int_{U\cap\left|\frac{Du_j}{\mu^s}\right|\geq R}\right)\left(f\left(u_j,\frac{Du_j}{\mu^s}\right)-f^\infty\left(u_j,\frac{Du_j}{\mu^s}\right)\right)d\mu^s\\
&=\int_{U\cap\left|\frac{Du_j}{\mu^s}\right|<R} f\left(u_j,\frac{Du_j}{\mu^s}\right)-f^\infty\left(u_j,\frac{Du_j}{\mu^s}\right)d\mu^s\\
&\qquad -i\mu^s\left(U\cap\left|\frac{Du_j}{\mu^s}\right|<R\right)\\
&\geq -\int_{U\cap\left|\frac{Du_j}{\mu^s}\right|<R} f^\infty\left(u_j,\frac{Du_j}{\mu^s}\right)d\mu^s - i\mu^s\left(U\cap\left|\frac{Du_j}{\mu^s}\right|\geq R\right)\geq -(CR+i)\mu^s(U).
\end{aligned}$$

Because $Du_j \ll \mu$, then $D^s u_j \ll \mu^s$ and therefore we can estimate

$$\int_U f\left(u_j,\frac{Du_j}{\mu^s}\right)d\mu^s \geq \int_U f\left(u_j,\frac{D^s u_j}{|D^s u_j|}\right)d|D^s u_j|.$$

Collecting the above inequalities we obtain

$$\begin{aligned}
&\int_U f\left(u_j,\frac{Du_j}{\mu}\right)d\mu\\
&\geq \int_U f\left(u_j,\frac{\nabla u}{a\wedge M}\right)a\wedge M dx + f^\infty\left(u_j,\frac{D^s u_j}{|D^s u_j|}\right)d|D^s u_j| - \varepsilon_M - (CR+i)\mu^s(U).
\end{aligned}$$

Using that $f_M(x,y,z) = f\left(y,\frac{z}{a(x)\wedge M}\right)a\wedge M$ satisfies the assumption of Theorem A.4.1, we first let $j\to\infty$ and then $M\uparrow\infty$, and recover

$$\begin{aligned}
&\liminf_j \int_U f\left(u_j,\frac{Du_j}{\mu}\right)d\mu\\
&\geq \int_U f\left(u,\frac{\nabla u}{a}\right)a dx + K_f[u]d\mathcal{H}^{n-1}\llcorner J_u + f^\infty\left(u,\frac{D^c u}{|D^c u|}\right)d|D^c u| - (CR+i)\mu^s(U).
\end{aligned}$$

□

5.2.2.2 The singular part

Here, we provide a lower bound for the singular part. In this proof, μ-Lifting Young Measures will be used, alongside fundamental properties of 1-homogeneous quasi-convex functions $f\colon \mathbb{R}^m\times\mathbb{R}^{m\times n}\to\mathbb{R}$, such as convexity of f at all $z\in\mathbb{R}^{m\times n}$ with $rank(z)\leq 1$.

Lemma 5.2.4. *Let $V\subset\Omega$ be open and bounded, $\mu\in\mathcal{M}^+(\Omega)$ p-admissible with $\mathcal{L}^n\ll\mu$ and $f\colon\mathbb{R}^m\times\mathbb{R}^{m\times n}\to\mathbb{R}$ be so that*

- *$f(y,\cdot)$ is quasi-convex for all $y \in \mathbb{R}^m$,*
- *f satisfies Assumption 1.*

If $(u_j)_j$ in $W^{1,1}_\mu(U,\mathbb{R}^m)$ and $u \in BV(\Omega,\mathbb{R}^m)$ are so that $u_j \rightharpoonup^* u$ in BV, then

$$\liminf_j \int_V f\left(u_j, \frac{Du_j}{\mu}\right) d\mu \geq \int_V f\left(u, \frac{D^s u}{\mu^s}\right) d\mu^s.$$

Proof. Consider the μ-Lifting Young Measure $\nu \in LY(\mu, V)$ generated by $\frac{\gamma[u_j]}{\mu}$, say

$$\frac{\gamma[u_j]}{\mu} \xrightarrow{LY(\mu)} (\iota, \nu, \lambda, \nu^\infty_x).$$

By [5], Lemma 2.6, find a measurable vector field $\eta(x) \in \partial\mathbb{R}^m$ and ξ_j so that

$$\frac{Du_j}{\mu^s} = \xi_j \otimes \eta \quad \text{and} \quad \frac{Du}{\mu^s} = \xi \otimes \eta.$$

We are now going to show that for μ^s almost every $x \in \Omega$, the measure $\nu_{x,u(x)}$ is carried by the hyperplane $\mathbb{R}^n \times \eta(x)$. This was first shown in [65] for classical Gradient Young Measures.

Let $N \subset \Omega$ with $\mu^s(N) = 0$ be so that $\Omega \setminus \mathbb{N}$ is the set of Lebesgue μ^s points for the μ^s integrable functions $x \mapsto \eta(x)$ and $x \mapsto \langle \nu_{x,u(x)}, f\rangle$. We can also assume that, for every $x \in \Omega \setminus \mathbb{N}$, the following inequality holds

$$\int_{\mathbb{B}_r(x)} a dx \leq \varepsilon \mu(\mathbb{B}_r(x)),$$

which can be achieved by using the Besicovitch differentiation Theorem. Fix any $x_0 \in \Omega \setminus N$. For $R > 0$, we consider a class of functions that vanish on the hyperplane $\mathbb{R}^n \times \eta(x_0)$ and outside of a ball of radius R, for example

$$f(z) = \min\left\{1, dist\Big(z, \mathbb{R}^n \times \eta(x_0) \cup \mathbb{B}^c_R\Big)\right\} \in C_c(\mathbb{R}^{m\times n}).$$

We can immediately compute

$$\left| \fint_{\mathbb{B}_r(x_0)} f(\xi_j \otimes \eta) d\mu^s - f(\xi_j \otimes \eta(x_0)) d\mu^s \right| \leq R \fint_{\mathbb{B}_r(x_0)} |\eta - \eta(x_0)| d\mu^s \leq R\varepsilon$$

assuming r is small enough. We can also assume that

$$\fint_{\mathbb{B}_r(x_0)} |\langle \nu_{x,u(x)}, f\rangle - \langle \nu_{x_0,u(x_0)}, f\rangle| d\mu < \varepsilon.$$

Using the Young Measure representation of the limit and that f is compactly supported, we have

$$f\left(\frac{Du_j}{\mu}\right)d\mu \rightharpoonup^* \int_{\mathbb{R}^m} \langle \nu_{x,y}, f\rangle d\iota(x,y)$$
$$= \langle \nu_{x,u(x)}, f\rangle d\mu = \langle \nu_{x,u(x)}, f\rangle a dx + \langle \nu_{x,u(x)}, f\rangle d\mu^s.$$

Because f is non-negative and bounded by 1, weak* convergence in the sense of measure implies lower semi-continuity on open sets, thus

$$\begin{aligned}
\langle \nu_{x_0,u(x_0)}, f\rangle &\leq \fint_{\mathbb{B}_r(x_0)} \langle \nu_{x,u(x)}, f\rangle d\mu + \varepsilon \leq \liminf_j \fint_{\mathbb{B}_r(x_0)} f\left(\frac{Du_j}{\mu}\right)d\mu + \varepsilon \\
&= \liminf_j \frac{1}{\mu(\mathbb{B}_r(x_0))}\int_{\mathbb{B}_r(x_0)} f\left(\frac{Du_j}{\mu^s}\right)d\mu^s + \frac{1}{\mu(\mathbb{B}_r(x_0))}\int_{\mathbb{B}_r(x_0)} f\left(\frac{\nabla u_j}{a}\right)a dx + \varepsilon \\
&\leq \frac{\mu^s(\mathbb{B}_r(x_0))}{\mu(\mathbb{B}_r(x_0))}\liminf_j \fint_{\mathbb{B}_r(x_0)} f(\xi_j \otimes \nu)d\mu^s + \frac{1}{\mu(\mathbb{B}_r(x_0))}\int_{\mathbb{B}_r(x_0)} a dx + \varepsilon \\
&\leq \liminf_j \frac{\mu^s(\mathbb{B}_r(x_0))}{\mu(\mathbb{B}_r(x_0))} \fint_{\mathbb{B}_r(x_0)} f(\xi_j \otimes \nu(x_0))d\mu^s + (R+2)\varepsilon = (R+2)\varepsilon.
\end{aligned}$$

Letting $\varepsilon \downarrow 0$ we get that $\langle \nu_{x_0,u(x_0)}, f\rangle = 0$ and letting then $R \to \infty$ we conclude that for μ^s almost every $x \in \Omega$, the measure $\nu_{x,u(x)}$ is carried by the hyperplane $R^m \times \eta(x_0)$.

Next, take $f(y,z) = z$, which is clearly an admissible integrand that satisfies Assumption 1. The Young Measure limit provides the relation

$$\big((\cdot, u(\cdot))_\# Du\big)(x,y) = \langle \nu_{x,y}, id\rangle d\iota(x,y) + \langle \nu^\infty_{x,y}, id\rangle d\lambda(x,y).$$

as measures on $\Omega \times \mathbb{R}^m$. Let

$$\iota^s = \mathrm{gr}^u_\# \mu^s.$$

We can apply the differentiation Theorem A.4.4 to both sides of the equation, and using that

$$\frac{\iota}{\iota^s} = \frac{\mu}{\mu^s} = 1 \quad \iota^s\text{-a.e.},$$

we obtain the relation

$$\frac{u_\# Du}{\iota^s} = \frac{Du}{\mu^s} = \langle \nu_{x,u(x)}, id\rangle + \overbrace{\langle \nu^\infty_{x,u(x)}, id\rangle}^{:=R(x)} \frac{\lambda}{\iota^s}(x,u(x)).$$

After projecting using $\pi\colon \Omega \times \mathbb{R}^m \to \Omega$, the above holds also at μ^s almost every $x \in \Omega$.

Because both $\frac{D^s u}{\mu^s}$ and $\langle \nu_{x,u(x)}, id\rangle$ belong to $\mathbb{R}^m \times \eta(x)$ for μ^s almost every x, then

$$R(x) \in \mathbb{R}^m \times \eta(x) \quad \frac{\lambda}{\iota^s}(x, u(x)) d\mu^s\text{-a.e.}$$

Next, because $f^\infty(y, \cdot)$ is quasi-convex and 1-homogeneous, by Theorem A.3.3 it is convex at all $z \in \mathbb{R}^{m\times n}$ so that $\text{rank}(z) = 1$. In particular, $f^\infty(y, z) = (f^\infty)^c(y, z)$ for all matrices z having rank 1, where $(f^\infty)^c(y, \cdot)$ is the convex envelope of $f^\infty(y, \cdot)$ in the variable z, for each y fixed. Then, if we restrict $f(u(x), \cdot)$ to the space $\mathbb{R}^m \times \eta(x)$, such function is indeed convex and so

$$f(u(x), w + z) \leq f(u(x), w) + f^\infty(u(x), z)$$

for all $x \in \Omega$.

Therefore, we have

$$\begin{aligned}
f\left(u(x), \frac{D^s u}{\mu^s}(x)\right) &\leq f(u(x), \langle \nu_{x,u(x)}, id\rangle) + f^\infty(u(x), R(x)) \\
&= f(u(x), \langle \nu_{x,u(x)}, id\rangle) + f^\infty(u(x), \langle \nu^\infty_{x,u(x)}, id\rangle)\frac{\lambda}{\iota^s}(x, u(x)) \\
&= f(u(x), \langle \nu_{x,u(x)}, id\rangle) + (f^\infty)^c(u(x), \langle \nu^\infty_{x,u(x)}, id\rangle)\frac{\lambda}{\iota^s}(x, u(x)) \\
&\overset{\text{Jensen}}{\leq} \langle \nu_{x,u(x)}, f(u(x), \cdot)\rangle + \langle \nu^\infty_{x,u(x)}, (f^\infty)^c(u(x), \cdot)\rangle\frac{\lambda}{\iota^s}(x, u(x)) \\
&= \langle \nu_{x,u(x)}, f(u(x), \cdot)\rangle + \langle \nu^\infty_{x,u(x)}, f^\infty(u(x), \cdot)\rangle\frac{\lambda}{\iota^s}(x, u(x))
\end{aligned}$$

Integrating everything in μ^s and using the positivity of f we conclude

$$\begin{aligned}
&\int_V f\left(u(x), \frac{D^s u}{\mu^s}(x)\right) d\mu^s \\
&\leq \int_V \langle \nu_{x,u(x)}, f(u(x), \cdot)\rangle d\mu^s + \int_V \langle \nu^\infty_{x,u(x)}, f^\infty(u(x), \cdot)\rangle\frac{\lambda}{\iota^s}(x, u(x)) d\mu^s \\
&= \int_V \langle \nu_{x,u(x)}, f(u(x), \cdot)\rangle d\mu^s + \int_{V\times\mathbb{R}^m} \langle \nu^\infty_{x,y}, f^\infty(y, \cdot)\rangle\frac{\lambda}{\iota^s} d\iota^s \\
&\leq \int_V \langle \nu_{x,u(x)}, f(u(x), \cdot)\rangle d\mu + \int_{V\times\sigma\mathbb{R}^m} \langle \nu^\infty_{x,y}, f^\infty(y, \cdot)\rangle d\lambda \\
&\leq \lim_j \int_V f\left(u, \frac{Du_j}{\mu}\right) d\mu.
\end{aligned}$$

□

5.2.2.3 Combining absolutely continuous and singular part

In this section, we combine the estimates of the absolutely continuous and the singular part of the lower bound. We divide the proof into two parts.

Theorem 5.2.5. *Let $\mu \in \mathcal{M}^+(\Omega)$ be p-admissible and so that $\mathcal{L}^n \ll \mu$, and let $f = f(y,z) \in C(\mathbb{R}^m \times \mathbb{R}^{m\times n})$ be so that*

1. *$f(y,\cdot)$ is quasi-convex for all $y \in \mathbb{R}^m$,*
2. *f satisfies Assumption 1.*

If $(u_j)_j$ in $W^{1,1}_\mu(\Omega,\mathbb{R}^m)$ and $u \in BV(\Omega,\mathbb{R}^m)$ are such that $u_j \rightharpoonup^* u$ in BV, then

$$\begin{aligned}&\liminf_j \int_\Omega f\left(u_j, \frac{Du_j}{\mu}\right) d\mu \\ &\quad\geq \int_\Omega f\left(u, \frac{Du}{\mu}\right) d\mu + K_f[u] d\mathcal{H}^{n-1} \llcorner J_u + f^\infty\left(u, \frac{D^{s,\mu}u}{|D^{s,\mu}u|}\right) d|D^{s,\mu}u|.\end{aligned}$$

We first prove the theorem for the subclass of special quasi-convex functions.

Lemma 5.2.6. *Theorem 5.2.5 holds provided $f \in SQ(\mathbb{R}^m \times \mathbb{R}^{m\times n})$.*

Proof. Let $f \in SQ(\mathbb{R}^m \times \mathbb{R}^{m\times n})$, with parameters i and R as in 5.2.1, where we can also assume that

$$0 \leq f(z) \leq (1 + |z|).$$

Fix $\varepsilon > 0$ and let $H \subset \Omega$ be so that $\mathcal{L}^n(H) = 0 = \mu^s(\Omega \setminus H)$. Also, let $D \subset \Omega$ be so that $\mu(D) = 0 = |D^{s,\mu}u|(\Omega \setminus D)$. Let $G \subset \Omega$ open so that $H \setminus D \subset G$ and

$$|D^{s,\mu}u|(G) + \int_G (a + |\nabla u|)dx < \varepsilon,$$

where we remind that $|D^j u| \leq |D^{s,\mu}u|$ and

$$\int_G K_f[u] d\mathcal{H}^{n-1} \llcorner J_u \leq |D^j u|(G).$$

The set of balls $\{\overline{\mathbb{B}_r(x)}\}_{x\in H\setminus D, r>0}$ such that $\mu(\partial\mathbb{B}_r(x)) = 0$ and $\overline{\mathbb{B}_r(x)} \subset G$ is a fine cover for $H \setminus D$, and so we can find countably many, pairwise disjoint $\mathbb{B}_i = \overline{\mathbb{B}_{r_i}(x_i)}$ such that $\mu^s((H \setminus D) \setminus \cup_i \mathbb{B}_i) = 0$. In particular by our choice of H,

$$\mu^s(\Omega \setminus \cup_i \mathbb{B}_i) = 0.$$

Then we can find $m \in \mathbb{N}$ big enough so that

$$\mu(\cup_{i\geq m}\mathbb{B}_i) + \int_{\cup_{i\geq m}\mathbb{B}_i} \left(\frac{|D^s u|}{\mu^s}\right) d\mu^s < \varepsilon.$$

By Lemma 5.2.4 and because $\mu(\partial\mathbb{B}_i) = 0$ for all i, we get

$$\begin{aligned}
&\liminf_j \int_{\cup_{1\leq i\leq m}\mathbb{B}_i} f\left(u, \frac{Du_j}{\mu}\right) d\mu \\
&\qquad \geq \int_{\cup_{1\leq i\leq m}\mathbb{B}_i} f\left(u, \frac{Du}{\mu^s}\right) d\mu^s = \int_\Omega f\left(u, \frac{D^s u}{\mu^s}\right) d\mu^s - \varepsilon.
\end{aligned}$$

On the other side, because $\Omega \setminus \cup_{1\leq i\leq m}\mathbb{B}_i$ is open, we can use the lower semi-continuity result 5.2.3 to get

$$\begin{aligned}
&\liminf_j \int_{\Omega\setminus_{1\leq i\leq m}\mathbb{B}_i} f\left(u, \frac{Du_j}{\mu}\right) d\mu \\
&\geq \int_{\Omega\setminus_{1\leq i\leq m}\mathbb{B}_i} f\left(u, \frac{\nabla u}{a}\right) a dx + K_f[u] d\mathcal{H}^{n-1} \llcorner J_u + f^\infty\left(u, \frac{D^c u}{|D^c u|}\right) d|D^c u| \\
&\qquad - (MR+i)\mu^s(\Omega \setminus_{1\leq i\leq m} \mathbb{B}_i) \\
&\geq \int_{\Omega\setminus_{1\leq i\leq m}\mathbb{B}_i} f\left(u, \frac{\nabla u}{a}\right) a dx + K_f[u] d\mathcal{H}^{n-1} \llcorner J_u + f^\infty\left(u, \frac{D^{c,\mu} u}{|D^{c,\mu} u|}\right) d|D^{c,\mu} u| \\
&\qquad - (MR+i)\varepsilon \\
&\geq \int_\Omega f\left(u, \frac{\nabla u}{a}\right) a dx + K_f[u] d\mathcal{H}^{n-1} \llcorner J_u + f^\infty\left(u, \frac{D^{c,\mu} u}{|D^{c,\mu} u|}\right) d|D^{c,\mu} u| \\
&\qquad - (M(R+1)+i+1)\varepsilon,
\end{aligned}$$

where we used that

$$\frac{D^c u}{\mu} = \frac{D^s u}{\mu}$$

given that μ is p-admissible, and that $f \geq 0$ and $|D^c u| \geq |D^{c,\mu} u|$. Combining the two inequalities we then get

$$\begin{aligned}
&\liminf_j \int_\Omega f\left(u_j, \frac{Du_j}{\mu}\right) d\mu \\
&\quad \geq \int_\Omega f\left(u, \frac{\nabla u}{a}\right) a dx + K_f[u] d\mathcal{H}^{n-1} \llcorner J_u + f^\infty\left(u, \frac{D^{c,\mu} u}{|D^{c,\mu} u|}\right) d|D^{c,\mu} u| \\
&\qquad + f\left(u, \frac{D^s u}{\mu^s}\right) d\mu^s - C(M,R,i)\varepsilon \\
&= \int_\Omega f\left(u, \frac{Du}{\mu}\right) d\mu + K_f[u] d\mathcal{H}^{n-1} \llcorner J_u + f^\infty\left(u, \frac{D^{s,\mu} u}{|D^{s,\mu} u|}\right) d|D^{s,\mu} u| \\
&\qquad - C(M,R,i)\varepsilon.
\end{aligned}$$

Let $\varepsilon \downarrow 0$ to obtain the sought lower bound. □

We now show how to connect the lower semi-continuity on functions $f \in SQ(\mathbb{R}^m \times \mathbb{R}^{m\times n})$ to functions $f \in C(\mathbb{R}^m \times \mathbb{R}^{m\times n})$ with $f(y,\cdot)$ quasi-convex for all $y \in \mathbb{R}^m$ and satisfying Assumption 1.

Let $(u_j)_j$ in $BV(\Omega,\mathbb{R}^m)$ generate the μ-Lifting Young Measure $\nu \in LY(\mu)$

$$\frac{\gamma[u_j]}{\mu} \xrightarrow{LY(\mu)} (\iota,\nu,\lambda,\nu^\infty).$$

For all special $f \in SQ(\mathbb{R}^m \times \mathbb{R}^{m\times n})$ that in addition satisfy 1 we have

$$\begin{aligned}
&\int_\Omega \langle \nu_{x,u(x)}, f\rangle d\mu + \int_{\overline{\Omega}\times\sigma\mathbb{R}^m} \langle \nu^\infty_{x,y}, \sigma f^\infty(y,\cdot)\rangle d\lambda \\
&\quad = \lim_j \int_\Omega f\left(u_j, \frac{Du_j}{\mu}\right) d\mu \\
&\quad \geq \int_\Omega f\left(u, \frac{Du}{\mu}\right) d\mu + K_f[u] d\mathcal{H}^{n-1} \llcorner J_u + f^\infty\left(u, \frac{D^{s,\mu}u}{|D^{s,\mu}u|}\right) d|D^{s,\mu}u|.
\end{aligned}$$

So to prove the main result 5.2.5 we only have to recall the approximation Lemma 5.2.2.

Proof. (of Theorem 5.2.5).

Consider the μ-Lifting Young Measure generated by $(u_j)_j$ in $BV(\Omega,\mathbb{R}^m)$

$$\frac{\gamma[u_j]}{\mu} \xrightarrow{LY(\mu)} (\iota,\nu,\lambda,\nu^\infty).$$

For all f satisfying the main Assumption 1 we have

$$\lim_j \int_\Omega f\left(u_j, \frac{Du_j}{\mu}\right) d\mu = \int_\Omega \langle \nu_{x,u(x)}, f\rangle d\mu + \int_{\overline{\Omega}\times\sigma\mathbb{R}^m} \langle \nu^\infty_{x,y}, \sigma f(y,\cdot)^\infty\rangle d\lambda.$$

By Lemma 5.2.2 we can find f_i such that $f_i \downarrow f$ and $f_i^\infty \downarrow f^\infty$. Therefore,

$$\begin{aligned}
&\int_\Omega \langle \nu_{x,u(x)}, f\rangle + \int_{\overline{\Omega}\times\sigma\mathbb{R}^m} \langle \nu^\infty_{x,y}, \sigma f^\infty(y,\cdot)\rangle d\lambda \\
&= \lim_i \int_\Omega \langle \nu_{x,u(x)}, f_i\rangle d\iota + \int_{\overline{\Omega}\times\sigma\mathbb{R}^m} \langle \nu^\infty_{x,y}, \sigma f_i^\infty(y,\cdot)\rangle d\lambda \\
&\geq \liminf_i \int_\Omega f_i\left(u, \frac{Du}{\mu}\right) d\mu + K_{f_i}[u] d\mathcal{H}^{n-1} \llcorner J_u + f_i^\infty\left(u, \frac{D^{c,\mu}u}{|D^{c,\mu}u|}\right) d|D^{c,\mu}u| \\
&\geq \int_\Omega f\left(u, \frac{Du}{\mu}\right) d\mu + K_f[u] d\mathcal{H}^{n-1} \llcorner J_u + f^\infty\left(u, \frac{D^{c,\mu}u}{|D^{c,\mu}u|}\right) d|D^{c,\mu}u|,
\end{aligned}$$

where in the last passage we just used that $f_i \geq f$. □

This concludes the proof of the lower bound for any limit $u_j \rightharpoonup^* u$ in BV. We shall now prove the upper bound.

5.2.3 Bound from above

In this section, we show that the formula for F_{**} is actually the lower semi-continuous relaxation in $BV(\Omega, \mathbb{R}^m)$ with respect to sequential weak* convergence.

Theorem 5.2.7. *Let $\mu \in \mathcal{M}^+(\Omega)$ be p-admissible and so that $\mathcal{L}^n \ll \mu$, and let $f\colon \mathbb{R}^m \times \mathbb{R}^{m\times n} \to \mathbb{R}$ be so that*

- *$f(y,\cdot)$ is quasi-convex for all $y \in \mathbb{R}^m$,*
- *f satisfies Assumption 1.*

For every $u \in BV(\Omega, \mathbb{R}^m)$ we have

$$F_{**}(u) \le \int_\Omega f\left(u, \frac{Du}{\mu}\right) d\mu + K_f[u] d\mathcal{H}^{n-1} \llcorner J_u + f^\infty\left(u, \frac{D^{c,\mu}u}{|D^{c,\mu}u|}\right) d|D^{c,\mu}u|.$$

Similarly to the previous section, we first prove the theorem for special quasi-convex functions and then use an approximation argument.

Theorem 5.2.8. *Theorem 5.2.7 holds for $f \in SQ(\mathbb{R}^m \times \mathbb{R}^m)$.*

Proof. Let f in $SQ(\mathbb{R}^m \times \mathbb{R}^{m\times n})$ and R and i be given as in Definition 5.2.1, and without loss of generality assume that

$$0 \le f(y,z) \le 1 + |z|.$$

Fix $\varepsilon > 0$ and find a Borel set $D \subset \Omega$ so that $|D^{s,\mu}u|(\Omega \setminus D) = \mu(D) = 0$ and let $G \supset D$ open so that

$$\int_G (R + i + 1 + |\nabla u|)dx + \left(1 + R + \left|\frac{Du}{\mu}\right|\right) d\mu < \varepsilon.$$

Next, let $(u_j)_j$ in $C^\infty(G, \mathbb{R}^m) \cap W^{1,1}(G, \mathbb{R}^m)$ be as in Theorem A.4.2, with ε as above.

For every $j \in \mathbb{N}$, because $u_j \in C^\infty(G, \mathbb{R}^m)$, then

$$\frac{Du_j}{\mu} = \frac{\nabla u_j}{a} \quad \mu\text{-a.e.},$$

and so $a\,dx$-a.e. as well. Thus,

$$\begin{aligned}\int_G f\left(u_j, \frac{Du_j}{\mu}\right) d\mu &= \int_G f\left(u_j, \frac{\nabla u_j}{a}\right) a dx \\ &\le \int_{G\cap|\nabla u_j|>Ra} f\left(u_j, \frac{\nabla u_j}{a}\right) a dx + (1+R)\mu(G).\end{aligned}$$

Because $f(y,z) = f^\infty(y,z) - i \le f^\infty(y,z)$ for all $|z| \ge R$, uniformly in $y \in \mathbb{R}^m$, then

$$\begin{aligned}\int_{G\cap|\nabla u_j|>Ra} f\left(u_j, \frac{\nabla u_j}{a}\right) a dx &\le \int_{G\cap|\nabla u_j|>Ra} f^\infty\left(u_j, \frac{\nabla u_j}{a}\right) a dx \\ &\le \int_G f^\infty(u_j, \nabla u_j) dx \\ &\le \int_{G\cap|\nabla u_j|>R} f^\infty(u_j, \nabla u_j) dx + R\mathcal{L}^n(G) \\ &= \int_{G\cap|\nabla u_j|>R} f(u_j, \nabla u_j) dx + (R+i)\mathcal{L}^n(G).\end{aligned}$$

Taking the liminf as $j \to \infty$ we get

$$\begin{aligned}&\liminf_j \int_G f\left(u_j, \frac{Du_j}{\mu}\right) d\mu \\ &\le \liminf_j \int_G f(u_j, \nabla u_j) dx + (R+i)\mathcal{L}^n(G) + (1+R)\mu(G).\end{aligned}$$

Using that $(u_j)_j$ is an ε-recovery sequence on G, we have

$$\begin{aligned}&\liminf_j \int_G f(u_j, \nabla u_j) dx \\ &\le \int_G f(u, \nabla u) dx + K_f[u] d\mathcal{H}^{n-1} \llcorner J_u + f^\infty\left(u, \frac{D^c u}{|D^c u|}\right) d|D^c u| + \varepsilon.\end{aligned}$$

Putting the estimates together we obtain the upper bound

$$\begin{aligned}&\liminf_j \int_\Omega f(u_j, \nabla u_j(x)) dx \\ &\le \int_G f(u, \nabla u) dx + K_f[u] d\mathcal{H}^{n-1} \llcorner J_u + f^\infty\left(u, \frac{D^c u}{|D^c u|}\right) d|D^c u| \\ &\quad + (R+i)\mathcal{L}^n(G) + (1+R)\mu(G) + C\varepsilon.\end{aligned}$$

The absolutely continuous part is immediately upper bounded by

$$\int_G f(u, \nabla u) dx \le \int_G (1 + |\nabla u|) dx,$$

and on the singular part we have

$$\begin{aligned}\int_G f^\infty\left(u, \frac{D^c u}{|D^c u|}\right) d|D^c u| &= \int_G f^\infty\left(u, \frac{D^c u}{\mu}\right) d\mu + \int_G f^\infty\left(u, \frac{D^{c,\mu} u}{|D^{c,\mu} u|}\right) d|D^{c,\mu} u| \\ &\le \int_G \frac{|D^c u|}{\mu} d\mu + \int_G f^\infty\left(u, \frac{D^{c,\mu} u}{|D^{c,\mu} u|}\right) d|D^{c,\mu} u| \\ &\le \int_G f^\infty\left(u, \frac{D^{c,\mu} u}{|D^{c,\mu} u|}\right) d|D^{c,\mu} u| + \varepsilon.\end{aligned}$$

So, altogether, we obtain

$$\begin{aligned}&\liminf_j \int_G f\left(u_j, \frac{Du_j}{\mu}\right) d\mu \\ &\quad \leq \int_\Omega K_f[u] d\mathcal{H}^{n-1} \llcorner J_u + f^\infty\left(u, \frac{D^{c,\mu}u}{|D^{c,\mu}u|}\right) d|D^{c,\mu}u| + C\varepsilon,\end{aligned}$$

where $C > 0$ is a universal constant.

Next, we need to adjust for the absolutely continuous part to μ of the integral functional. As remarked in A.4.3, the sequence $(u_j)_j$ in $C^\infty(G, \mathbb{R}^m)$ can be taken to have trace u in $BV(G, \mathbb{R}^m)$. Let $v_j = u\chi_{\Omega\setminus G} + u_j\chi_G$. Then, for all $j \in \mathbb{N}$, $v_j \in BV(\Omega, \mathbb{R}^m)$ with

$$Dv_j = Du \llcorner \Omega \setminus G + \nabla u_j(x)dx \llcorner G,$$

and so

$$\begin{aligned}&\liminf_j \int_\Omega f\left(v, \frac{Dv_j}{\mu}\right) d\mu = \int_{\Omega\setminus G} f\left(u, \frac{Du}{\mu}\right) d\mu + \liminf_j \int_G f\left(u_j, \frac{Du_j}{\mu}\right) d\mu \\ &\quad \leq \int_\Omega f\left(u, \frac{Du}{\mu}\right) d\mu + \int_\Omega K_f[u] d\mathcal{H}^{n-1} \llcorner J_u + f\left(u, \frac{D^{s,\mu}u}{|D^{s,\mu}u|}\right) d|D^{s,\mu}u| + C\varepsilon\end{aligned}$$

Conclude by the arbitrariness of $\varepsilon > 0$. □

Similarly to the case of the lower bound, we can use μ-Lifting Young Measures and the approximation by special quasi-convex functions to prove the upper bound when f satisfies the Assumption 1.

Proof. (of Theorem 5.2.7).

Let f_i be an approximation of f as in Lemma 5.2.2. For $\mathcal{H}^{n-1}$-a.e. $x \in \Omega$ we have

$$\limsup_i K_{f_i} = \limsup_i \inf_\phi \int_{\mathbb{B}^n} f_i^\infty(\phi, \nabla\phi) \leq \limsup_i \int_{\mathbb{B}^n} f_i^\infty(\phi, \nabla\phi) = \int_{\mathbb{B}^n} f^\infty(\phi, \nabla\phi)$$

and so passing to the inf again we get

$$\limsup_i K_{f_i}[u] \leq K_f[u].$$

But because we also have $K_{f_i}[u] \geq K_{f_{i+1}}[u] \geq K_f[u]$, then $K_{f_i}[u] \downarrow K_f[u]$ and applying the dominated convergence theorem we get

$$\begin{aligned}
&F_{**}(u) \leq \liminf_j \int_\Omega f\left(v_j, \frac{Dv_j}{\mu}\right) d\mu \\
&\leq \liminf_j \int_\Omega f_i\left(v_j, \frac{Dv_j}{\mu}\right) d\mu \\
&\leq \int_\Omega f_i\left(u, \frac{Du}{\mu}\right) d\mu + K_{f_i}[u](x) d\mathcal{H}^{n-1} \llcorner J_u + f_i^\infty\left(u, \frac{D^{c,\mu}u}{|D^{c,\mu}u|}\right) d|D^{c,\mu}u| + C\varepsilon \\
&\xrightarrow{i\to\infty} \int_\Omega f\left(u, \frac{Du}{\mu}\right) d\mu + K_f[u](x) d\mathcal{H}^{n-1} \llcorner J_u + f^\infty\left(u, \frac{D^{c,\mu}u}{|D^{c,\mu}u|}\right) d|D^{c,\mu}u| + C\varepsilon.
\end{aligned}$$

Because $C > 0$ is a universal constant, conclude by letting $\varepsilon \downarrow 0$. □

The proof of Theorem 5.0.1 is now a corollary of Theorem 5.2.5 and 5.2.7.

Remark 5.2.9. *Looking at the end of the previous proof, one can take a sequence $(u_i)_i$ in $W^{1,1}_\mu(\Omega, \mathbb{R}^m)$ with trace u so that*

$$\int_\Omega f\left(u_i, \frac{Du_i}{\mu}\right) d\mu \leq F_{**}(u) + \frac{1}{i}$$

and

$$\|u_i - u\|_{1,\mathcal{L}^n} \leq \frac{1}{i},$$

and obtain a full L^1 recovery sequence.

We end this chapter with a simple but important consequence of the upper bound.

Corollary 5.2.10. *Let $\mu \in \mathcal{M}^+(\Omega)$ be p-admissible and so that $\mathcal{L}^n \ll \mu$. Then for all $u \in BV(\Omega, \mathbb{R}^m)$ there is $(u_j)_j$ in $W^{1,1}_\mu(\Omega, \mathbb{R}^m)$ so that $u_j \to u$ μ-strictly in BV and u_j has the same trace as u.*

Proof. The function $f(z) = \sqrt{1 + |z|^2}$ satisfies all the assumptions of the previous theorems and it is so that $f^\infty(z) = |z|$ for all $z \in \mathbb{R}^{m\times n}$. Moreover, the surface energy is upper bounded by the jump part of u, and so, according to the remark, the recovery sequence $u_j \to u$ in L^1 is so that

$$\limsup_j \int_\Omega \sqrt{1 + \left|\frac{|Du_j|}{\mu}\right|^2} d\mu \leq \int_\Omega \sqrt{1 + \left|\frac{|Du|}{\mu}\right|^2} d\mu + |D^{s,\mu}u|(\Omega),$$

proving μ-strict density of $W^{1,1}_\mu(\Omega, \mathbb{R}^m)$ in $BV(\Omega, \mathbb{R}^m)$. □

Chapter 6

Regularity of quasi-convex envelopes and equi-integrability of minimising sequences when $p > 1$

In this chapter, we consider generalised Young Measures generated by sequences $(u_j)_j$ bounded in $W^{1,p}(\Omega, \mathbb{R}^m)$, which we call *p-Gradient Young Measures*. The higher integrability, which corresponds to $1 < p < +\infty$, prevents the diffuse concentration from interacting with the oscillation part. This chapter is divided into three parts. First, we prove the regularity of quasi-convex envelopes of C^1 functions $f\colon \mathbb{R}^{m\times n} \to \mathbb{R}$ of p-growth that are $p-1$ *mean-coercive*, meaning that there is $C > 0$ so that

$$(f - C|\cdot|^{p-1})^{qc} > -\infty.$$

Secondly, we study minimising sequences $(u_j)_j$ in $W^{1,p}_g(\Omega, \mathbb{R}^m)$ of the Dirichlet problem for

$$u \mapsto \int_\Omega f(\nabla u(x))dx,$$

where $g \in W^{1,p}(\mathbb{R}^n, \mathbb{R}^m)$ acts as boundary trace and $f\colon \mathbb{R}^{m\times n} \to \mathbb{R}$ is quasi-convex and of p-growth. We are able to characterise when a minimising sequence is so that $(\nabla u_j)_j$ is p-equi-integrable by asking that there is $z_0 \in \mathbb{R}^{m\times n}$ so that $f - C|\cdot|^p$ is quasi-convex at z_0. Notice that this condition is stronger than $p-1$-mean coercivity.

Finally, we apply both results to the function

$$K_t(z) = t|z + z^T|^p - |z|^p, \tag{6.0.1}$$

where $p > 1$, $z \in \mathbb{R}^{n\times n}$ and $t > 0$ is big enough. This integrand comes from Korn's inequality, which asserts the existence of $c > 0$, depending on $p > 1$ and $n \in \mathbb{N}$, so

that

$$\int_{\mathbb{R}^n} \left|\nabla\phi\right|^p dx \leq c \int_{\mathbb{R}^n} \left|\nabla\phi + (\nabla\phi)^T\right|^p dx \tag{6.0.2}$$

holds for all $\phi \in \mathcal{D}(\mathbb{R}^n, \mathbb{R}^n)$, which means that K_t is quasi-convex. Since (6.0.1) is unbounded from below, classical results would fail. However, we are now able to study the minima of the Dirichlet problem for (6.0.1) via the equi-integrability result 6.3.2 and represent them in virtue of the regularity of its envelope via Theorem 6.1.3.

This chapter is part of a paper, in the process of being finalised, in collaboration with Prof. Jan Kristensen and Prof. Bernd Kirchheim.

6.1 Regularity of quasi-convex envelope

In order to study the regularity of the quasi-convex envelope, we use the representation via Dacorogna's formula,

$$f^{qc}(z) = \inf\left\{\int_Q f(z + \nabla\phi(x))dx, \quad \phi \in W_0^{1,\infty}(Q, \mathbb{R}^m)\right\},$$

where $Q = \left[-\frac{1}{2}, \frac{1}{2}\right]^n$ is the unit cube in $\mathbb{R}^n$. Taking a minimizing sequence $(\phi_j)_j$ and suitably using our assumptions we can represent the difference quotients of the envelope in terms of differences of the original function. Employing properties of rank-one convex functions as in [20] we may conclude.

The problem is non-trivial, as one can notice by looking at the regularity of the convex envelope.

Example 6.1.1. *Let $f(x_1, x_2) = \sqrt{x_1^2 + e^{-x_2^2}} \in C^\infty(\mathbb{R}^2)$. Because $f \geq f^{c,x_2} \geq f^c$, where f^{c,x_2} is the convexification of f in the variable x_2 for each fixed x_1, then $f^c(x_1, x_2) = |x_1|$ which is not differentiable at 0.*

However, as proven in [61], if f satisfies some minimal growth assumption at infinity, the regularity of its convex envelope is always guaranteed.

Theorem 6.1.1 ([61])**.** *Let $f\colon \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ be a continuous function, differentiable on its effective domain $dom_e(f) = \{x \in \mathbb{R}^d : f(x) < +\infty\}$ and assume that*

$$f(x) \to +\infty \text{ as } x \to \infty.$$

Then f^c is a C^1 function on $dom_e(f)$.

The above theorem also provides Holder's regularity of the envelope f^c when $f \in C^{1,\alpha}$. We refer to [51, 25, 86, 87, 34] for more results on the smoothness of convex envelopes and the properties of their subdifferentials.

In the case of quasi-convex integrands, the regularity of the envelope was first proven in [20] in the form of the following theorem.

Theorem 6.1.2 ([20], Theorem A)**.** *Let $1 < p < +\infty$, and $f\colon \mathbb{R}^{m\times n} \to \mathbb{R}$ be C^1 and have p-growth. Assume that*

$$\frac{f(z)}{1+|z|^{p-1}} \to \infty \text{ as } |z| \to \infty.$$

Then, the quasi-convex envelope $f^{qc}\colon \mathbb{R}^{m\times n} \to \mathbb{R}$ is also of class C^1.

The assumption that f has p-growth is necessary even in the case $n = m = 2$, for a counterexample see [20]

We state now the main theorem of this chapter.

Theorem 6.1.3 (Main)**.** *Let $f\colon \mathbb{R}^{m\times n} \to \mathbb{R}$ of class C^1 be so that, for some $1 < p < +\infty$,*

$$\limsup_{z\to\infty} \frac{|f(z)|}{1+|z|^p} = 0$$

and f is $p-1$ mean-coercive*, meaning*

$$(f - C|\cdot|^{p-1})^{qc} > -\infty \tag{6.1.1}$$

for some $C > 0$. Then $f^{qc}\colon \mathbb{R}^{m\times n} \to \mathbb{R}$ is also of class C^1

6.1.1 Auxiliary notions and results

Here we mention some auxiliary results necessary to prove the regularity theorem. We say that $f\colon \mathbb{R}^{m\times n} \to \mathbb{R}$ is rank-1 convex if for all $z \in \mathbb{R}^{m\times n}$ and all $a \in \mathbb{R}^m, b \in \mathbb{R}^n$ we have

$$t \mapsto f\big(z + ta\otimes b\big) \text{ is convex.}$$

Lemma 6.1.4. *Let $f\colon \mathbb{R}^{m\times n} \to \mathbb{R}$ be rank-1 convex. If f is differentiable at z_0, then f is strictly differentiable at z_0, meaning that*

$$\lim_{z\to z_0, h\to 0} \frac{|f(z+h) - f(z) - \nabla f(z_0)\cdot h|}{|h|} = 0.$$

In particular, the above implies that if f is both differentiable on an open set and rank-1 convex, then it is C^1 in such a set.

Proof. The proof uses the fact that rank-1 convex functions are locally Lipschitz, see for example [62], Lemma 2.3. Consider the rank-1 convex function

$$g(z) = f(z) - \nabla f(z_0) \cdot z.$$

Its Lipschitz constant is given by

$$\mathrm{Lip}(g, \mathbb{B}_r(z_0)) = \sup_{z,w \in \mathbb{B}_r(z_0), z \neq w} \frac{|f(w) - f(z) - \nabla f(z_0) \cdot (w - z)|}{|w - z|}.$$

As for its oscillation, we calculate

$$osc(g, \mathbb{B}_{2r}(z_0)) = \sup_{z,w \in \mathbb{B}_{2r}(z_0)} (g(w) - g(z)) = \sup_{z,w \in \mathbb{B}_{2r}(z_0)} \Big(f(w) - f(z) - \nabla f(z_0) \cdot (w - z) \Big)$$

$$\leq 2 \sup_{z \in \mathbb{B}_{2r}(z_0)} |f(z) - f(z_0) - \nabla f(z_0) \cdot (z - z_0)|,$$

and the conclusion now follows because, since g is rank-1 convex, there is a constant $C = C(m, n)$ such that

$$\mathrm{Lip}(g, \mathbb{B}_r(z_0)) \leq C \frac{osc(g, \mathbb{B}_{2r}(z_0))}{r}.$$

□

Notice that the above can be easily extended to functions that are convex along directions that span the entire space.

Next, we obtain a sup-inf inequality for rank-1 convex functions.

Lemma 6.1.5. *Let $f \colon \mathbb{R}^{m \times n} \to \mathbb{R}$ be rank-1 convex and $a \colon \mathbb{R}^{m \times n} \to \mathbb{R}$ affine with $f(z_0) = a(z_0)$. Then*

$$- \inf_{w \in Q_r(z_0)} (f - a)(w) \leq (2^{m \times n} - 1) \sup_{w \in Q_r(z_0)} (f - a)(w)$$

Proof. Pick $\overline{z} \in \overline{Q_r(z_0)}$ so that $(f - a)(\overline{z}) = \inf_{w \in Q_r(z_0)} (f - a)(w)$. Consider the cube, centred at z_0, to which z_0 is a vertex, and select an enumeration for the other $2^{m \times n} - 1$ vertices:

$$\overline{z} = (z_1, z_2, \ldots, z_{2^{m \times n}}).$$

Using rank-1 convexity of $f-a$, which implies separate convexity, we estimate

$$0 = (f-a)(z_0) \leq 2^{-m\times n} \sum_{j=1}^{2^{m\times n}} (f-a)(z_j),$$

hence

$$0 \leq (f-a)(\overline{z}) + (2^{m\times n}-1) \sup_{w\in Q_r(z_0)} (f-a)(w).$$

□

Remark 6.1.6. *The above bound is sharp for separately convex functions. To see this, take*

$$f(z) = 1 - \Pi_{j=1}^{d}(1-z_j),\ z\in\mathbb{R}^d.$$

Then $f(0)=0$ *and*

$$-\inf_{w\in Q_1(0)} f(w) = (2^d-1) \sup_{w\in Q_1(0)} f(w).$$

As for rank-1 convex functions, the constant is not sharp and can be replaced by $2^{min(m,n)}-1$. *The proof of this is similar to the previous one and uses a decomposition of* z_0 *along rank-1 directions instead.*

Definition 6.1.7. *A function* $f\colon \mathbb{R}^{m\times n}\to\mathbb{R}$ *is upper semi-differentiable at* z_0 *if there exists* $\xi_0\in\mathbb{R}^{m\times n}$ *such that*

$$\limsup_{z\to 0} \frac{f(z_0+z)-f(z_0)-\xi_0\cdot z}{|z|} \leq 0$$

The set of all such ξ_0 *is* $\partial^* f(z_0)$.

Example 6.1.2. *A notorious example of a function which is upper semi-differentiable but not differentiable at* 0 *is*

$$x\mapsto -|x| \quad \textit{with} \quad \partial^*(-|\cdot|)(0) = [-1,1].$$

The following result combines the previous ideas and shows how a rank-1 convex function supporting an upper semi-differentiable function forces the gradients of both functions to exist and be equal.

Corollary 6.1.8. *Suppose that* $f,g\colon \mathbb{R}^{m\times n}\to\mathbb{R}$ *are so that* $g\leq f$ *and* $g(z_0)=f(z_0)$. *If* g *is rank-1 convex and* f *is upper semi-differentiable at* z_0, *then both* f *and* g *are differentiable at* z_0 *with* $\nabla f(z_0)=\nabla g(z_0)$.

Proof. If $\xi_0 \in \partial^* f(z_0)$, then

$$g(z) - g(z_0) - \xi_0 \cdot (z - z_0) \leq f(z) - f(z_0) - \xi_0 \cdot (z - z_0) \quad \forall z \in \mathbb{R}^d,$$

so g is also upper semi-differentiable at z_0. In Lemma 6.1.5, using

$$g(z) = g(z_0) + \xi_0 \cdot (z - z_0)$$

for the affine function, we recover the differentiability of g at z_0 with $\nabla g(z_0) = \xi_0$. This implies that f is also differentiable at z_0, with $\nabla f(z_0) = \xi_0$ as well. □

At the core of the proof of Theorem 6.1.3 lays the fact that $f^{qc}(z_0)$ can be obtained as the limit of a sequence in $W_0^{1,\infty}(Q, \mathbb{R}^m)$. This is known as Dacorogna's formula.

Lemma 6.1.9 ([32], Theorem 6.9)**.** *Let $f\colon \mathbb{R}^{m\times n} \to \mathbb{R}$ be locally bounded and Borel measurable. Then, for every $z \in \mathbb{R}^{m\times n}$ we have*

$$f^{qc}(z) = \inf \left\{ \int_Q f(z + \nabla\phi(x))dx,\ \phi \in W_0^{1,\infty}(Q, \mathbb{R}^m) \right\}, \tag{6.1.2}$$

where Q is the unit cube in $\mathbb{R}^n$.

In particular, if f^{qc} is real valued at a point, then it is real valued everywhere and formula (6.1.2) holds everywhere as well, see also [58].

Lemma 6.1.10. *Let $f\colon \mathbb{R}^{m\times n} \to \mathbb{R}$ be continuous and $1 < p < +\infty$ so that*

$$\limsup_{|z|\to\infty} \frac{|f(z)|}{1 + |z|^p} = 0.$$

Then, there exists an increasing and concave function $\theta\colon (0,\infty) \to (0,\infty)$ so that $\theta(0) = 0$, $\theta(t)/t \to 0$ as $t \to \infty$ and

$$|f(z)| \leq C(1 + \theta(|z|^p)).$$

Moreover, there is a constant $C_2 > 0$ so that

$$\left| \frac{f^{qc}(z + th) - f^{qc}(z)}{t} \right| \leq C_2(1 + \theta(|z|^{p-1}) + \theta(|h|^{p-1}))|h|$$

for all $z, h \in \mathbb{R}^{m\times n}$ and $0 \leq t \leq 1$.

We omit the standard proof and refer to [32].

Proof. (of Theorem 6.1.3).

Fix $z_0 \in \mathbb{R}^{m\times n}$. By Dacorogna's formula, find $(\phi_j)_j$ in $W_0^{1,\infty}(Q,\mathbb{R}^m)$, where Q is the unit square, so that $\phi_j \to 0$ in L^∞ and

$$\int_Q f(z_0+\nabla\phi_j(x))dx \to f^{qc}(z_0).$$

By $p-1$-coercivity and convergence of the above quantity, we get

$$\sup_j \int_Q |z_0+\nabla\phi_j(x)|^{p-1}dx < +\infty.$$

Set

$$\nu_j = (z_0+\nabla\phi_j(\cdot))_\# \mathcal{L}^n \llcorner Q,$$

where we remind that the push-forward operator $(\cdot)_\#(\cdot)$ maps the functions

$$z_0+\nabla\phi_j \colon X \to \mathbb{R}^{m\times n}$$

and the measure $\mathcal{L}^n \llcorner Q$ to measures in $\mathcal{M}^+(\mathbb{R}^{m\times n})$ via the action

$$\int_{\mathbb{R}^{m\times n}} \Phi(y)d\nu_j(y) = \int_Q \Phi(z+\nabla\phi(x))dx.$$

In this case, the barycentre of ν_j is constant

$$\overline{\nu_j} = \int_{\mathbb{R}^{m\times n}} z d\nu_j(z) = \int_Q (z_0+\nabla\phi(x))dx = z_0,$$

and also

$$\sup_j \int_{\mathbb{R}^{m\times n}} |z|^{p-1}d\nu_j(z) = \sup_j \int_Q |z_0+\nabla\phi(x)|^{p-1}dx < \infty.$$

By the Riesz-Markov Theorem, there is a subsequence, identified with ν_j, and $\nu \in \mathcal{M}^+(\mathbb{R}^{m\times n})$ so that the $p-1$-moment is bounded:

$$\int_{\mathbb{R}^{m\times n}} |z|^{p-1}d\nu(z) \le \liminf_j \int_{\mathbb{R}^{m\times n}} |z|^{p-1}d\nu_j(z) < +\infty.$$

Because f^{qc} is continuous and $f - f^{qc} \ge 0$, then

$$\begin{aligned}
0 &\le \int_{\mathbb{R}^n} f - f^{qc} d\nu \le \liminf_j \int_Q (f-f^{qc})(z_0+\nabla\phi_j(x))dx \\
&\le \liminf_j \int_Q f(z_0+\nabla\phi_j(x))dx - \int_Q f^{qc}(z_0+\nabla\phi_j(x))dx \\
&\le \liminf_j \int_Q f(z_0+\nabla\phi_j(x))dx - \int_Q f^{qc}(z_0)dx = 0,
\end{aligned}$$

where we use the quasi-convexity of f^{qc}. In particular, equality holds throughout, and

$$\int_{\mathbb{R}^{m\times n}} f - f^{qc} d\nu = 0.$$

Then, $f = f^{qc}$ ν-almost everywhere, and by the very definition of the support of ν and continuity of f and f^{qc}, $f = f^{qc}$ on the support of ν. We then apply Corollary 6.1.8 to deduce that $\nabla(f^{qc})(z)$ exists and equals $\nabla f(z)$ for all $z \in \operatorname{supp}(\nu)$. Notice that, at this point, z_0 needs not to be included in such a set.

Let $t > 0$ and $h \in \mathbb{R}^{m\times n}$:

$$\begin{aligned} f^{qc}(z_0 + th) &\leq \int_Q f^{qc}(z_0 + \nabla\phi_j(x) + th)dx \\ &\leq \int_Q f(z_0 + \nabla\phi_j(x) + th)dx = \int_{\mathbb{R}^{m\times n}} f(z + th)d\nu_j(z). \end{aligned}$$

Therefore, if

$$\Lambda_t(h) = \frac{f^{qc}(z_0 + th) - f^{qc}(z_0)}{t},$$

then

$$\Lambda_t(h) \leq \int_{\mathbb{R}^{m\times n}} \frac{f^{qc}(z + th) - f^{qc}(z)}{t} d\nu_j(z) + \frac{O_j(1)}{t},$$

where $O_j(1)$ is a positive function that goes to 0 as $j \to 0$, uniformly in h. For a fixed $t > 0$ and $h \in \mathbb{R}^{m\times n}$, the function

$$z \mapsto \frac{f^{qc}(z + th) - f^{qc}(z)}{t}$$

is continuous. Apply Lemma 6.1.10 to find a concave function $\theta\colon (0,\infty) \to (0,\infty)$ so that $\theta(t)/t \to 0$ as $t \to \infty$ and

$$\left|\frac{f^{qc}(z + th) - f^{qc}(z)}{t}\right| \leq C(1 + \theta(|z|^{p-1}) + \theta(|h|^{p-1}))|h| \quad \forall\ 0 < t \leq 1. \tag{6.1.3}$$

Notice that, for any such θ we have that

$$\lim_j \int_{\mathbb{R}^{m\times n}} \theta(|z|^{p-1})d\nu_j(z) = \int_{\mathbb{R}^{m\times n}} \theta(|z|^{p-1})d\nu(z),$$

and also

$$\lim_{|z|\to\infty} \frac{f^{qc}(z + th) - f^{qc}(z)}{t|z|^{p-1}} = 0.$$

But then we can pass to the limit and obtain

$$\Lambda_t(h) \leq \int_{\mathbb{R}^{m\times n}} \frac{f^{qc}(z+th)-f^{qc}(z)}{t} d\nu(z),$$

and since f^{qc} is differentiable ν-a.e.,

$$\frac{f^{qc}(z+th)-f^{qc}(z)}{t} \xrightarrow{t\to 0} \nabla f(z)\cdot h$$

for ν-almost all $z \in \mathbb{R}^d$. Using the Lipschitz bound (6.1.3) and the dominated convergence theorem, we deduce that $\nabla f \in L^1(\nu)$ with

$$\Lambda(h) := \limsup_{t\to 0} \Lambda_t(h) \leq \int_{\mathbb{R}^{m\times n}} \nabla f d\nu \cdot h.$$

Note that the function Λ is positively 1-homogeneous

$$\Lambda(sh) = s\Lambda(h) \quad s \geq 0, h \in \mathbb{R}^{m\times n}$$

and rank-1 convex. By Corollary 6.1.8, Λ is differentiable at 0, and consequently

$$\Lambda(h) = \int_{\mathbb{R}^{m\times n}} \nabla f d\nu \cdot h.$$

Thus

$$\limsup_{t\to 0} \frac{f^{qc}(z_0+th)-f^{qc}(z_0)-\Lambda(th)}{t} \leq 0,$$

and, using once again the local Lipschitz estimate (6.1.3), we deduce that f^{qc} is then upper semi-differentiable at z_0 and

$$\int_{\mathbb{R}^{m\times n}} \nabla f d\nu \in \partial^* f^{qc}(z_0).$$

By the corollary, f^{qc} is differentiable at z_0 with

$$\nabla(f^{qc})(z_0) = \int_{\mathbb{R}^{m\times n}} \nabla f d\nu.$$

Conclude using Lemma 6.1.4. □

6.1.2 Generalisation

The above argument works also when f is merely upper semi-differentiable. It then yields information about the measure ν:

$$\operatorname{supp}(\nu) \subset \{z \in \mathbb{R}^{m\times n} : f(z) = f^{qc}(z)\} \subset \{z \in \mathbb{R}^{m\times n} : f \text{ is differentiable at } z\}.$$

In particular, formula

$$\nabla(f^{qc})(z_0) = \int_{\mathbb{R}^{m\times n}} f d\nu$$

remains valid.

Corollary 6.1.11. *Assume that $f\colon \mathbb{R}^{m\times n} \to \mathbb{R}$ is continuous, upper semi-differentiable,*

$$\limsup_{|z|\to\infty} \frac{f(z)}{1+|z|^p} = 0,$$

and $p-1$ mean-coercive like in (6.1.1)*. Then $f^{qc} \in C^1$, and relaxed minimisers satisfy the Euler-Lagrange equation. If it also holds that*

$$|\nabla f(z)| \leq C(|z|^p + 1) \quad \forall z \in \mathbb{R}^{m\times n},$$

then for any minimising sequence $(u_j)_j$ in $W^{1,p}_g(\Omega, \mathbb{R}^m)$ for the primal problem, such that $u_j \rightharpoonup u$ in $W^{1,p}$, we have that $u \in W^{1,p}_g(\Omega, \mathbb{R}^m)$ is a relaxed minimiser and

$$\nabla f(\nabla u_j) \rightharpoonup \nabla(f^{qc})(\nabla u) \quad \text{in } L^1.$$

6.2 Gradient Young Measures when $p > 1$

In this section, we briefly discuss generalised Young Measures, as defined in Chapter 2, when they are generated by a sequence of gradients $(\nabla u_j)_j$ with $(u_j)_j$ bounded in $W^{1,p}(\Omega, \mathbb{R}^m)$ for some $1 < p < +\infty$. Thanks to the lower semi-continuity result proven in [56], we are able to show that Jensen-type inequalities as those proven in 3 hold for Young Measures generated by p integrable gradients as well.

Consider a family $\{f_i, \in \mathbb{N}\}$ of continuous functions $f_i\colon \mathbb{R}^{m\times n} \to \mathbb{R}$ having p-growth. According to the theory developed in 2, if a sequence $(u_j)_j$ is bounded in $W^{1,p}(\Omega, \mathbb{R}^m)$, then there is (non-relabelled) subsequence and a Young Measure $\nu \in Y^p(e_{f_i, i\in\mathbb{N}})$ so that

$$\nabla u_j \xrightarrow{Y^p(e_{f_i,i\in\mathbb{N}})} \nu = (\nu_x, \lambda, \nu_x^\infty),$$

where we remind that

- $\nu_x \in \mathcal{M}_1^+(\mathbb{R}^{m\times n})$ $\mathcal{L}^n$-a.e. $x \in \Omega$,
- $\lambda \in \mathcal{M}^+(\overline{\Omega})$,
- $\nu_x^\infty \in \mathcal{M}_1^+(\partial e_{f_i,i\in\mathbb{N}})$ for λ-a.e. $x \in \overline{\Omega}$.

In our notation, we can rewrite the main result in [56] in the following form.

Theorem 6.2.1 ([56], Theorem 2.9 - revisited). *Let $u \in W^{1,p}(\Omega, \mathbb{R}^m)$ and $(u_j)_j$ in $W^{1,p}_u(\Omega, \mathbb{R}^m)$ be so that $u_j \rightharpoonup u$ in $W^{1,p}$ and*

$$\nabla u_j \xrightarrow{Y(e_{f_i, i\in\mathbb{N}})} (\nu_x, \lambda, \nu_x^\infty).$$

Then, for all $\phi \in C(\overline{\Omega})$ with $\phi \geq 0$ and all quasi-convex functions $f \in e_{f_i, i\in\mathbb{N}}$ we have

$$\int_\Omega \phi(x)\langle \nu_x, f\rangle dx + \int_{\overline{\Omega}} \phi(x)\langle \nu_x^\infty, f^\infty\rangle d\lambda \geq \int_\Omega \phi(x) f(\nabla u(x)) dx.$$

In conclusion, when $p > 1$ it holds

$$\begin{cases} \langle \nu_x, f\rangle \geq f(\nabla u(x)) & \mathcal{L}^n \llcorner \Omega\text{-a.e.} \\ \langle \nu_x^\infty, f^\infty\rangle \geq 0 & \lambda\text{-a.e.} \end{cases} \tag{6.2.1}$$

for all $f \in e_{f_i, i\in\mathbb{N}}$ that are quasi-convex. Moreover, if $f, g \in e_{f_i, i\in\mathbb{N}}$ are so that $f(z) \geq g(z)$ for all $z \in \mathbb{R}^{m\times n}$, then

$$\langle \nu_x^\infty, f^\infty\rangle \geq \langle \nu_x^\infty, g^\infty\rangle$$

at λ-a.e. $x \in \overline{\Omega}$.

It is worth noting that, when $p > 1$, the first inequality in (6.2.1) does not include the diffuse concentration. This is because if $(u_j)_j$ and u in $W^{1,p}(\Omega, \mathbb{R}^m)$ are so that $u_j \rightharpoonup u$ in $W^{1,p}$ and

$$\nabla u_j \xrightarrow{Y^p(\mathbb{B}^{m\times n})} (\nu_x, \lambda, \nu_x^\infty),$$

one can find $(o_j)_j$ and $(c_j)_j$ in $W^{1,p}(\Omega, \mathbb{R}^m)$ such that $u_j = o_j + c_j$ and

$$\nabla o_j \xrightarrow{Y^p(\mathbb{B}^{m\times n})} (\nu_x, 0, N/A) \quad \text{and} \quad \nabla c_j \xrightarrow{Y^p(\mathbb{B}^{m\times n})} (\delta_0, \lambda, \nu_x^\infty).$$

This decomposition was proven in [48], Lemma 1.2, and, independently, in [63]. Roughly speaking, when $p > 1$, one can separate oscillation and concentration in weakly convergent sequences. This is not possible, in general, when $p = 1$, and makes this latter case much harder to work with.

6.3 Equi-integrability of minimising sequences when $p > 1$

We start this section with a result that provides a characterization of the coercivity of a variational integral defined on a Dirichlet class in terms of a quasiconvexity condition.

Proposition 6.3.1. *Let $f\colon \mathbb{R}^{m\times n} \to \mathbb{R}$ be a continuous integrand and $1 \leq p < +\infty$. The following are equivalent:*

1. *There exist $\Omega \subset \mathbb{R}^n$ open and bounded and a function $g \in W^{1,\infty}(\mathbb{R}^n, \mathbb{R}^m)$ such that all minimizing sequences $(u_j)_j$ in $W^{1,p}_g(\Omega, \mathbb{R}^m)$ of*

$$u \mapsto \int_\Omega f(\nabla u(x))dx$$

are bounded in $W^{1,p}$.

2. *For all $\Omega \subset \mathbb{R}^n$ open and bounded and all $g \in W^{1,\infty}(\mathbb{R}^n, \mathbb{R}^m)$, all minimizing sequences $(u_j)_j$ in $W^{1,p}_g(\Omega, \mathbb{R}^m)$ of*

$$u \mapsto \int_\Omega f(\nabla u(x))dx$$

are bounded in $W^{1,p}$.

3. *There exist $C_1 > 0$ and $C_2 \in \mathbb{R}$ such that*

$$\int_\Omega f(\nabla u(x))dx \geq \int_\Omega C_1|\nabla u(x)|^p + C_2 dx$$

for all $u \in W^{1,\infty}_g(\Omega)$.

4. *There exist $c > 0$ and a point $z_0 \in \mathbb{R}^{m\times n}$ such that $f - c|\cdot|^p$ is quasiconvex at z_0.*

We omit the details of the proof here and only remark that the equivalence of (3) and (4) is established in [31] and the equivalence of (1), (2) and (4) is established in [50] in the special case $p = 1$. The reader can easily adapt the proof given there to also cover the more general case considered here.

We turn to the main result of the section that provides a good illustration of the use of Young Measures. It is a generalization of [59] where a similar result was established for nonnegative integrands and under more restrictive conditions. The fact that we consider signed integrands here necessitates the use of Young Measures that also quantify concentration effects for integrands with p-growth and general asymptotic behaviour at infinity.

Theorem 6.3.2. *Let $f\colon \mathbb{R}^{m\times n} \to \mathbb{R}$ be continuous and have p-growth for some $1 < p < \infty$, and let $g \in W^{1,p}(\mathbb{R}^n, \mathbb{R}^m)$. Then, any minimising sequence $(u_j)_j$ in $W^{1,p}_g(\Omega, \mathbb{R}^m)$ for*

$$u \mapsto \int_\Omega f(\nabla u(x))dx \tag{6.3.1}$$

has the property that $(\nabla u_j)_j$ is p-equi-integrable if and only if there exists $c > 0$ and $z_0 \in \mathbb{R}^{m\times n}$ such that $f - c|\cdot|^p$ is quasi-convex at z_0.

We emphasize in particular that the result applies to *any* minimizing sequence and that no condition is imposed on the set Ω that is merely assumed to be bounded and open. The Dirichlet boundary condition g is however assumed in $W^{1,p}(\mathbb{R}^n, \mathbb{R}^m)$, that is, it is defined on all of $\mathbb{R}^n$. The standard approach would not apply to all minimizing sequences but would use Ekeland's variational principle to construct a sequence of quasi-minimizers, and for these, the higher integrability follows via Caccioppoli inequalities and Gehring-type Lemmas. As shown in [57], such an approach requires a suitable density condition for the complement of Ω when the exponent $p \in (1, n]$. We also remark that a related interesting result was obtained in [53] for more general variational integrals and within the more general set-up of Orlicz-Sobolev spaces.

Proof. " $\Rightarrow$ " This implication follows from the previous result, see also [31].

" $\Leftarrow$ " By the previous result, point (3), we find $C_1 > 0$ and $C_2 \in \mathbb{R}$ such that

$$\int_\Omega f(\nabla v(x))dx \geq \int_\Omega C_1|\nabla v(x)|^p + C_2 \; dx$$

for all $v \in W^{1,p}_g(\Omega, \mathbb{R}^m)$. Let $(u_j)_j$ in $W^{1,p}_g(\Omega, \mathbb{R}^m)$ be a minimising sequence for (6.3.1) and find a (non-relabelled) weakly convergent subsequence $u_j \rightharpoonup u$ in $W^{1,p}$ for some $u \in W^{1,p}_g(\Omega, \mathbb{R}^m)$. In virtue of the Vitali Theorem, we only have to show that $(|\nabla u_j|^p)_j$ is weakly pre-compact in $L^1(\Omega)$. Extend u_j to $\mathbb{R}^n$ by setting it to be g outside of Ω and let $\mathbb{B}_R(0) = \mathbb{B} \supset \overline{\Omega}$, so that

$$u_j \in W^{1,p}(\mathbb{B}, \mathbb{R}^m), \quad u_j \rightharpoonup u\chi_\Omega + g\chi_{\mathbb{B}\setminus\Omega} \quad \text{in } W^{1,p}.$$

Let $e_{f,f^{qc},(f-c|\cdot|^p)^{qc}} = K$ be the compactification generated by the functions f, f^{qc} and $(f - c|\cdot|^p)^{qc}$, as they are all continuous and have p-growth. Since

$$\sup_j \int_{\mathbb{B}} |\nabla u_j(x)|^p dx < +\infty,$$

we can extract a subsequence (again non-relabelled) so that

$$\nabla u_j \xrightarrow{Y^p(K)} (\nu_x, \lambda, \nu_x^\infty),$$

and

$$\nabla u_j \rightharpoonup \nabla u = \overline{\nu} = \overline{\nu_x}dx + \overline{\nu_x^\infty}d\lambda \quad \text{in } L^p.$$

By assumption,

$$\int_\Omega f(\nabla u_j(x))dx \to \inf_{v\in W^{1,p}_g}\int_\Omega f(\nabla v(x))dx = \inf_{v\in W^{1,p}_g}\int_\Omega f^{qc}(\nabla v(x))dx,$$

see for example [32], Chapter 9, and so, by Meyers' Theorem [81], we also have the inequality

$$\begin{aligned}\inf_{v\in W^{1,p}_g}\int_\Omega f(\nabla v(x))dx =& \liminf_j \int_{\mathbb{B}} f(\nabla u_j(x))dx\\ \geq& \liminf_j \int_\Omega f^{qc}(\nabla u_j(x))dx \geq \int_\Omega f^{qc}(\nabla u(x))dx,\end{aligned}$$

and the above are, a posteriori, all equalities. Using the Young Measure representation and the fact both f and f^{qc} are admissible integrands, we get

$$\int_\Omega \langle \nu_x, f\rangle dx + \int_{\overline{\Omega}} \langle \nu_x^\infty, f^\infty\rangle d\lambda = \int_\Omega \langle \nu_x, f^{qc}\rangle dx + \int_{\overline{\Omega}} \langle \nu_x^\infty, (f^{qc})^\infty\rangle d\lambda.$$

Because $f \geq f^{qc}$ and $f^\infty \geq (f^{qc})^\infty$, then there are $N \subset \Omega$, $\mathcal{L}^n(\Omega\setminus N) = 0$ and $M \subset \Omega$, $\lambda(\Omega \setminus M) = 0$ such that

$$f = f^{qc} \text{ on } \cup_{x\in N}\operatorname{supp}(\nu_x) \quad \text{ and } \quad f^\infty = (f^{qc})^\infty \text{ on } \cup_{x\in M}\operatorname{supp}(\nu_x^\infty).$$

Because $p > 1$ we also have that $\nabla u(x) = \overline{\nu_x}$ $\mathcal{L}^n$-a.e. and $\overline{\nu_x^\infty} = 0$ λ-a.e.

So, the following equalities hold

$$\langle \nu_x, f^{qc}\rangle = f^{qc}(\overline{\nu_x})\ \mathcal{L}^n\text{-a.e.} \quad \text{ and } \quad \langle \nu_x^\infty, f^\infty\rangle = 0\ \lambda\text{-a.e.}$$

Because $x \mapsto |z|^p$ is convex then

$$\begin{aligned}(f - c|\cdot|)^{qc}(z) =& \inf_{\phi\in W^{1,\infty}_0(Q,\mathbb{R}^m)}\int_Q f(z+\nabla\phi(x))dx - c|z+\nabla\phi(x)|^p dx\\ \leq& \inf_{\phi\in W^{1,\infty}_0(Q,\mathbb{R}^m)}\int_Q f(z+\nabla\phi(x))dx - c|z|^p dx = f^{qc}(z) - c|z|^p,\end{aligned}$$

where we also used that, if $f - c|\cdot|^p$ is quasi-convex at z_0, it is finite everywhere as a function $\mathbb{R}^{m\times n} \to \mathbb{R}$. Letting z converge in K, the same equality holds for their recession functions on K, that is

$$((f - c|\cdot|)^{qc})^\infty \leq (f^{qc}(z) - c|z|^p)^\infty = (f^{qc})^\infty - c.$$

So, we can estimate

$$\begin{aligned}0 = \langle \nu_x^\infty, (f^{qc})^\infty\rangle \geq& \langle \nu_x^\infty, ((f - c|\cdot|^p)^{qc})^\infty\rangle + \langle \nu_x^\infty, c\rangle\\ =& \langle \nu_x^\infty, ((f - c|\cdot|^p)^{qc})^\infty\rangle + c \geq c > 0 \quad \lambda\text{-a.e.}\end{aligned}$$

where we used inequality (6.2.1). The above inequality means that $\lambda = 0$, which shows that $(\nabla u_j)_j$ is p-equi-integrable. □

6.4 Example

Let $n \in \mathbb{N}$ and $1 < p < +\infty$. It is well-known that Korn's inequality asserts the existence of a constant $c = c(n,p)$ such that

$$\int_{\mathbb{R}^n} \big|\nabla\phi\big|^p \, dx \leq c \int_{\mathbb{R}^n} \big|\nabla\phi + (\nabla\phi)^T\big|^p \, dx \tag{6.4.1}$$

holds for all $\phi \in \mathcal{D}(\mathbb{R}^n, \mathbb{R}^n)$. Here we use the standard Hilbert-Schmidt norm on $n \times n$ matrices. Let $\gamma = \gamma(n,p)$ denote the smallest constant c for which (6.4.1) holds and note that $\gamma(n,2) = \frac{1}{2}$ (indeed, the inequality (6.4.1) with $c = \frac{1}{2}$ holds with equality when the vector field ϕ is divergence-free), whereas the precise value of $\gamma(n,p)$ is unknown when $p \neq 2$. Denoting

$$K_t(z) := t\big|z + z^T\big|^p - \big|z\big|^p, \quad z \in \mathbb{R}^{n\times n},$$

where $t > 0$ is a parameter, we see that (6.4.1) amounts to $K_c^{qc}(0) = 0$, and that $\gamma = \gamma(n,p)$ is the smallest constant for which (6.4.1) holds amounts to

$$K_t^{qc}(0) = \begin{cases} 0 & \text{when } t \geq \gamma \\ -\infty & \text{when } t < \gamma. \end{cases}$$

We now consider the integrand K_t for $t > \gamma$. Because $p > 1$ we have that K_t is C^1 and because $\big(K_t - \varepsilon|\cdot|^p\big)^{qc}(0) = 0 > -\infty$ for $\varepsilon \in (0, \gamma - t]$ theorem implies that K_t is p mean-coercive. Because K_t is not quasiconvex at non-zero skew-symmetric matrices the variational principle

$$\text{Minimize } \int_\Omega K_t(\nabla u)\, dx \text{ subject to } u \in W_g^{1,p}(\Omega, \mathbb{R}^n) \tag{6.4.2}$$

will not admit minimizers for some boundary data $g \in W^{1,p}(\mathbb{R}^n, \mathbb{R}^n)$. However, any minimizing sequence will be bounded in $W^{1,p}(\Omega, \mathbb{R}^n)$ (in fact will have p-equi-integrable derivatives by theorem) and any of its $W^{1,p}$-weak cluster points will be a minimizer for the relaxed problem

$$\text{Minimize } \int_\Omega K_t^{qc}(\nabla u)\, dx \text{ subject to } u \in W_g^{1,p}(\Omega, \mathbb{R}^n) \tag{6.4.3}$$

Note that K_t^{qc} is a quasiconvex, p-homogeneous and p mean-coercive integrand. By Theorem 6.1.3 the envelope K_t^{qc} is C^1. We remark that the result in [20] does not apply to the integrand K_t since $K_t(z) = -|z|^p$ when z is skew-symmetric. Furthermore,

when $(u_j)_j$ is a minimizing sequence for (6.4.2) then we have for a subsequence (non-relabelled) that $u_j \rightharpoonup u$ in $W^{1,p}(\Omega,\mathbb{R}^n)$ for some $u \in W^{1,p}_g(\Omega,\mathbb{R}^n)$ that is then a minimizer for the relaxed problem (6.4.3). It is not difficult to see that

$$\nabla K_t(\nabla u_j) \rightharpoonup \nabla(K_t^{qc})(\nabla u) \text{ in } L^{p'}(\Omega,\mathbb{R}^{n\times n}). \tag{6.4.4}$$

Indeed, each subsequence of $(u_j)_j$ admits a further subsequence, say $(u_{j_k})_k$, such that $(\nabla u_{j_k})_k$ generates a Young Measure of oscillation $\big(\nu_x\big)_{x\in\Omega}$ and since the sequence is p-equi-integrable we have in particular that $\nabla K_t(\nabla u_{j_k}) \rightharpoonup \langle \nu_x, \nabla K_t\rangle$ in $L^{p'}(\Omega,\mathbb{R}^{n\times n})$. Here the proof of Theorem 6.1.3 yields, since $K_t^{qc}(\overline{\nu}_x) = \langle \nu_x, K_t\rangle$ for a.e. $x\in\Omega$, that $\langle \nu_x, \nabla K_t\rangle = \langle \nu_x, \nabla(K_t^{qc})\rangle = \nabla(K_t^{qc})(\overline{\nu}_x)$ for a.e. $x\in\Omega$. Because $\overline{\nu}_x = \nabla u(x)$ a.e. a subsequence argument then completes the proof of (6.4.4). It is not difficult to see that also

$$\nabla(K_t^{qc})(\nabla u_j) \rightharpoonup \nabla(K_t^{qc})(\nabla u) \text{ in } L^{p'}(\Omega,\mathbb{R}^{n\times n})$$

holds. In fact, this result holds more generally, that is beyond homogeneous integrands such as K_t, because the derivative of the envelope inherits a growth condition from the envelope.

A standard argument yields the Euler-Lagrange equation

$$\int_\Omega \nabla(K_t^{qc})(\nabla u)\cdot\nabla\phi\,dx = 0 \quad \forall \phi \in W^{1,p}_0(\Omega,\mathbb{R}^n)$$

as a necessary condition for $u\in W^{1,p}_g(\Omega,\mathbb{R}^n)$ to be a minimizer for the relaxed problem (6.4.3). If we combine this with (6.4.4) then we get in particular that the sequence of row-wise divergences converges weakly: $\mathrm{div}\nabla K_t(\nabla u_j) \rightharpoonup 0$ in $W^{-1,p'}(\Omega,\mathbb{R}^{n\times n})$ and similarly for the sequence $\big(\nabla(K_t^{qc})(\nabla u_j)\big)_j$. It is an interesting question as to whether, in fact, the sequence $\big(\mathrm{div}\nabla(K_t^{qc})(\nabla u_j)\big)_j$ converges strongly in $W^{-1,p'}(\Omega,\mathbb{R}^{n\times n})$. If so, then it would be a feature of quasiconvexity, where we note that it is true if the quasiconvex envelope happened to be convex. In these connections, we remark that using Ekeland's variational principle one can construct minimizing sequences with this added feature, so the question is not about the existence of such a minimizing sequence, but whether all minimizing sequences have this property.

Chapter 7

Conclusions and an open problem

This thesis aims at building a robust framework for working with limits of integrals

$$\int_\Omega f(u_j(x))dx \tag{7.0.1}$$

when $(u_j)_j$ is only bounded in some subspace of $L^p(\Omega, \mu, \mathbb{R}^d)$ and f is continuous and has p-growth. Existence of such limits as measures $\nu \in \mathcal{M}^+(K)$, where K is some abstract compactification of $\mathbb{R}^d$, were known to exist since [36], in the form of

$$\lim_j \int_\Omega f(u_j(x))dx = \lim_j \int_{\mathbb{R}^d} f(z)d\nu_j(z) = \nu(f).$$

However, these Young Measures were too abstract and concealed how ν acts as a dual element of functions f that are admissible in the compact space K. The first systematic attempt of providing a geometric representation for ν was carried out in [6]. However, it was limited to one specific compactification of $\mathbb{R}^d$, called the sphere compactification, and so enables one to work with, exclusively, functions f that have a homogeneous-like behaviour at infinity.

We successfully managed to improve such representation to separable compactifications and also showed that, in many cases of interest, the compact space K has the topology of a finite-dimensional Euclidean space. We used the notation $e_{f_i, i\in\mathbb{N}}$ in place of K to refer to the compact space for which f_i is admissible for all $i \in \mathbb{N}$.

Our geometric representation of Young Measures reveals that many properties of these generalised measures are inherited by their representation on the sphere. For example, in Chapter 4, we show that we can generate certain Young Measures by recalling results that hold only on the sphere compactification, and then extending the sphere representation to general compactifications $e_{f_i, i\in\mathbb{N}}$ in virtue of a good choice of the class $\{f_i, \in \mathbb{N}\}$.

On the other side, it is interesting to compare the characterisation of Gradient Young Measures in Chapter 3 to the characterisation on the sphere compactification proven in [68] and the tools we deploy to prove the lower semi-continuity of measure derivatives in Chapter 5 with [65] and the Lifting Young Measures constructed in [94]. To prove our statements, we cannot always merely recall previous results from the aforementioned papers, but we are often able to adapt them to our setting because the arithmetic involved in the proofs, in terms of symbolic computations, is almost unchanged from classical results on the sphere compactification. Similarly, this happens also in the case of p-equi-integrability of minimizing sequences in Chapter 6.

However, it is important to underline that this generalisation of Young Measures only allows representing accumulation points of (7.0.1), but does not, intrinsically, bring any more insight on how to tackle problems in the calculus of variations. In fact, with generalised Young Measures, we are only able to break down the study, in our case, of lower semi-continuity to a model case where more is known, and then use classical tools. Therefore, on a personal note, one should look at generalised Young Measures mainly as a tool to bridge different theories together, rather than trying to extract pieces of information solely from these representations.

To conclude this thesis, we mention the main problem that motivated us to study both the characterisation result of generalised Gradient Young Measures in Chapter 3 and the lower semi-continuity result of measure derivatives in Chapter 5. We were originally interested in the lower semi-continuity extension to $BV(\Omega, \mathbb{R}^m)$ of

$$u \mapsto \int_\Omega f\left(\frac{Du}{\mu}\right) d\mu, \tag{7.0.2}$$

where $\mu \in \mathcal{M}^+(\Omega)$, $u \in W^{1,1}_\mu(\Omega, \mathbb{R}^m) = \{u \in BV(\Omega, \mathbb{R}^m) : |Du| \ll \mu\}$ and f is non-negative, quasi-convex and has linear growth. A relaxation formula was established in [65] for the case of f having regular recession at infinity and being coercive and $\mathcal{L}^n \ll \mu$, which means that $\mu = a dx + \mu^s$, with $\mu^s \perp \mathcal{L}^n$ and $a > 0$ $\mathcal{L}^n$ almost everywhere, via a methodology similar to what we used in Chapter 5. Writing

$$Du = \frac{Du}{\mu} d\mu + \frac{D^{s,\mu}u}{|D^{s,\mu}u|} d|D^{s,\mu}u|,$$

where Du is decomposed into its absolutely continuous part with respect to μ, $\frac{Du}{\mu} d\mu$, and its singular part with respect to μ, $\frac{D^{s,\mu}u}{|D^{s,\mu}u|} d|D^{s,\mu}u|$, the authors showed that the lower semi-continuous envelope with respect to L^1 convergence in $BV(\Omega, \mathbb{R}^m)$ is given by the formula

$$u \mapsto \int_\Omega f\left(\frac{Du}{\mu}\right) d\mu + f^\infty\left(\frac{D^{s,\mu}u}{|D^{s,\mu}u|}\right) d|D^{s,\mu}u|. \tag{7.0.3}$$

Looking at the model case $\mu = a dx$, $a \in L^1(\Omega, \mathcal{L}^n)$, we see that the existence of $f^\infty(z)$ for all $z \in \partial \mathbb{B}^d$ (on the sphere compactification) guarantees that if a sequence $(u_j)_j$ bounded in $W^{1,1}$ generates the Young Measure

$$\frac{\nabla u_j}{a} \xrightarrow{Y(adx)} (\eta_x, \lambda, \eta_x^\infty) \quad \text{and} \quad \nabla u_j \xrightarrow{Y(dx)} (\nu_x, \lambda, \nu_x^\infty),$$

then $\eta_x^\infty = \nu_x^\infty$ λ almost everywhere (see Proposition 2.4.10). However, this is not true anymore if one takes the Young Measures generated on a separable compactification $e_{f_i, i \in \mathbb{N}} \geq \mathbb{B}^s$, and so even if no compactification appears in the expression (7.0.3), the result cannot be easily adapted to prove this general case.

Question 1. *Given a measure $\mu \in \mathcal{M}^+(\Omega)$ with $\mathcal{L}^n \ll \mu$ and $f \colon \mathbb{R}^{m \times n} \to \mathbb{R}$ non-negative, quasi-convex and of linear growth (not necessarily on the sphere compactification!), is*

$$u \in BV(\Omega, \mathbb{R}^m) \mapsto \int_\Omega f\left(\frac{Du}{\mu}\right) d\mu + f^\infty\left(\frac{D^{s,\mu}u}{|D^{s,\mu}u|}\right) d|D^{s,\mu}u|$$

weak sequentially lower semi-continuous in BV?*

It would interesting to know whether generalised Young Measures can reveal new properties of the Gradient Young Measures $\nu \in Y(\mu, e_{f_i, \in \mathbb{N}})$ generated by $\left(\frac{Du_j}{\mu}\right)_j$, and whether one can actually use the lower semi-continuous extension of (7.0.2) when $\mu \neq \mathcal{L}^n$ to say something on the case $\mu = \mathcal{L}^n$, or new tools need to be developed to tackle this problem.

Appendix A

Appendix

In this appendix, the references to the following theorems are not necessarily attributed to the original author of such results but are just references for the reader.

A.1 Measure theory

Vitali convergence Theorem.

Theorem A.1.1 ([27], Theorem 4.5.4)**.** *Let $\mu \in \mathcal{M}^+(\Omega)$ and $(f_n)_n$ and f in $L^1(\Omega, \mu)$. Then $f_n \to f$ in $L^1(\Omega, \mu)$ if and only if $f_n \to f$ in measure and f_n is uniformly integrable.*

Stone-Weierstrass Theorem.

Theorem A.1.2 ([39], Theorem 4.45)**.** *Let X be a compact Hausdorff space. If $\mathbb{A}$ is a closed subalgebra of $C(X)$ that separates points, then either $\mathbb{A} = C(X)$ or there is $x_0 \in X$ so that $\mathbb{A} = \{f \in C(X) : f(x_0) = 0\}$. In particular, $\mathbb{A} = C(X)$ if and only if $\mathbb{A}$ contains the constant functions.*

Tychonoff Theorem.

Theorem A.1.3. *If $\{X_\lambda\}_{\lambda \in \Lambda}$ is an arbitrary family of compact spaces, $\Pi_{\lambda \in \Lambda} X_\lambda$ is compact in the product topology.*

Banach-Alaoglu sequential version.

Theorem A.1.4 ([75], Theorem 12)**.** *Let X be a separable Banach space and $\mathbb{B} \subset X^*$ the closed unit ball of the dual. Then $\mathbb{B}$ is weakly* sequentially compact.*

Chacon biting Lemma.

Lemma A.1.5 ([21]). *Let $\mu \in \mathcal{M}^+(\Omega)$ and $(v_j)_j$ in $L^1(\Omega,\mu)$ be a sequence such that $\sup_j \|v_j\| < +\infty$. There are sets $E_k \subset E_{k+1}, \mu(\Omega \setminus E_k) \to 0$ and a subsequence $(v_{j_i})_i$ of $(v_j)_j$ and $v \in L^1(\mu)$ so that $v_{j_i} \rightharpoonup v$ in $L^1(E_k,\mu)$ for all k.*

Kantorovich metric.

Lemma A.1.6 ([67], Lemma 2.1). *Let (X,d) be a metric space. The Kantorich metric on $\mathcal{M}^+(X)$ generates the same topology as the weak* topology of measures.*

Lemma A.1.7. *Let $\eta \in \mathcal{M}(\Omega,\mathbb{R}^d)$, $\Omega \subset \mathbb{R}^n$ open bounded set, and $(\phi_\varepsilon)_{0<\varepsilon\leq 1}$ be a family of standard mollifiers,* $\operatorname{supp}(\phi_1) \subset \mathbb{B}^n$. *Then*

$$\eta \star \phi_\varepsilon \xrightarrow{\text{area-strictly}} \eta.$$

Proof. Because $(\mathcal{L}^n \llcorner \Omega, \eta_\varepsilon) \rightharpoonup^* (\mathcal{L}^n \llcorner \Omega, \eta)$, we immediately obtain the lower bound

$$\liminf_{\varepsilon\to 0} |(\mathcal{L}^n, \rho_\varepsilon)|(\Omega) \geq |(\mathcal{L}^n, \rho)|(\Omega).$$

To prove the upper semi-continuity of the above quantity, notice the following equality:

$$(\mathcal{L}^n \llcorner \Omega, \rho_\varepsilon) = (\mathcal{L}^n \llcorner \Omega, \rho) \star \phi_\varepsilon - (\mathcal{L}^n \llcorner \Omega_{-\varepsilon} \star (\delta_0 - \phi_\varepsilon), 0),$$

where

$$\Omega_{-\varepsilon} = \{x \in \Omega : d(x, \partial\Omega) > \varepsilon\}.$$

Using Jensen inequality we get

$$\begin{aligned}\limsup_{\varepsilon\to 0} |(\mathcal{L}^n, \rho_\varepsilon)|(\Omega) &\leq \limsup_{\varepsilon\to 0} |(\mathcal{L}^n \llcorner \Omega, \rho) \star \phi_\varepsilon|(\Omega) + |(\mathcal{L}^n \llcorner \Omega_{-\varepsilon} \star (\delta_0 - \phi_\varepsilon), 0)|(\Omega) \\ &\leq |(\mathcal{L}^n \llcorner \Omega, \rho)|(\Omega) + \limsup_{\varepsilon\to 0} 2\mathcal{L}^n(\Omega \setminus \Omega_{-\varepsilon}) = |(\mathcal{L}^n, \rho)|(\Omega)\end{aligned}$$

□

Atomic decomposition.

Theorem A.1.8 ([42], Proposition 1.22). *Let $\mu \in \mathcal{M}(\Omega)$. Then there exists a purely atomic measure μ^a and a non-atomic measure μ^{n-a} such that $\mu = \mu^a + \mu^{n-a}$.*

Besicovich covering.

Theorem A.1.9 ([10], Theorem 2.19)**.** *$E \subset \mathbb{R}^n$ be a bounded Borel set, and let $\mathcal{F}$ be a fine cover of E. Then, for every positive Radon measure in $\mathbb{R}^n$, there is a disjoint family $\mathcal{F}' \subset \mathcal{F}$ such that*

$$\mu\left(E \setminus \bigcup \mathcal{F}'\right) = 0.$$

Lemma A.1.10. *Let $E = \{E_{x,r}\}$ be a covering of Ω such that there is $0 < \eta < 1$, independent of x and r, such that*

$$\mathbb{B}_{\eta r}(x) \subset E_{x,r} \subset \mathbb{B}_r(x), \quad \mathcal{L}^n(\partial E_{x,r}) = 0,$$

and for all x we have

$$\inf\{r > 0 : E_{x,r} \subset \mathbb{B}_r(x)\} = 0.$$

Then we can find a countable family of pairwise disjoint sets $(E_i)_{i\in\mathbb{N}} \subset E$ such that $\mathcal{L}^n(\Omega \setminus \cup_i E_i) = 0$.

Proof. Apply Besicovich Theorem to find finitely many disjoint balls $\mathbb{B}_{r_i}(x_i)$ so that

$$\mathcal{L}^n(\Omega \setminus \cup_i \mathbb{B}_{r_i}(x_i)) \leq \eta^n/2\mathcal{L}^n(\Omega).$$

We can estimate

$$\begin{aligned}\mathcal{L}^n(\Omega \setminus \cup_i E_{r_i}(x_i)) &\leq \mathcal{L}^n(\Omega \setminus \cup_i \mathbb{B}_{\eta r_i}(x_i)) \\ &\leq \mathcal{L}^n(\Omega) - \sum_i \eta^n \mathcal{L}^n(\mathbb{B}_{r_i}(x_i)) \leq \mathcal{L}^n(\Omega)(1 - \eta^n/2).\end{aligned}$$

Letting $\Omega \setminus \cup_i \overline{E_i} = \Omega^1$, then $\mathcal{L}^n(\Omega^1) \leq \mathcal{L}^n(\Omega)(1 - \eta^n/2)$ we can repeat the same argument on the interior Ω^1 and find disjoint new sets E_i^1. Iterate to find a countable disjoint collection $\{E_i^j, i \in \mathbb{N}\}_{j\in\mathbb{N}} \equiv \{E_k, k \in \mathbb{N}\}$. Then clearly

$$\mathcal{L}^n(\Omega \setminus \cup_k E_k) \leq \mathcal{L}^n(\Omega \setminus \cup_{k=1}^j E_k) \leq \mathcal{L}^n(\Omega)(1 - \eta^n/2)^j \xrightarrow{j\to\infty} 0.$$

I.e. E_k is a covering of Ω with respect to the Lebesgue measure. □

Lieb-Brezis.

Theorem A.1.11 ([28])**.** *Let (Ω, μ) be an arbitrary measure space and suppose that $(f_n)_n$ in $L^p(\Omega, \mu)$ is so that*

$$f_n \to f \quad \mu\text{-a.e. and} \quad \sup_n \|f_n\|_{p,\mu} < +\infty.$$

Then

$$\lim_n \int_\Omega |f_n|^p - |f_n - f|^p d\mu = \int_\Omega |f|^p d\mu.$$

A.2 BV functions

Theorem A.2.1 ([10], Theorem 3.23)**.** *Let $(u_n)_n$ be a sequence in $BV(\Omega, \mathbb{R}^m)$ such that $\sup_n \|u_n\|_{BV} < +\infty$. There exists a subsequence $(u_{n_j})_j$ and $u \in BV(\Omega, \mathbb{R}^m)$ so that $u_{n_j} \rightharpoonup^* u$ in BV, meaning*

$$u_j \to u \text{ in } L^1(\Omega, \mathbb{R}^m) \quad \text{and} \quad Du_j \rightharpoonup^* Du \text{ in } C_0(\Omega, \mathbb{R}^{m\times n})^*.$$

Here we list a few important properties of functions of bounded variations. Consult [10], Chapter 3, or [101], for proof of these results.

Theorem A.2.2. *Let $u \in BV(\Omega, \mathbb{R}^m)$, the following properties hold:*

- *if $E \subset \Omega$ is Borel, $\mathcal{H}^{n-1}(E) = 0$ implies $|Du|(E) = 0$,*
- *there is a Borel set $J_u \subset \Omega$, called the jump set, which is $n-1$ rectifiable and such that for all $x \in J_u$ there are $\eta(x) \in \partial\mathbb{B}^n$, $u^+(x) \in \mathbb{R}^m$ and $u^-(x) \in \mathbb{R}^m$ so that*

$$⨍_{\mathbb{B}_r(x)\cap\{\langle x-y,\eta(x)\rangle\geq 0\}} u(y)dy \to u^+(x)$$

and

$$⨍_{\mathbb{B}_r(x)\cap\{\langle x-y,\eta(x)\rangle\leq 0\}} u(y)dy \to u^-(x).$$

- *There is a set $N \subset \Omega \setminus J_u, \mathcal{H}^{n-1}(N) = 0$ so that*

$$⨍_{\mathbb{B}_r(x)} u(y)dy \to \frac{u^+(x)+u^-(x)}{2} =: \tilde{u}(x) \quad \forall x \in J_u$$
$$⨍_{\mathbb{B}_r(x)} u(y)dy \to \tilde{u}(x) \quad x \in \Omega \setminus (J_u \cup N)$$

and $\tilde{u} = u$ $\mathcal{L}^n$-a.e.

Every time we take a function $u \in BV(\Omega, \mathbb{R}^m)$ we tacitly consider its *precise representative* $\tilde{u}$ which is defined $\mathcal{H}^{n-1}$-a.e. in Ω. Also, we call $Du \llcorner J_u = D^j u$ its *jump part.*

It is possible to compute the gradient of $u \in BV(\Omega, \mathbb{R}^m)$ on its jump set J_u

$$D^j u = (u^+(x) - u^-(x)) \otimes \nu(x) d\mathcal{H}^{n-1} \llcorner J_u.$$

It is clear that, after writing Du in its polar representation, $\frac{Du}{|Du|}$ has rank-1 for $\mathcal{H}^{n-1}$ almost all $x \in J_u$. The same holds on the singular part of Du to $\mathcal{L}^n$ and $D^j u$, called its Cantor part $D^c u$.

The following is known as the co-area formula.

Theorem A.2.3 ([38], Theorem 5.9)**.** *Let $u \in BV(\Omega)$. Then $\{u > t\}$ has a finite perimeter for almost all $t \in \mathbb{R}$, and, calling its perimeter $\partial\{u > t\}$, we have*

$$\|Du\|(\Omega) = \int_{-\infty}^{+\infty} \mathcal{H}^{n-1}(\partial\{u > t\})dt.$$

Theorem A.2.4 ([4], Corollary 4.6)**.** *For all $u \in BV(\Omega, \mathbb{R}^m)$, $\frac{Du}{|Du|}(x)$ has rank-1 for $|D^s u|$ almost all $x \in \Omega$.*

Something similar is true in higher dimensions.

Theorem A.2.5 ([4], Theorem 4.13)**.** *Let $u \in W^{1,k-1}(\Omega, \mathbb{R}^m)$ be so that $D^k u \in \mathcal{M}(\Omega, \mathbb{R}^{m\times n\times\ldots\times n})$. Then for $|D^k u|^s$-a.e. $x \in \Omega$ we have that*

$$\frac{D^k u}{|D^k u|} = a \otimes \overbrace{b \otimes \ldots \otimes b}^{k-times}$$

for some $a \in \mathbb{R}^m, b \in \mathbb{R}^n$.

See also [35], Theorem 1.6 for a more general result.

Lemma A.2.6. *Let $\mu \in \mathcal{M}^+(\Omega)$. Then $BV(\Omega, \mathbb{R}^m)$ endowed with either the strict topology is a metric space, which metric is given by:*

- *Strict:*

$$d(u, v) = \|u - v\|_{1,\Omega} + \Big||Du|(\Omega) - |Dv|(\Omega)\Big|,$$

- *μ-strict:*

$$d(u, v) = \|u - v\|_{1,\Omega} + \Big|(|Du|, \mu)(\Omega) - (|Dv|, \mu)(\Omega)\Big|.$$

Proof. If U is an open neighbourhood in the strict topology, then is it bounded in BV norm because and therefore it is metrisable. We can then just check equivalence with respect to the convergence of sequences, which is clear from the expressions above. □

Lemma A.2.7 ([10], Theorem 3.88)**.** *Let $\Omega \subset \mathbb{R}^n$ have a Lispchitz boundary. Then the trace operator is continuous in the strict topology, as a function $BV(\Omega, \mathbb{R}^m) \to L^1(\partial\Omega, \mathcal{H}^{n-1}, \mathbb{R}^m)$.*

Lemma A.2.8 ([92], Lemma 11.1)**.** *Let $\Omega \subset \mathbb{R}^n$ be an open, bounded domain. For all $u \in BV(\Omega)$ there exists a sequence $(u_j)_j$ in $C^\infty(\Omega) \cap W^{1,1}(\Omega)$ such that u_j has the same trace as u and $u_j \to u$ area-strictly in BV.*

Lemma A.2.9. *Let $f \in \mathcal{M}(\Omega, \mathbb{R}^{m\times n})$ be such that* $\operatorname{curl}(f) = 0$ *in $\mathcal{D}'$ and for finitely many xs there is a convex domain $E(x)$ such that $f \llcorner E(x)$ has compact support, and $E(x) \cap E(y) = \emptyset$ if $x \neq y$. We can find $C = C(\Omega)$ and $v \in BV(\Omega, \mathbb{R}^m)$ so that $Dv = f$ and $\|v\|_1 \leq C|f|(\Omega)$.*

Proof. Fix any such x and let $f = f \llcorner E(x)$. By convexity, we can find $u \in BV(E(x))$ such that $Du = f$. Because f is compactly supported in $E(x)$, then $Du = 0$ on $\{y : d(y, \partial E(x)) < d(\operatorname{supp}(f), \partial E(x))\}$ which is connected and so, upon translation, we have $u \in BV(\mathbb{R}^n)$ and $\operatorname{supp}(u) \subset E(x)$. Repeat the same argument on every $E(x)$ and sum all potentials together to u. Then, such u is so that $Du = f$ and it is compactly supported in Ω, so by the Sobolev embedding we can find C such that $\|u\|_1 \leq C|f|(\Omega)$, C depending only on Ω. □

A.3 $\mathcal{A}$ quasi-convexity

Definition A.3.1. *A function is said to be $\mathcal{A}$ quasi-convex if*

$$f(z) \leq \int_Q f(U(x) + z)dx$$

for all periodic $U \in C^\infty(Q, W), Q \subset \mathbb{R}^n$ being the unit cubit in $\mathbb{R}^n$, so that

$$\int_Q U(x)dx = 0 \quad \text{and} \quad \mathcal{A}U = 0 \text{ in } \mathcal{D}'(Q).$$

Definition A.3.2. *We say that $\mathcal{A}$ satisfies the* spanning condition *if*

$$W = \operatorname{span}(\cup_{|\xi|=1} \ker(\mathcal{A}(\xi))).$$

Assumption 3. *In the following results, $\mathcal{A}$ is always assumed to be a constant coefficient, constant rank, linear differential operator satisfying the spanning condition. Also, refer to Chapter 4 regarding definitions and properties of $\mathcal{A}$ and $\mathcal{B}$ and associated spaces.*

The following result shows that $\mathcal{A}$ quasi-convex functions that are homogeneous of degree 1 are automatically convex on the wave cone.

Theorem A.3.3 ([62], Theorem 1.1)**.** *Let $\mathcal{A}$ satisfy 3 and let $f\colon \mathbb{R}^d \to \mathbb{R}$ be $\mathcal{A}$ quasi-convex and positively 1-homogeneous. Then f is convex at all $z \in \Lambda_{\mathcal{A}}$.*

On the other side, when the function does not have a regular recession at infinity we can still derive the following convex-like inequality.

Theorem A.3.4 ([62], Lemma 2.5)**.** *Let $\mathcal{A}$ satisfy 3 and let $f\colon \mathbb{R}^d \to \mathbb{R}$ be $\mathcal{A}$ quasi-convex. Then for all $z \in \mathbb{R}^d$ and $w \in \Lambda_{\mathcal{A}}$ we have*

$$f(z+w) \leq f(z) + f^{\infty}(w).$$

Here $f^{\infty}(w)$ exists because f is convex at directions $w \in \Lambda_{\mathcal{A}}$.

Theorem A.3.5 ([67], Theorem 1.2)**.** *Let $\mathcal{A}$ satisfy 3 and let $(\nu_x, \lambda, \nu_x^{\infty}) \in Y(\Omega, \mathbb{B}^d)$ be so that $\lambda(\partial\Omega) = 0$ and barycenter $v \in \mathcal{M}(\Omega)$. Then there exists a sequence of mollifiers $\phi_j \rightharpoonup^* 0$ in $C_0(\Omega)^*$ and a sequence of maps $(u_j)_j$ in $\mathcal{D}(\Omega, W)$ so that*

$$\begin{aligned} &\phi_j \star v + \mathcal{B}(u_j) \xrightarrow{Y} (\nu_x, \lambda, \nu_x^{\infty}) \\ &\|u_j\|_{W^{k-1,1}} \to 0 \end{aligned}$$

if and only if $\mathcal{A}v = 0$ and for all $\mathcal{A}$ quasi-convex $f\colon V \to \mathbb{R}$ of linear growth it holds that

$$\langle \nu_x, f\rangle + \frac{\lambda}{\mathcal{L}^n}(x)\langle \nu_x^{\infty}, f^{\infty}\rangle \geq f\left(\overline{\nu_x} + \overline{\nu_x^{\infty}}\frac{\lambda}{\mathcal{L}^n}(x)\right)$$

for $\mathcal{L}^n$-a.e. $x \in \Omega$.

Theorem A.3.6 ([67], Proposition 1.1)**.** *Let $\mathcal{A}$ satisfy 3 and suppose that $(U_j)_j$ in $L^1(\Omega, W)$ is so that $\mathcal{A}(U_j) = 0$ and*

$$U_j \xrightarrow{Y(e_{\mathbb{B}^d})} (\nu_x, \lambda, \nu_x^{\infty}).$$

Then, for all quasi-convex f of linear growth, we have

$$\langle \nu_x, f\rangle + \frac{\lambda}{\mathcal{L}^n}(x)\langle \nu_x^{\infty}, f^{\infty}\rangle \geq f\left(\overline{\nu_x} + \overline{\nu_x^{\infty}}\frac{\lambda}{\mathcal{L}^n}(x)\right)$$

for almost every $x \in \Omega$.

See also [13] for the original result. Theorems A.3.5 and A.3.6 apply to the examples in 4.3.3.

A.4 Lifting Young Measures

We only state a partial result here as we don't need the full result. First, we define the main class of functions:

Assumption 4. *$f \in C(\Omega \times \mathbb{R}^m \times \mathbb{R}^{m\times n}$ satisfied:*

- $0 \leq f(x,y,z) \leq C(1+|y|^{\frac{n}{n-1}}+|z|)$ *for some* $C>0$ *and all* $(x,y,z) \in \Omega \times \mathbb{R}^m \times \mathbb{R}^{m\times n}$,

- *the limit*

$$\lim_{x'\to x, y'\to y, z'\to z, t\uparrow\infty} \frac{f(x',y',z't)}{t} = f^\infty(x,y,z) \in C(\Omega \times \mathbb{R}^m \times \mathbb{R}^{m\times n})$$

exists locally uniformly on all sequences $x' \to x \in \Omega$, $y' \to y \in \mathbb{R}^m$, $z' \to z \in \mathbb{R}^{m\times n}$ *and* $t \uparrow \infty$.

Theorem A.4.1 ([94], Theorem A)**.** *Let* $f\colon \overline{\Omega} \times \mathbb{R}^m \times \mathbb{R}^{m\times n} \to \mathbb{R}$ *be so that:*

1. f *satisfied 4,*

2. $f(x,y,\cdot)$ *is quasi-convex for all* $(x,y) \in \Omega \times \mathbb{R}^m$.

Then if $(u_j)_j$ *in* $W^{1,1}(\Omega,\mathbb{R}^m), u \in BV(\Omega,\mathbb{R}^m)$ *is so that* $u_j \rightharpoonup^* u$ *in* BV*, then*

$$\begin{aligned}&\liminf_j \int_\Omega f(x,u_j(x),\nabla u_j(x))dx\\ &\geq \int_\Omega f(\nabla u(x))dx + K_f[u](x)d\mathcal{H}^{n-1}\llcorner J_u(x) + f^\infty\left(x,u(x),\frac{D^c u}{|D^c u|}\right)d|D^c u|.\end{aligned}$$

In the next lemma, we report that such an extension can be recovered via smooth functions.

Theorem A.4.2 ([94], Theorem 6.4)**.** *Let* $u \in BV(\Omega,\mathbb{R}^m)$*, and* f *as in A.4.1. For every* $\varepsilon > 0$ *there exists a sequence* $(u_j)_j$ *in* $C^\infty(\Omega,\mathbb{R}^m) \cap W^{1,1}(\Omega,\mathbb{R}^m)$, u_j *having the same trace as* u *and with* $|u_j|^{\frac{n}{n-1}}$ *is* $\mathcal{L}^n$ *equi-integrable, so that*

$$\begin{aligned}&\limsup_j \int_\Omega f(x,u_j(x),\nabla u_j(x))dx\\ &\leq \int_\Omega f(x,u(x),\nabla u(x))dx + K_f[u](x)d\mathcal{H}^{n-1}\llcorner J_u + f^\infty\left(x,u(x),\frac{D^c u}{|D^c u|}\right)d|D^c u| + \varepsilon.\end{aligned}$$

The proof in the original paper does not remark that u_j can be constructed having the same trace as u, so we write a separate remark to show how to get this piece of information.

Remark A.4.3. *In the proof of [94], Proposition 6.3, the recovery sequence* $(u_j)_j$ *in* $BV(\Omega,\mathbb{R}^m)$ *has trace* u*. Subsequently, in [94], Theorem 6.4, such a sequence is further approximated, in the area-strict topology, by smooth functions* $(v_j)_j$ *in* $C^\infty(\Omega,\mathbb{R}^m)$ *to obtain an* ε*-recovery sequence for non-homogeneous* f*s (see page 96 of the same*

paper). The area-strict approximating sequences can be taken to belong to the same trace class, in this case $(v_j)_j$ in $C^\infty(\Omega,\mathbb{R}^m)\cap BV_u(\Omega,\mathbb{R}^m)$, in virtue of Lemma A.2.8. This fact has also been discussed with one of the authors of the paper.

Also, the recovery sequence $(u_j)_j$ in $BV(\Omega,\mathbb{R}^m)$ constructed in [94] does not concentrate at the boundary, meaning that (up to a subsequence)

$$|\nabla u_j|dx \rightharpoonup^* \lambda \quad \text{in } C(\overline{\Omega})^*$$

for some $\lambda\in\mathcal{M}^+(\overline{\Omega})$ with $\lambda(\partial\Omega)=0$. We claim that this implies that if

$$|u_j|^p d\mu \rightharpoonup^* |u|^p d\mu + \delta^s \quad \text{in } C(\overline{\Omega}),$$

then $\delta^s(\partial\Omega)=0$. To see this, fix $\phi\in\mathcal{D}(\Omega)$ with $0\le\phi\le 1$ and assume that $u_j\rightharpoonup^* 0$ in $BV(\mathbb{R}^n,\mathbb{R}^m)$. Write $\psi=1-(1-\phi)^{\frac{1}{p}}\in\mathrm{Lip}(\mathbb{R}^n)$, then

$$\begin{aligned}\int_{\mathbb{R}^n}|v_j|^p(1-\phi)d\mu &= \int_{\mathbb{R}^n}|(v_j)(1-\psi)|^p d\mu \le C\|v_j(1-\psi)\|_{BV}\\ &\le C\int_{\mathbb{R}^n}|v_j|(1-\psi)dx + C\int_{\mathbb{R}^n}|v_j||\nabla\psi|dx + C\int_{\mathbb{R}^n}(1-\psi)|\nabla v_j|dx.\end{aligned}$$

By the Rellich-Kondrachov embedding Theorem $v_j\to 0$ strongly in L^1. So, we can pass to the limit in the above inequality and obtain that, in light of the fact that $\mathrm{supp}(v_j)\subset\Omega$,

$$\limsup_j \int_{\mathbb{R}^n}|v_j|^p(1-\phi)d\mu \le C\int_\Omega(1-\psi)d\lambda,$$

and conclude by letting $\phi\uparrow 1$.

This fact is not written down in the original paper because it is nowhere used there.

In the following theorem, we slightly change the statement so that it is actually just a corollary of the original theorem.

Theorem A.4.4 ([94], 5.1)**.** *Let $\mu\in\mathcal{M}^+(\Omega)$ be p-admissible and $u\in BV(\Omega,\mathbb{R}^m)$ and set $\iota=\mathrm{gr}^u_\#\mu\in\mathcal{M}^+(\Omega\times\mathbb{R}^m)$. If $\lambda\in\mathcal{M}(\Omega\times\mathbb{R}^m,\mathbb{R}^{m\times n})$, for μ-a.e. every $x\in\Omega$ it holds that*

$$\frac{\lambda}{\iota}(x,u(x)) = \lim_{r\to 0}\frac{\lambda(\overline{\mathbb{B}_r(x)\times\mathbb{B}_r(u(x))})}{\iota(\overline{\mathbb{B}_r(x)\times\mathbb{B}_r(u(x))})} = \lim_{r\to 0}\frac{\lambda(\overline{\mathbb{B}_r(x)\times\mathbb{B}_r(u(x))})}{\mu(\overline{\mathbb{B}_r(x)})}.$$

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