On the existence of Hamiltonian stationary Lagrangian submanifolds in symplectic manifolds

Dominic Joyce, Yng-Ing Lee and Richard Schoen

Abstract

Let (M, ω) be a compact symplectic $2n$-manifold, and g a Riemannian metric on M compatible with ω. For instance, g could be Kähler, with Kähler form ω. Consider compact Lagrangian submanifolds L of M. We call L *Hamiltonian stationary*, or *H-minimal*, if it is a critical point of the volume functional Vol_g under Hamiltonian deformations, computing $\mathrm{Vol}_g(L)$ using $g|_L$. It is called *Hamiltonian stable* if in addition the second variation of Vol_g under Hamiltonian deformations is nonnegative.

Our main result is that if L is a compact, Hamiltonian stationary Lagrangian in $\mathbb{C}^n$ which is *Hamiltonian rigid*, then for any M, ω, g as above there exist compact Hamiltonian stationary Lagrangians L' in M contained in a small ball about some $p \in M$ and locally modelled on tL for small $t > 0$, identifying M near p with $\mathbb{C}^n$ near 0. If L is Hamiltonian stable, we can take L' to be Hamiltonian stable.

Applying this to known examples L in $\mathbb{C}^n$ shows that there exist families of Hamiltonian stable, Hamiltonian stationary Lagrangians diffeomorphic to T^n, and to $(\mathcal{S}^1 \times \mathcal{S}^{n-1})/\mathbb{Z}_2$, and with other topologies, in every compact symplectic $2n$-manifold (M, ω) with compatible metric g.

1 Introduction

Let (M, ω) be a symplectic manifold of real dimension $2n$, and g a Riemannian metric on M compatible with ω, and J the associated almost complex structure, so that $\omega(v, w) = g(Jv, w)$ for vector fields v, w on M. For example, J could be an integrable complex structure, g a Kähler metric on (M, J), and ω the Kähler form. This paper concerns some special classes of compact Lagrangian submanifolds L in (M, ω):

- We call L *Hamiltonian stationary* (or *H-minimal*) if L has stationary volume amongst Hamiltonian equivalent Lagrangians L'. The Euler–Lagrange equation for Hamiltonian stationary Lagrangians is $\mathrm{d}^*\alpha_H = 0$, where H is the mean curvature vector on L, α_H the 1-form on L defined by $\alpha_H(\cdot) = \omega(H, \cdot)$, and d^* the Hodge dual of the exterior derivative d.
- If L is Hamiltonian stationary, we call L (*Hamiltonian*) *stable* if the second variation of volume at L amongst Hamiltonian equivalent Lagrangians L' is nonnegative.

Now let M be $\mathbb{C}^n$ with its Euclidean Kähler structure J_0, g_0, ω_0, and L be a compact Lagrangian in $\mathbb{C}^n$. Then

- If L is Hamiltonian stationary in $\mathbb{C}^n$, we call L (*Hamiltonian*) *rigid* if all infinitesimal Hamiltonian deformations of L as a Hamiltonian stationary Lagrangian are induced by the action of the Lie algebra $\mathfrak{u}(n) \oplus \mathbb{C}^n$ of the automorphism group $\mathrm{U}(n) \ltimes \mathbb{C}^n$ of $(\mathbb{C}^n, J_0, g_0)$.

More details are given in §2. Hamiltonian stationary Lagrangians were defined and studied by Oh [9,10], in the Kähler case. In [10, Th. IV] he proves that for $a_1, \dots, a_n > 0$, the torus $T^n_{a_1,\dots,a_n}$ in $\mathbb{C}^n$ given by

$$T^n_{a_1,\dots,a_n} = \big\{(z_1, \dots, z_n) \in \mathbb{C}^n : |z_j| = a_j,\ j = 1, \dots, n\big\} \tag{1}$$

is a stable, rigid, Hamiltonian stationary Lagrangian in $\mathbb{C}^n$. In Example 2.10 we show that the Lagrangian L_n diffeomorphic to $(\mathcal{S}^1 \times \mathcal{S}^{n-1})/\mathbb{Z}_2$ given by

$$L_n = \big\{(x_1 e^{is}, \dots, x_n e^{is}) : 0 \le s < \pi,\ \textstyle\sum_{j=1}^n x_j^2 = 1,\ (x_1, \dots, x_n) \in \mathbb{R}^n\big\}, \tag{2}$$

is a stable, rigid, Hamiltonian stationary Lagrangian in $\mathbb{C}^n$, and Example 2.12 gives more examples due to Amarzaya and Ohnita [1].

Hamiltonian stationary Lagrangians are interesting as they can be viewed as the 'best' representatives of a Hamiltonian isotopy class of Lagrangians, and therefore studying Hamiltonian stationary Lagrangians may give us some understanding of the family of all Lagrangians (see §7 on this point). Also, for compact, nonsingular, graded Lagrangians in a Calabi–Yau manifold, to be Hamiltonian stationary is equivalent to being special Lagrangian. Thus, Hamiltonian stationary Lagrangians are a generalization of special Lagrangians.

The goal of this paper is to prove the following:

Theorem A. *Suppose (M, ω) is a compact symplectic $2n$-manifold, g a Riemannian metric on M compatible with ω, and L is a compact, Hamiltonian rigid, Hamiltonian stationary Lagrangian in $\mathbb{C}^n$. Then there exist compact, Hamiltonian stationary Lagrangians L' in M which are diffeomorphic to L, such that L' is contained in a small ball about some point $p \in M$, and identifying M near p with $\mathbb{C}^n$ near 0 in geodesic normal coordinates, L' is a small deformation of tL for small $t > 0$. If L is also Hamiltonian stable, we can take L' to be Hamiltonian stable.*

Applying Theorem A with $L = T^n_{a_1,\dots,a_n}$ or L_n yields:

Corollary B. *Suppose (M, ω) is a compact symplectic $2n$-manifold, and g a Riemannian metric on M compatible with ω. Then there exist stable, Hamiltonian stationary Lagrangians L' in M diffeomorphic to T^n and to $(\mathcal{S}^1 \times \mathcal{S}^{n-1})/\mathbb{Z}_2$, which are locally modelled on $tT^n_{a_1,\dots,a_n}$ in (1) or on tL_n in (2) for small $t > 0$ near some point $p \in M$, identifying M near p with $\mathbb{C}^n$ near 0.*

We note that a special case of Corollary B has been proved independently by Butscher and Corvino [2], using a similar method. They fix $n = 2$, suppose

(M,J,g) is Kähler, take $L=T^2_{a_1,a_2}$, and also assume a nondegeneracy condition on the metric g near the point p which we do not need. One difference between our approach and theirs is that they fix the point $p\in M$ where they glue in L in advance and make assumptions about it, whereas we show we can glue in L near p for some unknown point $p\in M$. A different nondegeneracy condition for Kähler manifolds of any dimension and $L=T^n_{a_1,\ldots,a_n}$ is obtained by the second author recently [5]. Butscher and Corvino believe that their method can also be generalized to higher dimensions.

Another approach to the construction of Hamiltonian stationary Lagrangians in general Kähler and symplectic manifolds is the variational approach introduced by Schoen and Wolfson [11]. The idea there is to minimize volume among Lagrangian cycles representing a given homology (or homotopy) class to produce a minimizer in a class of singular Lagrangian submanifolds. One then hopes to study the regularity properties of these minimizing cycles. This minimization can be done in general dimensions in the class of Lagrangian integral currents, but the regularity theory is still missing in general. For the two dimensional problem, one can minimize in the class of surfaces which are images of a fixed surface (under $W^{1,2}$ maps), and the paper [11] develops the existence and regularity theory for this problem. It is shown that such minimizers are smooth branched Hamiltonian stationary surfaces outside a finite number of singular points at which the possible tangent cones can be described.

We begin in §2 with some background material from symplectic geometry, the definition of Hamiltonian stationary, Hamiltonian stable, and Hamiltonian rigid Lagrangians, and examples of stable and rigid Hamiltonian stationary Lagrangians in $\mathbb{C}^n$. Given a compact symplectic manifold (M,ω) with compatible metric Riemannian g, §3 constructs a smooth family of Darboux coordinate systems $\Upsilon_{p,\upsilon}:B_\epsilon\to M$ for each $p\in M$ such that the metric $\Upsilon^*_{p,\upsilon}(g)$ on $B_\epsilon\subset\mathbb{C}^n$ is close to Euclidean metric g_0 on B_ϵ in C^k for all $k\geqslant 0$, uniformly in $p\in M$.

Section 4 sets up the notation for the proof of Theorem A, recasting it as solving one of a family of fourth-order nonlinear elliptic p.d.e.s $P^t_{p,\upsilon}(f)=0$ on L', for small $t>0$ and (p,υ) in the $\mathrm{U}(n)$-frame bundle U of M, where for small t,f the equation $P^t_{p,\upsilon}(f)=0$ approximates a fourth-order linear elliptic equation $\mathcal{L}f=0$. Section 5 proves that we can solve this equation mod $\operatorname{Ker}\mathcal{L}$ for all (p,υ) in U, and there is a unique solution $f^t_{p,\upsilon}$ which is L^2-orthogonal to $\operatorname{Ker}\mathcal{L}$ and small in $C^{4,\gamma}$. Section 6 shows that $P^t_{p,\upsilon}(f^t_{p,\upsilon})=0$ if and only if (p,υ) is a stationary point of a smooth function K^t on the compact manifold U, and deduces Theorem A. Finally, §7 speculates on the possibility of defining invariants 'counting' Hamiltonian stationary Lagrangians in a fixed Hamiltonian isotopy class in (M,ω).

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Lagrangians in $\mathbb{C}^n$.

2 Background material

2.1 Background from symplectic geometry

We start by recalling some elementary symplectic geometry, which can be found in McDuff and Salamon [7]. Here are the basic definitions.

Definition 2.1. Let M be a smooth manifold of even dimension $2n$. A closed 2-form ω on M is called a *symplectic form* if the $2n$-form ω^n is nonzero at every point of M. Then (M,ω) is called a *symplectic manifold*. A submanifold L in M is called *Lagrangian* if $\dim L = n = \frac{1}{2}\dim M$ and $\omega|_L \equiv 0$.

The simplest example of a symplectic manifold is $\mathbb{C}^n$.

Example 2.2. Let $\mathbb{C}^n$ have complex coordinates $(z_1,\dots,z_n)$, where $z_j = x_j + iy_j$ with $i=\sqrt{-1}$. Define the standard Euclidean metric g_0, symplectic form ω_0, and complex structure J_0 on $\mathbb{C}^n$ by

$$\begin{aligned} g_0 &= \textstyle\sum_{j=1}^n |\mathrm{d}z_j|^2 = \sum_{j=1}^n (\mathrm{d}x_j^2 + \mathrm{d}y_j^2),\\ \omega_0 &= \textstyle\frac{i}{2}\sum_{j=1}^n \mathrm{d}z_j\wedge \mathrm{d}\bar z_j = \sum_{j=1}^n \mathrm{d}x_j\wedge\mathrm{d}y_j, \quad \text{and}\\ J_0 &= \textstyle\sum_{j=1}^n \bigl(i\mathrm{d}z_j\otimes\frac{\partial}{\partial z_j} - i\mathrm{d}\bar z_j\otimes\frac{\partial}{\partial \bar z_j}\bigr) = \sum_{j=1}^n\bigl(\mathrm{d}x_j\otimes\frac{\partial}{\partial y_j} - \mathrm{d}y_j\otimes\frac{\partial}{\partial x_j}\bigr), \end{aligned} \tag{3}$$

noting that $\mathrm{d}z_j = \mathrm{d}x_j + i\mathrm{d}y_j$ and $\frac{\partial}{\partial z_j} = \frac{1}{2}\bigl(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\bigr)$. Then $(\mathbb{C}^n,\omega_0)$ is a symplectic manifold, and g_0 is a Kähler metric on $(\mathbb{C}^{2n},J)$ with Kähler form ω_0. For $\epsilon>0$, write B_ϵ for the open ball of radius ϵ about 0 in $\mathbb{C}^n$.

Darboux' Theorem [7, Th. 3.15] says that every symplectic manifold is locally isomorphic to $(\mathbb{C}^n,\omega_0)$.

Theorem 2.3. *Let (M,ω) be a symplectic $2n$-manifold and $p\in M$. Then there exist $\epsilon>0$ and an embedding $\Upsilon: B_\epsilon\to M$ with $\Upsilon(0)=p$ such that $\Upsilon^*(\omega)=\omega_0$, where ω_0 is the standard symplectic form on $\mathbb{C}^n\supset B_\epsilon$.*

Let L be a real n-manifold. Then its tangent bundle T^*L has a *canonical symplectic form* $\hat\omega$, defined as follows. Let $(x_1,\dots,x_n)$ be local coordinates on L. Extend them to local coordinates $(x_1,\dots,x_n,y_1,\dots,y_n)$ on T^*L such that $(x_1,\dots,y_n)$ represents the 1-form $y_1\mathrm{d}x_1+\cdots+y_n\mathrm{d}x_n$ in $T^*_{(x_1,\dots,x_n)}L$. Then $\hat\omega = \mathrm{d}x_1\wedge\mathrm{d}y_1+\cdots+\mathrm{d}x_n\wedge\mathrm{d}y_n$. Identify L with the zero section in T^*L. Then L is a *Lagrangian submanifold* of $(T^*L,\hat\omega)$. The *Lagrangian Neighbourhood Theorem* [7, Th. 3.33] shows that any compact Lagrangian submanifold L in a symplectic manifold looks locally like the zero section in T^*L.

Theorem 2.4. *Let (M,ω) be a symplectic manifold and $L\subset M$ a compact Lagrangian submanifold. Then there exists an open tubular neighbourhood T of the zero section L in T^*L, and an embedding $\Phi: T\to M$ with $\Phi|_L = \mathrm{id}: L\to L$ and $\Phi^*(\omega)=\hat\omega$, where $\hat\omega$ is the canonical symplectic structure on T^*L.*

We shall call T, Φ a *Lagrangian neighbourhood* of L. Such neighbourhoods are useful for parametrizing nearby Lagrangian submanifolds of M. Suppose that $\tilde{L}$ is a Lagrangian submanifold of M which is C^1-close to L. Then $\tilde{L}$ lies in $\Phi(T)$, and is the image $\Phi(\Gamma_\alpha)$ of the graph Γ_α of a unique C^1-small 1-form α on L. As $\tilde{L}$ is Lagrangian and $\Phi^*(\omega) = \hat{\omega}$ we see that $\hat{\omega}|_{\Gamma_\alpha} \equiv 0$. But $\hat{\omega}|_{\Gamma_\alpha} = -\pi^*(\mathrm{d}\alpha)$, where $\pi : \Gamma_\alpha \to L$ is the natural projection. Hence $\mathrm{d}\alpha = 0$, and α is a *closed* 1*-form*. This establishes a 1-1 correspondence between C^1-small closed 1-forms on L and Lagrangian submanifolds $\tilde{L}$ close to L in M.

Let (M, ω) be a compact symplectic manifold and $F : M \to \mathbb{R}$ a smooth function. The *Hamiltonian vector field* v_F of F is the unique vector field satisfying $v_F \cdot \omega = \mathrm{d}F$. The Lie derivative satisfies $\mathcal{L}_{v_F}\omega = v_F \cdot \mathrm{d}\omega + \mathrm{d}(v_F \cdot \omega) = 0$, so the trajectory of v_F gives a 1-parameter family of symplectomorphisms $\mathrm{Exp}(sv_F) : M \to M$ for $s \in \mathbb{R}$, called the *Hamiltonian flow* of F. If L is a compact Lagrangian in M then $\mathrm{Exp}(sv_F)L$ is also a compact Lagrangian in M.

Two compact Lagrangians L, L' in (M, ω) are called *Hamiltonian equivalent* if there exist Lagrangians $L = L_0, \ldots, L_k = L'$ and functions $F_1, \ldots, F_k$ on M with $L_j = \mathrm{Exp}(v_{F_j})L_{j-1}$ for $j = 1, \ldots, k$. In the situation of Theorem 2.4, a Lagrangian $\tilde{L}$ which is C^1 close to L is Hamiltonian equivalent to L if it corresponds to the graph $\Gamma_{\mathrm{d}f}$ of an *exact* 1-form $\mathrm{d}f$ on L.

2.2 The volume functional on Lagrangians

Now let (M, ω) be a symplectic manifold and g a Riemannian metric on M compatible with ω. For a compact Lagrangian L in M, write $\mathrm{Vol}_g(L)$ for the volume of L, computed using the induced Riemannian metric $h = g|_L$. The *volume functional* is $\mathrm{Vol}_g : L \mapsto \mathrm{Vol}_g(L)$, regarded as a functional on the infinite-dimensional manifold of all compact Lagrangians in (M, ω).

Definition 2.5. A compact Lagrangian submanifold L in M is called *Hamiltonian stationary*, or *H-minimal*, if it is a critical point of the volume functional on its Hamiltonian equivalence class of Lagrangians. That is, L is Hamiltonian stationary if

$$\tfrac{\mathrm{d}}{\mathrm{d}s} \mathrm{Vol}_g\bigl(\mathrm{Exp}(sv_F)L\bigr)\big|_{s=0} = 0 \tag{4}$$

for all smooth $F : M \to \mathbb{R}$. By Oh [10, Th. I], who attributes the result to Weinstein, (4) is equivalent to the Euler–Lagrange equation

$$\mathrm{d}^*\alpha_H = 0, \tag{5}$$

where H is the mean curvature vector of L, and $\alpha_H = (H \cdot \omega)|_L$ is the associated 1-form on L, and d^* is the Hodge dual of the exterior derivative d on L, computed using the metric $h = g|_L$. Oh assumed that M is a Kähler manifold in his paper. However, the proof for (5) also works when M is a symplectic manifold with a compatible metric. Explicitly, if $(x_1, \ldots, x_n)$ are local coordinates on L and $h = h_{ab}\,\mathrm{d}x_a\mathrm{d}x_b$, $\alpha = \alpha_a\mathrm{d}x_a$, we have

$$\mathrm{d}^*\alpha = -\frac{\partial h^{ab}}{\partial x_b}\,\alpha_a - h^{ab}\,\frac{\partial \alpha_a}{\partial x_b} - \tfrac{1}{2}\,h^{ab}\alpha_a\,\frac{\partial}{\partial x_b}\bigl(\ln\det(h_{cd})\bigr). \tag{6}$$

Here we use the convention that repeated indices stand for a summation whenever there is no confusion.

Definition 2.6. Following Oh [9, Def. 2.5], a compact Hamiltonian stationary Lagrangian L in M is called *Hamiltonian stable*, or just *stable*, if the second variation of Vol_g is nonnegative for all Hamiltonian variations of L, that is, if

$$\tfrac{\mathrm{d}^2}{\mathrm{d}s^2}\,\mathrm{Vol}_g\big(\mathrm{Exp}(sv_F)L\big)\big|_{s=0} \geqslant 0 \tag{7}$$

for all smooth $F : M \to \mathbb{R}$. A geometric expression for the left hand side of (7) is computed by Oh [10, §3] when M is a Kähler manifold. Define a tensor $S = S_{jkl}\mathrm{d}x_j\mathrm{d}x_k\mathrm{d}x_l$ on L by $S(u, v, w) = \big\langle J(B(u, v)), w\big\rangle$ for all vector fields u, v, w on L, where B is the second fundamental form of L in M, so that $B(u, v)$ is a normal vector field to L in M and $J(B(u, v))$ is a vector field on L. Then S is symmetric by [10, Lem. 3.1]. Let $F : M \to \mathbb{R}$ be smooth and $f = F|_L$. Then Oh [10, Th. 3.4] proves:

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}s^2}\,\mathrm{Vol}_g\big(\mathrm{Exp}(sv_F)L\big)\big|_{s=0} = \int_L \big(&\langle\Delta_H \mathrm{d}f, \mathrm{d}f\rangle - \mathrm{Ric}(J\mathrm{d}f, J\mathrm{d}f)\\ &- 2\langle \mathrm{d}f\otimes \mathrm{d}f\otimes\alpha_H, S\rangle + \langle \mathrm{d}f, \alpha_H\rangle^2\big)\,\mathrm{d}V,\end{aligned} \tag{8}$$

where $\Delta_H = \mathrm{d}\mathrm{d}^* + \mathrm{d}^*\mathrm{d}$ is the Hodge Laplacian, and Ric is the Ricci curvature of g on M.

By a similar computation, one can derive the second variation formula of volume for Hamiltonian deformation at a Lagrangian L (not necessarily Hamiltonian stationary) in a Kähler manifold. From this expression, one can write down the linearized operator of $-\mathrm{d}^*\alpha_H$. More precisely, given a smooth function f on L, we extend it to a smooth function F on M and consider $L_s = \mathrm{Exp}(sv_F)L$ whose mean curvature vector is denoted by H_s. Then we have

$$\begin{aligned}\mathcal{L}f = -\frac{\mathrm{d}}{\mathrm{d}s}\left(\mathrm{d}^*\alpha_{H_s}\right)\big|_{s=0} = \Delta^2 f + \mathrm{d}^*\alpha_{\mathrm{Ric}^\perp(J\nabla f)} &- 2\mathrm{d}^*\alpha_{B(JH,\nabla f)}\\ &- JH(JH(f)),\end{aligned} \tag{9}$$

which is a fourth-order linear elliptic operator from $C^\infty(L)$ to $C^\infty(L)$. Here $\Delta f = \mathrm{d}^*\mathrm{d}f$ is the Hodge Laplacian on functions, and the normal vector $\mathrm{Ric}^\perp(v)$ for a normal vector v is characterized by $\mathrm{Ric}(v, w) = \langle \mathrm{Ric}^\perp(v), w\rangle$ for any normal vector w.

When M is just a symplectic manifold with a compatible metric, the second variation formula of volume and the linearized operator $\mathcal{L}$ do not have such nice expressions. However, since we will work on small balls in Darboux coordinates, the linearized operator $\mathcal{L}$ at L will be very close to the corresponding linearized operator at L in $\mathbb{C}^n$. This is made more precise in Proposition 4.1. The estimate is good enough to pursue our argument and prove the theorems. The expression (9) for $\mathcal{L}$ at a Lagrangian in a Kähler manifold is helpful in understanding the general symplectic picture.

When L is a compact Hamiltonian stationary Lagrangian in a Kähler manifold, we have

$$\tfrac{\mathrm{d}^2}{\mathrm{d}s^2}\,\mathrm{Vol}_g\big(\mathrm{Exp}(sv_F)L\big)\big|_{s=0} = \big\langle \mathcal{L}f, f\big\rangle_{L^2(L)}. \tag{10}$$

Thus L is Hamiltonian stable if and only if the eigenvalues of $\mathcal{L}$ are all nonnegative. The interpretation of the kernel $\operatorname{Ker}\mathcal{L}$ will be important below. From (10) we can see that if $f \in C^\infty(L)$ and v is the normal vector field to L in M with $\alpha_v = \mathrm{d}f$, then v is an infinitesimal Hamiltonian deformation of L as a Hamiltonian stationary Lagrangian, if and only if $\mathcal{L}f = 0$.

Now suppose that (M, J, g) is a *Calabi–Yau n-fold*. Then we can choose a holomorphic $(n, 0)$-form Ω on M with $\nabla\Omega = 0$, normalized so that

$$\omega^n/n! = (-1)^{n(n-1)/2}(i/2)^n\Omega \wedge \bar{\Omega}.$$

If L is an oriented Lagrangian in M, then $\Omega|_L \equiv e^{i\theta}\mathrm{d}V_{g_L}$, where $\mathrm{d}V_{g_L}$ is the volume form of L defined using the metric $g|_L$ and the orientation, and $e^{i\theta} : L \to \mathrm{U}(1) = \{z \in \mathbb{C} : |z| = 1\}$ is the *phase function* of L. We call L *special Lagrangian* if it has constant phase.

In the Calabi–Yau case, the picture above simplifies in two ways. Firstly, as g is Ricci-flat, the Ricci curvature terms in (8) and (9) are zero. Secondly, the 1-form α_H associated to the mean curvature H of L is given by $\alpha_H = -\mathrm{d}\theta$. (This does not imply that α_H is exact, as θ maps $L \to \mathbb{R}/2\pi\mathbb{Z}$ rather than $L \to \mathbb{R}$.) Thus, the condition (5) that L be Hamiltonian stationary becomes $\mathrm{d}^*\mathrm{d}\theta = 0$, that is, the phase function $e^{i\theta} : L \to \mathrm{U}(1)$ is harmonic as a map into $\mathrm{U}(1) = \mathbb{R}/2\pi\mathbb{Z}$, though it is not harmonic as a map into $\mathbb{C}$.

If $e^{i\theta}$ lifts continuously to $\theta : L \to \mathbb{R}$, that is, if L is *graded*, and also L is compact, then the maximum principle implies that θ is constant, so L is special Lagrangian. Hence, any compact, graded, Hamiltonian stationary Lagrangian in a Calabi–Yau n-fold is special Lagrangian. Also, $e^{i\theta}$ lifts continuously to $\theta : L \to \mathbb{R}$ if and only if $-[\alpha_H] = [\mathrm{d}\theta] = 0$ in $H^1(L;\mathbb{R})$.

2.3 Rigid Hamiltonian stationary Lagrangians in $\mathbb{C}^n$

We now discuss Lagrangians in $\mathbb{C}^n$, with g_0, ω_0, J_0 as in Example 2.2. This is Calabi–Yau, so as in §2.2, a compact Hamiltonian stationary Lagrangian L in $\mathbb{C}^n$ is special Lagrangian if $-[\alpha_H] = [\mathrm{d}\theta] = 0$ in $H^1(L;\mathbb{R})$. But there are no compact special Lagrangians L in $\mathbb{C}^n$ for $n > 0$, as a (co)homological calculation shows that $\mathrm{Vol}_{g_0}(L) = 0$. Hence any immersed, compact, Hamiltonian stationary Lagrangian L in $\mathbb{C}^n$ has $[\mathrm{d}\theta] \neq 0$ in $H^1(L;\mathbb{R})$, so that $H^1(L;\mathbb{R}) \neq 0$. This constrains the possible topologies of Hamiltonian stationary Lagrangians in $\mathbb{C}^n$.

We define some notation.

Definition 2.7. The Lie group $\mathrm{U}(n) \ltimes \mathbb{C}^n$ acts on $\mathbb{C}^n$ preserving g_0, ω_0, J_0. For x in the Lie algebra $\mathfrak{u}(n) \oplus \mathbb{C}^n$, write v_x for the vector field on $\mathbb{C}^n$ induced by the action of $\mathrm{U}(n) \ltimes \mathbb{C}^n$ on $\mathbb{C}^n$, and let $\mu_x : \mathbb{C}^n \to \mathbb{R}$ be a moment map for v_x, that is, $\mathrm{d}\mu_x = v_x \cdot \omega$. Each such moment map is a real quadratic polynomial on $\mathbb{C}^n$ whose homogeneous quadratic part is of type $(1, 1)$. Define W_n to be the vector space of such moment maps, that is, elements Q of W_n are of the form

$$Q(z_1, \ldots, z_n) = a + \textstyle\sum_{j=1}^n (b_j z_j + \bar{b}_j \bar{z}_j) + \sum_{j,k=1}^n c_{jk} z_j \bar{z}_k$$

for $a \in \mathbb{R}$ and $b_j, c_{jk} \in \mathbb{C}$ with $\bar{c}_{jk} = c_{kj}$. Then $\dim W_n = n^2 + 2n + 1$, which is $\dim(\mathfrak{u}(n) \oplus \mathbb{C}^n) + 1$, since as moment maps are unique up to the addition of constants we have $W_n/\langle 1 \rangle \cong \mathfrak{u}(n) \oplus \mathbb{C}^n$.

Lemma 2.8. *Let L be a compact Hamiltonian stationary Lagrangian in $\mathbb{C}^n$, and $\mathcal{L} : C^\infty(L) \to C^\infty(L)$ be as in* (9)*. Then*

$$\{Q|_L : Q \in W_n\} \subseteq \operatorname{Ker} \mathcal{L}. \tag{11}$$

If also L is connected then $\dim\{Q|_L : Q \in W_n\} = n^2 + 2n + 1 - \dim G$, where G is the Lie subgroup of $\mathrm{U}(n) \ltimes \mathbb{C}^n$ preserving L.

Proof. As in §2.2, we have $f \in \operatorname{Ker} \mathcal{L}$ if and only if $\mathrm{d}f$ is the 1-form associated to an infinitesimal Hamiltonian deformation of L as a Hamiltonian stationary Lagrangian. If x lies in $\mathfrak{u}(n) \oplus \mathbb{C}^n$ with vector field v_x and moment map μ_x then v_x is the Hamiltonian vector field of μ_x, so flow by v_x induces a Hamiltonian deformation of L. Since the action of $\mathrm{U}(n) \ltimes \mathbb{C}^n$ takes Hamiltonian stationary Lagrangians to Hamiltonian stationary Lagrangians, $v_x|_L$ is an infinitesimal Hamiltonian deformation of L as a Hamiltonian stationary Lagrangian. The associated 1-form on L is $\mathrm{d}\mu_x|_L$. So $\mu_x|_L$ lies in $\operatorname{Ker} \mathcal{L}$. As every $Q \in W_n$ is a moment map μ_x for some $x \in \mathfrak{u}(n) \oplus \mathbb{C}^n$, equation (11) follows.

For the second part, we have $\dim\{Q|_L : Q \in W_n\} = \dim W_n - \dim\{Q \in W_n : Q|_L \equiv 0\}$, where $\dim W_n = n^2 + 2n + 1$. Let $\mathfrak{g}$ be the Lie algebra of G, and let $x \in \mathfrak{g}$ and μ_x be a moment map for x. Then $v_x|_L$ is tangent to L, as x preserves L, so $\mathrm{d}(\mu_x|_L) = v_x \cdot (\omega|_L) = 0$ as L is Lagrangian. Since L is connected this implies that $\mu_x|_L$ is constant, and there is a unique choice of moment map μ_x for x such that $\mu_x|_L \equiv 0$. This yields an isomorphism between $\mathfrak{g}$ and $\{Q \in W_n : Q|_L \equiv 0\}$, so $\dim\{Q \in W_n : Q|_L \equiv 0\} = \dim \mathfrak{g} = \dim G$, and the lemma follows. □

The following definition is new, as far as the authors know.

Definition 2.9. Let L be a compact, Hamiltonian stationary Lagrangian in $\mathbb{C}^n$. We call L *Hamiltonian rigid*, or just *rigid*, if equality holds in (11). That is, L is Hamiltonian rigid if all infinitesimal Hamiltonian deformations of L as a Hamiltonian stationary Lagrangian in $\mathbb{C}^n$ come from the action of $\mathfrak{u}(n) \oplus \mathbb{C}^n$. By Lemma 2.8, L is Hamiltonian rigid if $\dim \operatorname{Ker} \mathcal{L} = n^2 + 2n + 1 - \dim G$.

Note that any Hamiltonian rigid L must be *connected*, since otherwise we could apply different elements of $\mathfrak{u}(n) \oplus \mathbb{C}^n$ to different connected components of L to prove equality does not hold in (11). It seems likely that in some sense, *most* compact, connected, Hamiltonian stationary Lagrangians in $\mathbb{C}^n$ are rigid, since in generic situations one expects the kernels of elliptic operators to be as small as possible, given any geometric constraints on index, etc.

We now give examples of stable, rigid, Hamiltonian stationary Lagrangians in $\mathbb{C}^n$. In the first, for completeness, we give a full proof of rigidity and stability.

Example 2.10. Define a submanifold L_n in $\mathbb{C}^n$ by

$$L_n = \left\{(x_1 e^{is}, \ldots, x_n e^{is}) : 0 \leq s < \pi,\ \textstyle\sum_{j=1}^n x_j^2 = 1,\ (x_1, \ldots, x_n) \in \mathbb{R}^n\right\},$$

which is diffeomorphic to $(\mathcal{S}^1 \times \mathcal{S}^{n-1})/\mathbb{Z}_2$ by identifying (s, x) and $(s+\pi, -x)$ in $\mathcal{S}^1 \times \mathcal{S}^{n-1}$. Lee and Wang [6] prove that L_n is a Hamiltonian stationary Lagrangian and its mean curvature vector satisfies $H = -nF$, where F is the position vector. Moreover, the induced metric on L_n computed there is a product metric. More precisely, assume that $\{v_j\}_{j=1}^{n-1}$ is a local orthonormal basis for $\mathcal{S}^{n-1}$ and $v_0 = (x_1, \ldots, x_n)$, then $e_0 = \frac{\partial}{\partial s} = i\, e^{is} v_0 = JF$ and $e_j = e^{is} v_j$, $j = 1, \ldots, n-1$, will form a local orthonormal basis for L_n. We will prove that L_n is Hamiltonian stable and rigid.

The linearized operator (9) for L_n in $\mathbb{C}^n$ and $f \in C^\infty(L_n)$ has the form

$$\mathcal{L}f = \Delta^2 f - 2\mathrm{d}^* \alpha_{B(JH,\nabla f)} - JH(JH(f)).$$

Note that $JH = -ne_0 = -n\frac{\partial}{\partial s}$ and

$$\langle B(e_i, e_j), H\rangle = -n\langle B(e_i, e_j), F\rangle = n\delta_{ij}.$$

Hence

$$\langle B(JH, \nabla f), Je_j\rangle = -\langle B(e_j, \nabla f), H\rangle = -n\langle e_j, \nabla f\rangle = -ne_j(f),$$

that is, $B(JH, \nabla f) = -nJ\nabla f$ and $\alpha_{B(JH,\nabla f)} = n\mathrm{d}f$. Therefore, we have

$$\mathcal{L}f = \Delta^2 f - 2n\Delta f - n^2 \frac{\partial^2 f}{\partial s^2}. \tag{12}$$

We can lift a function on L_n to $\mathcal{S}^1 \times \mathcal{S}^{n-1}$ and consider f as a $\mathbb{Z}_2$-invariant function on $\mathcal{S}^1 \times \mathcal{S}^{n-1}$ instead. Since the induced metric is a product metric, the products of eigenfunctions on $\mathcal{S}^1$ and eigenfunctions on $\mathcal{S}^{n-1}$ respectively form a complete basis for functions on $\mathcal{S}^1 \times \mathcal{S}^{n-1}$. That is, $f = \sum_{k,l} a_{kl} \cos ks\, \varphi_l + b_{kl} \sin ks\, \varphi_l$, where a_{kl} and b_{kl} are constants, k is a nonnegative integer, and φ_l is an eigenfunction of the Laplacian on $\mathcal{S}^{n-1}$ with eigenvalue λ_l. Since the eigenfunctions of the Laplacian on $\mathcal{S}^{n-1}$ are homogenous polynomials in $\mathbb{R}^n$, and f is $\mathbb{Z}_2$-invariant, the sum of k and the degree $\deg \varphi_l$ of φ_l must be even.

From (12), it follows that $\cos ks\, \varphi_l$ and $\sin ks\, \varphi_l$ are eigenfunctions for $\mathcal{L}$ and form a complete basis. To study $\operatorname{Ker} \mathcal{L}$, we only need to check these functions. Rewrite (12) as $\mathcal{L}f = (\Delta - n)^2 f + n^2(-\frac{\partial^2}{\partial s^2} - 1)f$. Suppose $\cos ks\, \varphi_l$ or $\sin ks\, \varphi_l$ is in $\operatorname{Ker} \mathcal{L}$. It follows that

$$(k^2 + \lambda_l - n)^2 + n^2(k^2 - 1) = 0,$$

which implies $k \leqslant 1$. When $k = 1$, we must have $\lambda_l = n - 1$ and thus $\deg \varphi_l$ is 1. These solutions and their combinations come from the restriction functions on L_n of elements in W_n with the form $\sum_{j=1}^n b_j z_j + \bar{b}_j \bar{z}_j$ in Definition 2.7. When $k = 0$, we must have $\lambda_l(\lambda_l - 2n) = 0$ and thus $\lambda_l = 0$ or $\lambda_l = 2n$. Hence $\deg \varphi_l = 0$ or 2. These solutions and their combinations come from the restriction functions of elements in W_n with the form $a + \sum_{j,k=1}^n c_{jk} z_j \bar{z}_k$ in Definition 2.7. This completes the proof that L_n is Hamiltonian rigid.

Now we show that L_n is Hamiltonian stable. From (10), this is equivalent to the eigenvalues for $\mathcal{L}$ all being nonnegative. The eigenfunctions for $\mathcal{L}$ are $\cos ks\,\varphi_l$ and $\sin ks\,\varphi_l$ with eigenvalue $(k^2+\lambda_l-n)^2+n^2(k^2-1)$, which is nonnegative for $k\geqslant 1$. It is also nonnegative when $k=0$ and $\deg\varphi_l=0$ or $\deg\varphi_l>1$. The case $k=0$ and $\deg\varphi_l=1$ does not occur as it is not $\mathbb{Z}_2$-invariant. Therefore, L_n is Hamiltonian stable.

Example 2.11. Let $a_1,\ldots,a_n>0$. Define a torus $T^n_{a_1,\ldots,a_n}$ in $\mathbb{C}^n$ by

$$T^n_{a_1,\ldots,a_n}=\bigl\{(z_1,\ldots,z_n)\in\mathbb{C}^n:\lvert z_j\rvert=a_j,\ j=1,\ldots,n\bigr\}. \tag{13}$$

Then Oh [10, Th. IV] proves that $T^n_{a_1,\ldots,a_n}$ is a Hamiltonian stationary Lagrangian, and is stable and rigid. He also remarks that two tori $T^n_{a_1,\ldots,a_n}$ and $T^n_{a'_1,\ldots,a'_n}$ are not Hamiltonian isotopic to one another if $(a_1,\ldots,a_n)\neq(a'_1,\ldots,a'_n)$ (this needs caution: if $a'_1,\ldots,a'_n$ are a permutation of $a_1,\ldots,a_n$, then $T^n_{a'_1,\ldots,a'_n}$ is Hamiltonian isotopic to $T^n_{a_1,\ldots,a_n}$, but after a diffeomorphism of T^n not isotopic to the identity), and conjectures that the $T^n_{a_1,\ldots,a_n}$ are globally volume-minimizing under Hamiltonian deformations.

Example 2.12. Amarzaya and Ohnita study Lagrangians with parallel second fundamental form in [1]. These examples must be Hamiltonian stationary since their mean curvature vectors are parallel. They prove in the paper that the following irreducible symmetric R-spaces are Hamiltonian stable and rigid:

(i) $Q_{2,p+1}(\mathbb{R})\cong(\mathcal{S}^1\times\mathcal{S}^{p+2})/\mathbb{Z}_2\subset\mathbb{C}^{p+3}$ for $p\geqslant 1$;

(ii) $\mathrm{U}(p)\subset\mathbb{C}^{p^2}$ for $p\geqslant 2$;

(iii) $\mathrm{U}(p)/\mathrm{O}(p)\subset\mathbb{C}^{\frac{p(p+1)}{2}}$ for $p\geqslant 3$;

(iv) $\mathrm{U}(2p)/\mathrm{Sp}(p)\subset\mathbb{C}^{p(2p-1)}$ for $p\geqslant 3$; and

(v) $T\cdot(E_6/F_4)\subset\mathbb{C}^{27}$.

We refer to [1] for the details of these examples. Example 2.10 is the same as $Q_{2,p+1}(\mathbb{R})$ in (i), but their proof of stability and rigidity is different to ours.

3 Darboux coordinates with estimates

3.1 Families of Darboux coordinate systems for all $p\in M$

We will need the following notation.

Definition 3.1. Let (M,ω) be a symplectic $2n$-manifold, and g a Riemannian metric on M compatible with ω. Then at each point $p\in M$ the structures $\omega|_p$, $g|_p$ on T_pM are isomorphic to ω_0,g_0 on $\mathbb{C}^n$, where ω_0,g_0 are as in (3). The linear automorphism group of $(\mathbb{C}^n,\omega_0,g_0)$ is the unitary group $\mathrm{U}(n)$.

Write U for the $\mathrm{U}(n)$ *frame bundle* of M, with projection $\pi:U\to M$. That is, points of U are pairs (p,υ) with $p\in M$ and $\upsilon:\mathbb{C}^n\to T_pM$ an isomorphism of real vector spaces with $\upsilon^*(\omega|_p) = \omega_0$, and $\upsilon^*(g|_p) = g_0$, and $\pi : (p, \upsilon) \mapsto p$. Then $\mathrm{U}(n)$ acts freely on the right on U by $\gamma : (p, \upsilon) \mapsto (p, \upsilon \circ \gamma)$ for $\gamma \in \mathrm{U}(n)$ and $(p, \upsilon) \in U$. The orbits of $\mathrm{U}(n)$ in U are fibres of π, and $\pi : U \to M$ is a *principal* $\mathrm{U}(n)$*-bundle*. Thus U is a real manifold of dimension $n^2 + 2n$, which is compact if M is compact.

We now show that for compact M we can choose Darboux coordinate systems as in Theorem 2.3 for all $(p, \upsilon) \in U$, smoothly and $\mathrm{U}(n)$-equivariantly in (p, υ).

Proposition 3.2. *Let (M, ω) be a compact symplectic $2n$-manifold, g a Riemannian metric on M compatible with ω, and U the $\mathrm{U}(n)$ frame bundle of M. Then for small $\epsilon > 0$ we can choose a family of embeddings $\Upsilon_{p,\upsilon} : B_\epsilon \to M$ depending smoothly on $(p, \upsilon) \in U$, where B_ϵ is the ball of radius ϵ about 0 in $\mathbb{C}^n$, such that for all $(p, \upsilon) \in U$ we have:*

(i) $\Upsilon_{p,\upsilon}(0) = p$ *and* $\mathrm{d}\Upsilon_{p,\upsilon}|_0 = \upsilon : \mathbb{C}^n \to T_pM$;

(ii) $\Upsilon_{p,\upsilon\circ\gamma} \equiv \Upsilon_{p,\upsilon} \circ \gamma$ *for all* $\gamma \in \mathrm{U}(n)$;

(iii) $\Upsilon^*_{p,\upsilon}(\omega) = \omega_0 = \frac{i}{2}\sum_{j=1}^n \mathrm{d}z_j \wedge \mathrm{d}\bar{z}_j$; *and*

(iv) $\Upsilon^*_{p,\upsilon}(g) = g_0 + O(|\boldsymbol{z}|) = \sum_{j=1}^n |\mathrm{d}z_j|^2 + O(|\boldsymbol{z}|)$.

Proof. Let $\epsilon' > 0$, and for each $(p, \upsilon) \in U$ define $\Upsilon'_{p,\upsilon} : B_{\epsilon'} \to M$ by $\Upsilon'_{p,\upsilon} = \exp_p \circ \upsilon|_{B_{\epsilon'}}$, where $\upsilon|_{B_{\epsilon'}} : B_{\epsilon'} \to T_pM$ and $\exp_p : T_pM \to M$ is geodesic normal coordinates at p using g. Then $\Upsilon'_{p,\upsilon}$ is a diffeomorphism near $0 \in B_{\epsilon'}$ and $p \in M$, so as M, U are compact, if we take $\epsilon' > 0$ small enough $\Upsilon'_{p,\upsilon}$ is an embedding for all $(p, \upsilon) \in U$. It is immediate that $\Upsilon'_{p,\upsilon}$ is smooth in p, υ, and $\Upsilon'_{p,\upsilon}(0) = p$, $\mathrm{d}\Upsilon'_{p,\upsilon}|_0 = \upsilon$, and $\Upsilon'_{p,\upsilon\circ\gamma} \equiv \Upsilon'_{p,\upsilon} \circ \gamma$ for all $\gamma \in \mathrm{U}(n)$. Also $(\Upsilon'_{p,\upsilon})^*(\omega)|_0 = \omega_0$ and $(\Upsilon'_{p,\upsilon})^*(g)|_0 = g_0$, so by Taylor's Theorem we have $(\Upsilon'_{p,\upsilon})^*(\omega) = \omega_0 + O(|\boldsymbol{z}|)$ and $(\Upsilon'_{p,\upsilon})^*(g) = g_0 + O(|\boldsymbol{z}|)$. Thus the ϵ' and $\Upsilon'_{p,\upsilon}$ satisfy all the proposition except (iii), which is replaced by $(\Upsilon'_{p,\upsilon})^*(\omega) = \omega_0 + O(|\boldsymbol{z}|)$.

We shall use Moser's method for proving Darboux' Theorem in [8] to modify the $\Upsilon'_{p,\upsilon}$ to $\Upsilon_{p,\upsilon}$ with $\Upsilon^*_{p,\upsilon}(\omega) = \omega_0$, preserving the other properties (i),(ii),(iv). Define closed 2-forms $\omega^s_{p,\upsilon}$ on $B_{\epsilon'}$ for $(p, \upsilon) \in U$ and $s \in [0, 1]$ by $\omega^s_{p,\upsilon} = (1 - s)\omega_0 + s(\Upsilon'_{p,\upsilon})^*(\omega)$. As $\omega_0|_0 = (\Upsilon'_{p,\upsilon})^*(\omega)|_0$, we have $\omega^s_{p,\upsilon}|_0 = \omega_0$, so $\omega^s_{p,\upsilon}$ is symplectic near 0 in $B_{\epsilon'}$. Since $[0, 1] \times U$ is compact, by making $\epsilon' > 0$ smaller we can suppose $\omega^s_{p,\upsilon}$ is symplectic on $B_{\epsilon'}$ for all $s \in [0, 1]$ and $(p, \upsilon) \in U$.

As $\omega_0, (\Upsilon'_{p,\upsilon})^*(\omega)$ are closed, we can choose a family of 1-forms $\zeta_{p,\upsilon}$ on $B_{\epsilon'}$, smooth in p, υ, such that $\mathrm{d}\zeta_{p,\upsilon} = \omega_0 - (\Upsilon'_{p,\upsilon})^*(\omega)$. Since $\omega_0|_0 = (\Upsilon'_{p,\upsilon})^*(\omega)|_0$, we can choose the $\zeta_{p,\upsilon}$ to satisfy $|\zeta_{p,\upsilon}| = O(|\boldsymbol{z}|^2)$. We also choose the family $\zeta_{p,\upsilon}$ to be $\mathrm{U}(n)$-equivariant, in the sense that $\zeta_{p,\upsilon\circ\gamma} = \gamma|^*_{B_{\epsilon'}}(\zeta_{p,\upsilon})$ for all $\gamma \in \mathrm{U}(n)$, noting that $\gamma : \mathbb{C}^n \to \mathbb{C}^n$ restricts to $\gamma|_{B_{\epsilon'}} : B_{\epsilon'} \to B_{\epsilon'}$. To do this, we first choose $\zeta'_{p,\upsilon}$ not necessarily $\mathrm{U}(n)$-equivariant, and then take $\zeta_{p,\upsilon}$ to be the average of $(\gamma^{-1})^*(\zeta'_{p,\upsilon\circ\gamma})$ over $\gamma \in \mathrm{U}(n)$.

Now let $v^s_{p,\upsilon}$ be the unique vector field on $B_{\epsilon'}$ with $v^s_{p,\upsilon} \cdot \omega^s_{p,\upsilon} = \zeta_{p,\upsilon}$, which is well-defined as $\omega^s_{p,\upsilon}$ is symplectic, and varies smoothly in s, p, υ. Then $\mathrm{d}(v^s_{p,\upsilon} \cdot \omega^s_{p,\upsilon}) = \omega_0 - (\Upsilon'_{p,\upsilon})^*(\omega)$. As $|\zeta_{p,\upsilon}| = O(|\boldsymbol{z}|^2)$ we have $|v^s_{p,\upsilon}| = O(|\boldsymbol{z}|^2)$. For

$0 < \epsilon \leqslant \epsilon'$ we construct a family of embeddings $\varphi^s_{p,\upsilon} : B_\epsilon \to B_{\epsilon'}$ with $\varphi^0_{p,\upsilon} = \mathrm{id} : B_\epsilon \to B_\epsilon \subset B_{\epsilon'}$ by solving the o.d.e. $\frac{\mathrm{d}}{\mathrm{d}s}\varphi^s_{p,\upsilon} = v^s_{p,\upsilon} \circ \varphi^s_{p,\upsilon}$. By compactness of $[0,1] \times U$, this is possible provided $\epsilon > 0$ is small enough. Then $(\varphi^s_{p,\upsilon})^*(\omega^s_{p,\upsilon}) = \omega_0$ for all s, so that $(\varphi^1_{p,\upsilon})^*\big((\Upsilon'_{p,\upsilon})^*(\omega)\big) = \omega_0$. Also, as $|v^s_{p,\upsilon}| = O(|\boldsymbol{z}|^2)$ we see that $\varphi^s_{p,\upsilon}(0) = 0$ and $\mathrm{d}\varphi^s_{p,\upsilon}|_0 = \mathrm{id} : \mathbb{C}^n \to \mathbb{C}^n$ for all $s \in [0,1]$. And the $\mathrm{U}(n)$-equivariance of the $\zeta_{p,\upsilon}$ implies $\mathrm{U}(n)$-equivariance of the $v^s_{p,\upsilon}$, which in turn implies that $\varphi^s_{p,\upsilon\circ\gamma} = \gamma^{-1} \circ \varphi^s_{p,\upsilon} \circ \gamma$ for all $s \in [0,1]$, $(p,\upsilon) \in U$ and $\gamma \in \mathrm{U}(n)$.

Define $\Upsilon_{p,\upsilon} = \Upsilon'_{p,\upsilon} \circ \varphi^1_{p,\upsilon}$. Then $\Upsilon_{p,\upsilon}$ depends smoothly on p, υ. Part (i) holds as $\varphi^1_{p,\upsilon}(0) = 0$, $\mathrm{d}\varphi^1_{p,\upsilon}|_0 = \mathrm{id}$, $\Upsilon'_{p,\upsilon}(0) = p$ and $\mathrm{d}\Upsilon'_{p,\upsilon}|_0 = \upsilon$. Part (ii) holds by $\mathrm{U}(n)$-equivariance of the $\Upsilon'_{p,\upsilon}$, and $\varphi^1_{p,\upsilon\circ\gamma} = \gamma^{-1} \circ \varphi^1_{p,\upsilon} \circ \gamma$ for all $(p,\upsilon) \in U$ and $\gamma \in \mathrm{U}(n)$. Part (iii) follows from $(\varphi^1_{p,\upsilon})^*\big((\Upsilon'_{p,\upsilon})^*(\omega)\big) = \omega_0$, and (iv) from $\varphi^1_{p,\upsilon}(0) = 0$, $\mathrm{d}\varphi^1_{p,\upsilon}|_0 = \mathrm{id}$, and $(\Upsilon'_{p,\upsilon})^*(g) = g_0 + O(|\boldsymbol{z}|)$. □

Remark 3.3. When (M, J, g) is a Kähler manifold with Kähler form ω, by applying the same argument to holomorphic normal coordinates, we can obtain the better approximation $\Upsilon^*_{p,\upsilon}(\omega) = \omega_0 = \frac{i}{2}\sum_{j=1}^n \mathrm{d}z_j \wedge \mathrm{d}\bar{z}_j$ and

$$\Upsilon^*_{p,\upsilon}(g) = \textstyle\sum_{j=1}^n |\mathrm{d}z_j|^2 + \frac{1}{2}\sum_{i,j,k,l} \mathrm{Re}\big(R_{i\bar{j}k\bar{l}}(p) z^i z^k \mathrm{d}\bar{z}^j \mathrm{d}\bar{z}^l\big) + O\big(|\boldsymbol{z}|^3\big).$$

3.2 Dilations, and uniform estimates of $t^{-2}(\Upsilon_{p,\upsilon} \circ t)^*(g)$

We continue to use the notation of §3.1 and Proposition 3.2. Let $R > 0$ be given. For $0 < t \leqslant R^{-1}\epsilon$, consider the dilation map $t : B_R \to B_\epsilon$ mapping $t : (z_1, \ldots, z_n) \mapsto (tz_1, \ldots, tz_n)$. Then $\Upsilon_{p,\upsilon} \circ t$ is an embedding $B_R \to M$, so we can consider the pullbacks $(\Upsilon_{p,\upsilon} \circ t)^*(\omega)$, $(\Upsilon_{p,\upsilon} \circ t)^*(g)$. Since $t^*(\omega_0) = t^2\omega_0$ and $t^*(g_0) = t^2 g_0$, Proposition 3.2(iii),(iv) give

$$(\Upsilon_{p,\upsilon} \circ t)^*(\omega) = t^2\omega_0 \quad \text{and} \quad (\Upsilon_{p,\upsilon} \circ t)^*(g) = t^2 g_0 + O(t^3|\boldsymbol{z}|), \tag{14}$$

where the power $t^3 = t^2 \cdot t$ in $O(t^3|\boldsymbol{z}|)$ combines the fact that $t^*(\mathrm{d}z_i \mathrm{d}\bar{z}_j) = t^2 \mathrm{d}z_i \mathrm{d}\bar{z}_j$, that is, tensors of type (0,2) scale by t^2 under t^*, and regarding $|\boldsymbol{z}|$ as a function on $\mathbb{C}^n$ we have $t^*(|\boldsymbol{z}|) = t|\boldsymbol{z}|$. Multiplying (14) by t^{-2} gives

$$t^{-2}(\Upsilon_{p,\upsilon} \circ t)^*(\omega) = \omega_0 \quad \text{and} \quad t^{-2}(\Upsilon_{p,\upsilon} \circ t)^*(g) = g_0 + O(t|\boldsymbol{z}|) \quad \text{on } B_R. \tag{15}$$

Define a Riemannian metric $g^t_{p,\upsilon}$ on B_R by $g^t_{p,\upsilon} = t^{-2}(\Upsilon_{p,\upsilon} \circ t)^*(g)$. This depends smoothly on $t \in (0, R^{-1}\epsilon]$ and $(p,\upsilon) \in U$, and satisfies $g^t_{p,\upsilon} = g_0 + O(t|\boldsymbol{z}|)$ by (15). Since g is compatible with ω, $t^{-2}(\Upsilon_{p,\upsilon} \circ t)^*(g)$ is compatible with $t^{-2}(\Upsilon_{p,\upsilon} \circ t)^*(\omega)$, and thus $g^t_{p,\upsilon}$ is compatible with the fixed symplectic form ω_0 on B_R for all t, p, υ. We prove *uniform estimates* on these metrics $g^t_{p,\upsilon}$.

Proposition 3.4. *There exist positive constants $C_0, C_1, C_2, \ldots$ such that for all $t \in (0, \frac{1}{2}R^{-1}\epsilon]$ and $(p,\upsilon) \in U$, the metric $g^t_{p,\upsilon} = t^{-2}(\Upsilon_{p,\upsilon} \circ t)^*(g)$ on B_R satisfies the estimates*

$$\|g^t_{p,\upsilon} - g_0\|_{C^0} \leqslant C_0 t \quad \textit{and} \quad \|\partial^k g^t_{p,\upsilon}\|_{C^0} \leqslant C_k t^k \quad \textit{for } k = 1, 2, \ldots, \tag{16}$$

where norms are taken w.r.t. g_0, and ∂ is the Levi-Civita connection of g_0.

Proof. We will prove (16) by first estimating the metrics $\Upsilon_{p,\upsilon}^*(g)$ for all $(p,\upsilon) \in U$. As B_ϵ is noncompact, we cannot do this on the whole of B_ϵ, since $\Upsilon_{p,\upsilon}^*(g)$ might have bad behaviour approaching the boundary of B_ϵ, forcing norms to be unbounded. Instead, write $\overline{B}_{\epsilon/2}$ for the closure of $B_{\epsilon/2}$ in B_ϵ. Then $\overline{B}_{\epsilon/2}$ is compact. For each fixed $(p,\upsilon) \in U$, Proposition 3.2(iv) implies that there exists $C > 0$ with $\bigl|\Upsilon_{p,\upsilon}^*(g) - g_0\bigr| \leqslant C|z|$ on $\overline{B}_{\epsilon/2}$, where $|\,.\,|$ is taken using g_0. Since U is compact, we can choose $C > 0$ such that this holds for all $(p,\upsilon) \in U$. Now let $t \in (0, \frac{1}{2}R^{-1}\epsilon]$. Then $t(B_R) \subseteq B_{\epsilon/2} \subset \overline{B}_{\epsilon/2}$. By the proof of the second equation of (15), $\bigl|\Upsilon_{p,\upsilon}^*(g) - g_0\bigr| \leqslant C|z|$ on $\overline{B}_{\epsilon/2}$ implies that $|g_{p,\upsilon}^t - g_0| \leqslant Ct|z| \leqslant CRt$ on B_R, as $|z| \leqslant R$ on B_R. Setting $C_0 = CR$ proves the first equation of (16).

Fix $k = 1, 2, \ldots$, and consider $\bigl|\partial^k \Upsilon_{p,\upsilon}^*(g)(\boldsymbol{z})\bigr|$. This is a continuous function of $(p,\upsilon) \in U$ and $\boldsymbol{z} \in \overline{B}_{\epsilon/2}$, so as $U, \overline{B}_{\epsilon/2}$ are compact there exists $C_k > 0$ with $\bigl|\partial^k \Upsilon_{p,\upsilon}^*(g)(\boldsymbol{z})\bigr| \leqslant C_k$ for all $(p,\upsilon) \in U$ and $\boldsymbol{z} \in \overline{B}_{\epsilon/2}$. Now let $t \in (0, \frac{1}{2}R^{-1}\epsilon]$ and $\boldsymbol{z} \in B_R$. Then an easy scaling argument shows that $\bigl|\partial^k\bigl(t^{-2}(\Upsilon_{p,\upsilon} \circ t)^*(g)\bigr)(\boldsymbol{z})\bigr| = t^k\bigl|\partial^k \Upsilon_{p,\upsilon}^*(g)(t\boldsymbol{z})\bigr|$. As $t\boldsymbol{z} \in B_{\epsilon/2} \subset \overline{B}_{\epsilon/2}$ we have $\bigl|\partial^k \Upsilon_{p,\upsilon}^*(g)(t\boldsymbol{z})\bigr| \leqslant C_k$, and the second equation of (16) follows. □

The proposition implies that by taking t sufficiently small, we can make $g_{p,\upsilon}^t$ arbitrarily close to g_0 on B_R uniformly for all $(p,\upsilon) \in U$, in the C^k norm for any $k \geqslant 0$, and hence also in the Hölder $C^{k,\gamma}$ norm for any $k \geqslant 0$ and $\gamma \in (0,1)$.

4 Setting up the problem

In §4–§6 we will prove Theorem A. This section will set up a lot of notation, and formulate a family of fourth-order nonlinear elliptic partial differential operators $P_{p,\upsilon}^t : C^\infty(L) \to C^\infty(L)$ depending on $(p,\upsilon) \in U$ and small $t > 0$, such that C^1-small $f \in C^\infty(L)$ correspond to Lagrangians $L_{p,\upsilon}^{t,f}$ in M, and $L_{p,\upsilon}^{t,f}$ is Hamiltonian stationary when $P_{p,\upsilon}^t(f) = 0$.

Section 5 will show that for sufficiently small, fixed $t > 0$ and all $(p,\upsilon) \in U$ we can find a family of functions $f_{p,\upsilon}^t \in C^\infty(L)$ with $P_{p,\upsilon}^t(f_{p,\upsilon}^t) \in \operatorname{Ker}\mathcal{L}$ and $f_{p,\upsilon}^t \perp \operatorname{Ker}\mathcal{L}$, which are unique for $\|f_{p,\upsilon}^t\|_{C^{4,\gamma}}$ (the Hölder $C^{4,\gamma}$ norm) small, and depend smoothly on $(p,\upsilon) \in U$. Finally, §6 shows that the smooth map $H^t : U \to \operatorname{Ker}\mathcal{L}$ defined by $H^t : (p,\upsilon) \mapsto P_{p,\upsilon}^t(f_{p,\upsilon}^t)$ can be interpreted in terms of the exact 1-form $\mathrm{d}K^t$ on U, where $K^t : U \to \mathbb{R}$ is defined by $K^t(p,\upsilon) = t^{-n}\operatorname{Vol}_g\bigl(L_{p,\upsilon}^t\bigr)$, with $L_{p,\upsilon}^t = L_{p,\upsilon}^{t,f_{p,\upsilon}^t}$. Thus, if (p,υ) is a critical point of K^t then $L' = L_{p,\upsilon}^t$ is Hamiltonian stationary, as we want.

Let L be a nonempty, compact, rigid, Hamiltonian stationary Lagrangian in $\mathbb{C}^n$. Then L is connected as in §2.3. Let G be the subgroup of $\mathrm{U}(n) \ltimes \mathbb{C}^n$ preserving L. Then G is compact, as L is compact, so it is a Lie subgroup of $\mathrm{U}(n) \ltimes \mathbb{C}^n$, which acts on L. As G is compact it must fix a point in $\mathbb{C}^n$, the centre of gravity of L. Translating L in $\mathbb{C}^n$ if necessary, we suppose G fixes 0 in $\mathbb{C}^n$, so that $G \subset \mathrm{U}(n)$.

By the Lagrangian Neighbourhood Theorem, Theorem 2.4, we can choose an open tubular neighbourhood T of the zero section L in T^*L, and an embedding $\Phi : T \to \mathbb{C}^n$ with $\Phi|_L = \mathrm{id} : L \to L$ and $\Phi^*(\omega_0) = \hat{\omega}$, where $\hat{\omega}$ is the canonical symplectic structure on T^*L. Making T smaller if necessary, we suppose T is of the form

$$T = \{(p,\alpha) : p \in L,\, \alpha \in T_p^*L,\, |\alpha| < \delta\} \tag{17}$$

for some small $\delta > 0$, where $|\alpha|$ is computed using the metric $g_0|_L$. The action of G on L induces an action of G on T^*L, and as $g_0|_L$ is G-invariant, T is G-invariant. We suppose Φ is chosen to be equivariant under the actions of G on T and $\mathbb{C}^n$; this can be done following the proof of the dilation-equivariant Lagrangian Neighbourhood Theorem in [3, Th. 4.3].

Fix $R > 0$ such that $L \subset \Phi(T) \subset B_R \subset \mathbb{C}^n$. Now let $f \in C^\infty(L)$ with $\|\mathrm{d}f\|_{C^0} < \delta$. Define the graph $\Gamma_{\mathrm{d}f}$ of $\mathrm{d}f$ in T^*L to be $\Gamma_{\mathrm{d}f} = \{(q, \mathrm{d}f|_q) : q \in L\}$. Then $\Gamma_{\mathrm{d}f}$ is an embedded Lagrangian submanifold of T^*L diffeomorphic to L. As $\|\mathrm{d}f\|_{C^0} < \delta$ we have $|\mathrm{d}f|_q| < \delta$ for all $q \in L$, and $\Gamma_{\mathrm{d}f} \subset T$. Hence $\Phi(\Gamma_{\mathrm{d}f})$ is a submanifold of $\Phi(T) \subset B_R \subset \mathbb{C}^n$ diffeomorphic to L. Since $\Gamma_{\mathrm{d}f}$ is Lagrangian in $(T, \hat\omega)$ and $\Phi^*(\omega_0) = \hat\omega$, we see that $\Phi(\Gamma_{\mathrm{d}f})$ is Lagrangian in (B_R, ω_0).

Suppose (M, ω) is a compact symplectic $2n$-manifold, and g a Riemannian metric on M compatible with ω. Define U as in Definition 3.1 and $\epsilon > 0$, $\Upsilon_{p,\upsilon} : B_\epsilon \to M$ for $(p,\upsilon) \in U$ as in Proposition 3.2. For $0 < t \leqslant R^{-1}\epsilon$ and $(p,\upsilon) \in U$, define metrics $g^t_{p,\upsilon}$ on B_R as in §3.2.

Let $0 < t \leqslant \frac{1}{2}R^{-1}\epsilon$. For each $f \in C^\infty(L)$ with $\|\mathrm{d}f\|_{C^0} < \delta$, define $L^{t,f}_{p,\upsilon} = \Upsilon_{p,\upsilon} \circ t \circ \Phi(\Gamma_{\mathrm{d}f})$. Then $L^{t,f}_{p,\upsilon}$ is an embedded submanifold of M diffeomorphic to L, and as $\Phi(\Gamma_{\mathrm{d}f})$ is Lagrangian in (B_R, ω_0) and $(\Upsilon_{p,\upsilon} \circ t)^*(\omega) = t^2\omega_0$ by (14), we see that $L^{t,f}_{p,\upsilon}$ is Lagrangian in (M, ω). Define a functional

$$F^t_{p,\upsilon} : \{f \in C^\infty(L) : \|\mathrm{d}f\|_{C^0} < \delta\} \to \mathbb{R} \quad \text{by} \quad F^t_{p,\upsilon}(f) = t^{-n}\,\mathrm{Vol}_g\bigl(L^{t,f}_{p,\upsilon}\bigr). \tag{18}$$

We have

$$F^t_{p,\upsilon}(f) = t^{-n}\,\mathrm{Vol}_g\bigl(L^{t,f}_{p,\upsilon}\bigr) = t^{-n}\,\mathrm{Vol}_{(\Upsilon_{p,\upsilon}\circ t)^*(g)}\bigl(\Phi(\Gamma_{\mathrm{d}f})\bigr) = \mathrm{Vol}_{g^t_{p,\upsilon}}\bigl(\Phi(\Gamma_{\mathrm{d}f})\bigr), \tag{19}$$

since $g^t_{p,\upsilon} = t^{-2}(\Upsilon_{p,\upsilon} \circ t)^*(g)$ and $\dim L = n$. Proposition 3.4 shows that $g^t_{p,\upsilon}$ approximates g_0 when t is small. Thus (19) implies that

$$F^t_{p,\upsilon}(f) \approx \mathrm{Vol}_{g_0}\bigl(\Phi(\Gamma_{\mathrm{d}f})\bigr) \tag{20}$$

for small $t > 0$, where the right hand side is independent of t, p, υ.

Observe that for fixed t, p, υ and varying f, the Lagrangians $L^{t,f}_{p,\upsilon}$ in (M, ω) are all Hamiltonian equivalent, and furthermore, for any fixed f the $L^{t,f'}_{p,\upsilon}$ for f' close to f in $C^\infty(L)$ realize all Hamiltonian equivalent Lagrangians close to $L^{t,f}_{p,\upsilon}$. This parametrization of Lagrangians $L^{t,f}_{p,\upsilon}$ by functions f is degenerate, since $L^{t,f}_{p,\upsilon} = L^{t,f+c}_{p,\upsilon}$ for $c \in \mathbb{R}$. We can make it a local isomorphism by restricting to $f \in C^\infty(L)$ with $\int_L f\,\mathrm{d}V_{g_0|_L} = 0$. Anyway, $L^{t,f}_{p,\upsilon}$ is Hamiltonian stationary in M if and only if f is a critical point of the functional $F^t_{p,\upsilon}$, and $L^{t,f}_{p,\upsilon}$ is Hamiltonian stable if in addition the second variation of $F^t_{p,\upsilon}$ at f is nonnegative.

From (19) we see that

$$F^t_{p,\upsilon}(f) = \int_{\boldsymbol{\Phi}(\Gamma_{\mathrm{d}f})} \mathrm{d}V_{g^t_{p,\upsilon}|_{\Phi(\Gamma_{\mathrm{d}f})}} = \int_L (\Phi_f)^*\bigl(\mathrm{d}V_{g^t_{p,\upsilon}|_{\Phi(\Gamma_{\mathrm{d}f})}}\bigr). \tag{21}$$

Here in the first step $g^t_{p,\upsilon}|_{\Phi(\Gamma_{\mathrm{d}f})}$ is the restriction of the metric $g^t_{p,\upsilon}$ on B_R to the submanifold $\Phi(\Gamma_{\mathrm{d}f})$, and $\mathrm{d}V_{g^t_{p,\upsilon}|_{\Phi(\Gamma_{\mathrm{d}f})}}$ is the induced volume form on $\Phi(\Gamma_{\mathrm{d}f})$. Now $\Phi(\Gamma_{\mathrm{d}f})$ is the diffeomorphic image of L under the map $\Phi_f : q \mapsto \Phi(q, \mathrm{d}f|_q)$. In the second step of (21), we pull back the volume form by Φ_f and do the integration on L.

At each $q \in L$, the integrand $(\Phi_f)^*\big(\mathrm{d}V_{g^t_{p,\upsilon}|_{\Phi(\Gamma_{\mathrm{d}f})}}\big)$ in (21) is a positive multiple of the volume form $\mathrm{d}V_{g_0|_L}$ of the metric $g_0|_L$ on L at q, and this multiple depends only on $t, p, \upsilon, q, \mathrm{d}f|_q$ and $\nabla\mathrm{d}f|_q$, where ∇ is the Levi-Civita connection of $g_0|_L$. This is because the tangent space to $\Phi(\Gamma_{\mathrm{d}f})$ at $\Phi(q, \mathrm{d}f|_q)$ depends only on $q, \mathrm{d}f|_q, \nabla\mathrm{d}f|_q$, and the volume form depends on this tangent space and on $g^t_{p,\upsilon}$, which depends on t, p, υ. Thus we may write

$$F^t_{p,\upsilon}(f) = \int_L G^t_{p,\upsilon}\big(q, \mathrm{d}f|_q, \nabla\mathrm{d}f|_q\big)\, \mathrm{d}V_{g_0|_L}. \tag{22}$$

Here $G^t_{p,\upsilon}$ maps

$$G^t_{p,\upsilon} : \big\{(q,\alpha,\beta) : q \in L,\ \alpha \in T^*_qL,\ |\alpha| < \delta,\ \beta \in S^2T^*_qL\big\} \longrightarrow \mathbb{R}, \tag{23}$$

where the condition $|\alpha| < \delta$ is because we restrict to f with $\|\mathrm{d}f\|_{C^0} < \delta$ so that $\Gamma_{\mathrm{d}f} \subset T$, and we take $\beta \in S^2T^*_qL \subset \otimes^2T^*_qL$ because the second derivative $\nabla\mathrm{d}f$ is a symmetric tensor. Clearly, $G^t_{p,\upsilon}(q,\alpha,\beta)$ is a smooth, nonlinear function of its arguments q, α, β, and also depends smoothly on $t \in (0, \frac{1}{2}R^{-1}\epsilon]$ and $(p,\upsilon) \in U$.

From above, $L^{t,f}_{p,\upsilon}$ is Hamiltonian stationary in M if and only if f is a critical point of the functional $F^t_{p,\upsilon}$. Applying the Euler–Lagrange method to (22), we see that $L^{t,f}_{p,\upsilon}$ is Hamiltonian stationary in M if and only if $P^t_{p,\upsilon}(f) = 0$, where

$$P^t_{p,\upsilon} : \big\{f \in C^\infty(L) : \|\mathrm{d}f\|_{C^0} < \delta\big\} \longrightarrow C^\infty(L) \tag{24}$$

is defined by

$$\begin{aligned} P^t_{p,\upsilon}(f)(q) = \mathrm{d}^*\big(&(\tfrac{\partial}{\partial\alpha}G^t_{p,\upsilon})(q, \mathrm{d}f|_q, \nabla\mathrm{d}f|_q)\big) \\ &+ \mathrm{d}^*\big(\nabla^*\big((\tfrac{\partial}{\partial\beta}G^t_{p,\upsilon})(q, \mathrm{d}f|_q, \nabla\mathrm{d}f|_q)\big)\big). \end{aligned} \tag{25}$$

Here we consider $G^t_{p,\upsilon}(q,\alpha,\beta)$ as a function of $q \in L$ and $\alpha \in T^*_qL$ and $\beta \in S^2T^*_qL$, so that $\frac{\partial}{\partial\alpha}G^t_{p,\upsilon}$, $\frac{\partial}{\partial\beta}G^t_{p,\upsilon}$ are the partial derivatives of $G^t_{p,\upsilon}$ in the α, β directions. Having defined these partial derivatives, we then set $\alpha = \mathrm{d}f|_q$, $\beta = \nabla\mathrm{d}f|_q$ and regard $(\frac{\partial}{\partial\alpha}G^t_{p,\upsilon})(q, \mathrm{d}f|_q, \nabla\mathrm{d}f|_q)$, $(\frac{\partial}{\partial\beta}G^t_{p,\upsilon})(q, \mathrm{d}f|_q, \nabla\mathrm{d}f|_q)$ as tensor fields on L depending on $q \in L$, and we apply d^* and $\mathrm{d}^* \circ \nabla^*$ to them to get functions on L. The first term on the right hand side of (25) involves three derivatives of f, and the second term four derivatives. Hence $P^t_{p,\upsilon}$ is a fourth-order nonlinear partial operator, which is in fact quasilinear and elliptic at f for all $t \in (0, \frac{1}{2}R^{-1}\epsilon]$ and $f \in C^\infty(L)$ with $\|\mathrm{d}f\|_{C^0} < \delta$.

As for (18), define

$$F_0 : \big\{f \in C^\infty(L) : \|\mathrm{d}f\|_{C^0} < \delta\big\} \to \mathbb{R} \quad \text{by} \quad F_0(f) = \mathrm{Vol}_{g_0}\big(\Phi(\Gamma_{\mathrm{d}f})\big). \tag{26}$$

The proof of (22)–(23) shows that we may write

$$F_0(f)=\int_{q\in L} G_0\bigl(q,\mathrm{d}f|_q,\nabla\mathrm{d}f|_q\bigr)\,\mathrm{d}V_{g_0|_L}, \tag{27}$$

where G_0 is a smooth nonlinear map

$$G_0:\bigl\{(q,\alpha,\beta):q\in L,\ \alpha\in T_q^*L,\ |\alpha|<\delta,\ \beta\in S^2T_q^*L\bigr\}\longrightarrow\mathbb{R}. \tag{28}$$

As for (24)–(25), define

$$\begin{aligned}&P_0:\bigl\{f\in C^\infty(L):\|\mathrm{d}f\|_{C^0}<\delta\bigr\}\longrightarrow C^\infty(L)\quad\text{by}\\ &P_0(f)(q)=\mathrm{d}^*\bigl((\tfrac{\partial}{\partial\alpha}G_0)(q,\mathrm{d}f|_q,\nabla\mathrm{d}f|_q)\bigr)\\ &\qquad\qquad+\mathrm{d}^*\bigl(\nabla^*\bigl((\tfrac{\partial}{\partial\beta}G_0)(q,\mathrm{d}f|_q,\nabla\mathrm{d}f|_q)\bigr)\bigr).\end{aligned} \tag{29}$$

Then (20) says that $F^t_{p,\upsilon}(f)\approx F_0(f)$ for small t. A similar proof shows that $G^t_{p,\upsilon}(q,\alpha,\beta)\approx G_0(q,\alpha,\beta)$ and $P^t_{p,\upsilon}(f)\approx P_0(f)$ for small t. In fact, the difference between $P^t_{p,\upsilon}$ and P_0 depends only on the difference between $g^t_{p,\upsilon}$ and g_0, and on finitely many derivatives of this. Therefore, from Proposition 3.4 we deduce:

Proposition 4.1. *Let any $k\geqslant 0$, $\gamma\in(0,1)$, and small $C>0$ and $\zeta>0$ be given. Then if $t>0$ is sufficiently small, for all $f\in C^{k+4,\gamma}(L)$ with $\|\mathrm{d}f\|_{C^0}\leqslant\frac12\delta$ and $\|\nabla\mathrm{d}f\|_{C^0}\leqslant C$, and all $(p,\upsilon)\in U$ we have*

$$\bigl\|P^t_{p,\upsilon}(f)-P_0(f)\bigr\|_{C^{k,\gamma}}\leqslant\zeta\quad\text{and}\quad\bigl\|\mathcal{L}^t_{p,\upsilon}(f)-\mathcal{L}(f)\bigr\|_{C^{k,\gamma}}\leqslant\zeta\|f\|_{C^{k+4,\gamma}}, \tag{30}$$

where $\mathcal{L}^t_{p,\upsilon}$ denotes the linearization of $P^t_{p,\upsilon}$ at 0. That is, by taking t small we can suppose $P^t_{p,\upsilon}$ and its linearization at 0 are arbitrarily close to P_0 and its linearization at 0 as operators $C^{k+4,\gamma}(L)\to C^{k,\gamma}(L)$ (on their respective domains) uniformly in $(p,\upsilon)\in U$.

We impose the conditions $\|\mathrm{d}f\|_{C^0}\leqslant\frac12\delta$ and $\|\nabla\mathrm{d}f\|_{C^0}\leqslant C$ so that we restrict to a compact subset of the domains of $G^t_{p,\upsilon},G_0$ in (23), (28), and then we use Proposition 3.4 to bound the difference between $G^t_{p,\upsilon}$ and G_0 in $C^{k+2,\gamma}$ on this compact subset.

Finally, we note that $P_0(f)=0$ is the Euler–Lagrange equation for stationary points of the functional $F_0(f)$. Thus, $P_0(f)=0$ if and only if $\Phi(\Gamma_{\mathrm{d}f})$ is Hamiltonian stationary in $\mathbb{C}^n$. But when $f=0$, $\Phi(\Gamma_0)=L$ which is Hamiltonian stationary in $\mathbb{C}^n$, by assumption. Hence $P_0(0)=0$. Also, as in §2.2, the linearization of P_0 at $f=0$ is $\mathcal{L}$ in (9).

5 Solving the family of p.d.e.s mod $\operatorname{Ker}\mathcal{L}$

Our ultimate goal is to show that there exists $f^t_{p,\upsilon}\in C^\infty(L)$ with $P^t_{p,\upsilon}(f^t_{p,\upsilon})=0$ for all small $t>0$ and some $(p,\upsilon)\in U$ depending on t. As an intermediate step we will show that we can solve the equation $P^t_{p,\upsilon}(f^t_{p,\upsilon})=0$ mod $\operatorname{Ker}\mathcal{L}$ for all small $t>0$ and all $(p,\upsilon)\in U$, and the solution is unique provided $f^t_{p,\upsilon}$ is orthogonal to $\operatorname{Ker}\mathcal{L}$ and small in $C^{4,\gamma}$.

Theorem 5.1. *In the situation of §4, suppose $0 < t \leqslant \frac{1}{2}R^{-1}\epsilon$ is sufficiently small and fixed. Then for all $(p, \upsilon) \in U$, there exists $f^t_{p,\upsilon} \in C^\infty(L)$ satisfying*

$$P^t_{p,\upsilon}(f^t_{p,\upsilon}) \in \operatorname{Ker}\mathcal{L} \quad \textit{and} \quad f^t_{p,\upsilon} \perp \operatorname{Ker}\mathcal{L}, \tag{31}$$

where $f^t_{p,\upsilon} \perp \operatorname{Ker}\mathcal{L}$ means $f^t_{p,\upsilon}$ is L^2-orthogonal to $\operatorname{Ker}\mathcal{L}$. Furthermore $f^t_{p,\upsilon}$ is the unique solution of (31) *with $\|f^t_{p,\upsilon}\|_{C^{4,\gamma}}$ small, and $f^t_{p,\upsilon}$ depends smoothly on $(p, \upsilon) \in U$.*

Proof. Let X_1 denote the Banach space of functions $f \in C^{4,\gamma}(L)$ which are orthogonal to $\operatorname{Ker}\mathcal{L}$, and let X_2 denote the Banach subspace of $C^{0,\gamma}(L)$ consisting of functions which are orthogonal to $\operatorname{Ker}\mathcal{L}$. The starting point of the proof is the observation that the operator $\mathcal{L}$ is a bounded linear isomorphism from X_1 to X_2 with bounded inverse. This follows directly from the self-adjointness and ellipticity of $\mathcal{L}$. We let Π denote orthogonal projection from $L^2(L)$ to the orthogonal complement of $\operatorname{Ker}\mathcal{L}$. We will show that for t sufficiently small and for all $(p, \upsilon) \in U$ there is a unique small solution $f^t_{p,\upsilon} \in X_1$ of $\Pi \circ P^t_{p,\upsilon}(f^t_{p,\upsilon}) = 0$, which depends smoothly on (t, p, υ).

The first step is to show that there are $t_0, r_0 > 0$ sufficiently small so that for all $t \in (0, t_0)$ and for all $(p, \upsilon) \in U$ there is a unique solution $f^t_{p,\upsilon} \in B_{r_0}(0)$ of $\Pi \circ P^t_{p,\upsilon}(f^t_{p,\upsilon}) = 0$. To accomplish this we consider the smooth map $F = \Pi \circ P^t_{p,\upsilon}$ from a neighbourhood of the origin in X_1 to X_2. The derivative of F at 0 is $\Pi \circ \mathcal{L}^t_{p,\upsilon}$, and by Proposition 4.1 this is close in the operator norm to $\mathcal{L}$ for t sufficiently small, and therefore is a linear isomorphism with bounded inverse (note that $\Pi \circ \mathcal{L} = \mathcal{L}$). The standard contraction mapping argument for proving the Inverse Function Theorem implies that the map F is a diffeomorphism from a ball of radius r_0 centered at 0 in X_1 onto a domain containing the ball of radius λr_0 about $F(0)$, where $\lambda = (2\|F'(0)^{-1}\|)^{-1}$, and the radius r_0 can be estimated below in terms of the norm of $F'(0)^{-1}$ and the modulus of continuity of F'. This result may be found in [4, §VI.1]. Thus by the first inequality of (30) with $f = 0$ (note that $P_0(0) = 0$), there is a t_0 sufficiently small so that 0 lies in the ball of radius λr_0 centered at $F(0)$ for $t \in (0, t_0)$. This gives us a unique small solution $f^t_{p,\upsilon}$ of $\Pi \circ P^t_{p,\upsilon}(f^t_{p,\upsilon}) = 0$, as claimed.

The next step is to show that the solutions $f^t_{p,\upsilon}$ depend smoothly on the parameters (t, p, υ). We will do this by using the Implicit Function Theorem (see [4, §VI.2]). Precisely, we consider the smooth map G from $(0, t_0) \times U \times B_{r_0}(0)$ to X_2 given by $G(t, p, \upsilon, f) = \Pi \circ P^t_{p,\upsilon}(f)$. We need to analyze the set $G(t, p, \upsilon, f) = 0$, and we observe that the derivative in the f variable is a linear isomorphism with bounded inverse. Thus the Implicit Function Theorem implies that this zero set is a smooth graph $(t, p, \upsilon) \to f^t_{p,\upsilon}$ in a neighbourhood of any chosen point $(s, q, \upsilon_1, f^s_{q,\upsilon_1})$ of the zero set.

Finally it follows from elliptic regularity theory that the solutions $f^t_{p,\upsilon}$ are actually in $C^\infty(L)$. This is because they are $C^{4,\gamma}$ solutions of the quasilinear elliptic equation $P^t_{p,\upsilon}(f) = k$, where $k \in \operatorname{Ker}\mathcal{L}$ is a smooth function, and we may improve the regularity by using linear elliptic estimates in a standard way. This completes the proof. □

Note that nothing in §3–§5 uses the assumption that L is Hamiltonian rigid, only that it is Hamiltonian stationary. So Theorem 5.1 holds for general Hamiltonian stationary L in $\mathbb{C}^n$. We will use Hamiltonian rigidity in §6.

6 Completing the proof of Theorem A

We work in the situation of §3–§5. Let $t > 0$ be sufficiently small and fixed, and $f^t_{p,\upsilon} \in C^\infty(L)$ for $(p,\upsilon) \in U$ be as in Theorem 5.1. Define $L^t_{p,\upsilon} = L^{t,f^t_{p,\upsilon}}_{p,\upsilon}$ for $(p,\upsilon) \in U$. Define a smooth function $K^t : U \to \mathbb{R}$ by $K^t(p,\upsilon) = t^{-n}\operatorname{Vol}_g\bigl(L^t_{p,\upsilon}\bigr)$. Define a smooth map $H^t : U \to \operatorname{Ker}\mathcal{L}$ by $H^t : (p,\upsilon) \mapsto P^t_{p,\upsilon}(f^t_{p,\upsilon})$. We will show that we can express H^t in terms of the exact 1-form $\mathrm{d}K^t$ on U.

Recall that G is the Lie subgroup of $\mathrm{U}(n)$ preserving L, and that U is a principal $\mathrm{U}(n)$-bundle over M, so that $\mathrm{U}(n)$ and hence G act on U. Also the operator $\mathcal{L}$ of (9) is equivariant under the action of G on L, since G preserves all the geometric data used to define $\mathcal{L}$, so the action of G on $C^\infty(L)$ restricts to an action of G on $\operatorname{Ker}\mathcal{L}$.

Lemma 6.1. *For all $(p,\upsilon) \in U$ and $\gamma \in G$ we have $f^t_{p,\upsilon\circ\gamma} \equiv f^t_{p,\upsilon} \circ \gamma$ as maps $L \to \mathbb{R}$. The function $K^t : U \to \mathbb{R}$ is G-invariant, and the function $H^t : U \to \operatorname{Ker}\mathcal{L}$ is G-equivariant, under the natural actions of G on U and $\operatorname{Ker}\mathcal{L}$.*

Proof. Let $(p,\upsilon) \in U$, $\gamma \in G$ and $f \in C^\infty(L)$. Then

$$\begin{aligned} F^t_{p,\upsilon}(f\circ\gamma^{-1}) &= t^{-n}\operatorname{Vol}_g\bigl(L^{t,f\circ\gamma^{-1}}_{p,\upsilon}\bigr) = t^{-n}\operatorname{Vol}_g\bigl(\Upsilon_{p,\upsilon}\circ t\circ\Phi(\Gamma_{\mathrm{d}(f\circ\gamma^{-1})})\bigr)\\ &= t^{-n}\operatorname{Vol}_g\bigl(\Upsilon_{p,\upsilon}\circ t\circ\gamma\circ\Phi(\Gamma_{\mathrm{d}f})\bigr) = t^{-n}\operatorname{Vol}_g\bigl(\Upsilon_{p,\upsilon}\circ\gamma\circ t\circ\Phi(\Gamma_{\mathrm{d}f})\bigr)\\ &= t^{-n}\operatorname{Vol}_g\bigl(\Upsilon_{p,\upsilon\circ\gamma}\circ t\circ\Phi(\Gamma_{\mathrm{d}f})\bigr) = t^{-n}\operatorname{Vol}_g\bigl(L^{t,f}_{p,\upsilon\circ\gamma}\bigr) = F^t_{p,\upsilon\circ\gamma}(f), \end{aligned} \tag{32}$$

using (18) in the first and seventh steps, the definition of $L^{t,f}_{p,\upsilon}$ in the second and sixth, G-equivariance of Φ in the third, that γ and the dilation t commute in the fourth, and Proposition 3.2(ii) in the fifth. Since $P^t_{p,\upsilon}$ is the Euler–Lagrange variation of $F^t_{p,\upsilon}$, we deduce that

$$P^t_{p,\upsilon}(f\circ\gamma^{-1}) = \bigl(P^t_{p,\upsilon\circ\gamma}(f)\bigr)\circ\gamma^{-1}. \tag{33}$$

Applying (33) to $f = f^t_{p,\upsilon\circ\gamma}$, since $P^t_{p,\upsilon\circ\gamma}(f^t_{p,\upsilon\circ\gamma}) \in \operatorname{Ker}\mathcal{L}$ which is G-invariant, we see that $P^t_{p,\upsilon}(f^t_{p,\upsilon\circ\gamma}\circ\gamma^{-1}) \in \operatorname{Ker}\mathcal{L}$. Also $f^t_{p,\upsilon\circ\gamma} \perp \operatorname{Ker}\mathcal{L}$, so $f^t_{p,\upsilon\circ\gamma}\circ\gamma^{-1} \perp \operatorname{Ker}\mathcal{L}$, by G-invariance of $\operatorname{Ker}\mathcal{L}$ and the L^2-inner product. Hence both $f^t_{p,\upsilon}$ and $f^t_{p,\upsilon\circ\gamma}\circ\gamma^{-1}$ satisfy (31), so by uniqueness in Theorem 5.1 we have $f^t_{p,\upsilon} = f^t_{p,\upsilon\circ\gamma}\circ\gamma^{-1}$, which proves $f^t_{p,\upsilon\circ\gamma} \equiv f^t_{p,\upsilon}\circ\gamma$ as we want.

Substituting $f = f^t_{p,\upsilon\circ\gamma}$ into (32) and using $f^t_{p,\upsilon} = f^t_{p,\upsilon\circ\gamma}\circ\gamma^{-1}$ now gives

$$K^t(p,\upsilon) = t^{-n}\operatorname{Vol}_g\bigl(L^{t,f^t_{p,\upsilon}}_{p,\upsilon}\bigr) = F^t_{p,\upsilon}(f^t_{p,\upsilon}) = F^t_{p,\upsilon\circ\gamma}(f^t_{p,\upsilon\circ\gamma}) = K^t(p,\upsilon\circ\gamma),$$

so K^t is G-invariant. Equivariance of H^t follows from $f^t_{p,\upsilon\circ\gamma} \equiv f^t_{p,\upsilon}\circ\gamma$ and equation (33). □

Let $(p,\upsilon)\in U$, and consider the tangent space $T_{(p,\upsilon)}U$. Now U is a principal $\mathrm{U}(n)$-bundle over M, and the metric g induces a natural connection on this principal bundle, so we have a splitting $T_{(p,\upsilon)}U=V_p\oplus H_p$, where V_p,H_p are the vertical and horizontal subspaces. The projection $\pi:U\to M$ induces $\mathrm{d}\pi|_p:T_{(p,\upsilon)}U\to T_pM$ which has kernel V_p and induces an isomorphism $H_p\to T_pM$. Also V_p is the tangent space to the fibre of π over p, which is a free orbit of $\mathrm{U}(n)$. Thus the $\mathrm{U}(n)$-action induces an isomorphism $\mathfrak{u}(n)\cong V_p$. But υ is an isomorphism $\mathbb{C}^n\to T_pM$, so we have isomorphisms $H_p\cong T_pM\cong\mathbb{C}^n$. Putting these together gives a natural isomorphism $T_{(p,\upsilon)}U\cong\mathfrak{u}(n)\oplus\mathbb{C}^n$, where $\mathfrak{u}(n)\oplus\mathbb{C}^n$ is the Lie algebra of the symmetry group $\mathrm{U}(n)\ltimes\mathbb{C}^n$ of $\mathbb{C}^n$.

The 1-form $\mathrm{d}K^t$ on U lies in $T^*_{(p,\upsilon)}U\cong(\mathfrak{u}(n)\oplus\mathbb{C}^n)^*$ at $(p,\upsilon)\in U$. As K^t is G-invariant, $\mathrm{d}K^t$ contracts to zero with the vector fields of the Lie algebra $\mathfrak{g}$ of G. Hence under the identification $T^*_{(p,\upsilon)}U\cong(\mathfrak{u}(n)\oplus\mathbb{C}^n)^*$, $\mathrm{d}K^t|_{(p,\upsilon)}$ lies in the annihilator $\mathfrak{g}^\circ$ of $\mathfrak{g}$ in $(\mathfrak{u}(n)\oplus\mathbb{C}^n)^*$. Thus, we can regard $\mathrm{d}K^t$ as a smooth function $U\to\mathfrak{g}^\circ$.

Now consider the function $H^t:U\to\operatorname{Ker}\mathcal{L}$ defined by $H^t:(p,\upsilon)\mapsto P^t_{p,\upsilon}(f^t_{p,\upsilon})$. From (25) we see that $P^t_{p,\upsilon}(f^t_{p,\upsilon})$ is of the form $\mathrm{d}^*\eta$ for some 1-form η on L. Hence $\int_L P^t_{p,\upsilon}(f^t_{p,\upsilon})\,\mathrm{d}V_{g_0|_L}=\langle 1,\mathrm{d}^*\eta\rangle_{L^2}=\langle \mathrm{d}1,\eta\rangle_{L^2}=0$. Therefore H^t maps U to the subspace $\bigl\{f\in\operatorname{Ker}\mathcal{L}:\int_L f\,\mathrm{d}V_{g_0|_L}=0\bigr\}$ in $\operatorname{Ker}\mathcal{L}$. Since $\operatorname{Ker}\mathcal{L}$ contains the constants, this subspace has codimension 1 in $\operatorname{Ker}\mathcal{L}$.

We now for the first time use the assumption that L is Hamiltonian rigid. By definition, equality holds in (11), so Lemma 2.8 implies that $\dim\operatorname{Ker}\mathcal{L}=n^2+2n+1-\dim G$. Therefore

$$\begin{aligned}\dim\bigl\{f\in\operatorname{Ker}\mathcal{L}:\textstyle\int_L f\,\mathrm{d}V_{g_0|_L}=0\bigr\}&=n^2+2n-\dim G\\&=\dim(\mathfrak{u}(n)\oplus\mathbb{C}^n)-\dim\mathfrak{g}=\dim\mathfrak{g}^\circ.\end{aligned}\tag{34}$$

We will construct an isomorphism $\psi^t:\bigl\{f\in\operatorname{Ker}\mathcal{L}:\int_L f\,\mathrm{d}V_{g_0|_L}=0\bigr\}\to\mathfrak{g}^\circ$, which explains (34). Define a linear map $\xi^t:\mathfrak{u}(n)\oplus\mathbb{C}^n\to\bigl\{f\in\operatorname{Ker}\mathcal{L}:\int_L f\,\mathrm{d}V_{g_0|_L}=0\bigr\}$ by $\xi^t:x\mapsto\mu_x\circ t|_L$, where t acts by dilations on $\mathbb{C}^n$, and $\mu_x:\mathbb{C}^n\to\mathbb{R}$ is the unique moment map for the vector field v_x associated with $x\in\mathfrak{u}(n)\oplus\mathbb{C}^n$ with $\int_L(\mu_x\circ t)\,\mathrm{d}V_{g_0|_L}=0$. Since $G\subset\mathrm{U}(n)$, it commutes with the dilation t, and as G is the subgroup of $\mathrm{U}(n)\ltimes\mathbb{C}^n$ fixing L, it is also the subgroup of $\mathrm{U}(n)\ltimes\mathbb{C}^n$ fixing tL. Hence $\operatorname{Ker}\xi^t$ is the Lie algebra $\mathfrak{g}$ of G. As L is Hamiltonian rigid, ξ^t is surjective. Using the L^2-inner product to identify $\bigl\{f\in\operatorname{Ker}\mathcal{L}:\int_L f\,\mathrm{d}V_{g_0|_L}=0\bigr\}$ with its dual, define $\psi^t:\bigl\{f\in\operatorname{Ker}\mathcal{L}:\int_L f\,\mathrm{d}V_{g_0|_L}=0\bigr\}\to(\mathfrak{u}(n)\oplus\mathbb{C}^n)^*$ to be the dual map of ξ^t. Then ψ^t maps to $\mathfrak{g}^\circ$, as $\operatorname{Ker}\xi^t=\mathfrak{g}$, and is injective, as ξ^t is surjective. Equation (34) thus implies that ψ^t is an isomorphism.

Proposition 6.2. *Regard* $\mathrm{d}K^t$ *as a smooth function* $U\to\mathfrak{g}^\circ$, *and* H^t *as a smooth function* $U\to\bigl\{f\in\operatorname{Ker}\mathcal{L}:\int_L f\,\mathrm{d}V_{g_0|_L}=0\bigr\}$, *as above. Then for small* t *and all* $(p,\upsilon)\in U$ *there is an isomorphism* $\Psi^t_{p,\upsilon}:\bigl\{f\in\operatorname{Ker}\mathcal{L}:\int_L f\,\mathrm{d}V_{g_0|_L}=0\bigr\}\to\mathfrak{g}^\circ$ *such that* $\mathrm{d}K^t|_{(p,\upsilon)}=\Psi^t_{p,\upsilon}\circ H^t(p,\upsilon)$. *Furthermore* $\Psi^t_{p,\upsilon}$ *approximates* $\psi^t:\bigl\{f\in\operatorname{Ker}\mathcal{L}:\int_L f\,\mathrm{d}V_{g_0|_L}=0\bigr\}\to\mathfrak{g}^\circ$ *above, and depends smoothly on* p,υ.

Proof. For all $(p,\upsilon)\in U$, define $\iota^t_{p,\upsilon}:L\to M$ by $\iota^t_{p,\upsilon}(q)=\Upsilon_{p,\upsilon}\circ t\circ\Phi(q,\mathrm{d}f^t_{p,\upsilon}|_q)$. Then $\iota^t_{p,\upsilon}$ is a Lagrangian embedding with $\iota^t_{p,\upsilon}(L)=L^t_{p,\upsilon}$. By construction, with t fixed, the family of Lagrangians $\iota^t_{p,\upsilon}(L)$ for (p,υ) are all Hamiltonian equivalent. Let $(p,\upsilon)\in U$ and $x\in T_{(p,\upsilon)}U$. Consider the derivative of the family of maps $\iota^t_{p',\upsilon'}:L\to M$ for $(p',\upsilon')\in U$ in direction x at $(p',\upsilon')=(p,\upsilon)$ in U. This gives $\partial_x\iota^t_{p,\upsilon}\in C^\infty\bigl((\iota^t_{p,\upsilon})^*(TM)\bigr)$, that is, $\partial_x\iota^t_{p,\upsilon}$ is a section of the vector bundle $(\iota^t_{p,\upsilon})^*(TM)\to L$.

Since the family $\iota^t_{p,\upsilon}:L\to M$ for $(p,\upsilon)\in U$ are Hamiltonian equivalent Lagrangian embeddings, $\partial_x\iota^t_{p,\upsilon}$ is a Hamiltonian variation of $\iota^t_{p,\upsilon}(L)$. Hence $(\partial_x\iota^t_{p,\upsilon}\cdot\omega)|_{\iota^t_{p,\upsilon}(L)}$ is an exact 1-form on $\iota^t_{p,\upsilon}(L)$, and $(\iota^t_{p,\upsilon})^*(\partial_x\iota^t_{p,\upsilon}\cdot\omega)$ is an exact 1-form on L. Since L is connected, as in §2.3, there is a unique function $h^t_{p,\upsilon}(x)\in C^\infty(L)$ with $\int_L h^t_{p,\upsilon}(x)\,\mathrm{d}V_{g_0|_L}=0$ such that $(\iota^t_{p,\upsilon})^*(\partial_x\iota^t_{p,\upsilon}\cdot\omega)=\mathrm{d}(h^t_{p,\upsilon}(x))$. This $h^t_{p,\upsilon}(x)$ depends linearly on $x\in T_{(p,\upsilon)}U$, so we have defined a linear map $h^t_{p,\upsilon}:T_{(p,\upsilon)}U\to C^\infty(L)$.

By construction, if $\gamma\in G$ then $\iota^t_{p,\upsilon\circ\gamma}(L)=\iota^t_{p,\upsilon}(L)$ as submanifolds of M, although the actual parametrizations $\iota^t_{p,\upsilon\circ\gamma},\iota^t_{p,\upsilon}$ may differ. It follows under the identification $T_{(p,\upsilon)}U\cong\mathfrak{u}(n)\oplus\mathbb{C}^n$, if x lies in the Lie subalgebra $\mathfrak{g}$ of $\mathfrak{u}(n)\oplus\mathbb{C}^n$, then $h^t_{p,\upsilon}(x)\equiv 0$, since $\partial_x\iota^t_{p,\upsilon}$ is an infinitesimal reparametrization of a fixed Lagrangian $\iota^t_{p,\upsilon}(L)$ in M.

We have

$$\begin{aligned}(\mathrm{d}K^t|_{(p,\upsilon)})\cdot x&=\partial_x\bigl[t^{-n}\operatorname{Vol}_g\bigl(\Upsilon_{p,\upsilon}\circ t\circ\Phi(\Gamma_{\mathrm{d}f^t_{p,\upsilon}})\bigr)\bigr]\\&=\bigl\langle h^t_{p,\upsilon}(x),P^t_{p,\upsilon}(f^t_{p,\upsilon})\bigr\rangle_{L^2},\end{aligned}\tag{35}$$

by definition of K^t, and using the fact that $P^t_{p,\upsilon}(f)$ is the Euler–Lagrange variation of $f\mapsto t^{-n}\operatorname{Vol}_g\bigl(\Upsilon_{p,\upsilon}\circ t\circ\Phi(\Gamma_{\mathrm{d}f^t_{p,\upsilon}})\bigr)$, so that for $h\in C^\infty(L)$ we have

$$\tfrac{\mathrm{d}}{\mathrm{d}s}\bigl[t^{-n}\operatorname{Vol}_g\bigl(\Upsilon_{p,\upsilon}\circ t\circ\Phi(\Gamma_{\mathrm{d}(f+sh)})\bigr)\bigr]\big|_{s=0}=\bigl\langle h,P^t_{p,\upsilon}(f)\bigr\rangle_{L^2}.$$

Define a linear map $\Psi^t_{p,\upsilon}:\bigl\{f\in\operatorname{Ker}\mathcal{L}:\int_L f\,\mathrm{d}V_{g_0|_L}=0\bigr\}\to\bigl(\mathfrak{u}(n)\oplus\mathbb{C}^n\bigr)^*$ by

$$\Psi^t_{p,\upsilon}(f):x\longmapsto\bigl\langle h^t_{p,\upsilon}(x),f\bigr\rangle_{L^2}\tag{36}$$

for $f\in\operatorname{Ker}\mathcal{L}$ with $\int_L f\,\mathrm{d}V_{g_0|_L}=0$ and $x\in\mathfrak{u}(n)\oplus\mathbb{C}^n$, using the identification $T_{(p,\upsilon)}U\cong\mathfrak{u}(n)\oplus\mathbb{C}^n$. From above, if $x\in\mathfrak{g}$ then $h^t_{p,\upsilon}(x)\equiv 0$, so $\Psi^t_{p,\upsilon}(f)|_{\mathfrak{g}}=0$, and $\Psi^t_{p,\upsilon}(f)\in\mathfrak{g}^\circ$. Thus $\Psi^t_{p,\upsilon}$ is a linear map $\bigl\{f\in\operatorname{Ker}\mathcal{L}:\int_L f\,\mathrm{d}V_{g_0|_L}=0\bigr\}\to\mathfrak{g}^\circ$. Equations (35)–(36) and $H^t(p,\upsilon)=P^t_{p,\upsilon}(f^t_{p,\upsilon})$ imply that $\mathrm{d}K^t|_{(p,\upsilon)}=\Psi^t_{p,\upsilon}\circ H^t(p,\upsilon)$, as we want. Clearly $\Psi^t_{p,\upsilon}$ depends smoothly on p,υ.

It remains to show that $\Psi^t_{p,\upsilon}$ is an isomorphism, with $\Psi^t_{p,\upsilon}\approx\psi^t$. We claim that for small t and all $(p,\upsilon)\in U$ we have

$$h^t_{p,\upsilon}(x)\approx t^2\cdot\partial_x\bigl[f^t_{p,\upsilon}\bigr]+\xi^t(x)\tag{37}$$

in $C^\infty(L)$, where $\partial_x\bigl[f^t_{p,\upsilon}\bigr]$ is the derivative of the function $(p,\upsilon)\mapsto f^t_{p,\upsilon}$ in direction $x\in T_{(p,\upsilon)}U$ at $(p,\upsilon)\in U$. To see this, note that $h^t_{p,\upsilon}(x)$ measures the variation of the family of Hamiltonian equivalent Lagrangians $(p,\upsilon)\mapsto\Upsilon_{p,\upsilon}\circ$

$t\circ\Phi(\Gamma_{\mathrm{d}f^t_{p,\upsilon}})$ in direction x in $T_{(p,\upsilon)}U$. By the chain and product rules, we can write this variation as the sum of two contributions: (a) that due to varying $f^t_{p,\upsilon}$ as a function of (p,υ) in direction x, with $\Upsilon_{p,\upsilon}$ fixed; and (b) that due to varying $\Upsilon_{p,\upsilon}$ as a function of (p,υ) in direction x, with $f^t_{p,\upsilon}$ fixed.

The contributions of type (a) are $t^2\cdot\partial_x\bigl[f^t_{p,\upsilon}\bigr]$. This is because if we fix $\Upsilon_{p,\upsilon}$ and vary $f^t_{p,\upsilon}$ in direction x in the Lagrangian $\Upsilon_{p,\upsilon}\circ t\circ\Phi(\Gamma_{\mathrm{d}f^t_{p,\upsilon}})$, then $\Gamma_{\mathrm{d}f^t_{p,\upsilon}}$ in $(T,\hat\omega)$ changes by a Hamiltonian variation from the function $\partial_x\bigl[f^t_{p,\upsilon}\bigr]$; so $\Phi(\Gamma_{\mathrm{d}f^t_{p,\upsilon}})$ in (B_R,ω_0) changes by a Hamiltonian variation from $\partial_x\bigl[f^t_{p,\upsilon}\bigr]$, as $\Phi^*(\omega_0)=\hat\omega$; so $t\circ\Phi(\Gamma_{\mathrm{d}f^t_{p,\upsilon}})$ in (B_{tR},ω_0) changes by a Hamiltonian variation from $t^2\cdot\partial_x\bigl[f^t_{p,\upsilon}\bigr]$, since $t^*(\omega_0)=t^2\cdot\omega_0$; so $\Upsilon_{p,\upsilon}\circ t\circ\Phi(\Gamma_{\mathrm{d}f^t_{p,\upsilon}})$ in (M,ω) changes by a Hamiltonian variation from $t^2\cdot\partial_x\bigl[f^t_{p,\upsilon}\bigr]$, as $\Upsilon^*_{p,\upsilon}(\omega)=\omega_0$.

To understand the contributions of type (b), consider the smooth family of embeddings $\Upsilon_{p,\upsilon}:B_\epsilon\to M$ for $(p,\upsilon)\in U$. The derivative $\partial_x\Upsilon_{p,\upsilon}$ in direction $x\in T_{(p,\upsilon)}U$ at (p,υ) in U is a section of the vector bundle $\Upsilon^*_{p,\upsilon}(TM)\to B_\epsilon$. But $\mathrm{d}\Upsilon_{p,\upsilon}:TB_\epsilon\to\Upsilon^*_{p,\upsilon}(TM)$ is an isomorphism, so $(\mathrm{d}\Upsilon_{p,\upsilon})^{-1}(\partial_x\Upsilon_{p,\upsilon})$ is a vector field on B_ϵ. Since $\Upsilon^*_{p,\upsilon}(\omega)\equiv\omega_0$ for $(p,\upsilon)\in U$, and B_ϵ is simply-connected, this is a Hamiltonian vector field on (B_ϵ,ω_0), so there exists a smooth function $N^x_{p,\upsilon}:B_\epsilon\to\mathbb{R}$, unique up to addition of constants, such that $(\mathrm{d}\Upsilon_{p,\upsilon})^{-1}(\partial_x\Upsilon_{p,\upsilon})\cdot\omega_0\equiv\mathrm{d}N^x_{p,\upsilon}$ in 1-forms on B_ϵ. Following the definitions through shows that the contributions of type (b), from varying $\Upsilon_{p,\upsilon}$ with $f^t_{p,\upsilon}$ fixed, are

$$q\longmapsto N^x_{p,\upsilon}\circ t\circ\Phi(q,\mathrm{d}f^t_{p,\upsilon}\vert_q)+c, \tag{38}$$

for $q\in L$, where $c\in\mathbb{R}$ is such that (38) integrates to zero over L.

Now Proposition 3.2(i) says that $\Upsilon_{p,\upsilon}(0)=p$ and $\mathrm{d}\Upsilon_{p,\upsilon}\vert_0=\upsilon$. It follows that near 0 in B_ϵ, the vector field $(\mathrm{d}\Upsilon_{p,\upsilon})^{-1}(\partial_x\Upsilon_{p,\upsilon})$ on B_ϵ approximates the $\mathfrak{u}(n)\oplus\mathbb{C}^n$ vector field v_x on $\mathbb{C}^n$ corresponding to x under the identification $T_{(p,\upsilon)}U\cong\mathfrak{u}(n)\oplus\mathbb{C}^n$, and thus $N^x_{p,\upsilon}\approx\mu_x$ near 0 in $\mathbb{C}^n$, where $\mu_x:\mathbb{C}^n\to\mathbb{R}$ is a moment map for v_x, and is unique up to the addition of constants. Since $t\circ\Phi(q,\mathrm{d}f^t_{p,\upsilon}\vert_q)$ lies in B_{tR}, for small t we see that (38) approximates $q\mapsto\mu_x\circ t\circ\Phi(q,\mathrm{d}f^t_{p,\upsilon}\vert_q)+c$. Also, the proof of Theorem 5.1 shows that when t is small $\Vert f^t_{p,\upsilon}\Vert_{C^{4,\gamma}}$ is small, so we can approximate $f^t_{p,\upsilon}$ by zero, and (38) approximates $\mu_x\circ t\circ\mathrm{id}_L$. The definition of ξ^t above now implies that (38) approximates $\xi^t(x)$, which proves (37).

Since $f^t_{p,\upsilon}\perp\operatorname{Ker}\mathcal{L}$ for all $(p,\upsilon)\in U$, it follows that $\partial_x\bigl[f^t_{p,\upsilon}\bigr]\perp\operatorname{Ker}\mathcal{L}$. Thus, substituting (37) into (36) and noting that $f\in\operatorname{Ker}\mathcal{L}$ shows that

$$\Psi^t_{p,\upsilon}(f)(x)\approx\bigl\langle\xi^t(x),f\bigr\rangle_{L^2}$$

for small t, for all $f\in\operatorname{Ker}\mathcal{L}$ with $\int_L f\,\mathrm{d}V_{g_0\vert_L}=0$ and $x\in\mathfrak{u}(n)\oplus\mathbb{C}^n$. Since ψ^t is the dual map to ξ^t, using the L^2 inner product to identify $\bigl\{f\in\operatorname{Ker}\mathcal{L}:\int_L f\,\mathrm{d}V_{g_0\vert_L}=0\bigr\}$ with its dual, it follows that $\Psi^t_{p,\upsilon}\approx\psi^t$ for small t. But ψ^t is an isomorphism, which is an open condition, so $\Psi^t_{p,\upsilon}$ is also an isomorphism for small t. By compactness of U, for small enough t these hold uniformly for all $(p,\upsilon)\in U$. $\square$

We can now complete the proof of Theorem A. As $\Psi^t_{p,\upsilon}$ is an isomorphism, $\mathrm{d}K^t|_{(p,\upsilon)} = \Psi^t_{p,\upsilon} \circ H^t(p,\upsilon)$ implies that $\mathrm{d}K^t|_{(p,\upsilon)} = 0$ if and only if $P^t_{p,\upsilon}(f^t_{p,\upsilon}) = H^t(p,\upsilon) = 0$. But $L^t_{p,\upsilon}$ is a Hamiltonian stationary Lagrangian in (M,ω) if and only if $P^t_{p,\upsilon}(f^t_{p,\upsilon}) = 0$. Hence for small t, $L^t_{p,\upsilon}$ is a Hamiltonian stationary Lagrangian if and only if (p,υ) is a stationary point of $K^t : U \to \mathbb{R}$. But U is a compact manifold without boundary and K^t is a smooth function, so K^t must have at least one stationary point (p,υ) in U. Then $L' = L^t_{p,\upsilon}$ satisfies the first part of Theorem A.

For the second part, suppose also that L is Hamiltonian stable. Take (p,υ) to be a local minimum of K^t on U, which must exist as U is compact. Then $L' = L^t_{p,\upsilon}$ is Hamiltonian stationary, as above. We claim that L' is also Hamiltonian stable. To see this, for $f \in C^\infty(L)$ and small $s \in \mathbb{R}$ write

$$\mathrm{Vol}_g\bigl(\Upsilon_{p,\upsilon}\circ t\circ\Phi(\Gamma_{\mathrm{d}f^t_{p,\upsilon}+s\mathrm{d}f})\bigr)=\mathrm{Vol}_g\bigl(\Upsilon_{p,\upsilon}\circ t\circ\Phi(\Gamma_{\mathrm{d}f^t_{p,\upsilon}})\bigr)+s^2Q^t_{p,\upsilon}(f)+O(|s|^3),$$

where the homogeneous quadratic form $Q^t_{p,\upsilon} : C^\infty(L) \to \mathbb{R}$ is the second variation of Vol_g at L'. We must show that $Q(f) \geqslant 0$ for all $f \in C^\infty(L)$.

Divide Hamiltonian variations of L' into two kinds: (i) those coming from functions $f \in (\mathrm{Ker}\,\mathcal{L})^\perp$, that is, $f \in C^\infty(L)$ with $f \perp \mathrm{Ker}\,\mathcal{L}$; and (ii) those coming from the family of Lagrangians $L^t_{p',\upsilon'}$ in (M,ω), for (p',υ') in U close to (p,υ). The vector space of functions $f \in C^\infty(L)$ corresponding to Hamiltonian variations of type (ii) turns out to be $W^t_{p,\upsilon} = \bigl\{h^t_{p,\upsilon}(x) : x \in T_{(p,\upsilon)}U\bigr\} \oplus \langle 1\rangle$, for $h^t_{p,\upsilon}$ as in the proof of Proposition 6.2. We have $C^\infty(L) = (\mathrm{Ker}\,\mathcal{L})^\perp \oplus W^t_{p,\upsilon}$.

For Hamiltonian variations of type (i), the second variation of the volume functional Vol_g at $L^t_{p,\upsilon}$ approximates the second variation of the volume functional Vol_{g_0} at tL in $\mathbb{C}^n$ on functions $f \in (\mathrm{Ker}\,\mathcal{L})^\perp$, or equivalently, the second variation of $t^n \cdot \mathrm{Vol}_{g_0}$ at L in $\mathbb{C}^n$ on functions $f \in (\mathrm{Ker}\,\mathcal{L})^\perp$. Using this we can show that

$$Q^t_{p,\upsilon}(f) = t^n\langle f,\mathcal{L}f\rangle_{L^2} + O\bigl(t^\gamma\|f\|^2_{L^2_2}\bigr) \quad \text{for all } f \in (\mathrm{Ker}\,\mathcal{L})^\perp, \tag{39}$$

for some $\gamma > n$. As the second variation of Vol_{g_0} at L is nonnegative, $\mathcal{L}$ is a nonnegative fourth-order linear elliptic operator on a compact manifold L. Using this we can show that there exists $\lambda > 0$ such that $\langle f,\mathcal{L}f\rangle_{L^2} \geqslant \lambda\|f\|^2_{L^2_2}$ for all $f \in (\mathrm{Ker}\,\mathcal{L})^\perp$. Thus the term $t^n\langle f,\mathcal{L}f\rangle_{L^2}$ in (39) dominates the term $O\bigl(t^\gamma\|f\|^2_{L^2_2}\bigr)$ for small t, as $\gamma > n$. Hence the second variation of Vol_g on variations of type (i) is positive definite, for small t.

For variations of type (ii), the second variation of Vol_g is the second variation of the function $(p',\upsilon') \mapsto \mathrm{Vol}_g(L^t_{p',\upsilon'}) = t^n \cdot K^t(p',\upsilon')$ at (p,υ). But (p,υ) is a local minimum of K^t at (p,υ), so this second variation is nonnegative. Therefore $Q^t_{p,\upsilon}(f) \geqslant 0$ for all $f \in W^t_{p,\upsilon}$.

A general variation f may be written uniquely as $f = f_1 + f_2$ for $f_1 \in (\mathrm{Ker}\,\mathcal{L})^\perp$ of type (i) and $f_2 \in W^t_{p,\upsilon}$ of type (ii). We claim that $Q^t_{p,\upsilon}(f) = Q^t_{p,\upsilon}(f_1) + Q^t_{p,\upsilon}(f_2)$, that is, the bilinear terms in $f_1 \otimes f_2$ in $Q^t_{p,\upsilon}(f_1 + f_2)$ are zero. This holds because the definition of the family of Lagrangians $L^t_{p',\upsilon'}$ with $P^t_{p',\upsilon'}(f^t_{p',\upsilon'}) \in \mathrm{Ker}\,\mathcal{L}$ means that the volume of $L^t_{p',\upsilon'}$ is stationary under

Hamiltonian variations coming from $f_1 \in (\operatorname{Ker}\mathcal{L})^\perp$ not just at the single point (p,υ), but for all $(p',\upsilon') \in U$. That is, if $f_1 \in (\operatorname{Ker}\mathcal{L})^\perp$ then

$$\tfrac{\mathrm{d}}{\mathrm{d}s}\bigl[\mathrm{Vol}_g\bigl(\Upsilon_{p',\upsilon'}\circ t\circ\Phi(\Gamma_{\mathrm{d}f^t_{p',\upsilon'}+s\mathrm{d}f_1})\bigr)\bigr]\big|_{s=0}=0.$$

Differentiating this identity at $(p',\upsilon')=(p,\upsilon)$ in the direction in $T_{(p,\upsilon)}U$ induced by the Hamiltonian variation $f_2 \in W^t_{p,\upsilon}$ implies that the $f_1\otimes f_2$ term in $Q^t_{p,\upsilon}(f_1+f_2)$ is zero. Therefore $Q^t_{p,\upsilon}(f) = Q^t_{p,\upsilon}(f_1) + Q^t_{p,\upsilon}(f_2) \geqslant 0$, since $Q^t_{p,\upsilon}(f_1) \geqslant 0$ and $Q^t_{p,\upsilon}(f_2) \geqslant 0$. So L' is Hamiltonian stable. This completes the proof.

7 Conclusions

We finish with a question about the family of Hamiltonian stationary Lagrangians L' in a fixed Hamiltonian isotopy class $\mathcal{HI}$ in a compact symplectic manifold (M,ω). Although in §4–§6 we dealt with a family of Lagrangians $L^t_{p,\upsilon}$ parametrized by $(p,\upsilon) \in U$, as the whole construction is G-equivariant we can think of this family of Lagrangians as parametrized by $(p,\upsilon)G$ in U/G. The G-invariant function $K^t : U \to \mathbb{R}$ descends to $K^t_G : U/G \to \mathbb{R}$. In §6 we proved that $L^t_{p,\upsilon}$ is a Hamiltonian stationary Lagrangian if and only if $(p,\upsilon)G$ is a stationary point of K^t_G on the compact manifold U/G.

Suppose the only Hamiltonian stationary Lagrangians L' in this Hamiltonian isotopy class $\mathcal{HI}$ are of the form $L^t_{p,\upsilon}$. Then we would have identified the family of Hamiltonian stationary Lagrangians L' in $\mathcal{HI}$, which is the critical locus of a real function on an infinite-dimensional, noncompact manifold $\mathcal{HI}$, with the critical locus of a real function on a finite-dimensional, compact manifold U/G.

This is suggestive. There are several important areas in geometry, dealing either with counting invariants such as Donaldson, Gromov–Witten, or Donaldson–Thomas invariants, or with Floer homology theories, for which the original motivation comes from considering some infinite-dimensional, noncompact moduli space $\mathcal{M}$ of connections or submanifolds, and then treating $\mathcal{M}$ as if it were a finite-dimensional compact manifold.

If Y is a compact manifold and $f : Y \to \mathbb{R}$ is a Morse function, then the number of critical points of f, counted with signs, is $\chi(Y)$, and using the gradient flow lines of f between critical points one can construct the (Morse) homology $H_*(Y;\mathbb{R})$. The invariants and homology theories mentioned above work by counting critical points or gradient flow lines of a functional $F : \mathcal{M} \to \mathbb{R}$ on an infinite-dimensional, noncompact manifold $\mathcal{M}$; the answers turn out to be independent of most of the geometric choices in the definition of F, even though $\mathcal{M}$ is neither finite-dimensional nor compact. This motivates the following:

Question 7.1. *Let (M,ω) be a compact symplectic manifold, g a Riemannian metric on M compatible with ω, and $\mathcal{HI}$ a Hamiltonian isotopy class of compact Lagrangians L in (M,ω). Write $\mathrm{Vol}_g : \mathcal{HI} \to \mathbb{R}$ for the volume functional.*

Can one define some invariant $I(\mathcal{HI}) \in \mathbb{Z}$ which 'counts' (with multiplicity and sign) Hamiltonian stationary Lagrangians L in $\mathcal{HI}$, that is, stationary points of Vol_g, and gives an answer independent of the choice of g?

Can one define some kind of Floer homology theory $HF_*(\mathcal{HI})$ by studying the gradient flow of Vol_g *between critical points, whose Euler characteristic is* $I(\mathcal{HI})$*, and which is independent of* g *up to canonical isomorphism?*

In the case of Theorem A, since the family of Hamiltonian stationary Lagrangians L' we have constructed corresponds to the critical points of a function $K^t : U/G \to \mathbb{R}$, we would expect the answers $I(\mathcal{HI}) = \chi(U/G)$ and $HF_*(\mathcal{HI}) \cong H_*(U/G;\mathbb{R})$. Since U/G is a fibre bundle over M with fibre $\mathrm{U}(n)/G$ we have $\chi(U/G) = \chi\bigl(\mathrm{U}(n)/G\bigr)\chi(M)$. If L is $T^n_{a_1,\ldots,a_n}$ in (13) with $a_1,\ldots,a_n > 0$ distinct, then G is the maximal torus T^n in $\mathrm{U}(n)$, and $\chi\bigl(\mathrm{U}(n)/G\bigr) = n!$, so we expect $I(\mathcal{HI}) = n!\,\chi(M)$.

The Mathematical Institute, 24-29 St. Giles, Oxford, OX1 3LB, U.K.

E-mail: joyce@maths.ox.ac.uk

Department of Mathematics and Taida Institute of Mathematical Sciences, National Taiwan University, Taipei 10617, Taiwan

E-mail: yilee@math.ntu.edu.tw

Department of Mathematics, Stanford University, Stanford, CA 94305-2125, U.S.A.

E-mail: schoen@math.stanford.edu

References

[1] A. Amarzaya and Y. Ohnita, *'Hamiltonian stability of parallel Lagrangian submanifolds in complex space forms'*, preprint.

[2] A. Butscher and J. Corvino, *'Hamiltonian stationary tori in Kähler manifolds'*, arXiv:0811.2829, 2008, revised 2010.

[3] D. Joyce, *'Special Lagrangian submanifolds with isolated conical singularities. I. Regularity'*, Ann. Global Anal. Geom. 25 (2004), 201–251. math.DG/0211294.

[4] S. Lang, *'Analysis II'*, Addison-Wesley, 1969.

[5] Y.-I. Lee, *'The existence of Hamiltonian stationary Lagrangian tori in Kähler manifolds of any dimension'*, arXiv:1001.3861, 2010.

[6] Y.-I. Lee and M.-T. Wang, *'Hamiltonian stationary cones and self-similar solutions in higher dimension'*, Trans. A.M.S. 362 (2010), 1491–1503.

[7] D. McDuff and D. Salamon, *'Introduction to Symplectic Topology'*, second edition, OUP, Oxford, 1998.

[8] J.K. Moser, *'On the volume elements on manifolds'*, Trans. A.M.S. 120 (1965), 280–296.

[9] Y.-G. Oh, *'Second variation and stabilities of minimal Lagrangian submanifolds in Kähler manifolds'*, Invent. math. 101 (1990), 501–519.

[10] Y.-G. Oh, *'Volume minimization of Lagrangian submanifolds under Hamiltonian deformation'*, Math. Z. 212 (1993), 175–192.

[11] R. Schoen and J. Wolfson, *'Minimizing area among Lagrangian surfaces: the mapping problem'*, J. Diff. Geom. 58 (2001), 1–86. math.DG/0008244.

[12] R.P. Thomas and S.-T. Yau, *'Special Lagrangians, stable bundles and mean curvature flow'*, Comm. Anal. Geom. 10 (2002), 1075–1113. math.DG/0104197.